Sample paths of some Gaussian processes via Malliavin calculus

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This thesis is dedicated to

my beloved parents.

Thank you for always being there for me.

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Abstract

In this thesis, we study the sample paths of some Gaussian processes using the methods from Malliavin calculus. To be more specific, we consider several interesting properties of fractional Brownian motion sample paths in the context of both probability measures and capacities. We are in particular interested in the non-differentiability, the modulus of continuity, the law of the iterated logarithm and self-avoiding properties. The capacities we use here are those induced by Brownian motions on the classical Wiener space, that is, we regard fractional Brownian motions with distinct Hurst parameters as a collection of Wiener functionals on the classical Wiener space and use the classical Wiener capacities as uniform measurements. We also formulate a capacity version of the large deviation principles for these functionals and determine the corresponding rate functions.

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Chapter 1

Introduction

1.1 General background

In past decades, Gaussian analysis has been one of the prevalent topics in stochastic analysis, probability theory as well as functional analysis due to its wide range of applications in various fields, including finance, statistical physics, quantum physics. Typical examples of Gaussian processes are Brownian motions, Ornstein-Uhlenbeck processes, and fractional Brownian motions.

Among all these Gaussian processes, fractional Brownian motions are in particular of much interest as they have been extensively used in the modelling of stock prices in financial mathematics, and have also found applications in other areas such as hydrodynamics and communication networks etc., see e.g. [3], [56]. Fractional Brownian motions are centred Gaussian processes with covariance function given by

$$R(s,t) = \mathbb{E}[B_s B_t] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad \forall s,t \geq 0,$$

where the constant H, taking values in $(0,1)$, is called the Hurst parameter corresponding to this process. The properties of fractional Brownian motion heavily depend on the value of H. In particular, it is worth noting that when this Hurst parameter H is equal to $\frac{1}{2}$, this process becomes a standard Brownian motion.

There have been a lot of studies concerning fractional Brownian motions. It was Kolmogorov who first considered this type of processes in [37] in early 1940s. The name "fractional Brownian motion" was introduced by Mandelbrot and Van Ness [50] in 1968. In the same paper, a stochastic integral representation of fractional Brownian motions was provided, which is given by

$$B_t = \frac{1}{C(H)^{1/2}} \left(\int_{-\infty}^{0} \left((t-s)^{H-1/2} - (-s)^{H-1/2}\right) \mathrm{d}W_s + \int_0^t (t-s)^{H-1/2} \mathrm{d}W_s \right),$$

where $(W_t)_{t\geq 0}$ is a two-sided Brownian motion, and $C(H)$ is some constant depending only on the parameter H. More recently, another integral representation of fractional Brownian motions was found by Decreusefond and Üstünel in [10], which characterises fractional Brownian motions over finite time intervals by a stochastic integral of a singular kernel against a standard Brownian motion:

$$B_t = \int_0^t K(t,s)\mathrm{d}W_s, \quad \forall t \geq 0, \tag{1.1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion, and $K(t,s)$ is the singular kernel (we will given the explicit definition of K in next chapter).

Though fractional Brownian motions may look similar to Brownian motion in many ways at the first glance, they are neither semi-martingales, nor Markov processes unless the Hurst parameter $H = \frac{1}{2}$. Therefore, when studying problems concerning fractional Brownian motions, many powerful tools that are effective when dealing with Brownian motions, such as potential theory, are no longer applicable.

Nonetheless, fractional Brownian motions are nice Gaussian processes. One robust theory for Gaussian measures and processes is the theory of Malliavin calculus, also known as the stochastic calculus of variations, which was initiated by Malliavin in late 1970s. In [46] and [47], Malliavin established a differential structure on infinite-dimensional spaces, which allows us to study Gaussian processes on their paths spaces – abstract Wiener spaces. The concept of abstract Wiener spaces was introduced by Gross in [26]. Gaussian measures on separable Banach spaces are defined via these abstract spaces, which are generalisations of the classical Wiener space. Apart from Gaussian measures, another type of outer measures can be constructed on abstract Wiener spaces using Malliavin calculus, which appear useful when exploring the properties of Gaussian processes. In [48] (see also [49]), (p, r)-capacities were introduced as outer measures on abstract Wiener spaces by Malliavin, where $p \in (1, \infty)$ and r is a positive integer. It turns out that on the same abstract Wiener space, capacities are indeed finer than the corresponding Gaussian probability measure, that is, a null set may have strictly positive capacity. Therefore, by replacing Gaussian measures with capacities, we are able to take a closer look and describe the properties of fractional Brownian motion more precisely.

Among all properties of Gaussian processes, sample path properties of these processes are of much interest, and many questions concerning these properties arose from statistical physics while studying interacting particle systems. In fact the behaviour of these paths reflects their Gaussian feature. For instance, a standard Brownian motion is a centred Gaussian process with stationary and independent increments, and almost all of its sample paths are non-differentiable. These results were proved by Paley, Wiener and Zygmund [60] in 1933. Another result related to the regularity of Brownian motion paths is the modulus of continuity, proved by Lévy in [40], which states that almost all sample paths of Brownian motions are α-Hölder continuous with the component $\alpha < \frac{1}{2}$. In 1933, Khintchine [35] described the asymptotic behaviour of Brownian motion paths near time zero and as time goes to infinity via the law of the iterated logarithm. Another rather interesting question that comes from statistical field theory is whether a Brownian motion path intersects with itself. Intuitively, one would expect that it becomes more likely for a path to be self-avoiding as the dimension of the Brownian motion increases. In 1944, Kakutani proved that almost all sample paths are self-avoiding when the dimension of the Brownian motion is greater than or equal to 5 in his work [32], and later in [16], Dvoretzky, Erdős and Kakutani showed that $d = 4$ is the critical dimension of this phenomenon.

In addition, these results can be formulated in the framework of capacities as well. Fukushima [21] studied sample path properties of Brownian motions with respect to the capacity defined using Dirichlet form, which is indeed equivalent to the one constructed by Malliavin when $p = 2$ and $r = 1$. He established the results on non-differentiability, modulus of continuity, the law of the iterated logarithm, and self-avoiding properties for Brownian motion paths under the capacity he defined. It is worth mentioning that Fukushima proved that when the dimension of a Brownian motion is greater than or equal to 7, then apart from on a zero capacity set, Brownian motion paths are self-avoiding, and later Lyons [43] identified that the optimal dimension of self-avoiding property under this capacity is 6. We would like to note here that the increase of the optimal dimension for self-avoiding property reflects the fact that capacities are finer than probability measures. After Malliavin introduced the (p, r)-capacity, Takeda [67] generalised Fukushima's results and proved the above properties for Brownian motion under the (p, r)-capacity.

These properties of fractional Brownian motion sample paths have also been investigated in the setting of probability. Almost everywhere non-differentiability of fractional Brownian motion sample paths was proved in [50] by Mandelbrot and Van Ness. The modulus of continuity result was established by Decreusefond and Üstünel in [10] and they proved that the sample paths of fractional Brownian motions are α-Hölder continuous for all $\alpha < H$ almost surely. As for the law of the iterated logarithm, Coutin [9] mentioned the following result

$$\limsup_{\varepsilon \to 0^+} \frac{B_{t+\varepsilon} - B_t}{\sqrt{2\varepsilon^{2H} \log\log(1/\varepsilon)}} = 1, \quad \text{a.s.}$$

While a written proof does not exist for the case that $H < \frac{1}{2}$, the functional version of the law of the iterated logarithm for Gaussian processes implies the above result for fractional Brownian motion (see for example [1] for a proof). For the case that $H \in (0, \frac{1}{2}]$, the law of the iterated logarithm was established by Cohen and Istas using Slepian's lemma in [8].

Another interesting topic related to Gaussian measures and Gaussian processes is the theory of large deviations, which has plenty of applications in statistics as well as statistical mechanics. The theory of large deviations completes the central limit theorem by telling us that the convergence of tail distributions is exponentially fast. The most essential result in this theory is the celebrated Cramér's theorem, in which the rate of decay of the tail probabilities are provided explicitly. The theory of large deviation principles experienced a rapid development in 1970s due to the work done by Donsker and Varadhan, see [13]. The theory can be established for probability measures on both finite-dimensional and infinite-dimensional spaces. Schilder's theorem describes the exponential decay of large perturbations of a Brownian motion from its mean trajectory, and Freidlin–Wentzell theorem extends the result to case of Itô diffusions, see e.g. [11] and [12] for proofs. One may refer to [12], [13] and [70] for a comprehensive introduction.

Large deviation principles may be established for not only probability measures, but also for capacities. In [72], a version of large deviation principles for (p, r)-capacity was formulated by Yoshida on abstract Wiener spaces. Gao and Ren considered the capacity version of Freidlin-Wentzell's theory (see [23, 24]). After Lyons introduced the analysis of rough paths (see e.g. [18, 20, 44, 45] for details), similar large deviation principles results were established via rough paths theory by Ledoux, Qian and Zhang in [38] (see also [19, 55]). In the case of Gaussian rough paths, large deviation principles with respect to capacities were established in [5] and [30].

Our goal is to study fractional Brownian motion in view of these topics, and explore sample paths properties with respect to both probability measures and capacities. Though capacities are finer measurements compared to probabilities, its disadvantage is obvious, that is when using capacities, fractional Brownian motions with different Hurst parameters live on distinct abstract Wiener spaces, and their properties will be studied with different capacities. Thus, instead of using distinct capacities, we use one uniform measurement – the (p,r)-capacity induced by Brownian motions. We regard a family of fractional Brownian motions with distinct Hurst parameters as Wiener functionals on the classical Wiener space, and explore the properties shared by this family of processes, therefore fractional Brownian motions with different Hurst parameters are compared.

1.2 Main results

The necessary background knowledge for the topics we study, including abstract Wiener spaces, Malliavin calculus, Wiener chaos, fractional Brownian motions, as well as large deviation principles will be introduced in Chapter 2. In Chapter 3, we will introduce two sample path properties of fractional Brownian motions under probability measures. We revisit one standard result on the law of the iterated logarithm for fractional Brownian motions, following the argument by Cohen and Istas in [8]. The reason why we include this result here is to show how to use Slepian's lemma to tackle the negative correlation between the increments over different time intervals when the Hurst parameter is less than $\frac{1}{2}$. The result is stated in two propositions as follows.

Proposition 1.2.1 (Theorem 3.2.4, [8]). *Let $(B_t)_{t\geq 0}$ be a fractional Brownian motion of dimension one on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$ with Hurst parameter $H \in (0,1)$. Then*

$$\mathbb{P}\left(\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t^{2H}\log\log(1/t)}} \leq 1\right) = 1.$$

Proposition 1.2.2 (Theorem 3.2.4, [8])**.** *Let $B = (B_t)_{t\geq 0}$ be a fractional Brownian motion of dimension one on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$ with Hurst parameter $H \in \left(0, \frac{1}{2}\right]$, then*

$$\mathbb{P}\left(\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t^{2H}\log\log(1/t)}} \geq 1\right) = 1.$$

Combining these two results, the law of the iterated logarithm when $H \leq \frac{1}{2}$ follows.

The second property we consider is the self-avoiding property of fractional Brownian motion paths. We adopt the classical method due to Kakutani [32] and prove the following result:

Proposition 1.2.3. *Let $(B_t)_{t\geq 0}$ be an n-dimensional fractional Brownian motion with Hurst parameter H. Then B has no double point almost surely if $\frac{2}{n} < H$.*

Nevertheless, as potential theory no longer applies to the case of fractional Brownian motions, the critical dimension of self-intersection for fBMs still remains an open question.

The main contribution and new results of this thesis are presented in Chapter 4 and 5.

From Chapter 4 and on, we use capacities instead of probability measures, and the integral representation (1.1) allows us to view fractional Brownian motions as a family of measurable functionals on the classical Wiener space. We first consider the regularity of fractional Brownian motion paths, and establish the modulus of continuity for these paths under capacities:

Theorem 1.2.1. *Let $(B_t)_{t\geq 0}$ be a fractional Brownian motion with Hurst parameter H. Then it holds that*

$$\limsup_{\delta\downarrow 0} \frac{1}{\sqrt{2\delta^{2H}\log(1/\delta)}} \max_{\substack{0\leq s<t\leq 1 \\ t-s\leq\delta}} |B_t - B_s| \leq 1, \quad q.s.$$

when $H \in (0,1)$ *and*

$$\limsup_{\delta \downarrow 0} \frac{1}{\sqrt{2\delta^{2H}\log(1/\delta)}} \max_{\substack{0 \le s < t \le 1 \\ t-s \le \delta}} |B_t - B_s| \ge 1, \quad q.s.$$

when $H \in \left(0, \frac{1}{2}\right]$.

Accordingly, we deduce the following corollary:

Corollary 1.2.1. $(B_t)_{t\ge 0}$ *is* α*-Hölder-continuous for* $\alpha < H$ *quasi-surely with respect to the classical Wiener capacity.*

Next, we prove that apart from a capacity zero set, all fractional Brownian motion paths are nowhere differentiable.

Theorem 1.2.2. *Let* $H \in (0,1)$*. Then for a fractional Brownian motion* $(B_t)_{t\ge 0}$ *with Hurst parameter* H*,*

$$\limsup_{h \downarrow 0} \frac{|B_{t+h} - B_t|}{h} = \infty \quad \text{for all } t \in [0,1] \quad q.s.$$

Then we establish the law of the iterated logarithm under capacities, but only for the case when $p = 2$ and $r = 1$ due to technical reasons.

Theorem 1.2.3. *Let* $(B_t)_{t\ge 0}$ *be a one-dimensional fractional Brownian motion on the classical Wiener space* $(\boldsymbol{W}, \mathcal{H}, P)$ *with Hurst parameter* $H \in \left(0, \frac{1}{2}\right]$*. Then it holds that*

$$c_{2,1}\left(\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t^{2H}\log\log(1/t)}} \neq 1\right) = 0.$$

At the end of Chapter 4, we study the self-intersection problem of fractional Brownian motions with respect to the classical Wiener capacity when $p = 2$ and $r = 1$. What we obtain is the following:

Theorem 1.2.4. *Let* $B = (B_t)_{t\ge 0}$ *be a* d*-dimensional fractional Brownian motion with Hurst parameter* H*. When* $H \le \frac{1}{2}$ *and* $d > \frac{2}{H} + 2$*,* B *has no double point under*

$(2,1)$-*capacity on the classical Wiener space; when* $H \geq \frac{1}{2}$ *and* $d > 6$, B *has no double point under the* $(2,1)$-*capacity.*

In Chapter 5, we consider sample paths over time interval $[0,1]$. We first prove that fractional Brownian motions are Wiener functionals that are quasi-surely defined, that is, they are defined apart from on a set of zero capacity, with the restriction that the Hurst parameter H is greater than or equal to $\frac{1}{2}$. Then we establish the large deviation principles for such functionals and identify the corresponding rate function.

