ASPECTS OF FOUR-DIMENSIONAL BLACK HOLES WITH HAIR: STABILITY AND ENTROPY CONSIDERATIONS

ELIZABETH WINSTANLEY

ST. HUGH'S COLLEGE, OXFORD

Trinity Term, 1996

Thesis submitted for the Degree of Doctor of Philosophy
in the University of Oxford

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ABSTRACT

In recent years, a large number of black holes have been presented as candidates for an evasion of the "no-hair" conjecture. These examples typically have two features: a non-Abelian gauge field and instability. A large part of this thesis is devoted to a detailed study of the Einstein-Yang-Mills-Higgs (EYMH) black holes, including the analytic proof of the evasion of the "no-hair" theorem in this case and proving that the black holes are unstable. We also consider an example of a "hairy" black hole *not* involving a non-Abelian gauge field, which arises in a higher derivative model of gravity derived from string theory, and prove analytically how the "no-hair" theorem is evaded.

The rest of this thesis is concerned with the thermodynamics and quantum field theory of these black holes. In a first order approximation to the unknown theory of quantum gravity, we calculate the entropy of the "hairy" black holes. This turns out to be divergent, and parts of the divergences are attributed to the effect of hair on information loss processes occurring as the black hole evolves in time. We pursue this idea further by making a preliminary estimate of the magnitude of the quantum de-coherence effects on the state of the quantum field as time proceeds. These processes may be of interest phenomenologically in the future.

The extension of the theory to non-static geometries is also discussed, by describing the results of bringing rotation into the picture. We prove that the Hartle-Hawking state is not regular everywhere outside the event horizon of a Kerr black hole, with the result that quantum field theory on rotating black hole space-times is more complicated than on static geometries.

Contents

Acknowledgements

First and foremost, I would like to express a deep gratitude to all those, without whose supervisory expertise, time and consideration I would not have been able to complete this thesis. Especially I would like to thank Dr. N. E. Mavromatos, for his limitless enthusiasm, for having the project idea in the first place, and for providing me with so many rewarding academic opportunities. Dr. A. C. Ottewill supervised the work on Kerr black holes, and I am appreciative of the way he introduced me to research work and sparked my interest in black hole physics. I would also like to thank him for his continuing interest in my work. I am grateful to Prof. R. Penrose for starting off the whole process and Dr. S. Sarkar for help with administration.

Secondly, I would like to thank everyone with whom I have had helpful discussions and guidance. In particular, I thank Profs. J. Ellis and D. V. Nanopoulos for the ideas and collaboration which led to the work of chapter 4. I am also grateful to Profs. K. Tamvakis and J. Rizos and Miss P. Kanti for finding the black hole solutions described in chapter 5, and for their collaboration in this area. The people who have helped me informally are too numerous to list here: I mention only Dr. G. W. Gibbons' helpful comments and thank Prof. J. D. Bekenstein for a useful correspondence. I am also grateful to all the organizations which have given me an opportunity to present my work, including the relativity group at the Department of Applied Mathematics and Theoretical Physics, Cambridge; the Institute of Physics High Energy Particle Physics Group and the Theoretical Particle Physics Group here in Oxford.

On the practical side of things, I have been supported financially by the Engineering and Physical Sciences Research Council, and am particularly grateful to them for generous support while I was at CERN. I thank St. Hugh's College for the provision of a Searle Graduate Scholarship, and for assistance with travel, together with the Department of Physics, Oxford. I should also mention the Ettore Majorana Centre for Scientific Culture, Sicily and thank them for awarding me the J. J. Sakurai scholarship at the International School of Sub-Nuclear Physics in 1995. I am grateful to the Theory Division, CERN, for hospitality during the summer of 1994 and at the end of 1995, and for providing a stimulating and exciting environment in which to study. I would like to thank the EPSRC for the opportunity at the British Universities Summer School in Theoretical Elementary Particle Physics in 1994 to learn about current topics of research in particle physics. Finally I would like to thank, in anticipation, Oriel College, for the opportunity to continue my researches.

Introduction

The study of hairy black holes is a fairly new field, and is now of great interest due to the plethora of such objects which have been discovered in recent years. In this thesis we address two fundamental questions regarding hairy black holes.

1. Firstly, in what sense is the "no-hair" conjecture violated, and should it now be abandoned?

2. Secondly, what are the physical consequences of the existence of hair?

We begin, in chapter 1, by reviewing critically the history and current status of the "no-hair" conjecture. We also outline a recent modern "proof" of a no-hair theorem [1], which applies to a restricted class of scalar field theories. This analysis will be important for later considerations. In this chapter we introduce one of the main, important ideas of the thesis, namely that since many of the examples of hairy black holes are unstable, the "spirit" of the no-hair conjecture may be regarded as intact, although its "letter" has been violated.

The hairy black holes known to date typically involve a non-Abelian Yang-Mills gauge field coupled to gravity. In chapter 2 we begin our study of one such example, the $SU(2)$ Einstein-Yang-Mills-Higgs (EYMH) system. Black hole solutions of the field equations were found numerically in [2], but otherwise this system has been little studied to date. We review the field equations and the properties of these numerical solutions, before proving analytically the evasion of the no-hair theorem in this case [3]. This is important because it is the first *analytical* evidence for the existence of these black holes, as well as providing some valuable physical insight.

One of the important, and previously unanswered, questions regarding the EYMH black holes was whether or not they are stable. This is a particularly pertinent question in the light of the discussion of chapter 1. We answer this question in chapter 3, showing that in fact the black holes are unstable [4]. The instabilities fall into two types: gravitational and sphaleronic. The sphaleronic stabilities are topological in nature, and are similar to those exhibited in the flat space "sphaleron". The analysis of this chapter uses various techniques: a variational method; analyticity and continuity arguments; and an important application of catastrophe theory.

We address some of the deep issues inherent in question 2 in chapter 4. We study the quantum field theory of a scalar field perturbation on the hairy black hole space-times, and calculate the entropy and time-evolution of the density matrix describing the state of the field [3, 5]. The entropy is found to be divergent. Part of this divergence cannot be absorbed into a renormalization of the gravitational constant, and is attributed to information loss processes in the black hole, these processes being affected by the presence of hair. Using the Gibbs formula for the density matrix as a first approximation, and a new proposal for the definition of a suitable time co-ordinate for the system, we find terms in the evolution equation of the density matrix which do not correspond to the usual unitary evolution of quantum mechanics.

In chapter 5 we discuss a string-inspired model of gravity which involves a higher derivative Gauss-Bonnet curvature term. Even without a non-Abelian gauge field, hairy black hole solutions exist in this model, and are described briefly. We also prove analytically how the no-hair theorem is evaded in this situation, and draw similarities between the mechanism here and in the EYMH case [6].

Up to this point, we have considered only spherically symmetric geometries. We introduce rotation in chapter 6, and consider the Kerr space-time in order to separate the effects of rotation explicitly, without having hair present to cloud the picture. The quantum field theory on this background geometry is more complex than in the static case, and we prove, using an important approach based on basic physical principles rather than detailed calculations, that the Hartle-Hawking state is pathological [7, 8].

The appendices contain background material on catastrophe theory (appendix A); the field equations for the string-inspired gravity model (appendix B); and some background on the definition and properties of the standard vacua on Kerr space-time (appendix C).

The greater part of the contents of this thesis is the author's original work, and has appeared in the following publications:

Chapter 3, sections 2.3 and 4.2 [3] N. E. Mavromatos and E. Winstanley *Aspects of hairy black holes in spontaneously broken Einstein-Yang-Mills systems: Stability analysis and entropy considerations* Physical Review **D53** 3190 (1996)

Sections 3.2, 3.3 [4] E. Winstanley and N. E. Mavromatos *Instability of hairy black holes in spontaneously broken Einstein-Yang-Mills-Higgs systems* Physics Letters **B352** 242 (1995).

Chapter 4, especially section 4.5 [5] J. Ellis, N. E. Mavromatos, D. V. Nanopoulos and E. Winstanley *Quantum de-coherence in a four-dimensional black hole background* submitted to Modern Physics Letters **A** (1996)

Chapter 5 [6] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis and E. Winstanley *Dilatonic black holes in higher curvature string gravity* submitted to Physical Review **D** (1996)

Chapter 6 [7] A. C. Ottewill and E. Winstanley *General results on the renormalized stress tensor in Kerr space-time* (to appear)

Chapter 6 [8] E. Winstanley *Quantum field theory in black hole space-times* Unpublished dissertation. Oxford University (1993)

Chapter 1 and sections 2.1 and 2.2 contain review material, presented in a manner differing from that currently available in the literature. The analysis of section 2.3 is original. Chapter 3 is original in its entirety, as is chapter 4, with the exception of sections 4.1 and 4.4, which contain introductory discussions. The analysis of section 5.2 is the author's, although the black holes discussed in the rest of chapter 5 were found numerically by Kanti, Tamvakis and Rizos. The work of chapter 6 is original, apart from the introductory sections 6.1, 6.2.

Notation and Conventions

In common with much of the literature, we use the sign conventions of Misner, Thorne and Wheeler [9]; in particular the metric has signature $(-,+,+,+)$. We also use Planck units in which $G_N = \hbar = c = k_B = 1$ throughout, with the exception of sections 4.1.3, 4.3.2 and 4.4.3, where we have explicitly retained the Newton constant G_N. Every attempt has been made to use standard notation wherever possible. This means that occasionally the same symbol will denote two different quantities, although we have sometimes strayed from standard notation where confusion would otherwise result.

Chapter 1

Status of the "No-Hair" Conjecture

We begin by describing what is meant by "hair" and review critically the history of the "no-hair" conjecture. We also discuss whether, in the context of the recent discovery of "hairy" black holes, the no-hair conjecture should now be abandoned. It should be stressed from the outset that this thesis is concerned with *classical* rather than *quantum* hair [10]. Quantum hair is not covered by the "no-hair" conjecture, and we shall not consider here the physical consequences of its existence [10].

1.1 What is hair?

The phrase "a black hole has no hair" was coined in the early 1970s by John Wheeler [11]. He was referring to the remarkable simplicity of black hole solutions of general relativity. Powerful mathematical theorems (see [12] for a review) had shown that the only black hole solutions of Einstein-Maxwell theory are those of the Kerr-Newman type, the geometry being characterized by just three quantities: mass, angular momentum and charge. This led to the formulation of the following conjecture:

Conjecture 1.1 (Israel-Carter conjecture) *All exteriors of stationary, bare, single black holes are of the Kerr-Newman type; they have associated with them no independent properties other than mass, charge and angular momentum.*

The term "bare" in the above conjecture does not imply that the black holes are solutions of the vacuum Einstein equations (although the conjecture is true for such black holes). However, it does exclude the possibility of having dust, radiation or field sources outside the event horizon. This conjecture is now known as the *"no-hair"* conjecture.

The motivation for the term *"hair"* is physical. All that remains visible of a drowning victim is the hair, floating on the surface of the water, the rest of the body being submerged. According to the Israel-Carter conjecture, a particle which vanishes down the event horizon of a black hole leaves no trace apart from its mass, angular momentum, and charge. Thus the black hole has "no hair". In the case of a black hole formed by gravitational collapse, all other characterizing features must either disappear down the event horizon or be radiated away to infinity.

"Hair" is therefore any property of a black hole which violates the Israel-Carter conjecture, and we shall focus here on classical, rather than quantum, hair. Hair can be considered in two complementary ways.

1. Firstly, "hair" is thought of as additional structure outside the event horizon. In other words, the geometry outside the event horizon does not coincide with the Kerr-Newman geometry. The existence of hair of this type requires parameters other than simply mass, angular momentum and charge in order to characterize completely the space-time.

2. Secondly, "hair" refers to additional, conserved quantum numbers carried by the black hole.

For Kerr-Newman black holes, the mass, angular momentum and charge act in both these roles: knowledge of their values completely determines the geometry and they are also conserved quantum numbers. For hairy black holes, the roles diverge a little. For example, the EYMH black holes of chapter 2 possess additional structure as described in (1) but no global charge.

For the remainder of this thesis we shall be concerned primarily with viewpoint (1) and shall not focus attention on viewpoint (2), despite the considerable interest in the literature, particularly in the context of non-critical string theory (see for example [13, 14, 15, 16]). At this point we should stress that we are concerned only with *classical* hair, i.e. solutions of the classical coupled Einstein and matter field equations, rather than the quantum hair associated with string theory. This should be borne in mind when it comes to the discussion of chapter 4.

1.2 History of the no-hair conjecture

The initial, affirmative, work on the no-hair conjecture was completed in the early 1970s, inspired by Wheeler's original remark. The seminal paper from this era is that of Bekenstein [17], who ruled out non-trivial hair of a scalar field ϕ having potential V such that $\frac{\partial V}{\partial \phi} \geq 0$. As in all such theorems, "trivial" field configurations are allowed, i.e. solutions where $\phi \equiv$ constant, and the geometry is that of Schwarzschild space-time. Bekenstein's theorem implies that black holes cannot be ascribed quantities such as baryon or strangeness number, in accordance with our second notion of "hair". The remarkable thing about [17] is the simplicity of the argument, which uses only the scalar field equation and the geometric properties of a black hole space-time at the event horizon and at infinity. It does not require detailed analysis of possible solutions or the Einstein equations.

Bekenstein's original theorem confirmed the Israel-Carter conjecture for scalar and massive vector fields. This result, coupled with a similar one for neutrino fields [11], led to a fairly widespread belief that the Israel-Carter conjecture was in fact true for all physically relevant fields, and the subject went quiet.

The question of "hair" was re-opened in the late 1980s. Bizon [18] was attempting to find a proof of the no-hair theorem for $SU(2)$ Einstein-Yang-Mills (EYM) systems when Bartnik and McKinnon [19] announced their discovery of regular, particle-like solutions of the theory. Such solitons do not exist in Einstein-Maxwell theory, or indeed, in Yang-Mills theory in flat space [20], and Bizon investigated the possibility of non-trivial black hole solutions in the $SU(2)$ EYM theory. The "coloured" black hole solutions he found were to be the first of many such violations of the no-hair conjecture.

Bizon's work triggered not only research into various hairy black hole solutions and their properties (to which we return shortly) but also further efforts to prove "no-hair" theorems and derive conditions for the existence of hair. Heusler and Straumann [21] used general scaling arguments to derive conditions on the Lagrangian of a general field theory necessary for hair to exist. These conditions are, of course, satisfied by the EYM Lagrangian. The theorems that have been proved concentrate largely on various types of scalar field theories. Sudarsky [22] used the Einstein and field equations to extend the original result of [17] to scalar fields with a positive semi-definite potential, i.e. $V \geq 0$. Bekenstein [1] extended his result even further, proving that a scalar field with a very general Lagrangian, and minimally coupled to gravity, cannot sustain non-trivial hair. In the next section we outline his proof as it will be important for considerations in later chapters (see especially sections 2.3 and 5.2).

The main assumption of [1] is that the matter field should satisfy the *weak energy condition*, namely

$$T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq 0 \tag{1.1}$$

for all time-like vectors ξ^{μ}. In other words, the energy density of matter measured by an observer with four-velocity ξ^{μ} is positive. From this assumption, [1] uses a physical argument based on the conservation of the stress-energy tensor.

The theorem, in common with much recent work, focuses on spherically symmetric geometries only. A notable exception is [23], where the authors found that the inclusion of angular momentum into the picture made little difference as far as classical hair is concerned. We shall discuss some of the subtleties involved in working with rotating black holes in chapter 6, but shall otherwise be concerned only with the spherically symmetric case. The technical simplification afforded by this choice should not be underestimated, while the pertinent physics is unchanged.

We note, in passing, an interesting result due to Heusler [24], who considered the dominant energy condition and strong energy condition. The *dominant energy* condition states that for all future directed, time-like vectors ξ^{μ}, the vector

$$-T^{\mu}_{\nu}\xi^{\nu} \tag{1.2}$$

should be a future directed time-like or null vector. Physically this means that the speed of the energy flow of matter seen by an observer should be less that the speed of light. The dominant energy condition implies the weak energy condition described above. The *strong energy condition*, despite its name, does not necessarily imply either of the other two energy conditions. It states that, for all unit time-like vectors ξ^μ,

$$R_{\mu\nu}\xi^\mu\xi^\nu \geq 0. \tag{1.3}$$

Heusler proved that for spherically symmetric black holes having a time-like Killing vector ξ^μ, and coupled to matter which satisfies the dominant energy condition for ξ but violates the strong energy condition for ξ, only the Schwarzschild geometry is possible. Thus hairy black holes must either satisfy both conditions or neither.

In fact, it is not necessarily the case that even the weak energy condition is satisfied by physically relevant matter. For example, it will be violated for a scalar field non-minimally coupled to gravity [25]. In this case, the proof of [1] is no longer valid. This gap has been filled in [25, 26], so that no-hair theorems now exist for a wide variety of non-minimally coupled scalar field theories.

1.3 "Modern" proof of a no-hair theorem

We now outline Bekenstein's "modern" proof of the no-hair theorem for scalar fields minimally coupled to gravity [1]. The analysis should be compared to that of sections 2.3 and 5.2 where we show how the no-hair theorem is evaded for the EYMH and Gauss-Bonnet black holes. In this section we draw the reader's attention to those aspects of the analysis which break down in each of the hairy black hole cases we shall consider later.

Consider for simplicity a single scalar field ϕ minimally coupled to gravity. The theorem can easily be extended to a multiplet of scalar fields [1]. For our purposes, a single scalar field will suffice. The action is

$$\mathcal{S}_\phi = -\int \mathcal{E}(\mathcal{B},\phi)\sqrt{-g}\, d^4x \tag{1.4}$$

where $\mathcal{E}$ is a function and

$$\mathcal{B} = g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}. \tag{1.5}$$

We make no assumptions on the functional form of $\mathcal{E}$, except that it does not contain the curvature in any form, so that the coupling to gravity is minimal. This assumption is valid for the EYMH black holes of chapter 2, but is violated in the Gauss-Bonnet gravity case (chapter 5).

The energy-momentum tensor corresponding to the action (1.4) is

$$T_{\mu\nu} = -\mathcal{E}g_{\mu\nu} + 2\frac{\partial\mathcal{E}}{\partial\mathcal{B}}\phi_{;\mu}\phi_{;\nu}. \tag{1.6}$$

Consider an observer with a four-velocity U^α, where

$$U_\alpha U^\alpha = -1, \tag{1.7}$$

so that the observer moves along a time-like world-line. The observer sees the following energy density ρ

$$\begin{aligned} \rho &= T_{\mu\nu} U^\mu U^\nu \\ &= \mathcal{E} + 2\frac{\partial \mathcal{E}}{\partial \mathcal{B}} \left(\phi_{;\mu} U^\mu\right)^2 . \end{aligned} \tag{1.8}$$

Suppose that the space-time possesses a time-like Killing vector (this will be the case for a static black hole geometry), and furthermore that the observer moves along this Killing vector. Then

$$\phi_{;\mu} U^\mu = 0 \tag{1.9}$$

and

$$\rho = \mathcal{E}. \tag{1.10}$$

Hence, in order for the observer to measure a positive energy density, it must be the case that

$$\mathcal{E} \geq 0. \tag{1.11}$$

This key condition (which is equivalent to the weak energy condition (see section 1.2)) is also violated in the Gauss-Bonnet case (section 5.2), but holds for the EYMH black holes (section 2.3). Now consider a second observer, moving relative to the first (Killing vector) observer, with a three-velocity $\boldsymbol{v}$. In the instantaneous rest-frame of the first observer, the four-velocity of the second observer has components

$$U^\mu = \frac{1}{(1-\boldsymbol{v}^2)^{\frac{1}{2}}}(1, \boldsymbol{v}). \tag{1.12}$$

In the limit $|\boldsymbol{v}| \to 1$, the second term in (1.8) dominates the energy density ρ, and in order for this quantity to be non-negative, we have

$$\frac{\partial \mathcal{E}}{\partial \mathcal{B}} \geq 0. \tag{1.13}$$

In order to derive a contradiction, assume that there exists a self-consistent, asymptotically flat solution of the coupled Einstein and scalar field equations representing a static, spherically symmetric, black hole. Outside the (outermost) event horizon $r = r_h$, the metric takes the form

$$ds^2 = -e^\Gamma dt^2 + e^\Lambda dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) \tag{1.14}$$

where Γ and Λ are functions of r alone. For a non-trivial scalar field, ϕ depends on r only. The action (1.4) is invariant under co-ordinate transformations, and hence the energy-momentum tensor is conserved:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{1.15}$$

The r component of this equation reads

$$\left(\sqrt{-g}T^r_r\right)' = \frac{1}{2}\sqrt{-g}\frac{\partial g_{\alpha\beta}}{\partial r}T^{\alpha\beta} \tag{1.16}$$

where $'$ denotes $\frac{\partial}{\partial r}$. Since the space-time is static and spherically symmetric, T^ν_μ must be diagonal, and further

$$T^\theta_\theta = T^\varphi_\varphi. \tag{1.17}$$

Equation (1.16) then becomes

$$\left(e^{\frac{1}{2}(\Gamma+\Lambda)}r^2T^r_r\right)' = \frac{1}{2}e^{\frac{1}{2}(\Gamma+\Lambda)}r^2\left[\Gamma' T^t_t + \Lambda' T^r_r + \frac{4}{r}T^\theta_\theta\right]. \tag{1.18}$$

The terms containing Λ' cancel to give

$$\left(e^{\frac{1}{2}\Gamma}r^2T^r_r\right)' = \frac{1}{2}e^{\frac{1}{2}\Gamma}r^2\left[\Gamma' T^t_t + \frac{4}{r}T^\theta_\theta\right]. \tag{1.19}$$

From the form (1.6) of the stress tensor and the symmetries of the geometry,

$$T^t_t = T^\theta_\theta = -\mathcal{E}, \tag{1.20}$$

which will turn out not to hold for the two hairy black hole scenarios in chapters 2 and 5. Thus equation (1.19) simplifies down to

$$\left(e^{\frac{1}{2}\Gamma}r^2T^r_r\right)' = -\left(e^{\frac{1}{2}\Gamma}r^2\right)'\mathcal{E}. \tag{1.21}$$

The first step is to integrate (1.21) from the event horizon r_h to a generic radius r. At the horizon $e^\Gamma = 0$ and, further, physical invariants such as

$$T_{\alpha\beta}T^{\alpha\beta} = \left(T^t_t\right)^2 + \left(T^r_r\right)^2 + \left(T^\theta_\theta\right)^2 + \left(T^\varphi_\varphi\right)^2 \tag{1.22}$$

must be finite, otherwise the horizon will not be regular. Thus T^r_r and $T^t_t = -\mathcal{E}$ are finite at $r = r_h$. This means that the boundary term at the horizon vanishes, leaving

$$T^r_r(r) = -\frac{e^{-\frac{\Gamma}{2}}}{r^2}\int_{r_h}^r \left(r^2 e^{\frac{\Gamma}{2}}\right)'\mathcal{E}\,dr. \tag{1.23}$$

Secondly, we may explicitly carry out the differentiation in (1.21) to give

$$(T^r_r)'(r) = -\frac{e^{-\frac{\Gamma}{2}}}{r^2}\left(r^2 e^{\frac{\Gamma}{2}}\right)'\left(\mathcal{E} + T^r_r\right). \tag{1.24}$$

From the fundamental form of the stress tensor (1.6) we have

$$\mathcal{E} + T^r_r = 2e^{-\Lambda}\frac{\partial\mathcal{E}}{\partial\mathcal{B}}\left(\phi_{;r}\right)^2. \tag{1.25}$$

We now consider the behaviour of the stress tensor in the two asymptotic regions of the space-time. Firstly, near the event horizon, e^{Γ} must be positive. Since $e^{\Gamma} = 0$ at $r = r_h$, the function $r^2 e^{\Gamma/2}$ must grow with r sufficiently close to the horizon. Since we already have that $\mathcal{E} \geq 0$ everywhere, it follows from (1.23) that $T_r^r \leq 0$ for r sufficiently close to r_h. Similarly, since

$$\frac{\partial \mathcal{E}}{\partial \mathcal{B}} \geq 0, \tag{1.26}$$

from (1.25) we have $T_r^r + \mathcal{E} \geq 0$, giving $(T_r^r)' \leq 0$ for r sufficiently close to the event horizon.

At infinity, $e^{\Gamma} \to 1$ and hence, from (1.24), we have that $(T_r^r)' \leq 0$. In order to assess the behaviour of T_r^r from (1.23), we first need to know how $\mathcal{E}$ behaves, and for this we turn to the Einstein equations.

The relevant Einstein equations are

$$\begin{aligned} e^{-\Lambda}\left(\frac{1}{r^2} - \frac{\Lambda'}{r}\right) - \frac{1}{r^2} &= 8\pi T_t^t = -8\pi\mathcal{E}, && (1.27) \\ e^{-\Lambda}\left(\frac{1}{r^2} + \frac{\Gamma'}{r}\right) - \frac{1}{r^2} &= 8\pi T_r^r. && (1.28) \end{aligned}$$

The first of these equations, (1.27), can be integrated immediately to give

$$e^{-\Lambda} = 1 - \frac{2M_B}{r} - \frac{8\pi}{r}\int_{r_h}^{r} \mathcal{E} r^2 \, dr \tag{1.29}$$

where M_B is a constant of integration. Imposing the condition $e^{\Lambda} \to \infty$ at the event horizon $r = r_h$ fixes M_B as

$$2M_B = r_h, \tag{1.30}$$

so that it can be interpreted as the "bare" mass of the black hole. Asymptotic flatness of the space-time implies that $\mathcal{E} \sim O(r^{-3})$ as $r \to \infty$, so that $e^{\Lambda} \to 1$.

The consequence of this for T_r^r is that the integral in (1.23) converges and $|T_r^r| \sim O(r^{-2})$ as $r \to \infty$. Since $(T_r^r)' \leq 0$ as $r \to \infty$, it must be the case that T_r^r is positive and decreasing at infinity. From our earlier analysis, T_r^r is negative and decreasing close to the horizon. Thus, there is some interval $[r_a, r_b]$ on which $(T_r^r)' \geq 0$, with T_r^r changing sign from negative to positive at r_c, where $r_a < r_c < r_b$.

This conclusion is incompatible with the Einstein equations. Equation (1.27) implies that $e^{\Lambda} \geq 1$ everywhere outside the event horizon. The second Einstein equation can be rewritten as

$$\begin{aligned} \frac{e^{-\frac{\Gamma}{2}}}{r^2}\left(e^{\frac{\Gamma}{2}} r^2\right)' &= \left[4\pi r T_r^r + \frac{1}{2r}\right] e^{\Lambda} + \frac{3}{2r} \\ &\geq 4\pi r T_r^r e^{\Lambda} + \frac{2}{r}. && (1.31) \end{aligned}$$

In the interval $[r_c, r_b]$, from above, $T^r_r \geq 0$ which implies that

$$\frac{e^{-\frac{\Gamma}{2}}}{r^2}\left(e^{\frac{\Gamma}{2}}r^2\right)' \geq 0 \tag{1.32}$$

in this interval also. Then, from equation (1.24), we have $(T^r_r)' \leq 0$ in the interval $[r_c, r_b]$. However, we have already observed that $(T^r_r)' \geq 0$ in the larger interval $[r_a, r_b]$. Thus we have a contradiction, unless $\mathcal{E} \equiv 0$ everywhere outside the event horizon, so that all components of the stress tensor vanish identically. In other words, the scalar field ϕ is constant and

$$\mathcal{E}(0, \phi) = 0. \tag{1.33}$$

There must be at least one such value for ϕ in order for a trivial solution of the scalar field equation to exist in flat space, this solution forming the asymptotic boundary conditions for the space-time. Since $T_{\mu\nu} \equiv 0$, the space-time is vacuum and must therefore coincide with the Schwarzschild geometry.

1.4 Evading the no-hair conjecture

Bizon's [18] discovery of hairy black holes in $SU(2)$ Einstein-Yang-Mills theory has led to a succession of other examples of hair being found. Most of these examples have in common a non-Abelian gauge field. The non-Abelian nature of the field is vital to the existence of hair. For example, in [17] the existence of hair for the Abelian Proca field was ruled out, although non-Abelian Proca fields may possess hair [2]. This has been linked to the mathematical structure of the gauge group in [23], and discussed from a more physical point of view in [2]. In section 2.3 we argue that the gauge field repulsion balances the attraction of the gravitational field, allowing hair to exist. A similar mechanism arises in theories of gravity involving a Gauss-Bonnet term, which are discussed in chapter 5. This is one of the few examples of hairy black holes *not* involving a Yang-Mills field, and the Gauss-Bonnet term plays a similar role to the Yang-Mills field strength in the analysis of that chapter.

However, the vast majority of the hairy black holes in the literature contain an $SU(2)$ gauge field, either on its own, as in the case of coloured black holes (either with [27] or without [18] a cosmological constant) and the Proca [2] or Einstein-Skyrme system [28]; or coupled to another matter field, whether it be a Higgs [2] or dilaton field [29]. Work has concentrated on $SU(2)$ theories, although similar solutions exist in the $SU(3)$ case [30]. Where the Yang-Mills field is coupled to an additional field, the Yang-Mills field is essential for the existence of hair (there is no Higgs hair without it). For this reason, the gauge field hair is known as *primary hair* [10], since it exists independently of any other matter fields. The primary hair acts as a source for the scalar field hair, which is therefore called *secondary hair.* It is the primary hair which truly violates the no-hair conjecture, for without it there would be no secondary hair.

Note that, although these hairy black hole solutions are known only numerically, their existence has been rigorously established [31, 32]. We shall present analytical evidence for the existence of the EYMH and Gauss-Bonnet black holes in, respectively, sections 2.3 and 5.2. However, their physical relevance depends to some extent on their stability. The coloured black hole solutions have been shown to be unstable gravitationally [20] (see chapter 3 for a full, technical explanation of what this means) and also to possess topological instabilities [33], similar to those found for the flat-space "sphaleron" [34, 35]. The Einstein-Yang-Mills-Higgs black holes are shown to be unstable in a similar fashion in chapter 3, where we make a full stability analysis. Some sort of topological instability seems to be typical of the hairy black holes; a notable exception is the Einstein-Skyrme system, which has been shown to be at least linearly stable [36]. This instability is not unexpected if we return to our physical picture of the gauge and gravitational forces balancing each other. If, say, the gauge field force were a little stronger, then all the energy would be radiated to infinity, whereas if the gravitational field were a little stronger, everything would be absorbed down the event horizon. Thus we would expect the equilibrium solutions to be unstable.

What is the importance of hairy black holes, apart from proving a long-standing conjecture wrong? In this thesis we focus primarily on the EYMH system, whose physical relevance can hardly be doubted, since it forms part of the standard model of particle physics. These black holes may be important in cosmological particle creation processes [37], as is the case for the flat space sphaleron. It is beyond the scope of the present work to address this interesting question further. However, hairy black holes are important not only because they are physically realistic, but also because they have profound implications for the deep issues involved in information loss and quantum de-coherence. We shall discuss this topic in some detail in chapter 4. Here it suffices to say that hairy black holes may provide a window on the strange effects of quantum gravity, and tell us something of how the final (unknown) theory will behave.

Summary

In this chapter we have discussed the analytic proof of the no-hair conjecture for a restricted class of scalar field theories. In later chapters we shall perform a similar analysis to prove how the no-hair conjecture is evaded by the EYMH (chapter 2) and higher-derivative gravity (chapter 5) models.

The existence of a large number of counter-examples to the no-hair conjecture has introduced an additional consideration: stability. This is important because the no-hair theorems refer only to existence, and make no mention of stability. The issue will be one of the main foci of this thesis, and will be important when considering the physical relevance of the black holes.

Chapter 2

The Einstein-Yang-Mills-Higgs System

This chapter begins with a detailed review of the Einstein-Yang-Mills-Higgs system, motivating the ansatz employed in [2] and describing the properties of the black hole solutions found by [2]. This information will be useful in section 2.3 when we study carefully the mechanism whereby these solutions evade the no-hair theorem of [1]. As well as being important analytical support for the existence of the EYMH black holes [3], the concepts in section 2.3 can be applied to other hairy black hole systems, for example the higher curvature gravity black holes discussed in chapter 5.

2.1 Basic equations

We begin by deriving the general equations of the $SU(2)$ EYMH system. Consider a spherically symmetric (though not necessarily static) space-time. Outside the event horizon (or, in the case of more than one horizon, exterior to the outermost horizon) the metric can be written in terms of Schwarzschild co-ordinates (t, r, θ, φ) as [38]

$$ds^2 = -NS^2dt^2 + N^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{2.1}$$

where N and S are functions of t and r only, and have the form

$$\begin{aligned} N(r,t) &= 1 - \frac{2m(r,t)}{r} \\ S(r,t) &= \exp[-\delta(r,t)]. \end{aligned}$$

The existence of a regular event horizon at $r = r_h$ requires that

$$m(r_h, t) = \frac{r_h}{2} \qquad \text{and} \qquad \delta(r_h, t) < \infty, \tag{2.2}$$

so that $N(r_h, t) = 0$; and in order for the space-time to be asymptotically flat, we also require that

$$\delta(r,t) \to 0 \qquad \text{as } r \to \infty \tag{2.3}$$

namely, $N(r,t)$, $S(r,t) \to 1$ as $r \to \infty$. The most general spherically symmetric gauge potential can be taken in the form [39]

$$gA = a\hat{\tau}_r dt + b\hat{\tau}_r dr + [d\hat{\tau}_\theta - c\hat{\tau}_\varphi]d\theta + [c\hat{\tau}_\theta + d\hat{\tau}_\varphi]\sin\theta\, d\varphi \tag{2.4}$$

where a, b, c and d are functions of t and r alone, g is the gauge coupling and $(\hat{\tau}_r, \hat{\tau}_\theta, \hat{\tau}_\varphi)$ are given in terms of the Pauli matrices (τ_1, τ_2, τ_3) as follows [2]:

$$\begin{aligned}
\hat{\tau}_r = \boldsymbol{e_r} \cdot \boldsymbol{\tau} &= \tau_1 \sin\theta\cos\varphi + \tau_2 \sin\theta \sin\varphi + \tau_3\cos\theta \\
\hat{\tau}_\theta = \boldsymbol{e_\theta} \cdot \boldsymbol{\tau} &= \tau_1 \cos\theta\cos\varphi + \tau_2 \cos\theta \sin\varphi - \tau_3\sin\theta \\
\hat{\tau}_\varphi = \boldsymbol{e_\varphi} \cdot \boldsymbol{\tau} &= -\tau_1 \sin\varphi + \tau_2 \cos\varphi.
\end{aligned} \tag{2.5}$$

Using the notation $\dot{} = \partial_t$, and $' = \partial_r$, the non-zero components of the $SU(2)$ Yang-Mills field strength are calculated using the expression [2]

$$F = dA + gA \wedge A \tag{2.6}$$

that is,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc}A^b_\mu A^c_\nu \tag{2.7}$$

giving

$$\begin{aligned}
gF_{tr} &= (\dot{b} - a')\hat{\tau}_r \\
gF_{t\theta} &= [\dot{d} + a(c-1)]\hat{\tau}_\theta - (\dot{c} - ad)\hat{\tau}_\varphi \\
gF_{t\varphi} &= (\dot{c} - ad)\hat{\tau}_\theta \sin\theta + [\dot{d} + a(c-1)]\hat{\tau}_\varphi \sin\theta \\
gF_{r\theta} &= [d' + b(c-1)]\hat{\tau}_\theta - (c' - bd)\hat{\tau}_\varphi \\
gF_{r\varphi} &= (c' - bd)\hat{\tau}_\theta \sin\theta + [d' + b(c-1)]\hat{\tau}_\varphi \sin\theta \\
gF_{\theta\varphi} &= (c^2 - 2c + d^2)\hat{\tau}_r \sin\theta.
\end{aligned} \tag{2.8}$$

The most general complex doublet Higgs field may be parametrized as [2]

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_2 + i\psi_1 \\ \phi - i\psi_3 \end{pmatrix} \tag{2.9}$$

where three of the degrees of freedom have been grouped together in the vector $\boldsymbol{\psi}$. A suitable spherically symmetric ansatz is to take

$$\phi = \phi(r,t) \qquad \text{and} \qquad \boldsymbol{\psi} = \psi(r,t)\boldsymbol{e_r}. \tag{2.10}$$

The coupled EYMH system is described by the Lagrangian [2] (note that we explicitly gauge-fix the potential A (see section 2.2.1) and hence do not include a gauge-fixing term in the Lagrangian)

$$\begin{aligned}
\mathcal{L}_{EYMH} &= -\frac{1}{4\pi}\left[\frac{1}{4}|F|^2 + \frac{g^2}{8}(\phi^2 + |\boldsymbol{\psi}|^2)|A|^2 + \frac{1}{2}g^{\mu\nu}\left[\partial_\mu\phi\partial_\nu\phi + (\partial_\mu\boldsymbol{\psi})\cdot(\partial_\nu\boldsymbol{\psi})\right]\right. \\
&\left. + V_{EYMH}(\phi^2 + |\boldsymbol{\psi}|^2) + \frac{1}{2}g^{\mu\nu}A_\mu \cdot [\boldsymbol{\psi} \times \partial_\nu\boldsymbol{\psi} + \boldsymbol{\psi}\partial_\nu\phi - \phi\partial_\nu\boldsymbol{\psi}]\right]
\end{aligned} \tag{2.11}$$

where

$$V_{EYMH}(\phi^2) = \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{2.12}$$

is the Higgs potential, with v the vacuum expectation value (v.e.v.) of the Higgs field; and

$$\begin{aligned}
|F|^2 &= F^a_{\mu\nu}F^{\mu\nu}_a \\
&= -\frac{2}{S^2}(\dot{b} - a')^2 - \frac{4}{NS^2r^2}\left[(\dot{d} - a + ac)^2 + (\dot{c} - ad)^2\right] \\
&\quad + \frac{4N}{r^2}\left[(d' - b + bc)^2 + (c' - bd)\right] + \frac{2}{r^4}(c^2 - 2c + d^2)^2 \\
|A|^2 &= A^a_\mu A^\mu_a \\
&= -\frac{a^2}{NS^2} + b^2N + \frac{2}{r^2}(c^2 + d^2) \\
|\psi|^2 &= \psi_a\psi^a = \psi^2 \\
|\partial\phi|^2 &= g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\
&= -\frac{1}{NS^2}\dot{\phi}^2 + N\phi'^2 \\
|\partial\psi|^2 &= g^{\mu\nu}\partial_\mu\psi^a\partial_\nu\psi^a \\
&= -\frac{1}{NS^2}\dot{\psi}^2 + N\psi'^2 + \frac{2}{r^2}\psi^2
\end{aligned} \tag{2.13}$$

using the ansatz (2.4) and (2.10). Here, and throughout this section, lower-case Greek letters denote space-time indices, whilst lower-case Roman letters (a, b, c, etc) denote $SU(2)$ indices. Variation of the full action

$$\mathcal{S}_{EYMH} = \int \mathcal{L}_{EYMH}\sqrt{-\det g_{\mu\nu}}\, d^4x \tag{2.14}$$

gives the stress-energy tensor

$$\begin{aligned}
8\pi T_{\mu\nu} &= 2F^a_{\mu\gamma}F^{a\gamma}_\nu - \frac{1}{2}g_{\mu\nu}|F|^2 + \frac{1}{4}g^2(\phi^2 + |\psi|^2)(2A^a_\mu A^a_\nu - g_{\mu\nu}|A|^2) \\
&\quad + 2\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}|\partial\phi|^2 + 2\partial_\mu\psi^a\partial_\nu\psi^a - g_{\mu\nu}|\partial\psi|^2 - 2g_{\mu\nu}V_{EYMH}(\phi^2 + |\psi|^2) \\
&\quad + gA^a_\mu \cdot (\epsilon^{abc}\psi^b\partial_\nu\psi^c + \psi^a\partial_\nu\phi - \phi\partial_\nu\psi^a) \\
&\quad + gA^a_\nu \cdot (\epsilon^{abc}\psi^b\partial_\mu\psi^c + \psi^a\partial_\mu\phi - \phi\partial_\mu\psi^a) \\
&\quad - gg_{\mu\nu}A^{\rho a} \cdot (\epsilon^{abc}\psi^b\partial_\rho\psi^c + \psi^a\partial_\rho\phi - \phi\partial_\rho\psi^a).
\end{aligned} \tag{2.15}$$

The stress-energy tensor forms the right-hand-side of the Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}. \tag{2.16}$$

The relevant components of the Einstein tensor are [40]

$$G_{tt} = 2\frac{NS^2}{r^2}m'$$

$$\begin{aligned}
G_{tr} &= -\frac{\dot{N}}{Nr} = \frac{2\dot{m}}{Nr^2} \\
G_{rr} &= -\frac{2m'}{Nr^2} + \frac{2S'}{Sr}.
\end{aligned} \tag{2.17}$$

The relevant components of the stress-energy tensor are, for the spherically symmetric ansatz outlined above (2.4, 2.10),

$$\begin{aligned}
8\pi T_{tt} &= \frac{N}{g^2}(\dot{b}-a')^2 + \frac{2}{g^2r^2}\left\{[\dot{d}+a(c-1)]^2 + (\dot{c}-ad)^2\right\} \\
&+\frac{2N^2S^2}{g^2r^2}\left\{[d'+b(c-1)]^2 + (c'-bd)^2\right\} + \frac{NS^2}{g^2r^4}(c^2-2c+d^2)^2 \\
&+\frac{1}{4}(\phi^2+\psi^2)\left[a^2 + N^2S^2b^2 + \frac{2NS^2}{r^2}(c^2+d^2)\right] + (\dot{\phi}^2+\dot{\psi}^2) \\
&+N^2S^2(\phi'^2+\psi'^2) + \frac{2NS^2}{r^2}\psi^2 + 2NS^2V_{EYMH}(\phi^2+\psi^2) \\
&+a(\psi\dot{\phi}-\phi\dot{\psi}) + bN^2S^2(\psi\phi'-\phi\psi') - \frac{2NS^2}{r^2}d\phi\psi
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
8\pi T_{tr} &= \frac{4}{g^2r^2}\left\{[\dot{d}+a(c-1)][d'+b(c-1)] + (\dot{c}-ad)(c'-bd)\right\} \\
&+\frac{1}{2}ab(\phi^2+\psi^2) + 2(\dot{\phi}\phi'+\dot{\psi}\psi') + a(\psi\dot{\phi}-\phi\dot{\psi}) + b(\psi\phi'-\phi\psi')
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
8\pi T_{rr} &= -\frac{1}{g^2NS^2}(\dot{b}-a')^2 + \frac{2}{g^2r^2N^2S^2}\left\{[\dot{d}+a(c-1)]^2 + (\dot{c}-ad)^2\right\} \\
&+\frac{2}{g^2r^2}\left\{[d'+b(c-1)]^2 + (c'-bd)^2\right\} - \frac{1}{g^2r^4N}(c^2-2c+d^2)^2 \\
&+\frac{1}{4}(\phi^2+\psi^2)\left[\frac{a^2}{N^2S^2} + b^2 - \frac{2}{Nr^2}(c^2+d^2)\right] + \frac{1}{N^2S^2}(\dot{\phi}^2+\dot{\psi}^2) \\
&+(\phi'^2+\psi'^2) - \frac{2}{Nr^2}\psi^2 - \frac{2}{N}V_{EYMH}(\phi^2+\psi^2) \\
&+\frac{a}{N^2S^2}(\psi\dot{\phi}-\phi\dot{\psi}) + b(\psi\phi'-\phi\psi') - \frac{2}{r^2}d\phi\psi.
\end{aligned} \tag{2.20}$$

Substituting in the Einstein equations (2.16), we obtain

$$\begin{aligned}
\dot{m} &= N\mathtt{H} \\
m' &= \frac{1}{2}[N\mathtt{G} + r^2p_\theta] \\
S' &= \frac{\mathtt{G}S}{r}
\end{aligned} \tag{2.21}$$

where

$$\mathtt{H} = \frac{r^2}{2}(8\pi T_{tr})$$

$$\begin{aligned}
\mathtt{G} &= \frac{r^2}{2}\left(\frac{1}{N^2S^2}8\pi T_{tt} + 8\pi T_{rr}\right) \\
p_\theta &= \frac{N}{2}\left(\frac{1}{N^2S^2}8\pi T_{tt} - 8\pi T_{rr}\right) \qquad (2.22)
\end{aligned}$$

so that

$$\begin{aligned}
\mathtt{H} &= \frac{2}{g^2}\left\{[\dot{d} + a(c-1)][d' + b(c-1)] + (\dot{c} - ad)(c' - bd)\right\} \\
&+\frac{r^2}{4}ab(\phi^2+\psi^2) + r^2(\dot{\phi}\phi' + \dot{\psi}\psi') + \frac{r^2}{2}a(\psi\dot{\phi} - \phi\dot{\psi}) + \frac{r^2}{2}b(\psi\phi' - \phi\psi') \qquad (2.23) \\
\mathtt{G} &= +\frac{2}{g^2N^2S^2}\left\{[\dot{d} + a(c-1)]^2 + (\dot{c} - ad)^2\right\} + \frac{2}{g^2}\left\{[d' + b(c-1)]^2 + (c' - bd)^2\right\} \\
&+\frac{r^2}{4}(\phi^2+\psi^2)\left[\frac{a^2}{N^2S^2} + b^2\right] + \frac{r^2}{N^2S^2}(\dot{\phi}^2 + \dot{\psi}^2) + r^2(\phi'^2 + \psi'^2) \\
&+\frac{r^2a}{N^2S^2}(\psi\dot{\phi} - \phi\dot{\psi}) + br^2(\psi\phi' - \phi\psi') - 2d\phi\psi \qquad (2.24) \\
p_\theta &= \frac{1}{g^2S^2}(\dot{b} - a')^2 + \frac{1}{g^2r^4}(c^2 - 2c + d^2)^2 + \frac{1}{2r^2}(\phi^2+\psi^2)(c^2+d^2) \\
&+\frac{2}{r^2}\psi^2 + 2V_{EYMH}(\phi^2+\psi^2). \qquad (2.25)
\end{aligned}$$

Varying $\mathcal{L}_{EYMH}$ (2.11) with respect to A^a_μ, ϕ and $\boldsymbol{\psi}$ leads to the Yang-Mills and Higgs field equations:

$$\begin{aligned}
0 &= \nabla_\mu F^{\mu\nu}_a + g\epsilon^{abc}F^{\nu\mu}_b A^c_\mu - \frac{1}{4}g^2A^\nu_a(\phi^2 + |\boldsymbol{\psi}|^2) \\
&+\frac{1}{2}gg^{\nu\mu}[\epsilon^{abc}\psi^b\partial_\mu\psi^c + \psi^a\partial_\mu\phi - \phi\partial_\mu\psi^a] \qquad (2.26) \\
0 &= \Box\phi - \frac{1}{4}g^2|A|^2\phi - \lambda\phi(\phi^2 + |\boldsymbol{\psi}|^2 - v^2) \\
&+\frac{1}{2}g\nabla_\mu(A^\mu_a\psi^a) + \frac{1}{2}gA^\mu_a\partial_\mu\psi^a \qquad (2.27) \\
0 &= \Box\psi^a - \frac{1}{2}g\nabla_\mu(A^\mu_a\phi) + \frac{1}{2}g\epsilon^{abc}\nabla_\mu(A^\mu_b\psi^c) \\
&-\frac{1}{4}g^2|A|^2\psi^a - \lambda\psi^a(\phi^2 + |\boldsymbol{\psi}|^2 - v^2) - \frac{1}{2}gA^\mu_a\partial_\mu\phi \qquad (2.28)
\end{aligned}$$

where $\Box\phi = \nabla_\mu\nabla^\mu\phi$ is the Laplacian. The Yang-Mills equations take the form

$$\begin{aligned}
0 &= \left(\frac{2}{rS^2} - \frac{S'}{S^3}\right)(\dot{b} - a') + \frac{1}{S^2}(\dot{b}' - a'') - \frac{2}{r^2NS^2}[\dot{d} + a(c-1)] \\
&-\frac{2}{r^2NS^2}\left\{d(\dot{c} - ad) - c[\dot{d} + a(c-1)]\right\} \\
&+\frac{g^2}{4NS^2}a(\phi^2+\psi^2) - \frac{g^2}{2NS^2}(\psi\dot{\phi} - \phi\dot{\psi}) \qquad (2.29)
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{\dot{S}}{S^3}(\dot{b}-a') - \frac{1}{S^2}(\ddot{b}-\dot{a}') - \frac{2N}{r^2}\left\{(c-1)[d'+b(c-1)] - d(c'-bd)\right\} \\
&\quad -\frac{g^2N}{4}b(\phi^2+\psi^2) + \frac{g^2N}{2}(\psi\phi'-\phi\psi') && (2.30) \\
0 &= \frac{(NS)^{\cdot}}{r^2N^2S^3}[\dot{d}+a(c-1)] - \frac{1}{r^2NS^2}[\ddot{d}+\dot{a}(c-1)+a\dot{c}] \\
&\quad -\frac{1}{r^2NS^2}a(\dot{c}-ad) + \frac{(NS)'}{r^2S}[d'+b(c-1)] + \frac{N}{r^2}[d''+b'(c-1)+bc'] \\
&\quad +\frac{N}{r^2}b(c'-bd) - \frac{d}{r^4}(c^2-2c+d^2) - \frac{g^2}{4r^2}d(\phi^2+\psi^2) + \frac{g^2}{2r^2}\phi\psi && (2.31) \\
0 &= \frac{(NS)^{\cdot}}{r^2N^2S^3}(\dot{c}-ad) - \frac{1}{r^2NS^2}(\ddot{c}-\dot{a}d-a\dot{d}) + \frac{1}{r^2NS^2}a[\dot{d}+a(c-1)] \\
&\quad +\frac{(NS)'}{r^2S}(c'-bd) + \frac{N}{r^2}(c''-b'd-bd') - \frac{N}{r^2}b[d'+b(c-1)] \\
&\quad -\frac{1}{r^4}(c^2-2c+d^2)(c-1) - \frac{g^2}{4r^2}c(\phi^2+\psi^2) - \frac{g^2}{2r^2}\psi^2 && (2.32)
\end{aligned}$$

whilst the Higgs equations are given by

$$\begin{aligned}
0 &= \frac{1}{NS^2}\left[-\ddot{\phi}+\frac{(NS)^{\cdot}}{NS}\dot{\phi}\right] + N\left[\phi''+\frac{2}{r}\phi'+\frac{(NS)'}{NS}\phi'\right] \\
&\quad +\left[\frac{a^2}{4NS^2}-\frac{Nb^2}{4}-\frac{1}{2r^2}(c^2+d^2)\right]\phi - \lambda\phi(\phi^2+\psi^2-v^2) + \frac{d\psi}{r^2} \\
&\quad +\frac{1}{NS^2}\left[\frac{(NS)^{\cdot}}{2NS}a\psi - \frac{1}{2}\dot{a}\psi - a\dot{\psi}\right] + N\left[\frac{1}{2}b'\psi+\psi'b+\frac{b\psi}{r}+\frac{(NS)'}{2NS}b\psi\right] && (2.33) \\
0 &= \frac{1}{NS^2}\left[-\ddot{\psi}+\frac{(NS)^{\cdot}}{NS}\dot{\psi}\right] + N\left[\psi''+\frac{2}{r}\psi'+\frac{(NS)'}{NS}\psi'\right] \\
&\quad +\left[\frac{a^2}{4NS^2}-\frac{Nb^2}{4}-\frac{1}{2r^2}(c^2+d^2)+\frac{2c}{r^2}\right]\psi \\
&\quad -\lambda\psi(\phi^2+\psi^2-v^2) + \frac{1}{NS^2}\left[-\frac{(NS)^{\cdot}}{2NS}a\phi + \frac{1}{2}\dot{a}\phi - a\dot{\phi}\right] \\
&\quad +\frac{d\phi}{r^2} - \frac{2}{r^2}\psi + N\left[\frac{1}{2}b'\phi+\phi'b+\frac{b\phi}{r}+\frac{(NS)'}{2NS}b\phi\right]. && (2.34)
\end{aligned}$$

2.2 Static equilibrium solutions

We now turn to static (equilibrium) solutions of the EYMH field equations, and see how the field parametrizations (2.4) and (2.10) are simplified in this situation. A similar analysis was performed for the Einstein-Yang-Mills system in [20]. We then discuss the behaviour of the black hole solutions of [2].

2.2.1 Field equations

For equilibrium field configurations, all variables are time-independent. In general, the $SU(2)$ gauge potential (2.4) possesses a residual gauge freedom [2]

$$A \to \mathcal{R}A\mathcal{R}^{-1} + \frac{1}{g}\mathcal{R}d\mathcal{R}^{-1} \tag{2.35}$$

under unitary transformations of the form

$$\mathcal{R} = \exp[j(r,t)\hat{\tau}_r] \tag{2.36}$$

where $j(r,t)$ is an arbitrary real function. Under this transformation, the potential functions transform as

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \to \begin{pmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \\ \tilde{d} \end{pmatrix} = \begin{pmatrix} a - \partial_t j \\ b - j' \\ (c-1)\cos j - d\sin j + 1 \\ d\cos j + (c-1)\sin j \end{pmatrix}. \tag{2.37}$$

Hence we can set $b = 0$ without loss of generality for the static solutions [2]. With this choice, the Yang-Mills equation for a (2.29) can be written in the form

$$\frac{d}{dr}\left[\frac{r^2}{S}aa'\right] = \frac{r^2}{S}a'^2 + \frac{2a^2}{NS}\left[d^2 + (c-1)^2 + \frac{g^2r^2}{8}(\phi^2+\psi^2)\right] \tag{2.38}$$

which can be integrated over the region exterior to the event horizon to give

$$\left[\frac{r^2}{S}aa'\right]_{r_h}^{\infty} = \int_{r_h}^{\infty}\left[\frac{r^2}{S}a'^2 + \frac{2a^2}{NS}\left\{(c-1)^2 + d^2 + \frac{g^2r^2}{8}(\phi^2+\psi^2)\right\}\right]dr. \tag{2.39}$$

The energy density of the field configuration is

$$\begin{aligned} -8\pi T_t^t &= \frac{a'^2}{g^2}S^2 + \frac{2a^2}{g^2r^2NS^2}[(c-1)^2+d^2] \\ &+\frac{2N}{g^2r^2}(c'^2+d'^2) + \frac{1}{g^2r^4}(c^2-2c+d^2)^2 + \frac{g^2a^2}{4NS^2}(\phi^2+\psi^2) - \frac{2d\phi\psi}{r^2} \\ &+\frac{g^2}{2r^2}(c^2+d^2)(\phi^2+\psi^2) + N(\phi'^2+\psi'^2) + \frac{2}{r^2}\psi^2 + 2V(\phi^2+\psi^2). \end{aligned} \tag{2.40}$$

Hence, in order for the energy density to be regular at the event horizon, it must be the case that

$$a(r_h) = 0, \qquad a'(r_h) < \infty \tag{2.41}$$

whilst for finite total energy solutions we require

$$aa' \sim O(r^{-3}) \quad \text{as} \quad r \to \infty. \tag{2.42}$$

Thus the integral on the right-hand-side of (2.39) must vanish, which is only possible if either

$$a \equiv \text{constant}, \quad c \equiv 1, \quad d \equiv 0, \quad \phi \equiv 0, \quad \psi \equiv 0 \tag{2.43}$$

or $a \equiv 0$. The first choice is only possible if the Higgs v.e.v. $v = 0$, otherwise the energy will not be finite. In this case the solution reduces to the Reissner-Nordström solution [40]. The second case corresponds to the t'Hooft-Polyakov ansatz [41] for purely magnetic field configurations.

The Gauss constraint equation (2.30) reads

$$-\frac{2N}{r^2}[(c-1)d' - dc'] - \frac{Ng^2}{2}(\psi\phi' - \phi\psi') = 0, \tag{2.44}$$

that is,

$$2(c-1)^2\frac{d}{dr}\left[\frac{d}{c-1}\right] = \frac{r^2g^2\phi^2}{2}\frac{d}{dr}\left[\frac{\psi}{\phi}\right] \tag{2.45}$$

together with the equations for ψ and d;

$$\begin{aligned}
0 &= \frac{N}{r^2}d'' + \frac{(NS)'}{Sr^2}d' - \frac{d}{r^4}(c^2 - 2c + d^2) - \frac{g^2}{4r^2}d(\phi^2 + \psi^2) + \frac{g^2}{2r^2}\phi\psi \\
0 &= N\psi'' + \frac{2N}{r}\psi' + \frac{(NS)'}{S}\psi' - \frac{2}{r^2}\psi + \frac{d\phi}{r^2} \\
&\quad -\frac{\psi}{2r^2}(c^2 + d^2) + \frac{c}{r^2}\psi - \lambda\psi(\phi^2 + \psi^2 - v^2).
\end{aligned} \tag{2.46}$$

Thus, following [2], we can consistently set $\psi \equiv 0$ and $d \equiv 0$ for the equilibrium solutions.

