

# Sensitivity analysis of generalised eigenproblems and application to wave and finite element models

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**Abstract.** First order sensitivity analysis of the eigenvalue problem of generalised, non-symmetric matrices using perturbation theory is presented. These results are then applied to sensitivity analysis of free wave propagation estimates (wavenumbers and wave mode shapes) using the wave and finite element (WFE) method with respect to system parameters. Three formulations of the WFE eigenvalue problem are presented: the transfer matrix method, the projection method and Zhong's method. Numerical results for a thin rod are given as an example. Sensitivities can be calculated at negligible computational cost.

## 1. Introduction

Perturbation theory can be used to evaluate the sensitivity of the eigencharacteristics of a system with respect to changes in the material or geometrical properties, without requiring the eigenproblem to be solved multiple times. In particular, this approach yields approximate eigenvalues of the perturbed system which are computed via matrix multiplications, reducing computational cost drastically. When applied to structural dynamics, the system is frequently characterized by symmetric matrices, and therefore much research effort has focused on the analysis of so-called symmetric systems which are characterized by symmetric eigenvalue problems. Solutions for first order perturbations of the eigenproblem for real symmetric matrices with respect to a parameter can be found in [1,2].

In this paper the first order derivatives of the eigensolutions with respect to the system's parameters are derived for the generalized, asymmetric eigenproblem and applied to a wave and finite element (WFE) model. In the WFE eigenproblem the matrices involved are non-symmetric and normally complex. The analysis of real, non-symmetric matrices for the case of weak and strong interaction with equal eigenvalues was considered by Seyranian [3], but the problem of distinct eigenvalues was not addressed. Recently, some results have been presented in [4].

The generalised eigenproblem and eigensolution sensitivities are discussed in section 2. Three forms of the WFE eigenvalue problem are summarised in section 3. In section 4 a numerical example is presented, and the theory is illustrated with respect to analytical results for a rod.

## 2. The generalised eigenproblem and eigensolution sensitivities

### 2.1. The generalised eigenproblem for fixed system parameters

Consider the generalized eigenvalue problem (EP) expressed as

$$\mathbf{B}(\mathbf{p})\mathbf{u} = \lambda\mathbf{C}(\mathbf{p})\mathbf{u}, \quad (1)$$

where  $\mathbf{B}(\mathbf{p})$  and  $\mathbf{C}(\mathbf{p})$  are complex, non-symmetric system matrices of dimensions  $m \times m$  and are functions of  $n$  system parameters in the  $n \times 1$  vector  $\mathbf{p}$ . Here  $\lambda$  and  $\mathbf{u}$  are the eigenvalue and right eigenvector. The left EP involves the left eigenvector  $\mathbf{z}$  and is given by

$$\mathbf{z}^T \mathbf{B}(\mathbf{p}) = \lambda \mathbf{z}^T \mathbf{C}(\mathbf{p}). \quad (2)$$

For some specific values of the parameters  $\mathbf{p} = \mathbf{p}_0$ , the two EPs can be written as

$$\mathbf{B}_0 \mathbf{u}^{(0)} = \lambda^{(0)} \mathbf{C}_0 \mathbf{u}^{(0)}; \quad \mathbf{z}^{(0)T} \mathbf{B}_0 = \lambda^{(0)} \mathbf{z}^{(0)T} \mathbf{C}_0. \quad (3)$$

Left and right eigenvectors are normalised such that

$$\mathbf{z}_i^{(0)T} \mathbf{C}_0 \mathbf{u}_j^{(0)} = \delta_{ij}. \quad (4)$$

## 2.2. Sensitivity of the eigenvalues of the perturbed system

Consider a perturbation of order  $\varepsilon$  of the system parameters  $\mathbf{p}_0$  in an arbitrary vector of variation  $\mathbf{e} = (e_1, \dots, e_n)$ , such that

$$\mathbf{p} = \mathbf{p}_0 + \varepsilon \mathbf{e}; \quad \|\mathbf{e}\| = 1. \quad (5)$$

As a result, the system matrices can be expressed as

$$\mathbf{B} = \mathbf{B}_0 + \varepsilon \mathbf{B}_1 + \varepsilon^2 \mathbf{B}_2 + \dots; \quad \mathbf{C} = \mathbf{C}_0 + \varepsilon \mathbf{C}_1 + \varepsilon^2 \mathbf{C}_2 + \dots \quad (6)$$

where

$$\mathbf{B}_1 = \sum_{r=1,n} \frac{\partial \mathbf{B}_0}{\partial p_r} e_r; \quad \mathbf{C}_1 = \sum_{r=1,n} \frac{\partial \mathbf{C}_0}{\partial p_r} e_r \quad (7)$$

Consequently, the eigenvalues and eigenvectors can be expressed as

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \dots; \quad \mathbf{u} = \mathbf{w}_0 + \varepsilon \mathbf{w}_1 + \varepsilon^2 \mathbf{w}_2 + \dots \quad (8)$$