Theorem 1.2.5. *Let* $r \in \mathbb{N}$, $1 < p < \infty$, *and* $H \in \left[\frac{1}{2}, 1\right)$. *Then for a fractional Brownian motion* $(B_t)_{t\in[0,1]}$ *with Hurst parameter* H *defined on the classical Wiener space via the integral representation (1.1), there is a modification of* B *that is defined* (p,r)-*quasi-surely.*

Let us abuse our notations, and still denote the quasi-sure modification given in the above theorem by B, then we have the following result on large deviations of such functionals:

Theorem 1.2.6. *Let* $r \in \mathbb{N}$, $1 < p < \infty$ *and* $\frac{1}{2} \leq H < 1$. *Let* $X_t^{\varepsilon}(\omega) = B_t(\varepsilon\omega)$ *for all* ω *expect for a* (p,r)-*capacity zero subset (for all* $t \in [0,1]$, $\varepsilon > 0$*). Then* $\{X^{\varepsilon} : \varepsilon > 0\}$ *(which are scaled fractional Brownian motions with Hurst parameter* $H \in \left[\frac{1}{2}, 1\right)$*) satisfies the large deviation principle with respect to* (p,r)-*capacity, with the good rate function*

$$I(\omega) = \begin{cases} \frac{\|\omega\|_{\hat{\mathcal{H}}}^2}{2}, & \omega \in \hat{\mathcal{H}}, \\ \infty, & \text{otherwise}, \end{cases}$$

where $\hat{\mathcal{H}}$ *denotes the Cameron-Martin space corresponding to the law of* $(B_t)_{t\in[0,1]}$, *and* $\|\cdot\|_{\hat{\mathcal{H}}}$ *denotes the norm induced by the inner product on this Hilbert space.*

The content of Chapter 4 is published in [42], and Chapter 5 is a submitted paper, see [41].

Chapter 2

Preliminaries

This chapter is devoted to the introduction to all definitions and notions that will be used throughout this thesis.

Firstly, we give a brief introduction to the theory of Gaussian measures and abstract Wiener spaces. Then we introduce several basic notions in Malliavin calculus as well as the definition of capacities in the sense of Malliavin. In addition, we will review several important results related to the fine properties of fractional Brownian motions and end this chapter with a guide to the theory of large deviation principles. For convenience, we shall omit the Hurst parameter H in our notation and denote fractional Brownian motions by $B = (B_t)_{t\geq 0}$ if no confusion arises. A standard Brownian motion will be denoted by $W = (W_t)_{t\geq 0}$, or simply just $\omega = \omega(t)$ as we will mainly work with the classical Wiener space.

2.1 Gaussian measures and abstract Wiener spaces

Let us first consider the finite-dimensional case. Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a real-valued random variable ξ on this space is a Gaussian random variable or has normal distribution $N(\mu, \sigma^2)$ if its law is given by

$$\mathbb{P}[\xi \in A] = \int_A \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{d}x,$$

for all $A \in \mathscr{B}(\mathbb{R})$, the Borel σ-algebra on $\mathbb{R}$. Here, μ is the mean value of ξ, and σ^2 is its variance. The characteristic function of this random variable ξ is given by

$$\mathbb{E}\left[e^{it\xi}\right] = e^{-\frac{1}{2}t^2\sigma^2 + it\mu}, \quad \forall t \in \mathbb{R}.$$

An n-dimensional vector-valued random variable $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$ is called Gaussian if for all $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, $\sum_{i=1}^n \lambda_i \xi_i$ is a real Gaussian random variable. It is characterised by its mean value and covariance matrix. Its mean value is an n-dimensional vector $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n)$, where $\mu_i = \mathbb{E}[\xi_i]$, $1 \le i \le n$, and its covariance matrix $\Sigma = (\sigma_{ij})_{1 \le i,j \le n}$ is a positive semi-definite matrix, whose components are given by

$$\sigma_{ij} = \mathbb{E}[(\xi_i - \mu_i)(\xi_j - \mu_j)], \quad \forall 1 \le i, j \le n.$$

An n-dimensional Gaussian random variable induces a Borel probability measure ν on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ in that

$$\nu(A) = \mathbb{P}[\boldsymbol{\xi} \in A], \quad \forall A \in \mathscr{B}(\mathbb{R}^n),$$

which is said to be a Gaussian measure.

A family of random variables $\{\xi_\alpha\}_{\alpha \in \Lambda}$ with index set Λ (which can be uncountable) is said to be Gaussian if every finite linear combination $\sum_{i \in I} \lambda_i \xi_i$ is a real Gaussian random variable whose mean value is $\sum_{i \in I} \lambda_i \mathbb{E}[\xi_i]$, and variance is

$$\sum_{i,j} \lambda_i \lambda_j \mathbb{E}\left[(\xi_i - \mathbb{E}[\xi_i])(\xi_j - \mathbb{E}[\xi_j])\right],$$

where I a finite index set, and $\lambda_i \in \mathbb{R}$. A good reference for further details on Gaussian random variables and Gaussian measures is [6].

Now we would like to generalise Gaussian measures to the infinite-dimensional setting. According to Gross [26], we may define Gaussian measures on a real separable Banach space by the construction of abstract Wiener spaces. Indeed, following the same idea as in the finite-dimensional case, Gaussian measures may be characterised via Gaussian random variables: let $(X, ||\cdot||)$ be a real separable Banach space, and $\mathscr{B}(X)$ be its Borel σ-algebra. Let X^* denote its topological dual space. Then a probability measure γ on $(X, \mathscr{B}(X))$ is Gaussian if for all continuous linear functional $l \in X^*$, $l(\omega)$ with $\omega \in X$ is a real Gaussian random variable under this measure, i.e. $\gamma \circ l^{-1}$ is a Gaussian measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ for all $l \in X^*$.

Let $(X, ||\cdot||)$ be a real separable Banach space, and $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a real separable Hilbert space which may be embedded into X via a continuous and dense embedding J. Let $J^* : X^* \hookrightarrow \mathcal{H}^*$ be the dual map of J, where X^* and $\mathcal{H}^*$ denote the topological dual of X and $\mathcal{H}$ respectively, so that we have the following embeddings:

$$X^* \overset{J^*}{\hookrightarrow} \mathcal{H}^* \cong \mathcal{H} \overset{J}{\hookrightarrow} X.$$

A Gaussian measure γ on $(X, \mathscr{B}(X))$ is determined by the corresponding characteristic functional, namely

$$\int_X \exp\left(il(x)\right) \gamma(\mathrm{d}x) = \exp\left(-\frac{1}{2}||J^* l||_{\mathcal{H}}\right), \quad \forall l \in X^*, \tag{2.1}$$

then the triple $(X, \mathcal{H}, \gamma)$ is called an abstract Wiener space. For all $l \in X^*$, set

$$W_l(x) = l(x), \quad \forall x \in X,$$

and using (2.1), we deduce that $\{W_l\}_{l \in X^*}$ is a family of centred Gaussian random variables on $(X, \mathscr{B}(X), \gamma)$, satisfying

$$\mathbb{E}[W_x W_z] = \langle J^* x, J^* z\rangle_{\mathcal{H}}, \quad \forall x, z \in X^*.$$

As $J^*(X^*)$ is dense in $\mathcal{H}^* \cong \mathcal{H}$, one may extend W from X^* to $\mathcal{H}$ such that for each element $h \in \mathcal{H}$, W_h is defined to be a random variable on $(X, \mathscr{B}(X), \gamma)$. Let $\mathscr{F}$ denote the γ-completion of $\mathscr{B}(X)$, then the extended process $W = \{W_h\}_{h\in\mathcal{H}}$ on $(X, \mathscr{B}(X), \gamma)$ is called an isonormal Gaussian process, see [58] for further details.

One concrete example of abstract Wiener spaces is the classical Wiener space. We shall follow the notation in the book by Ikeda and Watanabe (see Section 8, Chapter V, [29]). Let $\boldsymbol{W}_0^d$ denote the space of all continuous paths valued in the Euclidean space $\mathbb{R}^d$ that start from the origin, that is, $\omega(0) = 0$ for all $\omega \in \boldsymbol{W}_0^d$. The space $\boldsymbol{W}_0^d$ is equipped with the norm $\|\cdot\|$, defined by

$$\|\omega\| = \sum_{n=1}^{\infty} 2^{-n} \max_{0\le t\le n} \left(|\omega(t)| \wedge 1\right),$$

which induces the topology of uniform convergence over every compact subset of $[0, \infty)$, and therefore $(\boldsymbol{W}_0^d, \|\cdot\|)$ is a real separable Banach space. The Borel σ-algebra on $\boldsymbol{W}_0^d$ is denoted by $\mathscr{B}(\boldsymbol{W}_0^d)$ or by $\mathscr{B}$ if no confusion may arise.

We will use ω to denote a general element of $\boldsymbol{W}_0^d$, so that for any fixed $t \geq 0$, $\omega(t)$ denotes the value of a path ω at time t. The same notation $\omega(t)$ denotes also the coordinate mapping $\omega \mapsto \omega(t)$, and the parametrised family $\{\omega(t) : t \geq 0\}$ is the coordinate mapping process on $\boldsymbol{W}_0^d$ (see e.g. Section 2.2, Chapter 2, [33]). The coordinate mapping $\omega(t)$ may be denoted by ω_t (for $t \geq 0$) as well. Then the Borel σ-algebra, denoted by $\mathscr{B}(\boldsymbol{W}_0^d)$, is the smallest σ-algebra on $\boldsymbol{W}_0^d$ with which all coordinate mappings $\omega(t)$ (for $t \geq 0$) are measurable (for a proof, see e.g. Stroock and Varadhan [66]).

Let P^W be the Wiener measure on $(\boldsymbol{W}_0^d, \mathscr{B}(\boldsymbol{W}_0^d))$. The Wiener measure is the unique probability measure on $(\boldsymbol{W}_0^d, \mathscr{B}(\boldsymbol{W}_0^d))$ such that the coordinate mapping process $(\omega(t))_{t\geq 0}$ is a d-dimensional standard Brownian motion. For simplicity, $\boldsymbol{W}_0^d$ and P^W will be abbreviated as $\boldsymbol{W}$ and P respectively, if no ambiguity may occur.

To complete the construction of classical Wiener space, one should identify the Hilbert space $\mathcal{H}$, which is the Cameron-Martin space associated with the Wiener measure P. Let $\mathcal{H}$ be the space of all $h \in \boldsymbol{W}$ such that $t \to h(t)$ is absolutely continuous, and its generalised derivative $\dot{h}$ is square-integrable on $[0, \infty)$. Then given the norm

$$\|h\|_{\mathcal{H}} = \sqrt{\int_0^\infty |\dot{h}(t)|^2 \mathrm{d}t},$$

$\mathcal{H}$ is a Hilbert space, and the dual space $\boldsymbol{W}^*$, consisting of all continuous linear functionals on $\boldsymbol{W}$, can be identified as a subset of $\mathcal{H}$, so that we have the continuous dense embeddings $\boldsymbol{W}^* \hookrightarrow \mathcal{H} \hookrightarrow \boldsymbol{W}$ with respect to their corresponding norms respectively.

Furthermore, P is the unique measure on $(\boldsymbol{W}, \mathscr{B}(\boldsymbol{W}))$ such that every continuous linear functional $l \in \boldsymbol{W}^*$ has a normal distribution with mean value equal to zero and variance $\|l\|^2_{\mathcal{H}}$. In other words, P is the unique probability measure on $\boldsymbol{W}$ such that

$$\int_{\boldsymbol{W}} e^{il(\omega)} P(\mathrm{d}\omega) = \exp\left[-\frac{1}{2}\|l\|^2_{\mathcal{H}}\right]$$

for every $l \in \boldsymbol{W}^*$.

The isonormal Gaussian process can also be introduced in the setting of classical Wiener space. Every $h \in \mathcal{H}$ corresponds (unique up to a P-null set) to a random variable on $\boldsymbol{W}$, which will be denoted by $[h]$. This random variable $[h]$ has a normal distribution $N(0, \|h\|^2_{\mathcal{H}})$. In fact, for every $h \in \mathcal{H}$, the corresponding Gaussian random variable $[h]$ can be identified using the Itô integral $[h] = \int_0^\infty \dot{h} \mathrm{d}\omega$, which is the stochastic integral of $\dot{h}$ against a standard Brownian motion $(\omega(t))_{t\geq 0}$. In this sense, the triple $(\boldsymbol{W}, \mathcal{H}, P)$ is an example of abstract Wiener spaces, called the classical Wiener space.

The completion of the Borel σ-algebra $\mathscr{B}(\boldsymbol{W})$ with respect to the Wiener measure P is denoted by $\mathscr{F}$. An $\mathscr{F}$-measurable function on $\boldsymbol{W}$, valued in a separable Hilbert space, is called, according to the convention in literature, a Wiener functional.

Remark 2.1.1. *Though in this section, when introducing the classical Wiener space, we take our real separable Banach space $\boldsymbol{W}$ to be the collection of all continuous paths over $[0,\infty)$, equipped with the norm defined as in (2.1). This space $\boldsymbol{W}$ could also be taken to be $C_0([0,1])$, the space of all continuous paths over time interval $[0,1]$ starting from the origin, and equipped with the norm*

$$\|\omega\| = \sup_{0\leq t\leq 1} |\omega(t)|, \quad \forall \omega \in \boldsymbol{W}.$$

Accordingly, the corresponding Cameron-Martin space $\mathcal{H}$ is taken to be all absolutely continuous functions over $[0,1]$ whose generalised derivatives are elements of $L^2([0,1])$. In Chapter 4, we use continuous paths over $[0,\infty)$, and in Chapter 5, to simplify our computation, we adopt $\boldsymbol{W} = C_0([0,1])$.