The remaining Yang-Mills and Higgs equations are then, respectively, [2]

$$0 = \frac{N}{r^2}c'' + \frac{(NS)'}{Sr^2}c' - \frac{1}{r^4}(c^2 - 2c)(c-1) - \frac{g^2}{4r^2}\phi^2 c \tag{2.47}$$

$$0 = N\phi'' + \frac{2N}{r}\phi' + \frac{(NS)'}{S}\phi' - \frac{c^2}{2r^2}\phi - \lambda\phi(\phi^2 - v^2) \tag{2.48}$$

and the Einstein equations reduce to [2]

$$m' = \frac{1}{2}[N\mathtt{G} + r^2 p_\theta] \tag{2.49}$$

$$S' = \frac{\mathtt{G}S}{r} \tag{2.50}$$

$$\Rightarrow \quad \delta' = -\frac{\mathtt{G}}{r} \tag{2.51}$$

where

$$\mathtt{G} = \frac{2c'^2}{g^2} + r^2\phi'^2 \tag{2.52}$$

$$p_\theta = \frac{(c^2-2c)^2}{g^2r^4} + \frac{\phi^2c^2}{2r^2} + \frac{\lambda}{2}(\phi^2 - v^2)^2. \tag{2.53}$$

Finally in this section we rewrite the static field equations in terms of the variable $\omega \equiv c - 1$, setting $g = 1$ in anticipation of section 2.2.2 [2]:

$$0 = \frac{N}{r^2}\omega'' + \frac{(NS)'}{r^2 S}\omega' - \frac{\omega}{r^4}(\omega^2 - 1) - \frac{\phi^2}{4r^2}(1+\omega) \tag{2.54}$$

$$0 = N\phi'' + \frac{2N}{r}\phi' + \frac{(NS)'}{S}\phi' - \frac{\phi}{2r^2}(1+\omega)^2 - \lambda\phi(\phi^2 - v^2) \tag{2.55}$$

$$m' = \left(1 - \frac{2m}{r}\right)\left(\omega'^2 + \frac{1}{2}r^2\phi'^2\right) + \frac{1}{2r^2}(\omega^2-1)^2 \tag{2.56}$$

$$+\frac{1}{4}\phi^2(1+\omega)^2 + \frac{\lambda}{4}r^2(\phi^2 - v^2)^2 \tag{2.57}$$

$$\delta' = -\frac{1}{r}\left(2\omega'^2 + r^2\phi'^2\right). \tag{2.58}$$

2.2.2 Scaling behaviour

Without loss of generality, we may take the radius of the event horizon to be $r_h = 1$ (as in [2]) using the following transformation:

$$x = \frac{r}{r_h}, \qquad \hat{m}(x) = \frac{m(r_h x)}{r_h}, \qquad \hat{\delta}(x) = \delta(r_h x) \tag{2.59}$$

$$\begin{aligned} \hat{c}(x) &= c(r_h x), & \hat{\phi}(x) &= \phi(r_h x), \\ \hat{N}(x) &= N(r_h x), & \hat{S}(x) &= S(r_h x). \end{aligned} \tag{2.60}$$

The field equations then take the form

$$0 = \frac{\hat{N}}{x^2}\frac{d^2\hat{c}}{dx^2} + \frac{1}{x^2\hat{S}}\frac{d}{dx}(\hat{N}\hat{S})\frac{d\hat{c}}{dx} - \frac{1}{x^4}(\hat{c}^2 - 2\hat{c})(\hat{c}-1) - \frac{g^2 r_h^2}{4x^2}\hat{\phi}^2\hat{c} \tag{2.61}$$

$$0 = \hat{N}\frac{d^2\hat{\phi}}{dx^2} + \frac{2\hat{N}}{x}\frac{d\hat{\phi}}{dx} + \frac{1}{\hat{S}}\frac{d}{dx}(\hat{N}\hat{S})\frac{d\hat{\phi}}{dx} - \frac{1}{2x^2}\hat{c}^2\hat{\phi} - \lambda r_h^2\hat{\phi}(\hat{\phi}^2 - 1) \tag{2.62}$$

$$\frac{d\hat{\delta}}{dx} = -\frac{1}{x}\left[\frac{2}{g^2 r_h^2}\left(\frac{d\hat{c}}{dx}\right)^2 + x^2\left(\frac{d\hat{\phi}}{dx}\right)^2\right] \tag{2.63}$$

$$\frac{d\hat{m}}{dx} = \frac{\hat{N}}{2}\left[\frac{2}{g^2 r_h^2}\left(\frac{d\hat{c}}{dx}\right)^2 + x^2\left(\frac{d\hat{\phi}}{dx}\right)^2\right]$$

$$+\frac{1}{2g^2 r_h^2 x^2}(\hat{c}^2 - 2\hat{c})^2 + \frac{1}{4}\hat{\phi}^2\hat{c}^2 + \frac{\lambda}{4}r_h^2 x^2(\hat{\phi}^2 - v^2)^2 \tag{2.64}$$

where

$$\frac{d}{dx}\hat{N} = r_h\frac{d}{dr}\hat{N} \tag{2.65}$$

and similarly for the other functions. These are identical with the original equations (2.47–2.51) if we make the following identifications:

$$\hat{g} = g r_h, \qquad \hat{\lambda} = \lambda r_h^2, \qquad \hat{v} = v. \tag{2.66}$$

From here on we drop the ˆ notation, and fix the gauge coupling constant $g = 1$ and the parameter $\lambda = 0.15$, leaving the Higgs v.e.v. v free to vary. (See sections 2.2.8 and 3.6.4 for discussion on this choice of λ.)

2.2.3 Analytic solutions

The Higgs equation (2.48) possesses two constant solutions for ϕ:

$$\phi \equiv 0 \qquad \text{and} \qquad \phi \equiv \pm v. \tag{2.67}$$

In the case $\phi \equiv \pm v$, the Yang-Mills equation (2.47) reduces to

$$\frac{1}{4r^2} v^2 c = 0 \tag{2.68}$$

which implies that

$$c \equiv 0. \tag{2.69}$$

The metric functions then satisfy

$$\delta \equiv 0, \qquad m \equiv M = \frac{r_h}{2}, \tag{2.70}$$

giving a Schwarzschild black hole geometry. In the case $\phi \equiv 0$, equation (2.47) is satisfied by

$$c \equiv 0, 1, 2, \tag{2.71}$$

any of which lead to

$$\delta \equiv 0. \tag{2.72}$$

For any of these values of c, as $r \to \infty$, (2.49) gives

$$m' \sim \frac{\lambda}{4} r^2 v^4 \tag{2.73}$$

so that the spacetime will not be asymptotically flat unless $v = 0$. In this case, $\phi \equiv 0$ everywhere and the solutions with $c \equiv 0$ or $c \equiv 2$ correspond to the Schwarzschild metric (2.70), whilst that with $c \equiv 1$ is the Reissner-Nordström black hole (cf. [20]).

2.2.4 Global behaviour of the equilibrium solutions

As $r \to \infty$, the equation (2.48) for ϕ reduces to

$$\lambda\phi(\phi^2 - v^2) = 0. \tag{2.74}$$

Hence either $\phi \equiv 0$ or $\phi \equiv \pm v$. If $\phi \equiv 0$, then (2.49) takes the form

$$m' = \frac{\lambda}{4} r^2 v^4 \tag{2.75}$$

which will not give an asymptotically flat solution unless $v = 0$. For non-zero v, therefore, $\phi \equiv \pm v$. The local energy density of the system is given by

$$-8\pi T_t^t = \frac{2}{r^2}Nc'^2 + N\phi'^2 + \frac{1}{r^4}(c^2-2c)^2 + \frac{1}{2r^2}\phi^2c^2 + \frac{\lambda}{2}(\phi^2-v^2)^2. \tag{2.76}$$

In order to have finite total energy solutions, it is clear that

$$\phi \to \text{constant} \qquad c \to \text{constant} \qquad \text{as } r \to \infty \tag{2.77}$$

and further that

$$\begin{array}{lll} \phi & \to & \pm v \quad \text{as } r \to \infty \\ c & \to & 0 \quad\;\; \text{as } r \to \infty. \end{array} \tag{2.78}$$

We rewrite the equilibrium Yang-Mills (2.47) and Higgs (2.48) equations in the form [2]:

$$[NSc'(c-1)]' = NSc'^2 + \frac{1}{4}S\phi^2(c-1) + \frac{1}{r^2}Sc(c-2)(c-1)^2 \tag{2.79}$$

$$[NS\phi'\phi]' = NSr^2\phi'^2 + \frac{1}{2}Sc^2\phi^2 + \lambda S\phi^2r^2(\phi^2-v^2). \tag{2.80}$$

At the critical points of c, where $c' = 0$, (2.79) gives

$$NSc''(c-1) = \frac{1}{4}\phi^2S(c-1) + \frac{1}{r^2}Sc(c-2)(c-1)^2 \tag{2.81}$$

whilst at the critical points of ϕ we have, from (2.80),

$$NS\phi''\phi = \frac{1}{2}S\phi^2c^2 + \lambda Sr^2\phi^2(\phi^2-v^2). \tag{2.82}$$

If $|\phi^2| > v^2$, then ϕ'' has the same sign as ϕ and hence $|\phi|$ diverges away from $|v|$. Hence $|\phi|^2 \leq v^2$ for finite energy solutions. If we ignore temporarily the first term in (2.82), then

$$\phi\phi'' < 0 \tag{2.83}$$

indicating that ϕ oscillates about the value $\phi = 0$, which is a solution of infinite energy, unless the initial value of ϕ' is large enough for $\phi \to \pm v$ as $\phi' \to 0$. The gauge field coupling thus plays an important rôle in avoiding solutions of the field equations with infinite energy [2].

Turning now to the Yang-Mills field, since $\phi \sim O(v)$ for $r \geq 1$, there are two characteristic length scales. Firstly, for $r \lesssim \frac{1}{v}$, at the critical points of c,

$$NSc''(c-1) \sim \frac{1}{r^2}Sc(c-2)(c-1)^2. \tag{2.84}$$

Hence, if $c < 0$, then $c'' < 0$, so that c diverges away from 0. If $c > 2$, then $c'' > 0$, so that c diverges away from 2. For $r \gtrsim \frac{1}{v}$, when $c' = 0$, we have

$$NSc''(c-1) \sim \frac{1}{r^2}S\phi^2(c-1) \tag{2.85}$$

so that if $c < 0$, then $c'' < 0$ which gives an infinite energy solution, whereas if $c > 0$, then $c'' > 0$, and c diverges away from 0 also. Hence there can be no turning points in this region. Thus we require $0 < c < 2$ for finite energy solutions. In this case, c will oscillate about the value $c = 1$ in the region $r \lesssim \frac{1}{v}$ and we denote by k the number of times the function c crosses the line $c = 1$, or alternatively, the number of nodes of the function $\omega \equiv c - 1$ [2].

2.2.5 Boundary conditions at infinity

Substituting the vacuum values $c = 0$ and $\phi = v$ into the field equations, we obtain the following asymptotic behaviour of the fields at infinity [2]:

$$
\begin{aligned}
m(r) &\sim M - \frac{1}{2}\phi_\infty(\sqrt{2}\mu r) r e^{-2\sqrt{2}\mu r} && (2.86)\\
\delta(r) &\sim \frac{1}{2}\phi_\infty^2 \sqrt{2}\mu r e^{-2\sqrt{2}\mu r} && (2.87)\\
c(r) &\sim \omega_\infty e^{-\frac{v}{2}r} && (2.88)\\
\phi(r) &\sim v + \phi_\infty e^{-\sqrt{2}\mu r} && (2.89)
\end{aligned}
$$

for constants ω_∞ and ϕ_∞, where $\sqrt{2}\mu$ is the Higgs field mass, given by [2]

$$\sqrt{2}\mu = \sqrt{2\lambda}v. \tag{2.90}$$

2.2.6 Boundary conditions at the event horizon

Here we use a Taylor series expansion for the variables [2]:

$$
\begin{aligned}
m(r) &\sim m(r_h) + m'(r_h)(r - r_h) + O(r - r_h)^2 && (2.91)\\
\delta(r) &\sim \delta(r_h) + \delta'(r_h)(r - r_h) + O(r - r_h)^2 && (2.92)\\
c(r) &\sim c(r_h) + c'(r_h)(r - r_h) + O(r - r_h)^2 && (2.93)\\
\phi(r) &\sim \phi(r_h) + \phi'(r_h)(r - r_h) + O(r - r_h)^2 && (2.94)
\end{aligned}
$$

where the field equations evaluated at $r = r_h$ give:

$$
\begin{aligned}
m_h' = m'(r_h) &= \frac{1}{2r_h^2}(c_h^2 - 2c_h)^2 + \frac{1}{4}c_h^2\phi_h^2 + \frac{\lambda}{4}r_h^2(\phi_h^2 - v^2)^2 && (2.95)\\
c_h' = c'(r_h) &= \frac{1}{N_h'}\left[\frac{1}{r_h^2}(c_h - 1)(c_h^2 - 2c_h) + \frac{1}{4}c_h\phi_h^2\right] && (2.96)\\
\phi_h' = \phi'(r_h) &= \frac{1}{N_h'}\left[\frac{1}{2r_h^2}c_h^2\phi_h + \lambda\phi_h(\phi_h^2 - v^2)\right] && (2.97)\\
\delta_h' = \delta'(r_h) &= -\frac{1}{r_h}\left[2c_h'^2 + r_h^2\phi_h'^2\right] && (2.98)
\end{aligned}
$$

where

$$c_h = c(r_h), \qquad \phi_h = \phi(r_h) \tag{2.99}$$

and

$$N'_h = N'(r_h) = \frac{1}{r_h} - \frac{2m'_h}{r_h}. \tag{2.100}$$

2.2.7 Limiting behaviour of the solutions as $v \to 0$

For finite energy solutions we have $|\phi| \leq v$, so, as $v \to 0$, then $\phi \to 0$ everywhere. Let $\phi = v\tilde{\phi}$; then the equations of motion become

$$\begin{aligned}
0 &= Nc'' + \frac{(NS)'}{S}c' - \frac{1}{r^2}(c^2 - 2c)(c-1) - \frac{1}{4}v^2\tilde{\phi}^2 c && (2.101)\\
0 &= N\tilde{\phi}'' + \frac{2N}{r}\tilde{\phi}' + \frac{(NS)'}{S}\tilde{\phi}' - \frac{1}{2r^2}c^2\tilde{\phi} - \lambda v^2\tilde{\phi}(\tilde{\phi}^2 - 1) && (2.102)\\
m' &= \frac{1}{2}\left[N\mathtt{G} + r^2 p_\theta\right] && (2.103)\\
\delta' &= -\frac{1}{r}\mathtt{G}, && (2.104)
\end{aligned}$$

where

$$\begin{aligned}
\mathtt{G} &= 2c'^2 + r^2v^2\tilde{\phi}'^2 && (2.105)\\
p_\theta &= \frac{1}{r^4}(c^2 - 2c)^2 + \frac{1}{2r^2}v^2c^2\tilde{\phi}^2 + \frac{1}{2}\lambda v^4(\tilde{\phi}^2 - 1)^2. && (2.106)
\end{aligned}$$

In the limit $v \to 0$, we see that $\tilde{\phi}$ decouples from the other field equations and we are left with

$$\begin{aligned}
0 &= Nc'' + \frac{(NS)'}{S}c' - \frac{1}{r^2}(c^2 - 2c)(c-1)\\
m' &= \frac{1}{2}\left[2Nc'^2 + \frac{1}{r^2}(c^2 - 2c)^2\right] && (2.107)\\
\delta' &= -\frac{2}{r}c'^2
\end{aligned}$$

and

$$N\tilde{\phi}'' + \frac{2N}{r}\tilde{\phi}' + \frac{(NS)'}{S}\tilde{\phi}' - \frac{1}{2r^2}c^2\tilde{\phi} = 0. \tag{2.108}$$

The equations (2.107) are the coupled Einstein-Yang-Mills equations studied by Bizon [18] which have the coloured black holes as solutions, as well as the Schwarzschild solution

$$c \equiv 2, \qquad m \equiv \frac{r_h}{2}, \qquad \delta \equiv 0 \tag{2.109}$$

which can be regarded as the $k = 0$ coloured black hole solution.

Branches of solutions Hence, as $v \to 0$, the EYMH black hole solutions approach one of the coloured black hole geometries. For each fixed $k = n$, there are two observed behaviours. Firstly, as $v \to 0$, the solutions approach those of the $k = n$ coloured black hole, with $\tilde{\phi}$ approaching a finite non-zero limit. These EYMH solutions are said to lie on the $k = n$ branch of solutions. Secondly, as $v \to 0$, the functions approach those of the $k = n - 1$ coloured black hole (or Schwarzschild black hole in the case $n = 1$) and $\tilde{\phi} \to 0$ everywhere. These solutions form the *quasi-*$k = n - 1$ branch, as the nth node of ω gradually moves further and further away from the horizon as v decreases [2].

Explanation of branches This branching of the solution space can be understood heuristically as follows [2]. The parameters of the theory produce two length scales:

$$L_1 = \frac{\sqrt{G_N}}{g}, \qquad L_2 = \frac{1}{g}\sqrt{\frac{\lambda}{\mu^2}} \tag{2.110}$$

where G_N is the Newton gravitational constant, and $\sqrt{2}\mu = \sqrt{2\lambda}v$ is the Higgs field mass. The scale L_1 is determined by the gravitational field, but the scale L_2 will vary as the Higgs mass μ varies. We define the scale of a particular static solution of the field equations to be the radius r after which the solutions rapidly approach their asymptotic values. The $k = n$ branch of solutions has its scale set by L_1, and the quasi-$k = n - 1$ branch of solutions has its scale set by L_2. As $v \to 0$ for this latter branch, the nth node has its position determined by the scale L_2, and is pushed out to infinity. In this limit the solutions approach the $k = n - 1$ coloured black hole solution, with scale set by L_1. In particular, for the quasi-$k = 0$ solutions, the scale is set entirely by L_2, since there is only one node, and the solutions approach the Schwarzschild geometry as $v \to 0$.

2.2.8 Numerical solution of the field equations

Apart from the Schwarzschild and Reissner-Nordström solutions, there are no solutions of the field equations (2.47–2.51) in closed form. Hence a numerical method of solution is necessary as in [2].

From the field equations (2.47–2.51), if the function $\delta(r)$ satisfies (2.51) then $\delta(r) +$ constant will also be a valid solution. In order to make the numerical solution easier, we set $\delta(r_h) = 0$ (so that $\delta(\infty) = 0$ will not be satisfied) when integrating outwards from r_h. An appropriate constant can then be added to $\delta(r)$, after the field equations have been solved, to ensure that the boundary condition at infinity is satisfied.

With this transformation, there are two unknowns at the event horizon, $c(r_h)$ and $\phi(r_h)$. Solving the field equations (2.47–2.51) is therefore a two-parameter shooting problem [42]. The procedure is to take initial 'guesses' for the unknowns c_h and ϕ_h and then integrate the differential equations out from r_h using a standard ordinary differential equation solver, attempting to satisfy the boundary conditions for large r. The initial starting values for c_h and ϕ_h are then adjusted until these boundary conditions are satisfied. The algorithm used was that of [42] and was programmed in Fortran using subroutines from [42].

For each fixed value of the Higgs v.e.v. v, the many possible numerical solutions are indexed by k, the number of nodes of the potential function $\omega(r) \equiv c(r)-1$. Here we concentrate on the case $k=1$. Then, for each v, there are two solutions which can be ascribed to one of two families of solutions: the $k=1$ branch or the quasi-$k=0$ branch, depending on the behaviour of the families as $v \to 0$, as described in the previous subsection. The quasi-$k=0$ branch of solutions approaches the Schwarzschild solution (where $c \equiv 2$ and $\phi \equiv 0$) as $v \to 0$, whereas the $k=1$ branch of solutions approaches the first coloured black hole of [18] as $v \to 0$, with $\phi \equiv 0$. As v increases, with $\lambda = 0.15$, the two branches of solutions join up at $v = v_{max} = 0.352$. This phenomenon does not occur for $\lambda = 0.125$, as found by Greene, Mathur and O'Neill [2]. However, they conjectured that the two branches of solutions would converge for some value of λ.

In the following graphs (figures 2.1–2.7), we sketch the functions for the $k=1$ and quasi-$k=0$ branches of solutions, for various values of the Higgs v.e.v. v, with the gauge coupling $g=1$, the event horizon radius fixed at $r_h=1$ and the parameter $\lambda = 0.15$. All the graphs have $\log r$ along the horizontal axis, so that the event horizon is situated where $\log r = 0$. Note that there is little variation in the field variables for the $k=1$ branch as v varies, except that for decreasing v, the variables decay more slowly to their values at infinity. There is greater variation for the quasi-$k=0$ branch, and we therefore give more plots for this branch of solutions.

2.2.9 Non-extremality of the solutions

One issue that is important, especially when we come to consider the thermodynamics and entropy of the black holes, is whether or not they are extremal. An extremal black hole occurs when N has a double zero at the event horizon, and is caused physically by an inner horizon moving outwards until it coincides with the outermost event horizon. Mathematically, the condition for extremality is (with $r_h = 1$),

$$m'(1) = \frac{1}{2}. \tag{2.111}$$

From the field equations (2.56) we have, with $\omega \equiv c-1$,

$$m'(1) = \frac{1}{2}\left[(1-\omega_h^2)^2 + \frac{1}{2}\phi_h^2(1+\omega_h)^2 + \frac{\lambda}{2}(\phi_h^2 - v^2)^2\right]. \tag{2.112}$$

There is thus no a priori reason why this quantity should not be equal to $\frac{1}{2}$ for some equilibrium solution. For the solutions on the $k=1$ and quasi-$k=0$ branches, we can however place the following numerical bounds on $m'(1)$. The first term in (2.112) is decreasing for ω_h positive and increasing, and hence is bounded above by

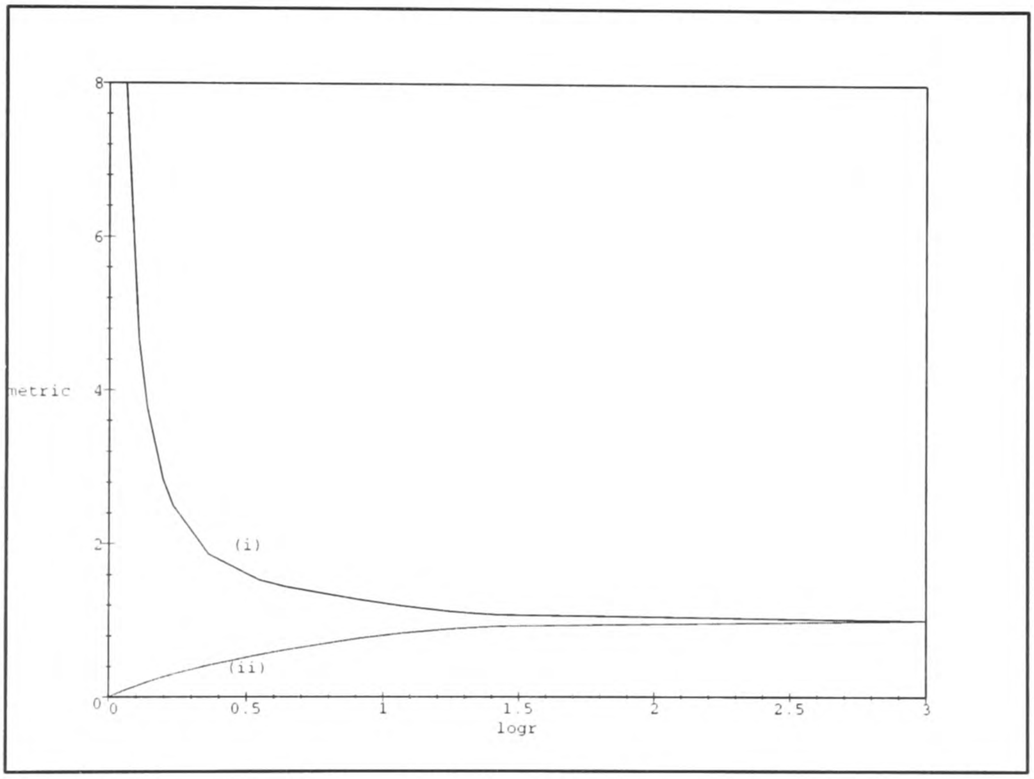

Figure 2.1: The metric functions (i) $g_{rr}(r)$ and (ii) $-g_{tt}(r)$ plotted against $\log r$ for the $v = 0.3$, quasi-$k = 0$ solution. The behaviour of the metric functions is typically the same for both branches and all values of v.

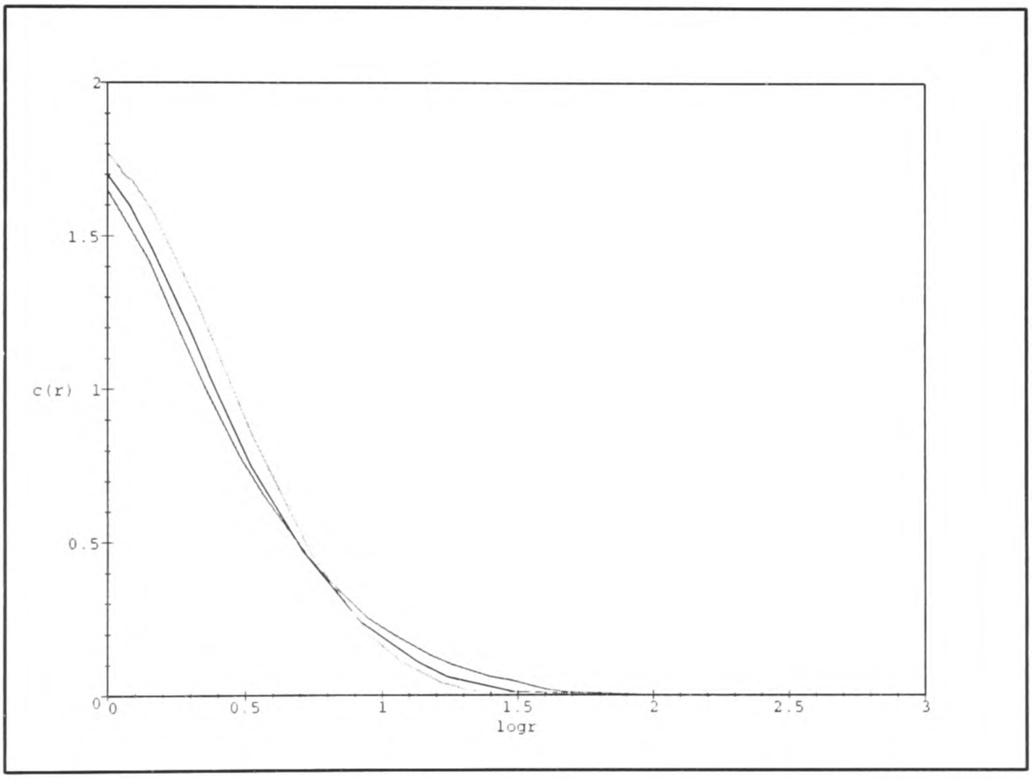

Figure 2.2: The gauge field $c(r) = 1 + \omega(r)$ for the $\kappa = 1$ branch of solutions, and for $v = 0.1$, 0.2 and 0.3. As v increases, the value of c at the event horizon increases. Otherwise there is little variation in the functions as v varies.

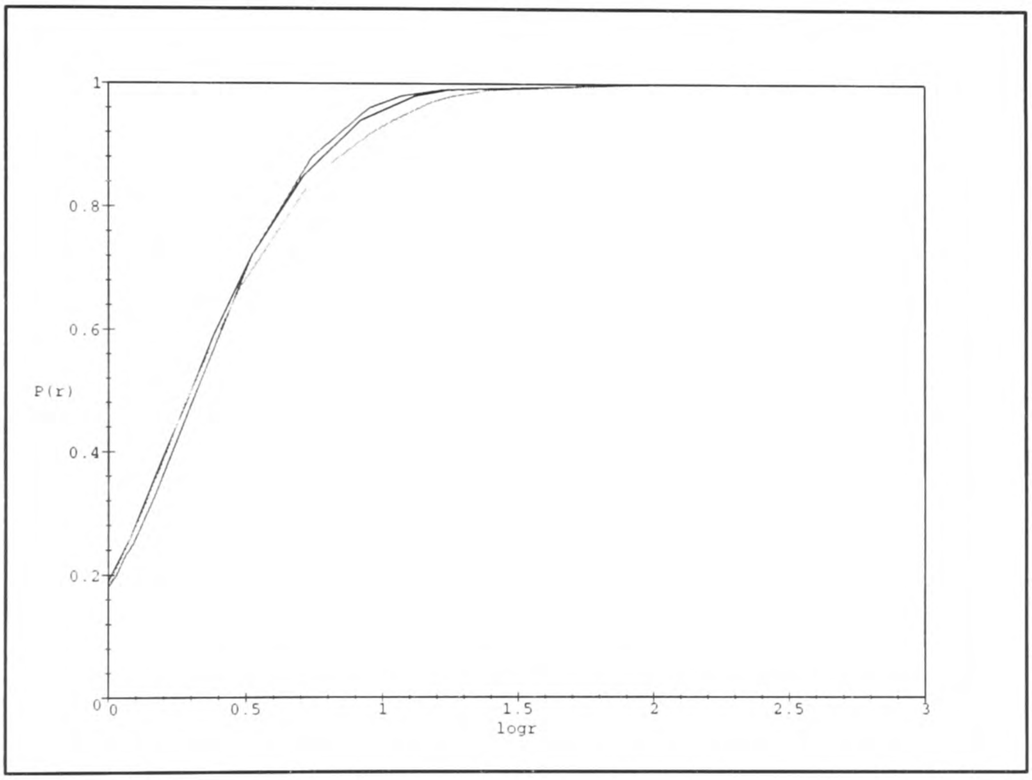

Figure 2.3: The Higgs field $P(r) = \frac{\phi}{v}$ for the $\kappa = 1$ branch of solutions.

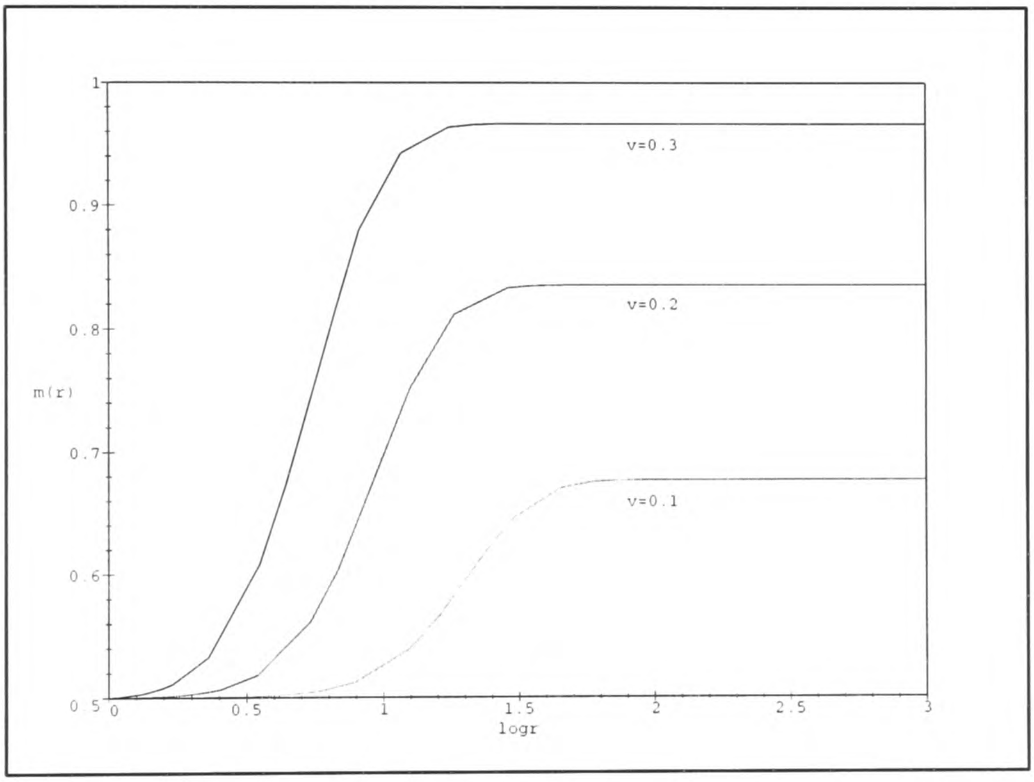

Figure 2.4: The metric function $m(r)$ for the quasi-$k = 0$ black holes. The ADM mass decreases as v decreases, approaching the Schwarzschild mass $0.5r_h$ as $v \to 0$.

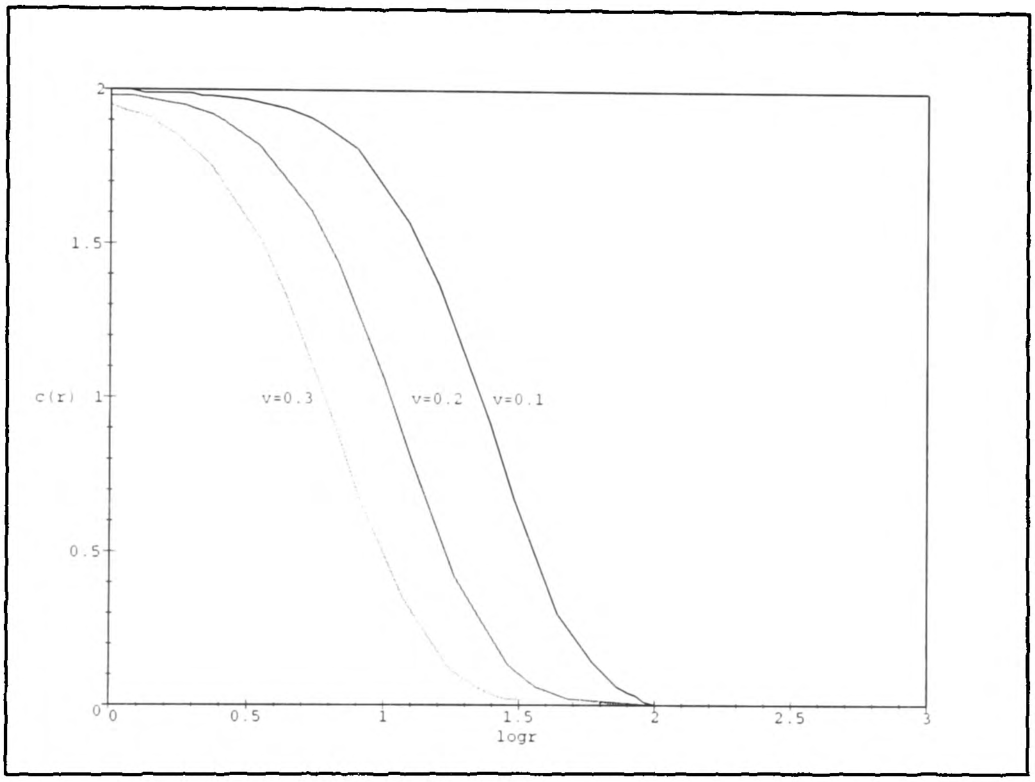

Figure 2.5: The gauge field $c(r)$ for the quasi-$k=0$ solutions. In the limit $v \to 0$, observe that $c(r) \equiv 2$ everywhere.

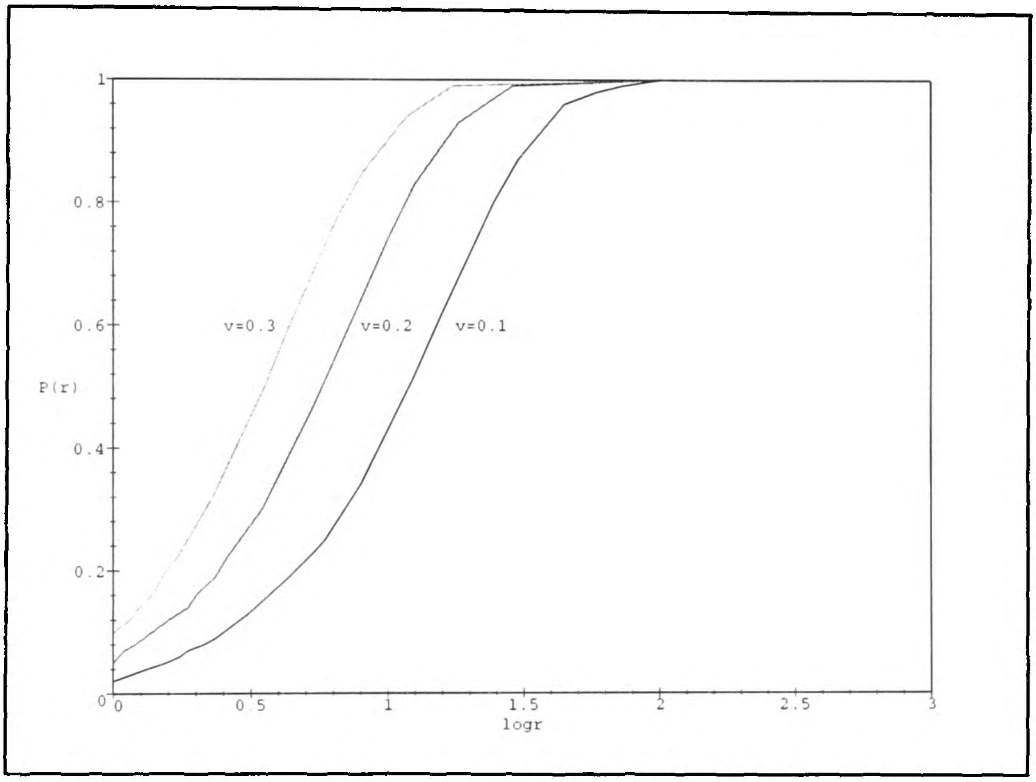

Figure 2.6: The Higgs field $P(r)$ for the quasi-$k = 0$ solutions. In this case $P(r) \to 0$ outside the event horizon in the limit $v \to 0$.

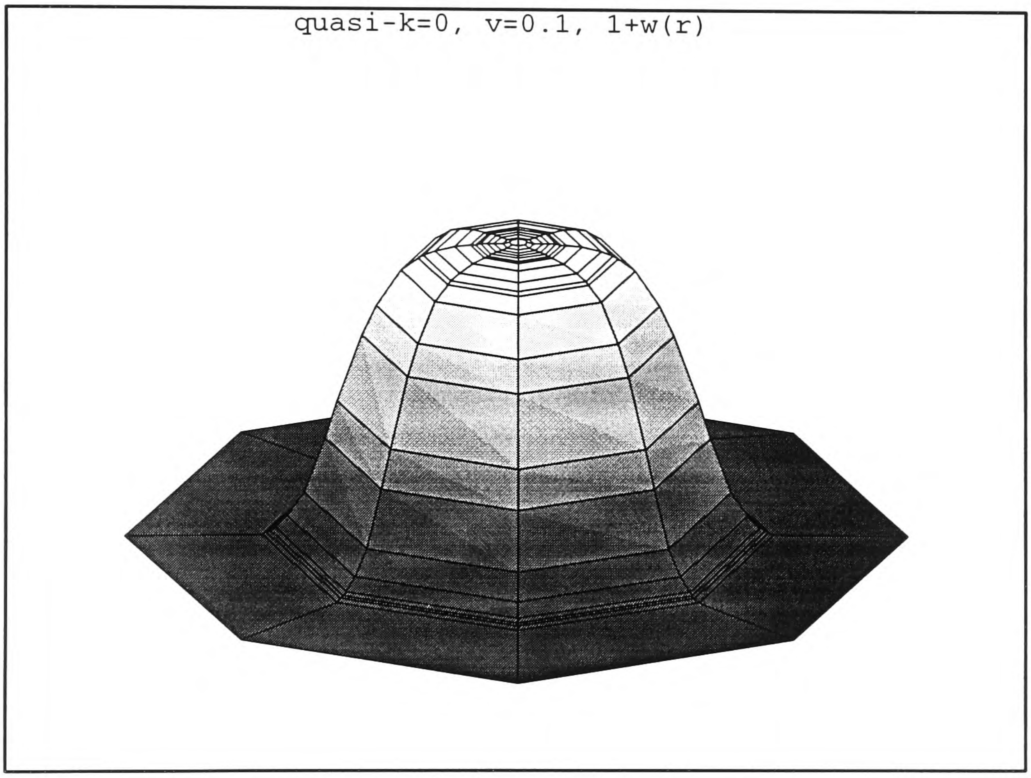

Figure 2.7: A three-dimensional representation of the gauge field hair for the $v = 0.1$, quasi-$k = 0$ black hole. The value of $1 + \omega(r)$ is plotted vertically as a function of the variables r and θ used as plane polar co-ordinates in the horizontal plane. The co-ordinate φ is suppressed. The black hole event horizon is situated at the top of the peak, and from this picture we can see how the gauge field decays away at infinity.

its value for the smallest value of ω_h along these branches, which is $\omega_h = 0.632$, whence

$$(1-\omega_h^2)^2 \leq 0.360. \tag{2.113}$$

Along both these branches,

$$\phi_h \leq 0.19v \tag{2.114}$$

which gives the following bound on the second term in (2.112)

$$\begin{aligned} \frac{1}{2}\phi_h^2(1+\omega_h)^2 &\leq 2\times(0.19v)^2 \\ &\leq 2\times 0.19^2 \times 0.352^2 \\ &= 8.95\times 10^{-3}. \end{aligned} \tag{2.115}$$

Finally, for the last term we have

$$\begin{aligned} \frac{\lambda}{2}(\phi_h^2 - v^2)^2 &\leq \frac{\lambda}{2}v^4 \leq 0.15\times 0.5\times 0.352^4 \\ &= 1.15\times 10^{-3}. \end{aligned} \tag{2.116}$$

Adding together all these contributions, we find that

$$\begin{aligned} m'(1) &\leq 0.5\times\left[0.360 + 8.95\times 10^{-3} + 1.15\times 10^{-3}\right] \\ &= 0.185 \leq 0.5, \end{aligned} \tag{2.117}$$

and hence all the black holes considered in this chapter are non-extremal.

For higher-node solutions, v is extremely small [2] and $|\omega_h|$ also becomes very small [18]. Hence, as $n \to \infty$, the black hole solutions come ever closer to extremality.

2.3 Eluding Bekenstein's no-hair theorem in EYMH systems

The "no-hair" theorem of Bekenstein [1] rules out the existence of hair due to a Higgs field with a double-well potential. In this section we follow the method of [3], and study carefully the mechanism by which the theorem is evaded in the EYMH case, which corresponds physically to a balancing between the repulsive gauge field force and the attraction of gravity. This analytical evidence for the existence of the solutions is important because the black holes have only been found numerically to date [3].

2.3.1 Analysis of the energy-momentum tensor

Solution of the conservation equations Consider EYMH theory with the Lagrangian (2.11)

$$\mathcal{L}_{EYMH} = -\frac{1}{4\pi}\left\{\frac{1}{4}|F_{\mu\nu}|^2 + \frac{1}{8}\phi^2|A_\mu|^2 + \frac{1}{2}|\partial_\mu\phi|^2 + V(\phi)\right\}. \tag{2.118}$$

The energy-momentum tensor of this Lagrangian is then (2.15):

$$\begin{aligned} 8\pi T_{\mu\nu} &= 2F^a_{\mu\gamma}F^{a\gamma}_{\mu} - \frac{1}{2}g_{\mu\nu}|F|^2 + \frac{1}{2}\phi^2 A^a_\mu A^a_\nu \\ &\quad -\frac{1}{4}g_{\mu\nu}\phi^2|A|^2 + 2\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}|\partial\phi|^2 - 2g_{\mu\nu}V(\phi) \end{aligned} \tag{2.119}$$

which can be rewritten in the form

$$T_{\mu\nu} = -\mathcal{E}g_{\mu\nu} + \frac{1}{4\pi}\left\{F^a_{\mu\gamma}F^{a\gamma}_{\nu} + \frac{\phi^2}{4}A^a_\mu A^a_\nu + \partial_\mu\phi\partial_\nu\phi\right\} \tag{2.120}$$

where

$$\mathcal{E} = -\mathcal{L}_{EYMH}. \tag{2.121}$$

Next consider an observer moving with 4-velocity U^α. The observer sees a local energy density

$$\rho = \mathcal{E} + \frac{1}{4\pi}\left\{U^\mu F^a_{\mu\gamma}F^{a\gamma}_{\nu}U^\nu + \frac{\phi^2}{4}(U^\mu A_\mu)^2 + (U^\mu\partial_\mu\phi)^2\right\}, \tag{2.122}$$

where $U^\alpha U_\alpha = -1$. For space-times possessing a time-like Killing vector, the expression (2.122) can be simplified for observers moving along this Killing vector. In this case $U^\alpha\partial_\alpha\phi = 0$, and by an appropriate choice of gauge we may set

$$U^\mu A_\mu = 0 = U^\mu F_{\mu\nu}. \tag{2.123}$$

This choice of gauge is compatible with the ansatz of section 2.2 and [2] for the gauge potential A_μ. Under these conditions, then,

$$\rho = \mathcal{E} \tag{2.124}$$

and the requirement that the local energy density measured by any observer is non-negative implies that

$$\mathcal{E} \geq 0. \tag{2.125}$$

This is precisely the energy condition stipulated in section 1.3, which holds here as well.

We now turn to the particular case of a static, spherically symmetric metric of the form [40]

$$ds^2 = -e^\Gamma dt^2 + e^\Lambda dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{2.126}$$

where Γ and Λ are functions of r alone and in addition we assume that space-time is asymptotically flat. The action

$$\mathcal{S}_{EYMH} = \int \mathcal{L}_{EYMH}\sqrt{-g}\, d^4x \tag{2.127}$$

is invariant under general co-ordinate transformations and hence the stress tensor is conserved [38]:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.128}$$

The r-component of equation (2.128) is

$$\left[\sqrt{-g}\, T^r_r\right]' - \frac{1}{2}\sqrt{-g}(\partial_r g_{\alpha\beta})T^{\alpha\beta} = 0. \tag{2.129}$$

The spherical symmetry of the space-time implies that

$$T^\theta_\theta = T^\varphi_\varphi \tag{2.130}$$

and then (2.129) becomes

$$\left(e^{\frac{1}{2}(\Gamma+\Lambda)} r^2 T^r_r\right)' - \frac{1}{2}e^{\frac{1}{2}(\Gamma+\Lambda)} r^2 \left[\Gamma' T^t_t + \Lambda' T^r_r + \frac{4}{r}T^\theta_\theta\right] = 0. \tag{2.131}$$

The terms containing Λ' cancel to give

$$\left(e^{\frac{1}{2}\Gamma} r^2 T^r_r\right)' = \frac{1}{2}e^{\frac{1}{2}\Gamma} r^2 \left[\Gamma' T^t_t + \frac{4}{r}T^\theta_\theta\right]. \tag{2.132}$$

Integrate from the horizon radius r_h to generic r to obtain

$$e^{\frac{1}{2}\Gamma} r^2 T^r_r(r) = \frac{1}{2}\int_{r_h}^{r} e^{\frac{1}{2}\Gamma} r'^2 \left[\Gamma' T^t_t + \frac{4}{r'}T^\theta_\theta\right] dr' + e^{\frac{1}{2}\Gamma} r^2 T^r_r\Big|_{r=r_h}. \tag{2.133}$$

In order to have a regular event horizon at $r = r_h$, it must be the case that scalar invariants such as

$$T_{\alpha\beta}T^{\alpha\beta} = (T^t_t)^2 + (T^r_r)^2 + (T^\theta_\theta)^2 + (T^\varphi_\varphi)^2 \tag{2.134}$$

are finite there. Hence T^r_r is finite at $r = r_h$ and since $e^\Gamma = 0$ at the horizon, the boundary term in (2.133) vanishes. With this fact, and performing the differentiation in (2.132) explicitly, we have two equations for $T^r_r(r)$ and $(T^r_r)'(r)$:

$$\begin{aligned} T^r_r(r) &= \frac{e^{-\frac{1}{2}\Gamma}}{2r^2}\int_{r_h}^{r} e^{\frac{1}{2}\Gamma} r'^2 \left[\Gamma' T^t_t + \frac{4}{r'}T^\theta_\theta\right] dr', \\ (T^r_r)'(r) &= \frac{1}{2}\left[\Gamma' T^t_t + \frac{4}{r}T^\theta_\theta\right] - \frac{e^{-\frac{1}{2}\Gamma}}{r^2}\left(e^{\frac{1}{2}\Gamma} r^2\right)' T^r_r. \end{aligned} \tag{2.135}$$

Form of $T_{\mu\nu}$ in the EYMH case Next we restrict attention to Yang-Mills fields of the form [2] (cf. section 2.2)

$$A = [1 + \omega(r)]\,[-\hat{\tau}_\varphi d\theta + \hat{\tau}_\theta \sin\theta\, d\varphi]\,. \tag{2.136}$$

Substituting in (2.119) gives the functional forms of the components of the stress tensor as

$$\begin{aligned} T_t^t &= -\mathcal{E} \\ T_r^r &= -\mathcal{E} + \mathcal{F} \\ T_\theta^\theta &= -\mathcal{E} + \mathcal{G} \end{aligned} \tag{2.137}$$

where

$$\begin{aligned} \mathcal{E} &= \frac{1}{4\pi}\left[\frac{\omega'^2}{r^2}e^{-\Lambda} + \frac{(1-\omega^2)^2}{2r^4} + \frac{\phi^2(1+\omega)^2}{4r^2} + \frac{1}{2}\phi'^2 e^{-\Lambda} + \frac{\lambda}{4}(\phi^2 - v^2)^2\right]. \\ \mathcal{F} &= \frac{e^{-\Lambda}}{4\pi}\left[\frac{2\omega'^2}{r^2} + \phi'^2\right] \\ \mathcal{G} &= \frac{1}{4\pi}\left[\frac{\omega'^2}{r^2}e^{-\Lambda} + \frac{(1-\omega^2)^2}{r^4} + \frac{\phi^2}{4r^2}(1+\omega^2)\right]. \end{aligned} \tag{2.138}$$

Note that in this situation it is not the case that

$$T_t^t = T_\theta^\theta, \tag{2.139}$$

which formed an important part of the analysis of section 1.3. Substituting (2.137) in (2.135) gives

$$\begin{aligned} T_r^r(r) &= \frac{e^{-\frac{\Gamma}{2}}}{r^2}\int_{r_h}^r \left[-(e^{\frac{\Gamma}{2}}r^2)'\mathcal{E} + \frac{2}{r}\mathcal{G}\right] dr \\ (T_r^r)'(r) &= -\frac{e^{-\frac{\Gamma}{2}}}{r^2}(e^{\frac{\Gamma}{2}}r^2)'\mathcal{F} + \frac{2}{r}\mathcal{G}. \end{aligned} \tag{2.140}$$

At this point a couple of comments are in order. Firstly, $\mathcal{E}$ is always positive (due the energy consideration given earlier), as also is $\mathcal{F}$. The term $\mathcal{G}$ is dependent on the gauge field and vanishes in the case where the gauge term is not present in the Lagrangian (2.118), i.e. in the pure Higgs theory case.

Solution of the Einstein equations We now turn to the Einstein equations, following [1], the relevant ones for this system being [40]

$$\begin{aligned} e^{-\Lambda}\left(\frac{1}{r^2} - \frac{\Lambda'}{r}\right) - \frac{1}{r^2} &= 8\pi T_t^t = -8\pi\mathcal{E} \\ e^{-\Lambda}\left(\frac{1}{r^2} + \frac{\Gamma'}{r}\right) - \frac{1}{r^2} &= 8\pi T_r^r. \end{aligned} \tag{2.141}$$

Integrating the first of these gives

$$e^{-\Lambda} = 1 - \frac{8\pi}{r}\int_{r_h}^{r} \mathcal{E} r^2 dr - \frac{2\mathcal{M}_B}{r} \tag{2.142}$$

where $\mathcal{M}_B$ is a constant of integration.

The required asymptotic flatness of space-time implies that $\mathcal{E} = O(r^{-3})$ as $r \to \infty$, so that $\Lambda \sim r^{-1}$. In order that $e^{\Lambda} \to \infty$ at the event horizon $r \to r_h$, the constant $\mathcal{M}_B$ is fixed by

$$\mathcal{M}_B = \frac{r_h}{2}, \tag{2.143}$$

the "bare" mass of the black hole. The second Einstein equation can now be rewritten in the form

$$\frac{e^{-\frac{\Gamma}{2}}}{r^2}\left(r^2 e^{\frac{\Gamma}{2}}\right)' = \left(4\pi r T^r_r + \frac{1}{2r}\right)e^{\Lambda} + \frac{3}{2r}. \tag{2.144}$$

2.3.2 Behaviour of $T_{\mu\nu}$ at infinity

We now consider the behaviour of T^r_r as $r \to \infty$. Asymptotically, $e^{\Gamma} \to 1$ and so the leading behaviour of $(T^r_r)'(r)$ is

$$(T^r_r)'(r) = \frac{2}{r}(\mathcal{G} - \mathcal{F}). \tag{2.145}$$

The asymptotic behaviour of the fields (ω, ϕ) is (2.89):

$$\begin{aligned} \omega(r) &\sim -1 + \omega_\infty e^{-\frac{v}{2}r} + \frac{\omega_\infty}{vr} e^{-\frac{v}{2}r} \\ \phi(r) &\sim v + \phi_\infty e^{-\sqrt{2}\mu r} \end{aligned} \tag{2.146}$$

where ϕ_∞, ω_∞ are constants and $\sqrt{2}\mu = \sqrt{2\lambda}v$ is the Higgs field mass. Hence the leading asymptotic behaviour of $\mathcal{F}$ and $\mathcal{G}$ is

$$\begin{aligned} \mathcal{F} &\sim \frac{1}{4\pi}\left[\frac{\omega_\infty^2 v^2}{2r^2} e^{-vr} + \frac{\omega_\infty^2 v}{r^3} e^{-vr} - \frac{\omega_\infty^2 v^2 M}{r^3} e^{-vr} + 2\phi_\infty^2 \mu^2 e^{-2\sqrt{2}\mu r}\right] \\ \mathcal{G} &\sim \frac{1}{4\pi}\left[\frac{\omega_\infty^2 v^2}{2r^2} e^{-vr} + \frac{\omega_\infty^2 v}{r^3} e^{-vr} - \frac{\omega_\infty^2 v^2 M}{2r^3} e^{-vr}\right] \end{aligned} \tag{2.147}$$

since $e^{-\Lambda} \to 1$ asymptotically. The leading order behaviour of $(T^r_r)'(r)$ is then

$$(T^r_r)'(r) \sim \frac{1}{4\pi}\left[\frac{\omega_\infty^2 v^2 M}{2r^3} e^{-vr} - 2\phi_\infty^2 \mu^2 e^{-2\sqrt{2}\mu r}\right]. \tag{2.148}$$

There are two possible cases:

1. $2\sqrt{2}\mu > v$ (corresponding to $\lambda > \frac{1}{8}$), in which case $(T^r_r)'(r) > 0$ asymptotically;

2. $2\sqrt{2}\mu \leq v$ (corresponding to $\lambda \leq \frac{1}{8}$), when $(T_r^r)'(r) < 0$ asymptotically;

Since $\mathcal{G}$ vanishes exponentially (2.147), and $\mathcal{E} = O(r^{-3})$ as $r \to \infty$, the integral (2.140) defining T_r^r converges and $|T_r^r| = O(r^{-2})$ as $r \to \infty$. Thus, in case (1) above, T_r^r is negative and increasing as $r \to \infty$, whereas in case (2) T_r^r is positive and decreasing.

2.3.3 Behaviour of $T_{\mu\nu}$ at the horizon

Now turn to the behaviour of T_r^r at the horizon. When $r \sim r_h$, both $\mathcal{E}$ and $\mathcal{G}$ are finite, and Γ' diverges as $(r - r_h)^{-1}$. Thus the main contribution to T_r^r as $r \sim r_h$ is

$$T_r^r = -\frac{e^{-\frac{\Gamma}{2}}}{2r^2}\int_{r_h}^{r} e^{\frac{\Gamma}{2}} r^2 \Gamma' \mathcal{E}\, dr \tag{2.149}$$

which is finite. At the horizon $e^{\Gamma} = 0$; outside the horizon $e^{\Gamma} > 0$. Hence $\Gamma' > 0$ sufficiently close to the horizon, and since $\mathcal{E} > 0$, then $T_r^r < 0$ for r sufficiently close to the horizon.

Since $\mathcal{F} = O(r - r_h)$ for $r \sim r_h$, it follows that $(T_r^r)'$ is finite at the horizon and the leading contribution is

$$(T_r^r)'(r) = -\frac{\Gamma'}{2}\mathcal{F} + \frac{2}{r}\mathcal{G} \tag{2.150}$$

where Γ', $\mathcal{F}$ and $\mathcal{G}$ are all positive sufficiently close to the horizon. In order to determine whether (2.150) is positive or negative, more detailed analysis is required.

2.3.4 Determination of the sign of $(T_r^r)'(r_h)$

We begin with [2] (see also section 2.2),

$$re^{-\Lambda}\frac{\Gamma'}{2} = e^{-\Lambda}\left[\omega'^2 + \frac{1}{2}r^2\phi'^2\right] - \frac{1}{2}\frac{(1-\omega^2)^2}{r^2} - \frac{1}{4}\phi^2(1+\omega^2)^2 + \frac{m}{r} - \frac{\lambda}{4}(\phi^2 - v^2)^2 r^2 \tag{2.151}$$

where

$$e^{-\Lambda} \equiv 1 - \frac{2m(r)}{r}. \tag{2.152}$$

Hence, near the horizon,

$$\begin{aligned}(T_r^r)'(r) &= -\frac{1}{4\pi r}\left[\frac{2\omega'^2}{r^2} + \phi'^2\right]\left\{-\frac{1}{2}\frac{(1-\omega^2)}{r^2} - \frac{\phi^2}{4}(1+\omega)^2 + \frac{m}{r} - \frac{\lambda}{4}(\phi^2 - v^2)r^2\right\} \\ &\quad + \frac{1}{2\pi r}\left[\frac{(1-\omega^2)^2}{r^4} + \frac{\phi^2}{4r^2}(1+\omega)^2\right]\end{aligned} \tag{2.153}$$

which simplifies, since $m(r_h) = \frac{1}{2}r_h$, to

$$(T_r^r)'(r_h) = \tilde{\mathcal{F}}(r_h)\left[\frac{(1-\omega_h^2)^2}{2r_h^2} + \frac{\phi_h^2}{4r_h}(1+\omega_h)^2 + \frac{\lambda}{4}r_h(\phi_h^2 - v^2)^2 - \frac{1}{2r_h}\right] + \frac{2}{r_h}\mathcal{G}_h \tag{2.154}$$

where

$$\tilde{\mathcal{F}}(r_h) = e^{\Lambda}\mathcal{F}(r)\,|_{r=r_h} = \frac{1}{4\pi}\left[\frac{2\omega_h'^2}{r_h^2} + \phi_h'^2\right] \tag{2.155}$$

and $\mathcal{G}_h = \mathcal{G}(r_h)$.

Consider for simplicity the case $r_h = 1$. From the field equations (2.54) and (2.55) we have

$$\begin{aligned} \omega_h' &= \frac{1}{\mathcal{D}_h}\left[\frac{1}{4}\phi_h^2(1+\omega_h) - \omega_h(1-\omega_h^2)\right] = \frac{\mathcal{A}_h}{\mathcal{D}_h} \\ \phi_h' &= \frac{1}{\mathcal{D}_h}\left[\frac{1}{2}\phi_h(1+\omega_h)^2 + \lambda\phi_h(\phi_h^2 - v^2)\right] = \frac{\mathcal{B}_h}{\mathcal{D}_h} \end{aligned} \tag{2.156}$$

where

$$\mathcal{D}_h = 1 - (1-\omega_h^2)^2 - \frac{\phi_h^2}{2}(1+\omega_h)^2 - \frac{\lambda}{2}(\phi_h^2 - v^2)^2. \tag{2.157}$$

Then (2.154) becomes

$$(T_r^r)'(r_h) = \frac{1}{4\pi\mathcal{D}_h}\left[8\pi\mathcal{D}_h\mathcal{G}_h - \mathcal{A}_h^2 - \frac{1}{2}\mathcal{B}_h^2\right] = \frac{\mathcal{C}_h}{4\pi\mathcal{D}_h}. \tag{2.158}$$

From the field equations (2.56),

$$\mathcal{D}_h = 1 - 2m_h' \tag{2.159}$$

which is always positive because the black holes are non-extremal (see section 2.2.9). Thus the sign of $(T_r^r)'(r_h)$ is the same as that of $\mathcal{C}_h$.

