Algorithm for Iwahori-Matsumoto duality for tempered unipotent representations of geometric Hecke algebras of type B, C, D

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Abstract

The Iwahori–Matsumoto involution is an involutive operation on the Grothendieck group of complex finite-dimensional representations of an affine Hecke algebra, or a graded Hecke algebra. The Aubert–Zelevinsky involution is another involutive operation on the Grothendieck group of complex finite-length representations of p-adic groups. The representation theory of p-adic groups can be described by certain affine and graded Hecke algebras, and the involutions mentioned above are related via this. The main goal of the thesis it to provide an explicit algorithm for the Iwahori–Matsumoto involution for irreducible tempered representations of certain affine Hecke algebras, for irreducible tempered representations of certain graded Hecke algebras which additionally have real infinitesimal character, and an algorithm for AZ for tempered unipotent representations for certian p-adic groups. The affine and geometric Hecke algebras that we consider are those coming from the p-adic groups $\mathrm{SO}(N)$ and $\mathrm{Sp}(2n)$. For AZ, we only consider the p-adic group $\mathrm{SO}(2n+1)$. We will consider the cases $\mathrm{SO}(2n)$ and $\mathrm{Sp}(2n)$ in a forthcoming paper. The algorithm will be obtained by completely different methods than the methods used by Atobe and Minguez, who obtained an algorithm for AZ for $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(2n)$.

Contents

Introduction

Let k be a nonarchimedean local field of characteristic 0 with ring of integers $\mathfrak{o}$ and finite residue field $k := \mathbb{F}_q$ of cardinality q. Let $\mathbf{G}$ be a connected reductive algebraic group defined and split over k. The representation theory of $\mathbf{G}(\mathsf{k})$ is of major interest for the local Langlands correspondence. There are various ways of studying the representations of $\mathbf{G}(\mathsf{k})$. One such way is via 'arithmetic Hecke algebras'. These arise in the following way. The category of smooth complex representations $\mathcal{R}(\mathbf{G}(\mathsf{k}))$ of $\mathbf{G}(\mathsf{k})$ splits into a direct product of full subcategories called Bernstein blocks. Consider the Hecke algebra $\mathcal{H}(\mathbf{G}(\mathsf{k}))$ of $\mathbf{G}(\mathsf{k})$, which is the non-unital $\mathbb{C}$-algebras consisting of locally constant and compactly supported complex functions $\mathbf{G}(\mathsf{k}) \to \mathbb{C}$. For each Bernstein block, there exists a certain idempotent $e \in \mathcal{H}(\mathbf{G}(\mathsf{k}))$ such that the category of $e\,\mathcal{H}(\mathbf{G}(\mathsf{k}))e$-modules is isomorphic to the Bernstein block. The unital algebra $e\,\mathcal{H}(\mathbf{G}(\mathsf{k}))e$ is called an 'arithmetic' Hecke algebra.

Lusztig has done a large amount of work on unipotent representations for $\mathbf{G}(\mathsf{k})$ under the assumption that $\mathbf{G}$ is of adjoint type. They form the simple objects of a finite number of *unipotent Bernstein blocks*. In the case that $\mathbf{G}(\mathsf{k})$ is adjoint, the arithmetic Hecke algebras for such Bernstein blocks were shown to be of affine type in [Lus95a, §1], i.e. they have an explicit Iwahori–Matsumoto and Bernstein presentation. Lusztig furthermore defined geometric affine Hecke algebras associated to the Langlands dual $G^\vee$, i.e. the complex reductive group whose root datum is the dual of that of $\mathbf{G}(\mathsf{k})$, and remarkably matched up each geometric affine Hecke algebra with an isomorphic arithmetic affine Hecke algebra in [Lus95a], giving a bijection between the set of arithmetic affine Hecke algebras and the set of geometric affine Hecke algebras. Lusztig also introduced the notion of graded Hecke algebras associated to these geometric affine Hecke algebras in [Lus89] and related their representation theories to each other. Tracing all the equivalences and bijective correspondences described above, we obtain a bijection between the unipotent representations of $\mathbf{G}(\mathsf{k})$ and irreducible representations of various graded Hecke algebras.

The main goal of the thesis is to obtain explicit algorithms for various involutions on the Grothendieck group of certain unipotent representations of $\mathbf{G}(\mathsf{k})$ as well as certain representations of the related arithmetic and geometric affine Hecke algebras, and the graded Hecke algebras. The first two involutions of interest are the following. The Iwahori–Matsumoto involution IM (resp. graded Iwahori–Matsumoto involution $\mathbb{IM}$) is an involution on the Grothendieck group of finite-dimensional representations of an affine Hecke algebras (resp. graded Hecke algebra). We aim to obtain an algorithm for IM for tempered representations of the arithmetic and geometric affine Hecke algebras, and an algorithm for $\mathbb{IM}$ for tempered representations with real infinitesimal character of the geometric graded Hecke algebras that show up in Lusztig's classification of unipotent representations of $\mathrm{SO}(N, \mathsf{k})$ and $\mathrm{Sp}(2n, \mathsf{k})$. The next involution is the Aubert–Zelevinsky involution AZ on the Grothendieck group of finite-length representations of $\mathbf{G}(\mathsf{k})$. Our goal is to obtain an algorithm for AZ for tempered unipotent representations of $\mathrm{SO}(2n+1, \mathsf{k})$. We note here that an algorithm for AZ for all irreducible representations of $\mathrm{SO}(2n+1, \mathsf{k})$ and $\mathrm{Sp}(2n, \mathsf{k})$ was given in [AM23], but the approach is completely different.

We describe the level of 'explicitness' of the algorithms for $\mathbb{IM}$, IM and AZ that we will obtain. For a connected complex reductive group G, Lusztig classified the irreducible representations of the geometric graded Hecke algebras in [Lus88] and [Lus95b], showing that they are in bijection with G-conjugacy classes of triples (e, σ, ψ) with $e, \sigma \in \mathfrak{g}^\vee = \mathrm{Lie}(G)$ such that $\mathrm{ad}(s)e = qe$ and ψ is an irreducible representation of the group $A_G(e, \sigma)$ of components of the simultaneous centraliser of σ and e. Using this as one of the main inputs, Lusztig classified all unipotent representations for $\mathbf{G}(\mathsf{k})$ (as well as all its inner twists) in [Lus95a] in the case that $\mathbf{G}(\mathsf{k})$ is adjoint in terms of the so-called Deligne–Langlands–Lusztig triples (e, s, ϕ), where $s \in G^\vee$ is semisimple, called the *infinitesimal character*, $e \in \mathfrak{g}^\vee = \mathrm{Lie}(G^\vee)$ is a nilpotent element such that $\mathrm{ad}(s)e = qe$ and ϕ is an irreducible representation of the group $A_{G^\vee}(s, e)$ of connected components of the simultaneous centraliser of s and e. The sought algorithm for $\mathbb{IM}$ mentioned above will be stated in terms of these triples for G when $G = \mathrm{SO}(N)$ and $G = \mathrm{Sp}(2n)$.

Using the reduction theorems in [Lus89] mentioned in the previous paragraph and a result from [Cha16] (which was based on [Kat93]), we can use the algorithm for $\mathbb{IM}$ to obtain an algorithm for IM for the associated geometric affine Hecke algebra for irreducible tempered representations, and in turn for the corresponding arithmetic affine Hecke algebra via the isomorphisms coming from Lusztig's matching. Finally, a result of Kato [Kat93] is used to relate IM to AZ, giving an algorithm for AZ for all *unipotent representations* as well in the case that $\mathbf{G}(\mathsf{k})$ is adjoint.

We note that the adjointness assumption is the reason why we only consider the $\mathrm{SO}(2n+1, \mathsf{k})$ case for the algorithm for AZ. The $\mathrm{SO}(2n, \mathsf{k})$ and $\mathrm{Sp}(2n, \mathsf{k})$ case will be studied in a subsequent paper, where we will use the results from [Sol18], which are strongly based on results from [AMS17], [AMS18a], and [AMS18b].

Main result: obtaining an algorithm for $\mathbb{IM}$

As noted in the previous paragraph, we aim to obtain an algorithm for $\mathbb{IM}$, and then use general theory (most of it due to Lusztig) to obtain an algorithm for IM and AZ.

Our approach towards obtaining the algorithm for $\mathbb{IM}$ for $G = \mathrm{SO}(N)$ and $G = \mathrm{Sp}(2n)$ is as follows. As mentioned above, the irreducible representations of the geometric graded Hecke algebras attached to $G^\vee$ are parametrised by triples (e, σ, ψ) of data of the Lie algebra. To each such triple, one can attach a standard module $Y(e, \sigma, \psi)$, which is a representation of one of the geometric Hecke algebras, $\mathbb{H}$ say. This standard module has a unique irreducible quotient that we shall denote by $\bar{Y}(e, \sigma, \psi)$. We assume that $Y(e, \sigma, \psi)$ is tempered, this then implies that $Y(e, \sigma, \psi) = \bar{Y}(e, \sigma, \psi)$ is irreducible (however, $\mathbb{IM}(Y(e, \sigma, \psi))$ may be non-tempered). Let W_L denote the relative Weyl group associated to $\mathbb{H}$. If σ is furthermore a real infinitesimal character, it turns out that the module $Y(e, \sigma, \psi)|_{W_L}$ is isomorphic to the ψ-isotypic component of the cohomology of a certain 'generalised Springer fibre', i.e. $Y(e, \sigma, \psi)|_{W_L}$ is completely described by Green functions. As a consequence of the Lusztig–Shoji algorithm [Lus86a, Theorem 24.8], the generalised Springer representation $\mathrm{GSpr}(e, \psi)$ corresponding to (e, ψ) in $Y(e, \sigma, \psi)|_{W_L}$ is the unique 'minimal representation', i.e. its Springer support – that is, the nilpotent class in $\mathfrak{g}^\vee$ containing e – is strictly smaller than the Springer support of any other irreducible subrepresentation of $Y(e, \sigma, \psi)|_{W_L}$, and furthermore its multiplicity in $Y(e, \sigma, \psi)|_{W_L}$ is 1. Furthermore, we have $\mathbb{IM}(Y(e, \sigma, \psi))|_{W_L} = \bar{Y}(e, \sigma, \psi)|_{W_L} \otimes \mathrm{sgn}$. The key idea to find $\mathbb{IM}$ is the following

1. We show that $Y(e, \sigma, \psi)|_{W_L}$ has a unique 'maximal subrepresentation' $\bar{\rho} = \mathrm{GSpr}(\bar{e}, \bar{\psi})$ with multiplicity 1 in the same sense as above (Theorem 3.2.2 for $\mathrm{SO}(N)$ and Waldspurger's result Theorem 3.2.12 for $\mathrm{Sp}(2n)$),

2. We show that $\bar{\rho} \otimes \mathrm{sgn}$ is the minimal representation in $Y(e, \sigma, \psi)|_{W_L} \otimes \mathrm{sgn}$ (Theorem 3.2.6 for $\mathrm{SO}(N)$ and Waldspurger's result Theorem 3.2.13 for $\mathrm{Sp}(2n)$),

3. We show that this implies that $\mathbb{IM}(Y(e, \sigma, \psi)) = \bar{Y}(\bar{e}, \sigma, \bar{\psi})$ (Section 4.1).

Thus we have reduced the problem to finding an algorithm to determine $\bar{\rho}$, i.e. to determine $(\bar{e}, \bar{\psi})$. For $G = \mathrm{Sp}(2n, \mathbb{C})$, this result was proved in [Wal19], with the condition that e is parametrised by a symplectic partition with only even parts. For $G = \mathrm{SO}(N, \mathbb{C})$, this was done in [La22], with the condition that e is parametrised by an orthogonal partition with only odd parts. Using these results, we obtain the desired algorithm for $\mathbb{IM}$, but only when e satisfies the conditions above. We will use the results from [Wal18a, §3.4] to prove Theorem 4.3.1, which extends our algorithm to arbitrary e, giving us an algorithm for all tempered representations with real infinitesimal character.

Structure of the thesis

We will discuss the theory of Bernstein blocks, their corresponding Hecke algebras, affine Hecke algebras, graded Hecke algebras, the generalised Springer correspondence and unipotent representations and in Chapter 1. We will define the involutions AZ, IM, and $\mathbb{IM}$ in Chapter 2, as well as the correspondences between them coming from the correspondence between unipotent representations of $\mathbf{G}(\mathsf{k})$ and representations of the affine and graded Hecke algebras. In Chapter 3 we will discuss the maximality and minimality result for certain generalised Springer representations of and $\mathrm{SO}(N,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$ as mentioned above. Finally, in Chapter 4 we will give the explicit algorithm for $\mathbb{IM}$ tempered representations with real infinitesimal character and AZ for tempered unipotent representations with arbitrary infinitesimal character. For $\mathbb{IM}$, the main result will be obtaining the algorithm for geometric graded Hecke algebras of $\mathrm{SO}(N,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$, although we also give an algorithm for $\mathrm{SL}(n,\mathbb{C})$ as an easy example. We briefly comment on the exceptional types, for which we intend to include all the tables for $\mathbb{IM}$ for the tempered representations with real infinitesimal character in a subsequent paper. For AZ for tempered unipotent representations, we will give the algorithm for $\mathrm{SO}(2n+1,\mathsf{k})$, and briefly explain the steps that we will take in a subsequent paper to obtain an algorithm for $\mathrm{SO}(N,\mathsf{k})$ with odd N and $\mathrm{Sp}(2n,\mathsf{k})$.

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Chapter 1

Preliminaries

1.1 Notation

We fix the following notation for the remainder of the paper. Let k be a nonarchimedean local field (of arbitrary characteristic) with ring of integers $\mathfrak{o}$ with finite residue field $k := \mathbb{F}_q$ of cardinality q. Let $\mathbf{G}$ be a connected reductive algebraic group defined and split over k. Denote $\mathfrak{g}$ for the Lie algebra of $\mathbf{G}$, $\mathbf{G}^\vee$ for the Langlands dual group associated to $\mathbf{G}$ and $\mathfrak{g}^\vee$ for its Lie algebra. We also denote by $G^\vee$ and $\mathfrak{g}^\vee$ the group of $\mathbb{C}$-points of $\mathbf{G}^\vee$ and $\mathfrak{g}^\vee$ respectively.

Fix a minimal parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$ defined over k and let $\mathcal{Q}$ denote the set of parabolic subgroups of $\mathbf{G}$ defined over k containing $\mathbf{B}$.

Fix an Iwahori subgroup $\mathbf{I}$ of $\mathbf{G}$. Let $\mathscr{P}$ be the set of parahoric subgroups $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{I}$. These are certain group schemes defined over $\mathrm{Spec}(\mathfrak{o})$. The modern definition of parahoric subgroups and the Iwahori subgroup uses the theory of Bruhat-Tits buildings (see [BT84, Définition 5.2.6]; see also [RL00, §1.1] for a short overview of the definition). An older definition for $\mathbf{I}$ in the case that $\mathbf{G}$ is adjoint traces back to [IM65] and coincides with the modern definition. For $\mathbf{P} \in \mathscr{P}$ of $\mathbf{G}$, let $\mathbf{U_P}$ be its pro-unipotent radical. The reductive quotient $\bar{\mathbf{P}} = \mathbf{P}\,/\,\mathbf{I}$ of $\mathbf{P}$ is a connected reductive algebraic group defiend over the residue field k of $\mathsf{k} = \mathbb{F}_q$.

Denote by $\mathrm{Rep}(\mathbf{G}(\mathsf{k}))$ the category of finite-length complex smooth representations of $\mathbf{G}(\mathsf{k})$ and let $\mathcal{R}(\mathbf{G}(\mathsf{k}))$ denote its Grothendieck group. Similarly, for a $\mathbb{C}$-algebra A, denote by $\mathcal{R}(A)$ the Grothendieck group of finite-dimensional A-modules.

In general, given a group Γ, we write $\Gamma^\wedge$ for the set of isomorphism classes of irreducible representations of Γ. If Γ acts on a set X, we write X^Γ for the set of fixed points of X.

For the generalised Springer correspondence, as is well-known, one could also look at the set of nilpotent orbits N on $\mathrm{Lie}(G)$ instead of $\mathcal{U}$ and consider G-equivariant local systems on N. In particular the definitions of the objects later that will involve nilpotent orbits or unipotent orbits such as (the generalised) Springer fibres are 'independent' of whether we choose to work with nilpotent or unipotent orits, i.e. if N is a nilpotent orbit of $\mathrm{Lie}(G)$ corresponding to the unipotent orbit U of G, then the definitions involving N will be the same as the one involving U. In this thesis, we shall use both notations. Particularly for the generalised Springer correspondence in Section 1.2 and Chapter 3 we shall use the notation with unipotent orbits, to be consistent with [Wal19] and [La22]. In the context of unipotent representations and standard modules, especially in Section 1.3, Section 1.4 and Chapter 4, we shall use the notation with nilpotent orbits.

1.2 Generalised Springer correspondence

Let G be a connected complex reductive group. We give an overview of the generalised Springer correspondence for G as studied by Lusztig in [Lus84b], omitting details regarding perverse sheaves.

Let $\mathcal{U}$ be the set of unipotent classes of G and let $\mathcal{N}_{\mathrm{G}}$ be the set of pairs $(C, \mathcal{E})$, where $C \in \mathcal{U}$ and $\mathcal{E}$ is an irreducible G-equivariant local system on C. Let L be a Levi subgroup of G containing a cuspidal pair $(C_L, \mathcal{L})$ (see for instance [Lus84b, 2.4 Definition]). Denote by S_G the set of such triples $(L, C_L, \mathcal{L})$. Lusztig showed that $W_L := N_G(L)/L$ is a Coxeter group. We call W_L a *relative Weyl group* of G corresponding to $(L, C_L, \mathcal{L})$. For any group Γ, denote by $\Gamma^\wedge$ the set of equivalence classes of irreducible representations of Γ. The generalised Springer correspondence is a certain bijection

$$\mathrm{GSpr}\colon \mathcal{N}_{\mathrm{G}} \to \bigsqcup_{(L, C_L, \mathcal{L}) \in S_G} W_L^\wedge.$$

We call the representations of the W_L *generalised Springer representations of* G. For each representation of W_L, we say that $(L, C_L, \mathcal{L})$ is its *cuspidal support*. We have a partition $\mathcal{N}_{\mathrm{G}} = \bigsqcup_{(L, C_L, \mathcal{L}) \in S_G} \mathcal{N}_{\mathrm{G}}(L)$ where $\mathcal{N}_{\mathrm{G}}(L) = \mathrm{GSpr}^{-1}(W_L^\wedge)$. The restriction $\mathrm{GSpr}\colon \mathcal{N}_{\mathrm{G}}(T) \to W$ is the original Springer correspondence due to Springer [Spr76].

Next, we will describe the generalised Springer correspondence for $G = \mathrm{Sp}(2n, \mathbb{C})$ and $G = \mathrm{SO}(N, \mathbb{C})$ in terms of symbols. The notation will be the same as in [Wal19, §4] and is slightly different than the notation originally used in [Lus84b].

1.2.1 Combinatorics of partitions

Let $\mathcal{R}$ be the set of decreasing sequences $\lambda = (\lambda_1, \lambda_2, \dots)$ of real numbers such that for each $r \in \mathbb{R}$, there exist only finitely many $i \in \mathbb{N}$ such that $\lambda_i \geq r$. For each $r \in \mathbb{R}$ and $\lambda \in \mathcal{R}$, we define $\mathrm{mult}_\lambda(r) = \#\{i \in \mathbb{N} \colon \lambda_i = r\}$. For each $c \in \mathbb{N}$, we define $S_c(\lambda) = \lambda_1 + \cdots + \lambda_c$. We define an ordering $\leq$ on $\mathcal{R}$ as follows: for $\lambda, \lambda' \in \mathcal{R}$, we denote $\lambda \leq \lambda'$ if $S_c(\lambda) \leq S_c(\lambda')$ for all

$c \in \mathbb{N}$. For $\lambda, \lambda' \in \mathcal{R}$, we define $\lambda + \lambda' = (\lambda_1 + \lambda'_1, \lambda_2 + \lambda'_2, \dots)$, and we define $\lambda \sqcup \lambda'$ to be the unique element of $\mathcal{R}$ such that for each $n \in \mathbb{N}$, we have $\operatorname{mult}_{\lambda \sqcup \lambda'}(n) = \operatorname{mult}_\lambda(n) + \operatorname{mult}_{\lambda'}(n)$.

Let $\mathcal{R}_f$ be the set of finite decreasing sequences $\lambda = (\lambda_1, \dots, \lambda_n)$ of non-negative real numbers. For $m, n \in \mathbb{N}$ with $m < n$ and two elements $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\nu = (\nu_1, \dots, \nu_n)$ of $\mathcal{R}_f$, we identify λ and ν if $\lambda_i = \nu_i$ for $i = 1, \dots, m$ and $\nu_j = 0$ for $j = m+1, \dots, n$. Similarly as above, we can define $\operatorname{mult}_\lambda$ and S_c for $c \in \mathbb{N}$, and we similarly define an ordering $\leq$ on $\mathcal{R}_f$. For $\lambda \in \mathcal{R}$ and $\lambda' \in \mathcal{R}_f$, we define $\lambda + \lambda' \in \mathcal{R}$ and $\lambda \sqcup \lambda' \in \mathcal{R}$ similarly as above as well. For $\lambda \in \mathcal{R}_f$, let $t(\lambda)$ be the largest integer i such that $\lambda_i \neq 0$ and let $S(\lambda) = S_{t(\alpha)}(\lambda)$

A sequence $\lambda \in \mathcal{R}_f$ is called a *partition* if it is a sequence of non-negative integers and let $\mathcal{P} \subseteq \mathcal{R}_f$ be the set of all partitions. For $N \in \mathbb{N}$, we say that λ is a *partition of* N if $N = S(\lambda)$. Let $\mathcal{P}(N)$ denote the set of partitions of N.

1.2.2 Sp($2n$)

Let $n \in \mathbb{Z}_{\geq 0}$. A partition λ of $2n$ is called *symplectic* if each odd part of λ occurs with even multiplicity. We write $\mathcal{P}^{\mathrm{symp}}(2n)$ for the set of symplectic partitions of $2n$. Let $\Delta(\lambda) = \{i \in \mathbb{N} \colon i \text{ is even, } \operatorname{mult}_\lambda(i) \neq 0\}$. Denote by $\boldsymbol{\mathcal{P}}^{\mathbf{symp}}(N)$ the set of pairs (λ, ε) with $\lambda \in \mathcal{P}^{\mathrm{ort}}(2n)$ and $\varepsilon \in \{\pm 1\}^{\Delta(\lambda)}$.

It is well-known that $\mathcal{P}^{\mathrm{symp}}(2n)$ is in 1–1 correspondence with $\mathcal{U}$. If $C \in \mathcal{U}$ is parametrised by $\lambda \in \mathcal{P}^{\mathrm{symp}}(2n)$, then $\{\pm 1\}^{\Delta(\lambda)}$ is in 1–1 correspondence with the set of irreducible G-equivariant local systems on C. Let $\mathcal{N}_{\mathrm{G}}$ be the set of pairs $(C, \mathcal{E})$ where C is a unipotent class in G and $\mathcal{E}$ is an irreducible G-equivariant local system on C. We thus have a bijection $\mathcal{N}_{\mathrm{G}} \to \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$.

Suppose $C \in \mathcal{U}$ is parametrised by $\lambda \in \mathcal{P}^{\mathrm{symp}}(2n)$ and let $u \in C$. Then $\{\pm 1\}^{\Delta(\lambda)}$ is also in 1–1 correspondence with the set of irreducible representations of the component group $A(u) = Z_G(u)/Z_G^\circ(u)$ of u in G. Up to isomorphism, $A(u)$ does not depend on the choice of $u \in C$, hence we write $A(\lambda) = A(C) = A(u)$.

Let $(\lambda, \varepsilon) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$. Let $\lambda' = \lambda + [-1, -\infty[_1 = (\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \dots)$. There exist sequences $z, z' \in \mathcal{R}$ with integer terms such that $\lambda' = (2z_1, 2z_2, \dots) \sqcup (2z'_1 + 1, 2z'_2 + 1, \dots)$. Let $A^{\#} = z' + [1, -\infty[_1$ and $B^{\#} = z + [0, -\infty[_1$. Then $A^{\#}_1 \geq B^{\#}_1 \geq A^{\#}_2 \geq B^{\#}_3 \geq \dots$. A finite subset of $\mathbb{Z}$ is called an interval if it is of the form $\{i, i+1, i+2, \dots, j\}$ for some $i, j \in \mathbb{Z}$. Let C be the collection of intervals I of $A \triangle B = (A \cup B) \setminus (A \cap B)$, with the property that for any $i \in (A \Delta B) \setminus I$, the set $I \cup \{i\}$ is not an interval. There is an obvious ordering on C: for $I, I' \in C$, I is larger than I' if any element of I is larger than any element of I'. Then C is in bijection with $\Delta(\lambda)$. Let $\Delta(\lambda) \to C \colon i \mapsto C_i$ be the unique increasing bijection. Let $t = t(\lambda)$. For $i \in \{1, \dots, t\}$, write $\varepsilon(i) = \varepsilon_{\lambda_i}$. For $u \in \{\pm 1\}$, define

$$J^u = \{i \in \{1, \dots, t\} \colon \varepsilon(i)(-1)^i = u\}.$$

Define the *symbol* of (λ, ε) to be the ordered pair $S_{\lambda,\varepsilon} = (A_{\lambda,\varepsilon}, B_{\lambda,\varepsilon})$, where

$$A_{\lambda,\varepsilon} = (A^{\#} \setminus \bigcup_{i\in\Delta(\lambda);\varepsilon_i=-1} (A^{\#} \cap C_i)) \cup (\bigcup_{i\in\Delta(\lambda);\varepsilon_i=-1} (B^{\#} \cap C_i)),$$
$$B_{\lambda,\varepsilon} = (B^{\#} \setminus \bigcup_{i\in\Delta(\lambda);\varepsilon_i=-1} (B^{\#} \cap C_i)) \cup (\bigcup_{i\in\Delta(\lambda);\varepsilon_i=-1} (A^{\#} \cap C_i)).$$

For $m \in \mathbb{N}$, let $W(B_m) = W(C_m) = S_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$. The set $W(C_m)^\vee$ is in bijection with $\mathcal{P}_2(m)$. For $(\alpha, \beta) \in \mathcal{P}_2(n)$, we write $\rho_{(\alpha,\beta)}$ for the corresponding representation of W_n.

Let $k \in \mathbb{N}$ such that $k(k+1) \leq 2n$. Let $(\alpha, \beta) \in \mathcal{P}_2(n - \frac{k(k-1)}{2})$. If k is even (resp. odd), let $A_{\alpha,\beta;k} = \alpha + [k, -\infty[_2$ and $B_{\alpha,\beta;k} = \beta + [-k-1, -\infty[_2$ (resp. $A_{\alpha,\beta;k} = \beta + [-k-1, -\infty[_2$ and $B_{\alpha,\beta;k} = \alpha + [k, -\infty[_2$) and define the *symbol of* (α, β) to be $S_{\alpha,\beta;k} = (A_{\alpha,\beta;k}, B_{\alpha,\beta;k})_k$. Let

$$\mathcal{W} = \bigsqcup_{k\in\mathbb{N}, k(k+1)\leq 2n} W(C_{n-\frac{k(k+1)}{2}})^\wedge.$$

The $W(C_{n-\frac{k(k+1)}{2}})$ form all the relative Weyl groups of G (see the start of this section). Hence the generalised Springer correspondence is a bijection

$$\mathrm{GSpr}\colon \mathcal{N}_G \to \mathcal{W}.$$

Using the parametrisations of $\mathcal{N}_G$ and $\mathcal{W}$ described above, we rephrase the generalised Springer correspondence as follows.

Theorem 1.2.1 (Generalised Springer correspondence for $\mathrm{Sp}(2n)$)**.** *Let $n \in \mathbb{N}$. For each $(\lambda, \varepsilon) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$, there exists a unique $k \in \mathbb{Z}_{\geq 0}$ and a unique pair $(\alpha, \beta) \in \mathcal{P}_2(n - \frac{k(k+1)}{2})$ such that $(A_{\lambda,\varepsilon}, B_{\lambda,\varepsilon}) = (A_{\alpha,\beta;k}, B_{\alpha,\beta;k})$. Conversely, for each $k \in \mathbb{Z}_{\geq 0}$ such that $k(k+1) \leq 2n$, and for each $(\alpha, \beta) \in \mathcal{P}_2(n - \frac{k(k+1)}{2})$, there exists a unique $(\lambda, \varepsilon) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$ such that $(A_{\lambda,\varepsilon}, B_{\lambda,\varepsilon}) = (A_{\alpha,\beta;k}, B_{\alpha,\beta;k})$. Thus we have a bijection*

$$\Phi_N\colon \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N) \to \bigsqcup_{k\in\mathbb{Z}_{\geq 0},\, k(k+1)\leq 2n} \mathcal{P}_2(n - \frac{k(k+1)}{2}).$$

1.2.3 SO(N)

Let $N \in \mathbb{Z}_{\geq 0}$. A partition λ of N is called *orthogonal* if each even part of λ occurs with even multiplicity. We write $\mathcal{P}^{\mathrm{ort}}(N)$ for the set of orthogonal partitions of N. Let $\Delta(\lambda) = \{i \in \mathbb{N}\colon i \text{ is odd},\ \mathrm{mult}_\lambda(i) \neq 0\}$. Let F_2 be the field with two elements. Let $F_2[\Delta(\lambda)] = \{\pm 1\}^{\Delta(\lambda)}$ be the set of maps $\varepsilon\colon \Delta(\lambda) \to \{\pm 1\}\colon \lambda_i \mapsto \varepsilon_{\lambda_i}$, considered as an F_2-vector space. Let $F_2[\Delta(\lambda)]'$ be the quotient of $F_2[\Delta(\lambda)]$ by the line spanned by the sum of the canonical basis of this F_2-vector space. Denote by $\boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ the set of pairs $(\lambda, [\varepsilon])$

where $\lambda \in \mathcal{P}^{\mathrm{ort}}(N)$ and $[\varepsilon] \in F_2[\Delta(\lambda)]'$ is the image of $\varepsilon \in F_2[\Delta(\lambda)]$ under the quotient map. We define $\boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N)$ to be the set of pairs (λ, ε) with $\lambda \in \mathcal{P}^{\mathrm{ort}}(N)$ and $\varepsilon \in F_2[\Delta(\lambda)]$.

Let $\mathcal{U}$ be the set of unipotent classes of G. It is well-known that $\mathcal{P}^{\mathrm{ort}}(N)$ is in 1–1 correspondence with $\mathcal{U}$, except the orthogonal partitions of N that only have even parts correspond to precisely two (degenerate) unipotent classes. This correspondence has the following property. Suppose $C, C' \in \mathcal{U}$ are parametrised by $\lambda, \lambda' \in \mathcal{P}^{\mathrm{ort}}(N)$ respectively. Then $C \subseteq \bar{C}'$ (i.e. $C \preceq C'$ where $\preceq$ is the closure ordering), if and only if $\lambda \leq \lambda'$. We say that λ is *degenerate* if λ only has even parts and we call λ *non-degenerate* otherwise. If $C \in \mathcal{U}$ is parametrised by $\lambda \in \mathcal{P}^{\mathrm{ort}}(N)$, then $F_2[\Delta(\lambda)]'$ is in 1–1 correspondence with the set of irreducible G-equivariant local systems on C. Note that if C is degenerate, then $F_2[\Delta(\lambda)]'$ only contains the empty map, and note that there exist no non-trivial G-equivariant local systems on C. Let $\mathcal{N}_G$ be the set of pairs $(C, \mathcal{E})$ where C is a unipotent class in G and $\mathcal{E}$ is an irreducible G-equivariant local system on C. We obtain a surjective map $\mathcal{N}_G \to \boldsymbol{\mathcal{P}^{\mathrm{ort}}}(N)$ such that the preimage of (λ, ε) has one element if λ is non-degenerate, and two elements if λ is non-degenerate. We denote the preimage of $(\lambda, [\varepsilon])$ by $\{(C^+_\lambda, \mathcal{E}^+_{[\varepsilon]}), (C^-_\lambda, \mathcal{E}^-_{[\varepsilon]})\}$. If λ is non-degenerate, we write $(C_\lambda, \mathcal{E}_{[\varepsilon]}) = (C^+_\lambda, \mathcal{E}^+_{[\varepsilon]}) = (C^-_\lambda, \mathcal{E}^-_{[\varepsilon]})$, i.e. we may drop the $+$ and $-$ from the notation.

Suppose $C \in \mathcal{U}$ is parametrised by $\lambda \in \mathcal{P}^{\mathrm{ort}}(N)$ and let $u \in C$. Then $F_2[\Delta(\lambda)]'$ is also in 1–1 correspondence with the set of irreducible representations of the component group $A(u) = Z_G(u)/Z^\circ_G(u)$ of u in G.

Let $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}^{\mathrm{ort}}}(N)$. Let $\lambda' = \lambda + [0, -\infty[_1 = (\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \dots)$. There exist sequences $z, z' \in \mathcal{R}$ with integer terms such that $\lambda' = (2z_1, 2z_2, \dots) \sqcup (2z'_1 - 1, 2z'_2 - 1, \dots)$. Let $A^\# = z' + [0, -\infty[_1$ and $B^\# = z + [0, -\infty[_1$. Then $A^\#_1 \geq B^\#_1 \geq A^\#_2 \geq B^\#_3 \geq \dots$. A finite subset of $\mathbb{Z}$ is called an interval if it is of the form $\{i, i+1, i+2, \dots, j\}$ for some $i, j \in \mathbb{Z}$. Let C be the collection of intervals I of $(A \cup B) \setminus (A \cap B)$, with the property that for any $i \in (A \cup B) \setminus ((A \cap B)$ such that $i \notin I$, we have that $I \cup \{i\}$ is not an interval. There is an obvious ordering on C: for $C', C'' \in C$, C' is larger than C'' if any element of C' is larger than any element of C''. Then C is in bijection with $\Delta(\lambda)$. Let $\Delta(\lambda) \to C \colon i \mapsto C_i$ be the unique increasing bijection. Let $t = t(\lambda)$. For $i \in \{1, \dots, t\}$, write $\varepsilon(i) = \varepsilon_{\lambda_i}$. For $u \in \{\pm 1\}$, define

$$J^u = \{i \in \{1, \dots, t\} \colon \varepsilon(i)(-1)^i = u\}.$$

Let $M(\lambda, \varepsilon) = M = |J^1| - |J^{-1}|$ and let $w(\lambda, \varepsilon) = w = \mathrm{sgn}(M + 1/2)$ so that $w = 1$ if

$M=0$ and $w=\operatorname{sgn} M$ if $M\neq 0$. Define

$$A_{\lambda,\varepsilon}=(A^{\#}\setminus\bigcup_{i\in\Delta(\lambda);\varepsilon_i=-w}(A^{\#}\cap C_i))\cup(\bigcup_{i\in\Delta(\lambda);\varepsilon_i=-w}(B^{\#}\cap C_i)),$$
$$B_{\lambda,\varepsilon}=(B^{\#}\setminus\bigcup_{i\in\Delta(\lambda);\varepsilon_i=-w}(B^{\#}\cap C_i))\cup(\bigcup_{i\in\Delta(\lambda);\varepsilon_i=-w}(A^{\#}\cap C_i)).$$

If $M\neq 0$ and $-\,\varepsilon$ is the other representative of $[\varepsilon]$, we have $A_{\lambda,\varepsilon}=A_{\lambda,-\,\varepsilon}$ and $B_{\lambda,\varepsilon}=B_{\lambda,-\,\varepsilon}$, so we define the *symbol* of $(\lambda,[\varepsilon])$ to be the ordered pair $S_{\lambda,[\varepsilon]}=(A_{\lambda,\varepsilon},B_{\lambda,\varepsilon})=(A_{\lambda,-\,\varepsilon},B_{\lambda,-\,\varepsilon})$. If $M=0$, $A_{\lambda,\varepsilon}=B_{\lambda,-\,\varepsilon}$ and $B_{\lambda,\varepsilon}=A_{\lambda,-\,\varepsilon}$, we define the *symbol* of $(\lambda,[\varepsilon)]$ to be the unordered pair $S_{\lambda,[\varepsilon]}=\{A_{\lambda,\varepsilon},B_{\lambda,\varepsilon}\}$. Let X be a set, $d\in\mathbb{Z}_{\geq 0}$ and $(x,y)\in X\times X$. Then let

$$(x,y)_k=\begin{cases}(x,y)\in X\times X & \text{if } k>0,\\ \{x,y\}\subseteq X & \text{if } k=0.\end{cases}$$

Following this notation, we see that $S_{\lambda,[\varepsilon]}=(A_{\lambda,\varepsilon},B_{\lambda,\varepsilon})_k$ where $k=|M|$.

We define $p_{\lambda,[\varepsilon]}=p_{\lambda,\varepsilon}=A_{\lambda,\varepsilon}\sqcup B_{\lambda,\varepsilon}=A^{\#}_{\lambda,\varepsilon}\sqcup B^{\#}_{\lambda,\varepsilon}$. Two pairs $(A,B),(A',B')\in\mathcal{R}\times\mathcal{R}$ are *similar* if $A\sqcup B=A'\sqcup B'$. For any $(\lambda,[\varepsilon]),(\lambda',[\varepsilon'])\in\boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$, their symbols are similar if and only if $\lambda=\lambda'$.

Remark 1.2.2. Let $t=t(\lambda)$, $s=\lfloor t/2\rfloor$ and $k=|M|$. Let $A'=(A_{\frac{t+d}{2}}+s,A_{\frac{t+d}{2}-1}+s,\ldots,A_1+s)$, $B'=(B_{\frac{t-d}{2}}+s,B_{\frac{t-d}{2}-1}+s,\ldots,B_1+s)$ and note that these are increasing sequences. Then $(A',B')_k$ is the usual symbol in the literature. The *defect* of (A,B) is defined to be $|\#A'-\#B'|$ and this is equal to $k=|M|$.

Let $n\in\mathbb{N}$. Recall that $W(B_n)=W(C_n)$ and let $W(D_n)=S_n\ltimes(\mathbb{Z}/2\mathbb{Z})^{n-1}$. Define $\theta\colon\mathcal{P}_2(n)\to\mathcal{P}_2(n)$ by $\theta(\alpha,\beta)=(\beta,\alpha)$ for $(\alpha,\beta)\in\mathcal{P}_2(n)$. Let $\mathcal{P}_2(n)/\theta$ be the set of θ-orbits and let $c_{(\alpha,\beta)}$ be the size of the θ-orbit of (α,β). We identify $\mathcal{P}_2(n)$ with the set of unordered pairs $\{\alpha,\beta\}$ where $(\alpha,\beta)\in\mathcal{P}_2(n)$. Then $W(D_n)^\vee$ is parametrised as

$$W(D_n)^\vee=\{\rho_{\{\alpha,\beta\},i}\colon\{\alpha,\beta\}\in\mathcal{P}_2(n)/\theta,i\in\{1,2/c_{(\alpha,\beta)}\}\}.$$

We sometimes write $\rho_{(\alpha,\beta),i}$ or $\rho_{(\beta,\alpha),i}$ if it is clear from the context that we are talking about $\rho_{\{\alpha,\beta\},i}$. When $c_{(\alpha,\beta)}=1$, we also write $\rho_{\{\alpha,\beta\}}=\rho_{\{\alpha,\beta\},1}$.

Let $k\in\mathbb{N}$ such that $k\equiv N\mod 2$ and $k^2\leq N$. Let $(\alpha,\beta)\in\mathcal{P}_2((N-k^2)/2)$. We define $A_{\alpha,\beta;k}=\alpha+[k,-\infty[_2$ and $B_{\alpha,\beta;k}=\beta+[-k,-\infty[_2$ and define the *symbol of* (α,β) to be $S_{\alpha,\beta;k}=(A_{\alpha,\beta;k},B_{\alpha,\beta;k})_k$.

Let W be the Weyl group of G. Then $W=W(B_{(N-1)/2})$ (resp. $W=W(D_{N/2})$) if N is odd (resp. even). Let

$$\mathcal{W}=W^\vee\sqcup\bigsqcup_{k\in\mathbb{Z}_{\geq 2},k^2\leq N,k\equiv N\mod 2}W(B_{(N-k^2)/2})^\vee.$$

The Weyl groups W and $W(B_{(N-k^2)/2})$ with $k \in \mathbb{Z}_{\geq 2}$, $k \equiv N \mod 2$ and $k^2 \leq N$ form all the relative Weyl groups of G (see the start of this section). Hence the generalised Springer correspondence is a bijection

$$\mathrm{GSpr} \colon \mathcal{N}_G \to \mathcal{W}.$$

Using the parametrisations of $\mathcal{N}_G$ and $\mathcal{W}$ described above, we rephrase the generalised Springer correspondence as follows.

Theorem 1.2.3 (Generalised Springer correspondence for $\mathrm{SO}(N)$)**.** *Let $N \in \mathbb{N}$.*

1. *Suppose $N = 2n+1$ is odd. For each $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$, there exists a unique odd integer $k(\lambda, [\varepsilon]) = k \in \mathbb{N}$ and a unique pair $(\alpha, \beta) \in \mathcal{P}_2((N-k^2)/2)$ such that $(A_{\lambda,\varepsilon}, B_{\lambda,\varepsilon}) = (A_{\alpha,\beta;k}, B_{\alpha,\beta;k})$. Conversely, for each odd $k \in \mathbb{N}$ such that $k^2 \leq N$, and for each $(\alpha, \beta) \in \mathcal{P}_2((N-k^2)/2)$, there exists a unique $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ such that $(A_{\lambda,[\varepsilon]}, B_{\lambda,[\varepsilon]}) = (A_{\alpha,\beta;k}, B_{\alpha,\beta;k})$. Thus we have a bijection*

$$\Phi_N \colon \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N) \to \bigsqcup_{k \in \mathbb{N}\,\mathrm{odd},\, k^2 \leq N} \mathcal{P}_2((N-k^2)/2).$$

2. *Suppose $N = 2n$ is even. Let $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$. Then one of the following is true:*

   - *There exists a unique $\{\alpha, \beta\} \in \mathcal{P}_2(N/2)/\theta$ with $\alpha \neq \beta$ such that $\{A_{\lambda,[\varepsilon]}, B_{\lambda,[\varepsilon]}\} = \{A_{\alpha,\beta;0}, B_{\alpha,\beta;0}\}$,*
   - *There exists a unique even integer $k(\lambda, [\varepsilon]) = k \in \mathbb{N}$ and a unique pair $(\alpha, \beta) \in \mathcal{P}_2((N-k^2)/2)$ such that $(A_{\lambda,[\varepsilon]}, B_{\lambda,[\varepsilon]}) = (A_{\alpha,\beta;k}, B_{\alpha,\beta;k})$.*

   Conversely, we have

   - *For each $\{\alpha, \beta\} \in \mathcal{P}_2(N/2)/\theta$, there exists a unique $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ such that $\{A_{\lambda,[\varepsilon]}, B_{\lambda,[\varepsilon]}\} = \{A_{\alpha,\beta;0}, B_{\alpha,\beta;0}\}$,*
   - *For each even $k \in \mathbb{N}$ such that $k^2 \leq N$ and for each $(\alpha, \beta) \in \mathcal{P}_2((N-k^2)/2)$, there exists a unique $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ such that $(A_{\lambda,[\varepsilon]}, B_{\lambda,[\varepsilon]}) = (A_{\alpha,\beta;k}, B_{\alpha,\beta;k})$.*

   Thus we have a bijection

$$\Phi_N \colon \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N) \to (\mathcal{P}_2(N/2)/\theta) \sqcup \bigsqcup_{k \in \mathbb{N}\,\mathrm{even},\, k^2 \leq N} \mathcal{P}_2((N-k^2)/2).$$

Remark 1.2.4. Note that in Theorem 1.2.3, we have $N \equiv t(\lambda) \equiv k(\lambda, [\varepsilon]) \mod 2$. Furthermore, it is well-known that $k(\lambda, [\varepsilon])$ is equal to the defect of $(\lambda, [\varepsilon])$ (see [Lus84b, §13]), hence $k(\lambda, [\varepsilon]) = |M| = |J^1| - |J^{-1}|$ by Remark 1.2.2.

Remark 1.2.5. Let $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ such that λ only has odd parts. Let $t = t(\lambda)$, $t^+ = \lceil t/2 \rceil$ and $t^- = \lfloor t/2 \rfloor$. Then

$$A^\# = \left(\frac{\lambda_{2i-1} + 2i - 1}{2} : i = 1, \dots, t^+ \right) \sqcup (-t - 2i : i \in \mathbb{Z}_{\geq 0}), \tag{1.2.1}$$
$$B^\# = \left(\frac{\lambda_{2i} + 2i - 1}{2} : i = 1, \dots, t^- \right) \sqcup (-t - 2i : i \in \mathbb{Z}_{\geq 0}),$$

and $A_1^\# > B_1^\# > A_2^\# > B_2^\# > \cdots > A_{t^+}^\# > B_{t^+}^\#$. Hence the largest t terms of $p_{\lambda,\varepsilon} = A_{\lambda,\varepsilon} \sqcup B_{\lambda,\varepsilon} = A^\# \sqcup B^\#$ are all distinct. Suppose $(\alpha, \beta)_k = \Phi_N(\lambda, [\varepsilon])$ with $k = k(\lambda, [\varepsilon])$. Note that $t \equiv k \mod 2$ and let $m_0 = \frac{t+k}{2}$ and $m_1 = \frac{t-k}{2}$. It is easy to check that m_0 and m_1 are the same as in §3.1.3, i.e. m_0 is the smallest integer such that $m_0 \geq t(\alpha)$ and $m_0 - k \geq t(\beta)$, so $p_{\lambda,\varepsilon} = \Lambda_{k,-k;2}(\alpha, \beta) \in H_S((N - k^2)/2, k)$, i.e. $(\alpha, \beta) \in H((N - k^2)/2, k)$.

1.3 Hecke algebras

1.3.1 Hecke algebra of a Bernstein component

We give a brief overview of [BK98, §1–§3]. For a k-rational character $\phi\colon \mathbf{G}(\mathsf{k}) \to \mathsf{k}^\times$ and $s \in \mathbb{C}$, define a smooth one-dimensional complex representation

$$\mathbf{G}(\mathsf{k}) \to \mathbb{C}^\times : g \mapsto \|\phi(g)\|_{\mathsf{k}}^s,$$

where $\|\cdot\|_{\mathsf{k}}^s$ denotes the usual normalised absolute value of k. Let $X_{\mathbb{C}}(\mathbf{G}(\mathsf{k}))$ be the group generated by these smooth representations; it has the structure of a complex torus. Elements of $X_{\mathbb{C}}(\mathbf{G}(\mathsf{k}))$ are called *unramified quasicharacters of* $\mathbf{G}(\mathsf{k})$.

Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$ defined over k and σ a supercuspidal representation of $\mathbf{L}(\mathsf{k})$. Two pairs $(\mathbf{L}_1, \sigma_1)$ and $(\mathbf{L}_2, \sigma_2)$ are called *inertially equivalent* if there exists a $g \in \mathbf{G}(\mathsf{k})$ and $\chi \in X_{\mathbb{C}}(\mathbf{L}_2(\mathsf{k}))$ such that

$$\mathbf{L}_1(\mathsf{k}) = g\, \mathbf{L}_2(\mathsf{k}) g^{-1} \quad \text{and} \quad \sigma_1^g \cong \sigma_2 \otimes \chi,$$

where $\sigma_1^g\colon x \mapsto \sigma_1(gxg^{-1})$. We write $[\mathbf{L}, \sigma]_{\mathbf{G}(\mathsf{k})}$ for the *inertial equivalence class* of a pair $(\mathbf{L}, \sigma)$ and $\mathscr{B}(\mathbf{G}(\mathsf{k}))$ for the set of inertial equivalence classes of $\mathbf{G}(\mathsf{k})$. For each irreducible $\pi \in \mathrm{Rep}(\mathbf{G}(\mathsf{k}))$ there exists a $\mathbf{Q} \in \mathcal{Q}$ with Levi component $\mathbf{L}$ and a supercuspidal representation of $\mathbf{L}(\mathsf{k})$ such that π is a subquotient of $\mathrm{Ind}_{\mathbf{Q}(\mathsf{k})}^{\mathbf{G}(\mathsf{k})}(\sigma)$. In this case, we say that $[\mathbf{L}, \sigma]_{\mathbf{G}(\mathsf{k})}$ is the *support* of π.

Definition 1.3.1. For $\mathfrak{s} \in \mathscr{B}(\mathbf{G}(\mathsf{k}))$, let $\mathrm{Rep}^{\mathfrak{s}}(\mathbf{G}(\mathsf{k}))$ be the full subcategory of $\mathrm{Rep}(\mathbf{G}(\mathsf{k}))$ consisting of all $\pi \in \mathrm{Rep}(\mathbf{G}(\mathsf{k}))$ such that each irreducible subquotient of π has support $\mathfrak{s}$. Such a $\mathrm{Rep}^{\mathfrak{s}}(\mathbf{G}(\mathsf{k}))$ is called a *Bernstein component of* $\mathbf{G}(\mathsf{k})$.

Proposition 1.3.2 ([Ber84, Proposition 2.10])**.** *We have a direct product decomposition*

$$\mathrm{Rep}(\mathbf{G}(\mathsf{k})) = \prod_{\mathfrak{s}\in\mathscr{B}(\mathbf{G}(\mathsf{k}))} \mathrm{Rep}^{\mathfrak{s}}(\mathbf{G}(\mathsf{k})).$$

Definition 1.3.3. Fix a Haar measure on $\mathbf{G}(\mathsf{k})$. The *Hecke algebra* $\mathcal{H}(\mathbf{G}(\mathsf{k}))$ *of* $\mathbf{G}(\mathsf{k})$ is the space of locally constant, compactly supported functions $f\colon G \to \mathbb{C}$ on $\mathbf{G}(\mathsf{k})$ with $\mathbb{C}$-algebra structure given by convolution with respect to some fixed Haar measure.

