

Turbulent damping of fast tidal oscillations by three-dimensional Rayleigh–Bénard convection with a radiating free surface

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Accepted 2026 May 6. Received 2026 May 3; in original form 2026 February 17

ABSTRACT

We present three-dimensional Dedalus simulations of Rayleigh–Bénard convection with a blackbody-radiating free upper surface, subject to a low-amplitude oscillatory forcing that mimics tidal perturbations in convective envelopes of stars and planets. The forcing period is 10–100 times shorter than the convective time-scale, t_{conv} . Using a Reynolds decomposition of the velocity field averaged over one oscillation period, in which the tidal oscillations naturally constitute the fluctuating field and convection the mean flow, we elucidate the kinetic energy exchange between the two. Provided the oscillatory Reynolds number exceeds a modest threshold, we find that the oscillations systematically transfer kinetic energy to the mean flow at a volume-averaged rate $D_R \sim u'^2 t_{\text{conv}}^{-1}$, where u' is the rms fluctuation velocity. This reflects strong, order-unity correlations between the fluctuation velocities and the mean flow. These arise because the oscillatory forcing displaces fluid elements that are then redirected by buoyancy and incompressibility in the same manner as the mean flow. The transfer is dominated by correlations involving vertical velocity fluctuations and vertical gradients of the mean flow. The resulting energy transfer rate is consistent, within the equilibrium-tide framework, with the observed tidal circularization of solar-type binaries and with the orbital evolution of moons of Jupiter and Saturn. This validates the formalism proposed by C. Terquem (2021) for the dissipation of fast tides, a long-standing problem. Replacing the free surface with a rigid upper boundary significantly and artificially modifies the correlations.

Key words: convection – hydrodynamics – Sun: general – planets and satellites: dynamical evolution and stability – planet–star interactions – binaries: close.

1 INTRODUCTION

Tidal dissipation in convective flows plays a key role in the evolution of stellar binaries, star–planet systems, and the satellite systems of giant planets (G. I. Ogilvie 2014). In many of these systems, the tidal forcing varies on time-scales much shorter than the convective turnover time. How such fast tides interact with turbulent convection, and how efficiently they are dissipated, remains an open problem.

Starting with J. P. Zahn (1966), tidal dissipation in convective regions was traditionally described using an effective turbulent viscosity acting on the large-scale oscillatory flow. In this model, based on mixing length theory, dissipation is assumed to arise from correlations between convective velocity fluctuations and

gradients of the tidal flow. When the tidal period is much shorter than the convective overturning time, convective eddies are assumed to respond inefficiently to the rapidly varying shear. This behaviour is implemented through a prescribed reduction of the effective turbulent viscosity at short forcing periods (J. P. Zahn 1966, P. Goldreich & P. D. Nicholson 1977), rather than emerging directly from the mixing-length formalism itself. This leads to dissipation rates that are typically orders of magnitude too small to explain the observations.

C. Terquem (2021, 2023) showed that this discrepancy reflects a more fundamental limitation of the turbulent-viscosity approach in the fast-tide regime. When the forcing period is short, the tidal flow must instead be treated as part of the fluctuating velocity field, with the mean flow defined by temporal averaging over an oscillation period. Dissipation then arises from correlations between the oscillatory velocity fluctuations and gradients of the convective flow, rather than from convective motions being sheared by the tide. This leads to a qualitatively different dissipation law: the rate still scales with the inverse convective time-scale but,

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crucially, it does not suffer the strong reduction at short forcing periods that is artificially imposed in traditional turbulent-viscosity models.

Despite this theoretical progress, the existence and robustness of such Reynolds-stress correlations in fully three-dimensional convection remain debated. Most numerical studies have instead focused on estimating an effective turbulent viscosity (K. Penev et al. 2009; G. I. Ogilvie & G. Lesur 2012; C. D. Duguid, A. J. Barker & C. A. Jones 2020; J. Vidal & A. J. Barker 2020a, b). To date, the only direct numerical attempt to evaluate the dissipation rate proposed by C. Terquem (2021) in a fully convective flow is that of A. J. Barker & A. A. V. Astoul (2021). However, the use of rigid boundaries at both the top and bottom of the domain precludes a fully consistent computation of the work done by a potential tidal force in an incompressible fluid. As a consequence, they prescribe the tidal response kinematically as an irrotational velocity field, appearing only through advection terms in the mean-flow equations. While this set-up captures some aspects of tide-convection interactions, it does not allow the oscillatory tidal flow to emerge self-consistently as the dynamical response to an external forcing.

In contrast, the present study models tides as an externally applied oscillatory forcing, allowing the full velocity field to adjust dynamically and to generate secondary fluctuations as part of the convective flow itself. This distinction is crucial: the Reynolds-stress correlations responsible for energy transfer arise precisely because oscillatory displacements are redirected by buoyancy and incompressibility in the same way as the mean flow itself, rather than being prescribed *a priori* as a fixed velocity field. The forcing period is chosen to be one to two orders of magnitude shorter than the convective time-scale, placing the system firmly in the fast-tide regime. In this limit, the mean flow is naturally defined by averaging over the oscillation period rather than over the convective time-scale. Accordingly, we adopt a Reynolds decomposition based on temporal averaging over one oscillation period and analyse the kinetic-energy exchange between the oscillatory fluctuations and the mean convective flow.

To the best of our knowledge, this is the first three-dimensional numerical study of turbulent Rayleigh–Bénard convection with a dynamically deformable free surface and radiative cooling, subject to fast oscillatory forcing. Previous numerical studies of convection with deformable free surfaces have largely focused on weakly non-linear Marangoni-driven or mixed Rayleigh–Bénard–Marangoni regimes, primarily concerned with instability and pattern formation, rather than on turbulent convection and the energy transfer between oscillatory motions and the mean flow considered here (e.g. K. A. Cliffe & S. J. Tavener 1998; C. Dang et al. 2021). The free surface is essential for capturing the large-scale vertical structure of convective plumes and the associated mean-flow gradients that drive the Reynolds-stress correlations. As we show below, imposing a rigid upper boundary qualitatively alters these correlations and can suppress the net energy transfer altogether. Moreover, when the flow is incompressible and the forcing derives from a scalar potential, a free surface is required for the forcing to perform net work on the flow.

We show that, above a modest threshold in the fluctuating Reynolds number, the oscillations systematically transfer kinetic energy to the mean flow at a rate $D_R \sim u'^2 \tau_{\text{conv}}^{-1}$, independent of the forcing period over the range explored, where u' is the rms fluctuation velocity. We identify the Reynolds-stress components responsible for this transfer and discuss how viscous

diffusion, thermal diffusion and boundary conditions control the establishment of the underlying correlations. Our results provide direct numerical support for the fast-tide dissipation mechanism proposed by C. Terquem (2021, 2023).

The outline of the paper is as follows. In Section 2, we introduce the governing equations, define the equilibrium state and describe the Reynolds decomposition appropriate for fast oscillatory forcing. We also derive the mean-flow and fluctuation momentum equations, together with the associated local and global kinetic-energy balances that form the theoretical framework of the study. Section 3 presents the boundary conditions, including the formulation of the radiating, deformable free surface adopted here. In Section 4, we describe the numerical set-up. We first present unforced reference simulations used to characterise the convective flow, and then introduce the oscillatory forcing. We consider both a purely vertical forcing, which provides a simple framework for understanding how correlations arise within the convective flow, and a forcing that derives from a scalar potential and mimics tidal forcing. We then analyse the resulting energy budgets and Reynolds-stress transfer, and examine their dependence on viscosity and boundary conditions. In Section 5, we discuss the physical origin of the Reynolds-stress correlations. Finally, Section 6 summarises the main results and discusses their implications for tidal dissipation in stellar and planetary convective envelopes.

2 GOVERNING EQUATIONS

We consider a fluid with velocity \mathbf{u} , mass density ρ , pressure P and temperature T subject to an external gravitational field \mathbf{g} and an imposed force per unit mass \mathbf{f} . The fluid has constant kinematic viscosity ν and thermal diffusivity κ . We adopt Cartesian coordinates (x, y, z) with corresponding unit vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$, such that $\mathbf{g} = -g\hat{\mathbf{z}}$. The force \mathbf{f} represents an externally imposed perturbation and is taken to be zero in the equilibrium state defined below.

The system is governed by the continuity, Navier–Stokes and thermal energy equations which, in the Boussinesq approximation introduced below, take the form:

$$\nabla \cdot \mathbf{u} = 0, \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{u} - g\hat{\mathbf{z}} + \mathbf{f}, \quad (2)$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T. \quad (3)$$

2.1 Equilibrium state

In equilibrium, the fluid is at rest ($\mathbf{u} = \mathbf{0}$), the external forcing vanishes ($\mathbf{f} = \mathbf{0}$) and the domain spans $0 \leq z \leq H$. The temperature is fixed at T_1 at the bottom and T_2 at the surface, with $T_2 < T_1$. We define $\Delta T \equiv T_1 - T_2$ and assume that the interior temperature profile is linear:

$$T_0(z) = T_1 - \frac{T_1 - T_2}{H} z. \quad (4)$$

With this choice, the thermal equation (3) is automatically satisfied. The mass density is taken to be uniform, equal to ρ_0 , and the equilibrium pressure profile $P_0(z)$ satisfies the hydrostatic balance condition from equation (2):

$$\nabla P_0 = -\rho_0 g \hat{\mathbf{z}}. \quad (5)$$

2.2 Perturbation and Boussinesq approximation

Deviations from the equilibrium state are characterized by a velocity field $\mathbf{u} = (u_x, u_y, u_z)$ and perturbations in density, pressure and temperature: $\rho = \rho_0 + \delta\rho$, $P = P_0 + \delta P$ and $T = T_0 + \delta T$. We assume that density variations are small and arise solely from temperature perturbations, with pressure-induced density variations neglected. Specifically, we write:

$$\rho = \rho_0 (1 - \alpha \delta T), \quad (6)$$

where α is the (constant) coefficient of thermal expansion and $|\alpha \delta T| \ll 1$. Within the Boussinesq approximation, density variations are neglected in the mass conservation equation, yielding the incompressibility equation (1). In the momentum equation (2), density variations affect the dynamics only through the pressure-gradient term. A first-order expansion of $1/\rho$ in δT , combined with the hydrostatic balance (5), produces the buoyancy term $g\alpha\delta T\hat{\mathbf{z}}$. The governing equations for the perturbed quantities then become

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0} \nabla \delta P + \nu \nabla^2 \mathbf{u} + g\alpha\delta T\hat{\mathbf{z}} + \mathbf{f}, \quad (7)$$

$$\frac{\partial \delta T}{\partial t} + \mathbf{u} \cdot \nabla (T_0 + \delta T) = \kappa \nabla^2 \delta T. \quad (8)$$

In the absence of external forcing ($\mathbf{f} = \mathbf{0}$), this system reduces to the classical Rayleigh–Bénard problem. Here, we focus on the turbulent convective regime (unstable equilibrium) and apply \mathbf{f} as an oscillatory forcing to model perturbations within an already convective flow.

2.3 Dimensionless form

The Rayleigh and Prandtl numbers are defined as

$$Ra = \frac{\alpha \Delta T g d^3}{\nu \kappa}, \quad Pr = \frac{\nu}{\kappa}, \quad (9)$$

where d is a characteristic length-scale. The Rayleigh number measures the relative strength of buoyancy-driven advection compared to thermal and viscous diffusion. When Ra exceeds a critical value Ra_c , the equilibrium state becomes unstable and thermal convection develops, corresponding to the classical Rayleigh–Bénard instability. In non-turbulent regimes, d is typically taken to be the depth of the convective layer (here, H). However, in turbulent flows, a more appropriate length-scale may be the pressure scale height. We therefore formulate the equations using a general characteristic length d . For the numerical simulations presented below, where the pressure scale height and layer depth are comparable, we take $d = H$.

We also define dimensionless viscosity and thermal diffusivity as:

$$\bar{\kappa} \equiv \frac{\kappa}{\sqrt{\alpha \Delta T g d^3}} = Ra^{-1/2} Pr^{-1/2}, \quad (10)$$

$$\bar{\nu} \equiv \frac{\nu}{\sqrt{\alpha \Delta T g d^3}} = Ra^{-1/2} Pr. \quad (11)$$

Next, we introduce a characteristic velocity and time-scale:

$$U = \sqrt{\alpha \Delta T g d}, \quad \tau = \frac{d}{U} = \sqrt{\frac{d}{\alpha \Delta T g}}. \quad (12)$$

Using these, we define the following dimensionless variables: velocity $\tilde{\mathbf{u}} = \mathbf{u}/U$, time $\tilde{t} = t/\tau$, pressure $\tilde{P} = \delta P/(\rho_0 U^2)$, temperature perturbation $\tilde{b} = \delta T/\Delta T$ and external force $\tilde{\mathbf{f}} = \mathbf{f}/(\alpha \Delta T g)$. We also rescale the spatial coordinates as $\tilde{x} = x/d$, $\tilde{y} = y/d$, $\tilde{z} =$

z/d , which gives the dimensionless gradient operator $\tilde{\nabla} = d\nabla$, and $L_z = H/d$.

Substituting these expressions into equations (1), (7) and (8) yields the dimensionless equations:

$$\tilde{\nabla} \cdot \tilde{\mathbf{u}} = 0, \quad (13)$$

$$\frac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + \tilde{\mathbf{u}} \cdot \tilde{\nabla} \tilde{\mathbf{u}} = -\tilde{\nabla} \tilde{P} + \bar{\nu} \tilde{\nabla}^2 \tilde{\mathbf{u}} + \tilde{b} \hat{\mathbf{z}} + \tilde{\mathbf{f}}, \quad (14)$$

$$\frac{\partial \tilde{b}}{\partial \tilde{t}} + \tilde{\mathbf{u}} \cdot \tilde{\nabla} \tilde{b} - \frac{\tilde{u}_z}{L_z} = \bar{\kappa} \tilde{\nabla}^2 \tilde{b}. \quad (15)$$

The quantity $b \equiv \delta T$ is often loosely referred to as the *buoyancy*, although the actual buoyancy force per unit mass is $g\alpha\delta T$.

We also introduce the dimensionless viscous stress tensor:

$$\tilde{\sigma}_{ij} = \tilde{\sigma}_{ji} = \bar{\nu} \left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i} \right), \quad (16)$$

where indices i and j take values in (1,2,3), corresponding to $x_1 = x$, $x_2 = y$ and $x_3 = z$. Because the flow is incompressible, the viscous term in equation (14) may equivalently be written as $\bar{\nu} \tilde{\nabla}^2 \tilde{u}_i = \partial \tilde{\sigma}_{ij} / \partial \tilde{x}_j$.

2.4 Time-scales and Reynolds decomposition

In stellar models, the convective time-scale t_{conv} at a radius r is the time required for energy to be transported across the local mixing-length (eddy turnover length). It is commonly defined as $t_{\text{conv}} \sim H_P/V_{\text{conv}}$, where H_P is the local pressure scale height and V_{conv} is the horizontally averaged convective velocity on the sphere of radius r . The Rayleigh–Bénard simulations presented here model a local patch of a stellar convective envelope, with the vertical extent of the domain corresponding to the eddy turnover length. Accordingly, we define the convective time-scale in the volume \mathcal{V} of the simulation as:

$$t_{\text{conv}} = \frac{H}{u_{z,\text{rms}}}, \quad (17)$$

where:

$$u_{z,\text{rms}} = \sqrt{\frac{1}{\mathcal{V}} \int_{\mathcal{V}} u_z^2 dv}, \quad (18)$$

is the volume-averaged rms vertical velocity. Both upward and downward motions contribute to $u_{z,\text{rms}}$, consistent with the fact that energy is transported towards the surface by the correlation between u_z and the temperature fluctuation δT , not by the mean vertical flow.

Another relevant time-scale is the crossing time:

$$t_{\text{cross}} = \frac{H}{u_{z,\text{max}}}, \quad (19)$$

defined as the time it takes for the fastest plumes to traverse the layer. A defining feature of convection, especially at high Rayleigh numbers, is that $t_{\text{cross}} \ll t_{\text{conv}}$, because heat transport is dominated by rare, rapid plumes that occupy only a small fraction of the flow volume.

We consider a force \mathbf{f} that oscillates in time with period t_{osc} such that $t_{\text{osc}} \ll t_{\text{conv}}$. We perform a Reynolds decomposition:

$$\mathbf{u} = \mathbf{V} + \mathbf{u}', \quad \text{where } \mathbf{V} \equiv \langle \mathbf{u} \rangle \quad (20)$$

with the angle brackets denoting a time average over one oscillation period t_{osc} . By construction, at a fixed point in the flow, \mathbf{u}' oscillates about zero on the time-scale t_{osc} . The

mean velocity \mathbf{V} evolves on a time-scale $\tau_V \gg t_{\text{osc}}$, determined by the intrinsic convective dynamics. The decomposition (20) is therefore well-defined. At high Rayleigh numbers, where convection is fully turbulent, $\tau_V \sim t_{\text{cross}}$. In contrast, at the moderate Rayleigh numbers explored in this study, convective structures are more persistent and $\tau_V > t_{\text{conv}}$, so that the mean flow appears quasi-steady on time-scales small compared to t_{conv} .

