Geometric and probabilistic aspects of groups with hyperbolic features

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Abstract

The main objects of interest in this thesis are relatively hyperbolic groups. We will study some of their geometric properties, and we will be especially concerned with geometric properties of their boundaries, like linear connectedness, avoidability of parabolic points, etc. Exploiting such properties will allow us to construct, under suitable hypotheses, quasi-isometric embeddings of hyperbolic planes into relatively hyperbolic groups and quasi-isometric embeddings of relatively hyperbolic groups into products of trees. Both results have applications to fundamental groups of 3-manifolds.

We will also study probabilistic properties of relatively hyperbolic groups and of groups containing "hyperbolic directions" despite not being relatively hyperbolic, like mapping class groups, $Out(F_n)$, $CAT(0)$ groups and subgroups of the above. In particular, we will show that the elements that generate the "hyperbolic directions" (hyperbolic elements in relatively hyperbolic groups, pseudo-Anosovs in mapping class groups, fully irreducible elements in $Out(F_n)$ and rank one elements in $CAT(0)$ groups) are generic in the corresponding groups (provided at least one exists, in the case of $CAT(0)$ groups, or of proper subgroups). We also study how far a random path can stray from a geodesic in the context of relatively hyperbolic groups and mapping class groups, but also of groups acting on a relatively hyperbolic space. We will apply this, for example, to show properties of random triangles.

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Contents

Chapter 1

Introduction

Gromov-hyperbolicity is a central notion in Geometric Group Theory. Recall that a geodesic metric space is (Gromov-)hyperbolic if there exists δ so that any geodesic triangle is δ-thin, i.e. any side is contained in the δ-neighbourhood of the union of the other two. Such definition captures an important feature of fundamental groups of compact negatively curved Riemannian manifolds, in particular hyperbolic manifolds, whose Cayley graphs are the main examples of hyperbolic spaces.

The theory of hyperbolic groups is so developed and rich that much research has been devoted to generalising hyperbolicity and describing hyperbolic features in interesting but non-hyperbolic groups, especially in recent years.

1.1 Relatively hyperbolic groups

One of the first generalisations of hyperbolicity to be considered has been relative hyperbolicity, which will be the central notion of this thesis. While hyperbolic groups are modelled on fundamental groups of *compact* manifolds of negative curvature, relatively hyperbolic groups are modelled on fundamental groups of *finite volume* manifolds of negative curvature. Such manifolds are especially interesting in dimension three, where Thurston showed that they play a central role. For example, the complements of many knots in S^3 admit hyperbolic structures of finite volume, and more generally every, say, closed 3-manifold has a canonical decomposition, called JSJ-decomposition, where every piece is either Seifert-fibred or hyperbolic (of finite volume if the decomposition is non-trivial). Also, Dehn fillings of finite volume hyperbolic manifolds are widely studied and especially interesting.

The idea behind relative hyperbolicity is that all the non-hyperbolic behaviour of the group is confined in a given collection of subgroups, the peripheral subgroups.

Examples of relatively hyperbolic groups include:

1. Free products, or more generally free products with amalgamation over finite subgroups, are hyperbolic relative to the factors. This provides a good toy example to keep in mind.

2. As mentioned above, the main example are fundamental groups of finite volume negatively curved manifolds of pinched negative curvature. Such a manifold M has a compact core, called thick part, and finitely many *cusps*, each diffeomorphic to a product $N_i \times [0, \infty)$. For each i, $\pi_1(N_i)$ injects in $\pi_1(M)$ and $\pi_1(M)$ is hyperbolic relative to the peripheral subgroups $\pi_1(N_1), \dots, \pi_1(N_n)$ [76]. If M is real hyperbolic, then the manifolds N_i are flat, so that if the dimension of M is 3 they are either tori or Klein bottles. As evidence that such objects are abundant, we remark that among the 1.701.936 prime knots in S^3 with up to 16 crossings, the complements of all but 32 of them admit finite volume hyperbolic metrics [91]. Also, a manifold is infranil if and only if it appears as the cross-section of one of the cusps of a negatively curved manifold [135].

3. Any prime closed 3-manifold M can be either geometric, meaning that it admits a Riemannian metric whose universal cover is isometric to one of 8 possible homogeneous manifolds, or non-geometric, in which case it admits a decomposition along a family of tori and Klein bottles into pieces each of which either admits a hyperbolic metric of finite volume or is Seifert-fibred. In this case, $\pi_1(M)$ is hyperbolic relative to the fundamental groups of the manifolds obtained removing from M the interior of the hyperbolic pieces [60]. A shorter way of saying this is: The fundamental group of a non-geometric 3-manifold is hyperbolic relative to subgroups each of which is either $\mathbb{Z}^2$, the fundamental group of a Klein bottle, or the fundamental group of a special class of manifolds called graph manifolds. This fact is very useful to prove results about all fundamental groups of 3-manifolds by reducing to studying the geometric case and the case of graph manifolds. We will see instances of this approach in Part II, see also [152].

4. Hyperbolic groups are hyperbolic relative to the empty list of peripheral subgroups. However, they are also hyperbolic relative to any almost malnormal collection of quasi-convex subgroups [35]. A collection of subgroups $\{H_i\}$ is almost-malnormal if $|H_i \cap gH_jg^{-1}| = \infty$ implies $i = j$ and $g \in H_i$, while a subgroup is quasi-convex, in this context, if it is undistorted in the ambient group meaning that the inclusion is a quasi-isometric embedding. Considering

such non-trivial structures is actually sometimes useful, and indeed it is used, for example, in the proof of the Virtual Haken Conjecture [2].

5. Limit groups, a class of groups introduced by Sela in his solution of the Tarski problem [150], are hyperbolic relative to (representatives of the conjugacy classes of) maximal Abelian subgroups of rank at least 2 [60].

6. There are constructions that preserve relative hyperbolicity. One of them is taking graphs of groups with certain properties, and this construction is actually implicit in the non-geometric 3-manifolds and limit groups examples. See, e.g., [60, 4, 129].

7. A small cancellation theory over relatively hyperbolic groups has been developed, allowing once again to construct more relatively hyperbolic groups starting from given ones. For example, there is a notion of $C'(1/6)$ quotient of a free-product, and such quotients are relatively hyperbolic in a natural way [141]. Such small cancellation theory has been used to construct groups with exotic properties, like infinite groups with exactly two conjugacy classes [139].

8. Yet another construction one can perform is the so-called peripheral, or Dehn, filling [138, 85]. This is a group theoretic version of the Dehn filling in the context of hyperbolic 3-manifolds, which is an important construction in the theory of hyperbolic 3-manifolds. We will describe this construction in more detail in Section 4.2.2. We remark that this construction is also used in the proof of the Virtual Haken Conjecture [2].

Relatively hyperbolic groups were originally introduced by Gromov [83]. He defined them, roughly speaking, as those groups admitting an action on a hyperbolic space similar to the action of the fundamental group of a finite volume hyperbolic manifold on $\mathbb{H}^n$. Farb [76], taking a different approach, defined them in terms of their coned-off graph, which is the space obtained by "crushing" the left cosets of the peripheral subgroups to points. Such space is hyperbolic, but this requirement alone is quite weak ($\mathbb{Z}^2$ satisfies it with respect to any of its infinite cyclic subgroups) and hence a further property, the bounded coset penetration property, is required alongside hyperbolicity of the coned-off graph.

Bowditch [35] and Groves-Manning [85] elaborated instead on Gromov's ideas, describing and identifying properties of the (proper) action of a relatively hyperbolic group on a certain hyperbolic space with a specified collection of horoballs. A characterisation in terms of topological properties of the corresponding action on the boundary of such hyperbolic space has also been given [165].

Another approach that has been very fruitful is due to Osin [141], who defined a notion of relative isoperimetric inequality (again using the coned-off graph).

It has to be noticed that all the equivalent characterisations of relative hyperbolicity mentioned above rely on properties of some space associated to a relatively hyperbolic group which is not its Cayley graph. As a consequence, geometric properties of (Cayley graphs of) relatively hyperbolic groups are often not easy to deduce nor prove using these approaches.

It is due to work of Druţu and Sapir [68] and subsequent work that geometric features of relatively hyperbolic groups such as behaviour of individual quasi-geodesics and quasi-geodesic polygons have been clarified and used to show, for example, quasi-isometric rigidity results. We will actually use several facts arising from their work.

1.2 Actions on (relatively) hyperbolic spaces

Many interesting groups, like mapping class groups (say of closed hyperbolic surfaces) and $Out(F_n)$ (for $n \geq 3$), are known not to be relatively hyperbolic [10], but still they contain "directions" along which they behave in a hyperbolic fashion. A very basic example of this phenomenon is the fact that pseudo-Anosovs have virtually cyclic centraliser, just like infinite order elements in hyperbolic groups. More geometrically, pseudo-Anosovs are Morse, meaning that every quasi-geodesic in the Cayley graph of a mapping class group connecting points in the subgroup generated by a pseudo-Anosov stays close to such subgroup.

In recent years several ways of axiomatising these properties have been found [24, 87, 58, 153], and very recently have been shown to be equivalent [137]. The common idea is to exploit "nice" actions on, typically, hyperbolic spaces. Such actions encode the hyperbolic features of the group one wants to study and do not "see" the non-hyperbolic behaviour. It is worth mentioning, as in the literature this key point is often only implicit, that in most cases the actions being considered are actions on hyperbolic spaces that carry a natural relatively hyperbolic structure (the peripheral sets being orbits of cosets of certain subgroups). In order to clarify this point, we give an equivalent characterisation of *hyperbolically embedded* subgroup, a notion introduced in [58]. The reader may wish to keep in mind that any pseudo-Anosov

g is contained in a certain virtually cyclic subgroup $E(g)$ which is hyperbolically embedded in the corresponding mapping class group.

Definition 1. (cfr. [58, Theorem 4.42],[154]) Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a finite collection of subgroups of the group G and let X be a (possibly infinite) generating system for G. Then $\{H_\lambda\}$ is hyperbolically embedded in (G, X) if and only if the Cayley graph $\Gamma = Cay(G, X)$ is hyperbolic relative to the left cosets of the H_λ's and $d_\Gamma|_{H_\lambda}$ is proper (i.e. balls of finite radius contain finitely many points).

In the case of $E(g)$ as above, the Cayley graph Γ will be hyperbolic (but the relative hyperbolicity requirement is stronger than just this!).

We report below a list of motivating examples of groups together with the elements that generate the "hyperbolic" directions.

Hyperbolic group	infinite order elements
Relatively hyperbolic group	infinite order elements not conjugate into a peripheral subgroup
Mapping class group (of a closed surface of genus ≥ 2)	pseudo-Anosovs
$Out(F_n)$ $(n \geq 3)$	fully irreducible elements
fundamental group of an acylindrical graph of groups	element acting hyperbolically on the Bass-Serre tree
$CAT(0)$ group	rank-one elements
Subgroup of one of the above	same elements

Table 1.1: Hyperbolic-like elements in several kinds of groups

In Chapter 6 we present the way of describing this behaviour that has been introduced in [153], which is inspired by closest-point projections on peripheral sets in relatively hyperbolic spaces.

1.3 Geometric results

In Part I we will cover background material and give some general results about relative hyperbolicity, which will be used as tools in later chapters. In particular, we describe geodesic triangles in relatively hyperbolic spaces, give a characterisation of relative hyperbolicity in terms of the set of "coarse cut-points", the transient set, along geodesics and give an axiomatic description of such transient sets (the Guessing Geodesics Lemma). Such results are meant to clarify basic, but important, geometric features of relatively hyperbolic spaces that are sometimes only implicit in the literature. Most of these results can be traced back to [68], even though most of them do not appear in that paper in the form they will be presented in here.

Moreover, we will give a distance formula for relatively hyperbolic groups in terms of closest-point projections on cosets of peripheral groups.

The machinery we introduce will be used in a sample application to the divergence of relatively hyperbolic groups. Roughly speaking, the divergence of a metric space measures the length of detours avoiding a specified ball as a function of the radius of the ball. We will show the following.

Theorem 1.3.1. *[154] Suppose that the one-ended group G is hyperbolic relative to the (possibly empty) collection of proper subgroups $H_1, \dots, H_k$. Then*

$$e^n \preceq Div^G(n) \preceq \max\{e^n, e^n Div^{H_i}(n)\}.$$

Moreover, if G is finitely presented then $Div^G(n) \asymp e^n$.

Notice that a relatively hyperbolic group is finitely presented if and only its peripheral subgroups are, see [57] and [141].

In Part I we also describe a space which will be referred to as Bowditch space and, following [35, 85], describe its basic properties. The Bowditch space, and especially its boundary, will be the main object of interest in Part II. Studying its properties, we will be able to prove results about relatively hyperbolic groups, and we hope that that properties we prove will be useful for further applications.

The first main result of Part II is the following, which extends a result of Bonk and Kleiner [29], as well as addressing a question of Papazoglou in the context of relatively hyperbolic groups.

Theorem. *[117] Let $(G, \mathcal{P})$ be a relatively hyperbolic group, where all peripheral subgroups are virtually nilpotent. Then there is a quasi-isometrically embedded copy of $\mathbb{H}^2$ in G if and only if G does not split as a non-trivial graph of groups where the edge groups are finite and the vertex groups virtually nilpotent.*

The existence of surface subgroups in (non-virtually-free) hyperbolic groups is an important problem posed by Gromov, whose solution in the context of hyperbolic 3-manifolds [98] has been a major breakthrough. The metric analogue of this problem is what the theorem addresses. Further results will be given in Chapter 4, for example the persistence of such hyperbolic planes when performing peripheral fillings and the characterisation of fundamental groups of 3-manifolds containing a quasi-isometrically embedded copy of $\mathbb{H}^2$. We emphasise that the proof requires showing metric properties of the boundary of G hopefully of independent interest, like linear connectedness, avoidability of parabolic points, etc.

The main result of Chapter 5, reported below, relies on further properties of the boundary, and in particular on a metrically controlled notion of dimension.

Theorem. *[118] A relatively hyperbolic group quasi-isometrically embeds in a product of finitely many metric trees if and only if its peripheral subgroups do.*

The case of hyperbolic groups had been studied previously [46, 42]. It is a common strategy throughout mathematics to embed an object one would like to understand into some "simple" object in the same category. For example, it is useful to embed or immerse manifolds in $\mathbb{R}^n$, which is a natural target as $\mathbb{R}$ is in some sense the simplest manifold, and then one has to take products to have enough "space". Requiring the existence of a quasi-isometric embedding in $\mathbb{R}^n$ is too strong a condition, for the only groups satisfying it are virtually Abelian. So one replaces $\mathbb{R}$ with more general (1-dimensional) objects, trees, and this is why products of trees are natural targets for a quasi-isometric embedding.

Besides being natural, quasi-isometric embeddings in products of trees provide bounds for several large-scale notions of dimension, as well as an intermediate step for constructing embeddings in other spaces, like Banach spaces.

We will actually prove a more quantitative statement, and we will use it to show the following (not stated in its most general form).

Theorem. *Let M be a closed 3-manifold that does not admit a decomposition $M = N_1 \# N_2$ where N_1 has geometry Nil. Then $\pi_1(M)$ quasi-isometrically embeds in the product of at most 8 trees.*

1.4 Probabilistic results

In Part III we will be concerned with generic objects and configurations in groups. Given any mathematical object it is a natural question to ask how a typical one looks like. The question can often be made precise (sometimes in different ways) by constructing a randomised model for the object one wants to study, and figuring out properties that hold with high probability. The most famous example within Geometric Group Theory is the notion of random group, and as it turns out most (finitely presented) groups are hyperbolic, where "most" can be defined in terms of several randomised models like the few-relators and the density models [84, 132].

It is useful to study randomised models at the very least because in order to show that an object of a certain kind exists sometimes the most convenient or even the only feasible way to proceed is showing that such objects exist with positive probability.

The first probabilistic result we give is about generic elements in certain groups. The model for an element in a group that we will use is a simple random walk. Informally, a simple random walk consists in fixing a Cayley graph of the group under consideration, starting at the identity and at every step moving to a neighbouring vertex with uniform probability.

We state the result here in terms of weakly contracting elements, that will be defined later, for the moment we will only mention that in the groups from Table 1.1, an element is weakly contracting if and only if it is of the type described in the corresponding row (in the case of $CAT(0)$ group this is slightly imprecise).

Theorem. *Suppose that G is a non-virtually cyclic subgroup containing a weakly contracting element. Then for any simple random walk $\{X_n\}$ on G there exists $C \geq 1$ so that*

$$\mathbb{P}[X_n \textit{ not weakly contracting}] \leq Ce^{-n/C}.$$

So, generic elements are weakly contracting. Something worth pointing out is that in all groups from Table 1.1 the weakly contracting elements are the most interesting ones, and often they are also the hardest ones to construct, despite being generic.

Related results have been obtained by, e.g., Maher [120].

We now briefly give two sample applications, the (very simple) proofs of which are given in Chapter 6. The first one does not rely on the exponential decay, convergence to 0 would be good enough, while the second one does.

Corollary 1. *If G_1, G_2 are as in the theorem, and $\phi : G_1 \to G_2$ is an isomorphism, then there exists a weakly contracting $g \in G$ so that $\phi(g)$ is also weakly contracting.*

Corollary 2. *If G is as in the theorem, H is amenable, and $\phi : G \to H$ is a homomorphism, then for every $h \in \phi(G)$ there exists a weakly contracting $g \in \phi^{-1}(h)$.*

The second main result of Part III concerns the efficiency of random paths, i.e. how close they are to being geodesics.

We denote the Hausdorff distance by d_{Haus} and the set of all geodesics from x to y (in a given Cayley graph for the purposes of the theorem below) by $\Gamma(x, y)$. Also, we say that a relatively hyperbolic group is non-trivial if it is not virtually cyclic and the peripheral subgroups are proper subgroups. We denote the mapping class group of the closed surface of genus g by $MCG(S_g)$.

Theorem. *Let $\{X_n\}$ be a simple random walk on the group G. Then*

$$\mathbb{E}\left[\sup_{\gamma\in\Gamma(1,X_n)} d_{Haus}(\{X_i\}_{i\leq n},\gamma)\right]=\begin{cases} O(\log(n)) & \text{if } G \text{ is a non-trivial relatively hyperbolic group} \\ O(\sqrt{n\log(n)}) & \text{if } G=MCG(S_g), g\geq 2\end{cases}$$

Hence, a random path deviates from a geodesic at most logarithmically in a non-trivial relatively hyperbolic group. For hyperbolic groups the result is shown in [26] (with entirely different methods). In the case of mapping class groups, the author was not able to find a reference even just for sublinear tracking (in the Cayley graph). The theorem should extend to several other classes of groups, and in particular the proof in the mapping class group case relies on tools, most notably the distance formula, that should have analogues for other groups from, say, Table 1.1, like right-angled Artin groups. We will actually also show a more general result about actions on relatively hyperbolic spaces (and geodesics in such space), and give a polynomial decay estimate for the probability of a path giving an "off-range" Hausdorff distance.

Tracking properties like the one given by the theorem have been used to study Poisson boundaries, see e.g. [99]. However, such results typically rely on sublinear tracking and very little is known about effective rates like the ones we provide.

What originally motivated the theorem was showing the following corollary, which we only state informally for the moment.

Corollary 3. *In a non-trivial relatively hyperbolic group*

- *random geodesic triangles are logarithmically thin in terms of their perimeter, and*
- *the average Dehn function is subasymptotic to the Dehn function, if the Dehn function of the peripheral subgroups is at most polynomial.*

Hence, "most" triangles are better behaved than worst-case-scenario triangles, so that one can say that random configurations of points in a non-trivial relatively hyperbolic group look quite similar to configurations of points in hyperbolic groups.

The second result provides evidence that the average running time of an algorithm to solve the word problem in many non-trivial relatively groups is much lower than the worst-case running time. The average running time is often a more interesting measure of efficiency of an algorithm than the worst-case running time, and indeed randomised algorithms with good average running time are often preferred in applications to deterministic algorithms.

Notation

The notation $x \gtrsim_C y$ (occasionally abbreviated to $x \gtrsim y$) signifies $x \geq y - C$. Similarly, $x \lesssim_C y$ signifies $x \leq y + C$. If $x \lesssim_C y$ and $x \gtrsim_C y$ we write $x \approx_C y$.

For a metric space (Z, d), the *open ball* with centre $z \in Z$ and radius $r > 0$ is denoted by $B(z, r)$. The *closed ball* with the same centre and radius is denoted by $\overline{B}(z, r)$. We write $d(z, V)$ for the infimal distance between a subset $V \subset Z$ and a point $z \in Z$. The *open neighbourhood* of $V \subset Z$ of radius $r > 0$ is the set

$$N(V, r) = \{z \in Z : d(z, V) < r\}.$$

Part I

Geometry of relatively hyperbolic groups

The goal of this part is to give a definition and state the geometric properties of relatively hyperbolic spaces that will be exploited in the following chapters.

In Chapter 2 we will give the working definition that will be used in Part II. The definition is in terms of a certain hyperbolic space, which will be referred to as Bowditch space, that can be associated to a relatively hyperbolic space. The boundary of the Bowditch space will be of particular interest to us. Most of the theory in this chapter has been developed by Bowditch [35] and Groves-Manning [85].

In Chapter 3 we derive several metric properties of relatively hyperbolic spaces, and hence especially of Cayley graphs of relatively hyperbolic groups. The reader interested in Part II only can skip this chapter, as very few facts from it will be used in Part II. The material in this chapter is taken mostly from [151, 154] and is much influenced by [68, 70, 92].

We will study closest point projections on peripheral sets and the so-called transient points, which are the "well-behaved" points of a geodesic. We also give a distance formula for relatively hyperbolic spaces and groups analogous to the distance formula for mapping class groups.

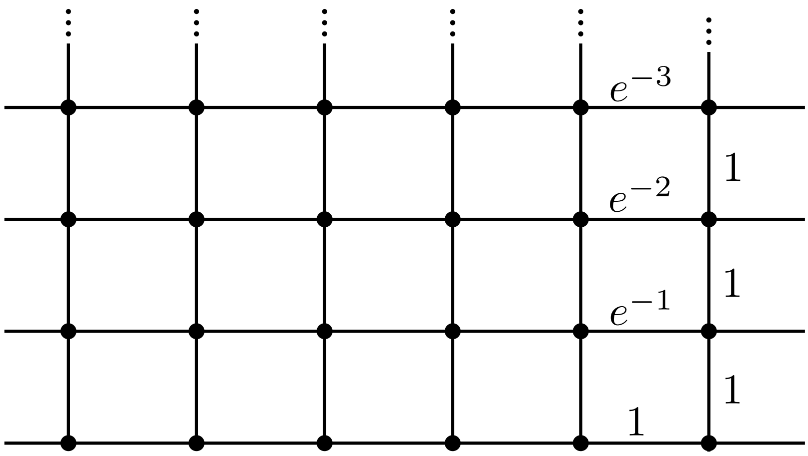
Chapter 2

The Bowditch space

2.1 Definitions for groups

Let us start by defining combinatorial horoballs. These are metric objects resembling horoballs in $\mathbb{H}^n$, except that the corresponding horosphere is an arbitrary metric graph rather than $\mathbb{R}^{n-1}$.

Definition 2.1.1. Suppose Γ is a connected graph with vertex set Γ^0 and edge set Γ^1, where every edge has length one. The *combinatorial horoball* (or simply horoball) $\mathcal{H}(\Gamma)$ is defined to be the graph with vertex set $\Gamma^0 \times \mathbb{N}$ and edges $((v,n),(v,n+1))$ of length 1, for all $v \in \Gamma^0$, $n \in \mathbb{N}$, and edges $((v,n),(v',n))$ of length e^{-n}, for all $(v,v') \in \Gamma^1$.



Note that $\mathcal{H}(\Gamma)$ is quasi-isometric to the metric space constructed from Γ by gluing to each edge in E a copy of the strip $[0,1] \times [1,\infty)$ in the upper half-plane model of $\mathbb{H}^2$, where the strips are attached to each other along $v \times [1,\infty)$, where $v \in \Gamma^0$. This is the construction used, e.g., in [35].

These horoballs are hyperbolic with boundary a single point, as we will show later.

Definition 2.1.2. Suppose G is a finitely generated group, and $\{H_i\}_{i=1}^n$ a collection of finitely generated subgroups of G. Let S be a finite generating set for G, so that $S \cap H_i$ generates H_i for each $i = 1, \ldots, n$.

Let $\Gamma(G, S)$ be the Cayley graph of G with respect to S. We let $Bow(G) = Bow(G, \{H_i\}, S)$ be the space resulting from gluing to $\Gamma(G, S)$ a copy of $\mathcal{H}(\Gamma(H_i, S \cap H_i))$ to each coset gH_i of H_i, for each $i = 1, \ldots, n$. We call $Bow(G)$ the *Bowditch space* associated to $(G, \{H_i\}, S)$.

We say that $(G, \{H_i\})$ is *relatively hyperbolic* if $Bow(G)$ is Gromov hyperbolic, and call the subgroups in $\{H_i\}$ *peripheral subgroups*.

Notice that any Bowditch space comes with a natural action of the corresponding group. Such action resembles the action of a non-uniform lattice in $SO(n, 1)$ (e.g., the fundamental group of a finite-volume non-compact hyperbolic n-manifold) on $\mathbb{H}^n$.

2.2 Definitions for metric spaces

The definition given in the previous section is actually a special case of the definition of relatively hyperbolic space.

Definition 2.2.1. A *maximal k-net* in a metric space X is a maximal collection of distinct points at reciprocal distance at least k. An *approximation graph* Γ with constants k, R for a metric space X is a connected metric graph whose vertex set is a maximal k-net in X and so that two distinct vertices are connected by an edge of length 1 if and only if their distance (as points of X) is at most some fixed R.

We emphasize that approximation graphs are required to be connected, so that they can be endowed with the path metric. Such metric is clearly quasi-isometric to that of X when X is, say, a length space.

Definition 2.2.2. Let X be a geodesic metric space and let $\mathcal{P}$ be a collection of subsets of X. Fix approximation graphs Γ_P for each $P \in \mathcal{P}$. The *Bowditch space* $Bow(X, \{\Gamma_P\}_{P \in \mathcal{P}})$ is the metric space obtained identifying $\Gamma_P^0 \subseteq \mathcal{H}(\Gamma_P)$ with the corresponding maximal net in P for each $P \in \mathcal{P}$.

We will often write $Bow(X)$ instead of $Bow(X, \{\Gamma_P\})$. Notice that the metric d_X of X and the restriction to X of the metric of $Bow(X)$ are coarsely equivalent: There exists a diverging function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ so that for each $x, y \in X$ we have

$$\rho(d_X(x, y)) \leq d_{Bow(X)}(x, y) \leq d_X(x, y).$$

We are now ready to state the generalised version of Definition 2.1.2.

Definition 2.2.3. The geodesic metric space X is *hyperbolic relative* to the collection of subsets $\mathcal{P}$, called *peripheral sets*, if some Bowditch space is hyperbolic.

Notice that the peripheral sets play the role of the cosets of the peripheral subgroups in a relatively hyperbolic group.

We will not need it, but it can be shown that if some Bowditch space is hyperbolic then they all are (this will be implicit in the proof of another characterisation of relative hyperbolicity we will give later).

Remark 2.2.4. Notice that the requirement for approximation graphs to be connected implies that each $P \in \mathcal{P}$ is coarsely connected, meaning that there exists K so that any pair of points on P can be joined by a chain of points in P with consecutive points at distance at most K. Also, K is independent of $P \in \mathcal{P}$.

In the remainder we will show basic properties of horoballs and Bowditch spaces and recall some concepts from the theory of hyperbolic spaces.

2.3 Metric features of combinatorial horoballs

This section contains basic lemmas about combinatorial horoballs, versions of which can be found in [35, 85].

We will call *vertical ray* a geodesic ray in $\mathcal{H}(\Gamma)$ obtained by concatenating all edges of type $((v,n),(v,n+1))$, for some $v \in \Gamma^0$. It is readily seen that it actually is a geodesic ray. We call *vertical segment* a geodesic contained in a vertical ray. A *horizontal segment* is instead a path contained entirely in a *level*, i.e. (the full subgraph spanned by) $\Gamma \times \{n\}$ for some $n \in \mathbb{N}$. It is easy to estimate distances in horoballs. Recall that we write $A \approx B$ if the quantities A and B differ by some constant.

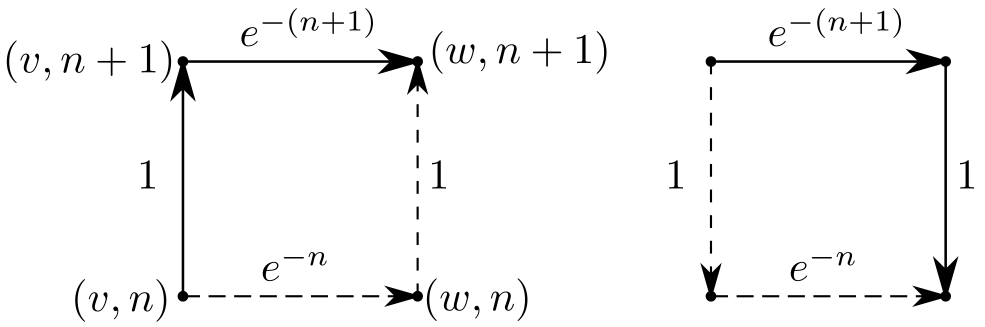
Lemma 2.3.1. *Suppose Γ is a metric graph and $\mathcal{H}(\Gamma)$ is the corresponding combinatorial horoball. Let d_Γ and $d_{\mathcal{H}}$ denote the corresponding path metrics. Then for each $(x,m),(y,n) \in \mathcal{H}(\Gamma)$, we have*

$$d_{\mathcal{H}}((x,m),(y,n)) \approx 2\ln(d_\Gamma(x,y)e^{-\min\{m,n\}}+1)+|m-n|.$$

Moreover, the following holds for $\rho = 2e/(e-1)$. Any geodesic γ between (x,m) and (y,n) in $\mathcal{H}(\Gamma)$ goes from (x,m) to (x,t) along a vertical segment, then follows a geodesic $\gamma' \subset \Gamma \times \{t\} \subset \mathcal{H}(\Gamma)$ to (y,t), then back to (y,n), where $t = \max\{\lceil \ln(d_\Gamma(x,y)/\rho)\rceil, m, n\}$ (the subgeodesics are allowed to be trivial).

Proof. We may assume that $m \leq n$.

Observe that, if $[v, w]$ is an edge of Γ, then, for any integer n, $[(v, n), (v, n+1)] \cup [(v, n+1), (w, n+1)]$ is the only geodesic from $(v, n) \in \mathcal{H}(\Gamma)$ to $(w, n+1) \in \mathcal{H}(\Gamma)$. Similarly, the only geodesic from $(v, n+1)$ to (w, n) is $[(v, n+1), (w, n+1)] \cup [(w, n+1), (w, n)]$.



The observations above imply that any geodesic γ between (x, m) and (y, n) must take the vertical-horizontal-vertical form described in the statement. Thus, for $t \geq m, n$,

$$d_{\mathcal{H}}((x, m), (y, n)) \leq (t - m) + (t - n) + e^{-t} d_\Gamma(x, y), \tag{2.1}$$

with equality for the best choice of t. It is readily seen that this value is attained for the least $t \geq m, n$ so that $l_t = e^{-t} d_\Gamma(x, y)$ satisfies $l_t/e + 2 \geq l_t$, that is, $l_t \leq 2e/(e-1) = \rho$. So the best choice of t is $t = \max\{\lceil \ln(d_\Gamma(x, y)/\rho) \rceil, m, n\}$, and the right hand side of (2.1) is $2(\ln d_\Gamma(x, y) - m) - 2 \ln \rho + |n - m| + \epsilon$, where $|\epsilon| \leq 2 + \rho$.

If $d_\Gamma(x, y) \geq e^m$ we are done as in this case $\ln d_\Gamma(x, y) - m \approx \ln(d_\Gamma(x, y)/e^m + 1)$. On the other hand, if $d_\Gamma(x, y) < e^m$ then $|m - n| \leq d_{\mathcal{H}}((x, m), (y, n)) \leq 1 + |m - n|$, so that the formula holds in this case as well. $\square$

The description of the geodesics makes it easy to show hyperbolicity of combinatorial horoballs.

Lemma 2.3.2. *For any metric graph Γ, $\mathcal{H}(\Gamma)$ is hyperbolic. Moreover, any geodesic ray in $\mathcal{H}(\Gamma)$ is within finite Hausdorff distance from any vertical ray.*

Proof. We will show that for any geodesic triangle there is a point at uniformly bounded distance from all its sides.

Let $(x, m), (y, n), (z, o)$ be points in $\mathcal{H}(\Gamma)$. Recall that by Lemma 2.3.1, all geodesics take a vertical-horizontal-vertical form. We can assume without loss of generality that the horizontal part of geodesics from (x, m) to (y, n) are at a "lower" level than those of geodesics from (y, n) to (z, o) and from (z, o) to (x, m) (if the first coordinate of the endpoints coincide we define the horizontal part to be the endpoint with larger second coordinate).

Any geodesic from (x, m) to (y, n) has two vertical segments γ_x, γ_y and a horizontal segment γ (of bounded length). Let p be any point on γ. Then the distance between p and any geodesic α from (x, m) to (z, o) can be bounded observing that γ_x is contained in α, and similarly for geodesics from (y, n) to (z, o).

In order to show the statement about geodesic rays just notice that subgeodesics of a geodesic ray cannot have a horizontal part, for otherwise they could not be extended indefinitely in view of the description of the geodesics. In particular, geodesic rays are vertical. Finally, any two vertical rays are within bounded Hausdorff as $d((x, m+k), (y, n+k))$ tends to $|m-n|$ as k goes to infinity for each $x, y \in \Gamma$ and $m, n \in \mathbb{N}$. $\square$

We now look at how geodesics behave within a Bowditch space of a relatively hyperbolic space.

Notation 2.3.3. When a Bowditch space Y, with a basepoint w, is fixed, we denote the collection of horoballs by $\mathcal{O}$. For each $O \in \mathcal{O}$ the set $\partial_\infty O$ consists of a single point $a_O \in \partial_\infty Y$ called a *parabolic point*. We also set $d_O = d(w, O)$.

For convenience, when we say that a space is δ-hyperbolic we will imply that geodesic triangles with possibly ideal vertices are δ-thin.

Lemma 2.3.4. *Let $(X, \mathcal{P})$ be relatively hyperbolic and $Bow(X)$ a δ-hyperbolic Bowditch space. For any $P \in \mathcal{P}$ and any $x, y \in P$, all geodesics from x to y are contained in the corresponding horoball $O \in \mathcal{O}$ except for, at most, an initial and a final segment of length at most $2\delta + 1$.*

Proof. For any $v \in P$ and $t \geq 1$, the closest point in $X \setminus O$ to $(v, t) \in O$ is $(v, 0) \in \overline{O}$.

Consider a geodesic triangle with sides the vertical rays above x, y and a geodesic γ in $Bow(X)$ from x to y. If $p \in \gamma$ has distance greater than $2\delta + 1$ from both x and y, then it is δ close to a point on one of the vertical rays which has distance at least $\delta + 1$ from $\{x, y\}$, so $d(p, Bow(X) \setminus O) \geq 1$ and $p \in O$. $\square$

We can extend the distance estimate of Lemma 2.3.1 to Bowditch spaces.

Lemma 2.3.5. *Let $(X, \mathcal{P})$ be relatively hyperbolic, let $Y = Bow(X)$ be a hyperbolic Bowditch space, and d the metric on it. There exists $A = A(Y) < \infty$ so that for any $P \in \mathcal{P}$, whose approximation graph has metric d_P, for any distinct $x, y \in P$, we have*

$$d(x, y) \approx_A 2 \log(d_P(x, y)).$$

Proof. Consider a geodesic γ from x to y. If $d(x,y) \leq 4\delta + 3$, then there is a uniform upper bound of $C = C(Y)$ on $d_P(x,y)$ as d_P is a proper function with respect to d, so this case is done. Otherwise, by Lemma 2.3.4, γ has a subgeodesic γ' connecting points $x', y' \in P$, with γ' entirely contained in the horoball O that corresponds to P, and $d(x,y) \approx_{4\delta+2} d(x',y') \geq 1$. In particular, $x' \neq y'$.

As before, $d_P(x,x')$ and $d_P(y,y')$ are both at most C. Let $d_{\mathcal{B}}$ denote the horoball distance for points in O. Notice that $d(x',y') = d_{\mathcal{B}}(x',y')$ as γ' is a geodesic in X connecting x' to y' entirely contained in O. Using Lemma 2.3.1 we have

$$\begin{aligned} d(x,y) &\approx_{4\delta+2} d(x',y') = d_{\mathcal{B}}(x',y') \\ &\approx_1 2\log(d_P(x',y')) \approx_{C'} 2\log(d_P(x,y)), \end{aligned}$$

for an appropriate constant $C' = C'(C)$. $\square$

We now recall the definition of horoball in a hyperbolic space and justify the name combinatorial horoball by showing that in a Bowditch space of a relatively hyperbolic space the combinatorial horoballs are actually within bounded Hausdorff distance from horoballs.

Definition 2.3.6. Let X be a proper geodesic hyperbolic metric space. Given a point $c \in \partial_\infty X$, and basepoint $w \in X$, the *Busemann function* corresponding to c and w is the function $\beta_c(\cdot, w) : X \to \mathbb{R}$ defined by

$$\beta_c(x,w) = \sup_\gamma \left\{ \limsup_{t\to\infty} (d(\gamma(t),x) - t) \right\},$$

where the supremum is taken over all geodesic rays $\gamma : [0,\infty) \to X$ from w to c.

The supremum in this definition is only needed to remove the dependence on the choice of γ, however for any two rays from w to c the difference between the corresponding $\limsup$'s is bounded as a function of the hyperbolicity constant only, see e.g. [81, Lemma 8.1].

See Notation 2.3.3 for the definition of a_O, d_O.

Lemma 2.3.7. *Let Y be a Bowditch space with basepoint w. There exists C so that for each $O \in \mathcal{O}$ we have*

$$\{x \in Y : \beta_{a_O}(x,w) \leq -d_O - C\} \subseteq O \subseteq \{x \in X : \beta_{a_O}(x,w) \leq -d_O + C\}.$$

Proof. We define the level of a point p in a horoball $\mathcal{H}(\Gamma)$ to be n if p is contained in the n-th level, and to be the distance from the 0-th level otherwise.

In the notation of the definition of Busemann function and for any $x \in O$, for t large enough the geodesic from x to $\gamma(t)$ is the concatenation just of a vertical segment and a horizontal segment of bounded length, see Lemma 2.3.1. The level in O of $\gamma(t)$ is, up to an appropriate multiple of the hyperbolicity constant, $t - d_O$. The second inclusion follows combining these observations.

In order to show the first one, just observe that if x satisfies $\beta_{a_O}(x, w) \leq -d_O - C$ for C large enough, then we can pick t so that the distance between $\gamma(t)$ and x is smaller than the level of t, so that x is in O. $\square$

2.4 Visual metrics

Let X be a proper geodesic hyperbolic metric space.

Choose a basepoint $w \in X$, and denote the Gromov product in $\partial_\infty X$ by $(\cdot|\cdot) = (\cdot|\cdot)_w$; as X is proper, this can be defined as $(a|b) = \inf\{d(w, \gamma)\}$, where the infimum is taken over all (bi-infinite) geodesic lines γ from a to b; such a geodesic is denoted by (a, b).

A *visual metric* ρ on $\partial_\infty X$ with parameters $C_0, \epsilon > 0$ is a metric so that for all $a, b \in \partial_\infty X$ we have $e^{-\epsilon(a|b)}/C_0 \leq \rho(a, b) \leq C_0 e^{-\epsilon(a|b)}$. As X is proper and geodesic, for every $\epsilon > 0$ small enough there exists C_0 and a visual metric with parameters C_0, ϵ, see e.g. [81, Proposition 7.10].

In order for the boundary of a Gromov hyperbolic space to control the geometry of the space itself, one typically requires the following standard property.

Definition 2.4.1. A proper, geodesic, Gromov hyperbolic space X is *visual* if there exists $w \in X$ and $C > 0$ so that for every $x \in X$ there exists a geodesic ray $\gamma : [0, \infty) \to X$, with $\gamma(0) = w$ and $d(x, \gamma) \leq C$.

A weaker version of this condition, suitable for spaces that are not proper, or not geodesic, is given in [28, Section 5].

We record the following observation for completeness.

Proposition 2.4.2. *If $(G, \mathcal{P})$ is a relatively hyperbolic group with every $P \in \mathcal{P}$ a proper subgroup of G, then $Bow(G, \mathcal{P})$ is visual.*

Proof. Let $w \in X = Bow(G, \mathcal{P})$ be the point corresponding to $1 \in G$. Let $x \in X$ be arbitrary.

First we assume that $\mathcal{P} \neq \emptyset$. The point x lies in $N(O, 1)$, for some (possibly many) $O \in \mathcal{O}$. Let $a_O = \partial_\infty O$ be the parabolic point corresponding to O, and let $b \in \partial_\infty X \setminus \{a_O\}$ be any other point. Such a point exists as the peripheral group corresponding to O has infinite index in G.

As X is proper and geodesic, there is a bi-infinite geodesic line $\gamma : (-\infty, \infty) \to X$ with endpoints a_O and b. An appropriate conjugate Q of a peripheral subgroup acts on X stabilising $a_O \in \partial_\infty X$ (if O corresponds to the coset gP of $P \in \mathcal{P}$, then $Q = gPg^{-1}$), so that some translate γ' of γ is at distance at most C from the point x. This is easy to see as γ contains a vertical ray in O, and using the action of Q one can move the starting point of such ray close to x.

Denote the endpoints of γ' by a_O and b', and let α be the geodesic ray from w to a_O, and β the geodesic ray from w to b'. As the geodesic triangle γ', α, β is δ-thin, x lies within a distance of $C + \delta$ of one of the geodesic rays α and β, and we are done.

Secondly, if $\mathcal{P} = \emptyset$, we have that $Bow(G, \emptyset)$ is a Cayley graph of G. Fix any $a \neq b$ in $\partial_\infty X$. As X is proper and geodesic, there is a bi-infinite geodesic γ from a to b. As the action of G on X is cocompact, some translate of γ passes within a uniformly bounded distance of x, and the proof proceeds as in the first case. $\square$

Chapter 3

Metric properties of relatively hyperbolic spaces

In this chapter we introduce transient sets, we relate them to closest point projections on peripheral sets and we show a distance formula for relatively hyperbolic spaces. The results we will need in further chapters are summarised in the next section.

However, instead of just proving these results, we will also show that there is a characterisation of relatively hyperbolic spaces in terms of transient sets and introduce the main technical result from [154], which provides a convenient criterion for a space to be relatively hyperbolic. To further motivate the introduction of these notions, we give a sample application to the divergence function of relatively hyperbolic groups. Further applications are found in [154].

The reader can go through the following subsection and skip everything else if he/she is interested in the applications but not in the proofs and the further material.

3.1 Transient sets, projections and distance formula

Throughout this section we fix a space X hyperbolic relative to the collection of subsets $\mathcal{P}$. The proofs of the results in this section are given in Section 3.3.

The following lemma, which can be found in [151], also follows by combining results on relative hyperbolicity from [68].

Lemma 3.1.1. *For each $P \in \mathcal{P}$ denote by $\pi_P : X \to P$ a coarse closest point projection, i.e. a map so that $d(x, \pi_P(x)) \leq d(x, P) + 1$. There exists C with the following properties.*

1. *For each distinct $P, Q \in \mathcal{P}$ we have $diam(\pi_P(Q)) \leq C$.*

2. For each $x, y \in X$ and $P \in \mathcal{P}$ so that $d(\pi_P(x), \pi_P(y)) \geq C$, and each geodesic $[x, y]$, we have

$$d([x,y], \pi_P(x)),\ d([x,y], \pi_P(y)) \leq C.$$

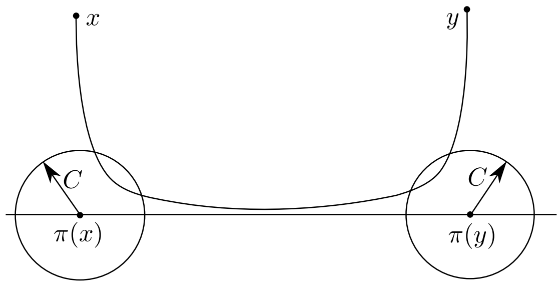


Figure 3.1: Lemma 3.1.1-(2)

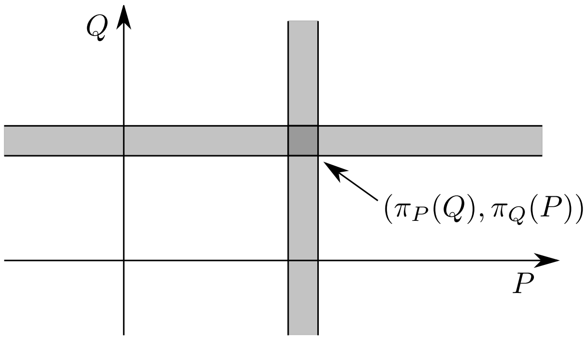
In other words, the projection of one peripheral set onto another one has bounded diameter and if two points project far away on some peripheral set then every geodesic connecting them passes close to the projection points.

The following corollary follows from the lemma by standard arguments. An analogue result has been shown for mapping class groups and subsurface projections by Behrstock in [13], see Theorem 7.4.3.

Corollary 3.1.2 (Projections estimate)**.** *There exists B with the following property. Let $P, Q \in \mathcal{P}$ be distinct and let $x \in X$. Then*

$$\min\{d(\pi_P(x), \pi_P(Q)),\ d(\pi_Q(x), \pi_Q(P))\} \leq B.$$

So, if x and Q have far away projections on P, then x and P have close projections on Q, providing a useful trick to control a projection. Here is another useful way of describing what the statement means. Consider the map $\pi_P \times \pi_Q : X \to (P \times Q)$. Then the image of such map is (contained in) a neighbourhood of two copies of P and Q as suggested by the following figure.



In [154] relative hyperbolicity has been characterised in terms of transient sets of geodesics, that were introduced in [92]. Roughly speaking, a point on a geodesic fails to be transient if it is well-within a subgeodesic that fellow-travels a peripheral set. The formal definition is below.

Let μ, R be constants and α a geodesic in X. Denote by $deep_{\mu,R}(\alpha)$ the set of points p of α that belong to some subgeodesic $[x, y]$ of α with endpoints in $\overline{N}_\mu(P)$ for some $P \in \mathcal{P}$ and so that $d(p, x), d(p, y) > R$. Denote $trans_{\mu,R}(\alpha) = \alpha \backslash deep_{\mu,R}(\alpha)$, the set of *transient points.*

The main properties of transient and deep sets are reported below and will be shown in Lemma 3.2.4. Some of them follow from results in [68] which are however not phrased in terms of these notions.

Lemma 3.1.3. *There exist μ, μ', R, D, C with the following properties.*

1. *[Relative Rips condition] For each $x, y, z \in X$ we have*

$$trans_{\mu,R}([x, y]) \subseteq \overline{N}_D(trans_{\mu,R}([x, z]) \cup trans_{\mu,R}([z, y])).$$

2. *$deep_{\mu,R}([x, y])$ is contained in a disjoint union of subgeodesics of $[x, y]$ each contained in $\overline{N}_{\mu'}(P)$ for some $P \in \mathcal{P}$, called* deep component along *P.*

3. *The endpoints of the deep component of $[x, y]$ along $P \in \mathcal{P}$ (if it exists) are C-close to $\pi_P(x), \pi_P(y)$.*

4. *If for some $P \in \mathcal{P}$ we have $d(\pi_P(x), \pi_P(y)) > C$, then $[x, y]$ has a deep component along P of length at least $d(\pi_P(x), \pi_P(y)) - C$.*

Convention 3.1.4. When we write *trans* instead of $trans_{\mu,R}$ we implicitly fix constants μ, R as in the lemma.

When x, L are real numbers, we will set $\{\{x\}\}_L$ to be x if $x \geq L$ and 0 otherwise. We use the symbol $A \approxeq_{\lambda,\mu} B$ to denote the inequalities

$$A/\lambda - \mu \leq B \leq \lambda A + \mu.$$

The following result provides an analogue to the distance formula for mapping class groups [127] (we give the statement as Theorem 7.4.4) and will be used to show quasi-isometric embeddings results for relatively hyperbolic groups.

Theorem 3.1.5 (Distance Formula, [151])**.** *Let $(X, \mathcal{P})$ be relatively hyperbolic. Then there exists L_0 with the property that for each $L \geq L_0$ there exist $\lambda, \mu \geq 1$ so that for each $x, y \in X$ we have*

$$d(x,y) \approxeq_{\lambda,\mu} \sum_{P \in \mathcal{P}} \{\{d(\pi_P(x), \pi_P(y))\}\}_L + d_{Bow(X)}(x,y).$$

3.2 Transient sets characterisation

We now state, after a preliminary definition, a characterisation of relative hyperbolicity that, keeping the notation in [154], will be denoted (RH3).

Definition 3.2.1. The collection $\mathcal{P}$ of subsets of the geodesic metric space X is said to be *geometrically separated* if for each K there exists B so that $diam(\overline{N}_K(P) \cap \overline{N}_K(Q)) \leq B$ for each distinct $P, Q \in \mathcal{P}$.

Definition 3.2.2. The pair $(X, \mathcal{P})$ satisfies (RH3) if there exist μ, R_0 so that

1. $\mathcal{P}$ is geometrically separated;
2. for each k and $R \geq R_0$ there exists K so that if $d(x, P), d(y, P) \leq k$ for some $P \in \mathcal{P}$ and $d(x, y) \geq K$ then $trans_{\mu,R}([x, y]) \subseteq B_K(x) \cup B_K(y)$ and there exists $z \in trans_{\mu,R}([x, y]) \cap \overline{N}_\mu(P)$;
3. (**Relative Rips condition**) For each $R \geq R_0$ there exists D so that if $\Delta = \gamma_0 \cup \gamma_1 \cup \gamma_2$ is a geodesic triangle then

$$trans_{\mu,R}(\gamma_0) \subseteq \overline{N}_D(trans_{\mu,R}(\gamma_1) \cup trans_{\mu,R}(\gamma_2)).$$

The content of (2) is that when x, y are close to a peripheral set then any transient point between x, y is close to either x or y, and also there is a transient point in a specified neighbourhood of the peripheral set.

Remark 3.2.3. We will show later that (RH3) is another characterization of relative hyperbolicity (Theorem 3.4.7). In particular, the following lemmas apply to relatively hyperbolic spaces despite being stated for pairs satisfying (RH3). The following lemma is a restatement of Lemma 3.1.3.

Lemma 3.2.4. *Suppose that $(X, \mathcal{P})$ satisfies (RH3). Then there exist μ, μ', R, D, C with the following properties.*

1. *For each $x, y, z \in X$ we have*

$$trans_{\mu,R}([x,y]) \subseteq \overline{N}_D(trans_{\mu,R}([x,z]) \cup trans_{\mu,R}([z,y])).$$

2. *$deep_{\mu,R}([x,y])$ is contained in a disjoint union of subgeodesics of $[x,y]$ each contained in $\overline{N}_{\mu'}(P)$ for some $P \in \mathcal{P}$, called* deep component along P.

3. *The endpoints of the deep component of $[x,y]$ along $P \in \mathcal{P}$ (if it exists) are C-close to $\pi_P(x), \pi_P(y)$.*

4. *If for some $P \in \mathcal{P}$ we have $d(\pi_P(x), \pi_P(y)) > C$, then $[x,y]$ has a deep component along P of length at least $d(\pi_P(x), \pi_P(y)) - C$.*

We listed the first property for completeness, but it is just part of the definition.

Proof. Let γ be a geodesic and suppose that for some $P \in \mathcal{P}$ we have $\gamma \cap \overline{N}_\mu(P) \neq \emptyset$. Suppose that $x \in \gamma$ is not the first nor the last point in $\gamma \cap \overline{N}_\mu(P)$ and that it lies outside $\overline{N}_\mu(P)$. Let $[x_1, x_2]$ be the minimal subgeodesic of γ containing x with endpoints in $\overline{N}_\mu(P)$. We want to show that $[x_1, x_2]$ cannot be too long, so that the maximal subgeodesic of γ with endpoints in $\overline{N}_\mu(P)$ is contained in $\overline{N}_{\mu'}(P)$ for an appropriate μ'. To show this, it is enough to consider the points $x_i' \in [x_1, x_2]$ with $0 < d(x_1, x_i') \leq 1$. If $d(x_1, x_2)$ was sufficiently large, we would get a contradiction with $(RH3)-(2)$ as $[x_1', x_2'] \cap \overline{N}_\mu(P) = \emptyset$.