Note that any $\pi \in \mathrm{Rep}(\mathbf{G}(\mathsf{k}))$ gives a representation of $\mathbf{G}(\mathsf{k})$ obtained from convolution. We denote this representation by π as well.

Let K be a compact open subgroup of $\mathbf{G}(\mathsf{k})$ and $(\rho, \mathcal{W})$ a smooth representation of K. Let $\mathscr{X}$ be the space of compactly supported and locally constant functions $f\colon G \to \mathcal{W}$ such that $f(kg) = \rho(k^{-1})f(g)$. Then $\mathbf{G}(\mathsf{k})$ acts on $\mathscr{X}$ by right translation, and we obtain a smooth representation which we denote by $\text{c-Ind}_{\mathbf{Q}(\mathsf{k})}^{\mathbf{G}(\mathsf{k})}(\rho)$. We call this a *compactly induced representation.*

Definition 1.3.4 ([BK98, (2.6)])**.** The *Hecke algebra of compactly supported ρ-spherical functions on* $\mathbf{G}(\mathsf{k})$ is defined to be the $\mathbb{C}$-algebra $\mathcal{H}(\mathbf{G}(\mathsf{k}), \rho) := \mathrm{End}_G(\text{c-Ind}_{\mathbf{Q}(\mathsf{k})}^{\mathbf{G}(\mathsf{k})}(\rho))$.

Let $(\pi, \mathcal{V}) \in \mathrm{Rep}(\mathbf{G}(\mathsf{k}))$ and let $e \in \mathcal{H}(\mathbf{G}(\mathsf{k}))$ be a non-zero idempotent element. Let $\mathrm{Rep}_e(\mathbf{G}(\mathsf{k}))$ denote the full subcategory of $\mathrm{Rep}(\mathbf{G}(\mathsf{k}))$ whose objects are representations $(\pi, \mathcal{V})$ for which $\mathcal{H}(\mathbf{G}(\mathsf{k})) \star e \star \mathcal{V} = \mathcal{V}$. Define

$$e_\rho := \begin{cases} \frac{\dim \rho}{\mathrm{vol}(K)}\, \mathrm{tr}_{\mathcal{W}}(\rho(x^{-1})) & \text{if } x \in K, \\ 0 & \text{if } x \in G,\ x \notin K. \end{cases}$$

It is well-known that $e_\rho \star \mathcal{H}(\mathbf{G}(\mathsf{k})) \star e_\rho$ and $H(\mathbf{G}(\mathsf{k}), \rho)$ are Morita equivalent.

Proposition 1.3.5. *In the notation above, suppose* $\mathrm{Rep}_{e_\rho}(\mathbf{G}(\mathsf{k}))$ *is closed relative to subquotients in* $\mathrm{Rep}(\mathbf{G}(\mathsf{k}))$*. Then*

1. *[BK98, §2, (3.3) Proposition] There is an equivalence of categories* $\mathrm{Rep}_{e_\rho}(\mathbf{G}(\mathsf{k})) \cong \mathcal{H}(\mathbf{G}(\mathsf{k}), \rho)$-mod*,*

2. *[BK98, (3.5) Proposition, (3.7) Lemma] There exists a finite set $\mathfrak{S} \subseteq \mathscr{B}(\mathbf{G}(\mathsf{k}))$ such that* $\mathrm{Rep}_{e_\rho}(\mathbf{G}(\mathsf{k})) = \prod_{\mathfrak{s}\in\mathfrak{S}} \mathrm{Rep}^{\mathfrak{S}}(\mathbf{G}(\mathsf{k}))$.

In this situation, we say that the pair (K, ρ) is an $\mathfrak{S}$-type.

Proposition 1.3.6. *For any finite subset (hence in particular also singletons) $\mathfrak{S} \subseteq \mathscr{B}(\mathbf{G}(\mathsf{k}))$, there exists an $\mathfrak{S}$-type.*

1.3.2 Affine Hecke algebras

A *root datum* is a quadruple $\Phi = (X^*, X_*, R, R^\vee)$, where X^* and X_* are free abelian groups of finite rank with a perfect pairing

$$\langle \cdot, \cdot \rangle \colon X^* \times X_* \to \mathbb{Z},$$

R, $R^\vee$ are finite subsets of X^*, X_* respectively with a bijection denoted by $R \to R^\vee \colon \alpha \mapsto \check{\alpha}$. These data satisfy the following conditions

1. $\langle \alpha, \check{\alpha} \rangle = 2$ for all $\alpha \in R$,
2. For each $\alpha \in R$, the reflections $s_\alpha \colon X^* \to X^* \colon x \mapsto x - \langle x, \check{\alpha} \rangle \alpha$ and $s_\alpha \colon X_* \to X_* \colon y \mapsto y - \langle \alpha, y \rangle \check{\alpha}$ leave R and $R^\vee$ stable, respectively.

The elements of R and $R^\vee$ are called *roots* and *coroots*, respectively. A *basis* of R is a subset Π such that each $\alpha \in R$ can uniquely be expressed as a linear combination of elements of Π with coefficients that are all either non-negative or all non-positive. Let R^+ be the set of *positive roots* with respect to Π, i.e. the roots in R that are linear combinations of elements of Π with strictly non-negative integer coefficients. Let $S := \{s_\alpha \colon \alpha \in \Pi\}$ be the set of *simple reflections*. Define the *length* $\ell(w)$ of $w \in W$ to be the smallest integer ℓ such that w can be written as a product of ℓ simple reflections. The tuple $(X^*, X_*, R, R^\vee, \Pi)$ is called a *based root datum*. The *Weyl group* of Φ is the finite group $W := \langle s_\alpha \colon \alpha \in R \rangle$. A *parameter set* of Φ is a pair (λ, λ^*) of W-equivariant functions $\lambda \colon R \to \mathbb{C}$, $\lambda^* \colon \{\alpha \in R \colon \check{\alpha} \in 2X_*\} \to \mathbb{C}$. We also write $\lambda(s_\alpha) = \lambda(\alpha)$ and $\lambda^*(s_\alpha) = \lambda^*(\alpha)$ for $\alpha \in R$.

Let $G = G(\Phi)$ denote the connected complex reductive group corresponding to Φ. Given a maximal torus T of G, we can canonically identify

$$X^* = \mathrm{Hom}(T, \mathbb{C}^\times), \quad X_* = \mathrm{Hom}(\mathbb{C}^\times, T),$$

such that if $\chi \in \mathrm{Hom}(T, \mathbb{C}^\times)$ and $\phi \in \mathrm{Hom}(\mathbb{C}^\times, T)$ are identified with $x \in X^*$ and $y \in X_*$, respectively, we have $(\chi \circ \phi)(z) = z^{\langle x, y \rangle}$ for all $z \in \mathbb{C}^\times$. In other words, X^* and X_* are identified with the character lattice and lattice of 1-parameter subgroups, respectively. Then we have an isomorphism $X_* \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong T, (y \otimes z) \mapsto y(z)$. We can also canonically identify $W = N_G(T)/T$. The choice of Π uniquely determines a Borel subgroup B of G containing T such that the root subgroups of the positive roots are precisely the ones that lie in B.

For the rest of the paper, let t be an indeterminate.

Definition 1.3.7 (Bernstein presentation, [Lus89, Prop. 3.6, 3.7])**.** The *affine Hecke algebra* $\mathcal{H} = \mathcal{H}^{\lambda, \lambda^*}(\Phi)$ associated to a based root datum $\Phi = (X^*, X_*, R, R^\vee, \Pi)$ with parameter set

(λ, λ^*) is defined to be the unique associative unital $\mathbb{C}[t, t^{-1}]$-algebra generated by $\{T_w : w \in W\}$ and $\{\theta_x : x \in X^*\}$ subject to the relations

$$\begin{aligned}
&(T_s + 1)(T_s - t^{2\lambda(s)}) = 0, && \text{for all } s \in S,\\
&T_w T_{w'} = T_{ww'}, && \text{for all } w, w' \in W \text{ with } \ell(ww') = \ell(w) + \ell(w'),\\
&\theta_x \theta_{x'} = \theta_{x+x'} && \text{for all } x, x' \in X^*,\\
&\theta_x T_s - T_s \theta_{s(x)} = (\theta_x - \theta_{s(x)})(\mathcal{G}(s) - 1) && \text{for } x \in X, s \in \Pi,
\end{aligned}$$

where for $s = s_\alpha$ with $\alpha \in \Pi$, we have

$$\mathcal{G}(s) = \mathcal{G}(\alpha) = \begin{cases} \frac{\theta_\alpha z^{2\lambda(\alpha)} - 1}{\theta_\alpha - 1}, & \text{if } \check{\alpha} \notin 2X_*,\\ \frac{(\theta_\alpha z^{\lambda(\alpha)+\lambda^*(\alpha)} - 1)(\theta_\alpha t^{\lambda(\alpha)-\lambda^*(\alpha)} + 1)}{\theta_{2\alpha} - 1}, & \text{if } \check{\alpha} \in 2X_*. \end{cases}$$

Let $\mathcal{A}$ be the $\mathbb{C}[t, t^{-1}]$-subalgebra of $\mathcal{H}$ generated by $\{\theta_x : x \in X^*\}$.

For the remainder of the section, we fix $\mathcal{H} = \mathcal{H}^{\lambda,\lambda^*}(\Phi)$. Note that up to isomorphism, $\mathcal{H}$ does not depend on the choice of Π; in fact the Iwahori-Matsumoto presentation of $\mathcal{H}$ is defined for any root datum without a specified root basis.

Proposition 1.3.8 ([Lus89, Prop. 3.11])**.** *We have $\mathcal{Z} = Z(\mathcal{H}) = \mathcal{A}^W$.*

1.3.3 Graded Hecke algebras

The graded Hecke algebra can be constructed from a filtration on $\mathcal{H}$ coming from a certain ideal of $\mathcal{A}$. However, we will define the graded Hecke algebra here by giving its presentation.

Fix a finite W-invariant set Σ in T. Let $\mathscr{C}$ be the commutative associative unital $\mathbb{C}$-algebra with vector space basis $\{E_\sigma : \sigma \in \Sigma\}$ subject to the relations

$$\sum_{\sigma \in \Sigma} E_\sigma = 1, \quad E_\sigma E_{\sigma'} = \delta_{\sigma,\sigma'} E_\sigma.$$

Let $\mathscr{S}$ be the symmetric algebra of $\mathfrak{t}^\vee = X^* \otimes_{\mathbb{Z}} \mathbb{C}$. Note that $\mathfrak{t}^\vee$ can be viewed as the Lie algebra of the dual torus $T^\vee := X^* \otimes_{\mathbb{Z}} \mathbb{C}^\times$ of $T = X_* \otimes_{\mathbb{Z}} \mathbb{C}^\times$. Let $\mathbb{A}$ be the commutative associative $\mathbb{C}$-algebra $\mathbb{C}[r] \otimes \mathscr{C} \otimes_{\mathbb{C}} \mathscr{S}$ with componentwise multiplication.

Consider the diagonal action of W on $\Sigma \times R$ and let $\mu : \Sigma \times R \to \mathbb{C}$ be a W-invariant function. Given (λ, λ^*) as above, consider the W-invariant function μ_{λ,λ^*} given by

$$\mu_{\lambda,\lambda^*}(\sigma, \alpha) = \begin{cases} 0 & \text{if } s_\alpha \sigma \neq \sigma,\\ 2\lambda(\alpha) & \text{if } s_\alpha \sigma = \sigma,\ \alpha \notin 2X^*,\\ \lambda(\alpha) + \lambda^*(\alpha)\theta_{-\alpha}(\sigma) & \text{if } s_\alpha \sigma = \sigma,\ \alpha \in 2X^*. \end{cases} \tag{1.3.1}$$

Definition 1.3.9 ([Lus89, Proposition 4.4], [BM93, Proposition 4.2])**.** The graded Hecke algebra $\mathbb{H}_\Sigma = \mathbb{H}^\mu_\Sigma(\Phi)$ is the unique associative unital $\mathbb{C}[r]$-algebra generated by $\{t_w \colon w \in \mathbb{C}[W]\}$ and $\mathbb{A}$, with relations given by

$$\begin{aligned} &t_w t_{w'} = t_{ww'} && \text{for all } w, w' \in W, \\ &E_\sigma t_\alpha = t_\alpha E_{s_\alpha(\sigma)} && \text{for all } \sigma \in \Sigma,\ \alpha \in \Pi, \\ &\omega \cdot t_s - t_s \cdot s(\omega) = r\langle \omega, \check{\alpha} \rangle \sum_{\sigma \in \Sigma} E_\sigma \mu(\sigma, \alpha) && \text{for all } s \in S,\ \omega \in \mathscr{S},\ \alpha \in \Pi. \end{aligned}$$

The grading of $\mathbb{H}_\Sigma$ is given by $\deg(t_w) = \deg(E_\sigma) = 0$ for $w \in W$ and $\sigma \in \Sigma$ and $\deg(r) = \deg(\omega) = 1$ for non-zero $\omega \in X_* \otimes_{\mathbb{Z}} \mathbb{C} \subseteq \mathbb{A}$. If $\mu = \mu_{\lambda,\lambda^*}$ as in (1.3.1), we also write $\mathbb{H}^{(\lambda,\lambda^*)}_\Sigma(\Phi)$ for $\mathbb{H}^\mu_\Sigma(\Phi)$.

When Σ is a singleton, Definition 1.3.9 is the same as Lusztig's original definition in [Lus89].

For $\sigma \in \Sigma$, let $W(\sigma) = \operatorname{stab}(\sigma) = \{w \in W \colon w(\sigma) = \sigma\}$.

Proposition 1.3.10. *Suppose Σ is a disjoint union of two W-invariant sets Σ_1 and Σ_2. Then $\mathbb{H}_\Sigma \cong \mathbb{H}_{\Sigma_1} \oplus \mathbb{H}_{\Sigma_2}$.*

Proposition 1.3.11 ([BM93, Proposition 3.2])**.** *It holds that*

1. *$Z(\mathbb{H}_\Sigma) = \mathbb{A}^W$,*
2. *Suppose $\Sigma = W \cdot \sigma$ is a single W-orbit. Then*
$$\begin{aligned} \mathbb{A}^W &\to E_\sigma \cdot (\mathbb{C}[r] \otimes_{\mathbb{C}} \mathscr{S}^{W(\sigma)}) \cong \mathbb{C}[r] \otimes_{\mathbb{C}} \mathscr{S}^{W(\sigma)} \colon \\ a &\mapsto E_\sigma \cdot a \end{aligned}$$
is an isomorphism of $\mathbb{C}[r]$-algebras.
3. *Write $\Sigma = \bigsqcup_{i=1}^m \Sigma_i$ as a finite union of W-orbits $\Sigma_i = W \cdot \sigma^i$. Then*
$$Z(\mathbb{H}_\Sigma) \cong \bigoplus_{i=1}^m Z(\mathbb{H}_{\Sigma_i}) \cong \bigoplus_{i=1}^m \mathbb{C}[r] \otimes_{\mathbb{C}} \mathscr{S}^{W(\sigma^i)}.$$

1.3.4 Reduction theorems

We will state two theorems that relate the representation theory of Hecke algebras with the representation theory of certain associated graded Hecke algebras. The theorems are originally due to Lusztig [Lus89], but we state the version given in [BM93].

Suppose Σ is a single W-orbit $W \cdot \sigma$ for some $\sigma \in T$ and consider

$$\begin{aligned}
R_\sigma &= \{\alpha \in R \colon \theta_\alpha(\sigma) = \begin{cases} 1 & \text{if } \check{\alpha} \notin 2X_*, \\ \pm 1 & \text{if } \check{\alpha} \in 2X_*. \end{cases}\}, \\
R_\sigma^+ &= R_\sigma \cap R^+, \\
\Pi_\sigma &= \text{ set of simple roots w.r.t. } R_\sigma^+, \\
W_\sigma &= \langle s_\alpha \colon \alpha \in \Pi_\sigma \rangle, \\
\Gamma_\sigma &= \{w \in W(\sigma) \colon w(R_\sigma^+) = R_\sigma^+\}.
\end{aligned}$$

Note that (λ, λ^*) restricts to a parameter set on the based root datum $\Phi_\sigma = (X^*, X_*, R_\sigma, R_\sigma^\vee, \Pi_\sigma)$. Let $\mathcal{H}_\sigma = \mathcal{H}^{(\lambda,\lambda^*)}(\Phi_\sigma)$. Note that $\mathrm{stab}_W(\sigma) = W(\sigma) = \Gamma_\sigma \ltimes W_\sigma$ and so $W_\sigma \cdot \sigma$ is a singleton. Let $\mathbb{H}_\sigma = \mathbb{H}_{\{\sigma\}}^{(\lambda,\lambda^*)}(\Phi_\sigma)$. The finite group Γ_σ acts as algebra automorphisms of $\mathbb{H}_\sigma$ via its action on W_σ and $\mathscr{T}$ (the action on $\mathscr{C}$ is trivial). Thus we can consider the semidirect product $\mathbb{H}'_\sigma := \Gamma_\sigma \ltimes \mathbb{H}_\sigma$. Note that it has a centre $Z(\mathbb{H}'_\sigma) = \mathbb{C}[r] \otimes \mathscr{S}^{W(\sigma)} \cong Z(\mathbb{H})$.

Write $\Sigma = \{\sigma = \sigma_1, \sigma_2, \dots, \sigma_m\}$. For each $i, j \in \{1, \dots, m\}$, let $w_i \in W$ such that $w_i \sigma_1 = \sigma_i$ and let

$$E_{i,j} = t_{w_i^{-1}} E_\sigma t_{w_j}.$$

Note that $E_{\sigma_i} = E_{i,i}$.

Theorem 1.3.12 (First reduction theorem [Lus89, Theorem 8.6], [BM93, Theorem 3.3])**.**

1. *For each $i \in \{1, \dots, m\}$, $E_{\sigma_i} \cdot \mathbb{H}_\sigma \cdot E_{\sigma_i}$ is canonically isomorphic to $\mathbb{H}'_\sigma$.*
2. *Let $\mathcal{M}_m$ be the $\mathbb{C}$-algebra generated by $\{E_{i,j}\}_{i,j}$. Then $\mathcal{M}_m \cong M_m(\mathbb{C})$ in the obvious way and we have an isomorphism $\mathbb{H}_\Sigma \cong M_n(\mathbb{H}'_\sigma) = \mathcal{M}_n \otimes_{\mathbb{C}} (\mathbb{H}'_\sigma)$ of $\mathbb{C}[r]$-algebras that maps $\mathbb{A}^W$ isomorphically to $\mathbb{C}[r] \otimes_{\mathbb{C}} \mathscr{S}^{W(\sigma)} \cdot I_n$.*

We consider something analogous to a completion of $\mathbb{H}_\Sigma$. For now, we drop the assumption that Σ is a single W-orbit. Rather than considering formal power series, consider the algebras $\hat{\mathbb{C}}[r]$, $\hat{\mathscr{S}}$ of convergent power series in $\mathbb{C}[r]$ and $\mathscr{S}$, respectively. Define

$$\hat{\mathbb{A}} = \hat{\mathbb{C}}[r] \otimes_{\mathbb{C}} \mathscr{C} \otimes_{\mathbb{C}} \hat{\mathscr{S}}, \quad \hat{\mathbb{H}}_\Sigma = \mathbb{C}[W] \otimes_{\mathbb{C}} \hat{\mathbb{A}}.$$

We have $Z(\hat{\mathbb{H}}_\Sigma) = \hat{\mathbb{A}}^W \cong \hat{\mathbb{C}}[r] \otimes_{\mathbb{C}} \hat{\mathscr{S}}^{W(\sigma)}$, similar to Proposition 1.3.11.

We have a polar decomposition $T = T_c \cdot T_r$, where

$$T_c = Y \otimes_{\mathbb{Z}} S^1 (\text{the compact part}), \quad T_r = Y \otimes_{\mathbb{Z}} \mathbb{R}_{>0} (\text{the real part}).$$

Recall that $\mathfrak{t} = Y \otimes_{\mathbb{Z}} \mathbb{C}$. We have a morphism of algebraic groups

$$\log \colon T_r \to \mathfrak{t} \colon y \otimes u \mapsto y \otimes \log u,$$

with inverse given by

$$\exp\colon \mathfrak{t} \to T_r : y \otimes v \mapsto y \otimes \exp(v).$$

For the following, we assume that $\sigma \in T_c$. Let $\mathscr{K}$ be the field of fractions of $\mathbb{C}[t] \otimes_{\mathbb{C}} \mathscr{S}$ and let $\hat{\mathbb{H}}_\Sigma(\mathscr{K}) = \mathbb{C}[W] \otimes (\mathscr{C} \otimes_{\mathbb{C}} \mathscr{K}) \supseteq \mathbb{H}_\Sigma$.

Proposition 1.3.13 ([BM93, Proposition 4.1])**.** *Recall that Σ is not necessarily a single W-orbit. We have a $\mathbb{C}$-algebra homomorphism*

$$\begin{aligned} \Psi\colon \mathcal{H} &\to \hat{\mathbb{H}}_\Sigma(\hat{\mathscr{K}}) \\ \theta_x &\mapsto \sum_{\sigma \in \Sigma} \theta_x(\sigma) \cdot E_\sigma \cdot e^x \quad (x \in X^* \subseteq \mathscr{S}), \\ z &\mapsto e^r, \\ T_s + 1 &\mapsto \sum_{\sigma \in \Sigma} E_\sigma (t_s + 1) \frac{\Psi(\mathcal{G}(\alpha))}{1 + \frac{r\mu(\sigma,\alpha)}{\alpha}}. \end{aligned}$$

Now, assume that $\Sigma = W \cdot \sigma \subseteq T_c$ is a single W-orbit again. Let $s_r \in T_r$ and let $s = \sigma s_r \in T$. Since $\mathcal{H}$ is a finitely generated algebra over its centre $\mathcal{Z} = Z(\mathcal{H}) = \mathcal{A}^W$, it follows by Dixmier's version of Schur's lemma that $\mathcal{H}$ has central characters. As $\mathcal{A}$ is isomorphic to the coordinate ring of $T \times \mathbb{C}^\times = (X^* \otimes_{\mathbb{Z}} \mathbb{C}^\times) \times \mathbb{C}^\times$, the set of central characters of $\mathcal{H}$ is in bijection with $W \backslash T \times \mathbb{C}^\times$. Similarly, $\mathbb{H}_\Sigma$, $\mathbb{H}'_\sigma$ and $\hat{\mathbb{H}}_\Sigma$ have central characters, and for each of the three algebras, the set of central characters is in bijection with $(W(\sigma) \backslash \mathfrak{t}) \times \mathbb{C}$, where $\mathfrak{t} = X^* \otimes_{\mathbb{Z}} \mathbb{C}$ is the Lie algebra of T. Let χ, $\bar{\chi}$, $\bar{\chi}'$, $\hat{\bar{\chi}}$ be central characters of $\mathcal{H}$, $\mathbb{H}$, $\mathbb{H}'_\sigma$, $\hat{\mathbb{H}}$, respectively, such that χ corresponds to $(W \cdot s, v_0)$ and $\bar{\chi}$, $\bar{\chi}'$ and $\hat{\bar{\chi}}$ correspond $(W(\sigma) \cdot s_r, r_0) \in (W(\sigma) \backslash \mathfrak{t}) \times \mathbb{C}$ where $e^{r_0} = v_0$. Let $I_\chi = \ker \chi$, $I_{\bar{\chi}} = \ker \bar{\chi}$, $I_{\bar{\chi}'} = \ker \bar{\chi}$, and $I_{\hat{\bar{\chi}}} = \ker \hat{\bar{\chi}}$ and define

$$\begin{aligned} &\mathbb{H}_{\bar{\chi}} = \mathbb{H}_\Sigma / (\mathbb{H}_\Sigma \cdot I_{\bar{\chi}}) \cong \hat{\mathbb{H}}_\Sigma / (\hat{\mathbb{H}}_\Sigma \cdot I_{\hat{\bar{\chi}}}) = \hat{\mathbb{H}}_{\hat{\bar{\chi}}} (\text{ isomorphism of } \mathbb{C}[r] \otimes_{\mathbb{C}} \mathscr{S}^{W(\sigma)}\text{-algebras}), \\ &\mathbb{H}'_{\sigma,\bar{\chi}} = \mathbb{H}'_\sigma / \mathbb{H}'_\sigma I_{\bar{\chi}'}, \quad \mathbb{H}_{\sigma,\bar{\chi}} = \mathbb{H}_\sigma / \mathbb{H}_\sigma \cdot I_{\bar{\chi}}, \quad \mathcal{H}_\chi = \mathcal{H} / \mathcal{H} \cdot I_\chi, \end{aligned}$$

where $\mathbb{H}_{\bar{\chi}} \cong \hat{\mathbb{H}}_{\bar{\chi}}$ as $\mathbb{C}[r] \otimes_{\mathbb{C}} \mathscr{S}^{W(\sigma)}$-algebras. Theorem 1.3.12 gives an isomorphism of $\mathbb{C}[r] \otimes_{\mathbb{C}} \mathscr{S}^{W(\sigma)}$-algebras

$$\mathbb{H}_{\bar{\chi}} \cong M_m(\mathbb{H}'_{\sigma,\bar{\chi}}). \tag{1.3.2}$$

Let $t \in \mathfrak{t}$ and $r \in \mathbb{C}$ and consider the central character χ (resp. $\bar{\chi}$) of $\mathcal{H}$ (resp. $\mathbb{H}_\Sigma$) corresponding to $(W \cdot \sigma e^t, e^z) \in W \backslash T \times \mathbb{C}^\times$ (resp. $(W(\sigma) \cdot t, z) \in (W(\sigma) \backslash \mathfrak{t}) \times \mathbb{C}$).

Theorem 1.3.14 (Second reduction theorem [Lus89, Theorem 9.3], [BM93, Theorem 4.3])**.** *Recall that $\Sigma = W \cdot \sigma$ is a single orbit. Let χ and $\bar{\chi}$ be as above, and suppose that $z \in \mathbb{R}$ and $t \in Y \otimes_{\mathbb{Z}} \mathbb{R}^\times \subseteq \mathfrak{t}$. The map Ψ in Proposition 1.3.13 induces an isomorphism of $\mathbb{C}$-algebras*

$$\Psi_\chi\colon \mathcal{H}_\chi \to \hat{\mathbb{H}}_{\bar{\chi}} (\cong \mathbb{H}_{\bar{\chi}}).$$

Corollary 1.3.15 (cf. [Lus89, Corollary 10.8]). *1. We have an isomorphism of* $\mathbb{C}$*-algebras*

$$\Psi'_\chi \colon \mathcal{H}_\chi \to M_m(\mathbb{H}'_{\sigma,\bar{\chi}})$$

obtained by composing the isomorphisms (1.3.2) *and* Ψ_χ.

2. For $x \in X^*$, *we have* $\Psi'_\chi(\theta_x) = \operatorname{diag}(\theta_x(\sigma_1)e^x, \dots, \theta_x(\sigma_m)e^x)$.

Let $m \in \mathbb{N}$ and $s^1, \dots, s^m \in T$. For each $i \in \{1, \dots, m\}$, let σ^i be the compact part of s^i and s^i_r the real part of s^i. Consider the W-invariant set $\mathcal{S} = \bigsqcup_{i=1}^m \mathcal{S}_i$, where $\mathcal{S}_i = W \cdot s^i$. Also consider the W-invariant set $\Sigma = \bigsqcup_{i=1}^m \Sigma_i$ where $\Sigma_i = W \cdot \sigma^i$ in T. Let χ_i be the central character of $\mathcal{H}$ corresponding to $(\mathcal{S}_i, v_0)$. Recall that $\mathbb{H}_\Sigma \cong \bigoplus_{i=1}^m \mathbb{H}_{\Sigma_i}$ and $\hat{\mathbb{H}}_\Sigma \cong \bigoplus_{i=1}^m \hat{\mathbb{H}}_{\Sigma_i}$. Let $\bar{\chi}_i$ (resp. $\hat{\bar{\chi}}_i$) be the central characer of $\mathbb{H}_{\Sigma_i}$ (resp. $\hat{\mathbb{H}}_{\Sigma_i}$) corresponding to $(\bar{\mathcal{S}}_i, r_0)$ where $\bar{\mathcal{S}}_i$ is the $W(\sigma^i)$-orbit $W(\sigma^i) \cdot \log(s^i_r)$ in $\mathfrak{t}$. Consider the ideals

$$\begin{aligned}
I_{\mathcal{S}_i,v_0} &:= \{f \in Z(\mathcal{H}) \colon f(s', v_0) = 0 \text{ for all } s' \in \mathcal{S}_i\} = \ker(\chi_i),\\
I_{\mathcal{S},v_0} &:= \{f \in Z(\mathcal{H}) \colon f(s', v_0) = 0 \text{ for all } s' \in \mathcal{S}\} = \prod_{i=1}^m I_{\mathcal{S}_i,v_0},\\
\mathbb{I}_{\bar{\mathcal{S}}_i,r_0} &:= \{g \in Z(\mathbb{H}_{\Sigma_i}) \colon g(\bar{s}', r_0) = 0 \text{ for all } \bar{s}' \in \bar{\mathcal{S}}_i\} = \ker(\bar{\chi}_i),\\
\hat{\mathbb{I}}_{\bar{\mathcal{S}}_i,r_0} &:= \{g \in Z(\hat{\mathbb{H}}_{\Sigma_i}) \colon g(\bar{s}', r_0) = 0 \text{ for all } \bar{s}' \in \bar{\mathcal{S}}_i\} = \ker(\hat{\bar{\chi}}_i).
\end{aligned}$$

Abusing notation, we denote the ideal of $\mathbb{H}_\Sigma$ (resp. $\hat{\mathbb{H}}_\Sigma$) generated by $\mathbb{I}_{\bar{\mathcal{S}}_i,r_0}$ (resp. $\boldsymbol{I}_{\bar{\mathcal{S}}_i,r_0}$) and the $\mathbb{H}_{\Sigma_j}$ (resp. $\hat{\mathbb{H}}_{\Sigma_j}$) with $j \neq i$ by $\mathbb{I}_{\bar{\mathcal{S}}_i,r_0}\mathbb{H}_\Sigma$ (resp. $\hat{\mathbb{I}}_{\bar{\mathcal{S}}_i,r_0}\hat{\mathbb{H}}_\Sigma$).

Proposition 1.3.16. *We have an isomorphism of -algebras*

$$\Psi'_{\mathcal{S},v_0} \colon \mathcal{H} / I_{\mathcal{S},v_0}\mathcal{H} \to \hat{\mathbb{H}}_\Sigma \Big/ \prod_{i=1}^m \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i,r_0}\hat{\mathbb{H}}_\Sigma \cong \mathbb{H}_\Sigma \Big/ \prod_{i=1}^m \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i,r_0}\mathbb{H}_\Sigma$$

induced by the map $\Psi \colon \mathcal{H} \to \mathbb{H}_\Sigma(\hat{\mathscr{K}})$ *in Proposition 1.3.13.*

Proof. We will first show that $\Psi(I_{\mathcal{S},v_0}) \subseteq \prod_{i=1}^m \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i,r_0}$ so that $\Psi'_{\mathcal{S},v_0}$ is well-defined. The centre $Z(\mathcal{H})$ is spanned by elements of the form $h = \sum_{x\in A}\theta_x$ for a W-orbit A in X^*. We have

$$\Psi(h) = \sum_{x\in A}\sum_{\sigma\in\Sigma}\theta_x(\sigma)\cdot e^x \cdot E_\sigma \in \hat{\mathbb{H}}_\Sigma(\hat{\mathscr{K}}).$$

Let $y \otimes \zeta \in X_* \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{t}$ with $y \in X_*$ and $z \in \mathbb{C}$. Note that $v_0 = e^{r_0}$, so we have

$$(\Psi(h))(y \otimes z, r_0) = \sum_{x\in A}\sum_{\sigma\in\Sigma}\theta_x(\sigma)\cdot e^{\langle x,y\rangle r_0}\cdot E_\sigma = \sum_{x\in A}\sum_{\sigma\in\Sigma}\theta_x(\sigma\cdot(y\otimes v_0))\cdot E_\sigma.$$

Hence we see that for each $i \in \{1, \dots, m\}$ and $\bar{s}' \in \bar{\mathcal{S}}_i$, we have

$$(\Psi(h))(\bar{s}', r_0) = \sum_{\sigma \in \Sigma} \sum_{x \in A} \theta_x(\sigma \cdot \exp(\bar{s}')) \cdot E_\sigma = \sum_{\sigma \in \Sigma} h(\sigma \cdot \exp(\bar{s}')) \cdot E_\sigma. \tag{1.3.3}$$

Note that for each $\sigma \in \Sigma$, $\sigma \cdot \exp(\bar{s}')$ lies in $W(\sigma) \cdot s_i$, so if $h \in I_{\mathcal{S},v_0}$, then $h(\sigma \cdot \exp(\bar{s}')) = 0$, and so $(\Psi(h))(\bar{s}', r_0) = 0$ by (1.3.3). Thus $\Psi(h) \in \bigcap_{i=1}^m \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i, r_0} = \prod_{i=1}^m \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i, r_0}$, and so $\Psi'_{\mathcal{S},v_0}$ is well-defined.

The Chinese remainder theorem gives

$$\mathcal{H} / I_{\mathcal{S},v_0} \mathcal{H} \cong \prod_{i=1}^m \mathcal{H} / I_{\mathcal{S}_i,v_0} \mathcal{H}, \tag{1.3.4}$$

$$\mathbb{H}_\Sigma \Big/ \prod_{i=1}^m \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i, r_0} \mathbb{H}_\Sigma \cong \prod_{i=1}^m \mathbb{H}_\Sigma \Big/ \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i, r_0} \mathbb{H}_\Sigma \cong \prod_{i=1}^m \mathbb{H}_{\Sigma_i} / \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i, r_0} \mathbb{H}_{\Sigma_i}. \tag{1.3.5}$$

By Corollary 1.3.15(1), we have for $i = 1, \dots, m$ isomorphisms

$$\Psi'_i \colon \mathcal{H} / I_{\mathcal{S}_i,v_0} \to \hat{\mathbb{H}}_{\Sigma_i} / \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i, r_0} \hat{\mathbb{H}}_{\Sigma_i}. \tag{1.3.6}$$

The products of these maps together with the isomorphisms (1.3.4) and (1.3.5) give an isomorphism

$$\Psi' \colon \mathcal{H} / I_{\mathcal{S},v_0} \mathcal{H} \to \hat{\mathbb{H}}_\Sigma \Big/ \prod_{i=1}^m \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i, r_0} \hat{\mathbb{H}}_\Sigma.$$

It is easily checked that the image of the Bernstein generators of $\mathbb{H}$ under Ψ' and $\Psi'_{\mathcal{S},v_0}$ are the same, hence $\Psi'_{\mathcal{S},v_0} = \Psi'$ is an isomorphism, as desired.

Finally, the isomorphism $\hat{\mathbb{H}}_\Sigma / \prod_{i=1}^m \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i, r_0} \hat{\mathbb{H}}_\Sigma \cong \mathbb{H}_\Sigma / \prod_{i=1}^m \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i, r_0} \mathbb{H}_\Sigma$ quickly follows from the Chinese remainder theorem and the fact that $\hat{\mathbb{H}}_{\Sigma_i} / \hat{\mathbb{I}}_{\bar{\mathcal{S}}_i, r_0} \hat{\mathbb{H}}_\Sigma \cong \mathbb{H}_{\Sigma_i} / \mathbb{I}_{\bar{\mathcal{S}}_i, r_0} \mathbb{H}_\Sigma$. □

1.3.5 Graded Hecke algebras attached to cuspidal local systems

Let H be a connected complex reductive algebraic group and fix a Borel subgroup B and a maximal torus T with Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$. Let S_H be the set of triples $(L, C_L, \mathcal{L})$ where L is a Levi subgroup of a standard parabolic subgroup P (containing B) of H, C_L is a nilpotent orbit in $\mathrm{Lie}(L)$ and $\mathcal{L}$ is a cuspidal L-equivariant local system on C_L. To each $(L, C_L, \mathcal{L}) \in S_H$, Lusztig attached a graded Hecke algebra, denoted by $\mathbb{H}(H, L, C_L, \mathcal{L})$, and classified all their representations in [Lus88] and [Lus95b].

We will define $\mathbb{H}(H, L, C_L, \mathcal{L})$ as in the two papers of Lusztig mentioned above, but we will present it in notation that is more similar to the notation in [Ciu08, §2,§3]. Let

$$\begin{aligned}\mathfrak{h} &= \mathrm{Lie}(H),\\ \mathfrak{z}_L &= \mathrm{Lie}(Z(L)^\circ),\\ W_L &= N_H(L)/L,\\ R_L &= \{\alpha \in X^*(Z(L)^\circ)\colon \alpha \text{ appears in the adjoint action of } Z(L)^\circ \text{ on } \mathfrak{h}\}.\end{aligned}$$

Let $R_L^\vee$ be any subset of $X_*(Z(L)^\circ)$ such that it forms a set of coroots of R, i.e. $R_L^\vee = \{\check{\alpha}\colon \alpha \in R_L\}$ and $s_\alpha(\beta) = \beta - \langle \beta, \check{\alpha}\rangle\alpha$. Since $X_*(Z(L)^\circ)$ and $X^*(Z(L)^\circ)$ are in perfect pairing, this uniquely determines $\langle\beta,\check{\alpha}\rangle$ for all $\alpha,\beta \in R_L$, and hence this value is independent of the choice of $R_L^\vee$.

By [Lus84b, Theorem 9.2], W_L is the Weyl group with (possibly non-reduced) root system R_L. Let Π_L be the set of roots in R_L that are simple with respect to the parabolic P associated to L and containing B. Define a function $\mu_L\colon \Pi_L \to \mathbb{Z}_{\geq 2}$ as follows. Let $u \in C_L$. For each $\alpha \in \Pi_L$, let $\mu_L(\alpha)$ be the unique integer in $\mathbb{Z}_{\geq 2}$ such that

$$\begin{aligned}\mathrm{Ad}(u)^{\mu_L(\alpha)-2}\colon\ \mathfrak{h}_\alpha \oplus \mathfrak{h}_{2\alpha} \to \mathfrak{h}_\alpha \oplus \mathfrak{h}_{2\alpha} &\text{ is non-zero},\\ \mathrm{Ad}(u)^{\mu_L(\alpha)-1}\colon\ \mathfrak{h}_\alpha \oplus \mathfrak{h}_{2\alpha} \to \mathfrak{h}_\alpha \oplus \mathfrak{h}_{2\alpha} &\text{ is zero}.\end{aligned}$$

Then μ_L is W-equivariant by [Lus88, Prop. 2.12] and thus extends W-equivariantly to a function $\mu_L\colon R_L \to \mathbb{Z}_{\geq 2}$.

Definition 1.3.17. Let $(L, C_L, \mathcal{L}) \in S_H$. The *graded Hecke algebra* $\mathbb{H}(H, L, C_L, \mathcal{L})$ *attached to* $(L, C_L, \mathcal{L})$ is defined to be the graded Hecke algebra $\mathbb{H}_{\{1\}}(X^*(Z(L)^\circ), X_*(Z(L)^\circ), R_L, R_L^\vee, \Pi_L)$.

Fix $(L, C_L, \mathcal{L}) \in S_H$, write $\mathbb{H} = \mathbb{H}(H, L, C_L, \mathcal{L})$, and let $P = LU$ be a parabolic subgroup of G with Levi subgroup L and unipotent radical U. Let $\mathfrak{n} = \mathrm{Lie}(U)$. Let

$$\dot{\mathfrak{h}}_U = \{(x, hP) \in \mathfrak{h} \times H/P\colon\ \mathrm{Ad}(h^{-1})x \in \mathfrak{n}\}.$$

The L-equivariant local system $\mathcal{L}$ gives a $G \times \mathbb{C}^\times$-local system $\dot{\mathcal{L}}$ on $\dot{\mathfrak{h}}_U$ via the map

$$\dot{\mathfrak{h}}_U \to C_L\colon (x, hP) \mapsto \mathrm{pr}_{C_L}(\mathrm{Ad}(h^{-1})x).$$

Let $r_0 \in \mathbb{C}$ and $e \in \mathfrak{h}$ be a nilpotent element. Consider the variety

$$\mathcal{P}_e = \{hP \in H/P\colon\ \mathrm{Ad}(h^{-1})e \in C_L + \mathfrak{n}\}.$$

The centraliser $Z_{H\times\mathbb{C}^\times}(e)$ acts on $\mathcal{P}_e$ by $(h', \lambda)hP = (h'h)P$. Lusztig constructed a $\mathbb{H} \times A_{H\times\mathbb{C}^\times}(e)$-action on the $Z_{H\times\mathbb{C}^\times}(e)^\circ$-equivariant homology $H_\bullet^{Z^\circ_{H\times\mathbb{C}^\times}(e)}(\mathcal{P}_e, \dot{\mathcal{L}})$, see [Lus88, §8.5, Theorem 8.13]. By [Lus88, §8.7], the $Z_{H\times\mathbb{C}^\times}(e)^\circ$-equivariant cohomology $H^\bullet_{Z^\circ_{H\times\mathbb{C}^\times}(e)}(\mathcal{P}_e, \dot{\mathcal{L}})$ is the coordinate ring of the affine variety $\mathcal{V}_e$ of semisimple $Z_{H\times\mathbb{C}^\times}(e)^\circ$-orbits on the Lie algebra

$$\mathrm{Lie}(Z_{H\times\mathbb{C}^\times}(e)) = \mathfrak{z}_{H\times\mathbb{C}^\times}(e) = \{(x, r_0) \in \mathfrak{h} \oplus \mathbb{C}\colon [x, e] = 2r_0 e\}.$$

For $(\sigma, r_0) \in \mathcal{V}_e$, let $\mathbb{C}_{\sigma,r_0}$ be the $H^\bullet_{Z^\circ_{H\times\mathbb{C}^\times}(e)}$-module given by the evaluation map $H^\bullet_{Z^\circ_{H\times\mathbb{C}^\times}(e)} \to \mathbb{C}$ at (σ, r_0), and define the $\mathbb{H}$-module

$$Y(e, \sigma, r_0) = \mathbb{C}_{\sigma,r_0} \otimes_{H^\bullet_{Z^\circ_{H\times\mathbb{C}^\times}(e)}} H_\bullet^{Z^\circ_{H\times\mathbb{C}^\times}(e)}(\mathcal{P}_e, \dot{\mathcal{L}}).$$

Let $A_{H\times\mathbb{C}^\times}(e, \sigma, r_0)$ denote the stabiliser of (σ, r_0) in $A_{H\times\mathbb{C}^\times}(e)$. For each $\psi \in A_{H\times\mathbb{C}^\times}(e, \sigma, r_0)^\wedge$, define $Y(e, \sigma, r_0, \psi)$ to be the ψ-isotypic component of $Y(e, \sigma, r_0)$, also denoted by

$$Y(e, \sigma, r_0, \psi) := Y(e, \sigma, r_0)^\psi := \operatorname{Hom}_{A_{H\times\mathbb{C}^\times}(e,\sigma,r_0)}(\psi \colon Y(e, \sigma, r_0)).$$

When $(\sigma, r_0) = (0, 0)$, we have the following isomorphism of $W \times A_H(e)$-modules by [Lus88, Proposition 7.2,§8.9] and [Lus95b, 10.12(d)]

$$Y(e, 0, 0) \cong H_\bullet^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}}) \quad (\{1\}\text{-equivariant homology}). \tag{1.3.7}$$

We have a canonical isomorphism $A_{H\times\mathbb{C}^\times}(e, \sigma, r_0) \cong A_H(e, \sigma)$. Let $A_H(e, \sigma)_0^\wedge$ denote the set of representations which appear in the restriction of $H_\bullet^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})$ to $A_H(e, \sigma)$.

Define

$$\begin{aligned}\mathcal{M}_{\sigma,r_0} &:= Z_H(\sigma)\text{-conjugacy classes on } \{(e, \psi) \in \mathfrak{h} \times A_H(e, \sigma)_0^\wedge \colon [\sigma, e] = 2r_0 e\}.\\ \mathcal{M} &:= \{(e, \sigma, r_0, \psi) \colon (\sigma, r_0) \in \mathcal{V}_e, (e, \psi) \in \mathcal{M}_{\sigma,r_0}\}.\end{aligned}$$

Recall from Section 1.3.4 that the central characters of $\mathbb{H}$ are in bijection with $W\backslash\, \mathfrak{t} \otimes \mathbb{C}^\times$. Note that each semisimple element $\sigma \in H$ is H-conjugate to an element $t \in \mathfrak{t}$. Let χ be the central character corresponding to $(W \cdot t, r_0)$ define

$$\operatorname{Irr}_{\sigma,r_0} \mathbb{H} := \operatorname{Irr}_{t,r_0} \mathbb{H} := \operatorname{Irr} \mathbb{H}_\chi\,.$$

Theorem 1.3.18 ([Lus88, Proposition 8.10, Theorem 8.15], [Lus95b, Corollary 8.18])**.**

1. *Let $e \in \mathfrak{h}$ be nilpotent, $(\sigma, r_0) \in \mathcal{V}_e$, and $\psi \in A_H(e, \sigma)^\wedge$. Then $Y(e, \sigma, r_0, \psi) \neq 0$ if and only if $\psi \in A_H(e, \sigma)_0^\wedge$.*
2. *Suppose M is a simple $\mathbb{H}$-module on which r acts as multiplication by $r_0 \in \mathbb{C}$. Then there exists a unique $(e, r_0, \sigma, \psi) \in \mathcal{M}$ such that M is a quotient $\bar{Y}(e, \sigma, r_0, \psi)$ of $Y(e, \sigma, r_0, \psi)$.*
3. *We have a bijection*

$$\begin{aligned}\mathcal{M}_{\sigma,r_0} &\leftrightarrow \operatorname{Irr}_{\sigma,r_0} \mathbb{H}\\ (e, \psi) &\mapsto \bar{Y}(e, \sigma, r_0, \psi).\end{aligned}$$

Let M be an irreducible $\mathbb{H}$-module. We have a weight space decomposition

$$M = \bigoplus_{\lambda \in \mathfrak{z}_L} M_\lambda.$$

We say that $\lambda \in \mathfrak{z}_L$ is a weight if $V_\lambda \neq 0$.

Definition 1.3.19. An irreducible module M is called *tempered* if $\omega(\lambda) \neq 0$ for all fundamental weights $\omega \in \mathfrak{z}_L^\vee$ and all weights λ of M. We write

$$\begin{aligned} \mathcal{M}_{\text{temp}} &= \{(e, \sigma, \psi) \colon (e, \sigma, q, \psi) \in \mathcal{M},\ \bar{Y}(e, \sigma, q, \psi) \text{ is tempered}\}. \qquad (1.3.8) \\ \mathcal{M}_{\text{temp,real}} &= \{(e, \sigma, \psi) \colon (e, \sigma, q, \psi) \in \mathcal{M}_{\text{temp}},\ s_c = 0\}. \end{aligned}$$

This is Casselman's notion of temperedness and agrees with the notion of τ-temperedness in [KL87] and [Lus02] where we take the group homomorphism $\tau \colon \mathbb{C} \to \mathbb{R}$ to be given by $z \mapsto \log |z|$.

Proposition 1.3.20 ([Lus02, Theorem 1.21])**.** *Let $(e, \sigma, r_0, \psi) \in \mathcal{M}$. The following are equivalent:*

1. $(e, \sigma, r_0, \psi) \in \mathcal{M}_{\text{temp}}$,
2. *There exists an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{h}$ such that the semisimple element $\sigma - r_0 h$ is compact.*

If any of these conditions are satisfied, then $Y(e, \sigma, r_0, \psi) = \bar{Y}(e, \sigma, r_0, \psi)$ is irreducible.

Proposition 1.3.21. *Let $(e, \sigma, r_0, \psi) \in \mathcal{M}_{\text{temp,real}}$. Then $A_H(e, \sigma) = A_H(e)$.*

The definition of the standard module $Y(e, \sigma, r_0, \psi)$ depends on the choice of cuspidal support $(L, C_L, \mathcal{L}) \in S_H$. For now, denote the standard module by $X_L(e, \sigma, r_0, \psi)$. By [Lus95b, Proposition 8.16a], there exists precisely one triple $(L, C_L, \mathcal{L})$ such that $Y_L(e, \sigma, r_0, \psi) \neq 0$ is non-zero. By Theorem 1.3.18(1), we have

$$0 \neq H_\bullet^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})^\psi = \bigoplus_{\phi \in A_H(e)^\wedge} [\phi : \psi] H_\bullet^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})^\phi,$$

where $[\phi : \psi]$ denotes the multiplicity of ψ in $\phi|_{A_H(e,\sigma)}$. There exists a $\phi \in A_H(e)^\wedge$ such that $[\phi : \psi] \neq 0$ and $H_\bullet^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})^\phi \neq 0$. Thus $(L, C_L, \mathcal{L})$ is the cuspidal support of (e, ϕ) in the generalised Springer correspondence. To summarise, we have the following.