By substituting Eqs. (6) and (8) in Eq. (1), using the normality and orthogonality conditions (Eqs. 3 and 4) and keeping the first order terms, the following expressions are obtained [4]:

$$\mathbf{w}_0 = \mathbf{u}_0; \quad (9)$$

$$\lambda_1 = \mathbf{z}_0^T [\mathbf{B}_1 - \lambda_0 \mathbf{C}_1] \mathbf{u}_0; \quad \mathbf{w}_1 = -[\mathbf{B}_0 - \lambda_0 \mathbf{C}_0]^{-1} [\mathbf{B}_1 - \lambda_0 \mathbf{C}_1 - \lambda_1 \mathbf{C}_0] \mathbf{u}_0. \quad (10)$$

Eqs. (10) can be used to obtain the first-order perturbations of the eigencharacteristics of an asymmetric system by vector and matrix multiplications [4]. These results reduce to those for symmetric eigenproblems for which  $\mathbf{B}, \mathbf{C}$  are symmetric and the left and right eigenvectors are equal. The derivative of an eigenvalue with respect to a single parameter  $p$  is therefore

$$\frac{\partial \lambda}{\partial p} = \mathbf{z}_0^T [\mathbf{B}_1 - \lambda_0 \mathbf{C}_1] \mathbf{u}_0; \quad \mathbf{B}_1 = \frac{\partial \mathbf{B}_0}{\partial p}; \quad \mathbf{C}_1 = \frac{\partial \mathbf{C}_0}{\partial p}. \quad (11)$$

These results are used in the following sections to compute the eigencharacteristics of a system modelled using the WFE approach.

## 3. The WFE eigenproblem and sensitivity analysis

The WFE method for free wave propagation [7,8] involves determining the mass and stiffness matrices  $\mathbf{M}$  and  $\mathbf{K}$  of a short segment of the structure of length  $\Delta$ , forming the dynamic stiffness matrix  $\mathbf{D} = \mathbf{K} - \omega^2 \mathbf{M}$  and applying periodicity equations. Damping can be included by a viscous damping matrix  $\mathbf{C}$  or by  $\mathbf{K}$  being complex. An eigenproblem follows, the solutions yielding the eigenvalues  $\lambda = \exp(-ik\Delta)$ , with  $k$  being the wavenumbers, generally complex-valued. The WFE eigenproblem can be phrased in at least 3 different ways. The results derived in the previous section can then be applied to determine the sensitivity of the eigenvalues with respect to a parameter.

### 3.1. The transfer matrix of the segment

The degrees of freedom (DOFs)  $\mathbf{q}$  and nodal forces  $\mathbf{f}$  at the left and right ( $L$  and  $R$ ) ends of the segment are related by the transfer matrix

$$\mathbf{T} = \begin{bmatrix} -\mathbf{E}\mathbf{D}_{LL} & \mathbf{E} \\ -\mathbf{D}_{RL} + \mathbf{D}_{RR}\mathbf{E}\mathbf{D}_{LL} & -\mathbf{D}_{RR}\mathbf{E} \end{bmatrix}, \quad (12)$$

where  $\mathbf{D}_{LR}$ ,  $\mathbf{D}_{RR}$ ,  $\mathbf{D}_{LL}$  and  $\mathbf{D}_{RL}$  are the partitions of  $\mathbf{D}$  and  $\mathbf{E} = \mathbf{D}_{LR}^{-1}$ . In the notation of section 2,  $\mathbf{B} = \mathbf{T}$  and  $\mathbf{C} = \mathbf{I}$ . The matrix derivatives are

$$\mathbf{B}' = \begin{bmatrix} -\mathbf{E}'\mathbf{D}_{LL} - \mathbf{E}\mathbf{D}_{LL}' & \mathbf{E}' \\ -\mathbf{D}_{RL}' + \mathbf{D}_{RR}'\mathbf{E}\mathbf{D}_{LL} + \mathbf{D}_{RR}\mathbf{E}'\mathbf{D}_{LL} + \mathbf{D}_{RR}\mathbf{E}\mathbf{D}_{LL}' & -\mathbf{D}_{RR}'\mathbf{E} - \mathbf{D}_{RR}\mathbf{E}' \end{bmatrix}, \quad \mathbf{C}' = \mathbf{0}, \quad (13)$$

where  $\mathbf{E}' = -\mathbf{E}\mathbf{D}_{LR}'\mathbf{E}$ . While the transfer matrix approach is perhaps the simplest intuitively, it is prone to poor numerical conditioning [11] and this is exacerbated for the case of sensitivity estimation.