2.2 Malliavin calculus and capacities

In this section, we introduce several definitions and results in Malliavin calculus. A differential structure on the classical Wiener space $(\boldsymbol{W}, \mathcal{H}, P)$ that is compatible with the Wiener measure P was introduced by Malliavin (see [47] and [46]). We first introduce the definition of smooth cylindrical random variables, which are random variables of the form

$$F = f([h_1], \cdots, [h_n]), \quad h_i \in \mathcal{H},$$

where $f \in C_p^\infty(\mathbb{R}^n)$ is a function whose partial derivatives have polynomial growth. The Malliavin derivative of F is defined to be an $\mathcal{H}$-valued random variable given by

$$DF = \sum_{i=1}^{n} \partial_i f([h_1], \cdots, [h_n]) h_i,$$

where $\partial_i f(x_1, \cdots, x_n)$, $1 \le i \le n$, is the partial derivative of f with respect to the i-th variable. The second order Malliavin derivative of F is given by

$$D^2 F = \sum_{i,j=1}^{n} \partial_{i,j}^2 f([h_1], \cdots, [h_n]) h_i \otimes h_j,$$

where $\partial_{i,j}^2 f$ denotes the second order partial derivative of f in the i-th and j-th components. Thus, higher order Malliavin derivatives $D^k F$ for $k \ge 1$ may be defined inductively. We denote the collection of all such smooth cylindrical random variables by $\mathcal{S}$.

For all $r \in \mathbb{N}$ and $1 < p < \infty$, the Sobolev norm of a smooth cylindrical random variable $F \in \mathcal{S}$ is defined to be

$$\|F\|_{\mathbb{D}_r^p} = \left(\mathbb{E}\left[|F|^p\right] + \sum_{k=1}^{r} \mathbb{E}\left[\left|\|D^k F\|_{\mathcal{H}^{\otimes k}}\right|^p\right] \right)^{1/p}.$$

Let $\mathbb{D}_r^p$ be the completion of $\mathcal{S}$ with respect to the Sobolev norm $\|\cdot\|_{\mathbb{D}_r^p}$. The (p,r)-capacity of an open subset O of $\boldsymbol{W}$ is defined by (see e.g. [49]):

$$c_{p,r}(O) = \inf\left\{\|\varphi\|_{\mathbb{D}_r^p} : \varphi \in \mathbb{D}_r^p,\ \varphi \ge 1 \text{ a.e. on } O,\ \varphi \ge 0 \text{ a.e. on } \boldsymbol{W}\right\},$$

and for an arbitrary subset A of $\boldsymbol{W}$, its (p,r)-capacity is

$$c_{p,r}(A) = \inf\left\{c_{p,r}(O) : A \subset O,\ O \text{ is open}\right\}.$$

A set $A \subset \boldsymbol{W}$ is said to be slim if $c_{p,r}(A) = 0$ for all $r \in \mathbb{N}$ and $1 < p < \infty$. A property π defined over $\boldsymbol{W}$ is said to hold $c_{p,r}$-quasi-surely or (p, r)-quasi-surely if the set on which this property is not satisfied has (p, r)-capacity zero, and a property π defined over $\boldsymbol{W}$ is said to hold quasi-surely (q.s.) if it holds $c_{p,r}$-quasi-surely for all $r \in \mathbb{N}$ and $1 < p < \infty$.

The notion of slim sets on the classical Wiener space $(\boldsymbol{W}, \mathcal{H}, P)$ can be studied via the Ornstein-Uhlenbeck operator, which gives rise to a different but equivalent approach to the definition of (p, r)-capacity. For a given $p \in [1, \infty]$, let $(T_t)_{t\geq 0}$ denote the Ornstein-Uhlenbeck semigroup on $L^p(\boldsymbol{W}, P)$, which is a one-parameter semigroup of contractions on $L^p(\boldsymbol{W}, P)$ given by Mehler's formula:

$$T_t u(x) = \int_{\boldsymbol{W}} u\left(e^{-t}x + \sqrt{1 - e^{-2t}}\omega\right) P(\mathrm{d}\omega). \tag{2.2}$$

For each fixed $t \geq 0$, T_t is a contraction operator on $L^p(\boldsymbol{W}, P)$, $p \geq 1$. Indeed,

$$\int_{\boldsymbol{W}} |T_t f(\omega)|^p P(\mathrm{d}\omega) \leq \int_{\boldsymbol{W}} \int_{\boldsymbol{W}} |f(e^{-t}x + \sqrt{1 - e^{-2t}}\omega)|^p P(\mathrm{d}\omega) P(\mathrm{d}x).$$

Consider the mapping $\varphi_t : \boldsymbol{W} \times \boldsymbol{W} \to \boldsymbol{W}$, given by

$$\varphi_t(x, y) = e^{-t}x + \sqrt{1 - e^{-2t}}y,$$

which induces a measure $\widetilde{P} = (P \otimes P) \circ \varphi_t^{-1}$ on $\boldsymbol{W}$. As for each $t \geq 0$, write $e^{-t} = \sin\theta$, and $\sqrt{1 - e^{-2t}} = \cos\theta$ for some $\theta \in [0, \pi]$. Since Gaussian measures are invariant under rotation, we deduce that $\widetilde{P} = P$, and thus

$$\begin{aligned} \int_{\boldsymbol{W}} |T_t f(\omega)|^p P(\mathrm{d}\omega) &\leq \int_{\boldsymbol{W} \times \boldsymbol{W}} |f(\varphi_t(x, y))|^p (P \otimes P)(\mathrm{d}x, \mathrm{d}y) \\ &= \int_{\boldsymbol{W}} |f(z)|^p \widetilde{P}(\mathrm{d}z) \\ &= \int_{\boldsymbol{W}} |f|^p \mathrm{d}P, \end{aligned}$$

which implies that, T_t is a contraction on $L^p(\boldsymbol{W}, P)$ for all $t \geq 0$ and $p \geq 1$.

The Ornstein-Uhlenbeck semigroup defined above enjoys several properties:

1. For each $t \geq 0$, T_t is positivity preserving in that if $f \geq 0$, then $T_t f \geq 0$.

2. $(T_t)_{t\geq 0}$ is symmetric with respect to the underlying probability measure P, which is

$$\int_{\boldsymbol{W}} (T_t f) g \mathrm{d}P = \int_{\boldsymbol{W}} f(T_t g) \mathrm{d}P, \quad \forall t \geq 0.$$

3. The third property is the hypercontractivity: let $p > 1$ and $t > 0$, and set $q(t) = e^{2t}(p-1) + 1$, which is, by definition, always strictly greater than p. Suppose $F \in L^p(\boldsymbol{W})$, then it holds that

$$\|T_t F\|_{L^{q(t)}} \leq \|F\|_{L^p}$$

for all t.

As the hypercontractivity property is rather important and will be used in the proof of other results, we shall include the proof of this result here.

Proof of 3, Theorem 1.4.1, Chapter 1, [58]. The following proof is contained in Section 1.4.3, Chapter 1, [58]. It suffices to verify that for any $G \in L^{q^*}(\boldsymbol{W})$, where $q^* \in (1, \infty)$ satisfying $1/q^* + 1/q = 1$, i.e.

$$q^* = q^*(t) = \frac{1 + e^{2t}(p-1)}{e^{2t}(p-1)},$$

it holds that

$$\mathbb{E}\left[(T_t F) G\right] \leq \|F\|_{L^p} \|G\|_{L^{q^*}}. \tag{2.3}$$

It holds that

$$|T_t F| \leq T_t(|F|).$$

We assume that F and G are positive bounded measurable functions on $\boldsymbol{W}$ (or otherwise argue by approximation) and of the forms

$$F(\omega) = f([h_1](\omega), \cdots, [h_n](\omega)) = f \circ W(\omega)$$

and

$$G(\omega) = g([h_1](\omega), \cdots, [h_n](\omega)) = g \circ W(\omega)$$

respectively, where $f, g : \mathbb{R}^n \to \mathbb{R}$ are positive bounded measurable functions, the functions $h_1, \cdots, h_n$ are orthonormal in $\mathcal{H}$, and $W = ([h_1], \cdots, [h_n])$ is an n-dimensional standard Gaussian random variable defined on $\boldsymbol{W}$. Therefore using Mehler's formula, we may rewrite the left-hand side as

$$\begin{aligned} \mathbb{E}\left[(T_t F)G\right] &= \int_{\boldsymbol{W}} \int_{\boldsymbol{W}} F(e^{-t}x + \sqrt{1-e^{-2t}}y) G(x) P(\mathrm{d}y) P(\mathrm{d}x) \\ &= \mathbb{E}\left[f(e^{-t}W_1 + \sqrt{1-e^{-2t}}W_1') g(W_1) \right], \end{aligned}$$

where $(W_s)_{0 \le s \le 1}$ and $(W_s')_{0 \le s \le 1}$ are two independent n-dimensional Brownian motions over the time interval $[0, 1]$. Notice that

$$\widetilde{W}_s = e^{-t}W_s + \sqrt{1-e^{-2t}}W_s', \quad \forall s \in [0, 1]$$

is still an n-dimensional Brownian motion with respect to the filtration generated by W and W'. Moreover, by the definition of F and G, we observe that

$$\|F\|_{L^p}^p = \mathbb{E}\left[|f(\widetilde{W}_1)|^p \right],$$

and

$$\|G\|_{L^{q^*}}^{q^*} = \mathbb{E}\left[|g(W_1)|^{q^*} \right].$$

For convenience, denote $X = f(\widetilde{W}_1)$ and $Y = g(W_1)$, and notice that X and Y are non-negative, bounded as f and g are, and measurable with respect to the σ-algebra generated by $\widetilde{W}_1$ and W_1.

With the above observations, the inequality (2.3) can be reduced to the following inequality:

$$\mathbb{E}[XY] \leq \mathbb{E}[X^p]^{1/p}\mathbb{E}[Y^{q^*}]^{1/q^*}.$$

By the martingale representation theorem, there exist two predictable adapted n-dimensional stochastic processes $(\varphi_s)_{s\geq 0}$ and $(\psi_s)_{s\geq 0}$ such that

$$X^p = \mathbb{E}[X^p] + \int_0^1 \varphi_u \mathrm{d}\widetilde{W}_u,$$

and

$$Y^{q^*} = \mathbb{E}[Y^{q^*}] + \int_0^1 \psi_u \mathrm{d}W_u.$$

Define

$$M_s = \mathbb{E}[X^p] + \int_0^s \varphi_u \mathrm{d}\widetilde{W}_u$$

and

$$N_s = \mathbb{E}[Y^{q^*}] + \int_0^s \psi_u \mathrm{d}W_u$$

for $s \in [0, 1]$, then $(M_s)_{0\leq s\leq 1}$ and $(N_s)_{0\leq s\leq 1}$ are positive martingales with respect to the filtration generated by $\widetilde{W}$ and W respectively, and thus Itô's formula applies. It folllows that

$$\mathrm{d}M_s = \varphi_s \mathrm{d}\widetilde{W}_s,$$

$$\mathrm{d}N_s = \psi_s \mathrm{d}W_s,$$

and by the definition of $\widetilde{W}$,

$$\mathrm{d}\langle M, N\rangle_s = \varphi_s\psi_s \mathrm{d}\langle \widetilde{W}, W\rangle_s = e^{-t}\varphi_s\psi_s \mathrm{d}s$$

for all $s \in [0,1]$. Applying Itô's formula to $M^{1/p}N^{1/q^*}$, we get that

$$
\begin{aligned}
XY &= M_1^{1/p}N_1^{1/q^*} \\
&= M_0^{1/p}N_0^{1/q^*} + \int_0^1 \frac{1}{p}M_s^{1/p-1}N_s^{1/q^*}\mathrm{d}M_s + \int_0^1 \frac{1}{q^*}M_s^{1/p}N_s^{1/q^*-1}\mathrm{d}N_s \\
&\quad + \frac{1}{2}\Bigg[\int_0^1 \frac{1}{p}\left(\frac{1}{p}-1\right)M_s^{1/p-2}N_s^{1/q^*}\mathrm{d}\langle M\rangle_s \\
&\quad + 2\int_0^1 \frac{1}{pq^*}M_s^{1/p-1}N_s^{1/q^*-1}\mathrm{d}\langle M,N\rangle_s \\
&\quad + \int_0^1 \frac{1}{q^*}\left(\frac{1}{q^*}-1\right)M_s^{1/p}N_s^{1/q^*-2}\mathrm{d}\langle N\rangle_s\Bigg] \\
&= \mathbb{E}[X^p]^{1/p}\mathbb{E}[Y^{q^*}]^{1/q^*} + \frac{1}{p}\int_0^1 M_s^{1/p-1}N_s^{1/q^*}\varphi_s\mathrm{d}\widetilde{W}_s \\
&\quad + \frac{1}{q^*}\int_0^1 M_s^{1/p}N_s^{1/q^*-1}\psi_s\mathrm{d}W_s \\
&\quad + \frac{1}{2p}\left(\frac{1}{p}-1\right)\int_0^1 M_s^{1/p-2}N_s^{1/q^*}\varphi_s^2\mathrm{d}s \\
&\quad + \frac{1}{2q^*}\left(\frac{1}{q^*}-1\right)\int_0^1 M_s^{1/p}N_s^{1/q^*-2}\psi_s^2\mathrm{d}s \\
&\quad + \frac{1}{pq^*}\int_0^1 M_s^{1/p-1}N_s^{1/q^*-1}e^{-t}\varphi_s\psi_s\mathrm{d}s.
\end{aligned}
$$

Therefore, after taking expectation on both sides what we need to prove becomes the following:

$$
\begin{aligned}
\mathbb{E}\Bigg[&\frac{1}{2p}\left(\frac{1}{p}-1\right)\int_0^1 M_s^{1/p-2}N_s^{1/q^*}\varphi_s^2\mathrm{d}s \\
&+\frac{1}{2q^*}\left(\frac{1}{q^*}-1\right)\int_0^1 M_s^{1/p}N_s^{1/q^*-2}\psi_s^2\mathrm{d}s \\
&+\frac{1}{pq^*}\int_0^1 M_s^{1/p-1}N_s^{1/q^*-1}e^{-t}\varphi_s\psi_s\mathrm{d}s\Bigg] \le 0
\end{aligned} \tag{2.4}
$$

Since $p > 1$ and $1/p - 1 < 0$, by setting

$$A_s = M_s^{1/2p-1}N_s^{1/2q^*}\varphi_s,$$

and

$$B_s = M_s^{1/2p} N_s^{1/2q^*-1} \psi_s,$$

we may write the left-hand side of the above inequality (2.4) as

$$\frac{1}{2}\int_0^1 \mathbb{E}\left[\frac{1}{p}\left(\frac{1}{p}-1\right)A_s^2 + \frac{2}{pq^*}e^{-t}A_sB_s + \frac{1}{q^*}\left(\frac{1}{q^*}-1\right)B_s^2\right]\mathrm{d}s,$$

and hence it remains to show that

$$\frac{1}{(pq^*)^2}e^{-2t} - \frac{1}{p}\left(\frac{1}{p}-1\right)\frac{1}{q^*}\left(\frac{1}{q^*}-1\right) \leq 0,$$

which may be easily verified by plugging in the value of q^*. □

Now we introduce the generator of semigroup (T_t), denoted by L in the sequel. Let

$$\mathcal{D}(L) = \left\{u \in L^p(\boldsymbol{W}, P) : \lim_{t\downarrow 0}\frac{T_t u - u}{t} \text{ exists in } L^p\text{-space}\right\},$$

and define

$$Lu = \lim_{t\downarrow 0}\frac{T_t u - u}{t} \text{ for } u \in \mathcal{D}(L).$$

For each $r > 0$, $(I-L)^{-\frac{r}{2}}$ is a contraction on $L^p(\boldsymbol{W}, P)$, and is given by the following integral

$$(I-L)^{-\frac{r}{2}} = \frac{1}{\Gamma(r/2)}\int_0^\infty t^{\frac{r}{2}-1}e^{-t}T_t\mathrm{d}t,$$

(defined in the sense of Bochner's integrals). The Sobolev norm can also be defined. The corresponding Sobolev norm, denoted by $\|\cdot\|_{r,p}$ $(1 < p < \infty)$, is then defined to be

$$\|u\|_{r,p} = \|(I-L)^{-\frac{r}{2}}u\|_p.$$

The corresponding (p,r)-capacity $C_{r,p}$, following Fukushima's convention in [22], can be defined in a similar manner as before, namely, for an open subset O of $\boldsymbol{W}$,

$$C_{r,p}(O) = \inf\left\{\|\phi\|_p^p : (I-L)^{-\frac{r}{2}}\phi \geq 1 \text{ a.e. on } O,\ (I-L)^{-\frac{r}{2}}\phi \geq 0 \text{ a.e. on } \boldsymbol{W}\right\},$$

(with convention that $\inf \emptyset = \infty$) and

$$C_{r,p}(A) = \inf\left\{\mathrm{C}_{r,p}(O) : A \subset O,\ O \text{ is open}\right\}$$

for an arbitrary subset A of $\boldsymbol{W}$.