Proposition 1.3.22. *Let $(e, \sigma, r_0, \psi) \in \mathcal{M}$ and $(L, C_L, \mathcal{L}) \in S_H$. Let $\phi \in A_H(e)^\wedge$ with $[\phi : \psi] > 0$. Then $Y_L(e, \sigma, r_0, \psi) \neq 0$ if and only if $(L, C_L, \mathcal{L})$ is the cuspidal support of (e, ϕ) in the generalised Springer correspondence of H.*

1.4 Unipotent representations

1.4.1 Unipotent representations and Bernstein components

We define an important class of irreducible admissible representations of $\mathbf{G}(\mathsf{k})$ analogous to the notion of unipotent representations of a finite group of Lie type. Let $\mathbf{H}$ be a connected reductive group split over $\mathbb{F}_q$ and let $\mathbf{H}(\mathbb{F}_q)$ be a finite group of Lie type with corresponding Frobenius map $\mathsf{Fr}\colon \mathbf{H}(\bar{\mathbb{F}}_q) \to \mathbf{H}(\bar{\mathbb{F}}_q)$. Let $\mathbf{T}$ be a Fr-stable maximal torus in $\mathbf{H}$. Given a character $\theta\colon \mathbf{T}(\mathbb{F}_q) \to \mathbb{C}^\times$, Deligne and Lusztig define a virtual representation $R_{\mathbf{T}}(\theta)$ of $\mathbf{H}(\mathbb{F}_q)$. They proved that all finite-dimensional irreducible representations of $\mathbf{H}(\mathbb{F}_q)$ appear in $R_{\mathbf{T}}(\theta)$ for some Fr-stable maximal torus $\mathbf{T}$ and some character $\theta\colon \mathbf{T}(F_q)$. An irreducible representation of $\mathbf{G}(\mathbb{F}_q)$ is called unipotent if it appears in $R_{\mathbf{T}}(1)$ for some Fr-stable maximal torus $\mathbf{T}$. Lusztig proved that the classification of the finite dimensional representations of $\mathbf{H}(\mathbb{F}_q)$ reduces to the classification of the unipotent representations, which was done in [Lus84a].

Lusztig defined unipotent representations for a connected reductive group $\mathbf{G}(\mathsf{k})$ in [Lus95a]. Similarly to unipotent representations for finite groups of Lie type, they should form a 'small' set of representations that play an important role in the classification of irreducible smooth representations of $\mathbf{G}(\mathsf{k})$ and are parametrised in terms of geometric data of the complex Langlands dual $G^\vee$.

Definition 1.4.1 ([Lus95a])**.** An irreducible representation X of $\mathbf{G}(\mathsf{k})$ is called *unipotent* (or is said to have *unipotent cuspidal support*) if there exists a $\mathbf{P} \in \mathscr{P}$ such that $X^{\mathbf{U}_{\mathbf{P}}(\mathfrak{o})}$ has a $\mathbf{L}(\mathbb{F}_q)$-subrepresentation that is unipotent in the sense above. Write $\mathrm{Unip}(\mathbf{G}(\mathsf{k}))$ for the set of unipotent representations of $\mathbf{G}(\mathsf{k})$.

An important result is that the unipotent representations form the set of irreducible representations of a finite product of Bernstein components of $\mathbf{G}(\mathsf{k})$. We shall call each of these Bernstein components a *unipotent.*

The classification of unipotent representations of $\mathbf{G}(\mathsf{k})$ (recall that $\mathbf{G}$ splits over k) was first done in [Lus95a] for the case that $\mathbf{G}$ is adjoint. The classification is best described by the *Deligne–Langlands–Lusztig correspondence* (see Theorem 1.4.2). The classification in the case that $\mathbf{G}$ is of arbitrary isogeny was later done in [Sol18]. We will state the DLL-correspondence and give some useful properties regarding tempered, elliptic and square integrable representations of $\mathbf{G}(\mathsf{k})$.

Let $\Phi(G^\vee)$ be the set of triples (e, s, ϕ) where $s \in G^\vee$ is a semisimple element, $e \in \mathfrak{g}^\vee$ is a nilpotent element such that $\mathrm{ad}(s)e = qe$, and $\phi \in A_{G^\vee}(s,e)^\wedge$, where $A(s,e)$ is the component group $A_{G^\vee}(s,e) = Z_{G^\vee}(s,e)/Z_{G^\vee}(s,e)^\circ$, where $Z_{G^\vee}(s,e)$ is the set of elements in $G^\vee$ that centralise both s and e. We have an action of $G^\vee$ on $\Phi(G^\vee)$ given by the conjugation with $G^\vee$. The set $\Phi(G^\vee)$ is equivalent to the set of so-called 'enhanced unramified L-parameters' of $\mathbf{G}(\mathsf{k})$, which we will note define here, and motivates the connection with the local Langlands correspondence. It is known that there exists an $\mathfrak{sl}_2$-triple $\{e^-, h, e\}$ such that $h \in \mathfrak{t}^\vee_{\mathbb{R}}$. Then for $s_0 := sq^{-h/2}$, we have $\mathrm{Ad}(s_0)e = e$. Furthermore, we have $A_{G^\vee}(s_0, e) = A_{G^\vee}(s, e)$. Note that e thus lies in the Lie algebra of $Z_{G^\vee}$. We can therefore also write $A_{G^\vee}(s, e) = Z_{Z_{G^\vee}(s_0)}(e)/Z_{Z_{G^\vee}(s_0)}(e)^\circ = A_{Z_{G^\vee}(s_0)}(e)$.

1.4.2 Classification of unipotent representations

We briefly discuss Lusztig's classification of unipotent representations [Lus95a] as well as some of the properties of this classification related to tempered, elliptic, and square-integrable representations.

The Deligne-Langlands-Lusztig correspondence for the unipotent representations gives an injection of $\mathrm{Unip}(\mathbf{G}(\mathsf{k}))$ into $\Phi(G^\vee)$. To account for the non-surjectiveness, we look at the disjoint union of the unipotent representations of all the inner twists of the split form of $\mathbf{G}(\mathsf{k})$, analogous to Vogan's idea to consider these twists in [Vog93]. The set of these inner twists is in bijection with the the set of irreducible characters of $Z(G^\vee)$. The trivial character of $Z(G^\vee)$ corresponds to the split form $\mathbf{G}(\mathsf{k})$. We now state the correspondence, as well as a related result regarding tempered representations.

Theorem 1.4.2 (Deligne–Langlands–Lusztig correspondence)**.** *Suppose* $\mathbf{G}$ *is split over* k. *Then there is a bijection*

$$\mathrm{LLC}_{\mathrm{unip}} \colon \Phi(G^\vee) \to \bigsqcup_{\zeta \in Z(G^\vee)^\wedge} \mathrm{Unip}(\mathbf{G}(\mathsf{k})_\zeta) \colon (e, s, \phi) \mapsto X(e, s, \phi),$$

such that

1. For all $(e, s, \phi) \in \Phi(G^\vee)$*, we have* $X(e, s, \phi) \in \mathrm{Unip}(\mathbf{G}(\mathsf{k})_\zeta)$*, where* $\zeta \in Z(G^\vee)^\wedge$ *is determined by the action of the cuspidal local system corresponding to* (e, s, ϕ) *on* $Z(G^\vee)$*,*

2. $X(e, s, \rho)$ *is tempered if and only if* $s_0 \in T^\vee_c$ *and*

$$\overline{G^\vee(s)e} = \{x \in \mathfrak{g}^\vee \colon \mathrm{Ad}(s)x = qx\}.$$

Definition 1.4.3. Denote by $\Phi_{\mathrm{temp}}(G^\vee)$ the subset of $\Phi(G^\vee)$ consisting of the triples corresponding to tempered unipotent modules.

We briefly give an overview of some of the ideas in the proof of the classification of unipotent representations of $\mathbf{G}$ in the case the $\mathbf{G}$ is adjoint in [Lus95a].

The first step is to describe each unipotent Bernstein component in terms of an associated *arithmetic affine Hecke algebra* as in [Lus95a, §1] and [BK98, §2]. The set of unipotent Bernstein components were parametrised by a certain set of 'arithmetic diagrams' in [Lus95a, §1]. To each such arithmetic diagram, Lusztig attaches an affine Hecke algebra, which turns out to be almost the same Hecke algebra corresponding to the relevant Bernstein block (recall that each Bernstein block has a type by Proposition 1.3.6): the difference is that the Hecke algebra corresponding to the type of the Bernstein block is a direct sum of copies of Lusztig's arithmetic affine Hecke algebra. We refer to [Lus95a, §7] for a table with all possible arithmetic affine Hecke algebras, whose Iwahori-Matsumoto presentations are further described in [Lus95a, §1.18,§1.19].

On the other side, the set $\Phi(G^\vee)$ is shown to be a parametrising set for the irreducible representations of a finite collection of affine Hecke algebras in [Lus95a, §5] called *geometric affine Hecke algebras*. The parametrisation is roughly as follows. Suppose $s_c \in T_c$ is compact. The centraliser $Z_{G^\vee}(s_c)$ is a pseudo-Levi subgroup of $G^\vee$, and corresponds to a subset of the extended Dynkin diagram associated to the (root datum of) $G^\vee$, i.e. a subset J of the set of affine simple reflections of the extended Weyl group $\tilde{W}$ of $G^\vee$. Since G is adjoint, its pseudo-Levi subgroups are connected. The generalised Springer correspondence (which was studied for connected groups in [Lus84b]) for $Z_{G^\vee}(s_c)$ attaches to each $(e,\phi) \in \mathcal{N}_{G^\vee}$ a triple $(L^\vee, C, \mathcal{L})$, where $L^\vee$ is a Levi-subgroup of $Z_{G^\vee}(s_c)$ (hence also a Levi-subgroup of $G^\vee$), C is a unipotent class in $L^\vee$ and $\mathcal{L}$ is a cuspidal $L^\vee$-equivariant local system on $\mathcal{L}$. We shall not give a description of the geometric affine Hecke algebra $\mathcal{H}(G, L^\vee, C, \mathcal{L})$, and instead refer to [Lus95a, §7] for a table for all the possibilities. Note that $L^\vee$ corresponds to a subset $I \subseteq J$. Let $K = J \setminus I$ and let

$$P^0 = \{x \in H^0 \colon \langle x, \alpha_i \rangle = 0 \text{ for all } i \in I\}.$$

Let $\Phi(s_c, L^\vee, C, \mathcal{L})$ be the set of triples (ξ, e, ϕ) where $\xi \in Z_{G^\vee}(s_c)$ is semisimple and real, n is a nilpotent element of $\mathrm{Lie}(Z_{G^\vee}) \subseteq \mathfrak{g}$ such that $\mathrm{Ad}(\xi)e = qe$ and ϕ is an irreducible representation of the component group $A_{Z_G^\vee}(e)$. We then have a bijection

$$\bigsqcup_{s_c \in T_c} \Phi(s_c, L^\vee, C, \mathcal{L})/Z_{G^\vee}(s_c) \leftrightarrow \Phi(G^\vee)/G^\vee$$
$$(e, \xi, \phi) \mapsto (e, s_c \exp(\xi), \phi)$$

where we mod out the conjugation action of $G^\vee$ and $Z_{G^\vee}(s_c)$, respectively. If (ξ, e, ϕ) corresponds to (e, s, ϕ) as above, then we have $s = s_c \exp(\xi)$, i.e. $s_r = \exp(\xi)$. To each $(L^\vee, C, \mathcal{L})$ as above, Lusztig defined the geometric affine Hecke algebra $\mathcal{H}(G^\vee, L^\vee, C, \mathcal{L})$. We shall not give a description of $\mathcal{H}(G^\vee, L^\vee, C, \mathcal{L})$: from [Lus95a, §7], one can find all possibilities for $\mathcal{H}(G, L^\vee, C, \mathcal{L})$. Furthermore, Lusztig constructed a bijection between the set of geometric affine Hecke algebras and the set of arithmetic affine Hecke algebras such that the corresponding Hecke algebras are isomorphic. The representations of the geometric algebras $\mathcal{H}(G^\vee, L^\vee, C, \mathcal{L})$ can be studied via the representation theory of the corresponding graded Hecke algebras $\mathbb{H}(Z_{G^\vee}(s_c), L^\vee, C, \mathcal{L})$ via Lusztig's reduction theorems.

Applying the results of [Lus95b, §8] to the group $Z_{G^\vee}(s_c)$ and applying Corollary 1.3.15, we obtain by [Lus95b, Corollary 8.18] and the discussion in [Lus95a, §5.20] a bijection between

$$\bigsqcup_{(L^\vee,C,\mathcal{L})\in} \bigsqcup_{(\Sigma,r_0)\in W\backslash T\times\mathbb{C}^\times} \mathrm{Irr}_{\Sigma,r_0} \mathbb{H}(Z_{G^\vee}(t_c), L^\vee, C, \mathcal{L})$$

and $\Phi(s_c, L^\vee, C, \mathcal{L})/Z_{G^\vee}(s_c)$.

Chapter 2

Aubert-Zelevinsky duality and Iwahori-Matsumoto duality

2.0.1 Aubert-Zelevinsky involution

Recall the set $\mathcal{Q}$ of parabolic subgroups of G defined over k containing B. It is well-known that $\mathcal{Q}$ is canonically in bijection with the power set of S. For $\mathsf{Q} \in \mathcal{Q}$, let $\mathrm{Ind}_{\mathsf{Q}(\mathsf{k})}^{\mathsf{G}(\mathsf{k})}$ and $\mathrm{Res}_{\mathsf{Q}(\mathsf{k})}^{\mathsf{G}(\mathsf{k})}$ denote the normalised induction and Jacquet functor, respectively (see for instance [BR96, §III.1] for the definitions). Suppose Q corresponds to the subset $J \subseteq S$. Let $r_{\mathsf{Q}} = |J|$, i.e. r_{Q} is the semisimple rank of Q.

Definition 2.0.1 ([Aub95, §1])**.** The *Aubert-Zelevinsky involution* on $\mathcal{R}(\mathsf{G}(\mathsf{k}))$ is the group homomorphism $\mathsf{AZ}_{\mathsf{G}(\mathsf{k})} \colon \mathcal{R}(\mathsf{G}(\mathsf{k})) \to \mathcal{R}(\mathsf{G}(\mathsf{k}))$, defined as

$$\mathsf{AZ}_{\mathsf{G}(\mathsf{k})} = \sum_{\mathsf{Q} \in \mathcal{Q}} (-1)^{r_{\mathsf{Q}}} \mathrm{Ind}_{\mathsf{Q}(\mathsf{k})}^{\mathsf{G}(\mathsf{k})} \circ \mathrm{Res}_{\mathsf{Q}(\mathsf{k})}^{\mathsf{G}(\mathsf{k})}.$$

We often drop $\mathsf{G}(\mathsf{k})$ from the notation.

The map AZ is an involution [Aub95, Theorem 1.7].

2.0.2 Iwahori-Matsumoto involution

Let $\Phi = (X_*, X^*, R, R^\vee, \Pi)$ be a based root datum with parameter set (λ, λ^*) and let $\mathcal{H} = \mathcal{H}^{(\lambda,\lambda^*)}$ be the corresponding affine Hecke algebra. We retain the notation from Section 1.3. Let w_0 be the longest element of W.

Definition 2.0.2. The *Iwahori-Matsumoto involution* of $\mathcal{H}$ is a $\mathbb{C}[t, t^{-1}]$-algebra homomorphism $\mathsf{IM} \colon \mathcal{H} \to \mathcal{H}$ defined by

$$\mathsf{IM}(T_w) = (-t)^{\ell(w)} T_{w^{-1}}^{-1} \qquad \text{for } w \in W,$$
$$\mathsf{IM}(\theta_x) = T_{w_0} \theta_{w_0(x)} T_{w_0}^{-1} \qquad \text{for } x \in X.$$

Clearly, $\mathsf{IM} \circ \mathsf{IM}$ is the identity homomorphism on $\mathcal{H}$, so IM is in fact an involution. Thus IM induces an involution on $\mathcal{R}(\mathcal{H})$: for $M \in \mathcal{R}(\mathcal{H})$, $\mathsf{IM}(M)$ has the same underlying vector space as M and the action of $\mathcal{H}$ is twisted by IM. From the definition, it is furthermore easy to see that IM restricts to the identity on $Z(\mathcal{H}) = \mathcal{A}^W$, and so IM preserves central characters.

Let $J \subseteq S$, let W_J be the corresponding parabolic subgroup of W and let $\mathcal{H}_J$ be the parabolic subgalgebra of $\mathcal{H}$, i.e. the subalgebra generated by $\{T_w : w \in W_J\}$ and $\mathcal{A}$. Note that $\mathcal{H}_J$ is itself the Hecke algebra of some smaller root datum with roots determined by J. Let $\mathrm{Res}_J : \mathcal{H}\text{-mod} \to \mathcal{H}_J\text{-mod}$ denote the restriction functor and $\mathrm{Ind}_J : \mathcal{H}_J\text{-mod} \to \mathcal{H}\text{-mod}$ denote the induction functor, i.e. $\mathrm{Ind}_J = \mathcal{H} \otimes_{\mathcal{H}_J} -$. This defines abelian Group morphisms $\mathrm{Res}_J : \mathcal{R}(\mathcal{H}) \to \mathcal{R}(\mathcal{H}_J)$ and $\mathrm{Ind}_J : \mathcal{R}(\mathcal{H}_J) \to \mathcal{R}(\mathcal{H})$.

Theorem 2.0.3 ([Kat93, Theorem 2])**.** *We have the following equality of homomorphisms of* $\mathcal{R}(\mathcal{H})$

$$\mathsf{IM} = \sum_{J \subseteq S} (-1)^{|J|} \mathrm{Ind}_J \circ \mathrm{Res}_J .$$

2.0.3 Graded Iwahori-Matsumoto involution

Recall that $T = X_* \otimes_{\mathbb{Z}} \mathbb{C}^\times$ and let $\Sigma = W \cdot \sigma$ be a single W-orbit in T. Consider the graded Hecke algebra $\mathbb{H} = \mathbb{H}_\Sigma^{(\lambda,\lambda*)}(\Phi)$.

Definition 2.0.4. The *graded Iwahori-Matsumoto involution* of $\mathbb{H}$ is a $\mathbb{C}[r]$-algebra homomorphism $\mathbb{IM} : \mathbb{H} \to \mathbb{H}$ defined by

$$\begin{aligned} \mathbb{IM}(t_w) &= (-1)^{\ell(w)} t_w && \text{for } w \in W, \\ \mathbb{IM}(E_\sigma) &= E_\sigma && \text{for } \sigma \in \Sigma, \\ \mathbb{IM}(\omega) &= t_{w_0} w_0(\omega) t_{w_0}^{-1} && \text{for } \omega \in \mathscr{S}. \end{aligned}$$

We obtain an involution on $\mathcal{R}(\mathbb{H})$, also denoted by $\mathbb{IM}$, such that $\mathbb{IM}(M)$ has the same underlying vector space as M, but whose $\mathbb{H}$-action is twisted by $\mathbb{IM}$. From Proposition 1.3.11, we see easily that $\mathbb{IM}$ restricts to the identity map on $Z(\mathbb{H})$, so $\mathbb{IM}$ preserves central characters.

Remark 2.0.5. Let M be a representation of $\mathbb{H}$. We identify W with the subset $\{t_w : w \in W\}$ of $\mathbb{H}$. Note that $\mathbb{H}$ contains the group algebra $\mathbb{C}[W]$ of W. Let χ and χ' be the character of $M|_{\mathbb{C}[W]}$ and $\mathbb{IM}(M)|_{\mathbb{C}[W]}$, respectively. Then $\chi'(t_w) = (-1)^{\ell(w)} \chi(t_w)$, and so we see that $\mathbb{IM}(M)|_{\mathbb{C}[W]} = M|_{\mathbb{C}[W]} \otimes \mathrm{sgn}$.

Given $J \subseteq S$, let $\mathbb{H}_J$ be the subalgebra of $\mathbb{H}$ generated by $\{t_w : w \in W_J\}$ and $\mathbb{A}$. Let $\operatorname{Ind}_J : \mathbb{H}_J\text{-mod} \to \mathbb{H}\text{-mod}$ be the induction functor $\mathbb{H} \otimes_{\mathbb{H}_J} -$, and let $\operatorname{Res}_J : \mathbb{H}\text{-mod} \to \mathbb{H}_J\text{-mod}$ denote the restriction functor.

Theorem 2.0.6. *We have the following equality of homomorphisms of* $\mathcal{R}(\mathbb{H})$

$$\mathbb{IM} = \sum_{J \subseteq S} (-1)^{|J|} \operatorname{Ind}_J \circ \operatorname{Res}_J .$$

Theorem 2.0.6 was proved in [Cha16, Theorem 6.5] for the case that Σ is a singleton and for a certain specialisation of r, and the proof was based on results and ideas from [Kat93]. Our proof will almost be identical and will include some comments regarding the differences that arise from the fact that Σ here is an arbitrary finite W-orbit.

Write $\Pi = \{\alpha_1, \ldots, \alpha_m\}$. In particular, we have fixed an ordering on the simple roots. For $J \subseteq \Pi$ and M a $\mathbb{H}$-module, let

$$C_J(M) = \operatorname{Ind}_J \operatorname{Res}_J X = \mathbb{H} \otimes_{\mathbb{H}_J} (\operatorname{Res}_J M).$$

For $i \in \mathbb{Z}_{\geq 0}$, let

$$C_i(M) = \bigoplus_{J \subseteq \Pi, |J| = i} C_J(M).$$

Given $J \subseteq J' \subseteq \Pi$ with $|J'| = |J| + 1$, write $\Pi \setminus J = \{\alpha_{i_1}, \ldots, \alpha_{i_{m-|J|}}\}$ with $1 \leq i_1 < \cdots < i_{m-J} \leq m$. Let j be the unique integer such that $\alpha_{i_j} \in J' \setminus J$. Let $\varepsilon_J^{J'} = (-1)^{j+1}$. We then define

$$\pi_J^{J'} : C_J(M) \to C_{J'}(M) \colon h \otimes x \mapsto \varepsilon_J^{J'} h \otimes x$$

and for $i \in \mathbb{Z}_{\geq 0}$, we define

$$\pi_i = \bigoplus_{\substack{J \subseteq \Pi \\ |J| = i}} \bigoplus_{\substack{J \subseteq J' \subseteq \Pi \\ |J'| = i+1}} \pi_J^{J'} : C_i(M) \to C_{i+1}(M).$$

The proof of the following proposition is identical to the proof of [Cha16, Proposition 6.2], even for the case that Σ is not a singleton.

Proposition 2.0.7 ([Cha16, Proposition 6.2], cf. [Kat93, Theorem 1])**.** *We have an exact sequence of* $\mathbb{H}$*-modules*

$$0 \longrightarrow \ker \pi_0 \longrightarrow C_0(M) \xrightarrow{\pi_0} C_1(M) \xrightarrow{\pi_1} \ldots \xrightarrow{\pi_{m-2}} C_{m-1}(M) \xrightarrow{\pi_{m-1}} X \longrightarrow 0.$$

In particular, as elements of $\mathcal{R}(\mathbb{H})$*, we have*

$$[\ker \pi_0] = \sum_{i=0}^{m} (-1)^i [C_i(M)] = \sum_{J \subseteq \Pi} (-1)^{|J|} [\operatorname{Ind}_J \operatorname{Res}_J M].$$

Let M be an $\mathbb{H}$-module and let

$$\begin{aligned}\chi\colon M &\to \mathbb{H} \otimes_{\mathbb{A}} (\operatorname{Res}_{\mathbb{A}} M),\\ x &\mapsto \sum_{w\in W} (-1)^{\ell(w)} t_w \otimes t_w^{-1} \cdot x.\end{aligned}$$

Clearly, χ is injective.

Lemma 2.0.8 ([Cha16, Lemma 6.3])**.** *We have an isomorphism of* $\mathbb{H}$*-modules*

$$\operatorname{im}\chi \cong \mathbb{IM}(M).$$

Proof. The proof of [Cha16, Lemma 6.3] already gives us

$$\begin{aligned}t_s \cdot \chi(x) &= \chi(\mathbb{IM}(t_s) \cdot x),\\ v \cdot \chi(x) &= \chi(\mathbb{IM}(v) \cdot x).\end{aligned}$$

Furthermore, it is easy to see that

$$\begin{aligned}r \cdot \chi(x) &= \chi(r \cdot x) = \chi(\mathbb{IM}(r) \cdot x),\\ E_\sigma \cdot \chi(x) &= \chi(E_\sigma \cdot x) = \chi(\mathbb{IM}(E_\sigma) \cdot x),\end{aligned}$$

and thus we have now proved that $\operatorname{im}\chi \cong \mathbb{IM}(M)$. □

Lemma 2.0.9 ([Cha16, Lemma 6.4], cf. [Kat93, Theorem 1, Lemma 1])**.** *We have* $\ker \pi_0 = \operatorname{im}\chi$*.*

Proof. We have $\ker\pi_0 = \bigcap_{s\in S} L_s$ where $L_s = \{ht_s \otimes x - h \otimes t_s x \colon h \in \mathbb{H}, x \in M\} \subseteq C_0(M) = \mathbb{H} \otimes_{\mathbb{A}} M$. Note that $M^W = \mathbb{A}^W \otimes_{\mathbb{A}} M$ and consider the $\mathbb{A}$-linear map

$$\phi\colon M^W \to \mathbb{H} \otimes_{\mathbb{A}} M \colon (m_w)_{w\in W} \mapsto \sum_{w\in W} t_w \otimes t_w^{-1} m_w.$$

For $s \in S$, let

$$\mathbb{A}_s^W = \{(x_w)_{w\in W} \in \mathbb{A}^W \colon x_{ws} = -x_w \text{ for all } w \in W\}.$$

Note that

$$\bigcap_{s\in S} \phi(\mathbb{A}_s^W \otimes_{\mathbb{A}} M) = \operatorname{im}\chi. \tag{2.0.1}$$

We will show that $L_s = \phi(\mathbb{A}_s^W \otimes_{\mathbb{A}} M)$. Clearly, we have $L_s \supseteq \phi(\mathbb{A}_s^W \otimes_{\mathbb{A}} M)$. As an $\mathbb{A}$-algebra, L_s is generated by elements of the form $t_y \otimes x - t_{ys} \otimes t_s x$ with $y \in W$, $x \in M$. Let $-x_{ys} := x_y := t_y x$. For $w \in W \setminus \{y, ys\}$, let $x_w = 0$. Then

$$t_y \otimes x - t_{ys} \otimes t_s x = t_y \otimes t_y^{-1} x_y + t_{ys} \otimes t_{ys}^{-1} x_{ys} = \phi((x_w)_{w\in W}) \in \phi(\mathbb{A}_s^W \otimes_{\mathbb{A}} M).$$

Thus we have $L_s \subseteq \phi(\mathbb{A}_s^W \otimes_{\mathbb{A}} M)$, hence equality.

By the above and (2.0.1), we have

$$\ker \pi_0 = \bigcap_{s\in S} L_s = \bigcap_{s\in S} \phi(\mathbb{A}_s^W \otimes_{\mathbb{A}} M) = \operatorname{im}\chi. \qquad \square$$

Proof of Theorem 2.0.6. By Proposition 2.0.7, Lemma 2.0.8, and Lemma 2.0.9, we have

$$[\mathbb{IM}(M)] = [\operatorname{im}\chi] = [\ker \pi_0] = \sum_{i=0}^{m} (-1)^{i+1} [\operatorname{Ind}_J \operatorname{Res}_J M]. \qquad \square$$

2.0.4 A correspondence between AZ, IM, and $\mathbb{IM}$

Let $s \in T$ and let σ be its compact part and s_r its real part. Let χ, $\bar{\chi}$, $\bar{\chi}'$, be central characters of $\mathcal{H}$, $\mathbb{H}_\Sigma$, $\mathbb{H}'_\sigma$, respectively, such that χ corresponds to $(W\cdot s, v_0)$ and $\bar{\chi}$ and $\bar{\chi}'$ correspond to $(W(\sigma)\cdot \log(s_r), r_0) \in (W(\sigma)\backslash \mathfrak{t}) \times \mathbb{C}$ where $e^{r_0} = v_0$. Let $\Omega'\colon \operatorname{Irr}(\mathcal{H}_\chi) \to \operatorname{Irr}(\mathbb{H}_{\bar{\chi}})$ be the map induced from the composition of Ω in Corollary 1.3.15(1) with the isomorphism $M_n(\Gamma_\sigma \ltimes \mathbb{H}_{\sigma,\bar{\chi}}) \cong \Gamma_\sigma \ltimes \mathbb{H}_{\sigma,\bar{\chi}}$ coming from Morita equivalence.

Theorem 2.0.10. *We have a commutative diagram*

$$\begin{array}{ccc} \operatorname{Irr}(\mathcal{H}_\chi) & \xrightarrow{\mathsf{IM}} & \operatorname{Irr}(\mathcal{H}_\chi) \\ \downarrow{\scriptstyle \Psi'_\chi} & & \downarrow{\scriptstyle \Psi'_\chi} \\ \operatorname{Irr}(\mathbb{H}_{\bar{\chi}}) & \xrightarrow{\mathbb{IM}} & \operatorname{Irr}(\mathbb{H}_{\bar{\chi}}). \end{array}$$

Proof. We can write $W\cdot s$ as a disjoint union of W_J-orbits $W\cdot s = \bigsqcup_{i=1}^m W_J \cdot s^i$. For $i = 1,\dots,m$, let σ^i be the compact part of s^i and s_r^i, let χ_i be the central character of $\mathcal{H}_J$ corresponding to $(W_J\cdot s^i, v_0)$, and let $\bar{\chi}_i$ be the central character of $\mathbb{H}_J$ corresponding to $(W(\sigma^i)\cdot\log(s_r^i), r_0)$.

Let M be an irreducible representation of $\mathcal{H}$ with central character χ. Then M is a representation of $\mathcal{H}/I_{W\cdot s,v_0}\mathcal{H}$. The representation $\operatorname{Res}_J(M)$ is annihilated by $I_{W\cdot s,v_0}$, so it is also a representation of $\mathcal{H}_J/I_{W\cdot s,v_0}\mathcal{H}_J$. Similarly, $\Psi'_\chi(M)$ is an irreducible representation of $\mathbb{H}$ with central character $\bar{\chi}$, hence a representation of $\mathbb{H}_\Sigma/\mathbb{I}_{W(\sigma)\cdot\log(s_r),r_0}\mathbb{H}_\Sigma$, and $\operatorname{Res}_J(\Psi'_\chi(M))$ is a representation of $\mathbb{H}_{J,\Sigma}/\mathbb{I}_{W(\sigma)\cdot\log(s_r),r_0}\mathbb{H}_{J,\Sigma}$. Write

$$\Psi'_J\colon \mathcal{H}_J/I_{W\cdot s,v_0}\mathcal{H}_J \to \mathbb{H}_{J,\Sigma}/\mathbb{I}_{W(\sigma)\cdot\log(s_r),r_0}\mathbb{H}_{J,\Sigma}$$

for the map in Proposition 1.3.16 for $\mathcal{H}_J$ and $\mathbb{H}_J$ instead of $\mathcal{H}$ and $\mathbb{H}$. Consider the diagram

$$\begin{array}{ccc} \mathcal{R}(\mathcal{H}\,/I_{W\cdot s,v_0}\,\mathcal{H}) & \xrightarrow{\mathrm{Res}_J} & \mathcal{R}(\mathcal{H}_J\,/I_{W\cdot s,v_0}\,\mathcal{H}_J) \\ \downarrow{\Psi'_\chi} & & \downarrow{\Psi'_J} \\ \mathcal{R}(\mathbb{H}\,/\mathbb{I}_{W(\sigma)\cdot\log(s_r),r_0}\,\mathbb{H}) & \xrightarrow{\mathrm{Res}_J} & \mathcal{R}(\mathbb{H}_{J,\Sigma}\,/\mathbb{I}_{W(\sigma)\cdot\log(s_r),r_0}\,\mathbb{H}_{J,\Sigma}), \end{array}$$

It is easy to check that this diagram commutes by checking that the action of the generators (in the sense of Definition 1.3.9) of $\mathbb{H}_{J,\Sigma}$ on $\Psi'_J \circ \mathrm{Res}_J(M)$ is the same as on $\mathrm{Res}_J \circ \Psi'_\chi(M)$. From (1.3.4), we see that each irreducible representation of $\mathcal{H}\,/I_{W\cdot s,v_0}\,\mathcal{H}$ has central character corresponding to $(W_J \cdot s^i, v_0) = (\mathcal{S}_i, v_0)$ for some $i \in \{1,\ldots,m\}$, which we denote by χ_i. Thus we obtain an isomorphism of abelian groups

$$\mathcal{R}(\mathcal{H}_J\,/I_{W\cdot s,v_0}\,\mathcal{H}_J)\tilde{\to}\bigoplus_{i=1}^{m}\mathcal{R}(\mathcal{H}_{J,\chi_i}),$$

such that each irreducible representation of $\mathcal{H}_J\,/I_{W\cdot s,v_0}\,\mathcal{H}_J$ is mapped to the appropriate direct summand according to its central character. Similarly, by (1.3.5), irreducible representations of $\mathbb{H}_{J,\Sigma}\,/\mathbb{I}_{W(\sigma)\cdot\log(s_r),r_0}\,\mathbb{H}_{J,\Sigma}$ have central characters corresponding to $(W(\sigma^i)\cdot\log(s_r^i), r_0)$ for some $i \in \{1,\ldots,m\}$, which we denote by $\bar{\chi}_i$. We similarly obtain an isomorphism of abelian groups

$$\mathcal{R}(\mathbb{H}\,/\mathbb{I}_{W(\sigma)\cdot\log(s_r),r_0}\,\mathbb{H})\tilde{\to}\bigoplus_{i=1}^{m}\mathcal{R}(\mathbb{H}_{J,\bar{\chi}_i}).$$

For $i = 1,\ldots,m$, let Ψ'_i be the map as in (1.3.6) for $\mathcal{H}_J$ and $\mathbb{H}_J$ instead of $\mathcal{H}$ and $\mathbb{H}$. By [BM93, Theorem 6.2], we have a well-defined commutative diagram

$$\begin{array}{ccc} \mathcal{R}(\mathcal{H}_{J,\chi_i}) = \mathcal{R}(\mathcal{H}_J\,/I_{W_J\cdot s^i,v_0}\,\mathcal{H}_J) & \xrightarrow{\mathrm{Ind}_J} & \mathcal{R}(\mathcal{H}\,/I_{W\cdot s,v_0}\,\mathcal{H}) = \mathcal{R}(\mathcal{H}_\chi) \\ \downarrow{\Psi'_i} & & \downarrow{\Psi'_\chi} \\ \mathcal{R}(\mathbb{H}_{J,\bar{\chi}_i}) = \mathcal{R}(\mathbb{H}_{J,\Sigma_i}\,/\mathbb{I}_{W(\sigma^i)\cdot\log(s_r^i),r_0}\,\mathbb{H}_{J,\Sigma_i}) & \xrightarrow{\mathrm{Ind}_J} & \mathcal{R}(\mathbb{H}_\Sigma\,/\mathbb{I}_{W(\sigma)\cdot\log(s_r),r_0}\,\mathbb{H}_\Sigma) = \mathcal{R}(\mathbb{H}_{\bar{\chi}}). \end{array}$$

We noted in the proof of Proposition 1.3.16 that Ψ'_J is the same as the sum of the Ψ'_i, and so we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{R}(\mathcal{H}_J\,/I_{W\cdot s,v_0}\,\mathcal{H}_J) & \xrightarrow{\mathrm{Ind}_J} & \mathcal{R}(\mathcal{H}_\chi) \\ \downarrow{\Psi'_J} & & \downarrow{\Psi'_\chi} \\ \mathcal{R}(\mathbb{H}_{J,\Sigma}\,/\mathbb{I}_{W(\sigma)\cdot\log(s_r),r_0}\,\mathbb{H}_{J,\Sigma}) & \xrightarrow{\mathrm{Ind}_J} & \mathcal{R}(\mathbb{H}_{\bar{\chi}}). \end{array}$$

The result follows from Theorem 2.0.3, Theorem 2.0.6. □

Remark 2.0.11. The proof of Theorem 2.0.10 is a result of joint work with Jonas Antor and Emile Okada, who noticed an error in the proof in a previous version of the thesis.

We can also consider the involution $\mathbb{IM}$ defined on $\mathbb{H}_\sigma$ and extend it to an involution $\mathbb{IM}$ on $\mathbb{H}'_\sigma$. Consider the bijection (from Morita equivalence)

$$\begin{aligned} F\colon \operatorname{Irr}(\mathbb{H}_{\bar{\chi}}) &\to \operatorname{Irr}(\mathbb{H}'_{\sigma,\bar{\chi}'}) \\ M &\mapsto E_{1,1}\cdot M = E_\sigma \cdot M \end{aligned}$$

and consider the composition $\Omega = F \circ \Psi'_\chi$.

Corollary 2.0.12. *We have a commutative diagram*

$$\begin{array}{ccc} \operatorname{Irr}(\mathcal{H}_\chi) & \xrightarrow{\mathsf{IM}} & \operatorname{Irr}(\mathcal{H}_\chi) \\ \downarrow{\scriptstyle\Omega} & & \downarrow{\scriptstyle\Omega} \\ \operatorname{Irr}(\mathbb{H}'_{\sigma,\bar{\chi}}) & \xrightarrow{\mathbb{IM}} & \operatorname{Irr}(\mathbb{H}'_{\sigma,\bar{\chi}}). \end{array}$$

Proof. By Theorem 2.0.10, it suffices to show that the following diagram commutes

$$\begin{array}{ccc} \operatorname{Irr}(\mathbb{H}_{\bar{\chi}}) & \xrightarrow{\mathbb{IM}} & \operatorname{Irr}(\mathbb{H}_{\bar{\chi}}) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ \operatorname{Irr}(\mathbb{H}'_{\sigma,\bar{\chi}'}) & \xrightarrow{\mathbb{IM}} & \operatorname{Irr}(\mathbb{H}'_{\sigma,\bar{\chi}'}), \end{array}$$

Let $M \in \operatorname{Irr}(\mathbb{H}_{\bar{\chi}})$ and let $m \in M$, $w \in W(\sigma)$, and $\omega \in \mathscr{S}$. Note that $w(\sigma) = \sigma$ for such w. Since $\mathbb{IM}(E_\sigma) = E_\sigma$, the underlying vector space of $F \circ \mathbb{IM}(M)$ and $\mathbb{IM} \circ F(M)$ are both equal to $E_\sigma \cdot M$, and it is easy to see from that t_w and ω acts on $F \circ \mathbb{IM}(M)$ in the same way as on $\mathbb{IM} \circ F(M)$. □

Finally, we will only briefly mention the relation between AZ on $\mathbf{G}(\mathsf{k})$ to IM on the geometric Hecke algebras: if $\bar{X}$ is an irreducible representation of $\mathbf{G}(\mathsf{k})$ corresponding to the irreducible representation π of the relevant geometric Hecke algebra, then we have $\mathsf{AZ}(\bar{X}) = \mathsf{IM}(\pi)$. The proof for this statement will be given in a forthcoming paper of the author together with Emile Okada and Jonas Antor.

Chapter 3

Maximal representations in cohomology of generalised Springer fibres

The contents of this chapter form the majority of [La22].

3.1 Green functions

We recall some notions regarding certain multiplicites from [Lus86a]. Let G be a complex connected reductive group and let $(C, \mathcal{E}), (C', \mathcal{E}') \in \mathcal{N}_{\mathrm{G}}$. Consider the intersection cohomology complex $\mathrm{IC}(\bar{C}', \mathcal{E}')$ on $\bar{C}'$ associated to $(C', \mathcal{E}')$. For each $m \in \mathbb{N}$, the restriction of $H^{2m}(\mathrm{IC}(\bar{C}', \mathcal{E}'))$ to C is a direct sum of irreducible G-equivariant local systems on C. Consider the polynomial $P_{(C',\mathcal{E}';C,\mathcal{E})}(t) \in \mathbb{Z}[t]$ whose m-th coefficient is the multiplicity of $\mathcal{E}$ in $H^{2m}(\mathrm{IC}(\bar{C}', \mathcal{E}'))|_C$. Let $\mathrm{mult}(C, \mathcal{E}; C', \mathcal{E}') = P_{(C',\mathcal{E}';C,\mathcal{E})}(1)$, i.e. the multiplicity of $\mathcal{E}$ in $\bigoplus_{m\in\mathbb{Z}} H^{2m}(\mathrm{IC}(\bar{C}', \mathcal{E}'))|_C$. The Lusztig-Shoji algorithm [Lus86a, Theorem 24.8] describes a linear algebraic method to determine $P_{(C',\mathcal{E}';C,\mathcal{E})}(t)$. It furthermore holds that $P_{(C',\mathcal{E}';C,\mathcal{E})}(t) = 0$ if $(C, \mathcal{E})$ and $(C', \mathcal{E}')$ have different cuspidal supports (i.e. there are certain orthogonality relations). Let $\rho = \mathrm{GSpr}(C, \mathcal{E})$ and $\rho' = \mathrm{GSpr}(C', \mathcal{E}')$. We also write $P_{(C',\mathcal{E}';C,\mathcal{E})}(t) = P_{\rho',\rho}(t)$.

Despite the existence of an algorithm to compute $P_{(C',\mathcal{E}';C,\mathcal{E})}(t)$, it is still often difficult to determine some of its properties. Particularly, we are interested in determining whether there exists a unique $(\mathrm{C}^{\max}, \mathcal{E}^{\max}) \in \mathcal{N}_{\mathrm{G}}$ such that $P_{(\mathrm{C}^{\max},\mathcal{E}^{\max};C,\mathcal{E})} = 1$ and such that for any other $(C', \mathcal{E}') \in \mathcal{N}_{\mathrm{G}}$ with $P_{(C',\mathcal{E}';C,\mathcal{E})}$, we have either $\mathrm{C}^{\max} > C'$ or $(C', \mathcal{E}') = (\mathrm{C}^{\max}, \mathcal{E}^{\max})$. We prove the existence of such a $(\mathrm{C}^{\max}, \mathcal{E}^{\max})$ in this chapter for $\mathrm{SO}(N)$. The result for $\mathrm{Sp}(2n)$ was proved in [Wal19, §4] and we shall only state the result without proof at the end of the chapter in Section 3.4. They key idea in the proof will be to use the fact that the polynomials $P_{(C',\mathcal{E}';C,\mathcal{E})}(t)$ are very closely related to so-called *Green functions* (in fact they are the same up to normalisation of the powers of t), which for $\mathrm{SO}(N)$ (as well as for instance $\mathrm{Sp}(2n)$, $\mathrm{SL}(N)$, $\mathrm{PGL}(N)$) have combinatorial descriptions.

3.1.1 Combinatorial results

We describe certain combinatorial results from [Sho01] and [Wal19] that will be used to obtain certain results about Green functions for $\mathrm{SO}(N,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$, but we shall mostly only focus on the $\mathrm{SO}(N,\mathbb{C})$ case; the $\mathrm{Sp}(2n,\mathbb{C})$ case is fully described in [Wal19].

Let $\alpha,\beta \in \mathcal{P}$ and consider integers $m_0 \geq t(\alpha)$ and $n_1 \geq t(\beta)$. Let $I := I_{m_0,m_1} := \{(i,0),(j,1)\colon i \in \{1,\dots,m_0\}, j \in \{1,\dots,m_1\}\}$, which we call the *set of indices of* (α,β). Consider a total order $<$ on I such that for all $e \in \{0,1\}$ and $i,j \in \{1,\dots,m_e\}$, we have $(i,e) < (j,e)$ if and only if $i < j$. We call the triple $(m_0,m_1,<)$ an *order on the set of indices of* (α,β). Any other order $(m_0',m_1',<')$ is said to be equivalent to $(m_0,m_1,<)$ if for each $i \in \mathbb{N}$ such that $\alpha_i > 0$, we have $\{j \in \{1,\dots,m_1\}\colon (j,1) < (i,0)\} = \{j \in \{1,\dots,m_1'\}\colon (j,1) <' (i,0)\}$ and for each $j \in \mathbb{N}$ such that $\beta_j > 0$, we have $\{i \in \{1,\dots,m_0\}\colon (i,0) < (j,1)\} = \{i \in \{1,\dots,m_0'\}\colon (i,0) <' (j,1)\}$. Often, we simply write $<$ for the triple $(m_0,m_1,<)$, since the m_0 and m_1 were required to define $I = I_{m_0,m_1}$. We also often write the equivalence class of $(m_0,m_1,<)$ by just $<$, especially in situations where m_0 and m_1 are not important. Any equivalence class of orders has a 'minimal' order $(m_0^<,m_1^<,<)$ in the sense that $m_0^< \leq m_0$ and $m_1^< \leq m_1$ for all $(m_0,m_1,<)$ in the equivalence class.

Let $X \subseteq \mathbb{N}$ be finite. Let $\phi_X\colon \mathbb{N} \to \mathbb{N}\backslash X$ be the unique increasing bijection. Define $\alpha' \in \mathcal{P}$ by setting $\alpha_i' = \alpha_{\phi_X(i)}$ for all $i \in \mathbb{N}$. For a finite set $Y \subseteq N$, we define ϕ_Y and $\beta' \in \mathcal{P}$ similarly. Given an order $(m_0,m_1,<)$ on the set of indices of (α,β), we uniquely obtain an order $(m_0',m_1',<')$ on the set of indices of (α',β') such that $(i,0) <' (j,1)$ if and only if $(\phi_X(i),0) < (\phi_Y(j),1)$. Let $(\tilde{m}_0,\tilde{m}_1,\tilde{<})$ be another order on the set of indices of (α,β) equivalent to $<$. It is shown in [Wal19, §1] that $(\tilde{m}_0',\tilde{m}_1',\tilde{<}')$ is equivalent to $<'$.

Fix an equivalence class $<$ of orders on the set of indices of (α,β). We shall describe two procedures (a), resp. (b). For procedure (a), fix a representative $(m_0,m_1,<)$ such that $m_0 \geq 1$. Let $a_1 = 1$. Suppose a_i is defined for some $i \in \mathbb{N}$. If $B_i := \{j \in \{1,\dots,m_1\}\colon (a_i,0) < (j,1)\}$ is non-empty, let $b_i = \min B_i$. For $i \in \mathbb{N}_{\geq 2}$, if $A_i := \{j \in \{1,\dots,m_0\}\colon (b_{i-1},1) < (j,0)\}$ is non-empty, let $a_i = \min A_i$. This defines integers $a_1,\dots,a_p,b_1,\dots,b_q \in \mathbb{N}$ for some $p,q \in \mathbb{N}$. Let $\mu_1 = \alpha_{a_1} + \beta_{b_1} + \alpha_{a_2} + \beta_{b_2} + \dots$. Let $(\alpha',\beta') \in \mathcal{P} \times \mathcal{P}$ as above for $X = \{a_1,\dots,a_p\}$ and $Y = \{b_1,\dots,b_q\}$. For procedure (b), fix a representative $(m_0,m_1,<)$ such that $m \geq 1$. We define $b_1 = 1$, and we recursively define $a_1,b_2,a_2,b_3,\dots$ similarly as in procedure (a) to obtain integers $a_1,\dots,a_p,b_1,\dots,b_q \in \mathbb{N}$ for some $p,q \in \mathbb{N}$, and a pair of partitions (α',β'). We also define $\nu_1 = \beta_{b_1} + \alpha_{a_1} + \dots$.

It is shown in [Wal19, §1] that each of the objects defined in both procedures are independent of the choice of representative $(m_0, m_1, <)$.

3.1.2 The sets $P(\alpha, \beta, <)$ and $P_{A,B;s}(\alpha, \beta, <)$

Let $(\alpha, \beta) \in \mathcal{P} \times \mathcal{P}$ and consider an order $<$ on the set of its indices. We will define a set $P(\alpha, \beta, <)$ by induction on $m_0^< + m_1^<$. First, define $P(\varnothing, \varnothing, <) = \{(\varnothing, \varnothing)\}$. Suppose that $m_0^< + m_1^< > 0$. If $m_0^< > 0$ (resp. $m_1^< > 0$) we obtain $\mu_1, (\alpha', \beta'), <'$ (resp. $\nu_1, (\alpha'', \beta''), <''$) from procedure (a) (resp. (b)). By induction, $P(\alpha', \beta', <')$ and $P(\alpha'', \beta'', <'')$ have been defined. We define $P(\alpha, \beta, <) = P^a(\alpha, \beta, <) \cup P^b(\alpha, \beta, <)$, where

$$P^a(\alpha, \beta, <) = \begin{cases} \varnothing & \text{if } m_0^< = 0, \\ \{((\mu_1) \sqcup \mu', \nu') \colon (\mu', \nu') \in P(\alpha', \beta', <')\} & \text{if } m_0^< \neq 0, \end{cases}$$

$$P^b(\alpha, \beta, <) = \begin{cases} \varnothing & \text{if } m_1^< = 0, \\ \{(\mu'', (\nu_1) \sqcup \nu'') \colon (\mu'', \nu'') \in P(\alpha'', \beta'', <'')\} & \text{if } m_1 \neq 0. \end{cases}$$

Let $A, B \in \mathbb{R}$, $s \in \mathbb{R}_{>0}$. We define a subset $P_{A,B;s}(\alpha, \beta, <)$ of $P(\alpha, \beta, <)$ by recursion on $m_0^< + m_0^<$. Let $P_{A,B;s}(\varnothing, \varnothing, <) = \{(\varnothing, \varnothing)\}$. Suppose $m_0^< + m_1^< > 0$. If the conditions

1. $m_0^< > 0$;

2. $m_1^< = 0$, or $(1,0) < (1,1)$ and $\alpha_1 + A \geq B$, or $(1,1) < (1,0)$ and $\beta_1 + B \leq A$.

are satisfied, consider $\mu_1, (\alpha', \beta'), <'$ from procedure (a) and let

$$P^a_{A,B;s}(\alpha, \beta, <) = \{((\mu_1) \sqcup \mu', \nu') \colon (\mu', \nu') \in P_{A-s,B;s}(\alpha', \beta', <')\}.$$

If neither conditions hold, we set $P^a_{A,B;s}(\alpha, \beta, <) = \varnothing$. Similarly, if the conditions

1. $m_1^< > 0$;

2. $m_0^< = 0$, or $(1,0) < (1,1)$ and $\alpha_1 + A \leq B$, or $(1,1) < (1,0)$ and $\beta_1 + B \geq A$.

are satisfied, consider $\nu_1, (\alpha'', \beta''), <''$ from procedure (b) and let

$$P^b_{A,B;s}(\alpha, \beta, <) = \{(\mu'', (\nu_1) \sqcup \nu'') \colon (\mu'', \nu'') \in P_{A,B-s;s}(\alpha'', \beta'', <'')\}.$$

If neither conditions hold, we set $P^b_{A,B;s}(\alpha, \beta, <) = \varnothing$. Finally, let $P_{A,B;s}(\alpha, \beta, <) = P^a_{A,B;s}(\alpha, \beta, <) \cup P^b_{A,B;s}(\alpha, \beta, <)$.