As a consequence, we see that, given $R \geq R_0$ large enough, (the closure of) $deep_{\mu,R}(\gamma)$ is a disjoint union of subgeodesics each with both endpoints in $\overline{N}_{\mu'}(P)$ for some $P \in \mathcal{P}$, for any geodesic γ. In fact, if γ_1, γ_2 are subgeodesics of γ so that the endpoints of γ_i are in $\overline{N}_\mu(P_i)$ and $P_1 \neq P_2$ then we can assume by $(RH3)-(1)$ (geometric separation of $\mathcal{P}$) that $diam(\gamma_1 \cap \gamma_2) \leq R$.

In order to show the third and fourth property we will use geodesic quadrangles. Notice that transient parts of geodesic quadrangles satisfies the obvious generalisation of the Relative Rips condition with constant $2D$.

Let x, y be points in X and let x', y' be their projections on P. It is easily seen that the union of the transient sets of $[x,x'], [x',y']$ and $[y,y']$ is contained in $B_{d(x,P)+L}(x) \cup B_{d(y,P)+L}(y)$, for a suitable L.

Suppose now that $[x,y]$ is a deep component along P. Then the endpoints of such deep component are within distance $d(x,P) + 2D + L$ and $d(y,P) + 2D + L$ respectively from x, y, as well as being μ'-close to P. This easily implies that they are actually close to $\pi_P(x), \pi_P(y)$, proving the third property.

We now show (4). By symmetry, we only need to show that $[x, y]$ passes close to $\pi_P(x)$, the existence of the required deep component will then follow for the analogue fact for $\pi_P(y)$ and an application of $(RH3) - (2)$.

Let x'' be a point along $[x, x']$ at distance k from x', where k will be determined by the following argument. Consider a geodesic quadrangle with vertices x'', y', y, x, see Figure 3.2.

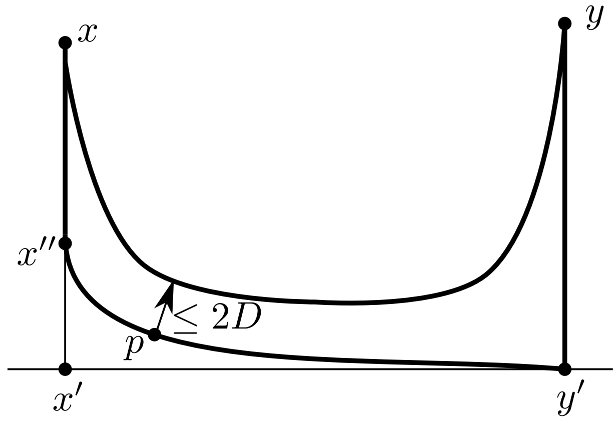


Figure 3.2: Proof of the contraction property for projections

There exists a transient point p along $[x'', y']$ at distance at most μ from P and at most K from x'', where K is provided by $(RH3) - (2)$. For k large enough, such point is not $2D$-close to $[x, x'']$. Also, p is not $2D$-close to any $y'' \in [y, y']$ if $d(x', y')$ is large enough. In fact, if this was the case, then we would have

$$d(y, P) \leq d(y, y'') + 2D + \mu = d(y, y') - d(y', y'') + 2D + \mu \leq$$

$$d(y, P) + 1 - d(y', x') + K + k + 4D + \mu,$$

a contradiction.

In particular, p is $2D$-close to $[x, y]$, and hence $[x, y]$ passes $K + k + 2D$-close to $\pi_P(x)$. $\square$

An easy consequence of part (2) of the lemma is the following fact that was first shown in [68].

Corollary 3.2.5. *Suppose that $(X, \mathcal{P})$ satisfies $(RH3)$. Then any $P \in \mathcal{P}$ is* quasi-convex, *i.e. any geodesic connecting points in P stays within uniformly bounded distance from P.*

Proof. $(RH3)-(2)$ implies that a geodesic γ as in the statement is the concatenation of two geodesics of uniformly bounded length containing the endpoints and a deep component. Such deep component must be a deep component along P if γ is long enough due to geometric separation of $\mathcal{P}$. $\square$

We now give a more accurate description of geodesics triangles in a relatively hyperbolic space. Figure 3.3 illustrates the content of the lemma. The reader might wish to compare this lemma with the characterisation of relative hyperbolicity given in [70].

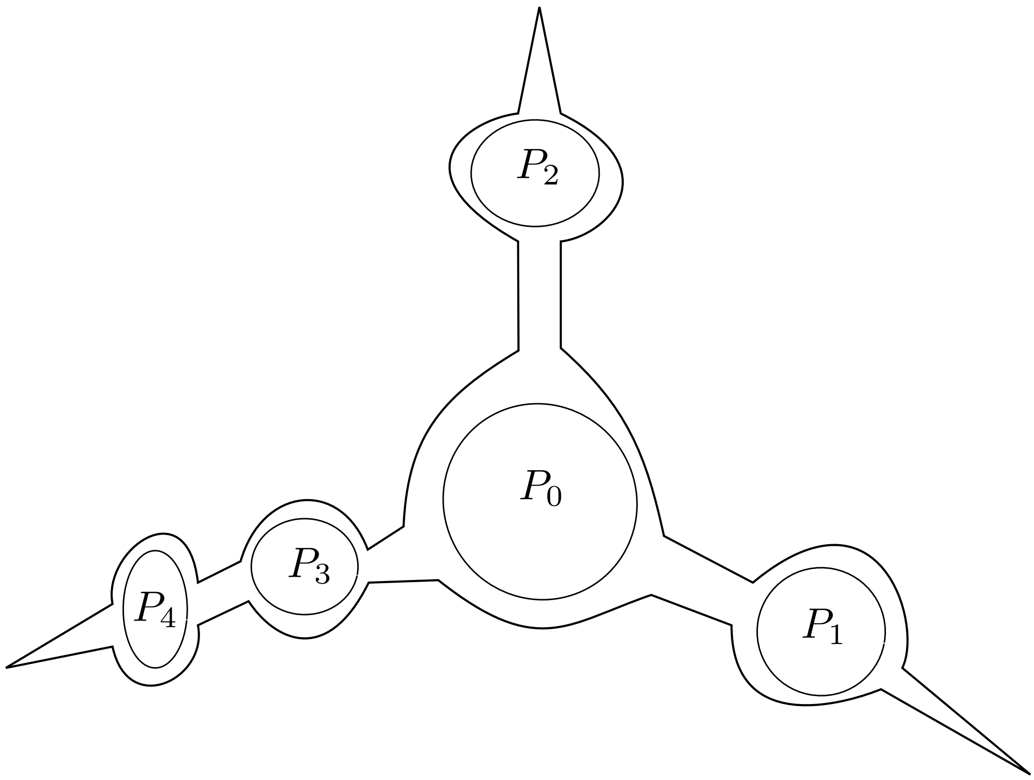


Figure 3.3: A triangle in a relatively hyperbolic space

Lemma 3.2.6. *Suppose that $(X,\mathcal{P})$ satisfies (RH3). There exists C with the following property, where μ,μ' are as in Lemma 3.2.4. For every geodesic triangle with sides $\gamma_1,\gamma_2,\gamma_3$ there exist $P_1,\dots,P_n\in\mathcal{P}$ and possibly $P_0\in\mathcal{P}$ so that, for $\gamma_i'=\gamma_i\backslash\bigcup\overline{N}_{\mu'}(P_i)$, we have*

$$\gamma_1'\subseteq\overline{N}_C(\gamma_2'\cup\gamma_3').$$

Also, for each $i\geq 1$, two of $\{\gamma_1,\gamma_2,\gamma_3\}$ intersect $\overline{N}_\mu(P_i)$ with corresponding entrance and exit points within distance C of each other. For $i=0$ all γ_i's intersect $\overline{N}_\mu(P_i)$, *again with corresponding entrance and exit points within distance C of each other. If some γ_i has a deep component along some $Q \in \mathcal{P}$, then Q is one of the P_i's.*

Proof. We fix a choice of constants as in Lemma 3.2.4. We will reduce to the case when one of the sides has length bounded by some large enough constant. This case can be easily dealt with using the fact that the endpoints of all sufficiently long deep components are determined by closest point projections by Lemma 3.1.3-(3)-(4), combined with the fact that closest point projections are coarsely Lipschitz, again a consequence of Lemma 3.1.3-(4).

For later purposes, we need the following lemma.

Lemma 3.2.7. *Let $\alpha_i = [y_i, z_i]$ be, for $i = 0, 1, 2$, a geodesic with endpoints in $\overline{N}_\mu(Q_i)$ for some $Q_i \in \mathcal{P}$. Suppose $d(z_i, y_{i+1}) \leq K$, with indices running cyclically. Then if $l(\alpha_i) \geq A = A(K)$ we have $Q_0 = Q_1 = Q_2$.*

Proof. If A is large enough we can argue as follows. By $(RH3) - (2)$ there exists L so that for all $w \in [y_0, z_0]$ we have $trans_{\mu,R}([w, z_0]) \subseteq \overline{N}_L(\{w, z_0\})$. Consider $w_0 \in [y_0, z_0]$ so that

$$d(w_0, y_0) = 1 + \sup\{diam(\overline{N}_B(P) \cap \overline{N}_B(P'))\},$$

where the supremum is taken over all $P, P' \in \mathcal{P}$ with $P \neq P'$ and $B = \max\{K + \mu, L + 3D + \mu'\}$. Consider a pentagon with vertices w_0, z_0, y_1, z_1, y_2. By $(RH3) - (2)$ there exists $p_0 \in trans([w_0, y_2]) \cap \overline{N}_\mu(Q_2)$. It is readily checked that p_0 cannot be $3D$-close to $trans_{\mu,R}([z_0, y_1]) \cup trans_{\mu,R}([y_1, z_1]) \cup trans_{\mu,R}([z_1, y_2])$ (for A large enough) and can be $3D$-close to $trans_{\mu,R}([w_0, z_0])$ only if $d(w_0, p_0) \leq L + 3D$. In particular, $w_0, p_0 \in \overline{N}_B(Q_0) \cap \overline{N}_B(Q_2)$, which implies $Q_0 = Q_2$ by the choice of $d(w_0, y_0)$. □

We now consider two cases, and in each we subdivide the geodesic triangles in smaller triangles with one short side. Let $A = A(3D)$.

Case 1: There exists i and $y \in trans_{\mu,R}(\gamma_i)$ so that $d(y, trans_{\mu,R}(\gamma_{i\pm1})) \leq A + 10D + 10R$ (recall that R is the second constant involved in the definition of transient sets).

In this case there are $y_i \in trans(\gamma_i)$ all within bounded distance, and we reduce to studying the triangle with vertices y_i, y_{i+1} and the appropriate vertex of the main geodesic triangle. It is readily checked that if the lemma holds for such smaller triangles then it holds for the triangles we started with.

Case 2: There is no such y. This easily implies that we can subdivide each γ_i into subgeodesics $\gamma_i^-, [y_i, z_i], \gamma_i^+$ with the following properties (see Figure 3.4).

- $[y_i, z_i]$ is a deep component,
- $d(y_i, z_i) \geq A$,
- $trans(\gamma_i) \cap \gamma_i^{\pm} \subseteq \overline{N}_D(trans(\gamma_{i\pm1}) \cap \gamma_{i\pm1}^{\mp})$,
- $d(z_i, y_{i+1}) \leq 3D$.

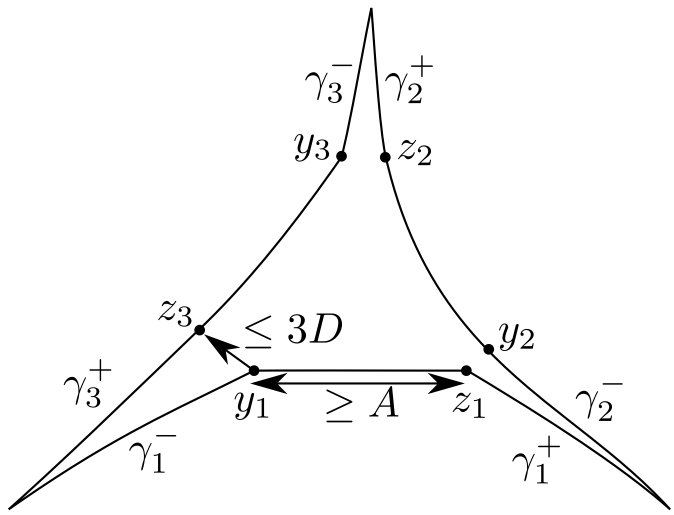


Figure 3.4: Notation for Case 2

We let y_i be the last transient point along $\gamma_i = [p_i, q_i]$ which is D-close to to γ_{i-1}, and z_i the first transient point close to γ_{i+1}. We need the following technical claim.

Claim: y_i is closer to p_i than z_i.

Suppose that the claim does not hold. Let $z_i' \in \gamma_{i+1}$ be a point D-close to z_i. Observe that for every subgeodesic $\alpha = [p, q]$ of a geodesic β we have

$$d_{Haus}(trans(\alpha), trans(\beta) \cup \{p, q\}) \leq R.$$

In particular, we see that as y_i is a transient point of $[z_i, q_i]$, it is either $(D+R)$-close to $[z_i, z_i']$, but this contradicts the standing assumption of Case 2, or D-close to a transient point of γ_{i+1}, again contradicting the standing assumption. The claim is proved.

There are no transient points strictly between y_i and z_i, so that the first property holds, while the second property holds by the standing assumption. Also, by the definition and the fact that y_i, z_i appear in this order along γ_i the third property is also satisfied.

The argument for the fourth property is the following. We know that z_i is D-close to (y_{i+1} or) some point along γ_{i+1} before y_{i+1}. Such point has to be within distance

$2D$ of y_{i+1}. Otherwise, if p_i is the final point of γ_i^+, y_{i+1} would be at distance more than $d(p_i, z_i) + D$ from p_i, and hence cannot be D-close to γ_i^+.

Let Q_i be so that $y_i, z_i \in \overline{N}_\mu(Q_i)$. Lemma 3.2.7 implies $Q_0 = Q_1 = Q_2$. We see now that $P_0 = Q_0$ is as required. Also, we reduced to studying triangles with vertices z_i, y_{i+1} and the appropriate vertex of the main triangle. $\square$

3.3 Proof of projection properties

We now make a brief digression to point out that for a pair $(X, \mathcal{P})$ satisfying (RH3) (which we will show later to be equivalent to relative hyperbolicity), the properties of closest-point projections stated in Lemma 3.1.1 and Corollary 3.1.2 hold. All statements follow rather easily from Lemma 3.2.4. For the convenience of the reader, the statements are reported below.

Lemma 3.3.1. *Let $(X, \mathcal{P})$ be a pair satisfying (RH3). For each $P \in \mathcal{P}$ denote by $\pi_P : X \to P$ a coarse closest point projection, i.e. a function so that $d(x, \pi_P(x)) \leq d(x, P) + 1$. There exists C with the following properties.*

1. *For each distinct $P, Q \in \mathcal{P}$ we have $diam(\pi_P(Q)) \leq C$.*
2. *For each $x, y \in X$ and $P \in \mathcal{P}$ so that $d(\pi_P(x), \pi_P(y)) \geq C$ we have*
$$d([x, y], \pi_P(x)),\ d([x, y], \pi_P(y)) \leq C.$$
3. *Let $P, Q \in \mathcal{P}$ be distinct and let $x \in X$. Then*
$$\min\{d(\pi_P(x), \pi_P(Q)),\ d(\pi_Q(x), \pi_Q(P))\} \leq C.$$

Proof. Part (2) follows immediately from Lemma 3.2.4-(4).

Let us now show (1). By contradiction, if we had two point $x, y \in Q$ so that $d(\pi_P(x, \pi_P(y))$ is large, for some distinct $P, Q \in \mathcal{P}$, then a geodesic from x, y, besides being contained in a suitable neighbourhood of Q, e.g. by Lemma 3.2.4-(2), will have a long subgeodesic contained in a neighbourhood of P by Lemma 3.2.4-(4)-(2). This contradicts the geometric separation of $\mathcal{P}$.

We are left to show (3). Let C be a constant large enough that (1),(2) hold for C. If we had $d(\pi_P(x), \pi_P(Q)) > C$, then any geodesic from x to Q would pass C-close to P by (2). In particular, this holds for a geodesic γ from x to $\pi_Q(x)$. We claim that any point y within distance C on such geodesic satisfies, say, $d(\pi_Q(y), \pi_Q(x)) \leq 10C + 10$, which then implies $d(\pi_Q(x), \pi_Q(P)) \leq 10C + 10$ as required.

Let y' be a point on γ within distance C of y. If $d(y', \pi_Q(x)) \leq 2C + 1$, and hence $d(y, Q) \leq 3C + 1$, then

$$d(\pi_Q(y), \pi_Q(x)) \leq d(\pi_Q(y), y) + d(y, y') + d(y', \pi_Q(x)) \leq (3C+2) + C + (2C+1) \leq 6C+3.$$

So, we can assume $d(y', \pi_Q(x)) > 2C + 1$. In particular, the subgeodesic of γ from x to y' does not intersect $\overline{N}_C(Q)$, so that $d(\pi_Q(x), \pi_Q(y')) \leq C$ by (2). What is more, any geodesic from y to y' also does not intersect $\overline{N}_C(Q)$, so that once again $d(\pi_Q(y), \pi_Q(y')) \leq C$. Hence,

$$d(\pi_Q(x), \pi_Q(y)) \leq d(\pi_Q(x), \pi_Q(y')) + d(\pi_Q(y'), \pi_Q(y)) \leq 2C,$$

as required. □

3.4 Guessing geodesics

The aim of this section is to give an "axiomatic" characterisation of the pairs transient sets/geodesics in a relatively hyperbolic space. Informally, we will prove the following.

Guessing Geodesics Lemma: *Let X be a geodesic metric space and $\mathcal{P}$ a collection of subsets of X. Suppose that for each pair of points x, y a path $\eta(x, y)$ connecting them and a subset $trans(x, y) \subseteq \eta(x, y)$ have been assigned. Suppose that the pairs $trans(x, y)/\eta(x, y)$ behave like the pairs transient points/geodesics in a relatively hyperbolic space (e.g. they satisfy the relative Rips condition). Then X is hyperbolic relative to $\mathcal{P}$ and the set $trans(x, y)$ coarsely coincides with the set of transient points on a geodesic from x to y.*

We report a version of this result for hyperbolic spaces, similar to a an earlier result due to Bowditch [36]. For a path α and $p, q \in \alpha$ we denote by $\alpha|_{[p,q]}$ the subpath of α joining p to q.

Proposition 3.4.1. *[86, Proposition 3.5] Let X be a geodesic metric space. Suppose that for all $x, y \in X$ there is an arc $\eta(x, y)$ connecting them so that the following hold for some D.*

1. *$diam(\eta(x, y)) \leq D$ whenever $d(x, y) \leq 1$;*
2. *for all $x', y' \in \eta(x, y)$ we have $d_{Haus}(\eta(x, y)|_{[x',y']}, \eta(x', y')) \leq D$;*
3. *$\eta(x, y) \subseteq \overline{N}_D(\eta(x, z) \cup \eta(z, y))$ for all x, y, z.*

Then X is hyperbolic and there exists κ so that $d_{Haus}(\eta(x, y), [x, y]) \leq \kappa$ for all x, y.

The second part of the conclusion does not appear in the statement of [86, Proposition 3.5], but the proof strategy, as explained in the first paragraph of the proof, is to show the second part and then deduce the first one. The proposition is a special case of Theorem 3.4.3 below.

Before stating the generalisation to relative hyperbolicity, let us show how to use it to show that (RH3) implies relative hyperbolicity.

Proposition 3.4.2. *If $(X, \mathcal{P})$ satisfies (RH3) then it is relatively hyperbolic.*

Proof. We have to show that a Bowditch space $Bow(X)$ is hyperbolic. To illustrate the idea of the proof, we first define paths $\eta(x, y)$ connecting points $x, y \in X \subseteq Bow(X)$. We fix constants as in Lemmas 3.2.4, 3.2.6, and we refer to deep components and transient parts for the given constants. Pick a geodesic $[x, y] \subseteq X$ and substitute all deep components with a concatenation of two paths of uniformly bounded length connecting to the appropriate $P \in \mathcal{P}$ and a geodesic in the corresponding horoball. Notice that property 1) from Proposition 3.4.1 is immediate.

For $x, y, z \in X$, we see that in view of Lemma 3.2.6 we substituted all the possible "fat parts" of a geodesic triangles in X with either two or three geodesics in a horoball with corresponding endpoints at uniformly bounded distance (and some paths of uniformly bounded diameter). We then see that the triangle $\eta(x, y), \eta(x, z), \eta(z, y)$ is thin, establishing property 3). Property 2) easily follows from the fact that, for any $x', y' \in [x, y] \subseteq X$, $[x', y'] \subseteq [x, y]$ and any geodesic γ from x to y, $trans(\gamma)$ is within uniformly bounded Hausdorff distance from $\{x', y'\} \cup (trans([x, y]) \cap [x', y'])$. This is obvious for $\gamma = [x', y']$, and otherwise one first considers $[x', y']$ and then applies the Relative Rips condition to the geodesic bigon formed by γ and $[x', y']$.

We now define the path $\eta(x, y)$ for general x, y.

If x, y are in the same horoball, then let $\eta(x, y)$ just be a geodesic connecting them. If they are in distinct horoballs, corresponding to $P, Q \in \mathcal{P}$, then let $\eta(x, y)$ be the concatenation of geodesics in the horoballs connecting x, y to $x' \in \pi_P(Q), y' \in \pi_Q(P)$ respectively and $\eta(x', y')$ as defined earlier (recall that $\pi_P(Q), \pi_Q(P)$ have uniformly bounded diameter). If x is in a horoball, say corresponding to $P \in \mathcal{P}$, and y is in X proceed similarly, concatenating a geodesic in the horoball from x to $\pi_P(x)$ and $\eta(\pi_P(x), y)$.

Properties $2), 3)$ can be checked analysing separately the cases when x, y, z are/are not in X using an argument similar to the one we gave above. □

Theorem 3.4.3. *Let (X, d) be a geodesic metric space and $\mathcal{P}$ a collection of subsets. Suppose that for each pair of points $x, y \in X$ a path $\eta(x, y)$ connecting them and a closed subset $trans(x, y) \subseteq \eta(x, y)$ have been assigned so that the following conditions hold for some large enough D.*

1. *if $d(x, y) \leq 2$ then $diam(trans(x, y)) \leq D$;*
2. *for all $x', y' \in \eta(x, y)$, we have*
$$d_{Haus}(trans(x', y'), trans(x, y)|_{[x',y']} \cup \{x', y'\}) \leq D,$$
where $trans(x, y)|_{[x',y']} = trans(x, y) \cap \eta(x, y)|_{[x',y']}$;
3. *$trans(x, y) \subseteq \overline{N}_D(trans(x, z) \cup trans(z, y))$ for all $x, y, z \in X$;*
4. *if $x', y' \in \eta(x, y)$ do not both lie on $P \in \mathcal{P}$ for any P, then there exists $z \in trans(x, y)$ between x', y';*
5. *$\mathcal{P}$ is geometrically separated;*
6. *for every k there exists K so that if $d(x, P), d(y, P) \leq k$ and $d(x, y) \geq K$ then $trans(x, y) \subseteq B_K(x) \cup B_K(y)$ and there exists $z \in trans(x, y) \cap \overline{N}_D(P)$.*

Then

(a) X is hyperbolic relative to $\mathcal{P}$;

(b) there exist μ, c_0, L with the following property. For any $c \geq c_0$ and geodesic β with endpoints x, y the Hausdorff distance between $trans(x, y)$ and $trans_{\mu,c}(\beta)$ is at most some $L + c$.

We will proceed as follows. We first show part (b), which implies that $(X, \mathcal{P})$ satisfies (RH3) (in view of properties 3,5,6). Also, using part (b), we will be able to show that (RH3) is a characterization of relative hyperbolicity, so that part (a) follows immediately combining these two facts.

Proof of part (b). Let us start with some general remarks. First of all, for each n-gon $\eta(x_1, x_2) \cup \cdots \cup \eta(x_n, x_1)$, with $n \geq 3$, we have

$$trans_{\mu,R}(\eta(x_1, x_2)) \subseteq \overline{N}_{(n-2)D}(trans(x_2, x_3) \cup \cdots \cup trans(x_n, x_1)).$$

In particular, subdividing a geodesic from x to y and using 1, we get

$$diam(trans(x, y)) \leq D \max\{(d(x, y)/2 - 1, 0\} + D \leq Dd(x, y) + D.$$

Notice that by 2) with $x' = x, y' = y$ for each x, y there is $p \in trans(x, y)$ with $d(p, x) \leq D$. For convenience we will always assume $x, y \in trans(x, y)$.

Lemma 3.4.4. *Up to increasing D, we can substitute* 6) *with the following:*

6′) *There exists $M \geq 1$ so that for every $k \geq 1$ if $d(x,P), d(y,P) \leq k$ and $d(x,y) \geq Mk$ then $trans(x,y) \subseteq B_{Mk}(x) \cup B_{Mk}(y)$ and there exists $z \in trans(x,y) \cap \overline{N}_D(P)$.*

Proof. Fix the notation of 6′), where M is large enough to carry out the following argument. Consider $x', y' \in P$ so that $d(x,x') \leq d(x,P)+1, d(y,y') \leq d(y,P)+1$. Fix K as in 6) for $k = 4D+1$. Then by 6) with $k=0$ we have that $trans(x',y')$ is contained in the K-neighbourhood of $\{x',y'\}$, so that keeping into account $diam(trans(x,x')) \leq Dd(x,x')+D$ and the same estimate for y,y' we have

$$A = trans(x,x') \cup trans(x',y') \cup trans(y',y) \subseteq \overline{N}_{Dk+K+2D}(\{x,y\}),$$

and we get $trans(x,y) \subseteq \overline{N}_{Mk}(\{x,y\})$ as $trans(x,y) \subseteq \overline{N}_{2D}(A)$. The second part is obvious for $d(x,P) \leq 4D+1$ or $d(y,P) \leq 4D+1$ (as we allow an increase of D), so assume that this is not the case. Consider the first points x'', y'' in $\eta(x,x'), \eta(y,y')$ at distance $4D+1$ from P. By 6) there exists $z \in trans(x'',y'') \cap \overline{N}_D(P) \cap B_K(x'')$. Such z cannot be $2D$-close to $trans(x,x'')$ (keeping 2) into account) and $trans(y,y'')$ (as we can bound $d(x,z)$ linearly in $d(x,P)$). Hence, it is $2D$-close to $trans(x,y)$, so that $trans(x,y)$ intersects $\overline{N}_{3D}(P)$. We are done up to increasing D to $4D+1$. □

Lemma 3.4.5. *Let α be any rectifiable path connecting, say, x to y. Then we have $trans(x,y) \subseteq \overline{N}_{f(\alpha)}(\alpha)$, where $f(\alpha) = D\log_2 \max\{l(\alpha),1\} + D$.*

Proof. Indeed, this holds if α has length at most 2 by condition 1). Suppose $l(\alpha) \geq 2$. Let α_1, α_2 be subpaths of α of equal length and let z be the common endpoint. Then, by 3), $trans(x,y) \subseteq \overline{N}_D(trans(x,z) \cup trans(z,y))$. On the other hand, we can assume by induction that $trans(x,z), trans(z,y)$ are both contained in the $(D\log_2(l(\alpha)/2) + D)$-neighbourhood of α. So, $trans(x,y)$ is contained in the $(D\log_2(l(\alpha)/2) + 2D)$-neighbourhood of α, and we are done as

$$D\log_2(l(\alpha)/2) + 2D = D\log_2 l(\alpha) + D.$$

□

We will also need "logarithmic quasi-convexity", i.e. the following estimate.

Lemma 3.4.6. *Suppose that α is a geodesic joining x to y. Denote*

$$\xi(\alpha) = \sup\{d(p, \alpha|_{[x',y']}) | x', y' \in \alpha, p \in trans(x',y')\}.$$

Then for every $P \in \mathcal{P}$ we have $\alpha \subseteq \overline{N}_L(P)$, where $L = \max\{Md(x,P)+M, Md(y,P)+M, \xi(\alpha)+D\}$ and M is as in 6′).

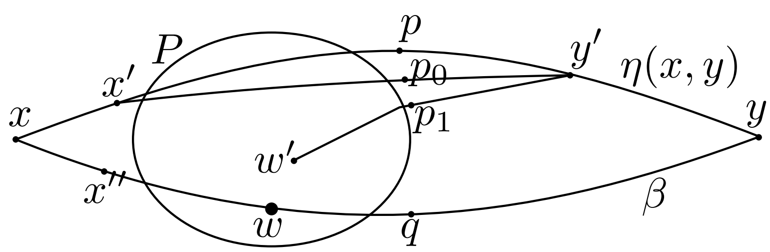
Notice that $\xi(\alpha)$ is bounded logarithmically in $l(\alpha)$ by Lemma 3.4.5.

Proof. Set $k = \max\{d(x,P), d(y,P)\}$. Let $\beta = \alpha|_{[x',y']}$ be so that $\beta \cap \overline{N}_k(P) = \{x', y'\}$. Suppose by contradiction that there exists $p \in \beta$ with $d(p,P) > L$. In particular, $d(x',y') \geq 2(L-k) \geq Mk$. Let $z \in trans(x',y')$ be as in $6'$). Clearly, $d(z,\beta) \geq d(p,P) - D > \xi(\alpha)$, a contradiction. □

We are now ready for the core of the proof and we will proceed as follows. We first show (I) that any point in $trans(x,y)$ is close to any geodesic β connecting x to y. Then we show (II) that points in $trans_{\mu,c'}(\beta)$, for suitable μ, c', are close to $trans(x,y)$, and in order to do so we will actually show that if $p \in \beta$ is far from $trans(x,y)$ then $p \in deep_{\mu,c'}(\beta)$. Finally, we show (III) that points in $trans(x,y)$ are close to $trans_{\mu,c_0}(\beta)$ for some $c_0 \geq c'$, but we will actually show that points in $deep_{\mu,c_0}(\beta)$ are far from $trans(x,y)$. Furthermore, μ, c' will be chosen so that each $deep_{\mu,c'}(\beta)$ is a disjoint union of subpaths each contained in a large subpath with endpoints in $\overline{N}_\mu(P)$ for some $P \in \mathcal{P}$. It is easily deduced then that $d_{Haus}(trans_{\mu,c}(\beta), trans_{\mu,c'}(\beta)) \leq c - c'$ for all $c \geq c'$ (Q-C).

Let now β be a geodesic connecting x to y.

Step (I). Let $p \in trans(x,y)$ be the point at maximal distance from β. We want to give a bound on $d(p,\beta)$. Let q be a closest point to p in β and set $d(p,q) = \xi$. We can assume $\xi \geq 4D+1$ and $l(\beta) \geq 1$, for otherwise we are done. Also, up to substituting β with a suitable subpath, we can assume $\xi \geq \xi(\beta) - 1$, as defined in Lemma 3.4.6.



Consider the closest $x' \in trans(x,y)$ before p so that $d(x',p) \geq \min\{2\xi, d(p,x)\}$, and define y' similarly. By 2) we have $d(p,p_0) \leq D$ for some $p_0 \in trans(x',y')$. The proof in the hyperbolic setting (in the case $x' \neq x, y' \neq y$) relies on $d(x',p), d(y',p) = 2\xi$, but a linear upper bound in ξ would make the proof work as well. So, we would like to reduce to the case when x' is either x or is not too far from p_0 (and similarly for y') by substituting x' with some w' if necessary. Set $w' = x'$ and $p' = p_0$ if $d(x',p) < M_0\xi$, where $M_0 = M_0(D,M)$ is large enough to allow us to carry out the following argument. If $d(x',p)$ is large, by 4) we have that x',p lie in $\overline{N}_{2\xi}(P)$ for some $P\in\mathcal{P}$. Let $x''\in\beta$ be so that $d(x',x'')\leq\xi$. By Lemma 3.4.6, $\beta|_{[x'',q]}\subseteq\overline{N}_L(P)$, where $L\leq 3M\xi+M+D$. We can choose $w\in\beta|_{[x'',q]}$, $w'\in P$ so that $d(w,w')\leq L+1$, $M+D<d(w',p_0)\leq M_0L$. From 6′) we have $d(p_0,trans(w',x'))>D$, so that by 3) (with x',w',y')) we have $d(p_0,trans(w',y'))\leq D$. Fix $p_1\in trans(w',y')$ with $d(p_0,p_1)\leq D$. Up to running the same argument on a final subpath of $\eta(w',y')$ in case $d(y',p)$ is large, we get a point p_2 (possibly $p_1=p_2$) with $d(p_1,p_2)\leq D$ and a point z' so that $p_2\in trans(w',z')$ and $d(z',p_2),d(w',p_2)\leq M_0\xi+4D$, $d(z',\beta),d(w',\beta)\leq L+1$, but either $d(w',p_2)\geq 2\xi-4D$ or $w'=x$, and similarly for z'. Consider now a path α obtained concatenating in the suitable order a geodesic of length at most $L+1$ from w' to $a\in\beta$, a geodesic of length at most $L+1$ from $b\in\beta$ to y' and $\beta|_{[a,b]}$ (choose $a=x$ and/or $b=y$ if $x'=x$ and/or $y'=y$). The length of α is at most $A=2L+2+(2L+2+2M_0\xi+8D)$, which, upon fixing M,D,M_0, can be bounded linearly in ξ. Notice that $\xi=d(p,\beta)\leq d(p_2,\alpha)+4D\leq D\log_2(A)+4D$, by Lemma 3.4.5. This gives a bound μ for ξ in terms of D,M (recall that M_0 depends on D,M).

Step (Q-C). We now show that geodesics with endpoints in $\overline{N}_\mu(P)$ for some $P\in\mathcal{P}$ are contained in $\overline{N}_{\mu'}(P)$, a fact that will be needed later. As a consequence of this and 5), for each c' large enough $deep_{\mu,c'}(\beta)$ is a disjoint union of subpaths each contained in a large subpath with endpoints in $\overline{N}_\mu(P)$ for some $P\in\mathcal{P}$, as subpaths of β with endpoints in $\overline{N}_\mu(P)$ and $\overline{N}_\mu(P')$ for $P,P'\in\mathcal{P}$ with $P\neq P'$ have controlled intersection. Suppose that a geodesic α as above contains a subpath α' outside $\overline{N}_{\mu+D+1}(P)$ with endpoints x,y. Then, if by contradiction $d(x,y)$ is larger than some suitable R, by 6) there exists $z\in trans(x,y)\cap\overline{N}_D(P)$. But such point cannot be μ-close to α. So, $d(x,y)\leq R$ and hence $\alpha'\subseteq\overline{N}_{\mu'=\mu+D+R+1}(P)$, which in turn gives $\alpha\subseteq\overline{N}_{\mu'}(P)$.

Step (II). Fix c' as determined in (Q-C). Let now $p\in\beta$ be at distance at least $\max\{c'+\mu,2\mu\}+1$ from $trans(x,y)$. We wish to show that $p\in deep_{\mu,c'}(\beta)$. Let p_1,p_2 be the closest points on the sides of p in β at distance at most μ from $trans(x,y)$. Clearly, $d(p_1,p_2)\geq\max\{2c',2\mu\}+1$. Let q_1 be the last point on $trans(x,y)$ so that $d(q_1,\beta_1)\leq\mu$, where β_1 is the subgeodesic of β with final point p_1. Define q_2 and β_2 similarly. Notice that there are no points in $trans(x,y)$ between q_1 and q_2, as any point in $trans(x,y)$ is μ-close to either a point in β_1 or a point in β_2 and $d(\beta_1,\beta_2)\geq d(p_1,p_2)>2\mu$. In particular, q_1,q_2 both lie on some $P\in\mathcal{P}$ by 4), which implies (considering $\beta|_{[p_1,p_2]}$) that p is (μ,c')-deep, as required.

Step (III). Finally, we have to show that if $p\in deep_{\mu,B}(\beta)$, where B will be determined later, then $d(p,trans(x,y))>\mu$. Let p_1,p_2 be the endpoints of the subpath of β contained in the neighbourhood of some $P \in \mathcal{P}$ as in the definition of $deep_{\mu,B}(\beta)$ (in particular, $d(p_1, p_2) \geq 2B$). Notice that $p_1, p_2 \in trans_{\mu,B}(\beta)$ if $B \geq diam(\overline{N}_{\mu'}(P) \cap \overline{N}_{\mu'}(P'))$ for any $P, P' \in \mathcal{P}$ with $P \neq P'$ and for k' as in (Q-C). We have, by Step (II), $d(p_i, q_i) \leq \max\{c' + \mu, 2\mu\} + 1 = A$ for some $q_i \in trans(x, y)$. In particular, $d(q_i, P) \leq A + \mu$ so that for B large enough we have, by 6) and 2),

$$trans(x, y) \subseteq \overline{N}_{K+D}(\eta|_{[x,q_1]} \cup \eta|_{[q_2,y]}),$$

for K as in 6) with $k = A + \mu$. This easily implies the claim, up to further increasing B, because for B large enough

$$d(x, p) \notin [0, d(x, q_1) + \mu + K + D] \cup [d(x, q_2) - \mu - K - D, d(x, y)]$$

as β is a geodesic.

This concludes the proof of part (b). $\square$

Theorem 3.4.7. *Let X be a geodesic metric space and let $\mathcal{P}$ be a collection of subsets of X. Then $(X, \mathcal{P})$ satisfies (RH3) if and only if it is relatively hyperbolic.*

Moreover, if $(X, \mathcal{P})$ is relatively hyperbolic, given a model of $Bow(X)$ there exist K, μ, R_0 so that for each geodesic γ in $Bow(X)$ connecting $x, y \in X$ we have that $\gamma \cap X$ is within Hausdorff distance $K + R$ from $trans_{\mu,R}([x, y])$ for each $R \geq R_0$.

Proof. We showed the implication $\Rightarrow$ in Proposition 3.4.2.

$\Leftarrow$: Recall that combinatorial horoballs are within bounded Hausdorff distance from horoballs in the sense of Busemann functions, see Lemma 2.3.7.

Recall also that each $P \in \mathcal{P}$ is coarsely connected so that for any pair of points on P there is a path in X connecting them and staying within bounded distance of P. We can use Theorem 3.4.3 applied to $\eta(x, y)$ constructed substituting each subpath of $[x, y] \subseteq Bow(X)$ contained in some combinatorial horoball with a path in P with the same endpoints, with $trans(x, y)$ being $[x, y] \cap X$. In view of the previous paragraph and the fact that the metric $d_{Bow(X)}$ is coarsely equivalent to d_X on X, all conditions are readily checked. More precisely, 1) and 4) are clear, 5) and 6) follow from the corresponding statements in $Bow(X)$ and 2) can be proven in a similar way to 3), which we are about to show. Consider a geodesic triangle in $Bow(X)$ with vertices $x, y, z \in X$. Consider a point $w \in [x, y] \cap X$ and assume without loss of generality that there exists $p \in [y, z]$ with $d_{Bow(X)}(p, w) \leq \delta$, the hyperbolicity constant. As $y, z \in X$ and we can regard the levels of H as horospheres, there is on $[y, z] \cap X$ a point q not too far away from p, meaning that $d_{Bow(X)}(p, q)$ is bounded by some constant depending on $Bow(X)$ only. Given this, 3) follows from the coarse equivalence of $d_{Bow(X)}$ and d_X on X. $\square$

3.5 Proof of the distance formula

Recall that the symbol $\{\{A\}\}_L$ denotes A if $A \geq L$ and 0 otherwise, and that we use $\approxeq$ to denote equality up to multiplicative constant at most λ and additive constant at most μ.

Theorem 3.5.1. *Let $(X, \mathcal{P})$ be relatively hyperbolic and let $Bow(X)$ be a Bowditch space for X. Then there exists L_0 with the property that for each $L \geq L_0$ there exist $\lambda, \mu \geq 1$ so that for each $x, y \in X$ we have*

$$d(x,y) \approxeq_{\lambda,\mu} \sum_{P \in \mathcal{P}} \{\{d(\pi_P(x), \pi_P(y))\}\}_L + d_{Bow(X)}(x,y).$$

Proof. As usual, we will refer to transient parts and deep components implying a choice of constants as in Lemma 3.2.4.

Given a geodesic γ from x to y, its length is at least $d_{Bow(X)}(x,y)$ and, if L is large enough, corresponding to every term contributing to the sum on the right hand side (RHS) there is a deep component in γ of comparable length (meaning up to bounded multiplicative error). In particular, the RHS gives a coarse lower bound for the LHS.

In order to show the other inequality we use the "moreover" part of Theorem 3.4.7. Start with a geodesic γ in $Bow(X)$ from x to y. We can form a path α in X from x to y by substituting each subpath of γ contained in a horoball by a geodesic in X. We have $d(x,y) \leq l(\alpha)$, so we now estimate the length of α. Let $\gamma_1, \dots, \gamma_k, \beta_1, \dots, \beta_k$ be the geodesics we used to replace subpaths of γ, enumerated in such a way that the γ_i's are the ones of length at most some suitable K to be determined. The sum of the lengths of the γ_i's is at most $Kl(\gamma)$, as each of them replaces a path of length at least 1. Also, if K is large enough, we see that the endpoints of each β_i are close to the endpoints of a deep component, say along P_i, of some fixed geodesic in X from x to y, and such deep components are disjoint. By Lemma 3.2.4-(3), β_i has length at most, say, $2d(\pi_{P_i}(x), \pi_{P_i}(y)) \geq 2L$ when K is large enough. To sum up, the length of α is at most $(K+1)l(\gamma)$ (what we get by replacing short horoball paths only) plus

$$2 \sum_i \{\{d(\pi_{P_i}(x), d_{P_i}(y))\}\}_L.$$

In particular, the RHS provides a coarse upper bound for $l(\alpha)$ and hence for $d(x,y)$. $\square$

3.6 Divergence of relatively hyperbolic groups

In this section we give a simple application of the notions developed in this chapter, and determine properties of the divergence function of a relatively hyperbolic group. Roughly speaking, the divergence of a metric space measures the length of detours avoiding a specified ball as a function of the radius of the ball. It has been first studied in [84] and [80], and in recent years in [104, 13, 134, 73, 71, 12, 9].

We now recall the definition of divergence of a metric space X, following [71]. Choose constants $0 < \delta < 1$ and $\gamma \geq 0$. For a triple of points $a, b, c \in X$ with $d(c, \{a, b\}) = r > 0$, let $div_\gamma(a, b, c; \delta)$ be the infimum of the lengths of paths connecting a, b that avoid $B(c, \delta r - \gamma)$. (If no such path exists, set $div_\gamma(a, b, c; \delta) = \infty$.)

Definition 3.6.1. The divergence function $Div^X_\gamma(n, \delta)$ of the space X is defined as the supremum of all numbers $div_\gamma(a, b, c; \delta)$ with $d(a, b) \leq n$.

For functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ write $f \preceq g$ if there exists C so that $f(n) \leq Cg(Cn + C) + Cn + C$, and define $\succeq, \asymp$ similarly. By [71, Corollary 3.12], there exist δ_0, γ_0 so that when X is a Cayley graph we have $Div^X_\gamma(n, \delta) \asymp Div^X_{\gamma_0}(n, \delta_0)$ whenever $0 < \delta \leq \delta_0$ and $\gamma \geq \gamma_0$. Also, the $\asymp$-equivalence class of $Div^X_\gamma(n, \delta)$ is a quasi-isometry invariant (of Cayley graphs). We will write $Div^X(n)$ for (the $\asymp$-equivalence class of) $Div^X_{\gamma_0}(n, \delta_0)$.

We now show the following.

Theorem 3.6.2. *Suppose that the one-ended group G is hyperbolic relative to the (possibly empty) collection of proper subgroups $H_1, \dots, H_k$. Then*

$$e^n \preceq Div^G(n) \preceq \max\{e^n, e^n Div^{H_i}(n)\}.$$

(We write $\max\{e^n, e^n Div^{H_i}(n)\}$ instead of $\max\{e^n Div^{H_i}(n)\}$ because we allow the collection of subgroups to be empty.)

Not all super-exponential functions $f(n)$ satisfy $f(n) \asymp e^n f(n)$, hence the following questions arise.

Question 3.6.3. Does there exist a group G of super-exponential divergence so that $e^n Div^G(n) \not\asymp Div^G(n)$? If so, is it true that $Div^G(n) \preceq \max\{e^n, Div^{H_i}(n)\}$? Do there exist one-ended properly relatively hyperbolic groups with isomorphic peripheral subgroups but different divergence?

In the case of finitely presented groups we have a cleaner statement, due to the following lemma.

Lemma 3.6.4. *The divergence of a finitely presented one-ended group is at most exponential.*

Proof. All pairs of points x, y, say with $r_x = d(x,1) \leq d(y,1) = r_y$, can be joined by a path in $\overline{B}(1, 2r_y) \backslash B(1, r_x/3)$, see e.g. Lemma 4.6.6. The intersection of such subset and G has $\preceq e^{r_y}$ points, and the conclusion easily follows. $\square$

A relatively hyperbolic group is finitely presented if and only if its peripheral subgroups are, see [57] and [141]. Hence we have the following.

Corollary 3.6.5. *Suppose that the one-ended group G is hyperbolic relative to its proper subgroups $H_1, \dots, H_k$. Then $Div^G(n) \asymp e^n$.*

We split the proof of the theorem in two propositions.

Proposition 3.6.6. *Let the group G be hyperbolic relative to its proper subgroups $H_1, \dots, H_k$. Then $Div^G(n) \succeq e^n$.*

Proof. Denote $\mathcal{H} = \{H_1, \dots, H_n\}$. By [140] there exists an infinite virtually cyclic subgroup $E(g)$ in G so that $(G, \mathcal{H} \cup \{E(g)\})$ is relatively hyperbolic. In particular, $E(g)$ is within bounded Hausdorff distance from a bi-infinite geodesic α so that, for each μ, $diam(\alpha \cap \overline{N}_\mu(gH_i))$ is uniformly bounded for each $g \in G$ and each H_i (by geometric separation and quasi-convexity of peripheral sets). We would like to show that if the path β, which we assume to have length at least 1 for simplicity, connects points $a, b \in \alpha$ on opposite sides of some $c \in \alpha$ then $d(c, \beta) \leq C \log_2(l(\beta)) + C$ for some constant C. In order to do so, we modify a standard argument which can be found, e.g., in [37, Proposition III.H.1.6] (and which we already used in Lemma 3.4.5). For μ as in (RH3) and R large enough we have $trans_{\mu,R}(\alpha) = \alpha$, and similarly for all subgeodesics of α. Let $q \in \beta$ be so that $l(\beta|_{[a,q]}) = l(\beta|_{[q,b]})$ and consider a geodesic triangle with vertices a, b, q. We have some D so that $\alpha|_{[a,b]} = trans_{\mu,R}(\alpha|_{[a,b]}) \subseteq \overline{N}_D(trans_{\mu,R}([a,q]) \cup trans_{\mu,R}([q,b]))$. We can assume by induction (starting with paths of length at most, say, 100), that $trans_{\mu,R}([a,q]) \subseteq \overline{N}_{D\log_2(l(\beta)/2)+100}(\beta)$ and similarly for $[q,b]$, and we get $\alpha_{[a,b]} \subseteq \overline{N}_{D\log_2(l(\beta))+100}(\beta)$ as required. $\square$

The author was not able to find a reference for the following fact in the case of hyperbolic groups.

Proposition 3.6.7. *Let the one-ended group G be hyperbolic relative to $H_1, \dots, H_k$. Then $Div^G(n) \preceq \max\{e^n, e^n Div^{H_i}(n)\}$.*

Proof. It is enough to find a bound on $div_\gamma(a,b,c;\delta)$ when a,b,c are elements of G (as opposed to points in the interior of edges). Fix $a,b,c\in G$, and suppose without loss of generality $r=d(a,c)\leq d(b,c)$. We would like to find a "short" path connecting a to b outside $B=B(c,r/\delta-\gamma)$, for $\delta>0$ small enough and γ large enough. We can assume $r\leq 10n$ for $n=d(a,b)$ (for otherwise a geodesic from a to b would avoid B). As G is one-ended, by [71, Lemma 3.4] there exists a path β connecting a,b outside B. Consider $a=x_0,\dots,x_k=b$ on β so that $d(x_i,x_{i+1})\leq 1$, where $x_i\in G$. Define $y_i\in G$ to be the first point of $[x_i,c]$ in $B'=B(c,d(b,c))$. We then have a sequence of points $a=y_0,\dots,y_k=b$ in $(B'\cap G)\backslash B$, which contains $\preceq e^n$ elements.

If G was hyperbolic, we would have a uniform bound on $d(y_i,y_{i+1})$. Then it would be readily seen that we can choose $0=j_0<\dots<j_l=k$ so that $d(y_{j_i},y_{j_{i+1}})$ is bounded and $l\preceq e^n$, so that the desired path can be obtained concatenating geodesics of the form $[y_{j_i},y_{j_{i+1}}]$.

In the relatively hyperbolic case, it is easily seen by the relative Rips condition (and the deep components being contained in neighbourhoods of peripheral sets) that for an appropriate D we have either $d(y_i,y_{i+1})\leq D$ or there exists a left coset P_i of some H_i so that $y_i,y_{i+1}\in\overline{N}_D(P_i)$. Also, all geodesics from c to P_i pass D-close to some $z_i\in P_i$, see Lemma 3.1.1-(2). Assume for convenience that we are considering a Cayley graph with respect to a generating set containing generating sets for the H_i's, so that P_i is connected. As $d(y_i,y_{i+1})\leq 2d(c,b)$ and peripheral subgroups are undistorted, there exists a path α in P_i of length $\preceq Div^{P_i}(2d(c,b))$ connecting points y'_i,y'_{i+1} at distance at most D from y_i,y_{i+1} which avoids a P_i-ball of radius $\delta_0\min\{d(y'_i,z_i),d(y'_{i+1},z_i)\}-\gamma_0$ around z_i. Again as peripheral subgroups are undistorted, we have that α avoids B, if we chose δ,γ large enough. Now, we can choose $0=j_0<\dots<j_l=k$ so that $l\preceq e^n$ and y_{j_i} can be connected to $y_{j_{i+1}}$ by a path avoiding B and of length $\preceq\max\{Div^{H_i}(n)\}$. So, we can construct a path of length $\preceq\max\{e^nDiv^{H_i}(n)\}$ connecting a to b. $\square$

3.7 Relative hyperbolicity of hyperbolic spaces

We record the following result, essentially due to Bowditch, for later purposes and also because it might provide useful examples of relatively hyperbolic spaces for the reader to keep in mind.

Lemma 3.7.1. *Suppose that X is hyperbolic and $\mathcal{P}$ is a collection of subsets of X. Then X is hyperbolic relative to $\mathcal{P}$ if and only if $\mathcal{P}$ is geometrically separated and all $P\in\mathcal{P}$ are quasi-convex with the same constants.*

Proof. The following is well-known, but we provide a proof for the convenience of the reader. As usual, we denote (almost) closest-point projections on P by π_P.

Lemma 3.7.2. *Suppose that X is hyperbolic and $P \subseteq X$ is quasi-convex. Then there exists C, depending on the quasi-convexity constants only, so that for each $x, y \in X$ if $d(\pi_P(x), \pi_P(y)) \geq C$ then $d([x,y], \pi_P(x)), d([x,y], \pi_P(y)) \leq C$ for each geodesic $[x,y]$ from x to y.*

Proof. Set $\pi = \pi_P$ and let δ be a hyperbolicity constant for X. As quadrangles in X are 2δ-thin, we have that $[\pi(x), \pi(y)]$ is contained in the 2δ-neighbourhood of $[x,y] \cup [x, \pi(x)] \cup [y, \pi(y)]$. Due to our choice of π, points $[\pi(x), \pi(y)]$ farther than K from the endpoints are not 2δ-close to $[x, \pi(x)] \cup [y, \pi(y)]$, where K depends on the quasi-convexity constants of P. Hence, for $d(\pi(x), \pi(y)) > 2K + 1$, there is a point on $[x,y]$ which is 2δ-close to the point on $[\pi(x), \pi(y)]$ at distance $K+1$ from $\pi(x)$, and so we have $d(\alpha, \pi(x)) \leq K + 2\delta + 1$. $\square$

The first part of (RH3)-(2) follows directly form the lemma, for a suitable choice of constants. The second part follows from the fact that, for any geodesic γ and a suitable μ, the entrance point p in the μ-neighbourhood of P is a transient point, by the following argument we already used elsewhere: If not, p would be contained in a long subgeodesic of γ with endpoints close to some $Q \in \mathcal{P}$ with $Q \neq P$ and would be far from the endpoints of such geodesic, and this would contradict geometric separation of $\mathcal{P}$. Also, notice that by the same argument we can define the deep components and that they are disjoint.