In three dimensions, convection supports a broad spectrum of eddies with varying velocities, length-scales and time-scales, reflecting the cascade of kinetic energy from large to small scales. Since the largest eddies have both the highest velocities and the longest intrinsic time-scales, the mean flow \mathbf{V} is dominated by these large-scale coherent structures. The fluctuation field \mathbf{u}' comprises (i) the direct velocity response to the oscillating force, (ii) secondary oscillations generated by the interaction of this response with convection, and (iii) convective motions whose intrinsic time-scales are comparable to or shorter than t_{osc} .

2.5 Averaged equations

We focus on cases where $t_{\text{osc}} \ll t_{\text{conv}}$. In this regime, mean quantities can be considered approximately constant over a single oscillation period. This assumption remains valid even when t_{osc} is comparable to, or exceeds, t_{cross} , provided that $t_{\text{cross}} \ll t_{\text{conv}}$.

Substituting the Reynolds decomposition into the incompressibility equation (13) and averaging over \tilde{t}_{osc} gives:

$$\bar{\nabla} \cdot \bar{\mathbf{V}} = 0. \quad (21)$$

Subtracting this from equation (13) yields:

$$\bar{\nabla} \cdot \bar{\mathbf{u}}' = 0, \quad (22)$$

indicating that both the mean flow and the fluctuations are incompressible.

2.5.1 Momentum equations

Substituting the Reynolds decomposition into the momentum equation (14) and averaging over \tilde{t}_{osc} gives the mean-flow momentum equation:

$$\begin{aligned} \frac{\partial \bar{V}_i}{\partial \tilde{t}} + \bar{V}_j \frac{\partial \bar{V}_i}{\partial \bar{x}_j} + \left\langle \bar{u}'_j \frac{\partial \bar{u}'_i}{\partial \bar{x}_j} \right\rangle \\ = -\frac{\partial \langle \bar{P} \rangle}{\partial \bar{x}_i} + \frac{\partial \langle \bar{\sigma}_{ij} \rangle}{\partial \bar{x}_j} + \langle \bar{b} \rangle \delta_{3i}, \end{aligned} \quad (23)$$

with summation over repeated indices implied. We have used the identity $\langle \partial \bar{u}'_i / \partial \tilde{t} \rangle = 0$, which holds because \bar{u}'_i is periodic over the averaging time interval.

Equation (23) retains all non-linear terms and does not assume that the fluctuations are small compared to the mean quantities. Since the external forcing $\bar{\mathbf{f}}$ does not appear explicitly, it influences the mean flow only through the term $\langle \bar{u}'_j (\partial \bar{u}'_i / \partial \bar{x}_j) \rangle$. As long as this term remains small compared to the dominant buoyancy contribution $\langle \bar{b} \rangle$ and/or the mean advection term $\bar{V}_j (\partial \bar{V}_i / \partial \bar{x}_j)$, the amplitude of the forcing may be chosen freely without altering the leading-order balance of the mean flow.

Subtracting equation (23) from equation (14) gives the fluctuation momentum equation:

$$\begin{aligned} \frac{\partial \bar{u}'_i}{\partial \tilde{t}} + \bar{u}'_j \frac{\partial \bar{V}_i}{\partial \bar{x}_j} + \bar{V}_j \frac{\partial \bar{u}'_i}{\partial \bar{x}_j} + \bar{u}'_j \frac{\partial \bar{u}'_i}{\partial \bar{x}_j} - \left\langle \bar{u}'_j \frac{\partial \bar{u}'_i}{\partial \bar{x}_j} \right\rangle \\ = -\frac{\partial \bar{P}'}{\partial \bar{x}_i} + \frac{\partial \bar{\sigma}'_{ij}}{\partial \bar{x}_j} + \bar{b}' \delta_{3i} + \bar{f}'_i, \end{aligned} \quad (24)$$

where the fluctuating components are defined as $\bar{P}' = \bar{P} - \langle \bar{P} \rangle$, $\bar{b}' = \bar{b} - \langle \bar{b} \rangle$ and $\bar{\sigma}'_{ij} = \bar{\sigma}_{ij} - \langle \bar{\sigma}_{ij} \rangle$.

2.5.2 Local energy equations

Multiplying equation (23) by \bar{V}_i and summing over i yields the kinetic energy equation for the mean flow:

$$\frac{\partial \bar{E}_k}{\partial \tilde{t}} = -\frac{\partial \bar{F}_j}{\partial \bar{x}_j} + \bar{D}_R - \bar{D}_v + \langle \bar{b} \rangle \bar{V}_z, \quad (25)$$

where $\bar{E}_k \equiv \frac{1}{2} (\bar{V}_x^2 + \bar{V}_y^2 + \bar{V}_z^2)$ is the kinetic energy of the mean flow. The transfer and dissipation terms are given by

$$\bar{D}_R = \left\langle \bar{u}'_i \bar{u}'_j \right\rangle \frac{\partial \bar{V}_i}{\partial \bar{x}_j}, \quad (26)$$

$$\bar{D}_v = \langle \bar{\sigma}_{ij} \rangle \frac{\partial \bar{V}_i}{\partial \bar{x}_j}. \quad (27)$$

The associated mean energy flux $\bar{\mathbf{F}}$ is

$$\bar{F}_j = \bar{E}_k \bar{V}_j + \left\langle \bar{u}'_i \bar{u}'_j \right\rangle \bar{V}_i - \langle \bar{\sigma}_{ij} \rangle \bar{V}_i + \langle \bar{P} \rangle \bar{V}_j. \quad (28)$$

In deriving equation (25), we have used the incompressibility of both $\bar{\mathbf{V}}$ and $\bar{\mathbf{u}}'$. For later reference, we define the kinetic contribution to the flux as

$$\bar{F}_{\text{kin},j} = \bar{E}_k \bar{V}_j + \left\langle \bar{u}'_i \bar{u}'_j \right\rangle \bar{V}_i. \quad (29)$$

Similarly, multiplying equation (24) by \bar{u}'_i , summing over i and averaging over \tilde{t}_{osc} yields the kinetic energy equation for the fluctuations:

$$\frac{\partial \langle \bar{e}'_k \rangle}{\partial \tilde{t}} = -\frac{\partial \langle \bar{F}'_j \rangle}{\partial \bar{x}_j} - \bar{D}_R - \bar{D}'_v + \langle \bar{b}' \bar{u}'_z \rangle + \langle \bar{\mathbf{f}} \cdot \bar{\mathbf{u}}' \rangle, \quad (30)$$

where $\bar{e}'_k \equiv \frac{1}{2} (\bar{u}'_x^2 + \bar{u}'_y^2 + \bar{u}'_z^2)$ is the kinetic energy of the fluctuations. The viscous dissipation rate is

$$\bar{D}'_v = \left\langle \bar{\sigma}'_{ij} \frac{\partial \bar{u}'_i}{\partial \bar{x}_j} \right\rangle, \quad (31)$$

and the mean fluctuation energy flux satisfies:

$$\langle \bar{F}'_j \rangle = \langle \bar{e}'_k \rangle \bar{V}_j + \langle \bar{e}'_k \bar{u}'_j \rangle - \langle \bar{\sigma}'_{ij} \bar{u}'_i \rangle + \langle \bar{P}' \bar{u}'_j \rangle. \quad (32)$$

In deriving equation (30), we have used the identity $\langle \partial \bar{e}'_k / \partial \tilde{t} \rangle = \partial \langle \bar{e}'_k \rangle / \partial \tilde{t}$, which holds because the averaging is performed over one oscillation cycle of \bar{e}'_k .

The term \bar{D}_R represents the rate of energy exchange between the mean flow and the fluctuations via the Reynolds stress, i.e. correlations between the components of the fluctuation velocity. When $\bar{D}_R > 0$, energy is transferred from the fluctuations to the mean flow. The objective of this study is to determine the sign of \bar{D}_R . We emphasise that \bar{D}_R is a *transfer term*, not a dissipation rate. When $\bar{D}_R > 0$, the fluctuations are effectively damped because they lose energy to the mean flow. However, this energy is not dissipated directly at the fluctuation scale. Instead, it is subsequently removed either by viscosity acting on the mean flow or by transport through the boundaries *via* the energy flux.

2.5.3 Global energy balances

Integrating equation (25) over the flow volume yields the global energy balance for the mean flow:

$$-\int \tilde{\mathbf{F}} \cdot \hat{\mathbf{n}} \, ds + \int (\tilde{D}_R - \tilde{D}_v + \langle \tilde{b} \rangle \tilde{V}_z) \, dv = 0, \quad (33)$$

where the surface integral is taken over the boundaries of the domain, with $\hat{\mathbf{n}}$ the outward unit normal. We have assumed a statistically steady state ($\partial \tilde{E}_k / \partial \tilde{t} = 0$), although the time variation can be retained if necessary.

Similarly, integrating equation (30) gives the global energy balance for the fluctuations:

$$-\int \langle \tilde{\mathbf{F}}' \rangle \cdot \hat{\mathbf{n}} \, ds + \int (-\tilde{D}_R - \tilde{D}_v + \langle \tilde{b}' \tilde{u}'_z \rangle + \langle \tilde{\mathbf{f}} \cdot \tilde{\mathbf{u}}' \rangle) \, dv = 0. \quad (34)$$

Again, the time variation of $\langle \tilde{e}'_k \rangle$ may be included if needed.

3 BOUNDARY CONDITIONS

Periodic boundary conditions are applied in the horizontal directions.

3.1 Lower boundary ($z = 0$)

By analogy with the Sun, the convectively unstable region considered here may overlie a stably stratified layer, with $z = 0$ representing the interface between the two. In such configurations, convective overshooting can lead to a non-zero vertical velocity at this boundary. However, this effect is not included in the present study, and we therefore impose $\tilde{u}_z = 0$ at $z = 0$.

To complete the velocity boundary conditions, we assume that the tangential stresses vanish at the interface:

$$\tilde{\sigma}_{xz} = 0, \quad \tilde{\sigma}_{yz} = 0. \quad (35)$$

Such a stress-free interface approximates the behaviour of a fluid-fluid boundary with negligible resistance to shear.

For the temperature, we impose a fixed-flux condition at the lower boundary, corresponding to constant heating from below. The radiative flux at $z = 0$ is held equal to its hydrostatic equilibrium value, denoted F_0 :

$$-k \nabla T \cdot \hat{\mathbf{n}} = F_0, \quad (36)$$

where k is the thermal conductivity and $\hat{\mathbf{n}}$ is the unit normal to the boundary. At $z = 0$, we have $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, so the condition becomes $\partial T / \partial z = -F_0/k$. Writing $T = T_0 + \delta T$, and noting that F_0 sets the equilibrium gradient $dT_0/dz = -F_0/k$, we obtain the final boundary condition:

$$\frac{\partial \tilde{b}}{\partial \tilde{z}} = 0. \quad (37)$$

3.2 Free surface ($z = H + \eta$)

In the simulations presented below, the upper surface is free and located at $z = H + \eta(x, y)$, where η represents the departure from the equilibrium height $z = H$. The surface radiates as a blackbody at temperature $T(H + \eta)$.

In Appendix A, we derive the kinematic condition (A3), the continuity of tangential stresses (A8) and (A9), the continuity of normal stress (A14) and the radiative condition (A20). Together, these form the surface boundary conditions, valid to first order in $|\eta|/H$.

These equations include terms involving products of $\tilde{\eta} = \eta/d$ or its derivatives with dimensionless perturbed quantities. Since $|\tilde{\eta}| \ll 1$ in all simulations presented here, such non-linear terms are neglected in comparison to the perturbations themselves. This leads to the following boundary conditions at $\tilde{z} = L_z \equiv H/d$:

$$\tilde{u}_z = \frac{\partial \tilde{\eta}}{\partial \tilde{z}}, \quad (38)$$

$$\tilde{\sigma}_{xz} = 0, \quad \tilde{\sigma}_{yz} = 0, \quad (39)$$

$$\tilde{\sigma}_{zz} + \frac{\tilde{\eta}}{\alpha \Delta T} - \tilde{p} = 0, \quad (40)$$

$$L_z \frac{\partial \tilde{b}}{\partial \tilde{z}} = 1 + \frac{4\tilde{\eta}}{L_z \tilde{T}_2} - \left(1 + \frac{\tilde{b}}{\tilde{T}_2}\right)^4. \quad (41)$$

4 NUMERICAL SIMULATIONS

This section describes the numerical set-up and diagnostics used to quantify the transfer of kinetic energy between fluctuations and the mean flow. We first discuss the non-dimensional parameters, present the time-averaging procedure used to separate the mean flow from the fluctuations and characterize the unforced convective state that serves as the baseline reference. We then introduce an external oscillatory forcing and discuss the conditions required to ensure it probes the coupling between fluctuations and the mean flow without altering the leading-order dynamics of the convection. Finally, we analyse the forced simulations through their global energy budgets, dissipation scales and the detailed structure of the Reynolds-stress transfer term D_R , demonstrating its robustness across different oscillation time-scales and forcing configurations.

We solve equations (13), (14), and (15) using the spectral code Dedalus (K. J. Burns et al. 2020). For our Cartesian Rayleigh-Bénard set-up, the horizontal directions are expanded in Fourier bases and the vertical direction in Chebyshev polynomials.

4.1 Parameter choices and non-dimensional numbers

As discussed in Section 2.3, we adopt $d = H$ (the vertical extent of the domain) as the characteristic length-scale, yielding $L_z \equiv H/d = 1$. The horizontal dimensions are $L_x = L_y = 3$, sufficient to accommodate two convection cells while remaining computationally tractable. We set the Prandtl number to $Pr = 1$ and, in most simulations, the Rayleigh number to $Ra = 10^6$, corresponding to $\tilde{\nu} = \tilde{\kappa} = 10^{-3}$. These are the only dimensionless control parameters entering the governing equations. For this parameter set and the range of oscillation periods t_{osc} considered, numerical convergence of the forced simulations required 160–192 grid points in the vertical direction and 60–70 grid points in each horizontal direction.

For the boundary conditions, two additional quantities must be specified: $\alpha \Delta T$ and \tilde{T}_2 . The product $\alpha \Delta T$ represents the fractional density change across the imposed temperature difference ΔT and must be small to justify the Boussinesq approximation. We therefore set $\alpha \Delta T = 0.1$.

Because the flow is heated from below, convection cools the lower region and warms the upper boundary layer through the upward transport of heat. As a result, the horizontally averaged temperature perturbation δT at the upper surface is positive, implying a radiative flux σT^4 that exceeds the equilibrium value F_0 . This boundary condition, which provides a more realistic representation of a stellar envelope or giant-planet atmosphere, makes convection slightly more vigorous than in the case of a fixed-flux

upper boundary. We have verified that this choice does not alter our main conclusions by performing additional simulations with a fixed-flux condition at the top. In the simulations presented here, we fix the surface temperature $\bar{T}_2 = 0.25$, corresponding to $T_2/T_1 = 0.2$. Varying \bar{T}_2 within a reasonable range does not affect the qualitative behaviour of the flow.

4.2 Averaging procedure

To track the energy budget of the fluctuations, we compute time averages over one oscillation period t_{osc} not only of the full fields (e.g. velocity components), but also of products of fluctuating quantities (e.g. $u'_i u'_j$). Using equation (20), we compute the average Q of any quantity q as

$$Q(nt_{\text{osc}}) \equiv \langle q \rangle (nt_{\text{osc}}) = \frac{1}{t_{\text{osc}}} \int_{nt_{\text{osc}}}^{(n+1)t_{\text{osc}}} q dt. \quad (42)$$

Consider two quantities q and r whose averages Q and R vary on time-scales much longer than t_{osc} . We then have

$$\langle qr \rangle = \langle (Q + q')(R + r') \rangle = QR + \langle q'r' \rangle, \quad (43)$$

which implies $\langle q'r' \rangle = \langle qr \rangle - QR$. This identity allows us to compute $\langle q'r' \rangle$ without storing the fluctuating fields q' and r' , relying only on time-averaged quantities. The same decomposition readily extends to products of three or more fluctuating fields, as needed for the flux terms in the energy budgets.

An important consequence of this finite-window block average is that, even in the absence of external forcing, the residual field u' contains contributions from the temporal evolution of the convective flow on time-scales longer than t_{osc} within each averaging window. This is distinct from what we would obtain with, say, a Fourier decomposition at the forcing frequency, which would isolate only the oscillatory response. As we will see below, in the forced case, the externally driven oscillations dominate the fluctuation budget and this distinction is inconsequential. However, in the unforced case, the fluctuation field is dominated by this finite-window artefact. Keeping this in mind is essential for correctly interpreting the sign and magnitude of D_R in the two cases, as discussed below.

4.3 Simulations without forcing

We begin by considering unforced Rayleigh–Bénard convection, i.e. by setting $\mathbf{f} = 0$ in equation (14).

Fig. 1 shows instantaneous snapshots of the velocity field, illustrating the characteristic flow structures through the components \tilde{u}_x and \tilde{u}_z and their vertical gradients in the plane $\tilde{y} = L_y/2$. All simulations with $L_x = L_y = 3$ and $L_z = 1$ exhibit one ascending and one descending plume, accompanied by two vortices.

Fig. 2 shows the total energy budget, with all terms expressed as volume-averaged quantities (i.e. integrated over the domain and divided by the total volume).

To provide a baseline against which to compare simulations with external forcing, we apply the Reynolds decomposition introduced in Section 2.4, using a temporal block average over non-overlapping windows of duration t_{osc} , as described in Section 4.2. In the absence of forcing, t_{osc} acts as a temporal filter that separates the flow into a slowly varying mean component $\mathbf{V}(t)$ and a residual field $\mathbf{u}' = \mathbf{u} - \mathbf{V}$, which includes both motions on time-scales shorter than t_{osc} and contributions from the temporal evolution of the mean flow within each averaging window. Since $|\mathbf{u}'| \ll |\mathbf{u}|$ in all cases considered here, the mean flow \mathbf{V} is nearly

identical to the full velocity field \mathbf{u} , and the top-left panel of Fig. 2 is therefore essentially identical for all values of t_{osc} .