Proposition 3.1.1. *For any $C \in \mathbb{R}$, we have $P_{A+C,B+C;s}(\alpha, \beta, <) = P_{A,B;s}(\alpha, \beta, <)$.*

For $C \in \mathbb{R}$ and $\mu, \nu \in \mathcal{P}$, define

$$[C, -\infty[_s = (C, C-s, C-2s, \dots) \in \mathcal{R},$$
$$\Lambda_{A,B;s}(\mu, \nu) = (\mu + [A, -\infty[_s) \sqcup (\nu + [B, -\infty[_s) \in \mathcal{R}\,.$$

Lemma 3.1.2. *Let* $(\mu,\nu) \in P(\alpha,\beta,<)$ *and* $(\boldsymbol{\mu},\boldsymbol{\nu}) \in P_{A,B;s}(\alpha,\beta,<)$. *Then*

(a) $\Lambda_{A,B;s}(\mu,\nu) \leq \Lambda_{A,B;s}(\boldsymbol{\mu},\boldsymbol{\nu})$,

(b) $\Lambda_{A,B;s}(\mu,\nu) = \Lambda_{A,B;s}(\boldsymbol{\mu},\boldsymbol{\nu})$ *if and only if* $(\mu,\nu) \in P_{A,B;s}(\alpha,\beta,<)$.

Definition 3.1.3. Let $(\mu,\nu) \in P_{A,B;s}(\alpha,\beta,<)$. By Lemma 3.1.2 we can define

$$p_{A,B;s}(\alpha,\beta,<) := \Lambda_{A,B;s}(\mu,\nu).$$

3.1.3 Some results for $P_{k,-k;2}(\alpha,\beta,<_{\alpha,\beta,k})$

Let $n \in \mathbb{N}$. A pair $(\alpha,\beta) \in \mathcal{P}\times\mathcal{P}$ is called a *bipartition* of n if $S(\alpha)+S(\beta)=n$. Denote by $\mathcal{P}_2(n)$ the set of bipartitions of n. Let $(\alpha,\beta) \in \mathcal{P}_2(n)$, $k \in \mathbb{Z}$ and write $\Lambda = \Lambda_{k,-k;2}(\alpha,\beta)$. Let $m_0 \in \mathbb{N}$ be the smallest integer such that $m_0 \geq t(\alpha)$ and $m_0 - k \geq t(\beta)$ and let $m_1 = m_0 - k$. Let $r = m_0 + m_1$. Consider the condition

1. $\Lambda^r := (\Lambda_1,\ldots,\Lambda_r) \in \mathcal{R}_f$ has 'no multiplicities', i.e. each term of Λ^r has multiplicity 1.

Let $H(n,k)$ be the set of $(\alpha,\beta) \in \mathcal{P}_2(n)$ such that $\Lambda_{k,-k;2}(\alpha,\beta)$ satisfies condition 1 above and let $H_S(n,k) = \{\Lambda_{k,-k;2}(\alpha,\beta) \in \mathcal{P}_2(n) \colon (\alpha,\beta) \in H(n,k)\}$.

Remark 3.1.4. Note that $m_0 = \frac{r+k}{2}$ and $m_1 = \frac{r-k}{2}$ and

$$\begin{aligned}
\alpha_{m_0} + k + 2 - 2m_0 &= \alpha_{\frac{r+k}{2}} + 2 - r > -r,\\
\beta_{m_1} - k + 2 - 2m_1 &= \beta_{\frac{r-k}{2}} + 2 - r > -r,\\
\alpha_{m_0+1} + k + 2 - 2(m_0+1) &= -r,\\
\beta_{m_1+1} - k + 2 - 2(m_1+1) &= -r,
\end{aligned}$$

so Λ^r consists of the first m_0 terms of $\alpha + [k,-\infty[_2$ and the first m_1 terms of $\beta + [-k,-\infty[_2$.

If $(\alpha,\beta) \in H(n,k)$, there exists a unique order $<_{\alpha,\beta,k}$ on the indices of (α,β) such that for $i < m_0$, $j < m_1$, we have $(i,0) <_{\alpha,\beta,k} (j,1)$ if and only if $\alpha_i + k + 2 - 2i > \beta_j - k + 2 - 2j$.

Lemma 3.1.5. *Let* $k \in \mathbb{Z}$, $(\alpha,\beta) \in H(n,k)$ *and consider the order* $<_{\alpha,\beta,k}$. *Let* (α',β') *be a bipartition obtained from* (α,β) *via procedure (a) or (b) and let* $<'$ *be the induced ordering on the indices of* (α',β'). *Then* $P_{k,-k;2}(\alpha,\beta,<_{\alpha,\beta,k})$ *has a unique element* (μ,ν),

Proof. Let m_0 and $m_1 = m_0 - k$ be as above and consider the representative $(m_0, m_1, <)$ of the order $<_{\alpha,\beta,k}$. We will prove the result by induction on $m_0^< + m_1^<$. Clearly, the result holds if $m_0^< = 0$ or $m_1^< = 0$. Suppose $m_0^< > 0$ and $m_1^< > 0$. We may assume that $(1,0) < (1,1)$, otherwise we consider $(\beta,\alpha,-k)$. From procedure (a) we obtain a bipartition $(\alpha', \beta') \in \mathcal{P}_2(n')$ for some $n' \in \mathbb{N}$. Since $(\alpha, \beta) \in H(n, k)$, we have $\alpha_1 + k > \beta_1 - k > -k$, so $P_{k,-k;2}(\alpha, \beta, <) = P^a_{k,-k;2}(\alpha, \beta, <)$ is in bijection with $P_{k-2,-k;2}(\alpha', \beta', <') = P_{k-1,-k+1;2}(\alpha', \beta', <')$. Hence it suffices to show that $(\alpha', \beta') \in H(n', k-1)$ and that $<'$ is equivalent to $<_{\alpha',\beta',k-1}$. By the induction hypothesis, $P_{k-1,-k+1;2}(\alpha', \beta', <')$ has a unique element (μ, ν). For Lemma 3.1.7, we will also prove that $<'$ is equivalent to $<_{\alpha',\beta',k}$.

Let $k' \in \{k-1, k\}$. Procedure (a) applied to (α, β) also creates $\mu_1, a_1, \ldots, a_T, b_1, \ldots, b_t \in \mathbb{N}$ where $T \in \{t, t+1\}$. Let $m_0' = m_0 - T$ and $m_1' = m_1 - t$ and let $\phi\colon \{1, \ldots, m_0'\} \to \{1, \ldots, m_0\} \setminus \{a_1, \ldots, a_T\}$ and $\psi\colon \{1, \ldots, m_1'\} \to \{1, \ldots, m_1\} \setminus \{b_1, \ldots, b_t\}$ be the unique increasing bijections. For each $i \in \{1, \ldots, m_0'\}$, resp. $j \in \{1, \ldots, m_1'\}$, let $h(i) \in \{1, \ldots, T\}$, resp. $g(j) \in \{1, \ldots, t\}$ be the unique element for which $a_{h(i)} < \phi(i) < a_{h(i)+1}$, resp. $b_{g(j)} < \psi(j) < b_{g(j)+1}$. We have $\phi(i) = h(i) + i$, $\psi(j) = g(j) + j$ and the maps $i \mapsto h(i)$, $j \mapsto g(j)$ are weakly increasing.

Let $i, j \in \mathbb{N}$. To show that $(\alpha', \beta') \in H(n', k')$, we prove the following:

(a) if $i \le m_0'$, $j \le m_1'$ and $(i, 0) <' (j, 1)$, then

$$\alpha_i' + k' + 2 - 2i > \beta_j' - k' + 2 - 2j,$$

(b) if $i \le m_0'$, $j \le m_1'$ and $(j, 1) <' (i, 0)$, then

$$\beta_j' - k' + 2 - 2j > \alpha_i' + k' + 2 - 2i,$$

(c) if $i \le m_0'$ and $j > m_1'$, then

$$\alpha_i' + k' + 2 - 2i \ge -k' + 2 - 2j,$$

(d) if $i > m_0'$ and $j \le m_1'$, then

$$\beta_j' - k' + 2 - 2j \ge k' + 2 - 2i,$$

(e) The inequality in (d) is strict if $\beta_j' \ne 0$.

Assume for now that these inequalities hold. Let $\Lambda' = \Lambda_{k'-k';2}(\alpha', \beta')$ and for $m \in \mathbb{N}$, let $\Lambda'^m = (\Lambda_1', \ldots, \Lambda_m')$. Let $\boldsymbol{m}_0' \in \mathbb{N}$ be the smallest integer such that $\boldsymbol{m}_0' \ge t(\alpha')$ and $\boldsymbol{m}_1' := \boldsymbol{m}_0' - k' \ge t(\beta')$. Note that $\Lambda'^{m_0'+m_1'}$ has no multiplicities by (a) – (d), so if we have $m_0' + m_1' \ge \boldsymbol{m}_0' + \boldsymbol{m}_1'$, then $\Lambda'^{\boldsymbol{m}_0'+\boldsymbol{m}_1'}$ has no multiplicities and so $(\alpha', \beta') \in H(n', k')$.

If $T = t$, then $m_0' - m_1' = k \ge k'$. Hence $m_0' \ge t(\alpha')$ and $m_0' - k' \ge m_1' \ge t(\beta')$, so we have $m_0' \ge \boldsymbol{m}_0'$ and $m_1' \ge \boldsymbol{m}_1$, so $(\alpha', \beta') \in H(n', k')$.

Suppose $T = t + 1$. Then $m_0' - m_1' = k - 1$. If $k' = k - 1$, then similarly as above, we have $m_0' \ge \boldsymbol{m}_0'$ and $m_1' \ge \boldsymbol{m}_1'$ and so $(\alpha', \beta') \in H(n', k')$.

Suppose $T = t+1$ and $k' = k$. If $\beta_{m_1'} = 0$, then $m_0' \geq t(\alpha')$ and $m_0' - k' = m_1' - 1 \geq t(\beta')$, so $m_0 \geq \boldsymbol{m}_0'$ and $m_1' > \boldsymbol{m}_1'$, so $(\alpha', \beta') \in H(n', k')$.

Suppose that $T = t+1$, $k' = k$, and $\beta_{m_1'} \neq 0$. Then $m_0' + 1 \geq t(\alpha')$ and $(m_0' + 1) - k' = m_1' \geq t(\beta')$, so $m_0' + 1 \geq \boldsymbol{m}_0'$ and $m_1' \geq \boldsymbol{m}_1'$. Now by (a) – (e), it follows that $\Lambda'^{m_0'+m_1'+1}$ has no multiplicities and since $\boldsymbol{m}_0' + \boldsymbol{m}_1' \leq m_0' + m_1' + 1$, we then have $(\alpha', \beta') \in H(n', k')$.

We conclude that $(\alpha', \beta') \in H(n', k')$, and so we can define $<_{\alpha', \beta', k'}$. We show that $(m_0', m_1', <')$ and $(\boldsymbol{m}_0', \boldsymbol{m}_1', <_{\alpha,\beta,k'})$ are equivalent. Suppose $i \in \mathbb{N}$ such that $\alpha_i' > 0$. Suppose $j \leq m_1'$ such that $(j, 1) <' (i, 0)$. Then by (b), we have $(j,1) <_{\alpha',\beta',k'} (i, 0)$, and by Remark 3.1.4, we have $j \leq \boldsymbol{m}_1'$. Suppose $j \leq \boldsymbol{m}_1'$ and $(j,1) <_{\alpha',\beta',k'} (i,0)$. Then $\beta_j' - k' + 2 - 2j > \alpha_i' + k' + 2 - 2i$. By (c), we have $j \leq m_1'$ and by (a), we have $(j,1) <' (i,0)$. Similarly, we can show that for $j \in \mathbb{N}$ such that $\beta_j' > 0$, we have $i \leq m_0'$ and $(i,0) <' (j,1)$ if and only if $i \leq \boldsymbol{m}_0'$ and $(i,0) <_{\alpha',\beta',k'} (j,1)$. Thus $<'$ and $<_{\alpha',\beta',k'}$ are equivalent.

It remains to prove (a) - (e) Let $i \in \{1, \ldots, m_0'\}$, $j \in \{1, \ldots, m_1'\}$. Suppose that $(i,0) <' (j,1)$, i.e. $(\phi(i), 0) < (\psi(j), 1)$. Note that $(\phi(i), 0) < (b_{h(i)}, 1)$, and since $(\alpha, \beta) \in H(n,k)$, we have

$$a_{\phi(i)} + k + 2 - 2\phi(i) > \beta_{b_{h(i)}} - k + 2 - 2b_{h(i)}.$$

Since $(\phi(i), 0) < (\psi(j), 1)$, we have $b_{h(i)} \leq \psi(j)$, so $\beta_{b_{h(i)}} \geq \beta_{\psi(j)} = \beta_j'$. We have $\alpha_{\phi(i)} = \alpha_i'$, so

$$\alpha_i' + k' + 2 - 2i > \beta_j' - k' + 2 - 2j + 2X,$$

where

$$X = k' - k + j - b_{h(i)} + \phi(i) - i.$$

We have $\phi(i) - i = h(i)$. Since $b_{h(i)} \leq \psi(j)$, we have $h(i) \leq g(j)$. Furthermore, $q \mapsto b_q$ is strictly increasing, so $q \mapsto q - b_q$ is weakly decreasing, so $g(j) - b_{g(j)} \leq h(i) - b_{h(i)}$. Hence we have

$$X = k' - k + j + h(i) - b_{h(i)} \geq -1 + j + g(j) - b_{g(j)} = -1 + \psi(j) - b_{g(j)} \geq 0,$$

where the last inequality follows from the fact that $\psi(j) > b_{g(j)}$. Thus we have proved (a).

Suppose that $(j,1) <' (i,0)$, i.e. $(\psi(j), 1) < (\phi(i), 0)$. We have $(b_{g(j)}, 1) < (a_{g(j)+1}, 0)$, and since $(\alpha, \beta) \in H(n,k)$, we have

$$\beta_{\psi(j)} - k + 2 - 2\psi(j) > \alpha_{a_{g(j)+1}} + k + 2 - 2a_{g(j)+1}.$$

Since $(\psi(j), 1) < (\phi(i), 0)$, we have $a_{g(j)+1} < \phi(i)$, so $h(i) \geq g(j) + 1$ and so

$$\beta_j' - k + 2 - 2j > \alpha_i' + k' + 2 - 2i + 2Y,$$

where

$$Y = k - k' + i - a_{g(j)+1} + \psi(j) - j = k - k' + i - a_{g(j)+1} + g(j).$$

Note that $k - k' \geq 0$. Similarly as before, the map $p \mapsto p - a_p$ is weakly decreasing, so $(g(j)+1) - a_{g(j)+1} \geq h(i) - a_{h(i)}$, and so

$$g(j) - a_{g(j)+1} = (g(j)+1) - a_{g(j)+1} - 1 \geq h(i) - a_{h(i)} - 1 = \phi(i) - i - a_{h(i)} - 1 \geq -i,$$

where the last inequality holds since $\phi(i) > a_{h(i)}$. Thus we have $Y \geq 0 + i - i \geq 0$ and (b) follows.

Let $i \leq m'_0$ and $j > m'_1$. Note that $m'_0 - m'_1 = k$ if $T = t$ and $m'_0 - m'_1 = k - 1$ if $T = t+1$, so $m'_0 - m'_1 \leq k' + 1$. Thus

$$2(k' + j - i) \geq 2(k' + m'_1 + 1 - m'_0) \geq 2(k' - k') \geq 0.$$

Thus $k' + 2 - 2i \geq -k' + 2 - 2j$, and (c) follows. Similarly, we can show that $k' + 2 - 2j \geq k' + 2 - 2i$, and so (d) and (e) follow. □

We state a consequence of [Wal19, (7), p.387] without proof.

Proposition 3.1.6. *Let $k \in \mathbb{Z}$ and let $(\alpha, \beta) \in H(n,k)$ and let $(\mu, \nu) \in P_{k,-k;2}(\alpha, \beta, <_{\alpha,\beta,k})$. Then the largest term of $\Lambda_{k,-k;2}(\mu, \nu)$ is strictly larger than the other terms of $\Lambda_{k,-k;2}(\mu, \nu)$.*

We can repeatedly use Proposition 3.1.6 to show that $(\mu, \nu) \in H(n,k)$. However, this is rather heavy on the notation, and it will later follow that $(\mu, \nu) \in H(n,k)$ from Theorem 3.3.1, so it is not necessary to prove this now.

Lemma 3.1.7. *Let $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $(\alpha, \beta) \in H(n,k)$. Write $<$ for $<_{\alpha,\beta,k}$. We have*

$$P_{k/2,-k/2;1/2}(\alpha, \beta, <) = P_{k,-k;2}(\alpha, \beta, <).$$

Proof. Suppose there exist $h, k \in \mathbb{Z}$ such that $(\alpha, \beta) \in H(n,k)$ and $(\alpha, \beta) \in H(n,h)$ and such that $<_{\alpha,\beta,h}$ and $<_{\alpha,\beta,k}$ are equivalent. Denote $<_{\alpha,\beta,k}$ by $<$. We shall show that

$$P_{h,-h;2}(\alpha, \beta, <) = P_{h/2,-k/2;1/2}(\alpha, \beta, <) = P_{k,-k;2}(\alpha, \beta, <). \tag{3.1.1}$$

This obviously holds if $m_0^< = 0$ or $m_1^< = 0$, so we assume that $m_0^< > 0$ and $m_1^< > 0$. We may assume that $(1,0) < (1,1)$, otherwise we consider $(\beta, \alpha, -k)$. From applying procedure (a) to (α, β), we obtain μ_1 and $(\alpha', \beta') \in \mathcal{P}_2(n')$ for some $n' \in \mathbb{N}$. For $j \in \{k, h\}$, we have $(\alpha, \beta) \in H(n,j)$ and so $\alpha_1 + j > \beta_1 - j > -j$. Thus we have

$$P_{j,-j;2}(\alpha, \beta, <) = P^a_{j,-j;2}(\alpha, \beta, <) \tag{3.1.2}$$

We show that

$$P_{h/2,-k/2;1/2}(\alpha,\beta,<) = P^a_{h/2,-k/2;1/2}(\alpha,\beta,<). \tag{3.1.3}$$

Since $(1,0) < (1,1)$, we have to show that $\alpha_1 + h/2 > -k/2$, i.e. $\alpha_1 > -(h+k)/2$. If $h+k \geq 0$, this holds. Suppose $h+k < 0$. We have $\alpha_1 + k > -h$, so $\alpha_1 > -h-k > -(h+k)/2$ since $h+k < 0$. Thus we have (3.1.3) and so $P_{h/2,-k/2;1/2}(\alpha,\beta,<)$ is in bijection with $P_{(h-1)/2,-k/2;1/2}(\alpha',\beta',<')$.

We showed in the proof of Lemma 3.1.5 that $(\alpha',\beta') \in H(n',j)$ and that $<'$, $<_{\alpha',\beta',j}$ are equivalent for $j = h, h-1, k, k-1$. By the induction hypothesis, (3.1.1) holds (α',β') for $(h-1,k)$ and $(h,k-1)$. Applying the first equality of (3.1.1) for $(h-1,k)$ (resp. the second equality of (3.1.1) for $(h,k-1)$) gives the first (resp. second) equality in the following:

$$P_{h-1,-h+1;2}(\alpha',\beta',<') = P_{(h-1)/2,-k/2;1/2}(\alpha',\beta',<') = P_{k-1,-k+1;2}(\alpha',\beta',<') =: P'.$$

By (3.1.2) and (3.1.3), any element of the three sets in (3.1.1) is of the form $(\mu_1 \sqcup \mu, \nu)$ for some $(\mu,\nu) \in P'$, and so (3.1.1) follows. The lemma then follows from (3.1.1) for $h = k$. □

3.1.4 Green functions for SO(N)

Recall that for $N \in \mathbb{N}$ odd (resp. even), the Weyl group of $\mathrm{SO}(N)$ is $W(B_{(N-1)/2})$ (resp. $W(D_{N/2})$) and the other relative Weyl groups are of the form $W(B_{(N-k^2)/2})$ for some odd (resp. even) integer $k \in \mathbb{Z}_{\geq 0}$ such that $k^2 \leq N$. In other words, for each $n \in \mathbb{Z}_{\geq 0}$, we can interpret $W(B_n)$ as a relative Weyl group of $\mathrm{SO}(2n+k^2)$ for any $k \in \mathbb{N}$, and $W(D_n)$ as the Weyl group of $\mathrm{SO}(2n)$. Let $n \in \mathbb{Z}_{\geq 0}$ and let W_n be $W(B_n)$ or $W(D_n)$. Let t be an indeterminate and let $d = 1$ if $W_n = W(D_n)$ and $d = 2$ if $W_n = W(B_n)$. The Poincaré polynomial of W_n is given by

$$P_{W_n}(t) = \frac{t^{dn}-1}{t-1} \cdot \prod_{i=1}^{n-1} \frac{t^{2i}-1}{t-1}.$$

Let V be the reflection representation of W. For a class function f of W_n, define $R(f) \in \mathbb{Z}[t]$ by

$$R(f) = \frac{(t-1)^{\dim V} P_W(t)}{|W|} \sum_{w \in W_n} \frac{\det_V(w) f(w)}{\det_V(t \cdot \mathrm{id}_V - w)}.$$

If f is an irreducible character of W_n, then $R(f)$ is called the fake degree of f. For $\rho \in W^\wedge$, let $\chi(\rho)$ denote its character.

Let N^* be the number of reflections of W_n. Let $\Omega = (\omega_{\rho,\rho'})_{\rho,\rho' \in W_n^\wedge}$ be the $|W_n| \times |W_n|$ matrix over $\mathbb{Z}[t]$ defined by

$$\omega_{\rho,\rho'} = t^{N^*} R(\chi(\rho) \otimes \chi(\rho') \otimes \overline{\det}_V).$$

Let $(\alpha, \beta) \in \mathcal{P}_2(n)$. If $W_n = W(B_n)$, let $\rho = \rho_{(\alpha,\beta)}$, and if $W_n = W(D_n)$, let $i \in \{1, 2/c_{(\alpha,\beta)}\}$ and let $\rho = \rho_{\{\alpha,\beta\},i}$. Let $k \in \mathbb{Z}_{\geq 0}$, $N = 2n + k^2$, and $r \in \mathbb{N}$ such that $r \geq t(\alpha) + t(\beta) + k$ (so r is as in Remark 3.1.4). If $W_n = W(D_n)$, we assume that $k = 0$. Let $\Lambda := \Lambda_{\rho;k} := \Lambda_{k,-k;2}(\alpha, \beta)$ and $\Lambda^{(0)} := [k, -\infty[_2 \sqcup [-k, -\infty[_2$. Let $(C, \mathcal{E}) = \mathrm{GSpr}^{-1}(\rho)$ and suppose C is parametrised by an orthogonal partition λ of N. We define

$$a_k(C) := a_k(\lambda) := a_k(\rho) := a_k(\alpha, \beta) := \sum_{1 \leq i < j \leq r} \min(\Lambda_i, \Lambda_j) - \sum_{1 \leq i < j \leq r} \min(\Lambda_i^{(0)}, \Lambda_j^{(0)}).$$

We will usually drop k from the notation. Note in particular that $a_k(\rho)$ is well-defined when $W_n = W(D_n)$, since in this case, we assumed that $k = 0$. Also note that $a_k(\rho)$ does not depend on r. We write $\rho \sim \rho'$ if the symbols $\Lambda_{\rho;k}$ and $\Lambda_{\rho';k}$ are similar. Then $a_k(\rho) = a_k(\rho')$ if $\rho \sim \rho'$, so $a_k(C)$ and $a_k(\lambda)$ are well-defined. Note that a_k is the same as in [Sho01, (1.2.2)] and if $k = 1$, then a_k is the same as the function b in [Lus86b, 4.4].

Define a total order $\prec_k$ on $W_n^\wedge$ such that for $\rho, \rho' \in W_n^\wedge$ we have $a_k(\rho) \geq a_k(\rho')$ if $\rho \prec_k \rho'$, and such that each similarity class forms an interval. Again, we will usually drop k from the notation. The order $\prec_k$ uniquely defines a total order $\prec_k$ on $\mathcal{P}_2(n)$. In [Sho01] (for type B) and in [Sho02] (for type D) the following theorem is proved:

Theorem 3.1.8. *Let $n \in \mathbb{N}$, $k \in \mathbb{Z}_{\geq 0}$, let $W_n = W(B_n)$ or $W_n = W(D_n)$, and write $\prec$ for $\prec_k$. If W_n is of type D_n, we assume that $k = 0$. There exist unique $|W_n^\wedge| \times |W_n^\wedge|$ matrices $P^{(k)} = P = (p_{\rho,\rho'})_{\rho,\rho' \in W_n^\wedge}$ and $\Lambda^{(k)} = \Lambda = (\lambda_{\rho,\rho'})_{\rho,\rho' \in W_n^\wedge}$ over $\mathbb{Q}(t)$ such that*

$$\begin{aligned} &P\lambda P^t = \Omega, && \\ &\lambda_{\rho,\rho'} = 0 && \text{if } \rho \not\sim \rho', \\ &p_{\rho,\rho'} = 0 && \text{unless } \rho' \prec \rho \text{ and } \rho \not\sim \rho', \text{ or } \rho = \rho', \\ &p_{\rho,\rho} = t^{a(\rho)}. && \end{aligned}$$

Furthermore, the entries of P and Λ lie in $\mathbb{Z}[t]$. The polynomials $p_{\rho,\rho'}$ are called Green functions of W_n for k.

Except when $W_n = W(B_n)$ and $k = 0$, we can view W_n as a relative Weyl group of $\mathrm{SO}(N) = \mathrm{SO}(2n + k^2)$. Regardless, we considered the Green functions of $W_n = W(B_n)$ for $k = 0$ in Theorem 3.1.8, as we will later see from Proposition 3.1.13 that they are related to the Green functions of $W(D_n)$ for $k = 0$.

The matrix $P^{(k)}$ is closely related to the solutions of a similar matrix equation considered in [Lus86a, §24]. We will briefly discuss this relation. Let $\rho \in W_n^\wedge$ and let $(C, \mathcal{E}) = \mathrm{GSpr}^{-1}(\rho) \in \mathcal{N}_{\mathrm{SO}(N)}$. Define the *Springer support* of ρ to be $\operatorname{supp} \rho := C$. Fix a maximal torus T of G. Recall that $W_n \cong N_G(L)/L$ for some Levi subgroup L of G with a cuspidal pair. Let D be the diagonal $|W_n^\wedge| \times |W_n^\wedge|$ matrix over $\mathbb{Q}(t)$ such that for all $\rho \in W_n^\wedge$, $D_{\rho,\rho} = t^{a(\rho)}$. Let $\tilde{\Omega} = D^{-1}\Omega D^{-1}$. Let $\rho, \rho' \in W_n^\wedge$, $C = \operatorname{supp}\rho$ and $C' = \operatorname{supp}\rho'$. The following result can be verified using [CM93, Cor. 6.1.4.].

Proposition 3.1.9. *We have $2a(\rho) = \dim G - \dim Z_L^\circ - \dim C$, where Z_L° is the connected component of the centre of L.*

Denote by $\tilde{\omega}_{\rho,\rho'}$ the polynomial $\Omega_{i',i}$ in [Lus86a, §24.7], where $i = \mathrm{GSpr}^{-1}(\rho)$ and $i' = \mathrm{GSpr}^{-1}(\rho')$. By Proposition 3.1.9, there exists a $c \in \mathbb{Z}$ such that for all $\rho, \rho' \in W_n^\wedge$, we have

$$\omega_{\rho,\rho'} = t^{a(\rho)+a(\rho')+c}\tilde{\omega}_{\rho,\rho'}.$$

Note that $\prec$ is compatible with the closure order on $\mathcal{U}$: if C is strictly smaller than C' in the closure order, then $a(\rho) > a(\rho')$ by Proposition 3.1.9, hence $\rho \prec \rho'$ and $\rho \not\sim \rho'$, and if $C = C'$, then $\Lambda_{\rho,k} = \Lambda_{\rho',k}$ by the generalised Springer correspondence, and so $\rho \sim \rho'$. Thus we can apply [Lus86a, Theorem 24.8(b)] with this order $\prec$, and by taking the transpose of the matrix equation in [Lus86a, Theorem 24.8(b)], this theorem becomes:

Theorem 3.1.10. *Let $n, k, W_n, \prec$ be as in Theorem 3.1.8, except we assume that $k \neq 0$ if $W_n = W(B_n)$. Then there exist unique $|W_n| \times |W_n|$ matrices $\tilde{P} = \tilde{P}^{(k)}$ and $\tilde{\Lambda} = \tilde{\Lambda}^{(k)}$ over $\mathbb{Q}(t)$ such that*

$$\begin{aligned}
&\tilde{P}\tilde{\lambda}\tilde{P}^t = \tilde{\Omega}, && \\
&\tilde{\lambda}_{\rho,\rho'} = 0 && \text{if } \rho \not\sim \rho', \\
&\tilde{p}_{\rho,\rho'} = 0 && \text{unless } \rho' \prec \rho \text{ and } \rho \not\sim \rho', \text{ or } \rho = \rho', \\
&\tilde{p}_{\rho,\rho} = 1. &&
\end{aligned}$$

Furthermore, the entries of $\tilde{P}, \tilde{\Lambda}$ lie in $\mathbb{Z}[t]$.

Consider $P = P^{(k)}$ and Λ from Theorem 3.1.8. Then it is easy to see that $\tilde{P} = D^{-1}P$ and $\tilde{\Lambda} = t^{-c}\Lambda$ are the solutions to the matrix equation in Theorem 3.1.10. Thus we have now related P in Theorem 3.1.8, which has a combinatorial interpretation as shown in [Sho01], to $\tilde{P}$ in Theorem 3.1.10, which has a geometric interpretation as shown in [Lus86a]. We shall discuss the geometric interpretation later in §3.1.6.

Remark 3.1.11. If $W_n = W(B_n)$ and $\rho = \rho_{\boldsymbol{\alpha}}$ and $\rho' = \rho_{\boldsymbol{\alpha}'}$ for $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathcal{P}_2(n)$, we will also write $p_{\boldsymbol{\alpha},\boldsymbol{\alpha}'} = p_{\rho,\rho'}$ and $\tilde{p}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'} = \tilde{p}_{\rho,\rho'}$. If $W_n = W(D_n)$ and $\rho = \rho_{\boldsymbol{\alpha},i}$ and $\rho' = \rho_{\boldsymbol{\alpha}',i'}$ for $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathcal{P}_2(n)/\theta$ and $i \in \{1, 1/c_{\boldsymbol{\alpha}}\}$, $i' \in \{1, 1/c_{\boldsymbol{\alpha}'}\}$, we will similarly write $p_{\boldsymbol{\alpha}^i,\boldsymbol{\alpha}'^{i'}} = p_{\rho,\rho'}$ and $\tilde{p}_{\boldsymbol{\alpha}^i,\boldsymbol{\alpha}'^{i'}} = \tilde{p}_{\rho,\rho'}$. We will furthermore drop the i (or i') from the notation if $c_{\boldsymbol{\alpha}} = 1$ (or $c_{\boldsymbol{\alpha}'} = 1$).

3.1.5 Symmetric polynomials

The matrix of Green functions $P^{A_{n-1}}$ for S_n has an interpretation in terms of symmetric functions, see for instance [Mac98, Ch. III §7], where it is shown that $P^{A_{n-1}}$ is equal to the Kostka matrix, which is the transition matrix between Schur functions and Hall-Littlewood functions. Shoji extended these results to complex reflection groups in [Sho01] and [Sho01]. In particular, the connection between the Green functions of type B/C and D and symmetric functions were described. The result [Sho01, p.685] allows one to compute the Green functions for Weyl groups of type B under some conditions. Waldspurger proved a certain generalisation [Wal19, Proposition 4.2] of this result for the relative Weyl groups of $\mathrm{Sp}(2n)$. Proposition 3.1.17 and Lemma 3.1.20 together are a direct analogue of [Wal19, Proposition 4.2] for $\mathrm{SO}(N)$.

We briefly describe the symmetric functions used by Shoji, but we shall not include their definitions here. The main purpose of this subsection is to highlight the main ideas in [Sho01] are relevant for Waldspurger's proof of [Wal19, Proposition 4.2] and the proof of Proposition 3.1.17.

Let $(\alpha, \beta) \in \mathcal{P} \times \mathcal{P}$ and let $\boldsymbol{m} = (m_0, m_1) \in \mathbb{N} \times \mathbb{N}$ such that $m_0 \geq t(\alpha)$ and $m_1 \geq t(\beta)$ and let $k = m_0 - m_1$. For $d = 1, 2$ and $j = 1, \ldots, m_d$ define indeterminates $x_j^{(d)}$ and write $x^{(d)} = (x_j^{(d)})_j$ and $x = (x_j^{(d)})_{d,j}$, Consider the following symmetric polynomials in $\mathbb{Z}[x]$ and $\mathbb{Z}[x, t]$:

(I) (Schur functions) $s_{(\alpha,\beta)}(x) = s_\alpha(x^{(0)}) s_\beta(x^{(1)}) \in \mathbb{Z}[x]$ as in [Sho01, §2.1], where $s_\alpha(x^{(0)}) \in \mathbb{Z}[x^{(0)}]$ and $s_\beta(x^{(1)}) \in \mathbb{Z}[x^{(1)}]$ are the usual Schur functions attached to the partitions α and β,

(II) (Monomial symmetric functions) $m_{(\alpha,\beta)}(x) \in \mathbb{Z}[x]$ as in [Sho01, §2.1],

(III) (Complete symmetric functions) $h_{(\alpha,\beta)}(x) \in \mathbb{Z}[x]$ as in [Sho01, §6.7],

(IV) $q_{(\alpha,\beta)}(x, t) \in \mathbb{Z}[x, t]$ ([Sho01, §2.4]),

(V) $R_{(\alpha,\beta)}(x, t) = R_{(\alpha,\beta),<}(x, t) \in \mathbb{Z}[x, t]$ as in [Sho01, (3.2.1)], where $<$ stands for the order $(m_0, m_1, <)$ on the set of indices of (α, β),

(VI) (Hall-Littlewood functions) $P'_{(\alpha,\beta)}(x, t) = P'_{(\alpha,\beta),k}(x, t) \in \mathbb{Z}[x, t]$ as in [Sho01, Theorem 4.4]. We write P' instead of P to the notation to avoid confusion with P in Theorem 3.1.8). Note that in *loc. cit.*, the Hall-Littlewood functions are indexed by certain symbols that are equivalent to $\Lambda_{k,-k;2}(\alpha, \beta)$, so we can also index them by $(\alpha, \beta), k$,

(VII) $Q_{(\alpha,\beta)}(x) = Q_{(\alpha,\beta),k}(x) \in \mathbb{Z}[x, t]$ as in [Sho01, Theorem 4.4 (ii)].

The polynomials above all satisfy a 'stability property'. The polynomials (I) – (V), can be defined replacing m_0 with $m_0 + 1$ (resp. m_1 with $m_1 + 1$). Evaluating $x^{(0)}_{m_0+1} = 0$ (resp. $x^{(1)}_{m_1+1} = 0$) yields the same polynomial as in the definition for (m_0, m_1). The polynomials $P'_{(\alpha,\beta),k}$ and $Q_{(\alpha,\beta),k}$ depend on $k = m_0 - m_1$, and they can be defined when we replace (m_0, m_1) by $(m_0 + 1, m_1 + 1)$, and evaluating $x^{(0)}_{m_0+1} = 0$ and $x^{(1)}_{m_1+1} = 0$ yields the same polynomial as in the definition for (m_0, m_1). As such, the polynomials above can be viewed as functions in infinitely many variables. These functions form bases for certain $\mathbb{Z}$ and $\mathbb{Q}(t)$-modules as follows. Denote by

$$\Xi_{(m_0,m_1)} = \mathbb{Z}[x^{(0)}_i : i = 1, \dots, m_0]^{S_{m_0}} \otimes \mathbb{Z}[x^{(1)}_i : i = 1, \dots, m_1]^{S_{m_1}}$$

the ring of symmetric polynomials with variables x with respect to $S_{m_0} \times S_{m_1}$. Then $\Xi_{(m_0,m_1)}$ has a graded ring structure $\Xi_{(m_0,m_1)} = \bigoplus_{i\geq 0} \Xi^i_{(m_0,m_1)}$, where each $\Xi^i_{(m_0,m_1)}$ consists of homogeneous symmetric polynomials of degree i and the zero polynomial. Consider the inverse limit

$$\Xi^i = \varprojlim_{(m_0,m_1)} \Xi^i_{(m_0,m_1)}$$

with respect to the obvious restrctions $\Xi^i_{(m_0+\ell,m_1+\ell)} \to \Xi^i_{(m_0,m_1)}$ where $\ell \in \mathbb{N}$. Define

$$\Xi = \bigoplus_{i\geq 0} \Xi^i.$$

For $n, m_0, m_1 \in \mathbb{N}$ with $m_0, m_1 \geq n$, we have that $(s_{(\alpha,\beta)}(x))_{(\alpha,\beta)\in\mathcal{P}_2(n)}$ is a basis for the $\mathbb{Z}$-module $\Xi^n_{(m_0,m_1)}$, as well as the $\mathbb{Q}(t)$-vector space $\Xi^n_{\mathbb{Q},(m_0,m_1)}[t] := \mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n_{m_0,m_1}$. By the stability property, $(s_{(\alpha,\beta)}(x))_{(\alpha,\beta)\in\mathcal{P}_2(n)}$ is also a $\mathbb{Z}$-basis (resp. $\mathbb{Q}(t)$-basis) of Ξ^n (resp. $\Xi^n_{\mathbb{Q}}[t] := \mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$). As polynomials (resp. functions in infinitely many variables), the functions in (II) – (VII) are shown in [Sho01] to be bases of the $\mathbb{Q}(t)$-vector space $\Xi^n_{\mathbb{Q},(m_0,m_1)}[t]$ (resp. $\Xi^n_{\mathbb{Q}}[t]$), where (α, β) runs through $\mathcal{P}_2(n)$. We want to study the transition matrices between some of these bases. By the stability properties, we see that the transition matrices do not depend on m_0 and m_1 as long as $m_0 \geq n$, $m_1 \geq n$ and $m_0 - m_1 = k$, and that the transition matrix between any two bases in $\Xi^n_{\mathbb{Q},(m_0,m_1)}[t]$ is the same as the transition matrix between between these two bases in $\Xi^n_{\mathbb{Q}}[t]$. Given any two bases X and Y of a vector space, denote by $M(X, Y)$ the transition matrix from Y to X.

Theorem 3.1.12 ([Sho01, Theorem 5.4])**.** *Let $n \in \mathbb{N}$, $k \in \mathbb{Z}_{\geq 0}$ and let $P = P^{(k)}$ be as in Theorem 3.1.8 for the Weyl group $W(B_n)$. Then $P = K^{B_n}(t^{-1})T_k$, where T_k is the diagonal $|W(B_n)| \times |W(B_n)|$ matrix with entries $(T_k)_{\rho,\rho} = t^{a_k(\rho)}$ and $K^{B_n}(t) = M(s, P')$ is the transition matrix between the Hall-Littlewood functions $P'_{(\alpha,\beta),k}$ and the Schur functions $s_{(\alpha,\beta)}$, where $(\alpha, \beta) \in \mathcal{P}_2(n)$. We call $K^{B_n}(t)$ a* Kostka matrix *of type B, and the entries of $K^{B_n}(t)$ are called* Kostka polynomials.

Note that the Kostka polynomials in Theorem 3.1.12 depend on k, but we drop it from the notation for simplicity.

The Green functions of $W(D_n)$ for $k=0$ are described in [Sho02]. Let T_0' be the diagonal $|W(D_n)| \times |W(D_n)|$ matrix with diagonal entries $(T_0')_{\rho,\rho} = t^{a_0(\rho)}$. Similar to the type B case, it is shown in *loc. cit.* that the matrix of $P = P^{(0)}$ of Green functions of $W(D_n)$ for $k=0$ is equal to the transition matrix $K^{D_n}(t^{-1})T_0'$, where $K^{D_n}(t)$ is a transition matrix between certain Schur functions and Hall-Littlewood functions 'of type D', defined slightly different than in (I) and (VI). Using [Sho02, Proposition 4.9], we find the following result for $K^{D_n}(t)$. If n is even, let $K^{A_{n/2-1}}(t)$ be the Kostka matrix of $S_{n/2}$ as in [Mac98, III.6]. From Theorem 3.1.12, we see that K^{B_n} depends on k. For the following, we assume that $k=0$. We will index the Kostka polynomials of type $A_{n/2-1}$ and B_n by partitions of $n/2$ and bipartitions of n, respectively, and we will index the Kostka polynomials of type D_n by $\boldsymbol{\alpha}^i$ where $\boldsymbol{\alpha} \in \mathcal{P}_2(n)/\theta$ and $i \in \{1, 2/c_{\boldsymbol{\alpha}}\}$, and we drop the i if $c_{\boldsymbol{\alpha}} = 2$ (cf. Remark 3.1.11).

Proposition 3.1.13. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{P}_2(n)$.*

1. Suppose $\theta(\boldsymbol{\alpha}) \neq \boldsymbol{\alpha}$. Then for $j = 1, 2$, we have

$$K^{D_n}_{\boldsymbol{\beta}^j,\boldsymbol{\alpha}}(t) = \frac{c_{\boldsymbol{\beta}}}{2}(K^{B_n}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t) + K^{B_n}_{\boldsymbol{\beta},\theta\boldsymbol{\alpha}}(t)).$$

2. Suppose $\theta(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ and $\theta(\boldsymbol{\beta}) \neq \boldsymbol{\beta}$. Then for $i = 1, 2$ we have

$$K^{D_n}_{\boldsymbol{\beta},\boldsymbol{\alpha}^i}(t) = K^{B_n}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t) = K^{B_n}_{\theta\boldsymbol{\beta},\boldsymbol{\alpha}}(t).$$

3. Suppose $\theta(\boldsymbol{\alpha}) = \boldsymbol{\alpha} = (\alpha, \alpha)$ and $\theta(\boldsymbol{\beta}) = \boldsymbol{\beta} = (\beta, \beta)$. Then n is even and for $i = 1, 2$ and $j = 1, 2$, we have

$$K^{D_n}_{\boldsymbol{\beta}^j,\boldsymbol{\alpha}^i}(t) = (-1)^{i+j} K^{A_{n/2-1}}_{\beta,\alpha}(t^2) + K^{B_n}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t).$$

Remark 3.1.14. It may not seem obvious from [Sho02, Proposition 4.9] that we have to evaluate $K^{A_{n/2-1}}_{\boldsymbol{\beta},\boldsymbol{\alpha}}$ in t^2 in Proposition 3.1.13(3), as it comes from a subtlety involved in defining the Hall-Littlewood functions. In the notation of *loc. cit.*, when $j=1$, then the P_z^j are the usual Hall-Littlewood functions of $S_{n/2}$. Since $h_j = h_1 = 2$, the term t^{h_j} in [Sho02, (3.3.7)] is t^2 and so the P_z^j are the Hall-Littlewood functions of $S_{n/2}$ evaluated in t^2 rather than t. As such, the Kostka matrix of $S_{n/2}$ that appears in [Sho02, Proposition 4.9] should also be evaluated in t^2. Similarly, when $j=0$, we have $h_j = 1$, so the Kostka matrices of type B appearing in [Sho02, Proposition 4.9] are simply evaluated in t.

Furthermore, we will only need Proposition 3.1.13(1) later, but we include the other two statements as well for completeness. In this case, we thus have a description of the Green functions of $W(D_n)$ for $k=0$ in terms of Green functions of $W(B_n)$ for $k=0$, which are described by Theorem 3.1.8 for $W_n = W(B_n)$ and $k=0$.

The following is a direct consequence of [Sho02, (3.3.2), §3.9]. Suppose $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{P}_2(n)/\theta$ and $i \in \{1, 2/c_{\boldsymbol{\alpha}}\}$, $j \in \{1, 2/c_{\boldsymbol{\beta}}\}$. Then

$$K^{B_n}_{\theta\boldsymbol{\beta},\theta\boldsymbol{\alpha}}(t) = K^{B_n}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t). \tag{3.1.4}$$

Note that the condition $k = 0$ is important here.

3.1.6 Multiplicities

Let $N \in \mathbb{N}$, $G = \mathrm{SO}(N)$, and $(C, \mathcal{E}), (C', \mathcal{E}') \in \mathcal{N}_G$. Consider the intersection cohomology complex $\mathrm{IC}(\bar{C}', \mathcal{E}')$ on $\bar{C}'$ associated to $(C', \mathcal{E}')$. Recall that for each $m \in \mathbb{N}$, the restriction of $H^{2m}(\mathrm{IC}(\bar{C}', \mathcal{E}'))$ to C is a direct sum of irreducible G-equivariant local systems on C. Let $\mathrm{mult}(C, \mathcal{E}; C', \mathcal{E}')$ be the multiplicity of $\mathcal{E}$ in $\bigoplus_{m\in\mathbb{Z}} H^{2m}(\mathrm{IC}(\bar{C}', \mathcal{E}'))|_C$. Let $\rho = \mathrm{GSpr}(C, \mathcal{E})$ and $\rho' = \mathrm{GSpr}(C', \mathcal{E}')$. Suppose $(C, \mathcal{E})$ and $(C', \mathcal{E}')$ are parametrised by $(\lambda, [\varepsilon]), (\lambda', [\varepsilon']) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$, respectively. If $k(\lambda, [\varepsilon]) \neq k(\lambda', [\varepsilon'])$, then $\mathrm{mult}(C, \mathcal{E}; C', \mathcal{E}') = 0$. If $k(\lambda, [\varepsilon]) = k(\lambda', [\varepsilon'])$, then $\rho = \mathrm{GSpr}(C, \mathcal{E})$ and $\rho' = \mathrm{GSpr}(C', \mathcal{E}')$ are representations of the same relative Weyl group W_{rel}. In this case, let $\tilde{P} = \tilde{P}^{(k(\lambda,[\varepsilon]))}$ be as in Theorem 3.1.10 for W_{rel}. For each $m \in \mathbb{N}$, the coefficient of t^m in $\tilde{p}_{\rho',\rho}(t)$ is equal to the multiplicity of $\mathcal{E}$ in $H^{2m}(\mathrm{IC}(\bar{C}', \mathcal{E}'))|_C$ by [Lus86a, (24.8.2)] (recall that $\tilde{P}$ in Theorem 3.1.10 is the transpose of the matrix Π in [Lus86a, (24.8.2)]). Let $\mathrm{mult}(C, \mathcal{E}; C', \mathcal{E}')$ be the multiplicity of $\mathcal{E}$ in $\bigoplus_{m\in\mathbb{Z}} H^{2m}(\mathrm{IC}(\bar{C}', \mathcal{E}'))|_C$. Then we have $\mathrm{mult}(C, \mathcal{E}; C', \mathcal{E}') = \tilde{p}_{\rho',\rho}(1)$.

We want to give a combinatorial description of $\mathrm{mult}(C, \mathcal{E}; C', \mathcal{E}')$. Let $(\alpha, \beta) \in \mathcal{P} \times \mathcal{P}$ and let $(m_0, m_1, <)$ be any order on the set of indices $I = I_{m_0,m_1}$ of (α, β). Let $J = \{((i, e), (j, f)) \in I^2 \colon (i, e) < (j, f), e \neq f\}$. Let $X = \mathbb{N}^J$ and write its elements as $x = (x_{(i,e),(j,f)})_{((i,e),(j,f))\in J}$ with $x_{(i,e),(j,f)} \in \mathbb{N}$. For $x \in X$, we define $(\alpha[x], \beta[x]) \in \mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1}$ by

$$\begin{aligned}
\alpha[x]_i &= \alpha_i + \sum_{j\in\{1,\dots,m_1\},(i,0)<(j,1)} x_{(i,0),(j,1)} - \sum_{j\in\{1,\dots,m_1\},(j,1)<(i,0)} x_{(j,1),(i,0)},\\
\beta[x]_j &= \beta_i + \sum_{i\in\{1,\dots,m_0\},(j,1)<(i,0)} x_{(j,1),(i,0)} - \sum_{i\in\{1,\dots,m_0\},(i,0)<(j,1)} x_{(i,0),(j,1)},
\end{aligned}$$

for $i = 1, \dots, m_0$, $j = 1, \dots, m_1$. For each $(\mu, \nu) \in \mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1}$, let $X(\alpha, \beta, <; \mu, \nu) = \{x \in X \colon \alpha[x] = \mu, \beta[x] = \nu\}$.

Remark 3.1.15.

1. For all $x \in X$, we have $S(\alpha[x]) + S(\beta[x]) = S(\alpha) + S(\beta)$. Hence $X(\alpha, \beta, <; \mu, \nu)$ is empty if $S(\mu) + S(\nu) \neq S(\alpha) + S(\beta)$.
2. For all $(\mu, \nu) \in \mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1}$, $X(\alpha, \beta, <; \mu, \nu)$ is finite.