It was Meyer [53] who proved that two Sobolev norms $\|\cdot\|_{\mathbb{D}_r^p}$ and $\|\cdot\|_{r,p}$ are equivalent to each other, and consequently we obtain that there exists a constant $\alpha_{r,p} > 0$ such that

$$\frac{1}{\alpha_{r,p}} C_{r,p}(A) \leq (c_{p,r}(A))^p \leq \alpha_{r,p} C_{r,p}(A) \tag{2.5}$$

for every $A \subset \boldsymbol{W}$. For further details about the Sobolev norm $\|\cdot\|_{r,p}$ and the corresponding capacity, one should refer to [22], [71] and [67].

Both $c_{p,r}$ and $C_{r,p}$ capacities have several rather important properties, which will be used in our arguments frequently:

1. Firstly, capacities $c_{p,r}$ and $C_{r,p}$ are outer measures in the sense that $c_{p,r}$ and $C_{r,p}$ are monotonic and sub-additive. In other words,

$$c_{p,r}(A) \leq c_{p,r}(B)$$

for any $A \subseteq B$, and

$$c_{p,r}(A) \leq \sum_n c_{p,r}(A_n)$$

if $A \subset \bigcup_n A_n$. These properties hold for $C_{r,p}$ as well. Let us point out that the sub-additivity of $c_{p,r}$ follows from the localisation of $\|\cdot\|_{\mathbb{D}_r^p}$, while the sub-additivity of $C_{r,p}$ follows from the triangle inequality for norms.

2. It follows that the first Borel-Cantelli applies to these capacities (see e.g. Corollary 1.2.4, Chapter IV, [49]). More precisely, if $\{A_n\}_{n=1}^{\infty}$ is a sequence of subsets of $\boldsymbol{W}$ such that $\sum_{n=1}^{\infty} c_{p,r}(A_n) < \infty$, then

$$c_{p,r}(\limsup_{n\to\infty} A_n) = 0.$$

The capacity version of the Borel-Cantelli lemma, together with the concept of the Malliavin derivative, are major tools in our arguments.

3. A version of Chebyshev's inequality can be established in the context of capacities. In fact, the definition of capacity $c_{p,r}$ implies the following Chebyshev's inequality (see e.g. Corollary 1.2.5, Chapter IV, [49]). If $\varphi \in \mathbb{D}_r^p$ and φ is lower-semi continuous, then

$$c_{p,r}\left(\varphi > \lambda\right) \leq \lambda^{-1} \|\varphi\|_{\mathbb{D}_r^p}$$

for every $\lambda > 0$.

4. We also have the following generalised sub-additivity. Lemma 1.1 in [22], together with Meyer's inequality, implies a stronger version of the sub-additivity for $c_{p,r}$, which says that

$$(c_{p,r}(A))^p \leq M_{p,r} \sum_{n=1}^{\infty} (c_{p,r}(A_n))^p \tag{2.6}$$

for some constant $M_{p,r}$ depending only on p and r, for any $A \subset \bigcup_n A_n$.

5. Lower continuity: $c_{p,r}$ is lower continuous (see e.g. Theorem 5.1, Chapter IV, [49]) in the sense that for an increasing sequence of sets $\{A_n\}_{n=1}^{\infty}$,

$$c_{p,r}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} c_{p,r}(A_n). \tag{2.7}$$

2.3 Wiener chaos

In this section, we introduce the theory of Wiener chaos decomposition and prove a useful estimate on the Sobolev norm $\|\cdot\|_{\mathbb{D}_r^p}$.

On the Wiener space, a Hermite polynomial is random variable given by

$$H_{\mathbf{m}}(\omega) = \prod_{k\in\mathbb{N}} H_{m_k}([h_k]), \quad \mathbf{m} = (m_1, \cdots)$$

with $\|\mathbf{m}\| = \sum_{k\in\mathbb{N}} m_k$. The n-th Wiener chaos is the closed subspace of $L^2(\boldsymbol{W}, \mathscr{F}, P)$ generated by $H_{\mathbf{m}}$ such that $\|\mathbf{m}\| = n$.

Then we have the following Wiener chaos decomposition of L^2-space:

$$L^2(\boldsymbol{W}, \mathscr{F}, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

The projection from $L^2(\boldsymbol{W})$ to $\mathcal{H}_n$ is denoted by J_n. Let $\mathcal{P}_n^0$ denote the space of polynomial random variables of the form

$$F = p([h_1], \cdots [h_n]), \quad \forall h_1, \cdots h_n \in \mathcal{H},$$

where p is a polynomial with degree less than or equal to n, and $\mathcal{P}_n$ be its closure in $L^2(\boldsymbol{W})$. Then

$$\mathcal{P}_n = \oplus_{m=0}^n \mathcal{H}_m.$$

If $F \in \mathbb{D}_r^p$ where $p \geq 2$, then for $l \leq r$,

$$\big\| \|D^l F\|_{\mathcal{H}^{\otimes l}} \big\|_2^2 = \sum_{n=l}^{\infty} n(n-1)\cdots(n-l+1)\|J_n F\|_2^2. \tag{2.8}$$

For a proof of this result, one may refer to Proposition 1.2.2, Section 1.2, Chapter 1, [58] and the comment afterwards.

We recall that the Ornstein-Uhlenbeck semigroup $(T_t)_{t\geq 0}$ may be defined via the Mehler's formula (2.2). Alternatively, this semigroup $(T_t)_{t\geq 0}$ can be defined to be a family of contraction operators on $L^2(\boldsymbol{W})$ satisfying

$$T_t(f) = \sum_{n=0}^{\infty} e^{-nt} J_n f,$$

for all $f \in L^2(\boldsymbol{W})$. These two definitions coincide on L^2-space, and we shall use this new definition to prove two useful estimates as follows.

Proposition 2.3.1. *If $F \in \mathcal{P}_n$, then*

$$\|F\|_q \leq (n+1)(q-1)^{\frac{n}{2}} \|F\|_2 \tag{2.9}$$

and

$$\left\| \|D^l F\|_{\mathcal{H}^{\otimes l}} \right\|_2 \leq n^{\frac{l}{2}} \|F\|_2 \tag{2.10}$$

for any $q > 2$ and $l \leq n$.

Proof. The first inequality (2.9) is due to the hypercontractivity property of the Ornstein-Uhlenbeck semigroup $(T_t)_{t\geq 0}$. Take $F \in \mathcal{H}_n$. Then by definition

$$T_t F = e^{-nt} F.$$

Set $p = 2$, and $q = q(t) = 1 + e^{2t}$, so

$$t = \frac{1}{2} \log(q-1),$$

and hence by the hypercontractivity property of the Ornstein-Uhlenbeck semigroup,

$$\|T_t F\|_q = \|e^{-\frac{n}{2}\log(q-1)} F\|_q = (q-1)^{-\frac{n}{2}} \|F\|_q \leq \|F\|_2,$$

which implies that

$$\|F\|_q \le (q-1)^{\frac{n}{2}} \|F\|_2.$$

If $F = \sum_{m=0}^{n} J_m F \in \mathcal{P}_n$, then

$$\begin{aligned}
\|F\|_q &\le \sum_{m=0}^{n} \|J_m F\|_q \\
&\le \sum_{m=0}^{n} (q-1)^{\frac{m}{2}} \|J_m F\|_2 \\
&\le (n+1)(q-1)^{\frac{n}{2}} \|F\|_2.
\end{aligned}$$

To prove (2.10), applying (2.8) to $F \in \mathcal{P}_n$, where $F \in \mathbb{D}_l^2$ for $l \le n$, and $J_m F$'s vanish when $m > n$. Notice that

$$\|F\|_2^2 = \sum_{m=0}^{n} \|J_m F\|_2^2,$$

so we deduce that

$$\begin{aligned}
\big\| \|D^l F\|_{\mathcal{H}^{\otimes l}} \big\|_2^2 &= \sum_{m=l}^{n} m(m-1)\cdots(m-l+1) \|J_m F\|_2^2 \\
&\le n^l \sum_{m=l}^{n} \|J_m F\|_2^2 \\
&\le n^l \|F\|_2^2,
\end{aligned}$$

and the proof is complete. □

2.4 Fractional Brownian motions

In this section, we will define fractional Brownian motion, introduce its integral representation, and then list several important properties of this process.

A 1-dimensional fractional Brownian motion (fBM for short) $(B_t)_{t\ge 0}$ with Hurst parameter $H \in (0,1)$ is a centred Gaussian process on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$ whose covariance function is given by

$$R(t,s) = \mathbb{E}\left[B_t B_s\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad \forall s,t \geq 0.$$

A d-dimensional fractional Brownian motion is d-copies of independent one-dimensional fBMs. As a centred Gaussian process is fully determined by its covariance function, the properties of fBMs vary according to the value of Hurst parameter H. In particular, when $H = \frac{1}{2}$, an fBM becomes a standard Brownian motion.

In Chapter 4 and 5, fBMs will be realised as Wiener functionals on the classical Wiener space $(\boldsymbol{W}, \mathcal{H}, P)$, in terms of the following integral representation due to Decreusefond and Üstünel [10]:

$$B_t = \int_0^t K(t,s)\mathrm{d}\omega(s), \tag{2.11}$$

where the integral on the right-hand side have to be interpreted as an Itô integral against standard Brownian motion $\{\omega(t) : t \geq 0\}$ under the Wiener measure P. Here, for each pair $t > s \geq 0$, define K to be the reproducing kernel

$$K(t,s) = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}\, s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u,$$

if $H > \frac{1}{2}$, and for $H < \frac{1}{2}$,

$$K(t,s) = \sqrt{\frac{2H}{(1-2H)B(1-2H, H+\frac{1}{2})}} \cdot \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} \mathrm{d}u\right],$$

where $B(\cdot,\cdot)$ denotes the Beta function. We define $K = 1$ when $H = \frac{1}{2}$, so that our results are compatible with the classical results established for standard Brownian motions. We notice that K is a non-negative but singular kernel and it satisfies that

$$\int_0^{t\wedge u} K(t,s)K(u,s)\mathrm{d}s = R(t,u), \quad \forall t,u \geq 0.$$

For further details on the above integral representation and the reproducing kernel K, one may refer to [10] and Chapter 5 in [58].

One may check that fBMs have the following properties:

1. Fractional Brownian motions have stationary increments, i.e. $B_t - B_s$ and B_{t-s} have the same distribution for any $0 \leq s < t$. We note here that one major difference between fBMs and Brownian motions is that the increments of fBMs over different time intervals are no longer independent. Indeed,

$$\mathbb{E}\left[(B_t - B_s)(B_r - B_u)\right] = \frac{1}{2}\left[(t-u)^{2H} - (t-r)^{2H} - \left((s-u)^{2H} - (s-r)^{2H}\right)\right],$$

where $0 \leq u < r < s < t$. As $f(x) = x^{2H}$ is convex when $H \in \left(\frac{1}{2}, 1\right)$, and $t - u > s - u$, we have

$$f(t-u) - f(t-r) \geq f(s-u) - f(s-r),$$

so the increments over two distinct time intervals are positively correlated; while $f(x) = x^{2H}$ is concave if $H \in \left(0, \frac{1}{2}\right)$, which implies that the increments are negatively correlated in this case. Due to this property, the sample paths of fractional Brownian motions with distinct Hurst parameters look different. The paths of fBMs with Hurst parameter $H < \frac{1}{2}$ tend to be more wiggly than those of fBMs with Hurst parameter $H > \frac{1}{2}$. See Figure 2.1 for the case when $H < \frac{1}{2}$ and Figure 2.2 when $H > \frac{1}{2}$.