4.3.1 Mean flow

The mean-flow energy budget (top-left) satisfies global energy conservation in the statistically steady state, as expressed by equation (33). The surface flux term is negligible: energy is dissipated locally by viscosity. The cyan curve (buoyancy work plus viscous dissipation) remains near zero, confirming this balance. The term \bar{D}_R is negative but small in magnitude compared to buoyancy work and may be neglected in the mean-flow budget. The time derivative of the mean kinetic energy is also negligible, indicating that no significant energy accumulation occurs over the integration time. Equation (33) therefore holds without a storage term.

Surface temperature perturbations are larger at early times and decrease as the system evolves towards a statistically steady state, with the maximum value of $|\tilde{b}|$ at the surface dropping from 0.2 at $\tilde{t} = 30$ to 0.06 at $\tilde{t} = 200$. As a result, convection is more vigorous at early times, since positive surface temperature perturbations δT enhance the radiative flux as $\propto (T_2 + \delta T)^4$. In contrast, the rms value of \tilde{b} within the flow volume increases from 0.32 to 0.48 over the course of the simulation. This behaviour reflects the fact that, as convection becomes established, thermal fluctuations are redistributed from the boundary into the interior, reducing surface temperature contrasts while amplifying temperature variations within the bulk of the flow. To isolate the role of the radiative boundary condition, we performed an otherwise identical simulation in which the upper surface is held at a fixed radiative flux rather than obeying a blackbody condition. In this case, surface temperature perturbations remain large throughout the simulation, with a maximum $|\tilde{b}| \simeq 0.5$ at all times, while temperature fluctuations in the bulk are weaker, with an rms value of 0.29. This contrast demonstrates that the blackbody radiative boundary condition promotes the redistribution of thermal variability from the surface into the interior of the flow. Similar behaviour is obtained for both free and rigid upper boundaries.

The dimensionless convective time-scale $\tilde{t}_{\text{conv}} = L_z/\tilde{u}_{z,\text{rms}}$ (see equation 17) increases from approximately 9 at $\tilde{t} = 30$ to about 13 at $\tilde{t} = 200$. The dimensionless crossing time \tilde{t}_{cross} (equation [19]) remains of order unity throughout. The rms values of the velocity components satisfy $\tilde{u}_{x,\text{rms}} \simeq \tilde{u}_{y,\text{rms}} \simeq \tilde{u}_{z,\text{rms}} \simeq 0.1$, while the rms values of the velocity gradients $\partial\tilde{u}_x/\partial\tilde{z}$ and $\partial\tilde{u}_z/\partial\tilde{x}$ are approximately 0.5, with comparable values obtained by interchanging x and y . These scalings imply a characteristic dimensionless convective length-scale $\tilde{\lambda}_{\text{conv}} \sim 0.2$.

All simulations presented here use a free upper surface. The maximum surface displacement satisfies $|\tilde{\eta}| \lesssim 0.01$ at all times, validating the assumption that surface deformations remain small compared to L_z . We have additionally verified that the quantities shown in Fig. 2 depend only weakly on whether a free or rigid upper boundary is used.

4.3.2 Fluctuations

The fluctuation budgets (other panels) show the energy of the motions on the filtered scale set by \tilde{t}_{osc} . Thus, $\tilde{t}_{\text{osc}} = 1$ retains all variability faster than unity but filters slower motions. Correspondingly, the fluctuation kinetic energy is much smaller than

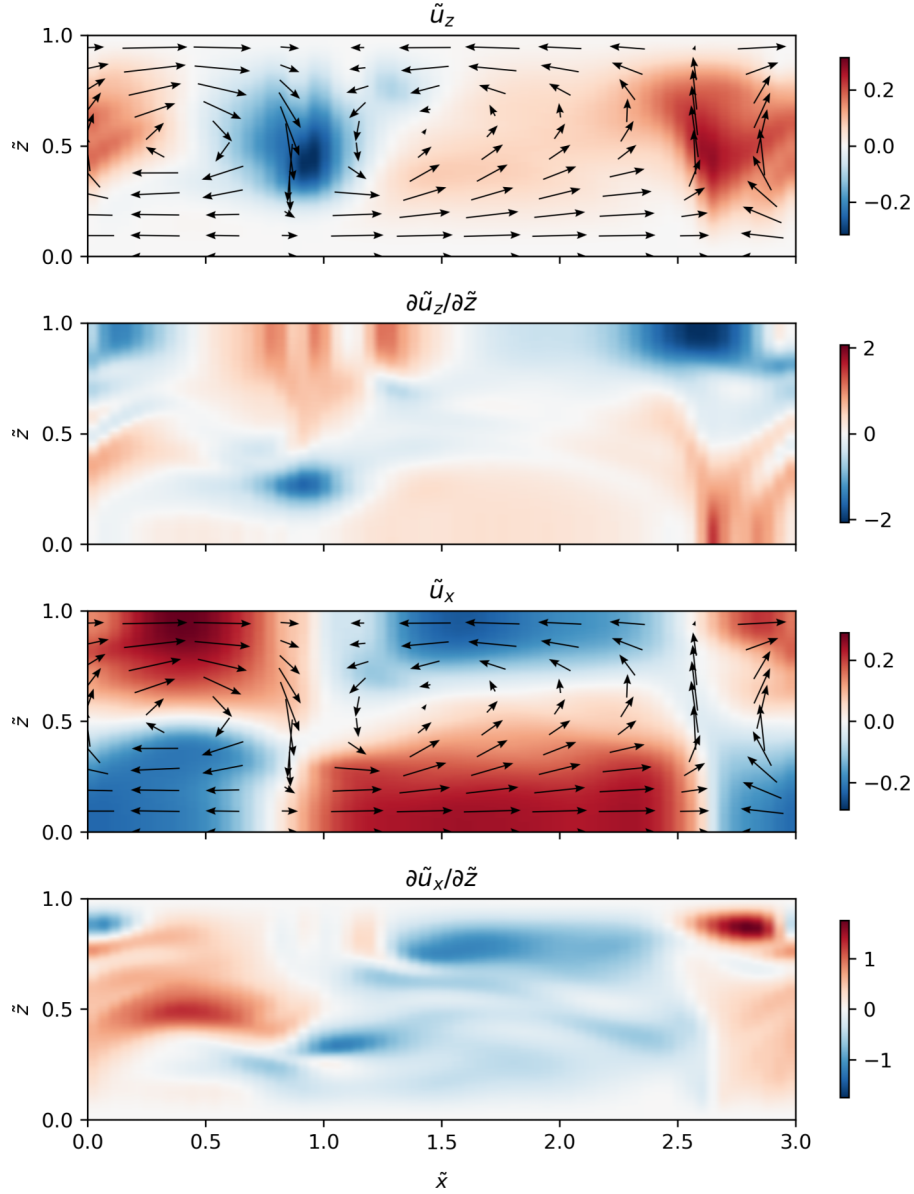


Figure 1. Instantaneous snapshots of the velocity field in the plane $\bar{y} = L_y/2$ at $\bar{t} \simeq 120$. From top to bottom, the panels show \bar{u}_z , $\partial \bar{u}_z / \partial \bar{z}$, \bar{u}_x , and $\partial \bar{u}_x / \partial \bar{z}$. Colour shading indicates scalar magnitude (red and blue corresponding to positive and negative values, respectively). Arrows in the velocity panels represent the in-plane velocity vector (\bar{u}_x, \bar{u}_z) , with lengths proportional to the velocity magnitude. Coordinates are shown in units of the domain dimensions, $0 \leq \bar{x} \leq L_x$ and $0 \leq \bar{z} \leq L_z$.

the mean-flow energy, indicating that most of the energy resides in motions with time-scales longer than 1, consistent with $\bar{t}_{\text{conv}} \sim 10$.

Because the fluctuations are defined as the residual with respect to a finite-time block average, they include not only motions produced by the turbulent cascade on time-scales shorter than t_{osc} , but also contributions from the temporal evolution of the slowly varying flow within each averaging window. To illustrate this, noting that t_{osc} is small compared to the time-scale on which the convective flow varies, we expand the convective velocity in a Taylor series for t between nt_{osc} and $(n+1)t_{\text{osc}}$. Denoting $t_m = (n+1/2)t_{\text{osc}}$, this yields $\mathbf{u}(t) \simeq \mathbf{u}(t_m) + (t - t_m)\dot{\mathbf{u}}(t_m)$, where the *dot* is a time-derivative. The linear term does not contribute to the block-averaged velocity \mathbf{V} over the interval

$[nt_{\text{osc}}, (n+1)t_{\text{osc}}]$, so that the residual within the block is, to leading order, $\mathbf{u}' \equiv \mathbf{u} - \mathbf{V} = (t - t_m)\dot{\mathbf{u}}(t_m)$. This contributes $\dot{u}^2(t_m)t_{\text{osc}}^2/24$ to $\langle e'_k \rangle$, consistent with the scaling $\langle e'_k \rangle \propto \bar{t}_{\text{osc}}^2$ observed in Fig. 2 and indicating dominance over the turbulent cascade. This interpretation is further supported by an analysis of the temporal power spectrum of the convective flow, obtained via Fourier transform, which shows that the kinetic energy contained at the corresponding high frequencies is significantly smaller than the fluctuation energy measured using the block-averaging procedure. In other words, the fluctuations isolated by the averaging procedure do not primarily represent energy injected at small spatial or temporal scales by a turbulent cascade, but rather energy already present in the large-scale flow that is assigned to the fluctuation field by the finite-time temporal filter.

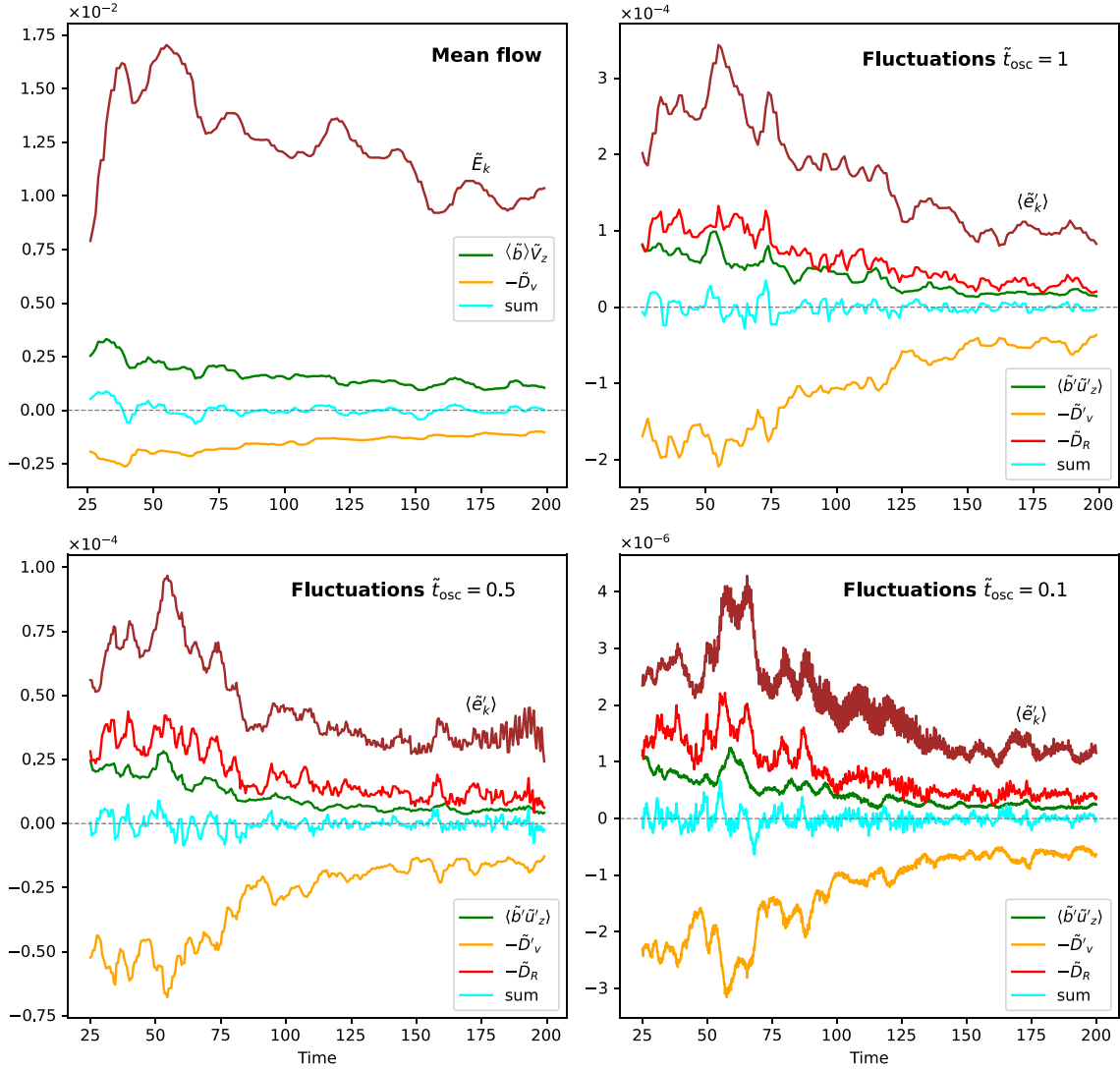


Figure 2. Time evolution of the volume-averaged total energy budget for unforced convection at $Ra = 10^6$. Top-left: mean flow (identical for all \tilde{t}_{osc}). Brown: \bar{E}_k (mean kinetic energy); green: $\langle \tilde{b} \tilde{V}_z \rangle$ (buoyancy work); orange: $-\bar{D}_v$ (viscous dissipation); cyan: sum of the last two terms. Other panels: fluctuations obtained using averaging windows $\tilde{t}_{\text{osc}} = 1, 0.5, 0.1$. Brown: $\langle \tilde{e}'_k \rangle$ (fluctuation kinetic energy); green: $\langle \tilde{b}' \tilde{u}'_z \rangle$ (buoyancy work); orange: $-\bar{D}'_v$ (viscous dissipation); red: $-\bar{D}'_R$ (net kinetic energy assigned to the fluctuation field); cyan: sum of the last three terms. *Note:* Kinetic energy is shown as energy, other terms are rates. All quantities are dimensionless and share the same axis for visual comparison (despite different physical dimensions). With $\tilde{t}_{\text{conv}} \sim 10$, the chosen \tilde{t}_{osc} values filter motions well below the dominant convective time-scale. The clear \tilde{t}_{osc}^2 scaling of fluctuation kinetic energy confirms that the fluctuations are dominated by large-scale temporal variability rather than a forward energy cascade to small scales.

The budgets of these fluctuations satisfy global energy conservation, as given by equation (34). Because the fluctuation kinetic energy is dominated by the temporal variability of the large-scale convective flow within the averaging window, rather than by a forward cascade to small spatial scales, the term D_R should not be interpreted as a classical turbulent energy transfer. Both the residual kinetic energy and the exchange term D_R arise simultaneously from this variability. From the expression of \mathbf{u}' above, we can write:

$$D_R = \frac{t_{\text{osc}}^2}{12} \dot{u}_i(t_m) \dot{u}_j(t_m) \frac{\partial V_i}{\partial x_j}(t_m).$$

The observed $\bar{D}_R < 0$ reflects a systematic anticorrelation between the local acceleration $\dot{\mathbf{u}}$ and the block-mean gradients, which arises because the evolution of the flow is dominated by

relaxation towards a statistically steady state. Thus, $-D_R > 0$ exactly accounts for the portion of large-scale kinetic energy that the finite averaging window reassigns from the block-mean to the residual field.

The surface flux term is negligible, and there is no significant accumulation of fluctuation kinetic energy over time: viscous dissipation $-\bar{D}'_v$ balances the sum of buoyancy work on fluctuations and $-\bar{D}'_R$.

4.4 Forcing prescriptions and constraints

We now introduce an external forcing in equation (14). We consider the following dimensionless prescriptions:

$$\tilde{\mathbf{f}}_1 = \tilde{f}_1 \hat{\mathbf{z}}, \quad \tilde{f}_1 = -a\tilde{z}\tilde{x}(L_x - \tilde{x})\tilde{y}(L_y - \tilde{y}) \sin\left(\frac{2\pi\tilde{t}}{\tilde{t}_{\text{osc}}}\right), \quad (44)$$

$$\tilde{\mathbf{f}}_2 = \tilde{f}_2 \hat{\mathbf{z}}, \quad \tilde{f}_2 = -a\tilde{z}^2(L_z - \tilde{z})\tilde{x}(L_x - \tilde{x})\tilde{y}(L_y - \tilde{y}) \sin\left(\frac{2\pi\tilde{t}}{\tilde{t}_{\text{osc}}}\right), \quad (45)$$

$$\tilde{\mathbf{f}}_3 = \tilde{f}_1(\hat{\mathbf{x}} + \hat{\mathbf{z}}), \quad (46)$$

$$\tilde{\mathbf{f}}_4 = -\nabla\Psi, \quad \Psi = \frac{a}{2}\tilde{z}^2 \sin\left(\frac{2\pi\tilde{x}}{L_x}\right) \sin\left(\frac{2\pi\tilde{y}}{L_y}\right) \sin\left(\frac{2\pi\tilde{t}}{\tilde{t}_{\text{osc}}}\right), \quad (47)$$

where a is a constant amplitude varied across simulations. All forcing functions satisfy the periodic boundary conditions in the horizontal directions. The form (44) is not intended to model tidal forcing in a dynamical sense, since it does not derive from a scalar potential. Rather, it provides a deliberately simple test case in which the only similarity with tides is the decay of the forcing amplitude with depth. Because it directly excites only vertical velocity fluctuations, it offers a clean framework for isolating how correlations arise within the convective flow. The expression (45) has a similar structure but vanishes at the upper boundary, allowing us to assess the sensitivity of the results to near-surface forcing. Case (46) extends (44) by including a horizontal component. This configuration is particularly useful for comparing simulations with free and rigid upper boundaries and for assessing whether horizontal forcing modifies the resulting correlations. Finally, the prescription (47) derives explicitly from a scalar potential and decreases with depth, and therefore provides the most physically faithful representation of tidal forcing considered in this study.