Now, we have to check the Relative Rips Condition. It is clear that any transient point p on a side γ_1 of a geodesic triangle is close to a point q on one of the other two sides γ_2, we only need to show that q is close to a transient point (possibly on the third side γ_3 of the triangle). If R, the second constant in the definition of transient point, is large enough then we can argue as follows.

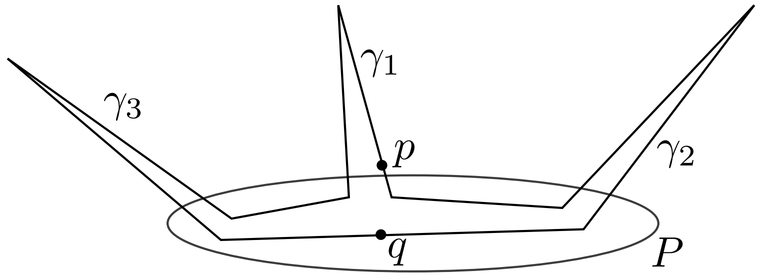


Figure 3.5: The configuration we are describing.

If q is in the deep component of γ_2 along P, then by Lemma 3.7.2 the projections of the endpoints of γ_2 are far from p. As p is close to P and is transient along γ_1, we claim that it is close to the projection on P of one of the endpoints of γ_1. Let us conclude the proof assuming that this is true. As p cannot be close to the projection on P of the common endpoint of γ_1 and γ_2, it is close to the projection of the common endpoint r of γ_1 and γ_3. There is a transient point along γ_3 close to such projection again by Lemma 3.7.2 (the common endpoint of γ_2 and γ_3 projects far away from the projection of r), so that we are done.

We are left to prove the claim. If γ_1 has no deep component along P, then p cannot be far from the projection of an endpoint of γ_1 for this would be easily seen to force the existence of a deep component. If γ_1 has a deep component along P, we apply the same argument to the initial or final subgeodesic of γ_1 before or after the deep component, depending on which of those contains p. $\square$

Part II

Boundaries of relatively hyperbolic groups

In this part we will study geometric properties of boundaries of relatively hyperbolic groups in order to show quasi-isometric embeddings results. In Chapter 4, which is taken from [117], we will construct quasi-isometric embeddings of $\mathbb{H}^2$ into suitable relatively hyperbolic groups, and in order to do so we will construct arcs in the boundary satisfying appropriate geometric conditions.

The material in Chapter 5 is taken from [118]. We will construct quasi-isometric embeddings of relatively hyperbolic groups into products of trees, and estimate the number of trees we need. We will then apply this in the context of fundamental groups of 3-manifolds.

Chapter 4

Quasi-hyperbolic planes

4.1 Introduction

A well known question of Gromov asks whether every hyperbolic group which is not virtually free contains a surface group. While this question is still open, its geometric analogue has a complete solution. Bonk and Kleiner [29], answering a question of Papazoglou, showed the following.

Theorem 4.1.1 (Bonk-Kleiner [29])**.** *A hyperbolic group contains a quasi-isometrically embedded copy of $\mathbb{H}^2$ if and only if it is not virtually free.*

In this chapter, we study when a relatively hyperbolic group admits a quasi-isometrically embedded copy of $\mathbb{H}^2$. Our main result is the following.

Theorem 4.1.2. *Let $(G, \mathcal{P})$ be a finitely generated relatively hyperbolic group, where all peripheral subgroups are virtually nilpotent. Then there is a quasi-isometrically embedded copy of $\mathbb{H}^2$ in G if and only if G does not split as a non-trivial graph of groups where the edge groups are finite and the vertex groups virtually nilpotent.*

This result generalizes Theorem 4.1.1 since a finitely generated group is virtually free if and only if it admits a graph of groups decomposition with all vertex and edge groups finite.

For a hyperbolic group G, quasi-isometrically embedded copies of $\mathbb{H}^2$ in G correspond to quasisymmetrically embedded copies of the circle $S^1 = \partial_\infty \mathbb{H}^2$ in the boundary of the group. Bonk and Kleiner build such a circle when a hyperbolic group has connected boundary by observing that the boundary is *doubling* (there exists N so that any ball can be covered by N balls of half the radius) and *linearly connected* (there exists L so that any points x and y can be joined by a continuum of diameter at most $Ld(x, y)$). For such spaces, a theorem of Tukia applies to find quasisymmetrically embedded arcs, or *quasi-arcs* [158].

We note that this proof relies on the local connectedness of boundaries of one ended hyperbolic groups, a deep result following from work of Bestvina and Mess, and Bowditch and Swarup [20, Proposition 3.3], [31, Theorem 9.3], [33, Corollary 0.3], [155].

Our strategy is similar to that of Bonk and Kleiner, but to implement this we have to prove several basic results regarding the geometry of the boundary of a relatively hyperbolic group, i.e. the boundary $\partial_\infty(G, \mathcal{P}) = \partial_\infty Bow(G, \mathcal{P})$ of its Bowditch space, which we believe are of independent interest.

We fix a choice of $Bow(G, \mathcal{P})$ and, for suitable conditions on the peripheral subgroups, we show that the boundary $\partial_\infty(G, \mathcal{P})$ has good geometric properties. For example, using work of Dahmani and Yaman, such boundaries will be doubling if and only if the peripheral subgroups are virtually nilpotent. We establish linear connectedness when the peripheral subgroups are one ended. (See Sections 4.4 and 4.5 for precise statements.)

At this point, by Tukia's theorem, we can find quasi-isometrically embedded copies of $\mathbb{H}^2$ in $Bow(G, \mathcal{P})$, but these can stray far away from G into horoballs in $Bow(G, \mathcal{P})$. To prove our theorem we must ensure that this does not occur, and we do so by building a quasi-arc in the boundary that suitably avoids the parabolic points.

To accomplish this, we require additional geometric properties of the boundary (see Section 4.6), and also a generalisation of Tukia's theorem which builds quasi-arcs that avoid certain kinds of obstacles (Theorem 4.7.2).

Our methods are able to find embeddings that avoid more subgroups than just virtually nilpotent peripheral groups.

Theorem 4.1.3. *Suppose both $(G, \mathcal{P}_1)$ and $(G, \mathcal{P}_1 \cup \mathcal{P}_2)$ are finitely generated relatively hyperbolic groups, where all peripheral subgroups in $\mathcal{P}_1$ are virtually nilpotent and non-elementary, and all peripheral subgroups in $\mathcal{P}_2$ are hyperbolic. Suppose G is one-ended and does not split over a subgroup of a conjugate of some $P \in \mathcal{P}_1$.*

Finally, suppose that $\partial_\infty H \subset \partial_\infty(G, \mathcal{P}_1)$ does not locally disconnect the boundary, for any $H \in \mathcal{P}_2$ (see Definition 4.4.2). Then there is a transversal *quasi-isometric embedding of $\mathbb{H}^2$ in G.*

If both $\mathcal{P}_1$ and $\mathcal{P}_2$ are empty the group is hyperbolic and the result is a corollary of Theorem 4.1.1. If $\mathcal{P}_1$ is empty, but $\mathcal{P}_2$ is not, then the group is hyperbolic, but the quasi-isometric embeddings we find avoid the hyperbolic subgroups conjugate to those in $\mathcal{P}_2$.

Example 4.1.4. Let M be a compact hyperbolic 3-manifold with a single, totally geodesic surface as boundary ∂M. The fundamental group $G = \pi_1(M)$ is hyperbolic, and also is hyperbolic relative to $F = \pi_1(\partial M)$ (see, for example, [14, Proposition 13.1]).

The hypotheses of Theorem 4.1.3 are satisfied for $\mathcal{P}_1 = \emptyset$ and $\mathcal{P}_2 = \{F\}$, since $\partial_\infty G = \partial_\infty(G, \emptyset)$ is a Sierpiński carpet, with the boundary of conjugates of F corresponding to the peripheral circles of the carpet. Thus, we find a transversal quasi-isometric embedding of $\mathbb{H}^2$ into G.

Roughly speaking, a quasi-isometric embedding fails to be transversal if the image contains points which are far apart from each other, but both close to the same left coset of a peripheral subgroup (see Definition 4.2.1). The notion of transversality is interesting for us in view of the following results. First, our embeddings persist in the coned-off (or "electrified") graph of G.

Proposition 4.2.3. *Let $\hat{\Gamma}$ be the coned-off graph of a relatively hyperbolic group $(G, \mathcal{P})$ and let $c : \Gamma \to \hat{\Gamma}$ be the natural map. Suppose Z is a geodesic metric space. If a quasi-isometric embedding $f : Z \to G$ is transversal then $c \circ f : Z \to \hat{\Gamma}$ is a quasi-isometric embedding.*

Second, the embeddings constructed above also persist in certain peripheral fillings of G.

Proposition 4.2.4. *Let G be hyperbolic relative to $P_1, \dots P_n$ (with a fixed system of generators) and let $N_i \lhd P_i$. Let Z be a geodesic metric space, and suppose that $f : Z \to G$ is a transversal quasi-isometric embedding. There exists R_0, depending only on suitable data, so that if the ball of radius R_0 around the identity e satisfies $B(e, R_0) \cap N_i = \{e\}$ for each i, then $p \circ f : Z \to G/\!\ll\!\{N_i\}\!\gg$ is a quasi-isometric embedding, where $p : G \to G/\!\ll\!\{N_i\}\!\gg$ is the quotient map.*

Recall that $G/\!\ll\!\{N_i\}\!\gg$ is usually referred to as a peripheral (or Dehn) filling of G, and these are relatively hyperbolic (with peripheral groups $\{P_i/N_i\}$) under the hypotheses described above [138, 85] (see Section 4.2.2).

When combined, Theorem 4.1.3 and Proposition 4.2.4 provide interesting examples of relatively hyperbolic groups containing quasi-isometrically embedded copies of $\mathbb{H}^2$ that do not have virtually nilpotent peripheral subgroups. A key point here is that Theorem 4.1.3 provides embeddings transversal to hyperbolic subgroups, and so one can find many interesting peripheral fillings. See Example 4.2.6 for details.

Using our results, we describe when the fundamental group of a closed, oriented 3-manifold contains a quasi-isometrically embedded copy of $\mathbb{H}^2$. Determining which 3-manifolds (virtually) contain immersed or embedded π_1-injective surfaces [98, 54, 109, 50] is a very difficult problem. The following theorem essentially follows from known results, in particular work of Masters and Zhang [124, 125]. However, our proof is a simple consequence of Theorem 4.1.2 and the geometrisation theorem.

Theorem 4.9.2. *Let M be a closed 3-manifold. Then $\pi_1(M)$ contains a quasi-isometrically embedded copy of $\mathbb{H}^2$ if and only if M does not split as the connected sum of manifolds each with geometry $S^3, \mathbb{R}^3, S^2 \times \mathbb{R}$ or* Nil.

Notice that the geometries mentioned above are exactly those that give virtually nilpotent fundamental groups.

As a final observation, we note that Leininger and Schleimer recently proved a result similar to Theorem 4.1.2 for Teichmüller spaces [114], using very different techniques.

4.1.1 Outline

In Section 4.2 we discuss transversality and its consequences. In Section 4.3 we give preliminary results linking the geometry of the boundary of a relatively hyperbolic group to that of its model space. Further results on the boundary itself are found in Sections 4.4–4.6, in particular, how sets can be connected, and avoided, in a controlled manner.

The existence of quasi-arcs that avoid obstacles is proved in Section 4.7. The proofs of Theorems 4.1.2 and 4.1.3 are given in Section 4.8. Finally, connections with 3-manifold groups are explored in Section 4.9.

4.2 Transversal quasi-hyperplanes

4.2.1 Transversality and coned-off graphs

Our goal here is to define transversality of quasi-isometric embeddings, and show that a transversal quasi-isometric embedding of $\mathbb{H}^2$ in $\Gamma = \Gamma(G)$ will persist if we cone-off the Cayley graph.

We continue with the notation of Definition 2.1.2.

Definition 4.2.1. Let $(G, \mathcal{P})$ be a relatively hyperbolic group. Let Z be a geodesic metric space. Given a function $\eta : [0, \infty) \to [0, \infty)$, a quasi-isometric embedding

$f : Z \to (G, d_G)$ is *η-transversal* if for all $M \geq 0$, $g \in G$, and $P \in \mathcal{P}$, we have $\mathrm{diam}(f(Z) \cap N(gP, M)) \leq \eta(M)$.

A quasi-isometric embedding $f : Z \to G$ is *transversal* if it is η-transversal for some such η.

Let $\hat{\Gamma}$ be the coned-off graph of G and let $c : \Gamma \to \hat{\Gamma}$ be the natural map. Essentially, $\hat{\Gamma}$ is the graph obtained by adding an edge of length one between any two members of the same coset of the same peripheral subgroup, and it is hyperbolic. This is part of the definition of relative hyperbolicity given in [76], which is equivalent to Definition 2.1.2 and can also be shown using, e.g. Proposition 3.4.1 using the description of geodesic triangles in relatively hyperbolic groups, Lemma 3.2.6.

We call λ-quasi-geodesic a (λ, λ)-quasi-isometric embedding of an interval.

Lemma 4.2.2. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group. If $\alpha : \mathbb{R} \to (G, d_G)$ is an η-transversal λ-quasi-geodesic, then $c(\alpha)$ is a λ'-quasi-geodesic in $\hat{\Gamma}$, where $\lambda' = \lambda'(\lambda, \eta, \hat{\Gamma})$. Moreover, α is also a λ'-quasi-geodesic in $Bow(G, \mathcal{P})$.*

Proof. Notice that α is coarsely Lipschitz in both $Bow(G)$ and $\hat{\Gamma}$ as on Γ we have $d_{\hat{\Gamma}} \leq d_{Bow(G)} \leq d_G$. Therefore, it suffices to show that for some $\lambda' = \lambda'(\lambda, \eta, \hat{\Gamma})$, for any $x_1, x_2 \in \alpha$, we have

$$d_{\hat{\Gamma}}(x_1, x_2) \geq \tfrac{1}{\lambda'} d_G(x_1, x_2) - \lambda'. \tag{4.1}$$

Let γ be the subgeodesic of α with endpoints x_1, x_2. Let $\hat{\gamma}$ be a geodesic in $\hat{\Gamma}$ with endpoints $c(x_1), c(x_2)$.

Now let $\pi : \hat{\Gamma} \to \hat{\gamma}$ be a closest point projection map, fixing x_1, x_2. As $\hat{\Gamma}$ is hyperbolic, such a projection map is coarsely Lipschitz: there exists $C_1 = C_1(\hat{\Gamma})$ so that for all $x, y \in \hat{\Gamma}$, $d_{\hat{\Gamma}}(\pi(x), \pi(y)) \leq C_1 d_{\hat{\Gamma}}(x, y) + C_1$.

By [92, Lemma 8.8], there exists $C_2 = C_2(\Gamma, \lambda)$ so that every vertex of $\hat{\gamma}$ is at most a distance C_2 from γ in Γ (not just $\hat{\Gamma}$). Let $\pi' : (\hat{\gamma}, d_{\hat{\Gamma}}) \to (\gamma, d_G)$ be a map so that for all $x \in \hat{\gamma}$, $d_G(\pi'(x), x) \leq C_2$, and assume that π' fixes x_1, x_2. This map is coarsely Lipschitz also. It suffices to check this for points $x, y \in \hat{\gamma}$ connected by an edge e. If e is an edge of Γ, then clearly $d_G(\pi'(x), \pi'(y)) \leq 2C_2 + 1$. Otherwise, $\pi'(x), \pi'(y)$ are both in γ and within distance C_2 of the same left coset of a peripheral group, so by transversality, $d_G(\pi'(x), \pi'(y)) \leq \eta(C_2)$.

Thus, for $C_3 = \max\{2C_2 + 1, \eta(C_2)\}$, (4.1) with $\lambda' = C_3 C_1$ follows from

$$\begin{aligned} d_G(x_1, x_2) = d_G(\pi' \circ \pi(x_1), \pi' \circ \pi(x_2)) &\leq C_3 d_{\hat{\Gamma}}(\pi(x_1), \pi(x_2)) \\ &\leq C_3(C_1 d_{\hat{\Gamma}}(x_1, x_2) + C_1). \end{aligned}$$

□

Proposition 4.2.3. *Suppose Z is a geodesic metric space, and $(G, \mathcal{P})$ a relatively hyperbolic group. If a quasi-isometric embedding $f : Z \to G$ is transversal then $c \circ f : Z \to \hat{\Gamma}$ is a quasi-isometric embedding.*

Proof. By Lemma 4.2.2, whenever γ is a geodesic in Z, $c \circ f(\gamma)$ is a quasi-geodesic with uniformly bounded constants in $\hat{\Gamma}$. $\square$

4.2.2 Stability under peripheral fillings

We now consider peripheral fillings of $(G, \{P_1, \ldots, P_n\})$.

Suppose $N_i \triangleleft P_i$ are normal subgroups. The *peripheral filling* of G with respect to $\{N_i\}$ is defined as

$$G(\{N_i\}) = G/ \ll \bigcup_i N_i \gg .$$

(See [138] and [85].) Let $p : G \to G(\{N_i\})$ be the quotient map.

Proposition 4.2.4. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group, and $G(\{N_i\})$ a peripheral filling of G, as defined above.*

Let Z be a geodesic metric space, and suppose that $f : Z \to G$ is an η-transversal (λ, λ)-quasi-isometric embedding. There exists $R_0 = R_0(\eta, \lambda, G, \mathcal{P})$ so that if $B(e, R_0) \cap N_i = \{e\}$ for each i, then $p \circ f : Z \to G(\{N_i\})$ is a quasi-isometric embedding.

Proof. We will use results from [58] (see also [63, 55]). For any sufficiently large R_0 we can combine Propositions 7.6 and 5.20 in [58] to show that $G(\{N_i\})$ acts on a certain δ'-hyperbolic space Y (namely $Y = X'/Rot$ for X' and $Rot = (C, \mathcal{R})$ as in Proposition 7.6), with the following properties:

1. δ' only depends on the hyperbolicity constant of $X = Bow(G, \mathcal{P})$,
2. there is a 1-Lipschitz map $\psi : G(\{N_i\}) \to Y$ (this follows from the construction of Y and the fact that the inclusion of G in X is 1-Lipschitz),
3. there is a map $\phi : (N(G, L) \subseteq X) \to Y$ such that, for each $g \in G$, $\phi|_{B(g,L)}$ is an isometry, where $L = L(R_0, X) \to \infty$ as $R_0 \to \infty$.
4. $\psi \circ p = \phi \circ \iota$, where $\iota : G \to X$ is the natural inclusion.

Let γ be any geodesic in Z. By Lemma 4.2.2, $f(\gamma)$ is a λ'-quasi-geodesic in X, for $\lambda' = \lambda'(\lambda, \eta, G, \mathcal{P})$. Let α be a geodesic in X connecting the endpoints of $f(\gamma)$. Let $C_1 = C_1(\delta', \lambda')$ bound the distance between each point on α and $G \subseteq Bow(G)$.

Suppose that L as in (3) satisfies $L \geq C_1 + 8\delta' + 1$. Then for each $x \in \alpha$ we have that $\phi|_{B(x,8\delta'+1)}$ is an isometry, and so [37, Theorem III.H.1.13-(3)] gives that $\phi(\alpha)$ is a C_2-quasi-geodesic, where $C_2 = C_2(\delta')$. This implies that $(\phi \circ \iota \circ f)(\gamma)$ is a C_3-quasi-geodesic, with $C_3 = C_3(C_1, C_2)$. Let x_1, x_2 be the endpoints of γ. Using (4) above, we see that

$$\begin{aligned} d_{G(\{N_i\})}(p \circ f(x_1), p \circ f(x_2)) &\geq d_Y(\psi \circ p \circ f(x_1), \psi \circ p \circ f(x_2)) \\ &= d_Y(\phi \circ \iota \circ f(x_1), \phi \circ \iota \circ f(x_2)) \\ &\geq \tfrac{1}{C_3} d(x_1, x_2) - C_3. \end{aligned}$$

On the other hand, recall that p is 1-Lipschitz, so

$$d_{G(\{N_i\})}(p \circ f(x_1), p \circ f(x_2)) \leq d_G(f(x_1), f(x_2)) \leq \lambda d_Z(x_1, x_2) + \lambda.$$

As γ was arbitrary, we are done. $\square$

As discussed in the introduction, we can use Proposition 4.2.4 to find interesting examples of relatively hyperbolic groups with quasi-isometrically embedded copies of $\mathbb{H}^2$, but whose peripheral groups are not virtually nilpotent. We note the following lemma.

Lemma 4.2.5. *Let F_4 be the free group with four generators, and let R be fixed. Then there are normal subgroups $\{K_\alpha\}_{\alpha \in \mathbb{R}}$ of F_4 so that the quotient groups F_4/K_α are amenable but not virtually nilpotent, so that if $\alpha \neq \beta$ then F_4/K_α and F_4/K_β are not quasi-isometric, and so that $K_\alpha \cap B(e, R) = \{e\}$.*

Proof. It is shown in [82] that there is an uncountable family of 4-generated groups $\{F_4/K'_\alpha\}_{\alpha \in \mathbb{R}}$ of intermediate growth with distinct growth rates. In particular, these groups are amenable, not virtually nilpotent and pairwise non-quasi-isometric. To conclude the proof, let K be a finite index normal subgroup of F_4 so that $K \cap B(e, R) = \{e\}$, and let $K_\alpha = K'_\alpha \cap K \lhd F_4$. As F_4/K_α is a finite extension of F_4/K'_α, it inherits all the properties above. $\square$

Example 4.2.6. Let M be a closed hyperbolic 3-manifold so that $G = \pi_1(M)$ is hyperbolic relative to a subgroup $P \leq G$ that is isomorphic to F_4. For example, let M' be a compact hyperbolic manifold whose boundary $\partial M'$ is a totally geodesic surface of genus 3, and let M be the double of M' along $\partial M'$. Observe that $\pi_1(M)$ is hyperbolic relative to $\pi_1(\partial M')$, and $\pi_1(\partial M')$ is hyperbolic relative to a copy of $\pi_1(S') = F_4$, where $S' \subset \partial M'$ is a punctured genus 2 subsurface. Thus $\pi_1(M)$ is hyperbolic relative to $\pi_1(S')$.

Since G is hyperbolic with 2-sphere boundary, and P is quasi-convex in G with Cantor set boundary, the hypotheses of Theorem 4.1.3 are satisfied for $\mathcal{P}_1 = \emptyset$ and $\mathcal{P}_2 = \{P\}$. Therefore, we find a transversal quasi-isometric embedding of $\mathbb{H}^2$ into G.

Let R_0 be chosen by Proposition 4.2.4. As P is quasi-convex in G, we choose R so that for $x \in P$, if $d_P(e, x) \geq R$ then $d_G(e, x) \geq R_0$. Now let $\{K_\alpha\}$ be the subgroups constructed in Lemma 4.2.5. By Proposition 4.2.4, for each $\alpha \in \mathbb{R}$ the peripheral filling $G_\alpha = G/ \ll K_\alpha \gg$ is relatively hyperbolic and contains a quasi-isometrically embedded copy of $\mathbb{H}^2$.

As P/K_α is non-virtually cyclic and amenable, it does not have a non-trivial relatively hyperbolic structure. Therefore, G_α is not hyperbolic relative to virtually nilpotent subgroups, for in any peripheral structure $\mathcal{P}$, some peripheral group $H \in \mathcal{P}$ must be quasi-isometric to P/K_α by [10, Theorem 4.8].

Finally, if $\alpha \neq \beta$ then G_α is not quasi-isometric to G_β by [10, Theorem 4.8] as P/K_α and P/K_β are not quasi-isometric.

4.3 Separation of parabolic points and horoballs

In this section we study how the boundaries of peripheral subgroups are separated in $\partial_\infty X$. We also establish a preliminary result on quasi-isometrically embedding copies of $\mathbb{H}^2$.

4.3.1 Separation estimates

We begin with the following lemma.

Lemma 4.3.1. *Let $(G, \mathcal{P}_1)$ and $(G, \mathcal{P}_1 \cup \mathcal{P}_2)$ be relatively hyperbolic groups, where all peripheral subgroups in $\mathcal{P}_2$ are hyperbolic groups ($\mathcal{P}_2$ is allowed to be empty), and set $X = Bow(G, \mathcal{P}_1)$. Let $\mathcal{H}$ denote the collection of the horoballs of X and the left cosets of the subgroups in $\mathcal{P}_2$; more precisely, the images of those left cosets under the natural inclusion $G \hookrightarrow X$. For $H \in \mathcal{H}$, let $d_H = d(w, H)$.*

Then the collection of subsets $\mathcal{H}$ has the following properties.

1. *For each $r \geq 0$ there is a uniform bound $b(r)$ on $\operatorname{diam}(N(H, r) \cap N(H', r))$ for each $H, H' \in \mathcal{H}$ with $H \neq H'$.*

2. *Each $H \in \mathcal{H}$ is uniformly quasi-convex in X.*

3. *There exists R such that, given any $H \in \mathcal{H}$ and any geodesic ray γ connecting w to $a \in \partial_\infty H$, the subray of γ whose starting point has distance d_H from w is entirely contained in $N(H, R)$.*

Proof. Claim (1) follows from the corresponding fact in the Cayley graph (see [68, Theorem 4.1(α_1)] or Section 3.2).

Let us show (2). Uniform quasi-convexity of the horoballs is a consequence of Lemma 2.3.4. If H is a left coset of a peripheral subgroup in $\mathcal{P}_2$, then it is quasi-convex in the Cayley graph Γ of G ([68, Lemma 4.15] or Corollary 3.2.6). What is more, by [68, Theorem 4.1(α_1)], geodesics in Γ connecting points of H are transversal with respect to $\mathcal{P}_1$. Therefore, by Lemma 4.2.2 they are quasi-geodesics in X. We conclude that H is quasi-convex in X since pairs of points of H can be joined by quasi-geodesics (with uniformly bounded constants) which stay uniformly close to H.

We now show (3). As $a \in \partial_\infty H$ and H is quasi-convex, there exists $C = C(X)$ so that $d(x, H) \leq C$ for each $x \in \gamma$ with $d(x, w)$ large enough. Fix such an x.

Let $y \in \gamma$ be such that $d(y, w) = d_H$. Let $p \in H$ be chosen so that $d(w, p) \leq d_H + 1$. By the defining property of y and the thinness of the geodesic triangle with vertices w, p, x, we have $d(y, H) \leq d(y, p) \leq C'(\delta_X)$. The desired property now follows from (2). □

From this lemma, we can deduce separation properties for the boundaries of sets in $\mathcal{H}$.

Lemma 4.3.2. *We make the assumptions of Lemma 4.3.1. Then there exists $C = C(X)$ so that for each $H, H' \in \mathcal{H}$ with $H \neq H'$ and $d_H \geq d_{H'}$ we have*

$$\rho(\partial_\infty H, \partial_\infty H') \geq e^{-\epsilon d_H}/C.$$

Proof. Let $a \in \partial_\infty H, a' \in \partial_\infty H'$. We have to show that $(a|a') \lesssim d_H$. Let γ, γ' be rays connecting w to a, a', respectively. For each $p \in \gamma$ such that $d(w, p) \leq (a|a')$ there exists $p' \in \gamma'$ such that $d(p', w) = d(p, w)$ and $d(p, p') \leq C' = C'(\delta_X)$. With $b(r)$ and R as found by Lemma 4.3.1, set $B = b(R + C')$.

Suppose that $(a|a') \geq d_H + B + 1$. Consider $p \in \gamma$, $p' \in \gamma$ such that $d(p, w) = d(p', w) = d_H$. By Lemma 4.3.1(3), we have $p \in N(H, R)$ and $p' \in N(H', R)$, as $d_H \geq d_{H'}$. So, $p \in N(H, R + C') \cap N(H', R + C')$. The same holds also when $p \in \gamma$ is such that $d(p, w) = d_H + B + 1$. Therefore we have $\mathrm{diam}(N(H, R+C') \cap N(H', R+C')) > B$, contradicting Lemma 4.3.1(1). Hence $(a|a') < d_H + B + 1$, and we are done. □

Conversely, we show that separation properties of certain points in the boundary $\partial_\infty X$ have implications for the intersection of sets in X.

Lemma 4.3.3. *We make the assumptions of Lemma 4.3.1.*

Let γ be a geodesic line connecting w to $a \in \partial_\infty X$. Suppose that for some $H \in \mathcal{H}$ and $r \geq 1$ we have $d(a, \partial_\infty H) \geq e^{-\epsilon d_H}/r$. Then γ intersects $N(H, L)$ in a set of diameter bounded by $K + 2L$, for $K = K(r, X)$ and any $L \geq 0$.

In order to prove the lemma we will use [81, Lemma 2.12], which states that any finite configuration of points and geodesics in a δ_X-hyperbolic space X is approximated by a tree, up to an additive error that depends only on δ_X and the number of points and geodesics involved. So, we only need to show a result in the case of trees, and (up to controlled error) we will have obtained the same result for all hyperbolic spaces. In the notation of Figure 1, notice that $\beta_c(x, w) = d(x, p) - d(w, p)$ (the point p can be chosen to be the point on a geodesic from x to w such that $d(w, p) = (c|x)$; this makes sense in all hyperbolic spaces, not just trees.)

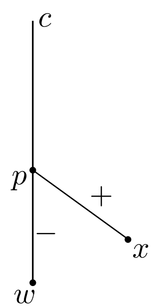


Figure 4.1: How to compute the Busemann function in a tree.

Proof of Lemma 4.3.3. We will treat the horoball case and the left coset case separately, beginning with the latter.

We can assume that H is not bounded, for otherwise the lemma is trivially true. Due to quasi-convexity, each point on H is at uniformly bounded distance from a geodesic line connecting points a_1, a_2 in $\partial_\infty H$, and thus also at uniformly bounded distance, say $C_1 = C_1(X)$, from either a ray connecting w to a_1 or a ray connecting w to a_2.

Let $c \in \partial_\infty H$ and let γ' be a ray connecting w to c. We have that $(a|c) \leq d_H + r' + C_2$, for $r' = \log(r)/\epsilon$ and $C_2 = C_2(\delta_X, \epsilon, C_0)$.

There exists $C_3 = C_3(C_1, X)$ so that any point x on γ such that $d(x, w) \geq (a|c) + L + C_3$ has the property that $d(x, \gamma') \geq L + C_1$. This applies to all rays connecting w to some $c \in \partial_\infty H$, and so $d(x, H) > L$.

Also, if $x \in \gamma$ and $d(x,w) < d_H - L$ then clearly $d(x,H) > L$. Thus if $\gamma \cap N(H,L) \neq \emptyset$ we have $(a|c) + L + C_3 \geq d_H - L$, and

$$\operatorname{diam}(\gamma \cap N(H,L)) \leq (a|c) + L + C_3 - (d_H - L) \lesssim_{C_2+C_3} r' + 2L.$$

We are left to deal with the horoball case. Let c and γ' be as above. Once again, $(a|c) \leq d_H + r' + C_2$, for $r' = \log(r)/\epsilon$. In an approximating tree for γ and γ' (see Figure 1), if x is any point on γ such that $d(x,w) > 2(a|c) - d_H + L$, one has $d(x,H) > L$ because

$$\beta_c(x,w) = d(x,p) - d(p,w) = d(x,w) - 2(a|c) > -d_H + L.$$

Therefore, using Lemma 2.3.1 and [81, Lemma 2.12], one finds $C_4 = C_4(X)$ so that if $d(x,w) > 2(a|c) - d_H + L + C_4$ then $d(x,H) > L$. Arguing as before, one sees that, for $C_5 = 2C_2 + C_4$,

$$\operatorname{diam}(\gamma \cap N_L(H)) \leq 2(a|c) - d_H + L + C_4 - (d_H - L) \lesssim_{C_5} 2r' + 2L. \qquad \square$$

4.3.2 Embedded planes

In order to find a quasi-isometrically embedded copy of $\mathbb{H}^2$ in a relatively hyperbolic group, we only need to embed a half-space of $\mathbb{H}^2$ into our model space X, provided that the embedding does not go too far into the horoballs. (Compare with [29].) As we see later, this means that we do not need to embed a quasi-circle into the boundary of X, but merely a quasi-arc.

Definition 4.3.4. The *standard half-space* in $\mathbb{H}^2$ is the set $Q = \{(x,y) : x^2 + y^2 < 1, x \geq 0\}$ in the Poincaré disk model for $\mathbb{H}^2$.

Let G, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{H}$ and $X = Bow(G, \mathcal{P}_1)$ be as in Lemma 4.3.1.

Proposition 4.3.5. *Let $f : Q \to X$ be a (λ,λ)-quasi-isometric embedding of the standard half-space $\mathbb{H}^2$ into X, with $f((0,0)) = w$. Suppose there exists $r > 0$ so that for each $a \in \partial_\infty Q$, $H \in \mathcal{H}$ we have*

$$d(f(a), \partial_\infty H) \geq e^{-\epsilon d_H}/r.$$

Then there exists a transversal (with respect to $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$) quasi-isometric embedding $g : \mathbb{H}^2 \to \Gamma(G)$.

Proof. Each point $x \in Q \setminus \{(0,0)\}$ lies on a unique geodesic γ_x connecting $(0,0)$ to a point a_x in $\partial_\infty Q$. As $f(\gamma_x)$ is a λ-quasi-geodesic and X is hyperbolic, $f(\gamma_x)$ lies within distance $C_1 = C_1(\lambda, \delta_X)$ from a geodesic γ'_x from w to $f(a_x)$. By Lemma 4.3.3 any point on γ'_x is $C_2 = C_2(r, X)$ close to a point in $\Gamma(G)$, and therefore any point in $f(Q)$ is $C_1 + C_2$ close to a point in $\Gamma(G)$.

Notice that Q contains balls $\{B_n\}$ of $\mathbb{H}^2$ of arbitrarily large radius, each of which admits a (λ, λ)-quasi-isometric embedding $f_n : B_n \to X$ so that each point in $f_n(B_n)$ is a distance $C_1 + C_2$ close to a point in $\Gamma(G)$. In particular, translating those embeddings appropriately using the action of G on X we can and do assume that the centre of B_n is mapped at uniformly bounded distance from w. As X is proper, we can use Arzelà-Ascoli to obtain a (λ', λ')-quasi-isometric embedding $\hat{g} : \mathbb{H}^2 \to X$ as the limit of a subsequence of $\{f_n\}$ (more precisely $\{f_n|_N\}$, where N is a maximal 1-separated net in $\mathbb{H}^2$), for $\lambda' = \lambda'(\lambda, C_1, C_2)$. Notice that $\hat{g}$ is transversal.

We now define $g : \mathbb{H}^2 \to \Gamma(G)$ so that for each $x \in \mathbb{H}^2$ we have $d(\hat{g}(x), g(x)) \leq C_3$, for $C_3 = C_3(C_1, C_2)$.

Pick $x, y \in \mathbb{H}^2$. Notice that

$$d_G(g(x), g(y)) \geq d(g(x), g(y)) \gtrsim_{2C_3} d(\hat{g}(x), \hat{g}(y)) \gtrsim_{\lambda'} d(x,y)/\lambda',$$

so it is enough to show that for some λ'',

$$d_G(g(x), g(y)) \leq \lambda'' d(\hat{g}(x), \hat{g}(y)) + \lambda''.$$

Let γ be the geodesic in $\mathbb{H}^2$ connecting x to y. Let γ' be a geodesic in X from $\hat{g}(x)$ to $\hat{g}(y)$, which is at Hausdorff distance at most $C_4 = C_4(\lambda', C_3, \delta_X)$ from $\hat{g}(\gamma)$.

Each maximal subpath $\beta \subseteq \gamma'$ contained in some horoball $O \in \mathcal{O}$ has length $l(\beta)$ at most $C_5 = C_5(C_4)$. If $z, z' \in G$ are the endpoints of β, then $d_G(z, z') \leq M d(z, z')$, for some $M = M(C_5, X)$, as d_G is a proper function of d, and if $z \neq z'$ then $d(z, z')$ is uniformly bounded away from zero. So we can substitute β by a subpath in $\Gamma(G)$ of length at most $Ml(\beta)$.

Let α be the path in $\Gamma(G)$ obtained from γ' by substituting each such β in this way. Clearly we have $l(\alpha) \leq Ml(\gamma')$, and so

$$d_G(\hat{g}(x), \hat{g}(y)) \leq l(\alpha) \leq M d(\hat{g}(x), \hat{g}(y)).$$

The desired estimate follows as $d(\hat{g}(x), g(x)), d(\hat{g}(y), g(y)) \leq C_3$. $\square$

4.4 Geometry of boundaries

We now begin our study of the geometry of the boundary of a relatively hyperbolic group, endowed with a visual metric ρ as in Section 2.4. In this section, we study the properties of being doubling and having partial self-similarity.

First, we summarize some known results about the topology of such boundaries.

Theorem 4.4.1 (Bowditch)**.** *Suppose $(G, \mathcal{P})$ is a one-ended relatively hyperbolic group which does not split over a subgroup of a conjugate of some $P \in \mathcal{P}$, and every group in $\mathcal{P}$ is finitely presented, one or two ended, and contains no infinite torsion subgroup. Then $\partial_\infty(G, \mathcal{P})$ is connected, locally connected and has no global cut points.*

Proof. Connectedness and local connectedness follow from [35, Prop. 10.1], [32, Thm. 0.1].

Any global cut point must be a parabolic point [33, Thm. 0.2], but the splitting hypothesis ensures that these are not global cut points either [32, Prop. 5.1, Thm. 1.2]. □

Recall that a point p in a connected, locally connected, metrisable topological space Z is *not a local cut point* if for every connected neighbourhood U of p, the set $U \setminus \{p\}$ is also connected. If, in addition, Z is compact, then Z is locally path connected, so p is not a local cut point if and only if every neighbourhood U of p contains an open V with $p \in V \subset U$ and $V \setminus \{p\}$ path connected.

More generally, we have the following definition, used in the statement of Theorem 4.1.3.

Definition 4.4.2. A closed set $V \subsetneq Z$ in a connected, locally connected, metrisable topological space Z *does not locally disconnect* Z if for any open connected $U \subset Z$, the set $U \setminus V$ is also connected.

For relatively hyperbolic groups, we note the following.

Proposition 4.4.3. *Suppose $(G, \mathcal{P})$ is relatively hyperbolic with connected and locally connected boundary. Let p be a parabolic point in $\partial_\infty(G, \mathcal{P})$ which is not a global cut-point. Then p is a local cut point if and only if the corresponding peripheral group has more than one end.*

Proof. The lemma follows, similarly to the proof of [56, Proposition 3.3], from the fact that the parabolic subgroup P corresponding to p acts properly discontinuously and cocompactly on $\partial_\infty(G,\mathcal{P})\backslash\{p\}$, which is connected and locally connected. Let us make this precise.

Choose an open set K_0 with compact closure in $\partial_\infty(G,\mathcal{P})\backslash\{p\}$, so that $PK_0 = \partial_\infty(G,\mathcal{P})\backslash\{p\}$. Then define K_1 as the union of all qK_0 for $q \in P$ with $d_P(q,e) \leq 1$. As $\partial_\infty(G,\mathcal{P})\backslash\{p\}$ is connected and locally path connected, and $\overline{K_1}$ is compact, one can easily find an open, path connected K so that $K_1 \subset K \subset \overline{K} \subset \partial_\infty(G,\mathcal{P})\backslash\{p\}$.

Now suppose that P has one end. Let U be a neighbourhood of p. As P acts properly discontinuously on $\partial_\infty(G,\mathcal{P})\backslash\{p\}$, there exists R so that if $d_P(e,g) > R$, then $gK \subset U$. Let Q be the unbounded connected component of $P \setminus B(e,R)$. Then QK is path connected as for $g,h \in P$, if $d_P(g,h) \leq 1$, $gK \cap hK \neq \emptyset$. Finally, observe that $V = QK \cup \{p\} \subset U$ is a neighbourhood of p so that $V \setminus \{p\} = QK$ is connected.

Conversely, suppose that p is not a local cut-point. Let D be so that if $qK \cap rK \neq \emptyset$ then $d_P(q,r) \leq D$. Suppose we are given $R > 0$. We can find a connected neighbourhood U of p in $\partial_\infty(G,\mathcal{P})$ so that $U \setminus \{p\}$ is path connected and $gK \cap U = \emptyset$ for all $g \in B(e, R+D) \subset P$. Let $R' \geq R + D$ be chosen so that $gK \cap U \neq \emptyset$ for all $g \in P \setminus B(e,R')$. Given $g,h \in P\backslash B(e,R')$ we can find a path in $U\backslash\{p\}$ connecting gK to hK. So, there exists a sequence $g = g_0, g_1, \ldots, g_n = h$ in $P \setminus B(e, R+D)$ so that $g_iK \cap g_{i+1}K \neq \emptyset$ for all $i = 0, \ldots, n-1$. Thus as $d_P(g_i, g_{i+1}) \leq D$, we have that g and h can be connected in P outside $B(e,R)$. As R was arbitrary, P is one ended. □

4.4.1 Doubling

Definition 4.4.4. A metric space (X,d) is *N-doubling* if every ball can be covered by at most N balls of half the radius.

Every hyperbolic group has doubling boundary, but this is not the case for relatively hyperbolic groups.

Proposition 4.4.5. *The boundary of a relatively hyperbolic group $(G,\mathcal{P})$ is doubling if and only if every peripheral subgroup is virtually nilpotent.*

Recall that all relatively hyperbolic groups we consider are finitely generated, with $\mathcal{P}$ a finite collection of finitely generated peripheral groups.

Proof. By [59, Theorem 0.1], every peripheral subgroup is virtually nilpotent if and only if $X = Bow(G,\mathcal{P})$ has bounded growth at all scales: for every $0 < r < R$ there exists some N so that every radius R ball in X can be covered by N balls of radius r.

If X has bounded growth at some scale then $\partial_\infty X$ is doubling [28, Theorem 9.2].

On the other hand, if $\partial_\infty X$ is doubling, then $\partial_\infty X$ quasisymmetrically embeds into some $\mathbb{S}^{n-1}$ (see [6], or [88, Theorem 12.1]). Therefore, X quasi-isometrically embeds into some $\mathbb{H}^n$ [28, Theorems 7.4, 8.2], which has bounded growth at all scales. We conclude that X has bounded growth at all scales (for small scales, the bounded growth of X follows from the finiteness of $\mathcal{P}$, and the finite generation of G and all peripheral groups). □

4.4.2 Partial self-similarity

The boundary of a hyperbolic group G with a visual metric ρ is self-similar: there exists a constant L so that for any ball $B(z,r) \subset \partial_\infty G$, with $r \leq \operatorname{diam}(\partial_\infty G)$, there is a L-bi-Lipschitz map f from the rescaled ball $(B(z,r), \frac{1}{r}\rho)$ to an open set $U \subset \partial_\infty G$, so that $B(f(z), \frac{1}{L}) \subset U$. (See [30, Proposition 3.3] or [49, Proposition 6.2] for proofs that omit the claim that $B(f(z), \frac{1}{L}) \subset U$. The full statement follows from the lemma below.)

There is not the same self-similarity for the boundary $\partial_\infty(G, \mathcal{P})$ of a relatively hyperbolic group $(G, \mathcal{P})$, because G does not act cocompactly on $Bow(G, \mathcal{P})$. However, as we see in the following lemma, the action of G does show that balls in $\partial_\infty(G, \mathcal{P})$ with centres suitably far from parabolic points are, after rescaling, bi-Lipschitz to large balls in $\partial_\infty(G, \mathcal{P})$. The proof of Lemma 4.4.6 follows [30, Proposition 3.3] closely.

Partial self-similarity is essential in the following two sections, as we use it to control the geometry of the boundary away from parabolic points. Near parabolic points we use the asymptotic geometry of the corresponding peripheral group to control the geometry of the boundary.

Lemma 4.4.6. *Suppose X is a δ_X-hyperbolic, proper, geodesic metric space with base point $w \in X$. Let ρ be a visual metric on the boundary $\partial_\infty X$ with parameters C_0, ϵ. Then for each $D > 0$ there exists $L_0 = L_0(\delta_X, \epsilon, C_0, D) < \infty$ with the following property:*

Whenever we have $z \in \partial_\infty X$ and an isometry $g \in \operatorname{Isom}(X)$ so that some $y \in [w, z)$ satisfies $d(g^{-1}w, y) \leq D$, then g induces an L_0-bi-Lipschitz map f from the rescaled ball $(B(z,r), \frac{1}{r}\rho)$, where $r = e^{-\epsilon(d(w,y)+\delta_X+1)}/2C_0$, to an open set $U \subset \partial_\infty G$, so that $B(f(z), \frac{1}{L_0}) \subset U$.

Proof. We assume that z, g and r are fixed as above. We use the following equality:

$$-\frac{1}{\epsilon}\log(2rC_0) = d(w,y) + \delta_X + 1. \tag{4.2}$$

For every $z_1, z_2 \in B(z, r)$, and every $p \in (z_1, z_2)$, one has

$$d(y, [w, p]) \leq 3\delta_X. \tag{4.3}$$

This is easy to see: $\rho(z_1, z_2) \leq 2r$, so $d(w, (z_1, z_2)) \geq -\frac{1}{\epsilon}\log(2rC_0)$. Let $y_1 \in [w, z_1)$ be so that $d(y_1, w) = d(y, w)$, and notice that $d(y_1, (z_1, z_2)) > \delta_X$ by (4.2). For any $p \in (z_1, z_2)$, the thinness of the geodesic triangle w, z_1, p implies that $d(y_1, [w, p]) \leq \delta_X$. In particular, for $z_2 = p = z$, we have $d(y_1, [w, z)) \leq \delta_X$, so $d(y_1, y) \leq 2\delta_X$, and the general case follows.

Let $g \in G$ be given so that $d(g^{-1}w, y) \leq D$. For any $z_1, z_2 \in B(z, r)$, by (4.2), (4.3) we have

$$\begin{aligned} d(w, (z_1, z_2)) &\approx_{6\delta_X} d(w, y) + d(y, (z_1, z_2)) \\ &\approx_{(7\delta_X + D + 1)} \frac{-1}{\epsilon}\log(2rC_0) + d(g^{-1}w, (z_1, z_2)). \end{aligned}$$

As $d(g^{-1}w, (z_1, z_2)) = d(w, (gz_1, gz_2))$, this gives that

$$L_0^{-1}\frac{\rho(z_1, z_2)}{r} \leq \rho(gz_1, gz_2) \leq L_0\frac{\rho(z_1, z_2)}{r},$$

for any choice of $L_0 \geq 2C_0^3 e^{\epsilon(7\delta_X + D + 1)}$.

Thus the action of g on $B(z, r)$ defines a L_0-bi-Lipschitz map f with image U, which is open because g is acting by a homeomorphism. It remains to check that $B(f(z), \frac{1}{L_0}) \subset U$.

Suppose that $z_3 \in B(f(z), \frac{1}{L_0})$. Then $d(w, (f(z), z_3)) > \frac{-1}{\epsilon}\log(C_0/L_0)$, but $d(w, (f(z), z_3)) = d(g^{-1}w, (z, g^{-1}z_3))$. So, for large enough L_0, we have

$$\begin{aligned} d(w, (z, g^{-1}z_3)) &\geq \frac{-1}{\epsilon}\log\left(\frac{C_0}{L_0}\right) + d(w, y) - C_1(\delta_X, D) \\ &> \frac{-1}{\epsilon}\log\left(\frac{C_0}{L_0}\right) - \frac{1}{\epsilon}\log(2rC_0) - C_2(C_1, \delta_X) \\ &> \frac{-1}{\epsilon}\log\left(\frac{r}{C_0}\right), \end{aligned}$$

where the last equality follows from increasing L_0 by an amount depending only on ϵ, C_0, C_2. We conclude that $\rho(z, g^{-1}z_3) < r$. $\square$

In our applications, it is useful to reformulate Lemma 4.4.6 so the input of the property is a ball in $\partial_\infty X$ rather than an isometry of X.

Corollary 4.4.7 (Partial self-similarity). *Let X, $\partial_\infty X$, ρ, C_0 and ϵ be as in Lemma 4.4.6. Suppose G acts on X by isometries. Then for each $D > 0$ there exists $L_0 = L_0(\delta_X, \epsilon, C_0, D) < \infty$ with the following property:*

Let $z \in \partial_\infty X$ and $r \leq \operatorname{diam}(\partial_\infty X)$ be given, and set $d_r = -\frac{1}{\epsilon}\log(2rC_0) - \delta_X - 1$. Then

1. *If $d_r \geq 0$, set $x \in [w, z)$ so that $d(w, x) = d_r$. Then for any $y \in [w, x]$ so that $d(g^{-1}w, y) \leq D$, for some $g \in G$, there exists a L_0-bi-Lipschitz map f (induced by the action of g on $\partial_\infty X$) from the rescaled ball $(B(z, r'), \frac{1}{r'}\rho)$, where $r' = re^{\epsilon d(x,y)}$, to an open set $U \subset \partial_\infty G$, so that $B(f(z), \frac{1}{L_0}) \subset U$.*

2. *If $d_r < 0$, then the identity map on $\partial_\infty X$ defines a L_0-bi-Lipschitz map from the rescaled ball $(B(z, r'), \frac{1}{r'}\rho)$, where $r' = r$, to an open set $U \subset \partial_\infty G$, so that $B(f(z), \frac{1}{L_0}) \subset U$.*

Proof. Let L_0' be the value of L_0 given by Lemma 4.4.6. Since $d(w, y) = d_r - d(x, y)$, and $e^{-\epsilon(d_r - d(x,y) + \delta_X + 1)}/2C_0 = r'$, part (1) follows from Lemma 4.4.6.

Note that if $d_r < 0$, then $r > 1/C_1 > 0$ for some $C_1 = C_1(\epsilon, C_0, \delta_X) < \infty$, so part (2) follows from setting $L_0 = \max\{L_0', C_1\}$. □

4.5 Linear Connectedness

Under the hypotheses of Theorem 4.4.1, we saw that $\partial_\infty(G, \mathcal{P})$ was connected and locally connected. In this section we show that $\partial_\infty(G, \mathcal{P})$ will satisfy a quantitatively controlled version of this property.

Definition 4.5.1. We say a complete metric space (X, d) is L-*linearly connected* for some $L \geq 1$ if for all $x, y \in X$ there exists a compact, connected set $J \ni x, y$ of diameter less than or equal to $Ld(x, y)$.

This is also called the L-bounded turning property in the literature. Up to slightly increasing L, we can assume that J is an arc, see [116, Page 3975].

Proposition 4.5.2. *If $(G, \mathcal{P})$ is relatively hyperbolic and $\partial_\infty(G, \mathcal{P})$ is connected and locally connected with no global cut points, then $\partial_\infty(G, \mathcal{P})$ is linearly connected.*

If $\mathcal{P}$ is empty then G is hyperbolic, and this case is already known by work of Bonk and Kleiner [28, Proposition 4]. Lemma 4.4.6 gives an alternate proof of this result, which we include to warm up for the proof of Proposition 4.5.2. Both proofs rely on the work of Bestvina and Mess, and Bowditch and Swarup cited in the introduction.

Corollary 4.5.3 (Bonk-Kleiner)**.** *If the boundary of a hyperbolic group G is connected, then it is linearly connected.*

Proof. Let $X = \Gamma(G)$ by a Cayley graph of G with visual metric ρ, and let L_0 be chosen by Corollary 4.4.7 for $D = 1$. The boundary of G is locally connected [20, 31, 33, 155], so for every $z \in \partial_\infty G$, we can find an open connected set V_z satisfying $z \in V_z \subset B(z, 1/2L_0)$. The collection of all V_z forms an open cover for the compact space $\partial_\infty G$, and so this cover has a Lebesgue number $\alpha > 0$.

Suppose we have $z, z' \in \partial_\infty G$. Let $r = \frac{2L_0}{\alpha}\rho(z, z')$. If $r > \mathrm{diam}(\partial_\infty X)$, we can join z and z' by a set of diameter $\mathrm{diam}(\partial_\infty X) < \frac{2L_0}{\alpha}\rho(z, z')$.

Otherwise, we apply Corollary 4.4.7, using either (1) with $y = x$ or (2), to find an L_0-bi-Lipschitz map $f : (B(z, r), \frac{1}{r}\rho) \to U$. Since $\rho(f(z), f(z')) \leq L_0\rho(z, z')/r = \frac{\alpha}{2}$, we can find a connected set $J \subset B(f(z), \frac{1}{L_0}) \subset U$ that joins $f(z)$ to $f(z')$. Therefore $f^{-1}(J) \subset B(z, r)$ joins z to z', and has diameter at most $2r = \frac{4L_0}{\alpha}\rho(z, z')$. So $\partial_\infty G$ is $4L_0/\alpha$-linearly connected. □

The key step in the proof of Proposition 4.5.2 is the construction of chains of points in the boundary.