2.3.5 Evaluation of $\mathcal{C}_h$

Simplifying, we have

$$\mathcal{C}_h = c_1 + c_2 + c_3 + c_4 + c_5 \tag{2.160}$$

where

$$\begin{aligned} c_1 &= (1-\omega_h^2)^2\omega_h^2(3 - 2\omega_h^2) \\ c_2 &= \frac{1}{8}\phi_h^2(1+\omega_h)(1-\omega_h^2)(12\omega_h^3 + 12\omega_h^2 - 7\omega_h - 9) \\ c_3 &= -\frac{1}{16}\phi_h^4(1+\omega_h)^2(4\omega_h^2 + 8\omega_h + 5) \\ c_4 &= -\lambda(\phi_h^2 - v^2)^2\left[\frac{1}{2}\lambda\phi_h^2 + (1-\omega_h^2)^2\right] \\ c_5 &= \frac{1}{4}\lambda\phi_h^2(v^2 - \phi_h^2)(1+\omega_h)^2(2 + \phi_h^2 - v^2). \end{aligned} \tag{2.161}$$

The first term is always positive since $|\omega_h| \leq 1$. The cubic in c_2 possesses a local maximum at $\omega_h = -0.886$ where it has the value -1.724 and a local minimum at $\omega_h = 0.219$ where it equals -9.831. The cubic has a single root at $\omega_h = 0.8215$, and thus is positive for $\omega_h > 0.8215$ and negative for $\omega_h < 0.8215$. The quadratic in c_3 is always positive and possesses no real roots. It has a minimum value of 1 when $\omega_h = -1$. The term c_4 is always negative and c_5 is always positive since $|\phi_h| \leq v$.

In order to assess whether or not $\mathcal{C}_h$ as a whole is positive or negative, we shall consider each branch of black hole solutions in turn. We restrict ourselves to the first two branches of solutions, with $k = 1$ and the case $\lambda = 0.15$.

The $k = 1$ branch of solutions Firstly, consider the $k = 1$ branch of solutions. As v increases from 0 up to $v_{max} = 0.352$, the parameter ω_h increases monotonically from 0.632 to 0.869. The derivative of the first term is

$$\frac{dc_1}{d\omega_h} = 2\omega_h(1 - \omega_h^2)(3 - 13\omega_h^2 + 8\omega_h^4) \tag{2.162}$$

where the quartic has roots given by $\omega_h^2 = 0.278$ or 1.347. The derivative (2.162) is negative for $\omega_h \in (0.632, 0.869)$ and hence the first term decreases as v increases, and is bounded below by its value at the bifurcation point $v = v_{max}$, which is

$$c_1 \geq 0.0674. \tag{2.163}$$

The cubic in c_2 increases as ω_h increases from 0.632 to 0.869 and is bounded below by its value when $\omega_h = 0.632$, which is

$$12\omega_h^3 + 12\omega_h^2 - 7\omega_h - 9 \geq -5.602. \tag{2.164}$$

Along this branch of solutions, it is also true that

$$\begin{aligned} 1 - \omega_h^2 &\leq 1 - (0.632)^2 = 0.601 \\ 1 + \omega_h &\leq 2 \\ \phi_h &\leq 0.19v \leq 0.0669. \end{aligned} \tag{2.165}$$

Altogether this gives

$$c_2 \geq -5.602 \times \frac{1}{8} \times (0.0669)^2 \times 2 \times 0.601 = -3.767 \times 10^{-3}. \tag{2.166}$$

The quadratic in c_3 increases as ω_h increases along the branch and so is bounded above by its value when $\omega_h = 0.869$, which is

$$4\omega_h^2 + 8\omega_h + 5 \leq 14.973. \tag{2.167}$$

Thus,

$$c_3 \geq -\frac{1}{16} \times (0.669)^4 \times 4 \times 14.973 = -7.498 \times 10^{-5}. \tag{2.168}$$

For the fourth term, since

$$(\phi_h^2 - v^2)^2 \leq v^4 \tag{2.169}$$

we have

$$\begin{aligned} c_4 &\geq -0.15 \times (0.352)^4 \times \left[(0.601)^2 + \frac{1}{2} \times 0.15 \times (0.0669)^2\right] \\ &= -8.326 \times 10^{-4}; \end{aligned} \tag{2.170}$$

and finally

$$c_5 \geq 0. \tag{2.171}$$

Thus, adding these expressions up, one obtains

$$\begin{aligned} \mathcal{C}_h &\geq 0.0674 - 3.767 \times 10^{-3} - 7.498 \times 10^{-5} - 8.326 \times 10^{-4} \\ &= 0.0627 \\ &> 0, \end{aligned} \tag{2.172}$$

so that $(T_r^r)'(r_h) > 0$ along the whole of the $k = 1$ branch of black hole solutions.

The quasi-$k = 0$ branch of solutions For the quasi-$k = 0$ branch, as v decreases from v_{max} down to 0, then ω_h increases monotonically from 0.869, subject to the inequality

$$0.869 \leq \omega_h \leq 1 - 0.1v^2 \tag{2.173}$$

(cf. [2], the above inequality being numerically verified in our case). Hence, along this branch,

$$(1 - \omega_h^2)^2 = (1 - \omega_h)^2(1 + \omega_h)^2 \geq (0.1v^2)^2 \times (1.869)^2 = 0.0349v^4. \tag{2.174}$$

Thus the first term in $\mathcal{C}_h$ is bounded below as follows:

$$c_1 \geq 0.0349v^4 \times (0.869)^2 \times 1 = 0.0264v^4. \tag{2.175}$$

Since $\omega_h \geq 0.869$, the cubic in c_2 is positive all along this branch, so that c_2 and c_5 are positive.

The quadratic in c_3 is bounded above by its value when $\omega_h = 1$, i.e.

$$4\omega_h^2 + 8\omega_h + 5 \leq 17. \tag{2.176}$$

All along the quasi-$k = 0$ branch,

$$\phi_h \leq 0.5v^2 \tag{2.177}$$

(again cf. [2]; we have checked this equality numerically here also). This gives

$$\begin{aligned} c_3 &\geq -\frac{1}{16} \times (0.5v^2)^4 \times 4 \times 17 \\ &= 0.266v^8. \end{aligned} \tag{2.178}$$

Since $v \le 0.352$, it follows that

$$\begin{aligned} c_3 &\ge -0.266 \times (0.352)^4 \times v^4 \\ &= -4.084 \times 10^{-3} v^4. \end{aligned} \tag{2.179}$$

Finally, for c_4 we have

$$\begin{aligned} c_4 &\ge -0.15v^4 \left[\frac{1}{2} \times 0.15 \times (0.5)^2 v^4 + (1-(0.869)^2)^2\right] \\ &= -8.992 \times 10^{-3} v^4 - 2.813 \times 10^{-3} v^8 \\ &\ge -8.992 \times 10^{-3} v^4 - 2.813 \times 10^{-3} \times (0.352)^4 v^4 \\ &= -9.035 \times 10^{-3} v^4. \end{aligned} \tag{2.180}$$

In total this gives

$$\begin{aligned} \mathcal{C}_h &\ge 0.0264v^4 - 4.084 \times 10^{-3} v^4 - 9.035 \times 10^{-3} v^4 \\ &= 1.328 \times 10^{-2} v^4 \\ &\ge 0. \end{aligned} \tag{2.181}$$

In conclusion then, $(T_r^r)'(r_h)$ is positive for all the black hole solutions having one node in ω, regardless of the value of the Higgs v.e.v. v.

Higher node solutions For higher node solutions, v is extremely small (for example, $v < 0.05$ for $k = 2$ [2]), and $|\omega_h|$ also becomes increasingly small [18]. In this case the expression for $(T_r^r)'(r_h)$ reduces to:

$$\begin{aligned} (T_r^r)'(r_h) &= \frac{1}{4\pi} \frac{(1-\omega_h^2)^2(3-2\omega_h^2)}{(2-\omega_h^2)} + O(v^2) \\ &\ge 0. \end{aligned} \tag{2.182}$$

Hence $(T_r^r)'(r_h) \ge 0$ for all black holes in EYMH theory.

2.3.6 Compatibility with Einstein's equations

We shall now check that there is no contradiction with Einstein's equations.

1. Firstly, consider the case $\lambda > \frac{1}{8}$. As $r \to \infty$, we saw that $T_r^r < 0$ and $(T_r^r)' > 0$. As $r \to r_h$, we have $T_r^r < 0$ and $(T_r^r)' > 0$. Hence there is no contradiction with Einstein's equations in this case.

2. Secondly, we turn to the case $\lambda \le \frac{1}{8}$. As $r \to \infty$, in this case $T_r^r > 0$ and $(T_r^r)' < 0$, whilst as $r \to r_h$, we have $T_r^r < 0$ and $(T_r^r)' > 0$. Hence there is an interval $[r_a, r_b]$ in which $(T_r^r)'$ is positive and a critical distance $r_c \in (r_a, r_b)$ at which T_r^r changes sign. However, unlike the situation for pure Higgs theory, when the gauge fields are absent [1], here there is no contradiction with Einstein's equations since $(T_r^r)' > 0$ in some open interval close to the event horizon.

2.3.7 Physical Interpretation

The preceding analysis shows that the method of section 1.3 cannot be used to prove a "no-hair" theorem for the EYMH system. This is the first analytic proof of the existence of the solutions discovered numerically [3]. The analysis provides some physical illumination as to why these solutions exist. The key difference between the scalar field and EYMH theories is the presence of the $\mathcal{G}$ term in T_θ^θ (2.137) and $(T_r^r)'$ (2.140), which vanishes when the gauge field is absent. When $\mathcal{G} = 0$, the relation $T_\theta^\theta = T_t^t$ holds, an important element of Bekenstein's proof.

From (2.150), $(T_r^r)'(r_h)$ is manifestly negative when $\mathcal{G} = 0$. The presence of the $\mathcal{G}$ term is thus crucial if we are to avoid a contradiction with Einstein's equations.

Physically this is interpreted as a type of "balancing" between the repulsive Yang-Mills gauge force and the attractive gravitational force, leading to non-trivial equilibrium black hole solutions. Heuristically, therefore, we may expect these solutions to be unstable. This will be discussed in the next chapter. Stability is an important issue in these circumstances, since the proofs of the "no-hair" theorem relate only to existence, not stability [43] (see also the discussion of section 1.4).

Summary

We have analyzed in some detail the physical mechanism whereby the EYMH black holes evade the no-hair theorem [3]. This analysis should be compared with the proof of the theorem for a scalar field *not* coupled to a non-Abelian gauge field (see section 1.3) and also with the corresponding work of section 5.2, which discusses black holes in a higher-derivative model of gravity. The evasion detailed in section 2.3 is therefore quite general in its application to various hairy black hole systems. This chapter also contains background discussion of the properties of the EYMH black holes, as found numerically in [2].

Chapter 3

Stability Analysis of the EYMH Black Holes

This chapter contains a detailed stability analysis of the EYMH black holes [3]. We begin by deriving the linearized perturbation equations, which decouple into two sectors, and then count the number of modes of instability in each sector. Firstly, we use a variational approach to show the existence of unstable modes. The exact counting of the number of such modes will involve continuity arguments and an application of catastrophe theory (a brief summary of the important concepts in catastrophe theory can be found in appendix A).

3.1 Linear perturbation equations

In this section we consider small-amplitude, spherically symmetric perturbations of the static equilibrium solutions, using the notation

$$m(r,t) = m_e(r) + \delta m(r,t) \tag{3.1}$$

where $m_e(r)$ is the static equilibrium function and $\delta m(r,t)$ is the perturbation, and similarly for all other quantities. Henceforth we drop the subscript e and assume that any quantity not prefaced by δ is an equilibrium quantity. For the whole of this section, we work to first order in the perturbations only, and follow the method of [44, 45].

Firstly, consider the perturbations of the metric functions. We have

$$\delta N = -2\frac{\delta m}{r}, \tag{3.2}$$

and hence from (2.49),

$$\begin{aligned}
\delta m' &= \frac{\mathtt{G}}{r}\delta m + \frac{1}{2}(N\delta\mathtt{G} + r^2\delta p_\theta) \\
&= -\frac{S'}{S}\delta m + \frac{1}{2}(N\delta\mathtt{G} + r^2\delta p_\theta)
\end{aligned} \tag{3.3}$$

using the equilibrium Einstein equation (2.51), where

$$\delta \mathtt{G} = 4c'\delta c' + 2r^2\phi'\delta\phi' \tag{3.4}$$

$$\delta p_\theta = \frac{4}{r^4}(c^2-2c)(c-1)\delta c + \frac{1}{r^2}\phi^2 c\delta c + \frac{1}{r^2}\phi c^2\delta\phi + 2\lambda\phi(\phi^2-v^2)\delta\phi. \tag{3.5}$$

From (3.3), then,

$$\begin{aligned}(S\delta m)' &= \frac{S}{2}(N\delta\mathtt{G} + r^2\delta p_\theta)\\ &= (2NSc'\delta c + NSr^2\phi'\delta\phi)'\end{aligned} \tag{3.6}$$

using (3.5) and the static field equations (2.47) and (2.48) for c and ϕ. Hence δm has the form

$$\delta m = 2Nc'\delta c + Nr^2\phi'\delta\phi + S^{-1}\mathcal{X}(t) \tag{3.7}$$

where $\mathcal{X}(t)$ is an arbitrary function of t. Using the t, r component of the Einstein equations (2.16),

$$\delta\dot{m} = N\delta\mathtt{H} \tag{3.8}$$

where

$$\delta\mathtt{H} = 2c'\delta\dot{c} + r^2\phi'\delta\dot{\phi}, \tag{3.9}$$

which gives

$$\delta m = 2Nc'\delta c + r^2N\phi'\delta\phi + \mathcal{Y}(r) \tag{3.10}$$

for $\mathcal{Y}(r)$ an arbitrary function of r. Comparing (3.7) and (3.10), we see that [45]

$$\mathcal{X}(t) \equiv 0 \equiv \mathcal{Y}(r) \tag{3.11}$$

and

$$\delta m = 2Nc'\delta c + Nr^2\phi'\delta\phi. \tag{3.12}$$

Variation of the final Einstein equation gives

$$\delta\left(\frac{S'}{S}\right) = \frac{1}{r}\delta\mathtt{G} = \frac{4c'}{r}\delta c' + 2r\phi'\delta\phi'. \tag{3.13}$$

The linearized Yang-Mills equations take the form [44, 45, 46]

$$\begin{aligned}0 &= \left(\frac{2}{rS^2} - \frac{S'}{S^3}\right)(\delta\dot{b} - \delta a') + \frac{1}{S^2}(\delta\dot{b}' - \delta a'') - \frac{2(c-1)}{NS^2r^2}(\delta\dot{d} + (c-1)\delta a)\\ &\quad + \frac{\phi^2}{4NS^2}\delta a - \frac{1}{2NS^2}\phi\delta\dot{\psi}\end{aligned} \tag{3.14}$$

$$\begin{aligned}0 &= -\frac{1}{S^2}(\delta\ddot{b} - \delta\dot{a}') - \frac{2N}{r^2}\left[(c-1)(\delta d' + (c-1)\delta b) - c'\delta d\right]\\ &\quad - \frac{N\phi^2}{4}\delta b + \frac{N}{2}(\phi'\delta\psi - \phi\delta\psi')\end{aligned} \tag{3.15}$$

$$\begin{aligned}
0 &= -\frac{1}{r^2NS^2}(\delta\ddot{d}+(c-1)\delta\dot{a})+\frac{(NS)'}{r^2S}(\delta d'+(c-1)\delta b) \\
&+\frac{N}{r^2}(\delta d''+(c-1)\delta b'+2c'\delta b)-\frac{1}{r^4}(c^2-2c)\delta d-\frac{\phi^2}{4r^2}\delta d+\frac{\phi}{2r^2}\delta\psi \quad (3.16)\\
0 &= -\frac{1}{r^2NS^2}\delta\ddot{c}+\frac{(NS)'}{r^2S}\delta c'+\frac{c'}{r^2}\delta\left(\frac{(NS)'}{S}\right)+\frac{c''}{r^2}\delta N+\frac{N}{r^2}\delta c'' \\
&-\frac{1}{r^4}(3c^2-6c+2)\delta c-\frac{\phi^2}{4r^2}\delta c-\frac{c\phi}{2r^2}\delta\phi \quad (3.17)
\end{aligned}$$

and the Higgs equations become [46]

$$\begin{aligned}
0 &= -\frac{1}{NS^2}\delta\ddot{\phi}+\left(\phi''+\frac{2}{r}\phi'\right)\delta N+N\delta\phi''+\frac{2N}{r}\delta\phi'+\frac{(NS)'}{S}\delta\phi' \\
&+\phi'\delta\left(\frac{(NS)'}{S}\right)-\frac{c^2}{2r^2}\delta\phi-\frac{c\phi}{r^2}\delta c-\lambda(3\phi^2-v^2)\delta\phi \quad (3.18)\\
0 &= -\frac{1}{NS^2}\delta\ddot{\psi}+N\delta\psi''+\left(\frac{2N}{r}+\frac{(NS)'}{S}\right)\delta\psi'+\frac{\phi}{2NS^2}\delta\dot{a}+N\phi'\delta b \\
&+\frac{N}{2}\phi\delta b'+\frac{N\phi}{r}\delta b+\frac{1}{2}\phi\frac{(NS)'}{S}\delta b+\frac{\phi}{r^2}\delta d \\
&-\frac{1}{2r^2}(c-2)^2\delta\psi-\lambda(\phi^2-v^2)\delta\psi. \quad (3.19)
\end{aligned}$$

We can set $\delta a \equiv 0$ following [46] by a choice of temporal gauge (this corresponds to a choice of the function in the remaining gauge freedom of the potential A (cf. section 2.2)). With this choice, the perturbation equations decouple into two sectors of coupled equations [47, 48]. Firstly, those for the perturbations δc and $\delta\phi$ are,

$$\begin{aligned}
0 &= -\frac{1}{r^2NS^2}\delta\ddot{c}+\frac{(NS)'}{r^2S}\delta c'+\frac{c'}{r^2}\delta\left(\frac{(NS)'}{S}\right)+\frac{c''}{r^2}\delta N+\frac{N}{r^2}\delta c'' \\
&-\frac{1}{r^4}(3c^2-6c+2)\delta c-\frac{\phi^2}{4r^2}\delta c-\frac{c\phi}{2r^2}\delta\phi \quad (3.20)\\
0 &= -\frac{1}{NS^2}\delta\ddot{\phi}+\left(\phi''+\frac{2}{r}\phi'\right)\delta N+N\delta\phi''+\frac{2N}{r}\delta\phi'+\frac{(NS)'}{S}\delta\phi' \\
&+\phi'\delta\left(\frac{(NS)'}{S}\right)-\frac{1}{2r^2}c^2\delta\phi-\frac{c\phi}{r^2}\delta c-\lambda(3\phi^2-v^2)\delta\phi. \quad (3.21)
\end{aligned}$$

These two equations govern the *gravitational sector* perturbations since, from (3.12) and (3.13) the perturbations δm and δS depend only on δc and $\delta\phi$ and not on any other perturbations.

The remaining equations reduce to a system of three coupled equations for δb, δd and $\delta\psi$,

$$0 = -\frac{1}{S^2}\delta\ddot{b}-\frac{2N}{r^2}(c-1)\delta d'-\frac{2N}{r^2}(c-1)^2\delta b+\frac{2N}{r^2}c'\delta d$$

$$-\frac{1}{4}\phi^2\delta b + \frac{1}{2}N\phi'\delta\psi - \frac{1}{2}N\phi\delta\psi' \tag{3.22}$$

$$\begin{aligned} 0 &= -\frac{1}{r^2NS^2}\delta\ddot{d} + \frac{(NS)'}{r^2S}\delta d' + \frac{(NS)'}{r^2S}(c-1)\delta b + \frac{N}{r^2}\delta d'' + \frac{N}{r^2}(c-1)\delta b' \\ &\quad + \frac{2N}{r^2}c'\delta b - \frac{1}{r^4}(c^2-2c)\delta d - \frac{1}{4r^2}\phi^2\delta d + \frac{1}{2r^2}\phi\delta\psi \end{aligned} \tag{3.23}$$

$$\begin{aligned} 0 &= -\frac{1}{NS^2}\delta\ddot{\psi} + N\delta\psi'' + \frac{2N}{r}\delta\psi' + \frac{(NS)'}{S}\delta\psi' + N\phi'\delta b \\ &\quad + \frac{N}{2}\phi\delta b' + \frac{N}{r}\phi\delta b + \frac{(NS)'}{2S}\phi\delta b + \frac{\phi}{r^2}\delta d \\ &\quad - \frac{1}{2r^2}(c-2)^2\delta\psi - \lambda(\phi^2-v^2)\delta\psi, \end{aligned} \tag{3.24}$$

together with the linearized Gauss constraint

$$0 = \left(\frac{2}{rS^2} - \frac{S'}{S^2}\right)\delta\dot{b} + \frac{1}{S^2}\delta\dot{b}' + \frac{2}{NS^2r^2}(c-1)\delta\dot{d} - \frac{1}{2NS^2}\phi\delta\dot{\psi}. \tag{3.25}$$

This system represents the *sphaleronic sector* since it does not involve any perturbations of the metric functions.

Consider firstly the gravitational sector. From the perturbations of the Einstein equations, we have

$$\delta\left(\frac{(NS)'}{S}\right) = \frac{2}{r^2}\delta m - r\delta p_\theta \tag{3.26}$$

and

$$\delta N = -\frac{2}{r}\delta m. \tag{3.27}$$

Using also the background field equations (2.47) and (2.48) gives

$$\begin{aligned} c'\delta\left(\frac{(NS)'}{S}\right) + c''\delta N &= \left(\frac{(NS)'}{S} + \frac{N}{r}\right)\left(\frac{4}{r}c'^2\delta c + 2\phi'c'r\delta\phi\right) \\ &\quad - \left[\frac{8c'}{r^3}(c^2-2c)(c-1) + \frac{2}{r}c'\phi^2\right]\delta c \\ &\quad - \left[\frac{2}{r}(c-1)(c^2-2c)\phi' + \frac{1}{2r^2}c\phi^2\phi' + \frac{c^2\phi}{r}c' + 2\lambda rc'\phi(\phi^2-v^2)\right]\delta\phi \end{aligned} \tag{3.28}$$

and similarly

$$\begin{aligned} \left(\phi'' + \frac{2}{r}\phi'\right)\delta N + \phi'\delta\left(\frac{(NS)'}{S}\right) &= \left(\frac{(NS)'}{S} + \frac{N}{r}\right)\left(\frac{4}{r}c'\phi'\delta c + 2r\phi'^2\delta\phi\right) \\ &\quad - \left[\frac{2}{r^3}c^2c'\phi + \frac{4}{r}\lambda\phi c'(\phi^2-v^2) + \frac{c\phi^2}{r^4}\phi' + \frac{4}{r^3}\phi'(c^2-2c)(c-1)\right]\delta c \end{aligned}$$

$$- \left[\frac{2}{r} c^2 \phi \phi' + 4\lambda \phi \phi' r (\phi^2 - v^2) \right] \delta\phi. \tag{3.29}$$

Substituting these expressions into (3.20) and (3.21) yields the equations

$$
\begin{aligned}
0 &= \delta\ddot{c} - p_*^2 \delta c + \mathcal{U}_{cc} \delta c + \mathcal{U}_{c\phi} \delta\phi && (3.30)\\
0 &= \delta\ddot{\phi} - p_*^2 \delta\phi - \frac{2NS}{r} p_* \delta\phi + \mathcal{U}_{\phi c} \delta c + \mathcal{U}_{\phi\phi} \delta\phi && (3.31)
\end{aligned}
$$

where the functions $\mathcal{U}$ are given by

$$
\begin{aligned}
\mathcal{U}_{cc} &= \frac{NS^2}{r^2} \left[3c^2 - 6c + 2 + \frac{1}{4} r^2 \phi^2 - 4rc'^2 \left(\frac{N}{r} + \frac{(NS)'}{S} \right) \right. \\
&\quad \left. + \frac{8}{r} c' (c^2 - 2c)(c-1) + 2rc'\phi^2 \right] && (3.32)\\
\mathcal{U}_{c\phi} &= \frac{NS^2}{r^2} \left[\frac{1}{2} r^2 c\phi - 2r^3 c'\phi' \left(\frac{N}{r} + \frac{(NS)'}{S} \right) + 2r\phi'(c^2 - 2c)(c-1) \right. \\
&\quad \left. + rc^2 c'\phi + \frac{1}{2} c\phi^2 \phi' + 2\lambda r^3 c'\phi(\phi^2 - v^2) \right] && (3.33)\\
\mathcal{U}_{\phi c} &= \frac{2}{r^2} \mathcal{U}_{c\phi} && (3.34)\\
\mathcal{U}_{\phi\phi} &= \frac{NS^2}{r^2} \left[\frac{1}{2} c^2 + \lambda r^2 (3\phi^2 - v^2) - 2r^3 \phi'^2 \left(\frac{N}{r} + \frac{(NS)'}{S} \right) \right. \\
&\quad \left. + 2rc^2 \phi\phi' + 4\lambda r^3 \phi\phi' (\phi^2 - v^2) \right]. && (3.35)
\end{aligned}
$$

The operator p_* is given by

$$p_* = NS \frac{d}{dr} = \frac{d}{dr^*} \tag{3.36}$$

where r^* is the usual *"tortoise"* co-ordinate defined by

$$\frac{dr^*}{dr} = \frac{1}{NS}. \tag{3.37}$$

In terms of this co-ordinate and the variable $\omega \equiv c - 1$ (whence $\delta\omega = \delta c$) the perturbation equations for the gravitational sector become

$$
\begin{aligned}
-\delta\ddot{\omega} &= -p_*^2 \delta\omega + \mathcal{U}_{\omega\omega} \delta\omega + \mathcal{U}_{\omega\phi} \delta\phi && (3.38)\\
-\delta\ddot{\phi} &= -p_*^2 \delta\phi - \frac{2NS}{r} p_* \delta\phi + \mathcal{U}_{\phi\omega} \delta\omega + \mathcal{U}_{\phi\phi} \delta\phi && (3.39)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{U}_{\omega\omega} &= \frac{NS^2}{r^2} \left[3\omega^2 - 1 + \frac{1}{4} r^2 \phi^2 - 4r^2 \omega'^2 \left(\frac{N}{r} + \frac{(NS)'}{S} \right) \right. \\
&\quad \left. + \frac{8\omega}{r} (\omega^2 - 1)\omega' + 2r\omega'\phi^2 \right] && (3.40)
\end{aligned}
$$

$$\begin{aligned}
\mathcal{U}_{\omega\phi} &= \frac{NS^2}{r^2}\left[\frac{1}{2}r^2(1+\omega)\phi - 2r^3\omega'\phi'\left(\frac{N}{r}+\frac{(NS)'}{S}\right) + 2r\omega(\omega^2-1)\phi'\right. \\
&\quad \left. + r\omega'(1+\omega)\phi + \frac{1}{2}(1+\omega)\phi^2\phi' + 2\lambda r^3\omega'\phi(\phi^2-v^2)\right] && (3.41) \\
\mathcal{U}_{\phi\omega} &= \frac{2}{r^2}\mathcal{U}_{\omega\phi} && (3.42) \\
\mathcal{U}_{\phi\phi} &= \frac{NS^2}{r^2}\left[\frac{1}{2}(1+\omega)^2 + \lambda r^2(3\phi^2-v^2) - 2r^3\phi'^2\left(\frac{N}{r}+\frac{(NS)'}{S}\right)\right. \\
&\quad \left. + 2r(1+\omega)^2\phi\phi' + 4\lambda r^3\phi\phi'(\phi^2-v^2)\right]. && (3.43)
\end{aligned}$$

Now turn to the sphaleronic sector perturbations. We adopt the following notation. The gauge potential (2.4) can be written in the equivalent form [2, 46]

$$A = a_0\hat{\tau}_r dt + a_1\hat{\tau}_r dr + (1+\omega)[-\hat{\tau}_\varphi d\theta + \hat{\tau}_\theta \sin\theta\, d\varphi] + \tilde{\omega}[\hat{\tau}_\theta d\theta + \hat{\tau}_\varphi \sin\theta\, d\varphi], \quad (3.44)$$

or, in other words,

$$a \equiv a_0, \qquad b \equiv a_1, \qquad c \equiv 1+\omega, \qquad d \equiv \tilde{\omega}. \quad (3.45)$$

With this change in notation, and using the operator p_*, the sphaleronic sector perturbation equations take the form [46]:

$$\begin{aligned}
-Nr^2\delta\ddot{a}_1 &= \mathcal{H}_{a_1a_1}\delta a_1 + \mathcal{H}_{a_1\tilde{\omega}}\delta\tilde{\omega} + \mathcal{H}_{a_1\psi}\delta\psi && (3.46) \\
-2\delta\ddot{\tilde{\omega}} &= \mathcal{H}_{\tilde{\omega}a_1}\delta a_1 + \mathcal{H}_{\tilde{\omega}\tilde{\omega}}\delta\tilde{\omega} + \mathcal{H}_{\tilde{\omega}\psi}\delta\psi && (3.47) \\
-r^2\delta\ddot{\psi} &= \mathcal{H}_{\psi a_1}\delta a_1 + \mathcal{H}_{\psi\tilde{\omega}}\delta\tilde{\omega} + \mathcal{H}_{\psi\psi}\delta\psi && (3.48)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{H}_{a_1a_1} &= 2(NS)^2\left(\omega^2 + \frac{r^2}{8}\phi^2\right) \\
\mathcal{H}_{\tilde{\omega}\tilde{\omega}} &= -2p_*^2 + 2NS^2\left(\frac{\omega^2-1}{r^2} + \frac{\phi^2}{4}\right) \\
\mathcal{H}_{\psi\psi} &= -2p_*\frac{r^2}{2}p_* + 2NS^2\left(\frac{(1-\omega)^2}{4} + \frac{r^2}{2}\lambda(\phi^2-v^2)\right) \\
\mathcal{H}_{a_1\tilde{\omega}} &= -2NS[(p_*\omega) - \omega p_*] \\
\mathcal{H}_{\tilde{\omega}a_1} &= -2[p_*NS\omega + NS(p_*\omega)] \\
\mathcal{H}_{a_1\psi} &= -\frac{r^2}{2}NS[(p_*\phi) - \phi p_*] \\
\mathcal{H}_{\psi a_1} &= -p_*\frac{r^2}{2}NS\phi - \frac{r^2}{2}NS(p_*\phi) \\
\mathcal{H}_{\tilde{\omega}\psi} = \mathcal{H}_{\psi\tilde{\omega}} &= -\phi NS^2. && (3.49)
\end{aligned}$$

Sphaleronic Sector Perturbations

Firstly we consider the sphaleronic sector perturbations, and prove the existence of instabilities in this sector, before proceeding to count the exact number of unstable modes.

3.2 Outline of the variational method

Consider a static equilibrium solution $\zeta_e(x)$ of a classical field theory which has finite energy. In order to study the stability of this configuration, we consider small, time-dependent perturbations $\delta\zeta(x,t)$ about $\zeta_e(x)$, and work to first order only in the perturbations. The linearized field equations are then cast into the form of a Schrödinger system:

$$\mathcal{H}\delta\zeta(x,t) = -\frac{d^2}{dt^2}\mathcal{A}\delta\zeta(x,t) \tag{3.50}$$

where the operators $\mathcal{H}$ and $\mathcal{A}$ do not contain any time derivatives. The time-dependence is then specified to be

$$\delta\zeta(x,t) = e^{-i\sigma t}\Psi(x) \tag{3.51}$$

so that we are concerned only with periodic perturbations. The perturbation equations then reduce to a time-independent Schrödinger equation

$$\mathcal{H}\Psi(x) = \sigma^2\mathcal{A}\Psi(x) \tag{3.52}$$

and the operators $\mathcal{H}$ and $\mathcal{A}$ are independent of the frequency σ. We assume further that the operator $\mathcal{H}$ is self-adjoint with respect to a suitably defined inner product $\langle,\rangle$ on the Hilbert space containing the domain $\mathcal{D}$ of the functions $\{\Psi\}$, and also that the operator $\mathcal{A}$ is positive definite, so that

$$\langle\Psi|\mathcal{A}|\Psi\rangle > 0 \qquad \text{for all } \Psi \in \mathcal{D}. \tag{3.53}$$

The equilibrium configuration will be unstable if the perturbation equations (3.52) possess solutions having a negative value of σ^2, because these solutions correspond to imaginary frequency σ and thus to exponentially growing perturbations, by (3.51). Hence, in order to show that the system in the configuration $\zeta_s(x)$ is unstable, it is sufficient to show that the Schrödinger equation (3.52) has bound state solutions [4].

The Schrödinger equation (3.52) is usually difficult to solve analytically, and numerical calculations are often required (see, for example, [43, 49]). The variational method outlined below uses an analytical technique to show the existence of negative eigenvalues, $\sigma^2 < 0$. It has the disadvantage that it does not yield information about the number or value of these eigenvalues.

Define the following functional for all $\Psi \in \mathcal{D}$, whether or not they are eigenfunctions:

$$\Omega^2(\Psi) = \frac{\langle\Psi|\mathcal{H}|\Psi\rangle}{\langle\Psi|\mathcal{A}|\Psi\rangle}; \tag{3.54}$$

that is, Ψ is a trial function here.

Lemma 3.1 *The lowest eigenvalue Ω_0^2 of (3.52) provides a lower bound for the functional (3.54).*

Proof Assume that $\mathcal{H}$ and $\mathcal{A}$ are linear. Since $\mathcal{H}$ is self-adjoint, we have a complete set of eigenfunctions Ψ_k of (3.52)

$$\mathcal{H}\Psi_k = \Omega_k^2\mathcal{A}\Psi_k \tag{3.55}$$

normalized according to

$$\langle\Psi_k|\mathcal{A}|\Psi_l\rangle = \delta_{kl} \tag{3.56}$$

for eigenfunctions in the discrete spectrum, and

$$\langle\Psi_k|\mathcal{A}|\Psi_l\rangle = \delta(k-l) \tag{3.57}$$

for eigenfunctions in the continuous spectrum. For any trial function Ψ, we can express Ψ as a linear combination of the Ψ_k,

$$\Psi = \sum_k z_k\Psi_k \tag{3.58}$$

for complex numbers z_k, where the summation symbol denotes summation over the eigenfunctions in the discrete spectrum and integration over the eigenfunctions in the continuous spectrum.

Then

$$\begin{aligned}\langle\Psi|\mathcal{H}|\Psi\rangle &= \sum_{k,m}\bar{z}_k z_m\langle\Psi_k|\mathcal{H}|\Psi_m\rangle \\ &= \sum_{k,m}\bar{z}_k z_m\Omega_m^2\langle\Psi_k|\mathcal{A}|\Psi_m\rangle \\ &= \sum_k |z_k|^2\Omega_k^2 \end{aligned}\tag{3.59}$$

using the normalization condition (3.56). Similarly,

$$\begin{aligned}\langle\Psi|\mathcal{A}|\Psi\rangle &= \sum_{k,m}\bar{z}_k z_m\langle\Psi_k|\mathcal{A}|\Psi_m\rangle \\ &= \sum_k |z_k|^2. \end{aligned}\tag{3.60}$$

Hence

$$\begin{aligned}\Omega^2(\Psi) &= \frac{\sum_k |z_k|^2\Omega_k^2}{\sum_k |z_k|^2} \\ &= \Omega_0^2 + \frac{\sum_{k\neq 0}(\Omega_k^2 - \Omega_0^2)|z_k|^2}{\sum_k |z_k|^2} \\ &\geq \Omega_0^2 \end{aligned} \tag{3.61}$$

since Ω_0 is the lowest eigenvalue of the Schrödinger equation (3.52). □

The upshot of this lemma is that, in order to show that the equilibrium configuration $\zeta_e(x)$ is unstable, it is sufficient to find a trial function $\Psi \in \mathcal{D}$ such that

$$\begin{aligned}\langle\Psi|\mathcal{H}|\Psi\rangle &< 0, \\ \langle\Psi|\mathcal{A}|\Psi\rangle &< \infty \end{aligned} \tag{3.62}$$

so that

$$\Omega^2(\Psi) < 0. \tag{3.63}$$

It will then be the case that the Schrödinger equation (3.52) possesses negative eigenvalues [4]. The condition (3.62) is required to ensure that $\Psi \in \mathcal{D}$, the domain on which $\mathcal{H}$ and $\mathcal{A}$ are defined.

The advantage of this variational approach is that it is easier to find a trial function satisfying (3.62), rather than to solve the eigenvalue problem (3.52) [4, 46].

3.3 Application of the variational method

3.3.1 Sphaleronic sector perturbation equations

From section 3.1 the linearized perturbation equations for the sphaleronic sector take the form

$$\mathcal{H}\Psi = -\mathcal{A}\ddot{\Psi} \tag{3.64}$$

where

$$\mathcal{A} = \begin{pmatrix} Nr^2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \delta a_1 \\ \delta\tilde{\omega} \\ \delta\psi \end{pmatrix}, \tag{3.65}$$

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_{a_1a_1} & \mathcal{H}_{a_1\tilde{\omega}} & \mathcal{H}_{a_1\psi} \\ \mathcal{H}_{\tilde{\omega}a_1} & \mathcal{H}_{\tilde{\omega}\tilde{\omega}} & \mathcal{H}_{\tilde{\omega}\psi} \\ \mathcal{H}_{\psi a_1} & \mathcal{H}_{\psi\tilde{\omega}} & \mathcal{H}_{\psi\psi} \end{pmatrix}, \tag{3.66}$$

and the components of $\mathcal{H}$ are (3.49):

$$\mathcal{H}_{a_1a_1} = 2(NS)^2\left(\omega^2 + \frac{r^2}{8}\phi^2\right)$$

$$\begin{aligned}
\mathcal{H}_{\tilde{\omega}\tilde{\omega}} &= -2p_*^2 + 2NS^2\left(\frac{\omega^2-1}{r^2} + \frac{\phi^2}{4}\right) \\
\mathcal{H}_{\psi\psi} &= -2p_*\frac{r^2}{2}p_* + 2NS^2\left(\frac{(1-\omega)^2}{4} + \frac{r^2}{2}\lambda(\phi^2 - v^2)\right) \\
\mathcal{H}_{a_1\tilde{\omega}} &= -2NS[(p_*\omega) - \omega p_*] \\
\mathcal{H}_{\tilde{\omega}a_1} &= -2[p_* NS\omega + NS(p_*\omega)] \\
\mathcal{H}_{a_1\psi} &= -\frac{r^2}{2}NS[(p_*\phi) - \phi p_*] \\
\mathcal{H}_{\psi a_1} &= -p_*\frac{r^2}{2}NS\phi - \frac{r^2}{2}NS(p_*\phi) \\
\mathcal{H}_{\tilde{\omega}\psi} &= \mathcal{H}_{\psi\tilde{\omega}} = -\phi NS^2.
\end{aligned} \tag{3.67}$$

The operator p_* is given by

$$p_* = NS\frac{d}{dr} = \frac{d}{dr^*} \tag{3.68}$$

where r^* is the "tortoise" co-ordinate. We specify the time dependence to be

$$\Psi(r,t) = e^{i\sigma t}\Psi(r) \tag{3.69}$$

to arrive at a Schrödinger equation of the form (3.52). The inner product on the domain $\mathcal{D}$ of the operators $\mathcal{H}$ and $\mathcal{A}$ is defined to be

$$\langle\Psi|\Upsilon\rangle \equiv \int_{r_h}^{\infty} \bar{\Psi}\Upsilon\frac{1}{NS}dr = \int_{-\infty}^{\infty} \bar{\Psi}\Upsilon dr^*. \tag{3.70}$$

Then $\mathcal{H}$ is symmetric with respect to this inner product [4] and in addition $\mathcal{A}$ is positive definite, as required for the application of the variational method.

Next consider the following trial functions (cf. [46]):

$$\begin{aligned}
\delta a_1 &= -\omega' Z(r) \\
\delta\tilde{\omega} &= (\omega^2 - 1)Z(r) \\
\delta\psi &= -\frac{1}{2}\phi(\omega+1)Z(r)
\end{aligned} \tag{3.71}$$

where $Z(r)$ is a function of r. At the moment it is arbitrary, and suitable Z will be discussed at the end of this section.

3.3.2 Calculation of $\langle\Psi|\mathcal{A}|\Psi\rangle$

This is straightforward and yields

$$\langle\Psi|\mathcal{A}|\Psi\rangle = \int_{r_h}^{\infty}\frac{dr}{NS}Z^2\left[Nr^2\omega'^2 + 2(\omega^2-1)^2 + \frac{1}{4}r^2\phi^2(1+\omega)^2\right]. \tag{3.72}$$

Clearly this expression is positive definite for all Z, and we will restrict our choice of Z so that this expectation value is finite.

3.3.3 Calculation of $\langle\Psi|\mathcal{H}|\Psi\rangle$

This calculation is also straightforward but rather lengthy. Substituting in (3.72) gives

$$\langle\Psi|\mathcal{H}|\Psi\rangle = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{3.73}$$

where

$$\begin{aligned}
I_1 &= \int_{r_h}^{\infty} \frac{dr}{NS} 2N^2S^2Z^2\omega'^2 \left[\omega^2 + \frac{1}{8}r^2\phi^2\right] && (3.74)\\
I_2 &= -\int_{r_h}^{\infty} \frac{dr}{NS} 2NS^2Z^2(\omega^2-1)\left[2\frac{(NS)'}{S}\omega\omega' + \frac{(NS)'}{S}\frac{1}{Z}\frac{dZ}{dr}(\omega^2-1)\right.\\
&\quad +2N\omega'^2 + 2N\omega\omega'' + 4N\omega\omega'\frac{1}{Z}\frac{dZ}{dr} + N(\omega^2-1)\frac{1}{Z}\frac{d^2Z}{dr^2}\\
&\quad \left. -\frac{1}{r^2}(\omega^2-1)^2 - \frac{1}{4}\phi^2(\omega^2-1)\right] && (3.75)\\
I_3 &= -\int_{r_h}^{\infty} \frac{dr}{NS}\frac{NS^2}{4} Z^2\phi(\omega+1)\left[\frac{(NS)'}{S}r^2\phi'(1+\omega) + \frac{(NS)'}{S}r^2\phi\omega'\right.\\
&\quad +\frac{(NS)'}{S}r^2\phi(\omega+1)\frac{1}{Z}\frac{dZ}{dr} + Nr^2\phi''(1+\omega) + Nr^2\phi\omega'' + 2Nr\phi(1+\omega)\frac{1}{Z}\frac{dZ}{dr}\\
&\quad +2Nr^2\phi'(1+\omega)\frac{1}{Z}\frac{dZ}{dr} + 2Nr^2\phi\omega'\frac{1}{Z}\frac{dZ}{dr} + Nr^2\phi(1+\omega)\frac{1}{Z}\frac{d^2Z}{dr^2} + 2Nr\phi\omega'\\
&\quad +2Nr\phi'(1+\omega) + 2Nr^2\phi'\omega' + \frac{1}{2}\phi(1-\omega)(\omega^2-1)\\
&\quad \left. -\lambda r^2\phi(1+\omega)(\phi^2-v^2)\right] && (3.76)\\
I_4 &= -\int_{r_h}^{\infty} \frac{dr}{NS} NS^2Z^2\phi^2(1+\omega)^2(1-\omega) && (3.77)\\
I_5 &= \int_{r_h}^{\infty} \frac{dr}{NS} NS^2Z^2\frac{1}{4}\phi\left[-Nr^2\phi\omega'^2 + 2Nr\phi\omega'(1+\omega) + \frac{(NS)'}{S}r^2\phi\omega'(1+\omega)\right.\\
&\quad \left. +2Nr^2\phi'\omega'(1+\omega) + Nr^2\phi\omega''(1+\omega)\right] && (3.78)\\
I_6 &= \int_{r_h}^{\infty} \frac{dr}{NS} 2NS^2Z^2\left[N\omega'^2(\omega^2-3) + N\omega\omega''(\omega^2-1)\right.\\
&\quad \left. +\frac{(NS)'}{S}\omega\omega'(\omega^2-1)\right]. && (3.79)
\end{aligned}$$

These expressions can be simplified in two ways. Firstly, by substituting for ω'' and ϕ'' from the field equations (2.54–2.55):

$$\begin{aligned}
N\omega'' + \frac{(NS)'}{S}\omega' &= \frac{1}{r^2}(\omega^2-1)\omega + \frac{1}{4}\phi^2(1+\omega) && (3.80)\\
N\phi'' + \frac{(NS)'}{S}\phi' + \frac{2N}{r}\phi' &= \frac{1}{2r^2}\phi(1+\omega)^2 + \lambda\phi(\phi^2-v^2). && (3.81)
\end{aligned}$$

The expressions then become:

$$\begin{aligned}
I_2 &= -\int_{r_h}^{\infty} \frac{dr}{NS} 2NS^2Z^2(\omega^2-1)\left[\frac{(NS)'}{S}\frac{1}{Z}\frac{dZ}{dr}(\omega^2-1) + 4N\omega\omega'\frac{1}{Z}\frac{dZ}{dr}\right. \\
&\quad \left. + N(\omega^2-1)\frac{1}{Z}\frac{d^2Z}{dr^2} + 2N\omega'^2 + \frac{1}{r^2}(\omega^4-1) + \frac{1}{4}\phi^2(\omega+1)^2\right] && (3.82) \\
I_3 &= -\int_{r_h}^{\infty} \frac{dr}{NS}\frac{NS^2}{4}Z^2\phi(1+\omega)\left[\frac{(NS)'}{S}r^2\phi(1+\omega)\frac{1}{Z}\frac{dZ}{dr} + 2Nr\phi(1+\omega)\frac{1}{Z}\frac{dZ}{dr}\right. \\
&\quad + 2Nr^2\phi'(1+\omega)\frac{1}{Z}\frac{dZ}{dr} + 2Nr^2\phi\omega'\frac{1}{Z}\frac{dZ}{dr} + Nr^2\phi(1+\omega)\frac{1}{Z}\frac{d^2Z}{dr^2} + 2Nr\phi\omega' \\
&\quad \left. + 2Nr^2\phi'\omega' + \phi\omega(1+\omega)^2 + \frac{1}{4}r^2\phi^3(1+\omega)\right] && (3.83) \\
I_5 &= \int_{r_h}^{\infty} \frac{dr}{NS}NS^2Z^2\frac{\phi}{4}\left[-Nr^2\phi\omega'^2 + 2Nr\phi\omega'(1+\omega) + 2Nr^2\phi'\omega'(1+\omega)\right. \\
&\quad \left. + \phi\omega(\omega^2-1)(1+\omega) + \frac{1}{4}r^2\phi^3(1+\omega)^2\right] && (3.84) \\
I_6 &= \int_{r_h}^{\infty} \frac{dr}{NS}2NS^2Z^2\left[N\omega'^2(\omega^2-3) + \frac{1}{r^2}\omega^2(\omega^2-1)^2\right. \\
&\quad \left. + \frac{1}{4}\phi^2\omega(1+\omega)(\omega^2-1)\right]. && (3.85)
\end{aligned}$$

Secondly, we may perform an integration by parts. For example, in the term I_2 we have

$$-\int_{r_h}^{\infty} \frac{dr}{NS}2NS^2Z^2N(\omega^2-1)^2\frac{1}{Z}\frac{d^2Z}{dr^2} = 2\int_{r_h}^{\infty}\frac{dr}{NS}NS^2Z^2(\omega^2-1) \tag{3.86}$$

$$\times\left[N\frac{1}{Z^2}\left(\frac{dZ}{dr}\right)^2(\omega^2-1) + \frac{(NS)'}{S}\frac{1}{Z}\frac{dZ}{dr}(\omega^2-1) + 4N\frac{1}{Z}\frac{dZ}{dr}\omega\omega'\right] \tag{3.87}$$

$$+\left[-2NSZ\frac{dZ}{dr}(\omega^2-1)^2\right]_{r_h}^{\infty}. \tag{3.88}$$

Hence I_2 simplifies down to

$$\begin{aligned}
I_2 &= -\int_{r_h}^{\infty}\frac{dr}{NS}2NS^2Z^2(\omega^2-1)\left[-N\frac{1}{Z^2}\left(\frac{dZ}{dr}\right)^2(\omega^2-1) + 2N\omega'^2\right. \\
&\quad \left. + \frac{1}{r^2}(\omega^4-1) + \frac{1}{4}\phi^2(\omega-1)^2\right] + B_2 && (3.89)
\end{aligned}$$

where

$$B_2 = \left[-2NSZ\frac{dZ}{dr}(\omega^2-1)^2\right]_{r_h}^{\infty}. \tag{3.90}$$

Similarly, for the term I_3 we use the fact that

$$-\int_{r_h}^{\infty} \frac{dr}{NS}\frac{1}{4}NS^2Z^2r^2N\phi^2(\omega-1)^2\frac{1}{Z}\frac{d^2Z}{dr^2} = \int_{r_h}^{\infty}\frac{dr}{NS}\frac{1}{4}NS^2Z^2\phi(\omega+1) \tag{3.91}$$

$$\times\left[-N\frac{1}{Z^2}\left(\frac{dZ}{dr}\right)^2 r^2\phi(1+\omega) + \frac{(NS)'}{S}\frac{1}{Z}\frac{dZ}{dr}r^2\phi(1+\omega) + 2Nr\frac{1}{Z}\frac{dZ}{dr}(1+\omega)\phi\right. \tag{3.92}$$

$$\left. +2Nr^2\frac{1}{Z}\frac{dZ}{dr}\phi'(1+\omega) + 2Nr^2\frac{1}{Z}\frac{dZ}{dr}\phi\omega'\right] \tag{3.93}$$

$$+\left[-\frac{1}{4}NSZr^2\phi^2(1-\omega)^2\frac{dZ}{dr}\right]_{r_h}^{\infty} \tag{3.94}$$

to give a simplified expression for I_3 which reads:

$$\begin{aligned} I_3 &= -\int_{r_h}^{\infty}\frac{dr}{NS}\frac{1}{4}NS^2Z^2\phi(1+\omega)\left[-N\frac{1}{Z^2}\left(\frac{dZ}{dr}\right)^2 r^2\phi(1+\omega) + 2Nr\phi\omega'\right. \\ &\quad \left. +2Nr^2\phi'\omega' + \phi\omega(1+\omega)^2 + \frac{1}{4}r^2\phi^3(1+\omega)\right] + B_3 \end{aligned} \tag{3.95}$$

where

$$B_3 = \left[-\frac{1}{4}NSZr^2\phi^2(1-\omega)^2\frac{dZ}{dr}\right]_{r_h}^{\infty}. \tag{3.96}$$

Adding up all the contributions to $\langle\Psi|\mathcal{H}|\Psi\rangle$ then gives the final result

$$\begin{aligned} \langle\Psi|\mathcal{H}|\Psi\rangle &= \int_{r_h}^{\infty} dr\, S\left[-N\omega'^2 - \frac{2}{r^2}(\omega^2-1)^2 - \frac{1}{2}\phi^2(1+\omega)^2\right] \\ &\quad + \int_{r_h}^{\infty} dr\, S(1-Z^2)\left[\frac{2}{r^2}(\omega^2-1)^2 + \phi^2(1+\omega)^2 + 2N\omega'^2\right] \\ &\quad + \int_{r_h}^{\infty} dr\, NS\left(\frac{dZ}{dr}\right)^2\left[2(\omega^2-1)^2 + \frac{1}{4}r^2\phi^2(1+\omega)^2\right] \\ &\quad + B_2 + B_3. \end{aligned} \tag{3.97}$$

3.3.4 Choice of Z

The first of these integrals is manifestly negative. For the remaining terms, introduce the "tortoise" co-ordinate r^* given by

$$\frac{dr^*}{dr} = \frac{1}{NS} \tag{3.98}$$

and consider the following sequence of functions $Z_k(r^*)$ [33]:

$$Z_k(r^*) = Z\left(\frac{r^*}{k}\right) \qquad \text{for } k = 1, 2, \ldots \tag{3.99}$$

where

$$\begin{aligned} Z(r^*) &= Z(-r^*) \\ Z(r^*) &= 1 \qquad \text{for } r^* \in [0,p] \\ Z(r^*) &= 0 \qquad \text{for } r^* > p+1 \end{aligned} \tag{3.100}$$

and

$$-q \leq \frac{dZ}{dr^*} \leq 0 \qquad \text{for } r^* \in [p, p+1] \tag{3.101}$$

where p and $q \geq 1$ are positive constants.

Near the horizon, for each k, since

$$r^* \sim \log(r - r_h) \tag{3.102}$$

we have

$$Z_k = 0 \qquad \text{for } r < r_h + e^{-(p+1)k}. \tag{3.103}$$

This is sufficient to ensure that $\langle\Psi|\mathcal{A}|\Psi\rangle$ is finite for each Z_k, since ω' and $\omega + 1$ vanish exponentially as $r \to \infty$. Similarly, the boundary terms B_2 and B_3 both vanish.

Lemma 3.2 *The integrands of the last two remaining integrals are uniformly convergent to zero.*

Proof Define

$$\begin{aligned} \mathcal{J}_{2k}(r^*) &= S(1 - Z_k^2)\left[\frac{2}{r^2}(\omega^2 - 1)^2 + \phi^2(1+\omega)^2 + 2N\omega'^2\right] NS \\ &= (1 - Z_k^2) J_2(r^*) NS \end{aligned} \tag{3.104}$$

$$\begin{aligned} \mathcal{J}_{3k}(r^*) &= NS\left(\frac{dZ_k}{dr}\right)^2 \left[2(\omega^2 - 1)^2 + \frac{1}{4} r^2\phi^2(1+\omega)^2\right] NS \\ &= \left(\frac{dZ_k}{dr}\right)^2 J_3(r^*) NS. \end{aligned} \tag{3.105}$$

Let $\delta > 0$. Now $J_2(r^*) \to 0$ as $r^* \to \infty$, so there is a $R_1 > 0$ such that

$$|NSJ_2(r^*)| < \delta \qquad \forall r^* > R_1. \tag{3.106}$$

Similarly, $J_2(r^*)$ remains bounded as $r^* \to -\infty$, so $NSJ_2(r^*) \to 0$ as $r^* \to -\infty$, and there is an $R_2 > 0$ such that

$$|NSJ_2(r^*)| < \delta \qquad \forall r^* < -R_2. \tag{3.107}$$

Now choose an integer k such that

$$pk > \max\{R_1, R_2\}. \tag{3.108}$$

Then

$$|\mathcal{J}_{2k}(r^*)| = 0 \qquad \text{for } -pk \le r^* \le pk, \tag{3.109}$$

$$|\mathcal{J}_{2k}(r^*)| \le |J_2(r^*)NS| < \delta \qquad \text{for } r^* > pk > R_1, \tag{3.110}$$

and

$$|\mathcal{J}_{2k}(r^*)| \le |J_2(r^*)NS| < \delta \qquad \text{for } r^* < -pk < -R_2. \tag{3.111}$$

Thus,

$$|\mathcal{J}_{2k}(r^*)| < \delta \qquad \forall r^*, \text{ and } \forall k > \frac{1}{p}\max\{R_1, R_2\}, \tag{3.112}$$

viz the sequence $\mathcal{J}_{2k}(r^*)$ of functions is uniformly convergent.

The proof is similar for the third integral. We use the fact that

$$\frac{dZ_k}{dr^*} = \frac{1}{k}\frac{dZ}{dr^*}\left(\frac{r^*}{k}\right). \tag{3.113}$$

Now $NSJ_3(r^*) \to 0$ as $r^* \to \infty$, and remains bounded as $r^* \to -\infty$. Thus $|NSJ_3(r^*)|$ is bounded for all r^*, which means that there exists a $R_3 > 0$ such that

$$|NSJ_3(r^*)| < R_3 \qquad \forall r^*. \tag{3.114}$$

By construction,

$$\left|\frac{dZ}{dr^*}\right| \le q. \tag{3.115}$$

Choose an integer k such that

$$k^2 > \frac{R_3 q^2}{\delta}. \tag{3.116}$$

Then

$$|\mathcal{J}_{3k}| < \frac{\delta}{R_3 q^2} \times R_3 q^2 = \delta \qquad \forall r^*, \text{ and } \forall k^2 > \frac{R_3 q^2}{\delta}, \tag{3.117}$$

so that the sequence $\mathcal{J}_{3k}$ is uniformly convergent to zero. □

From this lemma, the last two terms in (3.97) tend towards zero as $k \to \infty$. The first term is independent of k, and hence we can choose k sufficiently large that the dominant contribution to $\langle\Psi|\mathcal{H}|\Psi\rangle$ comes from this first term, which is negative.

We conclude that we have found a trial function satisfying the conditions (3.62) and hence, from the analysis of the previous section, we have the following [4]:

Result 3.3 *The EYMH black holes possess at least one mode of instability in the sphaleronic sector.*

3.3.5 Discussion

The method of this section was employed in [46] to show that regular, particle-like solutions of the EYMH theory are unstable in the sphaleronic sector. However, the result of [46] does not carry over directly to the black hole case because divergences occur at the event horizon in some of the expressions. Avoiding such divergences necessitated the use of the Z_k functions in (3.72) and (3.99–3.101).

It should be noted that the analysis of this section does not depend on the detailed structure of the black hole solutions, but only on their global properties. Hence, this instability may be considered in some sense "topological". It is precisely analogous to the saddle-point instability of the electroweak sphaleron in flat space (see, for example, [50, 51, 52]).

3.4 Counting the sphaleronic unstable modes

3.4.1 Outline of the method

Having shown that the sphaleronic sector contains at least one unstable mode, we now proceed to count the exact number of unstable modes in this sector for each of the equilibrium solutions [3]. We shall exploit the method of Volkov et al [48] who have shown that the kth coloured black hole solution of [18] possesses exactly k negative sphaleronic modes.