2. FBMs are known as examples of self-similar stochastic processes. A stochastic

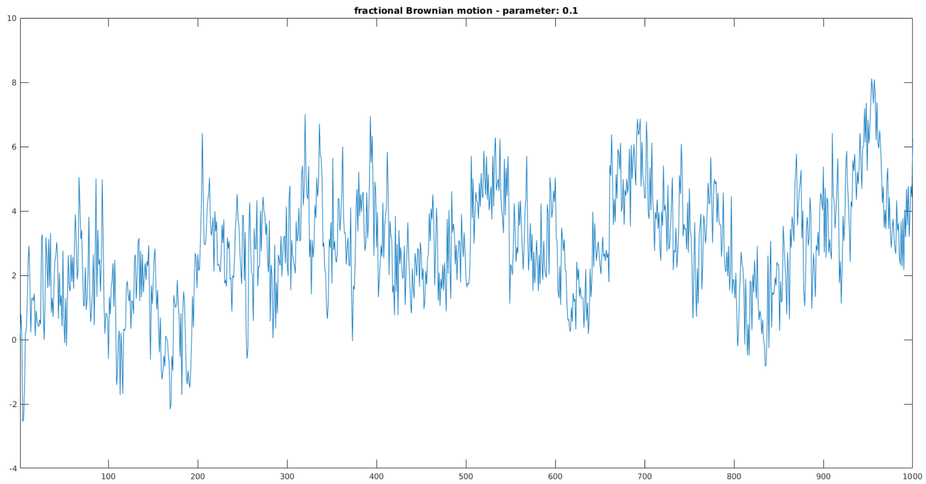

Figure 2.1: FBM with $H = 0.1$

process is said to be self-similar with parameter H if for any $h > 0$ and $t \geq 0$,

$$X_t \stackrel{\text{law}}{=} h^{-H} X_{ht}.$$

A fractional Brownian motion is self-similar with its Hurst parameter H, i.e. for any $\alpha > 0$, $\{B_t : t \geq 0\} = \left\{\alpha^{-H} B_{\alpha t} : t \geq 0\right\}$ in distribution. The scaling property of Brownian motion is simply just self-similarity property with parameter 1/2.

3. FBMs are almost surely continuous. For all $\alpha > 0$, it follows that

$$\mathbb{E}[|B_t - B_s|^\alpha] = \mathbb{E}[|B_1|^\alpha]|t - s|^{\alpha H}$$

by self-similarity and stationary increments properties, and hence Kolmogorov's continuity criterion indicates that for any $H \in (0, 1)$, $(B_t)_{t \geq 0}$ has a modification which is almost-surely continuous.

4. FBMs are nowhere differentiable. Here we present the outline of the proof, following Proposition 4.2 in [50]. Let $(B_t)_{t \geq 0}$ be a fractional Brownian motion

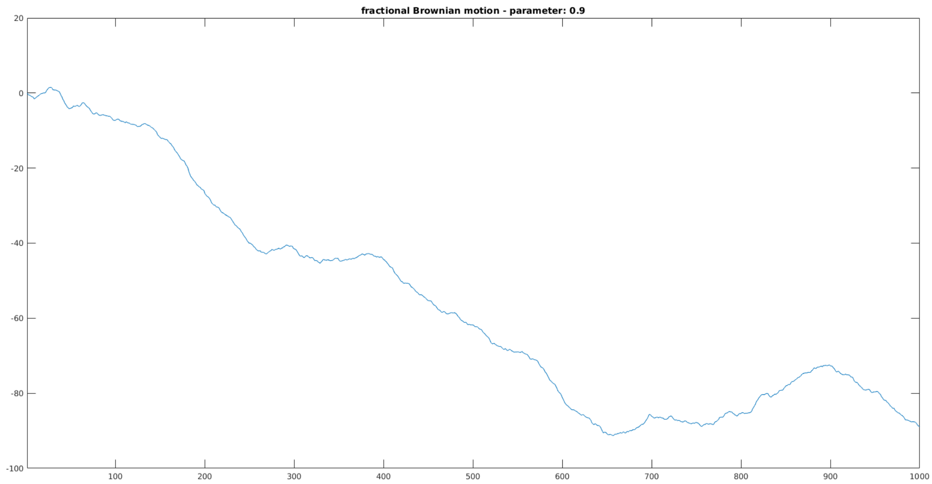

Figure 2.2: FBM with $H = 0.9$

on the probability space $(\Omega, \mathscr{G}, \mathbb{P})$ with Hurst parameter H. Consider the set

$$A := \left\{ \omega : \limsup_{t \to t_0} \frac{|B_t(\omega) - B_{t_0}(\omega)|}{|t - t_0|} = \infty \right\},$$

and let

$$A_{n,M} = \left\{ \omega : \sup_{s \in \left(t_0, t_0 + \frac{1}{n}\right)} \frac{|B_s(\omega) - B_{t_0}(\omega)|}{|s - t_0|} > M \right\}.$$

Then $\omega \in A$ if and only if there exists an M such that $\omega \in A_{n,M}$ for infinitely many n, that is,

$$\omega \in \bigcup_{M=1}^{\infty} \{A_{n,M} \text{ i.o.}\}.$$

However, we notice that $A_{n+1,M} \subset A_{n,M}$, and for each n, define

$$\begin{aligned} B_{n,M} &= \left\{ \omega : \frac{|B_{t_0 + \frac{1}{n}}(\omega) - B_{t_0}(\omega)|}{\frac{1}{n}} > M \right\} \\ &= \left\{ \omega : |B_{t_0 + \frac{1}{n}}(\omega) - B_{t_0}(\omega)| > \frac{M}{n} \right\}. \end{aligned}$$

It follows by definition that $B_{n,M} \subset A_{n,M}$, and since fractional Brownian motions are self-similar, we deduce that

$$\begin{aligned}\mathbb{P}(B_{n,M}) &= \mathbb{P}\left(\omega : |B_{\frac{1}{n}}(\omega)| > \frac{M}{n}\right)\\ &= \mathbb{P}\left(\omega : |B_1(\omega)| > Mn^{H-1}\right),\end{aligned}$$

so $\mathbb{P}(B_{n,M}) \to 1$ as $n \to \infty$. Furthermore, since

$$\mathbb{P}(\limsup_{n\to\infty} A_{n,M}) \geq \mathbb{P}(\limsup_{n\to\infty} B_{n,M}) \geq \limsup_{n\to\infty} \mathbb{P}(B_{n,M}) = 1,$$

we conclude that $\mathbb{P}(A) = 1$, which indicates that B is nowhere differentiable $\mathbb{P}$-almost surely.

5. Unless $H = \frac{1}{2}$, fBMs are not semimartingales. This result follows from the fact that every semimartingale has finite quadratic variation, and when its quadratic variation vanishes, it is of finite variation. Now for any partition $\Pi=\{0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_m = T\}$ of time interval $[0, T]$, where $T > 0$, define

$$V_T^p(\Pi) = \sum_{k=1}^{m} |B_{t_k} - B_{t_{k-1}}|^p,$$

where $p \geq 1$. Since

$$\mathbb{E}\left[|B_t - B_s|^p\right] = \mathbb{E}\left[|B_1|^p\right] |t - s|^{pH},$$

and convergence in L^1 implies convergence in probability, we conclude that B has p-variation equal to 0 when $pH > 1$ and equal to infinity when $pH < 1$. If $H \in \left(0, \frac{1}{2}\right)$, then there exists some $p \geq 2$ such that the p-variation of B is infinite, and so is its quadratic variation. On the other hand, when $H \in \left(\frac{1}{2}, 1\right)$, B has zero quadratic variation, but as $H < 1$, its variation is infinite. Therefore, in either case, B is not a semimartingale according to the criterion mentioned above. For details, see Proposition 5.1.1, Chapter 5, [58].

6. FBMs are not Markov processes when $H \neq 1/2$. Due to this fact, a lot of problems concerning fractional Brownian motions remain open, and many classical methods become ineffective. We include a proof of this result here. We adopt the proof of Theorem 2.3, Chapter 2, [57] here. Suppose B is Markovian when $H \neq 1/2$, and let $\{\mathscr{F}_t : t \geq 0\}$ denote the σ-algebra generated by B, i.e. $\mathscr{F}_s = \sigma(B_u : u \leq s)$. Set $X_t = B_t - aB_s$ with

$$a = \frac{\mathbb{E}\left[B_t B_s\right]}{\mathbb{E}\left[B_s^2\right]},$$

for $s \leq t$. Then we have $\mathbb{E}\left[X_t B_s\right] = 0$, and thus X_t and B_s are independent as they are Gaussian random variables. The Markov property of B implies that

$$\begin{aligned} \mathbb{E}\left[X_t B_u | \mathscr{F}_s\right] &= B_u \mathbb{E}\left[X_t | \mathscr{F}_s\right] \\ &= B_u \mathbb{E}\left[X_t | B_s\right] \\ &= 0, \end{aligned}$$

and hence $\mathbb{E}\left[X_t B_u\right] = 0$, i.e.

$$\mathbb{E}\left[B_t B_u - \frac{\mathbb{E}\left[B_t B_s\right]}{\mathbb{E}\left[(B_s)^2\right]} B_s B_u\right] = 0$$

for all $0 \leq u \leq s \leq t$. Plugging in the covariance function of fBM with Hurst parameter $H \neq 1/2$, we get that

$$\frac{\varphi\left(\frac{t}{u}\right)}{\varphi\left(\frac{s}{u}\right)} = \varphi\left(\frac{t}{s}\right),$$

where

$$\varphi(x) = \frac{1}{2}\left[x^{2H} + 1 - (x-1)^{2H}\right], \quad x \geq 1. \tag{2.12}$$

This implies that

$$\varphi(ab) = \varphi(a)\varphi(b)$$

for $a, b \geq 1$. Notice that $\varphi(1) = 1$ and $\varphi'(x) \geq 0$. We consider the function $f : [0, \infty) \to [0, \infty)$ given by $f(x) = \log \varphi(e^x)$. Then f satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y),$$

which implies that $f(x) = cx$ with some positive constant c, so

$$\varphi(x) = x^c. \tag{2.13}$$

This leads to a contradiction since

$$\lim_{x \downarrow 1} |\varphi''(x)| = \lim_{x \downarrow 1} |H(2H-1)(x^{2H-2} - (x-1)^{2H-2})| = \infty,$$

according to (2.12) while

$$\lim_{x \downarrow 1} |\varphi''(x)| = c|c-1| < \infty$$

by (2.13). Therefore, $(B_t)_{t \geq 0}$ is not a Markov process unless $H = 1/2$.

Remark 2.4.1. *Indeed, a non-constant self-similar Gaussian process with stationary increments and continuous variance must be a fractional Brownian motion. Self-similarity with parameter H and stationary increments force H to be less than 1, and continuous variance implies that $H \geq 0$. See Proposition 3.7 and Proposition 3.8 in [50] for a proof.*

2.5 Large deviation principles

In this section, we review several important results in the large derivation principle (LDP) theory without proofs. Let us first introduce several basic definitions in the large deviation principle theory following [70] and [12].

Let (E, d) be a Polish space, that is, a complete separable metric space, and $\mathscr{B}(E)$ be its Borel σ-algebra.

Definition 2.5.1. *We say that a family of probability measures $\{\mu_\varepsilon : \varepsilon > 0\}$ on $(E, \mathscr{B}(E))$ obeys the large deviation principle (LDP) with the rate function I if*

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log \mu_\varepsilon(F) \leq - \inf_{x \in F} I(x)$$

for each closed set $F \subset E$, and

$$\liminf_{\varepsilon \downarrow 0} \varepsilon \log \mu_\varepsilon(G) \geq - \inf_{x \in G} I(x)$$

for each non-empty open set $G \subset E$, where the rate function $I : E \to [0, \infty]$ is defined to be a lower semi-continuous function such that for every $c \in \mathbb{R} \cup \{\infty\}$, the level set

$$I_c = I^{-1}((-\infty, c]) = \{s \in X : I(s) \leq c\}$$

is closed. Moreover, a rate function is said to be a good rate function if for any $c \geq 0$, the level set $\{s : I(s) \leq c\}$ is compact.

Remark 2.5.1. *We say that a family of random variables $\{X^\varepsilon : \varepsilon > 0\}$ satisfies the LDP with a rate function I if their laws satisfies the LDP with I as its rate function.*

The most essential result in the large deviation principle theory is the celebrated Cramér's theorem. It provides us the explicit form of rate functions for finite dimensional case.

Let $\{X_i\}_{i\in\mathbb{N}}$ be a family of $\mathbb{R}^d$-valued independent identically distributed random variables, and let μ denote their law. Define the empirical mean $S_n = \frac{1}{n}\sum_{i=1}^n X_i$ for all $n \in \mathbb{N}$, and let μ_n denote the law of S_n. Cramér's theorem is stated as the following.

Proposition 2.5.1. *Suppose for $\lambda \in \mathbb{R}^d$ such that $\int_{\mathbb{R}^d} e^{\langle\lambda,z\rangle}\mu(\mathrm{d}z) < \infty$. Then the family $\{\mu_n : n \geq 1\}$ satisfies the LDP with a good rate function*

$$I(x) = \sup_{\lambda\in\mathbb{R}^d}\{\langle\lambda, x\rangle - \log\int_{\mathbb{R}^d} e^{\langle\lambda,z\rangle}\mu(\mathrm{d}z)\},$$

that is,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_n(F) \leq -\inf_F I$$

for each closed subset F of E, and

$$\liminf_{n\to\infty}\frac{1}{n}\log\mu_n(G) \geq -\inf_G I$$

for each open subset G of E.

The following example is a quick application of Cramér's theorem.

Example 2.5.1. *An easy example is to consider the family of measures induced by a collection of Gaussian random variables. Let $\xi_n = (\xi_n^{(1)}, \cdots, \xi_n^{(d)})$, $n = 1, 2, \cdots$, be independent standard Gaussian random variables in $\mathbb{R}^d$. We denote their law by μ and consider the family of laws μ_n of their empirical mean $S_n = \frac{1}{n}\sum_{i=1}^n \xi_i$. Since*

$$\begin{aligned}\int_{\mathbb{R}^d} e^{\langle\lambda,z\rangle}\mu(\mathrm{d}z) &= \int_{\mathbb{R}^d} e^{\langle\lambda,z\rangle}\frac{1}{(\sqrt{2\pi})^d}e^{-|z|^2/2}\mathrm{d}z \\ &= e^{|\lambda|^2/2} < \infty,\end{aligned}$$

Cramér's theorem applies, and hence

$$I(x) = \sup_{\lambda\in\mathbb{R}^d}\{\langle\lambda, x\rangle - \log\int_{\mathbb{R}^d} e^{\langle\lambda,x\rangle}\mu(\mathrm{d}x)\} = \sup_{\lambda\in\mathbb{R}^d} f(\lambda),\ \ \forall x\in\mathbb{R}^d$$

with

$$f(\lambda) = \langle\lambda, x\rangle - \frac{|\lambda|^2}{2}.$$

By setting the derivative of f equal to zero, we conclude that f reaches its supremum when $\lambda = x$. Therefore,

$$I(x) = f(x) = \frac{|x|^2}{2},\ \ x\in\mathbb{R}^d,$$

and the family of laws $\{\mu_n\}$ obeys the LDP with the rate function $I(x)$.