### 3.2. Projection of the equations of motion onto the left-hand DOFs

Projecting the equations of motion onto the left-hand DOFs  $\mathbf{q}_L$  leads to a quadratic eigenproblem that can be recast as the linear eigenproblem [7,8]

$$\left[ \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{D}_{RL} & -(\mathbf{D}_{LL} + \mathbf{D}_{RR}) \end{bmatrix} - \lambda \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{LR} \end{bmatrix} \right] \begin{Bmatrix} \mathbf{q} \\ \lambda \mathbf{q} \end{Bmatrix} = \mathbf{0}. \quad (14)$$

Hence

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{D}_{RL} & -(\mathbf{D}_{LL} + \mathbf{D}_{RR}) \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{LR} \end{bmatrix} \quad (15)$$

and the matrix derivatives follow straightforwardly.

### 3.3. Zhong's method

Zhong's method [9-10] is the most numerically robust approach. The eigenproblem becomes

$$[\mathbf{B} - \mu\mathbf{C}] \begin{Bmatrix} \mathbf{q} \\ \lambda \mathbf{q} \end{Bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{D}_{RL} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{LR} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -(\mathbf{D}_{LL} + \mathbf{D}_{RR}) & -(\mathbf{D}_{LR} - \mathbf{D}_{RL}) \\ (\mathbf{D}_{LR} - \mathbf{D}_{RL}) & -(\mathbf{D}_{LL} + \mathbf{D}_{RR}) \end{bmatrix}, \quad (16)$$

where

$$\mu = \frac{1}{(\lambda + 1/\lambda)}, \quad \frac{\partial \lambda}{\partial p} = \frac{(1 + \lambda^2)^2}{(1 - \lambda^2)} \frac{\partial \mu}{\partial p}, \quad \frac{\partial k}{\partial p} = \frac{i \exp(ik\Delta)}{\Delta} \frac{\partial \lambda}{\partial p} \quad (17)$$

## 4. Numerical example

A steel rod (Young Modulus  $Y = 210 \times 10^9$  N/m<sup>2</sup>, density  $\rho = 7850$  kg/m<sup>3</sup>) with rectangular cross-section  $b \times h = 3$  mm  $\times$  15 mm is considered. A two-noded rod element of length  $\Delta = 0.1$  m is taken. The wavenumber is given by [5]

$$k = \omega \sqrt{\frac{\rho}{Y}}, \quad (18)$$

so that

$$\frac{\partial k}{\partial \rho} = \frac{\omega}{2\sqrt{\rho Y}}, \quad \frac{\partial k}{\partial Y} = -\frac{\omega}{2Y} \sqrt{\frac{\rho}{Y}}, \quad \frac{\partial k}{\partial \Delta} = 0. \quad (19)$$

Various predictions of the sensitivity of the wavenumber with respect to the elastic modulus are shown in Figure 1. At the largest frequency shown  $k\Delta \approx 1$  and hence FE discretisation errors are significant. The numerical predictions are almost real-valued, with the imaginary part resulting from rounding errors etc. The projection method (section 3.2) and Zhong's method (section 3.3) give virtually identical

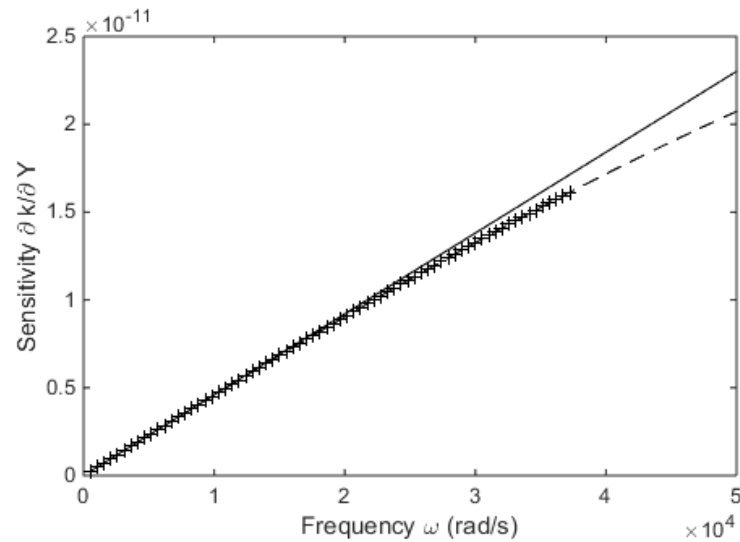


Figure 1. Sensitivity of wavenumber with respect to the Young Modulus for axial waves in a thin rod: \_\_\_\_ exact result; - - - Zhong's method and projection method; + transfer matrix method.

results, with the agreement being good but of decreasing accuracy as frequency increases due to FE discretisation effects. The transfer matrix approach, however, suffers from very poor numerical conditioning and yields accurate solutions only up to a frequency of approximately  $\omega = 3.75 \times 10^4$  rad/s, above which frequency results are not shown.

## 5. Conclusion

The perturbation approach has been applied to evaluate the first order sensitivity of the eigenvalues of complex non-symmetric matrices. These results were applied to wave and finite element models with respect to system parameters. Three different eigenformulations were presented. The case of a rod was presented as an illustrative example.

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