Lemma 4.5.4. *Suppose $(G, \mathcal{P})$ is as in Proposition 4.5.2. Then there exists K so that for each pair of points $a, b \in \partial_\infty(G, \mathcal{P})$ there exists a chain of points $a = c_0, \ldots, c_n = b$ such that*

1. *for each $i = 0, \ldots, n$ we have $\rho(c_i, c_{i+1}) \leq \rho(a, b)/2$, and*
2. *$\mathrm{diam}(\{c_0, \ldots, c_n\}) \leq K\rho(a, b)$.*

We defer the proof of this lemma.

Proof of Proposition 4.5.2. Given $a, b \in \partial_\infty(G, \mathcal{P})$, apply Lemma 4.5.4 to get a chain of points $J_1 = \{c_0, \ldots, c_n\}$. For $j \geq 1$, we define J_{j+1} iteratively by applying Lemma 4.5.4 to each pair of consecutive points in J_j, and concatenating these chains of points together. Notice that

$$\mathrm{diam}(J_{j+1}) \leq \mathrm{diam}(J_j) + \frac{2K}{2^j}\rho(a, b).$$

This implies that the diameter of $J = \overline{\bigcup J_j}$ is linearly bounded in $\rho(a, b)$, and J is clearly compact and connected as desired. □

We require two further lemmas before commencing the proof of Lemma 4.5.4. The first is an elementary lemma on the geometry of infinite groups.

Lemma 4.5.5. *Let P be an infinite, finitely generated group with Cayley graph $(\Gamma(P), d_P)$. Then for each $p, q \in P$ there exists a geodesic ray α starting from p and such that $d_P(q, \alpha) \geq d_P(p, q)/3$.*

Proof. As P is infinite, there exists a geodesic line γ through p, which can be subdivided into geodesic rays α_1, α_2 starting from p. We claim that either α_1 or α_2 satisfies the requirement. In fact, if that was not the case we would have points $p_i \in \alpha_i \cap B(q, d_P(p, q)/3)$. Notice that $d_P(p_i, p) \geq 2d_P(p, q)/3$. Now,

$$d_P(p_1, p_2) \leq d_P(p_1, q) + d_P(q, p_2) \leq 2d_P(p, q)/3,$$

but this contradicts

$$d_P(p_1, p_2) = d_P(p_1, p) + d_P(p, p_2) \geq 4d_P(p, q)/3. \qquad \square$$

The next lemma describes the geometry of geodesic rays passing through a horoball. If $a, b \in \partial_\infty X$, we use the notation $p_{a,b}$ for the centre of the quasi-tripod w, a, b, i.e. the point in $[w, a) \subset X$ such that $d(w, p_{a,b}) = (a|b)$.

Lemma 4.5.6. *Fix $O \in \mathcal{O}$ and $a, b \in \partial_\infty(G, \mathcal{P}) \backslash \{a_O\}$. Let γ_a, γ_b be geodesics from a_O to a, b and let q_a, q_b be the last points in $\gamma_a \cap \overline{O}, \gamma_b \cap \overline{O}$, which we assume to be both non-empty. Also, let γ be a geodesic from w to a_O and let q be the first point in $\gamma \cap \overline{O}$ (so that $d_O \approx d(w, q)$). Then there exists $E = E(X) < \infty$ so that the following holds.*

1. *If $(a|a_O) \geq d_O$ then*
$$(a|a_O) \approx_E d(q_a, q)/2 + d_O.$$
2. *If $(a|a_O), (b|a_O) \in [d_O, (a|b)]$ then*
$$(a|b) \gtrsim_E 2(a|a_O) - d_O - d(q_a, q_b)/2 \approx_E d(q_a, q) + d_O - d(q_a, q_b)/2.$$
Moreover, if $d(q_a, q_b) \geq E$ then $\approx_E$ holds in the equation above.
3. *If $(a|a_O) < d_O$ then $d(q_a, q) \approx_E 0$.*
4. *If $p_{a,b} \in O$ and $d(p_{a,b}, X \setminus O) \geq R \geq E$ then $d(p_{a,a_O}, X \setminus O)$ and $d(p_{b,a_O}, X \setminus O)$ are both at least $R - E$, and $d(q_a, q_b) \geq 2R - E$.*

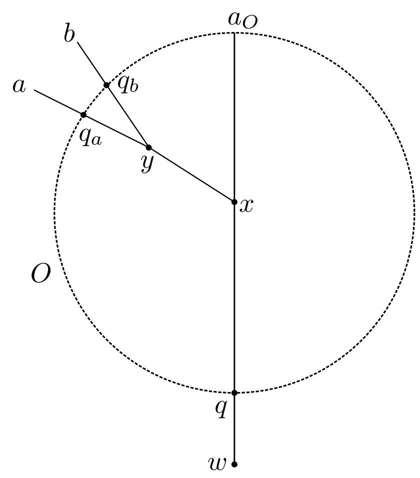


Figure 4.2: Geodesics passing through a horoball

Proof. As in Lemma 4.3.3 we only need to make the computations in the case of trees, illustrated by Figure 4.2, and an approximation argument gives in each case the desired inequalities.

(1) Keeping into account that x lies in O as $d(w,x) = (a|a_O) \geq d_O$, the computation in a tree yields

$$(a|a_O) = d(w,q) + d(q,x) = d_O + d(q_a,q)/2,$$

as $d(x,q) = d(x,q_a)$.

(2) The figure illustrates the first of the two possible types of tree approximating the configuration we are interested in. The second case to consider is when q_a, q_b are between x and y, and thus $q_a = q_b$ in the tree. Therefore, for a suitable choice of E, the "moreover" assumption ensures we are in the first case. In this first case we have the equalities:

$$(a|b) = d(w,q) + 2d(q,x) - d(q_a,q_b)/2 = d_O + 2((a|a_O) - d_O) - d(q_a,q_b)/2,$$

and also

$$(a|b) = d(w,q) + d(q,q_a) - d(q_a,q_b)/2 = d_O + d(q,q_a) - d(q_a,q_b)/2.$$

In the second case we can proceed similarly. We see that

$$\begin{aligned}(a|b) &\geq d(w,q_a) = d_O + d(q,q_a)\\ &= d_O + 2d(q,x) = d_O + 2((a|a_O) - d_O) = 2(a|a_O) - d_O,\end{aligned}$$

which is what we need as $d(q_a,q_b) = 0$.

(3) In the tree approximating this configuration the ray from w to a does not enter the horoball O, so that the bi-infinite geodesic γ_a exits O from q.

(4) There are two types of tree approximating this configuration. The first is given by Figure 4.2, where $p_{a,a_O} = p_{b,a_O} = x$ and $p_{a,b} = y$, so

$$d(p_{a,a_O}, X \setminus O) = d(p_{b,a_O}, X \setminus O) = d(x, q) = d(x, q_a) \geq d(y, q_a) \geq R.$$

In this case $d(q_a, q_b) = 2d(p_{a,b}, q_a) \geq 2R$.

The second configuration is when the geodesics $[w, a)$ and $[w, b)$ branch off from $[w, a_O)$ at different points. Suppose that $(a|a_O) < (b|b_O)$, so p_{a,a_O} lies on $[w, a_O)$ strictly between q and p_{b,b_O}, and thus $p_{a,b} = p_{a,a_O}$. In this case $d(q_a, q_b) \geq d(q_a, p_{a,b}) + d(p_{a,b}, q_b) \geq 2R$, and we have

$$d(p_{b,a_O}, X \setminus O) = d(p_{b,a_O}, q) > d(p_{a,b}, q) \geq R. \qquad \square$$

For each peripheral subgroup $P \in \mathcal{P}$ we denote by d_P the path metric on any left coset of P.

We are now ready to commence the proof of Lemma 4.5.4. This proof is somewhat delicate, splitting into two cases, depending on the position of the the points $a, b \in \partial_\infty(G, \mathcal{P})$. In the first case, we use Lemma 4.5.5 and the asymptotic geometry of a horoball to join a and b by a chain of points. The second case is similar to Corollary 4.5.3 for hyperbolic groups: the partial self-similarity of the boundary upgrades local connectedness to linear connectedness for a and b. In Case 2b, we use the group action and the no global cut point condition to cover the remaining configurations.

Proof of Lemma 4.5.4. We need to find chains of points joining distinct points $a, b \in \partial_\infty(G, \mathcal{P})$, as described in the statement of the lemma.

Recall that $p_{a,b} \in [w, a)$ denotes the centre of the quasi-tripod w, a, b, i.e. the point on $[w, a) \subset X = Bow(G, \mathcal{P})$ such that $d(w, p_{a,b}) = (a|b)$.

Let $R = R(X)$ be a large constant to be determined by Case 1 below. All constants may depend tacitly on C_0, ϵ, δ_X.

Case 1: We first assume that there exists $O \in \mathcal{O}$ such that $p_{a,b} \in O$ and $d(p_{a,b}, gP) > R$, where gP is the left coset of the peripheral subgroup P corresponding to O.

Case 1a: Suppose that $\rho(a, b) \leq \rho(a, a_O)/S$ and $\rho(a, b) \leq \rho(b, a_O)/S$, for some large enough $S > 1$ to be determined. In this case, we push an appropriate geodesic path in gP out to the boundary.

Let $S' = \log(S/C_0^2)/\epsilon$, and note that $(a|b) - (a|a_O) \geq S'$ and $(a|b) - (b|a_O) \geq S'$. We assume that $S' \geq 0$.

Let γ_a be a geodesic from a_O to a and let q_a be the last point in $\gamma_a \cap \overline{O}$. Define γ_b and q_b analogously. Let γ be a geodesic from w to a_O and let q be the first point in $\gamma \cap \overline{O}$.

Assuming $R \geq E$, by Lemma 4.5.6(4) we have

$$(a|a_O) = d(w, p_{a,a_O}) \geq d_O + d(p_{a,a_O}, X \setminus O) \geq d_O + R - E \geq d_O,$$

and likewise $(b|a_O) \geq d_O$. Using Lemma 4.5.6(1) and the approximate equality case of Lemma 4.5.6(2), we have

$$\begin{aligned} d(q_a, q) &\approx_{2E} 2((a|a_O) - d_O) \\ &\geq 2((a|a_O) - d_O) + 2(S' - (a|b) + (a|a_O)) \\ &= 2(2(a|a_O) - (a|b) - d_O + S') \approx_{2E} d(q_a, q_b) + 2S'. \end{aligned} \tag{4.4}$$

We now define our chain of points joining a to b. Let α be a geodesic in gP connecting q_a to q_b, and denote by $q_a = q_0, \ldots, q_n = q_b$ the points of $\alpha \cap gP$. For $i = 0, \ldots, n-1$ let $c_i = q_i q_0^{-1} a \in \partial_\infty(G, \mathcal{P})$ be the endpoint of $q_i q_0^{-1} \gamma_a$ other than a_O, and set $c_n = b$. Notice that

$$2\log(d_P(q, q_a)/d_P(q_a, q_b)) \approx_{2A} d(q, q_a) - d(q_a, q_b) \gtrsim_{4E} 2S'$$

by Lemma 2.3.5 and (4.4), so for $S = S(E, A)$ large enough,

$$2\log(d_P(q, q_a)/d_P(q_a, q_b) - 1) \geq S',$$

thus

$$\begin{aligned} d(q, q_i) &\approx_A 2\log(d_P(q, q_i)) \geq 2\log(d_P(q, q_a) - d_P(q_a, q_i)) \\ &\geq 2\log(d_P(q, q_a) - d_P(q_a, q_b)) \\ &\geq S' + 2\log(d_P(q_a, q_b)) \approx_A S' + d(q_a, q_b). \end{aligned}$$

In particular, if $S = S(E, A)$ is large enough we have $(c_i|a_O) \geq d_O$ for each i, by Lemma 4.5.6(3). By Lemma 2.3.5, as $d_P(q_a, q_i) \leq d_P(q_a, q_b)$ we have $d(q_a, q_i) \lesssim_{2A} d(q_a, q_b)$. Thus Lemma 4.5.6(2) gives

$$\begin{aligned} (a|c_i) &\gtrsim_E 2(a|a_O) - d_O - d(q_a, q_i)/2 \\ &\gtrsim_A 2(a|a_O) - d_O - d(q_a, q_b)/2 \quad \approx_E (a|b), \end{aligned}$$

which gives the distance bound $\rho(a, c_i) \leq C_1 \rho(a, b)$, for $C_1 = C_1(E, A)$. This gives the diameter bound $\operatorname{diam}(\{c_i\}) \leq K_1 \rho(a, b)$ for $K_1 = 2C_1$.

We saw that $(a|c_i) \gtrsim_{2E+A} (a|b) \geq S'+(a|a_O)$, and so $(c_i|a_O) \gtrsim \min\{(c_i|a),(a|a_O)\} \gtrsim (a|a_O)$, with error $C_2 = C_2(E,A)$. Applying Lemma 4.5.6(2) twice and Lemma 4.5.6(4) we see that

$$\begin{aligned}(c_i|c_{i+1}) &\gtrsim_E 2(c_i|a_O) - d_O - d(q_i,q_{i+1})/2 \gtrsim_{2C_2+1} 2(a|a_O) - d_O \\ &\approx_E (a|b) + d(q_a,q_b)/2 \gtrsim_E (a|b) + R,\end{aligned}$$

and so for $R \geq R_1(C_2,E)$ we have $\rho(c_i,c_{i+1}) \leq \rho(a,b)/2$.

Case 1b: Suppose that $b = a_O$. In this case, a chain of points joining a and a_O is found by using an appropriate geodesic ray in gP and pushing it out to the boundary. For a suitable choice of R, depending on the value of S fixed by Case 1a, we will actually ensure that the distance between subsequent points in the chain is at most $\rho(a,a_O)/2(S+1)$.

Let γ_a, q_a, γ and q be as above. Notice that q, q_a lie on gP, so by Lemma 4.5.6(1)

$$(a|a_O) \approx_E d(q_a,q)/2 + d_O \geq R + d_O. \tag{4.5}$$

By Lemma 4.5.5, there exists a geodesic ray α in gP starting at q_a such that $d_P(q,\alpha) \geq d_P(q_a,q)/3$. Therefore, by Lemma 2.3.5 and (4.5),

$$d(q,\alpha) \approx_A 2\log(d_P(q,\alpha)) \geq 2\log(d_P(q_a,q)/3) \approx_{A+3} d(q_a,q) \geq 2R. \tag{4.6}$$

Let $q_a = q_0, \dots, q_n, \dots$ be the points of $\alpha \cap gP$, and, as before, for each $i \geq 0$ let $c_i = q_i q_0^{-1}\gamma_a \in \partial_\infty(G,\mathcal{P})$.

By Lemma 4.5.6(3) and (4.6), for $R \geq R_2(E,A) \geq R_1$, we can assume that $(c_i|a_O) \geq d_O$ for each i.

Using Lemma 4.5.6(1) and (4.6), there exists $C_3 = C_3(A)$ so that

$$(c_i|a_O) \approx_E d(q_i,q)/2 + d_O \gtrsim_{C_3} d(q_a,q)/2 + d_O \approx_E (a|a_O).$$

And consequently there exists $C_4 = C_4(C_3,E)$ so that for each i

$$\rho(c_i,a_O) \leq C_4\rho(a,a_O); \tag{4.7}$$

this gives $\operatorname{diam}(\{c_i\}) \leq K_2\rho(a,a_O)$ for $K_2 = 2C_4$.

By Lemma 4.5.6(2), (4.5) and (4.6), we have for $C_5 = C_5(A)$

$$\begin{aligned}(c_i|c_{i+1}) &\gtrsim_E d(q_i,q) + d_O - d(q_i,q_{i+1})/2 \gtrsim_{C_5} d(q_a,q) + d_O \\ &= \big(d(q_a,q)/2 + d_O\big) + d(q_a,q)/2 \gtrsim_E (a|a_O) + R.\end{aligned}$$

So, taking $R \geq R_3(C_5, E, S) \geq R_2$, we have

$$\rho(c_i, c_{i+1}) \leq \rho(a, a_O)/2(S+1). \tag{4.8}$$

For each i, by Lemmas 2.3.5 and 4.5.6(1), $(c_i|a_O) \approx_{E+A} \log(d_P(q_i, q)) + d_O$, so for N large enough we have $\rho(c_N, a) \leq \rho(a, a_O)/2(S+1)$.

Therefore the chain of points $a = c_0, \ldots, c_N, a_O = b$ satisfies our requirements by (4.7), (4.8).

Case 1c: In this case, we have $\rho(a, b) \geq \rho(a, a_O)/S$ or $\rho(a, b) \geq \rho(b, a_O)/S$. Without loss of generality, we assume that $\rho(a, a_O) \leq \rho(b, a_O)$ and $\rho(a, a_O) \leq S\rho(a, b)$. Then $\rho(b, a_O) \leq \rho(b, a) + \rho(a, a_O) \leq (S+1)\rho(a, b)$.

Assume that $R \geq R_4 = R_3 + E$. Then by Lemma 4.5.6(4), $d(p_{a,a_O}, X \setminus O)$ and $d(p_{b,a_O}, X \setminus O)$ are both at least R_3, so by Case 1b there exist chains $a = c_0, c_1, \ldots, c_m = a_O$ and $a_O = c'_0, c'_1, \ldots c'_n = b$, with, for each i,

$$\rho(c_i, c_{i+1}) \leq \frac{\rho(a, a_O)}{2(S+1)} \leq \frac{S\rho(a, b)}{2(S+1)} < \frac{\rho(a, b)}{2}, \text{ and}$$
$$\rho(c'_i, c'_{i+1}) \leq \frac{\rho(b, a_O)}{2(S+1)} \leq \frac{\rho(b, a) + \rho(a, a_O)}{2(S+1)} \leq \frac{\rho(a, b)}{2}.$$

The diameter of $\{c_i\} \cup \{c'_i\}$ is at most $K_2\rho(a, a_O) + K_2\rho(b, a_O) \leq K_3\rho(a, b)$ for $K_3 = (2S+1)K_2$.

Case 2: We assume that $d(p_{a,b}, \Gamma(G)) < R$. In this case we can use the group action to find a connected set joining a and b directly.

Let $L_0 > 1$ be given by Corollary 4.4.7 applied to X with $D = R$. Since $\partial_\infty(G, \mathcal{P})$ is locally connected and compact, there exists $\alpha > 0$ so that any $B(z, \alpha) \subset \partial_\infty(G, \mathcal{P})$ is contained in an open, connected set of diameter less than $1/L_0$ (see the proof of Corollary 4.5.3).

Let $r_1 = 2\frac{L_0}{\alpha}\rho(a, b)$ and let $y_1 \in [w, a)$ be chosen so that $d(w, y_1) = -\frac{1}{\epsilon}\log(2r_1C_0) - \delta_X - 1$. If no such y_1 exists, then we are done as $\rho(a, b) \geq C_6 = C_6(L_0/\alpha) > 0$, so we can join a and b by a connected set of diameter $\leq K_4\rho(a, b)$, for $K_4 = \operatorname{diam}(\partial_\infty X)/C_6$.

Let $t \gg 0$ be a large constant to be determined by Case 2b.

Case 2a: If there exists $y \in [w, y_1]$ so that $d(y_1, y) \leq 3t$ and $d(y, Gw) \leq D$, then we argue as in the proof of Corollary 4.5.3.

By Corollary 4.4.7(1), using $z = a, r = r_1, x = y_1$ and y as given, there exists an L_0-bi-Lipschitz map $f : (B(a, r'), \frac{1}{r'}\rho) \to U$, where $r' = r_1e^{\epsilon d(y_1, y')}$, so that $B(f(a), 1/L_0) \subset U$.

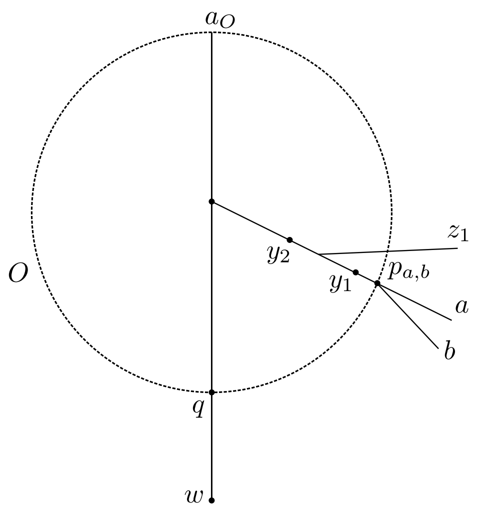


Figure 4.3: Lemma 4.5.4, Case 2b

Now,

$$\rho(f(a), f(b)) \leq L_0 \cdot \frac{1}{r'}\rho(a,b) \leq \frac{L_0}{r_1}\rho(a,b) = \frac{\alpha}{2},$$

so we can join $f(a)$ and $f(b)$ by a connected set $J \subset B(f(a), 1/L_0)$. Therefore we can join a and b by $f^{-1}(J) \subset B(a, r')$. As $r' \leq r_1 e^{\epsilon 3t}$, $f^{-1}(J)$ has diameter at most $2r' \leq K_5 \rho(a,b)$, for $K_5 = 4L_0 e^{\epsilon 3t}/\alpha$.

Case 2b: If no such y exists, we are in the situation of Figure 4.3. In this case, we use the absence of global cut points to find a connected set between a and b.

Let $y_2 \in [w, y_1)$ be chosen so that $d(y_1, y_2) = t$ and let $O \in \mathcal{O}$ be the horoball containing $[y_2, y_1]$, which corresponds to the coset gP. Let $d_O = d(w, O)$, and let $p_1 \in gP$ be chosen so that $d(p_1, p_{a,b}) < R$. (In the figure, $p_1 = p_{a,b}$.)

Let ρ_1 be a visual metric on $\partial_\infty X = \partial_\infty(G, \mathcal{P})$ based at p_1. We can assume that $(\partial_\infty X, \rho)$ and $(\partial_\infty X, \rho_1)$ are isometric, with the isometry induced by the action of p_1. In the metric ρ_1, we have that a, b and a_O are points separated by at least $\delta_0 = \delta_0(R)$.

The boundary $(\partial_\infty X, \rho) = (\partial_\infty X, \rho_1)$ is compact, locally connected and connected. Consequently, given a point $c \in \partial_\infty X$ that is not a global cut point, and $\delta_0 > 0$, there exists $\delta_1 = \delta_1(\delta_0, c, \partial_\infty X) > 0$ so that any two points in $\partial_\infty X \setminus B(c, \delta_0)$ can be joined by an arc in $\partial_\infty X \setminus B(c, \delta_1)$.

In our situation, δ_1 may be chosen to be independent of the choice of (finitely many) $c = a_O$ satisfying $d(O, p_1) = 0$, so $\delta_1 = \delta_1(\delta_0, \partial_\infty X)$. Therefore, a and b can be joined by a compact arc J in $(\partial_\infty X, \rho_1)$ that does not enter $B_{\rho_1}(a_O, \delta_1)$. So geodesic rays from p_1 to points in J are at least $2\delta_X$ from the geodesic ray $[p_{a,b}, a_O)$ outside the ball $B(p_1, t)$, for $t = -\frac{1}{\epsilon}\log(\delta_1) + C_7$, where $C_7 = C_7(C_0, \epsilon, \delta_X)$.

Translating this back into a statement about $(\partial_\infty X, \rho)$, we see that geodesics from w to points in J must branch from $[w, p_{a,b}]$ after y_2, that is, the set J lies in the ball $B(a, (K_6/2)d(a,b))$, for $K_6 = K_6(d(p_{a,b}, y_2)) = K_6(r_1, t)$.

From these connected sets of controlled diameter, it is easy to extract chains of points satisfying the conditions of the lemma, with $K = \max\{K_1, \ldots, K_6\}$. $\square$

4.6 Avoidable sets in the boundary

In order to build a hyperbolic plane that avoids horoballs, we need to build an arc in the boundary that avoids parabolic points. In Theorem 4.1.3, we also wish to avoid the specified hyperbolic subgroups. We have topological conditions such as the no local cut points condition which help, but in this section we find more quantitative control.

Given $p \in X$, and $0 < r < R$, the *annulus* $A(p, r, R)$ is defined to be $\overline{B}(p, R) \setminus B(p, r)$. More generally, we have the following.

Definition 4.6.1. Given a set V in a metric space Z, and constants $0 < r < R < \infty$, we define the *annular neighbourhood*

$$A(V, r, R) = \{z \in Z : r \leq d(z, V) \leq R\}.$$

If an arc passes through (or close to) a parabolic point in the boundary, we want to reroute it around that point. The following definition will be used frequently in the following two sections.

Definition 4.6.2 ([116])**.** For any x and y in an embedded arc A, let $A[x, y]$ be the closed, possibly trivial, subarc of A that lies between them.

An arc B *ι-follows* an arc A if there exists a (not necessarily continuous) map $p : B \to A$, sending endpoints to endpoints, such that for all x, $y \in B$, $B[x, y]$ is in the ι-neighbourhood of $A[p(x), p(y)]$; in particular, p displaces points at most ι.

We now define our notion of avoidable set, which is a quantitatively controlled version of the no local cut point and not locally disconnecting conditions.

Definition 4.6.3. Suppose (X, d) is a complete, connected metric space. A set $V \subset X$ is *L-avoidable on scales below* δ for $L \geq 1$, $\delta \in (0, \infty]$ if for any $r \in (0, \delta/2L)$, whenever there is an arc $I \subset X$ and points $x, y \in I \cap A(V, r, 2r)$ so that $I[x, y] \subset N(V, 2r)$, there exists an arc $J \subset A(V, r/L, 2rL)$ with endpoints x, y so that J $(4rL)$-follows $I[x, y]$.

The goal of this section is the following proposition.

Proposition 4.6.4. *Let $(G, \mathcal{P}_1)$ and $(G, \mathcal{P}_1 \cup \mathcal{P}_2)$ be relatively hyperbolic groups, where all groups in $\mathcal{P}_2$ are proper infinite hyperbolic subgroups of G ($\mathcal{P}_2$ may be empty), and all groups in $\mathcal{P}_1$ are proper, finitely presented and one ended. Let $X = Bow(G, \mathcal{P}_1)$, and let $\mathcal{H}$ be the collection of all horoballs of X and left cosets of the subgroups of $\mathcal{P}_2$. (As usual we regard G as a subspace of X.)*

Suppose that $\partial_\infty X$ is connected and locally connected, with no global cut points. Suppose that $\partial_\infty P$ does not locally disconnect $\partial_\infty X$ for each $P \in \mathcal{P}_2$. Then there exists $L \geq 1$ so that for every $H \in \mathcal{H}$, $\partial_\infty H \subset \partial_\infty X$ is L-avoidable on scales below $e^{-\epsilon d(w,H)}$.

This proposition is proved in the following two subsections.

4.6.1 Avoiding parabolic points

We prove Proposition 4.6.4 in the case H is a horoball. This is the content of the following proposition.

Proposition 4.6.5. *Suppose $(G, \mathcal{P})$ is relatively hyperbolic, $\partial_\infty(G, \mathcal{P})$ is connected and locally connected with no global cut points, and all peripheral subgroups are one ended and finitely presented. Then there exists $L \geq 1$ so that for any horoball $O \in \mathcal{O}$, $a_O = \partial_\infty O \in \partial_\infty(G, \mathcal{P})$ is L-avoidable on scales below $e^{-\epsilon d(w,O)}$.*

The reason for restricting to this scale is that this is where the geometry of the boundary is determined by the geometry of the peripheral subgroup. Recall from Proposition 4.4.3 that such parabolic points are not local cut points.

The first step is the following simple lemma about finitely presented, one ended groups. It essentially states that we can join two large elements of such a group without going too close or too far from the identity. Near a parabolic point, this allows us to prove Proposition 4.6.5 by joining two suitable points without going to far from or close to the parabolic point.

Lemma 4.6.6. *Suppose P is a finitely generated, one ended group, given by a (finite) presentation where all relators have length at most M, and let $\Gamma(P)$ be its Cayley graph. Then any two points $x, y \in \Gamma(P)$ such that $2M \leq r_x \leq r_y$, where $r_x = d(e, x)$ and $r_y = d(e, y)$, can be connected by an arc in $A(e, r_x/3, 2r_y) \subset \Gamma(P)$.*

Proof. By Lemma 4.5.5, we can find an infinite geodesic ray in $\Gamma(P)$ from x which does not pass through $B(e, r_x/3)$. Let x' be the last point on this ray satisfying $d(e, x') = r_y$. Do the same for y, and let y' denote the corresponding point. Note that x' and y' lie on the boundary of the unique unbounded component of $\{z \in \Gamma(P) : d(e, z) \geq r_y\}$, which we denote by Z. We prove the lemma by finding a path from x' to y' contained in $A(e, r_y - M, r_y + M)$.

Let β_1 be an arc joining x' and y' in Z. It suffices to consider the case when $\beta_1 \cap \overline{B}(e, r_y) = \{x', y'\}$. Let p be the first point of $[y', e]$ that meets $[e, x']$ in $\Gamma(P)$. Then the concatenation of $\beta_1, [y', p]$ and $[p, x']$ forms a simple, closed loop β_2 in $\Gamma(P)$.

As β_2 represents the identity in P, there exists a diagram $\mathcal{D}$ for β_2: a connected, simply connected, planar 2-complex $\mathcal{D}$ together with a map of $\mathcal{D}$ into the Cayley complex $\Gamma^2(P)$ sending cells to cells and $\partial\mathcal{D}$ to β_2.

Let $\mathcal{D}' \subset \mathcal{D}$ be the union of closed faces $B \subset \mathcal{D}$ which have a point $u \in \partial B$ with $d(u, e) = r_y$. Let $\mathcal{D}''$ be the connected component of x' in $\mathcal{D}'$. Let $\gamma : \mathbb{S}^1 \to \partial\mathcal{D}''$ be the outer boundary path of $\mathcal{D}'' \subset \mathbb{R}^2$.

If either β_1 or $[y', p] \cup [p, x']$ live in $A(p, r_y - M, r_y + M)$, we are done. Otherwise, as we travel around γ from x', in one direction we must take a value $> r_y$, and in the other a value $< r_y$, thus there is a point $v \in \gamma \setminus \{x'\}$ with $d(e, v) = r_y$. If v is in the interior of $\mathcal{D}$, the adjacent faces are in $\mathcal{D}'$, giving a contradiction. So $v \in \beta_2$, and v must be y'. Thus there is a path from x' to y' in $\mathcal{D}' \subset A(e, r_y - M, r_y + M)$. $\square$

We can now prove the proposition. The idea is similar to Lemma 4.5.4, Case 1: we find a suitable path using Lemma 4.6.6 and push it out to $\partial_\infty X$.

Proof of Proposition 4.6.5. By Proposition 4.4.3, parabolic points in the boundary are not local cut points.

We claim that there exists an $L \geq 1$ so that for any parabolic point a_O, any $r \leq e^{-\epsilon d(w,O)}/2L$, and any $x, y \in A(a_O, r, 2r)$, there exists an arc $J \subset A(a_O, r/L, 2rL)$ joining x to y.

This claim suffices to prove the proposition, because the $4rL$-following property automatically follows from $\mathrm{diam}(B(a_O, 2rL)) \leq 4rL$. We now proceed to prove the claim.

Using the notation of Lemma 4.5.6, let gP be the left coset of a peripheral group that corresponds to O, and let $q \in gP$ be the first point of $[w, a_O)$ in $\overline{O}$. Recall that $d_O = d(w, O) \approx d(w, q)$. Let q_a, q_b be the last points of $(a_O, a), (a_O, b)$ contained in $\overline{O}$.

We begin by describing the positions of q, q_a and q_b in the path metric d_P on $gP \subset X$. We write $x \asymp_C y$ if the quantities x, y satisfy $x/C \leq y \leq Cx$.

Since

$$e^{-\epsilon(a|a_O)} \asymp_{C_0} \rho(a, a_O) \asymp_2 r \leq e^{-\epsilon d_O}/2L,$$

we have, for some $C_1 = C_1(C_0, \epsilon)$,

$$(a|a_O) \approx_{C_1} \log(r^{-1/\epsilon}) \geq d_O + \log(2L)/\epsilon,$$

so for $L \geq L(C_1, \epsilon)$, we have $(a|a_O) \geq d_O$, and likewise $(b|a_O) \geq d_O$.

Lemmas 2.3.5 and 4.5.6(1), with A and E as before, give

$$\begin{aligned} 2\log(d_P(q_a, q)) \approx_A d(q_a, q) \approx_{2E} 2((a|a_O) - d_O) \\ \approx_{2C_1} 2\log(r^{-1/\epsilon}e^{-d_O}) \geq 2\log(2L)/\epsilon, \quad (4.9) \end{aligned}$$

so $d_P(q_a, q) \asymp_{C_2} r^{-1/\epsilon}e^{-d_O}$, for $C_2 = C_2(A, E, C_1)$. Let r_P be the smaller of $d_P(q_a, q)$ and $d_P(q_b, q)$, and notice that the larger value is at most $C_2^2 r_P$.

We now use Lemma 4.6.6 to find a chain of points $q_a = q_0, q_1, \ldots, q_n = q_b$ in gP joining q_a to q_b in gP, so that $q_i \in A(q, r_P/3, 2r_P C_2^2)$ in the metric d_P. Let $c_i = q_i q_0^{-1} a$, for $i = 0, \ldots, n-1$, and set $c_n = b$. This gives a chain of points $\{c_i\}$ in $\partial_\infty X$ joining a to b. (Observe that $(a_O, c_i) = q_i q_0^{-1}(a_O, a)$.)

Lemma 2.3.5 and (4.9) imply that

$$d(q_i, q) \approx 2\log(d_P(q_i, q)) \approx 2\log(d_P(q_a, q)) \approx d(q_a, q) \gtrsim 2\log(2L)/\epsilon,$$

with total error $C_3 = C_3(A, E, C_1, C_2)$. So if $L \geq L(C_3, \epsilon)$, we have $d(q_i, q) > E$, and therefore $(c_i|a_O) \geq d_O$ by Lemma 4.5.6(3).

Now Lemma 4.5.6(1) shows that

$$(c_i|a_O) \approx_E d(q_i, q)/2 + d_O \approx_{C_3} d(q_a, q)/2 + d_O \approx_E (a|a_O),$$

so $\rho(c_i, a_O) \asymp \rho(a, a_O) \asymp r$, with total error $C_4 = C_4(C_3, E)$, that is $c_i \in A(a_O, r/C_4, C_4 r)$.

We now wish to join each c_i and c_{i+1} in a suitable annulus around a_O. Consider the geodesics between w, c_i and c_{i+1}, and observe that $d(q_i, q) > E > \delta_X$, and $d_P(q_i, q_{i+1}) = 1$. From this, and (4.9), we see that

$$(c_i|c_{i+1}) \gtrsim d(w, q_i) \approx d_O + d(q, q_a) \approx 2\log(r^{-1/\epsilon}) - d_O,$$

and so, for suitable C_5,

$$\rho(c_i, c_{i+1}) \leq C_0 e^{-\epsilon(c_i|c_{i+1})} \leq C_5 r^2 e^{\epsilon d_O} \leq C_5 r/2L.$$

By Proposition 4.5.2, $\partial_\infty X$ is L'-linearly connected, so if $L \geq C_4 C_5 L'$ we can join c_i to c_{i+1} in $B(c_i, L' C_5 r/2L) \subset B(c_i, r/2C_4)$. Since $c_i \in A(a_O, r/C_4, C_4 r)$ we have joined c_i to c_{i+1} in $A(a_O, r/2C_4, 2C_4 r)$, and the claim follows. $\square$

4.6.2 Avoiding hyperbolic subgroups

In this section we complete the proof of Proposition 4.6.4 for $H = gP$, where $P \in \mathcal{P}_2$. By assumption, $\partial_\infty H \subset \partial_\infty X$ does not locally disconnect $\partial_\infty X$.

First, we show that boundaries of peripheral groups are porous.

Definition 4.6.7 (e.g. [88, 14.31])**.** A set V in a metric space (Z, ρ) is *C-porous on scales below δ* if for any $z \in V$ and $0 < r < \delta$, there exists $z' \in B(z, r)$ so that $\rho(z', V) \geq r/C$.

Lemma 4.6.8. *Under the assumptions of Proposition 4.6.4, there exists L_1 so that for every $H \in \mathcal{H}$, $\partial_\infty H \subset \partial_\infty X$ is L_1-porous on scales below $e^{-\epsilon d(w,H)}$.*

The proof follows from the partial self-similarity of Corollary 4.4.7 and the fact that for any $H \in \mathcal{H}$, $\partial_\infty H$ has empty interior in $\partial_\infty X$.

Proof. Observe that if H is a horoball, then $\partial_\infty H$ is a point in a connected space, and so is automatically porous.

If the conclusion is false, we can find a sequence of cosets $H_n = g_n P_n$, for $P_n \in \mathcal{P}_2$, points $a_n \in \partial_\infty H_n$ and values $r_n \leq e^{-\epsilon d(w,H_n)}$ so that $N(\partial_\infty H_n, r_n/n) \supset B(a_n, r_n)$.

Let $d_{r_n} \approx \log(r_n^{-1/\epsilon})$ be given as in Corollary 4.4.7 with $z = a_n$. If $d_{r_n} \geq 0$, let $x_n \in [w, a_n)$ be the point satisfying $d(w, x_n) = d_{r_n}$. Every $H \in \mathcal{H}$ is uniformly quasi-convex, see Lemma 4.3.1(2), and $r_n \leq e^{-\epsilon d(w,H_n)}$, so $d(x_n, H_n)$ is uniformly bounded for any such x_n. Therefore there exists $h_n \in G$ so that $d(h_n^{-1}w, x_n) \leq D$, for some uniform constant D.

Thus Corollary 4.4.7 implies that there exists $L_0 = L_0(D)$ and L_0-bi-Lipschitz maps $f_n : (B(a_n, r_n), \frac{1}{r_n}\rho) \to \partial_\infty X$ induced by the action of h_n, so that we have $B(f_n(a_n), 1/L_0) \subset f_n(B(a_n, r_n))$.

As $h_n H_n = H'_n$ for some $H'_n \in \mathcal{H}$, and $d(H'_n, w)$ is uniformly bounded, we may take a subsequence so that $H'_n = H' \in \mathcal{H}$ for all n, and moreover that $f_n(a_n) \in \partial_\infty H'$ converges to $a \in \partial_\infty H'$. Therefore, for all sufficiently large n,

$$\begin{aligned} B(a, 1/2L_0) &\subset B(f_n(a_n), 1/L_0) \subset f_n(B(a_n, r_n)) \\ &= f_n(B(a_n, r_n) \cap N(\partial_\infty H_n, r_n/n)) \subset N(\partial_\infty H', L_0/n), \end{aligned}$$

so a is in the interior of $\partial_\infty H' \subset \partial_\infty X$, since $\partial_\infty H'$ is closed in $\partial_\infty X$. This is a contradiction because $\partial_\infty H'$ is not all of $\partial_\infty X$ (proper peripheral subgroups of a relatively hyperbolic group are of infinite index), so if a is a point of $\partial_\infty H'$, one can use the action of H' to find points in $\partial_\infty X \setminus \partial_\infty H'$ that are arbitrarily close to a. $\square$

We continue with the proof of Proposition 4.6.4. The basic idea is to use partial self-similarity and a compactness argument to upgrade the topological condition of not locally disconnecting to the quantitative L-avoidable condition.

We begin with the following lemma.

Lemma 4.6.9. *Given $L_1 \geq 1$, there exists $L_2 = L_2(X, L_1)$ independent of $H = gP$, $P \in \mathcal{P}_2$, so that for any $r \leq e^{-\epsilon d(w,H)}/L_2$, and any two points $u, v \in A(\partial_\infty H, r/L_1, 2r)$ so that $\rho(u, v) \leq 4r$, there exists an arc*

$$K \subset A(\partial_\infty H, r/L_2, 2L_2 r)$$

joining u to v with $\operatorname{diam}(K) \leq 2L_2 r$.

Proof. As in Lemma 4.6.8, we assume the conclusion is false, and will use self-similarity to derive a contradiction.

If the conclusion is false, there is a sequence of $H_n = g_n P_n$ with $P_n \in \mathcal{P}_2$, $r_n \leq e^{-\epsilon d(w,H_n)}/n$, and points $a_n \in \partial_\infty H_n$ and $u_n, v_n \in B(a_n, 6r_n) \cap A(\partial_\infty H_n, r_n/L_1, 2r_n)$ so that there is no arc of diameter at most $2nr_n$ joining u_n to v_n in $A(\partial_\infty H_n, r_n/n, 2nr_n)$.

As before, the geodesic $[w, p_n)$ essentially travels from w straight to H_n then along H_n to $p_n \in \partial_\infty H_n$. More precisely, there are constants C_1 and D depending on the uniform quasi-convexity constant of H_n and δ_X so that for any $r \leq e^{-\epsilon d(w,H_n)}/C_1$, the point x, defined by Corollary 4.4.7(1) applied to $z = a_n$, lies within distance D of Gw. Let $L_0 = L_0(D)$ be the corresponding constant from Corollary 4.4.7.

Let L' be the linear connectivity constant of $\partial_\infty X$, and set $r'_n = 10L_0^2 L' r_n$. For n large enough, $r'_n \leq e^{-\epsilon d(w,H_n)}/C_1$, and so we find $h_n \in G$ that induces a L_0-bi-Lipschitz map $f_n : (B(a_n, r'_n), \frac{1}{r'_n}\rho) \to \partial_\infty X$, with $B(f_n(a_n), 1/L_0) \subset f_n(B(a_n, r'_n))$. Note that h_n maps H_n to some $H'_n \in \mathcal{H}$, with $f_n(a_n) \in \partial_\infty H'_n$. As $d(w, H'_n)$ is uniformly bounded, we can take a subsequence so that $H'_n = H' \in \mathcal{H}$.

The images $f_n(u_n), f_n(v_n)$ lie in $B(f_n(a_n), T) \setminus N(\partial_\infty H', t)$, where $T = 6r_n L_0/r'_n < 1/L_0 L'$ and $t = r_n/r'_n L_1 L_0$ are independent of n. Let

$$W = \{(u, v, a) : a \in \partial_\infty H', \{u, v\} \subset \overline{B}(a, T) \setminus N(\partial_\infty H', t)\} \subset (\partial_\infty X)^3,$$

and define $f : W \to (0, T)$ to be supremal so that for $(u, v, a) \in W$ there exists an arc joining u to v in $B(a, 1/L_0) \setminus N(\partial_\infty H', f(u, v, a))$. We know that f is positive because for any $(u, v, a) \in W$, $\overline{B}(a, T)$ lies in a connected open set $U \subset B(a, 1/L_0)$, and as $\partial_\infty H'$ does not locally disconnect, $U \setminus \partial_\infty H'$ is connected and we can join u to v in this set.

Observe that f is continuous, and W is compact, so $f(u, v, a) \geq 2C_2 > 0$ for some C_2 and all $(u, v, a) \in W$.

Now $(f_n(u_n), f_n(v_n), f_n(a_n)) \in W$, so there exists an arc K joining $f_n(u_n)$ to $f_n(v_n)$ with $K' \subset B(f_n(a_n), 1/L_0) \setminus N(\partial_\infty H', C_2)$. The preimage $K = f_n^{-1}(K')$ joins u_n to v_n so that

$$K \subset B(a_n, r'_n) \cap A(\partial_\infty H, r'_n C_2/L_0, r'_n),$$

which is a contradiction for large enough n. □

We now finish the proof of Proposition 4.6.4, fixing constants $L_1 \geq 2$ from Lemma 4.6.8 and $L_2 = L_2(X, L_1)$ from Lemma 4.6.9.

Let $H = gP$, $P \in \mathcal{P}_2$ be fixed. Suppose we are given $r \leq e^{-\epsilon d(w,H)}/L_2$, points $x, y \in A(\partial_\infty H, r, 2r)$, and an arc $I \subset N(H, 2r)$ with endpoints x and y.

We build our desired arc J from x to y in stages. First, let $x = z'_0, z'_1, \ldots, z'_m = y$ be a (finite) chain of points that 0-follows $I[x, y]$, so that $\rho(z_i, z_{i+1}) \leq r$. (The definition of ι-follows is extended from arcs to chains in the obvious way.) For each i, if $\rho(z'_i, \partial_\infty H) \leq r/L_1$, use Lemma 4.6.8 to find a point z_i at most $r/L_1 + r$ away from z'_i, and outside $N(\partial_\infty H, r/L_1)$. Otherwise let $z_i = z'_i$.

This new chain satisfies $\{z_i\} \subset A(\partial_\infty H, r/L_1, 2r)$, and for every i, $\rho(z_i, z_{i+1}) \leq r + 2(r/L_1 + r) \leq 4r$. It $2r$-follows $\{z'_i\}$, and thus $\{z_i\}$ also $2r$-follows I.

By Lemma 4.6.9 for each i there exist an arc J_i joining z_i and z_{i+1} which lies in $A(\partial_\infty H, r/L_2, 2L_2 r)$ and has $\operatorname{diam}(J_i) \leq 2L_2 r$. From this, we extract an arc J by cutting out loops: travel along J_0 until you meet J_j for some $j \geq 1$, and at that point cut out the rest of J_0 and all J_k for $1 \leq k < j$. Concatenate the remainders of J_0 and J_j together, and continue along J_j.

The resulting arc J will $2L_2 r$-follow the chain $\{z_i\}$, and so it will $4L_2 r$-follow I as desired. □

4.7 Quasi-arcs that avoid obstacles

In this section we build quasi-arcs in a metric space that avoid specified obstacles.

Definition 4.7.1. Let (Z, ρ) be a compact metric space. Let $\mathcal{V}$ be a collection of compact subsets of Z provided with some map $D : \mathcal{V} \to (0, \infty)$, which we call a *scale function*.

The *(modified) relative distance function* $\Delta : \mathcal{V} \times \mathcal{V} \to [0, \infty)$ is defined for $V_1, V_2 \in \mathcal{V}$ as

$$\Delta(V_1, V_2) = \frac{\rho(V_1, V_2)}{\min\{D(V_1), D(V_2)\}}.$$

We say $\mathcal{V}$ is *L-separated* if for all $V_1, V_2 \in \mathcal{V}$, if $V_1 \neq V_2$ then $\Delta(V_1, V_2) \geq \frac{1}{L}$.

As we saw in Section 4.6, we often only have control on topology on a sufficiently small scale. The purpose of the scale function is to determine the size of the neighbourhood of each $V \in \mathcal{V}$ on which we have this control. An example of a scale function is $D(V) = \operatorname{diam}(V)$, if every $V \in \mathcal{V}$ has $|V| > 1$. In this case, Δ is precisely the usual relative distance function, e.g. [88, page 59].

The goal of this section is the following result.

Theorem 4.7.2. *Let (Z, ρ) be a compact, N-doubling and L-linearly connected metric space. Suppose $\mathcal{V}$ is a collection of compact subsets of Z with scale function $D : \mathcal{V} \to (0, \infty)$.*

Suppose $\mathcal{V}$ is L-separated, and each $V \in \mathcal{V}$ is both L-porous and L-avoidable on scales below $D(V)$. Then for a constant $\lambda = \lambda(N, L)$ there exists a λ-quasi-arc γ in Z which satisfies $\operatorname{diam}(\gamma) \geq \frac{1}{2} \operatorname{diam}(Z)$, and $\rho(\gamma, V) \geq \frac{1}{\lambda} D(V)$ for each $V \in \mathcal{V}$.

A simple application of this theorem is the following.

Example 4.7.3. Let Z be the usual square Sierpiński carpet in the plane, with Euclidean metric d_{Euc}, and let $\mathcal{V}$ be the set of peripheral squares, i.e., boundaries of $[0, 1]^2$, $[1/3, 2/3]^2$, and so on. Define $D(V) = \operatorname{diam}(V)$ for each $V \in \mathcal{V}$.

The assumptions of Theorem 4.7.2 are satisfied for suitable N and L, so there exists some λ and a λ-quasi-arc γ in Z which satisfies $d_{Euc}(\gamma, V) \geq \frac{1}{\lambda} \operatorname{diam}(V)$ for each $V \in \mathcal{V}$.

It is not immediately obvious that there exists a point satisfying this last separation condition, let alone a quasi-arc, although in this example it is possible to build such an arc by hand. The way that Theorem 4.7.2 builds a quasi-arc is by an inductive process: starting with any arc in Z, push the arc away from the largest obstacles in $\mathcal{V}$, then push it away from the next largest, and so on. While this is going on, one also "cuts out loops" in order to ensure the limit arc is a quasi-arc. There is some delicacy involved in making sure the constants work out correctly.

Our proof adapts the methods of [116], and we now recall some terminology and results from that paper.

We say that an arc A in a doubling and complete metric space is an *ι-local λ-quasi-arc* if $\operatorname{diam}(A[x, y]) \leq \lambda d(x, y)$ for all x, $y \in A$ such that $d(x, y) \leq \iota$. (See Definition 4.6.2 for the notion of ι-following.)

Proposition 4.7.4 ([116, Proposition 2.1]). *Given a complete metric space (Z, ρ) that is L-linearly connected and N-doubling, there exist constants $s = s(L, N) > 0$ and $S = S(L, N) > 0$ with the following property: for each $\iota > 0$ and each arc $A \subset X$, there exists an arc J that ι-follows A, has the same endpoints as A, and satisfies*

$$\forall x, y \in J,\ \rho(x, y) < s\iota \implies \operatorname{diam}(J[x, y]) < S\iota. \tag{4.10}$$

Lemma 4.7.5 ([116, Lemma 2.2]). *Suppose (Z, ρ) is an L-linearly connected, N-doubling, complete metric space, and let s, S, ϵ and δ be fixed positive constants satisfying $\delta \leq \min\{\frac{s}{4+2S}, \frac{1}{10}\}$. Now, if we have a sequence of arcs $J_1, J_2, \ldots, J_n, \ldots$ in Z, such that for every $n \geq 1$*

- *J_{n+1} $\epsilon\delta^n$-follows J_n, and*
- *J_{n+1} satisfies (4.10) with $\iota = \epsilon\delta^n$ and s, S as fixed above,*

then the Hausdorff limit $J = \lim_{\mathcal{H}} J_n$ exists, and is an $\epsilon\delta^2$-local $\frac{4S+3\delta}{\delta^2}$-quasi-arc. Moreover, the endpoints of J_n converge to the endpoints of J, and J ϵ-follows J_1.

We now use these results to build our desired quasi-arc.

Proof of Theorem 4.7.2. Let $r = r(L, N) > 0$ be fixed sufficiently small as determined later in the proof. Let $D_0 = \sup\{D(V) : V \in \mathcal{V}\}$; if $\mathcal{V} = \emptyset$, set $D_0 = \operatorname{diam}(Z)$. Observe that as every $V \in \mathcal{V}$ is L-porous, we have $D_0 \leq L \operatorname{diam}(Z)$. (We assume $L \geq 10$.)

We filter $\mathcal{V}$ according to size. For $n \in \mathbb{N}$, let $\mathcal{V}_n = \{V \in \mathcal{V} : r^n < D(V)/D_0 \leq r^{n-1}\}$ and $\mathcal{C}_n = \{N(V, D_0 r^n/4L) : V \in \mathcal{V}_n\}$. As $\mathcal{V}$ is L-separated, each $\mathcal{C}_n$ consists of disjoint neighbourhoods. (Note that two neighbourhoods from different $\mathcal{C}_n$ may well intersect.)

We start with any arc $J_0 = J_0[x_0, x_0']$ in Z joining points distance $\operatorname{diam}(Z)$ apart, and build arcs J_n in Z by induction on n.

Inductive step: Assume we have been given an arc $J_{n-1} = J_{n-1}[x_{n-1}, x_{n-1}']$. We will modify J_{n-1} independently inside the (disjoint) sets in $\mathcal{C}_n$. Let $r_n' = D_0 r^n/16L^2$, and observe that for any $V \in \mathcal{V}_n$,

$$A(V, r_n'/L, 2r_n'/L) \subset A(V, r_n'/L^2, 3r_n'L) \subset N(V, D_0 r^n/4L) \in \mathcal{C}_n.$$

First, we modify the arc J_{n-1} to ensure that its endpoints lie outside $N(V, 2r_n'/L)$, for any $V \in \mathcal{V}_n$. If for some (unique) $V \in \mathcal{V}_n$ we have $\rho(x_{n-1}, V) \leq 2r_n'/L$, by porosity there exists $x_n \in A(V, 2r_n'/L, 2r_n')$ with $\rho(x_n, x_{n-1}) \leq 2r_n' + 2r_n'/L \leq 3r_n'$. By linear connectedness there exists an arc $\alpha[x_n, x_{n-1}]$ of diameter at most $3r'_nL$, and let w_n be the first point of α meeting J_{n-1}. If no such V exists, set $x_n = w_n = x_{n-1}$ and $\alpha = \{x_n\}$. Likewise, define x'_n, w'_n, and α'. From these paths we obtain a new arc $J'_n = \alpha[x_n, w_n] \cup J_{n-1}[w_n, w'_n] \cup \alpha'[w'_n, x'_n]$ (reversing the order on α'), which $3r'_nL$-follows J_{n-1}.