All forcings vary on length-scales comparable to the domain size, corresponding to a characteristic dimensionless oscillation length-scale $\tilde{\lambda}_{\text{osc}} \sim 0.5$ -1, several times larger than the convective length-scale $\tilde{\lambda}_{\text{conv}}$. This ordering is consistent with expectations for tidal oscillations in the convective envelopes of stars and giant planets.

To ensure reliable measurements of D_R , the forcing amplitude a must satisfy the following conditions:

(i) As discussed in section 2.5.1, the external forcing must not modify the leading-order balance of the mean flow. This requirement is satisfied if the mean buoyancy $\langle \tilde{b} \rangle$ remains large compared to the Reynolds stress term $\langle \tilde{u}'_j (\partial \tilde{u}'_i / \partial \tilde{x}_j) \rangle$. Multiplying both terms by \tilde{V}_z , this condition may be written as $\langle \tilde{b} \rangle \tilde{V}_z \gg |\tilde{D}_R| \tilde{\lambda}_{\text{conv}} / \tilde{\lambda}_{\text{osc}}$, where the factor $\tilde{\lambda}_{\text{conv}} / \tilde{\lambda}_{\text{osc}}$ accounts for the difference between the characteristic convective and forcing length-scales. Since $\tilde{\lambda}_{\text{conv}}$ is a few times smaller than $\tilde{\lambda}_{\text{osc}}$, it is sufficient to require that:

$$|\tilde{D}_R| \ll \langle \tilde{b} \rangle \tilde{V}_z, \quad (48)$$

which places the system safely within the desired asymptotic regime and therefore provides an upper bound on the forcing amplitude a .

(ii) A lower limit on a is set by viscous dissipation. For the oscillatory fluctuations to sustain Reynolds stress correlations $\langle u'_i u'_j \rangle \partial V_i / \partial x_j$ that yield significant D_R , advection of fluid elements by the oscillations across the scale of the mean velocity gradients must outpace viscous damping of the fluctuations. This requires the fluctuating Reynolds number:

$$Re' \equiv \frac{u' \lambda_{\text{conv}}}{\nu} = \frac{\tilde{u}' \tilde{\lambda}_{\text{conv}}}{\tilde{\nu}} \gtrsim 1-10, \quad (49)$$

where u' is the rms fluctuation velocity and λ_{conv} is the typical length-scale over which the mean flow varies. For $Ra = 10^6$ and

$Pr = 1$, $\tilde{\nu} = 10^{-3}$. With $\tilde{\lambda}_{\text{conv}} \simeq 0.2$, this criterion implies $\tilde{u}' \gtrsim$ a few times 5×10^{-3} .

(iii) To ensure that the block-averaged quantities isolate the externally forced oscillatory response, the forcing must produce oscillations whose kinetic energy is significantly larger than that associated with the large-scale variability that dominates the unforced case.

4.5 Simulations with external forcing

We have performed simulations with $\tilde{t}_{\text{osc}} = 1, 0.5$ and 0.1 , corresponding to ratios of the oscillation period to the convective time-scale in the range 0.1 - 0.01 . Because of the constraints discussed in the previous section, namely the lower and upper bounds on the forcing amplitude, the range of forcings that can be explored is necessarily limited.

We find that, whenever the time- and volume-averaged fluctuation kinetic energy $\langle \tilde{e}'_k \rangle$ exceeds approximately 2×10^{-3} , the volume-averaged transfer term \tilde{D}_R is positive, indicating a net transfer of kinetic energy from the fluctuations to the mean flow. Moreover, at fixed fluctuation kinetic energy, the magnitude of \tilde{D}_R is essentially independent of \tilde{t}_{osc} over the range of oscillation periods explored.

In the unforced case, the fluctuation kinetic energy scales as $\langle \tilde{e}'_k \rangle \propto \tilde{t}_{\text{osc}}^2$, reflecting the temporal variability of the slowly evolving mean flow within the averaging window rather than any distinct dynamical process. In the forced case, the externally driven oscillatory response dominates the fluctuation budget, with $\langle \tilde{e}'_k \rangle$ one to three orders of magnitude larger than in the unforced case at the same \tilde{t}_{osc} . This large separation in amplitude is the primary diagnostic confirming that the block-averaging procedure now isolates a physically meaningful oscillatory response rather than a filtering artefact. The sign and interpretation of \tilde{D}_R are correspondingly different: whereas the negative \tilde{D}_R of the unforced case reflects the finite-window accounting identity derived in Section 4.3.2, the positive \tilde{D}_R in the forced simulations reflects a genuine physical transfer of kinetic energy from the oscillatory fluctuations to the large-scale convective flow.

Since the fluctuating kinetic energy is distributed approximately equally among the three velocity components, we have $\tilde{u}'^2 \simeq 2 \langle \tilde{e}'_k \rangle / 3$. A value $\langle \tilde{e}'_k \rangle = 2 \times 10^{-3}$ therefore corresponds to a characteristic fluctuating velocity $\tilde{u}' \sim 0.036$, yielding a fluctuating Reynolds number $Re' \simeq 7$. The empirical threshold for the emergence of positive correlations is thus consistent with the condition expressed by equation (49).

We interpret the results below in terms of an effective dimensionless damping rate $\tilde{\gamma}$, defined as

$$\tilde{\gamma} \equiv \frac{\tilde{D}_R}{\tilde{u}'^2} \simeq \frac{3\tilde{D}_R}{2\langle \tilde{e}'_k \rangle}. \quad (50)$$

Although we refer to $\tilde{\gamma}$ as a damping rate, reflecting its role in depleting the fluctuation kinetic energy, it represents a transfer to the mean flow rather than a direct viscous sink.

4.5.1 Energy budgets

We now describe representative simulations at $Ra = 10^6$ that illustrate the energy budgets of the mean flow and fluctuations under external forcing. We begin with the purely vertical forcings $\tilde{\mathbf{f}}_1$ and $\tilde{\mathbf{f}}_2$, using amplitudes chosen in accordance with the constraints outlined above. Simulations employing the other forcings

($\tilde{\mathbf{f}}_3$ and the potential-derived $\tilde{\mathbf{f}}_4$) are discussed in subsequent subsections. Results for $\tilde{\mathbf{f}}_1$ are shown in Fig. 3. The viscous time-scale associated with a convective length-scale is $\tilde{\lambda}_{\text{conv}}^2/\bar{\nu} \simeq 40$. All simulations are run for at least $\tilde{t} = 100$, ensuring that viscous effects are fully captured over the convective scales of interest.

Table 1 summarizes the time- and volume-averaged fluctuating kinetic energy ($\langle \tilde{e}'_k \rangle$), transfer rate \tilde{D}_R and resulting damping rate $\tilde{\nu}$ for all the simulations.

The left column of Fig. 3 shows the time evolution of the volume-averaged kinetic energies of the mean flow and fluctuations. The panels in the middle and right columns show the cumulative time integrals, from $\tilde{t} = 25$, of the terms appearing in the global energy equations (33) and (34), after normalization by the volume. In the fluctuation energy budget, energy input is dominated by the external forcing, while the contribution from fluctuating buoyancy is small for $\tilde{t}_{\text{osc}} = 1$ and negligible at shorter oscillation time-scales. The fraction of the fluctuation energy transferred to the mean flow via the \tilde{D}_R term is approximately 30 per cent, 40 per cent and 74 per cent for $\tilde{t}_{\text{osc}} = 1, 0.5$ and 0.1 , respectively, with the remainder dissipated viscously.

For $\tilde{t}_{\text{osc}} = 1$ and 0.5 , the residual of the fluctuation energy budget (cyan curves) is one to two orders of magnitude smaller than \tilde{D}_R , indicating excellent energy conservation. No energy flux through the upper surface is included in the fluctuation budget, confirming that such fluxes are negligible in all cases considered.

For $\tilde{t}_{\text{osc}} = 1$ and 0.5 , the time- and volume-averaged external work ($\langle \tilde{\mathbf{f}} \cdot \tilde{\mathbf{u}}' \rangle$) is 4.7×10^{-4} and 5.1×10^{-4} , respectively. For $\tilde{t}_{\text{osc}} = 0.1$, however, the net work becomes highly sensitive to extremely small phase differences between $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{u}}'$. At such short periods, tiny numerical phase offsets accumulate over long integrations and dominate the inferred time average. We therefore do not rely on a direct evaluation of the forcing work in this case. Crucially, the transfer term \tilde{D}_R itself remains robust. Its value is insensitive to spatial resolution and to the procedure used to estimate the forcing work. Furthermore, for $\tilde{t}_{\text{osc}} = 0.1$, over $25 \leq \tilde{t} \leq 100$, the time-averaged residual of the mean-flow energy budget is more than an order of magnitude smaller than \tilde{D}_R , confirming that energy conservation is satisfied to high precision. The greater robustness of \tilde{D}_R , compared with the direct evaluation of the external work, arises because the latter depends on small phase offsets between $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{u}}'$ which are highly sensitive to numerical errors. By contrast, as will be shown below, \tilde{D}_R results from spatial correlations controlled by the large-scale convective dynamics and is therefore much less sensitive to discretization and time-stepping choices. This point is discussed further in Section 5.

The mean-flow energy budget is dominated by buoyancy work, whose time- and volume-averaged value remains approximately 2×10^{-3} for all values of \tilde{t}_{osc} considered here, similar to the corresponding value in the unforced case. The energy transfer from the fluctuations via the \tilde{D}_R term accounts for about 10 per cent of the total. Most of the mean-flow energy is dissipated viscously. For $\tilde{t}_{\text{osc}} = 1$, however, there is also a significant contribution from kinetic energy flux through the upper surface. In this case, essentially all of the energy gained from D_R , and slightly more, is transported to the surface by the kinetic component of the energy flux, $\tilde{\mathbf{F}}_{\text{kin}}$, defined in equation (29). Contributions to the surface flux arising from pressure and viscous stresses are not shown, as they are significantly smaller.

The fact that \tilde{D}_R remains more than an order of magnitude smaller than the buoyancy work acting on the mean flow con-

firms that the condition (48) is satisfied in a volume-averaged sense.

Among the three cases considered, the simulation with $\tilde{t}_{\text{osc}} = 1$ exhibits the largest level of noise and the largest residuals in the mean-flow energy budget. This occurs because, in this case, the time-scale of the externally driven fluctuations is comparable to that of the fastest convective plumes. Although such plumes are rare, this partial overlap makes the separation between mean flow and fluctuations less clean at the diagnostic level. Consistent with this interpretation, a substantial fraction of the energy transferred from the fluctuations to the mean flow is transported to the surface in this case.

Fig. 4 shows the energy budget in the same format as Fig. 3, but using the external forcing $\tilde{\mathbf{f}}_2$ (equation 45) at $\tilde{t}_{\text{osc}} = 1$. The time- and volume-averaged values of $\langle \tilde{e}'_k \rangle$ and \tilde{D}_R are listed in Table 1, with $\langle \tilde{\mathbf{f}} \cdot \tilde{\mathbf{u}}' \rangle = 6.3 \times 10^{-4}$. Approximately, 40 per cent of the fluctuation kinetic energy is transferred to the mean flow, a larger fraction than observed with forcing $\tilde{\mathbf{f}}_1$. Aside from this quantitative difference, the energy budgets are very similar to those found for $\tilde{\mathbf{f}}_1$, indicating that the results are not sensitive to whether the forcing vanishes at the upper boundary.

The maximum surface displacement $|\tilde{\eta}|$ remains small in all simulations, reaching values of at most 0.06 for $\tilde{t}_{\text{osc}} = 1$ and 0.02 for the shorter oscillation periods $\tilde{t}_{\text{osc}} = 0.5$ and 0.1 .

4.5.2 Magnitude of \tilde{D}'_v and dissipation length-scale

The rate \tilde{D}'_v of fluctuating kinetic energy loss through viscous dissipation, defined in equation (31), may be estimated as

$$\tilde{D}'_v \sim 2\bar{\nu} \frac{\tilde{u}'^2}{\tilde{\lambda}_{\text{dis}}^2}, \quad (51)$$

where $\tilde{\lambda}_{\text{dis}}$ is the characteristic dimensionless length-scale at which the fluctuating kinetic energy is dissipated.

The non-linear advection term $\mathbf{u} \cdot \nabla \mathbf{u}$ transfers energy from the forcing scale $\tilde{\lambda}_{\text{osc}}$ to smaller spatial scales, so that gradients of \mathbf{u}' are dominated by structures at scales $\tilde{\lambda}_{\text{dis}} < \tilde{\lambda}_{\text{osc}}$. However, when the oscillation time-scale \tilde{t}_{osc} becomes short, the time available for non-linear self-advection within one oscillation period is reduced. As a result, the fluctuations are less efficient at generating small-scale structures, and the cascade stops at progressively larger dissipation scales.

Using the measured values of $\langle \tilde{e}'_k \rangle$ and \tilde{D}'_v shown in Fig. 3, we infer dissipation length-scales $\tilde{\lambda}_{\text{dis}} = 0.07, 0.1$ and 0.2 for $\tilde{t}_{\text{osc}} = 1, 0.5$ and 0.1 , respectively, in the case of the forcing given by equation (44).

In the absence of external forcing, the dissipation length-scale associated with the fluctuations, as inferred from Fig. 2, is $\tilde{\lambda}_{\text{dis}} \sim 0.03$ for all values of \tilde{t}_{osc} , comparable to the grid scale. A similar dissipation length-scale is obtained for the mean flow. In this regime, the fluctuations arise from the temporal evolution of the convective flow within each averaging window rather than from a distinct dynamical cascade. The corresponding dissipation therefore reflects the viscous damping of this residual field at the smallest resolved scales, rather than a transfer of energy across an inertial range. The dissipation length-scale is thus set by the numerical resolution and is independent of \tilde{t}_{osc} .