For each $\mu \in \mathbb{Z}^{m_0}$ and $w \in S_{m_0}$, we define $\mu[w] \in \mathbb{Z}^{m_0}$ by $\mu[w]_i = \mu_{wi} + i - wi$ for $i = 1, \ldots, m_0$. We similary define $\nu[v]$ for $\nu \in \mathbb{Z}^{m_1}$ and $v \in S_{m_1}$. Let $(\mu, \nu) \in \mathcal{P} \times \mathcal{P}$. If $t(\mu) > m_0$ or $t(\nu) > m_1$, we define $\operatorname{mult}(\alpha, \beta, <; \mu, \nu) = 0$. Otherwise, we can consider (μ, ν) as an element of $\mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1}$ and we define

$$\operatorname{mult}(\alpha, \beta, <; \mu, \nu) := \sum_{w \in S_{m_0}, v \in S_{m_1}} \operatorname{sgn}(w) \operatorname{sgn}(v) |X(\alpha, \beta, <; \mu[w], \nu[v])|.$$

It is shown on [Wal19, §3.1, p. 412] that $\operatorname{mult}(\alpha, \beta, <; \mu, \nu)$ is independent of the choice of representative for $<$.

Recall $p_{A,B;s}(\alpha, \beta, <)$ from Definition 3.1.3. Let $Q_s(\alpha, \beta, <)$ be the set of pairs $(\mu, \nu) \in \mathcal{P} \times \mathcal{P}$ such that for all $A, B, s \in \mathbb{R}$ with $s > 0$, we have

$$\Lambda_{A,B;s}(\mu, \nu) \leq p_{A,B;s}(\alpha, \beta, <).$$

We have the following result [Wal19, Prop. 3.1]:

Proposition 3.1.16. *Let $(\mu, \nu) \in \mathcal{P} \times \mathcal{P}$ and $A, B, s \in \mathbb{R}$ with $s > 0$.*

1. *Suppose that* $\operatorname{mult}(\alpha, \beta, <; \mu, \nu) \neq 0$. *Then* $(\mu, \nu) \in Q_s(\alpha, \beta, <)$.
2. *Suppose that* $\Lambda_{A,B;s}(\mu, \nu) = p_{A,B;s}(\alpha, \beta, <)$. *Then* $\operatorname{mult}(\alpha, \beta, <; \mu, \nu) \neq 0$ *if and only if* $(\mu, \nu) \in P_{A,B;s}(\alpha, \beta, <)$.
3. *Suppose that* $(\mu, \nu) \in P_{A,B;s}(\alpha, \beta, <)$. *Then* $\operatorname{mult}(\alpha, \beta, <; \mu, \nu) = 1$.

Proposition 3.1.17 (cf. [Wal19, Proposition 4.2])**.** *Let $N \in \mathbb{N}$, let $(\lambda, [\varepsilon]), (\lambda', [\varepsilon']) \in \mathcal{P}^{\mathrm{ort}}(N)$ such that λ only has odd parts and such that $k := k(\lambda, [\varepsilon]) = k(\lambda', [\varepsilon'])$. Let $(\alpha, \beta)_k = \Phi_N(\lambda, [\varepsilon])$, $(\alpha', \beta')_k = \Phi_N(\lambda', [\varepsilon'])$ and write $<$ for $<_{\alpha,\beta,k}$.*

1. *If $k > 0$, then*

$$\operatorname{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C_{\lambda'}, \mathcal{E}_{[\varepsilon']}) = \operatorname{mult}(\alpha, \beta, <; \alpha', \beta').$$

2. *If $k = 0$ (note that N is even), then*

$$\begin{aligned}\operatorname{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^{\pm}_{\lambda'}, \mathcal{E}^{\pm}_{[\varepsilon']}) &= \frac{c_{(\alpha',\beta')}}{2}(\operatorname{mult}(\alpha, \beta, <; \alpha', \beta') + \operatorname{mult}(\alpha, \beta, <; \beta', \alpha')) \\ &= \begin{cases} \operatorname{mult}(\alpha, \beta, <; \alpha', \beta') & \text{if } \alpha' = \beta', \\ \operatorname{mult}(\alpha, \beta, <; \alpha', \beta') + \operatorname{mult}(\alpha, \beta, <; \beta', \alpha') & \text{if } \alpha' \neq \beta'. \end{cases}\end{aligned}$$

3.1.7 Proof of Proposition 3.1.17

We first prove some preliminary results.

Let $m = m_0 + m_1$ and let $e_1, \ldots, e_m$ be the unit vectors in $\mathbb{Z}^m$. For $i \neq j$ define an operator R_{ij} on $\mathbb{Z}^m$ by $R_{ij}\lambda = \lambda + e_i - e_j$. A *raising operator* (resp. *lowering operator*) R on $\mathbb{Z}^m$ is a product of R_{ij} with $i < j$ (resp. $i > j$). Given an identification of $\mathbb{Z}^m$ with $\mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1}$, we write R_{ij} as $R_{\boldsymbol{\mu},\boldsymbol{\nu}}$ for $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1}$ correspond to $i, j \in \mathbb{Z}^m$, respectively.

We define $s_{(\mu,\nu)}$, $h_{(\mu,\nu)}$ and $q_{(\mu,\nu)}$ for $(\mu, \nu) \in \mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1}$ as follows (so far, they were only defined for $(\mu, \nu) \in \mathcal{P} \times \mathcal{P}$). If any of the μ_i or ν_i is negative, we set $q_{(\mu,\nu)} = 0$ and $h_{(\mu,\nu)} = 0$. Otherwise, pick $w \in S_{m_0}$, $v \in S_{m_1}$ such that $w(\mu), v(\nu) \in \mathcal{P}$ and we set $q_{(\mu,\nu)} = q_{(w(\mu),v(\nu))}$ and $h_{(\mu,\nu)} = h_{(w(\mu),v(\nu))}$. We set $s_{(\mu,\nu)} = 0$ if $\mu_i - i$ for $i = 1, \ldots, m_0$ are not all distinct, or if $\nu_i - i$ for $i = 1, \ldots, m_1$ are not all distinct. Otherwise, there exist $(\alpha, \beta) \in \mathcal{P} \times \mathcal{P}$ and $w \in S_{m_0}$, $v \in S_{m_1}$ such that $\mu = \alpha[w]$ and $\nu = \beta[v]$ are partitions and we set

$$s_{(\mu,\nu)} = \operatorname{sgn}(w) \operatorname{sgn}(v) s_{(\alpha,\beta)}.$$

For a raising or lowering operator R on $\mathbb{Z}^m$, we define

$$\begin{aligned} Rq_{(\mu,\nu)} &= q_{R(\mu,\nu)}, \\ Rh_{(\mu,\nu)} &= h_{R(\mu,\nu)}, \\ Rs_{(\mu,\nu)} &= s_{R(\mu,\nu)}. \end{aligned}$$

Note that there may exist raising or lowering operators R, R' such that $R'q_{(\mu,\nu)} = 0$ but $(RR')q_{(\mu,\nu)} \neq 0$. The same problem can occur for $h_{(\mu,\nu)}$ and $s_{(\mu,\nu)}$.

Remark 3.1.18. For any raising or lowering operator R, the matrix of the linear map $q_{(\alpha,\beta)} \mapsto Rq_{(\alpha,\beta)}$ in the basis $(q_{(\alpha,\beta)})$ is the same as the matrix of the linear map $h_{(\alpha,\beta)} \mapsto Rh_{(\alpha,\beta)}$ in the basis $(h_{(\alpha,\beta)})$, but it is not generally the same as the matrix of $s_{(\alpha,\beta)} \mapsto Rs_{(\alpha,\beta)}$ in the basis $(s_{(\alpha,\beta)})$.

The following result follows from the proof of [Sho01, Corollary 6.8]. However, one of the steps in the proof that we present is different than what was done in *loc. cit.*, where we use ideas from [Gar92, §2].

Lemma 3.1.19. *Let $k \in \mathbb{Z}_{\geq 0}$, $n = \frac{N-k^2}{2}$. Let $\boldsymbol{\alpha} = (\alpha, \beta) \in \mathcal{P}_2(n)$ Let $m_0, m_1 \in \mathbb{N}$ such that $m_1 = m_0 - k \geq n$ and let $(m_0, m_1, <)$ be any order on the set of indices of (α, β) such that $\alpha_i + k + 2 - 2i \geq \beta_j - k + 2 - 2j$ if $(i, 0) < (j, 1)$. Suppose that $Q_{\boldsymbol{\alpha},k} = R_{\boldsymbol{\alpha},<}$ in $\Xi^n_{\mathbb{Q}}[t]$. Suppose $P = P^{(k)}$ is as in Theorem 3.1.8 for $W_n := W(B_n)$. Then*

$$p_{(\alpha',\beta'),(\alpha,\beta)}(1) = \operatorname{mult}(\alpha, \beta, <; \alpha', \beta').$$

Proof. Recall that the transition matrices between the bases of symmetric functions in (I) - (VII) in the space $\Xi_{\mathbb{Q}}[t]$ are the same as in the space $\Xi^n_{\mathbb{Q},(m_0,m_1)}[t]$, as $m_0 = m_1 + k$ and $m_0, m_1 \geq n$. Thus we can consider the symmetric functions in (I) - (VII) as elements of $\Xi^n_{\mathbb{Q},(m_0,m_1)}[t]$. Write $K(t)$ for $K^{B_n}(t) = M(s, P')$. By Theorem 3.1.12, we have $K(t^{-1})T_k = P(t)$. By [Sho01, (5.6.1)], we have

$$M(Q, q) = (M(P', m)^t)^{-1} = K(t)^t(M(s, m)^t)^{-1}.$$

Let $I = I_{m_0,m_1}$ be the set of indices of (α, β) and let $J = \{((i, e), (j, f)) \in I^2 \colon (i, e) < (j, f), e \neq f\}$ as in §3.1.6 and define $J' = \{((i, e), (j, e)) \in I^2 \colon (i, e) < (j, e), e = 0, 1\}$. We can identify I with $\{1, \ldots, m_0 + m_1\}$, respecting $<$ and the usual order on $\{1, \ldots, m_0 + m_1\}$. We identify $\mathbb{Z}^{m_0+m_1}$ with $\mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1}$ accordingly, and we define raising and lowering operators on $\mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1}$ as before. Then

$$s_{(\alpha,\beta)}(x) = \prod_{(a,b)\in J'} (1 - R_{a,b})h_{(\alpha,\beta)}(x) \tag{3.1.5}$$

by [Sho01, (6.7.3)]. Using $Q_{(\alpha,\beta)} = R_{(\alpha,\beta)}$ and [Sho01, (3.7.1)], we find

$$Q_{(\alpha,\beta)}(x, t) = \left(\prod_{(a,b)\in J'} (1 - R_{a,b}) \prod_{(a,b)\in J} (1 - tR_{a,b})^{-1} \right) q_{(\alpha,\beta)}(x, t). \tag{3.1.6}$$

Define a basis $(H_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathcal{P}_2(n)}$ of $\Xi^n_{\mathbb{Q},(m_0,m_1)}[t]$ by requiring that $M(H, h) = M(Q, q)$. By Remark 3.1.18, we have

$$H_{(\alpha,\beta)}(x) = \left(\prod_{(a,b)\in J'} (1 - R_{(i,e),(j,f)}) \prod_{(a,b)\in J} (1 - tR_{a,b})^{-1} \right) h_{(\alpha,\beta)}(x). \tag{3.1.7}$$

Note that $K(t)^t = M(Q, q)M(s, m)^t = M(H, h)M(s, h)^{-1} = M(H, s)$. We want to show that $K_{(\alpha',\beta'),(\alpha,\beta)}$ is the coefficient of $s_{(\alpha',\beta')}$ in

$$H_{(\alpha,\beta)}(x) = \prod_{(a,b)\in J} (1 - tR_{a,b})^{-1} s_{(\alpha,\beta)} = \prod_{(a,b)\in J} (\sum_{\ell=0}^{\infty} (tR_{a,b})^\ell) s_{(\alpha,\beta)}. \tag{3.1.8}$$

Note that (3.1.6) involves raising operators acting on $q_{(\alpha,\beta)}$, whereas in (3.1.8), the raising operators act on $s_{(\alpha,\beta)}$, and we noted that the matrices of these actions are not generally the same in Remark 3.1.18. We shall prove (3.1.8) using ideas from [Gar92, §2], where a similar result is proved for classical Schur functions, avoiding the use of raising operators.

Let $\boldsymbol{\alpha} = (\alpha^{(0)}, \alpha^{(1)}) \in \mathcal{P}_2(n)$. The complete symmetric functions $h_{(\alpha^{(0)},\alpha^{(1)})}$ are defined to be a product

$$h_{(\alpha^{(0)},\alpha^{(1)})}(x) = \prod_{d=0}^{1} \prod_{i=1}^{m_d} h^{(d)}_{\alpha^{(d)}_i}(x^{(d)}),$$

where $h^{(d)}_{\alpha_i^{(d)}}(x^{(d)})$ are the ordinary complete symmetric functions in the variables $x^{(d)} = (x_1^{(d)}, \dots, x_{m_d}^{(d)})$ of degree $a_i^{(d)}$. For $d = 0, 1$, consider the generating function of $\{h_m^{(d)}(x)\}_{m\in\mathbb{Z}_{\geq 0}}$:

$$\Omega^{(d)}(x^{(d)}, z) = \sum_{m=0}^{\infty} h_m^{(d)}(x^{(d)}) z^m.$$

For $d = 0, 1$ and $i = 1, \dots, m_d$, introduce the variables $y_i^{(d)}$. Then (3.1.5) is equivalent to the statement that $s_{\boldsymbol{\alpha}}(x) = s_{\alpha^{(0)}}(x^{(0)}) s_{\alpha^{(1)}}(x^{(1)})$ where for $d = 0, 1$, we have

$$s_{\alpha^{(d)}}(x^{(d)}) = \left(\prod_{i=1}^{m_d} \Omega^{(d)}(x^{(d)}, y_i^{(d)})\right) \left(\prod_{1\leq a<b\leq m_d} 1 - \frac{y_b^{(d)}}{y_a^{(d)}}\right)\Bigg|_{\prod_{i=1}^{m_d}(y_i^{(d)})^{\alpha_i^{(d)}}}, \tag{3.1.9}$$

where the $|_{\prod_{i=1}^{m_d} y_i^{(d)}}$ stands for taking the coefficient of the $(\prod_{i=1}^{m_d} y_i^{(d)})$ term. Let $\Delta_{m_d}(y^{(d)}) = \prod_{1\leq a<b\leq m_d}(y_a^{(d)} - y_b^{(d)})$, $\delta_{m_d} = (m_d - 1, m_d - 2, \dots, 0) \in \mathcal{P}$, and $\Omega_d(x^{(d)}, y^{(d)}) = \prod_{i=1}^{m_d} \Omega^{(d)}(x^{(d)}, y_i^{(d)})$. Then we can rewrite (3.1.9) to obtain

$$\begin{aligned} s_{\boldsymbol{\alpha}}(x) &= s_{\alpha^{(0)}}(x^{(0)}) s_{\alpha^{(1)}}(x^{(1)}) \\ &= \Omega_0(x^{(0)}, y^{(0)})\Delta_{m_0}(y^{(0)})\Big|_{(y^{(0)})^{\alpha^{(0)}+\delta_{m_0}}} \cdot \Omega_1(x^{(1)}, y^{(1)})\Delta_{m_1}(y^{(1)})\Big|_{(y^{(1)})^{\alpha^{(1)}+\delta_{m_1}}}. \end{aligned}$$

For $d = 0, 1$, since $\Omega_k(x^{(d)}, y^{(d)})\Delta_{m_d}(y^{(d)})$ is an alternating function in $y^{(d)}$, taking the coefficient of $(y^{(d)})^{\alpha^{(d)}+\delta_{m_d}}$ is the same as taking the coefficient of

$$\Delta_{\alpha^{(d)}}(y^{(d)}) := \sum_{\sigma\in S_{m_d}} \operatorname{sgn}(\sigma)(\sigma y^{(m_d)})^{\alpha^{(d)}+\delta_{m_d}} \tag{3.1.10}$$

in $\Omega_k(x, y)\Delta_{m_d}(y)$, so we see that

$$\Omega_0(x^{(0)}, y^{(0)})\Omega_1(x^{(1)}, y^{(1)}) = \sum_{\boldsymbol{\alpha}\in\mathcal{P}\times\mathcal{P}} s_{\alpha^{(0)}}(x^{(0)}) \frac{\Delta_{\alpha^{(0)}}(y^{(0)})}{\Delta_{m_0}(y^{(0)})} s_{\alpha^{(1)}}(x^{(1)}) \frac{\Delta_{\alpha^{(1)}}(y^{(1)})}{\Delta_{m_1}(y^{(1)})}. \tag{3.1.11}$$

Note that (3.1.7) is equivalent to

$$H_{\boldsymbol{\alpha}}(x) = \left(\prod_{d=0}^{1} \Omega_d(x^{(d)}, y^{(d)})\right) \left(\prod_{((a,e),(b,e))\in J'} 1 - \frac{y_b^{(e)}}{y_a^{(e)}}\right) \left(\prod_{((a,e),(b,1-e))\in J} 1 - t\frac{y_b^{(1-e)}}{y_a^{(e)}}\right)\Bigg|_{y^{\boldsymbol{\alpha}}}$$

and by (3.1.11), this becomes

$$H_{\boldsymbol{\alpha}}(x) = \sum_{\boldsymbol{\alpha}\in\mathcal{P}\times\mathcal{P}} s_{\boldsymbol{\alpha}}(x) \left(\prod_{d=0}^{1} \Delta_{\alpha^{(d)}}(y^{(d)})\right) \left(\prod_{((a,e),(b,1-e))\in J} 1 - t\frac{y_b^{(1-e)}}{y_a^{(e)}}\right)\Bigg|_{y^{\boldsymbol{\alpha}+(\delta_{m_0},\delta_{m_1})}}. \tag{3.1.12}$$

By (3.1.10) and the definition of the Schur functions $s_{(\mu,\nu)}$ for $(\mu, \nu) \in \mathbb{Z}\times\mathbb{Z}$, we see that (3.1.12) is equivalent to (3.1.8), as desired.

Thus (3.1.8) gives a way to compute $M(H,s) = K(t)^t$ in Ξ^n, hence a way to compute $p_{(\alpha',\beta'),(\alpha,\beta)}(1) = K_{(\alpha',\beta'),(\alpha,\beta)}(1)$. For each $w \in S_{m_0}$, $v \in S_{m_1}$, we want to count the number of ways we can create products R of raising operators $R_{\boldsymbol{j}}$ with $\boldsymbol{j} \in J$ such that $(\alpha'[w], \beta'[v]) = R(\alpha, \beta)$, i.e. we want to determine the size of the set

$$R(w,v) := \{(y_{\boldsymbol{j}})_{\boldsymbol{j}\in J} \in X \colon (\alpha'[w], \beta'[v]) = \prod_{\boldsymbol{j}\in J} R_{\boldsymbol{j}}^{y_{\boldsymbol{j}}}(\alpha,\beta)\}.$$

It holds that

$$p_{(\alpha',\beta'),(\alpha,\beta)}(1) = \sum_{w\in S_{m_0}, v\in S_{m_1}} \operatorname{sgn}(w)\operatorname{sgn}(v)|R(w,v)|.$$

Note that $R(w,v) = X(\alpha, \beta, <; \alpha'[w], \beta'[v])$, so

$$p_{(\alpha',\beta'),(\alpha,\beta)}(1) = \operatorname{mult}(\alpha, \beta, <; \alpha', \beta'). \qquad \square$$

The following lemma and [Wal19, Proposition 4.2] are a generalisation of [Sho01, Prop. 6.2], where it was shown that $Q_{\boldsymbol{\alpha}} = R_{\boldsymbol{\alpha}}$ for symbols of type B and C, but with strictly stronger conditions on $\boldsymbol{\alpha}$ than the conditions in Lemma 3.1.20 and [Wal19, Proposition 4.2].

Lemma 3.1.20. *Let $k \in \mathbb{Z}_{\geq 0}$, $n = \frac{N-k^2}{2}$ and let $\boldsymbol{\alpha} = (\alpha, \beta) \in \mathcal{P}_2(n)$ such that $\boldsymbol{\alpha} \in H(n,k)$. Let $m_0, m_1 \in \mathbb{N}$ such that $m_1 = m_0 - k \geq n$ and let $(m_0, m_1, <)$ be an order on the set of indices of (α, β) equivalent to $<_{\alpha,\beta,k}$. Then $Q_{\boldsymbol{\alpha},k} = R_{\boldsymbol{\alpha},<_{\alpha,\beta,k}}$ in $\Xi^n_{\mathbb{Q},(m_0,m_1)}[t]$.*

Proof. We show that $R_{\boldsymbol{\alpha}}$ satisfies the conditions (a) and (b) in [Sho01, Theorem 4.4 (ii)] that uniquely characterise $Q_{\boldsymbol{\alpha}}$. It was shown that $R_{\boldsymbol{\alpha}}$ satisfies condition (a) in [Sho01, Proposition 3.12 (i)]. Let $\mathscr{A}$ be the subring of $\mathbb{Q}(t)$ consisting of functions that have no pole at $t = 0$ and let $\mathscr{A}^*$ be the set of units of $\mathscr{A}$. Condition (b) can be formulated as

(b) $R_{\boldsymbol{\alpha}}$ can be expressed as

$$R_{\boldsymbol{\alpha}}(x,t) = \sum_{\boldsymbol{\alpha}'\in\mathcal{P}_2(n)} u_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}(t) s_{\boldsymbol{\alpha}'}(x),$$

where $u_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}(t) \in \mathscr{A}$ such that $u_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(t) \in \mathscr{A}^*$, and such that $u_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}(t) = 0$ if $\boldsymbol{\alpha} \prec \boldsymbol{\alpha}'$ and $\boldsymbol{\alpha} \not\sim \boldsymbol{\alpha}'$.

By [Sho01, Proposition 3.14], $R_{\boldsymbol{\alpha}}$ can be expressed as a linear combination of Schur functions with coefficients in $\mathbb{Z}[t]$, i.e.

$$R_{\boldsymbol{\alpha}}(x,t) = \sum_{(\alpha',\beta')\in\mathcal{P}_2(n)} r_{\boldsymbol{\alpha}}(\alpha',\beta')(t) s_{(\alpha',\beta')}(x), \quad r_{\boldsymbol{\alpha}}(\alpha',\beta')(t) \in \mathbb{Z}[t].$$

We show that for any $\boldsymbol{\alpha}' = (\alpha', \beta') \in \mathcal{P}_2(n)$ such that $r_{\boldsymbol{\alpha}}(\alpha', \beta') \neq 0$, we have

$$\Lambda_{k,-k;2}(\alpha', \beta') \leq \Lambda_{k,-k;2}(\alpha, \beta), \tag{3.1.13}$$

which implies that $a(\boldsymbol{\alpha}') \geq a(\boldsymbol{\alpha})$ and hence $\boldsymbol{\alpha}' \preceq \boldsymbol{\alpha}$ or $\boldsymbol{\alpha} \sim \boldsymbol{\alpha}'$, which in turn implies that $R_{\boldsymbol{\alpha}}$ satisfies condition (b) as above. An expression for $R_{\boldsymbol{\alpha}}$ is given in [Sho01, (3.13.1)] in terms of raising operators. Using this, a more explicit expression for $r_{\boldsymbol{\alpha}}(\alpha', \beta')$ is given in [Wal19, §4.3] as follows. Recall that we assume that $m_0 = m_1 - k \geq n$. Let $I = I_{m_0,m_1}$ be the set of indices of (α, β). For $e = 0, 1$, let $(v_i^e)_{i=1,\dots,m_e}$ denote the canonical basis for $\mathbb{Z}^{m_e}$. Let $\nu_0 = \max((t(\alpha), 0), (t(\beta), 1))$. Let $J = \{v_i^e - v_j^f \in \mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1} \colon ((i,e),(j,f)) \in I^2, e \neq f, (i,e) < (j,f), (i,e) \leq \nu_0\}$ and let $K = \{v_i^e - v_j^f \in \mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1} \colon ((i,e),(j,f)) \in I^2, e = f, (i,e) < (j,f), \nu_0 < (i,e)\}$. Let $\Delta_{m_e} = (m_e - 1, m_e - 2, \dots, 0)$ for $e = 1, 2$ and consider $(\alpha + \Delta_{m_0}, \beta + \Delta_{m_1}) \in \mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1}$. Consider the usual action of the symmetric groups S_{m_0} and S_{m_1} on $\mathbb{Z}^{m_0}$ and $\mathbb{Z}^{m_1}$, respectiveley. For $e \in \{0, 1\}$, $\sigma \in S_{m_e}$, and $x \in \mathbb{Z}^{m_e}$, write $x^\sigma = (x_{\sigma 1}, \dots, x_{\sigma m_e})$. For $(\alpha', \beta') \in \mathcal{P}_{m_0} \times \mathcal{P}_{m_1}$ and $d \in \mathbb{Z}_{\geq 0}$, define

$$\begin{aligned} r_{\boldsymbol{\alpha}}^d(\alpha', \beta') = &\sum_{\sigma \in S_{m_0}, \tau \in S_{m_1}} (\operatorname{sgn}\sigma \operatorname{sgn}\tau) \cdot \\ &|\{X \subseteq J \cup K \colon |X| = d, (\alpha + \Delta_{m_0}, \beta + \Delta_{m_1}) - \sum_{x \in X} x = ((\alpha' + \Delta_{m_0})^\sigma, (\beta' + \Delta_{m_1})^\tau)\}|. \end{aligned} \tag{3.1.14}$$

Then

$$r_{\boldsymbol{\alpha}}(\alpha', \beta') = v_{\alpha,\beta}(t)^{-1} \sum_{d \in \mathbb{Z}_{\geq 0}} (-1)^d r_{\boldsymbol{\alpha}}^d(\alpha', \beta') t^d, \tag{3.1.15}$$

for some non-zero $v_{\alpha,\beta}(t) \in \mathbb{Q}[t]$. We note that we can derive (3.1.15) by keeping track of the raising operators appearing in [Sho01, (3.13.1)]. Note in particular that [Sho01, (3.13.1)] requires that $m_0, m_1 \geq n$, which we indeed assume. Alternatively, we can write (3.1.14) as

$$r_{\boldsymbol{\alpha}}^d(\alpha', \beta') = \sum_{\sigma \in S_{m_0}, \tau \in S_{m_1}} \operatorname{sgn}\sigma \operatorname{sgn}\tau \sum_{X_J \subseteq J \colon |X_J| \leq d} |\mathscr{X}_K(X_J, \sigma, \tau)|,$$

where

$$\begin{aligned} \mathscr{X}_K(X_J, \sigma, \tau) = \{&X_K \subseteq K \colon |X_K| = d - |X_J|, \\ &(\alpha + \Delta_{m_0}, \beta + \Delta_{m_1}) - \sum_{x \in X_K} x = ((\alpha' + \Delta_{m_0})^\sigma, (\beta' + \Delta_{m_1})^\tau) + \sum_{x \in X_J} x\}. \end{aligned}$$

Let $(\alpha', \beta') \in \mathcal{P}_2(n)$ such that $r_{\boldsymbol{\alpha}}(\alpha', \beta') \neq 0$. Then there exist $X_J \subseteq J$, $\sigma \in S_{m_0}$ and $\tau \in S_{m_1}$ such that $\mathscr{X}_K(X_J, \sigma, \tau) \neq \varnothing$. Similar to [Wal19, §4.3 (11)], we can show that if $\mathscr{X}_K(X_J, \sigma, \tau)$ is non-empty, then we must have

$$(\alpha + \Delta_{m_0}, \beta + \Delta_{m_1}) = ((\alpha' + \Delta_n)^\sigma, (\beta' + \Delta_m)^\tau)) + \sum_{x \in X_J} x. \tag{3.1.16}$$

The result [Wal19, §4.3 (11)] is a priori specific for Sp($2n$), as the order on the indices of (α, β) comes from the ordering of the terms of the symbols of Sp($2n$). However, the arguments used in [Wal19, §4.3 (11)] work in the SO(N) setting as well. In particular, one of the arguments requires the use of [Sho01, (3.13.1)], and this result is applicable to our situation since $(m_0, m_1, <)$ satisfies the condition on [Sho01, §3.8 p.666]. In this paper, we shall not include the proof of (3.1.16). We shall mainly focus on the part of the proof of this proposition analogous to [Wal19, §4.4], as this part has some sublte differences compared to the proof in *loc. cit.*.

Let $r = m_0 + m_1$. We can view $\mathbb{R}^r$ as a root space of type A_{r-1} and we shall use the theory of the root system A_{r-1} to prove that $\Lambda(\alpha', \beta') \leq \Lambda(\alpha, \beta)$. Recall that I is the set of indices of (α, β). Similar as before, we identify I with $\{1, \ldots, r\}$, respecting the order $(m_0, m_1, <)$. This identification defines an isomorphism $\iota\colon \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \to \mathbb{R}^r$. For $h \in \mathbb{N}$, let $\mathcal{P}_h$ be the subset of $\mathcal{P}$ consisting of all the partitions of length at most h. Thus $\mathcal{P}_{m_0} \times \mathcal{P}_{m_1}$ is also identified with a subset of $\mathbb{R}^r$ via ι. For $i = 1, \ldots, m_0$ and $j = 1, \ldots, m_1$, let x_i and y_j be the elements of $\{1, \ldots, r'\}$ that are identified with the elements $(i, 0)$ and $(j, 1)$ in I, respectively. Let $I_a = \{x_1, \ldots, x_{m_0}\}$, $I_b = \{y_1, \ldots, y_{m_1}\}$, and let $\varepsilon_1, \ldots, \varepsilon_r$ be the unit vectors in $\mathbb{R}^r$. Then $\Sigma = \{\varepsilon_i - \varepsilon_j\}_{i \neq j}$ in $\mathbb{R}^r$ forms the root system A_{r-1} with positive roots $\Sigma^+ = \{\varepsilon_i - \varepsilon_j\}_{i<j}$. Denote by $W = S_r$ the Weyl group of Σ. Let $\Sigma^M = \{\varepsilon_i - \varepsilon_j \colon i \neq j; i, j \in I_a \text{ or } i, j \in I_b\}$ and $\Sigma^{M,+} = \Sigma^M \cap \Sigma^+$. For $e \in \mathbb{N}$, let $\delta_e = \{(e-1)/2, (e-3)/2, \ldots, (1-e)/2\}$. Then $\delta := \delta_r$ is the half-sum of the positive roots and $\delta^M = \iota(\delta_{m_0}, \delta_{m_1})$ is the half-sum of the roots in $\Sigma^{M,+}$. Let $\mathcal{J} = \iota(J) \subseteq \Sigma^+ \setminus \Sigma^{M,+}$. Let $\bar{C}^+ = \{x \in \mathbb{R}^r \colon x_1 \geq x_2 \geq \dots\}$ be the fundamental Weyl chamber. For all $x \in \mathbb{R}^r$, there exists a $w \in W$ such that $x^+ := wx \in \bar{C}^+$. Define $z \in \mathbb{R}^r$ by

$$z_i = \begin{cases} k + 1 - m_0 & \text{if } i \in I_a, \\ -k + 1 - m_1 & \text{if } i \in I_b. \end{cases}$$

Let

$$(A, B) = (A_{\alpha,\beta,k}, B_{\alpha,\beta,k}) \in \mathcal{R} \times \mathcal{R}, \quad (A', B') = (A_{\alpha',\beta',k}, B_{\alpha',\beta',k}) \in \mathcal{R} \times \mathcal{R}\,.$$

Let $\mathbf{\Lambda} = ((A_1, \ldots, A_{m_0}), (B_1, \ldots, B_{m_1}))$, $\mathbf{\Lambda'} = ((A'_1, \ldots, A'_{m_0}), (B'_1, \ldots, B'_{m_1})) \in \mathbb{Z}^{m_0} \times \mathbb{Z}^{m_1}$. Let $\Lambda = \iota(\mathbf{\Lambda})$ and $\underline{\Lambda} = \iota(\mathbf{\Lambda'})$. Then $\Lambda = \Lambda^+$ by definition of ι. Note that $\underline{\Lambda}^+$ and Λ each form the largest r terms of $\Lambda_{k,-k;2}(\alpha', \beta')$ and $\Lambda_{k,-k;2}(\alpha, \beta)$ respectively, so it suffices to show that $\underline{\Lambda}^+ \leq \Lambda$, since we then immediately get (3.1.13).

We first describe $\underline{\Lambda}$. Note that (α', β') can be considered as an element of $\mathcal{P}_{m_0} \times \mathcal{P}_{m_1}$, and so we can consider $\iota(\alpha', \beta')$. Let $X_J \subseteq J$, $\sigma \in S_{m_0}$ and $\tau \in S_{m_1}$ such that (3.1.16) holds. Since $\sigma \times \tau$ fixes $(\Delta_{m_0} - \delta_{m_0}, \Delta_{m_1} - \delta_{m_1})$, we can replace Δ_{m_0} and Δ_{m_1} by δ_{m_0} and δ_{m_1} in (3.1.16), respectively. Note that W^M is identified with $S_{m_0} \times S_{m_1}$ via ι. Let w^M be the element of W^M corresponding to $(\sigma \times \tau)^{-1} \in S_{m_0} \times S_{m_1}$ and let $X = \iota(X_J)$. Applying ι to both sides of (3.1.16) and rearranging, we find

$$\iota(\alpha', \beta') + \delta^M = w^M(\iota(\alpha, \beta) + \delta^M - \sum_{x \in X} x).$$

Note that $\underline{\Lambda} = \iota(\alpha', \beta') + 2\delta^M + z$ and $w^M z = z$, so we have

$$\begin{aligned}
\underline{\Lambda} &= w^M(\iota(\alpha, \beta) + \delta^M - \sum_{x \in X}) + \delta^M + z \\
&= w^M(\Lambda - \delta^M - z - \sum_{x \in X} x) + \delta^M + z \\
&= w^M(\Lambda - \delta^M - \sum_{x \in X} x) + \delta^M. \qquad (3.1.17)
\end{aligned}$$

Next, we want to prove (3.1.18), which is a ‘substitute’ for a certain property in [Wal19, Proposition 4.2] that does not hold in our setting. In the $\mathrm{Sp}(2n)$ setting of [Wal19, Proposition 4.2], it holds that $\Lambda - \delta \in C^+$, and this is a property that was used in the proof of the proposition. In our setting, we generally do not have $\Lambda - \delta \in \bar{C}^+$. In fact, let $m_0' \in \mathbb{N}$ be the smallest integer such that $m_0' \geq t(\alpha)$ and $m_1' := m_0' - k \geq t(\beta)$. Let $\bar{\Lambda} = \Lambda - \delta$ and $r' = m_0' + m_1' = t(\lambda)$. Since $(\alpha, \beta) \in H(n, k)$, we have $\Lambda_1 > \Lambda_2 > \cdots > \Lambda_{r'}$. Thus if $m_0 = m_0'$, and hence $m_1 = m_1'$, we have $\bar{\Lambda} \in \bar{C}^+$. However, we very well may have $m_0 > m_0'$, and thus $m_1 > m_1'$, in which case we have $\Lambda_{r'+1} = \Lambda_{r'+2}$, hence $\bar{\Lambda}_{r'+1} < \bar{\Lambda}_{r'+2}$ and so $\bar{\Lambda} \notin \bar{C}^+$.

Let $d_1 = m_0 - m_0' = m_1 - m_1' = \frac{1}{2}(r - r')$. Then

$$(\bar{\Lambda}_{r'+1}, \bar{\Lambda}_{r'+2}, \ldots, \bar{\Lambda}_r) = (x, x+1, x, x+1, \ldots, x, x+1)$$

where $x = \bar{\Lambda}_{r'+1}$. Let

$$\bar{X} = \{\varepsilon_{\bar{r}+2i-1} - \varepsilon_{r-2i+2} \colon i = 1, \ldots, \lceil d_1/2 \rceil\} \subseteq \Sigma^+.$$

For $i > \bar{m}_0$ and $j > \bar{m}_1$, we may assume that $(i, 0) < (j, 1) < (i+1, 0) < \ldots$, since the equivalence class of the order $<$ on the set of indices of (α, β) only depends on the ordering of the indices smaller than $\nu_0 = \max((t(\alpha), 0), (t(\beta), 1))$. Thus we have $\bar{X} \subseteq \Sigma^+ \setminus \Sigma^{M,+}$. Furthermore, note that $\Lambda_{r'} > \Lambda_{r'+1} + 1$ since $\lambda_r' > 0 = \lambda_{r'+1}$, so $\bar{\Lambda}_{r'} \geq \bar{\Lambda}_{r'+1} + 1 = x + 1$. Thus we have

$$(\Lambda - \delta) + \sum_{x \in \bar{X}} x = (\bar{\Lambda}_1 \geq \cdots \geq \bar{\Lambda}_{r'} \geq x + 1 = \cdots = x + 1 \geq x = \cdots = x) \in \bar{C}^+, \qquad (3.1.18)$$

where the $x + 1$ terms appear d_1 times and the x terms also appear d_1 times.

Next, we will further describe $\underline{\Lambda}$. Note that

$$\delta - \delta^M = \frac{1}{2} \sum_{x \in \Sigma^+ \setminus \Sigma^{M,+}} x = \frac{1}{2} \left(\sum_{x \in X} x + \sum_{x \in \Sigma^+ \setminus (\Sigma^{M,+} \cup X)} x \right).$$

Recall that $\bar{X} \subseteq \Sigma^+ \setminus \Sigma^{M,+}$ and let $X' = X \cup \bar{X} \subseteq \Sigma^+ \setminus \Sigma^{M,+}$. Note that $X \cap \bar{X} = \varnothing$, since for $\varepsilon_i - \varepsilon_j \in \bar{X}$, we have $\max\{x_{t(\alpha)}, y_{t(\beta)}\} \leq r' < i < j$, whereas for $\varepsilon_i - \varepsilon_j \in X$, we have $i < \max\{x_{t(\alpha)}, y_{t(\beta)}\}$. Thus we have

$$\begin{aligned} \mu := \delta^M - \delta + \sum_{x \in X} x + \sum_{x \in \bar{X}} x &= \frac{1}{2} \left(\sum_{x \in X} x - \sum_{x \in \Sigma^+ \setminus (\Sigma^{M,+} \cup X)} x \right) + \sum_{x \in \bar{X}} x \qquad (3.1.19) \\ &= \frac{1}{2} \left(\sum_{x \in X'} x - \sum_{x \in \Sigma^+ \setminus (\Sigma^{M,+} \cup X')} x \right). \end{aligned}$$

Let $\mu' = -\mu + (w^M)^{-1}(\delta^M)$. Now (3.1.17) and (3.1.19) together give

$$\underline{\Lambda} = w^M(\Lambda - \delta - \mu + \sum_{x \in \bar{X}} x) + \delta^M = w^M(\Lambda - \delta + \mu' + \sum_{x \in \bar{X}} x).$$

There exists a $w \in W$ such that

$$\begin{aligned} \underline{\Lambda}^+ &= w(w^M)^{-1}\underline{\Lambda} \\ &= w(\Lambda - \delta + \sum_{x \in \bar{X}} x + \mu') \\ &= w(\Lambda - \delta + \sum_{x \in \bar{X}} x) + w(\mu') + (\Lambda - \delta + \sum_{x \in \bar{X}} x) - (\Lambda - \delta + \sum_{x \in \bar{X}} x) \\ &= \Lambda + w(\Lambda - \delta + \sum_{x \in \bar{X}} x) - (\Lambda - \delta + \sum_{x \in \bar{X}} x) + w(\mu') - \delta + \sum_{x \in \bar{X}} x. \end{aligned}$$

We will now show that $\underline{\Lambda}^+ \leq \Lambda$. Since $(\Lambda - \delta + \sum_{x \in \bar{X}} x) \in \bar{C}^+$ by (3.1.18), we have

$$w(\Lambda - \delta + \sum_{x \in \bar{X}} x) - (\Lambda - \delta + \sum_{x \in \bar{X}} x) \in -{}^+\bar{C}. \qquad (3.1.20)$$

Next, we show that $w(\mu') - \delta \in -{}^+\bar{C}$. Let

$$\begin{aligned} E &:= \left\{ \frac{1}{2}(\sum_{y \in Y} y - \sum_{y \in \Sigma^+ \setminus Y} y) \in \mathbb{R}^r : Y \subseteq \Sigma^+ \right\}, \\ E^M &:= \left\{ \frac{1}{2}(\sum_{y \in Y} y - \sum_{y \in \Sigma^{M,+} \setminus Y} y) \in \mathbb{R}^r : Y \subseteq \Sigma^{M,+} \right\}. \end{aligned}$$

Note that E and E^M are fixed by the action of W and W^M on $\mathbb{R}^r$, respectively. For each $H^M \in E^M$, it is easy to see that $-\mu + H^M \in E$. Thus since $\delta^M \in E^M$, we have $\mu' \in E$, and so $w(\mu') \in E$. For any $H \in E$, we have $H - \delta \in -{}^+\bar{C}$, so in particular, we have

$$w(\mu') - \delta \in -{}^+\bar{C}.$$

We now have

$$\underline{\Lambda} - \sum_{x \in \bar{X}} x \in \Lambda - {}^+\bar{C} \quad \text{i.e.} \quad \underline{\Lambda} - \sum_{x \in \bar{X}} x \leq \Lambda.$$

As noted before, if $\varepsilon_i - \varepsilon_j \in \bar{X}$, then $\iota(\nu_0) < r' < i < j$. Thus for $\ell = 1, \ldots, r'$, the ℓ^{th} term of $\underline{\Lambda} - \sum_{x \in \bar{X}}$ is equal to $\underline{\Lambda}_\ell$, so

$$\sum_{i=1}^{\ell} \underline{\Lambda}_\ell \leq \sum_{i=1}^{\ell} \Lambda_\ell.$$

Now suppose $\ell \in \{r'+1, \ldots, r\}$. Let $\kappa = [k, -\infty[_2 \sqcup [-k, -\infty[_2$. As noted in Remark 3.1.4, the first $r' = m_0' + m_1'$ terms of Λ are the first m_0' terms of $\alpha + [k, -\infty[_2$ and the first m_1' terms of $\beta + [-k, -\infty[_2$. Thus

$$\begin{aligned}
\sum_{i=1}^{\ell} \Lambda_i &= \sum_{i=1}^{r'} \Lambda_i + \sum_{i=\bar{r}+1}^{\ell} \Lambda_i \\
&= \sum_{i=1}^{m_0'} (\alpha_i + k + 2 - 2i) + \sum_{i=1}^{m_1'} (\beta_i - k + 2 - 2i) + \sum_{i=r'+1}^{\ell} \kappa_i \\
&= \sum_{i=1}^{m_0'} \alpha_i + \sum_{i=1}^{m_1'} \beta_i + \sum_{i=1}^{\ell} \kappa_i \\
&= n + \sum_{i=1}^{\ell} \kappa_i,
\end{aligned}$$

where the last equality follows since $m_0' \geq t(\alpha)$ and $m_1' \geq t(\beta)$. The first ℓ terms of $\underline{\Lambda}^+$ are the first ℓ terms of $\Lambda_{k,-k;2}(\alpha', \beta') = (\alpha + [k, -\infty[_2) \sqcup (\beta + [-k, -\infty[_2)$. Thus we have

$$\sum_{i=1}^{\ell} \underline{\Lambda}_i^+ \leq \sum_{i=1}^{\ell} (\alpha \sqcup \beta)_i + \sum_{i=1}^{\ell} \kappa_i \leq n + \sum_{i=1}^{\ell} \kappa_i = \sum_{i=1}^{\ell} \Lambda_i.$$

To conclude, we have now shown that $\sum_{i=1}^{\ell} \underline{\Lambda}_i^+ \leq \sum_{i=1}^{\ell} \Lambda_i$ for $\ell = 1, \ldots, r$, i.e. $\underline{\Lambda}^+ \leq \Lambda$, as desired. □

Proof of Proposition 3.1.17. Since λ only has odd parts, we have $(\alpha, \beta) \in H((N - k^2)/2, k)$ by Remark 1.2.5, and so by Lemma 3.1.20, we have $Q_{(\alpha,\beta),k} = R_{(\alpha,\beta),<}$. Thus Proposition 3.1.17(1) follows from Lemma 3.1.19.

Suppose that $k = 0$. Then $N = 2n$ is even. Let P^B and P^D be the matrix of Green functions as in Theorem 3.1.8 for $k = 0$ and for $W(B_n)$ and $W(D_n)$, respectively. Note that the Green functions evaluated in $t = 1$ are the same as the Kostka polynomials evaluated in $t = 1$. Also note that $c_{(\alpha,\beta)} = 2$ since λ only has odd parts and hence non-degenerate. So for $i \in \{1, 2/c_{(\alpha',\beta')}\}$, Proposition 3.1.13(1) gives

$$p^D_{(\alpha',\beta')^i,(\alpha,\beta)}(1) = \frac{c_{(\alpha',\beta')}}{2}(p^B_{(\alpha',\beta'),(\alpha,\beta)}(1) + p^B_{(\alpha',\beta'),(\beta,\alpha)}(1)).$$

Since λ is non-degenerate, we have $\alpha \neq \beta$. By (3.1.4), we have $p^B_{(\alpha',\beta'),(\beta,\alpha)}(1) = p^B_{(\beta',\alpha'),(\alpha,\beta)}(1)$, so by Proposition 3.1.13(1) and Lemma 3.1.19, we have

$$\begin{aligned}\operatorname{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^\pm_{\lambda'}, \mathcal{E}^\pm_{[\varepsilon']}) = p^D_{(\alpha',\beta')^i,(\alpha,\beta)}(1) &= \frac{c_{(\alpha',\beta')}}{2}(p^B_{(\alpha',\beta'),(\alpha,\beta)}(1) + p^B_{(\beta',\alpha'),(\alpha,\beta)}(1)) \\ &= \frac{c_{(\alpha',\beta')}}{2}(\operatorname{mult}(\alpha, \beta, <; \alpha', \beta') + \operatorname{mult}(\alpha, \beta, <; \beta', \alpha')).\end{aligned}$$ □

3.2 Maximality and minimality theorems

3.2.1 Maximality and minimality results for SL(N)

Let $G = \mathrm{SL}(N)$. For details about the generalised Springer correspondence of this group, we refer to [Lus84b, §10.3] and [LS85, §5]; also see [CMO23, §5.6.1]. Let $(C, \mathcal{E}) \in \mathcal{N}_{\mathrm{G}}$ and suppose C is parametrised by some partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ of N. It is a well-known fact that $A_G(C) \cong \mathbb{Z}/n'\mathbb{Z}$ where $n' = \gcd(\lambda_1, \ldots, \lambda_t)$. Denote by ε the representation of $\mathbb{Z}/n'\mathbb{Z}$ corresponding to $\mathcal{E}$. Let d be the degree of ψ in $\mathbb{Z}/n'\mathbb{Z}$. Lusztig showed that $d \mid \lambda_i$ for $i = 1, \ldots, t$. Next, $\mathrm{GSpr}(e, \phi)$ is the representation $\rho_{\lambda/d}$ of the relative Weyl group $W_L \cong S_{n/d}$ corresponding to the partition $(\lambda_1/d, \ldots, \lambda_t/d)$ (denoted by λ/d) of n/d.

The combinatorial results regarding the Green functions for type A in [Mac98, III.6.] tell us that $(\mathrm{C}^{\max}, \mathcal{E}^{\max})$ exists and that it corresponds to the representation $\rho_{(n/d)}$ of $S_{n/d}$. From the generalised Springer correspondence, we then see that $\mathrm{C}^{\max}$ is parametrised by $\lambda^{\max} = (n)$ and $\mathcal{E}^{\max}$ corresponds to $\varepsilon^{\max} = \varepsilon$.

Tensoring a symmetric group representation with the sign representation corresponds to transposing the corresponding partition, so $\rho_{(n/d)} \otimes \mathrm{sgn} = \rho_{(1)^{n/d}}$ is the representation $E(N_0, \psi_0)$ in Proposition 4.1.1 in the current setting. Furthermore, this is an order-reversing operation, hence we may deduce that $\lambda^{\min} = (d)^{n/d}$ and $\varepsilon^{\min} = \varepsilon$ (here $\lambda^{\min}$ and $\varepsilon^{\min}$ are as in Theorem 3.2.6, but for SL(N) in an appropriate sense).

3.2.2 Maximality theorem for SO(N)

Lemma 3.2.1. *Let $(\lambda, [\varepsilon]), (\lambda', [\varepsilon']) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(2N)$. Then $\lambda \leq \lambda'$ if and only if $p_{\lambda,[\varepsilon]} \leq p_{\lambda',[\varepsilon']}$.*

Proof. Let $c \in \mathbb{N}$. We have

$$S_c(\lambda) = S_c(\lambda + [0, -\infty[_1) - S_c([0, -\infty[_1).$$

Recall z and z' from Section 1.2.3. Suppose c is even. If $S_c(\lambda)$ is odd, then the largest c terms of $\lambda + [0, -\infty[_1$ are $z_1, \dots, z_{c/2-1}$ and $z'_1, \dots, z'_{c/2+1}$, and we have that $\lambda_c = \lambda_{c+1}$ is even and $z'_{c/2+1} = z_{c/2}$. If $S_c(\lambda)$ is even, then $z_1, \dots, z_{c/2}$ and $z'_1, \dots, z'_{c/2}$. Suppose c is odd. If $S_c(\lambda)$ is even, then the largest c terms of $\lambda + [0, -\infty[_1$ are $z_1, \dots, z_{(c+1)/2}$ and $z'_1, \dots, z'_{(c-1)/2}$, and we have that $\lambda_c = \lambda_{c+1}$ is even and $z'_{(c+1)/2} = z_{(c+1)/2}$. If $S_c(\lambda)$ is odd, then $z_1, \dots, z_{(c-1)/2}$ and $z'_1, \dots, z'_{(c+1)/2}$. Let $c^+ = \lceil c/2 \rceil$, $c^- = \lfloor c/2 \rfloor$ and $\delta_c(\lambda) = 1$ if $S_c(\lambda) + c$ is odd and $\delta_c(\lambda) = 0$ otherwise. Then

$$S_c(\lambda + [0, -\infty[_1) = 2S_{c^+}(z') + 2S_{c^-}(z) - c^+ + \delta_c(\lambda).$$

Recall $A^\#$ and $B^\#$ from Section 1.2.3. By definition, we have

$$\begin{aligned} S_{c^+}(z') &= S_{c^+}(A^\#) - S_{c^+}([0, -\infty[_1), \\ S_{c^-}(z) &= S_{c^-}(B^\#) - S_{c^-}([0, -\infty[_1), \end{aligned}$$

and since $A^\#_1 \geq B^\#_1 \geq A^\#_2 \geq B^\#_2 \geq \dots$, we have

$$S_{c^+}(A^\#) + S_{c^-}(B^\#) = S_c(A^\# \sqcup B^\#) = S_c(p_{\lambda,\varepsilon}).$$

Hence we have

$$S_c(\lambda) = 2S_c(p_{\lambda,\varepsilon}) + \delta_c(\lambda) + C_c, \tag{3.2.1}$$

for some $C_c \in \mathbb{Z}$ independent of λ.