Second, given $V \in \mathcal{V}_n$, each time J'_n meets $N(V, r'_n/L^2)$, the arc J'_n travels through $A = A(V, r'_n/L, 2r'_n/L)$ both before and after meeting $N(V, r'_n/L^2)$. For each such meeting, we use that V is L-avoidable with "r" equal to r'_n/L to find a detour path in $A(V, r'_n/L^2, 2r'_n)$ which $4r'_n$-follows the previous path. After doing so, we concatenate the paths found into an arc J''_n, as at the end of the proof of Proposition 4.6.4. Note that J''_n will $4r'_n$-follow J'_n.

Third, apply Proposition 4.7.4 to J''_n with $\iota = r'_n/2L^2$. Call the resulting arc J_n: it ι-follows J''_n, so it $(D_0r^n/4L)$-follows J_{n-1}, as $\iota + 4r'_n + 3r'_nL \leq D_0r^n/4L$. Since J_n $(r'_n/2L^2)$-follows J''_n, which satisfies $\rho(J''_n, V) \geq r'_n/L^2$, we also have $\rho(J_n, V) \geq r'_n/2L^2 = D_0r^n/32L^4$.

Conclusion: We now find the limit of the arcs J_n. For every n, J_n $(D_0r^n/4L)$-follows J_{n-1}. Let $s' = s/8L^3$, where s is given by Proposition 4.7.4, then observe that J_n satisfies (4.10) with $\iota = D_0r^n/4L$, and where s is replaced by s'.

We can assume that $r \leq \min\left\{\frac{s'}{4+2S}, \frac{1}{10}\right\}$, where S is given by Proposition 4.7.4, since s' and S depend only on L and N.

Now apply Lemma 4.7.5 to the arcs J_n with s' replacing s, $\delta = r$ and $\epsilon = D_0/4L$, to find a quasi-arc γ. Provided $r \leq \frac{1}{2}$, the endpoints of γ are at most

$$\sum_{n=1}^{\infty} \frac{D_0r^n}{4L} = \frac{D_0r}{4L(1-r)} \leq \frac{D_0r}{2L} \leq \frac{\operatorname{diam}(Z)}{4},$$

from those of J_0, so $\operatorname{diam}(\gamma) \geq \frac{1}{2}\operatorname{diam}(Z)$.

Finally, for each n, γ lies in a neighbourhood of J_n of size at most

$$\frac{D_0r^{n+1}}{4L} + \frac{D_0r^{n+2}}{4L} + \cdots = \frac{D_0r^{n+1}}{4L(1-r)} \leq \frac{D_0r^n}{64L^4},$$

where this last inequality holds for $r \leq 1/32L^3$. We conclude by observing that for any $V \in \mathcal{V}_n \subset \mathcal{V}$, we have

$$\rho(\gamma, V) \geq \rho(J_n, V) - \frac{D_0r^n}{64L^4} \geq \frac{D_0r^n}{64L^4} \geq \frac{r}{64L^4}D(V). \qquad \square$$

4.8 Building hyperbolic planes

In this section we prove Theorems 4.1.2 and 4.1.3.

Proof of Theorem 4.1.3. Let $\mathcal{V} = \{\partial_\infty H : H \in \mathcal{H}\}$, where $\mathcal{H}$ is the collection of all horoballs of $X = Bow(G, \mathcal{P}_1)$ and left cosets of the subgroups of $\mathcal{P}_2$. Define the scale function $D : \mathcal{V} \to (0, \infty)$ by $D(\partial_\infty H) = e^{-\epsilon d(w,H)}$ for each $H \in \mathcal{H}$.

The boundary $\partial_\infty(G, \mathcal{P}_1)$ is N-doubling, for some N, by Proposition 4.4.5.

Theorem 4.4.1 implies that $\partial_\infty(G, \mathcal{P}_1)$ is connected and locally connected, with no global cut points. By Proposition 4.5.2 $\partial_\infty(G, \mathcal{P}_1)$ is L_2-linearly connected for some $L_2 \geq 1$. Proposition 4.6.4 implies that there exists $L_3 \geq 1$ so that for every $H \in \mathcal{H}$, $\partial_\infty H$ is L_3-avoidable on scales below $e^{-\epsilon d(w,H)}$, and (by Lemma 4.6.8) $\partial_\infty H$ is L_1-porous on scales below $e^{-\epsilon d(w,H)}$. Lemma 4.3.2 shows that $\mathcal{V}$ is L_4-separated, for some L_4.

We set L to be the maximum of L_1, L_2, L_3 and L_4. We apply Theorem 4.7.2 to build a λ-quasi-arc γ in $\partial_\infty(G, \mathcal{P}_1)$, which satisfies, for all $H \in \mathcal{H}$,

$$\rho(\gamma, \partial_\infty H) \geq \frac{1}{\lambda} e^{-\epsilon d(w,H)}. \tag{4.11}$$

In the Poincaré disc model for $\mathbb{H}^2$, denote the standard half-space by $Q = \{(x, y) : x^2 + y^2 < 1, x \geq 0\}$, and fix a basepoint $(0, 0)$. We endow the semi-circle $\partial_\infty Q$ with the chordal metric ρ_Q, which makes $\partial_\infty Q$ bi-Lipschitz to the interval $[0, 1]$. Therefore by [159, Theorem 4.9] there is a quasisymmetric map $f : \partial_\infty Q \to \gamma \subset \partial_\infty(G, \mathcal{P})$. In fact, as $\partial_\infty Q$ is connected, f is a "power quasisymmetry" by [159, Corollary 3.12]; see [28, Section 6] for this definition.

It is straightforward to check that, for some $C > 0$, ρ_Q is a visual metric on $\partial_\infty Q$ with basepoint $(0, 0)$, and parameters C and $\epsilon = 1$. Both Q and $Bow(G, \mathcal{P}_1)$ are visual (see Section 2.4). Thus there is an extension of f to a quasi-isometric embedding of Q in $Bow(G, \mathcal{P}_1)$, with boundary γ [28, Theorems 7.4, 8.2].

Finally, as we have (4.11), Proposition 4.3.5 gives us a transversal, quasi-isometric embedding of $\mathbb{H}^2$ in $Bow(G, \mathcal{P})$. □

It only remains to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. The only if direction of the proof is elementary, for if G admits a graph of groups decomposition with finite edge groups and virtually nilpotent vertex groups, then any quasi-isometrically embedded copy of $\mathbb{H}^2$ must map, up to finite distance, into (a left coset of) a vertex group, which is impossible.

We now reduce the if direction of Theorem 4.1.2 to a special case of Theorem 4.1.3. We can assume that $\mathcal{P}$ does not contain virtually cyclic subgroups as those can be removed from the list of peripheral subgroups preserving relative hyperbolicity. As the peripheral subgroups are finitely presented, G is finitely presented as well (see for example [141]), therefore it admits a graph of groups decomposition where all edge groups are finite and all vertex groups have at most one end [74, Theorem 5.1]. As all edge groups are finite, the vertex groups are undistorted and therefore they are themselves relatively hyperbolic with the obvious peripheral structure [68, Theorem 1.8]. By the assumptions of the theorem, one of the vertex groups, say G', is properly relatively hyperbolic, that is the peripheral subgroups are proper (virtually nilpotent) subgroups.

Bowditch shows [34, Theorem 1.4] that G' admits a maximal peripheral splitting, and so we can find a vertex group G'' of this splitting so that G'' does not admit a splitting over parabolic subgroups. Such a vertex group is one ended, hyperbolic relative to one-ended virtually nilpotent subgroups $\mathcal{P}''$ (we can remove all two-ended ones, if any). Therefore, all hypotheses of Theorem 4.1.3 are fulfilled for G_3 with $\mathcal{P}_1 = \mathcal{P}''$ and $\mathcal{P}_2 = \emptyset$. Unfortunately G'' may be distorted in G', but the statement of Theorem 4.1.3 gives the existence of a *transversal* quasi-isometric embedding of $\mathbb{H}^2$. As the coned-off graph of a vertex group isometrically embeds (given a suitable choice of generators) in the coned-off graph of the whole group, an easy argument based on Proposition 4.2.3 shows that a transversal quasi-isometric embedding of $\mathbb{H}^2$ in a vertex group gives a quasi-isometric embedding in the whole group. $\square$

4.9 Application to 3-manifolds

In this final section, we consider which 3-manifold groups contain a quasi-isometrically embedded copy of $\mathbb{H}^2$. Recall that an irreducible 3-manifold is a graph manifold if its JSJ decomposition contains Seifert fibred components only. A non-geometric graph manifold is one with non-trivial JSJ decomposition.

Lemma 4.9.1. *Let M be a non-geometric closed graph manifold. Then $\pi_1(M)$ contains a quasi-isometrically embedded copy of $\mathbb{H}^2$.*

Proof. All fundamental groups of closed non-geometric graph manifolds are quasi-isometric [11, Theorem 2.1], so we can choose M. Consider a splitting of the closed genus 2 surface S into an annulus A and a twice-punctured torus S', as in Figure 4.4 below.

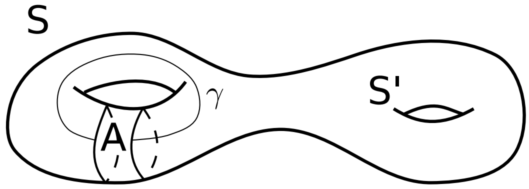


Figure 4.4: The surfaces S, S', A and the path γ.

The manifold $S' \times S^1$ has two boundary components homeomorphic to $S^1 \times S^1$. Let M be obtained from two copies M_1, M_2 of $S' \times S^1$ by gluing the corresponding boundary components together in a way that interchanges the two S^1 factors.

We now wish to find an embedding ι of S into M so that M retracts onto the image of ι. If we have such an embedding, then first of all $\pi_1(S)$ injects in $\pi_1(M)$, so that we get a map $f : \mathbb{H}^2 \to \widetilde{M}$. Also, $\pi_1(S)$ is undistorted in $\pi_1(M)$ and therefore f is a quasi-isometric embedding.

The specific embedding $\iota : S \to M$ which we describe is obtained from two embeddings $\iota_1 : A \to M_1$ and $\iota_2 : S' \to M_2$.

Let $\gamma : [0,1] \to S'$ be the path connecting the boundary components of S' depicted in Figure 4.4. As A can be identified with $[0,1] \times S^1$ we can define $\iota_1 : A \to M_1$ by $(t, \theta) \mapsto (\gamma(t), \theta)$. We can assume, up to changing the gluings, that there exists p such that $\iota_1(A) \cap M_2 \subseteq S' \times \{p\}$. We can then define ι_2 by $x \mapsto (x, p)$.

We now have an embedding $\iota : S \to M$ so that $\iota|_A = \iota_1, \iota|_{S'} = \iota_2$. We now only need to show that $\iota(S)$ is a retract of M. Define $g_2 : M_2 \to S' \times \{p\}$ simply as $(x, \theta) \mapsto (x, p)$. It is easy to see that there exists a retraction $g' : S' \to \gamma$ such that each boundary component of S' is mapped to an endpoint of γ. Let $g_1 : M_1 \to \gamma \times S^1$ be $(x, \theta) \mapsto (g'(x), \theta)$. There clearly exists a retraction $g : M \to \iota(S)$ which coincides with g_i on M_i. $\square$

Theorem 4.9.2. *Let M be a connected orientable closed* 3*-manifold. Then $\pi_1(M)$ contains a quasi-isometrically embedded copy of $\mathbb{H}^2$ if and only if M does not split as the connected sum of manifolds each with geometry $S^3, \mathbb{R}^3, S^2 \times \mathbb{R}$ or* Nil.

Proof. We will use the geometrisation theorem [144, 145, 108, 130, 52]. It is easily seen that $\pi_1(M)$ contains a quasi-isometrically embedded copy of $\mathbb{H}^2$ if and only if the fundamental group of one of its prime summands does. So, we can assume that M is prime. Suppose first that M is geometric. We list below the possible geometries, each followed by yes/no according to whether or not it contains a quasi-isometrically embedded copy of $\mathbb{H}^2$ in that case and the reason for the answer.

- S^3, no, it is compact.

- $\mathbb{R}^3$, no, it has polynomial growth.

- $\mathbb{H}^3$, yes, obvious.

- $S^2 \times \mathbb{R}$, no, it has linear growth.

- $\mathbb{H}^2 \times \mathbb{R}$, yes, obvious.

- $\widetilde{SL_2\mathbb{R}}$, yes, it is quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$ (see, for example, [147]).

- Nil, no, it has polynomial growth.

- Sol, yes, it contains isometrically embedded copies of $\mathbb{H}^2$.

If M is not geometric, then it has a non-trivial JSJ splitting, i.e. there is a canonical family of tori and Klein bottles that decomposes M into components each of which is either Seifert fibred or hyperbolic (meaning that it admits a finite volume hyperbolic metric). We will consider two cases.

- There are no hyperbolic components. By definition, M is a graph manifold. In this case we can apply Lemma 4.9.1 to find the quasi-isometrically embedded $\mathbb{H}^2$.

- There is at least one hyperbolic component, N. As $\pi_1(N)$ is one-ended and hyperbolic relative to copies of $\mathbb{Z}^2$, by Theorem 4.1.3 (or by [124, 125], upon applying Dehn filling to the manifold) it contains a quasi-isometrically embedded copy of $\mathbb{H}^2$. This is also quasi-isometrically embedded in $\pi_1(M)$ since $\pi_1(N)$ is undistorted in $\pi_1(M)$, because there exists a metric on M such that $\widetilde{N}$ is convex in $\widetilde{M}$ (see [112]). $\square$

Chapter 5

Embeddings in products of trees

5.1 Introduction

If a group admits a quasi-isometric embedding into the product of finitely many trees then one can say a lot about the large scale geometry of the group. For example, this bounds the (linearly controlled) asymptotic dimension of the group. It has implications for the Hilbert compression of the group, and also the topological dimension of any asymptotic cone. Such embeddings have recently been constructed for hyperbolic groups [46, 42].

In this chapter we show that if every peripheral group of a relatively hyperbolic group embeds into the product of finitely many trees, then the entire group does. Moreover, we give explicit bounds for the number of trees required.

We begin by recalling what is known for hyperbolic groups. In this case, the situation has been completely understood by work of Buyalo and Lebedeva [49, Theorems 1.5, 1.6]. (For further refinements, see [42].)

Theorem 5.1.1. *Let G be a Gromov hyperbolic group. Then G admits a quasi-isometric embedding into the product of $n+1$ metric trees, where $n = \dim \partial_\infty G$ is the topological dimension of the boundary. Moreover, G does not embed into any product of n metric trees.*

Our main result is the following.

Theorem 5.1.2. *Suppose the group G is hyperbolic relative to subgroups $H_1, \dots, H_n$. If each H_i quasi-isometrically embeds into a product of m metric trees, then* $\operatorname{asdim}(G) < \infty$ *and G quasi-isometrically embeds into a product of M metric trees, where*

$$M = \max\{\operatorname{asdim}(G), m+1\} + m + 1 < \infty.$$

Conversely, if G quasi-isometrically embeds into a product of N metric trees, then each peripheral group does also.

The assertion $\operatorname{asdim}(G) < \infty$ follows from [136].

Both these theorems use results that study the relationship between the asymptotic dimension of a hyperbolic space and the linearly controlled metric dimension of its boundary. We now proceed to define these and related concepts. (See [44] for more discussion.)

Suppose $\mathcal{U}$ is a family of subsets of a metric space X. We say $\mathcal{U}$ is *D-bounded* if the diameter of every $U \in \mathcal{U}$ is at most D. The *r-multiplicity* of $\mathcal{U}$ is the infimal integer n so that every subset of X with diameter less than or equal to r meets at most n subsets of $\mathcal{U}$.

The *asymptotic dimension* of X, denoted by $\operatorname{asdim}(X)$, is the smallest $n \in \mathbb{N} \cup \{\infty\}$ so that for all $r > 0$, there exists $D(r) < \infty$ and a $D(r)$-bounded cover $\mathcal{U}$ of X with r-multiplicity at most $n + 1$ [84, 1.E]. The *linearly controlled asymptotic dimension* of X, denoted by $\ell\text{-}\operatorname{asdim}(X)$, is the smallest n so that there exists $L < \infty$ with the property that for all sufficiently large $r < \infty$, there exists an Lr-bounded cover $\mathcal{U}$ of X with r-multiplicity at most $n + 1$ [110, 38]. (This is sometimes called "asymptotic Assouad-Nagata dimension" in the literature.)

The following definition simplifies our discussion of how metric spaces embed in trees. (For the related notion of "t-rank", see [44, page 176].)

Definition 5.1.3. Given a metric space X, let $\text{eco-dim}(X)$ be the smallest $n \in \mathbb{N}$ so that X quasi-isometrically embeds in the product of n metric trees, and set $\text{eco-dim}(X) = \infty$ if no such embedding exists.

The inequalities $\operatorname{asdim}(X) \leq \ell\text{-}\operatorname{asdim}(X) \leq \text{eco-dim}(X)$ hold for any metric space X. These three quantities are equal for hyperbolic groups, as shown in [49]. Both equalities can fail in general: there are groups G with finite asymptotic dimension but infinite linearly controlled asymptotic dimension [131]. The discrete Heisenberg group H has $\operatorname{asdim}(H) = \ell\text{-}\operatorname{asdim}(H) = 3$, but $\text{eco-dim}(H) = \infty$ (see the discussion in Section 5.4).

As an aside, Lang and Schlichenmaier show that if $\ell\text{-}\operatorname{asdim}(X, d) < \infty$, then for sufficiently small $\epsilon > 0$, the snowflaked space (X, d^ϵ) has $\text{eco-dim}(X, d^\epsilon) \leq \ell\text{-}\operatorname{asdim}(X, d) + 1$ [110, Theorem 1.3].

For hyperbolic groups, these asymptotic invariants are related to the local properties of the boundary. The *linearly controlled metric dimension* of X, denoted by $\ell\text{-}\dim(X)$, is the smallest n so that there exists $L < \infty$ with the property that for all sufficiently small $r > 0$, there exists an Lr-bounded cover $\mathcal{U}$ of X with r-multiplicity at most $n + 1$ [46, Prop. 3.2]. (This is also referred to as "capacity dimension".) Buyalo shows the following embedding theorem.

Theorem 5.1.4 ([47, Theorem 1.1])**.** *Suppose X is a visual, Gromov hyperbolic metric space, and $\ell\text{-dim}(\partial_\infty X) < \infty$. Then* $\text{eco-dim}(X) \leq \ell\text{-dim}(\partial_\infty X) + 1$.

The following proposition gives a simple bound for the linearly controlled metric dimension of the boundary of X.

Proposition 5.2.6 *Suppose X is a Gromov hyperbolic geodesic metric space. Then $\ell\text{-dim}(\partial_\infty X) \leq \text{asdim}(X)$.*

This proposition does not seem to be recorded in the literature, possibly because in the case when the space admits a cocompact isometric action the stronger equality $\text{asdim}(X) = \ell\text{-dim}(\partial_\infty X) + 1$ holds [49]. Even without such an action, Buyalo shows the inequality $\text{asdim}(X) \leq \ell\text{-dim}(\partial_\infty X) + 1$ [46]. We remark that, in the case of hyperbolic groups, the analogous inequality for the topological dimension of the boundary rather than the linearly controlled dimension was previously known [156].

In the case of a relatively hyperbolic group $(G, \{H_i\})$, we apply this for $X = Bow(G)$, the Bowditch space. We bound the asymptotic dimension of $Bow(G)$ in terms of the asymptotic dimension of G and the linearly controlled asymptotic dimension of the peripheral groups.

Proposition 5.2.4 *Let G be hyperbolic relative to $H_1, \ldots, H_n$, and denote $m = \max_{i=1,\ldots,n} \ell\text{-asdim}(H_i)$. Then*

$$\max\{\text{asdim}(G), m\} \leq \text{asdim}(Bow(G)) \leq \max\{\text{asdim}(G), m + 1\}.$$

Osin had earlier shown that $\text{asdim}(G)$ is finite if $\text{asdim}(H_i) < \infty$ for each i [136, Theorem 1.2].

The product of n (unbounded) metric trees has linearly controlled asymptotic dimension n (see, e.g., [110]). Therefore, Theorem 5.1.4 and Propositions 5.2.6 and 5.2.4 combine to show the following.

Corollary 5.1.5. *Suppose the group G is hyperbolic relative to $H_1, \ldots, H_n$. Let $m = \max\{\text{eco-dim}(H_i)\}$. Then* $\text{eco-dim}(Bow(G)) \leq \max\{\text{asdim}(G), m + 1\} + 1$.

In order to prove Theorem 5.1.2, we use work of Bestvina, Bromberg and Fujiwara to combine the embeddings of the peripheral groups and of $Bow(G)$ into a single embedding of G into a product of trees.

Theorem 5.3.1 *Let G be hyperbolic relative to $H_1, \dots, H_n$, and suppose that each H_i admits a quasi-isometric embedding into the product of m trees. Then G admits a quasi-isometric embedding into the product of m trees and either $Bow(G)$ or the coned-off graph $\hat{G}$.*

This theorem completes the proof of Theorem 5.1.2. Note that the converse statement is automatic, as peripheral groups of a relatively hyperbolic group are undistorted in the ambient group [68, Lemma 4.15].

Rather than using the Bowditch space $Bow(G)$, to improve the bounds in Theorem 5.1.2 one might hope to use instead the (hyperbolic) coned-off graph $\hat{G}$. However, the non-locally finite nature of $\hat{G}$ leads to a non-compact boundary, and much of the machinery used above no longer applies.

The embeddings we consider are not and cannot required to be equivariant, because there exist hyperbolic groups with property (T) (for example, any cocompact lattice in $Sp(n, 1)$), which in particular cannot act interestingly on trees. We remark that the inverse problem of equivariantly embedding trees into hyperbolic spaces is treated in [41].

Theorem 5.1.2 has the following immediate corollary.

Corollary 5.1.6. *If the peripheral groups of a relatively hyperbolic group G each quasi-isometrically embed into the product of finitely many trees, then $\ell\text{-asdim}(G) < \infty$.*

If the group G satisfies $\ell\text{-asdim}(G) < \infty$ then it has Hilbert compression 1 [79, Theorem 1.1.1]. For more general estimates on the Hilbert compression of a relatively hyperbolic group, see [94].

In Section 5.4 we apply our results to 3-manifold groups, and show, amongst other results, the following.

Theorem 5.4.1 *Let $G = \pi_1(M)$, where M is a compact, orientable 3-manifold whose (possibly empty) boundary is a union of tori. Then $\text{eco-dim}(G) < \infty$ if and only if no manifold in the prime decomposition of M has* Nil *geometry; in this case, $\text{eco-dim}(G) \leq 8$.*

In any case, $\ell\text{-asdim}(G) \leq 8$.

Finally, in Appendix 5.5 we prove a bound on the asymptotic dimension of HNN extensions, following work of Dranishnikov.

5.1.1 Recent developments

Hume [95] has recently shown that the construction of Bestvina, Bromberg and Fujiwara that we use in Section 5.3 gives rise to spaces that are quasi-isometric to a tree-graded space. Tree-graded spaces are defined in [68]; roughly speaking, they are tree-like arrangements of certain specified subspaces called pieces (for example, consider a Cayley graph of a free product). It is known [40] that if X is tree-graded and $\mathcal{P}$ is its collection of pieces then

$$\ell\text{-asdim}(X) \leq \max_{P \in \mathcal{P}} \{1, \ell\text{-asdim}(P)\}$$

when, say, there are finitely many (L, C)-quasi-isometry types of pieces, for some given L, C.

As noticed in [95], the arguments in Section 5.3 (just considering π_Y instead of $f_{i,Y} \circ \pi_Y$ in Subsection 5.3.2), combined with [95], give the following.

Theorem 5.1.7 (cf. Theorem 5.3.1)**.** *Let G be hyperbolic relative to $H_1, \ldots, H_n$. Then G quasi-isometrically embeds in the product of a space $\mathcal{T}$ and either the coned-off graph $\hat{G}$ or the Bowditch space $Bow(G)$, where $\mathcal{T}$ is tree-graded and each piece is uniformly quasi-isometric to some H_i.*

In particular, once again as noticed in [95], using the aforementioned result from [40] one can get the following.

Corollary 5.1.8. *A relatively hyperbolic group has finite* ℓ-asdim *if and only if its peripheral subgroups do.*

5.2 Large scale and boundary dimension estimates

In this section we bound the asymptotic dimension of the Bowditch space of a relatively hyperbolic group. We also bound the linearly controlled metric dimension of the boundary of a Gromov hyperbolic space by its asymptotic dimension.

Observe that at the cost of a slight relaxation in the value of r, a collection of subsets $\mathcal{U}$ has r-multiplicity at most m if and only if we can write $\mathcal{U} = \bigcup_{i=1}^{m} \mathcal{U}_i$, where each $\mathcal{U}_i$ is *r-disjoint*: it is a collection of subsets pairwise separated by a distance of at least r. This follows from an application of Zorn's lemma. We will use this alternative characterization throughout this section.

5.2.1 Asymptotic dimension of Bowditch spaces

First we bound the asymptotic dimension of a horoball.

Proposition 5.2.1. $\operatorname{asdim}(\mathcal{H}(\Gamma)) \leq \ell\text{-asdim}(\Gamma) + 1$.

Proof. We will use the Hurewicz theorem for asymptotic dimension.

Theorem 5.2.2 ([17, Theorem 1])**.** *Let $f : X \to Y$ be a Lipschitz map between geodesic metric spaces, and suppose that for each R the family $\{f^{-1}(B_R(y))\}_{y \in Y}$ has uniform asymptotic dimension $\leq m$. Then* $\operatorname{asdim}(X) \leq \operatorname{asdim} Y + m$.

Choose some vertex $x \in \Gamma$, and let γ be the geodesic ray in $\mathcal{H}(\Gamma)$ obtained by concatenating the edges between (x, i) and $(x, i+1)$ for all $i \in \mathbb{N}$. Let $f : \mathcal{H}(\Gamma) \to \gamma$ be the natural 1-Lipschitz map given by $f(z, i) = (x, i)$. Fix some $R > 0$. The preimages under f of all balls of radius R in γ are quasi-isometric (with constants depending on R only) to (Γ, d_n) for some n, where $d_n(x, y) = 2\ln(d_\Gamma(x, y)e^{-n} + 1)$, see Lemma 2.3.1. So, we only need to show that the uniform asymptotic dimension of $\{(\Gamma, d_n)\}$ is at most m. Fix any R large enough. There exists $C = C(\Gamma)$ so that the following holds. Let $m = \ell\text{-asdim}(\Gamma)$. For each n there exists a covering $\mathcal{U}(n) = \mathcal{U}_1(n) \cup \cdots \cup \mathcal{U}_{m+1}(n)$ of (Γ, d_Γ) such that each $\mathcal{U}_i(n)$ is e^{n+R}-disjoint and Ce^{n+R}-bounded. So, $\mathcal{U}_i(n)$ is $(2R - M)$-disjoint and $(2R + M)$-bounded with respect to d_n, where M depends on Γ only. □

Proposition 5.2.1 has a weak converse.

Proposition 5.2.3. $\ell\text{-asdim}(\Gamma) \leq \operatorname{asdim}(\mathcal{H}(\Gamma))$.

Proof. Set $m = \operatorname{asdim}(\mathcal{H}(\Gamma))$. There exists a covering $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_{m+1}$ of $\mathcal{H}(\Gamma)$ such that each $\mathcal{U}_i$ is r-disjoint and R-bounded, for some r, R large enough. Up to increasing R and decreasing r we have that for each n there exists a covering $\mathcal{U}_1(n) \cup \cdots \cup \mathcal{U}_{m+1}(n)$ of (Γ, d_n) with the same properties, where $d_n(x, y) = 2\ln(d_\Gamma(x, y)e^{-n} + 1)$ as in the previous proof. So, each $\mathcal{U}_i(n)$ is $e^n(e^{r/2} - 1)$-disjoint and $e^n(e^{R/2} - 1)$-bounded in (Γ, d_Γ). We are done as

$$\frac{e^n(e^{R/2} - 1)}{e^n(e^{r/2} - 1)}$$

is bounded independently of n. □

Now we can bound the asymptotic dimension of the Bowditch space.

Proposition 5.2.4. *Let G be hyperbolic relative to $H_1, \ldots, H_n$. Also, denote $m = \max_{i=1,\ldots,n} \ell\text{-asdim}(H_i)$. Then*

$$\max\{\text{asdim}(G), m\} \leq \text{asdim}(Bow(G)) \leq \max\{\text{asdim}(G), m+1\}.$$

Proof. The lower bound by m follows from Proposition 5.2.3. The lower bound by $\text{asdim}(G)$ follows as $Bow(G)$ contains G with a proper metric.

For the upper bound, we will use the union theorem for asymptotic dimension.

Theorem 5.2.5 ([15, Theorem 1])**.** *Let Y be a geodesic metric space and suppose that $Y = \bigcup_{i\in\mathbb{N}} A_i$. Also, suppose that $\{A_i\}$ has uniform asymptotic dimension $\leq n$ and for each R there exists $Y_R \subseteq Y$ so that $\text{asdim}\, Y_R \leq n$ and $\{A_i \backslash Y_R\}$ is R-disjoint. Then $\text{asdim}\, Y \leq n$.*

We apply the theorem with $\{A_i\}$ the family of horoballs of $Bow(G)$ and for each R, take Y_R to be a suitable neighbourhood of an orbit of G. By Proposition 5.2.1, $\text{asdim}(A_i) \leq m+1$, and the constants are uniform as there are only finitely many different isometry types. Finally, $\text{asdim}(Y_R) = \text{asdim}(G)$ because the action of G on $Bow(G)$ is proper. □

5.2.2 Linearly controlled metric dimension estimate

Proposition 5.2.6. *Suppose X is a Gromov hyperbolic geodesic metric space. Then $\ell\text{-dim}(\partial_\infty X) \leq \text{asdim}(X)$.*

Proof. We fix the notation of Subsection 2.4 regarding visual metrics, and write d for the metric on X.

Each $x \in \partial_\infty X$ is the limit of geodesic γ. Given $R > 0$, define the projection $\pi_R : \partial_\infty X \to X$ by $\pi_R(x) = \gamma(R)$. This is well defined up to an error of $C_1 = C_1(\delta)$. By considering a geodesic triangle between a, b and 0, observe that there exists $C_2 = C_2(C_0, C_1, \delta)$ so that if $d(\pi_R(a), \pi_R(b)) \geq s > 2C_1$, then $\rho(a,b) \geq \frac{1}{C_2} e^{-\epsilon(R-s/2)}$. Similarly, if $d(\pi_R(a), \pi_R(b)) \leq t$ then $\rho(a,b) \leq C_2 e^{-\epsilon(R-t/2)}$.

If $\text{asdim}(X) \leq n$, then, given $s = 3C_1$, there exists $t < \infty$ and a cover $\mathcal{U} = \bigcup_0^n \mathcal{U}_i$ so that each $U \in \mathcal{U}$ has diameter at most t, and $\mathcal{U}_i$ is s-disjoint.

Suppose some small $r > 0$ is given. Let $R = -\frac{1}{\epsilon} \ln r$. For $U \in \mathcal{U}_i \subset \mathcal{U}$, let $\hat{U} \subset \partial_\infty X$ be the set of points z so that there exists a geodesic γ from 0 to z with $\gamma(R) \in U$. Let $\hat{\mathcal{U}}_i = \bigcup_{U\in\mathcal{U}_i} \hat{U}$, and $\hat{\mathcal{U}} = \bigcup_{i=1}^n \hat{\mathcal{U}}_i$.

By the above estimates, $\hat{\mathcal{U}}_i$ is $(e^{\epsilon s/2} r/C_2)$-disjoint, and $\hat{\mathcal{U}}$ is $C_2 r e^{\epsilon t/2}$-bounded. Since the ratio of these distances is bounded by $C_2^2 e^{\epsilon(t-s)/2}$, we have $\ell\text{-dim}(\partial_\infty X) \leq n$. □

5.3 Embedding in a product of quasi-trees and the Bowditch space

The aim of this section is to find an embedding of a given relatively hyperbolic group into a product of trees "stabilized" by the Bowditch space or the coned-off graph.

Theorem 5.3.1. *Let G be hyperbolic relative to $H_1, \ldots, H_n$. Suppose there exists k so that each H_i admits quasi-isometric embedding into the product of k trees. Then G admits a quasi-isometric embedding into the product of k trees and either $Bow(G)$ or the coned-off graph $\hat{G}$.*

We prove this theorem in subsection 5.3.2.

5.3.1 Quasi-trees of spaces

To prove Theorem 5.3.1 we use a result by Bestvina, Bromberg and Fujiwara described below.

Let $\mathbf{Y}$ be a set and for each $Y \in \mathbf{Y}$ let $\mathcal{C}(Y)$ be a geodesic metric space. For each Y let $\pi_Y : \mathbf{Y}\backslash\{Y\} \to \mathcal{P}(\mathcal{C}(Y))$ be a function (where $\mathcal{P}(\mathcal{C}(Y))$ is the collection of all subsets of $\mathcal{C}(Y)$). Define

$$d_Y^\pi(X, Z) = \operatorname{diam}\{\pi_Y(X) \cup \pi_Y(Z)\}.$$

Using the enumeration in [18, Sections 3.1, 2.1], consider the following Axioms. There exists $\xi < \infty$ so that:

(0) $\operatorname{diam}(\pi_Y(X)) < \xi$ for all distinct $X, Y \in \mathbf{Y}$,

(3) for all distinct $X, Y, Z \in \mathbf{Y}$ we have $\min\{d_Y^\pi(X, Z), d_Z^\pi(X, Y)\} \leq \xi$,

(4) $\{Y : d_Y^\pi(X, Z) \geq \xi\}$ is a finite set for each $X, Z \in \mathbf{Y}$.

Using the functions d_Y^π, the authors of [18] define a complex $\mathcal{P}_K(\{(\mathcal{C}(Y), \pi_Y)\}_{Y\in\mathbf{Y}})$ with vertex set $\mathbf{Y}$, which they denote as $\mathcal{P}_K(\mathbf{Y})$, and turns out to be a quasi-tree [18, Section 2.3].

Let $\mathcal{C}(\{(\mathcal{C}(Y), \pi_Y)\}_{Y\in\mathbf{Y}})$ be the path metric space consisting of the union of all $\mathcal{C}(Y)$'s and edges of length 1 connecting all points in $\pi_X(Z)$ to all points in $\pi_Z(X)$ whenever X, Z are connected by an edge in the complex $\mathcal{P}_K(\{(\mathcal{C}(Y), \pi_Y)\}_{Y\in\mathbf{Y}})$. More details of this construction are found in [18, Section 3.1].

The only result from [18] that we need is the following.

Theorem 5.3.2 ([18, Theorem 3.10])**.** *If* $\{(\mathcal{C}(Y),\pi_Y)\}_{Y\in\mathbf{Y}}$ *satisfies Axioms* $(0),(3)$ *and* (4) *and each* $\mathcal{C}(Y)$ *is a tree then* $\mathcal{C}(\{(\mathcal{C}(Y),\pi_Y)\}_{Y\in\mathbf{Y}})$ *is a quasi-tree (i.e. it is quasi-isometric to a tree).*

To apply this theorem in the context of Theorem 5.3.1, we fix a Cayley graph $\Gamma(G,S)$ with word metric d of the relatively hyperbolic group G as in Definition 2.1.2. Now let $\mathbf{Y}$ be the collection of all left cosets of peripheral subgroups in G. For each such left coset $Y = gH_i$ we denote by $\mathcal{C}(Y)$ the subgraph $g\Gamma(H_i, S\cap H_i)$ in $\Gamma(G,S)$, which contains Y and is a copy of the Cayley graph of H_i. Let π_Y be a closest point projection on $\mathcal{C}(Y)$ for each $Y\in\mathbf{Y}$. Namely, for each $x\in X$, let $\pi_Y(x)$ be the subset of $\mathcal{C}(Y)$ minimizing the function $d(x,\cdot)$, and let $\pi_Y(X)=\bigcup_{x\in X}\pi_Y(x)$.

As peripheral subgroups are undistorted in G [68, Lemma 4.15] we can, for our purposes, identify $\mathcal{C}(Y)$ with the corresponding subset of (the Cayley graph of) G.

Lemma 5.3.3. *The collection* $\{(\mathcal{C}(Y),\pi_Y)\}_{Y\in\mathbf{Y}}$ *satisfies Axioms* $(0),(3)$ *and* (4).

Proof. Lemma 3.1.1 and Corollary 3.1.2 imply Axioms (0) and (3). Axiom (4) follows, for example, from the fact that the the right hand side of the distance formula (Theorem 3.1.5) is finite. □

5.3.2 The proof of Theorem 5.3.1

Let us go back to the general setting of Theorem 5.3.2 for a moment, with $\{\mathcal{C}(Y),\pi_Y\}_{Y\in\mathbf{Y}}$ as in the statement of that theorem. Suppose that we are also given another collection of geodesic metric spaces $\{\mathcal{C}'(Y)\}_{Y\in\mathbf{Y}}$, indexed by the same set $\mathbf{Y}$, and coarsely Lipschitz maps $f_Y:\mathcal{C}(Y)\to\mathcal{C}'(Y)$, for $Y\in\mathbf{Y}$, with uniform constants. Observe that if axioms $(0),(3)$ and (4) hold for $\{(\mathcal{C}(Y),\pi_Y)\}_{Y\in\mathbf{Y}}$, then they also hold for $\{(\mathcal{C}'(Y),f_Y\circ\pi_Y)\}_{Y\in\mathbf{Y}}$.

Now suppose that each H_i admits a quasi-isometric embedding into the product of k trees. We then have coarsely Lipschitz maps $f_{i,Y}$, $i=1,\ldots,k$ from $\mathcal{C}(Y)$ to some tree $T_{i,Y}$. According to the observation we just made and Theorem 5.3.2, for each i we have a map $p_i : G\to\mathbf{T}_i=\mathcal{C}(\{(T_{i,Y},f_{i,Y}\circ\pi_Y)\})_{Y\in\mathbf{Y}}$, and $\mathbf{T}_i$ is a quasi-tree. The last step in the proof of Theorem 5.3.1 is the following proposition.

Proposition 5.3.4. *The map* $f=\prod p_i\times c : G\to\prod\mathbf{T}_i\times\hat{G}$ *is a quasi-isometric embedding, where* $c:G\to\hat{G}$ *is the inclusion.*

Proof. There are distance formulas for the $\mathbf{T}_i$'s [18, Lemma 3.3, Corollary 3.12] which summed up give, for L, λ, μ large enough,

$$d\left(\prod p_i(x), \prod p_i(y)\right) \approx_{\lambda,\mu} \sum_{Y \in \mathbf{Y}} \left\{\left\{d(\pi_Y(x), \pi_Y(y))\right\}\right\}_L.$$

Comparing this with the distance formula for relatively hyperbolic groups (Theorem 3.1.5) immediately gives the required estimates. □

The map c factors through Lipschitz maps $G \to Bow(G) \to \hat{G}$, so the proposition implies both versions of Theorem 5.3.1.

5.4 Three dimensional manifold groups

In this section we prove the following theorem.

Theorem 5.4.1. *Let $G = \pi_1(M)$, where M is a compact, orientable 3-manifold whose (possibly empty) boundary is a union of tori. Then* eco-dim$(G) < \infty$ *if and only if no manifold in the prime decomposition of M has* Nil *geometry; in this case,* eco-dim$(G) \leq 8$.

*In any case, ℓ-*asdim$(G) \leq 8$.

The second part of the following proposition is not needed for our purposes, but we think it is of independent interest.

Proposition 5.4.2. 1. *If M is a graph manifold with non-empty boundary then ℓ-*asdim$(\pi_1(M)) = 2$.

2. *If M is a Haken orientable 3-manifold and its boundary is non-empty then* asdim$(\pi_1(M)) \leq 2$.

Recall that a compact orientable 3-manifold M is *Haken* if it is irreducible and it contains a π_1-injective embedded surface S. Cutting M along S gives another Haken manifold, and moreover the new π_1-injective surface can be required to have non-empty boundary. Repeating the cutting procedure finitely many times yields a disjoint union of balls. Such sequence of cuts is called *Haken hierarchy.*

Proof. 1) The lower bound follows from the existence of undistorted copies of $\mathbb{Z}^2$ in $\pi_1(M)$. Up to passing to a finite-sheeted cover of M, we can assume that M fibers over S^1 by [160], so that we have a short exact sequence

$$1 \to F \to \pi_1(M) \to \mathbb{Z} \to 1,$$

where F is a free group. What is more, the content of [160, Theorem 0.7] is that a fiber intersects all Seifert components and it splits them open into products of a surface with an interval. In particular, the monodromy of the fiber bundle is a product of Dehn twists along disjoint simple closed curves (the connected components of the intersection of the fiber with the boundary of the Seifert components), and so F is undistorted in $\pi_1(M)$. The result then follows from a direct application of [39, Theorem 0.2].

2) We proceed by induction on the length of a Haken hierarchy for M so that all surfaces involved have non-empty boundary. If M is a ball, then the result trivially holds. Otherwise, $\pi_1(M)$ can be written as an amalgamated product $\pi_1(N_1) *_{\pi_1(S)} \pi_1(N_2)$ or an HNN extension $\pi_1(N_1)*_{\pi_1(S)}$, where N_i is a Haken manifold admitting a strictly shorter Haken hierarchy (of the type described above) than M and S is a compact surface with non-empty boundary. We can then use the induction step and the fact that, when A, B, C are finitely generated groups, $\operatorname{asdim}(A *_C B) \leq \max\{\operatorname{asdim} A, \operatorname{asdim} B, \operatorname{asdim} C + 1\}$ and $\operatorname{asdim}(A*_C) \leq \max\{\operatorname{asdim} A, \operatorname{asdim} C + 1\}$ (see [65] and Theorem 5.5.1). $\square$

Remark 5.4.3. The second part of the proposition (in the case of toric boundary) also follows from recent results about virtual specialness of fundamental groups of 3-manifolds [162, 115, 146] in view of a criterion for virtual fibering discovered by Agol [1].

We now show the following easy lemma and then prove the theorem.

Lemma 5.4.4. $\text{eco-dim}\, G_1 * G_2 = \max\{\text{eco-dim}\, G_1, \text{eco-dim}\, G_2\}$.

Proof. The inequality $\geq$ follows from the fact that G_1, G_2 are undistorted in $G = G_1 * G_2$. Suppose that, for $i = 1, 2$, $f_i : G_i \to \prod_{j=1}^n T_j^i$ are quasi-isometric embeddings. We have to show that G embeds in the product of n trees as well. Denote by T the Bass-Serre tree of G. For each vertex v of T denote by T_k^v a copy of $T_k^{i(v)}$, where $i(v)$ equals i if the stabilizer of v is conjugate to G_i. When e is an edge of T with endpoints v_1, v_2, we let p_e be the only element of G in the intersection of the left cosets of G_1, G_2 corresponding to v_1, v_2. Finally, we let T_k be the tree obtained from $\bigcup T_k^v$ by adding an edge of length, say, 1 connecting $f_{1,k}(p_e)$ to $f_{2,k}(p_e)$ for each edge e of T, where $f_{i,k}$ is the k-th component of f_i and we identify G_i with its left coset corresponding to an endpoint of e.

There is a natural map $f : G \to \prod_{i=1}^n T_k$, which (up to an error bounded by 1) restricts to f_i on every left coset of G_i. It is readily checked that this map is a quasi-isometric embedding (using more sophisticated technology than needed, one can use the distance formula and observe that, as we added edges of length 1 connecting T_{v_1} to T_{v_2} when v_1 is adjacent to v_2, we have $d(f(x), f(y)) \geq d_T(x, y)$ and $d_T(x, y)$ is approximately $d_{\hat{G}}(x, y)$). □

Proof of Theorem 5.4.1. Let G_i, for $i = 1, \ldots, n$, be the fundamental groups of the prime factors M_i of M. Then G is the free product of the G_i, so that eco-dim$(G) = \max\{\text{eco-dim}(G_i)\}$ in view of the previous lemma. Also ℓ-asdim$(G) = \max\{\ell\text{-asdim}(G_i)\}$ by [40]. In particular, we can just study the case when M is prime.

We report below a list of previously known cases. Except for the last two cases, the first column indicates the geometry (of the universal cover) of M. The values in the table are justified below.

M	ℓ-asdim$(\pi_1(M))$	eco-dim$(\pi_1(M))$
S^3	0	0
$\mathbb{R}^3$	3	3
$\mathbb{H}^3$, closed	3	3
$S^2 \times \mathbb{R}$	1	1
$\mathbb{H}^2 \times \mathbb{R}$, closed	3	3
$\mathbb{H}^2 \times \mathbb{R}$, non-closed	2	2
$\widetilde{SL_2\mathbb{R}}$	3	3
Nil	3	∞
Sol	3	3 or 4
graph manifold, closed	3	3
graph manifold, non-closed	2	2 or 3

The bounds on S^3, $S^2 \times \mathbb{R}$ and $\mathbb{R}^3$ are trivial. The calculation of ℓ-asdim$(\mathbb{H}^n)$ = eco-dim$(\mathbb{H}^n) = n$ is found in [43, 45]. This also gives eco-dim$(\mathbb{H}^2 \times \mathbb{R}) \leq 3$, and ℓ-asdim$(\mathbb{H}^2 \times \mathbb{R}) = 3$ [67, Theorem 4.3]. The spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL_2\mathbb{R}}$ are quasi-isometric. If M is not closed and has geometry $\mathbb{H}^2 \times \mathbb{R}$ then $\pi_1(M)$ is virtually the product of a free group and $\mathbb{Z}$.

The discrete Heisenberg group H is quasi-isometric to Nil. The result asdim(H) = ℓ-asdim$(H) = 3$ has been obtained by several people, for example see [75]. On the other hand, eco-dim$(H) = \infty$ as H does not admit a quasi-isometric embedding into the product of finitely many metric trees, or indeed any CAT(0) space [143]. Thus the proof in the "only if" direction is complete.

For ℓ-asdim(Sol) = 3 see [90]. As Sol quasi-isometrically embeds in $\mathbb{H}^2 \times \mathbb{H}^2$ (see, for example, [62, Section 9]), eco-dim(Sol) ≤ 4.

If M is a graph manifold then ℓ-asdim$(\pi_1(M)) \leq$ eco-dim$(\pi_1(M)) \leq 3$ by [96], and it is observed in the same paper that the equalities hold if M is closed. We handled the non-closed case for ℓ-asdim in Proposition 5.4.2.

There are only two cases left. First, when M is a finite volume, non-closed hyperbolic manifold it is well-known that $\pi_1(M)$ is hyperbolic relative to virtually $\mathbb{Z}^2$ subgroups [76]. (Notice also that $\operatorname{asdim}(\pi_1(M)) \leq 3$ as $\pi_1(M)$ admits a coarse embedding in $\mathbb{H}^3$.) As $Bow(\pi_1(M))$ is quasi-isometric to $\mathbb{H}^3$, by Theorem 5.3.1,

$$2 \leq \ell\text{-asdim}(\pi_1(M)) \leq \text{eco-dim}(\pi_1(M)) \leq 5.$$

The second case is when M is non-geometric and its geometric decomposition contains at least one hyperbolic component. In this case $\pi_1(M)$ is hyperbolic relative to virtually $\mathbb{Z}^2$ and graph manifold groups, as a consequence of the combination theorem [60, Theorem 0.1] and aforementioned fact that the hyperbolic component groups are hyperbolic relative to virtually $\mathbb{Z}^2$ groups (a full statement is given in [5, Theorem 9.2]). Therefore, by the graph manifold groups bound of [96], every peripheral group has eco-dim at most $m = 3$. Note too that $\operatorname{asdim}(\pi_1(M)) \leq 4$ holds by [16]. Thus by Theorem 5.1.2,

$$\begin{aligned}\text{eco-dim}(\pi_1(M)) &\leq \max\{\operatorname{asdim}(\pi_1(M)), m+1\} + m + 1 \\ &= \max\{4, 3+1\} + 3 + 1 = 8.\end{aligned}$$

Therefore, we have

$$2 \leq \ell\text{-asdim}(\pi_1(M)) \leq \text{eco-dim}(\pi_1(M)) \leq 8,$$

when M is non-closed, and

$$3 \leq \ell\text{-asdim}(\pi_1(M)) \leq \text{eco-dim}(\pi_1(M)) \leq 8,$$

when M is closed. The lower bound in the last case follows from the fact that the asymptotic dimension is greater or equal to the virtual co-homological dimension for groups of type FP [66]. □

5.5 Appendix: Asymptotic dimension of HNN extensions

The proof of the following theorem follows almost verbatim the arguments in [65].

Theorem 5.5.1. *Let A, C be finitely generated groups. Then*

$$\operatorname{asdim}(A*_C) \leq \max\{\operatorname{asdim}(A), \operatorname{asdim}(C)+1\}.$$

The inequality $\operatorname{asdim}(A*_C) \leq \operatorname{asdim}(A) + 1$ is shown in [16].

We require the following notation. For X a metric space, we will say that (r,d)-$\dim X \leq n$ if there exists a d-bounded cover of X with Lebesgue number at least r and 0-multiplicity at most $n+1$. We will say that a family $\{X_i\}$ of metric spaces satisfy $\operatorname{asdim} X_i \leq n$ uniformly if for every $r > 0$ there exists d so that (r,d)-$\dim X_i \leq n$. A partition of the metric space X is a presentation of X as a union of subspaces with pairwise disjoint interiors. The proof of Theorem 5.5.1 uses the criterion below.

Theorem 5.5.2 ([65, Partition Theorem])**.** *Let X be a geodesic metric space. Suppose that for every $R > 0$ there exists $d > 0$ and a partition $X = \bigcup_{i\in\mathbb{N}} W_i$ where $\operatorname{asdim} W_i \leq n$ uniformly and with the property that (R,d)-$\dim(\bigcup \partial W_i) \leq n-1$ (with the restriction of the metric of X). Then $\operatorname{asdim}(X) \leq n$.*

We now do some preliminary work.

Fix a generating system S_A of A and let t be the stable letter of the HNN extension. Let d be the associated word metric on $G = A*_C$. The group G acts on its Bass-Serre tree whose vertices are left cosets of A in G and whose edges are labelled by left cosets of C. The endpoints of the edge gC are gA and gtA. Denote by K the graph dual to the Bass-Serre tree. We will denote the simplicial metric in K by $|\cdot,\cdot|$, and we let $|u| = |u, C|$. Notice that for each pair of vertices in K there exists a unique geodesic connecting them. Let $\pi : G \to K$ be the map $g \mapsto gC$.

Remark 5.5.3. π extends to a simplicial map of the Cayley graph of G. In particular, π is 1-Lipschitz.

In fact, let $s \in S_A$. Then for each $g \in G$ we have $gsA = gA$, so that the edges gC and gsC of the Bass-Serre tree share the endpoint gA, which is exactly saying that the vertices gC, gsC of K are connected by an edge. Similarly, the edges gC and gtC of the Bass-Serre tree share the endpoint gtA.

We divide K into two parts K_0, K_1 intersecting only at the base vertex C, where K_1 contains the edges corresponding to tA. Let $\overline{d}$ be the graph metric on K, and denote by B^1_r the closed r-ball in K_1 centred at C, where r will always denote an integer. We will write $v \leq u$, where $u, v \in K^{(0)}$, if v lies on the geodesic segment $[C, u]$ (notice that this is a partial order). For $u \in K^{(0)}$ with $u \neq C$ and $r > 0$ denote

$$K^u = \{v \in K^{(0)} |\ v \geq u\},$$

and

$$B^u_r = \{v \in K^u |\ |v| \leq |u| + r\}.$$

Notice that if $u = gC \in K_1$ then $B_r^u = gB_r^1$ and $K^u = gK_1$. In particular, $\pi^{-1}(B_r^u) = g\pi^{-1}(B_r^1)$ and $\pi^{-1}(K^u) = g\pi^{-1}(K_1)$.

We say that $F \subseteq G$ separates $H_1, H_2 \subseteq G$ if all paths in the Cayley graph connecting H_1 to H_2 intersect F. Set $D_R = \{x \in G|\ d(x, C) = R\} \cap \pi^{-1}(K_1)$, for $R \in \mathbb{N}$. For $u = gC \in K_1$ denote $D_R^u = gD_R$ and notice that $\pi(D_R^u) \subseteq B_R^u$.