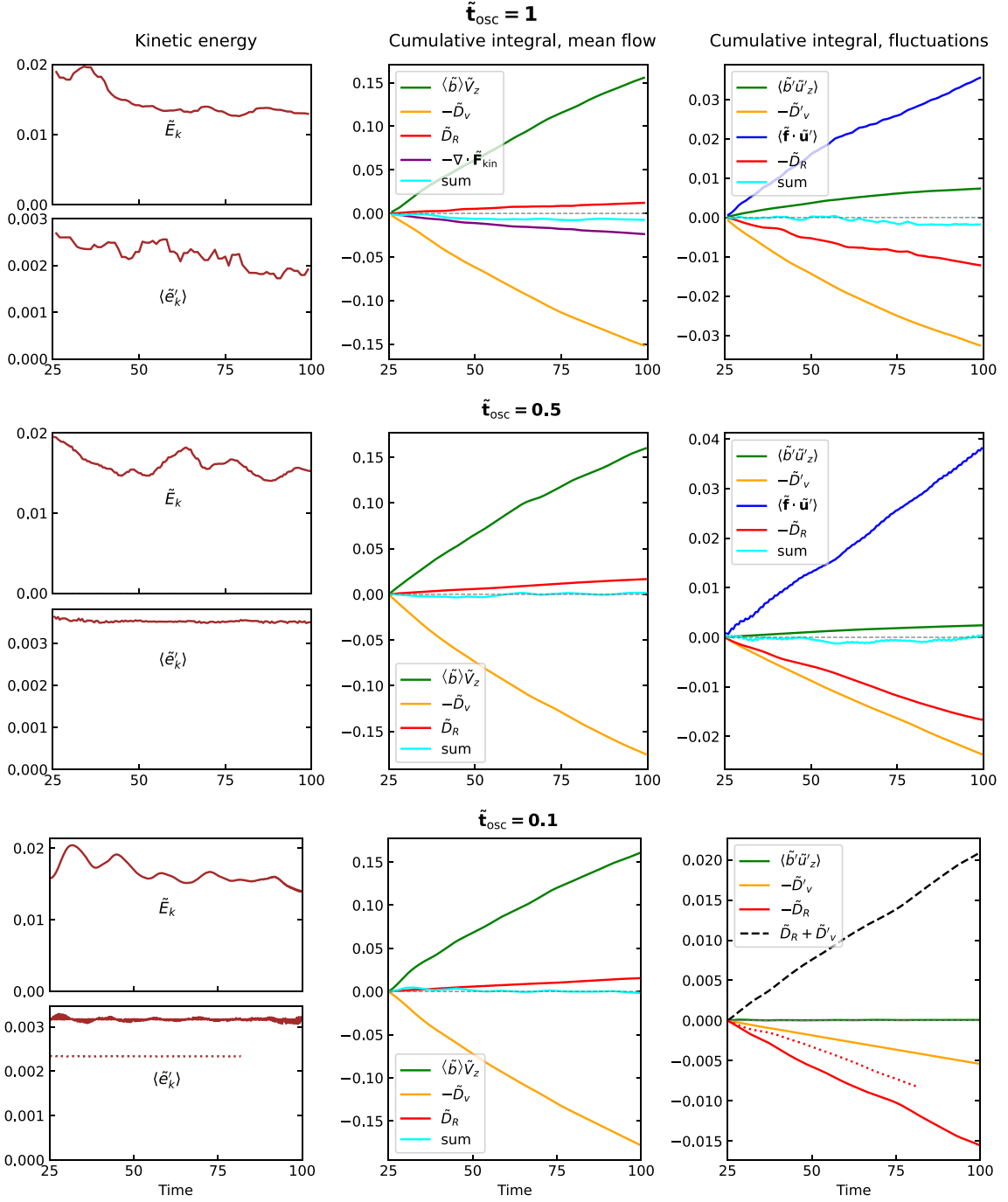


Figure 3. Time evolution of the volume-averaged total energy budget for forced convection with $\tilde{t}_{\text{osc}} = 1$ (upper row), 0.5 (middle row), and 0.1 (lower row). The forcing is $\tilde{\mathbf{f}}_1$ given by equation (44). The curves correspond to the local terms appearing in the global energy equations (33) and (34). Kinetic energies are volume-averaged, while all other curves show cumulative time integrals of the corresponding volume-averaged quantities. Left column: kinetic energy of the mean flow (upper panel) and of fluctuations (lower panel). Middle column: mean flow energy budget. *Green*: $\langle \tilde{b} \rangle \tilde{V}_z$ (buoyancy work); *orange*: $-\tilde{D}_v$ (viscous dissipation); *red*: \tilde{D}_R (kinetic energy gained from the fluctuations); *purple*: $-\nabla \cdot \tilde{\mathbf{F}}_{\text{kin}}$ (surface flux; negligible for $\tilde{t}_{\text{osc}} = 0.5$ and 0.1); *cyan*: sum of all terms. Right column: fluctuation energy budget. *Green*: $\langle \tilde{b}' \tilde{u}'_z \rangle$ (buoyancy work); *orange*: $-\tilde{D}'_v$ (viscous dissipation); *red*: $-\tilde{D}_R$ (kinetic energy transferred to the mean flow); *blue*: $\langle \tilde{\mathbf{f}} \cdot \tilde{\mathbf{u}}' \rangle$ (work of the external forcing); *cyan*: sum of all terms. For $\tilde{t}_{\text{osc}} = 0.1$, the work done by the external forcing is not shown, as it is sensitive to small phase errors at short oscillation periods; instead, we plot the cumulative time integral of $\tilde{D}'_v + \tilde{D}_R$ (black dashed curve) which, by the fluctuation energy budget, must balance the external work. The dotted curves in the $\tilde{t}_{\text{osc}} = 0.1$ row illustrate a case with smaller fluctuating kinetic energy. Overall, these budgets show that while the mean-flow energetics remain dominated by buoyancy work, a positive and well-converged transfer term \tilde{D}_R systematically channels a fraction of the fluctuation energy into the mean flow, with a time- and volume-averaged magnitude that is essentially independent of \tilde{t}_{osc} for the cases shown.

Table 1. Time- and volume-averaged fluctuating kinetic energy $\langle \tilde{e}'_k \rangle$, transfer rate \tilde{D}_R and resulting damping rate $\tilde{\gamma}$ for the simulations shown in Figs 3–8.

Forcing	\tilde{t}_{osc}	$\langle \tilde{e}'_k \rangle$	\tilde{D}_R	$\tilde{\gamma}$	Figures
\mathbf{f}_1	1.0	2.2×10^{-3}	1.6×10^{-4}	0.1	Figs 3, 5
–	0.5	3.5×10^{-3}	2.2×10^{-4}	0.1	–
–	0.1	3.2×10^{-3}	2.1×10^{-4}	0.1	–
–	–	2.3×10^{-3}	1.5×10^{-4}	0.1	–
\mathbf{f}_2	1.0	2.4×10^{-3}	3.0×10^{-4}	0.19	Fig. 4
\mathbf{f}_3	1.0	2.3×10^{-3}	1.9×10^{-4}	0.1	Fig. 6
$\mathbf{f}_4 = -\nabla\Psi$	0.5	1.8×10^{-3}	2.2×10^{-4}	0.19	Figs 7, 8

4.5.3 Reynolds-stress contributions to D_R and effective damping scaling

Fig. 5 shows the cumulative time integrals of the volume-averaged Reynolds-stress components $\tilde{D}_{ij} \equiv \langle \tilde{u}'_i \tilde{u}'_j \rangle (\partial \tilde{V}_i / \partial \tilde{x}_j)$, with no summation implied over i and j , and their sum \tilde{D}_R , for the same simulations as in Fig. 3, corresponding to the purely vertical forcing $\tilde{\mathbf{f}}_1$ given by equation (44). The upper, middle, and lower panels correspond to $\tilde{t}_{\text{osc}} = 1, 0.5$ and 0.1 , respectively. In all cases, the cumulative sum of the individual contributions yields a positive volume-averaged net transfer \tilde{D}_R , confirming that kinetic energy is systematically transferred from the fluctuations to the mean flow.

A component-wise decomposition shows that the dominant positive contributions arise from \tilde{D}_{zz} , \tilde{D}_{xz} and \tilde{D}_{yz} , indicating that correlations involving vertical velocity fluctuations and vertical gradients of the mean flow control the energy transfer across all oscillation time-scales examined. By contrast, the components \tilde{D}_{zx} and \tilde{D}_{zy} , associated with horizontal shear of the vertical velocity, contribute negatively. These terms typically offset the contribution from \tilde{D}_{zz} , such that the net volume-averaged transfer is well approximated by $\tilde{D}_R \sim \tilde{D}_{xz} + \tilde{D}_{yz}$. The physical origin of this dominance is discussed in Section 5.

Results obtained with the purely vertical forcing $\tilde{\mathbf{f}}_2$ that vanishes at the surface, given by equation (45) (not shown), are quantitatively similar, with the same Reynolds-stress components contributing positively and negatively to the volume-averaged \tilde{D}_R .

For all three oscillation periods shown in Figs 3 and 5, we find that the effective dimensionless damping rate (50) is $\tilde{\gamma} \simeq 0.1$. In the case where the forcing vanishes at the surface and $\tilde{t}_{\text{osc}} = 1$, the inferred value of $\tilde{\gamma}$ is larger, approaching $\tilde{\gamma} \simeq 0.2$. The theoretical basis for the period-independence of $\tilde{\gamma}$ is discussed in Section 6.2.

Consider one of the components \tilde{D}_{ij} that contributes positively to \tilde{D}_R . If the correlation between the fluctuations and the mean-flow velocity gradient were perfect, one would expect $\tilde{D}_{ij} \sim \tilde{u}'^2 (\partial \tilde{V}_i / \partial \tilde{x}_j)_{\text{rms}}$. In our simulations, the rms value of the mean-flow velocity gradient is approximately 0.5, which would imply a volume-averaged value of \tilde{D}_R several times larger than that observed.

The measured values therefore indicate that the correlations are not perfect, but remain of order unity. The observed magnitude of the volume-averaged \tilde{D}_R is instead consistent with replacing $(\partial \tilde{V}_i / \partial \tilde{x}_j)_{\text{rms}}$ by the characteristic mean-flow shear $\tilde{V}_{\text{rms}} / L_z = 1 / \tilde{t}_{\text{conv}}$. This leads to the scaling:

$$D_R \sim \frac{u'^2}{t_{\text{conv}}}, \quad (52)$$

which reflects strong correlations between the fluctuations and the large-scale mean flow. In terms of the effective damping rate, this scaling implies:

$$\gamma \sim \frac{1}{t_{\text{conv}}}. \quad (53)$$

4.5.4 The effect of changing the Rayleigh number

We investigate the effect of varying the Rayleigh number Ra while keeping the Prandtl number Pr fixed. Decreasing Ra increases both the viscosity and the thermal diffusivity, as implied by equations (10) and (11), thereby weakening buoyant driving relative to viscous and thermal diffusion.

For the reference case shown in Figs 3 and 5 with $\tilde{t}_{\text{osc}} = 1$ and $Ra = 10^6$, corresponding to the purely vertical forcing \mathbf{f}_1 , we find that, when Ra is reduced to 10^5 , the volume-averaged contributions D_{xz} , D_{yz} , and D_{zz} are initially positive but decay significantly after a time comparable to the viscous time-scale associated with a convective length-scale, $\tilde{\lambda}_{\text{conv}}^2 / \tilde{\nu}$.

For the alternative purely vertical forcing \mathbf{f}_2 , corresponding to the case shown in Fig. 4 at $Ra = 10^6$, we find that when Ra is reduced to 10^5 the volume-integrated transfer \tilde{D}_R remains comparable to its value at $Ra = 10^6$. However, its decomposition is significantly altered: the contribution from D_{zz} increases, while the relative contribution from horizontal shear, quantified by $(D_{xz} + D_{yz}) / D_R$, decreases accordingly.

4.5.5 Rigid upper surface

We now examine the effect of replacing the free upper surface by a rigid boundary. In this case, the boundary conditions at $\tilde{z} = L_z$ (see section 3.2) are changed to $\tilde{u}_z = 0$ and $L_z (\partial \tilde{b} / \partial \tilde{z}) = 1 - (1 + \tilde{b} / \tilde{T}_2)^4$. The normal stress condition (40) is replaced by a pressure constraint, enforced using a standard τ -formulation (K. J. Burns et al. 2020).

For a rigid upper surface, simulations with purely vertical forcing do not yield a positive volume-averaged transfer term \tilde{D}_R over the duration of the simulations, which extend up to $\tilde{t} \sim 500$. A positive \tilde{D}_R is obtained only when the forcing includes a horizontal component. Accordingly, we consider the forcing \mathbf{f}_3 defined in equation (46). In this configuration, the impermeability condition $\tilde{u}_z = 0$ at the upper boundary requires pressure forces to locally balance both buoyancy and the imposed vertical forcing near the surface, thereby strongly modifying the structure of the fluctuating response.

Fig. 6 shows the cumulative time integrals of the Reynolds-stress transfer terms, in the same format as Fig. 5, for this mixed forcing \mathbf{f}_3 and for both a free and a rigid upper surface. In the free-surface case (upper panel), the structure of the transfer terms is qualitatively similar to that obtained with purely vertical forcing: the dominant positive contributions arise from \tilde{D}_{zz} , \tilde{D}_{xz} , and \tilde{D}_{yz} , yielding an effective damping rate $\tilde{\gamma} \simeq 0.1$.

For a rigid upper surface (lower panel), the emergence of a positive \tilde{D}_R occurs only after a substantially longer transient. Moreover, the structure of the correlations responsible for the energy transfer is fundamentally different from that obtained in the free-surface case. Only the components \tilde{D}_{xx} and \tilde{D}_{zx} contribute positively to the volume-averaged \tilde{D}_R , while the contributions associated with vertical velocity fluctuations and vertical gradients of the mean flow are strongly reduced, with \tilde{D}_{xz} becoming negative. This behaviour, together with the absence

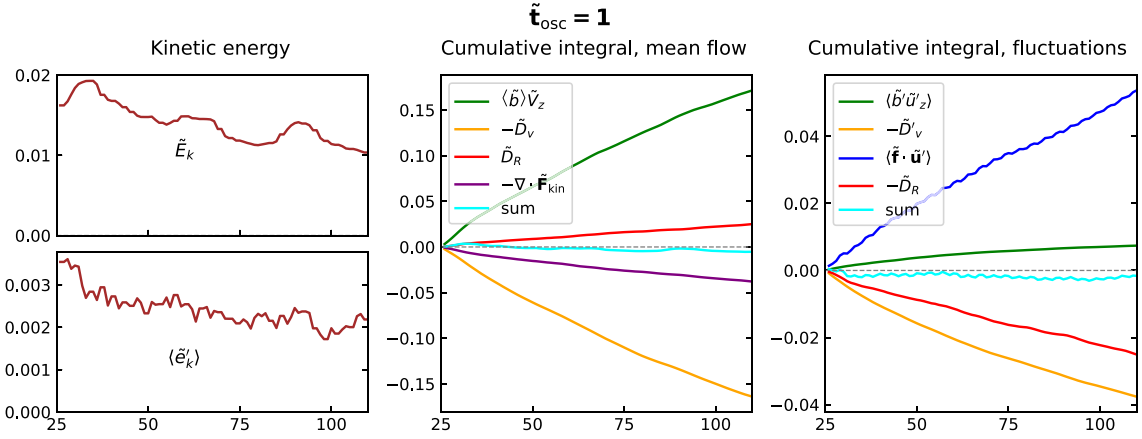


Figure 4. Same as Fig. 3, but for the external forcing $\tilde{\mathbf{f}}_2$ given by equation (45) and for $\tilde{t}_{\text{osc}} = 1$. The overall structure of the energy budget is very similar to that obtained with the forcing $\tilde{\mathbf{f}}_1$, although a larger fraction of the fluctuating kinetic energy is transferred to the mean flow.

of positive \bar{D}_R under purely vertical forcing, indicates that in the rigid-surface case the net energy transfer is dominated by correlations associated with the horizontally forced response, rather than by the vertical shear and buoyancy-driven interactions that dominate in the free-surface case. We have also verified that the same qualitative behaviour is obtained when $\mathbf{f}_3 = f_1(\hat{\mathbf{x}} + \hat{\mathbf{z}})$ is replaced by $f_2(\hat{\mathbf{x}} + \hat{\mathbf{z}})$, confirming that the results are not sensitive to the detailed vertical structure of the forcing.

These trends are consistent with additional simulations employing purely horizontal forcing in the x -direction (not shown). For a rigid upper surface, \bar{D}_{zx} contributes positively while \bar{D}_{xz} is negative, whereas the signs of these two contributions are reversed in the free-surface case.

4.5.6 Forcing deriving from a potential

The forcing $\mathbf{f}_4 = -\nabla\Psi$, derived from a scalar potential Ψ given by equation (47), mimics tidal forcing. Using incompressibility ($\nabla \cdot \mathbf{u}' = 0$), the net work done by the forcing on the flow can be rewritten as:

$$\int \langle \tilde{\mathbf{f}} \cdot \tilde{\mathbf{u}} \rangle dv = - \int \nabla \cdot \langle \Psi \tilde{\mathbf{u}} \rangle dv = - \int \langle \Psi \tilde{\mathbf{u}} \rangle \cdot \hat{\mathbf{n}} ds, \quad (54)$$

where the surface integral is taken over the domain boundaries and $\hat{\mathbf{n}}$ is the outward unit normal. For rigid boundaries ($\tilde{\mathbf{u}}' \cdot \hat{\mathbf{n}} = 0$ everywhere), the surface term vanishes, so the forcing performs no net work on the flow. Kinetic energy is injected locally in some regions and removed in others, but these contributions cancel in the volume integral. With a free surface present, however, the boundary term is generally non-zero, allowing the forcing to perform net work through the surface. In this case, the potential forcing acts effectively like a pressure perturbation applied at the boundary, enabling energy exchange between the exterior and the fluid domain.

Fig. 7 shows the energy budget for $\tilde{\mathbf{f}}_4$ at $\tilde{t}_{\text{osc}} = 0.5$, in the same format as Fig. 3. The time- and volume-averaged values of $\langle \tilde{e}'_k \rangle$ and \bar{D}_R are listed in Table 1, with $\langle \tilde{\mathbf{f}} \cdot \tilde{\mathbf{u}} \rangle = 4.1 \times 10^{-4}$. Approximately 50 per cent of the fluctuating kinetic energy is transferred to the mean flow, a larger fraction than in the cases with forcings $\tilde{\mathbf{f}}_1$ and $\tilde{\mathbf{f}}_2$. Unlike the purely vertical forcing case, both the mean-flow and fluctuation budgets now exhibit a significant surface-

flux contribution. For the mean flow, this is dominated by the kinetic component of the energy flux, $\tilde{\mathbf{F}}_{\text{kin}}$, whereas for the fluctuations the dominant contribution arises from pressure fluctuations, $\langle \tilde{P} \tilde{u}'_j \rangle$ in equation (32). The residual in both the mean-flow and fluctuation budgets is an order of magnitude smaller than \bar{D}_R , confirming that the transfer term satisfies energy conservation to high precision.

Fig. 8 presents the cumulative time integrals of the individual transfer components, together with the corresponding cycle-averaged values. As in the purely vertical forcing case, the positive volume-averaged transfer \bar{D}_R is primarily associated with the components \bar{D}_{zz} , \bar{D}_{xz} , and \bar{D}_{yz} , with \bar{D}_{xx} and \bar{D}_{yy} providing smaller additional contributions. In contrast to the vertical-forcing case, however, all components exhibit significantly stronger temporal variability. This enhanced variability, together with the significant surface contribution in the mean-flow and fluctuation budgets, reflects the fact that net work is delivered at the boundary rather than in the fluid interior.

To test the robustness of these results, we have also performed simulations with $\tilde{t}_{\text{osc}} = 1$ and with alternative polynomial (rather than sinusoidal) choices of $\Psi(x, y)$.