Suppose $\lambda < \lambda'$. Then for all $c \in \mathbb{N}$, we have $S_c(\lambda) \leq S_c(\lambda')$, and so by (3.2.3), we have

$$S_c(p_{\lambda,\varepsilon}) \leq S_c(p_{\lambda',\varepsilon'}) + \frac{\delta_c(\lambda') - \delta_c(\lambda)}{2} \leq S_c(p_{\lambda',\varepsilon'}) + \frac{1}{2},$$

hence $S_c(p_{\lambda,\varepsilon}) \leq S_c(p_{\lambda',\varepsilon'})$ since both are integers. Thus $p_{\lambda,[\varepsilon]} \leq p_{\lambda',[\varepsilon']}$.

Conversely, suppose that $p_{\lambda,[\varepsilon]} \leq p_{\lambda',[\varepsilon']}$. Then for all $c \in \mathbb{N}$, we have $S_c(p_{\lambda,\varepsilon}) \leq S_c(p_{\lambda',\varepsilon'})$ and hence by (3.2.3), we have

$$S_c(\lambda) \leq S_c(\lambda') + \delta_c(\lambda). \tag{3.2.2}$$

If $c = 1$, then obviously $S_c(\lambda) \leq S_c(\lambda')$, so assume $c > 1$. If $c + S_c(\lambda)$ is even, then $S_c(\lambda) \leq S_c(\lambda')$. Suppose $c + S_c(\lambda)$ is odd and suppose $S_c(\lambda) = S_c(\lambda') + 1$. Then $\lambda_c = \lambda_{c+1}$ is even. Thus $c - 1 + S_{c-1}(\lambda)$ is even and so $S_{c-1}(\lambda) \leq S_{c-1}(\lambda')$ by (3.2.2). Thus $\lambda_c \geq \lambda'_c + 1$ and so

$$\lambda_{c+1} = \lambda_c \geq \lambda'_c + 1 \geq \lambda'_{c+1} + 1.$$

Now we have

$$S_{c+1}(\lambda) = S_c(\lambda) + \lambda_{c+1} \geq (S_c(\lambda') + 1) + (\lambda'_{c+1} + 1) > S_{c+1}(\lambda'). \tag{3.2.3}$$

But $c + 1 + S_{c+1}(\lambda) = (c + S_c(\lambda) + (1 + \lambda_{c+1})$ is even since $\lambda_{c+1} = \lambda_c$ is even, and so $S_{c+1}(\lambda) \leq S_{c+1}(\lambda')$, which contradicts (3.2.3). Thus $S_C(\lambda) \leq S_c(\lambda')$. We conclude that $\lambda \leq \lambda'$. □

Recall that for $(\lambda, [\varepsilon]), (\lambda', [\varepsilon']) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ with λ non-degenerate, it follows from Proposition 3.1.17 that $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^+_{\lambda'}, \mathcal{E}^+_{[\varepsilon']}) = \mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^-_{\lambda'}, \mathcal{E}^-_{[\varepsilon']})$.

Theorem 3.2.2. *Suppose* $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ *such that* λ *only has odd parts. Then there exists a unique* $(\lambda^{max}, [\varepsilon^{\max}]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ *such that* $\lambda^{\max}$ *is non-degenerate and*

(1) $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C_{\lambda^{\max}}, \mathcal{E}_{[\varepsilon^{\max}]}) = 1,$

(2) For all $(\lambda', [\varepsilon']) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ *with* $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^+_{\lambda'}, \mathcal{E}^+_{[\varepsilon']}) = \mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^-_{\lambda'}, \mathcal{E}^-_{[\varepsilon']}) \neq 0,$ *we have* $\lambda' < \lambda^{max}$ *or* $(\lambda', [\varepsilon']) = (\lambda^{max}, [\varepsilon^{\max}])$.

Proof. Let $k = k(\lambda, [\varepsilon])$ and $(\alpha, \beta)_k = \Phi_N(\lambda, [\varepsilon])$. We have $(\alpha, \beta) \in H((N - k^2)/2, k)$ by Remark 1.2.5. Denote $<_{\alpha,\beta,k}$ by $<$. By Proposition 3.1.16 and Lemma 3.1.5, there exists a unique $(\alpha^{\max}, \beta^{\max}) \in \mathcal{P}_2((N - k^2)/2)$ such that

(i) $\mathrm{mult}(\alpha, \beta, <; \alpha^{\max}, \beta^{\max}) = 1,$

(ii) For all $(\alpha', \beta') \in \mathcal{P} \times \mathcal{P}$ with $\mathrm{mult}(\alpha, \beta, <; \alpha', \beta') \neq 0$, we have $\Lambda_{k,-k;2}(\alpha', \beta') < \Lambda_{k,-k;2}(\alpha^{\max}, \beta^{\max})$ or $(\alpha', \beta')_k = (\alpha^{\max}, \beta^{\max})_k$ (note that is an equality of ordered (resp. unordered) pairs if $k > 0$ (resp. $k = 0$)).

Furthermore, $(\alpha^{\max}, \beta^{\max})$ is the unique element of $P_{k,-k;2}(\alpha, \beta, <)$. By Proposition 3.1.6, the largest two terms of $\Lambda_{k,-k;2}(\alpha^{\max}, \beta^{\max})$ are distinct, so $\alpha^{\max} \neq \beta^{\max}$.

Suppose $k > 0$. Then Proposition 3.1.17(1) and Lemma 3.2.1 show that $\Phi_N^{-1}(\alpha^{\max}, \beta^{\max})$ satisfies (1) and (2).

Suppose $k = 0$ and let $(\bar{\lambda}, \bar{\varepsilon}) = \Phi_N^{-1}(\alpha^{\max}, \beta^{\max})$. By Proposition 3.1.17(2) and (i), we have

$$\begin{aligned}\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C_{\bar{\lambda}}, \mathcal{E}_{[\bar{\varepsilon}]}) &= \mathrm{mult}(\alpha, \beta, <; \alpha^{\max}, \beta^{\max}) + \mathrm{mult}(\alpha, \beta, <; \beta^{\max}, \alpha^{\max}) \\ &= 1 + \mathrm{mult}(\alpha, \beta, <; \beta^{\max}, \alpha^{\max}).\end{aligned}$$

Suppose that $\mathrm{mult}(\alpha, \beta, <; \beta^{\max}, \alpha^{\max}) \neq 0$. Since $\alpha^{\max} \neq \beta^{\max}$, we have $(\beta^{\max}, \alpha^{\max}) \notin P_{0,0;-2}(\alpha, \beta, <)$, but since $\Lambda_{0,0;2}(\beta^{\max}, \alpha^{\max}) = \Lambda_{0,0;2}(\alpha^{\max}, \beta^{\max})$, we have $\mathrm{mult}(\alpha, \beta, <; \beta^{\max}, \alpha^{\max}) = 0$ by Proposition 3.1.16(2). Hence $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C_{\lambda^{\max}}, \mathcal{E}_{[\varepsilon^{\max}]}) = 1$. Next, let $(\lambda', [\varepsilon']) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ such that $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^{\pm}_{\lambda'}, \mathcal{E}^{\pm}_{[\varepsilon']}) \neq 0$ and let $\{\alpha', \beta'\} = \Phi_N(\lambda', [\varepsilon'])$. Since $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^{\pm}_{\lambda'}, \mathcal{E}^{\pm}_{[\varepsilon']}) \neq 0$, we have either $\mathrm{mult}(\alpha, \beta, <; \alpha', \beta') \neq 0$ or $\mathrm{mult}(\alpha, \beta, <; \beta', \alpha') \neq 0$ by Proposition 3.1.17(2), and so $\Lambda_{0,0;2}(\alpha', \beta') = \Lambda_{0,0;2}(\beta', \alpha') < \Lambda_{0,-0;2}(\alpha^{\max}, \beta^{\max})$ or $\{\alpha', \beta'\} = \{\alpha^{\max}, \beta^{\max}\}$. Thus $(\bar{\lambda}, \bar{\varepsilon})$ satisfies (2) by Lemma 3.2.1.

Uniqueness follows easily from the fact that $P_{k,-k;2}(\alpha, \beta, <)$ has a unique element. Hence we conclude that $(\lambda^{\max}, [\varepsilon^{\max}]) = \Phi_N^{-1}(\alpha^{\max}, \beta^{\max})$ is the unique element of $\boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ that satisfies (1) and (2), and that $\lambda^{\max}$ is non-degenerate, since $\alpha^{\max} \neq \beta^{\max}$. □

Remark 3.2.3. We only showed that $\lambda^{\max}$ is non-degenerate, but we shall later see from a corollary of Theorem 3.3.1 that $\lambda^{\max}$ in fact only has odd parts.

3.2.3 Minimality theorem for $\mathrm{SO}(N)$

We state a result from [Wal19, §4.6] without proof. For $\mu \in \mathcal{P}$, write ${}^t\mu$ for the transpose of μ and let $R_\mu = \mu + [0, -\infty[_1$.

Lemma 3.2.4. *For all $\mu \in \mathcal{P}$ and $x \in \mathbb{Z}$, we have*

$$\mathrm{mult}_{R_\mu}(x) + \mathrm{mult}_{R_{{}^t\mu}}(1 - x) = 1.$$

For $\mu \in \mathcal{R}$, let $\mu^2 = \mu \sqcup \mu$. Let $n \in \mathbb{N}$ and let $(\alpha, \beta) \in \mathcal{P}_2(n)$. As representations of $W(B_n)$, we have $\mathrm{sgn} \otimes \rho_{(\alpha,\beta)} = \rho_{({}^t\beta, {}^t\alpha)}$, where sgn is the sign representation of $W(B_n)$. Let $i \in \{1, 2/c_{(\alpha,\beta)}\}$ and note that $c_{(\alpha,\beta)} = c_{({}^t\beta, {}^t\alpha)}$. As representations of $W(D_n)$, we have $\mathrm{sgn} \otimes \rho_{(\alpha,\beta)_i} = \rho_{({}^t\beta, {}^t\alpha)_i}$, where sgn is the sign representation of $W(D_n)$. Let $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$, $k = k(\lambda, [\varepsilon])$. If $(\alpha, \beta))_k = \Phi_N(\lambda, [\varepsilon])$, then let $({}^s\lambda, {}^s\varepsilon) = \Phi_N^{-1}(({}^t\beta, {}^t\alpha)_k)$.

Lemma 3.2.5. *We have*

$$2\Lambda_{k/2,-k/2;2}(\alpha, \beta) = {}^t({}^s\lambda) + ([0, -\infty[_1)^2.$$

Proof. Let $U = 2\alpha + [k, -\infty[_1$, $V = 2\beta + [-k, -\infty[_1$, $X = ({}^t\beta)^2 + [k, -\infty[_1$, $Y = ({}^t\alpha)^2 + [-k, -\infty[_1$. Then

$$2\Lambda_{k/2,-k/2;2}(\alpha, \beta) = U \sqcup V. \tag{3.2.4}$$

For $x \in \mathbb{Z}$, we have $\mathrm{mult}_U(x) = \mathrm{mult}_{R_{2\alpha}}(x - k)$ and $\mathrm{mult}_Y(x) = \mathrm{mult}_{R_{({}^t\alpha)^2}}(x + k)$. Hence by Lemma 3.2.4, we have

$$\mathrm{mult}_U(x) + \mathrm{mult}_Y(1 - x) = 1, \tag{3.2.5}$$

and similarly

$$\mathrm{mult}_V(x) + \mathrm{mult}_X(1 - x) = 1. \tag{3.2.6}$$

Write $(\mu, \tau) = ({}^s\lambda, {}^s\varepsilon)$. For $\nu \in \mathcal{R}$, write $\nu - 1 = (\nu_1 - 1, \nu_2 - 1 \ldots)$. We have

$$p_{\mu,\tau} = \Lambda_{k,-k;2}({}^t\beta, {}^t\alpha) = ({}^t\beta + [k, -\infty[_2) \sqcup ({}^t\alpha + [-k, -\infty[_2).$$

Note that $X \sqcup Y = p_{\mu,\tau} \sqcup (p_{\mu,\tau} - 1)$, so (3.2.4), (3.2.5) and, (3.2.6) give

$$\mathrm{mult}_{2\Lambda_{k/2,-k/2;2}(\alpha,\beta)}(x) + \mathrm{mult}_{p_{\mu,\tau} \sqcup (p_{\mu,\tau}-1)}(1 - x) = 2. \tag{3.2.7}$$

Define $\mu', \mu'' \in \mathcal{R}$ by $\mu'_j = \lfloor \mu_j/2 \rfloor$ and $\mu''_j = \lceil \mu_j/2 \rceil$ for all $j \in \mathbb{N}$. Then $\mu = \mu' + \mu''$. We show that for all $x \in \mathbb{Z}$, we have

$$p_{\mu,\tau} \sqcup (p_{\mu,\tau} - 1) = R_{\mu'} \sqcup R_{\mu''}. \tag{3.2.8}$$

Let $i \in \mathbb{N}$. Suppose μ_i is odd. If $i = 2j + 1$ is odd, then

$$(A^{\#}_\mu)_{j+1} = \frac{\mu_i + (1 - i) + 1}{2} + 1 - \frac{i + 1}{2} = \frac{\mu_i + 1}{2} + 1 - i = (R_{\mu''})_i$$

and so $(A^{\#}_\mu)_j - 1 = (R_{\mu'})_i$. If $i = 2j$ is even, then

$$(B^{\#}_\mu)_j = \frac{\mu_i + 1 - i}{2} + 1 - \frac{i}{2} = \frac{\mu_i + 1}{2} + 1 - i = (R_{\mu''})_i$$

and so $(B^{\#}_\mu)_j - 1 = (R_{\mu'})_i$. Since μ is orthogonal, we can partition the set consisting of the $i \in \mathbb{N}$ for which μ_i is even into pairs such that each pair contains consecutive integers $i, i + 1$ and such that $\mu_i = \mu_{i+1}$. For pairs of the form $(2j - 1, 2j)$, $S_{2j}(\lambda)$ is even and for pairs of the form $(2j, 2j + 1)$, $S_{2j+1}(\lambda)$ is odd.

Consider a pair $(i, i + 1)$ with $i = 2j - 1$, $i + 1 = 2j$ and $\mu_i = \mu_{i+1}$ even. Then

$$(A^{\#}_\mu)_j = (B^{\#}_\mu)_j = \frac{\mu_i + 1 - i}{2} + 1 - j = \frac{\mu_i}{2} + 1 - i = (R_{\mu'})_i = (R_{\mu''})_i,$$
$$(A^{\#}_\mu)_j - 1 = (B^{\#}_\mu)_j - 1 = (R_{\mu'})_{i+1} = (R_{\mu''})_{i+1}.$$

Consider a pair $(i, i + 1)$ with $i = 2j$, $i + 1 = 2j + 1$ and $\mu_i = \mu_{i+1}$ even. Then

$$(A^{\#}_\mu)_{j+1} = (B^{\#}_\mu)_j = \frac{\mu_i + (1 - i) + 1}{2} + 1 - j = \frac{\mu_{i+1}}{2} + 2 - i = (R_{\mu'})_{i+1} = (R_{\mu''})_{i+1},$$
$$(A^{\#}_\mu)_{j+1} - 1 = (B^{\#}_\mu)_j - 1 = (R_{\mu'})_i = (R_{\mu''})_i.$$

We conclude that (3.2.8) holds.

We apply Lemma 3.2.4 to μ' and μ'', which together with (3.2.7) and (3.2.8) gives $\mathrm{mult}_{2\Lambda_{k/2,-k/2;2}(\alpha,\beta)}(x) = \mathrm{mult}_{R_{{}^t\mu'}}(x) + \mathrm{mult}_{R_{{}^t\mu''}}(x)$ for all $x \in \mathbb{Z}$, hence

$$2\Lambda_{k/2,-k/2;2}(\alpha, \beta) = R_{{}^t\mu'} \sqcup R_{{}^t\mu''}. \tag{3.2.9}$$

We show that ${}^t\mu''_j \geq {}^t\mu'_j \geq {}^t\mu''_{j+1}$ for all $j \in \mathbb{N}$. Let $h = {}^t\mu'_j$. Then $\mu'_h \geq j$, and since $\mu''_h \geq \mu'_h$, we have ${}^t\mu''_j \geq h$. Now let $h = {}^t\mu''_{j+1}$. Then $\mu''_h \geq j + 1$, so $\mu'_h \geq \mu''_h - 1 \geq j$ and so ${}^t\mu'_j \geq h$, as desired.

Since $({}^t\mu' \sqcup {}^t\mu'') = {}^t(\mu' + \mu'') = {}^t\mu$, we now have

$$R_{{}^t\mu'} \sqcup R_{{}^t\mu''} = ({}^t\mu''_1, {}^t\mu'_1, {}^t\mu''_2 - 1, {}^t\mu'_2 - 1, \dots) = ({}^t\mu' \sqcup {}^t\mu'') + ([0, -\infty[_1)^2 = {}^t\mu + ([0, -\infty[_1)^2,$$

which together with (3.2.9) finishes the proof. □

For the following, note that for $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$, it holds that ${}^s\lambda^{\min}$ is degenerate if and only if $\lambda^{\min}$ is degenerate.

Theorem 3.2.6. *Let $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ with λ only consisting of odd parts. Then there exists a unique $(\lambda^{\min}, \varepsilon^{\min}) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ such that ${}^s\lambda^{\min}$ is non-degenerate and*

1. $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C_{{}^s\lambda^{\min}}, \mathcal{E}_{[{}^s\varepsilon^{\min}]}) = 1$,
2. *For all $(\lambda', [\varepsilon']) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ with* $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^+_{{}^s\lambda'}, \mathcal{E}^+_{[{}^s\varepsilon']}) = \mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^-_{{}^s\lambda'}, \mathcal{E}^-_{[{}^s\varepsilon']}) \neq 0$*, we have $\lambda^{\min} < \lambda'$ or $(\lambda', [\varepsilon']) = (\lambda^{\min}, [\varepsilon^{\min}])$.*

Furthermore, we have $(\lambda^{\max}, [\varepsilon^{\max}]) = ({}^s\lambda^{\min}, [{}^s\varepsilon^{\min}])$.

Proof. Let $k = k(\lambda, [\varepsilon])$. Let $(\lambda', [\varepsilon']), (\lambda'', \varepsilon'') \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ such that $k(\lambda', [\varepsilon']) = k(\lambda'', \varepsilon'') = k$. Let $(\alpha', \beta')_k = \Phi_N(\lambda', [\varepsilon'])$ and $(\alpha'', \beta'')_k = \Phi_N(\lambda'', \varepsilon'')$.

The theorem is equivalent to the following statement: there exists unique a $(\underline{\lambda}, \underline{\varepsilon}) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ such that $\bar{\lambda}$ is non-degenerate and

(a) $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C_{\underline{\lambda}}, \mathcal{E}_{\underline{\varepsilon}}) = 1$,

(b) For all $(\lambda', [\varepsilon']) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ with $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^\pm_{\lambda'}, \mathcal{E}^\pm_{[\varepsilon']}) \neq 0$, we have ${}^s\underline{\lambda} < {}^s\lambda'$ or $(\lambda', [\varepsilon']) = (\underline{\lambda}, \underline{\varepsilon})$,

and furthermore, we have $(\underline{\lambda}, \underline{\varepsilon}) = (\lambda^{\max}, [\varepsilon^{\max}])$

Since transposition is an order reversing operation on $\mathcal{P}$, it follows from Lemma 3.2.5 that

(I) ${}^s\lambda' \leq {}^s\lambda''$ if and only if $\Lambda_{k/2,-k/2;1/2}(\alpha'', \beta'') \leq \Lambda_{k/2,-k/2;1/2}(\alpha', \beta')$.

Denote $<_{\alpha,\beta,k}$ by $<$. By Lemma 3.1.7, we have

$$P_{k/2,-k/2;1/2}(\alpha, \beta, <) = P_{k,-k;2}(\alpha, \beta, <) = \{(\alpha^{\max}, \beta^{\max})\}$$

and we showed in the proof of Theorem 3.2.2 that $(\alpha^{\max}, \beta^{\max})_k = \Phi_N(\lambda^{\max}, [\varepsilon^{\max}])$. Thus by Proposition 3.1.16 (with $A = -B = k/2$, $s = 1/2$), $(\alpha^{\max}, \beta^{\max})$ is the unique element $(\underline{\alpha}, \underline{\beta}) \in \mathcal{P} \times \mathcal{P}$ such that

(A) $\mathrm{mult}(\alpha, \beta, <; \underline{\alpha}, \underline{\beta}) = 1$;

(B) For each $(\alpha', \beta') \in \mathcal{P}\times\mathcal{P}$ such that $\mathrm{mult}(\alpha, \beta, <; \alpha', \beta') \neq 0$, we either have $\Lambda_{k/2,-k/2;1/2}(\alpha', \beta') \leq \Lambda_{k/2,-k/2;1/2}(\underline{\alpha}, \underline{\beta})$ or $(\alpha', \beta')_k = (\underline{\alpha}, \underline{\beta})_k$ (this is an equality of ordered (resp. unordered) pairs if $k > 0$ (resp. $k = 0$).

Now $(\underline{\lambda}, \underline{\varepsilon}) = \Phi_N^{-1}(\underline{\alpha}, \underline{\beta}) = (\lambda^{\max}, [\varepsilon^{\max}])$ satisfies (a) by Theorem 3.2.2, and (b) by (B), Proposition 3.1.17 and (I). Uniqueness follows easily from the fact that $P_{k/2,-k/2;1/2}(\alpha, \beta, <)$ has a unique element. Thus we have shown that $(\lambda^{\max}, [\varepsilon^{\max}])$ is the unique element of $\boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ that satisfies (a) and (b). □

3.2.4 Orthogonal partitions with even parts

Let $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$, $k = k(\lambda, [\varepsilon])$ and $(\alpha, \beta)_k) = \Phi_N(\lambda, [\varepsilon])$ In Theorem 3.2.2, we assumed that $\lambda \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ only has odd parts. Note that λ only has odd parts if and only if $p_{\lambda,\varepsilon} \in H_S(n, k)$, where and $n = (N - k^2)/2$. If $(\alpha, \beta) \notin H(n, k)$ then we cannot define $<_{\alpha,\beta,k}$. However, consider the following order with a weaker condition: for m_0, m_1 as above, define $(m_0, m_1, \tilde{<})$ such that $(i, 0)\tilde{<}(j, 1)$ if $\alpha_i + k + 2 - 2i > \beta_j - k + 2 - 2j$. If we mimic the proof of Lemma 3.1.20 for (α, β), then (3.1.20) is no longer necessarily true. As such, we cannot show that $Q_{(\alpha,\beta)} = R_{(\alpha,\beta)}$, which was a crucial part in the proof of Theorem 3.2.2, since it allowed the use of Lemma 3.1.19.

In fact, there exists an orthogonal partition $\lambda \in \mathcal{P}$ with even parts for which $Q_{(\alpha,\beta)} \neq R_{(\alpha,\beta)}$. Although a tedious exercise, one can show that for $\lambda = (4\,4\,1) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(9)$ and any order $\tilde{<}$ defined above, we have $Q_{(\alpha,\beta)} \neq R_{(\alpha,\beta)}$.

For $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ appearing in the usual Springer correspondence (i.e. $k(\lambda, [\varepsilon]) \in \{0, 1\}$) for which λ has even parts, we shall study the Green functions via a different perspective using an induction theorem from Lusztig.

Let $G = \mathrm{SO}(N)$, T a maximal torus and B the Borel subgroup of G associated to T. Let $u \subseteq G$ be a unipotent element and $\phi \in A(u)^\wedge$ an irreducible representation of the component group $A(u) = Z_G(u)/Z_G^\circ(u)$ of u in G. Let $\mathcal{B} = G/B$ be the flag variety and $\mathcal{B}_u = \{B \in \mathcal{B} \colon u \in B\}$. Let W be the Weyl group of G. We have the following classical result.

Proposition 3.2.7. *Let $\rho \in W^\wedge$ and suppose it corresponds to (u, ϕ) via the original Springer correspondence. Let $H^\bullet(\mathcal{B}_u)^\phi$ be the ϕ-isotypic component of the total cohomology $H^\bullet(\mathcal{B}_u)$ of $\mathcal{B}_u$. Then we have the following isomorphism as W-representations*

$$H^\bullet(\mathcal{B}_u)^\phi \cong \bigoplus_{\rho' \in W^\wedge} p_{\rho',\rho}(1)\rho',$$

where $p_{\rho',\rho}$ are the Green functions of W as in Theorem 3.1.8 for $k = 1$ if N is odd, and $k = 0$ if N is even.

Let L be a Levi subgroup attached to some standard parabolic of G and let W' be the Weyl group of L (note that W' is not a relative Weyl group of G). Then W' can be canonically considered as a subgroup of W. Suppose that $u \in L$ is unipotent and $\phi \in A^L(u)^\wedge$ where $A^L(u) = Z_L(u)/Z_L^\circ(u)$. Since $G = \mathrm{SO}(N)$, we can identify $A^L(u) = A(u)$. Let $\mathcal{B}' = L/B$ and $\mathcal{B}'_u = \{B' \in \mathcal{B}' \colon u \in B'\}$. We have the following induction theorem. It was first stated in [AL82] without proof; a proof is given in [Lus04] (see also [Ree01, Proposition 3.3.3].

Proposition 3.2.8. *As W-representations, we have*

$$H^\bullet(\mathcal{B}_u)^\phi \cong \mathrm{ind}_{W'}^{W} H^\bullet(\mathcal{B}'_u)^\phi.$$

Let $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ with $k := k(\lambda, [\varepsilon]) \in \{0, 1\}$. Let $u \in G$ be a unipotent element parametrised by λ. Let $\varepsilon^{\mathrm{odd}} = \varepsilon$ and write $\lambda = \lambda^{\mathrm{odd}} \sqcup \lambda^{\mathrm{even}}$ where λ^{odd} (resp. λ^{even}) consists of all the odd (resp. even) parts of λ. Write $\lambda^{\mathrm{even}} = (a_1, a_1, a_2, a_2, \ldots, a_\ell, a_\ell)$ with $a_1 \geq a_2 \geq \cdots \geq a_\ell$ and let $a = a_1 + \cdots + a_\ell$. Let $N' = t(\lambda^{\mathrm{odd}})$ and $n' = \lfloor N'/2 \rfloor$. Let L be a Levi subgroup of G of a standard parabolic such that $u \in L$. Suppose $\phi \in A^L(u) = A(u)$ corresponds to ε. Then

$$L \cong \mathrm{SO}(N') \oplus \mathrm{GL}(a_1) \oplus \mathrm{GL}(a_2) \oplus \cdots \oplus \mathrm{GL}(a_\ell)$$

and $W' = W_{n'} \times \prod_{i=1}^{\ell} S_{a_i}$ where $W_{n'} = W(B_{n'})$ (resp. $W_{n'} = W(D_{n'})$) if N' is odd (resp. even). Note that $N' \equiv N \mod 2$, so W'_n and W are of the same type. Let $u' \in L$ be the element parametrised by λ^{odd}. By Proposition 3.2.8, we have

$$H^\bullet(\mathcal{B}_u)^\phi \cong \mathrm{ind}_{W'}^{W}(H^\bullet(\mathcal{B}_{\mathrm{u}^{\mathrm{odd}}})^\phi \boxtimes \prod_{i=1}^{\ell} \mathrm{triv}_{S_{a_i}}), \tag{3.2.10}$$

where $\mathrm{triv}_{S_{a_i}}$ is the trivial representation of S_{a_i}.

We can explicitly determine the subrepresentations of $H^\bullet(\mathcal{B}_u)^\phi$ using the Littlewood-Richardson rule for type B and D Weyl groups, see [GP00, §6] for type B, [Tay15] for type D, or [Ste06] for an overview of type B and D. We note that [Ste06] explicitly states the results that we will use. Let I_0 be the set of $(\alpha, \beta) \in \mathcal{P}_2(n')$ such that $\rho_{(\alpha,\beta)}$ is a $W_{n'}$-subrepresentation of $H^\bullet(\mathcal{B}_{\mathrm{u}^{\mathrm{odd}}})^\phi$. For $i = 0, \ldots, \ell$, let $n_i = n' + a_1 + \cdots + a_i$. We recursively define sets I_i as follows. Suppose $i \in \{1, \ldots, \ell\}$. Let I_i be the set of $(\gamma, \delta) \in \mathcal{P}_2(n_i)$ such that there exist a $(\tilde{\gamma}, \tilde{\delta}) \in I_{i-1}$ such that

$$\tilde{\gamma}_j^t \leq \gamma_j^t \leq \tilde{\gamma}_j^t + 1 \quad \text{and} \quad \tilde{\delta}_j^t \leq \delta_j^t \leq \tilde{\delta}_j^t + 1. \tag{3.2.11}$$

By Pieri's rule for type B and D (see [Ste06]), I_i consists of precisely all $(\gamma, \delta) \in \mathcal{P}_2(n_i)$ such that $\rho_{(\gamma,\delta)}$ is a W_{n_i}-subrepresentation of $\mathrm{ind}_{W_{n_{i-1}} \times S_{a_i}}^{W_{n_i}}(\rho_{(\tilde{\gamma},\tilde{\delta})} \boxtimes \mathrm{triv}_{S_{a_i}})$. Since induction and direct products are compatible, we see from (3.2.10) that I_ℓ consists of all irreducible constituents of $H^\bullet(\mathcal{B}_u)^\phi$.

Let $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ with $k := k(\lambda, [\varepsilon]) \in \{0, 1\}$. Let $((\lambda^{\mathrm{odd}})^{\max}, [(\varepsilon^{\mathrm{odd}})^{\max}])$ be as in Theorem 3.2.2 for $(\lambda^{\mathrm{odd}}, [\varepsilon^{\mathrm{odd}}])$. Define $(\lambda^{\max}, [\varepsilon^{\max}]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ such that for all $j \in \mathbb{Z}_{\geq 2}$, we have

$$\lambda_1^{\max} = (\lambda^{\mathrm{odd}})_1^{\max} + \sum_{i \in \mathbb{N}} \lambda_i^{\mathrm{even}} = (\lambda^{\mathrm{odd}})_1^{\max} + 2a,$$
$$\lambda_j^{\max} = (\lambda^{\mathrm{odd}})_j^{\max},$$

and such that $\varepsilon^{\max} = (\varepsilon^{\mathrm{odd}})^{\max}$. Note that $\lambda^{\max}$ is degenerate.

Theorem 3.2.9. *Let $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ with $k := k(\lambda, [\varepsilon]) \in \{0, 1\}$. Then $(\lambda^{\max}, [\varepsilon^{\max}])$ is the unique element of $\boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ such that*

1. $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C_{\lambda^{\max}}, \mathcal{E}_{[\varepsilon^{\max}]}) = 1$,
2. *For all $(\lambda', [\varepsilon']) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ with $\mathrm{mult}(C_\lambda, \mathcal{E}^{\pm}_{[\varepsilon]}; C_{\lambda'}, \mathcal{E}^{\pm}_{[\varepsilon']}) \neq 0$, we have $\lambda' < \lambda^{max}$ or $(\lambda', [\varepsilon']) = (\lambda^{\max}, [\varepsilon^{\max}])$.*

Proof. Let $(\alpha, \beta)_k = \Phi_N(\lambda^{\mathrm{odd}}, \varepsilon^{\mathrm{odd}})$ and let (μ, ν) be the unique element of $P_{k,-k;2}(\alpha, \beta, <_{\alpha,\beta,k})$. Then $(\mu, \nu)_k = \Phi_N((\lambda^{\mathrm{odd}})^{\max}, (\varepsilon^{\mathrm{odd}})^{\max})$, as shown in the proof of Theorem 3.2.2. If $(1, 0) <_{\alpha,\beta,k} (1, 1)$, let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\mu + (a, 0, 0, \dots), \mu)$ and if $(1, 1) <_{\alpha,\beta,k} (1, 0)$, let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\mu, \nu + (a, 0, 0, \dots))$. By definition of the I_i, we have $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in I_\ell$. Let $\Lambda = \Lambda_{k,-k;2}(\mu, \nu)$ and $\boldsymbol{\Lambda} = \Lambda_{k,-k;2}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Let $(\boldsymbol{\alpha}', \boldsymbol{\beta}') \in I_\ell$. Since (3.2.11) holds for $i = 1, \dots, \ell$, there exists an $(\alpha', \beta') \in I_0$ and $x \in \mathbb{Z}_{\geq 0}^{t(\alpha)}, y \in \mathbb{Z}_{\geq 0}^{t(\beta)}$ such that

$$\boldsymbol{\alpha}' = \alpha' + x \quad \boldsymbol{\beta}' = \beta' + y, \quad \sum_j (x_j + y_j) = a. \tag{3.2.12}$$

Clearly, if $(\alpha', \beta') \neq (\mu, \nu)$, then $(\boldsymbol{\alpha}', \boldsymbol{\beta}') \neq (\boldsymbol{\alpha}, \boldsymbol{\beta})$, and by Theorem 3.2.2, the multiplicity of $\rho_{(\mu,\nu)}$ in $H^\bullet(\mathcal{B}_u')^{\phi'}$ is 1. Furthermore, the multiplicity of $\rho_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ in

$$\mathrm{ind}_{W_{n'} \times \prod_{i=1}^{\ell} S_{a_i}}^{W_n} (\rho_{(\alpha,\beta)} \boxtimes \prod_{i=1}^{\ell} \mathrm{triv}_{S_{a_i}}) \tag{3.2.13}$$

can be computed using results from [Ste06, §2.A, §2.B] for type B and [Ste06, §3.A, §3.C] for type D, and it follows that this multiplicity is 1. Thus part 1 of the theorem follows. Let z be the sequence in $\mathcal{R}$ obtained by rearranging the terms of $x \sqcup y$. Let $\Lambda' = \Lambda_{k,-k;2}(\alpha', \beta')$ and $\boldsymbol{\Lambda}' = \Lambda_{k,-k;2}(\boldsymbol{\alpha}', \boldsymbol{\beta}')$. Let $i = 1, \dots, \ell$. By (3.2.12), we have $\Lambda_i' + S_i(z) \geq \boldsymbol{\Lambda}_i'$ and $a_1 + \cdots + a_\ell \geq S_i(z)$. Furthermore, $\Lambda_i \geq \Lambda_i'$ by Lemma 3.1.2, so

$$\boldsymbol{\Lambda}_i = \Lambda_i + a_1 + \cdots + a_\ell \geq \Lambda_i' + S_i(z) \geq \boldsymbol{\Lambda}_i'.$$

Hence $\mathbf{\Lambda} \geq \Lambda_{k,-k;2}(\boldsymbol{\alpha}', \boldsymbol{\beta}')$ for all $(\boldsymbol{\alpha}', \boldsymbol{\beta}') \in I_\ell$, so part 2 of the theorem follows from Lemma 3.2.1. To show that $(\lambda^{\max}, [\varepsilon^{\max}])$ is unique, suppose that $\mathbf{\Lambda} = \mathbf{\Lambda}'$. Then $\mathbf{\Lambda}'_1 = \mathbf{\Lambda}_1 = \Lambda_1 + a$. Since $\Lambda_1 \geq \Lambda'_1$ and $\mathbf{\Lambda}'_1 \leq \Lambda'_1 + a$, we have $\Lambda_1 = \Lambda'_1$. From this and (3.2.12), it follows that $\Lambda_i = \mathbf{\Lambda}_i = \mathbf{\Lambda}'_i = \Lambda'_i$ for all $i \in \mathbb{N}$, and so $\Lambda = \Lambda'$, hence $(\alpha', \beta') \in P_{k,-k;2}(\alpha, \beta, <_{\alpha,\beta,k})$ by Lemma 3.1.2. Since $P_{k,-k;2}(\alpha, \beta, <_{\alpha,\beta,k})$ has a unique element (μ, ν), we have $(\alpha', \beta') = (\mu, \nu)$, and using that $\mathbf{\Lambda}'_1 = \Lambda'_1 + a$, we find that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\boldsymbol{\alpha}', \boldsymbol{\beta}')$. Thus $(\lambda^{\max}, [\varepsilon^{\max}])$ is unique. □

Theorem 3.2.10. *Let $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ with $k := k(\lambda, [\varepsilon]) \in \{0, 1\}$ and consider the notation as above. Then there exists a unique $(\lambda^{\min}, [\varepsilon^{\min}]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ such that ${}^s\lambda^{\min}$ is non-degenerate and*

1. $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C_{{}^s\lambda^{\min}}, \mathcal{E}_{[{}^s\varepsilon^{\min}]}) = 1$,
2. *For all $(\lambda', [\varepsilon']) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ with $\mathrm{mult}(C_\lambda, \mathcal{E}_{[\varepsilon]}; C^{\pm}_{\lambda'}, \mathcal{E}^{\pm}_{[\varepsilon']}) \neq 0$, we have $\lambda^{\min} < \lambda'$ or $(\lambda', [\varepsilon']) = ({}^s\lambda^{\min}, {}^s[\varepsilon^{\min}])$.*

Furthermore, we have $(\lambda^{\max}, [\varepsilon^{\max}]) = ({}^s\lambda^{\min}, [{}^s\varepsilon^{\min}])$.

Proof. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}), (\boldsymbol{\alpha}', \boldsymbol{\beta}')$ as in the proof of Theorem 3.2.9. By the same arguments as in the proof of Theorem 3.2.9, we can show that $\Lambda_{k/2,-k/2;2}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \geq \Lambda_{k/2,-k/2;2}(\boldsymbol{\alpha}', \boldsymbol{\beta}')$, with equality if and only if $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\boldsymbol{\alpha}', \boldsymbol{\beta}')$. Hence

$$P_{k/2,-k/2;1/2}(\alpha, \beta, <_{\alpha,\beta,k}) = P_{k,-k;2}(\alpha, \beta, <_{\alpha,\beta,k}) = \{(\boldsymbol{\alpha}, \boldsymbol{\beta})\}.$$

By the arguments in Theorem 3.2.6, it then follows that $(\lambda^{\min}, [\varepsilon^{\min}]) = ({}^{\mathrm{s}}\lambda^{\max}, [{}^s\varepsilon^{\max}])$ is the unique element satisfying the two conditions. □

Remark 3.2.11. By similar arguments and using analogous results for $\mathrm{Sp}(2n)$ from [Wal19], we similarly obtain the analogues of Theorem 3.2.9 and 3.2.10 for $\mathrm{Sp}(2n)$ for any $(\lambda, [\varepsilon]) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$ such that $k(\lambda, [\varepsilon])) = 0$.

3.2.5 Maximality and minimality theorems for $\mathrm{Sp}(2n)$

As stated before, the $\mathrm{Sp}(2n)$ analogues of Theorem 3.2.2 and Theorem 3.2.6 were proved in [Wal19], and we shall state the results without proof.

Theorem 3.2.12. *Suppose $(\lambda, \varepsilon) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$ such that λ only has even parts. Then there exists a unique $(\lambda^{\max}, \varepsilon^{\max}) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$ such that $\lambda^{\max}$ is non-degenerate and*

(1) $\mathrm{mult}(C_\lambda, \mathcal{E}_\varepsilon; C_{\lambda^{\max}}, \mathcal{E}_{\varepsilon^{\max}}) = 1$,

(2) For all $(\lambda', \varepsilon') \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ *with* $\mathrm{mult}(C_\lambda, \mathcal{E}_\varepsilon; C_{\lambda'}, \mathcal{E}_{\varepsilon'}) \neq 0$*, we have* $\lambda' < \lambda^{\max}$ *or* $(\lambda', \varepsilon') = (\lambda^{\max}, \varepsilon^{\max})$*.*

Theorem 3.2.13. *Let* $(\lambda, \varepsilon) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$ *such that* λ *only has of even parts. Then there exists a unique* $(\lambda^{\min}, \varepsilon^{\min}) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$ *such that* ${}^s\lambda^{\min}$ *is non-degenerate and*

1. $\mathrm{mult}(C_\lambda, \mathcal{E}_\varepsilon; C_{{}^s\lambda^{\min}}, \mathcal{E}_{{}^s\varepsilon^{\min}}) = 1$,

2. *For all* $(\lambda', \varepsilon') \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$ *with* $\mathrm{mult}(C_\lambda, \mathcal{E}_\varepsilon; C_{{}^s\lambda'}, \mathcal{E}_{[{}^s\varepsilon']}) =\neq 0$*, we have* $\lambda^{\min} < \lambda'$ *or* $(\lambda', \varepsilon') = (\lambda^{\min}, \varepsilon^{\min})$*.*

Furthermore, we have $(\lambda^{\max}, \varepsilon^{\max}) = ({}^s\lambda^{\min}, {}^s\varepsilon^{\min})$*.*

3.3 Algorithm for $(\lambda^{\max}, \varepsilon^{\max})$ for $\mathrm{SO}(N)$

Let $N \in \mathbb{N}$. Recall $\boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N)$ defined in Section 1.2.3 and let $(\lambda, \varepsilon) \in \boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N)$. Suppose λ only has odd parts. We give an algorithm that outputs an element $(\bar{\lambda}, \bar{\varepsilon}) \in \boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N)$ such that $(\bar{\lambda}, [\bar{\varepsilon}]) = (\lambda^{\max}, [\varepsilon^{\max}]) \in \boldsymbol{\mathcal{P}}^{\mathbf{ort}}(N)$. The algorithm and its proof are inspired by the discussion of the algorithm for $\mathrm{Sp}(2n)$ in [Wal19, §5], but similarly as we have seen before, there will be some differences between the proofs since the symbols are different in our setting. In particular, when $k(\lambda, [\varepsilon]) = 0$, the symbols are unordered pairs, which we have to be especially careful about.

Let $t = t(\lambda)$. We view ε as a map $\{1, \ldots, t\} \to \{\pm 1\}$ by setting $\varepsilon(i) = \varepsilon_{\lambda_i}$ for all $i \in \mathbb{N}$. Let $u \in \{\pm 1\}$ and consider the finite sets

$$\begin{aligned} \mathfrak{S} &= \{1\} \cup \{i \in \{2, \ldots, t\} \colon \varepsilon(i) = \varepsilon(i-1)\} = \{s_1 < s_2 < \cdots < s_p\}, \text{ for some } p \in \mathbb{N}, \\ J^u &= \{i \in \{1, \ldots, t\} \colon \varepsilon(i)(-1)^{i+1} = u\}, \\ \tilde{J}^u &= J^u \setminus \mathfrak{S}. \end{aligned}$$

Recall from Section 1.2.3 that $M(\lambda, \varepsilon) = |J^1| - |J^{-1}|$ and that $k(\lambda, \varepsilon) = |M|$. Note that $\mathfrak{S}, J^u, M$ depend on the choice of representative ε of $[\varepsilon]$. In particular, we have $M(\lambda, \varepsilon) = -M(\lambda, -\varepsilon)$.

We define $(\bar{\lambda}, \bar{\varepsilon})$ by induction. For $N = 0, 1$. Let $(\bar{\lambda}, \bar{\varepsilon}) = (\lambda, \varepsilon)$. Suppose $N > 1$ and suppose that we have defined $(\bar{\lambda}', \bar{\varepsilon}')$ for any $(\lambda', \varepsilon') \in \boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N')$ with $N' < N$. Let

$$\begin{aligned} \bar{\lambda}_1 &= \sum_{i \in \mathfrak{S}} \lambda_i - 2|\tilde{J}^{-\varepsilon(1)}| - \frac{1 + (-1)^{|\mathfrak{S}|}}{2} = \begin{cases} \sum_{i \in \mathfrak{S}} \lambda_i - 2|\tilde{J}^{-\varepsilon(1)}| & \text{if } |\mathfrak{S}| \text{ is odd,} \\ \sum_{i \in \mathfrak{S}} \lambda_i - 2|\tilde{J}^{-\varepsilon(1)}| - 1 & \text{if } |\mathfrak{S}| \text{ is even,} \end{cases} \\ \bar{\varepsilon}(1) &= \varepsilon(1). \end{aligned}$$

Let $r' = |\tilde{J}^1| + |\tilde{J}^{-1}| = t(\lambda) - |\mathfrak{S}|$ and $\phi\colon \{1,\ldots,r'\} \to \tilde{J}^1 \cup \tilde{J}^{-1}$ be the unique increasing bijection. Let $N' = N - \bar{\lambda}_1$. We define $\lambda' \in \mathbb{Z}^{r'+(1+(-1)^{|\mathfrak{S}|})/2}$ and $\varepsilon'\colon \{1,\ldots,r'+\frac{1+(-1)^{|\mathfrak{S}|}}{2}\} \to \{\pm 1\}$ as follows.

$$(\lambda'_j, \varepsilon'(j)) = \begin{cases} (\lambda_{\phi(j)}, \varepsilon(\phi(j))) & \text{if } j \le r', \phi(j) \in \tilde{J}^{\varepsilon(1)}, \\ (\lambda_{\phi(j)} + 2, -\varepsilon(\phi(j))) & \text{if } j \le r', \phi(j) \in \tilde{J}^{-\varepsilon(1)}, \\ (1, (-1)^{r'} \varepsilon(1)) & \text{if } j = r'+1 \text{ and } |\mathfrak{S}| \text{ is even}, \\ 0 & \text{else.} \end{cases}$$

Note that we can similarly define $(\lambda', (-\varepsilon)')$, in which case we have $(-\varepsilon)' = -\varepsilon'$, i.e. $[-\varepsilon'] = [\varepsilon']$. We prove in Proposition 3.3.4(2) that $N' < N$. By induction, we have defined $(\bar{\lambda}', \bar{\varepsilon}') \in \boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N')$ by applying the algorithm to $(\lambda', \varepsilon') \in \boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N')$.

Let $\ell = t(\bar{\lambda}')$. We define $\bar{\lambda} \in \mathbb{Z}^{\ell+1}$ and $\bar{\varepsilon}\colon \{1,\ldots,\ell+1\} \to \{\pm 1\}$ as follows: for $i = 1,\ldots,\ell$, let

$$\bar{\lambda}_{i+1} := \bar{\lambda}'_{i-1},$$
$$\bar{\varepsilon}(i+1) := \bar{\varepsilon}'(i).$$

Theorem 3.3.1. *Let $(\lambda, \varepsilon) \in \boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N)$ such that λ only has odd parts and let $(\lambda^{\max}, \varepsilon^{\max}) \in \boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N)$ such that $(\lambda^{\max}, [\varepsilon^{\max}])$ is as in Theorem 3.2.2. Let $k = k(\lambda, \varepsilon)$ and let $(\alpha, \beta) \in \mathcal{P}_2((N-k^2)/2)$ so that $(\alpha, \beta)_k = \Phi_N(\lambda, [\varepsilon])$ and such that if $k = 0$, then $\alpha_1 = (A_{\lambda,\varepsilon})_1$ and $\beta_1 = (B_{\lambda,\varepsilon})_1$. Let $(\alpha^{\max}, \beta^{\max}) \in P_{k,-k;2}(\alpha, \beta, <_{\alpha,\beta,k})$. Then it holds that $(\bar{\lambda}, [\bar{\varepsilon}]) = (\lambda^{\max}, [\varepsilon^{\max}])$. Furthermore, we have*

$$M(\bar{\lambda}, \bar{\varepsilon}) = M(\lambda, \varepsilon), \quad (A_{\bar{\lambda},\bar{\varepsilon}}, B_{\bar{\lambda},\bar{\varepsilon}}) = (\alpha^{\max} + [k, -\infty[_2, \beta^{\max} + [-k, -\infty[_2).$$

Remark 3.3.2. For $k > 0$, the statement $(A_{\bar{\lambda},\bar{\varepsilon}}, B_{\bar{\lambda},\bar{\varepsilon}}) = (\alpha^{\max} + [k, -\infty[_2, \beta^{\max} + [-k, -\infty[_2)$ is equivalent to the statement $(\bar{\lambda}, [\bar{\varepsilon}]) = \Phi_N^{-1}(\alpha^{\max}, \beta^{\max}) = (\lambda^{\max}, [\varepsilon^{\max}])$. However, for $k = 0$, the equality $(\bar{\lambda}, [\bar{\varepsilon}]) = (\lambda^{\max}, [\varepsilon^{\max}])$ is equivalent to

$$\{A_{\bar{\lambda},\bar{\varepsilon}}, B_{\bar{\lambda},\bar{\varepsilon}}\} = \{\alpha^{\max} + [k, -\infty[_2, \beta^{\max} + [-k, -\infty[_2\},$$

which is weaker than an equality of ordered pairs. We include this equality of ordered pairs in the theorem, as the theorem will be proved by induction on N and this equality of ordered pairs will become useful for that.