The method involves mapping the linearized perturbation equations to a set of coupled Schrödinger equations [45, 46, 48]. The number of unstable modes (corresponding to negative eigenvalues) is then the number of bound states of the quantum-mechanical system described by these equations.

The equilibrium solutions are known only numerically, there being no closed analytic solution of the field equations of this type known to date. In order to calculate the number of bound states without using numerical computations, it will be necessary to make certain assumptions concerning the analyticity of these equilibrium solutions. These assumptions will remain unproved, although there can be little doubt physically that they will be true. (See [31, 32] for the analyticity theorems proved for the EYM case.)

3.4.2 The coupled Schrödinger equations

We begin with the linearized perturbation equations for the sphaleronic sector in the form (3.46–3.49):

$$\begin{aligned} 2N^2S^2\left(\omega^2+\frac{r^2}{8}\phi^2\right)\delta a_1 + 2NS\left[\omega(p_*\delta\tilde{\omega}) - (p_*\omega)\delta\tilde{\omega}\right] & \\ +\frac{1}{2}r^2NS\left[(p_*\phi)\delta\psi - \phi(p_*\delta\psi)\right] &= Nr^2\sigma^2\delta a_1 \quad (3.118) \end{aligned}$$

$$\begin{aligned} 2p_*(NS\omega\delta a_1) + 2NS(p_*\omega)\delta a_1 + p_*^2\delta\tilde{\omega} + NS^2\phi\delta\psi & \\ -\frac{2}{r^2}S^2\left[(\omega^2-1)+\frac{\phi^2}{4}\right]\delta\tilde{\omega} &= -2\sigma^2\delta\tilde{\omega} \quad (3.119) \\ \frac{1}{2}p_*(NSr^2\phi\delta a_1) + \frac{1}{2}r^2NS(p_*\phi)\delta a_1 - NS^2\phi\delta\tilde{\omega} & \\ -p_*[r^2\delta(p_*\psi)] + 2NS^2\left[\frac{(1-\omega)^2}{4}+\frac{1}{2}r^2\lambda(\phi^2-v^2)\right]\delta\psi &= r^2\sigma^2\delta\psi \quad (3.120) \end{aligned}$$

where periodic perturbations are considered. In addition, the perturbations must satisfy the linearized Gauss constraint:

$$\sigma\left\{p_*\left(\frac{r^2}{S}\delta a_1\right) + 2\omega\delta\tilde{\omega} - r^2\frac{\phi}{2}\delta\psi\right\} = 0. \quad (3.121)$$

Change variables and define

$$\delta\xi = r\delta\psi, \qquad \delta\alpha = \frac{r^2}{2S}\delta a_1. \quad (3.122)$$

The reason for this change of variables will become apparent at the end of this section. In terms of the new variables, (3.122), equation (3.118) reads

$$\sigma^2\delta\alpha = \mathcal{B}(r^*) \quad (3.123)$$

where

$$\begin{aligned} \mathcal{B}(r^*) &= NS\omega^2\delta a_1 + \frac{1}{8}NSr^2\phi^2\delta a_1 - (p_*\omega)\delta\tilde{\omega} + \omega(p_*\delta\tilde{\omega}) \\ &\quad +\frac{1}{4}NS\phi\delta\xi + \frac{1}{4}r\left[(p_*\phi)\delta\xi - \phi(p_*\delta\xi)\right]. \end{aligned} \quad (3.124)$$

Equations (3.119) and (3.120) can be arranged to give, using the static field equations for ω and ϕ,

$$p_*\mathcal{B}(r^*) = \sigma^2\left(-\omega\delta\tilde{\omega} + \frac{1}{4}r\phi\delta\xi\right) \quad (3.125)$$

and the Gauss constraint (3.121) is now

$$\sigma\left(p_*\delta\alpha + \omega\delta\tilde{\omega} - \frac{1}{4}r\phi\delta\xi\right) = 0. \quad (3.126)$$

When $\sigma^2 = 0$, the function $\mathcal{B}(r^*)$ is invariant under the transformation

$$\begin{aligned} \delta\alpha &\to \delta\alpha + \frac{r^2}{2NS^2}p_*\Theta \\ \delta\tilde{\omega} &\to \delta\tilde{\omega} - \omega\Theta \\ \delta\xi &\to \delta\xi + \frac{r\phi}{2}\Theta \end{aligned} \quad (3.127)$$

where $\Theta(r^*)$ is an arbitrary function of r^*. Furthermore, the equations (3.119) and (3.120) are also invariant under this transformation. When $\sigma^2 \neq 0$, the equation for $p_*\mathcal{B}(r^*)$ follows directly from the definition of $\mathcal{B}(r^*)$ and the Gauss constraint.

Next, following [48], a "strong" Gauss constraint is defined by requiring that

$$p_*\delta\alpha = -\omega\delta\tilde{\omega} + \frac{1}{4}r\phi\delta\xi \tag{3.128}$$

even in the case where $\sigma = 0$. This strong Gauss constraint can then be used together with (3.123) to write $\delta\tilde{\omega}$ and $\delta\xi$ in terms of $\delta\alpha$ only:

$$\begin{aligned} \delta\tilde{\omega} &= \frac{1}{\mathtt{P}}r^2\phi^2 p_*\left(\frac{p_*\delta\alpha}{r^2\phi^2}\right) - \frac{\mathtt{Q}}{\mathtt{P}}\delta\alpha \\ \delta\xi &= \frac{4\omega^2}{\mathtt{P}r^2\phi^2}\left(\frac{p_*\delta\alpha}{\omega^2}\right) - \frac{4\omega^2\mathtt{Q}}{\mathtt{P}r^2\phi^2}\delta\alpha \end{aligned} \tag{3.129}$$

where the functions $\mathtt{P}(r^*)$ and $\mathtt{Q}(r^*)$ are given by

$$\begin{aligned} \mathtt{P}(r^*) &= -2p_*\omega + 2\omega\frac{\omega p_*\phi}{\phi} + 2\omega\frac{NS}{r} \\ \mathtt{Q}(r^*) &= 2\frac{NS^2}{r^2}\omega^2 + \frac{1}{4}NS^2\phi^2 - \sigma^2. \end{aligned} \tag{3.130}$$

It is possible to proceed in two ways from here. Firstly, we may substitute the expressions for $\delta\tilde{\omega}$ and $\delta\xi$ (3.129) into the equation (3.119) for $\delta\tilde{\omega}$. This yields a single fourth-order differential equation for $\delta\alpha$:

$$\begin{aligned} 0 &= -p_*^4\delta\alpha + \left(\frac{2\mathtt{P}'}{\mathtt{P}} + \mathcal{P}\mathtt{P}\right)p_*^3\delta\alpha \\ &+ \left\{\frac{p_*^2\mathtt{P}}{\mathtt{P}} - \frac{2(p_*\mathtt{P})^2}{\mathtt{P}^2} + 2(p_*\mathcal{P})\mathtt{P} + \mathtt{Q} - \sigma^2 + NS^2\mathcal{Q}\right\}p_*^2\delta\alpha \\ &+ \left\{(p_*^2\mathcal{P})\mathtt{P} + 2\mathtt{P}p_*\left(\frac{\mathtt{Q}}{\mathtt{P}}\right) + \sigma^2\mathcal{P}\mathtt{P}\right. \\ &\left. + NS^2\left(-\frac{2\omega\mathtt{P}}{r^2} - \frac{\mathcal{P}}{r^2}(\omega^2-1)\mathtt{P} - \frac{\mathcal{P}}{4}\phi^2\mathtt{P} + \frac{4}{r^2}p_*\omega\right)\right\}p_*\delta\alpha \\ &+ \left\{\mathtt{P}p_*^2\left(\frac{\mathtt{Q}}{\mathtt{P}}\right) - 2\omega\mathtt{P}p_*\left(\frac{NS^2}{r^2}\right) - \frac{4\mathtt{P}NS^2}{r^2}p_*\omega + \sigma^2\mathtt{Q} - NS^2\mathtt{Q}\mathcal{Q}\right\}\delta\alpha \end{aligned} \tag{3.131}$$

where

$$\begin{aligned} \mathcal{P}(r^*) &= \frac{2NS}{r\mathtt{P}} + \frac{2\phi}{\mathtt{P}}p_*\phi \\ \mathcal{Q}(r^*) &= -\frac{2\omega}{r^2} + \frac{\omega^2-1}{r^2} + \frac{\phi^2}{4}. \end{aligned} \tag{3.132}$$

This rather unwieldy equation does not readily lend itself to analysis due to the presence of the eigenvalue σ in the coefficients of derivatives of $\delta\alpha$. However, it will be useful for determining the asymptotic form of $\delta\alpha$.

Alternatively, we may use (3.129) to eliminate the variable $\delta\xi$ from the original equations (3.118–3.120), which produces the required pair of standard coupled Schrödinger equations

$$\begin{aligned} \sigma^2\delta\alpha &= -p_*^2\delta\alpha + \left(\frac{2p_*\phi}{\phi} + \frac{2NS}{r}\right)p_*\delta\alpha \\ &\quad + \left(\frac{2NS^2}{r^2}\omega^2 + \frac{NS^2}{4}\phi^2\right)\delta\alpha + \mathrm{P}\delta\tilde{\omega} \qquad (3.133) \\ \sigma^2\delta\tilde{\omega} &= -p_*^2\delta\tilde{\omega} - \frac{2NS^2}{r^2}(1+\omega)p_*\delta\alpha - \left\{\frac{4NS^2}{r^2}p_*\omega + 2\omega p_*\left(\frac{NS^2}{r^2}\right)\right\}\delta\alpha \\ &\quad + \frac{NS^2}{r^2}\left\{(\omega-1)^2 + \frac{r^2\phi^2}{4}\right\}\delta\tilde{\omega}. \qquad (3.134) \end{aligned}$$

The perturbation equations are now in a form appropriate for the counting of the number of negative eigenvalues σ^2.

3.4.3 Analyticity assumptions

We wish to count the number of negative modes of the coupled system (3.133), (3.134), using continuity arguments rather than numerical calculations. (It should be borne in mind that numerical evaluation of eigenvalues can only give a lower bound on the number of negative eigenvalues in a particular case, although appropriate searches may provide good evidence for the non-existence of further modes [47].)

Consider a branch of solutions formed by fixing the number of nodes k of ω and the horizon radius r_h, and letting the Higgs v.e.v. v vary. The argument below requires that the eigenvalues σ^2 of the system

$$p_*^2\Psi + \mathcal{V}\Psi = \sigma^2\Psi \qquad (3.135)$$

are continuous in the parameter v, where

$$\Psi = \begin{pmatrix} \delta\alpha \\ \delta\tilde{\omega} \end{pmatrix} \qquad (3.136)$$

and $\mathcal{V}$ is a 2×2 matrix-valued function of r^* and the parameter v. (This standard form of the Schrödinger equation can be obtained from (3.133) and (3.134) by a change of variables.) This is not an unreasonable requirement, as may be motivated as follows. If we assume that the equilibrium solutions are themselves continuous in v, as is indicated by all numerical investigations [2], then the potential $\mathcal{V}$ will also be continuous in v. Bound-state perturbation theory [53] can be used to calculate the eigenvalues of the system (3.135) with potential

$$\mathcal{V}(v+\delta v) \simeq \mathcal{V}(v) + \delta\mathcal{V} \qquad (3.137)$$

in terms of the spectrum of the system with potential $\mathcal{V}(v)$. This implies that the corresponding shifts in the bound state energies $\delta\sigma^2$ will be small for infinitesimal $\delta\mathcal{V}$, so that the eigenvalues themselves are also continuous in v.

This heuristic explanation can be put on a more rigorous footing using functional analysis, appealing to the Kato-Rellich theorem [54].

Theorem 3.4 (Kato-Rellich) *Let $\mathcal{D}$ be a connected domain in the complex plane and let $\mathcal{T}(v)$ be a closed operator with non-empty* resolvent set, *given for each $v \in \mathcal{D}$. Suppose further that*

1. *the operator domain of $\mathcal{T}(v)$ is some set* $\mathtt{D}$ *independent of v, and*

2. *for each $\psi \in \mathtt{D}$, the vector $\mathcal{T}(v)\psi$ is a vector-valued analytic function of v.*

Let σ_0^2 be a non-degenerate discrete eigenvalue of $\mathcal{T}(v_0)$. Then, for v near v_0, there is exactly one point $\sigma^2(v)$ in the spectrum of $\mathcal{T}(v)$ near σ_0^2 and this point is isolated and non-degenerate. In addition, $\sigma^2(v)$ is an analytic function of v for v near v_0.

This is a somewhat technical theorem. However, most of its conditions are greatly simplified in our case since we are concerned only with a self-adjoint, linear operator $\mathcal{T}$ given by

$$\mathcal{T}(v) = -p_*^2 + \mathcal{V}(v). \tag{3.138}$$

The fact that $\mathcal{T}$ is self-adjoint and linear automatically tells us that it is closed (see, for example, [55] for details of the theorems invoked here), and further that its spectrum is a subset of the real line $\mathbb{R}$. The *resolvent set* of an operator is defined to be the set of all complex numbers z which do not lie in the spectrum [55]. Clearly this set is non-empty in our case.

Conditions (1) and (2) of the theorem can easily be seen to be satisfied, provided that the equilibrium solutions of the field equations are analytic in v. This is a strong condition, which, as with continuity, cannot be rigorously established. However, it is a legitimate assumption. The coloured black hole solutions of [18], which arise in the limit $v \to 0$, have been shown in [31] to be analytic in the parameter r_h which defines them, indicating that it may in the future be possible to prove a similar assertion about the more general EYMH black holes.

We shall not be concerned further with these issues, and shall, for the rest of the section, assume that the eigenvalues of the Schrödinger system (3.135) are in fact continuous in the Higgs v.e.v. v.

3.4.4 The spectrum of the Schrödinger system

The perturbation equations (3.133) and (3.134) are in the standard Schrödinger form and we will now proceed to count the number of eigenvalues σ^2 of this system satisfying $\sigma^2 < 0$.

Discrete and continuous spectrum The first step is to determine the regions of $\mathbb{R}$ corresponding to the discrete and continuous parts of the spectrum of the system:

$$\begin{aligned}
\sigma^2 \delta\alpha &= -p_*^2 \delta\alpha + \left(\frac{2p_*\phi}{\phi} + \frac{2NS'}{r} \right) p_* \delta\alpha \\
&+ \left(\frac{2NS^2}{r^2}\omega^2 + \frac{NS^2}{4}\phi^2 \right) \delta\alpha + \mathrm{P}\delta\tilde{\omega} \\
\sigma^2 \delta\tilde{\omega} &= -p_*^2 \delta\tilde{\omega} - \frac{2NS^2}{r^2}(1+\omega) p_* \delta\alpha - \left\{ \frac{4NS^2}{r^2} p_* \omega + 2\omega p_* \left(\frac{NS^2}{r^2} \right) \right\} \delta\alpha \\
&+ \frac{NS^2}{r^2} \left\{ (\omega - 1)^2 + \frac{r^2\phi^2}{4} \right\} \delta\tilde{\omega}.
\end{aligned} \tag{3.139}$$

As the event horizon is approached, $r \to r_h$ and $r^* \to -\infty$, which means that $N \to 0$ and p_* of any function goes to zero providing that derivatives with respect to r are finite at the horizon. This is true for the equilibrium field variables we are considering here. Then the system (3.139) reduces to:

$$\begin{aligned}
-p_*^2 \delta\alpha &= \sigma^2 \delta\alpha \\
-p_*^2 \delta\tilde{\omega} &= \sigma^2 \delta\tilde{\omega}.
\end{aligned} \tag{3.140}$$

At infinity, similarly, both r and $r^* \to \infty$, and the field variables approach their limiting values exponentially:

$$N \to 1, \qquad S \to 1 \tag{3.141}$$

(a requirement that the space-time be asymptotically flat) and

$$\omega \to -1, \qquad \phi \to v. \tag{3.142}$$

The leading behaviour of the system (3.139) is now

$$\begin{aligned}
-p_*^2 \delta\alpha &= \sigma^2 \delta\alpha + \frac{1}{4} v^2 \delta\alpha \\
-p_*^2 \delta\tilde{\omega} &= \sigma^2 \delta\tilde{\omega} + \frac{1}{4} v^2 \delta\tilde{\omega}.
\end{aligned} \tag{3.143}$$

Thus the region $\sigma^2 > 0$ corresponds to the continuous spectrum, and the region $\sigma^2 < 0$ corresponds to the discrete spectrum of these Schrödinger equations. The upshot is, then, that using the assumption that the eigenvalues σ^2 are continuous (as discussed in the previous subsection), the number of negative modes changes by one whenever a mode is either absorbed into the continuous spectrum or emerges from it. Hence we next need to consider the zero modes of (3.139).

Zero modes For $\sigma^2 = 0$, there are "pure gauge" (cf. [48]) solutions of the form

$$\delta\alpha = \frac{r^2 p_* \Theta}{2NS^2}, \qquad \delta\tilde{\omega} = -\omega\Theta, \qquad \delta\xi = \frac{r\phi}{2}\Theta \tag{3.144}$$

where $\Theta(r^*)$ satisfies the following second-order differential equation:

$$p_* \left(\frac{r^2 p_* \Theta}{2NS^2} \right) = \left(\omega^2 + \frac{1}{8} r^2 \phi^2 \right) \Theta \tag{3.145}$$

which follows from the strong Gauss constraint (3.128). The behaviour of Θ in the asymptotic regions is given by (3.145) to be

$$\Theta \sim O[(r - r_h)^{\varsigma}], \qquad \text{where } \varsigma = 0, 1, \qquad \text{as } r \to r_h, \tag{3.146}$$

$$\Theta \sim e^{\pm \frac{v}{2} r^*} \qquad \text{as } r \to \infty, \tag{3.147}$$

where we have stipulated that $S(\infty) = 1$ in order for the geometry to be asymptotically flat. The solution behaving like $e^{-\frac{v}{2} r^*}$ as $r^* \to \infty$ then produces a single, non-degenerate, non-normalizable zero mode of (3.139) via substitution in (3.144). The solution

$$\Theta \sim e^{\frac{v}{2} r^*} \qquad \text{as } r^* \to \infty \tag{3.148}$$

does not remain bounded at infinity and hence does not represent an eigenfunction.

In order to check that this is in fact the only zero mode of the system, turn to the fourth-order differential equation (3.131) for $\delta\alpha$. Solutions of this equation with $\sigma^2 = 0$ have the following behaviour near the event horizon:

$$\delta\alpha \sim O[(r - r_h)^{\varsigma}] \tag{3.149}$$

where $\varsigma = 0$ (twice, corresponding to the pure gauge mode solutions above), or

$$\varsigma = \frac{-1 \pm \sqrt{5}}{2}. \tag{3.150}$$

In the latter case, $\delta\tilde{\omega}$ and $\delta\xi$ can be calculated using (3.129), and satisfy

$$\delta\tilde{\omega} \sim O[(r - r_h)^{\varsigma - 1}], \qquad \delta\xi \sim O[(r - r_h)^{\varsigma - 1}]. \tag{3.151}$$

Hence $\delta\tilde{\omega}$ and $\delta\xi$ do not remain bounded as $r^* \to -\infty$, and the "pure gauge" mode solution found above is the only zero mode of the system (3.139), present for all values of $v \neq 0$.

From the above considerations, it follows that the number of negative eigenvalues of the system (3.139) cannot change at any non-zero value of v, including the bifurcation point $v = v_{max}$, from the continuity of the eigenvalues with v. Note that the negative eigenvalues (which form the discrete spectrum of the Schrödinger operator) are themselves non-degenerate and hence cannot bifurcate at any value of v.

Behaviour at $v = 0$ The only remaining possibility is that the number of negative modes may change at $v = 0$, the number of modes of instability otherwise remaining constant along each branch of solutions. To determine the behaviour of the perturbation equations as $v \to 0$ we re-scale the variables:

$$\phi = v\tilde{\phi}, \qquad \delta\xi = v\delta\tilde{\xi}. \tag{3.152}$$

The system (3.139) then reads

$$\begin{aligned}
\sigma^2\delta\alpha &= -p_*^2\delta\alpha + \left(\frac{2}{\tilde{\phi}}p_*\tilde{\phi} + \frac{2NS}{r}\right)p_*\delta\alpha + \left(\frac{2NS^2}{r^2}\omega^2 + \frac{1}{4}NS^2v^2\tilde{\phi}^2\right)\delta\alpha \\
&\quad + \left(-2p_*\omega + \frac{2\omega}{\tilde{\phi}}p_*\tilde{\phi} + \frac{2NS}{r}\omega\right)\delta\tilde{\omega} &(3.153)\\
\sigma^2\delta\tilde{\omega} &= -p_*^2\delta\tilde{\omega} - \frac{2NS^2}{r^2}(1+\omega)p_*\delta\alpha + \frac{NS^2}{r^2}\left[(1-\omega)^2 + \frac{1}{4}v^2r^2\tilde{\phi}^2\right]\delta\tilde{\omega} \\
&\quad - \left[\frac{4NS^2}{r^2}p_*\omega + 2\omega p_*\left(\frac{NS^2}{r^2}\right)\right]\delta\alpha &(3.154)
\end{aligned}$$

together with the Gauss constraint

$$p_*\delta\alpha = -\omega\delta\tilde{\omega} + \frac{1}{4}v^2r^2\tilde{\phi}\delta\tilde{\xi}. \tag{3.155}$$

Consider firstly the $k = n$ branch of equilibrium black hole solutions. Along this branch, $p_*\tilde{\phi}/\tilde{\phi}$ has a well-defined limit as $v \to 0$ [2], and the equations (3.153) are continuous at $v = 0$, taking the form

$$\begin{aligned}
\sigma^2\delta\alpha &= -p_*^2\delta\alpha + \left(\frac{2}{\tilde{\phi}}p_*\tilde{\phi} + \frac{2NS}{r}\right)p_*\delta\alpha + \frac{2NS^2}{r^2}\omega^2\delta\alpha \\
&\quad + \left(-2p_*\omega + \frac{2\omega}{\tilde{\phi}}p_*\tilde{\phi} + \frac{2NS}{r}\omega\right)\delta\tilde{\omega} &(3.156)\\
\sigma^2\delta\tilde{\omega} &= -p_*^2\delta\tilde{\omega} - \frac{2NS^2}{r^2}(1+\omega)p_*\delta\alpha + \frac{NS^2}{r^2}(1-\omega)^2\delta\tilde{\omega} \\
&\quad - \left[\frac{4NS^2}{r^2}p_*\omega + 2\omega p_*\left(\frac{NS^2}{r^2}\right)\right]\delta\alpha. &(3.157)
\end{aligned}$$

In this limit, the Gauss constraint reduces to

$$p_*\delta\alpha = -\omega\delta\tilde{\omega} \tag{3.158}$$

and substituting for $\delta\tilde{\omega}$ in (3.156) yields the single equation

$$-p_*^2\delta\alpha + \frac{2}{\omega}(p_*\omega)(p_*\delta\alpha) + \frac{2NS^2}{r^2}\omega^2\delta\alpha = \sigma^2\delta\alpha \tag{3.159}$$

where the field ω now has the form of the nth coloured black hole. This equation was studied by Volkov et al [48], who showed that it possesses exactly n negative eigenvalues. Their method is outlined below.

Instabilities of the coloured black holes Firstly, make the substitution [48]

$$\tilde{\alpha} = \frac{\delta\alpha}{\omega}, \tag{3.160}$$

to yield the equation

$$-p_*^2\tilde{\alpha} + \left(\frac{NS^2}{r^2}(1+\omega^2) + 2\left(\frac{p_*\omega}{\omega}\right)^2\right)\tilde{\alpha} = \sigma^2\tilde{\alpha} \tag{3.161}$$

where the equilibrium Yang-Mills equation has been used to eliminate $p_*^2\omega$. The presence of zeros of ω prevents this from being a standard Schrödinger equation. The method of Volkov et al [48] involves transforming to a "dual" Schrödinger equation for which the potential is regular, allowing the use of the node rule to evaluate the number of bound states.

As above, for $\sigma^2 = 0$ the "pure gauge" mode solutions take the form

$$\tilde{\alpha}_0 = \frac{r^2 p_*\Theta}{2\omega NS^2} \tag{3.162}$$

where a particular choice of $\Theta(r^*)$ will be effected later. The differential operator in (3.161) can then be factorized:

$$-p_*^2 + \mathcal{V}_1 = p_*^2 + \frac{p_*^2\tilde{\alpha}_0}{\tilde{\alpha}_0} = W^+W^- \tag{3.163}$$

where

$$W^\pm = \mp p_* - \frac{p_*\tilde{\alpha}_0}{\tilde{\alpha}_0}. \tag{3.164}$$

The differential equation then reads

$$W^+W^-\tilde{\alpha} = \sigma^2\tilde{\alpha}. \tag{3.165}$$

The dual equation is constructed by transforming variables:

$$\varrho = W^-\tilde{\alpha}. \tag{3.166}$$

For $\sigma^2 \neq 0$ this transformation has a unique inverse:

$$\tilde{\alpha} = \frac{1}{\sigma^2}W^+\varrho \tag{3.167}$$

which yields the dual equation

$$W^-W^+\varrho = \sigma^2\varrho. \tag{3.168}$$

The operator on the left-hand side of this equation is

$$W^-W^+ = -p_*^2 + \mathcal{V}_2 \tag{3.169}$$

where the potential $\mathcal{V}_2(r^*)$ is given by [48]

$$\mathcal{V}_2(r^*) = \frac{NS^2}{r^2}(3\omega^2 - 1) + 2p_*\mathcal{V}_3 \tag{3.170}$$

and

$$\mathcal{V}_3 = \frac{r^2}{NS^2}(p_*\Theta)p_*\left(\frac{NS^2}{r^2 p_*\Theta}\right). \tag{3.171}$$

The zero mode solutions of this equation (3.169) satisfy

$$W^+\varrho_0 = 0 \tag{3.172}$$

that is,

$$-\frac{p_*\varrho_0}{\varrho_0} = \frac{p_*\tilde{\alpha}_0}{\tilde{\alpha}_0} \tag{3.173}$$

which gives

$$\varrho_0 = \frac{1}{\tilde{\alpha}_0} = \omega \exp\left(\int_{r_0^*}^{r^*} \mathcal{V}_3\, dr^*\right). \tag{3.174}$$

Volkov et al [48] then showed that it is possible to choose Θ so that the Gauss constraint is satisfied, the potential $\mathcal{V}_2$ (3.170) is everywhere regular and bounded, and the factor

$$\exp\left(\int_{r_0^*}^{r^*} \mathcal{V}_3\, dr^*\right) \tag{3.175}$$

is also regular and bounded for all values of r^*. Further, the function ϱ_0 is normalizable and thus represents an eigenfunction of the dual Schrödinger equation with zero eigenvalue. This eigenfunction has precisely n zeros, where n is the number of zeros of the Yang-Mills field ω and then, by invoking the node rule, (see for example [56]) we see that there are exactly n negative eigenvalues of the equation (3.169), and hence, via the inverse transformation (3.167), of the original Schrödinger equation (3.161).

Application to the EYMH system Returning to the EYMH system, the result of Volkov et al [48] tells us that, for the limit $v \to 0$ along the $k = n$ branch of solutions, there are exactly n negative eigenvalues and a single, non-degenerate zero mode given by

$$\delta\alpha = \frac{r^2 p_*\Theta}{2NS^2} \tag{3.176}$$

where

$$p_*\left(\frac{r^2 p_*\Theta}{2NS^2}\right) = \omega^2\Theta \tag{3.177}$$

so that $\Theta \sim O(r - r_h)$ or $O(1)$ as $r \to r_h$, and $\Theta \sim O(r)$ or $O(r^{-2})$ as $r \to \infty$, whence $\Theta \sim O(r^{-2})$ for the zero mode. This means that the number of negative modes does not change at $v = 0$ for this branch of solutions.

It is a different story for the quasi-$k = n - 1$ branch of solutions. Here $(p_*\tilde{\phi})/\tilde{\phi}$ does not have a well-defined limit as $v \to 0$ and the system (3.153) is discontinuous at $v = 0$. We conclude that, starting at the $v = 0$ limit of the $k = n$ branch, and working continuously up to the bifurcation point, and down the quasi-$k = n - 1$ branch, the number of negative modes remains constant at n.

Result 3.5 *Both the $k = n$ and quasi-$k = n - 1$ black holes have exactly n unstable modes in the sphaleronic sector.*

Gravitational Sector Perturbations

We now consider the linearized perturbation equations for the gravitational sector, and count the number of modes of instability. The approach initially is similar to that employed in the sphaleronic sector, and makes use of the same assumptions of analyticity. However, in order to elucidate the behaviour at the bifurcation point, $v = v_{max}$, we shall appeal to an application of catastrophe theory.

3.5 The Schrödinger system

From (3.38–3.43), the linearized perturbation equations for the gravitational sector take the form:

$$\begin{aligned} -p_*^2\delta\omega + \mathcal{U}_{\omega\omega}\delta\omega + \mathcal{U}_{\omega\phi}\delta\phi &= \sigma^2\delta\omega \\ -p_*^2\delta\phi - \frac{2NS}{r}p_*\delta\phi + \mathcal{U}_{\phi\omega}\delta\omega + \mathcal{U}_{\phi\phi}\delta\phi &= \sigma^2\delta\phi \end{aligned} \tag{3.178}$$

where

$$p_* = \frac{d}{dr^*} = NS\frac{d}{dr} \tag{3.179}$$

with r^* the "tortoise" co-ordinate, and the potential functions $\mathcal{U}$ are given by:

$$\begin{aligned} \mathcal{U}_{\omega\omega} &= \frac{NS^2}{r^2}\left[3\omega^2 - 1 + \frac{1}{4}r^2\phi^2 - 4r^2\omega'^2\left(\frac{N}{r} + \frac{(NS)'}{S}\right)\right. \\ &\quad \left. + \frac{8\omega(\omega^2-1)}{r}\omega' + 2r\omega'\phi^2\right] \\ \mathcal{U}_{\omega\phi} &= \frac{NS^2}{r^2}\left[\frac{1}{2}(1+\omega)\phi r^2 - 2\phi'\omega' r^3\left(\frac{N}{r} + \frac{(NS)'}{S}\right) + 2r\phi'\omega(\omega^2-1)\right. \\ &\quad \left. + \phi\omega'(1+\omega)^2 r + \frac{1}{2}\phi'\phi^2(1+\omega) + 2\lambda r^3\phi\omega'(\phi^2 - v^2)\right] \\ \mathcal{U}_{\phi\omega} &= \frac{2}{r^2}\mathcal{U}_{\omega\phi} \\ \mathcal{U}_{\phi\phi} &= \frac{NS^2}{r^2}\left[\frac{1}{2}(1+\omega)^2 + \lambda r^2(3\phi^2 - v^2) - 2r^3\phi'^2\left(\frac{N}{r} + \frac{(NS)'}{S}\right)\right. \\ &\quad \left. + 2\phi'\phi(1+\omega)^2 r + 4\lambda\phi\phi' r^3(\phi^2 - v^2)\right]. \end{aligned} \tag{3.180}$$

These equations are in the standard Schrödinger form and we can now follow the procedure of section 3.4, in order to count the number of negative eigenvalues σ^2 of this system.

3.5.1 The spectrum

Firstly, we note that as $r^* \to -\infty$, the event horizon is approached and the equations (3.178) take the form

$$\begin{aligned} -p_*^2\delta\omega &= \sigma^2\delta\omega \\ -p_*^2\delta\phi &= \sigma^2\delta\phi \end{aligned} \tag{3.181}$$

whereas, at infinity the system reduces to

$$\begin{aligned} -p_*^2\delta\omega + \frac{1}{4}v^2\delta\omega &= \sigma^2\delta\omega \\ -p_*^2\delta\phi + 2\lambda v^2\delta\phi &= \sigma^2\delta\phi \end{aligned} \tag{3.182}$$

where we have used the fact that the space-time is asymptotically flat. This means that the region of the real line $\mathbb{R}$ with $\sigma^2 > 0$ forms the continuous spectrum of the system (3.178), whilst eigenvalues with $\sigma^2 < 0$ comprise the discrete spectrum.

The continuity argument presented in sections 3.4 can be applied here to imply that, as the Higgs parameter v varies, the number of negative eigenvalues of (3.178) can change only at those values of v for which there is a zero mode. The next step is therefore to determine what these values of v are.

3.5.2 Zero modes

Suppose that for some $v \neq 0$ the Schrödinger equations (3.178) possess a zero mode. This corresponds to an infinitesimal, static perturbation, in other words, both the background equilibrium fields (ω, ϕ) and the perturbed fields $(\omega + \delta\omega, \phi + \delta\phi)$, together with the corresponding metric functions, satisfy the static field equations.

Near the event horizon, as $r^* \to -\infty$, from (3.181) we see that

$$\begin{pmatrix} \delta\omega \\ \delta\phi \end{pmatrix} \sim \text{constant} \tag{3.183}$$

(note that since $\delta\omega$, $\delta\phi$ are eigenfunctions, they must remain bounded in the asymptotic regions, but are not necessarily normalizable). Similarly, as $r^* \to \infty$, we see, from (3.182), that

$$\begin{pmatrix} \delta\omega \\ \delta\phi \end{pmatrix} \sim \begin{pmatrix} O(e^{-\frac{1}{2}vr^*}) \\ O(e^{-\sqrt{2\lambda}vr^*}) \end{pmatrix}. \tag{3.184}$$

The field perturbations $\delta\omega$ and $\delta\phi$ give the change in the "mass" function $\delta m(r^*)$ via the equation (3.12):

$$\delta m(r^*) = 2N\omega'\delta\omega + Nr^2\phi'\delta\phi. \tag{3.185}$$

Hence $\delta m \to 0$ in the limit $r^* \to \pm\infty$, so that the geometry is not changed at the event horizon or at infinity. The variation of the remaining Einstein equation gives (3.13)

$$\delta\left(\frac{S'}{S}\right) = \frac{4\omega'}{r}\delta\omega + 2r\phi'\delta\phi \to 0 \qquad \text{as } r^* \to \pm\infty. \tag{3.186}$$

Thus, for fixed parameters r_h, λ, g, v, there are two solutions of the static EYMH field equations, satisfying the boundary conditions for a black hole with regular event horizon at r_h and a geometry which is asymptotically flat.

Here a further assumption is in order, namely that the black hole solutions of [2] are unique. Once again this assertion cannot be rigorously established, although there is no evidence to the contrary.

With this assumption, it can be seen that there are two equilibrium solutions arbitrarily close together only at the bifurcation point $v = v_{max}$, so that this is the only non-zero value of v for which a zero mode is present.

3.5.3 Behaviour at $v = 0$

In order to assess the situation at $v = 0$, perform the change of variables,

$$\delta\phi = v\delta\tilde{\phi}, \qquad \phi = v\tilde{\phi}, \tag{3.187}$$

under which the equations (3.178) take the form

$$\begin{aligned} -p_*^2\delta\omega + \mathcal{U}_{\omega\omega}\delta\omega + v^2\tilde{\mathcal{U}}_{\omega\phi}\delta\tilde{\phi} &= \sigma^2\delta\omega \\ -p_*^2\delta\tilde{\phi} - \frac{2NS}{r}p_*\delta\tilde{\phi} + \tilde{\mathcal{U}}_{\phi\omega}\delta\omega + \mathcal{U}_{\phi\phi}\delta\tilde{\phi} &= \sigma^2\delta\tilde{\phi} \end{aligned} \tag{3.188}$$

where

$$\mathcal{U}_{\omega\phi} = v\tilde{\mathcal{U}}_{\omega\phi}, \qquad \mathcal{U}_{\phi\omega} = v\tilde{\mathcal{U}}_{\phi\omega}. \tag{3.189}$$

For all branches of solutions, these equations (3.188) have a well-defined limit as $v \to 0$, and are continuous at $v = 0$, reducing to the single equation

$$-p_*^2\delta\omega + \mathcal{U}_{\omega\omega}\delta\omega = \sigma^2\delta\omega \tag{3.190}$$

where

$$\mathcal{U}_{\omega\omega} = \frac{NS^2}{r^2}\left[3\omega^2 - 1 - 4r\omega'^2\left(\frac{N}{r} + \frac{(NS)'}{S}\right) + \frac{8\omega}{r}(\omega^2 - 1)\omega'\right]. \tag{3.191}$$

Suppose equation (3.190) possesses a zero mode. At infinity, as $r \to \infty$,

$$\begin{aligned} N &\to 1 - \frac{2\mathcal{M}}{r} + O(e^{-r}) \\ S &\to 1 + O(e^{-r}) \\ \omega &\to -1 + O(e^{-r}) \end{aligned} \tag{3.192}$$

where

$$\mathcal{M} = \lim_{r\to\infty} m(r), \tag{3.193}$$

and (3.190), rewritten in terms of the Schwarzschild co-ordinate r becomes

$$\delta\omega'' = -\frac{2\mathcal{M}}{r^2}\left(1 - \frac{2\mathcal{M}}{r}\right)^{-1}\delta\omega' + \frac{2}{r^2}\left(1 - \frac{2\mathcal{M}}{r}\right)^{-1}\delta\omega. \tag{3.194}$$

(note that at infinity, $r^* \to r$ for an asymptotically flat space-time). Using the Frobenius method, suppose

$$\delta\omega = \sum_{n=0}^{\infty} \mathrm{a}_n r^{i-n}, \qquad \mathrm{a}_0 \neq 0 \qquad \text{as } r \to \infty. \tag{3.195}$$

Then $i = 2$ or -1, hence for a bounded eigenfunction we require $\delta\omega \sim r^{-1}$ as $r \to \infty$. As $r^* \to -\infty$, equation (3.190) reduces to $p_*^2\delta\omega = 0$, so that $\delta\omega \to$ constant as $r^* \to -\infty$ for a zero mode. The variations of the Einstein equations give in this case

$$\delta m = 2N\omega'\delta\omega \to 0 \qquad \text{as } r^* \to -\infty. \tag{3.196}$$

As $r^* \to \infty$, for the coloured black holes of [18], the gauge field ω decays polynomially rather than exponentially, so that $\omega' \sim r^{-2}$ as $r \to \infty$. Hence $\delta m \sim r^{-3}$ as $r \to \infty$, which means that the total mass of the black hole is unchanged by the perturbation $\delta\omega$. Thus the existence of a zero mode $\delta\omega$ would represent two coloured black holes, having the same values of the parameters r_h and g and the same mass. By an argument similar to that of the previous subsection, this situation cannot arise if the coloured black hole solutions of [18] are unique. It should be noted that the analytic work to date on the EYM system [31, 32] covers only existence and not uniqueness of solutions. However, all numerical evidence suggests that the coloured black holes are unique (certainly with fixed mass), and hence there are no zero modes for $v = 0$.

3.5.4 The number of negative modes

The existence of a single zero mode at the bifurcation point v_{max} and no zero modes for other values of v means that the number of negative eigenvalues of the system (3.178) remains constant along each branch of solutions, from $v = 0$ up to $v = v_{max}$. However, it may change (by at most one since there is a single zero mode at v_{max}) as we move from one branch to another at the bifurcation point.

In order to determine what happens at the bifurcation point, it is necessary to employ catastrophe theory. The reader is referred to appendix A for an outline of the important concepts in catastrophe theory, which we apply to the EYMH black holes in the next section.

3.6 Catastrophe theory and stability of the EYMH black holes

In this section, catastrophe theory will be applied to the EYMH black holes in order to determine the stability behaviour at the bifurcation point $v = v_{max}$, and from this information count the number of unstable modes of the gravitational sector perturbations. The ideas of catastrophe theory that we shall be using are described in appendix A.

3.6.1 Cautionary remarks

Before we begin the analysis, a few comments should be made concerning the application of catastrophe theory. Catastrophe theory is a potentially powerful mathematical tool, having been applied quite extensively to a universal stability study of various "hairy" black holes [57, 58], as well as boson stars [59, 60], solitons [61], and many other areas of physics (see [62] for reviews of some of the work completed in various fields).

However, catastrophe theory can only be employed subject to the following caveat. Catastrophe theory gives qualitative information concerning the instability of certain modes of the system. This means that results obtained using catastrophe theory relate only to the *relative* stability of the configuration in question, not to its *absolute* stability. There may be other modes of instability which are not covered by the catastrophe theory, as will be the case below for the topological instabilities exhibited in the sphaleronic sector of the EYMH theory. In other words, if catastrophe theory implies that a system is stable, this only means that the system is less unstable than another, rather than the system being absolutely stable. It is in this context that we wish to apply catastrophe theory.

Our approach has the advantage over that of [58] in that, coupled with the analysis of section 3.5, we are able to count the exact number of unstable modes for each branch of black hole solutions.

Catastrophe theory is also able to tell us something of the non-linear stability behaviour of the solutions, albeit only qualitatively. A full non-linear stability analysis would be a somewhat lengthy undertaking [63].

3.6.2 Catastrophe-theoretic analysis

The first step in the catastrophe-theoretic analysis is to construct an appropriate family of functions. We define the mass functional $\mathcal{M}$ of the black hole configurations to be the ADM (Arnowitt-Deser-Misner [64]) mass of the geometry [21]:

$$\begin{aligned}\mathcal{M} &= m(\infty) = m(r_h) + \int_{r_h}^{\infty} m'(r)dr \\ &= \frac{r_h}{2} + \int_{r_h}^{\infty} \frac{1}{2}\left\{\left(1 - \frac{2m}{r}\right)(2\omega'^2 + r^2\phi'^2)\right.\end{aligned}$$

$$+r^2\left[\frac{(1-\omega^2)^2}{r^4}+\frac{\phi^2}{2r^2}(1+\omega)^2+\frac{\lambda}{2}(\phi^2-v^2)^2\right]\Bigg\}\,dr \tag{3.200}$$

using the Einstein equation (2.49). This functional can be written as a functional of the matter fields (ω,ϕ) only, as follows [21]. Let

$$\varepsilon(r)=m(r)-m(r_h)=m(r)-\frac{r_h}{2}. \tag{3.201}$$

Then

$$\begin{aligned}
\varepsilon'(r) &= m'(r)\\
&= \frac{1}{2}\Bigg\{\left(1-\frac{2m}{r}\right)(2\omega'^2+r^2\phi'^2)\\
&\quad +r^2\left[\frac{(1-\omega^2)^2}{r^4}+\frac{\phi^2}{2r^2}(1+\omega)^2+\frac{\lambda}{2}(\phi^2-v^2)^2\right]\Bigg\}\\
&= \frac{1}{2}\Bigg\{\left(1-\frac{r_h}{r}\right)(2\omega'^2+r^2\phi'^2)\\
&\quad +r^2\left[\frac{(1-\omega^2)^2}{r^4}+\frac{\phi^2}{2r^2}(1+\omega)^2+\frac{\lambda}{2}(\phi^2-v^2)^2\right]\Bigg\}\\
&\quad -\frac{\varepsilon}{r}(2\omega'^2+r^2\phi'^2)
\end{aligned} \tag{3.202}$$

using the definition of $\varepsilon(r)$. The Einstein equations (2.51) also give

$$\delta'(r)=-\frac{1}{r}(2\omega'^2+r^2\phi'^2) \tag{3.203}$$

whence the first-order differential equation (3.202) for ε can be solved:

$$\varepsilon(r)=e^{\delta(r)}\int_{r_h}^{r}\mathcal{K}[\omega,\phi](r')e^{-\delta(r')}dr' \tag{3.204}$$

where

$$\begin{aligned}
\mathcal{K}[\omega,\phi](r) &\equiv \frac{1}{2}\Bigg\{\left(1-\frac{r_h}{r}\right)(2\omega'^2+r^2\phi'^2)\\
&\quad +r^2\left[\frac{(1-\omega^2)^2}{r^4}+\frac{\phi^2}{2r^2}(1+\omega)^2+\frac{\lambda}{2}(\phi^2-v^2)^2\right]\Bigg\}
\end{aligned} \tag{3.205}$$

is a functional of ω, ϕ only. In an asymptotically flat space-time, $\delta(\infty)=0$ and thus

$$\begin{aligned}
\mathcal{M}[\omega,\phi] &= \frac{r_h}{2}+\varepsilon(\infty)\\
&= \frac{r_h}{2}+\int_{r_h}^{\infty}\mathcal{K}[\omega,\phi](r')e^{-\delta(r')}dr'.
\end{aligned} \tag{3.206}$$

It can be shown [21] that varying this functional with respect to the fields ω, ϕ yields the static equations of motion (2.47), (2.48) for the EYMH system. Thus the equilibrium configurations will be stationary points of the functional $\mathcal{M}$.

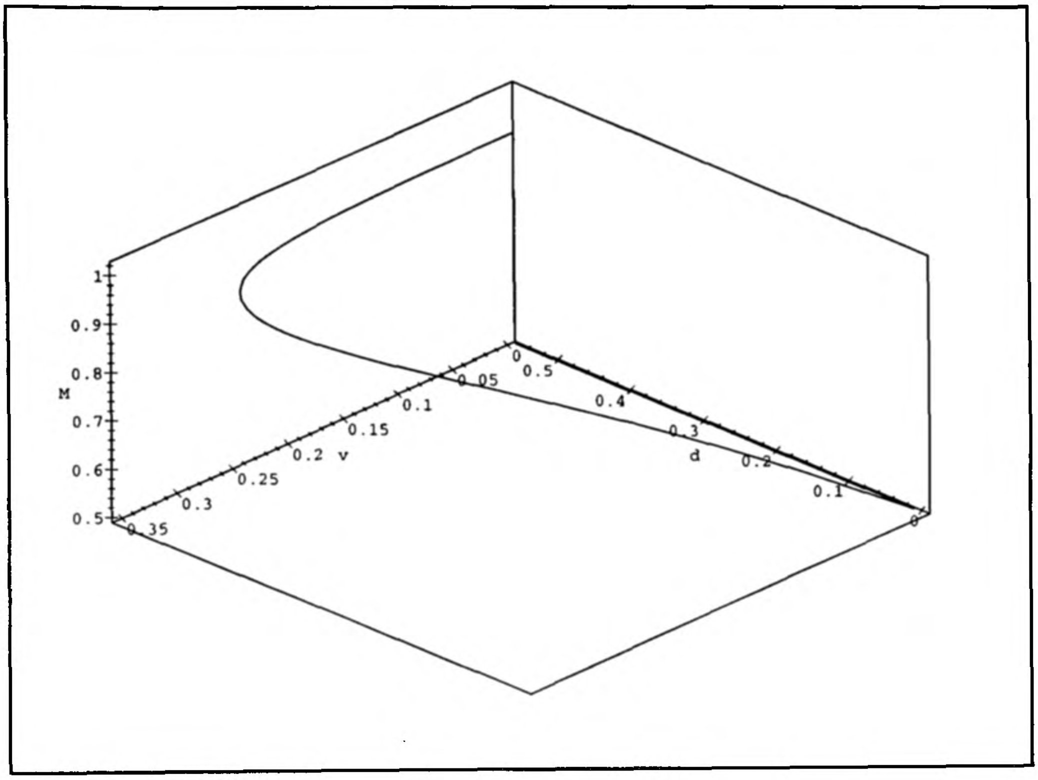

Figure 3.1: The smooth solution curve for the $k = 1$ and quasi-$k = 0$ black holes, plotted in $(v, \delta_0, \mathcal{M})$ space.

At present there are infinitely many degrees of freedom on which the functional $\mathcal{M}$ depends, corresponding to the freedom in choosing functions $\omega(r)$, $\phi(r)$ such that $\mathcal{M}$ is finite. In order to whittle down to just two degrees of freedom, define a functional δ_0 by

$$\delta_0[\omega, \phi] = \int_{r_h}^{\infty} \frac{1}{r} \left(2\omega'^2 + r^2\phi'^2\right) dr. \tag{3.207}$$

For each equilibrium black hole solution of the EYMH field equations, we can calculate $\mathcal{M}$ and δ_0 from the above expressions. In figure 3.1 we plot the solution curve $(v, \delta_0, \mathcal{M})$ representing the $k = 1$ and quasi-$k = 0$ solutions as v varies, with $\lambda = 0.15$ and $r_h = 1$ fixed. Notice that the two branches of solutions join up to give a smooth curve. The bifurcation point, where the branches meet, is at $(v_{max} = 0.352, \delta_0 = 0.332, \mathcal{M} = 1.027)$. The quasi-$k = 0$ family corresponds to that part of the curve from $v = v_{max}$ to the point $(0, 0, 0)$, the rest of the curve representing the $k = 1$ branch of solutions. The properties of this graph will be

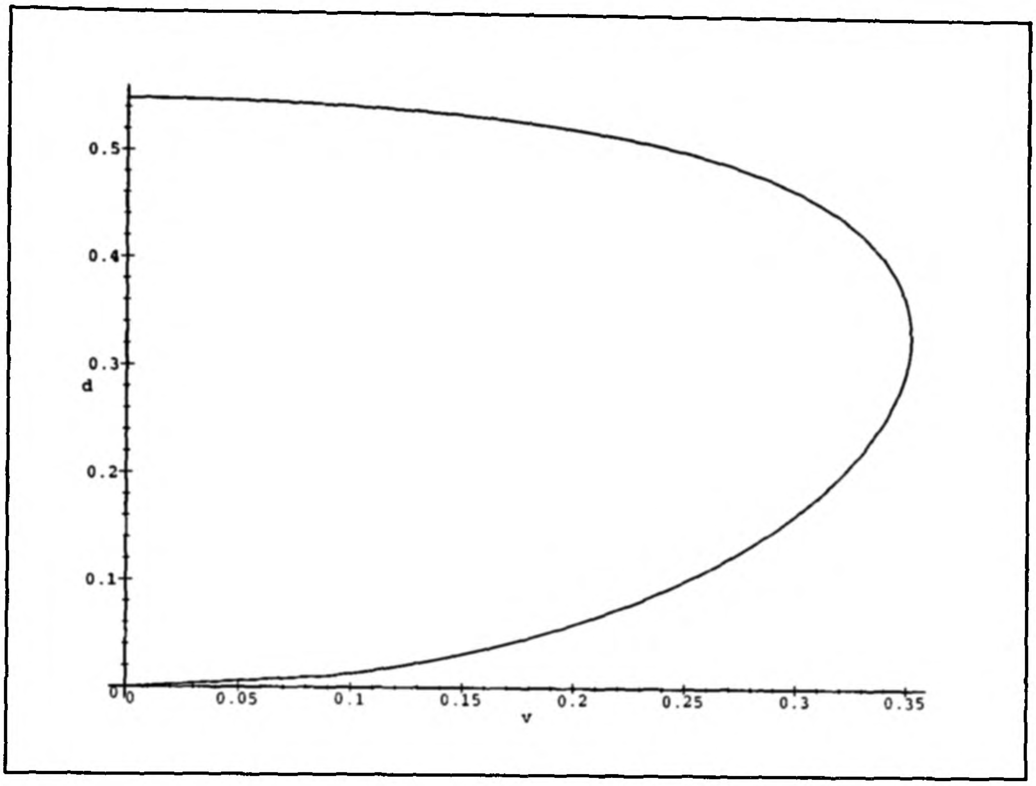

Figure 3.2: The projection of the curve in figure 3.1 onto the (v, δ_0) plane.

similar for the higher-node solutions.

The projections of the smooth curve on the (v, δ_0) and $(\delta_0, \mathcal{M})$ planes are also smooth, as may been seen from figures 3.2 and 3.3. However, the projection onto the $(v, \mathcal{M})$ plane has a cusp at the point $(v_{max}, \mathcal{M}_{max})$ (see figure 3.4). The presence of this cusp will be vital to our catastrophe theory analysis.

To define the Whitney surface, consider for each fixed v a smoothly varying set of functions ω_δ, ϕ_δ indexed by the value of δ_0 they give according to (3.207), and such that ω_δ, ϕ_δ are the appropriate solutions to the field equations when δ_0 lies on the solution curve. Then, for each v and δ_0 there is a unique choice of ω_δ and ϕ_δ from which $\mathcal{M}$ can be calculated. As v and δ_0 vary, so the values of $\mathcal{M}$ map out the Whitney surface. In addition, the solution curve is the curve of extremal points of this surface, where $\delta\mathcal{M} = 0$.

In our application of catastrophe theory, we shall be thinking of δ_0 as the variable, and v as the control parameter. Hence the Whitney surface of $\mathcal{M}$ describes a family of functions of the single variable δ_0, the members of the family being indexed by

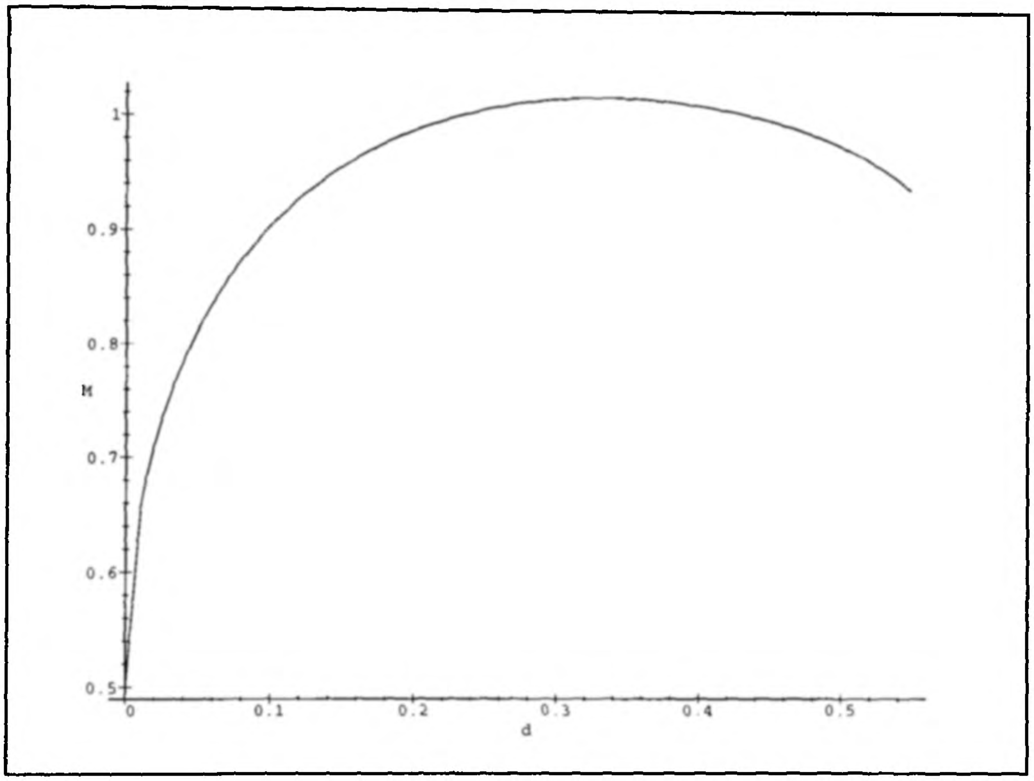

Figure 3.3: The projection of the curve in figure 3.1 onto the $(\delta_0, \mathcal{M})$ plane.

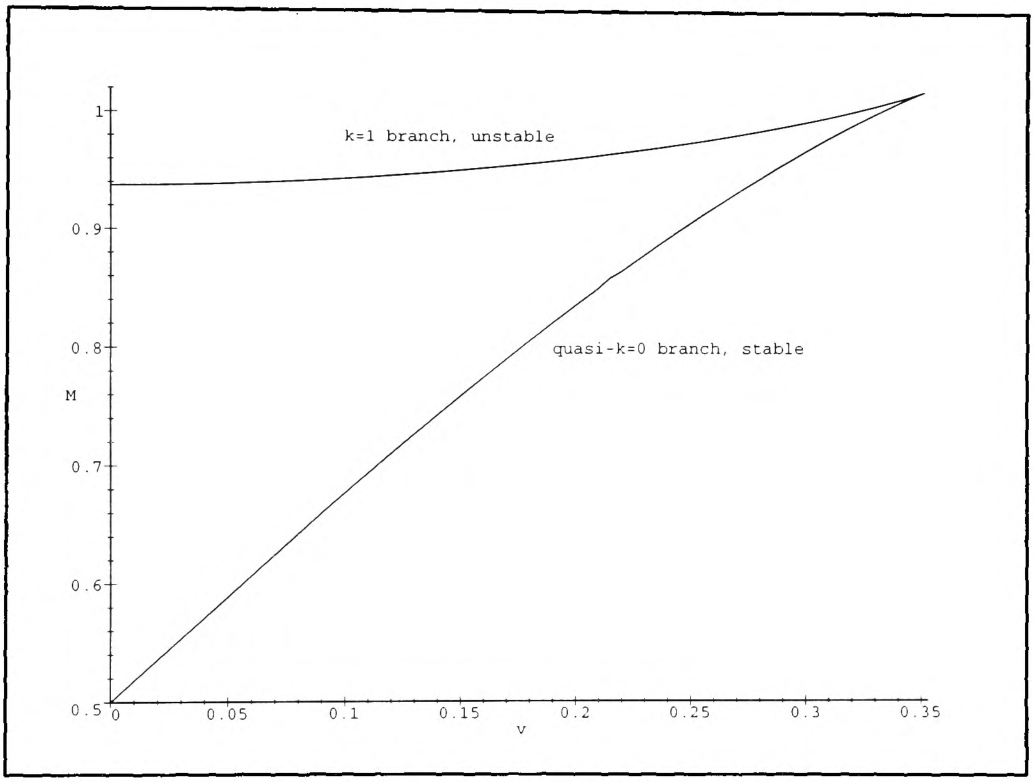

Figure 3.4: The projection of the curve in figure 3.1 onto the $(v, \mathcal{M})$ plane.

the parameter v.

The next step is to consider the catastrophe manifold $\mathcal{C}$, which is the projection of the solution curve onto the (v, δ_0) plane, shown in figure 3.2. The catastrophe map χ, which is the restriction to $\mathcal{C}$ of the natural projection

$$\pi : (v, \delta_0) \mapsto (v), \tag{3.208}$$

can easily be extended to the projection of the solution curve onto the $(v, \mathcal{M})$ plane:

$$\chi : (v, \delta_0, \mathcal{M}) \mapsto (v, \mathcal{M}). \tag{3.209}$$

The image of this map is shown in figure 3.4, where it can be easily seen that the map (3.209) is regular everywhere except at the point $(v = v_{max}, \mathcal{M} = \mathcal{M}_{max})$, where it is singular. This point therefore comprises the bifurcation set B.