Lemma 5.5.4. *Let $u \in K_0^{(0)}, v \in K^{(0)}$ be so that either $v < u$ or v is incomparable with u. Then D_R^u separates $\pi^{-1}(v)$ and $\pi^{-1}(u')$ whenever $u < u'$ and $|u'| - |u| > R$.*

Proof. We will show that D_R separates $\pi^{-1}(K_0)$ and $\pi^{-1}(u')$ if $|u'| > R$ and $u' \in K_1$, using the action of G then yields the required statement. As π is 1-Lipschitz we have $d(C, \pi^{-1}(u')) > R$ so that D_R separates C and $\pi^{-1}(u')$. To complete the proof notice that C separates $\pi^{-1}(K_0)$ and $\pi^{-1}(K_1)$ (as their images through π are separated by the vertex labelled C). □

Lemma 5.5.5. *Suppose $R \leq r/4$. Then $d(gD_R, g'D_R) \geq 2R$ whenever $g, g' \in G$ with $gC, g'C \in K_1$, $|gC|, |g'C| \in r\mathbb{N}$ and $gC \neq g'C$.*

Proof. Set $u = gC, u' = g'C$. Suppose first that $|u| \neq |u'|$. As $\pi(gD_R) \subseteq B_R^u$ and $\overline{d}(B_R^u, B_R^{u'}) \geq r - R \geq 3R$, and in view of the fact that π is 1-Lipschitz, we get $d(gD_R, g'D_R) \geq 3R$.

Suppose instead $|u| = |u'|$ and pick $x \in gD_R, y \in g'D_R$. Every path in K between $\pi(x)$ and $\pi(y)$ passes through u and u'. This applies in particular to the projection of a geodesic γ in the Cayley graph of G, so that γ intersects gC and $g'C$. Since $d(x, gC), d(y, g'C) = R$, we get $d(x, y) \geq 2R$. □

Lemma 5.5.6. *For $m \in \mathbb{N}$, let $(At)^m = \bigcup At^{\epsilon_1} \cdots At^{\epsilon_m} \subseteq A*_C$, where each ϵ_i equals 0 or 1. Then $\operatorname{asdim}(At)^m \leq \max\{\operatorname{asdim} A, \operatorname{asdim} C + 1\}$ for every m.*

Proof. $(At)^m$ admits a coarse embedding in $\pi_1(\mathcal{G})$, where $\mathcal{G}$ is a graph of groups so that all vertex groups are isomorphic to A, all edge groups are isomorphic to C and the underlying graph is a tree. Repeated applications of [65, Theorem 2.1] give $\operatorname{asdim} \pi_1(\mathcal{G}) \leq \max\{\operatorname{asdim} A, \operatorname{asdim} C + 1\}$, and hence the same holds for $(At)^m$. □

We are now ready for the proof of the theorem.

Proof of Theorem 5.5.1. Set $n = \max\{\operatorname{asdim}(A), \operatorname{asdim}(C) + 1\}$. Once we show $\operatorname{asdim}(\pi^{-1}(K_0)), \operatorname{asdim}(\pi^{-1}(K_1)) \leq n$, the conclusion follows from the Finite Union Theorem [15]. We will show the latter, using the Partition Theorem 5.5.2. Fix $R > 0$ and take $r > 4R$. By Lemma 5.5.4 we can write $G = X_+ \cup X_-$ with $X_+ \cap X_- = D_R$

so that $X_+ \subseteq \pi^{-1}(K_1)$, and $\pi^{-1}(K_0) \subseteq X_-$, and D_R separates $X_+ \backslash D_R$ and $X_- \backslash D_R$. For each $u \in K_1^{(0)}$ fix g_u so that $u = g_u C$. Set $X_\pm^u = g_u(X_\pm)$, $V_r = X_+ \cap \left(\bigcap_{|u|=r} X_-^u\right)$ and $V_r^u = g_u(V_r)$. It is readily seen that $\pi(V_r) \subseteq B^1_{r+R}$ and that

$$\pi^{-1}(K_1) = \bigcup_{|u| \in r\mathbb{N}^+} V_r^u \cup N_R^+(C),$$

where $N_R^+(C) = N_R(C) \cap \pi^{-1}(K_1)$. If $|u|, |w| \in r\mathbb{N}^+$, $V_r^u \cap V_r^w \neq \emptyset$ and $u \neq w$ then either $u < w$ and $|w| = |u| + r$ or, vice versa, $w < u$ and $|u| = |w| + r$. Also, if $V_r^u \cap V_r^w \neq \emptyset$ and $u < w$ then $V_r^u \cap V_r^w = D_R^w$ for $D_R^w = g_w D_R$. Putting these facts together we get

$$Z = \bigcup_{|u| \in r\mathbb{N}^+} \partial V_r^u = \bigcup_{|u| \in r\mathbb{N}^+} D_R^u.$$

As D_R is coarsely equivalent to C, there is, for some $d > 0$, an (R, d)-cover $\mathcal{U}$ of D_R (i.e. the Lebesgue number of $\mathcal{U}$ is at least R and $\mathcal{U}$ is d-bounded) with 0-multiplicity at most n. By Lemma 5.5.5, $\bigcup_{|u| \in r\mathbb{N}^+} g_u \mathcal{U}$ is an (R, d)-cover of Z, and this witnesses the fact that (R, d)-$\dim(Z) \leq n - 1$. Finally, notice that $\pi^{-1}(B_s^1) \subseteq (At)^{s+1}$ so that $\operatorname{asdim} B_s^1 \leq n$, for each $s \in \mathbb{N}$. In particular, $\operatorname{asdim} \pi^{-1}(B^1_{r+R}) \leq n$ and thus $\operatorname{asdim} \pi^{-1}(V_r^u) \leq n$ uniformly. Finally, $\operatorname{asdim} N_R^+(C) \leq n - 1 \leq n$, so that the Partition Theorem applies. $\square$

Part III

Random walks and actions on relatively hyperbolic spaces

In this chapter we will be concerned with properties of random walks on relatively hyperbolic groups, as well as on other groups from Table 1.1. In Chapter 6, which is taken from [153], we will describe the formalism of (weakly) contracting elements and show that (weakly) contracting elements are generic.

Chapter 7 is not yet available as a preprint. We will compare random paths and geodesics in relatively hyperbolic groups and mapping class groups, showing in particular that they lie with high probability within Hausdorff distance bounded by explicit sublinear functions.

Chapter 6

Genericity of contracting elements

6.1 Introduction

A Morse element in a group G is an element h such that $H = \langle h \rangle$ is undistorted in G and any quasi-geodesic with endpoints on H stays within bounded distance from H, and the bound depends on the quasi-isometry constants only. Examples of Morse elements include infinite order elements in hyperbolic groups [83], hyperbolic elements (of infinite order) in relatively hyperbolic groups [68], pseudo-Anosovs in mapping class groups [13], etc. See [71] for further details and examples. Indeed, in all mentioned cases a stronger property than being Morse holds. Namely, all elements are *contracting* with respect to a suitable collection, called *path system*, of quasi-geodesic paths in the respective groups. Roughly speaking, the main property that a contracting element g satisfies is the existence of a map π_g from the group onto $\langle g \rangle$ such that if two points x, y have far away projections then all special paths from x to y pass close to $\pi_g(x)$ and $\pi_g(y)$. (This definition is less general than the one we will give, which is stated in terms of group actions on a metric space.) Related properties have been considered in [25, 13, 3, 151]. These are the examples we can provide, see Section 6.3.

Theorem 6.1.1. *Let G be a relatively hyperbolic group (resp. mapping class group, group acting properly by isometries on a proper $CAT(0)$ space, graph manifold group). Then there exists a path system for G such that $g \in G$ is contracting if and only if g is hyperbolic (resp. is pseudo-Anosov, acts as a rank one isometry, acts hyperbolically on the Bass-Serre tree).*

Notice that not all graph manifolds are $CAT(0)$ [113, 48].

We also define a more general notion, that of being *weakly contracting*, which will be sufficient for our applications. This concept is defined using the notion of weak path system described in Definition 6.2.8.

Theorem 6.1.2 (Proposition 6.3.7, Proposition 6.3.8). *Let G be a group acting acylindrically on a hyperbolic space X, and let $\mathcal{PS}$ be the collection of all geodesics of X. Then $g \in G$ is weakly contracting for the weak path system $(X, \mathcal{PS})$ if and only if it acts hyperbolically on X.*

There exists a weak path system for $Out(F_n)$, $n \geq 3$, such that $g \in Out(F_n)$ is weakly contracting if and only if it is fully irreducible.

The disadvantage of the weak version of contractivity is that we were not able to show that weakly contracting elements are Morse (this is related to [58, Problem 9.4]).

We will treat all examples mentioned in Theorems 6.1.1 and 6.1.2 from a common perspective. As it turns out, this approach is closely related to the approach relying on the notion of hyperbolically embedded subgroup developed in [58]:

Theorem 6.1.3 (Theorem 6.4.6). *Each weakly contracting element g (for some action) is contained in a hyperbolically embedded elementary subgroup $E(g)$. Viceversa, every infinite order element contained in a virtually cyclic embedded subgroup is weakly contracting for an appropriate action.*

The subgroup $E(g)$ is defined in a natural way in terms of projections, see Corollary 6.4.4. Notice that in view of Theorem 6.1.1 we obtain new examples of hyperbolically embedded subgroups, i.e. $E(g)$ where g acts as a rank one isometry on a proper $CAT(0)$ space, solving [58, Problem 9.5].

Our main result is that (weakly) contracting elements are generic, from the random walks viewpoint. Let G be a group and $S \subseteq G$ a finite subset. Let $W_n(S)$ be the set of words of length n in the elements of S and their inverses. A *simple random walk* supported on $\langle S \rangle$ is a sequence of random variables $\{X_n\}$ taking values in G and with laws μ_n so that for each $g \in G$ and $n \in \mathbb{N}$ we have

$$\mu_n(\{g\}) = \frac{|\{w \in W_n(S) : w \text{ represents } g\}|}{|W_n(S)|}.$$

Theorem 6.1.4. *Let G be a group equipped with a path system (resp. weak path system) and let $H < G$ be a non-elementary finitely generated subgroup containing a contracting (resp. weakly contracting) element. Then the probability that a simple random walk supported on H gives rise to a non-contracting (resp. non-weakly-contracting) element decays exponentially in the length of the random walk.*

Roughly speaking, the theorem states that the probability that a long word written down choosing randomly generators of H represents a non-contracting element is very small, and in fact exponentially decaying in the length of the word.

For mapping class groups this was known already [120], and related results can be found in [148, 149, 121, 122]. We emphasize that we can give a self-contained proof of the theorem above modulo a classical result of Kesten.

Finally, we point out two sample applications of the theorem to give examples of how it can be used to show the existence of contracting elements with additional properties.

Corollary 6.1.5. *Let G_1, G_2 be a non-elementary group supporting the path system (resp. weak path system) $(X, \mathcal{PS})$ and containing a contracting (resp. weakly contracting) element.*

- *For any isomorphism $\phi : G_1 \to G_2$ there exists a (weakly) contracting $g \in G$ so that $\phi(g)$ is also (weakly) contracting.*
- *If H is an amenable group and $\phi : G_1 \to H$ is any homomorphism, for any $h \in \phi(G_1)$ there exists a (weakly) contracting $g \in \phi^{-1}(h)$.*

Outline

Section 6.2 contains the definitions we will use throughout. Section 6.3 is devoted to the proof of Theorems 6.1.1 and 6.1.2. In Section 6.4 we will show Theorem 6.1.3. Section 6.5 contains our main technical tool (Lemma 6.5.2), which gives us a way of constructing many contracting elements once we are given just one of them. Finally in Section 6.6 we will show our main result. In the Appendix we will sketch the proof of a "Weak Tits Alternative" not relying on Theorem 6.1.3 and [58].

6.2 Definitions and first properties

Let X be a metric space.

Definition 6.2.1. A *path system* $\mathcal{PS}$ in X is a collection of (μ, μ)-quasi-geodesics in X, for some μ, such that

1. any subpath of a path in $\mathcal{PS}$ is in $\mathcal{PS}$,
2. all pairs of points in X can be connected by a path in $\mathcal{PS}$.

Elements of $\mathcal{PS}$ will be called $\mathcal{PS}$*-special paths*, or simply special paths if there is no ambiguity on $\mathcal{PS}$.

Fix a path system $\mathcal{PS}$ on the metric space X.

Definition 6.2.2. A subset $A \subseteq X$ will be called $\mathcal{PS}$*-contracting* with constant C if there exists $\pi_A = \pi : X \to A$ such that

1. $d(\pi(x), x) \leq C$ for each $x \in A$,
2. for each $x, y \in X$, if $d(\pi(x), \pi(y)) \geq C$ then for any $\mathcal{PS}$-special path δ from x to y we have $d(\delta, \pi(x)), d(\delta, \pi(y)) \leq C$.

The map π will be called $\mathcal{PS}$*-projection* on A with constant C.

We point out a relation with properties that appeared in the literature in several contexts, see [25, 3, 151].

Lemma 6.2.3. [151, Lemma 4.24] *Let X be a geodesic metric space and $A \subseteq X$. Denote by $\mathcal{PS}$ the collection of all geodesics in X. Suppose that the map $\pi : X \to A$ satisfies the following properties for some C:*

- *$d(x, \pi(x)) \leq d(x, A) + C$ for each $x \in X$,*
- *$diam(\pi(B_d(x))) \leq C$ for each $x \in X$, where $d = d(x, A)$.*

Then π is a $\mathcal{PS}$-projection with constant depending on C only.

The following lemma will be used several times.

Lemma 6.2.4. *Let π be a $\mathcal{PS}$-projection with constant C on $A \subseteq X$. Then*

1. *whenever δ is a special path we have $diam(\pi(\delta)) \leq diam(\delta \cap N_C(A)) + 2C$ and more specifically $diam(\pi(\delta)) \leq C$ if $\delta \cap N_C(A) = \emptyset$.*

   Also, there exists $k = k(\mathcal{PS})$ such that

2. *for each $x, y \in X$, $d(\pi(x), \pi(y)) \leq kd(x, y) + k$,*
3. *for each $x \in X$ we have $diam(\pi(B_r(x))) \leq C$ for $r = \dfrac{d(x, A)}{k} - k$,*
4. *for each $x \in X$, $d(x, \pi(x)) \leq kd(x, A) + k$.*

Proof. Item 1) is clear from the second projection property.

Item 2) follows from special paths being quasi-geodesics and the fact that either $d(\pi(x),\pi(y)) \leq C$ or any special path from x to y passes C-close to $\pi(x), \pi(y)$.

Item 3) holds in view of item 1) because, for c large enough, for each $y \in B_r(x)$ there is a special path connecting x to y and not intersecting $N_C(A)$.

In order to show item 4), consider a special path γ from x to some $y \in A$ with $d(x,y) \leq d(x,A)+1$. If $d(y,\pi(x)) \leq C$, we are done. Otherwise, γ contains a point x' such that $d(x',\pi(x)) \leq C$. In particular,

$$d(x,\pi(x)) \leq d(x,x') + C \leq \mu d(x,y) + \mu + C \leq \mu d(x,A) + \mu + C + 1,$$

and we are done for c large enough. □

Lemma 6.2.5. *If k is as in the previous lemma, $A, B \subseteq X$ are $\mathcal{PS}$-contracting subsets with constant C and $x \in X$, then*

$$\min\{d(\pi_A(x),\pi_A(B)), d(\pi_B(x),\pi_B(A))\} \leq C'.$$

where $C' = kC + k + C$.

Proof. If $\max\{d(\pi_A(x),\pi_A(B)), d(\pi_B(x),\pi_B(A))\} \leq kC + k + C$ there is nothing to prove, so, up to swapping A and B, suppose that $d(\pi_A(x),\pi_A(B)) > kC + k + C$. Let δ be a special path from x to some $x' \in B$, and let p be the first point in $\delta \cap N_C(A)$. If we show that there is no point $q \in \delta \cap N_C(B)$ before p, we are done by Lemma 6.2.4$-(4)-(1)$. In fact, if there was such a point q, again by Lemma 6.2.4$-(1)-(4)$ we would have

$$d(\pi_A(x),\pi_A(B)) \leq d(\pi_A(x),\pi_A(q)) + d(\pi_A(q),\pi_A(B)) \leq C + kC + k.$$

□

We are ready for our main definitions.

Definition 6.2.6. Let G be a group. A *path system* $(X,\mathcal{PS})$ on the group G is a *proper* action of G on the metric space X preserving the path system $\mathcal{PS}$ on X.

An infinite order element g of G will be called $\mathcal{PS}$-contracting if for some (and hence any) $x_0 \in X$ and any integer n

1. the orbit of x_0 is a quasi-geodesic,
2. There exists a g-invariant contracting set $A \ni x_0$, called *axis*, so that g acts coboundedly on A.

The reason why we do not just take A to be the orbit is that we will be interested in contracting elements with translation length much larger than the contraction constant of their axes, and we will achieve this by taking powers while keeping the same axis.

Slightly extending a notion from [24] we say that, given a group G acting on the metric space X, the element $g \in G$ is *WPD* if the orbits of $\langle g \rangle$ are Morse quasi-geodesics and for every $R \geq 0$ and $x \in X$ there exists $N = N(R, x)$ such that

$$|\{h \in G : d(x, h(x)) \leq R, d(hg^N x, g^N x) \leq R\}| < +\infty.$$

We will need the following consequence of the WPD property.

Lemma 6.2.7. (cf. [24, Proposition 6−(2)]) *Suppose that G acts on X and $g \in G$ is WPD. Then for each $x \in X$, $R \geq 0$ there exists L so that*

$$\{h \in G : d(x, hx) \leq R, diam(N_R(h\langle g \rangle x) \cap \langle g \rangle x) \geq L\}$$

is a finite set.

The meaning of the lemma is illustrated in Figure 6.1. Roughly speaking, the definition of WPD gives us that there are finitely many elements h so that for some large N we have the situation depicted for $N' = N$. The lemma gives us the same condition for all large enough N and without the requirement $N' = N$.

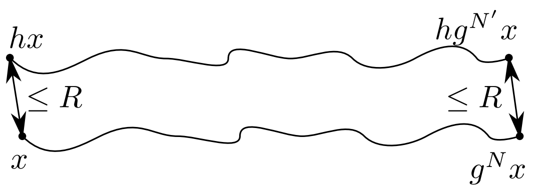


Figure 6.1

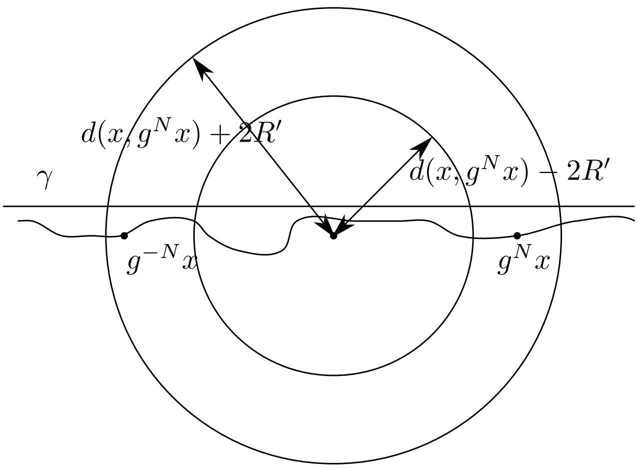


Figure 6.2

Proof. Fix x, R, g. First of all notice that, as $\langle g \rangle x$ is Morse, there exists $R' \geq R$ depending on R (and the data we fixed) so that if $d(x, hx) \leq R, d(g^N x, h\langle g \rangle x) \leq R$

then for each $0 \leq n \leq N$ we have $d(g^n x, h\langle g\rangle x) \leq R'$. We claim that it suffices to show that if

$$d(x, hx) \leq R, d(g^N x, hg^{N'} x) \leq R' \quad (*)$$

for some N, N' then either $d(g^N x, hg^N x) \leq C(R)$ or $d(g^N x, hg^{-N} x) \leq C(R)$, for some constant $C(R) \geq R$. In fact, if h, h' are as in the second case then hh' is easily seen to satisfy $d(hh'x, x) \leq 2R$, $d(g^N x, hh'g^N x) \leq 2C(R)$. So, the cardinality of the set as in the statement is at most the cardinality of

$$\{h \in G : d(x, hx) \leq 2R, d(hg^N x, g^N x) \leq 2C(R)\},$$

where for L large enough we can choose $N = N(C(R))$. By definition of $N(C(R))$ this set is finite, as required.

Now, suppose that $(*)$ holds. Then we have

$$|d(g^{N'} x, x) - d(g^N x, x)| \leq 2R'.$$

So, we just need to show that if N' satisfies this condition then $|N - N'| \leq K(R)$. In order to show this notice that, as the orbits of g are Morse quasi-geodesics, there exists a geodesic γ so that $d(g^N, \gamma), d(g^{N'}, \gamma), d(x, \gamma) \leq D$ (where D does not depend on N, N'), see Figure 6.2. In particular, if N' satisfies the condition then it is contained in a ball of radius, say, $2R'+10D$ around $g^N x$ or in a ball of the same radius around $g^{-N} x$. The existence of $K(R)$ then follows from the orbit of g being a quasi-geodesic. $\square$

Definition 6.2.8. Let G be a group. A *weak path system* $(X, \mathcal{PS})$ on the group G is any action of G on the metric space X which preserves the path system $\mathcal{PS}$ on X.

An infinite order element g of G will be called weakly $\mathcal{PS}$-contracting if it is WPD and some (hence any) orbit of g is contained in a contracting g-invariant set A on which g acts coboundedly.

Notice that we weakened the requirements for the action, by dropping the properness assumption, and "compensated" for this by requiring that the action is only proper "in the direction" of g.

Lemma 6.2.9.

1. *Special paths which are $\mathcal{PS}$-contracting with constant C are Morse with constants depending on C only.*

2. *Contracting elements are Morse.*

Proof. The first part can be proven in the same (actually, slightly simpler) way as the second part, so we will spell out the proof for the second part only.

Let $G, X, \mathcal{PS}, x_0$ be as in the definition above, and let g be a contracting element. The fact that $\langle g \rangle x_0$ is a quasi-geodesic easily implies that $\langle g \rangle$ is a quasi-geodesic in G.

In order to show that g is Morse we now need to show the following. Let α be a (μ, c)-quasi-geodesic in G connecting 1 to g^n and let γ be a special path from x_0 to $g^n x_0$. Then for each $h \in \alpha$ we have $d(\gamma x_0, hx_0) \leq K$, where K depends on g, μ, c and the action of G. A constant depending on the said data will be referred to as universal.

Let π be a projection on γ with constant C. Increase C suitably and define $\pi' : G \to \langle g \rangle$ in such a way that $d(\pi'(x)x_0, \pi(x))$ is bounded by C. Let ρ be a universal constant such that

$$d(\pi'(h), \pi'(h')) \leq \rho d(\pi(hx_0), \pi(h'x_0)) + \rho$$

for each $h, h' \in G$. Let $r \in \mathbb{N}$ be the least integer so that $d(\alpha(j), \alpha(j + r)) > \mu^2(\rho C + 1) + c$ for each integer j such that $j, j + r$ are in the domain of α. Whenever $h = \alpha(j), h' = \alpha(j + r)$ we will say that h, h' are consecutive. Notice that there is a universal bound L on $d(hx_0, h'x_0)$ whenever $h, h' \in \alpha$ are consecutive. Suppose that $h_0, \ldots, h_n \in \alpha$ is a maximal chain of consecutive points such that $d(hx_0, \gamma) \geq kL + k^2$, for k as in Lemma 6.2.4. Notice that we can bound the distance of h_0, h_n from γ again by a universal constant. We wish to show that n cannot be arbitrarily large. In fact, by Lemma 6.2.4$-$(3) we have $d(\pi(h_i(x_0)), \pi(h_{i+1}(x_0))) \leq C$, and hence $d(\pi(h_0(x_0)), \pi(h_n(x_0))) \leq nC$ so that $d(\pi'(h_0), \pi'(h_n)) \leq n\rho C + \rho$. For n large enough and in view of $d(h_i, h_{i+1}) > \mu^2(\rho C + 1) + c$ we get

$$d(h_0, h_n) \geq nr/\mu - c \geq \sum (d(h_i, h_{i+1}) - c)/\mu^2 - c >$$

$$n\rho C + \rho + d(h_0, \pi'(h_0)) + d(h_n, \pi'(h_n)) \geq d(h_0, h_n),$$

which is a contradiction. We can bound $d(p, \gamma)$ in terms of n, L, r, so we are done. $\square$

6.3 Examples

6.3.1 Relatively hyperbolic group

We will say that an infinite order element of a relatively hyperbolic group is *hyperbolic* if it not conjugate into any H_i.

Proposition 6.3.1. *Let G be a relatively hyperbolic group and let $\mathcal{PS}$ be the collection of the geodesics in $Bow(G)$. Then an element of G is contracting for the path system $(Bow(G), \mathcal{PS})$ if and only if it is hyperbolic.*

Proof. If an element is finite order or conjugate into a peripheral subgroup then clearly its orbit is not a bi-infinite quasi-geodesic.

On the other hand, hyperbolic elements are known to act hyperbolically on the Bowditch space, so that each orbit is a bi-infinite quasi-geodesic, see e.g. [35, Definition 1-(3)]. Here is a proof of this fact. Let g be a hyperbolic element. As g has infinite order we only have to show that it cannot act parabolically. Indeed, we claim that if an element of G acts parabolically on $Bow(G)$ then it fixes the parabolic point corresponding to some combinatorial horoball, and this easily implies that it must be conjugate into a peripheral subgroup. If g acted parabolically fixing a point that is not the limit point of a combinatorial horoball, then there would be a geodesic ray γ with $\gamma \cap G$ unbounded (we regard G as a subset of $Bow(G)$) so that for each n there is a point $h \in \gamma \cap G$ so that $d(h^{-1}g^i h, 1) = d(g^i h, h)$ is bounded by some constant C which depends on the hyperbolicity constant of $Bow(G)$. But this is not possible because there are finitely many elements of G in the ball of radius C in $Bow(G)$ around 1.

Finally, every geodesic in a given hyperbolic space (and hence every quasi-geodesic as quasi-geodesics are within bounded Hausdorff distance from geodesics) is contracting, see Lemma 3.7.2. $\square$

6.3.2 Mapping class group

In this subsection we will assume familiarity with the notion of curve complex, marking complex and hierarchy paths. We will use results from [126, 127], see also [36].

Proposition 6.3.2. *Let $\mathcal{M}(S)$ be the marking complex of the closed orientable punctured surface S of genus g with p punctures satisfying $3g + p \geq 5$ and let $\mathcal{H}(S)$ be the collection of all subpaths of hierarchy paths. An element of $MCG(S)$ is contracting for the path system $(\mathcal{M}(S), \mathcal{H}(S))$ if and only if it is pseudo-Anosov.*

Notice that the proposition and Lemma 6.2.9 give yet another proof that pseudo-Anosov elements are Morse [13, 71].

Remark 6.3.3. We remark that in view of the proof of [25, Proposition 8.1] (see the sixth to last line) and Lemma 6.2.3 the collection of all geodesics in Teichmüller space endowed with the Weil-Peterson metric has the property that all axes of pseudo-Anosov elements are contracting.

It is well known that hierarchy paths are quasi-geodesics with uniform constant, and that any non-pseudo-Anosov element is not Morse as, up to passing to a power, it has infinite index in its centralizer.

Recall from [13] that a pair of points $(\mu_1, \mu_2) \in \mathcal{M}(S)$ is D-transverse if for each $Z \subsetneq S$ we have $d_{C(Z)}(\mu_1, \mu_2) \leq D$. The proposition follows directly from the lemma below and the fact that pairs of points lying on the orbit of a pseudo-Anosov element are D-transverse.

Lemma 6.3.4. *For each D-transverse pair μ_1, μ_2 each hierarchy path $[\mu_1, \mu_2]$ is $\mathcal{H}(S)$-contracting with constant $C = C(D)$.*

Proof. Let B be such that if x, y lie on the (subpath of a) hierarchy path $[p, q]$ with main geodesic H then

1. for each subsurface $Y \subseteq S$ we have $d_{\mathcal{C}(Y)}(x, y) \leq d_Y(p, q) + B$,
2. $d_{\mathcal{C}(S)}(x, H) \leq B$,
3. for any $\gamma \in H$ there exists z such that $d_{\mathcal{C}(S)}(x, H) \leq B$.

Define $\pi : \mathcal{M}(S) \to [\mu_1, \mu_2]$ as follows. Consider $\mu \in \mathcal{M}(S)$ and let $\gamma_\mu = \pi_S(\mu)$. Pick $\alpha_\mu \in H$ (the main geodesic for the given hierarchy path) such that $d_{C(S)}(\gamma_\mu, \alpha_\mu) = d_{C(S)}(\gamma_\mu, H)$. Finally, choose $\nu \in [\mu_1, \mu_2]$ in such a way that $\pi_S(\nu)$ is B-close to α_μ and set $\pi(\mu) = \nu$.

Pick any $\nu_1, \nu_2 \in \mathcal{M}(S)$. Suppose $d(\pi(\nu_1), \pi(\nu_2)) \geq C(D)$, where $C(D)$ will be determined later. Notice that $d_{C(S)}(\alpha_{\nu_1}, \alpha_{\nu_2})$ is bounded from below by a linear function of $C(D)$, by the distance formula and D-transversality, so that for $C(D)$ large enough we can assume $d_{C(S)}(\alpha_{\nu_1}, \alpha_{\nu_2}) \geq 100\delta$, where δ is the hyperbolicity constant of $C(S)$. Consider a hierarchy path $[\nu_1, \nu_2]$ (more precisely containing ν_1, ν_2, but this does not affect what follows) and let H' be the main geodesic. We have that H' contains points β_1, β_2 which are B-close to the projections of $\nu_1', \nu_2' \in [\nu_1, \nu_2]$ on $C(S)$ and such that $d_{C(S)}(\alpha_{\nu_i}, \beta_i) \leq 10\delta$.

Our aim is to show that any point μ on $[\nu_1', \nu_2']$ such that $20\delta + 2B + 10 \leq d_{\mathcal{C}(S)}(\nu, \beta_1) \leq 20\delta + 3B + 20$ has the property that $d(\mu, \pi(\nu_1)) \leq C(D)$ (an analogous property holds for ν_2).

In order to show this, we will now analyse the K-large domains for the pair ν_1', ν_2', where $K > 3(D + B)$ (a L-large domain for a pair of markings ρ_1, ρ_2 is a subsurface $Y \subseteq S$ so that $d_{\mathcal{C}(Y)}(\rho_1, \rho_2) \geq L$). First, we claim that there are no common $(K/3)$-large domains for the pairs μ_1', ν_1' and μ_2', ν_2', where $\mu_i' \in [\mu_1, \mu_2]$ projects B-close to α_{ν_i}.

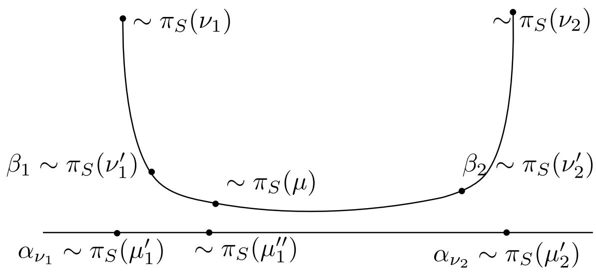


Figure 6.3

In fact, any $(K/3)$-large domain X_i for the pair μ_i', ν_i' is contained in some $S\backslash\gamma_i$, where γ_i is a simple closed curve appearing in a hierarchy path connecting $\pi_S(\mu_i')$ and ν_i'. This implies that $X_1 \neq X_2$ as $d_{C(S)}(\gamma_1, \gamma_2) \geq 3$ for $C(D)$ large enough.

We can now use the fact that there are no D-large domains for the pair μ_1', μ_2'. Suppose $d_{C(Y)}(\nu_1', \nu_2') \geq K$, for some subsurface $Y \subsetneq S$. Then (at least) one of the following must hold: $d_{C(Y)}(\nu_1', \mu_1') \geq K/3$, $d_{C(Y)}(\mu_1', \mu_2') \geq K/3$ or $d_{C(Y)}(\nu_1', \mu_1') \geq K/3$. However, the second inequality does not hold by hypothesis. Hence, any K-large domain for ν_1', ν_2', and so any $(K+B)$-large domain for any pair of points on $[\nu_1', \nu_2']$, is a $(K/3)$-large domain for μ_1', ν_1' or μ_2', ν_2' (but not both).

With a similar argument we get that for each $\nu \in [\nu_1', \nu_2']$ and each $\mu \in [\mu_1', \mu_2']$ all $(4K/3 + D + 2B)$-large domains $Y \subsetneq S$ for μ, ν are $(K/3)$-large domains for μ_1', ν_1' or μ_2', ν_2'. Choosing ν as above we have that $\pi_S(\nu)$ is far enough from β_1 and β_2 to guarantee that no $(4K/3 + D + 2B)$-large domain $Y \subsetneq S$ for ν, μ is a K-large domain for μ_i', ν_i', where $\mu \in [\mu_1', \mu_2']$ is such that $20\delta + 2B + 10 \leq d_{C(S)}(\mu, \mu_i') \leq 20\delta + 3B + 20$. By the distance formula $d(\nu, \mu_1')$ can be bounded in terms of δ and B, so we are done. $\square$

6.3.3 Graph manifolds

A graph manifold is a compact connected 3-manifold (possibly with boundary) which admits a decomposition into Seifert fibred surfaces, when cut along a collection of embedded tori and/or Klein bottles. In particular a graph manifold is a 3-manifold whose geometric decomposition admits no hyperbolic part.

Let M be a graph manifold. It is known [105] that its universal cover $\widetilde{M}$ is bi-Lipschitz equivalent to the universal cover $\widetilde{N}$ of a *flip* graph manifold N, that is to say a graph manifold with some special properties (among which a metric of nonpositive curvature). We will not need the exact definition of such manifolds, but we need to know that we can choose a bi-Lipschitz equivalence $\phi : \widetilde{M} \to \widetilde{N}$ that preserves the geometric components. In [152], a family of paths $\mathcal{PS}(N)$ in $\widetilde{N}$ has been defined, and those paths satisfy the following:

Lemma 6.3.5.

1. *All paths in $\mathcal{PS}(N)$ are bi-Lipschitz, with controlled constant;*

2. *any subpath of a path in $\mathcal{PS}(N)$ is again in $\mathcal{PS}(N)$;*

3. *if for $i = 1, 2$ α_i is a special path connecting some point in X_{w_i} to some point in $X_{w'_i}$ and the vertex v*

    - *lies on the geodesic connecting w_i, w'_i, and*

    - *$d(v, w_i), d(v, w'_i) \geq 2$,*

    then $\alpha_1 \cap X_v = \alpha_2 \cap X_v$.

Let $\mathcal{PS}(M) = \{g\phi^{-1}(\gamma) : g \in \pi_1(M), \gamma \in \mathcal{PS}(N)\}$.

Proposition 6.3.6. *$(\widetilde{M}, \mathcal{PS}(M))$ is a path system for $\pi_1(M)$. An element of $\pi_1(M)$ is contracting if and only if it acts hyperbolically on the Bass-Serre tree of M.*

Proof. The first part follows directly from Lemma 6.3.5$-(1)-(2)$, so we can focus on the second part.

Let us prove the "if" part. Let $g \in \pi_1(M)$ be an element acting hyperbolically on the Bass-Serre tree. Let $x_0 \in X_v$ for some vertex v of the Bass-Serre tree such that $\{g^i v\}$ is contained in a bi-infinite geodesic γ. Define $\pi : \widetilde{M} \to \{g^i x_0\}$ in the following way. For each $x \in \widetilde{M}$ define a vertex $v(x)$ in the Bass-Serre tree such that $x \in X_{v(x)}$ and let π' be the projection in the Bass-Serre tree on γ. Choose $i(x)$ so that $d(g^{i(x)}v, \pi'(v(x))) \leq d(v, gv)/2$. Finally, define $\pi(x)$ to be $g^{i(x)}x_0$.

Now, suppose that we have $x, y \in \widetilde{M}$ such that $d(\pi(x), \pi(y))$ is large enough. Then we can conclude that, say, $d(\pi'(v(x)), \pi'(v(y))) \geq 100$. We will now use Lemma 6.3.5 and the fact that ϕ preserves the geometric components to find a bound on the distance between any special path δ connecting x, y and $\pi(x), \pi(y)$. By hypothesis, ϕ induces a simplicial isomorphism from the Bass-Serre tree of M to that of N, which we will still denote by ϕ. Suppose that $\delta = h\phi^{-1}(\alpha)$. Then it is clear from Lemma 6.3.5 that δ shares a subpath with any special path $h\phi^{-1}(\beta)$ which connects some point in $X_{g^{i(x)}v}$ to some point in $X_{g^{i(y)}v}$. More precisely, those paths coincide in X_w whenever w is "well within" $[\pi'(v(x)), \pi'(v(y))]$. In particular, we can choose β so that the endpoints of $h^{-1}\phi(\beta)$ are $g^{i(x)}x_0, g^{i(y)}x_0$, so that any point in $h^{-1}\phi(\beta)$ is close to γ as γ is Morse [105]. It is now easy to see that points if $x', y' \in \delta \cap h^{-1}\phi(\beta)$ are so that $d(v(x'), g^{i(x)}v), d(v(y'), g^{i(y)}v) \leq 10$, then x', y' are within uniformly bounded distance from $\pi(x), \pi(y)$.

The "only if" part is easy. In fact, if $g \in \pi_1(M)$ is conjugate to some element in a vertex group, then it is clearly not even Morse in view of the fact that such groups are virtually products of a free group and $\mathbb{Z}$. □

6.3.4 Groups acting acylindrically on hyperbolic spaces and $Out(F_n)$

Recall that a group G is said to act *acylindrically* on the metric space X if for all d there exist R_d, N_d so that whenever $x, y \in X$ satisfy $d(x, y) \geq R_d$ we have

$$|\{g \in G : d(x, gx) \leq d, d(y, gy) \leq d\}| \leq N_d.$$

Graph manifold groups act acylindrically on their Bass-Serre tree. However, we are not able to prove the equivalent of Proposition 6.3.6 for groups acting acylindrically on trees or hyperbolic spaces, and indeed it might not be true. Nonetheless, we have the following.

Proposition 6.3.7. *Let G be a group acting acylindrically (by isometries) on the hyperbolic space X and let $\mathcal{PS}$ be the collection of all geodesics of X. Then $g \in G$ is weakly contracting for the weak path system $(X, \mathcal{PS})$ if and only if it acts hyperbolically on X.*

Proof. Given the following easy fact (Lemma 3.7.2), the proof just requires unwinding the definitions: if X is hyperbolic then all geodesics in X are contracting. □

Bestvina and Feighn proved that the complex of free factors for $Out(F_n)$ is hyperbolic and that an element acts hyperbolically if and only if it is fully irreducible, in which case it also satisfies the WPD property [19, Theorem 8.3]. In particular we have the following.

Proposition 6.3.8. *Let X be the complex of free factors of $Out(F_n)$ for some $n \geq 3$. Then $g \in Out(F_n)$ is weakly contracting for the weak path system $(X, \mathcal{PS})$ if and only it is fully irreducible.*

Remark 6.3.9. For our applications we could also use the complexes constructed in [21].

The present state of knowledge about the geometry of $Out(F_n)$ is not advanced enough to attempt imitating what we have done, say, for mapping class groups. It is not even known whether there is an algebraic characterization of Morse elements along the lines of the other cases, and hence the following is perhaps the most basic question arising at this point.

Question 6.3.10. Are all Morse elements of $Out(F_n)$ for $n \geq 3$ fully irreducible?

The converse is true by [3]. A standard way to show that an (infinite order) element is not Morse is to show that (the cyclic group generated by) it has infinite index in its commensurator. There might be infinite order non-fully irreducible elements of $Out(F_n)$ which have finite index in their commensurators. So, if the answer to the previous question is negative, one might ask the following.

Question 6.3.11. Is it true that all elements of $Out(F_n)$ that have finite index in their commensurators are Morse?

6.3.5 Groups acting properly on proper $CAT(0)$ spaces

Recall that an isometry of a $CAT(0)$ space is *of rank one* if it is hyperbolic and some (equivalently, every) axis of g does not bound a half-flat.

Remark 6.3.12. In the case when G is a right-angled Artin group and X its standard $CAT(0)$ cube complex, the elements of G that act as rank 1 isometries coincide with those not conjugated into a join subgroup, see [12].

Proposition 6.3.13. *Let G be a group acting properly by isometries on the proper $CAT(0)$ space X and let $\mathcal{PS}$ be the collection of all geodesics in X. Then $g \in G$ is contracting for the path system $(X, \mathcal{PS})$ if and only if it acts as a rank one isometry.*

Proof. It is clear that $(X, \mathcal{PS})$ is a path system on G, and the "only if" part follows from the fact that if g does not act as a rank one isometry then it is not Morse. Suppose that g acts as a rank one isometry. By [25, Theorem 5.4] the closest point projection on an axis l of g has the property that there exists B so that each ball disjoint from l projects onto a set of diameter at most B. Property 2) in Definition 6.2.2 then follows from Lemma 6.2.3. □

Remark 6.3.14. It is natural to ask when a $CAT(0)$ space admits a rank one isometry. This question is addressed by the Rank Rigidity Conjecture, formulated by Ballmann and Buyalo:

Conjecture 6.3.15. *[8] Let X be a locally compact geodesically complete $CAT(0)$ space and Γ be an infinite discrete group acting properly and cocompactly on X. If X is irreducible, then X is a higher rank symmetric space or a Euclidean building of dimension at least 2, or Γ contains a rank one isometry.*

The conjecture is known to hold for Hadamard manifolds, see e.g. [7], and $CAT(0)$ cube complexes [53].

6.4 Contracting vs hyperbolically embedded

Throughout this section we fix a weak path system $(X, \mathcal{PS})$ on the group G.

Suppose that A is an axis of a weakly contracting element of G. Given a constant b, for $x, y, z \in A$ we say that y is between x and z (with constant b) if any special path from x to z intersects $B_b(y)$. Similarly, for $h \in G$ and $hx, hy, hz \in hA$ we say that hy is between hx and hz if y is between x and z.

Lemma 6.4.1 (Projections are coarsely monotone)**.** *Let A be an axis of a weakly contracting element. Then there exist b, K so that the following hold for each $h \in G$. Suppose, for $i = 0, 1, 2$, that $y_i = \pi_A(x_i)$, for some $x_i \in hA$. Then if y_1 is between y_0 and y_2 with constant b and $d(y_0, y_1), d(y_1, y_2) \geq K$, then x_1 is between x_0 and x_2, again with constant b.*

Proof. Recall that special paths with endpoints on A stay uniformly close to A, see Lemma 6.2.9$-$(1). In particular, as A is quasi-isometric to $\mathbb{R}$, there exists b and K_0 so that if x, y, z are points at reciprocal distance at least K_0 then there exists one of them which is between the other two with constant b.

We can assume $K \geq K_0$ and apply this for $\{x, y, z\} = \{x_0, x_1, x_2\}$.

Suppose by contradiction that x_2 is between x_0 and x_1 (we can similarly handle the case when x_0 is assumed to be between x_1, x_2). Up to increasing K again, we have points $x, y \in hA$ lying, respectively, between x_0 and x_2 and between x_2 and x_1 such that $d(x, y_1), d(y, y_1)$ is bounded in terms of C. If K is large enough compared to a constant μ so that A and hence hA is (μ, μ)-quasi-isometric to $\mathbb{R}$ this gives a contradiction, see Figure 6.4. □

Let $(X, \mathcal{PS})$ be a weak path system on the group G, and let $x_0 \in X$.

Lemma 6.4.2. *Suppose that $g \in G$ is weakly contracting with axis A. Then there exists K such that for each $h \in G$ either $\pi_A(hA)$ has diameter bounded by K or it is K-dense in A. Moreover, there exists K' so that if $d(hx_0, A) \geq K'$ then the first case holds.*

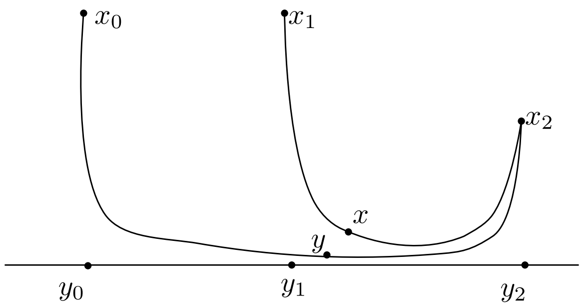


Figure 6.4

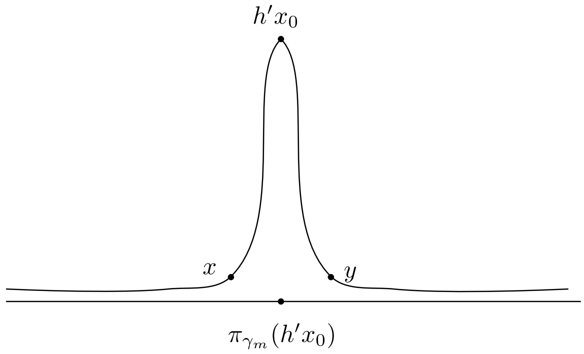


Figure 6.5

Proof. If A is contained in a neighbourhood of hA of finite radius, then it is easily seen that the second case holds. Otherwise, we can find $h'x_0 \in hA$ such that $h'x_0$ is as far as we wish from A. We claim that if $d(h'x_0, A)$ is large enough, then $\pi_A(h'x_0)$ cannot be between points p, q in $\pi_A(h'A)$ that are arbitrarily far from it.

In fact, by the previous lemma $h'x_0$ would be between points projecting to p, q, and hence one can find $x, y \in h'A$ with $h'x_0$ between x, y such that $d(x, \pi_A(h'x_0)) \leq C$ and $d(y, \pi_A(h'x_0)) \leq C$, where C is the projection constant. This is easily seen to contradict $h'A$ being a quasi-geodesic, see Figure 6.5.

Now, if $diam(\pi_A(hH_0))$ is large enough we can find (see Figure 6.6) a large family $\{N_i\}$ of integers such that

- $\pi_A(g^{N_i}h'A)$ has positive Hausdorff distance from $\pi_A(g^{N_j}h'A)$ whenever $i \neq j$. This gives in particular that $g^{N_i}h'$ and $g^{N_j}h'$ do not belong to the same left coset of $\langle g \rangle$ whenever $i \neq j$.

- $g^{N_i}h'A$ contains a point in some ball of fixed radius around $g^{m_0}x_0$, as well as a long subpath contained in a neighbourhood of A of radius depending on the projection constant and the quasi-geodesic constants of special paths only. This implies, by Lemma 6.2.7, that all left cosets $g^{N_i}h'\langle g \rangle$ contain an element from a certain fixed finite subset of G.

Therefore there is a bound on the cardinality of $\{N_i\}$, and hence a bound on $diam(\pi_A(hA))$. □

One can similarly prove the following.

Lemma 6.4.3. *Suppose that $H < G$ is not virtually cyclic and it contains the weakly contracting element g with axis A. Then for each K there exists a left coset $h\langle g \rangle \subseteq H$ such that $d(hA, A) \geq K$.*

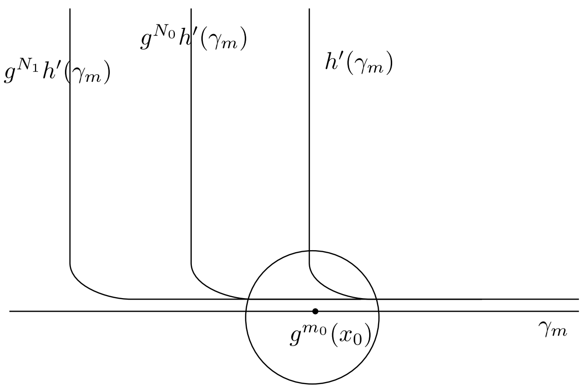


Figure 6.6

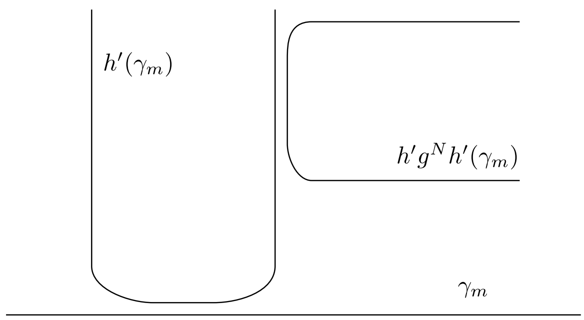


Figure 6.7

Proof. As H is not virtually cyclic, it is well-known that it is not quasi-isometric to $\mathbb{Z}$. In particular, we can find h' such that h' is as far as we wish from $\langle g \rangle$ *in a Cayley graph* of G. So, either $d(h'x_0, A)$ can be made arbitrarily large, and in this case we have $diam(\pi_A(h'A)) \leq K$ by the "moreover" part of Lemma 6.4.2, or we can use Lemma 6.2.7 to get that $\pi_A(h'A)$ is not K-dense in A, and hence once again $diam(\pi_A(h'A)) \leq K$ by Lemma 6.4.2. It is then easily seen that we can choose h of the form $h'g^N h'$ with N large enough so that $\pi_{h'A}(A)$ is far from $\pi_{h'A}(hA)$, see Figure 6.7. □

As a consequence of the previous lemmas we get:

Corollary 6.4.4. *Each weakly contracting element is contained in a virtually cyclic subgroup, denoted $E(g)$, such that there exists a uniform bound on $\pi_A(hE(g))$ for each $h \notin E(g)$.*

Proof. Just define $E(g)$ as the collection of all h such that $\pi_A(hA)$ has finite Hausdorff distance from A. This is clearly a subgroup containing g, and it is virtually cyclic by the previous lemma. The uniform bound is a consequence of Lemma 6.4.2. □

We point out some extra properties of $E(g)$.

Lemma 6.4.5. *There exists a positive integer n such that for each $h \in E(g)$ we either have $hg^n h^{-1} = g^n$ or $hg^n h^{-1} = g^{-n}$. In particular, for each integer k, $\langle g^{nk} \rangle$ is normal in $E(g)$ and $E(g)$ is the centralizer of g^{2n}.*

Proof. Choose representatives $h_1, \ldots, h_k$ of the left cosets of $\langle g \rangle$ in $E(g)$. Choose $n_i > 0$ so that $h_i g^{n_i} h_i^{-1} \in \langle g \rangle$. Such n_i exists because there are finitely many left cosets of $\langle g \rangle$, hence one can find $k_i > m_i$ such that $g^{k_i} h_i^{-1}$ and $g^{m_i} h_i^{-1}$ belong to the same left coset and set $n_i = k_i - m_i$. Choose $n = \prod n_i$. Then it is readily seen that

$hg^nh^{-1} \in \langle g\rangle$ for each $h \in E(g)$. In particular $hg^nh^{-1} = g^k$ for some k, and we would like to prove $k = \pm n$. Notice that there exists j such that $h^j \in \langle g\rangle$. Hence

$$g^{n^j} = h^j g^{n^j} h^{-j} = g^{k^j}.$$

In particular $n^j = k^j$, hence $k = \pm n$, as required. □

Theorem 6.4.6. *Let G be any group endowed with a fixed weak path system. For each weakly contracting element $g \in G$, $E(g)$ is hyperbolically embedded in G. Viceversa, every infinite order element contained in a virtually cyclic hyperbolically embedded subgroup is weakly contracting for an appropriate action.*

We will use work of Bestvina, Bromberg and Fujiwara [18], similarly to the proof of [58, Theorem 4.42]. We already described the setting elsewhere, we will now give the relevant definitions again for the convenience of the reader, and state the result we need. Let $\mathbf{Y}$ be a set and for each $Y \in \mathbf{Y}$ let $\mathcal{C}(Y)$ be a geodesic metric space. For each Y let $\pi_Y : \mathbf{Y}\backslash\{Y\} \to \mathcal{P}(\mathcal{C}(Y))$ be a function (where $\mathcal{P}(Y)$ is the collection of subsets of Y). Define

$$d^\pi_Y(X, Z) = diam\{\pi_Y(X) \cup \pi_Y(Z)\}.$$

Using the enumeration in [18], consider the following Axioms, for some $\xi \geq 0$:

(0) $diam(\pi_Y(X)) \leq \xi$,

(3) there exists ξ so that $\min\{d^\pi_Y(X, Z), d^\pi_Z(X, Y)\} \leq \xi$,

(4) there exists ξ so that $\{Y : d^\pi_Y(X, Z) \geq \xi\}$ is a finite set for each $X, Z \in \mathbf{Y}$.

For suitably chosen constants L, K, let $\mathcal{C}(\mathbf{Y})$ be the path metric space consisting of the union of all $\mathcal{C}(Y)$ and edges connecting all points in $\pi_X(Z)$ to all points in $\pi_Z(X)$ whenever X, Z are connected by an edge in a certain complex $\mathcal{P}_K(\mathbf{Y})$ whose definition we do not need.