3.3.1 Properties of (λ', ε') and $(\bar{\lambda}, \bar{\varepsilon})$

Recall that $\mathfrak{S} = \{s_1, \ldots, s_p\}$ and note that

$$J^{\varepsilon(1)} = \bigcup_i \{s_{2i-1}, \ldots, s_{2i} - 1\}, \qquad J^{-\varepsilon(1)} = \bigcup_i \{s_{2i}, \ldots, s_{2i+1} - 1\},$$
$$\tilde{J}^{\varepsilon(1)} = \bigcup_i \{s_{2i-1} + 1, \ldots, s_{2i} - 1\}, \qquad \tilde{J}^{-\varepsilon(1)} = \bigcup_i \{s_{2i} + 1, \ldots, s_{2i+1} - 1\}. \tag{3.3.1}$$

Throughout this subsection, we assume that $N > 1$ and so $\lambda \neq 0$.

Lemma 3.3.3. *Suppose $r' \neq 0$ and let $i, j \in \{1, \dots, r'\}$ with $i < j$. Then $\lambda_{\phi(i)} \geq \lambda_{\phi(j)} + 2(j-i)$.*

Proof. Suppose first that there exists no $s \in \mathfrak{S}$ such that $\phi(j-1) < s < \phi(j)$. Then $\phi(j-1) = \phi(j) - 1$, and since $\phi(j) \notin \mathfrak{S}$, we have $\varepsilon(\phi(j-1)) = -\varepsilon(\phi(j))$. Thus $\lambda_{\phi(j-1)} > \lambda_{\phi(j)}$, and since these are odd, we have $\lambda_{\phi(j-1)} \geq \lambda_{\phi(j)} + 2$. Suppose there exists an $s \in \mathfrak{S}$ such that $\phi(j-1) < s < \phi(j)$. Pick the largest such s so that $\phi(j) \geq s + 1 \notin \mathfrak{S}$. Then $\varepsilon(s+1) = -\varepsilon(s)$, hence $\lambda_s \geq \lambda_{s+1} + 2$. Thus we have

$$\lambda_{\phi(j-1)} \geq \lambda_s \geq \lambda_{s+1} + 2 \geq \lambda_{\phi(j)} + 2.$$

In both cases, we have $\lambda_{\phi(j-1)} \geq \lambda_{\phi(j)} + 2$. Thus we have

$$\lambda_{\phi(i)} \geq \lambda_{\phi(i+1)} + 2 \geq \lambda_{\phi(i+2)} + 4 \geq \cdots \geq \lambda_{\phi(j)} + 2(j-i). \qquad \square$$

Proposition 3.3.4. 1. *We have $(\lambda', \varepsilon') \in \boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N')$ and λ' only has odd parts.*

2. *We have $N' < N$.*

3. *By the first part, we can define J'^u, $\tilde{J}'^u$, $\mathfrak{S}'$ for (λ', ε') as we defined J^u, $\tilde{J}^u$, $\mathfrak{S}$ for (λ, ε). For $u \in \{\pm 1\}$, ϕ restricts to a bijection $\phi \colon J'^{-u} \cap \{1, \dots, r'\} \to \tilde{J}^u$.*

4. *Let $M = M(\lambda, \varepsilon)$, $M' = M(\lambda', \varepsilon')$, $k = k(\lambda, \varepsilon)$, and $k' = k(\lambda', \varepsilon')$. Let $w = \operatorname{sgn}(M + 1/2)$. Then $M' = -M + \varepsilon(1)$ and $k' = |k - \varepsilon(1)w|$.*

Proof. 1. By definition of λ' and Lemma 3.3.3, we have

$$\lambda'_i \geq \lambda_{\phi(i)} \geq \lambda_{\phi(i+1)} + 2 \geq \lambda'_{i+1},$$

so $\lambda' \in \mathcal{P}$. Clearly, λ'_i only has odd parts and $S(\lambda') = N'$, and so $\lambda' \in \mathcal{P}^{\mathrm{ort}}(N')$. Let $i \in \{1, \dots, t(\lambda')\}$ such that $\lambda'_i = \lambda'_{i+1}$. We have to show that $\varepsilon'(i) = \varepsilon'(i+1)$. If $\phi(i), \phi(i+1) \in \tilde{J}^u$ for some $u \in \{\pm 1\}$, then $\lambda_{\phi(i)} = \lambda_{\phi(i+1)}$, which contradicts Lemma 3.3.3. If $\phi(i) \in \tilde{J}^{-\varepsilon(1)}$ and $\phi(i+1) \in \tilde{J}^{\varepsilon(1)}$, then $\lambda_{\phi(i+1)} = \lambda'_{i+1} = \lambda'_i = \lambda_{\phi(i)} + 2$, which contradicts the fact that $\lambda_{\phi(i)} \geq \lambda_{\phi(i+1)}$. Hence $\phi(i) \in \tilde{J}^{\varepsilon(1)}$ and $\phi(i+1) \in \tilde{J}^{-\varepsilon(1)}$. Then there exists an $s \in \mathfrak{S}$ such that $\phi(i) < s < \phi(i+1)$, and since ϕ is an inreasing bijection, we have $\varnothing \neq \{\phi(i)+1, \phi(i)+2, \dots, \phi(i+1)-1\} \subseteq \mathfrak{S}$. Thus

$$\varepsilon(\phi(i)) = \varepsilon(\phi(i)+1) = \cdots = \varepsilon(\phi(i+1)-1) = -\varepsilon(\phi(i+1)),$$

and since $\phi(i) \in J^{\varepsilon(1)}$ and $\phi(i+1) \in J^{-\varepsilon(1)}$, we have $\varepsilon'(i) = \varepsilon(\phi(i)) = -\varepsilon(\phi(i+1) = \varepsilon'(i+1)$.

2. Since $N > 1$, we have $\lambda_1 \geq 1$. If $r' = 0$, then $|\tilde{J}^{-\varepsilon(1)}| = 0$ and so

$$\sum_{i \in \mathfrak{S}} \lambda_i \geq 1 + 2|\tilde{J}^{-\varepsilon(1)}|. \tag{3.3.2}$$

Suppose that $r' \neq 0$. Then $\lambda_{\phi(r')} \geq 1$ and $r' \geq |\tilde{J}^{-\varepsilon(1)}|$. We also have $\lambda_1 \geq \lambda_{\phi(1)} + 2$. Using Lemma 3.3.3, we find

$$\sum_{i \in \mathfrak{S}} \lambda_i \geq \lambda_1 \geq \lambda_{\phi(1)} + 2 \geq \lambda_{\phi(r')} + 2(r'-1) + 2 \geq 1 + 2|\tilde{J}^{-\varepsilon(1)}|. \tag{3.3.3}$$

Thus

$$N' = N - \sum_{i \in \mathfrak{S}} \lambda_i + 2|\tilde{J}^{-\varepsilon(1)}| + \frac{1+(-1)^{|\mathfrak{S}|}}{2} \leq N - 1 + \frac{1+(-1)^{|\mathfrak{S}|}}{2} \leq N.$$

If $|\mathfrak{S}|$ is odd, then the last inequality is strict. If $|\mathfrak{S}|$ is even, then the first inequality is strict, since the first inequality of (3.3.2) and (3.3.3) is strict. Thus we have $N' < N$.

3. Suppose $i \in J'^u$, i.e. $\varepsilon'(i)(-1)^{i+1} = u$. If $\phi(i) \in \tilde{J}^{-\varepsilon(1)}$, then $\varepsilon'(i) = -\varepsilon(\phi(i))$. Let $h \in \mathbb{N}$ be the unique integer such that $s_h < \phi(i) < s_{h+1}$. Then $s_h \in J^{-\varepsilon(1)}$, so h is even by (3.3.1). Note that $\phi(i) = h + i$, so i and $\phi(i)$ have the same parity. Thus $u = \varepsilon'(i)(-1)^{i+1} = -\varepsilon(\phi(i))(-1)^{\phi(i)+1} = \varepsilon(1)$. If $\phi(i) \in \tilde{J}^{\varepsilon(1)}$, we can similarly show that $u = -\varepsilon(1)$. Hence we have $\phi(J'^u) \subseteq \tilde{J}^{-u}$. Since $\phi \colon \{1, \ldots, r'\} \to \tilde{J}^1 \cup \tilde{J}^{-1}$ is a bijection and $\{1, \ldots, r'\} = J'^1 \cup J'^{-1}$, the result follows.

4. Let $u = \varepsilon(1)$. Suppose $|\mathfrak{S}| = 2q+1$ is odd. Then $|J^u| - |\tilde{J}^u| = q+1$, $|J^{-u}| - |\tilde{J}^{-u}| = q$ and $t(\lambda') = r'$. By the previous part, we then have $|J'^{\pm u}| = |\tilde{J}^{\mp u}|$. Thus

$$uM' = |J'^u| - |J'^{-u}| = |\tilde{J}^{-u}| - |\tilde{J}^u| = -(|J^u| - |J^{-u}|) + 1 = -uM + 1.$$

Suppose $|\mathfrak{S}| = 2q$ is even. Then $|J^u| - |\tilde{J}^u| = |J^{-u}| - |\tilde{J}^{-u}| = q$. Now, $t(\lambda') = r' + 1$ and $\varepsilon'(r'+1) = (-1)^{r'}u$, i.e. $r' + 1 \in J'^u$. So by the previous part, we have $|J'^u| = |\tilde{J}^{-u}| + 1$ and $|J'^{-u}| = |\tilde{J}^u|$. Thus

$$uM' = |J'^u| - |J'^{-u}| = |\tilde{J}^{-u}| - |\tilde{J}^u| + 1 = -(|J^u| - |J^{-u}|) + 1 = -uM + 1.$$

In both cases, we find $M' = -M + u$. Note that $k = |M| = wM$. Thus $k' = |M'| = |-M+u| = |-w||-M+u| = |wM - uw| = |k - uw|$. ☐

Proposition 3.3.5. *1. $\bar{\lambda}_1$ is odd and $\bar{\lambda}_1 \geq \lambda_1$.*

2. $(\bar{\lambda}, \bar{\varepsilon}) \in \boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N)$.

Proof. 1. It is easy to see from the definition that $\bar{\lambda}_1$ is odd. We have

$$\bar{\lambda}_1 - \lambda_1 = \sum_{i \in \mathfrak{S}, i \geq 2} \lambda_i - 2|\tilde{J}^{-\varepsilon(1)}| - \frac{1 + (-1)^{|\mathfrak{S}|}}{2}.$$

If $\mathfrak{S} = \{1\}$, then $\tilde{J}^{-\varepsilon(1)} = \varnothing$ and so $\bar{\lambda}_1 - \lambda_1 = 0$. Suppose $|\mathfrak{S}| \geq 2$. Then $s_2 < j$ for all $j \in \tilde{J}^{-\varepsilon(1)}$. Let $i = \min J'^{\varepsilon(1)}$ so that $r' - i + 1 \geq |J'^{\varepsilon(1)} \cap \{i, \ldots, r'\}| = |J'^{\varepsilon(1)} \cap \{1, \ldots, r'\}| = |\tilde{J}^{-\varepsilon(1)}|$, where the last equality holds by Proposition 3.3.4(3). By definition of i, we have $\phi(i) = \min \tilde{J}^{-\varepsilon(1)}$, and so we have $\lambda_{s_2} > \lambda_{\phi(i)}$. Lemma 3.3.3 then gives $\lambda_{s_2} \geq \lambda_{\phi(i)} + 2 \geq \lambda_{\phi(r')} + 2(r' - i) + 2 \geq 1 + 2|\tilde{J}^{-\varepsilon(1)}|$, hence $\bar{\lambda}_1 - \lambda_1 \geq 0$.

2. The assertion is clear if $N = 0, 1$. Suppose that $N > 1$. By the previous part, all the terms of $\bar{\lambda}$ are odd. It suffices to show that $\bar{\lambda}_1 \geq \bar{\lambda}'_1$, and that if $\bar{\varepsilon}(1) = -\bar{\varepsilon}(2)$, i.e. $\varepsilon(1) = -\varepsilon'(1)$, then $\bar{\lambda}_1 > \bar{\lambda}'_1$. It then follows that $(\bar{\lambda}, \bar{\varepsilon}) \in \boldsymbol{\mathcal{P}}^{\mathrm{ort},2}(N)$.

Write $\tilde{\mathfrak{S}}' = \{1, \ldots, r'\} \cap \mathfrak{S}' = \{s'_1 < s'_2 < \cdots < s'_q\}$. Note that

$$\bigcup_i \{s'_{2i-1}, \ldots, s'_{2i} - 1\} \subseteq J'^{\varepsilon'(1)}, \quad \bigcup_i \{s'_{2i}, \ldots, s'_{2i+1} - 1\} \subseteq J'^{-\varepsilon'(1)}. \tag{3.3.4}$$

Note that $\lambda'_{r'+1} = \frac{1+(-1)^{|\mathfrak{S}|}}{2}$ and $\mathfrak{S}' \subseteq \{1, \ldots, r' + 1\}$, so

$$\sum_{s' \in \mathfrak{S}'} \lambda'_{s'} \leq \lambda'_{r'+1} + \sum_{h'=1}^{q} \lambda'_{s'_{h'}} \leq \frac{1 + (-1)^{|\mathfrak{S}|}}{2} + \sum_{h'=1}^{q} \lambda'_{s'_{h'}}. \tag{3.3.5}$$

Define $\psi \colon \tilde{\mathfrak{S}}' \to \mathfrak{S}$ by $\psi(s') = \max\{s \in \mathfrak{S} \colon s < \phi(s')\} = \phi(s') - 1$. Then ψ is increasing and injective. Let $u \in \{\pm 1\}$ and let $s' \in \tilde{\mathfrak{S}}' \cap J'^{-u}$. By Proposition 3.3.4(3), we have $\phi(s') \in \tilde{J}^u$. Since $\phi(s') \notin \mathfrak{S}$, we have $\varepsilon(\psi(s')) = -\varepsilon(\phi(s'))$ and so $\lambda_{\psi(s')} \geq \lambda_{\phi(s')} + 2$. Thus if $u = \varepsilon(1)$, then $\lambda_{\psi(s')} \geq \lambda'_{s'} + 2$, and if $u = -\varepsilon(1)$ then $\lambda_{\psi(s')} \geq \lambda'_{s'}$. Hence we can further bound the right-hand side of (3.3.5):

$$\sum_{s' \in \mathfrak{S}'} \lambda'_{s'} \leq \frac{1 + (-1)^{|\mathfrak{S}|}}{2} + \sum_{h'=1}^{q} \lambda'_{s'_{h'}} \leq \frac{1 + (-1)^{|\mathfrak{S}|}}{2} + \sum_{h \in \mathrm{im}(\psi)} \lambda_{s_h} - 2|\tilde{\mathfrak{S}}' \cap J'^{-\varepsilon(1)}|.$$

From this, we find

$$\begin{aligned} &\bar{\lambda}_1 - \bar{\lambda}'_1 \\ &\geq \sum_{s \in \mathfrak{S} \setminus \mathrm{im}(\psi)} \lambda_s + 2|\tilde{\mathfrak{S}}' \cap J'^{-\varepsilon(1)}| + 2|\tilde{J}'^{-\varepsilon'(1)}| - 2|\tilde{J}^{-\varepsilon(1)}| - (1 + (-1)^{|\mathfrak{S}|}) + \frac{1 + (-1)^{|\mathfrak{S}'|}}{2} =: X. \end{aligned}$$

Suppose $\varepsilon(1) = \varepsilon'(1)$. If $\psi(1) = 1$, then $\phi(1) = 2 \in J^{\varepsilon(1)}$, so $1 \in J'^{-\varepsilon(1)}$ by Proposition 3.3.4(3), hence $\varepsilon'(1) = -\varepsilon(1) = -\varepsilon'(1)$, a contradiction. Thus $\psi(1) > 1$ and since ψ is increasing, we have $1 \notin \mathrm{im}(\psi)$, and so $|\mathfrak{S}| \geq 2$ and

$$X \geq \lambda_1 - 2|\tilde{J}^{-\varepsilon(1)}| - (1 + (-1)^{|\mathfrak{S}|}).$$

We showed in the previous part that $\lambda_{s_2} \geq 1 + 2|\tilde{J}^{-\varepsilon(1)}|$ when $|\mathfrak{S}| \geq 2$. Thus we have

$$X \geq \lambda_{s_2} - 2|\tilde{J}^{-\varepsilon(1)}| - (1 + (-1)^{|\mathfrak{S}|} \geq (-1)^{|\mathfrak{S}|} \geq -1,$$

and so $\bar{\lambda}_1 - \bar{\lambda}'_1 \geq -1$. By the previous part, $\bar{\lambda}_1$ and $\bar{\lambda}'_1$ are odd, so we have $\bar{\lambda}_1 - \bar{\lambda}'_1 \geq 0$.

Suppose $\varepsilon(1) = -\varepsilon'(1)$. We show that $X \geq 1$ so that $\bar{\lambda}_1 > \bar{\lambda}_2$. By Proposition 3.3.4(3), we have

$$\begin{aligned}\phi(\tilde{J}^{-\varepsilon(1)}) = J'^{\varepsilon(1)} \cap \{1, \ldots, r'\} &= (\tilde{\mathfrak{S}}' \cap J'^{\varepsilon(1)}) \sqcup (\tilde{J}'^{\varepsilon(1)} \cap \{1, \ldots, r'\}) \\ &= (\tilde{\mathfrak{S}}' \cap J'^{\varepsilon(1)}) \sqcup (\tilde{J}'^{\varepsilon(1)}) \setminus (\tilde{J}'^{\varepsilon(1)} \cap \{r' + 1\}),\end{aligned}$$

hence

$$|\tilde{J}'^{\varepsilon(1)}| = |\tilde{J}^{-\varepsilon(1)}| + |\tilde{J}'^{\varepsilon(1)} \cap \{r' + 1\}| - |\tilde{\mathfrak{S}}' \cap J'^{\varepsilon(1)}|.$$

Using that $\varepsilon(1) = -\varepsilon'(1)$, we find

$$\begin{aligned}X = \sum_{s \in \mathfrak{S} \setminus \mathrm{im}(\psi)} &\lambda_s + 2|\tilde{\mathfrak{S}}' \cap J'^{\varepsilon'(1)}| - 2|\tilde{\mathfrak{S}}' \cap J'^{-\varepsilon'(1)}| \\ &+ 2|\tilde{J}'^{-\varepsilon'(1)} \cap \{r' + 1\}| - (1 + (-1)^{|\mathfrak{S}|}) + \frac{1 + (-1)^{|\mathfrak{S}'|}}{2}.\end{aligned}$$

By (3.3.4), $|\tilde{\mathfrak{S}}' \cap J'^{\varepsilon'(1)}|$ (resp. $|\tilde{\mathfrak{S}}' \cap J'^{-\varepsilon'(1)}|$) is the number of odd (resp. even) integers in $\{1, \ldots, q\}$.

Suppose q is odd. Then

$$2|\tilde{\mathfrak{S}}' \cap J'^{\varepsilon'(1)}| - 2|\tilde{\mathfrak{S}}' \cap J'^{-\varepsilon'(1)}| = 2.$$

Suppose $|\mathfrak{S}|$ is odd. Then $1+(-1)^{|\mathfrak{S}|} = 0$, so $X \geq 2$. Suppose $|\mathfrak{S}|$ is even. Then $1+(-1)^{|\mathfrak{S}|} = 2$, $\lambda'_{r'+1} = 1$, and by definition of $\varepsilon'(r' + 1)$, we have $r' + 1 \in J'^{\varepsilon(1)} = J'^{-\varepsilon'(1)}$. Then either $r' + 1 \in \tilde{J}'^{-\varepsilon'(1)}$, in which case we have $|\tilde{J}'^{-\varepsilon'(1)} \cap \{r' + 1\}| = 1$, and so $X \geq 2 + 2 - 2 = 2$, or $r' + 1 \in \mathfrak{S}'$, in which case $|\mathfrak{S}'| = q + 1$ is even, and so $X \geq 2 - 2 + \frac{1+(-1)^{|\mathfrak{S}'|}}{2} = 1$.

Suppose q is even. Then

$$2|\tilde{\mathfrak{S}}' \cap J'^{\varepsilon'(1)}| - 2|\tilde{\mathfrak{S}}' \cap J'^{-\varepsilon'(1)}| = 0.$$

Suppose $|\mathfrak{S}|$ is odd. Then $\lambda_{r'+1} = 0$, so $|\mathfrak{S}'| = |\tilde{\mathfrak{S}}'| = q$ is even. Thus $1 + (-1)^{|\mathfrak{S}|} = 0$ and $(1 + (-1)^{|\mathfrak{S}'|})/2 = 1$ and so $X \geq 1$. Suppose $|\mathfrak{S}|$ is even. Then $\lambda_{r'+1} = 1$ and by definition of $\varepsilon'(r' + 1)$, we have $r' + 1 \in J'^{\varepsilon(1)} = J'^{-\varepsilon'(1)}$. Since q is even, we have $s'_q \in J'^{-\varepsilon'(1)}$ by (3.3.4), and so $r' + 1 \notin \mathfrak{S}'$, thus $r' + 1 \in \tilde{J}'^{-\varepsilon'(1)}$. Hence we have $|\mathfrak{S}'| = |\tilde{\mathfrak{S}}'| = q$ is even, so $X \geq 2|\tilde{J}'^{-\varepsilon'(1)} \cap \{r' + 1\}| - (1 + (-1)^{|\mathfrak{S}|}) + \frac{1+(-1)^{|\mathfrak{S}'|}}{2} = 2 - 2 + 1 = 1$.

We conclude that $\bar{\lambda}_1 > \bar{\lambda}_2$ if $\varepsilon(1) = -\varepsilon'(1)$, as desired. □

3.3.2 Proof of Theorem 3.3.1

Proof. We first set up some notation. Let $\eta \in \{\pm 1\}$. For $(x, y) \in \mathcal{R} \times \mathcal{R}$ or $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, let $(x^\eta, y^\eta) := (x, y)$ if $\eta = 1$, and $(x^\eta, y^\eta) := (y, x)$ if $\eta = -1$. Let

$$\phi^{(\eta)} \colon \{1, \ldots, |J^\eta|\} \to J^\eta$$

be the unique increasing bijection. Let $(A, B) = (A_{\lambda,\varepsilon}, B_{\lambda,\varepsilon})$. Let $(\alpha, \beta) \in \mathcal{P}_2(n)$ so that $(\alpha, \beta)_k = \Phi_N(\lambda, [\varepsilon])$ such that if $k = 0$, then $\alpha_1 = A_1$ and $\beta_1 = B_1$, which uniquely defines α and β, since λ only has odd parts, and so $A_1 \neq B_1$. Let $w = \operatorname{sgn}(M + 1/2)$ so that $w = \operatorname{sgn} M$ if $M \neq 0$ and $w = 1$ if $M = 0$. Using (1.2.1), we find

$$A = ((\lambda_j + 1)/2 + 1 - j \colon j \in J^w) \cup (-t(\lambda) - 2i + 2 \colon i \in \mathbb{N}) = (\alpha_i + k + 2 - 2i \colon i \in \mathbb{N}), \quad (3.3.6)$$
$$B = ((\lambda_j + 1)/2 + 1 - j \colon j \in J^{-w}) \cup (-t(\lambda) - 2j + 2 \colon j \in \mathbb{N}) = (\beta_j - k + 2 - 2j \colon j \in \mathbb{N}).$$

We note particularly that for $k = 0$, we defined α, β and w in such a way so that this holds. Thus for any $\eta \in \{\pm 1\}$ and for $i = 1, \ldots, |J^{\eta w}|$, $j = 1, \ldots, |J^{-\eta w}|$, and $h \in \mathbb{N}$, we have

$$A_i^\eta = \frac{\lambda_{\phi^{(\eta w)}(i)} + 1}{2} - \phi^{(\eta w)}(i) + 1 = \alpha_i^\eta + \eta k + 2 - 2i, \quad (3.3.7)$$
$$B_j^\eta = \frac{\lambda_{\phi^{(-\eta w)}(j)} + 1}{2} - \phi^{(-\eta w)}(j) + 1 = \beta_j^\eta - \eta k + 2 - 2j,$$
$$A^\eta_{|J^{\eta w}|+h} = B^\eta_{|J^{-\eta w}|+h} = -t(\lambda) - 2(h-1). \quad (3.3.8)$$

Let $u = \varepsilon(1)$ and $v = uw$. Then for $i = 1, \ldots, |J^u|$, $j = 1, \ldots, |J^{-u}|$, we have

$$A_i^v = \frac{\lambda_{\phi^{(u)}(i)} + 1}{2} - \phi^{(u)}(i) + 1 = \alpha_i^v + vk + 2 - 2i, \quad (3.3.9)$$
$$B_j^v = \frac{\lambda_{\phi^{(-u)}(j)} + 1}{2} - \phi^{(-u)}(j) + 1 = \beta_j^v - vk + 2 - 2j.$$

Let $m_0 = \frac{t(\lambda)+k}{2}$ and $m_1 = \frac{t(\lambda)-k}{2}$. As noted in Remark 1.2.5, m_0 is the smallest integer such that $m_0 \geq t(\alpha)$ and $m_0 - k \geq t(\beta)$. Thus we can consider the order $(m_0, m_1, <_{\alpha,\beta,k})$. We also consider the order $(m_0^v, m_1^v, <_{\alpha^v,\beta^v,vk})$ on the indices of (α^v, β^v). We denote $<_{\alpha,\beta,k}$ by $<$ and $<_{\alpha^v,\beta^v,vk}$ by $<^v$. Since λ only has odd parts, $P_{k,-k;2}(\alpha, \beta, <)$ has a unique element $(\alpha^{\max}, \beta^{\max})$ as shown in the proof of Theorem 3.2.2. We then have $\{(\alpha^{\max,v}, \beta^{\max,v})\} = P_{vk,-vk;2}(\alpha^v, \beta^v, <^v)$. We have $(1, 0) <^v (1, 1)$, since the largest term of $A^v \sqcup B^v$ is $\frac{\lambda_1+1}{2} = \frac{\lambda_{\phi^{(u)}(1)}+1}{2} = A_1^v$, so applying procedure (a) to (α^v, β^v) gives us a bipartition $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ with an order on the set of its indices equivalent to $<_{\boldsymbol{\alpha},\boldsymbol{\beta},vk-1}$ by Lemma 3.1.5, and positive integers $\bar{\mu}_1, a_1, a_2, \ldots, a_x, b_1, b_2, \ldots, b_y$ such that

$$(\alpha^{\max,v}, \beta^{\max,v}) = ((\bar{\mu}_1) \sqcup \boldsymbol{\mu}, \boldsymbol{\nu}),$$

where $\{(\boldsymbol{\mu},\boldsymbol{\nu})\} = P_{vk-2,-vk;2}(\boldsymbol{\alpha},\boldsymbol{\beta},<_{\boldsymbol{\alpha},\boldsymbol{\beta},vk-1}) = P_{vk-1,-vk+1;2}(\boldsymbol{\alpha},\boldsymbol{\beta},<_{\boldsymbol{\alpha},\boldsymbol{\beta},vk-1})$. From (3.3.9) and by definition of the a_i, we see that $\mathfrak{S}\cap J^u = \{\phi(a_1),\ldots,\phi(a_x)\}$ and so $\phi^{(u)}(a_i) = s_{2i-1}$ for $i=1,\ldots,x$. Similarly, $\phi^{(-u)}(b_j) = s_{2j}$ for $j=1,\ldots,y$.

Having set up the notation, we now prove the theorem by induction on N. The theorem is obvious for $N=0,1$, so assume $N>1$.

1. We will show that $\bar{\lambda}_1 = (\lambda^{\max})_1$. Let $q = |\,\mathfrak{S}\,|$. Using (3.3.9), we find that

$$\begin{aligned}\bar{\mu}_1 &= \sum_{h=1}^{\lceil q/2\rceil} \alpha^v_{a_h} + \sum_{h=1}^{\lfloor q/2\rfloor} \beta^v_{b_h}\\ &= \sum_{h=1}^{\lceil q/2\rceil} \frac{\lambda_{s_{2h-1}}+1}{2} - s_{2h-1} - 1 - vk + 2a_h + \sum_{h=1}^{\lfloor q/2\rfloor} \frac{\lambda_{s_{2h}}+1}{2} - s_{2h} - 1 + vk + 2b_h.\end{aligned}$$

Suppose first that q is odd. Then

$$\bar{\mu}_1 = -s_1 - 1 - vk + 2a_1 + \sum_{h=1}^{q} \frac{\lambda_{s_h}+1}{2} + \sum_{h=1}^{\frac{q-1}{2}} (2a_{h+1} - s_{2h+1} + 2b_h - s_{2h} - 2).$$

We have $2a_1 - s_1 - 1 = 2-1-1 = 0$. Since $a_{h+1} = |\{i\in J^u : i \le s_{2h+1}\}|$ and $b_h = |\{i \in J^{-u} : i \le s_{2h}\}|$, we also have $a_{h+1} + b_h = |\{1,\ldots,s_{2h}\}\cup\{s_{2h+1}\}| = s_{2h}+1$. Thus

$$\begin{aligned}\bar{\mu}_1 = -vk + \sum_{h=1}^{q}\frac{\lambda_{s_h}+1}{2} + \sum_{h=1}^{\frac{q-1}{2}} s_{2h} - s_{2h+1} &= \frac{1}{2} - vk + \sum_{h=1}^{q}\frac{\lambda_{s_h}}{2} + \sum_{h=1}^{\frac{q-1}{2}}(s_{2h} - s_{2h+1} + 1)\\ &= \frac{1}{2} - vk + \sum_{h=1}^{q}\frac{\lambda_{s_h}}{2} - |\tilde{J}^{-u}|.\end{aligned}$$

Proposition 3.1.6 implies that $(\lambda^{\max})_1$ is odd, so we also have $\bar{\mu}_1 = \frac{(\lambda^{\max})_1+1}{2} - 2vk$, hence

$$(\lambda^{\max})_1 = 2\bar{\mu}_1 + 2vk - 1 = \sum_{h=1}^{q}\lambda_{s_h} - 2|\tilde{J}^{-u}| = \bar{\lambda}_1.$$

Now suppose that q is even. Then

$$\bar{\mu}_1 = \left(\sum_{h=1}^{q}\frac{\lambda_{s_h}+1}{2}\right) + \sum_{h=1}^{q/2}(2a_h - s_{2h-1} + 2b_h - s_{2h} - 2).$$

Similarly as above, we find $a_h + b_h = |\{1,\ldots,s_{2h-1}\}\cup\{s_{2h}\}| = s_{2h-1}+1$. Thus

$$\begin{aligned}\bar{\mu}_1 = \left(\sum_{h=1}^{q}\frac{\lambda_{s_h}+1}{2}\right) + \sum_{h=1}^{q/2}(s_{2h-1} - s_{2h}) &= \left(\sum_{h=1}^{q}\frac{\lambda_{s_h}}{2}\right) + \sum_{h=1}^{q/2}(s_{2h-1} - s_{2h} + 1) \quad (3.3.10)\\ &= \left(\sum_{h=1}^{q}\frac{\lambda_{s_h}}{2}\right) - |\tilde{J}^{u}|.\end{aligned}$$

Since q is even, we have $|J^{\pm 1}| = |\tilde{J}^{\pm 1}| + q/2$, and so $M = |J^1| - |J^{-1}| = |\tilde{J}^1| - |\tilde{J}^{-1}|$. Thus we have $vk = uwk = uM = |\tilde{J}^u| - |\tilde{J}^{-u}|$, so by (3.3.10), we have

$$\bar{\mu}_1 = \left(\sum_{h=1}^{q} \frac{\lambda_{s_h}}{2} \right) - vk - |\tilde{J}^{-u}|,$$

hence

$$(\lambda^{\max})_1 = 2\bar{\mu}_1 + 2vk - 1 = -1 + \sum_{h=1}^{q} \lambda_{s_h} - 2|\tilde{J}^{-u}| = \bar{\lambda}_1.$$

2. We will show that $\Phi_{N'}(\lambda', [\varepsilon']) = (\boldsymbol{\alpha}^{-uw'}, \boldsymbol{\beta}^{-uw'})_{k'}$. Let $m'_u := |\tilde{J}^u| = |J'^{-u} \cap \{1, \ldots, r'\}|$ and let $\iota\colon \{1, \ldots, m'_u\} \to \mathbb{N} \setminus \{a_1, \ldots, a_x\}$ be the unique increasing injection. Define

$$\psi'^{(u)} = \phi^{-1} \circ \phi^{(u)} \circ \iota.$$

Then $\psi'^{(u)}$ is the unique increasing bijection from $\{1, \ldots, m'_u\}$ to $J'^{-u} \cap \{1, \ldots, r'\}$. Let $j \in \{1, \ldots, m'_u\}$ and $q = \psi'^{(u)}(j)$. For $h := \iota(j) - j$ we have $a_h < \iota(j) < a_{h+1}$, and applying $\psi'^{(u)}$ to the three terms gives $s_{2h-1} < \phi(q) < s_{2h+1}$. Thus we have $\phi(q) = q + 2h - 1$. Using (3.3.9), we find

$$\begin{aligned}
\boldsymbol{\alpha}_j = \alpha^v_{\iota(j)} &= \frac{\lambda_{\phi(q)} + 1}{2} - \phi(q) + 1 - (vk + 2 - 2\iota(j)) \\
&= \frac{\lambda_{\phi(q)} + 1}{2} - (q + 2h - 1) + 1 - (vk + 2 - 2j - 2h) \\
&= \frac{\lambda_{\phi(q)} + 1}{2} + 2j - vk - q.
\end{aligned}$$

Since $\phi(q) \in J^u$, we have $\lambda'_{\psi'^{(u)}(j)} = \lambda_{\phi(q)}$, and so

$$\boldsymbol{\alpha}_j + (vk - 1) + 2 - 2j = \frac{\lambda_{\phi(q)} + 1}{2} + 1 - q = \frac{\lambda'_{\psi'^{(u)}(j)} + 1}{2} + 1 - \psi'^{(u)}(j). \tag{3.3.11}$$

Let $u' = \varepsilon'(1)$, $w' = \mathrm{sgn}(M' + 1/2)$, $v' = u'w'$ and $(A', B') = (A_{\lambda', \varepsilon'}, B_{\lambda', \varepsilon'})$. Let $\phi'^{(\pm 1)}\colon \{1, \ldots, |J'^{\pm 1}|\} \to J'^{\pm 1}$ be the unique increasing bijections. We have $\phi'^{(-u)}|_{\{1, \ldots, m'_u\}} = \psi'^{(u)}$, and by the analogue of (3.3.7) for (λ', ε'), we have $A'^{-uw'}_j = \frac{\lambda'_{\phi'^{(-u)}(j)} + 1}{2} + 1 - \phi'^{(-u)}(j)$ for $j = 1, \ldots, m'_u$, so by (3.3.11), we get

$$A'^{-uw'}_j = \frac{\lambda'_{\phi'^{(-u)}(j)} + 1}{2} + 1 - \phi'^{(-u)}(j) = \frac{\lambda'_{\psi'^{(u)}(j)} + 1}{2} + 1 - \psi'^{(u)}(j) = \boldsymbol{\alpha}_j + (vk - 1) + 2 - 2j.$$

Thus the first m'_u terms of $\boldsymbol{\alpha} + [vk - 1, -\infty[_2$ are equal to the first m'_u terms of $A'^{-uw'}$. By similar arguments, it holds that the first m'_{-u} terms of $\boldsymbol{\beta} + [-vk + 1, -\infty[_2$ are equal to the first m'_{-u} terms of $B'^{-uw'}$. Let $h \in \mathbb{N}$. We consider two cases. Suppose $|\mathfrak{S}|$ is even. Then

$r'+1 \in J'^u$ and $t(\lambda') = r'+1$, so $|J'^u| = m'_{-u}+1$, and $|J'^{-u}| = m'_u$. By the analogue of (3.3.8) for (λ', ε'), we have

$$(A'^{-uw'})_{m'_u+h} = (A'^{-uw'})_{|J'^{-u}|+h} = -t(\lambda') - 2(h-1) = -r'+1-2h,$$
$$(B'^{-uw'})_{m'_{-u}+1+h} = (B'^{-uw'})_{|J'^u|+h} = -t(\lambda') - 2(h-1) = -r'+1-2h.$$

We have $r' = |\tilde{J}^u| + |\tilde{J}^{-u}| = m'_u + m'_{-u}$, and since $\mathfrak{S}$ is even, we have $m'_u - m'_{-u} = |\tilde{J}^u| - |\tilde{J}^{-u}| = |J^u| - |J^{-u}| = vk$. Thus $r' = -vk + 2m'_u = vk + 2m'_{-u}$. Since $\boldsymbol{\alpha}_{m'_u+h} = \boldsymbol{\beta}_{m'_{-u}+1+h} = 0$, we find

$$\begin{aligned}\boldsymbol{\alpha}_{m'_u+h} + (vk-1) + 2 - 2m'_u - 2h &= -r'+1-2h \\ &= \boldsymbol{\beta}_{m'_{-u}+1+h} + (1-vk) - 2m'_{-u} - 2h.\end{aligned}$$

It remains to consider the $(m'_{-u}+1)$-th term of $B'^{-uw'}$: we have $\boldsymbol{\beta}_{m'_{-u}+1} = 0$ and $\lambda'_{r'+1} = 1$, so

$$\boldsymbol{\beta}_{m'_{-u}+1} + (1-vk) - 2m'_{-u} = 1 - r' = \frac{\lambda'_{r'+1}+1}{2} - r' = (B'^{-uw'})_{m'_{-u}+1}.$$

Thus we have shown that if $|\mathfrak{S}|$ is even, then $(\boldsymbol{\alpha} + [vk-1, -\infty[_2, \boldsymbol{\beta} + [-vk+1, -\infty[_2) = (A'^{-uw'}, B'^{-uw'})$.

Suppose $|\mathfrak{S}|$ is odd. Then $\lambda'_{r'+1} = 0$, so $|J'^u| = m'_{-u}$, and $|J'^{-u}| = m'_u$. By similar arguments as in the previous case, we have $t(\lambda') = r'$, $r' = -vk+1+2m'_u = vk-1+2m'_{-u}$, so for all $h \in \mathbb{N}$, we have

$$\begin{aligned}(A'^{-uw'})_{m'_u+h} = (B'^{-uw'})_{m'_{-u}+h} &= -r'+2-2h, \\ \boldsymbol{\alpha}_{m'_u+h} + (vk-1) + 2 - 2m'_u - 2h = \boldsymbol{\beta}_{m'_{-u}+h} + (1-vk) + 2 - 2m'_{-u} - 2h &= -r'+2-2h.\end{aligned}$$

Thus we have shown that

$$(\boldsymbol{\alpha} + [vk-1, -\infty[_2, \boldsymbol{\beta} + [-vk+1, -\infty[_2) = (A'^{-uw'}, B'^{-uw'}).$$

Note that $-uw'(vk-1) = -ww'(k-v)$, so this gives

$$(\boldsymbol{\alpha}^{-uw'} + [-ww'(k-v), -\infty[_2, \boldsymbol{\beta}^{-uw'} + [ww'(k-v), -\infty[_2) = (A', B'). \tag{3.3.12}$$

We need to show that $k' = -ww'(k-v)$; it then follows from (3.3.12) that $\Phi_{N'}(\lambda', [\varepsilon']) = (\boldsymbol{\alpha}^{-uw'}, \boldsymbol{\beta}^{-uw'})_{k'}$, as desired. By Proposition 3.3.4(4), we have $k' = |k-v|$, so we need to show that $|k-v| = -ww'(k-v)$ This is obvious if $k-v=0$. Suppose $k-v \geq 1$. Then $M \neq 0$ and $M' \neq 0$. Since $M = -M'+u$ by Proposition 3.3.4(4), we have $\operatorname{sgn} M = -\operatorname{sgn} M'$, so $w = -w'$, and thus $-ww'(k-v) = k-v = |k-v|$. Suppose $k-v = -1$. Then $k=0$ and $v=1$. From $k=0$, we get $M=0$ and so $w=1$. Thus $u = v/w = 1$. Next, we get $M' = -M+u = 1$, so $w' = 1$. Thus $-ww'(k-v) = |k-v|$.

3. We will show that $(A_{\bar\lambda,\bar\varepsilon}, B_{\bar\lambda,\bar\varepsilon})_k = (A_{\lambda^{\max}}, B_{\varepsilon^{\max}})_k$, hence $(\bar\lambda, [\bar\varepsilon]) = (\lambda^{\max}, [\varepsilon^{\max}])$. We have

$$(A^v_{\lambda^{\max},\varepsilon^{\max}}, B^v_{\lambda^{\max},\varepsilon^{\max}})_k = ((\bar\mu_1 + vk) \sqcup (\boldsymbol{\mu} + [vk-2,-\infty[_2), \boldsymbol{\nu} + [-vk,-\infty[_2)_k. \quad (3.3.13)$$

Recall that $\{(\boldsymbol{\mu},\boldsymbol{\nu})\} = P_{vk-1,-vk+1;2}(\boldsymbol{\alpha},\boldsymbol{\beta},<_{\boldsymbol{\alpha},\boldsymbol{\beta},vk-1})$. We just saw that $-uw'(vk-1) = k'$, so $\{(\boldsymbol{\mu}^{-uw'},\boldsymbol{\nu}^{-uw'})\} = P_{k',-k';2}(\boldsymbol{\alpha}^{-uw'},\boldsymbol{\beta}^{-uw'},<_{\boldsymbol{\alpha}^{-uw'},\boldsymbol{\beta}^{-uw'},k'})$. So by the induction hypothesis, we have $(\bar\lambda', [\bar\varepsilon']) = (\lambda'^{\max}, [\varepsilon'^{\max}])$, $M(\bar\lambda',\bar\varepsilon') = M(\lambda',\varepsilon')$, and

$$A_{\bar\lambda',\bar\varepsilon'} = \boldsymbol{\mu}^{-uw'} + [k',-\infty[_2 \quad\text{and}\quad B_{\bar\lambda',\bar\varepsilon'} = \boldsymbol{\nu}^{-uw'} + [-k',-\infty[_2.$$

Let $\bar J^{\pm1} = \{i \in \mathbb{N} \colon \bar\varepsilon(i)(-1)^{i+1} = \pm1\}$ and $\bar J'^{\pm1} = \{i \in \mathbb{N} \colon \bar\varepsilon'(i)(-1)^{i+1} = \pm1\}$. Note that $M(\bar\lambda',\bar\varepsilon') = M(\lambda',\varepsilon')$ implies that $w' = \operatorname{sgn}(M(\bar\lambda',\bar\varepsilon') + 1/2)$. By Proposition 3.3.5, $\bar\lambda'$ and $\bar\lambda$ only have odd parts, so by the analogue of (3.3.6) for $(\bar\lambda',\bar\varepsilon')$, it holds that the largest $|\bar J'^{-u}|$ terms of $\boldsymbol{\mu} + [vk-1,-\infty[_2 = A^{-uw'}_{\bar\lambda',\bar\varepsilon'}$ are the largest $|\bar J'^{-u}|$ terms of

$$\begin{aligned}
&\left(\frac{\bar\lambda'_j+1}{2} + 1 - j \colon j \in \mathbb{N}, (-1)^{j+1}\bar\varepsilon'(j) = -uw'w' = -u\right)\\
&= \left(\frac{\bar\lambda_{j+1}+1}{2} + 1 - j \colon j \in \mathbb{N}, (-1)^{j+1}\bar\varepsilon(j+1) = -u\right)\\
&= \left(\frac{\bar\lambda_j+1}{2} + 2 - j \colon j \in \mathbb{N}_{\geq 2}, (-1)^{j}\bar\varepsilon(j) = -u\right)\\
&= \left(\frac{\bar\lambda_j+1}{2} + 2 - j \colon j \in \mathbb{N}_{\geq 2}, (-1)^{j+1}\bar\varepsilon(j) = u\right).
\end{aligned}$$

Thus the largest $|\bar J'^{-u}|+1$ terms of $(\bar\mu_1+vk)\sqcup(\boldsymbol{\mu}+[vk-2,-\infty[_2)$ are the largest $|\bar J'^{-u}|+1$ terms of

$$\left(\frac{\bar\lambda_1+1}{2}\right) \sqcup \left(\frac{\bar\lambda_j+1}{2} + 1 - j \colon j \in \mathbb{N}_{\geq 2}, (-1)^{j+1}\bar\varepsilon(j) = u\right). \quad (3.3.14)$$

Note that $\bar\varepsilon(1) = \varepsilon(1) = u$ and recall that $M(\bar\lambda',\bar\varepsilon') = M(\lambda',\varepsilon')$. We also have $M(\lambda',\varepsilon') = -M(\lambda,\varepsilon)+u$ by Proposition 3.3.4(4). By definition of $(\bar\lambda,\bar\varepsilon)$, we have $M(\bar\lambda,\bar\varepsilon) = -M(\bar\lambda',\bar\varepsilon')+u$, hence we have $M(\bar\lambda,\bar\varepsilon) = M(\lambda,\varepsilon)$ (and so also $k(\bar\lambda,\bar\varepsilon) = k$). Thus if we define $\bar u := \bar\varepsilon(1)$, $\bar w = \operatorname{sgn}(M(\bar\lambda,\bar\varepsilon)+1/2)$ and $\bar v := \bar u\bar w$, we find that $v = \bar v$, $u = \bar u$ and $w = \bar w$. Thus by the analogue of (3.3.9) for $(\bar\lambda,\bar\varepsilon)$, we see that the largest $|\bar J^u|$ terms of (3.3.14) are equal to the first $|\bar J^u|$ terms of $A^v_{\bar\lambda,\bar\varepsilon}$. By definition of $(\bar\lambda,\bar\varepsilon)$, we see that $|\bar J^u| = |\bar J'^{-u}|+1$, so the largest $|\bar J^u|$ terms of $A^v_{\bar\lambda,\bar\varepsilon}$ are equal to the largest $|\bar J^u| = |\bar J'^{-u}|+1$ terms of $(\bar\mu_1+vk)\sqcup(\boldsymbol{\mu}+[vk-2,-\infty[_2)$.

By similar arguments as above, we find that the largest $|\bar J^{-u}| = |\bar J'^{u}|$ terms of $(\boldsymbol{\nu} + [-vk,-\infty[_2)$ are equal to the largest $|\bar J'^{u}|$ terms of

$$\left(\frac{\bar\lambda_j+1}{2} + 1 - j \colon j \in \mathbb{N}, (-1)^{j+1}\bar\varepsilon(j) = -u\right),$$

hence equal to the largest $|\bar{J}^{-u}|$ terms of $B^v_{\bar{\lambda},\bar{\varepsilon}}$.