In particular, when $d = 1$, we have the following approximation due to the LDP:

$$P\{S_n > a\} \approx e^{-\frac{na^2}{2}}, \quad \forall a > 0.$$

For a proof of Cramér's theorem, one may refer to Varadhan's notes [70]. The following contraction principle is a very classical result in the LDP theory, which states that LDPs are preserved under continuous functions, see Theorem 4.2.1, Chapter 4, [11] for details.

Proposition 2.5.2. *Let X and Y be two Hausdorff topological spaces, and the mapping $f : X \to Y$ continuous. Suppose a family of probability measures $\{\mu_\varepsilon : \varepsilon > 0\}$ satisfies the LDP with a good rate function $I : X \to [0,\infty]$. Then the family of push-forward probability measures $\{\mu_\varepsilon\circ f^{-1} : \varepsilon > 0\}$ on Y satisfies the LDP with a good rate function given by*

$$J(y) := \inf\{I(x) : x\in X, y = f(x)\}.$$

Next result shows that LDPs are also preserved under exponentially fast convergence. Let us first introduce the definition of exponentially good approximations.

Definition 2.5.2. *Let* $\{X^{\varepsilon} : \varepsilon > 0\}$ *and* $\{X^{\varepsilon,(m)} : \varepsilon > 0\}_{m \in \mathbb{N}}$ *be two families of random variables on a probability space* $(\Omega, \mathscr{G}, \mu)$*, valued in a Polish space* (E, d)*. Random variables* $\{X^{\varepsilon,(m)} : \varepsilon > 0\}_{m \in \mathbb{N}}$ *are called exponentially good approximations of* $\{X^{\varepsilon} : \varepsilon > 0\}$ *if for every* $\delta > 0$*,*

$$\lim_{m\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \mu \left(d(X^{\varepsilon}, X^{\varepsilon,(m)}) > \delta \right) = -\infty.$$

Proposition 2.5.3. *Suppose that for each* $m \in \mathbb{N}$ *, the family of random variables* $\{X^{\varepsilon,(m)} : \varepsilon > 0\}$ *satisfies the LDP with rate function* $I_m : E \to [0, \infty]$*, and* $\{X^{\varepsilon,(m)} : \varepsilon > 0\}_{m \in \mathbb{N}}$ *are exponentially good approximations of* $\{X^{\varepsilon} : \varepsilon > 0\}$*. If* I *is a good rate function and for every closed set* F

$$\inf_{y \in F} I(y) \le \limsup_{m\to\infty} \inf_{y \in F} I_m(y),$$

then the LDP holds for $\{X^{\varepsilon} : \varepsilon > 0\}$ *with the good rate function* I*.*

The theory of LDPs may be generalised to the context of capacities, see for example [72] and [5]. Here we present the definition in the classical Wiener space setting only, but similar definition can be introduced for general abstract Wiener spaces as well.

Definition 2.5.3. *Let* $r \in \mathbb{N}$ *and* $p > 1$*, and* $\{X^{\varepsilon} : \varepsilon > 0\}$ *be a family of* $c_{p,r}$*-quasi-surely defined mappings from* $\boldsymbol{W}$ *to a Polish space* (Y, d)*. We say that the family* $\{X^{\varepsilon}\}$ *satisfies the* $c_{p,r}$*-LDP with a good rate function* $I : Y \to [0, \infty]$ *if*

(1) I *is a lower semi-continuous function and for every* $\alpha > 0$*, the level set*

$$\Psi_I(\alpha) = \{y \in Y : I(y) \le \alpha\}$$

is compact in Y; and

(2) for every closed subset $F \subset Y$,

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log c_{p,r} \{\omega \in \boldsymbol{W} : X^\varepsilon(\omega) \in F\} \leq -\frac{1}{p} \inf_{y \in F} I(y),$$

for open subset $G \subset Y$,

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log c_{p,r} \{\omega \in \boldsymbol{W} : X^\varepsilon(\omega) \in G\} \geq -\frac{1}{p} \inf_{y \in G} I(y).$$

The majority of conclusions in the theory of LDPs (see for example [11, 12] for details) remains valid in the context of capacities. Let us state some of them which will be useful in the sequel. Their proofs (see e.g. [5]) are routine and will be omitted. The following proposition is the counterpart of the contraction principle in the setting of capacities.

Proposition 2.5.4. *Let $\{X^\varepsilon : \varepsilon > 0\}$ be a family of $c_{p,r}$-quasi-surely defined maps from $\boldsymbol{W}$ to a Polish space (Y_1, d_1) satisfying the $c_{p,r}$-LDP with the good rate function I. Let F be a continuous map from (Y_1, d_1) to another Polish space (Y_2, d_2). Then the family $\{F \circ X^\varepsilon : \varepsilon > 0\}$ of $c_{p,r}$-quasi-surely defined maps satisfies the $c_{p,r}$-LDP with the good rate function*

$$J(z) = \inf_{y : F(y) = z} I(y),$$

where $\inf \emptyset = \infty$.

The natural modification of exponential tightness is formulated as the following.

Definition 2.5.4. *Let $\left\{X^{\varepsilon,(m)} : \varepsilon > 0\right\}$ (where $m = 1, 2, \cdots$) and $\{X^\varepsilon : \varepsilon > 0\}$ be two families of defined mappings from $\boldsymbol{W}$ to (Y, d). Then $\left\{X^{\varepsilon,(m)} : \varepsilon > 0\right\}$ is said to be a family of exponentially good approximations of $\{X^\varepsilon : \varepsilon > 0\}$ under (p, r)-capacity* *if for all* $\lambda > 0$,

$$\lim_{m\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 \log c_{p,r} \left\{\omega : d\left(X^{\varepsilon,(m)}(\omega), X^{\varepsilon}(\omega)\right) > \lambda\right\} = -\infty. \tag{2.14}$$

Proposition 2.5.5. *Suppose that for each* $m = 1, 2, \cdots$, *the family* $\left\{X^{\varepsilon,(m)} : \varepsilon > 0\right\}$, *consisting of* $c_{p,r}$*-quasi-surely defined mappings from* $\boldsymbol{W}$ *to a Polish space* (Y, d), *satisfies* $c_{p,r}$*-LDP with the good rate function* I_m, *and* $\left\{X^{\varepsilon,(m)} : \varepsilon > 0\right\}$ *are exponentially good approximations of* $c_{p,r}$*-quasi-surely defined mappings* $\{X^{\varepsilon} : \varepsilon > 0\}$. *Define the function*

$$J(y) = \sup_{\lambda>0} \liminf_{m\to\infty} \inf_{z\in B(y,\lambda)} J_m(z), \quad \forall y \in Y, \tag{2.15}$$

where $B(y, \lambda)$ *denotes the open ball in* (Y, d) *with centre* y *and radius* λ. *If* J *is a good rate function and for every closed subset* $C \subset Y$,

$$\inf_{y\in C} J(y) \leq \limsup_{m\to\infty} \inf_{y\in C} J_m(y), \tag{2.16}$$

then the family $\{X^{\varepsilon} : \varepsilon > 0\}$ *satisfies* $c_{p,r}$*-LDP with the good rate function* J.

Chapter 3

Fine properties of fractional Brownian motions under probability measure

In this chapter, we study the law of the iterated logarithm (LIL for short) and self-avoiding properties of fBMs in the context of probability. We prove the result on the LIL for fBMs following the argument in [8], and then prove the absence of double points using the classical argument due to Kakutani [32]. In this chapter, we consider fBMs on a general probability space $(\Omega, \mathscr{G}, \mathbb{P})$.

3.1 Law of the iterated logarithm for fBMs

The law of the iterated logarithm (LIL) was developed to describe the oscillations of stochastic processes near time zero and the behaviour of processes when time tends to infinity. The law of the iterated logarithm of random walks was firstly considered by Khintchine [34] in 1920s. Let $\{X_n\}_{n\in\mathbb{N}}$ be a family of independent identically distributed random variables with mean μ and variance σ^2, and let $S_n = \sum_{k=1}^{n} X_k$.

Then it holds that

$$\limsup_{n\to\infty} \frac{S_n - n\mu}{\sigma\sqrt{2n\log\log n}} = 1, \quad \text{a.s.}$$

and

$$\liminf_{n\to\infty} \frac{S_n - n\mu}{\sigma\sqrt{2n\log\log n}} = -1, \quad \text{a.s.}$$

In 1929, Kolmogorov [36] formulated a similar result for a family of independent random variables, but not necessarily having the same distribution: let $\{X_n\}_{n\in\mathbb{N}}$ be a family of such random variables and $S_n = \sum_{k=1}^{n} X_k$. If

$$s_n^2 = \mathrm{Var}(S_n) \to \infty$$

as $n \to \infty$, and

$$|X_n| \le \frac{\epsilon_n s_n}{\sqrt{\log\log s_n^2}}, \quad \text{a.s.}$$

for some sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ satisfying $\epsilon_n \to 0$ as $n \to \infty$, then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2\log\log s_n^2}} = 1, \quad \text{a.s.}$$

In 1933, Khintchine [35] extended the LIL to the case of standard Brownian motions $(W_t)_{t\ge 0}$. He proved that

$$\limsup_{t\downarrow 0} \frac{W_t}{\sqrt{2t\log\log(1/t)}} = 1, \quad \text{a.s.}$$

and by the scaling property and symmetry of Brownian motions, it follows that

$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t\log\log t}} = 1, \quad \text{a.s.},$$

$$\liminf_{t\downarrow 0} \frac{W_t}{\sqrt{2t\log\log(1/t)}} = -1, \quad \text{a.s.},$$

$$\liminf_{t\to\infty} \frac{W_t}{\sqrt{2t\log\log t}} = -1, \quad \text{a.s.}$$

See Theorem 9.23, Section 2.9, Chapter 2, [33] for a proof of this result.

In 1964, Strassen derived a functional version of the LIL for standard Brownian motions in his work [64]: let $\boldsymbol{W} = C_0([0,1])$, the space of all continuous functions on $[0,1]$ starting from the origin, and let P be the Wiener measure on $(\boldsymbol{W}, \mathscr{B}(\boldsymbol{W}))$ as defined in Chapter 2. Define the Strassen set K to be

$$K = \left\{ \omega \in \boldsymbol{W} : \omega \text{ is absolutely continuous, } \int_0^1 |\omega'(t)|^2 dt \le 1 \right\},$$

and let

$$\omega_n(t) = \frac{\omega(nt)}{\sqrt{2n\log\log n}}, \quad \forall n \ge 3,$$

then for almost all $\omega \in \boldsymbol{W}$, the family $\{\omega_n : n \ge 3\}$ is relatively compact in the topology of uniform convergence and K is the set of limit points of ω_n as $n \to \infty$.

In particular, the classical LIL for Brownian motions is a direct consequence of the functional version of LIL. Consider the continuous functional $\phi : \boldsymbol{W} \to \mathbb{R}$ given by $\phi(\omega) = \omega(1)$, then

$$\limsup_{n\to\infty} \phi(\omega_n) = \limsup_{n\to\infty} \omega_n(1) = \sup_{\omega\in K} \omega(1),$$

which yields that

$$\limsup_{n\to\infty} \frac{\omega(n)}{\sqrt{2n\log\log n}} = 1, \quad \text{a.s.}$$

as $\sup_{\omega\in K} \omega(1) = 1$. Moreover, one may set $\phi(\omega) = \sup_{s\in[0,1]} \omega(s)$ to obtain that

$$\limsup_{t\to\infty} \sup_{s\in[0,t]} \frac{\omega(s)}{\sqrt{2t\log\log t}} = 1, \quad \text{a.s.}$$

This result may be deduced from the large deviation principle theory for Brownian motions, see e.g. Theorem 1.17, [65] for further details.

In [51], Marcus established a similar result for real-valued centred Gaussian processes $(X_t)_{t\geq 0}$ with stationary increments under the condition that the variance function $\sigma(t) := \mathbb{E}[X_t^2]$ is concave and 2α-Hölder continuous with some $\alpha > 0$. We will use the method from his work to prove the LIL for fractional Brownian motions. In 1970s, Oodaira [59] generalised the result of Strassen and proved a functional version of LIL for centred Gaussian processes under some conditions imposed on covariance functions.

Now we will show the following result for fBM following the argument in Theorem 3.2.4, Chapter 3, [8]:

Proposition 3.1.1 (Theorem 3.2.4, [8])**.** *Let $(B_t)_{t\geq 0}$ be an fBM of dimension one with Hurst parameter $H \in (0,1)$. Then*

$$\mathbb{P}\left(\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t^{2H}\log\log(1/t)}} \leq 1\right) = 1. \tag{3.1}$$

Proof. Actually the upper bound of the iterated logarithm law has been established for general Gaussian processes under technical conditions on the covariance function, and the proof follows exactly the same arguments as for Brownian motions, see Theorem 9.23, Section 2.9, Chapter 2, [33]. Our proof is a modification of the approach used in [33].

Let us first introduce the following inequality, which is similar to Borell inequality. This inequality is due to Marcus and Shepp (see Theoreom 2.4 and Theorem 2.5 in [52]), and here we use a more friendly form in [62] (see (1.1)), that is, for every $\epsilon > 0$, there exists a constant $k(\epsilon) > 0$ such that

$$\mathbb{P}\left(\sup_{s\in(0,t)} B_s > \lambda\right) \leq k(\epsilon)e^{-\frac{(1-\epsilon)\lambda^2}{2t^{2H}}}, \qquad \forall t > 0. \tag{3.2}$$

For convenience, write $h(t) = \sqrt{2t^{2H} \log\log(1/t)}$, and consider the following event:

$$A_n^{\theta,\delta} := \left\{ \omega : \sup_{s\in(0,\theta^n)} B_s(\omega) > (1+\delta)h(\theta^n) \right\},$$

where $\theta \in (0,1)$, $\delta > 0$, and $n = 1, 2, \cdots$. The idea is to apply the first Borel-Cantelli lemma to $\{A_n^{\theta,\delta}\}_{n\in\mathbb{N}}$ for every given pair (θ, δ). By (3.2), we have

$$\mathbb{P}\left(A_n^{\theta,\delta}\right) \le k(\epsilon) e^{-\frac{(1-\epsilon)(1+\delta)^2 h^2(\theta^n)}{2\theta^{2nH}}} = k(\epsilon)\left(\frac{1}{n(\log(1/\theta))}\right)^{(1-\epsilon)(1+\delta)^2}.$$

Take ϵ small enough such that $\epsilon \in \left(0, 1 - \frac{1}{(1+\delta)^2}\right)$. Then as $(1-\epsilon)(1+\delta)^2 > 1$, it follows that

$$\sum_n \mathbb{P}\left(A_n^{\theta,\delta}\right) \le k(\epsilon) \sum_n \left(\frac{1}{n(\log(1/\theta))}\right)^{(1-\epsilon)(1+\delta)^2} < \infty,$$

which implies that, by the first Borel-Cantelli lemma,

$$\mathbb{P}\left(A_n^{\theta,\delta} \text{ i.o. }\right) = 0.$$

Let

$$A_{\theta,\delta} = \left\{A_n^{\theta,\delta} \text{ i.o. }\right\}^c.$$

Then $\mathbb{P}(A_{\theta,\delta}) = 1$ and for every $\omega \in A_{\theta,\delta}$,

$$\sup_{s\in(0,\theta^n)} B_s(\omega) \le (1+\delta)h(\theta^n) \tag{3.3}$$

for large n, by definition. Now for any $t \in (\theta^n, \theta^{n-1})$ and $\omega \in A_{\theta,\delta}$,

$$\begin{aligned} B_t(\omega) &\le \sup_{s\in(0,\theta^{n-1})} B_s(\omega) \\ &\le (1+\delta)\sqrt{2\theta^{2(n-1)H} \log\log(\theta^{1-n})} \end{aligned}$$

$$\leq \theta^{-H}(1+\delta)\sqrt{2\theta^{2nH}\log\log(\theta^{-n})}$$

$$= \theta^{-H}(1+\delta)h(\theta^n)$$

$$\leq \theta^{-H}(1+\delta)h(t),$$

where the last inequality follows from the fact that $h(t)$ is increasing for small t.