In the simulations shown, the maximum surface displacement is 0.04, confirming that the small-deformation assumption (relative to L_z) is well satisfied.

5 ORIGIN OF THE REYNOLDS-STRESS CORRELATIONS

The results presented in the previous section show that, over the parameter range investigated here, the volume-averaged transfer term D_R is consistently positive, implying a net transfer of kinetic energy from the fluctuations to the mean flow. This transfer arises from systematic correlations between the fluctuating velocity field and gradients of the mean flow. In this section, we discuss the physical origin of these correlations.

5.1 Forced and secondary fluctuations

For the forcing defined in equation (44), only the vertical velocity component is driven directly, generating a fluctuating motion u'_z . The horizontal velocity fluctuations u'_x and u'_y therefore arise

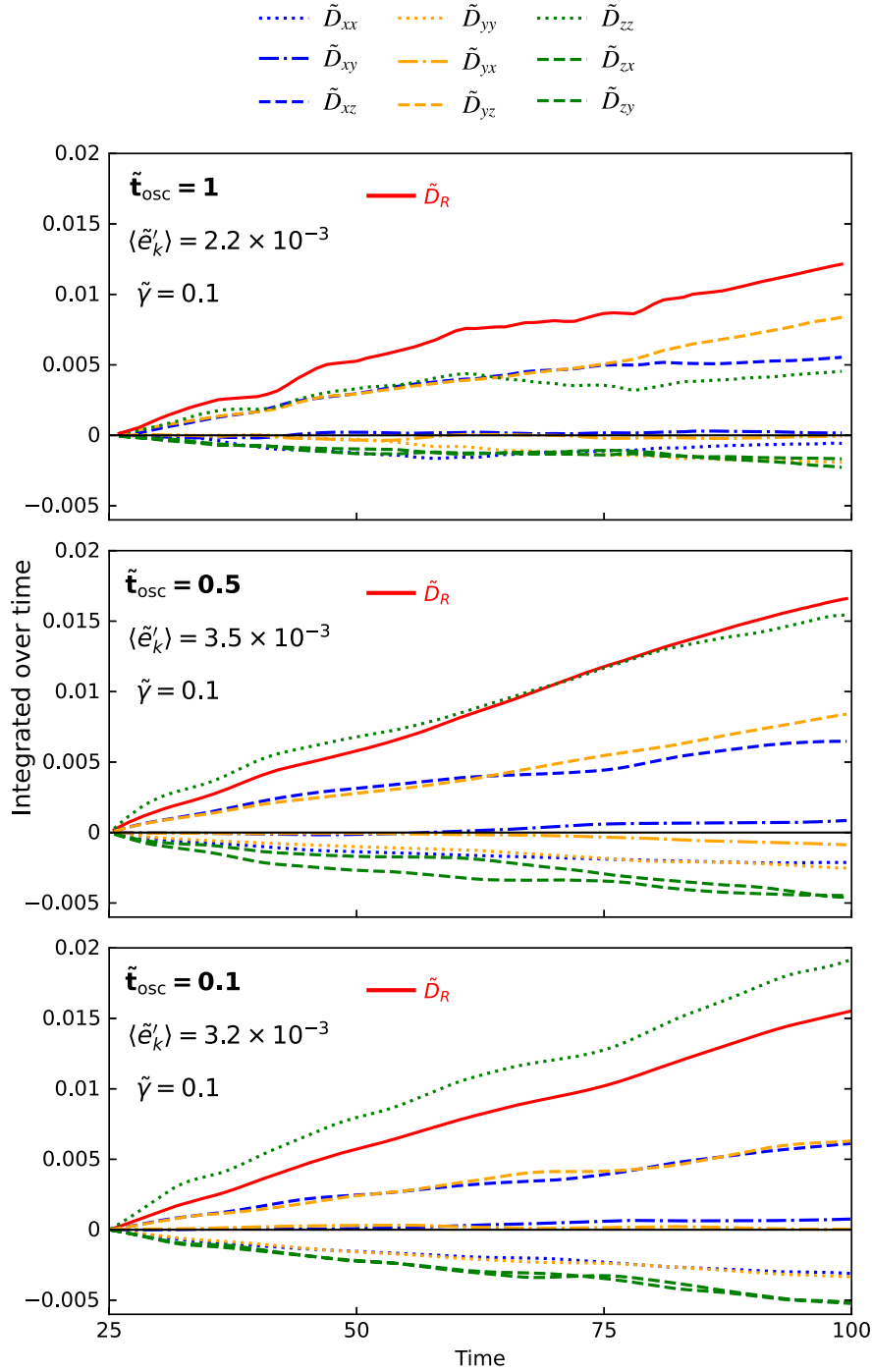


Figure 5. Cumulative time integrals of the volume-averaged transfer terms $\tilde{D}_{ij} \equiv \langle \tilde{u}'_i \tilde{u}'_j \rangle (\partial \tilde{V}_i / \partial \tilde{x}_j)$, defined with no summation over i and j , for the same simulations as in Fig. 3, corresponding to the purely vertical forcing $\tilde{\mathbf{f}}_1$ given by equation (44). The net transfer is $\tilde{D}_R = \sum_{i,j} \tilde{D}_{ij}$. Positive values of the volume-averaged \tilde{D}_{ij} correspond to a transfer of kinetic energy from the fluctuations to the mean flow. The upper, middle and lower panels correspond to $\tilde{t}_{\text{osc}} = 1, 0.5$ and 0.1 , respectively. Dotted blue, orange and green curves show \tilde{D}_{xx} , \tilde{D}_{yy} and \tilde{D}_{zz} . Dashed curves represent mixed components: \tilde{D}_{xz} and \tilde{D}_{yz} (blue and orange), and \tilde{D}_{zx} and \tilde{D}_{zy} (green). Dash-dotted curves show \tilde{D}_{xy} (blue) and \tilde{D}_{yx} (orange). The solid red curve shows the total transfer \tilde{D}_R . In each panel, the indicated value of $\langle \tilde{e}'_k \rangle$ corresponds to the time- and volume-averaged fluctuating kinetic energy over the interval shown, and $\tilde{\gamma} \equiv \tilde{D}_R / \tilde{u}'^2$ denotes the effective damping rate. In all cases, the net positive volume-averaged transfer \tilde{D}_R is dominated by the contributions \tilde{D}_{zz} , \tilde{D}_{xz} and \tilde{D}_{yz} , indicating that correlations involving vertical velocity fluctuations and vertical gradients of the mean flow play the primary role in transferring kinetic energy from fluctuations to the mean flow across oscillation time-scales. Despite the change in oscillation time-scale, $\tilde{\gamma}$ remains close to 0.1 in all three cases.

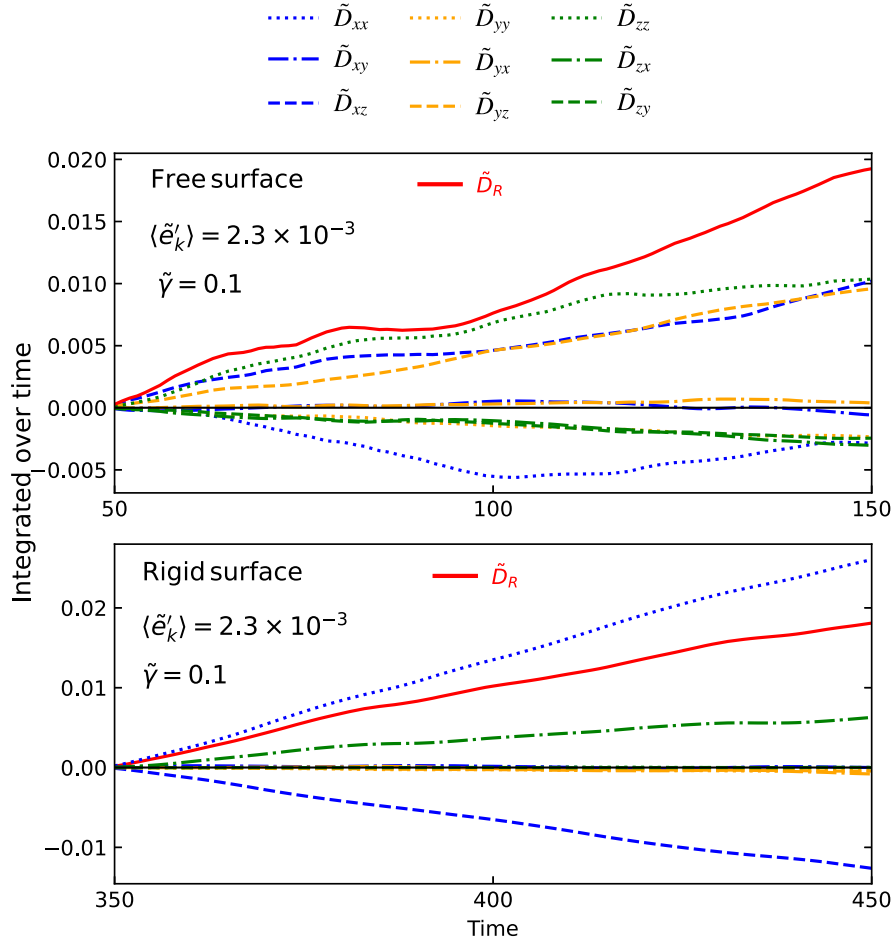


Figure 6. Same as Fig. 5, but for the external forcing \mathbf{f}_3 given by equation (46) with $\tilde{t}_{\text{osc}} = 1$, comparing a free upper surface (upper panel) and a rigid one (lower panel). In contrast to Fig. 5, the inclusion of a horizontal forcing component breaks the symmetry between the mixed vertical terms. As a result, \tilde{D}_{zx} (dash-dotted green curve) and \tilde{D}_{zy} (dashed green curve) are shown separately. The indicated values of $\langle \tilde{e}'_k \rangle$ denote the time- and volume-averaged fluctuating kinetic energy over the interval displayed. For a free upper surface, the structure of the transfer terms remains qualitatively similar to that obtained with purely vertical forcing. For a rigid upper surface subjected to the same mixed forcing, the behaviour differs markedly from the free-surface case, with correlations associated with the horizontal forcing dominating the net transfer. Note also that simulations with a rigid surface and purely vertical forcing do not yield a positive volume-averaged \tilde{D}_R .

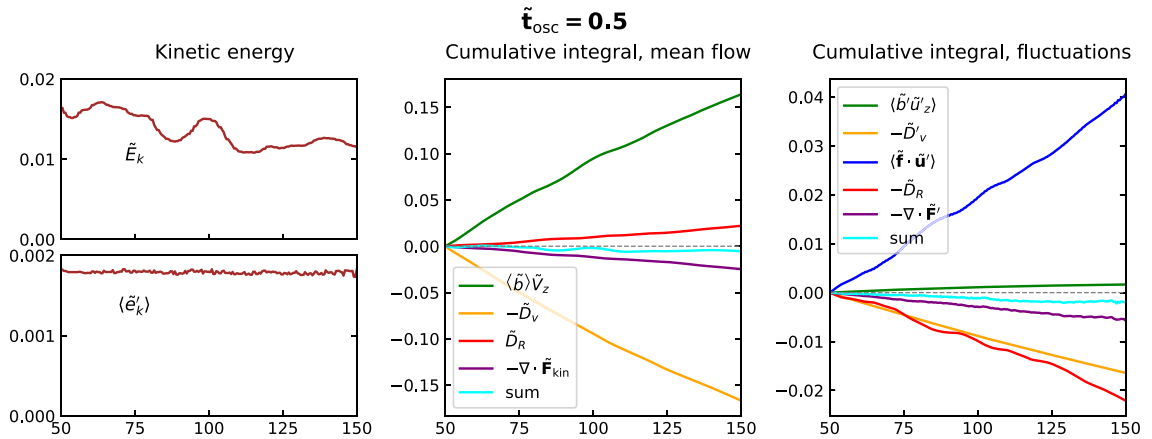


Figure 7. Energy budget for the potential-derived forcing $\tilde{\mathbf{f}}_4 = -\nabla\Psi$ (equation 47) at $\tilde{t}_{\text{osc}} = 0.5$, shown in the same format as Fig. 3. Both the mean-flow and fluctuation budgets include a significant surface-flux contribution, reflecting net work transmission through the boundary.

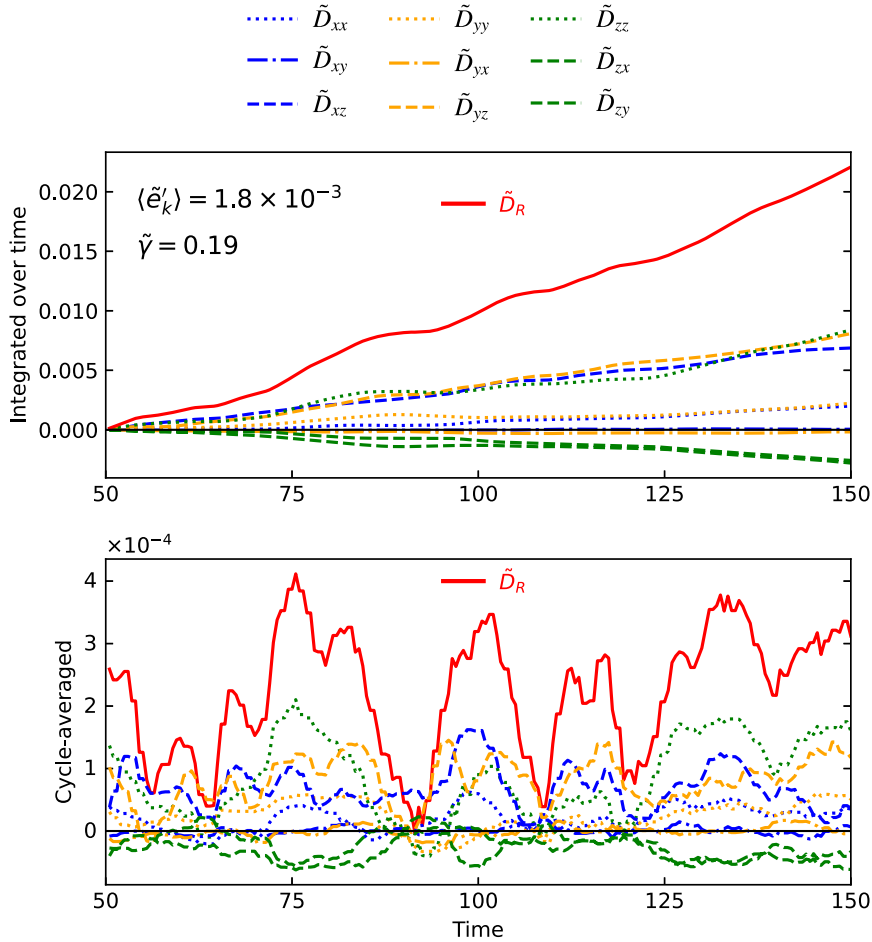


Figure 8. Cumulative time integrals (upper panel) and cycle-averaged values (lower panel) of the volume-averaged transfer components for the potential-derived forcing $\tilde{\mathbf{f}}_4 = -\nabla\Psi$ at $\tilde{t}_{\text{osc}} = 0.5$, shown in the same format as Fig. 5. The indicated value of $\langle \tilde{e}'_k \rangle$ denotes the time- and volume-averaged fluctuating kinetic energy over the interval displayed. As in the vertical-forcing case, the net positive transfer \tilde{D}_R is dominated by \tilde{D}_{zz} , \tilde{D}_{xz} , and \tilde{D}_{yz} . However, these components show much stronger temporal fluctuations.

indirectly: the forced vertical motion, being subject to buoyancy and incompressibility constraints, is redirected in the same manner as the mean convective flow, generating secondary horizontal fluctuations. As a result, correlations between the Reynolds stresses and the mean-flow velocity gradients naturally develop. Similar considerations apply when the forcing acts purely in the horizontal direction. More generally, even when the forcing includes both vertical and horizontal components, secondary fluctuations arise and establish correlations with the mean-flow gradients.

To clarify this mechanism, we decompose the fluctuating velocity as $\mathbf{u}' = \mathbf{u}'_f + \mathbf{u}'_s$, where \mathbf{u}'_f denotes the direct response to the external forcing and \mathbf{u}'_s represents secondary fluctuations. Although such a decomposition is not strictly valid in a fully nonlinear system, it provides a useful conceptual framework.

For example, a vertically oscillatory forced velocity $u'_{f,z}(x, y, z)$ induces horizontal velocity fluctuations $u'_{s,x}$ and $u'_{s,y}$ with amplitudes comparable to that of $u'_{f,z}$, through the incompressibility constraint and buoyancy. The Reynolds stress $\langle u'_x u'_z \rangle \approx \langle u'_{s,x} u'_{f,z} \rangle$ then couples efficiently to the mean vertical shear $\partial V_x / \partial z$.

Similarly, the forced vertical oscillation $u'_{f,z}$ generates a secondary vertical component $u'_{s,z}$ which couples to the mean vertical shear $\partial V_z / \partial z$. Together, these mechanisms provide a natural

explanation for the robust Reynolds-stress correlations observed between the fluctuating velocity field and the large-scale mean flow: the secondary fluctuations, being subject to the same buoyancy and incompressibility constraints as the mean flow, naturally inherit its spatial structure.