In summary we have a one-parameter (v) family of functions of a single variable (δ_0), whose bifurcation set B consists of a single point. From the classification theorem (see appendix A), therefore, we have a *fold catastrophe*. A fold catastrophe was also found by the authors of [57] from their study of the EYMH system, although their approach was somewhat different. This difference will be discussed at the end of this section.

The main result of catastrophe theory is that the stability of the system will change at the bifurcation point B. That branch of solutions (including the point B) having the higher mass $\mathcal{M}$ (for the same value of v) corresponds to a maximum of the functional (from the analysis of the fold catastrophe in subsection A.2) and hence will be unstable relative to the other branch, which corresponds to a minimum of the functional. From the continuity considerations of section 3.5, therefore, the $k = n$ branch of solutions (which has the higher mass) will have exactly one more unstable mode than the quasi-$k = n - 1$ branch (which has lower mass), since the number of negative modes may change by at most one as we pass from the quasi-$k = n - 1$ to the $k = n$ branch.

Before using this result to calculate the exact number of negative modes for each branch, a comment is in order. Catastrophe theory lends itself to analysis of the gravitational sector rather than the sphaleronic sector. This is because gravitational perturbations correspond to small changes in the functions ω and ϕ, whilst keeping the functional form of $\mathcal{M}$ fixed. In other words, our function $\mathcal{M}$ is fixed, and we are changing the values of the variable δ_0 slightly. On the other hand, for the sphaleronic sector perturbations, the functions ω and ϕ remain constant, but the functional form of $\mathcal{M}$ alters. This kind of transformation is not covered by catastrophe theory. We have already shown that both the $k = n$ and quasi-$k = n - 1$ branches of solutions possess n unstable modes in the sphaleronic sector. These modes are undetected by the catastrophe theory considerations, and the reader will now see why the caveat mentioned at the beginning of this section was necessary.

3.6.3 Inductive process

We shall now employ the process of induction to work out exactly how many gravitational modes of instability there are. Firstly note that the $v = 0$ limit of the $k = n$ branch is the same as that of the quasi-$k = n$ branch after replacing ω by $-\omega$ [2]. Hence, by continuity, the $k = n$ and quasi-$k = n$ branches have the same number of unstable modes in the gravitational sector (the quasi-$k = n$ branch has an additional unstable mode in the sphaleronic sector). So, working through the branches, the $k = n$ branch has $n + i$ negative modes, and the quasi-$k = n - 1$ branch has $n - 1 + i$ negative modes, where i is the number of negative modes of the quasi-$k = 0$ branch, and thus of its $v = 0$ limit.

In the limit $v \to 0$, the geometry of the quasi-$k = 0$ branch becomes that of the Schwarzschild black hole. The perturbation equation is (3.190)

$$-p_*^2 \delta\omega + \mathcal{U}_{\omega\omega} \delta\omega = \sigma^2 \delta\omega \tag{3.210}$$

where

$$\mathcal{U}_{\omega\omega} = \frac{NS^2}{r^2} \left[3\omega^2 - 1 - 4r\omega'^2 \left(\frac{N}{r} + \frac{(NS)'}{S} \right) + \frac{\omega(\omega^2 - 1)}{r} \omega' \right]. \tag{3.211}$$

For the Schwarzschild black hole,

$$N \equiv 1 - \frac{r_h}{r}, \qquad S \equiv 1, \qquad \omega \equiv 1, \tag{3.212}$$

and the equation (3.210) reduces to

$$-p_*^2 \delta\omega + \frac{2}{r^2} \left(1 - \frac{r_h}{r} \right) \delta\omega = \sigma^2 \delta\omega \tag{3.213}$$

which has the form of a standard one-dimensional Schrödinger equation, with potential

$$\mathcal{U}(r^*) = \frac{2}{r^2} \left(1 - \frac{r_h}{r} \right), \qquad \frac{dr^*}{dr} = \left(1 - \frac{r_h}{r} \right)^{-1}. \tag{3.214}$$

As $r^* \to \pm\infty$, corresponding to the event horizon and infinity, $\mathcal{U}(r^*) \to 0$. For finite values of r^*, the potential $\mathcal{U}(r^*) > 0$, since we are considering only points outside the event horizon, $r > r_h$. Thus the potential is positive definite.

A standard theorem of quantum mechanics (see for example [56]) then says that the Schrödinger equation (3.213) possesses no bound states, i.e. no negative modes, so that

$$i = 0. \tag{3.215}$$

This is in accordance with the known stability of the Schwarzschild solution [40].

Altogether we have the following:

Result 3.6 *The $k = n$ black holes have exactly n unstable modes in the gravitational sector, whilst the quasi-$k = n - 1$ black holes have exactly $n - 1$ unstable modes.*

3.6.4 Comparison with other results

Finally in this section we should compare our methodology with that of [57], where catastrophe theory was also used to draw conclusions about the stability of the EYMH black holes. There the authors fixed the parameter $\lambda = 0.125$ and also fixed the Higgs v.e.v. v. They then varied the horizon radius r_h and for each solution calculated the value of the mass functional $\mathcal{M}$, the field strength at the horizon F_h given by

$$F_h = |F^2|^{\frac{1}{2}}\Big|_{\text{horizon}} \tag{3.216}$$

and the Hawking-Bekenstein entropy $\mathrm{S} = \pi r_h^2$ [65, 66], to give a smooth solution curve in $(\mathcal{M}, F_h, \mathrm{S})$ space [57]. The projection of this curve onto the $(\mathcal{M}, \mathrm{S})$ plane has the same qualitative features as our figure 3.4. From this they concluded that the $k = 1$ branch of solutions is more unstable than the quasi-$k = 0$ branch. Here we have fixed $\lambda = 0.15$ and in addition $r_h = 1$, varying the Higgs mass. The advantage of our approach is that, by interpolating between the various coloured black hole solutions of [18], in order to exploit the known stability of the Schwarzschild solution, we have been able to calculate the exact number of unstable modes and not just give qualitative information concerning their stability.

Stability summary

The $k = n$ branch of solutions possesses:

1. n unstable modes in the sphaleronic sector
2. n unstable modes in the gravitational sector.

The quasi-$k = n - 1$ branch of solutions possesses:

1. n unstable modes in the sphaleronic sector
2. $n - 1$ unstable modes in the gravitational sector.

Chapter 4

Entropy and Quantum De-coherence

Having studied EYMH black holes within a purely classical context, we now regard them as fixed backgrounds for a quantum scalar field, i.e. using a semi-classical approximation. Much of the mathematical analysis of this chapter will be valid for a general black hole geometry; however we always have in the back of our minds the EYMH system. We concentrate on two aspects of the quantum field theory. Firstly, the entropy of the scalar field [3], and secondly, the quantum de-coherence of the state of the field as time proceeds [5]. Both these topics are exceedingly non-trivial, and we do not claim to have definitive answers. However, we hope that by finding new properties of entropy we may shed some light on the fundamental questions raised. The black holes we consider in this chapter are spherically symmetric. The introduction of rotation complicates the picture, and is considered in chapter 6.

4.1 Introduction to entropy

4.1.1 Historical note

The idea that a black hole has an associated entropy began with Bekenstein [65, 66], who, on noting that the area of a black hole cannot (classically) decrease [12], counted the unmeasurable information about the internal states of the black hole hidden behind the event horizon. His famous result is that the entropy due to this lost information is proportional to the area of the black hole event horizon.

The result of Hawking [67] that the black hole has an associated temperature

$$T = \frac{\kappa}{2\pi}, \tag{4.1}$$

where κ is the surface gravity of the black hole, coupled with the first law of black hole mechanics [68], was able to put Bekenstein's idea of entropy on a firmer footing and fix the constant of proportionality:

$$\mathtt{S} = \frac{1}{4}\mathtt{A} \tag{4.2}$$

where $\mathtt{A}$ is the area of the black hole event horizon.

These two seminal pieces of work laid the foundations for the idea of a thermodynamical entropy associated with a black hole, and an associated theory of statistical mechanics [69], which we now proceed to outline.

4.1.2 Statistical-mechanical entropy

The statistical-mechanical entropy $\mathtt{S}$ of a system is defined to be

$$\mathtt{S} = -\mathrm{Tr}\,[\rho \log \rho] \tag{4.3}$$

where ρ is the density operator which maximizes $\mathtt{S}$ subject to the constraints operating on the system. For black hole thermodynamics, it is necessary to consider the micro-canonical ensemble [70], for which the total energy E of the system is kept fixed. In other words, the system is fixed to be in an energy eigenstate of the Hamiltonian H with eigenvalue E. In general this eigenvalue will be degenerate, having degeneracy

$$\mathtt{N}(E) = \mathrm{Tr}\,\delta(E - H). \tag{4.4}$$

Then

$$\rho = \frac{1}{\mathtt{N}(E)} P(E) \tag{4.5}$$

where $P(E)$ is the projection operator for the eigenspace corresponding to E, which gives the formula

$$\mathtt{S} = \log \mathtt{N}(E). \tag{4.6}$$

This idea of counting the number of allowed states of the system (or, alternatively, the number of different unobservable internal configurations [71]) is Bekenstein's original formulation of black hole entropy.

The partition function $\mathtt{Z}$ is defined as the Laplace transform of the degeneracy $\mathtt{N}(E)$ [72]

$$\begin{aligned}
\mathtt{Z}(\beta) &= \int dE\, e^{-\beta E}\mathtt{N}(E) \\
&= \int dE\, e^{-\beta E}\mathrm{Tr}\,\delta(E - H) \\
&= \mathrm{Tr}\, e^{-\beta H}
\end{aligned} \tag{4.7}$$

where, for the moment, we have not assumed that β is the inverse temperature.

We now restrict attention to the case of a black hole in thermal equilibrium with a heat bath at temperature T (i.e. in the Hartle-Hawking state (section 6.2)). In this case the density operator ρ is given by the Gibbs formula [73]

$$\rho = \frac{e^{-\beta H}}{\mathsf{Z}} \tag{4.8}$$

where

$$\beta = \frac{1}{T} \tag{4.9}$$

is now the inverse temperature (note that we are using units in which $k_B = 1$ throughout this chapter). The Gibbs formula is only valid for a static black hole geometry, considered as a fixed background on which the quantum fields evolve. This is the approach we shall be taking throughout this chapter, in other words, we are ignoring the effect of the back reaction and the subsequent evolution of the geometry. The Helmholtz free energy F is defined by

$$\mathsf{Z} = \operatorname{Tr} e^{-\beta H} = e^{-\beta F} \tag{4.10}$$

from which the entropy is calculated via the formula

$$\mathsf{S} = \beta^2 \frac{\partial F}{\partial \beta}. \tag{4.11}$$

4.1.3 Path integral formulation of quantum field theory in curved space

It should be stressed from the outset that the author is *not* purporting to possess a theory of quantum gravity! Rather, in this chapter, we are concerned with first-order approximations to the theory.

For the rest of this chapter we shall focus our attention on a quantum mechanical scalar field ϕ, propagating on the background black hole geometry, which we assume to be static for simplicity (the case in which the space-time is stationary but not static is discussed in chapter 6). The metric can then be written in the form

$$ds^2 = -g_{00}dt^2 + g_{ij}dx^i dx^j \tag{4.12}$$

where $g_{\mu\nu}$ does not depend on t, and $g_{00} > 0$ with $i, j \in \{1, 2, 3\}$. Performing a Wick rotation $t = -i\tau$ (see also section 6.5), the metric of the Euclidean manifold is then

$$ds^2 = g_{00}d\tau^2 + g_{ij}dx^i dx^j. \tag{4.13}$$

The co-ordinate τ is periodic with period β (note that at this stage we have not necessarily fixed β to be the inverse of the Hawking temperature). The partition function for the field ϕ propagating in the space-time described by (4.13) is then

$$\mathsf{Z}(\beta) = \mathcal{N} \int \mathcal{D}[\phi] e^{-I[\phi]} \tag{4.14}$$

where $\mathcal{N}$ is a normalization factor and I is the action of the field theory on the Euclidean manifold.

If we now wish to describe first-order quantum gravity, the metric is one of the fields to be integrated over, and the partition function is defined, formally, as [69]

$$\mathtt{Z}(\beta) = \mathcal{N} \int_{\mathcal{W}} \mathcal{D}[g] \int \mathcal{D}[\phi] \exp\left(-I[g,\phi]\right) \tag{4.15}$$

where $\mathcal{W}$ is the space of Euclidean metrics to be integrated over, and is restricted by various boundary conditions, such as the total energy contained within the geometry and the space-time behaviour at infinity. The action I has the form

$$I[g,\phi] = I_{EH}[g] + I_{\phi}[g,\phi] \tag{4.16}$$

where I_ϕ is the matter field action and I_{EH} is the Euclidean Einstein-Hilbert action [69]

$$I_{EH} = \frac{1}{16\pi G_N}\left[-\int_{\mathtt{M}} \epsilon_g R + 2\int_{\partial\mathtt{M}} \epsilon_h \mathtt{K}\right] \tag{4.17}$$

with ϵ_g the volume form on the Euclidean manifold $\mathtt{M}$ with metric g, and ϵ_h is the volume form on the boundary sub-manifold $\partial\mathtt{M}$. The term $\mathtt{K}$ is chosen to be the difference in the trace of the second fundamental form of $\partial\mathtt{M}$ in the metric g and the flat space Minkowski metric [69]. The reason for this choice is so that $I_{EH} = 0$ for flat space. Note that in (4.17) we have explicitly retained the Newton constant G_N for reasons which will become apparent shortly.

The dominant contribution to the path integral (4.15) will come from geometries close to a background geometry $\hat{g}$ (on a manifold $\hat{\mathtt{M}}$) which is a stationary point of the classical action and satisfies the boundary conditions. We therefore perturb about this metric and write the general metric g in the form

$$g = \hat{g} + \delta g \tag{4.18}$$

and quantize the fluctuations δg and ϕ with the background metric $\hat{g}$ fixed. In this manner the action can be expanded in powers of δg as

$$\begin{aligned} I[g,\phi] &= I[\hat{g}+\delta g,\phi] \\ &= I_{EH}[\hat{g}] + I_\phi[\hat{g},\phi] + \int_{\hat{\mathtt{M}}} \epsilon_{\hat{g}} \left.\frac{\delta I}{\delta g}\right|_{\hat{g}} \delta g + \frac{1}{2}\int_{\hat{\mathtt{M}}} \epsilon_{\hat{g}} \left.\frac{\delta^2 I}{\delta g^2}\right|_{\hat{g}} \delta g^2 + \dots \end{aligned} \tag{4.19}$$

whence the partition function takes the form

$$\mathtt{Z} = \exp\left(-I_{EH}[\hat{g}]\right) \int \mathcal{D}[\phi] \exp\left(-I_\phi[\hat{g},\phi]\right) \int \mathcal{D}[\delta g] \exp\left(-\tilde{I}[\hat{g}+\delta g,\phi]\right) \tag{4.20}$$

where

$$\tilde{I}[\hat{g}+\delta g,\phi] = I[\hat{g}+\delta g,\phi] - I_{EH}[\hat{g}] - I_\phi[\hat{g},\phi]. \tag{4.21}$$

Since $\hat{g}$ satisfies the background field equations,

$$\left.\frac{\delta I}{\delta g}\right|_{\hat{g}} = 0 \tag{4.22}$$

and

$$\tilde{I}[\hat{g} + \delta g, \phi] = I_2[\delta g] + \text{higher order terms} \tag{4.23}$$

where $I_2[\delta g]$ is quadratic in the perturbations δg. As a first approximation we shall ignore the higher order terms. Thus the partition function has the simplified form

$$\mathrm{Z} = \mathrm{Z}_1\mathrm{Z}_2\mathrm{Z}_3 = \exp\left(-I_{EH}[\hat{g}]\right) \int \mathcal{D}[\phi] \exp\left(-I_\phi[\hat{g}, \phi]\right) \int \mathcal{D}[\delta g] \exp\left(-I_2[\delta g]\right) \tag{4.24}$$

where

$$\begin{aligned} \mathrm{Z}_1 &= \exp\left(-I_{EH}[\hat{g}]\right), \\ \mathrm{Z}_2 &= \int \mathcal{D}[\phi] \exp\left(-I_\phi[\hat{g}, \phi]\right), \\ \mathrm{Z}_3 &= \int \mathcal{D}[\delta g] \exp\left(-I_2[\delta g]\right). \end{aligned} \tag{4.25}$$

Using the formula

$$F = -\frac{1}{\beta} \log \mathrm{Z} \tag{4.26}$$

and (4.11), we then have a simple expression for the entropy:

$$\begin{aligned} \mathrm{S} &= \beta^2 \frac{\partial}{\partial \beta} \left[-\frac{1}{\beta} \log \mathrm{Z}_1 - \frac{1}{\beta} \log \mathrm{Z}_2 - \frac{1}{\beta} \log \mathrm{Z}_3 \right] \\ &= \mathrm{S}_1 + \mathrm{S}_2 + \mathrm{S}_3. \end{aligned} \tag{4.27}$$

The first term in (4.27) is purely classical, and has been calculated in, for example [69, 74]. For gravity theories governed by the Einstein-Hilbert action (4.17), the answer is simply the classical Hawking-Bekenstein entropy

$$\mathrm{S}_1 = \frac{\mathrm{A}}{4G_N} \tag{4.28}$$

where, again, we have explicitly retained the coupling constant. It should be noted that the formula (4.28) for S_1 becomes modified in theories of gravity which involve terms additional to the Einstein-Hilbert action (see for example [75]). We discuss one such example of a theory in chapter 5 and shall return to this issue briefly in section 4.3.

Terms S_2 and S_3 are the first-order contributions to the entropy from scalar and graviton fluctuations respectively. However, in the absence of a full theory of quantum gravity, we are not able to calculate S_3. For this reason we shall consider only S_2. The calculation of terms of this kind has been the focus of much recent research in non-hairy black hole space-times (see, for example, [72, 73, 76, 77, 78]). Two main approaches have emerged.

Firstly, the *conical singularity method* which directly uses the Euclidean manifold defined by (4.13). The partition function Z has to be calculated for "off-shell" temperatures not equal to the Hawking temperature, in which case the Euclidean manifold possesses a conical singularity. This procedure may have the apparent advantage of yielding finite answers [76], but the entropy so calculated is not statistical-mechanical in nature and so is difficult to interpret physically.

Here we shall favour the second method, known as the *brick wall* model [79]. This involves regulating the theory by placing an unphysical boundary just outside the event horizon of the Lorentzian black hole metric. The great advantage, for our purposes, of this approach is that the entropy calculated via this method has a clear statistical-mechanical interpretation and can easily be related to the density matrix, the fundamental object in which we shall be interested from section 4.4 onwards.

4.2 Calculation of the entropy

In this section we shall calculate the entropy of a quantum scalar field on a background hairy black hole space-time. The formalism we shall use is a semi-classical WKB method, as employed in [77, 79], which is valid for large black holes having small Hawking temperature. This is means that the space-time can be considered as a fixed background for the quantum field.

4.2.1 The WKB method

We begin with the hairy black hole metric in the form (section 2.1):

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right) e^{-2\delta(r)} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 \left[d\theta^2 + \sin^2\theta \, d\varphi^2\right] \quad (4.29)$$

and consider a scalar field ϕ of mass μ propagating in this geometry, and satisfying the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi\right) - \mu^2 \phi = 0. \quad (4.30)$$

The metric (4.29) is spherically symmetric, and so we consider solutions of (4.30) of the form

$$u_{Elm_l}(t, r, \theta, \varphi) = e^{-iEt} f_{El}(r) Y_{lm_l}(\theta, \varphi) \quad (4.31)$$

where $Y_{lm_l}(\theta, \varphi)$ is a spherical harmonic [80] and E is the energy of the wave (see section 4.5.3 below for further comment on the quantum-mechanical energy of the field). The wave equation (4.30) then separates to give the following radial equation for $f_{El}(r)$:

$$\begin{aligned} 0 &= \left(1-\frac{2m(r)}{r}\right)^{-1} E^2 f_{El}(r) + \frac{e^{\delta(r)}}{r^2}\frac{d}{dr}\left[e^{-\delta(r)}r^2\left(1-\frac{2m(r)}{r}\right)\frac{df_{El}(r)}{dr}\right] \\ &\quad - \left[\frac{l(l+1)}{r^2}+\mu^2\right] f_{El}(r). \end{aligned} \tag{4.32}$$

The "brick wall" boundary condition is assumed [79], namely the wave function is cut-off just outside the event horizon

$$\phi = 0 \qquad \text{at } r = r_h + \epsilon \tag{4.33}$$

where r_h is the black hole horizon radius and ϵ is a small, positive, fixed distance, which plays the role of an ultra-violet cut-off (see section 4.4.3 for further discussion on the role of ϵ). In order to regulate the theory (see section 6.1 for an explanation of the divergences arising in quantum field theory in curved space-time) we also impose an infra-red cut-off a very large distance L from the horizon:

$$\phi = 0 \qquad \text{at } r = L, \text{ where } L \gg r_h. \tag{4.34}$$

Conditions (4.33) and (4.34) mean that $f_{El}(r)$ must satisfy

$$f_{El} = 0 \qquad \text{at } r = r_h + \epsilon \text{ or } r = L. \tag{4.35}$$

In anticipation of being able to use a WKB approximation (whose validity is discussed below), next define functions $K_{El}(r)$ and $h(r)$ by

$$K_{El}^2(r) = \left(1-\frac{2m(r)}{r}\right)^{-1}\left[E^2\left(1-\frac{2m(r)}{r}\right)^{-1} - \frac{l(l+1)}{r^2} - \mu^2\right] \tag{4.36}$$

$$h(r) = e^{-\delta(r)}r^2\left(1-\frac{2m(r)}{r}\right). \tag{4.37}$$

In terms of these functions, the equation (4.32) for $f_{El}(r)$ takes the form:

$$\frac{1}{h(r)}\frac{d}{dr}\left[h(r)\frac{df_{El}(r)}{dr}\right] + K_{El}^2(r) f_{El}(r) = 0. \tag{4.38}$$

Now define a function $y_{El}(r)$ by

$$f_{El}(r) = \frac{y_{El}(r)}{\sqrt{h(r)}}. \tag{4.39}$$

Then $y_{El}(r)$ satisfies

$$\frac{d^2 y_{El}(r)}{dr^2} + \left[K_{El}^2 + \frac{1}{4h^2}\left(\frac{dh}{dr}\right)^2 - \frac{1}{2h}\frac{d^2h}{dr^2}\right] y_{El}(r) = 0. \tag{4.40}$$

The WKB approximation [53] for y_{El} will be valid if

$$\left|\frac{1}{4h^2}\left(\frac{dh}{dr}\right)^2 - \frac{1}{2h}\frac{d^2h}{dr^2}\right| \ll \left|K_{El}^2\right| \tag{4.41}$$

and

$$\left|\frac{dK_{El}}{dr}\right| \ll \left|K_{El}^2\right|. \tag{4.42}$$

The first inequality (4.41) is required so that y_{El} can be taken to satisfy the equation

$$\frac{d^2 y_{El}}{dr^2} + K_{El}^2 y_{El} = 0 \tag{4.43}$$

where K_{El} is now the radial wave number, and inequality (4.42) is required so that the approximation to the wave-function [53]

$$y_{El}(r) \sim \frac{1}{\sqrt{K_{El}(r)}} \exp\left[\pm i \int K_{El}(r)\, dr\right] \tag{4.44}$$

is valid.

Assuming, for the present, that the WKB approximation is valid (see section 4.2.5 below), define the radial wave number K_{El} as above (4.36) whenever the right hand side of (4.36) is non-negative. Define $K_{El} = 0$ otherwise, so that K_{El} is always real. Then the number of radial modes n_{El} is given by [79]

$$\pi n_{El} = \int_{r_h+\epsilon}^{L} K_{El}(r)\, dr \tag{4.45}$$

where the fact that n_{El} must be an integer restricts the possible values of E. For fixed energy E, the total number N_E of solutions with energy less than or equal to E is

$$\begin{aligned} \pi N_E &= \int (2l+1)\pi n_{El}\, dl \\ &= \int_{r_h+\epsilon}^{L} \left(1 - \frac{2m(r)}{r}\right)^{-1} dr \int (2l+1)\, dl \\ &\quad \times \left[E^2 - \left(\frac{l(l+1)}{r^2} + \mu^2\right)\left(1 - \frac{2m(r)}{r}\right)\right]^{\frac{1}{2}} \end{aligned} \tag{4.46}$$

where the l integration is performed over all values of l such that the argument of the square root is positive.

4.2.2 Free energy

The Hawking temperature [67] of the black hole having metric (4.29) is given by

$$T = \beta^{-1} = \frac{\kappa}{2\pi} = \frac{1 - 2m'_h}{4\pi r_h e^{\delta_h}} \tag{4.47}$$

where

$$\kappa = \frac{1 - 2m'_h}{2r_h e^{\delta_h}} \tag{4.48}$$

is the surface gravity of the black hole, $'$ denotes derivatives with respect to r, and the subscript h denotes quantities evaluated at the event horizon $r = r_h$. We shall assume here that $\beta^{-1} \ll 1$. (The validity of this assumption will be discussed later, in section 4.2.5.) For extreme black holes, $m'_h = 0.5$ and hence $T = 0$. We shall comment briefly on this case at the end of this section, but the reader should note that it does not arise for the EYMH black holes in which we are primarily interested (section 2.2.9).

The free energy F of the system is given by [79]

$$e^{-\beta F} = \sum e^{-\beta E} = \prod_{n_{El},l,m_l} \frac{1}{1 - e^{-\beta E}} \tag{4.49}$$

so that

$$\begin{aligned} \beta F &= \sum_{n_{El},l,m_l} \log(1 - e^{-\beta E}) \\ &\simeq \int (2l+1)\, dl \int dn_{El} \log(1 - e^{-\beta E}) \end{aligned} \tag{4.50}$$

for $\beta \gg 1$, integrating over appropriate l. The second integral can be integrated by parts:

$$\begin{aligned} F &= -\frac{1}{\beta}\int dl\,(2l+1)\int d(\beta E)\,\frac{n_{El}}{e^{\beta E} - 1} \\ &= -\frac{1}{\pi}\int dl\,(2l+1)\int dE\,\frac{1}{e^{\beta E} - 1}\int_{r_h+\epsilon}^{L} dr \\ &\quad \times \left(1 - \frac{2m(r)}{r}\right)^{-1}\left[E^2 - \left(\frac{l(l+1)}{r^2} + \mu^2\right)\left(1 - \frac{2m(r)}{r}\right)\right]^{\frac{1}{2}} \end{aligned} \tag{4.51}$$

where we have substituted for n_{El} from (4.45).

The l integration can be performed explicitly:

$$\begin{aligned} &\int dl\,(2l+1)\left[E^2 - \left(\frac{l(l+1)}{r^2} + \mu^2\right)\left(1 - \frac{2m(r)}{r}\right)\right]^{\frac{1}{2}} \\ &= \frac{2}{3}r^2\left(1 - \frac{2m(r)}{r}\right)^{-1}\left[E^2 - \left(1 - \frac{2m(r)}{r}\right)\mu^2\right]^{\frac{3}{2}}; \end{aligned} \tag{4.52}$$

which gives for F,

$$F = -\frac{2}{3\pi}\int dE\,\frac{1}{e^{\beta E}-1}\int_{r_h+\epsilon}^{L} dr\,r^2\left(1-\frac{2m(r)}{r}\right)^{-2}\left[E^2-\left(1-\frac{2m(r)}{r}\right)\mu^2\right]^{\frac{3}{2}}. \tag{4.53}$$

In order to facilitate the calculation of the leading behaviour of F, introduce a dimensionless radial co-ordinate x by

$$x = \frac{r}{r_h}. \tag{4.54}$$

The expression (4.53) for F then takes the form

$$F = -\frac{2r_h^3}{3\pi}\int dE\,\frac{1}{e^{\beta E}-1}\int_{1+\hat{\epsilon}}^{\hat{L}} dx\,x^2\left(1-\frac{2\hat{m}(x)}{x}\right)^{-2}\left[E^2-\left(1-\frac{2\hat{m}(x)}{x}\right)\mu^2\right]^{\frac{3}{2}} \tag{4.55}$$

where

$$\hat{\epsilon} = \frac{\epsilon}{r_h}, \qquad \hat{L} = \frac{L}{r_h}, \qquad \hat{m}(x) = \frac{m(xr_h)}{r_h}. \tag{4.56}$$

4.2.3 Contribution to F for large x

The contribution to F for large values of x is

$$F_0 = -\frac{2}{9\pi}L^3\int_{\mu^2}^{\infty} dE\,\frac{(E^2-\mu^2)^{\frac{3}{2}}}{e^{\beta E}-1}. \tag{4.57}$$

Equation (4.57) is the expression for the free energy in flat space. We shall not consider this expression further.

4.2.4 Contribution to F for small x

The contribution to (4.55) for x close to 1 diverges as $\hat{\epsilon} \to 0$. The leading order and next-to-leading order terms in the integrand of (4.55) are calculated using the expansion, near $x = 1$,

$$\hat{m}(x) = \hat{m}_h + \hat{m}_h'(x-1) + \frac{1}{2}\hat{m}_h''(x-1)^2 + \ldots \tag{4.58}$$

where

$$\hat{m}_h = \frac{1}{2}; \tag{4.59}$$

and using the trick

$$x = 1 + (x-1) \tag{4.60}$$

we have

$$\begin{aligned} x^{-1} &= 1-(x-1)+(x-1)^2+\ldots \\ x^2 &= 1+2(x-1)+(x-1)^2. \end{aligned} \tag{4.61}$$

Hence

$$\begin{aligned}1-\frac{2\hat{m}(x)}{x} &= (1-2\hat{m}_h')(x-1)+(2\hat{m}_h'-1-\hat{m}_h'')(x-1)^2+\dots \\ \left(1-\frac{2\hat{m}(x)}{x}\right)^{-2} &= (1-2\hat{m}_h')^{-2}(x-1)^{-2} \\ &\times\left[1+\frac{2(2\hat{m}_h'-1-\hat{m}_h'')}{2\hat{m}_h'-1}(x-1)+\dots\right]. \end{aligned} \tag{4.62}$$

Thus the leading order contribution to the integrand of (4.55) is

$$I_{LO}=E^3(x-1)^{-2}(1-2\hat{m}_h')^{-2} \tag{4.63}$$

and the next-to-leading order term is

$$I_{NLO}=2E^3\frac{(2-4\hat{m}_h'+\hat{m}_h'')}{(1-2\hat{m}_h')^3}(x-1)^{-1}-\frac{3}{2}\mu^2E\frac{1}{(1-2\hat{m}_h')}(x-1)^{-1}. \tag{4.64}$$

Using the standard integrals [81]

$$\begin{aligned}\int_0^\infty dE\,\frac{E^3}{e^{\beta E}-1} &= \frac{\pi^4}{15\beta^4} \\ \int_0^\infty dE\,\frac{E}{e^{\beta E}-1} &= \frac{\pi^2}{6\beta^2}\end{aligned} \tag{4.65}$$

the leading order divergence in F is thus

$$\begin{aligned}F_{LO} &= -\frac{2\pi^3}{45\hat{\epsilon}}\frac{r_h^3(1-2\hat{m}_h')^{-2}}{\beta^4} \\ &= -\frac{2\pi^3}{45\epsilon}\frac{r_h^4(1-2\hat{m}_h')^{-2}}{\beta^4}\end{aligned} \tag{4.66}$$

whilst the next-to-leading order contribution is

$$F_{NLO}=\frac{4\pi^3}{45}\frac{r_h^3}{\beta^4}\frac{(2-4\hat{m}_h'+\hat{m}_h'')}{(1-2\hat{m}_h')^3}\log\hat{\epsilon}-\frac{\pi}{6}\frac{r_h^3}{\beta^2}\mu^2(1-2\hat{m}_h')^{-1}\log\hat{\epsilon}. \tag{4.67}$$

4.2.5 Validity of the approximations

Before we calculate the entropy from the free energy (4.66–4.67), it is necessary to ascertain when the approximations we have used are valid. Firstly, from (4.47), $\beta \gg 1$ if

$$r_h \gg 1 \qquad \text{or} \qquad 1-2m_h' \ll 1. \tag{4.68}$$

In the first case, non-trivial (i.e. non-Schwarzschild) EYMH black holes exist only for very small values of the Higgs v.e.v. v [58], and these solutions will be very close to the Schwarzschild solution having mass $M=r_h/2$.

In the second case, the black hole is very nearly extremal. It should be noted that the analysis of this section is not valid for precisely extremal black holes (since then $T = 0$). However, as discussed in section 2.2.9, we have $1 - 2m'_h \ll 1$ for large $k = n$, and for any value of v for which a non-trivial black hole exists. We shall discuss the extremal case at the end of this section.

Secondly, consider the validity of the WKB approximation. The principal contribution to the free energy F comes from the region where K_{El} is large; here we are concerned primarily with the region where x is close to 1. The WKB approximation will be valid when K_{El} is large, and in this regime it may be approximated by

$$\begin{aligned} K_{El}(r) &\sim E\left(1 - \frac{2m(r)}{r}\right)^{-1} \\ &\sim E(1-2m'_h)^{-1}(r-r_h)^{-1} + O(1) \end{aligned} \tag{4.69}$$

for $r \sim r_h$. Then

$$\frac{dK_{El}}{dr} \sim -E(1-2m'_h)^{-1}(r-r_h)^{-2} \tag{4.70}$$

and

$$\left|\frac{dK_{El}}{dr}\right| \ll \left|K_{El}^2\right| \qquad \text{if} \qquad \left|\frac{E}{1-2m'_h}\right| \gg 1. \tag{4.71}$$

Similarly, near the event horizon we can expand $h(r)$:

$$h(r) = e^{-\delta_h} r_h^2 (1-2m'_h)(r-r_h) + O(r-r_h)^2 \tag{4.72}$$

so that

$$\frac{1}{4h^2}\left(\frac{dh}{dr}\right)^2 - \frac{1}{2h}\frac{d^2h}{dr^2} = \frac{1}{4(r-r_h)^2} + O(r-r_h)^{-1} \tag{4.73}$$

giving

$$\left|\frac{1}{4h^2}\left(\frac{dh}{dr}\right)^2 - \frac{1}{2h}\frac{d^2h}{dr^2}\right| \ll \left|K_{El}^2\right| \tag{4.74}$$

if

$$\left|\frac{4E}{1-2m'_h}\right| \gg 1. \tag{4.75}$$

Hence, for the EYMH black holes with large $k = n$, the WKB approximation is valid except for small values of E.

4.2.6 Entropy

The entropy S of the system is calculated from the free energy F by the formula [79]

$$\mathsf{S} = \beta^2 \frac{\partial F}{\partial \beta}. \tag{4.76}$$

Equations (4.66) and (4.67) then give the following expressions for the leading order divergence in S and next-to-leading order behaviour as follows:

$$\mathrm{S}_{LO} = \frac{8\pi^3}{45\epsilon}\frac{r_h^4}{\beta^3}(1-2\hat{m}_h')^{-2} \tag{4.77}$$

$$\mathrm{S}_{NLO} = -\frac{16\pi^3}{45}\frac{r_h^3}{\beta^3}\frac{(2-4\hat{m}_h'+\hat{m}_h'')}{(1-2\hat{m}_h')^3}\log\hat{\epsilon}+\frac{\pi}{3}\frac{r_h^3}{\beta}\mu^2(1-2\hat{m}_h')^{-1}\log\hat{\epsilon}. \tag{4.78}$$

Substituting for β from (4.47) gives the following final form of the entropy:

$$\mathrm{S}_{LO} = \frac{r_h}{360\epsilon}(1-2\hat{m}_h')e^{-3\delta_h} \tag{4.79}$$

$$\mathrm{S}_{NLO} = -\frac{1}{180}e^{-3\delta_h}(2-4\hat{m}_h'+\hat{m}_h'')\log\hat{\epsilon}+\frac{1}{12}r_h^2\mu^2e^{-\delta_h}\log\hat{\epsilon}. \tag{4.80}$$

The implications of these formulae will be considered in detail in the next section.

4.2.7 Extremal black holes

Finally, in this section, we compute the contribution to the entropy in the case of an extremal black hole. Then $\hat{m}_h' = 0.5$, so that S_{LO} vanishes identically. (It should be noted that the calculation of the free energy in section 4.2.2 is not valid for a precisely extremal black hole. However, we may consider the limiting behaviour as the black hole approaches extremality.) The next-to-leading order term (4.80) does not vanish, but reduces to:

$$\mathrm{S}_{NLO} = -\frac{1}{180}e^{-3\delta_h}\hat{m}_h''\log\hat{\epsilon}+\frac{1}{12}r_h^2\mu^2e^{-\delta_h}\log\hat{\epsilon}; \tag{4.81}$$

where we now evaluate an explicit simple form of $\hat{m}_h''$ in this case.

Calculation of $\hat{m}_h''$ Restricting our attention now to the EYMH black holes, in the extremal case we can derive a compact expression for m_h''. From the Einstein equations (2.56):

$$m'(r) = \frac{1}{2}N\left(\frac{2\omega'^2}{r^2}+\phi'^2\right)+\frac{1}{2}r^2\left[\frac{(1-\omega^2)^2}{r^4}+\frac{\phi^2}{2r^2}(1+\omega)^2+\frac{\lambda}{2}(\phi^2-v^2)^2\right] \tag{4.82}$$

where

$$N = 1-\frac{2m(r)}{r}. \tag{4.83}$$

For extremal black holes, $N(r_h) = 0 = N'(r_h)$ and $m'(r_h) = 0.5$. In this case

$$\begin{aligned} m''(r_h) &= r_h\left[\frac{(1-\omega_h^2)^2}{r_h^4}+\frac{\phi_h^2}{2r_h^2}(1+\omega_h)^2+\frac{\lambda}{2}(\phi_h^2-v^2)^2\right] \\ &\quad +\frac{1}{2}r_h^2\left[-\frac{4(1-\omega_h^2)^2}{r_h^5}-\frac{4}{r_h^4}(1-\omega_h^2)\omega_h\omega_h'-\frac{\phi_h^2}{r_h^3}(1+\omega_h)^2\right. \\ &\quad \left.+\frac{\phi_h\phi_h'}{r_h^2}(1+\omega_h)^2+\frac{\phi_h^2}{r_h^2}\omega_h'(1+\omega_h)+2\lambda\phi_h\phi_h'(\phi_h^2-v^2)\right]. \end{aligned} \tag{4.84}$$

Since $m'_h = 0.5$ this expression reduces to

$$\begin{aligned} m''_h &= \frac{1}{r_h} - \frac{2}{r_h^3}(1-\omega_h^2)^2 - \frac{2}{r_h^2}(1-\omega_h^2)\omega_h\omega'_h - \frac{\phi_h^2}{2r_h}(1+\omega_h)^2 \\ &+\frac{1}{2}\phi_h\phi'_h(1+\omega_h)^2 + \frac{1}{2}\phi_h^2\omega'_h(1+\omega_h) + \lambda r_h^2\phi_h\phi'_h(\phi_h^2 - v^2). \end{aligned} \quad (4.85)$$

Further simplification is possible using the field equations (2.54–2.55)

$$\begin{aligned} N\omega'' &= -\frac{(NS)'}{S}\omega' + \frac{1}{r^2}(\omega^2-1)\omega + \frac{\phi^2}{4}(1+\omega) \\ N\phi'' &= -\frac{(NS)'}{S}\phi' - \frac{2N}{r}\phi' + \frac{\phi}{2r^2}(1+\omega)^2 + \lambda\phi(\phi^2-v^2), \end{aligned} \quad (4.86)$$

where

$$S(r) = e^{-\delta(r)}, \quad (4.87)$$

giving, at the event horizon,

$$\begin{aligned} \frac{1}{r_h^2}(1-\omega_h^2)\omega_h &= \frac{\phi_h^2}{4}(1+\omega_h) \\ \frac{\phi_h}{2r_h^2}(1+\omega_h)^2 &= -\lambda\phi_h(\phi_h^2 - v^2). \end{aligned} \quad (4.88)$$

The expression for m''_h then reduces to:

$$\begin{aligned} m''_h &= \frac{1}{r_h} - \frac{2}{r_h^3}(1-\omega_h^2)^2 - \frac{\phi_h^2}{2r_h}(1+\omega_h)^2 \\ &= \frac{1}{r_h} - \frac{2}{r_h^3}(1-\omega_h^2)(1+\omega_h) \\ &= \frac{1}{r_h} - \frac{1}{2r_h}\frac{\phi_h^2}{\omega_h}(1+\omega_h)^2 \end{aligned} \quad (4.89)$$

providing $\omega_h \neq 0$.

4.3 Comments on the entropy calculation

4.3.1 Branch structure of the EYMH system

Consider the leading-order divergence in the entropy S:

$$\mathrm{S}_{LO} = \frac{r_h}{360\epsilon}(1-2m'_h)e^{-3\delta_h}. \quad (4.90)$$

As discussed in section 2.2, the black hole solutions of EYMH theory are indexed by k, the number of nodes of the gauge field ω, and for each $k = n$ there are two branches of solutions: the $k = n$ and quasi-$k = n - 1$ branches. For each fixed value of the Higgs v.e.v. v, the $k = n$ solution has larger m_h' and δ_h than the quasi-$k = n - 1$ solution. Hence the $k = n$ branch has a lower entropy (4.90) than the quasi-$k = n - 1$ branch. This is in agreement with [58], where the authors labelled the $k = n$ branch "low entropy" and the quasi-$k = n - 1$ branch "high entropy". The stability analysis of chapter 3 confirms that the "high entropy" quasi-$k = n - 1$ branch is more stable than the "low entropy" $k = n$ branch, as was conjectured but unproved in [58].

4.3.2 Renormalization of Newton's constant

Local quantum gravity is a fundamentally non-renormalizable theory, and this is one reason for ignoring graviton fluctuations in this chapter (section 4.1.3). Thus the entropy calculated in the previous section constitutes only part of the total entropy of the black hole (as explained in section 4.1.3). The entropy we have calculated is divergent due to the fact that we have ignored the back-reaction effect of the scalar field on the gravitational field. However, such divergences are easily tamed as follows.

We first focus on the leading order divergence in the entropy,

$$\mathrm{S}_{LO} = \left[\frac{1}{360}(1 - 2m_h')e^{-\delta_h}\right]\frac{r_h}{\epsilon} \tag{4.91}$$

which is proportional to the linear factor r_h/ϵ. Such a factor is typically found in calculations of entropy in non-hairy black hole space-times [77, 79]. The invariant distance of the brick wall from the event horizon is [79]

$$d_\epsilon = r_h \epsilon \tag{4.92}$$

and hence rewriting (4.91) as

$$\mathrm{S}_{LO} = \left[\frac{1}{360}(1 - 2m_h')e^{-\delta_h}\right]\frac{r_h^2}{d_\epsilon} \tag{4.93}$$

we see that S_{LO} is in fact proportional to the area of the event horizon. This means that, in the leading order approximation, the Hawking-Bekenstein formula for the entropy is still valid, but with a renormalized gravitational constant G_R replacing the bare Newton's constant G_N:

$$\mathrm{S} = \mathrm{S}_1 + \mathrm{S}_{LO} = \left[\frac{1}{4G_N} + O\left(\frac{1}{d_\epsilon}\right)\right]\mathrm{A} = \frac{1}{4G_R}\mathrm{A}. \tag{4.94}$$

This renormalization may be interpreted as expressing quantum back reaction effects.

The next-to-leading order, logarithmically divergent, terms are not so easily dismissed. They take the form (4.80)

$$\mathrm{S}_{NLO} = -\frac{1}{180}e^{-3\delta_h}(2 - 4\hat{m}_h' + \hat{m}_h'')\log\hat{\epsilon} + \left[\frac{1}{12}\mu^2 e^{-\delta_h}\log\hat{\epsilon}\right]r_h^2. \tag{4.95}$$

The second term in (4.95) *is* proportional to A, and hence can be absorbed into the renormalized G_R. However, the first term *cannot* be absorbed in this manner and we turn now to the interpretation of this term.

4.3.3 Interpretation of the additional terms in the entropy

We would like to be able to interpret the additional contribution to the black hole entropy in terms of information loss across the event horizon. We shall discuss the information loss problem for black holes and its consequences for quantum mechanics in the remaining sections of this chapter. For the time being we focus on the relationship of information loss with entropy.

The motivation for this interpretation is quite simple. Firstly, these contributions are present even for extremal black holes for which the Hawking temperature vanishes (section 4.2.7); indeed they are present even in the case of an extremal dilatonic black hole whose event horizon has zero area [77]. Secondly, the logarithmic divergences disappear for black holes whose event horizon is disappearing in the sense that $r_h \to \epsilon$ (for the dilatonic case [77] this corresponds to ADM mass $M \to \epsilon/2$).

Information can only be lost down and not retrieved from the event horizon. Hence any entropy associated with information loss must be positive. For the extremal EYMH black hole, the supposed information loss entropy term is (4.95)

$$\mathrm{S}_{IL} = -\frac{1}{180} e^{-3\delta_h} \hat{m}_h'' \log \hat{\epsilon}. \tag{4.96}$$

Since for $\epsilon < r_h$ the logarithm will be negative, in order for S_{IL} to be positive, we have (4.89)

$$\hat{m}_h'' = 1 - \frac{1}{2}\frac{\phi_h^2}{\omega_h}(1+\omega_h)^2 > 0. \tag{4.97}$$

Thus the condition of positivity places a mild restriction on the horizon values of the fields, namely

$$2\omega_h > \phi_h^2(1+\omega_h)^2 > 0. \tag{4.98}$$

From (4.97) it can be seen that any possible ambiguity in the sign of $\hat{m}_h''$ arises from the presence of the gauge field ω. In its absence $\hat{m}_h''$ is definitely positive.

This ambiguity in the sign of the entropy has arisen before, for spin one fields propagating on an ordinary (i.e. non-hairy) black hole background [78]. Here we have a gauge field as part of the background, associated with non-trivial hair. In this case, sign ambiguities are present already when we consider the quantum fluctuations of a spin zero field. Such problems do not arise for scalar fields on non-hairy black hole backgrounds, when the entropy is manifestly positive. The problems arise in the presence of gauge fields due to terms in the effective action which have a negative sign [78].

Higher curvature gravity Here we mention briefly that it may be possible to absorb those terms in the entropy which are *not* proportional to the area into a renormalization of the coupling constants in theories of gravity which involve higher-curvature terms than ordinary general relativity [75]. Such theories often arise as low-energy effective theories coming from string theory, and we consider one such theory in chapter 5. There we show that black hole solutions possess non-trivial hair, eluding the no-hair conjecture via a mechanism similar to that operating in the EYMH system. However, in string-inspired models the hair is present purely in the gravitational field, without the need for non-Abelian matter fields. The absorption of the "information loss" terms in the entropy may indicate that the extra hair reduces the amount of information lost down the event horizon, since more information is retained in the structure of the hair itself.

4.3.4 Entanglement versus thermodynamic entropy

At this stage we should discuss briefly the *entanglement entropy*, which has received much attention in the literature [78, 82, 83], and compare it with the *thermodynamical entropy*. Thermodynamical entropy is calculated via the methods of section 4.1, and hence is the type of entropy with which we have been concerned so far in this chapter. Classically, it corresponds to the Hawking-Bekenstein entropy (4.28), while in section 4.2 we have computed the first order quantum corrections to this formula.

Entanglement entropy is obtained from the density matrix of the field

$$\rho = \sum_{\text{states}} |\Psi\rangle\langle\Psi| \tag{4.99}$$

by tracing over the degrees of freedom which cannot be observed (in our case, because they lie inside the black hole). This process yields a modified density matrix $\hat{\rho}$, having eigenvalues p_n. The entanglement entropy is then given by the formula

$$\mathrm{S} = -\sum_n p_n \log p_n. \tag{4.100}$$

Thus entanglement entropy is closely associated with information loss, and, further, is always positive. Thus entanglement entropy is *not* the same as thermodynamic entropy for the spin one case described above [78]. However, for spin zero fields in non-hairy black hole backgrounds, the two ideas of entropy coincide [78]. For the EYMH black holes, if the two forms of entropy are identical, then it must be the case that $\hat{m}_h'' > 0$ (4.97), or, otherwise, the entanglement and thermodynamic entropies do not agree even for scalar fields on this background.

Entanglement entropy has been calculated explicitly for flat space-time with an artificial boundary [83], and also in cosmological geometries [82]. However, both calculations made explicit use of the forms of states that were being traced over. Although the results of [83] bear a striking resemblance to the black hole situation, the approach is somewhat different. The states inside the event horizon should be considered as intrinsically unobservable, in comparison with the artificial boundary case, where we have in effect chosen not to observe certain modes. Hence any calculation which explicitly involves these hidden states does not get to the heart of the black hole problem.

4.4 Quantum de-coherence

4.4.1 The information loss problem

The problem of information loss in the formation and subsequent evaporation of a black hole is a deep, and as yet unresolved, question in gravitation. One of the reasons it has not been resolved is that we do not have a complete theory of quantum gravity, and hence cannot say what the end-point of the evaporation process is. In this section we briefly describe what is meant by information loss in a black hole, and how it may affect quantum mechanics.

Suppose initially we have a pure quantum state describing matter which then collapses to form a black hole. According to the unitary evolution of quantum mechanics, this pure state will evolve in time, but will remain pure. However, once the event horizon has formed, an observer outside the black hole sees a mixed state because he cannot determine what is happening inside the event horizon. As far as the horizon is concerned, he has to apply Hawking's "Principle of Ignorance" [84], in other words, all configurations compatible with his limited knowledge (i.e. only what is going on outside the horizon) are equally probable, and the observer must trace over them accordingly. This procedure leads to a density matrix describing the mixed state observed, and, in effect, a pure state has evolved into a mixed state, resulting in a loss of information.

To describe this phenomenon mathematically, Hawking [85] suggested that the conventional $\mathcal{S}$-matrix description of scattering between asymptotic "in" and "out" states should be replaced by a super-scattering operator \$ which relates initial and final density matrices:

$$\rho_{out} = \$\rho_{in}. \tag{4.101}$$

If the operator \$ can be factorized as

$$\$ = \mathcal{S}\mathcal{S}^{\dagger}, \tag{4.102}$$

then (4.101) reduces to the conventional $\mathcal{S}$-matrix theory, with "in" and "out" states being described by density matrices. This is standard quantum mechanics; time-evolution is unitary and pure states evolve to pure states. Hence the operator \$ is non-factorizable for the black hole scenario we are interested in [85], where pure states can evolve to mixed states.

Proposition 4.1 *If pure states evolve to mixed states, then the operator \$ is not invertible.*

Proof [86] Let Ψ be a pure state which evolves to a mixed state described by the density matrix ρ;

$$\$(\Psi \otimes \bar{\Psi}) = \rho. \tag{4.103}$$

Suppose that $\$$ *is* invertible, then

$$\Psi \otimes \bar{\Psi} = \$^{-1}(\rho). \tag{4.104}$$

Next expand ρ in terms of its eigenvectors:

$$\rho = \sum_i p_i \Phi_i \otimes \bar{\Phi}_i \tag{4.105}$$

where

$$0 < p_i < 1 \tag{4.106}$$

and there is more than one Φ_i since ρ describes a mixed state. Then, since $\$$ is a linear operator, so too is $\$^{-1}$, giving

$$\Psi \otimes \bar{\Psi} = \sum_i p_i \$^{-1}(\Phi_i \otimes \bar{\Phi}_i). \tag{4.107}$$

Now let Υ be any state orthogonal to Ψ. Then we have

$$\sum_i p_i \langle \Upsilon | \$^{-1}(\Phi_i \otimes \bar{\Phi}_i) | \Upsilon \rangle = 0. \tag{4.108}$$

The operator $\$^{-1}(\Phi_i \otimes \bar{\Phi}_i)$ is a density matrix, and hence a positive operator. In addition, each $p_i > 0$, and so

$$\langle \Upsilon | \$^{-1}(\Phi_i \otimes \bar{\Phi}_i) | \Upsilon \rangle = 0 \tag{4.109}$$

for all Υ orthogonal to Ψ. This implies that

$$\$^{-1}(\Phi_i \otimes \bar{\Phi}_i) = \Psi \otimes \bar{\Psi} \qquad \forall i \tag{4.110}$$

and thus

$$\Phi_i \otimes \bar{\Phi}_i = \$(\Psi \otimes \bar{\Psi}) \qquad \forall i. \tag{4.111}$$

There is more than one Φ_i and hence we have a contradiction. In conclusion, then $\$$ is not an invertible operator. □

Proposition 4.2 *[87, 88] The fact that $\$$ is not invertible is incompatible with (strong) CPT invariance.*

Proof Suppose that the system is invariant under the CPT operator Θ. Let ρ_{in} be an "in" state which evolves to the state described by ρ_{out}, that is,

$$\rho_{out} = \$\rho_{in}. \tag{4.112}$$

By CPT invariance,

$$\rho_{in} = \Theta\rho'_{out} \tag{4.113}$$

for some "out" state ρ'_{out}, which gives

$$\rho_{out} = \$\Theta\rho'_{out}. \tag{4.114}$$

Now act on both sides with Θ:

$$\Theta\rho_{out} = \Theta\$\Theta\rho'_{out}, \tag{4.115}$$

where

$$\Theta\rho_{out} = \rho'_{in} \tag{4.116}$$

for an "in" density matrix ρ'_{in}:

$$\rho'_{in} = \Theta\$\Theta\rho'_{out}. \tag{4.117}$$

Since the "in" and "out" Hilbert spaces are the same for both primed and unprimed operators,

$$\rho'_{out} = \$\rho'_{in} \tag{4.118}$$

giving, finally,

$$\rho'_{in} = \Theta\$\Theta\$\rho'_{in} \qquad \forall\rho'_{in}. \tag{4.119}$$

This implies that $\Theta\$\Theta\$$ is the identity operator, which in turn implies that \$ is invertible. This is a contradiction with Proposition 4.1 above. Hence the system is not CPT invariant. □

The fact that CPT invariance is broken in black hole space-times is not that surprising; the CPT theorem is, after all, a theorem of field theory in flat space-time. The above proposition refers only to the violation of "strong" CPT invariance (i.e. the non-existence of the mathematical operator Θ [88]). It may be the case that such violation is suppressed by many powers of M_P^{-1}, where M_P is the Planck mass, and would therefore be unobservable in practice. We shall investigate the possible order of magnitude of these effects in the following sections. This violation of CPT invariance implies the existence of a fundamental arrow of time and, in particular, that although black holes may exist, white holes do not [88].

The information loss we have described when a black hole is formed is not, in itself, a "problem". Classically, there is no fundamental objection to the fact that objects can disappear forever down the event horizon; it is merely a rather puzzling observation when one first learns relativity. Quantum mechanically, the evolution of pure states to mixed states may be considered an artifact of our dealing with an open system. The real paradox arises when we consider black holes evaporating away completely to give a singularity-free geometry. Such processes are important when we consider microscopic, rather than macroscopic, black holes [89]. The net effect is that a pure state (the collapsing matter) has evolved to a mixed state (the Hawking radiation) but now there are no hidden states. Of course, this picture involves details of quantum gravity that are as yet unknown. However, what we have observed so far as a macro-physical phenomenon may have important consequences for the laws governing the micro-physical universe, and we now investigate one of the possibilities.

4.4.2 Modifying quantum mechanics

We shall not be concerned directly with the super-scattering operator \$ for the remainder of this chapter, but shall consider instead the time-evolution of the density matrix operator. For unitary evolution in ordinary quantum mechanics, this equation takes the form [89]

$$\frac{\partial \rho}{\partial t} = i[\rho, H]. \tag{4.120}$$

However, we have already observed that in the case at hand the system does not evolve unitarily in time; pure states can evolve to mixed states. Hence (4.120) must be modified, and has the new form [89]

$$\frac{\partial \rho}{\partial t} = i[\rho, H] + \not{\delta} H \rho. \tag{4.121}$$

The right hand side of equation (4.121) represents the most general linear operator mapping hermitian matrices to hermitian matrices. It is desirable to impose two constraints on $\not{\delta} H$ [89], firstly that

$$\mathrm{Tr}\, \rho = 1 \tag{4.122}$$

at all times, so that probability is conserved, and secondly that

$$\mathrm{Tr}\, \rho^2 \leq 1 \tag{4.123}$$

always, to avoid states with complex entropy. With these restrictions $\not{\delta} H$ can be considered to be a negative semi-definite quadratic form on the vector space of all possible density matrices ρ.

Equation (4.121) has the same structure as that typically found for the time evolution of open quantum systems, where unobservable degrees of freedom have to be effectively integrated over. We have stressed this striking similarity previously, but in the absence of a complete theory of quantum gravity, are unable to say how far the analogy can be pushed. It may be possible in string theory, a candidate theory of quantum gravity, to probe this question more deeply (see for example

[13, 14, 16]), but we are interested here in what macroscopic black holes can tell us, so we shall not pursue any string theoretic considerations, but shall rather take a model-independent approach.

Once we have proposed (4.121), the natural question is: what form does $\not{\partial} H$ take? This we cannot fully answer without a complete theory of Planck scale physics. For the remainder of this chapter, we shall consider the macroscopic hairy black holes, and link the logarithmic divergences in the entropy of a quantum scalar field with a possible $\not{\partial} H$ term. In a first approximation, we shall be able to see how $\not{\partial} H$ depends on the hairy black hole geometry, and on the quantum scalar field, and also give an estimate for its order of magnitude.