Therefore, for all $\omega \in A_{\theta,\delta}$, and every large n,

$$\sup_{t\in(\theta^n,\theta^{n-1})} \frac{B_t(\omega)}{h(t)} \leq \theta^{-H}(1+\delta),$$

so by letting $n \to \infty$, we obtain that

$$\limsup_{t\downarrow 0} \frac{B_t(\omega)}{h(t)} \leq \theta^{-H}(1+\delta), \qquad \forall \omega \in A_{\theta,\delta}.$$

As $A_{\theta_1,\delta_1} \subset A_{\theta_2,\delta_2}$ when $\delta_1 \leq \delta_2$, $\theta_1 \geq \theta_2$, we may take two sequences rationals $\{\delta_n\}$ and $\{\theta_n\}$ such that $\delta_n \downarrow 0$ and $\theta_n \uparrow 1$ as n tends to infinity, then $A = \lim_{n\to\infty} A_{\theta_n,\delta_n}$ has probability one, and

$$\limsup_{t\downarrow 0} \frac{B_t(\omega)}{h(t)} \leq 1$$

for all $\omega \in A$, which completes the proof of upper bound. □

The lower bound of the LIL for fBMs seems different from the upper bound, which is true for any H. In fact the lower bound was also proved for a general Gaussian process under some conditions on variance function (see [51]), which in turn requires that $H \leq 1/2$.

In order to show the lower bound of the LIL for fBM, we need a generalised version of the second Borel-Cantelli Lemma proved in [51], which weakens the independence condition in the classical second Borel-Cantelli lemma.

The proof of this generalised second Borel-Cantelli Lemma relies on one simple observation by Chung and Erdős [7], and it states as follows:

Lemma 3.1.1 ([7]). *Let* $\{B_k\}_{k=1}^n$ *be a sequence of events on a probability space* $(\Omega, \mathscr{G}, \mathbb{P})$. *If* $\mathbb{P}\left(\bigcup_{k=1}^n B_k\right) > 0$, *then*

$$2\sum_{1\le j<k\le n} \mathbb{P}\left(B_j \cap B_k\right) \ge \left(\mathbb{P}\left(\bigcup_{k=1}^n B_k\right)\right)^{-1} \left(\sum_{k=1}^n \mathbb{P}\left(B_k\right)\right)^2 - \sum_{k=1}^n \mathbb{P}\left(B_k\right).$$

Proof. This result follows from Cauchy-Schwarz inequality. Notice that

$$\mathbb{E}\left[\left(\sum_{k=1}^n \mathbb{1}_{B_k}\right)^2\right] \mathbb{E}\left[\mathbb{1}^2_{\cup_{k=1}^n B_k}\right] \ge \left(\mathbb{E}\left[\left(\sum_{k=1}^n \mathbb{1}_{B_k}\right) \mathbb{1}_{\cup_{k=1}^n B_k}\right]\right)^2,$$

and

$$\left(\sum_{k=1}^n \mathbb{1}_{B_k}\right) \mathbb{1}_{\cup_{k=1}^n B_k} = \sum_{k=1}^n \mathbb{1}_{B_k},$$

we deduce that

$$\begin{aligned}
2\sum_{1\le j<k\le n} \mathbb{P}\left(B_j \cap B_k\right) &= 2\mathbb{E}\left[\sum_{1\le j<k\le n} \mathbb{1}_{B_j}\mathbb{1}_{B_k}\right] \\
&= \mathbb{E}\left[\left(\sum_{k=1}^n \mathbb{1}_{B_k}\right)^2\right] - \sum_{k=1}^n \mathbb{E}\left[\mathbb{1}^2_{B_k}\right] \\
&\ge \left(\mathbb{E}\left[\mathbb{1}^2_{\cup_{k=1}^n B_k}\right]\right)^{-1} \left(\mathbb{E}\left[\left(\sum_{k=1}^n \mathbb{1}_{B_k}\right) \mathbb{1}_{\cup_{k=1}^n B_k}\right]\right)^2 - \sum_{k=1}^n \mathbb{P}\left(B_k\right) \\
&= \left(\mathbb{P}\left(\bigcup_{k=1}^n B_k\right)\right)^{-1} \left(\sum_{k=1}^n \mathbb{P}\left(B_k\right)\right)^2 - \sum_{k=1}^n \mathbb{P}\left(B_k\right).
\end{aligned}$$

□

Lemma 3.1.2 (Lemma 1, [51]). *Let* $\{B_n\}_{n\in\mathbb{N}}$ *be a sequence of sets such that*

$$\mathbb{P}\left(B_n \cap B_m\right) \le \mathbb{P}\left(B_n\right)\mathbb{P}\left(B_m\right)$$

for all $m > n$. If $\sum_n \mathbb{P}(B_n) = \infty$, then

$$\mathbb{P}(B_n \text{ i.o.}) = 1.$$

Proof. With assumption that $\mathbb{P}(B_n \cap B_m) \le \mathbb{P}(B_n)\mathbb{P}(B_m)$ for all $m > n$, we have

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{k=1}^{n} B_k\right) &\ge \frac{\left(\sum_{k=1}^{n} \mathbb{P}(B_k)\right)^2}{\sum_{k=1}^{n} \mathbb{P}(B_k) + 2\sum_{1\le j<k\le n} \mathbb{P}(B_j \cap B_k)} \\
&\ge \frac{\left(\sum_{k=1}^{n} \mathbb{P}(B_k)\right)^2}{\sum_{k=1}^{n} \mathbb{P}(B_k) + 2\sum_{1\le j<k\le n} \mathbb{P}(B_j)\mathbb{P}(B_k)} \\
&\ge \frac{\left(\sum_{k=1}^{n} \mathbb{P}(B_k)\right)^2}{\sum_{k=1}^{n} \mathbb{P}(B_k) + \left(\sum_{k=1}^{n} \mathbb{P}(B_k)\right)^2} \\
&= 1 - \frac{1}{1 + \sum_{k=1}^{n} \mathbb{P}(B_k)},
\end{aligned}$$

and now we may conclude Lemma 3.1.2 by applying the monotone convergence theorem. □

We also use the following criterion provided in Lemma 2 of [51]:

Lemma 3.1.3 (Lemma 2, [51])**.** *Let X and Y be jointly normal distributed random variables with mean value zero and $\mathbb{E}[XY] \le 0$. Then*

$$\mathbb{P}(X \ge a, Y \ge b) \le \mathbb{P}(X \ge a)\mathbb{P}(Y \ge b)$$

with $a, b \ge 0$.

Proof. Let Z be a random variable independent of X such that $Y = Z - cX$, where

$$c = -\frac{\mathbb{E}[XY]}{\mathbb{E}[X^2]} \ge 0,$$

and thus

$$\mathbb{E}[Y^2] = \mathbb{E}[Z^2] + c^2\mathbb{E}[X^2] \ge \mathbb{E}[Z^2].$$

Therefore, as X, Y, Z are all centred Gaussian random variables, we have that

$$\begin{aligned}\mathbb{P}(X \geq a, Y \geq b) &\leq \mathbb{P}(X \geq a, Z \geq b + ac) \\ &= \mathbb{P}(X \geq a)\,\mathbb{P}(Z \geq b + ac) \\ &\leq \mathbb{P}(X \geq a)\,\mathbb{P}(Z \geq b) \\ &\leq \mathbb{P}(X \geq a)\,\mathbb{P}(Y \geq b).\end{aligned}$$

□

Now we are ready to prove the other half of the LIL for fBMs with the restriction that Hurst parameter $H \leq \frac{1}{2}$.

Proposition 3.1.2 (Theorem 3.2.4, [8]). *Let $B = (B_t)_{t\geq 0}$ be an fBM of dimension one with Hurst parameter $H \in \left(0, \frac{1}{2}\right]$, then*

$$\mathbb{P}\left(\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t^{2H}\log\log(1/t)}} \geq 1\right) = 1. \tag{3.4}$$

Proof. Let $\theta \in (0,1)$ and define

$$G_n^\theta = \left\{\omega : B_{\theta^n}(\omega) - B_{\theta^{n+1}}(\omega) \geq (1-\theta)^H h(\theta^n)\right\},$$

where

$$h(t) = \sqrt{2t^{2H}\log\log(1/t)}.$$

Set

$$X_n^\theta = \frac{B_{\theta^n} - B_{\theta^{n+1}}}{(\theta^n - \theta^{n+1})^H}$$

for $n = 1, 2, \cdots$, so that X_n^θ is a standard Gaussian random variable. Then we have that

$$
\begin{aligned}
\mathbb{P}\left(G_n^\theta\right) &= \mathbb{P}\left(X_n^\theta \geq \sqrt{2\log\log(\theta^{-n})}\right) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2\log\log(\theta^{-n})}}^{\infty} e^{-\frac{u^2}{2}} \mathrm{d}u \\
&\geq \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2\log\log(\theta^{-n})}}{1+2\log\log(\theta^{-n})} e^{-\log\log(\theta^{-n})} \\
&\geq \frac{C}{n\sqrt{\log n}},
\end{aligned}
$$

where $C = C(\theta)$ is some constant depending on θ. It follows that

$$
\sum_n \mathbb{P}\left(G_n^\theta\right) = \infty.
$$

As $\mathbb{E}\left[X_n^\theta X_m^\theta\right] < 0$ for all $m > n$ when $H \in \left(0, \frac{1}{2}\right]$, according to Lemma 3.1.2, we have

$$
\mathbb{P}\left(G_n^\theta \cap G_m^\theta\right) \leq \mathbb{P}\left(G_n^\theta\right) \mathbb{P}\left(G_m^\theta\right).
$$

Now we may deduce that there exists G_θ with $\mathbb{P}\left(G_\theta\right) = 1$ such that for all $\omega \in G_\theta$, there are infinitely many n such that

$$
B_{\theta^n}(\omega) - B_{\theta^{n+1}}(\omega) \geq (1-\theta)^H h(\theta^n).
$$

Notice that by symmetry, $-B$ is also an fBM, and hence by (3.3) and taking $\delta = 1$, we have

$$
\begin{aligned}
-B_{\theta^{n+1}} &\leq (1+\delta) h(\theta^{n+1}) \\
&= 2\theta^H \sqrt{2\theta^{2nH} \log\log(\theta^{-n-1})} \\
&\leq 2\theta^H h(\theta^n), \qquad \text{a.s.}
\end{aligned}
$$

for large n. Therefore, for any sufficiently large N, there always exists some $n > N$ such that

$$
\frac{B_{\theta^n}}{h(\theta^n)} \geq (1-\theta)^H - 2\theta^H, \quad \text{a.s.}
$$

By letting $n \to \infty$, we have

$$\limsup_{t \downarrow 0} \frac{B_t}{h(t)} \geq (1-\theta)^H - 2\theta^H, \quad \text{a.s.}$$

Letting $\theta \downarrow 0$ via a rational sequence, the required result follows for the case when $H \in \left(0, \frac{1}{2}\right]$. □

We note here that the proof for the case when $H > \frac{1}{2}$ requires a different approach, and indeed it follows directly from the functional version of LIL for Gaussian processes as in Theorem 1.3, [2].

3.2 Self-intersection of fBMs

One interesting question concerning the sample paths of fBMs is whether an fBM path intersects itself or not. This question arose from statistical field theory when studying interacting fields. It dates back to 1944, when Kakutani [32] first proved that Brownian motion is self-avoiding almost surely when $n \geq 5$, where n is the dimension of Brownian motion. His proof based on a rather simple geometric observation. In [21], Fukushima showed that when the dimension $n \geq 7$, sample paths of Brownian motion are non-self-intersecting $c_{2,1}$-quasi-surely. Later, Takeda [67] extended Fukushima's result to the setting of Mallivin capacities for all r and p, and showed the non-self-intersecting property of n-dimensional Brownian motion paths with (p, r)-capacity when $n > 4 + rp$. In a recent work by H. Boedihardjo *et al.* [4], it was proved that the self-avoiding property also holds for signature paths of Brownian motion.

We shall adopt the method by Kakutani in [32] and establish a similar result for n-dimensional fBMs $(B_t)_{t \geq 0}$ with Hurst parameter H. However, the approach in literature to determine the optimal dimension of this property mainly relies on the

Markov property of underlying processes, and requires the potential theory techniques which are not applicable to non-Markovian processes such as fBMs.

Let $(B_t)_{t\geq 0}$ be an n-dimensional fBM with Hurst parameter $H \in (0,1)$ on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$. A point $x \in \mathbb{R}^d$ is called a double point if there exist two distinct time points s and t such that $B_s = B_t = x$.