This interpretation is supported by the fact that the forcings used in the simulations produce larger $u'_{f,z}$ in the upper regions of the domain, where $\partial V_z / \partial z$ is predominantly negative (see Fig. 1). Nevertheless, across all cases examined, we find $D_{zz} > 0$, implying that u'^2_z is systematically larger in regions where the mean shear $\partial V_z / \partial z$ is positive. This counterintuitive spatial preference reflects the fact that the secondary fluctuations, being subject to the same buoyancy constraints as the mean flow, are preferentially amplified in regions of stronger convective instability, where $\partial V_z / \partial z > 0$.

This picture also explains why the computation of D_R is comparatively insensitive to the details of the numerical scheme, in contrast to the external work $\langle \mathbf{f} \cdot \mathbf{u}' \rangle$, as noted earlier. The latter depends on small phase offsets between \mathbf{f} and \mathbf{u}' , which arise from the damping of \mathbf{u}' and are therefore very sensitive to numerical errors. By contrast, the volume-averaged transfer D_R results from spatial correlations between velocity fluctuations and slowly varying mean-flow gradients, which are controlled by the large-scale convective dynamics and are conse-

quently much more robust to discretization and time-stepping choices.

It is important to note that the directly forced oscillations alone may produce a Reynolds-stress tensor that does *not* correlate with the mean-flow velocity gradients. The energy transfer is therefore controlled primarily by the contribution from the secondary fluctuations. This explains why the detailed structure of the forcing has only a weak influence on the resulting transfer rates, as shown in the previous section. To illustrate this point, one may write:

$$\langle u'_i u'_j \rangle \approx \langle u'_{f,i} u'_{f,j} \rangle + \langle u'_{f,i} u'_{s,j} \rangle + \langle u'_{s,i} u'_{f,j} \rangle + \langle u'_{s,i} u'_{s,j} \rangle.$$

The first (purely forced) term on the right-hand side is expected to contribute little to D_{ij} , while the mixed terms involving both forced and secondary fluctuations provide the dominant source of correlation with the mean-flow velocity gradients. Since the amplitudes of the secondary fluctuations are comparable to those of the forced oscillations, these correlations naturally yield Reynolds stresses of order u_f^2 .

This mechanism applies equally to potential-derived forcings, such as $\tilde{\mathbf{f}}_4 = -\nabla\Psi$. Although the net work done by the forcing is achieved exclusively through the non-zero boundary term (see Section 4.5.6), the scalar potential Ψ induces oscillatory displacements of fluid elements throughout the fluid interior. These motions generate secondary fluctuations that develop the same correlations with the mean flow as in the non-potential cases. Consequently, Reynolds-stress correlations are established throughout the domain and sustain a net transfer to the mean flow. With a rigid upper boundary, the net work done by the forcing vanishes, so any Reynolds-stress correlations must cancel in the volume average and cannot produce a systematic transfer. When a free surface is present, the non-zero surface work provides a net energy input, allowing these correlations to organize into a sustained positive transfer.

5.2 How the convective flow produces the correlations

A detailed analysis of the simulations and description of the mechanisms responsible for all observed correlations will be presented in a subsequent paper. Here, we outline the physical processes underlying those Reynolds-stress components for which a clear interpretation can be identified from the simulations. To this end, it is helpful to refer to the snapshots in Fig. 1, which illustrate the typical velocity field and its vertical gradients. The full velocity field $\tilde{\mathbf{u}}$ shown in these snapshots is nearly identical to the mean-flow velocity $\tilde{\mathbf{V}}$ entering the transfer terms \tilde{D}_{ij} .

For the case of a free upper surface, for which the mean-flow velocity gradients can develop without direct kinematic constraints imposed by the boundary, we have found in all simulations that the volume-averaged D_{zz} , D_{xz} and D_{yz} are positive, whereas the volume-averaged D_{zx} and D_{zy} are negative.

5.2.1 $D_{zz} > 0$

Consider a plume moving upwards ($V_z > 0$). A positive vertical fluctuation $u'_z > 0$ displaces a fluid parcel upward faster than the mean flow into a cooler environment. As a result, it has not cooled as much (via diffusion or mixing) and is therefore more buoyant. This increased buoyancy then accelerates the parcel further upward and reinforces the original positive u'_z . By contrast, for a negative $u'_z < 0$, the parcel is displaced downward into a hotter

environment, reducing buoyancy, which opposes the fluctuation and tends to restore the parcel towards the mean velocity.

In regions where $\partial V_z / \partial z > 0$, the coupling between u'_z and δT is more efficient: the growth rate of the convective instability is higher. Thus, aligned fluctuations ($u'_z > 0$) grow faster, while misaligned ones ($u'_z < 0$) are damped more effectively. In contrast, where $\partial V_z / \partial z < 0$, the buoyancy drive is weaker, so the reinforcement/opposition is less pronounced. The same argument applies symmetrically to downward-moving plumes ($V_z < 0$), where negative fluctuations are preferentially reinforced in accelerating regions ($\partial V_z / \partial z > 0$).

This selective amplification in both rising and descending regions yields a robust positive volume-averaged $D_{zz} = \langle u_z'^2 \rangle (\partial V_z / \partial z)$.

Fig. 5 shows that a larger volume-averaged values of D_{zz} are obtained for smaller oscillation periods t_{osc} . Although decreasing t_{osc} at fixed fluctuation amplitude reduces the vertical displacement per cycle ($\delta z \sim u'_z t_{\text{osc}} / 2$), which would tend to weaken the buoyancy contrast and thus the selective reinforcement of u'_z , a competing effect dominates. As t_{osc} decreases, the dimensionless dissipation length-scale λ_{dis} increases (from ~ 0.07 to ~ 0.2 in the simulations) because non-linear interactions have less time to cascade energy to small scales. Consequently, the dimensionless diffusive time-scale $\tau_{\text{diff}} \sim \lambda_{\text{dis}}^2 / \bar{\kappa}$ grows and more readily exceeds t_{osc} , preserving buoyancy perturbations more effectively (closer to adiabatic conditions). This enhances the reinforcement mechanism despite the smaller displacements, resulting in stronger correlations and larger volume-averaged D_{zz} at lower t_{osc} .

5.2.2 D_{zx} and $D_{zy} < 0$

We focus on the term D_{zx} , with a similar argument applying to D_{zy} . Consider a rising plume in a region where $\partial V_z / \partial x < 0$, meaning the mean buoyancy δT decreases in the positive x -direction. A positive horizontal fluctuation $u'_x > 0$ displaces a fluid parcel towards lower mean buoyancy. Since the parcel approximately conserves its temperature perturbation over this short displacement, it becomes relatively hotter (more buoyant) than its new surroundings, accelerating upwards and acquiring a positive vertical fluctuation $u'_z > 0$. Conversely, a negative horizontal fluctuation $u'_x < 0$ displaces the parcel towards higher mean buoyancy, making it relatively cooler (less buoyant) and thus generating $u'_z < 0$. In both cases, u'_x and the induced u'_z have the same sign, yielding $D_{zx} = \langle u'_x u'_z \rangle (\partial V_z / \partial x) < 0$. An analogous argument applies where $\partial V_z / \partial x > 0$: displacements now induce opposite signs in u'_x and u'_z , yielding again $D_{zx} < 0$. The same logic holds for displacements along the y -direction.

5.2.3 D_{xz} and $D_{yz} > 0$

The physics mechanism responsible for the positive volume-averaged values of D_{xz} and D_{yz} is less clear than for the vertical contributions discussed above, and the interpretation we offer here remains speculative. A more thorough analysis will be carried out separately. Consider a rising plume in a region where $\partial V_x / \partial z > 0$, corresponding to mean-flow vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{V} = (\partial V_x / \partial z) \hat{\mathbf{y}}$. A positive vertical fluctuation $u'_z(z) > 0$ displaces a fluid parcel upward into layers where V_x is larger. One might therefore expect pressure and viscous stresses to adjust the parcel velocity towards that of the surrounding flow, leading to a vanishing horizontal fluctuation u'_x . However, the fact that $D_{xz} > 0$

indicates that the parcel acquires a positive horizontal fluctuation $u'_x > 0$, so that it locally overtakes the mean horizontal flow. One possible explanation is that the parcel experiences a lift-like acceleration associated with the background vorticity, analogous to the lift force acting on a body in a rotational flow (T. R. Auton 1987). To illustrate this idea, suppose the fluid parcel were replaced by a small bubble moving with velocity $V_z + u'_z$ in the laboratory frame. The surrounding mean flow has vertical velocity V_z , so in the bubble frame the flow moves past the bubble with relative velocity $\mathbf{U}_{\text{rel}} = -u'_z \hat{\mathbf{z}}$. The background vorticity then generates a lift force proportional to $\mathbf{U}_{\text{rel}} \times \boldsymbol{\omega}$, directed in the positive x -direction. Such a mechanism would accelerate the bubble horizontally beyond the local mean-flow speed. While a fluid parcel is not a rigid body and does not experience a literal Magnus force, this analogy suggests that interactions between vertical slip motions and mean-flow vorticity can bias the horizontal response of rising and sinking plumes, leading to positive correlations $\langle u'_x u'_z \rangle$ in regions where $\partial V_x / \partial z > 0$, and similarly positive $\langle u'_y u'_z \rangle$ where $\partial V_y / \partial z > 0$.

5.2.4 Viscous and thermal constraints on the transfer mechanism

These mechanisms are consistent with the results presented in Section 4.5.4, where the effect of varying the Rayleigh number was examined. Increasing viscous diffusion (i.e. decreasing Ra) reduces the relative contributions of D_{xz} and D_{yz} to the net transfer, because horizontal motions across a convective length-scale are increasingly damped by viscosity. By contrast, the D_{zz} contribution, which is primarily buoyancy-driven, is comparatively less affected. The criterion (49), involving the Reynolds number of the fluctuations, therefore reflects the conditions required to establish correlations in the D_{xz} and D_{yz} terms, rather than in D_{zz} .

The mechanisms described above also rely on fluid parcels retaining their temperature perturbations as they are displaced over a convective length-scale. This requires thermal diffusion to be sufficiently weak over the displacement time-scale. We therefore expect an additional constraint, analogous to equation (49), involving the fluctuating Péclet number:

$$Pe' \equiv \frac{u' \lambda_{\text{conv}}}{\kappa}, \quad (55)$$

which must be of order unity or larger for the buoyancy-mediated correlations to be sustained. This will be investigated in a subsequent paper.

5.2.5 The role of spatial coherence in the fluctuation-mean flow coupling

The mechanisms described above rely on fluid parcels being displaced by the fluctuating motions into regions where local mean-flow properties, such as buoyancy or velocity, differ from those at their original location. These displacements are small, of order $u' t_{\text{osc}}$, where u' is a characteristic amplitude of the fluctuating velocity. Nevertheless, because the fluctuations are coherent over large spatial scales comparable to convective plume widths, their cumulative effect becomes significant when integrated over the volume. This spatial coherence allows the resulting accelerations, whether buoyancy-driven or arising from other interactions with the mean flow, to act systematically in phase over extended regions. As a result, robust volume-averaged correlations develop between $\langle u'_i u'_j \rangle$ and the mean-flow gradients $\partial V_i / \partial x_j$, yielding finite contributions to the transfer terms D_{ij} .

5.2.6 Spatial coherence dominates over time-scale matching

It is often assumed that correlations between oscillatory fluctuations and the mean flow require a matching of time-scales between the forcing and the convective dynamics. Our results show that this is not the case. Instead, the emergence of a positive D_R reflects a match in spatial, rather than temporal, scales between the oscillatory motions and the convective structures. In particular, no resonance between the oscillation period and convective turnover times is required.

The oscillation period is largely irrelevant, provided that fluid elements can be displaced back and forth coherently over an oscillation period without viscous or thermal diffusion overwhelming advection. This condition is satisfied as long as the fluctuating Reynolds and Péclet numbers meet the criteria given by equations (49) and (55). By contrast, matching the relevant length-scales is essential: spatial coherence over a convective length-scale is required for the correlations to build up and yield a net energy transfer, as described above.

This perspective represents a significant shift in our understanding of tidal dissipation, emphasizing the primacy of spatial coherence over time-scale matching in mediating the interaction between oscillatory forcing and convective flows.

5.2.7 Effect of the boundary condition on the correlations

The free upper surface plays two conceptually distinct roles in the simulations, which we distinguish here. The first is energetic: as shown by equation (54), a forcing derived from a scalar potential can perform net work on the flow only through a non-zero surface term, which requires a free surface. This is a mathematical result independent of the details of the flow. The second role is dynamical: the free surface permits large-scale vertical plume structures and mean-flow gradients that differ qualitatively from those found with a rigid lid, and these differences affect the Reynolds-stress correlations, as discussed in Section 4.5.5.

With a rigid boundary, D_{xx} becomes positive when the forcing includes a horizontal component, while D_{zz} no longer contributes to the transfer. The impermeability constraint imposed by the rigid lid redirects part of the vertically forced response into strong, strain-dominated horizontal motions near the upper boundary. As a result, regions of enhanced u_x^2 preferentially coincide with positive horizontal strain. The signs of D_{zx} and D_{xz} are also reversed relative to the free-surface case. This indicates that, in the presence of a rigid boundary, the pressure forces required to enforce impenetrability substantially modify the mean-flow gradients and constrain the fluctuation dynamics. Consequently, the mechanisms operating in the free-surface case are no longer dominant. In this sense, the rigid-surface configuration represents a qualitatively distinct regime, in which coupling between the fluctuations and the mean flow is governed primarily by boundary constraints and pressure adjustments rather than by the intrinsic structure of the convective flow.

A more complete physical explanation of why the rigid boundary suppresses or modifies the correlations would require a detailed analysis of the mean-flow structure and plume morphology in the two configurations. This goes beyond what we can establish from the present simulations and we flag it as an important question for future work.

6 SUMMARY AND DISCUSSION

6.1 Summary

The simulations presented in this paper show that, when the upper surface is free and the fluctuating Reynolds number exceeds a modest threshold, the oscillations transfer kinetic energy systematically to the mean flow. The volume-averaged transfer rate is $D_R \sim u'^2 t_{\text{conv}}^{-1}$, where u' denotes a characteristic amplitude of the fluctuating velocity. This corresponds to an effective dimensionless damping rate $\tilde{\gamma} \simeq 0.1$, essentially independent of the oscillation period over the range explored. This behaviour reflects robust correlations between the Reynolds stresses of the fluctuations and the gradients of the large-scale convective flow.

A decomposition of the transfer by Reynolds-stress component shows that, for a free upper surface, the dominant positive contributions typically involve vertical velocity fluctuations interacting with vertical gradients of the mean flow. Contributions associated with horizontal shear tend to oppose this transfer and partially cancel the buoyancy-driven term, such that the net transfer is well approximated by the mixed components involving vertical motions. These correlations arise because the oscillations coherently displace fluid parcels over large spatial regions comparable to plume widths. The displaced parcels are then systematically redirected by buoyancy and incompressibility, producing sustained, volume-filling correlations.

Importantly, potential-derived forcings that mimic realistic tidal perturbations yield the same robust positive transfer rates and dominance of vertical correlations. This demonstrates that the mechanism remains effective even when the net energy input is localized at the surface.

Viscous and thermal diffusion place clear constraints on this mechanism. Positive transfer is observed only when the fluctuating Reynolds number satisfies $Re' \gtrsim \mathcal{O}(1-10)$, ensuring that advective displacements across convective length-scales are not erased by viscous damping. We expect an analogous constraint involving the fluctuating Péclet number, which reflects the need for fluid parcels to retain their temperature perturbations while being displaced.

Replacing the free upper surface with a rigid boundary has two distinct consequences. First, for forcings derived from a potential, a rigid boundary prevents the forcing from performing net work on the flow, as shown by equation (54). Secondly, and independently, the rigid boundary alters the large-scale structure of the mean flow and the associated Reynolds-stress correlations, suppressing the dominant transfer terms present in the free-surface case. The physical origin of this second effect remains unclear, and distinguishing the relative contributions of energetic consistency, plume morphology and mean-flow gradients is left for future work. In summary, the rigid-surface configuration represents a distinct regime that is not representative of stellar or planetary convective envelopes.

6.2 Discussion

A. J. Barker & A. A. V. Astoul (2021) argued that the volume-averaged transfer term D_R does not contribute to the damping of fast oscillations in convective flows. This conclusion was based on a combination of analytical arguments and numerical simulations. As already noted by C. Terquem (2023), their analytical argument is flawed, owing to a misidentification of the term responsible for energy exchange between the oscillatory

fluctuations and the mean flow. More fundamentally, their numerical simulations impose rigid upper and lower boundaries. In an incompressible fluid, a potential tidal forcing cannot perform net work in such a configuration, preventing a self-consistent determination of the energy exchange between the oscillatory flow and the mean flow. Consequently, the oscillatory tidal response is imposed through a prescribed irrotational velocity field rather than being generated dynamically by the forcing. As demonstrated in this paper, the correlations responsible for a positive D_R arise because fluid displacements driven by the forcing respond to buoyancy and incompressibility constraints, generating secondary oscillations. The rigid-boundary configuration suppresses the free-surface displacements required for these correlations to develop, while prescribing the oscillatory velocity field directly in the advection terms further interferes with the self-consistent development of the secondary responses. Consequently, the transfer mechanism identified here cannot arise in these simulations.

A key result of this study is that D_R is independent of the forcing period over the range explored. This period-independence is not merely an empirical observation. Rather, it follows from the physical argument developed in Section 5.2: the correlations responsible for D_R arise from spatial coherence between the oscillatory displacements and the convective structures, and do not require any matching of time-scales between the forcing and the convection. The scaling $D_R \sim u'^2 t_{\text{conv}}^{-1}$ contains no explicit dependence on t_{osc} , and period-independence is therefore expected to hold throughout the regime $t_{\text{osc}} \ll t_{\text{conv}}$, provided that the fluctuating Reynolds and Péclet number conditions given by equations (49) and (55) are satisfied. That said, our numerical simulations span only a factor of ten in t_{osc} , and direct verification over a broader range would be valuable.