We are ready to state the (special case of the) result we will use.

Theorem 6.4.7. *[18, Theorem 3.10, Lemma 3.1, Corollary 3.12] If $\mathbf{Y}$ and d^π satisfy axioms $(0), (3)$ and (4) and each $\mathcal{C}(Y)$ is (λ, μ)-quasi-isometric to $\mathbb{R}$ for some λ, μ not depending on Y, then $\mathcal{C}(\mathbf{Y})$ is a quasi-tree (in particular, it is hyperbolic).*

Moreover, each $\mathcal{C}(Y)$ is isometrically embedded in $\mathcal{C}(\mathbf{Y})$ and for each K there exists R so that

$$diam(N_K(\mathcal{C}(X)) \cap N_K(\mathcal{C}(Y))) \leq R$$

whenever $X \neq Y$ are elements of $\mathbf{Y}$.

We will use the following characterization of being hyperbolically embedded.

Theorem 6.4.8. *[58, Theorem 4.42],[154] Let G be group and $H < G$ a subgroup. Then H is is hyperbolically embedded in G if and only if it acts coboundedly on a space X which is hyperbolic relative to the orbits of the cosets of H (with respect to some basepoint) and the action restricted to H is proper.*

Proof of Theorem 6.4.6. For technical reasons, in this proof we will allow projections to take values in bounded subsets of the target space. More precisely, we consider a slightly generalized definition of the set A being $\mathcal{PS}$-contracting with constant C where the projection map π_A is allowed to take value in subsets of A of diameter bounded by C, while properties 1) and 2) in Definition 6.2.2 are left unchanged. If we have a map π_A with these properties we can define a projection π'_A in the sense of Definition 6.2.2 just by choosing some $\pi'_A(x) \in \pi_A(x)$ for each $x \in X$. In particular our results, including Lemma 6.2.5 and Corollary 6.4.4, hold for this more general notion (possibly with different constants).

Let $\mathbf{Y}$ be the collection of all left cosets of $E(g)$ in G and for each $Y \in \mathbf{Y}$ let $\mathcal{C}(Y)$ be a copy of $E(g)$ (more precisely, a copy of the Cayley graph of $E(g)$ with respect to a given finite set of generators), which we regard as identified with $E(g)x_0$, for some given x_0. Each orbit $hE(g)x_0$ is a contracting set, and the constant can be chosen uniformly. Actually, we have more. In fact, there exists a collection of equivariant projections on the orbits: choose a projection π'_e on $E(g)x_0$, define $\pi_e(x) = \bigcup_{h\in E(g)} h\pi'_e(h^{-1}x)$ and define a projection on $hE(g)x_0$ by $\pi_h(x) = h\pi_e(h^{-1}x)$. In order to show that π_h is actually a projection we just need to prove that we can uniformly bound $diam(\pi_h(x))$ for each x. This follows directly from the easily checked fact that there exist constants D_1, D_2 so that for each special path γ from x to $\pi_h(x)$ we have $diam(\gamma \cap N_{D_1}(E(g)x_0)) \leq D_2$. We can now define

$$\pi_{hE(g)}(h'E(g)) = h^{-1}\pi_h(h'E(g)x_0).$$

Axiom (0) follows from Corollary 6.4.4, while Lemma 6.2.5 implies Axiom (3). Axiom (4) is easy: if ξ is large enough then

$$|\{hE(g) : d_{hE(g)}(h_1E(g), h_2E(g)) \geq \xi\}| \leq d(h_1E(g)x_0, h_2E(g)x_0)$$

because all special paths from $h_1E(g)x_0$ to $h_2E(g)x_0$ have long disjoint subpaths each contained in some $hE(g)x_0$ for $hE(g)$ satisfying $d_{hE(g)}(h_1E(g), h_2E(g)) \geq \xi$. As all axioms are satisfied, Theorem 6.4.7 applies. Notice that all constructions are equivariant (including that of $\mathcal{P}_K(\mathbf{Y})$), so that there is a natural action of G on

$\mathcal{C}(\mathbf{Y})$. We wish to show that this action satisfies all properties of Theorem 6.4.8, with $H = E(g)$ and $X = \mathcal{C}(\mathbf{Y})$. In fact, in view of Theorem 6.4.7, $\mathcal{C}(\mathbf{Y})$ is hyperbolic, so that in order to check relative hyperbolicity we have to check that the orbits of the cosets of $E(g)$ are quasi-convex and geometrically separated by Lemma 3.7.1. There exists an orbit of $E(g)$ which is an isometrically embedded copy of (the vertices in a Cayley graph of) $E(g)$ and $E(g)$ acts on it in the natural way, so that the orbits of $E(g)$ are quasi-convex and $E(g)$ acts properly. Finally, the orbits of the cosets of $E(g)$ are geometrically separated by the "moreover" part of Theorem 6.4.7.

The "vice versa" is easily checked directly form the definitions, using X as in the definition of hyperbolically embedded subgroups and defining $\mathcal{PS}$ to be the collection of all geodesics in X (once again, one has to take into account the characterization of the peripheral structures of a relatively hyperbolic space). $\square$

6.5 Construction of contracting elements

We start with a lemma about concatenations of special paths. More specifically, we consider a concatenation of special paths so that every other path is contracting (more precisely, contained in a contracting set) and show that the resulting concatenation is contracting under suitable conditions. The most important such condition is that the projections of the previous and next contracting paths are close to the corresponding endpoints compared to the length of the contracting path under consideration, see Figure 6.8.

Lemma 6.5.1. *Let X be a metric space endowed with a path system. For each C there exist D with the following property. Suppose that the sequence of points $\{x_i, y_i\}$ satisfies the following properties for each i.*

(a) There exists a C-contracting set A_i containing x_i, y_i,

(b) $diam(\pi_{A_i}(A_{i-1})), diam(\pi_{A_i}(A_{i+1})) \le C$.

(c) $d(x_i, y_i) \ge d(x_i, \pi_{A_i}(A_{i-1})) + d(y_i, \pi_{A_i}(A_{i+1})) + D$.

(d) $\sup\{d(x_i, y_i), d(y_i, x_{i+1})\} < +\infty$.

Then $A : i \mapsto x_i$ (for $i \in \mathbb{Z}$) is a contracting bi-infinite quasi-geodesic. Moreover, any special path from from x_i to x_j has a subpath with endpoints C-close to A_k and at least $D/2$-far away from each other whenever $i < k < j$.

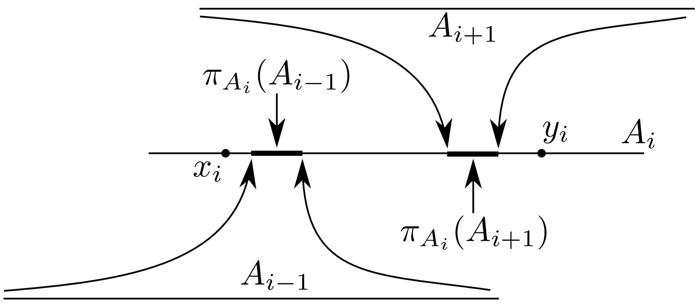


Figure 6.8: The hypotheses of Lemma 6.5.1

At the cost of making the last part of the proof more involved, it is possible to generalise the lemma to finite or one-sided infinite concatenations and substituting (d) with more general conditions (as well as obtaining a more explicit estimate of the contracting constants). We will not need this.

Proof. Denote $\pi_i = \pi_{A_i}$. We now show that everything coming after A_i in the concatenation projects close to y_i, and similarly in the other direction. Formally, setting $d_i = d(x_i, \pi_i(A_{i-1})), e_i = d(y_i, \pi_i(A_{i+1}))$, we would like to show that, for D large enough depending on C, we have

$$\pi_i(A_j) \subseteq B(y_i, e_i + C' + C), \pi_j(A_i) \subseteq B(x_j, d_j + C' + C) \tag{6.1}$$

for every $i < j$ and C' as in Lemma 6.2.5. We will show this inductively on $j - i$, the case $j = i + 1$ being an easy consequence of (b). For notational convenience, we will show the first containment, the other one can be shown using a symmetric argument. Suppose that (1) holds whenever $j \le i + n$ and set $j = i + n + 1$ instead. Using the inductive hypothesis we have

$$d(\pi_{i+1}(A_i), \pi_{i+1}(A_j)) \ge d(x_{i+1}, y_{i+1}) - d_{i+1} - e_{i+1} - C' - 2C \ge D - C' - 2C.$$

For D large enough we can apply Lemma 6.2.5 and get for each $x \in A_j$

$$d(\pi_i(A_{i+1}), \pi_i(x)) \le C',$$

and hence

$$d(y_i, \pi_i(x)) \le d(y_i, \pi_i(A_{i+1})) + diam(\pi_i(A_{i+1})) + C' \le e_i + C' + C,$$

as required.

The fact that A is a quasi-geodesic now readily follows from the fact that any special path from x_i to x_j has to pass $(d_k + C' + 2C)$-close to x_k and $(e_k + C' + 2C)$-close to y_k for each $i \le k \le j$, if we assume that D satisfies $D - 2(C' + 2C) \ge C$.

We are only left to show that A is contracting. The idea for defining a contraction is the following. Given x, there will be a critical "time" $i(x)$ so that the projection of x onto A_i switches from being close to y_i to being close to x_i. We define $\pi(x) = x_{i(x)}$. We then notice that, given x, y, the projections of x, y are far from each other at times intermediate between the critical ones for x and y, and it is not hard to conclude that the projection property holds using this fact.

Let us make this precise. For $x \in X$, define

$$i(x) = \min\{i \in I : d(\pi_i(x), y_i) > e_i + 2C' + C\},$$

and set $\pi(x) = x_{i(x)}$. Notice that the set on the right-hand side is non-empty because a special path from $x \in X$ to A_0 cannot be C-close to A_i for i large enough. In particular, $\pi_i(A_0)$ coarsely coincides with $\pi_i(x)$ for such i, and $\pi_i(A_0)$ is far from y_i. With a similar argument one sees that such set has a lower bound.

The fact that π is coarsely the identity map on $\{x_i\}$ easily follows from (1), which implies that $i(x_i) = i$. The fact that it is coarsely the identity on A follows from this and the contraction property shown below.

Consider $x, y \in X$ so that $d(\pi(x), \pi(y))$ is sufficiently large. We have to show that any special path from x to y passes close to $\pi(x), \pi(y)$. Up to exchanging x and y, we have $i(y) \geq i(x) + 2$. For $i(x) < i < i(y)$, in view of (1) we have

$$d(\pi_{i(x)}(A_i), \pi_{i(x)}(x)) \geq d(y_i, \pi_{i(x)}(x)) - (e_i + C' + C) > C',$$

so that keeping (c) and Lemma 6.2.5 into account we have

$$d(x_i, \pi_i(x)) \leq d(\pi_i(A_{i(x)}), \pi_i(x)) + (d_i + C' + C) \leq d_i + 2C' + C.$$

On the other hand,

$$d(\pi_i(y), y_i) \leq e_i + 2C' + C$$

as $i < i(y)$. For D sufficiently large this implies that any given special path γ from x to y passes $(d_i + C' + C)$-close to $\pi_i(x)$ and $(d_i + C' + C)$-close to $\pi_i(y)$. Recall that i was any integer strictly between $i(x)$ and $i(y)$. Let us now choose $i = i(x) + 1$. We have

$$d(\gamma, \pi(x)) \leq d(\gamma, \pi_i(x)) + d(\pi_i(x), x_i) + d(x_i, y_{i(x)}) + d(y_{i(x)}, x_{i(x)}).$$

We saw already that the first and second term on the right hand side can be uniformly bounded. The remaining two terms are uniformly bounded in view of (d).

Finally, a similar argument with $i = i(y) - 1$ provides a bound on $d(\gamma, \pi(y))$. $\square$

The following result is a consequence of Lemma 6.5.1 and gives us a way of constructing new contracting elements from a given one.

Lemma 6.5.2. *Let $(X, \mathcal{PS})$ be a weak path system on the group G. For each C there exists D with the following property. Let A be a C-contracting axis for the weakly contracting element $g \in G$, containing $p \in A$. Suppose that $h \in G \backslash E(g)$ is so that*

$$d(p, gp) \geq d(\pi_A(hp), p) + d(p, \pi_A(h^{-1}p)) + D.$$

Then hg, and hence any element conjugate to it, is weakly contracting.

It will be convenient to have a slight variation of this lemma, that can be proven in the same way (we leave the details to the reader).

Lemma 6.5.3. *Let $(X, \mathcal{PS})$ be a weak path system on the group G. For each C there exists D with the following property. Let A be a C-contracting axis for the weakly contracting element $g \in G$, containing $p \in A$. Suppose that $h_1, h_2 \in G \backslash E(g)$ and $n_1, b_2 \in \mathbb{Z}$ are so that*

$$d(p, g^{n_1}p) \geq d(\pi_A(h_1^{-1}p), p) + d(p, \pi_A(h_2p)) + D,$$

$$d(p, g^{n_2}p) \geq d(\pi_A(h_1p), p) + d(p, \pi_A(h_2^{-1}p)) + D,$$

Then $h_1g^{n_1}h_2g^{n_2}$, and hence any element conjugate to it, is weakly contracting.

Proof of Lemma 6.5.2. Let π_A be a C-contraction on A. We apply Lemma 6.5.1 with $x_i = (hg)^i hp$, $y_i = (hg)^{i+1}p$ and $A_i = (hg)^i hA$. We claim that all conditions in Lemma 6.5.1 are satisfied, up to suitably increasing the constants. Denote $h_i = (hg)^i h$ for convenience. Set $K = \sup\{b \notin E(g) : diam(\pi_A(bA))\} < +\infty$. Condition (d) clearly holds as $d(x_i, y_i) = d(p, gp)$ and $d(y_i, x_{i+1}) = d(p, hp)$. Condition (a) is clear setting $\pi_{A_i}(x) = h_i\pi_A(h_i^{-1}x)$. Condition (b) holds for with constant K instead of C. In order to show (c), first notice that there is a uniform bound on $d(\pi_A(gx), g\pi_A(x))$ for $x \in X$, as $\pi_A(y)$ coarsely coincides with the first point on any special path from y to A in a suitable neighbourhood of A. We can then make the computation:

$$\pi_{A_i}(A_{i+1}) = h_i\pi_A(ghA) \subseteq h_iB_K(\pi_A(ghp)) \subseteq (hg)^{i+1}B_{K'}(\pi_A(hp)),$$

and similarly we get $\pi_{A_i}(A_{i-1}) \subseteq h_iB_{K'}(\pi_A(h^{-1}p))$. So, using that G acts by isometries:

$$d(x_i, \pi_{A_i}(A_{i-1})) + d(y_i, \pi_{A_i}(A_{i+1})) \leq$$
$$d(p, \pi_A(h^{-1}p)) + d(p, \pi_A(hp)) + 2K' \leq d(p, gp) - D + 2K' \leq d(x_i, y_i) - D + 2K'.$$

Now, we are only left to check the WPD property. For a suitable B, there are finitely many representatives $k_1, \dots, k_n$ of left cosets of $E(g)$ so that a special path α from x_0 to some $(hg)^i x_0$ intersects $N_C(xA)$ in a set of diameter at least B (henceforth, fellow-travels for a long time) only if $x = (hg)^j k_i$ for some j, i.

To see this, we can first reduce to showing the analogous property for special paths α connecting a point C-close to x_0 to a point C-close to hgx_0 using the "moreover" part of Lemma 6.5.1 and using the action of the appropriate $(hg)^j x_0$. Such special path will fellow-travel a translate of A only if the projections of x_0, hgx_0 on such translate are far enough, and there are only finitely many such translates xA as any special path connecting x_0 to hgx_0 contains long disjoint subpaths each fellow-travelling some xA.

Suppose that $k \in G$ is so that $d(kx_0, x_0), d(k(hg)^N x_0, (hg)^N x_0) \leq K$, for N large compared to K, and let γ be a special path from x_0 to $(hg)^N x_0$. Then both $k\gamma$ and γ fellow-travel some A_j, and $|j|$ bounded in terms of K if we pick the smallest such j. This is because the projection of the endpoints of $k\gamma$ on A_j are close to those of the endpoints of γ whenever A_j has distance at least $K + C$ from the endpoints of γ.

Hence, k has the form $(hg)^j E(g)((hg)^{j'} k_i)^{-1}$, because kA' only fellow-travels for a long time translates of A of the form $k(hg)^{j'} k_i A$. Also, $|j'|$ can again be bounded in terms of K.

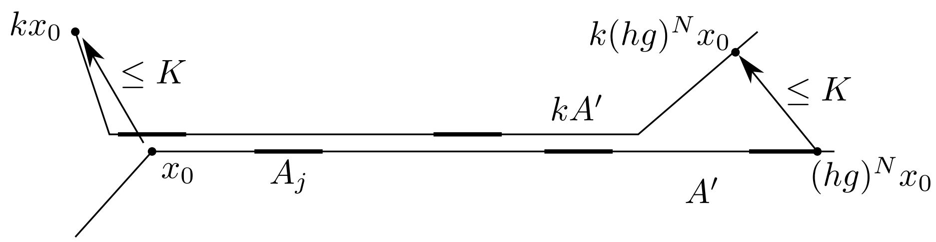


Figure 6.9: A', kA' both intersect far away K-balls and fellow-travel in between.

Given any a, b, there are only finitely elements g' of $E(g)$ so that $d(ag'b, x_0) \leq K$, so that we showed that k lies in one of finitely many double cosets of $E(g)$ and in each of them only finitely many elements satisfy a property satisfied by k. Hence, there are finitely many choices of k and we are done.

6.6 Random walks

Recall that our aim is to prove the following.

Theorem 6.6.1. *Let G be a non-elementary group supporting the path system (resp. weak path system) $(X, \mathcal{PS})$ and containing a contracting (resp. weakly contracting) element. Then the probability that a simple random walk $\{X_n\}$ supported on G gives rise to a non-contracting (resp. non-weakly-contracting) element decays exponentially in the length of the random walk.*

The reader may wish, on first reading, to take the short-cut suggested in Subsection 6.6.8, where we indicate how to prove the slightly weaker result that the probability of ending up in a non-contracting element goes to 0 without the extra information that it does so exponentially fast.

6.6.1 Notation

We fix the notation of the theorem throughout this section. Additionally, we let $W_n(S)$ be the set of words of length n in the generating system of G that defines the random walk $\{X_n\}$. For w a word in S, we denote by $g(w)$ the corresponding element of g. Finally, p is a fixed basepoint of X. To simplify the notation, for any contracting set A,

$$d_A(\cdot,\cdot) = d(\pi_A(\cdot), \pi_A(\cdot)).$$

6.6.2 Many contracting subwords

In order to keep the proof as elementary and self-contained as possible, rather than using more refined estimates on sums of independent random variables we will only use the following well-known inequality: for each positive integers $n \geq m$, we have

$$\binom{n}{m} \leq \left(\frac{ne}{m}\right)^m. \tag{6.2}$$

Let $w_0 \in W_m(S)$ be a word of length m (we will be interested in a word representing a weakly contracting element). For each k denote $W^k_{w_0}$ the set of words obtained concatenating k words of the form w_0 or w_0^{-1}. For a word w, denote by $w[j,k]$ the subword starting with the j-th letter and ending with the $(k-1)$-th letter. Let $l_k = km$ and denote $\mathcal{I}^k_{w_0}(w) = \{i \in 2\mathbb{N} : w[il_k, (i+1)l_k] \in W^k_{w_0}\}$. In words, we split w into subwords of length l_k, keep the even ones and count how many of those are concatenations of w_0 and w_0^{-1}.

The following lemma tells us that we expect the size of $\mathcal{I}^k_{w_0}(w)$ to be linear in the length n.

Lemma 6.6.2. *For each $w_0 \in W_m(S)$ and $k \geq 1$ there exists $C_0 \geq 1$ so that*

$$\mathbb{P}[\#\mathcal{I}^k_{w_0}(X_n) \leq n/C_0] \leq C_0 e^{-n/C_0}$$

for each $n \geq 1$.

Proof. For notational convenience, we drop the "w_0" subscripts. Let K be a large enough constant to be determined by the following argument. Let w be a word of length n and set $n_0 = \lfloor n/(2l_k) \rfloor$. We can assume that n is large enough that $\lfloor n/K \rfloor \geq n_0/(2K)$.

If $\#\mathcal{I}^k(w) \leq n_0/K$ then $\mathcal{I}^k(w)$ is contained in a set $I \subseteq \{0, 2, \dots, 2n_0\}$ of cardinality $\lfloor n_0/K \rfloor$, and for each $i \in \{0, \dots, 2n_0\} \backslash I$ we have that $w[il_k, (i+1)l_k]$ (is defined and) does not belong to W^k. The probability of a word of length l_k not belonging to W^k is some $\rho < 1$ (depending on k and the length of w_0). We now use inequality 6.2. Summing over all possible sets I we can bound the probability in the statement by

$$\binom{n_0}{\lfloor n_0/K \rfloor} \rho^{n_0 - \lfloor n_0/K \rfloor} \leq (2Ke)^{n_0/K} \rho^{(1-1/K)n_0} = \left((2Ke)^{1/K} \rho^{1-1/K}\right)^{n_0}.$$

If K is large enough, depending on ρ only, we have $(2Ke)^{1/K} \rho^{1-1/K} < 1$ (because the first term tends to 1 and the second one tends to ρ for $K \to \infty$). This gives us the exponential decay we were looking for. □

6.6.3 Avoiding $E(g)$

Lemma 6.6.3. *For any weakly contracting element g of G there exists $C_1 \geq 1$ so that*

$$\mathbb{P}[X_n \in E(g)] \leq C_1 e^{-n/C_1}.$$

Proof. As we point out in the Appendix, since H is non-elementary, it must contain a free group on two generators and so it is not amenable. Hence, by a result of Kesten, see e.g. [163, Corollary 12.12], there exists $K \geq 1$ so that for each $g \in G$ and $n \geq 1$ we have

$$\mathbb{P}[X_n = g] \leq K e^{-n/K}.$$

Also, up to increasing K, in the ball of radius n *in the Cayley graph* of G around 1 there are at most Kn elements of $E(g)$, because the inclusion of $E(g)$ in G is a quasi-isometric embedding (if it was not, the orbit of g could not be a quasi-geodesic in X). Hence,

$$\mathbb{P}[X_n \in E(g)] \leq (Kn)(K e^{-n/K}),$$

which decays exponentially. □

6.6.4 One small projection

In this section we show that we expect the projection of a random point on an axis of a fixed weakly contracting element to be close to p, the basepoint of X. This is the key fact we need to use Lemma 6.5.2 or 6.5.3, our methods for constructing weakly contracting elements. We use the standard notation $\mathbb{P}[\cdot|\cdot]$ for conditional probabilities.

Lemma 6.6.4. *Let w_0 be a word of even length representing a weakly contracting element $g \in G$, with axis A. For every k large enough depending on w_0 the following holds. There exists $C_2 \geq 1$ so that for each $j \geq 1$, $t \geq 0$ and $I \subseteq 2\mathbb{N}$ we have*

$$\mathbb{P}\left[d(p, \pi_A(X_n p)) \geq t | \mathcal{I}^k_{w_0^j}(X_n) = I\right] \leq C_2 e^{-t/C_2}.$$

The reason why we require w_0 to be even is just to make the existence of certain words slightly easier to prove. The reason why we consider a fixed $\mathcal{I}^k_{w_0^j}$ is merely technical: The final computation in the proof of the main theorem requires to consider a conditional probability, and this propagates backwards up to here.

We actually do not need the exponential decay, just a convergence to 0, but this is what the proof gives anyway.

Proof. We want to find $D > 0$ with the following property. Let w be a word with $\mathcal{I}^k_{w_0^j}(w) = I$ and $d(p, \pi_A(g(w)p)) \geq (i+1)D$ for some integer $i \geq 2$. We want to associate to w a new word w', again with $\mathcal{I}^k_{w_0^j}(w') = I$ and of the same length as w, so that $d(p, \pi_A(g(w')p)) \in [iD, (i+1)D)$. Also, we will make sure that at most N words w are mapped to the same w', where N does not depend on j. This is enough for our purposes. In fact, once we manage to do so we have, setting $C_n =$ "$\mathcal{I}^k_{w_0^j}(X_n) = I$",

$$\mathbb{P}\left[d(p, \pi_A(X_n p)) \geq (i+1)D \mid C_n\right] \leq$$

$$N\left(\mathbb{P}\left[d(p, \pi_A(X_n p)) \geq iD \mid C_n\right] - \mathbb{P}\left[d(p, \pi_A(X_n p)) \geq (i+1)D \mid C_n\right]\right),$$

and hence

$$\mathbb{P}\left[d(p, \pi_A(X_n p)) \geq (i+1)D \mid C_n\right] \leq (1+1/N)^{-1}\mathbb{P}\left[d(p, \pi_A(X_n p)) \geq iD \mid C_n\right].$$

A straightforward inductive argument completes the proof.

We are left to show how to construct w' with the required properties. Fix some word s so that $g(s) \notin E(g)$ and let C' be as in Lemma6.2.5. For reasons that will be clearer later, we need k large enough with the property that there exist words v_1, v_2, v_3, v_4 of length l_k (recall that $l_k = km$ for m the length of w_0) so that

1. $g(v_1), g(v_2) \in h_1 E(g) h_1^{-1}$, $g(v_3), g(v_4) \in h_2 E(g) h_2^{-1}$,

2. $h_1 E(g) \neq h_2 E(g)$,

3. $d(g(v_2)g(v_1)^{-1}x, x)$, $d(g(v_4)g(v_3)^{-1}y, y)$ are larger than a constant determined by the argument below for any $x \in h_1 A, y \in h_2 A$.

It is easy to see that such words exist. For example, one can start choosing any h_1, h_2 as required (recall that G is non-elementary so that $[G : E(g)] = \infty$) and words u_1, u_2 representing them. Then one finds words t_1, t_2 both of even length so that $g(t_i) \in E(g)$ and with long translation distance on A. One can then set $v_1' = u_1 t_1 u_1^{-1}$, etc. and prolong such words to get the v_i's by adding final subwords (of even length) that represent the identity, making sure that the prolonged words have the same length and that such length is a multiple of the length of w_0.

We will subdivide w into three subwords $w = w_1 w_2 w_3$, and substitute the middle subword with one of the v_i's. The length of w_2 will be l_k. Also, we let w_1 be the shortest initial subword of w whose length is an *odd* multiple of l_k and so that $d(p, \pi_A(w)) \geq (i + 0.5)D$. First of all, it is now readily seen that the property that $w \mapsto w'$ is bounded-to-1 is satisfied. It is also readily seen that $\mathcal{I}^k_{w_0^j}(w_1 v_i w_3) = I$ for each i (recall that $\mathcal{I}^k_{w_0^j}(\cdot)$ is a set of even numbers).

We now only have to gain control on the projection on A. In view of (2) at least one of $g(w_1)h_1, g(w_1)h_2$ is not in $E(g)$, let us assume the former. Translating by $g(w_1)^{-1}$ we get

$$d_{g(w_1)h_1 A}(g(w_1 v_1 w_3)p, g(w_1 v_2 w_3)p) = d_{h_1 A}(g(v_1 w_3)p, g(v_2 w_3)p),$$

and we would like to show that this quantity is large. Any special path γ from $g(v_1 w_3)p$ to any point in $h_1 A$ enters a suitable neighbourhood of $h_1 A$ close to the projection point. The special path $g(v_2)g(v_1)^{-1}\gamma$ connects $g(v_2 w_3)p$ to some point in $h_1 A$ because $g(v_1), g(v_2)$ stabilize $h_1 A$ in view of (1). Also, (3) implies that the entrance points in the neighbourhood of $h_1 A$ are far, and hence so are the projection points of $g(v_1 w_3)p$, $g(v_2 w_3)p$.

The argument above shows that at most one of $g(w_1 v_1 w_3)p, g(w_1 v_2 w_3)p$ can project closer than C' to $\pi_{g(w_1)h_1 A}(A)$, where C' is as in Lemma 6.2.5. In particular, if this holds for, say, $g(w_1 v_1 w_3)p$, we get a uniform bound on $d_A(g(w_1 v_1 w_3)p, (g(w_1)h_1 p))$ and hence on $d_A(g(w_1 v_1 w_3)p, g(w_1)p)$. Therefore, for D large enough the inequality $d_A(g(w_1 v_1 w_3)p, g(w_1)p) < D/2$ holds, and thus $d(\pi_A(g(w_1 v_1 w_3)p), p) \in [iD, (i+1)D)$, as required. $\square$

6.6.5 Many small projections

Let $\mathcal{A} = (A, w_0)$ be a pair where A is the axis of the weakly contracting element $g(w_0)$ represented by the word $w_0 \in W_m(S)$.

Standing assumptions: we assume that the length of w_0 is even. Also, we fix an odd $k \geq 10^9$ satisfying Lemma 6.6.4 and so that $\mathbb{P}[X_n \in E(g)] \leq 10^{-3}$ for each $n \geq k$ (see Lemma 6.6.3).

We denote $\mathcal{I}_w = \mathcal{I}_w^k$ for any word w. The reason why we fix an odd k is that any subword counted by $\mathcal{I}$ corresponds to a non-trivial power of $g(w_0^j)$, a fact that will be used later. Slightly extending the notation we already used, for $i < j \leq n$ and a word of length n we denote by $w[j, i]$ the concatenation of $w[j, n]$ and $w[n, i]$ (w is hence considered as a cyclic word). Given a pair as above, a word w and an integer $\hat{i}$ so that $\hat{i}l$ is not larger than the length of w, denote

$$\mathcal{J}^+_{\mathcal{A},L}(w, \hat{i}) = \left\{ i \neq \hat{i} : d\left(p, \pi_A\Big(g(w[(i+1)l, \hat{i}l])p\Big)\right) \leq L \right\} \cap \mathcal{I}_{w_0}(w),$$

for $l = km$.

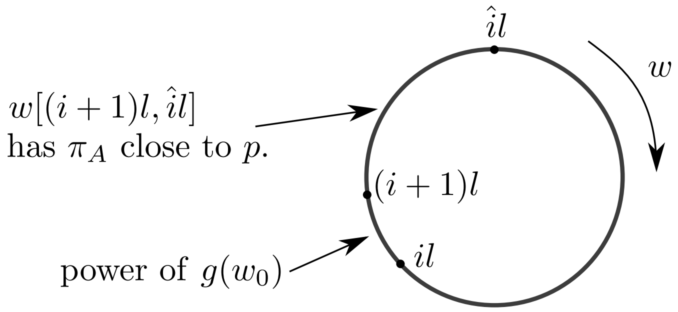


Figure 6.10: i is in $\mathcal{J}^+_{\mathcal{A},L}(w, \hat{i})$ if...

In words, we replace w by a cyclic permutation and we record the indices so that the projection of a final subword of w following one of the occurrences of powers of $g(w_0)$ counted by $\mathcal{I}_{w_0}$ has distance at most L from the basepoint p. Similarly, reading w backwards we set

$$\mathcal{J}^-_{\mathcal{A},L}(w, \hat{i}) = \left\{ i \neq \hat{i} : d\left(p, \pi_A\Big(g(w[(\hat{i}+1)l, il])^{-1}p\Big) \quad\right) \leq L \right\} \cap \mathcal{I}_{w_0}(w).$$

Suppose that some j is fixed and set $\mathcal{A}_j = (A, w_0^j)$. Then, given a word w, we say that $(\hat{i}_1, \hat{i}_2) \in \left(\mathcal{I}^k_{w_0^j}(w)\right)^2$ is a *good* L-pair if $\hat{i}_1 \in \mathcal{J}^+_{\mathcal{A}_j,L}(w, \hat{i}_2) \cap \mathcal{J}^-_{\mathcal{A}_j,L}(w, \hat{i}_2)$ and vice versa, i.e. if all projections suggested by the following picture are "smaller" than L.

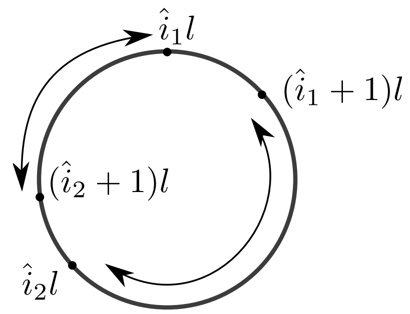


Figure 6.11: A good pair. Arrows correspond to controlled projections

The following proposition tells us that we expect to find a good pair of well-separated indices, once we replace w_0 by some large power to increase the translation distance on A.

Proposition 6.6.5. *There exists $L \geq 0$ with the following property. For each $\mathcal{A}$ as above and $D \geq 0$ there exists $j \geq 1$ so that $\mathcal{A}_j = (A, w_0^j)$ satisfies the following property for some $C_3 \geq 1$. For each $n \geq 1$, $I \subseteq \{0, 2, \ldots, 2\lfloor n/(2lj)\rfloor\}$ of cardinality at least $100C_3$ we have*

$$\mathbb{P}\left[\forall \hat{i}_1, \hat{i}_2 \in I,\ |\hat{i}_1 - \hat{i}_2| \geq \frac{\#I}{C_3} \Rightarrow (\hat{i}_1, \hat{i}_2) \text{ is not } L\text{-good} \mid \mathcal{I}_{w_0^j}(X_n) = I\right] \leq C_3 e^{-\#I/C_3}.$$

Moreover, $d(p, g^j p) > 2L + D$.

The proof is (essentially) an induction on $|I|$, the base case being Lemma 6.6.4.

Let us now choose j and L. Let C be a contraction constant for A. We fix $L > C' + C$ so that $\mathbb{P}[d(p, \pi_A(X_n p)) \geq L - C' - C|\mathcal{I}_{w_0^j}(X_n) = J] \leq 10^{-3}$ for any $J \subseteq 2\mathbb{N}$ and C' as in Lemma 6.2.5 (Lemma 6.6.4 guarantees that L exists). Also, we pick j so that $d(p, g(w_0)^j p) > 2L + D$. We now drop all subscripts in $\mathcal{I}, \mathcal{J}^{\pm}$ if they are the ones that can be inferred from the choices we just made.

Proposition 6.6.5 will follow easily from the following lemma.

Lemma 6.6.6. *There exists K so that*

$$\mathbb{P}\left[\exists \hat{i} \in I : \#\mathcal{J}^+(X_n, \hat{i}) \leq \frac{4}{5}\#I \mid \mathcal{I}(X_n) = I\right] \leq K e^{-\#I/K},$$

and the same holds for $\mathcal{J}^-$.

Proof. We prove the lemma for $\mathcal{J}^+$, the proof for $\mathcal{J}^-$ is symmetric.

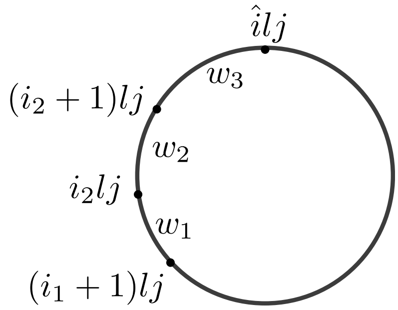
We can fix $\hat{i} \in I$ and just show the same conclusion with "$\exists \hat{i} \in I$" removed, the lemma will follow upon taking a sum over all $\hat{i} \in I$. For distinct $i_1, i_2 \in I$ we write

$i_1 < i_2$ if either $i_1 < i_2 \leq \hat{i}$, $\hat{i} < i_1 < i_2$ or $i_2 \leq \hat{i} < i_1$ (i.e., we read w in cyclic order starting after $\hat{i}lj$ and determine the order in this way).

Claim: There exists $\rho < 10^{-2}$ so that for each $i_1 < \cdots < i_l$ contained in I we have

$$\mathbb{P}[i_1 \notin \mathcal{J}^+(X_n, \hat{i}) | \mathcal{I}(X_n) = I, i_2, \dots, i_l \notin \mathcal{J}^+(X_n, \hat{i})] \leq \rho.$$

Proof of claim. If the conditional hypothesis holds for the word w and $i_1 \notin \mathcal{J}^+(w, \hat{i})$, then, as we are about to show, at least one of the conditions below hold. Denote $w_1 = w[(i_1+1)lj, i_2lj]$, $w_2 = w[i_2lj, (i_2+1)lj]$, $w_3 = w[(i_2+1)lj, \hat{i}lj]$.



1. $d(p, \pi_A(g(w_2w_3)p)) \leq L$. ($w_2w_3$ projects close to p.)
2. $d(\pi_{g(w_1)A}(p), g(w_1)p) > L - C' - C$. ($w_1^{-1}$ projects far from p.)
3. $d(p, \pi_A(g(w_1)p)) > L - C' - C$. ($w_1$ projects far from p.)
4. $g(w_1) \in E(g)$.

In fact, suppose that all conditions fail. Then the negation of (1) multiplied on the left by $g(w_1)$ and the negation of (2) imply

$$d_{g(w_1)A}(p, g(w_1w_2w_3)p) \geq$$

$$d(g(w_1)p, \pi_A(g(w_1w_2w_3)p)) - d(\pi_{g(w_1)A}(p), g(w_1)p) > C' + C.$$

Thus, by Lemma 6.2.5 (and $g(w_1) \notin E(g)$), $\pi_A(g(w_1w_2w_3)p)$ is C'-close to $\pi_A(g(w_1)A)$. But $\pi_A(g(w_1)A)$ has diameter at most C and contains a point at distance at most $L - C' - C$ from p by the negation of (3). In particular, $\pi_A(g(w_1w_2w_3)p)$ is L-close to p. As $w_1w_2w_3 = w[(i_1+1)lj, \hat{i}lj]$, we see that by the very definition this means that i_1 is in $\mathcal{J}^+(w, \hat{i})$.

We can conclude the proof of the Claim. The point about the conditions stated above is that they are (or, in the case of (1), are easily converted into) properties of w_1, w_2, so that they are "independent enough" from the conditional hypothesis, which is mostly about w_3.

The probability that (2) holds is bounded by 10^{-3} by our choice of L, and similarly for (3). The probability that (4) holds is also at most 10^{-3}, by our choice of k (see Standing Assumptions). Given any possible w_3, there is at most one power of $g(w_0)^j$ so that $d(\pi_A(g(w_2w_3)p), p) \leq C' + C$. The probability that an element of W^k represents such power of $g(w_0)^j$ is smaller than 10^{-3} (as can be seen for example using Inequality 6.2, using $k \geq 10^9$). This observation proves that the probability that 1 happens is smaller than 10^{-3} as well. Summing these probabilities we get the desired bound. $\square$

Using the claim, it is not hard to complete the proof with an argument similar to Lemma 6.6.2. For $w \in W_n(S)$ so that $\mathcal{I}(w) = I$ denote $\hat{I}(w) = I \backslash \mathcal{J}^+(w, \hat{\imath})$. For each $I_0 \subseteq I$ containing the elements $i_1 < \cdots < i_l$ we have

$$\mathbb{P}[\hat{I}(X_n) \supseteq I_0 | \mathcal{I}(X_n) = I] =$$

$$\Pi_{t \geq 1} \mathbb{P}[i_t \notin \mathcal{J}^+(X_n, \hat{\imath}) | \mathcal{I}(X_n) = I, i_{t+1}, \ldots, i_l \notin \mathcal{J}^+(X_n, \hat{\imath})] \leq \rho^l.$$

If the word w is so that $\#\mathcal{J}^+(w) \leq \frac{4}{5}\#\mathcal{I}(w)$ but $\mathcal{I}(w) = I$, then there exists $I_0 \subseteq \hat{I}(w)$ of cardinality $\lceil \#I/5 \rceil$. The final, straightforward, computation is just summing over all possible I_0's and using inequality 6.2:

$$\mathbb{P}\left[\#\mathcal{J}^+(X_n) \leq \frac{4}{5}\#\mathcal{I}(X_n) | \mathcal{I}(X_n) = I\right] \leq \binom{\#I}{\lceil \#I/5 \rceil} \rho^{\lceil \#I/5 \rceil} \leq$$

$$(100)^{\#I/5+1} \rho^{\#I/5} \leq 100(100\rho)^{\#I/5},$$

and the last term decays exponentially. (We used $(\#Ie)/\lceil \#I/5 \rceil \leq 100$, which follows from $\#I \geq 100C_3$.) $\square$

Proof of Proposition 6.6.5. Suppose we fixed a word w of length n so that $\mathcal{I}(w) = I$. We say that $\hat{\imath}_1 \in I$ is *good* for $\hat{\imath}_2 \in I$ if $\hat{\imath}_1 \in \mathcal{J}^+_{\mathcal{A}_j,L}(w, \hat{\imath}_2) \cap \mathcal{J}^-_{\mathcal{A}_j,L}(w, \hat{\imath}_2)$. We want to show that if w satisfies $\#\mathcal{J}^{\pm}(w, \hat{\imath}) \leq \frac{4}{5}\#I$ for each $\hat{\imath} \in I$ then there exists a pair of indices $\hat{\imath}_1$, $\hat{\imath}_2$ so that $|\hat{\imath}_1 - \hat{\imath}_2| \geq \#I/100$ but $\hat{\imath}_1$ is good for $\hat{\imath}_2$ and vice versa. The fraction of words without this property is exponentially small in $\#I$ by Lemma 6.6.6.

Fix instead a word satisfying the property. Then we can make the following counting argument. Given any index $\hat{\imath}_0$, at least $3\#I/5$ indices are good for $\hat{\imath}_0$. In particular, there is an index $\hat{\imath}_1$ which is good for at least $3\#I/5$ indices. Out of these, there are at least $\#I/5$ indices which are good for $\hat{\imath}_1$. If $\hat{\imath}_2$ the furthest such index from $\hat{\imath}_1$, then the pair $\hat{\imath}_1$, $\hat{\imath}_2$ is the one we were looking for.

6.6.6 Proof of main theorem

We are only left to put the pieces together. Then fix w_0, j, k, L as in Proposition 6.6.5 for D as in Lemma 6.5.3. An exponentially small fraction $\leq C_4 e^{-n/C_4}$ of the words w of any given length is so that $\#\mathcal{I}^k_{w_0^j}(w) \leq n/C_4$. Let C_3 be given by Proposition 6.6.5. Summing Lemma 6.6.3 over all subwords of length at least $n/(C_3C_4)$, we see that a fraction $\leq C_5 e^{-n/C_5}$ of words contains a subword of length at least $n/(C_3C_4)$ representing an element of $E(g(w_0))$.

Given a word w we say that $(\hat{i}_1, \hat{i}_2) \in \left(\mathcal{I}^k_{w_0^j}(w)\right)^2$ is a *very good* pair if it is a good pair and $|\hat{i}_1 - \hat{i}_2| \geq \#\mathcal{I}^k_{w_0^j}(w)/C_3$.

Recall that we chose k to be odd so that any subword counted by $\mathcal{I}^k_{w_0^j}$ corresponds to a non-trivial power of $g(w_0^j)$. Lemma 6.5.3 is a criterion for elements (conjugate to an element) of the form $h_1 g^{n_1} h_2 g^{n_2}$ to represent a weakly contracting element. Unravelling the definitions we see that if a word w has no subwords of length at least $n/(C_3C_4)$ representing an element in $E(g(w_0))$ and it admits a very good pair then w has, up to cyclic permutation, the form prescribed by Lemma 6.5.3 and in particular it represents a weakly contracting element.

Finally, keeping this and the estimates we gave above into account (which we use to get rid of words with few occurrences of power of $g(w_0^j)$ and long subwords in $E(g)$) we can make the following estimate for n large:

$$\mathbb{P}[g(X_n) \textit{ not weakly contracting}] \leq C_4 e^{-n/C_4} + C_5 e^{-n/C_5} +$$

$$\sum_{|I| \geq n/C_1} \mathbb{P}[\forall \hat{i}_1, \hat{i}_2 \in I,\ (\hat{i}_1, \hat{i}_2) \textit{ is not very good} | \mathcal{I}(X_n) = I]\ \mathbb{P}[\mathcal{I}(X_n) = I] \leq$$

$$C_6 e^{-n/C_6} + C_3 e^{-n/C_3} \sum \mathbb{P}[\mathcal{I}(X_n) = I] \leq C_7 e^{-n/C_7},$$

for appropriate C_6, C_7.

6.6.7 Proof of the corollaries

We now prove the following corollary of the main theorem.

Corollary 6.6.7. *Let G_1, G_2 be a non-elementary group supporting the path system (resp. weak path system) $(X, \mathcal{PS})$ and containing a contracting (resp. weakly contracting) element.*

- *For any isomorphism $\phi : G_1 \to G_2$ there exists a (weakly) contracting $g \in G$ so that $\phi(g)$ is also (weakly) contracting.*

- *If H is an amenable group and $\phi : G_1 \to H$ is any homomorphism, for any $h \in \phi(G_1)$ there exists a (weakly) contracting $g \in \phi^{-1}(h)$.*

Proof. The proof of both points relies on considering a simple random walk $\{X_n\}$ on G_1 and pushing it forward to a random walk on the target group.

In the first case, one obtains a simple random walk $\{Y_n\}$ on G_2. For n large enough, the probability that both X_n and Y_n represent a non-weakly contracting element are less than $1/2$, so that for such n there exists a (weakly) contracting element of G_1 of word length at most n that is mapped to a (weakly) contracting element of G_2. (A probability-free way of saying this is that, for S a generating system for G and n large enough, the sets of words of length n in both S and $\phi(S)$ that do not represent a (weakly) contracting element have cardinality less than a half of the cardinality of $W_n(S)$.)

In the case of 2), the push-forward of $\{X_n\}$ is a symmetric finitely supported random walk on H. It is well-known that the probability that such a random walk ends up in a given element $h \in H$ decays slower than exponentially, provided that h is in the group generated by the support of the random walk (which is in our case $\phi(G_1)$), see [163, Theorem 12.5]. Hence, for such h and each large enough n, $\mathbb{P}[X_n \text{ (weakly) contracting} | \phi(X_n) = h] > 0$, and we are done. □

6.6.8 A simpler proof of a simpler theorem

There is a shortcut in the proof of genericity of weakly contracting elements if one wants to show that the probability of ending up in a non-weakly contracting element goes to 0, without the extra information that it does so exponentially fast.

This is because there exists K so that, given a word w of length at most $\log(n)/K$, the probability that a random word of length n does not contain w goes to 0 for n going to infinity. The idea is then to look for powers of a word representing a weakly contracting element and using (the appropriate versions of) Lemmas 6.6.3 and 6.6.4 to show that the hypothesis of Lemma 6.5.2 are satisfied with high probability. More details are below.

Fix a word w_0 representing the weakly contracting element g, with axis A. We claim that for an appropriate $f(n) \in O(\log(n))$, "most" words w of length n (meaning all except for a fraction of size going to 0 as n goes to infinity) have a uniquely determined decomposition $w = w_1 w_2 w_3$ with the following properties:

1. w_2 is a power of w_0 with exponent $f(n)$,

2. either $w_1 = \emptyset$ and w_3 is any word of length $n - f(n)$ or $w_3 w_1$ is any word of length $n - f(n)$ not ending with w_0.

3. $g(w_3 w_1) \notin E(g(w_0))$,

4. $d(p, \pi_A(g(w_3 w_1)p)) \leq f(n)/3$.

We already commented on (1), and (2) just says that w_2 is the first occurrence of the $w_0^{f(n)}$.

Regard w as a cyclic word. Then the probability that any given subword of w of length $n - f(n)$ represents an element of $E(g(w_0))$ is exponentially small in n by Lemma 6.6.3, so that a stronger version of (3) can be shown summing over all possible such subwords.

Finally, using a small variation of Lemma 6.6.4 that can be proven with the same techniques (and fewer technical details), one can show (4). More precisely, one considers separately the case when $w_1 \neq \emptyset$ and $w_1 = \emptyset$. To deal with the first case, one checks that Lemma 6.6.4 still holds when replacing in the statement "$\mathcal{I}^k_{w_0^j}(X_n) = I$" with the new conditioning event that X_n does not terminate with w_0. To deal with the first case, one removes the conditioning event altogether and again checks that the proof goes through.

Lemma 6.5.2 applies to any element represented by a word satisfying the conditions above, so that all such elements are weakly contracting.

Appendix: Weak Tits Alternative

In this appendix we sketch the proof of a result implied by Theorem 6.4.6 and results in [58] (see, e.g., Theorem 2.23), that is to say that subgroups containing (weakly) contracting elements are either virtually cyclic or they contain a free group on two generators. We do so, as explained in the introduction, to show that our techniques can prove Theorem 6.1.4 in an almost self-contained way.

Theorem 6.6.8 (Weak Tits Alternative)**.** *Let G be a finitely generated group equipped with the weak path system $(X, \mathcal{PS})$. Suppose that G contains a weakly $\mathcal{PS}$-contracting element. Then either G is virtually cyclic or it contains a free group on two generators.*

The theorem above was known already in many cases. For relatively hyperbolic groups it essentially follows from the proofs in [164], for mapping class groups see [97, 128], for $Out(F_n)$ see [103] (the "full" Tits Alternative is also known [22, 23]), for right-angled Artin groups it follows from Tits' original result [157] in view of the fact that they are linear [93, 61] (and standard facts about $CAT(0)$ spaces), and for graph manifold groups one just needs to use some Bass-Serre theory.

Proof. [Sketch.] Let $g_0 \in G$ be a weakly contracting element. If G is not virtually cyclic, then there exists $h_0 \notin E(g)$. From Lemma 6.5.2 we see that for some appropriate N we have that $h_1 = h_0 g_0^N$ is again weakly contracting. We claim that for any sufficiently large m, $h = h_1^m$ and $g = g_0^m$ generate a free group on two generators.

If A_g, A_h are axes for g_0, h_0 (and hence also axes for g, h), then the diameters of $\pi_{A_g}(A_h), \pi_{A_h}(A_g)$ are finite. This follows, for example, from the proof of Lemma 6.5.1, in particular claim (1) and the description of the projection on A.

Suppose just for convenience that there exists $p \in A_g \cap A_h$, which can be arranged by taking neighbourhoods. Up to increasing m, the translation distance of any non-trivial power of g on A_g is much larger than the diameter of $\{p\} \cup \pi_{A_g}(A_h)$ and vice versa.

In order to show the non-triviality of any element of the form, e.g., $g^{a_1}h^{b_1}\dots g^{a_k}h^{b_k}$, where $k \geq 1$ and $a_i, b_i \neq 0$, one consider the sequence of points $x_i = g^{a_1}h^{b_1}\dots g^{a_i}h^{b_i}p$ and $y_i = g^{a_1}h^{b_1}\dots g^{a_i}h^{b_i}g^{a_{i+1}}p$. The non-triviality of such element (for m large) follows from the fact that the concatenation of special paths from x_i to y_i and from y_i to x_{i+1} forms a quasi-geodesic (and that the constants do not depend on m). This fact can be proven using the same strategy as in Lemma 6.5.1: x_i, y_i belong to an appropriate translate A_i of A_g, and one can show that $\pi_{A_i}(x_j), \pi_{A_i}(y_j)$ are close to y_i for $j > i$ and close to x_i for $j < i$. One can show this inductively, also considering translates of A_h containing y_j, x_{j+1}. $\square$

Chapter 7

Tracking rates

7.1 Introduction

It is known in several contexts that sample paths of random walks stay sublinearly close to geodesics [78, 100, 99, 101, 107, 157]. Such a property is useful, for example, to describe the Poisson boundary [99].

It seems that little is known about estimates on the tracking rates, i.e. the actual expected value of the Hausdorff distance between a random path and a corresponding geodesic. The tracking rate in non-Abelian free groups [111] and more generally non-elementary hyperbolic groups [26, Corollary 3.9] is logarithmic in the length of the walk. Our first main result is that the logarithmic rate holds for a more general class of groups, and we will show it with entirely different, more geometric, methods than [111] and [26]. We say that a relatively hyperbolic group is non-trivial if it is not virtually cyclic and all peripheral subgroups have infinite index. We denote the Hausdorff distance by d_{Haus}. Also, the supremum over all geodesics from x to y will be denoted by $\sup_{[x,y]}$ (i.e. if F is a real function depending on a geodesic from x to y, we denote by $\sup_{[x,y]} F([x,y])$ the supremum over all geodesics γ from x to y of $F(\gamma)$).

Theorem 7.1.1. *Let X_n be a simple random walk on the non-trivial relatively hyperbolic group G. There exists C so that for each $n \geq 2$ we have*

$$\mathbb{E}\left[\sup_{[1,X_n]} d_{Haus}(\{X_i\}_{i\leq n}, [1, X_n])\right] \leq C\log(n),$$

where the supremum is taken over all geodesics $[1, X_n]$ from 1 to X_n.

Notice in particular that the expected Hausdorff distance between two geodesics from 1 to X_n is at most logarithmic.

Sublinear tracking has been shown in [101] for hyperbolic groups and very recently in [157] for relatively hyperbolic groups (in both cases for random walks of finite first moment).