4.4.3 A new proposal

In section 4.2, the entropy of a quantum scalar field surrounding a black hole was calculated. Adding these first order corrections to the classical Hawking-Bekenstein entropy (as described in section 4.1.3) gives the quantum corrected black hole entropy to be

$$\begin{aligned} \mathrm{S} &= \mathrm{S}_1 + \mathrm{S}_2 = \mathrm{S}_1 + \mathrm{S}_{LO} + \mathrm{S}_{NLO} \\ &= \frac{\pi r_h^2}{G_N} + \frac{r_h^2}{360 d_\epsilon}(1 - 2\hat{m}_h')e^{-3\delta_h} - \frac{1}{180}(2 - 4\hat{m}_h' + \hat{m}_h'')e^{-3\delta_h}\log\hat{\epsilon} \\ &\quad + \frac{r_h^2}{12}\mu^2 e^{-\delta_h}\log\hat{\epsilon}. \end{aligned} \tag{4.124}$$

Suppose, formally, we may write (4.124) as

$$\mathrm{S} = \frac{\pi r_q^2}{G_R} \tag{4.125}$$

where G_R is the renormalized Newton's constant and r_q is a "quantum" radius. Then r_q is given by (again formally)

$$\begin{aligned} r_q^2 &= r_h^2 \left[\frac{G_R}{G_N} + \frac{G_R}{\pi}\frac{(1 - 2\hat{m}_h')}{360 d_\epsilon}e^{-3\delta_h} + \frac{G_R}{12\pi}\mu^2 e^{-3\delta_h}\log\hat{\epsilon} \right] \\ &\quad - \frac{G_R}{180\pi}(2 - 4\hat{m}_h' + \hat{m}_h'')e^{-3\delta_h}\log\hat{\epsilon}. \end{aligned} \tag{4.126}$$

As described in section 4.3.2, the renormalized gravitational constant G_R is related to the bare gravitational constant G_N via the renormalization

$$\frac{1}{G_R} = \frac{1}{G_N} + \frac{1}{\pi}\frac{(1 - 2\hat{m}_h')}{360 d_\epsilon}e^{-3\delta_h} + \frac{\mu^2}{12\pi}e^{-3\delta_h}\log\hat{\epsilon} \tag{4.127}$$

which gives a simplified expression for r_q^2;

$$r_q^2 = r_h^2 - \frac{G_R}{180\pi}(2 - 4\hat{m}_h' + \hat{m}_h'')e^{-3\delta_h}\log\hat{\epsilon}. \tag{4.128}$$

As the black hole evaporates due to Hawking radiation [67], its radius shrinks as the area of the event horizon shrinks. This will, in turn, alter the effective radius r_q, and also $\log \hat{\epsilon}$. Hence we formally identify $\log \hat{\epsilon}$ as giving a measure of time which is convenient for our purposes. This identification can also be motivated from considerations of "non-critical" string theory [5, 13, 14, 15, 16], but these issues will not be addressed here.

Our approach from here is to fix time $t = \log \hat{\epsilon}$, and with this choice of a time co-ordinate, calculate $\frac{\partial \rho}{\partial t}$, the time evolution of the density matrix. As discussed in section 4.3.4, we are not able to compute the density matrix by tracing explicitly over the unobservable degrees of freedom. Instead we shall assume that the density matrix for the quantum scalar field on the black hole background is given, for a system close to equilibrium, approximately by the Gibbs formula

$$\rho = \frac{e^{-\beta H}}{Z}. \tag{4.129}$$

This formula will be valid (at least approximately) for a black hole which is changing slowly with respect to time, so it is approximately static. This will be the case for large black holes whose emission via Hawking radiation is slow. These are the black holes for which the WKB approximation used in section 4.2 is valid. Evidently, corrections to the formula (4.129) will produce corresponding corrections in $\frac{\partial \rho}{\partial t}$, and so our calculation is only approximate.

If the formula (4.129) were exact, then for unitary evolution,

$$\frac{\partial \rho}{\partial t} = 0 \tag{4.130}$$

trivially since

$$[\rho, H] = 0. \tag{4.131}$$

However, since the system is (albeit slowly) evolving, this is not the case. From section 4.2, we see that Z depends on $\log \hat{\epsilon}$, and in the next section we shall compute the dependence of H on $\log \hat{\epsilon}$. Hence, even assuming the Gibbs formula (4.129), there are contributions to $\frac{\partial \rho}{\partial t}$ not stemming from the commutator term. These are precisely the $\not{\delta} H$ factors in (4.121), and hence the Gibbs formula will allow us to investigate this term, subject to the approximations we have made.

4.5 Calculation of the Hamiltonian

We now compute the $\hat{\epsilon}$ dependence of the Hamiltonian of the quantum scalar field. The calculation will make use of the WKB approximation described in section 4.2.

4.5.1 Form of the Hamiltonian

A scalar field of mass μ is described by the Lagrangian

$$\mathcal{L}_\phi = -\frac{1}{2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mu^2\phi^2\right] \tag{4.132}$$

and propagates in the black hole geometry described by the metric

$$ds^2 = -e^\Gamma dt^2 + e^\Lambda dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{4.133}$$

The canonical momentum π_ϕ conjugate to the field ϕ is

$$\pi_\phi = \frac{\partial \mathcal{L}_\phi}{\partial(\partial_t\phi)} = -g^{t\nu}\partial_\nu\phi = e^{-\Gamma}\partial_t\phi. \tag{4.134}$$

The Hamiltonian density $\mathcal{H}_\phi$ is defined by

$$\mathcal{H}_\phi = \pi_\phi\partial_t\phi - \mathcal{L}_\phi \tag{4.135}$$

and, for the metric (4.133) takes the form

$$\mathcal{H}_\phi = \frac{1}{2}\left[e^{-\Gamma}(\partial_t\phi)^2 + e^{-\Lambda}(\partial_r\phi)^2 + \frac{1}{r^2}(\partial_\theta\phi)^2 + \frac{1}{r^2\sin^2\theta}(\partial_\varphi\phi)^2 + \mu^2\phi^2\right] \tag{4.136}$$

from which we can evaluate the Hamiltonian

$$H_\phi = \int dr\, d\theta\, d\phi\, e^{\frac{1}{2}(\Gamma+\Lambda)} r^2 \sin\theta\, \mathcal{H}_\phi. \tag{4.137}$$

The first step in calculating (4.137) is to expand ϕ in normal modes:

$$u_{Elm_l}(t,r,\theta,\varphi) = e^{-iEt} f_{El}(r) Z_{lm_l}(\theta,\varphi), \qquad E > 0 \tag{4.138}$$

where $f_{El}(r)$ satisfies the radial equation (4.32)

$$0 = e^\Lambda E^2 f_{El}(r) + \frac{1}{r^2}e^{-\frac{1}{2}(\Gamma+\Lambda)}\frac{d}{dr}\left[e^{\frac{1}{2}(\Gamma-\Lambda)} r^2 \frac{df_{El}}{dr}\right] - \left[\frac{l(l+1)}{r^2} + \mu^2\right] f_{El}(r), \tag{4.139}$$

and $Z_{lm_l}(\theta,\varphi)$ is a real spherical harmonic [83] satisfying the differential equation

$$0 = \operatorname{cosec}\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Z_{lm_l}}{\partial\theta}\right) + \operatorname{cosec}^2\theta\frac{\partial^2 Z_{lm_l}}{\partial\varphi^2} + l(l+1)Z_{lm_l} \tag{4.140}$$

and normalized according to

$$\int d\theta\, d\varphi\, \sin\theta\, Z_{lm_l} Z_{l'm'_{l'}} = \delta_{ll'}\delta_{m_l m'_{l'}}. \tag{4.141}$$

As in section 4.2, we impose the "brick wall" boundary condition on ϕ, namely

$$f_{El} = 0 \qquad \text{at } r = r_h + \epsilon \tag{4.142}$$

and also an infra-red cut-off

$$f_{El} = 0 \qquad \text{at } r = L \gg r_h. \tag{4.143}$$

The Klein-Gordon inner product is [90]

$$\langle u_1 | u_2 \rangle = -i \int_\Sigma \sqrt{g_\Sigma} \left(u^*_{2,\mu} u_1 - u^*_2 u_{1,\mu} \right) d\Sigma^\mu \tag{4.144}$$

where $d\Sigma^\mu = n^\mu d\Sigma$, and n^μ is a future-directed unit vector orthogonal to the space-like hyper-surface Σ, and $\sqrt{g_\Sigma}\, d\Sigma$ is the volume element in Σ. If we take Σ to be a surface of constant time t, then

$$n^\mu = \left(e^{-\frac{1}{2}\Gamma}, 0, 0, 0 \right) \tag{4.145}$$

and

$$\sqrt{g_\Sigma} = e^{\frac{1}{2}\Lambda} r^2 \sin\theta. \tag{4.146}$$

The inner product between two normal modes (4.138) is then given by:

$$\begin{aligned} \langle u_{E_1 l_1 m_1} | u_{E_2 l_2 m_2} \rangle &= (E_1 + E_2) \int_{r=r_h+\epsilon}^{L} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} e^{-\frac{1}{2}(\Gamma-\Lambda)} r^2 \sin\theta \\ &\quad \times f_{E_1 l_1} f_{E_2 l_2} Z_{l_1 m_1} Z_{l_2 m_2}\, dr\, d\theta\, d\varphi \\ &= (E_1 + E_2) \delta_{l_1 l_2} \delta_{m_1 m_2} \int_{r_h+\epsilon}^{L} dr\, e^{-\frac{1}{2}(\Gamma-\Lambda)} r^2 f_{E_1 l_1} f_{E_2 l_2} \end{aligned} \tag{4.147}$$

using (4.141). We therefore impose the following normalization condition on the f_{EL}:

$$\int_{r_h+\epsilon}^{L} dr\, r^2 e^{-\frac{1}{2}(\Gamma-\Lambda)} f_{E_1 l_1} f_{E_2 l_2} = \frac{1}{2E_1} \delta_{E_1 E_2} \delta_{l_1 l_2} \tag{4.148}$$

so that

$$\langle u_{E_1 l_1 m_1} | u_{E_2 l_2 m_2} \rangle = \delta_{E_1 E_2} \delta_{l_1 l_2} \delta_{m_1 m_2}. \tag{4.149}$$

With condition (4.149) the modes (4.138) form an orthonormal basis of solutions of the Klein-Gordon equation (4.30), and hence we write the general field ϕ as a linear combination of these modes:

$$\phi(t, r, \theta, \varphi) = \sum_{E,l,m_l} a_{Elm_l} u_{Elm_l} + a^*_{Elm_l} u^*_{Elm_l} \tag{4.150}$$

where the a_{Elm_l} and $a^*_{Elm_l}$ are complex numbers. The field is then quantized by promoting the coefficients to operators a_{Elm_l}, $a^\dagger_{Elm_l}$ satisfying the canonical commutation relations [90]:

$$[a_{E_1 l_1 m_1}, a^\dagger_{E_2 l_2 m_2}] = \delta_{E_1 E_2} \delta_{l_1 l_2} \delta_{m_1 m_2}. \tag{4.151}$$

The operators a_{Elm_l}, $a^\dagger_{Elm_l}$ are, respectively, the *creation* and *annihilation* operators. The Fock space of particle states is built up from the vacuum $|0\rangle$ defined by

$$a_{Elm_l} |0\rangle = 0 \qquad \forall E, l, m_l. \tag{4.152}$$

The field expansion (4.150) is now substituted into the expression for the Hamiltonian (4.137), giving

$$\begin{aligned} H_\phi &= \frac{1}{2}\sum_{Elm_l}\left\{a_{Elm_l}a^\dagger_{Elm_l} + a^\dagger_{Elm_l}a_{Elm_l}\right\}\int_{r_h+\epsilon}^{L} dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \\ &\times \left[E^2 e^{-\frac{1}{2}(\Gamma-\Lambda)} r^2 f_{El}^2 Z_{lm_l}^2 \sin\theta + r^2 e^{\frac{1}{2}(\Gamma-\Lambda)} f_{El}'^2 Z_{lm_l}^2 \sin\theta \right. \\ &+ e^{\frac{1}{2}(\Gamma+\Lambda)} f_{El}^2 \left(\frac{\partial Z_{lm_l}}{\partial\theta}\right)^2 \sin\theta + e^{\frac{1}{2}(\Gamma+\Lambda)} f_{El}^2 \left(\frac{\partial Z_{lm_l}}{\partial\varphi}\right)^2 \operatorname{cosec}\theta \\ &\left. + r^2 e^{\frac{1}{2}(\Gamma+\Lambda)} f_{El}^2 \mu^2 Z_{lm_l}^2 \sin\theta\right]. \end{aligned} \tag{4.153}$$

This expression can be greatly simplified. Firstly, multiplying (4.139) by $f_{El} r^2 e^{\frac{1}{2}(\Gamma+\Lambda)}$ and integrating by parts, we have

$$\int_{r_h+\epsilon}^{L} dr\, r^2 e^{\frac{1}{2}(\Gamma-\Lambda)} f_{El}'^2 = \int_{r_h+\epsilon}^{L} dr\, r^2 e^{\frac{1}{2}(\Gamma+\Lambda)} \left[e^\Lambda E^2 - \frac{l(l+1)}{r^2} - \mu^2\right] f_{El}^2, \tag{4.154}$$

the boundary terms vanishing due to (4.142) and (4.143). Similarly, multiply (4.140) by $Z_{lm_l}\sin\theta$ to obtain

$$\begin{aligned} &\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} d\theta\, d\varphi \left[\sin\theta\left(\frac{\partial Z_{lm_l}}{\partial\theta}\right)^2 + \operatorname{cosec}\theta\left(\frac{\partial Z_{lm_l}}{\partial\varphi}\right)^2\right] \\ &\quad = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} d\theta\, d\varphi\, l(l+1) Z_{lm_l}^2 \sin\theta = l(l+1) \end{aligned} \tag{4.155}$$

from (4.141). Hence, using the normalization conditions (4.141) and (4.148), the Hamiltonian takes the form

$$H_\phi = \frac{1}{2}\sum_{Elm_l}\left\{a_{Elm_l}a^\dagger_{Elm_l} + a^\dagger_{Elm_l}a_{Elm_l}\right\}\left(\frac{E}{2} + E^2 J_{El}\right) \tag{4.156}$$

where

$$J_{El} = \int_{r_h+\epsilon}^{L} dr\, r^2 e^{\frac{1}{2}(\Gamma+3\Lambda)} f_{El}^2. \tag{4.157}$$

Note that all the ϵ dependence of H is contained in the integral J_{El} (4.157).

Before estimating the integral (4.157), a couple of comments are in order concerning the form of the Hamiltonian (4.156). Firstly, (4.156) ensures that the Fock basis states (built up from the vacuum (4.152) via the standard method [90]) are eigenstates of energy. This means that the Hamiltonian can alternatively be written as

$$\begin{aligned} H_\phi &= \sum_{\text{states}} \langle {}^1n_{k_1}, {}^2n_{k_2}, \ldots, {}^jn_{k_j} | H_\phi | {}^1n_{k_1}, {}^2n_{k_2}, \ldots, {}^jn_{k_j}\rangle \\ &\quad \times | {}^1n_{k_1}, {}^2n_{k_2}, \ldots, {}^jn_{k_j}\rangle\langle {}^1n_{k_1}, {}^2n_{k_2}, \ldots, {}^jn_{k_j} | \end{aligned} \tag{4.158}$$

where the expectation values are, from (4.156)

$$\langle {}^1n_{k_1}, {}^2n_{k_2}, \ldots, {}^jn_{k_j} | H_\phi | {}^1n_{k_1}, {}^2n_{k_2}, \ldots, {}^jn_{k_j} \rangle$$
$$= \frac{1}{2} \sum_{\text{modes}} \left[\frac{1}{2}E + E^2 J_{El}\right] + \frac{1}{2} \sum_i {}^in \left[\frac{1}{2}E + E^2 J_{El}\right]. \tag{4.159}$$

The first term in (4.159) is state-independent and contributes an infinite constant to the partition function $Z = \mathrm{Tr}e^{-\beta H_\phi}$. Hence it does not contribute to correlation functions, and so from here on it shall be ignored. At the end of this section we shall compare the second term with that used in section 4.2 to compute the free energy.

4.5.2 Estimation of the integral J_{El}

In order to find the ϵ dependence of the Hamiltonian (4.156), it is necessary to estimate the integral J_{El}. We shall use the WKB approximation as in section 4.2:

$$f_{El}(r) = \frac{1}{r} e^{-\frac{1}{4}(\Gamma - \Lambda)} \frac{C_{El}}{\sqrt{K_{El}(r)}} \sin\left(\int_{r_h+\epsilon}^{r} K_{El}(r')\, dr'\right) \tag{4.160}$$

where the radial wave number K_{El} is given by (4.36)

$$K_{El}(r) = e^{\Lambda} \left[E^2 - \frac{l(l+1)}{r^2} e^{-\Lambda} - \mu^2 e^{-\Lambda}\right]^{\frac{1}{2}}. \tag{4.161}$$

The form of f_{El} in (4.160) already satisfies the "brick wall" boundary condition (4.142), whilst the boundary condition (4.143), $f_{El}(L) = 0$, implies that

$$\int_{r_h+\epsilon}^{L} K_{El}(r)\, dr = n\pi \qquad \text{for some integer } n. \tag{4.162}$$

The constants C_{El} are determined by the normalization condition for the f_{El} (4.148), namely

$$C_{El}^2 = \frac{1}{2EI_{El}} \tag{4.163}$$

where

$$I_{El} = \int_{r_h+\epsilon}^{L} \frac{e^{-(\Gamma - \Lambda)}}{K_{El}(r)} \sin^2\left(\int_{r_h+\epsilon}^{r} K_{El}(r')\, dr'\right) dr. \tag{4.164}$$

Decompose the integral in (4.164) as

$$I_{El} = I_{1El} + I_{2El}, \tag{4.165}$$

where I_{1El} is the contribution to the integral for values of r very close to r_h, and I_{2El} is the remainder of the integral. Thus (4.163) becomes

$$C_{El}^2 = \frac{1}{2E(I_{1El} + I_{2El})}. \tag{4.166}$$

Similarly, write

$$\begin{aligned} J_{El} &= \int_{r_h+\epsilon}^{L} dr\, r^2 e^{\frac{1}{2}(\Gamma+3\Lambda)} f_{El}^2 \\ &= C_{El}^2 \int_{r_h+\epsilon}^{L} dr \frac{e^{2\Lambda}}{K_{El}(r)} \sin^2\left(\int_{r_h+\epsilon}^{r} K_{El}(r')\, dr'\right) \\ &= C_{El}^2 [J_{1El} + J_{2El}]. \end{aligned} \tag{4.167}$$

Next introduce a dimensionless co-ordinate x by (4.54)

$$x = \frac{r}{r_h} \tag{4.168}$$

and in addition define

$$\hat{\epsilon} = \frac{\epsilon}{r_h}. \tag{4.169}$$

For x very close to 1, we have

$$J_{1El} = e^{-2\delta_h} I_{1El} \tag{4.170}$$

where the metric functions have the alternative form (4.29)

$$\begin{aligned} e^{\Gamma} &= e^{-2\delta}\left(1 - \frac{2m(r)}{r}\right) \\ e^{-\Lambda} &= \left(1 - \frac{2m(r)}{r}\right). \end{aligned} \tag{4.171}$$

In order to compute these integrals near the event horizon, we shall require the expansion (section 4.2.4)

$$e^{-\Lambda} = (1 - 2\hat{m}_h')(x-1) + O(x-1)^2 \qquad \text{as } x \sim 1, \tag{4.172}$$

where

$$\hat{m}(x) = \frac{m(xr_h)}{r_h}. \tag{4.173}$$

Firstly, we need to evaluate, for $x \sim 1$,

$$\begin{aligned} \int_{1+\hat{\epsilon}}^{x} r_h K_{El}(x')\, dx' &= \int_{1+\hat{\epsilon}}^{x} dx' \left[\frac{Er_h}{1-2\hat{m}_h'}(x'-1)^{-1} + O(1)\right] \\ &= \frac{Er_h}{1-2\hat{m}_h'} \left[\log(x-1) - \log\hat{\epsilon}\right] \\ &\quad + O(x-1) + O(\hat{\epsilon}). \end{aligned} \tag{4.174}$$

Hence

$$\begin{aligned} J_{1El} &= \int_{1+\hat{\epsilon}}^{x} dx\, \frac{r_h(x-1)^{-1}}{E(1-2\hat{m}_h')} \sin^2\left(\frac{Er_h}{1-2\hat{m}_h'}\left[\log(x-1)-\log\hat{\epsilon}\right]\right) \\ &= \int_{1+\hat{\epsilon}}^{x} dx\, \frac{r_h(x-1)^{-1}}{2E(1-2\hat{m}_h')} \\ &\quad \times \left\{1-\cos\left(\frac{2Er_h}{1-2\hat{m}_h'}\left[\log(x-1)-\log\hat{\epsilon}\right]\right)\right\}. \end{aligned} \tag{4.175}$$

Now

$$\begin{aligned} &\int_{1+\hat{\epsilon}}^{x} dx\,(x-1)^{-1}\cos\left\{q\left[\log(x-1)-\log\hat{\epsilon}\right]\right\} \\ &\qquad = \int_{\log\hat{\epsilon}}^{\log(x-1)} d\tilde{x}\,\cos\left(q\left[\tilde{x}-\log\hat{\epsilon}\right]\right) \\ &\qquad = \frac{1}{q}\sin\left(q\left[\log(x-1)-\log\hat{\epsilon}\right]\right) \end{aligned} \tag{4.176}$$

so that

$$J_{1El} = \frac{1}{2E}\left[\frac{r_h}{1-2\hat{m}_h'}\log\hat{\epsilon} + \frac{1}{2E}\sin\left(\frac{2Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right)\right]. \tag{4.177}$$

Both I_{2El} and J_{2El} are also dependent on $\hat{\epsilon}$ due to the term

$$\sin^2\left(\int_{r_h+\epsilon}^{r} K_{El}(r')\,dr'\right). \tag{4.178}$$

Since

$$\int_{r_h+\epsilon}^{r} K_{El}(r')\,dr' = -\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon} + \text{terms independent of } \hat{\epsilon}, \tag{4.179}$$

the expression (4.178) contributes a factor to the integrands of I_{2El} and J_{2El} of the generic form

$$\sin^2\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right),$$

$$\cos^2\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right),$$

or

$$\sin\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right)\cos\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right). \tag{4.180}$$

We therefore write

$$I_{2El} = \mathcal{I}_{El}^{SS}\sin^2\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right) + \mathcal{I}_{El}^{CC}\cos^2\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right)$$

$$
\begin{aligned}
&+\mathcal{I}_{El}^{SC}\sin\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right)\cos\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right) \\
J_{2El} &= \mathcal{J}_{El}^{SS}\sin^2\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right)+\mathcal{J}_{El}^{CC}\cos^2\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right) \\
&+\mathcal{J}_{El}^{SC}\sin\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right)\cos\left(-\frac{Er_h}{1-2\hat{m}_h'}\log\hat{\epsilon}\right), \qquad (4.181)
\end{aligned}
$$

where the $\mathcal{I}$'s and $\mathcal{J}$'s are constants which are independent of $\hat{\epsilon}$ but do depend on L.

At this stage we should comment that the logarithmic terms in J_{El} have as their argument $\hat{\epsilon}=\epsilon/r_h$ rather than L/ϵ as in [91]. The reason for this is that the expansion we use in calculating these terms is valid only for values of x very close to 1. An alternative expansion in inverse powers of x has to be used for very large x.

Black holes without hair For space-times which represent black holes without hair (that is, the Schwarzschild geometry), $e^{\Gamma}=e^{-\Lambda}$ and in this case the integrals I_{El} (4.164) and J_{El} (4.157) are identical. Hence,

$$
J_{El}=\frac{1}{2E}, \qquad (4.182)
$$

the Hamiltonian (4.156) is independent of $\log\hat{\epsilon}$, and the form of the partition function Z is exactly as calculated in section 4.2.

4.5.3 Dependence of H_ϕ on $\log\hat{\epsilon}$

In order to find an order of magnitude estimate for the dependence of J_{El} (and hence of H_ϕ) on $\log\hat{\epsilon}$, we write J_{El} in the form (4.167)

$$
\begin{aligned}
J_{El} &= C_{El}^2\left[J_{1El}+J_{2El}\right] \\
&= \frac{1}{2E}\frac{\left[J_{1El}+J_{2El}\right]}{\left[I_{1El}+I_{2El}\right]} && \text{using (4.163)} \\
&= \frac{1}{2E}\frac{\left[J_{1El}+J_{2El}\right]}{\left[e^{2\delta_h}J_{1El}+I_{2El}\right]} && \text{using (4.170)} \\
&= \frac{1}{2E}\left[e^{-2\delta_h}+\frac{J_{2El}-e^{-2\delta_h}I_{2El}}{e^{2\delta_h}J_{1El}+I_{2El}}\right]. && (4.183)
\end{aligned}
$$

The second term in (4.183) is of the order of $(\log\hat{\epsilon})^{-1}$ and hence is very small. Using (4.183),

$$
\frac{\partial J_{El}}{\partial\log\hat{\epsilon}}=\frac{1}{2E}\frac{1}{e^{2\delta_h}J_{1El}+I_{2El}}\left[\frac{\partial J_{2El}}{\partial\log\hat{\epsilon}}-e^{-2\delta_h}\frac{\partial I_{2El}}{\partial\log\hat{\epsilon}}\right]+\text{smaller terms} \qquad (4.184)
$$

from which the $\log \hat{\epsilon}$ dependence of H_ϕ is given by

$$\frac{\partial H_\phi}{\partial \log \hat{\epsilon}} = \frac{1}{2} \sum_{E,l,m_l} \left[a_{Elm_l} a^\dagger_{Elm_l} + a^\dagger_{Elm_l} a_{Elm_l} \right] E^2 \frac{\partial J_{El}}{\partial \log \hat{\epsilon}}. \tag{4.185}$$

In order to estimate the order of magnitude of (4.185), first note that, from (4.181)

$$\begin{aligned} \frac{\partial J_{2El}}{\partial \log \hat{\epsilon}} &\sim r_h \mu J_{2El} \\ \frac{\partial I_{2El}}{\partial \log \hat{\epsilon}} &\sim r_h \mu I_{2El} \end{aligned} \tag{4.186}$$

and that

$$\begin{aligned} I_{1El} &\sim r_h \mu^{-1} \log \hat{\epsilon}, & I_{2El} &\sim r_h \mu^{-1}, \\ J_{1El} &\sim r_h \mu^{-1} \log \hat{\epsilon}, & J_{2El} &\sim r_h \mu^{-1}. \end{aligned} \tag{4.187}$$

Then, from (4.184),

$$\frac{\partial J_{El}}{\partial \log \hat{\epsilon}} \sim \frac{1}{\mu} r_h \mu \left(\log \hat{\epsilon} \right)^{-1} = r_h \left(\log \hat{\epsilon} \right)^{-1} \tag{4.188}$$

and finally we have, from (4.185),

$$\frac{\partial H_\phi}{\partial \log \hat{\epsilon}} \sim \mu^2 r_h \left(\log \hat{\epsilon} \right)^{-1}. \tag{4.189}$$

We complete this section by commenting briefly on the relationship between this analysis and that of section 4.2. The terms in H_ϕ which are dependent on $\log \hat{\epsilon}$ are very small, leaving in effect

$$H_\phi = \frac{1}{4} \sum_{E,l,m_l} \left[a_{Elm_l} a^\dagger_{Elm_l} + a^\dagger_{Elm_l} a_{Elm_l} \right] E (1 + e^{-2\delta_h}) \tag{4.190}$$

from (4.183). This should be compared with the expression

$$H_\phi = \frac{1}{2} \sum_{E,l,m_l} \left[a_{Elm_l} a^\dagger_{Elm_l} + a^\dagger_{Elm_l} a_{Elm_l} \right] E \tag{4.191}$$

used in section 4.2. As discussed previously, the WKB approximation on which our analysis is based is only valid for black hole solutions with large radii r_h, for which δ_h is very small [2]. In this case, the two expressions (4.190) and (4.191) are approximately identical.

4.6 Modified quantum-mechanical evolution

We are now in a position to estimate, in a first order approximation, the term $\not{\delta} H$ arising in the modified time evolution equation for the density matrix.

4.6.1 Form of $\not{\$}H$

We begin with the Gibbs formula:

$$\rho = \frac{e^{-\beta H_\phi}}{\mathsf{Z}}. \tag{4.192}$$

In this case,

$$[\rho, H_\phi] = 0 \tag{4.193}$$

and hence the time-evolution equation for ρ is (4.121):

$$\frac{\partial \rho}{\partial t} = \not{\$}H\rho. \tag{4.194}$$

Using (4.192),

$$\frac{\partial \rho}{\partial t} = \not{\$}H\rho = -\beta\frac{\partial H_\phi}{\partial t}\frac{e^{-\beta H_\phi}}{\mathsf{Z}} - \frac{\partial \mathsf{Z}}{\partial t}\frac{e^{-\beta H_\phi}}{\mathsf{Z}^2}. \tag{4.195}$$

The partition function Z is given by

$$\mathsf{Z} = e^{-\beta F} \tag{4.196}$$

where F is the free energy calculated in section 4.2.2. Thus

$$\frac{\partial \mathsf{Z}}{\partial t} = -\beta\frac{\partial F}{\partial t}e^{-\beta F} = -\beta\frac{\partial F}{\partial t}\mathsf{Z}, \tag{4.197}$$

which gives

$$\frac{\partial \rho}{\partial t} = -\beta\left[\frac{\partial H_\phi}{\partial t} - \frac{\partial F}{\partial t}\right]\rho, \tag{4.198}$$

which means that

$$\not{\$}H = -\beta\frac{\partial H_\phi}{\partial t} + \beta\frac{\partial F}{\partial t}. \tag{4.199}$$

Recall, from section 4.4.3, that we are making the identification

$$t = \log\hat{\epsilon}. \tag{4.200}$$

The free energy F was calculated in section 4.2.2 to be

$$\begin{aligned} F &= F_{LO} + F_{NLO} \\ &= -\frac{2\pi^3}{45\hat{\epsilon}}\frac{r_h^3}{\beta^4}(1-2\hat{m}_h')^{-2} + \frac{4\pi^3}{45}\frac{r_h^3}{\beta^4}\frac{(2-4\hat{m}_h'+\hat{m}_h'')}{(1-2\hat{m}_h')^3}\log\hat{\epsilon} \\ &\quad -\frac{\pi}{6}\frac{r_h^3}{\beta^2}\mu^2(1-2\hat{m}_h')^{-1}\log\hat{\epsilon}. \end{aligned} \tag{4.201}$$

Bearing in mind that

$$\frac{\partial}{\partial\log\hat{\epsilon}}\left(\frac{1}{\hat{\epsilon}}\right) = -\frac{1}{\hat{\epsilon}}, \tag{4.202}$$

then

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial \log \hat{\epsilon}} = \frac{2\pi^3}{45\hat{\epsilon}} \frac{r_h^3}{\beta^4}(1-2\hat{m}_h')^{-2} + \frac{4\pi^3}{45} \frac{r_h^3}{\beta^4} \frac{(2-4\hat{m}_h' + \hat{m}_h'')}{(1-2\hat{m}_h')^3} - \frac{\pi}{6} \frac{r_h^3}{\beta^2} \mu^2 (1-2\hat{m}_h')^{-1}. \tag{4.203}$$

Substituting for the inverse Hawking temperature

$$\beta = \frac{4\pi r_h e^{\delta_h}}{1-2\hat{m}_h'} \tag{4.204}$$

gives

$$\beta \frac{\partial F}{\partial t} = \frac{1}{1440} \frac{1}{\hat{\epsilon}} (1-2\hat{m}_h') e^{-3\delta_h} + \frac{1}{720}(2-4\hat{m}_h' + \hat{m}_h'') e^{-3\delta_h} - \frac{1}{24} r_h^2 \mu^2 e^{-\delta_h}. \tag{4.205}$$

From the previous section, we have

$$\frac{\partial H_\phi}{\partial t} = \frac{\partial H_\phi}{\partial \log \hat{\epsilon}} = O(\log \hat{\epsilon})^{-1}, \tag{4.206}$$

so that altogether we have

$$\not{H} = O(\log \hat{\epsilon})^{-1} + \frac{1}{1440} \frac{1}{\hat{\epsilon}} (1-2\hat{m}_h') e^{-3\delta_h} + \frac{1}{720}(2-4\hat{m}_h' + \hat{m}_h'') e^{-3\delta_h} - \frac{1}{24} r_h^2 \mu^2 e^{-\delta_h}. \tag{4.207}$$

4.6.2 Comments on $\not{H}$

Consistency of the approximation The first thing to note about the form of $\not{H}$ is that the terms obtained from the explicit time-dependence of the Hamiltonian are suppressed relative to the terms coming from the free energy. Thus the main time-dependence of ρ comes from the process of tracing over an infinite number of states in the calculation of Z. This is in accordance with the approximation we have used, in which the effect of quantum fluctuations on the Hawking temperature β^{-1} was neglected.

Order of magnitude of $\not{H}$ For non-extreme black holes, the most important term in $\not{H}$ is that proportional to $\hat{\epsilon}^{-1}$ as $\hat{\epsilon} \to 0$. This term corresponds, in the entropy, to a factor that can be absorbed into a renormalization of the gravitational constant (see section 4.3.2). For extreme black holes with $r_h \gg 1$, the dominant term is that containing μ^2. This is the only term which depends on the energy content of the scalar field, the remainder of $\not{H}$ depending only on the black hole geometry. The μ^2 term is the most interesting phenomenologically, since it is proportional to

$$\frac{\mu^2}{M_P} \tag{4.208}$$

where M_P is the Planck mass.

Of course, at this stage we cannot be precise about the numerical coefficient multiplying this factor, or, indeed, whether this term persists in a full theory of quantum gravity. However, it is interesting that in the approximation employed here, a term of order μ^2/M_P is present, and also that this term has not been further suppressed. These are the maximal gravitational effects that could be expected, and may even be observable in the not-too-distant future [92].

Entropy In section 4.2 we calculated the entropy of the quantum scalar field. The $\log\hat{\epsilon}$ dependence calculated in section 4.5, coupled with the Gibbs formula, exactly reproduces the $\hat{\epsilon}$ dependence found in section 4.2. Identifying $t = \log\hat{\epsilon}$, the time dependence of the entropy is

$$\frac{\partial \mathsf{S}}{\partial t} = \frac{\partial \mathsf{S}}{\partial \log\hat{\epsilon}} = -\frac{1}{360}\frac{1}{\hat{\epsilon}}(1 - 2\hat{m}_h')e^{-3\delta_h} - \frac{1}{180}(2 - 4\hat{m}_h' + \hat{m}_h'')e^{-3\delta_h} + \frac{1}{12}r_h^2\mu^2 e^{-\delta_h}. \tag{4.209}$$

For large extremal black holes, (4.209) is manifestly positive. Thus the quantum state of the scalar field becomes more mixed as time proceeds due to the $\oint H$ term, and this is accompanied by a corresponding increase in entropy. Note that the entropy increases in the extreme case even though the Hawking temperature vanishes. This is because the state of the field ϕ becomes increasingly entangled with the unobservable modes hidden behind the event horizon.

Summary

In this chapter we have made a proposal for how black holes may contribute to the modifications of quantum mechanics we may expect to find in a final theory of quantum gravity (at present unattainable!) [5]. Our estimate for the size of these effects is the largest that could be expected from a naive dimensional analysis of Planckian effects, hopefully spurring further phenomenological work in this area. It has been beyond the scope of this chapter to consider "non-critical" string theory, where such effects have been argued to arise naturally, through a proper mathematical formalism.

Chapter 5

Higher Curvature String Gravity

String theory is a promising candidate for the elusive, but much sought, theory of quantum gravity which will incorporate all known forces in the universe. It is therefore important to study black holes and in particular, black holes with hair, in string-inspired models of gravity. In this chapter we look at a particular model of gravity which involves higher order curvature terms than general relativity, and show that it is possible for black hole solutions with hair to exist. The mechanism whereby these black holes evade the "no-hair" theorem [1] (see also section 1.3) is elucidated in detail [6], and is similar to that for the EYMH system, discussed in section 2.3. Finally, we discuss briefly the properties of these solutions, found numerically in [6]. Some details on the field equations and their numerical solution can be found in appendix B.

5.1 String-inspired gravity

String theory possesses an infinite tower of massive string modes (see [93] for a review) as well as dilatons, axions, and Abelian or Yang-Mills gauge fields. In the presence of these low energy degrees of freedom, various black hole solutions are known (some examples of these solutions are studied in [29, 77, 94, 95, 96, 97, 98, 99, 100, 101]), but these will not concern us here. Instead we shall study an effective, low energy, Lagrangian which involves only the massless string modes and is derived from the full string theory by perturbation theory in the string tension α'. This effective Lagrangian contains higher-order curvature terms which are $O(\alpha')$ corrections to general relativity. Approximate (to $O(\alpha')$) black hole solutions of such a theory have been found recently [98, 99], and these solutions exhibit dilaton, axion and modulus field hair. A natural extension of this work is to look for solutions to all orders in α'.

To this end, consider the bosonic part of the gravitational multiplet consisting of the dilaton, graviton and antisymmetric tensor fields. Unlike [98], here we shall ignore the antisymmetric tensor for simplicity. Its presence is unlikely to modify drastically the results of this chapter. A generic form of the string-inspired $O(\alpha')$

corrections to general relativity is,

$$\mathcal{L}_{GB} = -\frac{1}{8\pi}\left[\frac{1}{2}R + \frac{1}{4}(\partial_\mu\Phi)^2 - \frac{\alpha'}{8g^2}e^{\Phi}\left(\eta_1 R^2_{GB} + \eta_2(\partial_\mu\Phi)^4\right)\right] \tag{5.1}$$

where Φ is the dilaton field, α' is the Regge slope, g^2 is a gauge coupling constant and R^2_{GB} is the Gauss-Bonnet invariant

$$R^2_{GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \tag{5.2}$$

The coefficients η_1 and η_2 are fixed by string scattering amplitudes. In the case $\eta_1 = 0$, only the fourth order dilaton terms in (5.1) are present. The modern proof of the no-hair theorem [1] (section 1.3) then applies to the theory described by (5.1), so that black hole solutions cannot sustain dilaton hair in this scenario. However, the presence of the Gauss-Bonnet term allows non-trivial hair to exist. This will be shown in the next section. For $\eta_1 \neq 0$, the presence of the $(\partial_\mu\Phi)^4$ term does not affect the analysis, and hence for the remainder of this chapter we set $\eta_2 = 0$ for simplicity.

5.2 Evading the no-hair conjecture

In analogy with the analysis of sections 1.3 and 2.3 we now consider in detail the mechanism whereby the black hole solutions of the higher curvature gravity evade the no-hair theorem of [1].

5.2.1 The field equations

Consider the Lagrangian for dilaton gravity with a Gauss-Bonnet term:

$$\mathcal{L}_{GB} = -\frac{1}{8\pi}\left[\frac{1}{2}R + \frac{1}{4}(\partial_\mu\Phi)^2 - \frac{\alpha'}{8g^2}e^{\Phi}R^2_{GB}\right] \tag{5.3}$$

where R^2_{GB} is given by (5.2). The dilaton Φ satisfies the field equation

$$\Box\Phi = \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\,\partial^\mu\Phi\right] = -\frac{\alpha'}{4g^2}e^{\Phi}R^2_{GB} \tag{5.4}$$

whilst the Einstein equations take the form [102]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \tag{5.5}$$

where

$$8\pi T_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{4}g_{\mu\nu}(\partial_\rho\Phi)^2 + \frac{\alpha'}{2g^2}e^{\Phi}K_{\mu\nu} \tag{5.6}$$

and

$$K_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu} \tag{5.7}$$

with

$$\begin{aligned} A_{\mu\nu} &= R\Phi_{;\mu\nu} - Rg_{\mu\nu}\Box\Phi - 2R_{\mu\rho}\Phi_{;\nu}{}^{;\rho} - 2R_{\nu\rho}\Phi_{;\mu}{}^{;\rho} \\ &\quad +2R^{\rho\sigma}\Phi_{;\rho\sigma}g_{\mu\nu} + 2R_{\mu\nu}\Box\Phi - 2R_{\mu\rho\nu\sigma}\Phi^{;\rho\sigma} \\ B_{\mu\nu} &= R\Phi_{;\mu}\Phi_{;\nu} - Rg_{\mu\nu}\Phi_{;\rho}\Phi^{;\rho} - 2R_{\mu\rho}\Phi_{;\nu}\Phi^{;\rho} - 2R_{\nu\rho}\Phi_{;\mu}\Phi^{;\rho} \\ &\quad +2R^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma}g_{\mu\nu} + 2R_{\mu\nu}\Phi_{;\rho}\Phi^{;\rho} - 2R_{\mu\rho\nu\sigma}\Phi^{;\rho}\Phi^{;\sigma}. \end{aligned} \tag{5.8}$$

The left hand side of Einstein's equation (5.5) is automatically conserved due to the Bianchi identities [38] and hence the right hand side is also conserved, i.e.

$$\nabla_\mu T^{\mu\nu} = 0. \tag{5.9}$$

However, it should be stressed that $T_{\mu\nu}$ is not a conventional "energy-momentum" tensor due to the presence of the $K_{\mu\nu}$ term in (5.6). This term involves contributions of the gravitational field itself to $T_{\mu\nu}$ which arise because the dilaton is part of the gravitational multiplet in string theory [93]. The consequence is that $-T^t_t$, which, in general relativity would correspond to the local energy density, is not necessarily positive. Indeed, there are regions in the black hole solutions where $-T^t_t$ is negative (see figure 5.3). The positivity of $-T^t_t$ is a key assumption in section 1.3 and hence we have an indication here that the theorem of [1] can be evaded in this situation.

Consider a spherically-symmetric space-time with metric

$$ds^2 = -e^\Gamma dt^2 + e^\Lambda dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{5.10}$$

where Γ and Λ are functions of r only. In addition, we assume that the dilaton field Φ depends only on r and denote derivatives with respect to r by a prime. In the metric (5.10) the Gauss-Bonnet curvature term is

$$R^2_{GB} = -\frac{4}{r^2}\left[e^\Lambda\Gamma''(1-e^{-\Lambda}) + \frac{1}{2}e^{-\Lambda}(\Gamma'^2 - \Gamma'\Lambda')(1-e^{-\Lambda}) + \Gamma'\Lambda' e^{-2\Lambda}\right]. \tag{5.11}$$

The relevant components of the Einstein tensor are [40]

$$\begin{aligned} R_{tt} - \frac{1}{2}Rg_{tt} &= \frac{\Lambda'}{r}e^{\Gamma-\Lambda} + \frac{e^\Gamma}{r^2}(1-e^{-\Lambda}) \\ R_{rr} - \frac{1}{2}Rg_{rr} &= \frac{\Gamma'}{r} - \frac{1}{r^2}(e^\Lambda - 1) \\ R_{\theta\theta} - \frac{1}{2}Rg_{\theta\theta} &= \frac{r^2}{2}\Gamma'' e^{-\Lambda} + (\Gamma'-\Lambda')e^{-\Lambda}\left[\frac{r^2}{4}\Gamma' + \frac{r}{2}\right]. \end{aligned} \tag{5.12}$$

We shall also require the following components of (5.8)

$$A_{tt} = \frac{1}{r^2}e^{\Gamma-\Lambda}\left[(1-e^{-\Lambda})(2\Phi'' - \Lambda'\Phi') + 2e^{-\Lambda}\Lambda'\Phi'\right]$$

$$
\begin{aligned}
A_{rr} &= -\frac{1}{r^2}\Gamma'\Phi'(1-3e^{-\Lambda}) \\
A_{\theta\theta} &= re^{-2\Lambda}\left[\Gamma'\Phi''+\Gamma''\Phi'+\frac{1}{2}\Gamma'\Phi'(\Gamma'-3\Lambda')\right] \\
B_{tt} &= \frac{2}{r^2}\Phi'^2 e^{\Gamma-\Lambda}(1-e^{-\Lambda}) \\
B_{rr} &= 0 \\
B_{\theta\theta} &= r\Phi'^2 e^{-2\Lambda}\Gamma'
\end{aligned} \tag{5.13}
$$

and

$$
\Box\Phi = e^{-\Lambda}\left[\frac{1}{2}(\Gamma'-\Lambda')\Phi'+\frac{2}{r}\Phi'+\Phi''\right]. \tag{5.14}
$$

Thus the dilaton equation (5.4) takes the form

$$
\Phi''+\Phi'\left(\frac{1}{2}(\Gamma'-\Lambda')+\frac{2}{r}\right) = \frac{\alpha'}{g^2r^2}e^{\Phi}\left\{\Gamma'\Lambda'e^{-\Lambda}+(1-e^{-\Lambda})\left[\Gamma''+\frac{\Gamma'}{2}(\Gamma'-\Lambda')\right]\right\} \tag{5.15}
$$

and the Einstein equations (5.5) can be written in the form

$$
\begin{aligned}
\Lambda'\left[1+\frac{\alpha'}{2g^2r}e^{\Phi}\Phi'(1-3e^{-\Lambda})\right] &= \frac{r\Phi'^2}{4}+\frac{1-e^{-\Lambda}}{r} \\
&\quad +\frac{\alpha'}{g^2r}e^{\Phi}\left(\Phi''+\Phi'^2\right)(1-e^{-\Lambda})
\end{aligned} \tag{5.16}
$$

$$
\Gamma'\left[1+\frac{\alpha'}{2g^2r}e^{\Phi}\Phi'(1-3e^{-\Lambda})\right] = \frac{r\Phi'^2}{4}+\frac{e^{\Lambda}-1}{r} \tag{5.17}
$$

$$
\begin{aligned}
\Gamma''+\frac{\Gamma'}{2}(\Gamma'-\Lambda')+\frac{1}{r}(\Gamma'-\Lambda') &= -\frac{\Phi'^2}{2}+\frac{\alpha'}{g^2r}e^{\Phi}e^{-\Lambda}\left[\Phi'\Gamma''+(\Phi''+\Phi'^2)\Gamma'\right. \\
&\quad \left.+\frac{1}{2}\Phi'\Gamma'(\Gamma'-3\Lambda')\right].
\end{aligned} \tag{5.18}
$$

Solutions without the Gauss-Bonnet term In the absence of the Gauss-Bonnet term, equation (5.15) can be integrated immediately to obtain

$$
\Phi' \sim \frac{1}{r^2}e^{\frac{1}{2}(\Lambda-\Gamma)}. \tag{5.19}
$$

For a black hole solution,

$$
e^{\Gamma}\to 0, \qquad e^{-\Lambda}\to 0 \qquad \text{as } r\to r_h. \tag{5.20}
$$

Therefore the radial derivative of the dilaton diverges on the event horizon, leading to a divergent stress tensor, since, in this case,

$$
T_t^t = -T_r^r = T_\theta^\theta = -\frac{e^{-\Lambda}}{32\pi}\Phi'^2 \to \infty \qquad \text{as } r\to r_h. \tag{5.21}
$$

Hence, in order to have a regular stress tensor, $\Phi' \equiv 0$ so that $\Phi \equiv$ constant everywhere, in accordance with the no-hair theorem [1], section 1.3.

5.2.2 Form of the stress tensor

Following section 2.3, the conservation equations (5.9) give the following two equations for T^r_r:

$$\begin{aligned} T^r_r(r) &= \frac{e^{-\frac{\Gamma}{2}}}{2r^2}\int_{r_h}^{r} e^{\frac{\Gamma}{2}} r^2 \left[\Gamma' T^t_t + \frac{4}{r} T^\theta_\theta\right] dr \\ (T^r_r)'(r) &= \frac{e^{-\frac{\Gamma}{2}}}{r^2}\left(e^{\frac{\Gamma}{2}} r^2\right)' (T^t_t - T^r_r) + \frac{2}{r}(T^\theta_\theta - T^r_r). \end{aligned} \tag{5.22}$$

Using the functional form of the "stress" tensor (5.6), we see that

$$\begin{aligned} 8\pi T^t_t &= -\frac{e^{-\Lambda}}{4}\Phi'^2 - \frac{\alpha'}{g^2 r^2} e^{\Phi} e^{-\Lambda}(\Phi'' + \Phi'^2)(1 - e^{-\Lambda}) + \frac{\alpha'}{2g^2 r^2} e^{\Phi} e^{-\Lambda}\Phi'\Lambda'(1 - 3e^{-\Lambda}) \\ 8\pi T^r_r &= \frac{e^{-\Lambda}}{4}\Phi'^2 - \frac{\alpha'}{2g^2 r^2} e^{\Phi} e^{-\Lambda}\Phi'\Gamma'(1 - 3e^{-\Lambda}) \\ 8\pi T^\theta_\theta &= -\frac{e^{-\Lambda}}{4}\Phi'^2 + \frac{\alpha'}{2g^2 r} e^{\Phi} e^{-2\Lambda}\left[\Gamma''\Phi' + \Gamma'(\Phi'' + \Phi'^2) + \frac{1}{2}\Gamma'\Phi'(\Gamma' - 3\Lambda')\right]. \end{aligned} \tag{5.23}$$

The equations (5.23) reveal a second reason for the evasion of the "no-hair" theorem, namely the fact that

$$T^t_t \neq T^\theta_\theta. \tag{5.24}$$

This represents the violation of another key relation in the proof of [1], mirroring the situation in the EYMH theory (cf. sections 1.3, 2.3).

5.2.3 Behaviour at infinity

Far from the event horizon, we require the geometry to be asymptotically flat and expand $\Phi(r)$, $e^{\Lambda(r)}$ and $e^{\Gamma(r)}$ as power series in r^{-1}. The field equations (5.15–5.18) ensure that these expansions have the form

$$\begin{aligned} e^{\Gamma(r)} &= 1 - \frac{2M}{r} + O\left(\frac{1}{r^3}\right) \\ e^{\Lambda(r)} &= 1 + \frac{2M}{r} + \frac{16M - D^2}{4r^2} + O\left(\frac{1}{r^3}\right) \\ \Phi(r) &= \Phi_\infty + \frac{D}{r} + \frac{MD}{r^2} + O\left(\frac{1}{r^3}\right). \end{aligned} \tag{5.25}$$

Here Φ_∞ is an arbitrary constant, M is the ADM [64] mass of the black hole and D is the dilaton charge, defined by [77, 95]

$$D = -\frac{1}{4\pi}\int \nabla_\mu \Phi \, d^2\Sigma^\mu \tag{5.26}$$

where the integral is performed over a two-sphere at spatial infinity. From (5.25),

$$\Gamma', \Lambda' \sim O\left(\frac{1}{r^2}\right) \qquad \text{as } r \to \infty \tag{5.27}$$

and hence the leading behaviour of the components of the stress tensor (5.23) is

$$\begin{aligned}8\pi T_t^t &= -\frac{1}{4}\Phi'^2 + O(r^{-5})\\ 8\pi T_r^r &= \frac{1}{4}\Phi'^2 + O(r^{-6})\\ 8\pi T_\theta^\theta &= -\frac{1}{4}\Phi'^2 + O(r^{-6})\end{aligned} \tag{5.28}$$

which, when substituted into (5.22) yields

$$(T_r^r)'(r) \sim -\frac{1}{4\pi r}\Phi'^2 < 0 \qquad \text{as } r \to \infty. \tag{5.29}$$

Thus T_r^r is positive and decreasing as $r \to \infty$.

5.2.4 Behaviour at the horizon

Near the event horizon, the field variables can be expanded as Taylor series:

$$\begin{aligned}e^{-\Lambda(r)} &= \Lambda_1(r-r_h) + \Lambda_2(r-r_h)^2 + \ldots\\ e^{\Gamma(r)} &= \Gamma_1(r-r_h) + \Gamma_2(r-r_h)^2 + \ldots\\ \Phi(r) &= \Phi_h + \Phi_h'(r-r_h) + \frac{1}{2}\Phi_h''(r-r_h)^2 + \ldots\end{aligned} \tag{5.30}$$

where $\Lambda_1, \Lambda_2, \ldots$ and $\Gamma_1, \Gamma_2, \ldots$ are constants and the subscript h denotes quantities evaluated on the event horizon. Equations (5.30) imply that

$$\Gamma' \sim \frac{1}{r-r_h} \qquad \text{and} \qquad \Lambda' \sim -\frac{1}{r-r_h} \tag{5.31}$$

as the event horizon is approached. The leading behaviour of the components of the energy momentum tensor (5.23) is then:

$$\begin{aligned}8\pi T_t^t &= \frac{\alpha'}{2g^2r^2}e^{\Phi}e^{-\Lambda}\Phi'\Lambda' + O(r-r_h)\\ 8\pi T_r^r &= -\frac{\alpha'}{2g^2r^2}e^{\Phi}e^{-\Lambda}\Phi'\Gamma' + O(r-r_h)\\ 8\pi T_\theta^\theta &= \frac{\alpha'}{2g^2r^2}e^{\Phi}e^{-2\Lambda}\left[\Gamma''\Phi' + \frac{1}{2}\Gamma'\Phi'(\Gamma'-3\Lambda')\right] + O(r-r_h).\end{aligned} \tag{5.32}$$

The above expressions (5.32), when substituted into the conservation equation solution (5.22), give the leading behaviour of T_r^r as

$$\begin{aligned}8\pi T_r^r(r) &= \frac{e^{-\frac{\Gamma}{2}}}{r^2}\int_{r_h}^r \frac{\alpha'}{4g^2}e^{\frac{\Gamma}{2}}\Gamma'\Lambda'\Phi'e^{-\Lambda}e^{\Phi}\,dr + O(r-r_h)\\ &= -\frac{e^{-\frac{\Gamma}{2}}}{r^2}\int_{r_h}^r \frac{\alpha'}{4g^2}e^{\frac{\Gamma}{2}}\Gamma'^2\Phi'e^{-\Lambda}e^{\Phi}\,dr + O(r-r_h)\end{aligned} \tag{5.33}$$

since

$$\Gamma' + \Lambda' \sim O(1) \qquad \text{as } r \to r_h. \tag{5.34}$$

From (5.33) we see that, for r sufficiently close to the horizon, T_r^r has the opposite sign to Φ_h'. Substituting (5.32) into the expression (5.22) for $(T_r^r)'(r)$ gives, near the horizon,

$$\begin{aligned} 8\pi(T_r^r)'(r) &= \frac{\alpha'}{2g^2r^2}e^{\Phi}e^{-\Lambda}\left\{-\Gamma'(\Phi''+\Phi'^2)+\Phi'\left[\frac{\Gamma'}{2}(\Gamma'+\Lambda')+2e^{-\Lambda}\Gamma''-\frac{2}{r}\Lambda'\right]\right\} \\ &\quad -\frac{1}{4}\Gamma' e^{-\Lambda}\Phi'^2 + O(r-r_h) \end{aligned} \tag{5.35}$$

which is finite by virtue of (5.34).

We now turn to Einstein's equations (5.16–5.18). Adding together (5.16) and (5.17) gives

$$(\Gamma'+\Lambda')\left[1+\frac{\alpha'}{2g^2r}e^{\Phi}\Phi'(1-3e^{-\Lambda})\right] = \frac{r}{2}\Phi'^2 + \frac{\alpha'}{g^2r}e^{\Phi}(\Phi''+\Phi'^2)(1-e^{-\Lambda}) \tag{5.36}$$

so that, near the event horizon,

$$\Gamma'+\Lambda' = \frac{1}{\mathcal{Z}}\left[\frac{1}{2}r_h\Phi_h'^2 + \frac{\alpha'}{g^2r_h}e^{\Phi_h}(\Phi_h''+\Phi_h'^2)\right] + O(r-r_h) \tag{5.37}$$

where

$$\mathcal{Z} = 1 + \frac{\alpha'}{2g^2r_h}e^{\Phi_h}\Phi_h'. \tag{5.38}$$

Equation (5.18) gives, as $r \to r_h$,

$$e^{-2\Lambda}\Gamma'' = -\frac{1}{2}e^{-2\Lambda}\Gamma'^2 + \frac{1}{2}e^{-2\Lambda}\Gamma'\Lambda' + O(r-r_h). \tag{5.39}$$

Subtracting (5.16) and (5.18), we obtain,

$$(\Gamma'-\Lambda')\left[1+\frac{\alpha'}{2g^2r}e^{\Phi}\Phi'(1-3e^{-\Lambda})\right] = \frac{2}{r}(e^{\Lambda}-1) - \frac{\alpha'}{g^2r}e^{\Phi}(\Phi''+\Phi'^2)(1-e^{-\Lambda}) \tag{5.40}$$

which implies that

$$(\Gamma'-\Lambda') = \frac{2e^{\Lambda}}{r_h\mathcal{Z}} + O(1) \qquad \text{as } r \to r_h. \tag{5.41}$$

From (5.39) we then see that

$$e^{-2\Lambda}\Gamma'' = -\frac{1}{r_h^2\mathcal{Z}^2} + O(r-r_h), \tag{5.42}$$

since, from (5.16) and (5.17),

$$\begin{aligned}\Gamma' &= \frac{e^{\Lambda}}{r_h \mathcal{Z}} + O(1) \\ \Lambda' &= -\frac{e^{\Lambda}}{r_h \mathcal{Z}} + O(1)\end{aligned} \tag{5.43}$$

as $r \to r_h$. Now substitute all the above formulae into (5.35) to yield, near the event horizon,

$$8\pi (T^r_r)'(r) = -\frac{1}{4}\frac{\Phi_h'^2}{r_h^2 \mathcal{Z}} - \frac{\alpha'}{g^2}\frac{e^{\Phi_h}}{r_h^3 \mathcal{Z}^2}(\Phi_h'' + \Phi_h'^2) - \frac{\alpha'}{4g^4}\frac{e^{2\Phi_h}}{r_h^5 \mathcal{Z}^2}\Phi_h'^2 + O(r - r_h). \tag{5.44}$$

Condition on Φ_h Finally we turn to the dilaton equation (5.15). As $r \to r_h$, it takes the form (assuming that Φ_h'' is finite),

$$\frac{\Phi_h'}{r_h}\mathcal{Z} = -\frac{3}{\mathcal{Z}^2}\frac{\alpha'}{g^2}\frac{e^{\Phi_h}}{r_h^4} + O(r - r_h). \tag{5.45}$$

If we substitute for $\mathcal{Z}$ from (5.38), and then rearrange, a quadratic equation for Φ_h' is obtained:

$$\frac{\alpha'}{2g^2}\frac{e^{\Phi_h}}{r_h}\Phi_h'^2 + \Phi_h' + \frac{3}{r_h^3}\frac{\alpha'}{g^2}e^{\Phi_h} = 0 \tag{5.46}$$

which has solutions

$$\Phi_h' = \frac{g^2}{\alpha'} r_h e^{-\Phi_h}\left[-1 \pm \sqrt{1 - \frac{6\alpha'^2}{g^4}\frac{e^{2\Phi_h}}{r_h^4}}\right]. \tag{5.47}$$

In appendix B is it shown, from a different perspective, that (5.47) guarantees that Φ_h'' is finite at the event horizon, which in turn implies that the "energy density" $-T^t_t$, given by (5.23), is also finite on $r = r_h$. Both the solutions given by (5.47) are negative.

5.2.5 Eluding the "no-hair" theorem

We have observed earlier that $T^r_r(r)$ has the opposite sign to Φ_h' for r sufficiently close to the event horizon r_h. Since Φ_h' is always negative, it must be the case that T^r_r is positive sufficiently close to the horizon. In addition, $T^r_r \geq 0$ at infinity, and so we have no contradiction with Einstein's equations. These observations on the sign of T^r_r are in agreement with numerical integrations (cf. figure 5.3).