Although our simulations were performed at Rayleigh and Prandtl numbers far from those relevant to stellar and planetary convection, they nevertheless isolate physical mechanisms that are expected to persist in more extreme regimes. In particular, the energy transfer quantified by D_R arises from correlations between oscillatory displacements and the large-scale convective flow, which depend primarily on spatial coherence and on the ability of the fluctuations to advect fluid across mean-flow gradients. These conditions are controlled by the fluctuating Reynolds and Péclet numbers, rather than by the absolute values of Ra and Pr , and are therefore expected to remain satisfied in astrophysical environments where Re' and Pe' are extremely large. In the outer convective envelope of a solar-type star, above roughly $0.8 R_{\odot}$, the region most relevant for tidal interactions with a companion, $Ra \sim 10^{20}-10^{22}$ and $Pr \sim 10^{-6}-10^{-4}$ (J. Schumacher & K. R. Sreenivasan 2020). These values correspond to thermal diffusivities $\kappa \sim 1-500 \text{ m}^2 \text{ s}^{-1}$ and kinematic viscosities $\nu \sim$ a few times $10^{-4} \text{ m}^2 \text{ s}^{-1}$. The characteristic convective length-scale in this region is $\lambda_{\text{conv}} \sim 0.08 R_{\odot}$ (C. Terquem 2021). The tidal velocity amplitude excited by a $1 M_{\odot}$ companion is typically in the range $0.01-1 \text{ m s}^{-1}$ for orbital periods between 4 and 12 days (C. Terquem et al. 1998, C. Terquem 2021). These values imply fluctuating Reynolds and Péclet numbers $Re' > 10^9$ and $Pe' > 10^3$, well above the thresholds required for the establishment of the correlations between the fluctuations and the mean flow. In this regime of extremely large fluctuating Reynolds numbers, we expect the vertical shear terms D_{xz} and D_{yz} to dominate tidal damping. At the same time, important uncertainties remain in extrapolating these results quantitatively to astrophysical conditions. The structure of convection at $Ra \sim 10^{20}-10^{22}$ is significantly more turbulent and intermittent than in the present simulations,

and may modify the detailed distribution of velocity gradients and the resulting correlations. In addition, the influence of very low Prandtl number, strong stratification, rotation and magnetic fields, all absent from our model, may affect both the magnitude and spatial structure of the energy transfer. While these effects are not expected to suppress the mechanism itself, they may either enhance or reduce its efficiency.

The results presented in this paper validate the formalism proposed by Terquem (2021, 2023), who demonstrated from first principles that the traditional turbulent-viscosity description breaks down when the external forcing varies on a time-scale much shorter than the convective time-scale. For fast tides, the relevant Reynolds stresses are set by correlations involving the tidal velocity fluctuations and gradients of the convective flow, rather than by correlations involving convective velocities and gradients of the tidal flow.

Adopting this formalism, C. Terquem & S. Martin (2021) explored the consequences of equilibrium-tide dissipation by assuming an energy transfer rate of the form investigated here. Under this assumption, they showed that dissipation of the equilibrium tide alone can account for the observed circularization periods of solar-type binaries. By incorporating, for the first time, the full time evolution of stellar structure, they found that tidal dissipation is efficient both before and after the main sequence, while remaining weak during the main-sequence phase, in significantly improved agreement with observations compared to previous theories. Using the same framework, C. Terquem (2023) derived an average phase lag between the equilibrium tide and the tidal potential due to moons, obtained by integrating over the convective envelope of giant planets, in good agreement with observational constraints for Jupiter and several of Saturn's moons.

The present results establish the existence and basic scaling of the mechanism by which fast tides transfer their energy to the mean convective flow. Further work is required to explore a broader region of parameter space and to develop a more detailed physical understanding of the correlations underlying this transfer. In particular, extending this study to a wider range of Rayleigh and Prandtl numbers, and to forcing prescriptions that more closely reflect tidal potentials in stratified convective envelopes, will help clarify the origin of these correlations and their dependence on viscosity, thermal diffusion and geometry.

Taken together, these results represent a substantial advance in our understanding of tidal dissipation, with broad implications for the evolution of stellar binaries, planetary systems and the dynamics of forced turbulent flows more generally.

ACKNOWLEDGEMENTS

AB is supported by a PhD studentship from the Science and Technology Facilities Council (STFC) under grant ST/Y509474/1. EM was supported by the Summer Undergraduate Research Opportunities Programme (UROP) hosted by the Rudolf Peierls Centre for Theoretical Physics. CT acknowledges the use of ChatGPT and Grok in developing the post-processing codes for data produced by the Dedalus simulations. CT also thanks the Isaac Newton Institute for Mathematical Sciences, Cambridge, for its support and hospitality during the programme *Antidiffusive dynamics: from sub-cellular to astrophysical scales*, where part of this work was undertaken. This programme was supported by EPSRC grant EP/R014604/1. CT is grateful to Sacha Brun, Stephan Fauve and Antoine Strugarek for illuminating discussions on convection, to

Geoff Vasil for advice on Dedalus and convection more broadly, and to Steven Balbus for sharing his invaluable expertise in fluid dynamics and for his unfailing optimism regarding the outcome of this work. Finally, we thank the referee for a thoughtful and insightful report and for suggestions that have improved the paper.

DATA AVAILABILITY

No new data were generated or analysed in support of this research.

REFERENCES

- Auton T. R., 1987, *J. Fluid Mech.*, 183, 199
 Barker A. J., Astoul A. A. V., 2021, *MNRAS*, 506, L69
 Burns K. J., Vasil G. M., Oishi J. S., Lecoanet D., Brown B. P., 2020, *ApJS*, 251, 17
 Cliffe K. A., Tavener S. J., 1998, *J. Comput. Phys.*, 145, 193
 Dang C., Peybernes M., Le Tellier R., Saas L., 2021, *Ann. Nucl. Energy*, 150, 107848
 Duguid C. D., Barker A. J., Jones C. A., 2020, *MNRAS*, 497, 3400
 Goldreich P., Nicholson P. D., 1977, *Icar*, 30, 301
 Ogilvie G. I., 2014, *ARA&A*, 52, 171
 Ogilvie G. I., Lesur G., 2012, *MNRAS*, 422, 1975
 Penev K., Sasselov D., Robinson F., Demarque P., 2009, *ApJ*, 704, 930
 Schumacher J., Sreenivasan K. R., 2020, *Rev. Modern Phys.*, 92, 041001
 Terquem C., 2021, *MNRAS*, 503, 5789
 Terquem C., 2023, *MNRAS*, 525, 508
 Terquem C., Martin S., 2021, *MNRAS*, 507, 4165
 Terquem C., Papaloizou J. C. B., Nelson R. P., Lin D. N. C., 1998, *ApJ*, 502, 788
 Vidal J., Barker A. J., 2020a, *MNRAS*, 497, 4472
 Vidal J., Barker A. J., 2020b, *ApJ*, 888, L31
 Zahn J. P., 1966, *AnAp*, 29, 489

APPENDIX A: DERIVATIONS OF FREE-SURFACE BOUNDARY CONDITIONS

We consider a free surface $z = H + \eta(x, y)$ which radiates as a blackbody at temperature T_2 .

A1 Kinematic boundary condition

At the surface $z = H + \eta$, the kinematic boundary condition is $u_z = d\eta/dt$. Defining dimensionless quantities $\tilde{\eta} = \eta/d$ and $L_z = H/d$, this becomes:

$$\tilde{u}_z = \tilde{\eta}_t + \tilde{u}_x \tilde{\eta}_x + \tilde{u}_y \tilde{\eta}_y, \quad (\text{A1})$$

where all velocity components are evaluated at $\tilde{z} = L_z + \tilde{\eta}$ and we define:

$$\tilde{\eta}_t \equiv \frac{\partial \tilde{\eta}}{\partial \tilde{t}}, \quad \tilde{\eta}_x \equiv \frac{\partial \tilde{\eta}}{\partial \tilde{x}}, \quad \tilde{\eta}_y \equiv \frac{\partial \tilde{\eta}}{\partial \tilde{y}}. \quad (\text{A2})$$

Assuming small deformation, i.e. $|\tilde{\eta}| \ll L_z$, we expand to first order:

$$\tilde{u}_z + \tilde{\eta} \frac{\partial \tilde{u}_z}{\partial \tilde{z}} = \tilde{\eta}_t + \tilde{u}_x \tilde{\eta}_x + \tilde{u}_y \tilde{\eta}_y, \quad (\text{A3})$$

evaluated at $\tilde{z} = L_z$.

A2 Continuity of the tangential stress

At the surface $z = H + \eta(x, y)$, we define tangent vectors:

$$\hat{\mathbf{i}}^{(x)} = \frac{1}{\sqrt{1 + \eta_x^2}} \begin{pmatrix} 1 \\ 0 \\ \eta_x \end{pmatrix}, \quad \hat{\mathbf{i}}^{(y)} = \frac{1}{\sqrt{1 + \eta_y^2}} \begin{pmatrix} 0 \\ 1 \\ \eta_y \end{pmatrix}, \quad (\text{A4})$$

and the surface normal:

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{1 + \eta_x^2 + \eta_y^2}} \begin{pmatrix} -\eta_x \\ -\eta_y \\ 1 \end{pmatrix}, \quad (\text{A5})$$

where $\eta_x = \partial\eta/\partial x = \tilde{\eta}_x$ and $\eta_y = \partial\eta/\partial y = \tilde{\eta}_y$.

The viscous stress is $T_i = \sigma_{ij}n_j$, where σ_{ij} are the components of the viscous stress tensor and n_j are the components of $\hat{\mathbf{n}}$. For an incompressible Newtonian fluid:

$$\sigma_{ij} = \sigma_{ji} = \rho\nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (\text{A6})$$

with $x_1 = x$, $x_2 = y$ and $x_3 = z$.

Continuity of tangential stress at the surface requires $\mathbf{T} \cdot \hat{\mathbf{i}}^{(x)} = 0$, which gives

$$\eta_x (\sigma_{zz} - \sigma_{xx}) + (1 - \eta_x^2) \sigma_{xz} - \eta_y (\sigma_{xy} + \eta_x \sigma_{zy}) = 0, \quad (\text{A7})$$

where the components of the stress tensor are evaluated at $z = H + \eta$. In dimensionless form and to first order in $|\tilde{\eta}|/L_z$, this becomes

$$\tilde{\eta}_x (\tilde{\sigma}_{zz} - \tilde{\sigma}_{xx}) + \tilde{\sigma}_{xz} + \tilde{\eta} \frac{\partial \tilde{\sigma}_{xz}}{\partial \tilde{z}} - \tilde{\eta}_y \tilde{\sigma}_{xy} = 0, \quad (\text{A8})$$

evaluated at $\tilde{z} = L_z$.

Similarly, the condition $\mathbf{T} \cdot \hat{\mathbf{i}}^{(y)} = 0$ yields

$$\tilde{\eta}_y (\tilde{\sigma}_{zz} - \tilde{\sigma}_{yy}) + \tilde{\sigma}_{yz} + \tilde{\eta} \frac{\partial \tilde{\sigma}_{yz}}{\partial \tilde{z}} - \tilde{\eta}_x \tilde{\sigma}_{xy} = 0, \quad (\text{A9})$$

also at $\tilde{z} = L_z$.

A3 Continuity of the normal stress

Let P_{atm} be the atmospheric pressure above the fluid. At $z = H + \eta$, the normal stress balance is

$$\mathbf{T} \cdot \hat{\mathbf{n}} - P = -P_{\text{atm}}, \quad (\text{A10})$$

We write $P = P_0 + \delta P$, with P_0 satisfying the hydrostatic equilibrium equation (5). This yields $P_0 = -\rho_0 g z + C$, where the constant C is determined by the boundary condition $P_0 = P_{\text{atm}}$ at $z = H$. Substituting in equation (A10) yields

$$\mathbf{T} \cdot \hat{\mathbf{n}} + \rho_0 g \eta - \delta P = 0, \quad (\text{A11})$$

evaluated at $z = H + \eta$. In dimensionless form,

$$\tilde{\mathbf{T}} \cdot \hat{\mathbf{n}} + \frac{\tilde{\eta}}{\alpha \Delta T} - \tilde{P} = 0, \quad (\text{A12})$$

evaluated at $\tilde{z} = L_z + \tilde{\eta}$, where $\tilde{T}_i = \tilde{\sigma}_{ij}n_j$. Expanding this using the stress components:

$$\frac{1}{1 + \tilde{\eta}_x^2 + \tilde{\eta}_y^2} \left(\tilde{\eta}_x^2 \tilde{\sigma}_{xx} + \tilde{\eta}_y^2 \tilde{\sigma}_{yy} + \tilde{\sigma}_{zz} + 2\tilde{\eta}_x \tilde{\eta}_y \tilde{\sigma}_{xy} - 2\tilde{\eta}_x \tilde{\sigma}_{xz} - 2\tilde{\eta}_y \tilde{\sigma}_{yz} \right) + \frac{\tilde{\eta}}{\alpha \Delta T} - \tilde{P} = 0, \quad (\text{A13})$$

To first order in $|\tilde{\eta}|/L_z$, this reduces to

$$\tilde{\sigma}_{zz} + \tilde{\eta} \frac{\partial \tilde{\sigma}_{zz}}{\partial \tilde{z}} - 2\tilde{\eta}_x \tilde{\sigma}_{xz} - 2\tilde{\eta}_y \tilde{\sigma}_{yz} + \frac{\tilde{\eta}}{\alpha \Delta T} - \tilde{P} - \tilde{\eta} \frac{\partial \tilde{P}}{\partial \tilde{z}} = 0, \quad (\text{A14})$$

evaluated at $\tilde{z} = L_z$.

A4 Radiative boundary condition

At the surface, the radiative flux is given by

$$-k \nabla T \cdot \hat{\mathbf{n}} = \epsilon \sigma (T^4 - T_{\text{atm}}^4), \quad (\text{A15})$$

where T is evaluated at $z = H + \eta$, T_{atm} is the atmospheric temperature above the fluid, σ is the Stefan-Boltzmann constant, ϵ is the emissivity and k is the thermal conductivity. We take $T_{\text{atm}} = 0$ and $\epsilon = 1$, corresponding to a blackbody radiating into a vacuum. Using $T = T_0 + \delta T$ and $\nabla T_0 = -(\Delta T/H) \hat{\mathbf{z}}$, the boundary condition becomes:

$$\frac{-k}{\sqrt{1 + \eta_x^2 + \eta_y^2}} \left(-\eta_x \frac{\partial \delta T}{\partial x} - \eta_y \frac{\partial \delta T}{\partial y} + \frac{\partial \delta T}{\partial z} - \frac{\Delta T}{H} \right) = \sigma T^4, \quad (\text{A16})$$

evaluated at $z = H + \eta$. To first order in $|\eta|/H$, we expand

$$T^4(H + \eta) = T^4(H) + 4\eta T^3(H) \frac{\partial T}{\partial z}(H). \quad (\text{A17})$$

With $T(H) = T_2 + \delta T(H)$, where $T_2 \equiv T_0(H)$, this becomes

$$T^4(H + \eta) = T_2^4 \left(1 + \frac{\delta T(H)}{T_2} \right)^4 + 4\eta T_2^3 \left(1 + \frac{\delta T(H)}{T_2} \right)^3 \left(-\frac{\Delta T}{H} + \frac{\partial \delta T}{\partial z}(H) \right). \quad (\text{A18})$$

At equilibrium, $-kdT_0/dz = k\Delta T/H = F_0$, and the surface temperature T_2 satisfies $\sigma T_2^4 = F_0$. Substituting these relations yields

$$\sigma T^4(H + \eta) = F_0 \left[\left(1 + \frac{\tilde{b}}{\tilde{T}_2} \right)^4 - 4\tilde{\eta} \frac{1}{\tilde{T}_2} \left(1 + \frac{\tilde{b}}{\tilde{T}_2} \right)^3 \left(\frac{1}{L_z} - \frac{\partial \tilde{b}}{\partial \tilde{z}} \right) \right], \quad (\text{A19})$$

where \tilde{b} and its derivative are evaluated at $\tilde{z} = L_z$ and we define $\tilde{T}_2 = T_2/\Delta T$.

Finally, substituting into equation (A16), we obtain, to first order in $|\eta|/H$:

$$1 + \left(\tilde{\eta}_x \frac{\partial \tilde{b}}{\partial \tilde{x}} + \tilde{\eta}_y \frac{\partial \tilde{b}}{\partial \tilde{y}} - \frac{\partial \tilde{b}}{\partial \tilde{z}} - \tilde{\eta} \frac{\partial^2 \tilde{b}}{\partial \tilde{z}^2} \right) L_z = \left(1 + \frac{\tilde{b}}{\tilde{T}_2} \right)^4 - 4\tilde{\eta} \frac{1}{\tilde{T}_2} \left(1 + \frac{\tilde{b}}{\tilde{T}_2} \right)^3 \left(\frac{1}{L_z} - \frac{\partial \tilde{b}}{\partial \tilde{z}} \right), \quad (\text{A20})$$

evaluated at $\tilde{z} = L_z$.

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