We will actually show a more general result, Theorem 7.3.2, which allows more general random walks, gives a polynomial decay of the probability that a sample path gives an "off-range" Hausdorff distance and deals with group actions on relatively hyperbolic spaces instead of relatively hyperbolic groups. The motivation for looking at such actions is on one hand that we will apply Theorem 7.3.2 to the action of a mapping class group on the corresponding curve complex, and on the other that such group actions are very much related to the notion of hyperbolically embedded subgroups as defined in [58], see [58, Theorem 4.42] and [154, Theorem 6.4]. Namely, any hyperbolically embedded subgroup gives an action of the ambient group on a relatively hyperbolic space (a Cayley graph with respect to a possibly infinite generating system), and vice versa a nice action on a relatively hyperbolic space gives hyperbolically embedded subgroups.

Our next result is that sublinear tracking holds in mapping class groups as well. Recall that the complexity of a surface of finite type is $3g+p-3$ where g is the genus of the surface and p the number of punctures.

Theorem 7.1.2. *Let S be a connected, orientable surface S of finite type, with empty boundary and complexity at least 2. Let $\mathcal{M}(S)$ be its mapping class group and let $\{X_n\}$ be a simple random walk on $\mathcal{M}(S)$. Then*

$$\mathbb{E}\left[\sup_{\gamma(X_n)} d_{Haus}(\{X_i\}_{i\leq n}, \gamma(X_n))\right] = O(\sqrt{n\log(n)}),$$

where the supremum is taken over all geodesics in a given word metric and hierarchy paths $\gamma(X_n)$ from 1 to X_n.

Once again, given any choice of a pair of hierarchy paths or geodesics from 1 to X_n the expected value of their Hausdorff distance is $O(\sqrt{n\log(n)})$.

We remark that not even sublinear tracking seems to appear in the literature. Once again, we will show a stronger result (Theorem 7.4.2). In order to prove the theorem we will need a refinement of the result [13, 73] that mapping class groups have (at least) quadratic divergence, which may be of independent interest. (Recall that the divergence is, roughly speaking, the minimal length of paths avoiding a ball as a function of the diameter of the ball, but we will not need the exact definition. The interested reader is referred to [71].)

D-bounded pairs will be defined in Subsection 7.4.2, for the moment we will just mention that all pairs of points on the orbit of a pseudo-Anosov g are D-bounded, but D depends on the choice of g.

Proposition 7.1.3. *Let S be a connected, orientable surface S of finite type, with empty boundary and complexity at least 2. Let $\mathcal{M}(S)$ be its mapping class group. Then there is a constant $C = C(S)$ so that the following holds. Let $D \geq 1$ and let $x_1, x_2 \in \mathcal{M}(S)$ be a D-bounded pair. Then for any path α of length at least 1 from x_1 to x_2 and any $p \in [x_1, x_2]$ we have*

$$d(p, \alpha) \leq C\sqrt{Dl(\alpha)},$$

where $[x_1, x_2]$ can denote either a hierarchy path or a geodesic in a given word metric from x_1 to x_2.

What is shown in [13, 73] is that for each D there exists $K(D)$ so that $d(p, \alpha) \leq K(D)\sqrt{l(\alpha)}$, so the improvement we make is showing that $K(D)$ can be chosen to be linear in $\sqrt{D}$. This result can also presumably be obtained using the techniques in, e.g., [13, Section 6]. However, our proof, which is inspired by arguments in [106, 71], is different and shorter. The coefficient $O(\sqrt{D})$ should be optimal.

Finally, we will give two applications of Theorem 7.3.2. We say that random triangles in a given group are ω-thin, where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, if the expected thinness constant of triangles joining the endpoints of three independent simple random walks is $O(\omega(n))$, where n is the number of steps of both walks. Also, we say that that random points have ω-small Gromov product if all geodesics joining the endpoints of two independent random walks pass $O(\omega)$-close to 1. Finally, given a combing of a group, we define the average Dehn function to be the expected area of loops obtained concatenating the sample path of a simple random walk and the path from the given combing joining the endpoint to the identity. All concepts will be formally defined and discussed in Section 7.3.2.

Theorem 7.1.4. *Let G be a non-trivial relatively hyperbolic group. Then*

1. *random triangles in G are $\log(n)$-thin.*
2. *random points in G have $\log(n)$-small Gromov product.*
3. *if all Dehn functions of the peripheral subgroups are bounded by $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ and are at most polynomial then for any geodesic combing the average Dehn function is $O(\, n\,\delta(\log(n))\,)$.*

In particular, we see that expected values of both the thinness constant and the Dehn function can be much lower than worst-case values. Also, notice that if all triangles in a group are $\log(n)$-thin then the group is hyperbolic (see for example [83], [69] or [77]) and that if the Dehn function is subquadratic then it is actually linear and the group is hyperbolic [133] (see also [142, pages 149-157]). In view of this, the theorem can be interpreted as indicating that random configurations in relatively hyperbolic groups resemble corresponding configurations in hyperbolic groups.

7.2 Outline

We emphasise that our methods of proof are completely different from other methods that have been used to show sublinear tracking. The proof of Theorem 7.1.1 uses three main ingredients, discussed in the following subsections.

7.2.1 Projections estimate

We will consider closest point projections on peripheral sets and show that it is unlikely that two random points project far away on some peripheral set (left coset of peripheral subgroup in the context of groups). The main tool we will use to show this is Corollary 3.1.2, an inequality for closest point projections on peripheral sets, which is similar to a very useful inequality due to Behrstock [13] in the context of subsurface projections. This step can be skipped if one wants to show Theorem 7.1.1 for hyperbolic groups only. The outcomes of this argument are Lemma 7.3.4 and Lemma 7.5.1. We record the latter as it could be useful in other contexts as well.

7.2.2 Exponential divergence

The divergence of non-trivial relatively hyperbolic groups is at least exponential (see [154] or Section 3.6). The same proof, using the consequence of the projections estimate, guarantees that points on a geodesic connecting the endpoints of a random path are close to the random path. For hyperbolic groups, this is a standard argument, see [37, Proposition III.H.1.6].

7.2.3 Drift estimates

The last ingredient is a fact about random walks on non-amenable groups, namely the fact that random walks on nonamenable groups make linear progress (see [163, Lemma 8.1(b)]). This allows us to exclude the existence of "large detours" in the random path.

Except for exponential divergence, all ingredients are available for mapping class groups. In that context instead of exponential divergence we have quadratic divergence, and this is why the rate we get for mapping class groups is worse than the one for relatively hyperbolic groups. We will actually proceed slightly differently in the mapping class group case in order to exhibit a variation on the argument for relatively hyperbolic groups, and we will use a drift estimate in the curve complex due to Maher [119].

7.3 Logarithmic tracking

The aim of this section is to show the following. As in the introduction, we denote a supremum over all geodesics from x to y by $\sup_{[x,y]}$. Recall that we say that a relatively hyperbolic group is non-trivial if it is not virtually cyclic and all peripheral subgroups have infinite index.

Theorem 7.3.1. *Let $\{X_n\}$ be a simple random walk on the non-trivial relatively hyperbolic group G. There exists C so that for each $n \geq 2$ we have*

$$\mathbb{E}\left[\sup_{[1,X_n]} d_{Haus}(\{X_i\}_{i\leq n}, [1, X_n])\right] \leq C\log(n).$$

We will actually show the following refinement. Following, e.g., [119], we say that a random walk $\{X_n\}$ on the group G acting on the pointed metric space (X, p) makes *linear progress* if there exists $C_0 \geq 1$ so that

$$\mathbb{P}\left[d(p, X_n p) \leq n/C_0\right] \leq C_0 e^{-n/C_0}.$$

As noticed in [51, Proposition 5.9], it follows from [163, Lemma 8.1(b)] that when G is a non-amenable group acting on itself, any symmetric random walk makes linear progress.

Theorem 7.3.2. *Suppose that the finitely generated group G acts by isometries on the relatively hyperbolic space X permuting the peripheral sets. Suppose also that there are at least 2 peripheral sets and that the stabiliser of each peripheral set has unbounded orbits. Let μ be a symmetric probability measure on G whose finite support generates*

G and let $\{X_n\}$ be the corresponding random walk, which we assume to make linear progress. Then for each $p \in X$ and for each $k \geq 1$ there exists C so that

$$\mathbb{P}\left[\sup_{[p,X_np]} d_{Haus}(\{X_ip\}_{i\leq n}, trans([p,X_np])) \geq C\log(n)\right] \leq Cn^{-k}.$$

Moreover, the same is true for $[p,X_np]$ substituting $trans([p,X_np])$.

Let us fix the notation of the theorem, including k. Let $\mathcal{H}$ be the collection of peripheral sets of X. We now state and prove the three lemmas we need, and we will combine them together in the next subsection.

The following general fact about relative hyperbolicity will immediately imply that each point on $trans([p,X_np])$ is close to $\{X_ip\}_{i\leq n}$.

Lemma 7.3.3. *There exists C_1 with the following property. Let α be a discrete path from x to y, where $x,y \in X$, and let $p \in trans([x,y])$. Then $d(p,\alpha) \leq C_1\log_2(l(\alpha)+1) + C_1$.*

Proof. The proof is the same as Proposition 3.6.6 and is an easy generalization of, e.g., [37, Proposition III.H.1.6]. We give the proof for completeness.

Let D be as in Lemma 3.1.3. If $l(\alpha) \leq 2$, then the lemma holds, with $C_1 = 2$. Otherwise, split α into paths α_i of length $l(\alpha)/2 \geq 1$ and let q be the common endpoint. Then p is D-close to some $p' \in trans([x,q]) \cup trans([q,y])$. By induction we can assume on the length of α we can assume $d(p',\alpha) \leq D\log_2(l(\alpha)/2)+2$, so that

$$d(p,\alpha) \leq d(p,p') + d(p',\alpha) \leq D\log_2(l(\alpha)) + 2. \qquad \square$$

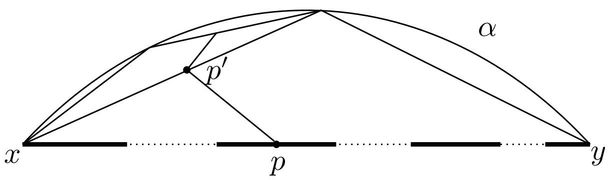


Figure 7.1: Proof of Lemma 7.3.3. The thick segments along $[x,y]$ represent the transient set.

Let us now show that the deep components of $[p,X_np]$ are expected to be logarithmically small. It will be convenient to set, for $H \in \mathcal{H}$,

$$d_H(\cdot,\cdot) = d(\pi_H(\cdot),\pi_H(\cdot)),$$

as it is customary for subsurface projections.

Lemma 7.3.4. *There exists C_2 so that, for each $n \geq 1$,*

$$\mathbb{P}\left[\exists H \in \mathcal{H} : d_H(p, X_n p) \geq C_2 \log(n)\right] \leq C_2 n^{-k}.$$

Proof. The usual notation $\mathbb{P}[\cdot|\cdot]$ will be used for the conditional probability. We will show that there exists K so that:

1. for all $l \geq 0$ and $H \in \mathcal{H}$ we have

$$\mathbb{P}\left[d_H(p, X_n p) \geq l |\, d_H(p, X_n p) \geq K\right] \leq K e^{-l/K}.$$

2. for any $g \in G$ the set $A(g) = \{H \in \mathcal{H} : d_H(p, gp) \geq K\}$ satisfies $|A(x)| \leq K d(p, gp)$.

Using these two facts we can make the estimate:

$$\mathbb{P}\left[\exists H \in \mathcal{H} : d_H(p, X_n p) \geq C_2 \log(n)\right] \leq$$

$$\textstyle\sum_{H \in \mathcal{H}} \mathbb{P}\big[d_H(p, X_n p) \geq C_2 \log(n) | H \in A(X_n)\big] \mathbb{P}[H \in A(X_n)] \leq$$

$$(K e^{-C_2 \log(n)/K}) \textstyle\sum_{H \in \mathcal{H}} \mathbb{P}[H \in A(X_n)] \leq (KTn)(K e^{-C_2 \log(n)/K}),$$

where $T = \max\{d(p, gp) : g \in supp(\mu)\}$. The last inequality follows from the observation that the random variable $|A(\cdot)|$ is the sum of the indicator functions $1_{H \in A(\cdot)}$, so that

$$\sum_{H \in \mathcal{H}} \mathbb{P}[H \in A(X_n)] = \mathbb{E}[\# A(X_n)].$$

We can then clearly choose C_2 large enough that is satisfies the lemma.

(1) The proof is similar to that of Lemma 6.6.4. We want to show that there exists K' so that

$$\mathbb{P}\left[d_H(p, X_n p) \geq l + K'\right] \leq K' \mathbb{P}\left[d_H(p, X_n p) \in [l - K', l + K')\right],$$

which then implies the exponential decay we are looking for (just as in [153, Lemma 6.2]). Namely, one readily shows inductively

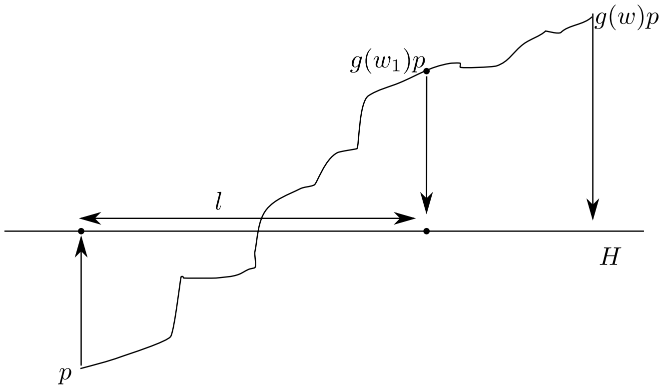
$$\mathbb{P}\left[d_H(p, X_n p) \geq (2i+1)K'\right] \leq (1 + 1/K')^{-i} \mathbb{P}[d_H(p, X_n p) \geq K'],$$

just using the inequality above for $l = (2i+2)K'$:

$$\mathbb{P}\left[d_H(p, X_n p) \geq (2i+3)K'\right] \leq$$

$$K'(\mathbb{P}\left[d_H(p, X_n p) \geq (2i+1)K'\right] - \mathbb{P}\left[d_H(p, X_n p) \geq (2i+3)K'\right]).$$

For later purposes, we fix distinct $H_1, H_2 \in \mathcal{H}$. For w a word in the elements of the support of μ, denote by $g(w)$ the corresponding element of g. Let w be a word of length n so that $d_H(p, g(w)p) \geq l + 100K_1$, for some large enough K_1. As π_H is coarsely Lipschitz, the minimal subword w_1 of w so that $d_H(p, g(w_1)p) \geq l$ actually satisfies $d_H(p, g(w_1)p) \in [l, l+K_1]$. Let w_2 be the subword of w starting right after w_1, of length K_2 (where K_2 is a large enough constant) and let w_3 be the final subword of w starting after w_2.



We claim that substituting w_2 by a suitable word w_2' of the same length we can make sure that $d_{g(w_1)H_i}(H, g(w_1w_2'w_3)p)$ is larger than B as in Corollary 3.1.2, where i is chosen so that $g(w_1)H_i \neq H$. (The word w_2' depends on w.)

Indeed, if w_2' represents an element in the stabiliser of H_i then (up to bounded additive error)

$$d_{g(w_1)H_i}(H, g(w_1w_2'w_3)p) = d(\pi_{H_i}(g(w_1)^{-1}H), g(w_2')\pi_{H_i}(g(w_3)p)),$$

so that we can just "push" $\pi_{H_i}(g(w_3)p)$ far from $\pi_{H_i}(g(w_1)^{-1}H)$ using $g(w_2')$ if they happen to be close, and choose w_2' so that $g(w_2')$ is at distance at most 1 from the identity otherwise. (The hypotheses that H_i has infinite diameter tells us that there is "enough space" to do so.)

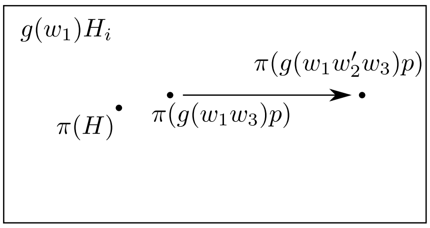


Figure 7.2: π denotes $\pi_{g(w_1)H}$.

In particular, keeping into account that $d(g(w_1)p, g(w_1)H_i)$ and therefore also $d_H(g(w_1)p, g(w_1)H_i)$ are bounded, we see that $d_H(g(w_1w_2'w_3)p, g(w_1)p)$ is bounded by Corollary 3.1.2. Hence $d_H(p, g(w_1w_2'w_3)p) \in [l - K_3, l + K_3)$, for a suitable K_3.

To sum up, we constructed for any word w of length n so that $d_H(p, g(w)p) \geq l + 100K_1$ another word $w' = w_1w_2'w_3$ of length n so that $d_H(p, g(w')p) \in [l - K_3, l + K_3)$, and the map $w \mapsto w'$ is easily seen to be bounded-to-1, as the decomposition of w' as $w_1w_2'w_3$ is uniquely determined by the definition of w_1 and the length of w_2', which is some fixed constant. This then gives the desired inequality as the support of μ is finite and hence for each s, s' in the support $\mu(s)/\mu(s')$ is uniformly bounded.

(2) For K large, we can assign to each $H \in A(g)$ a subgeodesic γ_H of any given geodesic γ from p to gp so that $l(\gamma_H) \geq 1$ and distinct γ_H's are disjoint, such subgeodesic being just the deep component along H, see Lemma 3.1.3-(2)-(4). □

Finally, we show that subwalks of at least logarithmic length are expected to make linear progress (the definition of linear progress is above Theorem 7.3.2). The lemma will also be used later.

Lemma 7.3.5. *Let G be a group acting on the metric space X and suppose that the random walk $\{X_n\}$ on G makes linear progress on X. Then for each $p \in X$ there exists C_3 so that for each $n \geq 1$*

$$\mathbb{P}\left[\exists i, j \leq n : |i - j| \geq C_3 \log(n), d(X_ip, X_jp) \leq |i - j|/C_3\right] \leq C_3 n^{-k}.$$

Proof. As $\{X_n\}$ makes linear progress there exists K_1 so that

$$\mathbb{P}\big[d(p, X_jp) \leq j/K_1\big] \leq K_1 e^{-j/K_1}.$$

Notice that for each $i < j$ we have $\mathbb{P}[d(X_ip, X_jp) \leq t] = \mathbb{P}[d(p, X_i^{-1}X_jp) \leq t] = \mathbb{P}[d(p, X_{j-i}p) \leq t]$. Summing the linear progress inequality for i ranging from 1 to n and $j \geq i + (k+1)K_1 \log n$ we get that the probability in the statement is at most

$$\sum_{i=1}^{n} \sum_{l \geq (k+1)K_1 \log(n)} K_1 e^{-l/K_1} = n \sum_{i \geq 0} K_1 n^{-(k+1)} e^{-i/K_1}. \qquad \square$$

7.3.1 Proof of Theorem 7.3.2

The statement for $[p, X_np]$ follows from the one for $trans([p, X_np])$ in view of Lemma 7.3.4 and Lemma 3.1.3-(3) (or otherwise directly showing that the transient set is logarithmically dense in the corresponding geodesic). Fix $n \geq 2$, and denote by C_i suitable constants that do not depend on n. Consider a sample path $\{w_i\}_{i \leq n}$ of the random walk. By Lemma 7.3.3, we have that each transient point on a geodesic from p to $w_n p$ is logarithmically close to $\{w_i p\}$, say at distance at most $C_4 \log(n)$. Now, assume that for all $H \in \mathcal{H}$ we have $d_H(p, w_n p) \leq C_2 \log(n)$ and for all i, j with $|i-j| \geq C_3 \log(n)$ we have $d(w_i p, w_j p) \geq |i-j|/C_3$, where C_2, C_3 are as in Lemma 7.3.4 and Lemma 7.3.5. The lemmas tell us that we can safely disregard sample paths not satisfying these properties. Suppose that for some j we have $d(w_j p, trans([p, w_n p])) > C_4 \log(n)$, so that any transient point on $[p, w_n p]$ is $C_4 \log(n)$-close to either $\{w_i p\}_{i<j}$ or $\{w_i\}_{i>j}$. If $p' \in trans([p, w_n p])$ is the closest point to $w_n p$ satisfying $d(p', \{w_i p\}_{i<j}) \leq C_4 \log(n)$, then we also have $d(p', \{w_i p\}_{i>j}) \leq C_5 \log(n)$, as $trans([p, w_n p])$ has at most logarithmic "gaps". Hence, there are $i_0 < j < i_1$ so that $d(w_{i_0} p, w_{i_1} p) \leq (C_4 + C_5) \log(n) = C_6 \log(n)$.

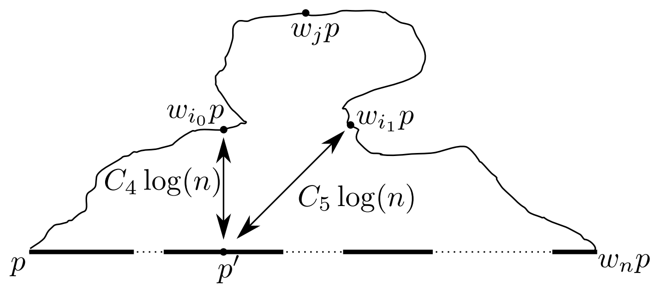


Figure 7.3: $i_1 - i_0$ cannot be large because $w_{i_0} p$ is close to $w_{i_1} p$.

By Lemma 7.3.5 either $i_1 - i_0 < C_3 \log(n)$ or $d(w_{i_0} p, w_{i_1} p) \geq (i_1 - i_0)/C_3$, in which case $C_6 \log(n) \geq d(w_{i_0} p, w_{i_1} p) \geq (i_1 - i_0)/C_3$. Hence, in any case we have $i_1 - i_0 \leq C_7 \log(n)$. Therefore,

$$\begin{aligned} d(w_j p, trans([p, w_n p])) &\leq d(w_j p, w_{i_0} p) + d(w_{i_0} p, trans([p, w_n p])) \\ &\leq (C_8 + C_4) \log(n), \end{aligned}$$

and this completes the proof. □

7.3.2 Applications

Given three points x_1, x_2, x_3 in a geodesic metric space denote by $\delta(x_1, x_2, x_3)$ the supremum of the thinness constants of geodesic triangles with vertices x_1, x_2, x_3, i.e.

$$\sup\{d(p, [x_{i-1}, x_i] \cup [x_{i+1}, x_{i-1}])\},$$

where the supremum is taken over all choices of geodesics $[x_i, x_{i+1}]$ and all $p \in [x_i, x_{i+1}]$ for $i = 1, 2, 3$ (we take indices modulo 3).

Definition 7.3.6. Let $\omega : \mathbb{N} \to \mathbb{N}$ be a function. We will say that *random triangles in the group G are ω-thin* if whenever $\{X_n\}, \{Y_n\}, \{Z_n\}$ are independent simple random walks on G with respect to the same generating system we have $\mathbb{E}[\delta(X_n, Y_n, Z_n)] = O(\omega)$. Also, we say that *the Gromov product of random points in G is ω-small* if whenever $\{X_n\}, \{Y_n\}$ are independent simple random walks on G, again with respect to the same generating system, we have $\mathbb{E}\left[\sup_{[X_n,Y_n]} d(1, [X_n, Y_n])\right] = O(\omega)$.

A related notion called statistical hyperbolicity has been considered in [72, 64], where it is proven for hyperbolic groups and Teichmüller spaces.

Theorem 7.3.7. *Random triangles in any given non-trivial relatively hyperbolic group G are $\log(n)$-thin. Also, the Gromov product of random points in G is $\log(n)$-small.*

Proof. As the random walks we are considering are independent and symmetric, we can concatenate them to obtain a longer random walk, meaning that the distribution of $X_j^{-1}Y_k$ is the same as that of X_{j+k}, and similarly for the other pairs. By Theorem 7.3.2 and the observation above we have an appropriate constant C so that

$$\mathbb{P}[d_{Haus}([X_n, Y_n], \{X_i, Y_i\}_{i \leq n}) \geq C \log(n)] =$$

$$\mathbb{P}[d_{Haus}([1, X_n^{-1}Y_n], \{X_n^{-1}X_i, X_n^{-1}Y_i\}_{i \leq n}) \geq C \log(n)]$$

goes to 0 faster than, say, $1/n^2$, and similarly for the other pairs. Both conclusions easily follow. $\square$

Remark 7.3.8. Notice that a similar statement holds for random polygons as well. Also, the theorem can be generalised to group actions on relatively hyperbolic spaces.

Recall that given a group G and a generating system for G, a (discrete) combing Γ is a choice, for each $x \in G$, of a discrete path $\Gamma(x)$ connecting 1 to x in the Cayley graph of G. Given a finitely presented group G and a (discrete) loop α in its Cayley graph, we will denote by $Fill(\alpha)$ the area of a minimal van Kampen diagram whose boundary is α.

Definition 7.3.9. Let Γ be a combing on the finitely presented group G and $\{X_i\}$ a simple random walk on G. The *average Dehn function* $\delta_{avg}^{G,\Gamma,\{X_i\}}$ of G with respect to Γ and $\{X_i\}$ is

$$\delta_{avg}^{G,\Gamma,\{X_i\}}(n) = \mathbb{E}[Fill(\{X_0, \ldots, X_n\} \cup \Gamma(X_n))].$$

In [166] another notion of average Dehn function is considered and it is shown that for most nilpotent groups this function is subasymptotic to the Dehn function. Other related results can be found in [102, 27].

Theorem 7.3.10. *Suppose that G is hyperbolic relative to proper subgroups with at most polynomial Dehn function. Then for each geodesic combing Γ and every simple random walk $\{X_i\}$ on G we have*

$$\delta_{avg}^{G,\Gamma,\{X_i\}}(n) = O(\, n\,\delta(\log(n))\,),$$

where δ is the maximum of the Dehn functions of the peripheral subgroups.

Proof. Fix $n \geq 2$. By Theorem 7.3.2, we can take the expected value that defines δ_{avg} conditioned on $d_{Haus}(trans(\Gamma(X_n)), \{X_i\}_{i\leq n}) \leq C\log(n)$ for some appropriate C. We are allowed to do so because the Dehn function of G is (equivalent to) δ [76], so that by choosing C large enough we can make sure that $\mathbb{P}[d_{Haus}(trans(\Gamma(X_n)), \{X_i\}_{i\leq n}) > C\log(n)]$ decays much faster than the inverse of the Dehn function of G (informally speaking, the loops we are disregarding are too few to contribute to the average Dehn function). Choose discrete geodesics α_i connecting X_i to $trans(\Gamma(X_n))$ of length at most $C\log(n)$ (choose α_0 and α_n to be trivial), and consider discrete loops l_i obtained concatenating α_i, a subgeodesic of $\Gamma(X_n)$ and α_{i+1}^{-1}. Each of these loops has area at most $K\delta(\log(n))$ for some suitable K. As there are $n-1$ such loops, and from fillings of all of them we can recover a filling of the full path, we get the desired bound. $\square$

It would be interesting to know whether the same result holds for the notion of average Dehn function defined in terms of the uniform distribution on loops.

7.4 Mapping class groups

Convention 7.4.1. From now on, all surfaces will be assumed to be orientable, connected, of finite type and to have empty boundary.

Recall that the complexity of a surface S is $3g + p - 3$, where g is the genus of S and p the number of punctures. In particular a surface has complexity at least 2 if it is not a sphere with at most four punctures or a torus with at most one puncture. In this section we show the following.

Theorem 7.4.2. *Let G be the mapping class group of a surface of complexity at least 2, let μ be a finitely supported probability measure on G and let $\{X_n\}$ be the corresponding random walk on G. Then*

$$\mathbb{E}\left[\sup_{\gamma(X_n)} d_{Haus}(\{X_i\}_{i\le n}, \gamma(X_n))\right] = O(\sqrt{n\log(n)}),$$

where the supremum is taken over all hierarchy paths and geodesics $\gamma(X_n)$ in a given word metric from 1 to X_n.

More precisely, for each k there exists C so that for all $n \ge 1$

$$\mathbb{P}\left[sup_{\gamma(X_n)} d_{Haus}(\{X_i\}_{i\le n}, \gamma(X_n)) \ge C\sqrt{n\log(n)}\right] \le Cn^{-k}.$$

7.4.1 Results from the literature

We will assume that the reader is somewhat familiar with hierarchies, subsurface projections and related notions from [126, 127], but we recall the main results we will need. We denote by S a surface of complexity at least 2, and let $\mathcal{M}(S)$ be its mapping class group (which for our purposes can be identified with the marking complex). With an abuse, when referring to a subsurface we will actually refer to its isotopy class and assume that it is connected and essential. For $Y \subseteq S$ a subsurface, $\pi_Y : \mathcal{M}(S) \to 2^{\mathcal{C}(Y)}$ will denote the subsurface projection on the curve complex $\mathcal{C}(Y)$ of Y, which is hyperbolic [126] (see [89] for a short and self-contained proof of this fact). Recall that π_Y is coarsely Lipschitz [126]. As customary, we also denote π_Y the subsurface projection as a map from $\mathcal{C}(S)$ to $2^{\mathcal{C}(Y)}$. The surfaces Y, Z are said to overlap if $\pi_Y(\partial Z), \pi_Z(\partial Y) \ne \emptyset$. We use the notation $d_Y(\mu,\nu) = diam_{\mathcal{C}(Y)}(\pi_Y(\mu) \cup \pi_Y(\nu))$.

Theorem 7.4.3 (Behrstock Inequality, [13, Theorem 4.3])**.** *There exists a constant C so that if the subsurfaces $Y, Z \subseteq S$ overlap then for each $m \in \mathcal{M}(S)$ we have*

$$\min\{d_Y(\partial Z, m), d_Z(\partial Y, m)\} \le C.$$

(The constant can be chosen to be 10 in view of a slick argument due to Leininger and written up in [123].)

We write $A \approx_{K,C} B$ if the quantities A, B satisfy

$$A/K - C \le B \le KA + C.$$

Let $\{\{A\}\}_L$ denote A if $A \ge L$ and 0 otherwise.

Theorem 7.4.4 (Distance Formula, [127, Theorem 6.12])**.** *There exists L_0 with the property that for each $L \geq L_0$ there are K, C so that, for each $\mu, \nu \in \mathcal{M}(S)$,*

$$d_{\mathcal{M}(S)}(\mu, \nu) \approx_{K,C} \sum_{Y} \{\{d_Y(\mu, \nu)\}\}_L,$$

where the sum is taken over all (isotopy classes of) subsurfaces Y.

For notational convenience, from now on we denote sums over all subsurfaces of a surface S simply by $\sum_{Y \subseteq S}$.

Theorem 7.4.5 (Bounded Geodesic Image Theorem, [127, Theorem 3.1])**.** *There exists C with the following property. If γ is a geodesic in $\mathcal{C}(S)$ so that for some proper subsurface Y we have that $\pi_Y(v)$ is non-empty for every vertex $v \in \gamma$ then $\pi_Y(\gamma)$ has diameter at most C.*

An elementary proof of this theorem can be found in [161].

Remark 7.4.6. The consequence of the theorem we will more often use is that if Y has complexity $\xi(S) - 1$ and $\pi_Y(\gamma)$ has sufficiently large diameter then a component of ∂Y appears in γ. More generally, if $\pi_Y(\gamma)$ has sufficiently large diameter then Y is contained in $S \backslash v$ for some $v \in \gamma$ or it is an annulus around some such v.

We also recall the following result on random walks due to Maher (not stated in full generality).

Theorem 7.4.7. *[119][120, Theorem 1.2] Random walks on a mapping class group $\mathcal{M}(S)$ of surfaces of complexity at least 2 whose finite support generates $\mathcal{M}(S)$ make linear progress in the curve complex.*

7.4.2 Quantitative quadratic divergence

Let L_0 be as in the Distance Formula and larger than the constant C in the Bounded Geodesic Image Theorem. Similarly to [13], we say that a pair $x, y \in \mathcal{M}(S)$ is D-bounded if for each proper subsurface Y of S we have $\sum_{Z \subseteq Y} \{\{d_Z(x, y)\}\}_{3L_0} \leq D$.

We set the threshold (almost) arbitrarily, but for our purposes different thresholds give equivalent notions of boundedness, as we can see from the following lemma.

Lemma 7.4.8. *For each $L \geq L_0$ there exists C so that for each $x, y \in \mathcal{M}(S)$ and $Y \subseteq S$ we have*

$$\sum_{Z \subseteq Y} \{\{d_Z(x, y)\}\}_L \leq \sum_{Z \subseteq Y} \{\{d_Z(x, y)\}\}_{L_0} \leq C \sum_{Z \subseteq Y} \{\{d_Z(x, y)\}\}_L + C.$$

This fact is implicit in the proof of the Distance Formula (as are a few facts that will appear in the proofs below). However, in order to make the proofs accessible to more readers we give will rely as little as possible on the machinery of hierarchies and use the Distance Formula instead.

Proof. The first inequality is obvious. In order to show the second one we would like to bound the number of subsurfaces $Z \subseteq Y$ so that $d_Z(x,y) \in [L_0, L)$. We proceed inductively on complexity. There is at most one such Z of complexity $\xi(Z) = \xi(Y)$, i.e. $Z = Y$. Suppose we are given inductively subsurfaces $\mathcal{Z}_k = \{Z_1, \dots Z_{i(k)}, Z'_1, \dots, Z'_{j(k)}\}$ of complexity $\xi(Y) - k$ so that $d_{Z_i}(x,y) < L$, $d_{Z'_j}(x,y) \geq L$ and so that any subsurface Z of positive complexity at most $\xi(Y) - k$ with $d_Z(x,y) \geq L_0$ is contained in some Z_i or in some Z'_i. Let $\hat{Z} \in \mathcal{Z}_k$. Choose a geodesic γ from $\pi_{\hat{Z}}(x)$ to $\pi_{\hat{Z}}(y)$ and choose at most $2d_{\hat{Z}}(x,y) + 2$ subsurfaces $\{Z''_i\}$ so that for every vertex $v \in \gamma$ any component of $\hat{Z} \backslash v$ is contained in one such subsurface. By the Bounded Geodesic Image Theorem (see also Remark 7.4.6), any subsurface Z' of positive complexity lower than that of Y and such that $d_{Z'}(x,y) \geq L_0$ is contained in some Z''_i. We then get a bound of the form

$$|i(k+1)| \leq |i(k)|(2L+2) + 4 \sum_j d_{Z'_j}(x,y),$$

where for convenience we used that $d_{Z'_j}(x,y) \geq 1$ and hence $2d_{Z'_j}(x,y) + 2 \leq 4d_{Z'_j}(x,y)$. It is also easy to bound the number of annuli $A \subseteq Y$ with $d_A(x,y) \in [L_0, L)$, again using Remark 7.4.6. Proceeding inductively one shows that the number of subsurfaces $Z \subseteq Y$ so that $d_Z(x,y) \in [L_0, L)$ is at most

$$p(L) + p(L) \sum_{Z \subseteq Y} \{\{d_Z(x,y)\}\}_L,$$

for an appropriate polynomial $p(x)$ (notice that each Z contributing a non-zero term to the sum appears at most once in the inductive procedure). $\square$

For $x, y \in \mathcal{M}(S)$, we denote by $[x,y]$ a hierarchy path joining them.

Proposition 7.4.9. *Let S be a surface of complexity at least 2. There is a constant $C = C(S)$ so that the following holds. Let $D \geq 1$ and let $x_1, x_2 \in \mathcal{M}(S)$ be a D-bounded pair. Then for any path α of length at least 1 from x_1 to x_2 and any $p \in [x_1, x_2]$ we have*

$$d(p, \alpha) \leq C\sqrt{Dl(\alpha)}.$$

The argument below can be somewhat simplified if one only wants to reprove quadratic divergence, as in this case one can use D as a threshold in the distance formula. We cannot do this because the error terms in the distance formula are not linear in the threshold.

Proof. We will denote by C_i suitable large enough constants, depending on S only. We fix L_0 as above and set $L = 3L_0$.

Let x_1, x_2, α be as in the statement and let $p \in [x_1, x_2]$ be a point maximising the distance from α. Set $d = d(p, \alpha)$, which we can safely assume to be larger than 1000δ, where δ is the maximum of the hyperbolicity constants of all curve complexes of subsurfaces of S. Consider a subpath $[x'_1, x'_2]$ of $[x_1, x_2]$ with the property that $d(x'_i, p) \in [d/2, 3d/4]$, so that in particular $d(x'_i, \alpha) \in [d/4, d]$. Also, consider $p_i \in \alpha$ so that $d(x'_i, p_i) \le d$. Denote α' the subpath of α from p_1 to p_2. Pick a maximal collection of points $y_1, \dots, y_n$ on $[x'_1, x'_2]$ so that $d_S(y_i, y_{i+1}) \ge C_0 \ge 100\delta$. Notice that any pair of points p, q on $[x_1, x_2]$ is $C_1 D$ bounded because for each $Y \subseteq S$ we have $d_Y(p, q) \le d_Y(x, y) + C_2$ (a standard property of hierarchy paths), and in view of Lemma 7.4.8.

We claim that we have

$$n \ge d/(C_3 D).$$

This follows from the lemma below, which we record for later purposes.

Lemma 7.4.10. *There exists C with the following property. Let $p, q \in \mathcal{M}(S)$ be an E-bounded pair for some $E \ge 1$. Then*

$$d_{\mathcal{M}(S)}(p, q) \le C d_S(p, q) E.$$

Proof. By the Bounded Geodesic Image Theorem (see also Remark 7.4.6) one can find $C_4 d_S(p, q)$ subsurfaces $Y_i \subsetneq S$ so that any subsurface $Y \subsetneq S$ with $d_Y(p, q) > L$ is contained in some Y_i, as for each such subsurface (a component of) ∂Y_i has to be contained in any given geodesic from $\pi_S(p)$ to $\pi_S(q)$. Using the Distance Formula we get

$$d_{\mathcal{M}(S)}(p, q) \le C_5 d_S(p, q) + C_5 \sum_i \sum_{Y \subseteq Y_i} \{\{d_Y(p, q)\}\}_L \le C_6 d_S(p, q) E. \qquad \square$$

There are points $q_1, \dots, q_n$ on α so that the closest point projection of $\pi_S(q_i)$ on a geodesic γ in $\mathcal{C}(S)$ from $\pi_S(x'_1)$ to $\pi_S(x'_2)$ is within bounded error $\pi_S(y_i)$ (and which appear in the given order along α).

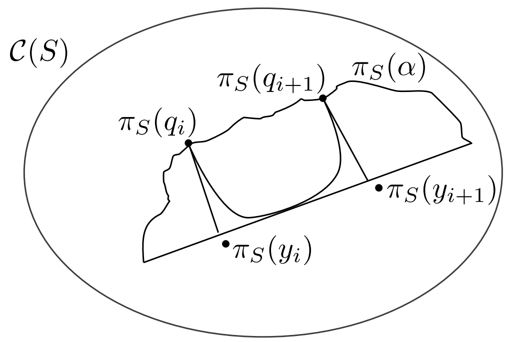


Figure 7.4: The situation as seen from the curve complex.

We will now show that $d_{\mathcal{M}(S)}(q_i, q_{i+1})$ can be bounded from below linearly in d. At the end of the proof we will just combine such lower bound with the lower bound on n.

Let $Y_1, \ldots, Y_k$ be subsurfaces so that

$$\sum_j \{\{d_{Y_j}(q_i, y_i)\}\}_{3L} \geq d/C_7,$$

and all terms in the sum are positive. Such subsurfaces exist by the Distance Formula, as $d_{\mathcal{M}(S)}(q_i, y_i) \geq d/4$. (Notice that we set the threshold to $3L$.) By Lemma 7.4.10, we know that $d_{\mathcal{M}(S)}(y_i, y_{i+1})$ is bounded linearly in D.

Once again by the Distance Formula we have

$$\sum_j \{\{d_{Y_j}(y_i, y_{i+1})\}\}_L \leq C_8 D.$$

Notice that if $l(\alpha) \leq \sqrt{D}$ then $d(p, x_1)$ is $O(\sqrt{D})$, so that we can assume $l(\alpha) \geq \sqrt{D}$. Hence, we can also assume that d is larger than, say, $100C_7C_8D$. We then claim that

$$\sum_j \{\{d_{Y_j}(q_i, y_{i+1})\}\}_{2L} \geq d/C_9.$$

(Notice that we lowered the threshold.) This follows combining the facts that for each j we have

$$\{\{d_{Y_j}(q_i, y_{i+1})\}\}_{2L} \geq \{\{d_{Y_j}(q_i, y_i)\}\}_{3L} - \{\{d_{Y_j}(y_i, y_{i+1})\}\}_L - 2L$$

and

$$\{\{d_{Y_j}(q_i, y_i)\}\}_{3L} - 2L \geq \frac{1}{3}\{\{d_{Y_j}(q_i, y_i)\}\}_{3L},$$

as the threshold is attained for each j by hypothesis.

By hyperbolicity of $\mathcal{C}(S)$, geodesics from $\pi_S(q_{i+1})$ to $\pi_S(y_{i+1})$ stay far from geodesics from $\pi_S(q_i)$ to $\pi_S(y_i)$, and hence $d_{Y_j}(q_{i+1}, y_{i+1})$ can be bounded by the Bounded

Geodesic Image Theorem for $Y_j \subsetneq S$. On the other hand, if for some j we have $Y_j = S$ then $d_{Y_j}(q_i, y_{i+1}) \leq d_{Y_j}(q_i, q_{i+1})/C_{10}$ because any geodesic from $\pi_S(q_i)$ to $\pi_S(q_{i+1})$ passes close to $\pi_S(y_i), \pi_S(y_{i+1})$. In particular

$$\sum_j \{\{d_{Y_j}(q_i, q_{i+1})\}\}_L \geq d/C_{11}.$$

and we can therefore use the Distance Formula to give the lower bound

$$d_{\mathcal{M}(S)}(q_i, q_{i+1}) \geq d/C_{12}.$$

Thus, we get

$$l(\alpha) \geq \sum d(q_i, q_{i+1}) \geq (n-1)d/C_{12} \geq d^2/(C_{13}D),$$

the inequality we were looking for. □

A standard argument now gives the following.

Corollary 7.4.11. *Fix the notation of Proposition 7.4.9 and assume furthermore that α is a geodesic in a given word metric. Then (up to increasing C)*

$$d_{Haus}(\alpha, [x_1, x_2]) \leq C\sqrt{Dd(x_1, x_2)}.$$

In particular, Proposition 7.4.9 still holds if $[x_1, x_2]$ denotes a geodesic in a given word metric rather than a hierarchy path.

Proof. The fact that any point on $[x_1, x_2]$ is close to α is the content of the proposition. Set $A = C\sqrt{Dd(x_1, x_2)}$. Any $q \in \alpha$ splits α into two subgeodesics α_1, α_2. All points on $[x_1, x_2]$ are A-close to either α_1 or α_2, so that there is a point $p \in [x_1, x_2]$ which is A-close to both (this is true up to bounded error if $[x_1, x_2]$ is regarded as a discrete path). Hence, q is contained in a subgeodesic of length at most $2A$ whose endpoints are A-close to $[x_1, x_2]$. In particular, q is $2A$-close to $[x_1, x_2]$. □

7.4.3 Proof of Theorem 7.4.2

Fix from now on the notation of Theorem 7.4.2. We now show that pairs $1, X_n$ are expected to be $O(\log(n))$-bounded.

Lemma 7.4.12. *For each $k \geq 1$ there exists C_0 so that, for each $n \geq 2$,*

$$\mathbb{P}\left[1, X_n \text{ is not } C_0 \log(n)\text{−bounded}\right] \leq C_0 n^{-k}.$$

Proof. Fix L_0 as in the distance formula and larger than the constant C in the Bounded Geodesic Image Theorem. For $Y \subsetneq S$ and $x, y \in \mathcal{M}(S)$ we denote $\sigma_Y(x,y) = \sum_{Y' \subseteq Y} \{\{d_{Y'}(x,y)\}\}_{3L_0}$ the contribution made to the Distance Formula by subsurfaces of Y. Notice that σ_Y is a coarsely Lipschitz function (in both variables).

Let $\{w_i\}_{i \le n}$ be a sample path. Consider a geodesic γ from $\pi_S(1)$ to $\pi_S(w_n)$. By Theorem 7.3.2, we can assume $d_{Haus}(\gamma, \{\pi_S(w_i)\}_{i \le n}) \le K_1 \log(n)$ (in $\mathcal{C}(S)$). Also, we can assume that for each $i, j \le n$ with $|i-j| \ge K_1 \log(n)$ we have $d_S(w_i, w_j) \ge |i-j|/K_1$ by Lemma 7.3.5. Both results apply in view of the linear progress in $\mathcal{C}(S)$, Theorem 7.4.7.

Let Y be any subsurface. If $d_S(\gamma, \partial Y)$ is positive then the projection of γ on any subsurface of Y is bounded by L_0, so assume that this is not the case. The idea is to split the random path into an initial, central and final part. The initial and final part will make bounded contribution to σ_Y in view of the Bounded Geodesic Image Theorem, while the contribution of the central part can be estimated using that σ_Y is coarsely Lipschitz.

Let $p \in \gamma$ be so that $d_S(p, \partial Y) \le 10$ and let w_i be so that $d_S(p, w_i) \le K_1 \log(n)$. Consider initial and final subgeodesics γ_1, γ_2 of γ at distance at least $K_1 \log(n) + 100$ from p. Then an initial subpath of the random path will be close to γ_1, while a final subpath will be close to γ_2, that it to say for K_2 large enough we have that if $|i-j| \ge K_2 \log(n)$ then $d(\gamma_l, w_j) \le K_1 \log(n)$, where $l = 1$ if $j < i$ and $l = 2$ if $j > i$. The Bounded Geodesic Image Theorem implies that the diameter of the projection of the γ_l's on any $Y' \subseteq Y$ is bounded, and likewise for all geodesics from w_j to the closest point to w_j in the corresponding γ_l, if $|i-j| \ge K_2 \log(n)$. Thus, we can give a uniform bound on $diam(\pi_{Y'}(\{w_j\}_{0 \le j \le i - K_2 \log(n)}))$ and $diam(\pi_{Y'}(\{w_j\}_{i + K_2 \log(n) \le j \le n}))$ for any subsurface $Y' \subseteq Y$. Define $\sigma'_Y(x,y) = \sum_{Y' \subseteq Y} \{\{d_{Y'}(x,y)\}\}_{L_0}$, which is again coarsely Lipschitz. We can now make the estimate

$$\sigma_Y(1, w_n) \le \sigma'_Y\left(1, w_{i-K_2\log(n)}\right) + \sigma'_Y\left(w_{i-K_2\log(n)}, w_{i+K_2\log(n)}\right)$$

$$+\sigma'_Y\left(1, w_{i+K_2\log(n)}\right) \le K_3 \log(n). \qquad \square$$

We can now conclude the proof of Theorem 7.4.2. Fix $n \ge 2$ (the constants C_i below do not depend on n). Consider a sample path $\{w_i\}_{i \le n}$ so that $1, w_n$ is $C_0 \log(n)$-bounded and for each i, j with $|i-j| \ge C_0 \log(n)$ we have $d_S(w_i, w_j) \ge |i-j|/C_0$, see Lemma 7.4.12 and Lemma 7.3.5. First of all, any point on $[1, w_n]$ has distance at most $C_1 \sqrt{n \log(n)}$ from the sample path by Proposition 7.4.9. Suppose that there is

w_i so that $d(w_i, [1, w_n]) > C_1\sqrt{n \log(n)}$. Notice that there is a point p on $[1, w_n]$ so that

$$d(p, \{w_j\}_{j<i}), d(p, \{w_j\}_{j>i}) \leq C_1\sqrt{n \log(n)},$$

for example the first point along $[1, w_n]$ that is $C_1\sqrt{n \log(n)}$-close to $\{w_j\}_{j>i}$. In particular, there are $j_0 < i < j_1$ so that $d(w_{j_0}, w_{j_1}) \leq 2C_1\sqrt{n \log(n)}$ and $d(w_{j_0}, [1, w_n]) \leq C_1\sqrt{n \log(n)}$. From Lemma 7.3.5 one easily gets (as in the proof of Theorem 7.3.2) that $j_1 - j_0 \leq C_2\sqrt{n \log(n)}$. From this, the inequality $d(w_i, [1, w_n]) \leq C_3\sqrt{n \log(n)}$ follows easily.

In view of Corollary 7.4.11 and Lemma 7.4.12, the statement for geodesics follows from the one for hierarchy paths.

7.5 Random projections estimate

The reason why the proof of Lemma 7.4.12 is different from that of Lemma 7.3.4 is just to illustrate two different techniques to get projection estimates for random points. We think it is worthwhile to state fact (1) in Lemma 7.3.4 as well as its counterpart in the mapping class group as a separate lemma.

Lemma 7.5.1 (Small Random Projections). *Let G be a non-trivial relatively hyperbolic group (resp. mapping class group of a surface of complexity at least 2). Consider a random walk generated by a symmetric probability measure whose finite support generates G. Then there exists K with the following property. If H is a left coset of a peripheral subgroup (resp. proper subsurface) then for all $l \geq 0$ and all $n \geq 1$ we have*

$$\mathbb{P}\left[d(\pi_H(1), \pi_H(X_n)) \geq l\right] \leq Ke^{-l/K}.$$

Proof. We proved this fact within Lemma 7.3.4 for relatively hyperbolic groups. All the properties of projections on peripheral sets that we used in the proof have analogues for subsurface projections. In particular, the same proof almost goes through, see below, except that an extra argument is needed in the part where we used Corollary 3.1.2. In fact, the Behrstock Inequality for subsurface projections holds for *overlapping* subsurfaces, so not for all pairs of disjoint subsurfaces. To solve this problem we can use the following fact [18, Section 4.3]. There exists a finite-index subgroup G' of the mapping class group so that whenever two distinct subsurfaces are in the same G'-orbit they overlap. Notice also that there are finitely many G'-orbits.

In order to show the lemma, we can follow the proof of Lemma 7.3.4(1) verbatim until the definition of w_3, where $\mathcal{H}$ now denotes the collection of all proper subsurfaces of S and we set $p = 1$. The last part of the proof can be substituted by the following argument.

Choose two distinct subsurfaces H_1^j, H_2^j from each G'-orbit. We claim that substituting w_2 by a suitable word $w_2' = u_2 v_2$ of the same length we can make sure that $d_{g(w_1u_2)H_i^j}(\partial H, g(w_1 w_2' w_3))$ is larger than C as in the Behrstock Inequality, where

1. u_2 is chosen so that $w_1 u_2$ represents an element of G',
2. j is chosen so that H_1^j, H_2^j are in the same G'-orbit as H,
3. $i \in \{1, 2\}$ is chosen so that $g(w_1u_2)H_i^j \neq H$.

Notice that u_2 can be chosen from a finite list of words as G' has finite index in the mapping class group, so that there is a uniform bound on the distance of $g(u_2)$ from 1 and hence from $g(w_1)$ to $g(w_1u_2)$. If v_2 represents an element in the stabiliser of H_i then (up to an additive constant)

$$d_{g(w_1u_2)H_i}(\partial H, g(w_1w_2'w_3)) = d(\pi_{H_i}(g(w_1u_2)^{-1}\partial H), g(v_2)\pi_{H_i}(g(w_3))).$$

In particular, for an appropriate choice of v_2, we have that $\pi_{H_i}(g(w_3))$ is far from $\pi_{H_i}(g(w_1u_2)^{-1}\partial H)$.

In particular, $d_H(g(w_1w_2'w_3), g(w_1u_2)\partial H_i^j)$ is bounded by the Behrstock Inequality. Also, $d = d_H(g(w_1u_2)\partial H_i^j, g(w_1u_2))$ can be uniformly bounded as $\pi_H(\partial H_i^j)$ coincides with the projection of a fixed marking m_i^j whose set of base curves contains ∂H_i^j. As π_H is coarsely Lipschitz and there are finitely many H_i^j's, we can give a bound d. To sum up, there is a uniform bound on $d_H(g(w_1w_2'w_3), g(w_1u_2))$ and hence on $d_H(g(w_1w_2'w_3), g(w_1))$ as the word u_2 has bounded length. Hence $d_H(1, g(w_1w_2'w_3)) \in [l - K_3, l + K_3)$, for a suitable K_3.

So, we constructed for any word w of length n so that $d_H(1, g(w)) \geq l$ another word $w' = w_1w_2'w_3$ of length n so that $d_H(1, g(w')) \in [l - K_3, l + K_3)$, and the map $w \mapsto w'$ is easily seen to be bounded-to-1. This then gives the desired inequality as the support of μ is finite and hence for each s, s' in the support $\mu(s)/\mu(s')$ is uniformly bounded. □

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