The form of T^t_t (5.32) near the horizon shows that it is positive, since both Λ' and Φ' are negative for $r \sim r_h$. Thus the quantity $-T^t_t$ (which here does not give the local matter energy density) is negative. Again, this fact is reflected in the black hole solutions (figure 5.3). The key observation in this section is that the two factors which enable us to evade the "no-hair" conjecture, the non-positivity of $-T^t_t$ and the fact that $T^t_t \neq T^\theta_\theta$, are both due to the $O(\alpha')$ corrections in the Lagrangian (5.3).

5.3 Properties of the black hole solutions

In this section, having shown that it is possible to evade the no-hair theorem for the Lagrangian (5.3), we now discuss the actual black holes of this theory, found numerically in [6]. Firstly we discuss the physical significance of the relation (5.47), before turning to the numerical solutions of the field equations. The field equations (5.15–5.18), when arranged into a form suitable for numerical integration, are rather unwieldy. Therefore we do not write them explicitly here, but they may be found in appendix B.

5.3.1 Absence of naked singularities

For general values of the coupling α'/g^2, we have from (5.47),

$$e^{\Phi_h} < \frac{g^2 r_h^2}{\sqrt{6}\alpha'}. \tag{5.48}$$

In the effective Lagrangian (5.3),

$$\beta_{eff} = \frac{1}{4}e^{\Phi_h} \tag{5.49}$$

can be viewed as the (appropriately normalized with respect to the Einstein term) coupling constant of the Gauss-Bonnet term. For a black hole with radius $r_h = 1$, solutions do not exist for $\beta_{eff} > \beta_{eff}^c$, where

$$\beta_{eff}^c = \frac{g^2}{4\sqrt{6}\alpha'}. \tag{5.50}$$

This can be compared to the situation for $SU(2)$ sphalerons coupled to Gauss-Bonnet gravity [103]. In that case, there is a critical value of the GB coefficient, above which solutions do not exist, the critical value depending on the number of nodes of the Yang-Mills field. Unlike [103], however, here we have an additional length scale, namely the horizon radius r_h. We therefore interpret condition (5.48) as being a necessary condition for the absence of naked singularities in the space-time geometry. This can be seen from the form of the scalar curvature as $r \to r_h$, which is

$$R = \frac{2}{r^2}(1 - \Lambda_1 r)^2 + O(r - r_h) \tag{5.51}$$

where the constant Λ_1 is given by (B.22). Substituting in this value of Λ_1 gives, at $r = r_h$,

$$R = \frac{2}{r_h^2}\left(\frac{1 \mp \sqrt{1 - \frac{6\alpha'^2}{g^4}\frac{e^{2\Phi_h}}{r_h^4}}}{1 \pm \sqrt{1 - \frac{6\alpha'^2}{g^4}\frac{e^{2\Phi_h}}{r_h^4}}}\right)^2 . \tag{5.52}$$

This expression becomes singular in the limit $r_h \to 0$. However, by virtue of (5.48), $r_h = 0$ can only be reached when $\Phi_h = -\infty$, so that finite Φ_h forbids r_h to shrink to zero and reveal a naked singularity.

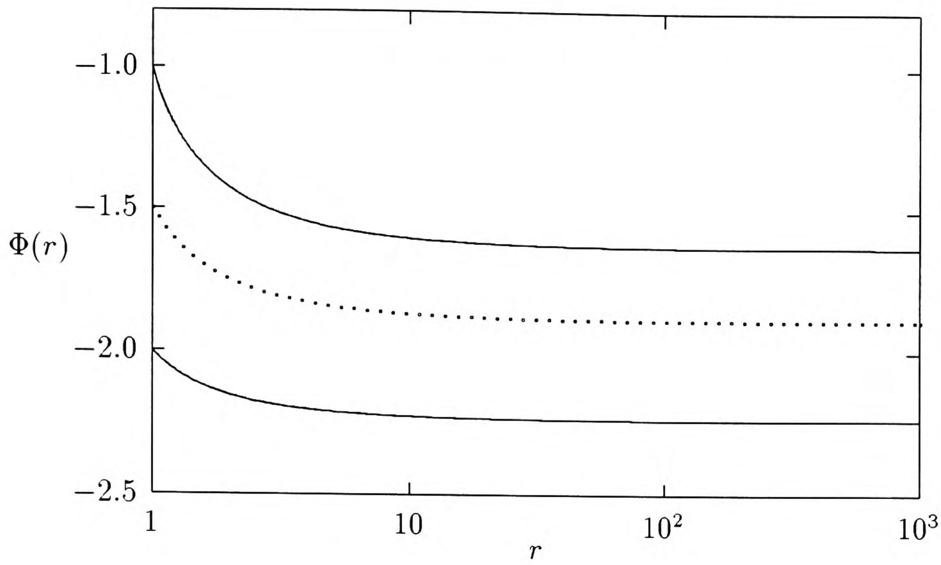

Figure 5.1: Dilaton field for the $r_h = 1$ black hole. Each curve corresponds to a different solution characterized by a different initial value of Φ_h.

5.3.2 Numerical analysis

Equations (B.6) have been integrated numerically in [6], using a Runge-Kutta procedure [42] with automatic step-size and accuracy 10^{-8}, integrating outward from r_h until the field variables have their asymptotic forms (5.25). It is found that the choice of a plus sign in (5.47) is necessary in order to give solutions having the correct behaviour for large r, and further that the solutions can be described by just two parameters, Φ_h and r_h (see appendix B).

The field equations are invariant under the transformation

$$\Phi \to \Phi + \Phi_0, \qquad r \to r e^{\frac{\Phi_0}{2}}. \tag{5.53}$$

Hence we can set $r_h = 1$ without loss of generality. Unlike the situation in EYMH theory (cf. section 2.2), a suitable black hole solution exists for every value of Φ_h satisfying (5.48), giving an infinite continuous family of black holes. The values of the dilaton field, metric tensor components and energy-momentum tensor components for these black hole solutions are shown in figures 5.1, 5.2 and 5.3 respectively.

5.3.3 Asymptotic properties of the solutions

Firstly, we note that beyond a certain value of Φ_h, the black hole quickly reaches the Schwarzschild geometry, with $r_h = 2M$, and non-vanishing dilaton charge, so that the dilaton ultimately behaves as a constant. In this case the Gauss-Bonnet term becomes irrelevant.

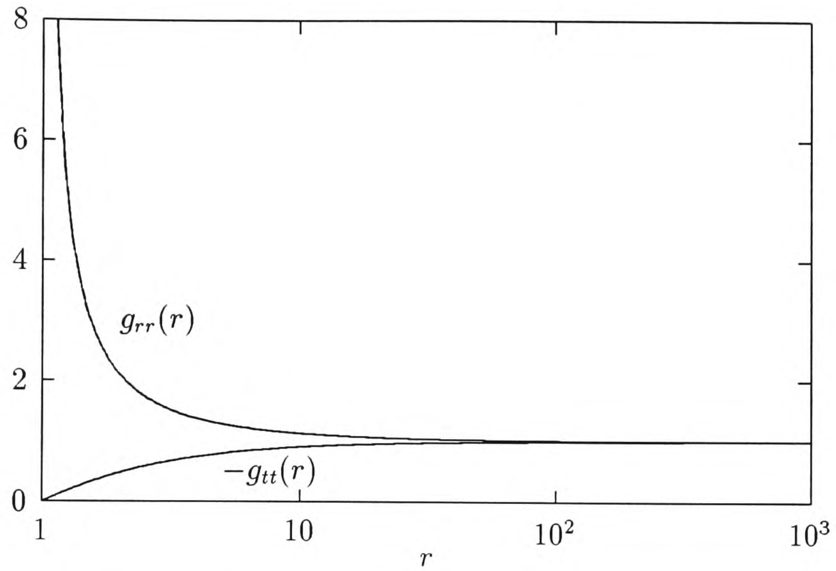

Figure 5.2: Components $-g_{tt}$ and g_{rr} of the metric tensor for the black hole solutions.

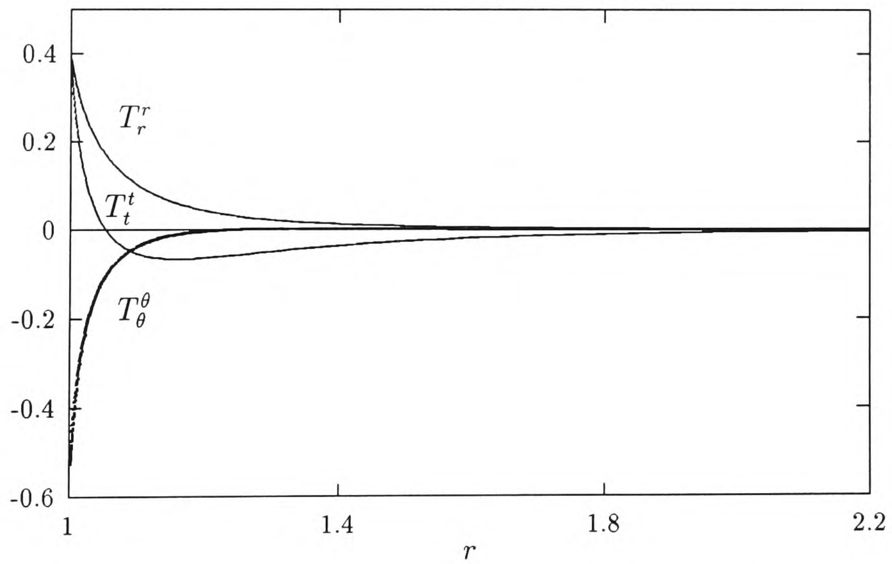

Figure 5.3: Components of the energy-momentum tensor for the black hole solutions.

Φ_h	Φ_∞	r_h	$2M$	D
−0.90	−1.59656	2.22172	2.40514	0.686968
−1.00	−1.62901	2.25806	2.41212	0.655905
−1.50	−1.88561	2.56717	2.63528	0.500778
−2.00	−2.23788	3.06161	3.09293	0.382030
−3.00	−3.08963	4.68710	4.69399	0.226401
−5.00	−5.01232	12.2578	12.2581	0.082250
−10.0	−10.0001	148.421	148.422	0.005937

Table 5.1: Values of the field parameters for the black hole solutions

In order to fix a unique scale on the solutions, we can impose the asymptotic condition $\Phi_\infty = 0$ (cf. (5.25)) on the dilaton field. This requires transforming the variables:

$$\Phi \to \Phi - \Phi_\infty, \qquad r \to r e^{-\frac{\Phi_\infty}{2}} \tag{5.54}$$

and results in a rescaling of the parameters M and D:

$$M \to M e^{-\frac{\Phi_\infty}{2}}, \qquad D \to D e^{-\frac{\Phi_\infty}{2}}. \tag{5.55}$$

The corresponding values Φ_h, Φ_∞ and the rescaled r_h, M and D are shown in table 5.1.

From above, we have a one-parameter family of black hole solutions, indexed by Φ_h. Hence, the two parameters M and D which characterize the solution at infinity (5.25) must be related in some way. The field equations (5.15-5.18) can be rearranged to give the following identity:

$$\frac{d}{dr}\left[r^2 e^{\frac{1}{2}(\Gamma-\Lambda)}(\Gamma' - \Phi') - \frac{\alpha'}{g^2} e^{\Phi} e^{\frac{1}{2}(\Gamma-\Lambda)}\left\{(1-e^{-\Lambda})(\Phi' - \Gamma') + e^{-\Lambda} r \Phi' \Gamma'\right\}\right] = 0. \tag{5.56}$$

Integrating this relation over (r_h, ∞) yields the equation

$$2M - D = \sqrt{\Gamma_1 \Lambda_1}\left(r_h^2 + \frac{\alpha'}{g^2} e^{\Phi_h}\right) \tag{5.57}$$

which relates the parameters describing the solution near the horizon and those describing it at infinity. This relationship is rather more complex than the relation $M = D$ found for the Einstein-Yang-Mills-Dilaton (EYMD) system [103]. In order to elucidate the precise relation between M and D, it is necessary to take into account the $O(\alpha'^2)$ expression for the dilaton charge [98]. The result is [6]

$$D = \frac{\alpha'}{g^2}\frac{e^{\Phi_\infty}}{2M} + \frac{\alpha'^2}{g^4}\frac{73 e^{2\Phi_\infty}}{60(2M)^3}. \tag{5.58}$$

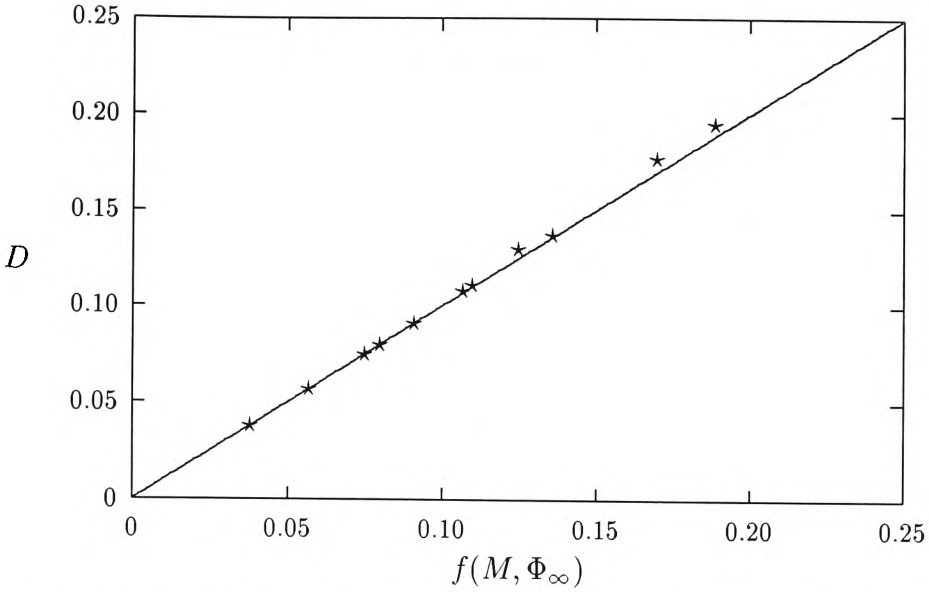

Figure 5.4: Relation between the dilaton charge D and M and Φ_∞ for the black hole solutions. The function $f(M, \Phi_\infty)$ stands for the right hand side of equation (5.58).

Figure 5.4 shows a numerical checking of this relation for the solutions found here, showing good agreement with the expression (5.58). Small deviations from (5.58) are expected due to higher order corrections. Relation (5.58) implies that the dilaton hair possessed by the black holes is of secondary type [10] (see also section 1.4), since the graviton field (which constitutes primary hair) is acting as a source for the dilaton field.

5.4 Additional solutions

We conclude this chapter by discussing briefly two further classes of solutions of the system of equations (5.15–5.18), which are generated by easing the restriction that Φ and its derivative Φ' remain finite everywhere.

The first class arises when we allow $\Phi'(r)$ to become infinite at a finite value r_i of r. This will turn out (see figure 5.7) not to be incompatible with finiteness of the energy-momentum tensor. The solutions have the same behaviour at infinity as the black hole solutions (see section 5.2.3), and for $r \sim r_i$ the functions have the expansions

$$\begin{aligned} e^{-\Lambda(r)} &= \Lambda_i(r - r_i) + \dots \\ \Gamma'(r) &= \frac{\Gamma_i}{\sqrt{r - r_i}} + \dots \\ \Phi(r) &= \Phi_i + \Phi_i'\sqrt{r - r_i} + \dots . \end{aligned} \tag{5.59}$$

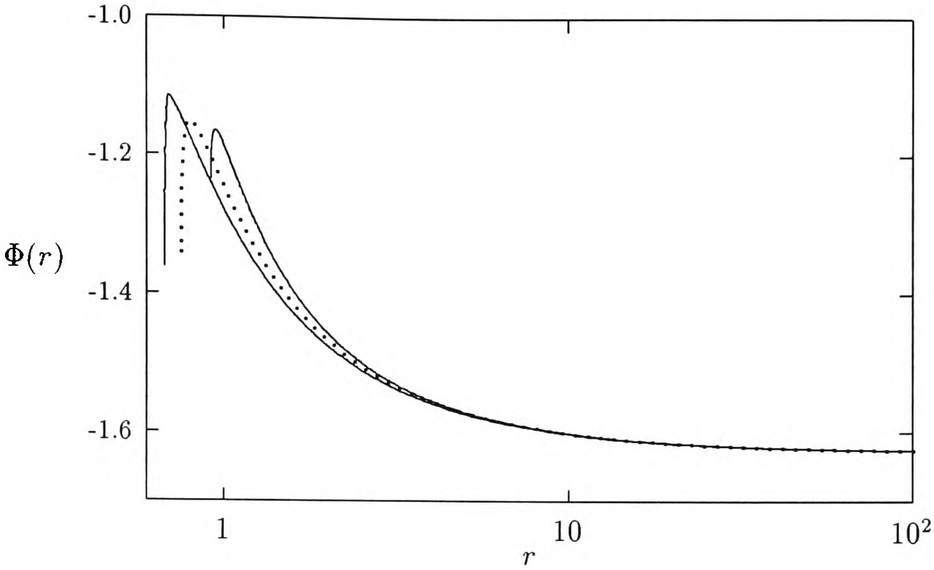

Figure 5.5: Dilaton field for the singular solution. Each curve corresponds to a different solution, with $r_i = (0.92, 0.75, 0.62)$ respectively.

The equations of motion (5.15-5.18) yield the following constraints:

$$\frac{\alpha'}{4g^2} e^{\Phi_i} \Phi_i' \Gamma_i = \frac{1}{\Lambda_i} + \frac{\Phi_i'^2 r_i^2}{16} \tag{5.60}$$

and

$$b_1 \Gamma_i^2 + b_2 \Gamma_i + b_3 = 0 \tag{5.61}$$

where

$$\begin{aligned} b_1 &= 16 e^{2\Phi_i} \left[2e^{\Phi_i} \Phi_i'^2 + 8 r_i + \Phi_i'^2 r_i^2 \right] \\ b_2 &= 4 e^{\Phi_i} \left[e^{2\Phi_i} \Phi_i'^3 - 4 e^{\Phi_i} \Phi_i'^3 r_i^2 - 16 \Phi_i' r_i^3 - 2 \Phi_i'^3 r_i^4 \right] \\ b_3 &= -8 e^{2\Phi_i} \Phi_i'^2 r_i - e^{2\Phi_i} \Phi_i'^4 r_i^2 + 2 e^{\Phi_i} \Phi_i'^4 r_i^4 + 8 \Phi_i'^2 r_i^5 + \Phi_i'^6. \end{aligned} \tag{5.62}$$

These constraints mean that the solutions form a two-parameter family, the behaviour of the dilaton field and metric components being shown in figures 5.5 and 5.6 respectively. It can be seen that these solutions do not represent black holes since g_{tt} is regular. In fact, both the scalar curvature R and the invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ are regular at $r = r_i$ [6], suggesting that the divergence in g_{rr} is merely a co-ordinate singularity. This is confirmed by the fact that the action (5.3) is also finite [6], as are the components of the stress tensor (figure 5.7).

The second class of solutions are regular everywhere and do not possess a horizon. However, the dilaton becomes infinite as $r \sim 0$. The behaviour of the field variables

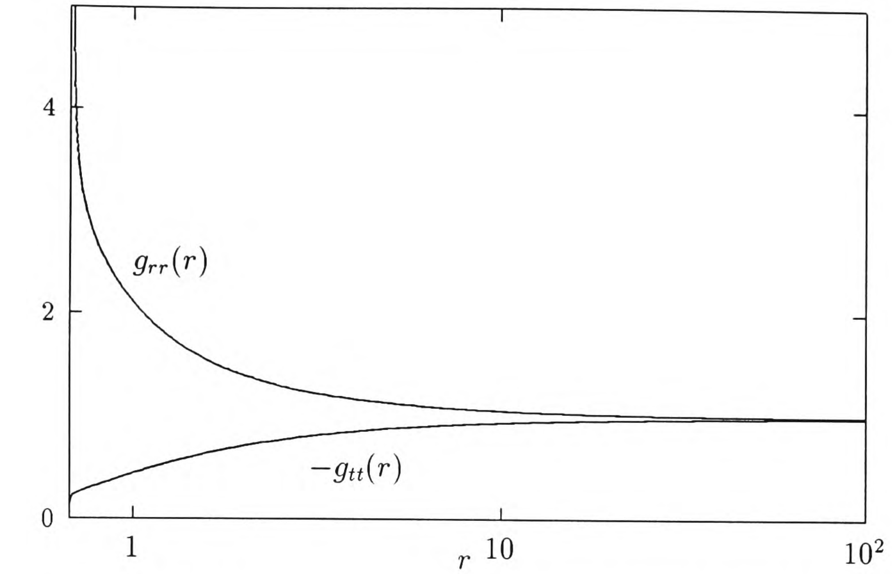

Figure 5.6: Metric components for the solution with $r_i = 0.68$.

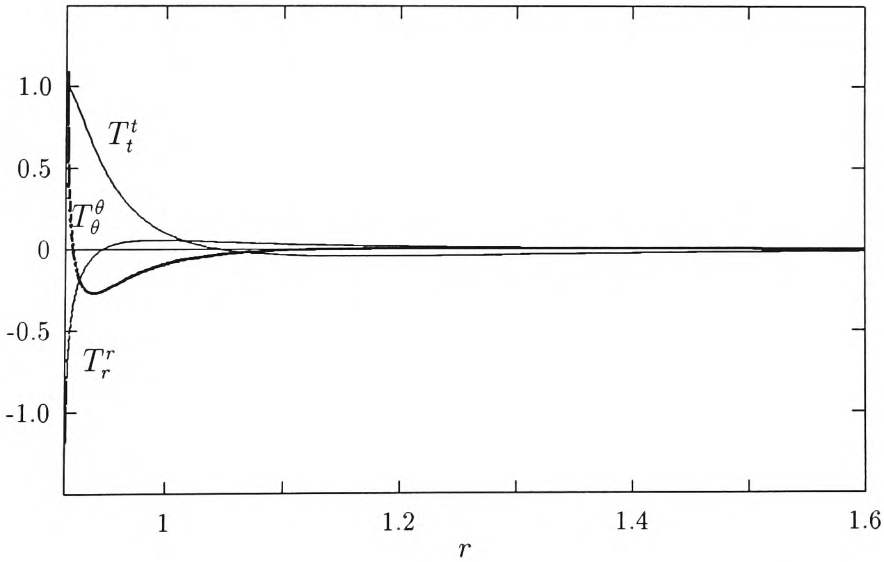

Figure 5.7: Components of the stress tensor for the solution with $r_i = 0.92$.

for $r \sim 0$ is

$$
\begin{aligned}
e^{\Lambda(r)} &= 1 + \Lambda_s e^{-\frac{4\Gamma_s}{r}} \\
\Gamma'(r) &= \Gamma_s e^{-\frac{\Phi_s}{r}} \\
\Phi(r) &= \frac{\Phi_s}{r}
\end{aligned}
\tag{5.63}
$$

giving a two-parameter family of solutions. The behaviour at infinity is the same as in section 5.2.3, being characterized by Φ_∞, M and D. These solutions do not appear to have any curvature singularities ($R \sim 0$ as $r \sim 0$), the components of the stress tensor becoming infinite as the origin is approached [6].

Finally, it is interesting to note that the black hole solutions of section 5.3 form a boundary surface in phase space, between the two classes of solutions discussed in this section. In other words, if Φ_0, Φ_0', Γ_0 and Γ_0' are the values of the field variables for the black hole solution with $r = r_0 \gg 1$, then integrating the equations (B.6) from r_0 with $\Phi'(r_0) > \Phi_0'$ leads to the solution (5.59), whilst if $\Phi'(r_0) < \Phi_0'$ a solution of the form (5.63) is constructed.

Summary

This chapter discusses the mechanism whereby black hole solutions of dilaton gravity with a Gauss-Bonnet term evade the "no-hair" theorem [6]. The mechanism is similar to that for the EYMH system (section 2.3). We also describe briefly the properties of the solutions, which may be important for absorbing logarithmic divergences in the quantum-mechanical entropy ([75], see also section 4.3.3, where this question is addressed in more detail).

Chapter 6

Quantum Field Theory in Kerr Space-time

Kerr space-time describes a black hole which does not possess any "hair" since it is surrounded by a vacuum. However, the quantum field theory of a field on this background contains many subtleties and is of great interest. We begin this chapter by motivating the physical approach we shall be taking, and then concentrate on the properties of the Hartle-Hawking vacuum state, which describes a black hole in thermal equilibrium with a heat bath at the Hawking temperature [7, 8]. We shall observe that the introduction of rotation produces some subtleties in the analysis of this state, so that the picture is rather more complicated than is the case in non-rotating space-times (see chapter 4).

6.1 The fundamental problem

As a first approximation to a full theory of quantum gravity one may consider the semi-classical Einstein equations

$$G_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle \tag{6.1}$$

where the right-hand-side is the expectation value of the stress-energy tensor of quantum fields, considered as perturbations of the fixed background geometry. Solution of (6.1) would give the effect of this quantum perturbation on the classical geometry, and is known as the *back reaction problem.* Calculation of the right-hand-side of (6.1) is therefore the first step in the solution of this problem, and has been the focus of research in quantum field theory in curved space. (See [90] for a review of the basic ideas of quantum field theory in curved space starting with the canonical quantization procedure, [104] for a more mathematical approach based on distributions and [105] for an introduction to the algebraic approach.)

However, such a calculation is beset with difficulties. In the canonical quantization approach, one takes the classical expression for the energy-momentum tensor, for example, for a mass-less scalar field:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(\partial_\mu\phi)^2 \tag{6.2}$$

and considers the fields as quantum-mechanical operators,

$$\phi(x) = \sum_{\text{modes}} u_k a_k + u_k^* a_k^\dagger \tag{6.3}$$

where the u_k are an orthonormal basis of modes of the field equation, and a_k and $a_k^\dagger$ are respectively the annihilation and creation operators for each mode, satisfying the canonical commutation relations

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}. \tag{6.4}$$

The quantum scalar field can be thought of as an infinite collection of simple harmonic oscillators, each oscillator mode having the form

$$u \propto e^{-iwt}. \tag{6.5}$$

We then obtain, for the Hamiltonian, in Minkowski space,

$$H_\phi \equiv \int T_{tt}\, d^3x = \frac{1}{2}\sum_{\text{modes}} \left(a_k^\dagger a_k + a_k a_k^\dagger\right) w. \tag{6.6}$$

This leads to divergences because there is a creation operator to the right of an annihilation operator. The vacuum expectation value of the energy of the quantum field is the sum

$$\langle 0|H_\phi|0\rangle = \frac{1}{2}\sum_{\text{modes}} w\, \delta^3(0) \tag{6.7}$$

which is infinite since w can be arbitrarily large. In Minkowski space this divergence can be removed consistently by the process of normal ordering, namely by writing each creation operator to the left of each annihilation operator. This is equivalent to subtracting an infinite constant (the vacuum expectation value of the stress tensor) from all the expectation values, thus rendering them finite.

Unfortunately, in curved space the situation is not so simple. There are two reasons for this. Firstly, in a general curved space-time there is no natural global definition of a vacuum state. In Minkowski space there is a natural set of basis modes

$$u_k = \frac{1}{(2w(2\pi)^3)^{\frac{1}{2}}} e^{i\mathbf{k}.\mathbf{x} - iwt}, \qquad w > 0 \tag{6.8}$$

which have positive frequency with respect to the Killing vector $\frac{\partial}{\partial t}$. The vacuum defined from this basis is then invariant under the action of the Poincaré group. In curved space, Poincaré invariance is broken and in general there will be no Killing vectors or "natural" co-ordinates with which to define positive frequency modes.

Even in black hole space-times, which possess a time-like Killing vector $\frac{\partial}{\partial t}$, there remains an inherent ambiguity in the choice of vacuum state. This is discussed in appendix C.

Secondly, the semi-classical Einstein equations (6.1) imply that absolute values of energy-momentum are important, whereas in non-gravitational physics, it is only differences in stress-energy that are of physical significance. It is therefore necessary to employ some kind of regularization and renormalization scheme to subtract the divergences in $\langle T_{\mu\nu} \rangle$ in a reasonable and consistent manner.

There are two approaches to this problem, which are compared in [106]. The standard method uses Green's functions defined at two separated points in space-time. Christensen [107, 108] calculated the appropriate terms which must be subtracted, before the two points are brought together, to give a finite answer. The alternative, axiomatic approach [109], is to specify the criteria that we wish $\langle T_{\mu\nu} \rangle_{ren}$ to satisfy, and construct a renormalization scheme which produces answers satisfying these criteria. The answers obtained by each of these two methods are identical [106].

Even with all this machinery in place, actually calculating $\langle T_{\mu\nu} \rangle_{ren}$ remains a long and complex process. Detailed calculations have been performed on the Schwarzschild geometry for the scalar [110], electro-magnetic [111] and graviton [112] fields, although the computational ingenuity required should not be underestimated. This is particularly true in Kerr space-time, where the extra complexity of the geometry is reflected in the stress tensor. A great deal of work was required to calculate $\langle T_{\mu\nu} \rangle_{ren}$ at a single point in this space-time [113] (corrected in [114]).

Therefore any information about the stress energy tensor which can be obtained from means other than direct computation is of great value, and should be a prerequisite for any future calculation. In this chapter we shall consider the Hartle-Hawking vacuum state, and work using this approach. Several, basic physical principles will be employed, of which the most notable is the conservation equations. Details on the definition of the Hartle-Hawking vacuum (and the other standard vacua) can be found in appendix C.

6.2 The Hartle-Hawking state

6.2.1 The importance of this state

The existing literature on the stress tensor in Kerr space-time concentrates largely on the Hartle-Hawking state (see, for example, [113, 115, 116]). From a calculational point of view, this is because $|H\rangle$ is the state which reflects most of the symmetries of the geometry, so that, for example, at the pole of the event horizon, one component of $\langle T_{\mu\nu} \rangle_{ren}$ is sufficient to determine the stress tensor [113]. The physical importance of this state has already been stressed in chapter 4, since it represents a black hole in thermal equilibrium with a heat bath at the Hawking temperature. The reason for studying the Hartle-Hawking state in Kerr geometry is to elucidate the effect of rotation without the extra effects of hair, which are expected to be similar (qualitatively at least) to those observed in spherically symmetric geometries in chapter 4.

There is an important theorem in algebraic quantum field theory [117] which states that a naive definition of $|H\rangle$ produces a state which does not exist in Kerr space-time. In this chapter we shall discuss the physical properties of the state $|H\rangle$ [115], the sense in which it is pathological, and how this difficulty might be resolved to yield a state which has all the desirable qualities mentioned above.

The standard construction of the state using mode expansions is reviewed in appendix C, which should be compared with the theory in non-rotating space-times (chapter 4). Here we are taking a more physical approach, considering the properties of the expectation value $\langle H|T_{\mu\nu}|H\rangle$ rather than how this might be calculated in practice.

6.2.2 Physical properties of the state

Here we consider the Hartle-Hawking state as seen by various observers in the Kerr space-time [115], and from this motivate the key property which is central to the analysis of this chapter.

The analogue for Kerr space-time of the static observers of Schwarzschild space-time are ZAMOs (zero angular momentum observers [115]) who rotate about the black hole at fixed r and θ with angular velocity

$$\frac{d\varphi}{dt} = \Omega_Z = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}. \tag{6.9}$$

Writing the Kerr metric in the form

$$ds^2 = -\Upsilon^2 dt^2 + \Sigma\Delta^{-1} dr^2 + \Sigma\, d\theta + \tilde{\Omega}_Z^2 (d\varphi - \Omega_Z dt)^2 \tag{6.10}$$

it can be seen that the orthonormal frames carried by the ZAMOs can be taken to have the following basis 1-forms:

$$\begin{aligned}
e_{(t)\mu} dx^\mu &= \Upsilon dt \\
e_{(r)\mu} dx^\mu &= (\Sigma\Delta^{-1})^{\frac{1}{2}} dr \\
e_{(\theta)\mu} dx^\mu &= \Sigma^{\frac{1}{2}} d\theta \\
e_{(\varphi)\mu} dx^\mu &= \tilde{\Omega}_Z (d\varphi - \Omega_Z dt).
\end{aligned} \tag{6.11}$$

The proper time τ_Z of a ZAMO is related to the time co-ordinate t by

$$\Upsilon \frac{\partial}{\partial \tau_Z} = \frac{\partial}{\partial t} + \Omega_Z \frac{\partial}{\partial \varphi} \tag{6.12}$$

where Υ is the lapse function, which vanishes at the horizon and tends to unity at infinity.

A particle detector measures states having positive frequency with respect to the detector's proper time, rather than with respect to some universal time co-ordinate [118]. Particle detectors carried by ZAMOs measure "Boulware particles" relative to the Boulware vacuum $|B\rangle$ [115]. From these measurements, ZAMOs can compute an expectation value of the stress-energy tensor $\langle T_{\mu\nu}\rangle_{ZAMO}$. Then, if the field is in a state $|\Psi\rangle$, we have [115]

$$\begin{aligned}\langle T_{\mu\nu}\rangle_{ZAMO} &= \langle\Psi|T_{\mu\nu}|\Psi\rangle_{ren} - \langle B|T_{\mu\nu}|B\rangle_{ren} \\ &= \langle\Psi|T_{\mu\nu}|\Psi\rangle - \langle B|T_{\mu\nu}|B\rangle,\end{aligned} \tag{6.13}$$

since the divergences subtracted in the renormalization process are the same for all states.

The crux of [115], is that when the field is in the state $|\Psi\rangle = |H\rangle$, then

$$\begin{aligned}\langle T_{\mu\nu}\rangle_{ZAMO} &= \langle H|T_{\mu\nu}|H\rangle - \langle B|T_{\mu\nu}|B\rangle \\ &= T^{th}_{\mu\nu},\end{aligned} \tag{6.14}$$

the stress tensor of a perfectly thermal atmosphere surrounding the black hole. In Schwarzschild space-time, the thermal nature of $|H\rangle$ can be easily interpreted [119] (see also chapter 4), as follows. For any observable $\mathcal{O}$, the expectation value in the Hartle-Hawking state is given by

$$\langle H|\mathcal{O}|H\rangle = \mathrm{Tr}(\rho\mathcal{O}) \tag{6.15}$$

where ρ is a density matrix describing black body radiation (see chapter 4 for the properties of this object in static geometries). This does not carry straight over to the rotating case, because the super-radiant modes render the density matrix ρ un-renormalizable. However, this difficulty can be circumvented [115], yielding a precisely thermal density matrix. In other words, for a field mode (cf. section C.1.1)

$$u_{wlm_l} = \frac{\mathtt{N(k)}}{(r^2+a^2)^{\frac{1}{2}}} e^{-iwt+im_l\varphi} S_{wlm_l}(\cos\theta) R_{wlm_l}(r) \tag{6.16}$$

the probability that the field contains n ZAMO-measured quanta is

$$p_n = p_n^{th} = (1 - e^{-\frac{\tilde{w}}{T}}) e^{-\frac{n\tilde{w}}{T}} \tag{6.17}$$

for $n \geq 0$, where $\tilde{w} = w - m_l\Omega_H > 0$ and $T = \frac{\kappa}{2\pi}$ is the Hawking temperature. The distribution (6.17) is precisely thermal.

Now consider an observer O_{ϖ} orbiting the black hole at fixed r and θ and with an angular velocity $\varpi(r,\theta)$, so that the observer resides at constant r, θ and $\bar{\varphi} = \varphi - \varpi t$. The Kerr metric can be written in the form

$$\begin{aligned}ds^2 &= -\Upsilon^2\gamma^{-2}\left[dt - \tilde{\Omega}_Z^2(\varpi - \Omega_Z)\Upsilon^{-2}\gamma^2(d\varphi - \varpi\,dt)\right]^2 \\ &\quad +\tilde{\Omega}_Z^2\gamma^2\left[d\varphi - \varpi dt\right]^2 + \Sigma\Delta^{-1}dr^2 + \Sigma\,d\theta\end{aligned} \tag{6.18}$$

where

$$\gamma = (1 - \nu^2)^{-\frac{1}{2}}, \tag{6.19}$$

with

$$\nu = \Upsilon^{-1}(\varpi - \Omega_Z)\tilde{\Omega}_Z, \tag{6.20}$$

is the Lorentz factor associated with the observer's angular velocity relative to the ZAMO at the same values of r and θ. Hence the orthonormal frame carried by the observer can be taken to have the basis 1-forms

$$\begin{aligned} e_{(t)\mu}dx^\mu &= \Upsilon\gamma^{-1}\left[dt - \tilde{\Omega}_Z^2(\varpi - \Omega_Z)\Upsilon^{-2}\gamma^2(d\varphi - \varpi\, dt)\right] \\ e_{(r)\mu}dx^\mu &= (\Sigma\Delta^{-1})^{\frac{1}{2}}dr \\ e_{(\theta)\mu}dx^\mu &= \Sigma^{\frac{1}{2}}d\theta \\ e_{(\varphi)\mu}dx^\mu &= \tilde{\Omega}_Z\gamma\,(d\varphi - \varpi\, dt)\,. \end{aligned} \tag{6.21}$$

The proper time τ_ϖ of the observer is related to the co-ordinate time t by

$$\begin{aligned} \Upsilon\gamma^{-1}\frac{\partial}{\partial\tau_\varpi} &= \frac{\partial}{\partial t} + \varpi\frac{\partial}{\partial\varphi} \\ &= \Upsilon\frac{\partial}{\partial\tau_Z} + (\varpi - \Omega_Z)\frac{\partial}{\partial\varphi} \end{aligned} \tag{6.22}$$

where τ_Z is the proper time of a ZAMO residing at the same r and θ co-ordinates as the observer.

If a particle in the mode

$$u_{wlm_l} \propto e^{-iwt+im_l\varphi} \tag{6.23}$$

has energy E_ϖ as measured by the observer, then

$$E_\varpi u_{wlm_l} = \frac{\partial}{\partial\tau_\varpi}u_{wlm_l} = \gamma\Upsilon^{-1}(w - m_l\varpi)u_{wlm_l}. \tag{6.24}$$

Hence the probability distribution (6.17) depends on the angular momentum of the quanta as measured by O_ϖ as well as the energy E_ϖ unless

$$\varpi = \Omega_H, \tag{6.25}$$

so that O_ϖ co-rotates with the event horizon. In this case,

$$p_n = \left[1 - \exp\left(\frac{E_\varpi}{T_\varpi}\right)\right]\exp\left(\frac{nE_\varpi}{T_\varpi}\right) \tag{6.26}$$

where

$$T_\varpi = \frac{\gamma}{\Upsilon}T \tag{6.27}$$

and

$$\gamma = (1 - \nu^2)^{-\frac{1}{2}}, \qquad \nu = \Upsilon^{-1}(\Omega_H - \Omega_Z)\tilde{\Omega}_Z. \tag{6.28}$$

Thus we expect that in the reference frame of an observer rotating rigidly with the horizon, the state $|H\rangle$ has an isotropic, perfectly thermal, distribution of quanta. In other words, we have the following:

Key Property 6.1 *The expectation value*

$$\langle H|T_{\mu\nu}|H\rangle - \langle B|T_{\mu\nu}|B\rangle \tag{6.29}$$

is isotropic in the reference frame of the observer O_{Ω_H}.

We shall discuss in section 6.4 to what extent this property is believed to hold, and it shall remain an important assumption in this chapter.

This key property suggests that $|H\rangle$ will exhibit pathological behaviour at the velocity of light surface $\mathcal{S}_L$. This is the surface where, in order to co-rotate with the horizon, O_{Ω_H} must move at the velocity of light. Outside $\mathcal{S}_L$, the observer O_{Ω_H} moves on a space-like world-line. The next section will show that this divergence does in fact occur.

6.3 Pathological behaviour

We shall now prove that the key property of the previous section, combined with the conservation equations, implies that the state $|H\rangle$ is pathological at the velocity of light surface.

We shall use the notation

$$\begin{aligned} \langle T_{\mu\nu}\rangle_{H-B} &= \langle H|T_{\mu\nu}|H\rangle_{ren} - \langle B|T_{\mu\nu}|B\rangle_{ren} \\ &= \langle H|T_{\mu\nu}|H\rangle - \langle B|T_{\mu\nu}|B\rangle. \end{aligned} \tag{6.30}$$

Proposition 6.2 *The expectation value* $\langle T_{\mu\nu}\rangle_{H-B}$ *is divergent at the velocity of light surface.*

Proof From our key property, $\langle T_{\mu\nu}\rangle_{H-B}$ is isotropic in the orthonormal frame carried by O_{Ω_H} i.e.

$$\langle T^{(b)}_{(a)}\rangle_{H-B} = \varepsilon(r,\theta)\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.31}$$

where bracketed indices denote components with respect to the orthonormal tetrad whose basis 1-forms are given by

$$\begin{aligned} e_{(t)\mu}dx^\mu &= \Upsilon\gamma^{-1}\left[dt - \tilde{\Omega}_Z^2(\Omega_H - \Omega_Z)\Upsilon^{-2}\gamma^2(d\varphi - \Omega_H dt)\right] \\ e_{(r)\mu}dx^\mu &= \left(\Sigma\Delta^{-1}\right)^{\frac{1}{2}} dr \\ e_{(\theta)\mu}dx^\mu &= \Sigma^{\frac{1}{2}} d\theta \\ e_{(\varphi)\mu}dx^\mu &= \tilde{\Omega}_Z\gamma(d\varphi - \Omega_H dt) \end{aligned} \tag{6.32}$$

and

$$\begin{aligned} \gamma &= (1-\nu^2)^{-\frac{1}{2}}, \\ \nu &= \Upsilon^{-1}(\Omega_H - \Omega_Z)\tilde{\Omega}_Z. \end{aligned} \tag{6.33}$$

The tensor (6.31) is traceless because it is the difference in expectation values for two states, and the trace anomaly is the same for all states [107, 108]. The function $\varepsilon(r,\theta)$ depends only on the position of the observer O_{Ω_H} in Boyer-Lindquist co-ordinates.

Tetrad indices are raised and lowered using the constant symmetric matrices (see [40] for a review of the tetrad formalism)

$$\eta_{(a)(b)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.34}$$

and

$$\eta^{(a)(b)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.35}$$

so that

$$\eta^{(a)(b)}\eta_{(b)(c)} = \delta^{(a)}_{(c)} \tag{6.36}$$

and

$$T^{(a)(b)} = \eta^{(b)(c)}T^{(a)}_{(c)} \tag{6.37}$$

etc. The tetrad components may be converted to co-ordinate components via

$$T_{\mu\nu} = e_{(a)\mu}e_{(b)\nu}T^{(a)(b)} \tag{6.38}$$

where the $e_{(a)\mu}$ are the 1-forms given above (6.32). This operation is the inverse of projecting the tensor field on to the tetrad frame. It is possible to write the conservation equations in the tetrad formalism, but solving them would require the calculation of the Ricci rotation coefficients $\gamma_{(a)(b)(c)}$ given by

$$\gamma_{(a)(b)(c)} = e^{\mu}_{(a)}e_{(b)\mu;\nu}e^{\nu}_{(c)}. \tag{6.39}$$

It is more straightforward to convert the tetrad components back to Boyer-Lindquist components and then solve the form of the conservation equations

$$\partial_\mu\left(\langle T^{\mu}_{\nu}\rangle_{H-B}\sqrt{-g}\right) = \frac{1}{2}\sqrt{-g}\left(\partial_\mu g_{\alpha\beta}\right)\langle T^{\alpha\beta}\rangle_{H-B} \tag{6.40}$$

in those co-ordinates.

The covariant components of the stress-energy tensor are now given by

$$\begin{aligned}
\langle T_{tt}\rangle_{H-B} &= \left[3\Upsilon^2\gamma^{-2}(1+\Omega_H\vartheta)^2 + \Omega_H^2\tilde{\Omega}_Z^2\gamma^2\right]\varepsilon(r,\theta) \\
\langle T_{t\varphi}\rangle_{H-B} &= \left[-3\Upsilon^2\gamma^{-2}\vartheta(1+\Omega_H\vartheta) - \Omega_H\tilde{\Omega}_Z^2\gamma^2\right]\varepsilon(r,\theta) \\
\langle T_{\varphi\varphi}\rangle_{H-B} &= \left[3\Upsilon^2\gamma^2\vartheta^2 + \tilde{\Omega}_Z^2\gamma^2\right]\varepsilon(r,\theta) \\
\langle T_{rr}\rangle_{H-B} &= \Sigma\Delta^{-1}\varepsilon(r,\theta) \\
\langle T_{\theta\theta}\rangle_{H-B} &= \Sigma\varepsilon(r,\theta)
\end{aligned} \tag{6.41}$$

where

$$\vartheta = \tilde{\Omega}_Z^2(\Omega_H - \Omega_Z)\Upsilon^{-2}\gamma^2, \tag{6.42}$$

with the other components vanishing. Hence the non-zero contravariant components are

$$\begin{aligned}
\langle T^{tt}\rangle_{H-B} &= \Upsilon^{-2}(4\gamma^2 - 1)\varepsilon(r,\theta) \\
\langle T^{t\varphi}\rangle_{H-B} &= \Upsilon^{-2}(4\gamma^2\Omega_H - \Omega_Z)\varepsilon(r,\theta) \\
\langle T^{\varphi\varphi}\rangle_{H-B} &= \left[\Upsilon^{-2}(4\gamma^2\Omega_H^2 - \Omega_Z^2) + \tilde{\Omega}_Z^2\right]\varepsilon(r,\theta) \\
\langle T^{rr}\rangle_{H-B} &= \Delta\Sigma^{-1}\varepsilon(r,\theta) \\
\langle T^{\theta\theta}\rangle_{H-B} &= \Sigma^{-1}\varepsilon(r,\theta).
\end{aligned} \tag{6.43}$$

The $\mu = t$ and $\mu = \varphi$ conservation equations are trivial, and the $\mu = r$ and $\mu = \theta$ equations give, respectively,

$$\begin{aligned}
\partial_r\left(\varepsilon(r,\theta)\Sigma\sin\theta\right) &= \frac{1}{2}\varepsilon(r,\theta)\Sigma\sin\theta\,\partial_r\left(\log|\Upsilon^{-8}\gamma^8\Sigma^2\sin^2\theta|\right) \\
\partial_\theta\left(\varepsilon(r,\theta)\Sigma\sin\theta\right) &= \frac{1}{2}\varepsilon(r,\theta)\Sigma\sin\theta\,\partial_\theta\left(\log|\Upsilon^{-8}\gamma^8\Sigma^2\sin^2\theta|\right).
\end{aligned} \tag{6.44}$$

These two equations are compatible and determine $\varepsilon(r,\theta)$ up to an arbitrary constant. The result is

$$\varepsilon(r,\theta) = \Xi\Upsilon^{-4}\gamma^4 \tag{6.45}$$

where Ξ is an arbitrary constant. This implies that:

1. The components of $\langle T_{\mu\nu}\rangle_{H-B}$ diverge as Δ^{-2} as the event horizon is approached.

2. The components of $\langle T_{\mu\nu}\rangle_{H-B}$ diverge as γ^4 as the velocity of light surface $\mathcal{S}_L$ is approached. (Note that $\gamma \to \infty$ at $\mathcal{S}_L$.)

□

Conclusion (2) implies that if $\langle B|T_{\mu\nu}|B\rangle_{ren}$ is regular on $\mathcal{S}_L$, then $\langle H|T_{\mu\nu}|H\rangle_{ren}$ diverges there. This is certainly true for slowly rotating black holes at least, when $\mathcal{S}_L$ is far from the horizon and $\langle B|T_{\mu\nu}|B\rangle_{ren}$ has the form (C.38).

Result 6.3 *For slowly rotating black holes at least, the Hartle-Hawking state is pathological at the velocity of light surface.*

6.4 Discussion of result 6.3

At this point there are a number of issues concerning the proof of proposition 6.2 which require elaboration.

6.4.1 Validity of the key property

Firstly, we should consider the appropriateness of the key property, which is an unproved assumption in this chapter. The corresponding property in Schwarzschild space-time, that $\langle T_{\mu\nu}\rangle_{H-B}$ is isotropic in the frame of static observers, was hypothesized in [120] but later shown not to be true in [121] for a mass-less scalar field. However, as may be seen from the numerical calculations of [121], the deviation from isotropy is in fact small. Hence our assumption of exact isotropy is a reasonable first approximation (at least for the spin 0 case) which, it is hoped, will be vindicated by future numerical calculations. Higher spin particles are in some sense directed objects, and so are expected to deviate further from isotropy than the spin-less case.

6.4.2 Behaviour at the event horizon

Conclusion (1) stated that the components of $\langle T_{\mu\nu}\rangle_{H-B}$ diverge as Δ^{-2} as the event horizon is approached. This is the behaviour found in [122] for the electro-magnetic case. There, the authors also found that the tensor was isotropic in the Carter orthonormal tetrad [123], given by

$$\begin{aligned}
e^{C}_{(t)\mu}dx^{\mu} &= \Sigma^{-\frac{1}{2}}\Delta^{\frac{1}{2}}\left(dt - a\sin^2\theta\, d\varphi\right) \\
e^{C}_{(r)\mu}dx^{\mu} &= \Sigma^{\frac{1}{2}}\Delta^{-\frac{1}{2}}dr \\
e^{C}_{(\theta)\mu}dx^{\mu} &= \Sigma^{\frac{1}{2}}d\theta \\
e^{C}_{(\varphi)\mu}dx^{\mu} &= \Sigma^{-\frac{1}{2}}\sin\theta\left[(r^2+a^2)d\varphi - a\,dt\right].
\end{aligned} \tag{6.46}$$

This is true only near the event horizon, where the Carter orthonormal tetrad (and also the ZAMO orthonormal tetrad (6.11)) coincide with the orthonormal tetrad constructed by O_{Ω_H}. Near the horizon, an observer at rest in the Carter tetrad has the same angular velocity as O_{Ω_H} and a ZAMO at the same (r,θ). However, this is not true away from the event horizon. Thus $\langle T_{\mu\nu}\rangle_{H-B}$ remains isotropic in the O_{Ω_H} tetrad, but will not be isotropic with respect to the other two tetrads.

Assuming that $|H\rangle$ is regular at the event horizons, then conclusion (1) gives us the divergent behaviour of the Boulware vacuum as the horizon is approached. Putting $a=0$, the ZAMOs and O_{Ω_H} now coincide, so that $\Omega_H = 0$, $\gamma = 1$, $\tilde{\Omega}_Z = r\sin\theta$, $\Omega_Z = 0$ and

$$\Upsilon^2 = \left(1 - \frac{2M}{r}\right). \tag{6.47}$$

In this case the stress tensor (6.41) becomes

$$\langle T_{\mu\nu}\rangle_{H-B} = \Xi\left(1-\frac{2M}{r}\right)^{-2}\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.48}$$

This reproduces the behaviour observed near the horizon of Schwarzschild space-time [119] for a massless conformally invariant field of spin 0, 1 or $\frac{1}{2}$, in the state $|B\rangle$, with

$$\Xi = -\frac{h(s)}{6\pi^2}\int_0^\infty \frac{w(w^2+s^2\kappa^2)}{e^{\frac{2\pi w}{\kappa}}-(-1)^{2s}}dw \tag{6.49}$$

where $h(s)$ is the number of helicity states of a field of spin s.

6.4.3 Behaviour outside $\mathcal{S}_L$

Unfortunately, our analysis does not tell us anything about the behaviour of $|H\rangle$ outside $\mathcal{S}_L$, since γ and hence the tetrad (6.32) are defined only inside $\mathcal{S}_L$. However, since our key property of $|H\rangle$ is that $\langle T_{\mu\nu}\rangle_{H-B}$ is isotropic with respect to the tetrad (6.32) and this tetrad is undefined outside $\mathcal{S}_L$, it seems to be the case that the state $|H\rangle$ cannot be defined outside $\mathcal{S}_L$, since the state $|B\rangle$ exists and is well defined outside $\mathcal{S}_L$. This is discussed from another point of view in the next section.

6.4.4 Comparison with known results

Result 6.3 is in complete agreement with the theorem of Kay and Wald [117], which states that there does not exist a generalized vacuum state on Kerr space-time which is invariant under the isometry generated by the Killing vector

$$\zeta = \frac{\partial}{\partial t} + \Omega_H\frac{\partial}{\partial\varphi} \tag{6.50}$$

(which is the null generator of the horizon) and is regular everywhere. The Hartle-Hawking state $|H\rangle$ satisfies the symmetry condition and is regular at the event horizons, and so the theorem leads us to expect some kind of singular behaviour somewhere outside the outer horizon.

The theorem [117] also holds for massive fields, and result 6.3 should apply to this case. However, numerical calculations [116] seem to suggest that in fact $\langle H|T_{\mu\nu}|H\rangle_{ren}$ is regular at $\mathcal{S}_L$ for massive fields. However, the approximation used in [116] is asymptotic for very large mass, in which case emission via the Hawking effect is negligible. Furthermore, the differences between the expectation values of the stress tensor in the Unruh, Boulware and Hartle-Hawking vacua are proportional to the factor

$$e^{-\frac{\mu}{T}} \tag{6.51}$$

(where μ is the mass of the quanta, and T is the Hawking temperature), except in regions where pathologies occur. In this regime the factor (6.51) is very small and so the calculations would not reveal a divergence at $\mathcal{S}_L$ unless $\langle H|T_{\mu\nu}|H\rangle_{ren}$ is calculated very close to $\mathcal{S}_L$.

In [115], the authors used an argument based on the form of the mode solutions to the conformal wave equation to show that, for a slowly rotating black hole at least, the renormalized stress energy tensor for the Hartle-Hawking vacuum state diverges as the velocity of light surface is approached, and remains singular everywhere outside it. The work in both [115] and [117] relies heavily on the existence of super-radiant modes, although divergences are expected for both fermions and bosons. Whilst fermions do not exhibit classical super-radiance phenomena, there is spontaneous emission of quanta via the Unruh-Starobinskii process [124]. It is this process which may well have a role to play in the pathological behaviour of $|H\rangle$. The advantage of our method of proof is that it does not rely on any detailed knowledge of the modes of the field, but only on the conservation of the stress tensor and the key property 6.1.

6.5 Construction of a well-behaved, thermal state

We now review the method of [125] and show how it can be exploited to modify the state $|H\rangle$ and produce a state $|H_M\rangle$ which has all the desirable physical properties of $|H\rangle$ (regularity at the event horizon and thermal nature) but which does not exhibit its pathological behaviour. This analysis will also shed light on why the state $|H\rangle$ is badly behaved at $\mathcal{S}_L$.

6.5.1 The Euclidean manifold

Following the method of section 4.1.3, the first step in constructing a thermal state is to consider the Euclidean manifold. We work in the Kruskal co-ordinates U_+, V_+, θ, φ_+ given by

$$U_+ = -e^{-\kappa u}, \qquad V_+ = e^{\kappa v}, \qquad \varphi_+ = \varphi - \Omega_H t \tag{6.52}$$

where

$$u = t - r^*, \qquad v = t + r^* \tag{6.53}$$

with r^* the usual "tortoise" co-ordinate (C.17) and

$$\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)} \tag{6.54}$$

is the surface gravity of the event horizon. These co-ordinates are regular everywhere outside the inner Cauchy horizon. Between the event horizon and the velocity of light surface $\mathcal{S}_L$, the Killing vector

$$\zeta = \left(\frac{\partial}{\partial t}\right)_{r,\theta,\varphi_+} \tag{6.55}$$

is time-like and coincides at the horizon with the null generators of the horizon. The geometry is invariant under the isometry generated by ζ, and the horizon and $\mathcal{S}_L$ are fixed points of ζ; that is, ζ vanishes on the horizon and $\mathcal{S}_L$.

Now introduce the co-ordinates

$$\begin{aligned} W &= \frac{1}{2}(V_+ + U_+) = e^{\kappa r^*} \sinh \kappa t \\ Z &= \frac{1}{2}(V_+ - U_+) = e^{\kappa r^*} \cosh \kappa t, \end{aligned} \tag{6.56}$$

in which case ζ becomes

$$\zeta = Z\frac{\partial}{\partial W} + W\frac{\partial}{\partial Z}. \tag{6.57}$$

The W, Z, θ, φ_+ co-ordinates are regular at and outside the event horizon, where $W = Z = 0$ and ζ vanishes.

Complexified Kerr space is constructed by allowing the co-ordinates W, Z, θ, φ_+ and the parameters a and M to be complex, with the space having the same metric [125]. The Euclidean Kerr black hole [125] is defined as the regular Euclidean section of complexified Kerr space, given by

$$t = -i\tau, \qquad a = ia_E \tag{6.58}$$

and τ, a_E, r, θ, φ_+, $M \in \mathbb{R}$ in Boyer-Lindquist co-ordinates, which is the same as introducing a real co-ordinate Y by

$$W = iY. \tag{6.59}$$

The metric in the Y, Z, θ, φ_+ co-ordinates is real and positive definite, and the Killing vector ζ now has the form

$$\zeta = Z\frac{\partial}{\partial Y} - Y\frac{\partial}{\partial Z} \tag{6.60}$$

and vanishes at $Y = 0 = Z$. The co-ordinates Y and Z are given by

$$\begin{aligned} Y &= e^{\kappa r^*} \sin \kappa\tau \\ Z &= e^{\kappa r^*} \cos \kappa\tau. \end{aligned} \tag{6.61}$$

Thus τ is an angular co-ordinate with period $\frac{2\pi}{\kappa}$ and, with this prescription, the manifold is regular at and outside the event horizon $Y = 0 = Z$, which has now become a symmetry axis [125].

6.5.2 Use of the Euclidean manifold

The Euclidean manifold is essential for the construction of a Green's function describing the thermal equilibrium between the black hole and the heat bath enclosed in the box. It is also the appropriate background on which to evaluate path integrals for scalar field perturbations, the approach taken in section 4.1.3. For our purposes here, the Green's function will suffice.

The Euclidean Green's function $G_E(x, x')$ is a suitably analytic function of the co-ordinates Y, Z, θ, φ_+ which satisfies the inhomogeneous wave equation

$$\Box G_E(Y, Z, \theta, \varphi_+; Y', Z', \theta', \varphi'_+) = -i\sqrt{g}\,\delta(x - x') \tag{6.62}$$

for a mass-less scalar field with conformal coupling, where g is the determinant of the positive definite Euclidean metric. When written as a function of τ, r, θ, φ_+ and τ', r', θ', φ'_+ it will be periodic in $\tau - \tau'$ with a period of $\frac{2\pi}{\kappa}$, which is the inverse of the Hawking temperature T. On substituting for τ and a_E,

$$G_H(t, r, \theta, \varphi_+; t', r', \theta', \varphi'_+) = G_E(it, r, \theta, \varphi_+; it', r', \theta', \varphi'_+)|_{a_E=-ia} \tag{6.63}$$

will be periodic in $t - t'$ with period iT^{-1}.