Clearly, we have $t(\bar{\lambda}) = t(\bar{\lambda}') + 1$. Let $h \in \mathbb{N}$ and $i := |\bar{J}^u| + h = |\bar{J}'^{-u}| + h + 1$. Then

$$(A^v_{\bar{\lambda},\bar{\varepsilon}})_i = -t(\bar{\lambda}) - 2(h-1).$$

Note that $\boldsymbol{\mu}_{i+h-1} \leq \boldsymbol{\mu}_{|\bar{J}'^{-u}|+1} = 0$, so the i^{th} term of $(\bar{\mu}_1 + vk) \sqcup (\boldsymbol{\mu} + [vk-2, -\infty[_2)$ is

$$\boldsymbol{\mu}_{i-1} + (vk-2) + 2 - 2(i-1) = vk - 2|\bar{J}^u| - 2h + 2.$$

We have $t(\bar{\lambda}) = |\bar{J}^u| + |\bar{J}^{-u}|$. We showed earlier that $\bar{u} = u$, $\bar{v} = v$,$\bar{w} = w$, and $k(\bar{\lambda},\bar{\varepsilon}) = k$, so we have $vk = |\bar{J}^u| - |\bar{J}^{-u}|$, hence $t(\bar{\lambda}) = 2|\bar{J}^u| - vk$. Thus $(A^v_{\bar{\lambda},\bar{\varepsilon}})_i$ is the i^{th} term of $(\bar{\mu}_1 + vk) \sqcup (\boldsymbol{\mu} + [vk-2, -\infty[_2)$. Hence we now have

$$A^v_{\bar{\lambda},\bar{\varepsilon}} = (\bar{\mu}_1 + vk) \sqcup (\boldsymbol{\mu} + [vk-2, -\infty[_2). \tag{3.3.15}$$

Similarly, for $j := |\bar{J}^{-u}| + h = |\bar{J}'^u| + h$, we find that $(B^v_{\bar{\lambda},\bar{\varepsilon}})_j$ is the j^{th} term of $(\boldsymbol{\nu} + [-vk, -\infty[_2)$, hence

$$B^v_{\bar{\lambda},\bar{\varepsilon}} = \boldsymbol{\nu} + [-vk, -\infty[_2. \tag{3.3.16}$$

By (3.3.13), we then have

$$(A_{\bar{\lambda},\bar{\varepsilon}}, B_{\bar{\lambda},\bar{\varepsilon}})_k = (A_{\lambda^{\max}}, B_{\varepsilon^{\max}})_k.$$

We conclude that $(\bar{\lambda}, [\bar{\varepsilon}]) = (\lambda^{\max}, [\varepsilon^{\max}])$. Note that $((\bar{\mu}_1) \sqcup \boldsymbol{\mu}, \boldsymbol{\nu}) = ((\alpha^{\max})^v, (\beta^{\max})^v)$, so we also have $(A_{\bar{\lambda},\bar{\varepsilon}}, B_{\bar{\lambda},\bar{\varepsilon}}) = (\alpha^{\max} + [k, -\infty[_2, \beta^{\max} + [-k, -\infty[_2)$ by (3.3.15) and (3.3.16). We showed earlier that $M(\bar{\lambda},\bar{\varepsilon}) = M(\lambda,\varepsilon)$, and so we completed the proof. □

Corollary 3.3.6. *It holds that $\lambda^{\max} = \bar{\lambda}$ only has odd parts.*

3.3.3 Example

Example 3.3.7. Let $(\lambda,\varepsilon) \in \boldsymbol{\mathcal{P}}^{\text{ort},2}(2n+1)$ for some $n \in \mathbb{Z}_{\geq 0}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_7 > \lambda_8 = 0$. Write $\varepsilon = (\varepsilon(1)\ \varepsilon(2)\ \ldots) = (-++-+-+)$. We compute $(\lambda^1, \varepsilon^1) := (\lambda', \varepsilon'), (\lambda^2, \varepsilon^2) := (\lambda'', \varepsilon''), \ldots$. For each $j \in \mathbb{N}$ for which $(\lambda^j, \varepsilon^j)$ is defined, we define $\tilde{\varepsilon}^j = (\varepsilon^j(1), -\varepsilon^j(2), \varepsilon^j(3), \ldots)$. Following the algorithm, we find

$$\begin{aligned} \bar{\lambda}_1 &= \lambda_1 + \lambda_3 - 9, & \bar{\varepsilon}(1) &= -1, \\ \bar{\lambda}_2 &= \lambda_2 + (\lambda_4 + 2) - 9, & \bar{\varepsilon}(2) &= 1. \end{aligned}$$

Next, we have $(\bar{\lambda}^2, \bar{\varepsilon}^2) = (\lambda^2, \varepsilon^2)$, since for any $N \in \mathbb{N}$ and $(\boldsymbol{\lambda}, \boldsymbol{\varepsilon}) \in \boldsymbol{\mathcal{P}}^{\text{ort},2}(N)$ such that $\boldsymbol{\lambda}$ only has odd parts and $\boldsymbol{\varepsilon}(i) = (-1)^{i+1}$ for all $i \in \{1, \ldots, t(\boldsymbol{\lambda})\}$, it is easy to see from the algorithm that $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\varepsilon}}) = (\boldsymbol{\lambda}, \boldsymbol{\varepsilon})$. Finally, we have $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2) \sqcup \bar{\lambda}^2$ and $\bar{\varepsilon} = (\bar{\varepsilon}(1), \bar{\varepsilon}(2), \varepsilon^2(1), \varepsilon^2(2), \varepsilon^2(3), \varepsilon^2(4), \varepsilon^2(5)) = (-++-+-+)$.

$$
\begin{array}{rccccccccc}
\lambda = & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & & \\
\varepsilon = & - & + & + & - & + & - & + & & \\
\tilde{\varepsilon} = & - & - & + & + & + & + & + & & \\
 & & & & & & & & & \\
\lambda^1 = & & \lambda_2 & & \lambda_4+2 & \lambda_5+2 & \lambda_6+2 & \lambda_7+2 & 1 & \\
\varepsilon^1 = & & + & & + & - & + & - & + & \\
\tilde{\varepsilon}^1 = & & + & & - & - & - & - & - & \\
 & & & & & & & & & \\
\lambda^2 = & & & & & \lambda_5+4 & \lambda_6+4 & \lambda_7+4 & 3 & 1 \\
\varepsilon^2 = & & & & & + & - & + & - & + \\
\tilde{\varepsilon}^2 = & & & & & + & + & + & + & +
\end{array}
$$

3.4 Algorithm for $(\lambda^{\max}, \varepsilon^{\max})$ for $\mathrm{Sp}(2n)$

The algorithm for $\mathrm{Sp}(2n)$ is given in [Wal19, §5]. For completeness, we shall state the algorithm here.

Suppose $(\lambda, \varepsilon) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$ such that λ only has even parts. If $n = 0$, let $(\bar{\lambda}, \bar{\varepsilon}) = (\lambda, \varepsilon)$. If $n > 0$. Similarly as we did previously for $\mathrm{SO}(N)$, we write ε as a function $\varepsilon\colon \mathbb{N} \to \{\pm 1\}$ with $\varepsilon(i) = \varepsilon_{\lambda_i}$ for all $i \in \mathbb{N}$. Let $\eta \in \{\pm 1\}$ and recall/define

$$
\begin{aligned}
&J^{\eta} = \{i \in \mathbb{N}\colon (-1)^{i+1}\, \varepsilon(i) = \eta\},\\
&\mathcal{O} := \{i \in \mathbb{N}_{\geq 2}\colon \varepsilon(i) = \varepsilon(i-1)\} \cup \{1\},\\
&\tilde{J}^{\eta} := J_1 \setminus \mathcal{O}\,.
\end{aligned}
$$

Note that $\mathbb{N} \setminus \mathcal{O}$ is finite, hence $\tilde{J}^{\eta}$ is also finite. Define

$$
\begin{aligned}
\bar{\lambda}_1 &= \sum_{i \in \mathcal{O}} \lambda_i - 2|\tilde{J}^{-\,\varepsilon(1)}|,\\
\bar{\varepsilon}_{\bar{\lambda}_1} &= \varepsilon(1).
\end{aligned}
$$

Also define

$$
\lambda' = (\lambda_j \colon j \in \tilde{J}^{\varepsilon(1)}) \sqcup (\lambda_j + 2 \colon j \in \tilde{J}^{-\,\varepsilon(1)}).
$$

For each $i \in \lambda'$, there exists a $j \in J^{\varepsilon(1)}$ such that $i = \lambda_j$ or a $j \in \tilde{J}^{-\,\varepsilon(1)}$ such that $i = \lambda_j + 2$. Fix such a j and set

$$
\varepsilon'_i = \begin{cases} \varepsilon(j) & \text{if } j \in \tilde{J}^{\varepsilon(1)},\\ -\,\varepsilon(j) & \text{if } j \in \tilde{J}^{-\,\varepsilon(1)}. \end{cases}
$$

This defines a pair $(\lambda', \varepsilon') \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n - \bar{\lambda}_1)$. We can repeat the steps above for (λ', ε') to obtain $\overline{\lambda'}_1 = \bar{\lambda}_2$ and $\overline{\varepsilon'}_{\overline{\lambda'}_1} = \bar{\varepsilon}_{\bar{\lambda}_2}$ as well as a pair $(\lambda'', \varepsilon'')$, etc. to obtain an element $(\bar{\lambda}, \bar{\varepsilon}) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$.

Theorem 3.4.1 ([Wal19, Théorème 5.1])**.** *We have* $(\bar{\lambda}, \bar{\varepsilon}) = (\lambda^{\max}, \varepsilon^{\max})$.

Chapter 4

Algorithm for $\mathbb{IM}$, IM, and AZ for tempered representations

4.1 Algorithm for $\mathbb{IM}$

We will now describe an explicit algorithm for $\mathbb{IM}$ for graded Hecke algebras attached to cuspidal local systems. Let G be a connected complex reductive group and suppose we are in the same setting as in Section 1.3.5. Recall $\mathcal{M}_{\text{temp}}$ from (1.3.8) and define

$$\mathcal{M}_{\text{temp,real,q}} = \{(e, \sigma, \psi) \colon (e, \sigma, \frac{1}{2} \log q, \psi) \in \mathcal{M}_{\text{temp,real}}\}.$$

Let $(e, \sigma, \psi) \in \mathcal{M}_{\text{temp,real,q}}$ and recall from Proposition 1.3.20 that the standard module $Y := Y(e, \sigma, \frac{1}{2}\log q, \psi)$ is irreducible. Since Y is tempered and σ is real, we have $A_G(e, \sigma) = A_G(e)$ by Proposition 1.3.21. Let $(L, C_L, \mathcal{L}) \in S_G$ be the cuspidal support of $(e, \sigma, \frac{1}{2}\log q, \psi)$. The graded Hecke algebra $\mathbb{H}(L, C_L, \mathcal{L})$ contains the relative Weyl group W_L by construction. From [Lus95b, §10.13] and (1.3.7), we have isomorphisms

$$Y|_{W_L} \cong Y(e, 0, 0, \psi) \cong H_\bullet^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})^\psi.$$

Note that $\psi \in A_G(e)^\wedge$ (the point we stress here is that this component group has no σ dependence), and from Proposition 1.3.22, it follows that $(L, C_L, \mathcal{L})$ is the cuspidal support of (e, ψ) in the generalised Springer correspondence. Hence the graded W_L-representation $H_\bullet^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})^\psi$ is fully described by Green functions of G associated to the relative Weyl group W_L: for $(C, \mathcal{E}), (C', \mathcal{E}') \in \mathcal{N}_{\mathrm{G}}$ with $(C, \mathcal{E})$ corresponding to (e, ψ) and (e', ψ') respectively, the multiplicity of $\mathrm{GSpr}(C', \mathcal{E}')$ in $H_\bullet^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})^\psi$ is precisely the multiplicity $\mathrm{mult}(C, \mathcal{E}; C', \mathcal{E}')$ from Section 3.1.6.

There exists an $(e', \psi') \in \mathcal{M}_{\sigma, \frac{1}{2}\log(q)}$ such that

$$\bar{Y}' := \bar{Y}(e', \sigma, \frac{1}{2}\log(q), \psi') = \mathbb{IM}(Y(e, \sigma, \frac{1}{2}\log(q), \psi)).$$

Write $Y' = Y(e', \sigma, \frac{1}{2}\log(q), \psi')$ for the corresponding standard module. Note that $\bar{Y}'|_{W_L}$ is a W_L-subrepresentation of

$$Y'|_{W_L} \cong H_\bullet^{\{1\}}(\mathcal{P}_{e'}, \dot{\mathcal{L}})^{\psi'} \cong \bigoplus_{\phi' \in A_G(e')^\wedge} [\phi' : \psi'] H_\bullet^{\{1\}}(\mathcal{P}_{e'}, \dot{\mathcal{L}})^{\phi'},$$

and each non-zero $H_\bullet^{\{1\}}(\mathcal{P}_{e'}, \dot{\mathcal{L}})^{\phi'}$ is a direct sum of generalised Springer representations of the form $E(e'', \phi'')$ with the property that the G-orbit of e'' is strictly larger than that of e' in the closure ordering, or $(e'', \phi'') = (e', \phi')$. Furthermore, $E(e', \phi')$ appears in $H_\bullet^{\{1\}}(\mathcal{P}_{e'}, \dot{\mathcal{L}})^{\phi'}$ with multiplicity 1, and furthermore also appears in $\bar{Y}'|_{W_L}$ (cf. last math display in [CMO21, Proof of Theorem 3.0.8]).

By Remark 2.0.5, we also have the following

$$\bar{Y}'|_{W_L} = Y|_{W_L} \otimes \mathrm{sgn} \cong H_\bullet^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})^{\psi} \otimes \mathrm{sgn},$$

which we can also view as a sum of generalised Springer representations of the relative Weyl group W_L.

From the above, we can deduce the following proposition.

Proposition 4.1.1. *We retain the notation above. Suppose that $Y'|_{W_L}$ contains a unique generalised Springer representation $E(e_0, \psi_0)$ with multiplicity 1 such that for any other generalised Springer representation $E(e'', \psi'')$ appearing in $Y'|_{W_L}$, we either have that e'' is strictly larger than e_0 in the closure ordering, or $(e_0, \psi_0) = (e'', \psi'')$. Then we have*

$$e' = e_0, \quad A_G(e') = A_G(e_0), \quad \psi' = \psi_0,$$

hence $\mathbb{IM}(Y(e, \sigma, \frac{1}{2}\log(q), \psi')) = \bar{Y}(e_0, \sigma, \frac{1}{2}\log(q), \psi_0)$.

In the following subsections, we explore cases where the condition in Proposition 4.1.1 is satisfied and where we are able to compute (e_0, ψ_0) from (e, ψ) explicitly.

4.1.1 Type A: $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$

Suppose $G = \mathrm{SL}(n, \mathbb{C})$. For details about the generalised Springer correspondence of G, we refer to [Lus84b, §10.3] and [LS85, §5]; also see [CMO23, §5.6.1] for a short overview. The nilpotent classes of G are parametrised by partitions of n. Let $(e, \sigma, \psi) \in \mathcal{M}_{\mathrm{temp,real}}$ and suppose e is parametrised by a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of n. Recall from Proposition 1.3.21 that we can identify $A_G(e) = A_G(e, \sigma)$, so $\psi \in A_G(e)^\wedge$. It is a well-known fact that $A_G(e) \cong \mathbb{Z}/n'\mathbb{Z}$ where $n' = \gcd(\lambda_1, \dots, \lambda_t)$. Let d be the degree of ψ in $\mathbb{Z}/n'\mathbb{Z}$. Lusztig showed that $d|\lambda_i$ for $i = 1, \dots, t$. The relative Weyl group W_L of the cuspidal support of (e, ϕ) is equal to $S_{n/d}$ and the generalised Springer representation corresponding to (e, ϕ) is the representation of $S_{n/d}$ parametrised by the partition $(\lambda_1/d, \ldots, \lambda_t/d)$ of n/d. We denote such a partition by λ/d and denote the corresponding representation by $\rho_{\lambda/d}$.

The combinatorial results regarding the Green functions for type A in [Mac98, III.6] tell us that $\rho_{(n/d)}$ appears with multiplicity 1 in $Y(e, s, \psi)|_W$, and is furthermore the unique subrepresentation with maximal Springer support. Tensoring a representation with the sign representation corresponds to transposing the corresponding partition, so $\rho_{(n/d)} \otimes \text{sgn} = \rho_{(1)^{n/d}}$ is the representation $E(e_0, \psi_0)$ in Proposition 4.1.1 in the current setting. Proposition 4.1.1 also tells us what e_0 and ψ_0 are, which from the generalised Springer correspondence turn out to be the orbit of a nilpotent element $e_{(d)^{n/d}}$ parametrised by $(d)^{n/d}$ and ψ. Recall that IM fixes infinitesimal characters, so we have

$$\mathbb{IM}(Y(e, \sigma, \psi)) = \bar{Y}(e_{(d)^{n/d}}, \sigma, \psi).$$

4.1.2 Type B and D: $\mathfrak{so}(N, \mathbb{C})$

Suppose $G = \text{SO}(N, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{so}(N, \mathbb{C})$. We identify $\mathcal{M}_{\text{temp,real}}$ with the set of G-conjugacy classes of triples $(\lambda, \sigma, [\varepsilon])$ with $\lambda \in \mathcal{P}^{\text{ort}}(N)$ and $[\varepsilon] \in F_2'[\Delta(\lambda)]$. We also write standard modules in terms of these triples (i.e. $Y(\lambda, \sigma, [\varepsilon])$) and generalised Springer representations in terms of the pairs (λ, ε) (i.e. $E(\lambda, [\varepsilon])$; this notation is consistent with the notation we used for the generalised Springer correspondence of $\text{SO}(N, \mathbb{C})$ earlier). Let $(\lambda, \sigma, [\varepsilon]) \in \mathcal{M}_{\text{temp,real}}$. Now if we suppose that λ only has odd parts, then by Theorem 3.2.2, it follows that $E(\lambda^{\min}, [\varepsilon^{\min}])$ satisfies the condition in Proposition 4.1.1. Theorem 3.3.1 furthermore allows us to compute $(\lambda^{\min}, [\varepsilon^{\min}])$. Hence we have

$$\mathbb{IM}(Y(\lambda, \sigma, [\varepsilon])) = \bar{Y}(\lambda^{\min}, \sigma, [\varepsilon^{\min}]).$$

We stress again that for now, we have only shown that this result holds for the case that λ only has odd parts.

4.1.3 Type C: $\mathfrak{sp}(2n, \mathbb{C})$

Suppose $G = \text{Sp}(2n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. We identify $\mathcal{M}_{\text{temp,real}}$ with the set of G-conjugacy classes of triples $(\lambda, \sigma, \varepsilon)$ with $\lambda \in \mathcal{P}^{\text{symp}}(2n)$ and $\varepsilon \in \{\pm 1\}^{\Delta(\lambda)}$ (G acts trivially on λ and ε and by conjugation on σ). We also write standard modules in terms of these triples (i.e. $X(\lambda, \sigma, \varepsilon)$) and generalised Springer representations in terms of the pairs (λ, ε) (i.e. $E(\lambda, \varepsilon)$). Let $(\lambda, \sigma, \varepsilon) \in \mathcal{M}_{\text{temp,real}}$. Now if we suppose that λ only has even parts, then by Theorem 3.2.12, it follows that $E(\lambda^{\min}, \varepsilon^{\min})$ satisfies the condition in Proposition 4.1.1. Theorem 3.4.1 furthermore allows us to compute $(\lambda^{\min}, [\varepsilon^{\min}])$. Hence we have

$$\mathbb{IM}(\lambda, \sigma, \varepsilon)) = \bar{Y}(\lambda^{\min}, \sigma, \varepsilon^{\min}).$$

We stress again that for now, we have only shown that this result holds for the case that λ only has even parts.

For direct products of symplectic groups, we shall use similar notation as above. For instance, for $G = \mathrm{Sp}(2n,\mathbb{C})\times\mathrm{Sp}(2m,\mathbb{C})$, we can identify $\mathcal{M}_{\text{temp,real}}$ with the set of conjugacy classes of tuples $(\lambda, \lambda', \sigma, \varepsilon, \varepsilon')$ in the obvious sense.

4.1.4 Comment on exceptional types

The strategy for a simply connected complex group G of exceptional type is completely the same as for type B, C, and D above. The corresponding analogous maximality and minimality results for any $(C, \mathcal{E}) \in \mathcal{N}_{\mathrm{G}}$ for G can be verified directly using GAP [Mic15] (without any additional constraints on the nilpotent orbit as in the $\mathrm{SO}(N,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$ cases). Using this, one can then again determine the Deligne–Langlands–Lusztig triple corresponding to $\mathbb{IM}(Y(e,\sigma,\psi))$ for any $(e,\sigma,\psi) \in \mathcal{M}_{\text{temp,real}}$ by using Proposition 4.1.1 – we show that the hypothesis in Proposition 4.1.1 is satisfied for the generalised Springer representations of the relevant pseudo-Levi subgroups of $G^\vee$, i.e. the centralisers of compact semisimple elements in $G^\vee$. There are only finitely many such pseudo-Levi subgroups, and so this is a matter of checking the hypothesis in Proposition 4.1.1 for a finite number of cases, which we aim to do with GAP. We intend to include all the tables and a description of the precise steps for the algorithm for $\mathbb{IM}$ in future work.

4.2 Algorithm for AZ

4.2.1 $\mathrm{SO}(2n+1, \mathsf{k})$

Let $\mathbf{G} = \mathrm{SO}(2n+1)$. Then $G^\vee = \mathrm{Sp}(2n,\mathbb{C})$ and $\Phi_{\text{temp}}(G^\vee)$ is in bijection with the set of $G^\vee$-conjugacy classes of triples $(\lambda, s, \varepsilon)$ with $\lambda \in \mathcal{P}^{\text{symp}}(2n)$ and $\varepsilon \in \{\pm 1\}^{\Delta(\lambda)}$. Let $\Phi_{\text{temp,quad}}(G^\vee)$ be the subset of $\Phi_{\text{temp}}(G^\vee)$ consisting of triples (e,s,ψ) such that $s^2 = 1$. In [Wal18a, §1.3] it is shown that $\Phi_{\text{temp,quad}}(G^\vee)$ is in bijection with the set $\boldsymbol{\mathcal{P}}_2^{\mathbf{symp}}(2n)$ of quadruples $(\lambda^+, \varepsilon^+, \lambda^-, \varepsilon^-)$ with $(\lambda^+, \varepsilon^+) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n^+)$ and $(\lambda^-, \varepsilon^-) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n^-)$ for some $n^+, n^- \in \mathbb{Z}_{\geq 0}$ such that $n^+ + n^- = n$. This bijection has the following properties. Suppose $(e,s,\psi) \in \Phi_{\text{temp,quad}}(G^\vee)$ such that (e,ψ) is corresponds to $(\lambda, \varepsilon) \in \boldsymbol{\mathcal{P}}^{\mathbf{symp}}(2n)$ and suppose that (e,s,ψ) correesponds to such a quadruple $(\lambda^+, \varepsilon^+, \lambda^-, \varepsilon^-)$ via the bijection above. Then $\lambda = \lambda^+ \sqcup \lambda^-$ and $\varepsilon^\pm$ is the restriction of ε to $(-1)^{\Delta(\lambda^\pm)}$. In this case we write

$$X(\lambda^+, \varepsilon^+, \lambda^-, \varepsilon^-) := X(\lambda^+ \sqcup \lambda^-, s, \varepsilon), \tag{4.2.1}$$

Let $(\lambda, s, \varepsilon) \in \Phi_{\text{temp}}(G^\vee)$. We intend to use the result from Section 4.1.3, and so we assume λ consists of only even parts, as was required in that section. Suppose $(\lambda, s, \varepsilon)$

corresponds to (e, s, ψ) with $e \in \mathfrak{g}^\vee$ nilpotent and $\psi \in A_{G^\vee}(s, e)$. We briefly explain how one obtains the corresponding quadruple $(\lambda^+, \varepsilon^+, \lambda^-, \varepsilon^-) \in \boldsymbol{\mathcal{P}}_2^{\mathbf{symp}}(2n)$. Fix an $\mathfrak{sl}_2$-triple (e^-, h, e) in $\mathfrak{g}^\vee$ such that $h \in \mathfrak{t}_\mathbb{R}^\vee$ and let $s_c = s_0 = sq^{-h/2}$ as above. Since λ only has even parts and s_c commutes with e, it follows that $Z_{G^\vee}(s_c)$ is of the form

$$Z_{G^\vee}(s_c) = \mathrm{Sp}(2n^+, \mathbb{C}) \times \mathrm{Sp}(2n^-, \mathbb{C}),$$

where $n^+, n^- \in \mathbb{Z}_{\geq 0}$ are the multiplicities of the eigenvalues 1 and -1 of s_c, respectively (note that $n^+ + n^- = n$), as shown on [Wal18a, p.8]. The element e is also an element of $\mathfrak{g}^\vee(s_c) := \mathrm{Lie}(Z_{G^\vee}(s_c)) \cong \mathfrak{sp}(2n^+, \mathbb{C}) \oplus \mathfrak{sp}(2n^-, \mathbb{C})$ and is parametrised by a pair of symplectic partitions $(\lambda^+, \lambda^-) \in \mathcal{P}^{\mathrm{symp}}(2n^+) \times \mathcal{P}^{\mathrm{symp}}(2n^-)$ such that $\lambda = \lambda^+ \sqcup \lambda^-$. By restricting $\varepsilon \colon \Delta(\lambda) \to \{\pm 1\}$, we obtain $\varepsilon^+ \in \{\pm 1\}^{\Delta(\lambda^+)}$ and $\varepsilon^- \in \{\pm 1\}^{\Delta(\lambda^-)}$.

Let $(L, C_L, \mathcal{L}) \in S_{Z_{G^\vee}}(s_c)$ be the cuspidal support of the element of $\mathcal{N}_{Z_{G^\vee}(s_c)}$ corresponding to $(\lambda^+, \varepsilon^+, \lambda^-, \varepsilon^-)$. As in Section 1.4.2, we attach an irreducible representation of the graded Hecke algebra $\mathbb{H}(Z_{G^\vee}(s_c), L, C_L, \mathcal{L})$ corresponding to the data $(\lambda^+, \varepsilon^+, \lambda^-, \varepsilon^-)$. Using results from Section 4.1.3 for the adjoint group $G = Z_{G^\vee(s_c)} = \mathrm{Sp}(2n^+, \mathbb{C}) \times \mathrm{Sp}(2n^-, \mathbb{C})$, we can compute the graded Iwahori-Matsumoto dual of this representations. Using the correspondence between $\mathbb{IM}$ and AZ under all of Lusztig's reductions in the classification of unipotent representations of adjoint groups, we then find that

$$\mathsf{AZ}(X(\lambda, s, \varepsilon)) = \mathsf{AZ}(X(\lambda^+, \varepsilon^+, \lambda^-, \varepsilon^-)) = X(\lambda^{+,\min}, \varepsilon^{+,\min}, \lambda^{-,\min}, \varepsilon^{-,\min}) = X(\underline{\lambda}, s, \underline{\varepsilon}),$$

where $(\underline{\lambda}, s, \underline{\varepsilon}) \in \Phi_{\mathrm{temp,quad}}(G^\vee)$ corresponds to $(\lambda^{+,\min}, \varepsilon^{+,\min}, \lambda^{-,\min}, \varepsilon^{-,\min}) \in \boldsymbol{\mathcal{P}}_2^{\mathbf{symp}}(2n)$. Note that in particular, $\underline{\lambda} = \lambda^{+,\min} \sqcup \lambda^{-,\min}$.

4.2.2 Further study: other types and non-adjoint groups over k

Both $\mathrm{Sp}(2n, \mathsf{k})$ and $\mathrm{SO}(2n, \mathsf{k})$ are non-adjoint groups, $\mathrm{Sp}(2n, \mathsf{k})$ being simply-connected and $\mathrm{SO}(2n, \mathsf{k})$ being neither adjoint nor simply-connected. The key results that we used for obtaining an algorithm for AZ was showing that $\mathbb{IM}$ on the graded geometric Hecke algebras corresponds to AZ through Lusztig's reduction theorems, the matching between the arithmetic affine Hecke algebras and geometric affine Hecke algebras, and finally the equivalence of categories between the representations of the arithmetic affine Hecke algebras and the corresponding Bernstein block. In Lusztig's classification of unipotent representations, $G^\vee$ was assumed to be adjoint. In particular the definitions of the arithmetic and geometric affine and graded Hecke algebras also relied on the fact that $G^\vee$. All of these notions were later generalised and proved for non-adjoint groups in [Sol18], which relies on results from [AMS17], [AMS18b], and [AMS18a]. In a subsequent paper, we intend to use these results to show that AZ for $\mathrm{Sp}(2n, \mathsf{k})$ and $\mathrm{SO}(2n, \mathsf{k})$ indeed corresponds to a version of $\mathbb{IM}$

of the 'twisted graded Hecke algebras'[1] in the non-adjoint case. We then use the result from Section 4.1.2 to obtain an algorithm for AZ in a similar fashion as in this thesis.

If $G^\vee$ is an adjoint exceptional group, all we have to do is identify all its pseudo-Levis (i.e. all the possibilities for $Z_{G^\vee}(s_c)$), which can be done using GAP. For all these pseudo-Levis, we use computations in GAP again to show that the condition in Proposition 4.1.1 is always satisfied for all tempered unipotent representations (the author has verified this for some, but not all exceptional groups yet). Following the same strategy as for the $\mathrm{SO}(2n+1,\mathsf{k})$ case, we find an algorithm for AZ for all tempered unipotent representations of $\mathbf{G}$.

For non-adjoint exceptional groups, we will have to use results from [AMS17], [AMS18b], and [AMS18a] as above, compute $\mathbb{IM}$ for all of graded Hecke algebras associated to the cuspidal pairs of the pseudo-Levis of $G^\vee$.

4.3 IM and AZ of tempered unipotent representations for $\mathrm{SO}(2n+1,\mathsf{k})$

Suppose $\mathbf{G} = \mathrm{SO}(2n+1)$ and $G^\vee = \mathrm{Sp}(2n,\mathbb{C})$. The unique split inner form of $\mathrm{SO}(2n+1,\mathsf{k})$ has two inner twists, which we denote by $G_{+,n}$ and $G_{-,n}$. Let $(\lambda, s, \varepsilon) \in \Phi(\mathrm{Sp}(2n,\mathbb{C}))$ and $\zeta \in \{+,-\}$. Suppose $X(\lambda, s, \varepsilon)$ is a representation of $G_{\zeta,n}$. It is noted in [Wal18b, §3.4], using results from [Wal18a, §1.3], that there exist

1. a parabolic subgroup P of $G_{\zeta,n}$ with Levi factor isomorphic to

$$M := \mathrm{GL}(m_1,\mathsf{k}) \times \cdots \times \mathrm{GL}(m_t,\mathsf{k}) \times G_{\zeta,n_0} \quad (G_{\zeta,n_0} \text{ inner twist of } \mathrm{SO}(2n_0+1,\mathsf{k}))$$

   for some $m_1, \dots, m_t, n_0 \in \mathbb{N}$ such that $m_1 + \dots m_t + n_0 = n$,

2. unramified characters $\chi_1, \dots, \chi_t$ of $\mathsf{k}^\times$,

3. a triple $(\lambda_0, s_0, \varepsilon_0) \in \Phi_{\mathrm{temp,quad}}(\mathrm{Sp}(2n_0,\mathbb{C}))$ such that λ_0 only has even parts such that $\lambda = (m_1,m_1) \sqcup (m_2,m_2) \sqcup \cdots \sqcup (m_t,m_t) \sqcup \lambda_0$ with $m_1,\dots,m_t \in \mathbb{N}$ odd, and

$$X(\lambda, s, \varepsilon) = \mathrm{Ind}_P^{G_{\zeta,n}}(\mathrm{st}_{m_1}(\chi_1 \circ \det) \otimes \cdots \otimes \mathrm{st}_{m_t}(\chi_t \circ \det) \otimes X(\lambda_0, s_0, \varepsilon_0)), \quad (4.3.1)$$

   where for each $m \in \mathbb{N}$, st_m is the Steinberg representation of $\mathrm{GL}(m,\mathsf{k})$.

Note that the complex Langlands dual of M is

$$M^\vee = M := \mathrm{GL}(m_1,\mathbb{C}) \times \cdots \times \mathrm{GL}(m_t,\mathbb{C}) \times \mathrm{Sp}(2n_0,\mathbb{C}).$$

[1]In [AMS18b], a certain twist of the graded Hecke algebra is considered as an analogue to Lusztig's geometric graded Hecke algebras.

4.3.1 Determining $\mathsf{AZ}(X(\lambda, s, \varepsilon))$

We retain the notation above. Our goal is to determine $(\underline{\lambda}, s, \underline{\varepsilon}) \in \Phi(G^\vee)$ such that $X(\underline{\lambda}, s, \underline{\varepsilon}) = \mathsf{AZ}_{G_{\zeta,n}}(X(\lambda, s, \varepsilon))$.

By [Wal18a, Théorème 1.7], we have

$$\mathsf{AZ}_{G_{\zeta,n}} \circ \mathrm{Ind}_P^{G_{\zeta,n}} = \mathrm{Ind}_P^{G_{\zeta,n}} \circ (\mathsf{AZ}_{\mathrm{GL}_{m_1}} \otimes \cdots \otimes \mathsf{AZ}_{\mathrm{GL}_{m_t}} \otimes \mathsf{AZ}_{G_{\zeta,n_0}}).$$

Applying this to (4.3.1) gives

$$X(\underline{\lambda}, s, \underline{\varepsilon}) = \mathrm{Ind}_P^{G_{\zeta,n}}((\chi_1 \circ \det) \otimes \cdots \otimes (\chi_t \circ \det) \otimes \mathsf{AZ}_{G_{\zeta,n_0}}(X(\lambda_0, s_0, \varepsilon_0))).$$

Note that

$$Z_{M^\vee}(s_c) \cong \mathrm{GL}(m_1) \times \cdots \times \mathrm{GL}(m_t) \times \mathrm{Sp}(2n_0^+) \times \mathrm{Sp}(2n_0^-), \tag{4.3.2}$$

and that $Z_{G^\vee}(s_c)$ is of the form

$$Z_{G^\vee}(s_c) \cong \prod_{i \in I} \mathrm{GL}(m_i) \times \mathrm{Sp}(2n^+) \times \mathrm{Sp}(2n^-), \tag{4.3.3}$$

where I is a subset of $\{1, \dots, t\}$, $n^\pm = n_0^\pm + \sum_{i \in J^\pm} m_i$ where $J^\pm \subseteq \{1, \dots, m\}$ such that $J^+ \sqcup J^- \sqcup I = \{1, \dots, m\}$. The isomorphisms (4.3.2) and (4.3.3) are such that for $i \in I$, the factor $\mathrm{GL}(m_i)$ in (4.3.2) are mapped to the $\mathrm{GL}(m_i)$ factor of (4.3.3), for $i \in J^\pm$, the factor $\mathrm{GL}(m_i)$ in (4.3.2) are mapped into the $\mathrm{Sp}(2n^\pm)$ factor of (4.3.3), and the $\mathrm{Sp}(2n_0^\pm)$ factor is also mapped into $\mathrm{Sp}(2n^\pm)$.

The pair (λ, ε) corresponds to an element of $\mathcal{N}_{Z_{G^\vee}(s_c)}$. Consider the graded Hecke algebra $\mathbb{H} := \mathbb{H}(Z_{G^\vee}(s_c), L, C_L, \mathcal{L})$ of the cuspidal support $(L, C_L, \mathcal{L}) \in S_{Z_{G^\vee}(s_c)}$ of this element of $\mathcal{N}_{Z_{G^\vee}(s_c)}$. Similarly, consider the graded Hecke algebra $\mathbb{H}_{M^\vee} := \mathbb{H}(Z_{M^\vee}(s_c), L', C'_L, \mathcal{L}')$ of the cuspidal support $(L', C'_L, \mathcal{L}') \in S_{Z_{M^\vee}(s_c)}$ of the element of $\mathcal{N}_{Z_{M^\vee}(s_c)}$ that corresponds to (λ, ε). The graded Hecke algebra $\mathbb{H}_{M^\vee}$ has Weyl group $W_{\mathcal{L}'} = N_{M^\vee}(L')/L'$ and $\mathbb{H}$ has Weyl group $W_{\mathcal{L}} = N_{G^\vee}(L)/L$. Note that the $\mathrm{GL}(m_i, \mathbb{C})$ have no non-trivial local systems, hence its only relative Weyl group is S_{m_i}. Thus we have

$$W_{\mathcal{L}'} = S_{m_1} \times \cdots \times S_{m_t} \times W(C_{n_0^{+\prime}}) \times W(C_{n_0^{-\prime}}), \tag{4.3.4}$$

for some $n_0^{\pm\prime} \in \mathbb{N}$ (in fact, we have $n_0^{\pm\prime} = n_0^\pm - k^\pm(k^\pm + 1)/2$ for some $k^\pm \in \mathbb{Z}_{\geq 0}$ from the generalised Springer correspondence for $\mathrm{Sp}(2n_0^\pm, \mathbb{C})$). Similarly, $W_{\mathcal{L}}$ is of the form

$$W_{\mathcal{L}} = \prod_{i \in I} S_{m_i} \times W(C_{n^{+\prime}}) \times W(C_{n^{-\prime}}), \tag{4.3.5}$$

where $n^{\pm\prime} = n_0^{\pm\prime} + \sum_{i \in J^\pm} m_i$. Furthermore, we can view $\mathbb{H}_{M^\vee}$ as a subalgebra of $\mathbb{H}$. We identify $W_{\mathcal{L}}$ and $W_{\mathcal{L}'}$ with the subsets $\{t_w \in \mathbb{H} \colon w \in W_{\mathcal{L}}\}$ and $\{t_w \in \mathbb{H}_{M^\vee} \colon w \in W_{\mathcal{L}'}\}$ in the obvious way, respectively, and the inclusion $\mathbb{H}_{M^\vee} \subseteq \mathbb{H}$ mentioned above and the inclusion $L' \subseteq L$ give an inclusion $W_{\mathcal{L}'} \subseteq W_{\mathcal{L}}$ such that for $i \in I$, each factor S_{m_i} of $W_{\mathcal{L}'}$ in (4.3.4) is mapped to the factor S_{m_i} of $W_{\mathcal{L}}$ in (4.3.5), for $i \notin J^\pm$, S_{m_i} is mapped to $W(C_{n^{\pm\prime}})$, and $W(C_{n_0^{\pm\prime}})$ is mapped into $W(C_{n^{\pm\prime}})$.

Let Y be the standard module of $\mathbb{H}$ whose unique irreducible quotient $\bar{Y}$ corresponds to $X(\underline{\lambda}, s, \underline{\varepsilon})$, and note that $(\chi_1 \circ \det) \otimes \cdots \otimes (\chi_t \circ \det) \otimes \mathsf{AZ}(X(\lambda_0, s_0, \varepsilon_0))$ corresponds to the irreducible representation $\mathrm{st}_{m_1} \otimes \cdots \otimes \mathrm{st}_{m_t} \otimes \mathbb{IM}(Y(\lambda_0^+, \lambda_0^-, \log(s_{0,r}), \varepsilon_0^+, \varepsilon_0^-))$ of $\mathbb{H}_{M^\vee}$. By [Sol18, Lemma 3.6] and [BM93, Theorem 6.2][2], $\mathrm{Ind}_P^{G_{\zeta,n}}$ and $\mathrm{Ind}_{\mathbb{H}}^{\mathbb{H}_{M^\vee}}$ correspond to each other when passing from unipotent representations of $G_{\zeta,n}$ to representations of $\mathbb{H}$, so we have

$$\bar{Y} = \mathrm{Ind}_{\mathbb{H}_{M^\vee}}^{\mathbb{H}}(\mathrm{st}'_{m_1} \otimes \cdots \otimes \mathrm{st}'_{m_t} \otimes \mathbb{IM}(Y(\lambda_0^+, \lambda_0^-, \log(s_{0,r}), \varepsilon_0^+, \varepsilon_0^-))), \tag{4.3.6}$$

where the st'_{m_i} are each some twist of the Steinberg representation st_{m_i} of the relevant graded Hecke algebra corresponding to $\mathrm{GL}(m_i, \mathbb{C})$ by an unramified character. Denote the sign and trivial representations of S_{m_i} by sgn_{m_i} and triv_{m_i}, respectively. Restricting $\bar{Y}$ to $W_{\mathcal{L}}$, we find using (4.3.6)

$$\begin{aligned}
&\bar{Y}|_{W_{\mathcal{L}}} \\
&= \mathrm{Ind}_{W_{\mathcal{L}'}}^{W_{\mathcal{L}}}(\mathrm{sgn}_{m_1} \otimes \cdots \otimes \mathrm{sgn}_{m_t} \otimes \\
&\qquad Y(\lambda_0^+, \lambda_0^-, \log(s_{0,r}), \varepsilon_0^+, \varepsilon_0^-)|_{W(C_{n_0^{+\prime}}) \times W(C_{n_0^{-\prime}})} \otimes \mathrm{sgn}_{W(C_{n_0^{+\prime}}) \times W(C_{n_0^{-\prime}})}) \\
&= \mathrm{Ind}_{W_{\mathcal{L}'}}^{W_{\mathcal{L}}}(\mathrm{triv}_{m_1} \otimes \cdots \otimes \mathrm{triv}_{m_t} \otimes Y(\lambda_0^+, \lambda_0^-, \log(s_{0,r}), \varepsilon_0^+, \varepsilon_0^-)|_{W(C_{n_0^{+\prime}}) \times W(C_{n_0^{-\prime}})}) \otimes \mathrm{sgn}_{W_{\mathcal{L}}}.
\end{aligned}$$

Note that $Y(\lambda_0^+, \lambda_0^-, \log(s_{0,r}), \varepsilon_0^+, \varepsilon_0^-)|_{W(C_{n_0^{+\prime}}) \times W(C_{n_0^{-\prime}})}$ is the tensor product of the generalised Springer representations $Y(\lambda_0^+, \log(s_{0,r}^+), \varepsilon_0^+)|_{W(C_{n_0^{+\prime}})}$ and $Y(\lambda_0^-, \log(s_{0,r}^-), \varepsilon_0^-)|_{W(C_{n_0^{-\prime}})}$ of $\mathrm{Sp}(2{n_0^+}', \mathbb{C})$ and $\mathrm{Sp}(2{n_0^-}', \mathbb{C})$, respectively. By Theorem 3.2.12, $Y(\lambda_0^\pm, \log(s_{0,r}^\pm), \varepsilon_0^\pm)|_{W(C_{n_0^{\pm\prime}})}$ has a unique maximal generalised Springer representation $\rho(\lambda^{\pm,\max}, \varepsilon^{\pm,\max})$, and so $Y(\lambda_0^+, \lambda_0^-, \log(s_{0,r}), \varepsilon_0^+, \varepsilon_0^-)|_{W(C_{n_0^{+\prime}}) \times W(C_{n_0^{-\prime}})}$ has a unique maximal generalised Springer representation $\rho(\lambda^{+,\max}, \varepsilon^{+,\max}) \otimes \rho(\lambda^{-,\max}, \varepsilon^{-,\max})$. Since $W_{\mathcal{L}'}$ is a subgroup of $W_{\mathcal{L}}$ and both are products of type A and B/C Weyl groups, we can show by similar arguments as in Section 3.2.4 (applied to $\mathrm{Sp}(2n, \mathbb{C})$ instead of $\mathrm{SO}(N, \mathbb{C})$), particularly the usage of the

[2] The way we use [BM93, Theorem 6.2] is briefly as follows. The representations $X(\lambda, s, \varepsilon)$ and $X(\lambda_0, s_0, \varepsilon_0)$ both correspond to some irreducible representation of an affine Hecke algebra $\mathcal{H}$ and $\mathcal{H}_{M^\vee}$ respectively, and it is well-known that $\mathcal{H}_{M^\vee}$ is a parabolic subgroup of $\mathcal{H}$ and that $\mathrm{Ind}_P^{G_{\zeta,n}}$ corresponds to $\mathrm{Ind}_{\mathcal{H}_{M^\vee}}^{\mathcal{H}}$ by for instance [Sol18, Lemma 3.6]. Let $\Sigma = W \cdot s_c$ and $\Sigma_J = W_J \cdot s_{0,c}$. Then $\mathbb{H}_{J,\Sigma_J}$ is a parabolic subgroup of $\mathbb{H}_\Sigma$, and [BM93, Theorem 6.2] states that $\mathrm{Ind}_{\mathbb{H}_{J,\Sigma_J}}^{\mathbb{H}_\Sigma}$ corresponds to $\mathrm{Ind}_{\mathcal{H}_J}^{\mathcal{H}}$ via the reduction theorems. Now $\mathbb{H}$ and $\mathbb{H}_{M^\vee}$ are Morita equivalent to $\mathbb{H}_\Sigma$ and $\mathbb{H}_{J,\Sigma_J}$ respectively, and via this Morita equivalence, we have that $\mathrm{Ind}_{\mathbb{H}_{M^\vee}}^{\mathbb{H}}$ corresponds to $\mathrm{Ind}_{\mathbb{H}_{J,\Sigma_J}}^{\mathbb{H}_\Sigma}$.

Littlewood-Richardson rule for type B/C, to compute $\bar{\lambda}$ as follows. First note that we can further write

$$\begin{aligned}
\bar{Y}|_{W_{\mathcal{L}}} &\cong \Big(\bigotimes_{i\in I} \mathrm{triv}_{m_i} \otimes \mathrm{Ind}_{\prod_{j\notin I} S_{m_j}\times W(C_{n_0^+{}'})\times W(C_{n_0^-{}'})}^{W(C_{n^+{}'})\times W(C_{n^-{}'})}\Big(\\
&\qquad \prod_{j\notin I} \mathrm{triv}_{m_j} \otimes Y(\lambda_0^+,\lambda_0^-,\log(s_{0,r}),\varepsilon_0^+,\varepsilon_0^-)|_{W(C_{n_0^+{}'})\times W(C_{n_0^-{}'})}\Big)\Big) \otimes \mathrm{sgn}_{W_{\mathcal{L}}} \\
&\cong \Big(\bigotimes_{i\in I} \mathrm{triv}_{m_i} \otimes \mathrm{Ind}_{\prod_{j\in J^+} S_{m_j}\times W(C_{n_0^+{}'})}^{W(C_{n^+{}'})}\Big(\prod_{j\in J^+} \mathrm{triv}_{m_j} \otimes Y(\lambda_0^+,\log(s_{0,r}^+),\varepsilon_0^+)|_{W(C_{n_0^+{}'})}\Big) \\
&\qquad \otimes \mathrm{Ind}_{\prod_{j\in J^-} S_{m_j}\times W(C_{n_0^-{}'})}^{W(C_{n^-{}'})}\Big(\prod_{j\in J^-} \mathrm{triv}_{m_j} \otimes Y(\lambda_0^-,\log(s_{0,r}^-),\varepsilon_0^-)|_{W(C_{n_0^-{}'})}\Big)\Big) \otimes \mathrm{sgn}_{W_{\mathcal{L}}}.
\end{aligned}$$

Compare

$$\mathrm{Ind}_{\prod_{j\in J^\pm} S_{m_j}\times W(C_{n_0^\pm{}'})}^{W(C_{n^\pm{}'})}\Big(\prod_{j\in J^\pm} \mathrm{triv}_{m_j} \otimes Y(\lambda_0^+,\log(s_{0,r}^\pm),\varepsilon_0^\pm)|_{W(C_{n_0^\pm{}'})}\Big) \tag{4.3.7}$$

with (3.2.13) – both are representations of a type B/C Weyl group induced from a product of type A and type B/C Weyl groups, and so by the exact same arguments as in the discussion above Theorem 3.2.9 and the proof of Theorem 3.2.9, we see that (4.3.7) has maximal generalised Springer representation $\rho(\bar{\lambda}^\pm,\varepsilon^{\pm,\max})$ where $\bar{\lambda}^\pm \in \mathcal{P}^{\mathrm{symp}}(2n^\pm)$ is defined by

$$\bar{\lambda}_1^\pm = \bar{\lambda}_1^{\pm,\max} + 2(n^{\pm\prime} - n_0^{\pm\prime}) = 2\sum_{i\in J^\pm} m_i = 2(n^\pm - n_0^\pm), \quad \bar{\lambda}_i^\pm = \bar{\lambda}_i^{\pm,\max} \text{ (for } i\geq 2).$$

Thus

$$\mathrm{Ind}_{\prod_{j\notin I} S_{m_j}\times W(C_{n_0^+{}'})\times W(C_{n_0^-{}'})}^{W(C_{n^+{}'})\times W(C_{n^-{}'})}\Big(\prod_{j\notin I} \mathrm{triv}_{m_j} \otimes Y(\lambda_0^+,\lambda_0^-,\log(s_{0,r}),\varepsilon_0^+,\varepsilon_0^-)|_{W(C_{n_0^+{}'})\times W(C_{n_0^-{}'})}\Big)$$

has maximal representation $\rho(\bar{\lambda}^+,\varepsilon^{+,\max})\otimes\rho(\bar{\lambda}^-,\varepsilon^{-,\max})$. Note that the nilpotent elements of $\mathrm{Lie}(Z_{G^\vee}(s_c)) \cong \prod_{i\in I}\mathfrak{gl}(m_i)\times\mathfrak{sp}(2n^+)\times\mathfrak{sp}(2n^-)$ are parametrised by a $(|I|+2)$-tuple of partitions of m_i for all $i\in I$ and symplectic partitions of $2n^+$ and $2n^-$, and so by the above, $\bar{Y}|_{W_{\mathcal{L}}}\otimes\mathrm{sgn}_{W_{\mathcal{L}}}$ has maximal representation

$$\rho((\prod_{i\in I}(m_i),\bar{\lambda}^+,\bar{\lambda}^-),(\prod_{i\in I}\mathrm{triv},\varepsilon^{+,\max},\varepsilon^{-,\max}))\in W_{\mathcal{L}}^\wedge.$$

Recall the notation using 'left superscript s' from Section 3.2.3. Tensoring $\bar{Y}|_{W_{\mathcal{L}}}\otimes\mathrm{sgn}_{W_{\mathcal{L}}}$ with $\mathrm{sgn}_{W_{\mathcal{L}}}$ again to obtain $\bar{Y}|_{W_{\mathcal{L}}}$, we find using the same arguments as in the proof of Theorem 3.2.10 that $\bar{Y}|_{W_{\mathcal{L}}}$ has minimal representation

$$\rho((\prod_{i\in I}(1^{m_i}),{}^s\bar{\lambda}^+,{}^s\bar{\lambda}^-),(\prod_{i\in I}\mathrm{triv},{}^s\varepsilon^{+,\max},{}^s\varepsilon^{-,\max}))\in W_{\mathcal{L}}^\wedge,$$

hence we can use Proposition 4.1.1 to find that

$$\bar{Y} = \bar{Y}((\prod_{i\in I}(1^{m_i}), {}^s\bar{\lambda}^+, {}^s\bar{\lambda}^-), \log(s_{0,r}), (\prod_{i\in I} \mathrm{triv}, {}^s\varepsilon^{+,\max}, {}^s\varepsilon^{-,\max})) \in \mathbb{H}\,.$$

Next, $(\prod_{i\in I}(1^{m_i}), {}^s\bar{\lambda}^+, {}^s\bar{\lambda}^-)$ corresponds to the nilpotent orbit of $\mathfrak{sp}(2n,\mathbb{C})$ parametrised by

$$\underline{\lambda} = {}^s\bar{\lambda}^+ \sqcup {}^s\bar{\lambda}^- \sqcup \bigsqcup_{i\in I}(1^{2m_i}),$$

and $(\prod_{i\in I} \mathrm{triv}, {}^s\varepsilon^{+,\max}, {}^s\varepsilon^{-,\max})$ corresponds to

$$\underline{\varepsilon} \in (-1)^{\Delta(\underline{\lambda})},$$

which is obtained from $({}^s\varepsilon^{+,\max}, {}^s\bar{\varepsilon}^{-,\max})$ in the same way as how ε is obtained from $(\varepsilon^+, \varepsilon^-)$ in (4.2.1).

Thus we have proved the following.

Theorem 4.3.1. *Suppose* $(\lambda, s, \varepsilon) \in \Phi(\mathrm{Sp}(2n,\mathbb{C}))$. *Let* $\underline{\lambda} = (\lambda_0)^{+,\min} \sqcup (\lambda_0)^{-,\min}$ *and* $\underline{\varepsilon}$ *be as above. It holds that*

$$\mathsf{AZ}_{G^\vee_{\varsigma,n}}(X(\lambda, s, \varepsilon)) = X(\underline{\lambda}, s, \underline{\varepsilon}).$$

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