This condition of periodicity in imaginary time means that G_H is a thermal Green's function, describing the thermal equilibrium between the black hole and a heat bath enclosed in a suitable box. In addition, G_H (and hence also the state it represents) will be regular at the event horizon since the Euclidean manifold is also regular there.

6.5.3 Properties of the state

The properties of G_H depend crucially on the global structure of the Euclidean manifold. If τ were not an angular co-ordinate, there would be a conical singularity at $Y = 0 = Z$ which would be reflected in the propagator G_E. The global structure also indicates why it is not possible to construct $|H\rangle$ for points outside $\mathcal{S}_L$. In Kerr space-time, the Killing vector ζ (6.55) has two zeros, one at the event horizon and one at $\mathcal{S}_L$. In the Euclidean manifold, it has just one, at the event horizon, and does not vanish at r and θ co-ordinates equivalent to the position of $\mathcal{S}_L$ in the physical geometry. Hence, at and outside $\mathcal{S}_L$, the Euclidean space-time ceases to represent a useful model of the situation.

To see what is happening physically, suppose the box is placed outside $\mathcal{S}_L$. The walls of the box are rigid and stationary in the co-ordinates t, r, θ, φ_+ since it is with respect to these co-ordinates (in effect) that the propagator G_E has been constructed. Thus the box rotates about the black hole with angular velocity Ω_H, as expected, since we have already seen that the state $|H\rangle$ represents a black hole in equilibrium with a thermal bath which is rigidly rotating with angular velocity Ω_H. Such a box cannot exist at or outside the velocity of light surface since it would be required to move on a null or space-like world-line.

Placing the box inside $\mathcal{S}_L$ produces a modified Hartle-Hawking state $|H_M\rangle$ which is regular at the horizon and everywhere outside it (as far as the field is concerned the geometry end at the reflecting walls of the box). The state $|H_M\rangle$ also has the entropy expected for a thermal state [126] (cf. section 4.2).

6.5.4 An alternative approach

If, instead of considering a black hole in an asymptotically flat space-time, we look at a rotating black hole in a space-time that is asymptotically anti-de Sitter, the above problems may not arise [127]. In Schwarzschild anti-de Sitter space-time, the gravitational potential performs the role of a finite box, and a stable equilibrium between the black hole and rotating thermal radiation is formed without the need for external constructs. Then, for Kerr anti-de Sitter space-time, provided the angular momentum of the hole is sufficiently small (so that the gravitational potential of anti-de Sitter space-time comes into play prior to the formation of a velocity of light surface), the state representing this physical situation may be regular throughout the space-time.

Summary

In this chapter we have considered in detail the effect of rotation on the Hartle-Hawking state (discussed in chapter 4), by working on the "hairless" Kerr geometry. Using basic physical ideas, in particular the conservation equations, we have proved that the Hartle-Hawking state is pathological at the velocity of light surface $\mathcal{S}_L$.

Appendix A

Catastrophe Theory

In this appendix we outline the basic concepts and results of catastrophe theory, and describe briefly one example of a catastrophe, the *fold catastrophe.* These results are important in the analysis of section 3.6.

A.1 Important definitions and results

Consider a family of functions

$$f : X \times C \to \mathbb{R}; \qquad f(x,c) = f_c(x) \tag{A.1}$$

where X and C are both manifolds known as the *state space* and *control space* respectively. In other words, we have a family of functions of the variable x, the members of the family being indexed by c. In the particular case of interest to our application of the theory, X and C are both intervals of the real line. Then f maps out a surface $z = f(x,c)$ in $\mathbb{R}^3$ which is known as the *Whitney surface.*

Definition A.1 *The* catastrophe manifold $\mathcal{C}$ *is defined as the subset of* $X \times C$ *at which*

$$Df_c(x) = 0, \tag{A.2}$$

where Df_c *is the derivative of the function* $f_c(x)$*; namely* $\mathcal{C}$ *is the set of all the critical points of the family of functions.*

The set $\mathcal{C}$ is not necessarily a manifold, despite its name. However, it will turn out that $\mathcal{C}$ is a manifold in our application (see sections A.2 and 3.6).

Definition A.2 *A* j-unfolding *of a function* $f_0 : \mathbb{R}^n \to \mathbb{R}$ *is a function*

$$f : \mathbb{R}^{n+j} \to \mathbb{R} \tag{A.3}$$

such that

$$f(x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0) = f_0(x_1, \ldots, x_n). \tag{A.4}$$

Alternatively, we can denote $f(x_1, \ldots, x_n, c_1, \ldots, c_j)$ as $f_{c_1 \ldots c_j}(x_1, \ldots, x_n)$ and think of f as a family of functions $\mathbb{R}^n \to \mathbb{R}$ parametrized by c.

Definition A.3 *A k-unfolding $\bar{f}$ of f_0 is* induced *from f by three mappings:*

1. *a smooth mapping*

$$\begin{aligned} e : \mathbb{R}^k &\to \mathbb{R}^j \\ (c_1, \ldots, c_k) &\to (e_1(c), \ldots, e_j(c)), \end{aligned} \tag{A.5}$$

2. *a c-independent local change of co-ordinates in $\mathbb{R}^n$, viz a mapping*

$$\begin{aligned} \iota : \mathbb{R}^{n+k} &\to \mathbb{R}^n \\ (x, c) &\to (\iota_1(x, c), \ldots, \iota_n(x, c)) \end{aligned} \tag{A.6}$$

   such that for each c the map

$$\iota_c : x \to \iota(x, c) \tag{A.7}$$

   is a local diffeomorphism around 0,

3. *a shear function $h : \mathbb{R}^k \to \mathbb{R}$,*

provided

$$\bar{f}(x, c) = f(\iota_c(x), e(c)) + h(c) \tag{A.8}$$

around 0 *in* $\mathbb{R}^{n+k}$.

Definition A.4 1. *A j-unfolding of f_0 is* versal *if all other unfoldings of f_0 can be induced from it.*

2. *It is* universal *if j is the smallest dimension for which a versal j-unfolding of f_0 exists.*

Theorem A.5 *If the family of functions f in (A.1) is universal as an unfolding, then the set C is a manifold.*

This condition is satisfied for the elementary catastrophes we shall be considering and hence the catastrophe manifold $\mathcal{C}$ is in fact a manifold [62].

Definition A.6 *The* catastrophe map *χ is the restriction to the catastrophe manifold of the natural projection*

$$\pi : X \times C \to C, \qquad \pi(x, c) = c. \tag{A.9}$$

Definition A.7 *The* singularity set *is the set of singular points of χ in the catastrophe manifold $\mathcal{C}$, and the image of the singularity set in C is called the* bifurcation set *B.*

In other words, the singularity set is the set of points in $\mathcal{C}$ where the rank of the derivative of χ is less than j.

Definition A.8 *Consider a smooth function $f_0 : \mathbb{R}^n \to \mathbb{R}$. A point $x \in \mathbb{R}^n$ is a* critical point *for f_0 if*

$$Df_0\,|_x = 0. \tag{A.10}$$

Further, x is a non-degenerate *critical point if $Df_0\,|_x = 0$ and $D^2 f_0\,|_x$ has rank n. Otherwise x is a* degenerate *critical point of f_0.*

Proposition A.9 *The singularity set is the set of points (x, c) where $f_c(x)$ has a degenerate critical point.*

Proof [62] We consider only the case of interest, namely where both X and C are intervals of the real line. Then $\mathcal{C}$ is a one-dimensional sub-manifold of $\mathbb{R}^2$, namely a curve, and can be parametrized by its arc-length $s \in \mathtt{U} \subseteq \mathbb{R}$. The catastrophe map $\chi : \mathtt{U} \to \mathbb{R}$ is given by

$$\chi(s) = \chi(x(s), c(s)) = c(s) \tag{A.11}$$

and has derivative

$$\frac{d}{ds}\chi(s) = \frac{\partial\chi}{\partial x}\frac{dx}{ds} + \frac{\partial\chi}{\partial c}\frac{dc}{ds} = \frac{d}{ds}c(s). \tag{A.12}$$

Thus $\frac{d}{ds}\chi(s) = 0$ if and only if $\frac{d}{ds}c(s) = 0$. If, for some s_0 it is the case that $\frac{d}{ds}c(s_0) = 0$, then $\frac{d}{ds}x(s_0) \neq 0$, since s is the arc-length of the curve $\mathcal{C}$. Consider two points $(x(s_0), c(s_0))$ and $(x(s_0 + \delta s), c(s_0 + \delta s))$ lying on the catastrophe manifold $\mathcal{C}$. Then by Taylor's theorem

$$\begin{aligned}\frac{\partial f}{\partial x}(x(s_0 + \delta s), c(s_0 + \delta s)) &= \frac{\partial f}{\partial x}(x(s_0), c(s_0)) + \frac{\partial^2 f}{\partial x^2}(x(s_0), c(s_0))\frac{dx}{du}(s_0)\delta s \\ &\quad + \frac{\partial^2 f}{\partial x \partial c}(x(s_0), c(s_0))\frac{dc}{ds}(s_0)\delta s + O(\delta s^2). \end{aligned} \tag{A.13}$$

Now

$$\frac{\partial f}{\partial x}(x(s_0), c(s_0)) = 0 = \frac{\partial f}{\partial x}(x(s_0 + \delta s), c(s_0 + \delta s)) \tag{A.14}$$

since these points lie on the catastrophe manifold. If, in addition, $\frac{dc}{ds}(s_0) = 0$, then

$$\frac{\partial^2 f}{\partial x^2}(x(s_0), c(s_0))\frac{dx}{ds}(s_0)\delta s = O(\delta s^2) \tag{A.15}$$

whence

$$\frac{d^2}{dx^2} f_{c(s_0)}(x(s_0)) = \frac{\partial^2 f}{\partial x^2}(x(s_0), c(s_0)) = 0. \tag{A.16}$$

In other words, $f_c(x)$ has a degenerate critical point whenever $\frac{dx}{ds} = 0$ and vice versa. This proof can readily be extended to the general case where $X = \mathbb{R}^n$ and $C = \mathbb{R}^j$. □

From this result it follows that the set B is the place where the number and nature of the critical points of the family of functions $f_c(x)$ change [62]. In order to see this, we need a number of important concepts in catastrophe theory.

Definition A.10 *Two families of functions $f, \tilde{f} : \mathbb{R}^n \times \mathbb{R}^j \to \mathbb{R}$ are* equivalent *if there exist*

1. *a diffeomorphism $e : \mathbb{R}^j \to \mathbb{R}^j$,*
2. *a smooth map $\iota : \mathbb{R}^n \times \mathbb{R}^j \to \mathbb{R}^n$ such that for each $c \in \mathbb{R}^n$ the map*
$$\iota_c : \mathbb{R}^n \to \mathbb{R}, \qquad \iota_c(x) = \iota(x, c) \tag{A.17}$$
is a diffeomorphism,
3. *a smooth map $h : \mathbb{R}^j \to \mathbb{R}$,*

each defined in a neighbourhood of 0*, such that*
$$\tilde{f}(x, c) = f(\iota_c(x), e(c)) + h(c) \tag{A.18}$$
for all $(x, c) \in \mathbb{R}^n \times \mathbb{R}^j$ in that neighbourhood.

Definition A.11 *A family of functions $f : \mathbb{R}^n \times \mathbb{R}^j \to \mathbb{R}$ is* structurally stable *if f is equivalent to any family $f + \tilde{f} : \mathbb{R}^n \times \mathbb{R}^j \to \mathbb{R}$ where $\tilde{f}$ is a sufficiently small family.*

The importance of these concepts lies in the following.

Theorem A.12 (Morse Lemma) *Let $f : \mathbb{R}^n \times \mathbb{R}^j \to \mathbb{R}$ be smooth and suppose that the matrix*
$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{1 \le i,j \le n} \tag{A.19}$$
is non-singular at $(x, c) = (0, 0)$. Then f is equivalent to a family of the form
$$\pm \tilde{x}_1^2 \pm \ldots \pm \tilde{x}_n^2. \tag{A.20}$$

This tells us that any family of functions having a non-degenerate critical point at $(x, c) = (0, 0)$ is structurally stable [62]. This is because the determinant of the matrix (A.19) is non-zero, and hence for sufficiently small perturbations $\tilde{f}$ it must be the case that
$$\det\left(\frac{\partial^2 (f + \tilde{f})}{\partial x_i \partial x_j}\right) \neq 0 \tag{A.21}$$
whence, from the Morse Lemma, f and $f + \tilde{f}$ have the same type of critical point at $(x, c) = (0, 0)$. In other words, the system does not change locally provided critical points remain non-degenerate, so that the bifurcation set B is the only place where the number and nature of the critical points of the family of functions $f_c(x)$ can change.

Now consider the case where X and C are subsets of $\mathbb{R}$. The following theorem then tells us how C and B behave [62]:

Theorem A.13 (Thom's classification theorem) *Typically a one-parameter family of smooth functions $\mathbb{R}^n \to \mathbb{R}$ for any n, is structurally stable and is equivalent around any point to one of the following forms:*

1. *non-critical:* x_1
2. *non-degenerate critical:* $x_1^2 + \cdots + x_i^2 - x_{i+1}^2 - \cdots - x_n^2 \qquad (0 \leq i \leq n)$
3. *fold catastrophe:* $x_1^3 + c_1 x_1 + x_2^2 + \cdots + x_i^2 - x_{i+1}^2 - \cdots - x_n^2 \qquad (1 \leq i \leq n)$.

A.2 The fold catastrophe

This is the only type of catastrophe that can occur in a one-parameter family of functions, and from Thom's theorem above, it is sufficient to consider the standard family

$$f_c(x) = \frac{1}{3}x^3 + cx. \tag{A.22}$$

This family is universal as an unfolding by the classification theorem, and hence $\mathcal{C}$ is a manifold. At points $(x, c) \in \mathcal{C}$, we have

$$0 = \frac{d}{dx} f_c(x) = x^2 + c \tag{A.23}$$

i.e.

$$c = -x^2. \tag{A.24}$$

Since

$$\frac{d^2}{dx^2} f_c(x) = 2x, \tag{A.25}$$

the only non-degenerate critical point occurs at $(0,0)$, the general point on $\mathcal{C}$ being $(x, -x^2)$. When $x > 0$, from (A.25),

$$\frac{d^2}{dx^2} f_c(x) > 0 \tag{A.26}$$

and f has a minimum; when $x < 0$, similarly,

$$\frac{d^2}{dx^2} f_c(x) < 0 \tag{A.27}$$

and f has a maximum. Hence the singularity set is simply the point $(0,0)$ and the bifurcation set B is a single point (see figure A.1).

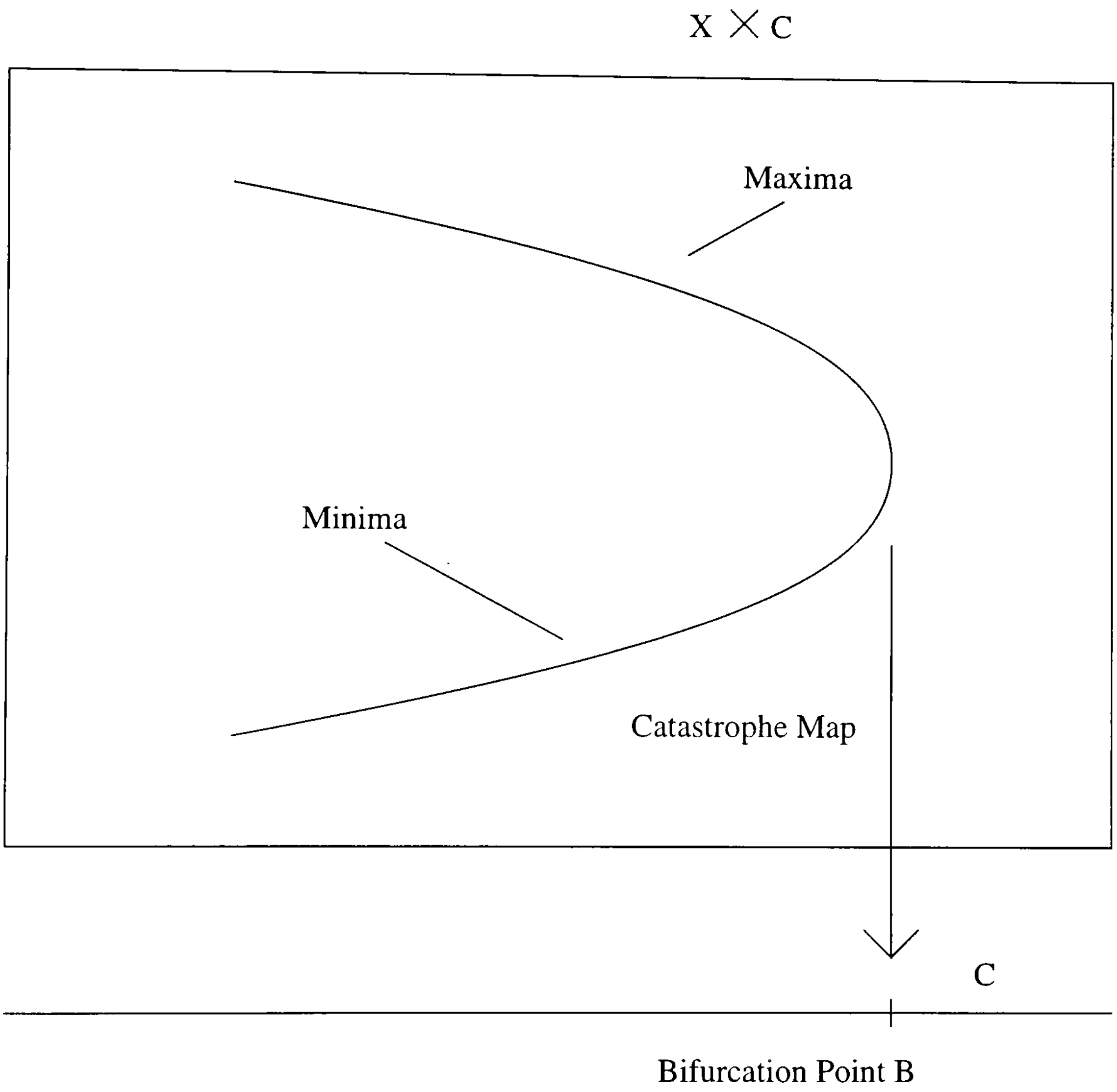

Figure A.1: The structure of a fold catastrophe

Appendix B

Equations for Gauss-Bonnet Black Holes

In this appendix we consider in detail the field equations for higher-curvature string gravity discussed in chapter 5. The somewhat complicated equations given below were integrated numerically in [6] to give the black holes which are described in section 5.3. The main result of this appendix is the confirmation, from another perspective, of the condition (5.47) satisfied by the dilaton field at the event horizon.

We consider the Gauss-Bonnet model of gravity described by the Lagrangian (5.3)

$$\mathcal{L}_{GB} = -\frac{1}{8\pi}\left[\frac{1}{2}R + \frac{1}{4}\left(\partial_\mu \Phi\right)^2 - \frac{\alpha'}{8g^2}e^{\Phi}R^2_{GB}\right]. \tag{B.1}$$

The part of the action given by the Gauss-Bonnet term R^2_{GB} is a topological invariant if the dilaton is not present in the Lagrangian (B.1). The model described by (B.1) is useful in string theory because it is ghost-free.

B.1 Behaviour near the horizon

Firstly, we make just one assumption: that, as $r \to r_h$, then $\Gamma' \to \infty$ and both Φ and Φ' remain finite. We do *not* assume that the field variables have the form (5.30), but rather will derive that asymptotic behaviour. By making the translation

$$\Phi \to \Phi - \log\left(\frac{\alpha'}{g^2}\right) \tag{B.2}$$

in the Lagrangian (B.1), we can, without loss of generality, set

$$\frac{\alpha'}{g^2} = 1. \tag{B.3}$$

Now note that the Einstein equation (5.17) possesses the analytic solution

$$e^{\Lambda} = \frac{1}{2}\left(-y \pm \sqrt{y^2 - 4z}\right) \tag{B.4}$$

where

$$\begin{aligned} y &= \frac{1}{4}r^2\Phi'^2 - 1 - \Gamma'\left(r + \frac{1}{2}e^{\Phi}\Phi'\right) \\ z &= \frac{3}{2}\Gamma'\Phi' e^{\Phi}. \end{aligned} \tag{B.5}$$

Relation (B.4) can then be used to eliminate Λ' from the remaining Einstein equations (5.16), (5.18), and the dilaton equation (5.15). This gives three equations in the two unknowns Γ and Φ. Fortunately they are compatible and give the following pair of (somewhat complex) second order differential equations [6]:

$$\Phi'' = -\frac{d_1}{d} \qquad \Gamma'' = -\frac{d_2}{d} \tag{B.6}$$

where

$$\begin{aligned} d &= 4e^{2\Lambda+\phi}r\left(-4 + 8e^{\Lambda} - 4e^{2\Lambda} - 4\Gamma' r + 4\Gamma' e^{\Lambda} r + 5\phi'^2 r^2 - \phi'^2 e^{\Lambda} r^2\right) \\ &\quad + 4\phi' e^{\Lambda+2\phi}\left(6 - 12e^{\Lambda} + 6e^{2\Lambda} + 6\Gamma' r - 8\Gamma' e^{\Lambda} r + 2\Gamma' e^{2\Lambda} r - 3\phi'^2 r^2\right. \\ &\quad \left. + \phi'^2 e^{\Lambda} r^2\right) - 12\Gamma'\phi'^2 e^{3\phi}(1 - e^{\Lambda})^2 - 8\phi' e^{3\Lambda} r^4 \end{aligned} \tag{B.7}$$

$$\begin{aligned} d_1 &= 2\Gamma'\phi'^3 e^{3\phi}\left(9\Gamma' - 6\phi' - 6\Gamma' e^{\Lambda} + 12\phi' e^{\Lambda} + \Gamma' e^{2\Lambda} - 6\phi' e^{2\Lambda}\right) \\ &\quad + \phi'^2 e^{\Lambda+2\phi}\left(24\phi' - 8\Gamma' e^{\Lambda} - 48\phi' e^{\Lambda} + 8\Gamma' e^{2\Lambda} + 24\phi' e^{2\Lambda} - 42\Gamma'^2 r\right. \\ &\quad - 30\phi'^2 r + 20\Gamma'^2 e^{\Lambda} r - 32\Gamma'\phi' e^{\Lambda} r + 16\phi'^2 e^{\Lambda} r - 2\Gamma'^2 e^{2\Lambda} r \\ &\quad \left. - 2\phi'^2 e^{2\Lambda} r + 3\Gamma'\phi'^2 r^2 - 3\Gamma'\phi'^2 e^{\Lambda} r^2 + 24\Gamma'\phi' r + 8\Gamma'\phi' e^{2\Lambda} r\right) \\ &\quad + \phi' e^{2\Lambda+\phi}\left(-24 + 48e^{\Lambda} - 24e^{2\Lambda} - 4\Gamma' r - 16\phi' r + 8\Gamma' e^{\Lambda} r\right. \\ &\quad + 32\phi' e^{\Lambda} r - 4\Gamma' e^{2\Lambda} r - 16\phi' e^{2\Lambda} r + 32\Gamma'^2 r^2 - 16\Gamma'\phi' r^2 + 38\phi'^2 r^2 \\ &\quad \left. + 16\Gamma'\phi' e^{\Lambda} r^2 - 6\phi'^2 e^{\Lambda} r^2 - 3\Gamma'\phi'^2 r^3 + \Gamma'\phi'^2 e^{\Lambda} r^3 - 8\Gamma'^2 e^{\Lambda} r^2\right) \\ &\quad + 2e^{3\Lambda} r\left(8 - 16e^{\Lambda} + 8e^{2\Lambda} + 4\Gamma' r - 4\Gamma' e^{\Lambda} r - 4\Gamma'^2 r^2 - 6\phi'^2 r^2\right. \\ &\quad \left. + \Gamma'\phi'^2 r^3 - 2\phi'^2 e^{\Lambda} r^2\right) \end{aligned} \tag{B.8}$$

$$\begin{aligned} d_2 &= \Gamma'\phi' e^{\Lambda+2\phi} r\left(18\Gamma'^2 + 6\phi'^2 - 4\Gamma'^2 e^{\Lambda} + 8\phi'^2 e^{\Lambda} + 2\Gamma'^2 e^{2\Lambda} + 2\phi'^2 e^{2\Lambda}\right. \\ &\quad \left. + 5\Gamma'\phi'^2 e^{\Lambda} r - 8\phi'^3 e^{\Lambda} r - 9\Gamma'\phi'^2 r\right) - 2\Gamma'^3\phi'^2 e^{3\phi}\left(3 + e^{2\Lambda}\right) \\ &\quad + \phi' e^{3\Lambda} r^2\left(8 - 8e^{\Lambda} - 4\Gamma' r - 4\Gamma' e^{\Lambda} r - 4\Gamma'^2 r^2 - 2\phi'^2 r^2 + \Gamma'\phi'^2 r^3\right) \\ &\quad + e^{2\Lambda+\phi}\left(8\Gamma' - 16\Gamma' e^{\Lambda} + 8\Gamma' e^{2\Lambda} - 4\Gamma'^2 r + 8\phi'^2 r + 8\Gamma'^2 e^{\Lambda} r\right. \\ &\quad - 4\Gamma'^2 e^{2\Lambda} r - 8\phi'^2 e^{2\Lambda} r - 12\Gamma'^3 r^2 - 10\Gamma'\phi'^2 r^2 - 8\phi'^3 r^2 + 4\Gamma'^3 e^{\Lambda} r^2 \\ &\quad + 2\Gamma'\phi'^2 e^{\Lambda} r^2 + 8\phi'^3 e^{\Lambda} r^2 + 13\Gamma'^2\phi'^2 r^3 + 4\Gamma'\phi'^3 r^3 + 6\phi'^4 r^3 \\ &\quad \left. + 4\Gamma'\phi'^3 e^{\Lambda} r^3 - 2\phi'^4 e^{\Lambda} r^3 - 3\Gamma'\phi'^4 r^4 - 3\Gamma'^2\phi'^2 e^{\Lambda} r^3\right). \end{aligned} \tag{B.9}$$

Choosing the minus sign in (B.4) leads to

$$e^{\Lambda} \sim O(1) \qquad \text{as } r \sim r_h \tag{B.10}$$

which does not give a regular black hole event horizon at $r = r_h$. Hence we choose the positive sign. With this choice, expanding (B.4) near $r = r_h$ gives

$$e^{\Lambda} = \frac{1}{2}\left(e^{\Phi}\Phi' + 2r\right)\Gamma' - \frac{8e^{\Phi}\Phi' - 8r + e^{\Phi}\Phi'^3 r^2 + 2\Phi'^2 r^3}{4(e^{\Phi}\Phi' + 2r)} + O\left(\frac{1}{\Gamma'}\right) \tag{B.11}$$

provided

$$e^{\Phi}\Phi' + 2r \neq 0. \tag{B.12}$$

If $e^{\Phi}\Phi' + 2r \sim 0$, then

$$e^{\Lambda} = \sqrt{3r}(\Gamma')^{\frac{1}{2}} + \ldots \tag{B.13}$$

and from (B.6),

$$\Phi'' = \sqrt{3r}e^{-\Phi}(\Gamma')^{\frac{1}{2}} + \ldots, \tag{B.14}$$

which means that Φ''_h is finite only if $\Phi_h \to \infty$, contrary to our initial assumption of finite Φ_h. Substituting (B.11) into (B.6) yields the following expansions close to the horizon:

$$\begin{aligned}
\Phi'' &= -\frac{1}{2}\frac{(e^{\Phi}\Phi' + 2r)(6e^{\Phi} + e^{\Phi}\Phi'^2 r^2 + 2\Phi' r^3)}{-6e^{2\Phi} + e^{\Phi}\Phi' r^3 + 2r^4}\Gamma' + O(1) \\
\Gamma'' &= -\frac{1}{2}\frac{-6e^{2\Phi} + e^{2\Phi}\Phi' r^2 + 4e^{\Phi}\Phi' r^3 + 4r^4}{-6e^{2\Phi} + e^{\Phi}\Phi' r^3 + 2r^4}\Gamma'^2 + O(\Gamma').
\end{aligned} \tag{B.15}$$

Since Γ' diverges at the horizon, and we have already observed that $e^{\Phi}\Phi' + 2r \neq 0$, the only way to keep Φ''_h finite is to impose an additional constraint, namely

$$6e^{\Phi} + e^{\Phi}\Phi'^2 r^2 + 2\Phi' r^3 = 0, \tag{B.16}$$

which gives

$$\Phi'_h = r_h e^{-\Phi_h}\left(-1 \pm \sqrt{1 - \frac{6e^{2\Phi}}{r_h^4}}\right) \tag{B.17}$$

so that

$$\Phi_h < \log\left(\frac{r_h^2}{\sqrt{6}}\right). \tag{B.18}$$

With this constraint, from (B.15), $\Phi'' \sim O(1)$ and

$$\Gamma'' = -\Gamma'^2 + O(1) \tag{B.19}$$

whence

$$\Gamma' = \frac{1}{r - r_h} + O(1) \tag{B.20}$$

as $r \to r_h$. This unique solution gives a regular "black hole" type metric for $r \sim r_h$ and follows from the requirement that Φ_h, Φ'_h and Φ''_h be finite as $\Gamma' \to \infty$. The expansions can also be written in the form

$$\begin{aligned}
e^{\Gamma(r)} &= \Gamma_1(r - r_h) + O(r - r_h)^2 \\
e^{-\Lambda(r)} &= \Lambda_1(r - r_h) + O(r - r_h)^2
\end{aligned} \tag{B.21}$$

which is exactly the ansatz used in section 5.2.4, where Γ_1 is an arbitrary constant and

$$\Lambda_1 = 2\left(e^{\Phi_h}\Phi_h' + 2r_h\right)^{-1}. \tag{B.22}$$

In addition, note that (B.17) is in complete agreement with the condition (5.47) derived in section 5.2.

B.2 Numerical analysis

We mention briefly a couple of points which are relevant for the discussion of the numerical solution of equations (B.6) in section 5.3. Firstly, it is found that the choice of a plus sign in (B.17) is necessary in order to give solutions having the correct behaviour for large r. Secondly, the solutions can be described by just two parameters. Due to relations (B.17) and (B.22) there are three independent parameters Φ_h, r_h and Γ_1, since the equations of motion do not give any constraint on Γ_1, due to the fact that they involve only $\Gamma'(r)$ and not $\Gamma(r)$. In fact, Γ_1 can be fixed by the boundary conditions at infinity (5.25), leaving just two parameters Φ_h and r_h.

Appendix C

Vacua in Kerr Space-time

In this appendix we describe how the standard Boulware, Unruh and Hartle-Hawking vacua are defined in Kerr space-time.

C.1 The Kerr geometry

Consider the Kerr-Newman space-time with metric, in Boyer-Lindquist co-ordinates:

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta\, d\varphi)^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\varphi - a\,dt\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2 \quad \text{(C.1)}$$

where

$$\begin{aligned} \Delta(r) &= r^2 - 2Mr + Q^2 + a^2 \\ \Sigma(r,\theta) &= r^2 + a^2\cos^2\theta, \end{aligned} \quad \text{(C.2)}$$

M being the ADM [64] mass of the black hole, Ma its angular momentum as viewed from infinity, and Q the charge.

In the non-extreme case, $a^2 < M^2 - Q^2$, the metric possesses two co-ordinate singularities at

$$\Delta = 0, \quad \text{(C.3)}$$

namely

$$r = r_\pm = M \pm \left(M^2 - a^2 - Q^2\right)^{\frac{1}{2}}, \quad \text{(C.4)}$$

where $r = r_+$ is the (outer) event horizon. The space-time is stationary and axisymmetric, possessing two Killing vectors,

$$\xi = \frac{\partial}{\partial t} \qquad \text{and} \qquad \chi = \frac{\partial}{\partial\varphi}. \quad \text{(C.5)}$$

The former is time-like at infinity, but becomes null when

$$r = r_s = M + \left(M^2 - a^2\cos^2\theta\right)^{\frac{1}{2}} \geq r_+. \quad \text{(C.6)}$$

This is known as the *stationary limit surface*, and between here and the event horizon is a region called the *ergo-sphere*. The event horizon rotates with an angular velocity Ω_H given by

$$\Omega_H = \frac{a}{r_+^2 + a^2} \tag{C.7}$$

due to the fact that the Killing field

$$\zeta = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \varphi} \tag{C.8}$$

rather than ξ is tangent to the null geodesic generators of the Kerr black hole event horizon. The Killing field ζ is time-like just outside the event horizon and becomes null again on a surface

$$r = r_v(\theta) \tag{C.9}$$

known as the *velocity of light surface*. This is the surface on which observers must move with the speed of light if they wish to rotate with the same angular velocity as the event horizon. It should be stressed that the velocity of light surface is not the same as the stationary limit surface.

C.1.1 Massless scalar field

Following [128], we consider a massless scalar field satisfying the Klein-Gordon equation

$$\Box \phi = 0, \tag{C.10}$$

the scalar curvature R vanishing in Ricci-flat Kerr space-time. This equation is separable in the Kerr metric [128], and the basis functions may be taken to be

$$u_{wlm_l} = \frac{\mathtt{N(k)}}{(r^2 + a^2)^{\frac{1}{2}}} e^{-iwt + im_l\varphi} S_{wlm_l}(\cos\theta) R_{wlm_l}(r) \tag{C.11}$$

where $\mathtt{k}$ is a function of the frequency w, the function $\mathtt{N(k)}$ is a normalization constant, l and m_l are integers and

$$|m_l| \leq l, \qquad l = 0, 1, 2, \ldots. \tag{C.12}$$

The spheroidal harmonic S_{wlm_l} satisfies the eigenvalue equation [80]:

$$\begin{aligned}\Bigg[\frac{d}{d\cos\theta}(1 - \cos^2\theta)\frac{d}{d\cos\theta} - \frac{m_l^2}{1 - \cos^2\theta} + 2m_l a w \\ -(aw)^2(1 - \cos^2\theta) + \lambda_{lm_l}(aw)\Bigg] S_{wlm_l}(\cos\theta) &= 0. \end{aligned} \tag{C.13}$$

The eigenvalue $\lambda_{lm_l}(aw)$ depends on the integers l and m_l and has the known value

$$\lambda_{lm_l}(0) = l(l+1). \tag{C.14}$$

The radial equation takes the form

$$\left[\frac{d^2}{dr^{*2}} - V_{wlm_l}(r)\right] R_{wlm_l}(r) = 0 \tag{C.15}$$

where r^* is the "tortoise" co-ordinate,

$$\frac{dr^*}{dr} = \frac{r^2+a^2}{\Delta}. \tag{C.16}$$

In other words,

$$r^* = r + \frac{M}{\sqrt{M^2-a^2-Q^2}}\left(r_+ \log|r-r_+| - r_- \log|r-r_-|\right), \tag{C.17}$$

and the potential V_{wlm_l} is given by

$$V_{wlm_l}(r) = -\left(w - \frac{m_l a}{r^2+a^2}\right)^2 + \lambda_{lm_l}(aw)\frac{\Delta}{(r^2+a^2)^2} + \frac{2\Delta(Mr^2-a^2)}{(r^2+a^2)^3} + \frac{3a^2\Delta^2}{(r^2+a^2)^4}. \tag{C.18}$$

In the asymptotic regions $r^* \to \pm\infty$, (C.18) reduces to

$$V_{wlm_l}(r) = \begin{cases} -(w - m_l\Omega_H)^2 & r^* \to -\infty \\ -w^2 & r^* \to \infty \end{cases} \tag{C.19}$$

where Ω_H is the angular velocity of the horizon (C.7).

We now consider two classes of solutions to (C.15). Waves incident from $\mathcal{I}^-$ will be partially scattered back to $\mathcal{I}^+$ by the potential barrier V, and partially transmitted to $\mathcal{H}^+$. Similarly, waves emanating from $\mathcal{H}^-$ will be scattered into either asymptotic region. These two classes therefore have the asymptotic forms

$$\begin{aligned} R^-_{wlm_l}(r) &= \begin{cases} e^{i\tilde{w}r^*} + A^-_{wlm_l}e^{-i\tilde{w}r^*} & \text{as } r^* \to -\infty \\ B^-_{wlm_l}e^{iwr^*} & \text{as } r^* \to \infty \end{cases} \\ R^+_{wlm_l}(r) &= \begin{cases} B^+_{wlm_l}e^{-i\tilde{w}r^*} & \text{as } r^* \to -\infty \\ e^{-iwr^*} + A^+_{wlm_l}e^{iwr^*} & \text{as } r^* \to \infty \end{cases} \end{aligned} \tag{C.20}$$

where $\tilde{w} = w - m_l\Omega_H$. We define an inner product on the mode functions u_{wlm_l} by

$$\langle u_1, u_2\rangle = \frac{1}{2}i\int_\Sigma \sqrt{-g}(u^*_{2,\mu}u_1 - u^*_2 u_{1,\mu})d\Sigma^\mu \tag{C.21}$$

where Σ is any Cauchy hyper-surface of the geometry. The basis functions are orthonormal provided one normalizes the spheroidal harmonics so that

$$\int_{-1}^{1} S_{wlm_l}(\cos\theta)S_{wl'm_l}(\cos\theta)\,d\cos\theta = \delta_{ll'} \tag{C.22}$$

and chooses the normalization constant according to

$$\mathtt{N}(\mathtt{k}) = \frac{1}{2\pi\sqrt{2|\mathtt{k}|}} \tag{C.23}$$

where $\mathtt{k} = w$ for incoming (+) modes and $k = \tilde{w}$ for outgoing (−) modes.

C.2 Properties of the vacua in Kerr space-time

We now extend analysis of [129] to the uncharged Kerr black hole, in order to derive the asymptotic properties of the candidate vacuum states without recourse to renormalization (for further details see [8]). Firstly, we need to consider precisely what is meant by the three vacuum states (Boulware [130], Unruh [118] and Hartle-Hawking [131]) in Kerr space-time.

In the Schwarzschild geometry, an outgoing mode which has positive frequency with respect to the retarded null co-ordinate $u = t - r^*$ at the past horizon $\mathcal{H}^-$ will also have positive frequency with respect to u at $\mathcal{I}^+$. Hence, if, for example, we wish to define the Boulware vacuum state, it does not matter whether we consider outgoing modes having positive frequency with respect to u at $\mathcal{H}^-$ or $\mathcal{I}^+$. Both definitions yield the same vacuum state.

This is not the case in Kerr geometry, due to super-radiant modes [124, 132, 133]. This can be seen from the asymptotic forms of the in-going and outgoing modes (C.20):

$$
\begin{aligned}
u^+_{wlm_l} &= \frac{S_{wlm_l}(\cos\theta)}{\sqrt{8\pi^2|w|(r^2+a^2)}} \times \begin{cases} \exp(-iwv + im_l\varphi) & \text{at } \mathcal{I}^- \\ A^+_{wlm_l}\exp(-iwu + im_l\varphi) & \text{at } \mathcal{I}^+ \\ 0 & \text{at } \mathcal{H}^- \\ B^+_{wlm_l}\exp(-i\tilde{w}v + im_l\varphi_+) & \text{at } \mathcal{H}^+ \end{cases} \\
u^-_{wlm_l} &= \frac{S_{wlm_l}(\cos\theta)}{\sqrt{8\pi^2|\tilde{w}|(r^2+a^2)}} \times \begin{cases} 0 & \text{at } \mathcal{I}^- \\ B^-_{wlm_l}\exp(-iwu + im_l\varphi) & \text{at } \mathcal{I}^+ \\ \exp(-i\tilde{w}u + im_l\varphi_+) & \text{at } \mathcal{H}^- \\ A^-_{wlm_l}\exp(-i\tilde{w}v + im_l\varphi_+) & \text{at } \mathcal{H}^+ \end{cases}
\end{aligned} \tag{C.24}
$$

where $v = t + r^*$ is the advanced null co-ordinate, $\varphi_+ = \varphi - \Omega_H t$ and $\tilde{w} = w - m_l\Omega_H$. Hence an outgoing mode having positive frequency with respect to u at $\mathcal{H}^-$ will have negative frequency with respect to u at $\mathcal{I}^+$ if it lies in the super-radiant regime, i.e. if $\tilde{w} > 0$ but $w < 0$.

The general consensus in the current literature is to define the Boulware, Unruh and Hartle-Hawking vacua so that the un-renormalized expectation values of the stress tensor in these states are, respectively [122]

$$
\begin{aligned}
\langle B|T_{\mu\nu}|B\rangle &= \sum_{l,m_l}\int_0^\infty d\tilde{w}\, T_{\mu\nu}[u^-_{wlm_l}, u^{-*}_{wlm_l}] + \sum_{l,m_l}\int_0^\infty dw\, T_{\mu\nu}[u^+_{wlm_l}, u^{+*}_{wlm_l}] \\
\langle U|T_{\mu\nu}|U\rangle &= \sum_{l,m_l}\int_0^\infty d\tilde{w}\coth\left(\frac{\pi\tilde{w}}{\kappa}\right) T_{\mu\nu}[u^-_{wlm_l}, u^{-*}_{wlm_l}] \\
&\quad + \sum_{l,m_l}\int_0^\infty dw\, T_{\mu\nu}[u^+_{wlm_l}, u^{+*}_{wlm_l}] \\
\langle H|T_{\mu\nu}|H\rangle &= \sum_{l,m_l}\int_0^\infty d\tilde{w}\coth\left(\frac{\pi\tilde{w}}{\kappa}\right) T_{\mu\nu}[u^-_{wlm_l}, u^{-*}_{wlm_l}]
\end{aligned}
$$

$$+\sum_{l,m_l}\int_0^\infty dw \coth\left(\frac{\pi\tilde{w}}{\kappa}\right) T_{\mu\nu}[u^{+}_{wlm_l}, u^{+*}_{wlm_l}] \tag{C.25}$$

where the contribution to the stress tensor for a single mode u_{wlm_l} and a conformally coupled, massless scalar field in the Ricci-flat Kerr space-time is [90]

$$T_{\mu\nu}[u_{wlm_l}, u^{*}_{wlm_l}] = \frac{1}{3}\left(u_{;\mu}u^{*}_{;\nu} + u_{;\nu}u^{*}_{;\mu}\right) - \frac{1}{6}\left(u_{;\mu\nu}u^{*} + uu^{*}_{;\mu\nu}\right) - \frac{1}{6}g_{\mu\nu}u^{*}_{;\rho}u^{;\rho}. \tag{C.26}$$

The definitions (C.25) correspond to choosing positive frequency, outgoing, modes on $\mathcal{H}^-$ and positive frequency, in-going, modes on $\mathcal{I}^-$. Below we shall investigate how this choice affects the properties of the vacua.

Firstly, it is expected that $\langle H|T_{\mu\nu}|H\rangle_{ren}$ will be regular on both the future and past outer event horizons, so that the divergent behaviour of $\langle U|T_{\mu\nu}|U\rangle_{ren}$ (if any) can be evaluated by renormalizing $\langle U|T_{\mu\nu}|U\rangle$ by subtracting $\langle H|T_{\mu\nu}|H\rangle$, namely, as $r\to r_+$,

$$\begin{aligned}\langle U|T_{\mu\nu}|U\rangle_{ren} &\sim \langle U|T_{\mu\nu}|U\rangle_{ren} - \langle H|T_{\mu\nu}|H\rangle_{ren}\\ &= \langle U|T_{\mu\nu}|U\rangle - \langle H|T_{\mu\nu}|H\rangle \qquad \text{(C.27)}\\ &= \sum_{l,m_l}\int_0^\infty dw\frac{-2}{e^{\frac{2\pi\tilde{w}}{\kappa}}-1}T_{\mu\nu}[u^{+}_{wlm_l}, u^{+*}_{wlm_l}]. \qquad \text{(C.28)}\end{aligned}$$

Note that the second line (C.27) holds because the subtraction terms used to renormalize $\langle T_{\mu\nu}\rangle$ are independent of the vacuum state [107, 108]. At infinity, the Boulware vacuum is expected to be as empty as possible. We shall calculate

$$\begin{aligned}\langle U|T_{\mu\nu}|U\rangle_{ren} - \langle B|T_{\mu\nu}|B\rangle_{ren} &= \langle U|T_{\mu\nu}|U\rangle - \langle B|T_{\mu\nu}|B\rangle\\ &= \sum_{l,m_l}\int_0^\infty dw\frac{2}{e^{\frac{2\pi\tilde{w}}{\kappa}}-1}T_{\mu\nu}[u^{-}_{wlm_l}, u^{-*}_{wlm_l}]. \qquad \text{(C.29)}\end{aligned}$$

C.2.1 Christoffel symbols

We record here for completeness the non-zero Christoffel symbols for the Kerr metric, which will be required in the calculation of (C.28) and (C.29):

$$\begin{aligned}\Gamma^t_{tr} &= \Delta^{-1}\Sigma^{-1}(r^2+a^2)M(r^2-a^2\cos^2\theta)\\ \Gamma^t_{t\theta} &= a^2\Sigma^{-2}\left[\Delta-(r^2+a^2)\right]\sin\theta\cos\theta\\ \Gamma^t_{r\varphi} &= a\Delta^{-1}\Sigma^{-2}\sin^2\theta\left[(r^2+a^2)M(r^2-a^2\cos^2\theta)+r\Sigma\left\{\Delta-(r^2+a^2)\right\}\right]\\ \Gamma^t_{\theta\varphi} &= -a^3\Sigma^{-2}\left[\Delta-(r^2+a^2)\right]\sin^3\theta\cos\theta\\ \Gamma^r_{tt} &= \Delta\Sigma^{-3}M(-r^2+a^2\cos^2\theta)\\ \Gamma^r_{t\varphi} &= a\Delta\Sigma^{-3}M(-r^2+a^2\cos^2\theta)\sin^2\theta\\ \Gamma^r_{rr} &= \Delta^{-1}\Sigma^{-1}(-Mr^2+ra^2\sin^2\theta+Ma^2\cos^2\theta)\end{aligned}$$

$$
\begin{aligned}
\Gamma^r_{r\theta} &= -a^2\Sigma^{-1}\sin\theta\cos\theta \\
\Gamma^r_{\theta\theta} &= -r\Delta\Sigma^{-1} \\
\Gamma^r_{\varphi\varphi} &= -\Delta\Sigma^{-3}\left[r\Sigma^2 - M\left(r^2 - a^2\cos^2\theta\right)a^2\sin^2\theta\right]\sin^2\theta \\
\Gamma^\theta_{tt} &= a^2\Sigma^{-3}\left[\Delta - (r^2+a^2)\right]\sin\theta\cos\theta \\
\Gamma^\theta_{t\varphi} &= -a\Sigma^{-3}\left[\Delta - (r^2+a^2)\right](r^2+a^2)\sin\theta\cos\theta \\
\Gamma^\theta_{rr} &= a^2\Delta^{-1}\Sigma^{-1}\sin\theta\cos\theta \\
\Gamma^\theta_{r\theta} &= -r\Sigma^{-1} \\
\Gamma^\theta_{\theta\theta} &= -a^2\Sigma^{-1}\sin\theta\cos\theta \\
\Gamma^\theta_{\varphi\varphi} &= -\Sigma^{-3}\left[(r^2+a^2)^3 - a^2\Delta(r^2+a^2+\Sigma)\sin^2\theta\right]\sin\theta\cos\theta \\
\Gamma^\varphi_{tr} &= a\Delta^{-1}\Sigma^{-2}M(r^2 - a^2\cos^2\theta) \\
\Gamma^\varphi_{t\theta} &= \Sigma^{-1}a\left[\Delta - (r^2+a^2)\right]\cot\theta \\
\Gamma^\varphi_{r\varphi} &= \Delta^{-1}\Sigma^{-2}\left[-a^2M(r^2-a^2\cos^2\theta)\sin^2\theta + r\Sigma(\Delta - a^2\sin^2\theta)\right] \\
\Gamma^\varphi_{\theta\varphi} &= \cot\theta. \qquad\qquad \text{(C.30)}
\end{aligned}
$$

C.2.2 Asymptotic behaviour at the event horizon

The behaviour of the components of the stress-energy tensor near the event horizon is given by [8]

$$
\langle U|T^\nu_\mu|U\rangle_{ren} \sim -\frac{1}{4\pi^2}\sum_{l,m_l}\int_0^\infty \frac{dw}{w(e^{\frac{2\pi\tilde{w}}{\kappa}}-1)}|B^+_{wlm_l}|^2|S_{wlm_l}(\cos\theta)|^2 \times\left\{\mathcal{Z} - \frac{1}{6}\mathcal{W}I\right\} \qquad \text{(C.31)}
$$

where $\mathcal{Z}$ is the matrix

$$
\begin{pmatrix}
-w\Delta^{-1}\Sigma^{-1}\mathcal{T} & w\tilde{w}\Sigma^{-1} & 0 & w\Delta^{-1}\Sigma^{-1}(r^2+a^2)^{-1}\mathcal{V} \\
-\tilde{w}\Delta^{-2}\Sigma^{-1}(r^2+a^2)\mathcal{T} & \tilde{w}^2\Delta^{-1}\Sigma^{-1}(r^2+a^2) & O(1) & \tilde{w}\Delta^{-2}\Sigma^{-1}\mathcal{V} \\
0 & O(\Delta) & O(1) & 0 \\
m_l\Delta^{-1}\Sigma^{-1}\mathcal{T} & -m_l\tilde{w}\Sigma^{-1} & 0 & -m_l\Delta^{-1}\Sigma^{-1}(r^2+a^2)^{-1}\mathcal{V}
\end{pmatrix} \qquad \text{(C.32)}
$$

and where

$$
\begin{aligned}
\mathcal{T} &= w(r^2+a^2) - m_l a - a\Delta(r^2+a^2)^{-1}(wa\sin^2\theta - m_l) \\
\mathcal{V} &= \Delta(wa - m_l\operatorname{cosec}^2\theta) - aw(r^2+a^2) + m_l a^2 \\
\mathcal{W} &= \Delta^{-1}\Sigma^{-1}(r^2+a^2)^{-1}\left\{-\left[(r^2+a^2)w - m_l a\right]^2 + \Delta(m_l\operatorname{cosec}\theta - wa\sin\theta)^2\right. \\
&\left. + r^2\Delta^2(r^2+a^2)^{-2} + \tilde{w}^2(r^2+a^2)^2\right\} \\
&+ \Sigma^{-1}(r^2+a^2)^{-1}\frac{|S'_{wlm_l}(\cos\theta)|^2}{|S_{wlm_l}(\cos\theta)|^2}\sin^2\theta. \qquad \text{(C.33)}
\end{aligned}
$$

The leading order behaviour of (C.31) is then given by

$$\langle U|T_{\mu}^{\nu}|U\rangle_{ren} \sim \frac{1}{4\pi^2\Sigma}\sum_{l,m_l}\int_0^{\infty}\frac{dw}{w(e^{\frac{2\pi\tilde{w}}{\kappa}}-1)}|B_{wlm_l}^{+}|^2|S_{wlm_l}(\cos\theta)|^2$$

$$\times\begin{pmatrix} \Delta^{-1}(r_+^2+a^2)w\tilde{w} & -w\tilde{w} & 0 & \Delta^{-1}aw\tilde{w} \\ \Delta^{-2}(r_+^2+a^2)^2\tilde{w}^2 & -\Delta^{-1}(r_+^2+a^2)^2\tilde{w}^2 & O(1) & -\Delta^{-2}a(r_+^2+a^2)\tilde{w}^2 \\ 0 & O(\Delta) & O(1) & 0 \\ \Delta^{-1}(r_+^2+a^2)m_l\tilde{w} & m_l\tilde{w} & 0 & -\Delta^{-1}am_l\tilde{w} \end{pmatrix} \quad \text{(C.34)}$$

since, as the horizon is approached,

$$\begin{aligned} \mathcal{T} &\sim \tilde{w}(r_+^2+a^2)+O(\Delta) \\ \mathcal{V} &\sim -a\tilde{w}(r_+^2+a^2)+O(\Delta) \end{aligned} \quad \text{(C.35)}$$

and $\mathcal{W}$ is finite.

C.2.3 Regularity on the horizon

The tensor (C.34) can be shown [8] to be regular at the future, but not on the past, event horizon, which means that, if $\langle H|T_{\mu}^{\nu}|H\rangle_{ren}$ is regular at the event horizon, then $\langle U|T_{\mu}^{\nu}|U\rangle_{ren}$ is regular on the future event horizon, but diverges on the past event horizon.

For non-zero a, the tensor (C.34) is in agreement with that obtained in [134] in the context of equivalence principle analysis. Unfortunately, in [134] the author does not state whether or not his tensor is regular on the future event horizon. This is probably because he was unable to calculate to the order of accuracy required. If this had been the case, we could have used the regularity of (C.34) to imply the regularity of $\langle H|T_{\mu}^{\nu}|H\rangle_{ren}$ at the horizon. All we can say is that the components of $\langle H|T_{\mu}^{\nu}|H\rangle_{ren}$ are at least one order of magnitude smaller than those of (C.34), which is sufficient to ensure that at most two components of $\langle H|T_{\mu}^{\nu}|H\rangle_{ren}$ diverge at the horizon.

C.2.4 Asymptotic behaviour at infinity

We now consider the asymptotic behaviour of the stress tensor at infinity. The leading behaviour is given by:

$$\langle U|T_{\mu}^{\nu}|U\rangle_{ren}-\langle B|T_{\mu}^{\nu}|B\rangle_{ren} \sim \sum_{l,m_l}\int_0^{\infty}\frac{d\tilde{w}}{4\pi^2\tilde{w}(e^{\frac{2\pi\tilde{w}}{\kappa}}-1)}|B_{wlm_l}^{-}|^2|S_{wlm_l}(\cos\theta)|^2$$

$$\times\begin{pmatrix} -w^2r^{-2} & w^2r^{-2} & 0 & m_lwr^{-2} \\ -w^2r^{-2} & w^2r^{-2} & 0 & m_lwr^{-2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad \text{(C.36)}$$

C.2.5 Properties of the Boulware vacuum

The tensor (C.36) represents a thermal flux of radiation at infinity. However, in the limit that the Hawking temperature $T \to 0$ (i.e. $\kappa \to 0$), (C.36) vanishes, whereas the Hawking flux [67, 135, 136]

$$\langle \mathrm{n}_w \rangle = \Gamma_w \left[\exp\left(\frac{2\pi \tilde{w}}{\kappa} \right) - 1 \right] \qquad w > 0 \tag{C.37}$$

reduces to the spontaneous Unruh-Starobinskii emission in the super-radiant modes. Hence, in this case, (C.36) does not give the form we expect of the Unruh vacuum state. The reason for this is that our definition of the Boulware vacuum state corresponds at infinity to the particles produced by the Unruh-Starobinskii effect, namely

$$\langle B|T_\mu^\nu|B\rangle_{ren} \sim \frac{1}{4\pi^2 r^2} \sum_{l,m_l} \int_0^{w_{min}} dw\, (1 - |A^+_{wlm_l}|^2)|S_{wlm_l}(\cos\theta)|^2$$
$$\times \begin{pmatrix} w & -w & 0 & -m_l \\ w & -w & 0 & -m_l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.38}$$

Using the Wronskian relations between the reflection and transmission coefficients [8] and adding (C.38) to (C.36) yields

$$\langle U|T_\mu^\nu|U\rangle_{ren} \sim \frac{1}{4\pi^2 r^2} \sum_{l,m_l} \int_0^{\infty} \frac{dw}{e^{\frac{2\pi\tilde{w}}{\kappa}} - 1} (1 - |A^+_{wlm_l}|^2)|S_{wlm_l}(\cos\theta)|^2$$
$$\times \begin{pmatrix} -w & w & 0 & m_l \\ -w & w & 0 & m_l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.39}$$

Equation (C.39) is the expected form of the stress tensor for the Unruh vacuum state, corresponding to a thermal distribution of quanta which reduces to the super-radiance effects in the limit $T \to 0$. It is also the form of $\langle U|T_\mu^\nu|U\rangle_{ren}$ calculated in [134] by an equivalence principle approach. This suggests that an alternative definition of the "Boulware vacuum" $|\tilde{B}\rangle$ may produce a state which resembles more closely the state $|B\rangle$ in Schwarzschild space-time.

C.2.6 Modifying the Boulware vacuum

We would like to construct a state which is as empty as possible at infinity, having no super-radiant flux. Consider the state $|\tilde{B}\rangle$, defined by considering outgoing modes with positive frequency with respect to u on $\mathcal{I}^+$, and in-going modes with positive frequency with respect to v on $\mathcal{I}^-$. Then

$$\langle\tilde{B}|T_{\mu\nu}|\tilde{B}\rangle = \sum_{l,m_l}\int_0^\infty dw\, T_{\mu\nu}[u^-_{wlm_l}, u^{-*}_{wlm_l}] + \sum_{l,m_l}\int_0^\infty dw\, T_{\mu\nu}[u^+_{wlm_l}, u^{+*}_{wlm_l}]. \quad \text{(C.40)}$$

At infinity,

$$\langle U|T^\nu_\mu|U\rangle_{ren} - \langle\tilde{B}|T_{\mu\nu}|\tilde{B}\rangle_{ren} = \langle U|T^\nu_\mu|U\rangle - \langle\tilde{B}|T_{\mu\nu}|\tilde{B}\rangle$$

$$\sim \frac{1}{4\pi^2 r^2}\sum_{l,m_l}\int_0^\infty \frac{dw}{e^{\frac{2\pi\tilde{w}}{\kappa}}-1}(1-|A^+_{wlm_l}|^2)|S_{wlm_l}(\cos\theta)|^2$$

$$\times\begin{pmatrix} -w & w & 0 & m_l \\ -w & w & 0 & m_l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{(C.41)}$$

which is the asymptotic behaviour exhibited by $\langle U|T^\nu_\mu|U\rangle_{ren}$ in (C.39). Hence the components of $\langle\tilde{B}|T_{\mu\nu}|\tilde{B}\rangle_{ren}$ are at most $O(r^{-3})$ at $r \to \infty$. Thus it may be expected that $|\tilde{B}\rangle$, rather than the state $|B\rangle$, exhibits the asymptotic properties of the Boulware vacuum as observed in Schwarzschild space-time.

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