Proposition 3.2.1. *Let $(B_t)_{t\geq 0}$ be an n-dimensional fBM with Hurst parameter H. Then B has no double point almost surely if $\frac{2}{n} < H$.*

Proof. Firstly, we notice that for a 1-dimensional fBM, as it has stationary increments, we have

$$\mathbb{P}\left(a < B_t - B_s < b\right) = \frac{1}{\sqrt{2\pi(t-s)^{2H}}} \int_a^b \exp\left(-\frac{x^2}{2(t-s)^{2H}}\right) dx$$

for any $s < t$ and $a < b$, and using the inequality (3.2) from [62] again, we have that for all $\epsilon \in (0,1)$, and $\eta > 0$,

$$\mathbb{P}\left(\sup_{s\in[t_0,t_1]} |B_s - B_{t_0}| > \eta\right) \leq 2k(\epsilon) \exp\left(-\frac{(1-\epsilon)\eta^2}{2(t_1-t_0)^{2H}}\right),$$

where $k(\epsilon) > 0$ is some constant depending only on ϵ. Therefore, to show an n-dimensional fBM $B_t = (B_t^1, \cdots, B_t^n)$ has no double point almost surely, we only need to prove that for any two disjoint time intervals $I = (s_0, s_1)$ and $J = (t_0, t_1)$ with $s_0 < s_1 < t_0 < t_1$,

$$\mathbb{P}\left(B_s = B_t \text{ for some } s \in I, t \in J\right) = 0.$$

Denote

$$A = \left\{\omega : B_s(\omega) = B_t(\omega) \text{ for some } s \in I, t \in J\right\},$$

then it holds that for any $\eta > 0$,

$$
\begin{aligned}
A \subset \bigcap_{i=1}^{n} \{|B^i_{s_1} - B^i_{t_0}| < 2\eta\} \cup \bigcup_{i=1}^{n} \left\{ \sup_{s\in I} |B^i_s - B^i_{s_1}| > \eta \right\} \\
\cup \bigcup_{i=1}^{n} \left\{ \sup_{t\in J} |B^i_t - B^i_{t_0}| > \eta \right\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathbb{P}(A) \leq & \prod_{i=1}^{n} \mathbb{P}\left(|B^i_{s_1} - B^i_{t_0}| < 2\eta\right) + \sum_{i=1}^{n} \mathbb{P}\left(\sup_{s\in I} |B^i_s - B^i_{s_1}| > \eta \right) \\
& + \sum_{i=1}^{n} \mathbb{P}\left(\sup_{t\in J} |B^i_t - B^i_{t_0}| > \eta \right) \\
\leq & \left(\frac{1}{\sqrt{2\pi (t_0 - s_1)^{2H}}} \int_{-2\eta}^{2\eta} \exp\left(-\frac{x^2}{2(t_0 - s_1)^{2H}} \right) \mathrm{d}x \right)^n \\
& + 2nk(\epsilon) \exp\left(-\frac{(1-\epsilon)\eta^2}{2|I|^{2H}} \right) + 2nk(\epsilon) \exp\left(-\frac{(1-\epsilon)\eta^2}{2|J|^{2H}} \right) \\
\leq & \left(\frac{4\eta}{\sqrt{2\pi (d(I,J))^{2H}}} \right)^n + 2nk(\epsilon) \exp\left(-\frac{(1-\epsilon)\eta^2}{2|I|^{2H}} \right) \\
& + 2nk(\epsilon) \exp\left(-\frac{(1-\epsilon)\eta^2}{2|J|^{2H}} \right).
\end{aligned}
$$

Divide I and J evenly into p subintervals, i.e. $I = \bigcup_{m=1}^{p} I_m$, and $J = \bigcup_{l=1}^{p} J_l$ with I_m and J_l disjoint, $|I_m| = |I|/p$ and $|J_l| = |J|/p$, then

$$
\begin{aligned}
\mathbb{P}(A) \leq & \sum_{m=1}^{p} \sum_{l=1}^{p} \mathbb{P}\left(B_s = B_t \text{ for some } s \in I_m, t \in J_l \right) \\
\leq & \sum_{m=1}^{p} \sum_{l=1}^{p} \left[\left(\frac{4\eta}{\sqrt{2\pi (d(I_m,J_l))^{2H}}} \right)^n + 2nk(\epsilon) \exp\left(-\frac{(1-\epsilon)\eta^2}{2|I_m|^{2H}} \right) \right. \\
& \left. + 2nk(\epsilon) \exp\left(-\frac{(1-\epsilon)\eta^2}{2|J_l|^{2H}} \right) \right] \\
\leq & \, p^2 \left[\left(\frac{4\eta}{\sqrt{2\pi (d(I,J))^{2H}}} \right)^n + 2nk(\epsilon) \exp\left(-\frac{(1-\epsilon)\eta^2 p^{2H}}{2|I|^{2H}} \right) \right. \\
& \left. + 2nk(\epsilon) \exp\left(-\frac{(1-\epsilon)\eta^2 p^{2H}}{2|J|^{2H}} \right) \right].
\end{aligned}
$$

Now set $\eta = p^{-\sigma}$, where σ satisfies

$$\frac{2}{n} < \sigma < H,$$

and hence the right-hand side of the above inequality converges to zero as p tends to infinity, which completes the proof of the required result. $\square$

Chapter 4

Sample path properties of fractional Brownian motions under the classical Wiener capacity

This chapter is organised as the following. In next section, we prove that fBMs defined by the Volterra integral representation are smooth in the sense of Malliavin, and compute their Malliavin derivatives explicitly, followed by several useful technical lemmas, which are similar to the estimates made by Fukushima in [21] and Takeda in [67].

In the third section, we establish the result on the modulus of continuity following the argument by Fukushima [21], and as a direct corollary, we deduce the quasi-sure Hölder continuity of fBMs on the classical Wiener space. This allows us to take continuous modifications of fBMs and prove the capacity version of non-differentiability for fBM sample paths based on the argument by Dvoretzky, Erdős and Kakutani in [17], as well as the law of the iterated logarithm (LIL) when $p = 2$ and $r = 1$ with the restriction $H \leq \frac{1}{2}$.

Finally, we prove the self-avoiding property of d-dimensional fBMs under $c_{2,1}$ when

$d > \frac{2}{H} + 2$ and $H \leq \frac{1}{2}$.

In this chapter, we consider fBM paths over $[0, \infty)$, and the classical Wiener space is taken to be $(\boldsymbol{W}, \mathcal{H}, P)$, where $\boldsymbol{W} = C_0([0, \infty))$, equipped with the norm

$$\|\omega\| = \sum_{n=1}^{\infty} 2^{-n} \max_{0 \leq t \leq n} |\omega(t)| \,, \quad \forall \omega \in \boldsymbol{W}.$$

4.1 FBMs as smooth random variables

In the first section of this chapter, we prove that fBMs given by the Volterra integral representation (2.11), which are regarded as Wiener functionals, are Malliavin differentiable random variables. Indeed, for every $t \geq 0$, B_t defined by

$$B_t = \int_0^t K(t, s) \mathrm{d}\omega(s)$$

is smooth according to the definition given by Malliavin, that is, it belongs to the Sobolev space $\mathbb{D}_r^p$ for all $r \in \mathbb{N}$ and $p \in (1, \infty)$. In fact, the Malliavin derivative of B_t is a function on $\boldsymbol{W}$, which can be calculated as in the following lemma.

Lemma 4.1.1. *Let $H \in (0, 1)$, $r \in \mathbb{N}$ and $p \in (1, \infty)$. Then for every $t > 0$, $B_t \in \mathbb{D}_r^p$, and its first order Malliavin derivative is given by*

$$DB_t(s) = \int_0^{s \wedge t} K(t, u) \mathrm{d}u. \tag{4.1}$$

The higher-order derivatives of B_t all vanish (which reflects the fact that B_t is an integral of a deterministic function against a standard Brownian motion).

This lemma is a corollary to the transfer principle given in Proposition 5.2.1, Chapter V, [58]. We provide an elementary proof here, which is quite different from the one in [58], based on the proof of Proposition 3.1 in Decreusefond and Üstünel [10].

We shall use the following elementary estimate from Theorem 3.2 in [10], which states that for any $H \in (0,1)$, there exists a constant c_H, which only depends on the Hurst parameter H, such that the kernel K has the following bound:

$$K(t,r) \leq c_H r^{-\left|H-\frac{1}{2}\right|}(t-r)^{-\left(\frac{1}{2}-H\right)_+}\mathbb{1}_{[0,t]}(r) \tag{4.2}$$

for any $t > r \geq 0$, where $x_+ = \max(x,0)$. The proof of this upper bound relies on the properties of hypergeometric functions and will be omitted here.

Proof of Lemma 4.1.1:

Proof. For each fixed $t > 0$, denote

$$u_t(s) = K(t,s)\mathbb{1}_{[0,t]}(s)$$

for simplicity, and for each $n \in \mathbb{N}$, set

$$u_t^{(n)}(s) = \sum_{i=0}^{2^n-1} \frac{2^n}{t}\left(\int_{i2^{-n}t}^{(i+1)2^{-n}t} u_t(r)\mathrm{d}r\right)\mathbb{1}_{(i2^{-n}t,(i+1)2^{-n}t]}(s).$$

Then u_t and $u_t^{(n)}$, $n \in \mathbb{N}$, belong to $L^2([0,\infty))$. For convenience, let

$$F_i^{t,(n)} = \frac{2^n}{t}\left(\int_{i2^{-n}t}^{(i+1)2^{-n}t} u_t(r)\mathrm{d}r\right), \quad 0 \leq i \leq 2^n - 1.$$

We want to apply the dominated convergence theorem to show that for each $t > 0$, $u_t^{(n)} \to u_t$ in $L^2([0,\infty))$ as n tends to infinity. Our first step is to find a control function of $(u_t^{(n)})_{n\in\mathbb{N}}$ in $L^2([0,\infty))$.

Notice that for each n, $u_t^{(n)}(s)$ vanishes outside of $(0,t]$, and by definition, it is a step function inside $(0,t]$, so we only need to check that within each "step", i.e. when $s \in (i2^{-n}t,(i+1)2^{-n}t]$, $0 \leq i \leq 2^{n-1}$, $u_t^{(n)}(s)$ is controlled by some square-integrable function, and this control function should be uniform in n.

When $H > \frac{1}{2}$, for each $s \in (i2^{-n}t, (i+1)2^{-n}t]$, $0 \le i \le 2^{n-1}$, by the estimate in (4.2),

$$\begin{aligned}
\left|u_t^{(n)}(s)\right| &= \frac{2^n}{t}\int_{i2^{-n}t}^{(i+1)2^{-n}t} K(t,r)\mathrm{d}r \\
&\le \frac{2^n}{t}\int_{i2^{-n}t}^{(i+1)2^{-n}t} c_H r^{\frac{1}{2}-H}\mathrm{d}r \\
&= c'_H\left(\frac{t}{2^n}\right)^{\frac{1}{2}-H}\left[(i+1)^{\frac{3}{2}-H} - i^{\frac{3}{2}-H}\right] \\
&\le c'_H t^{\frac{1}{2}-H}\left(2^n\right)^{H-\frac{3}{2}}\left[(i+1)\left(\frac{i+1}{2^n}\right)^{\frac{1}{2}-H} - i\left(\frac{i}{2^n}\right)^{\frac{1}{2}-H}\right] \\
&\le c'_H t^{\frac{1}{2}-H}\left[(i+1)\left(\frac{i+1}{2^n}\right)^{\frac{1}{2}-H} - i\left(\frac{i+1}{2^n}\right)^{\frac{1}{2}-H}\right] \\
&= c'_H t^{\frac{1}{2}-H}\left(\frac{i+1}{2^n}\right)^{\frac{1}{2}-H} \\
&\le c'_H t^{\frac{1}{2}-H} s^{\frac{1}{2}-H},
\end{aligned}$$

where

$$c'_H = \frac{c_H}{\left(\frac{3}{2}-H\right)}.$$

This implies that when $H > \frac{1}{2}$, we may take the control function to be

$$c'_H t^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\mathbb{1}_{(0,t]}(s),$$

which is independent of n.

When $H < \frac{1}{2}$, similar to above, we have that by (4.2),

$$\begin{aligned}
\left|u_t^{(n)}(s)\right| &= \frac{2^n}{t}\int_{i2^{-n}t}^{(i+1)2^{-n}t} K(t,r)\mathrm{d}r \\
&\le c_H\frac{2^n}{t}\int_{s}^{(i+1)2^{-n}t} r^{H-\frac{1}{2}}(t-r)^{H-\frac{1}{2}}\mathrm{d}r + c_H\frac{2^n}{t}\int_{i2^{-n}t}^{s} r^{H-\frac{1}{2}}(t-r)^{H-\frac{1}{2}}\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&\leq c_H \frac{2^n}{t} s^{H-\frac{1}{2}} \int_{i2^{-n}t}^{(i+1)2^{-n}t} (t-r)^{H-\frac{1}{2}} \mathrm{d}r + c_H \frac{2^n}{t} (t-s)^{H-\frac{1}{2}} \int_{i2^{-n}t}^{(i+1)2^{-n}t} r^{H-\frac{1}{2}} \mathrm{d}r \\
&= c_H'' \frac{2^n}{t} s^{H-\frac{1}{2}} \left[\left(t - \frac{i}{2^n}t\right)^{H+\frac{1}{2}} - \left(t - \frac{i+1}{2^n}t\right)^{H+\frac{1}{2}} \right] \\
&\quad + c_H'' \frac{2^n}{t} (t-s)^{H-\frac{1}{2}} \left[\left(\frac{i+1}{2^n}t\right)^{H+\frac{1}{2}} - \left(\frac{i}{2^n}t\right)^{H+\frac{1}{2}} \right] \\
&\leq c_H'' \frac{2^n}{t} s^{H-\frac{1}{2}} \left[\left(t - \frac{i}{2^n}t\right)\left(t - \frac{i}{2^n}t\right)^{H-\frac{1}{2}} - \left(t - \frac{i+1}{2^n}t\right)\left(t - \frac{i}{2^n}t\right)^{H-\frac{1}{2}} \right] \\
&\quad + c_H'' \frac{2^n}{t} (t-s)^{H-\frac{1}{2}} \left[\left(\frac{i+1}{2^n}t\right)\left(\frac{i+1}{2^n}t\right)^{H-\frac{1}{2}} - \left(\frac{i}{2^n}t\right)\left(\frac{i+1}{2^n}t\right)^{H-\frac{1}{2}} \right] \\
&= c_H'' s^{H-\frac{1}{2}} \left(t - \frac{i}{2^n}t\right)^{H-\frac{1}{2}} + c_H''(t-s)^{H-\frac{1}{2}} \left(\frac{i+1}{2^n}t\right)^{H-\frac{1}{2}} \\
&\leq 2c_H'' s^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}},
\end{aligned}
$$

where

$$
c_H'' = \frac{c_H}{H + \frac{1}{2}}.
$$

Therefore, when $H < \frac{1}{2}$, the control function is

$$
2c_H'' s^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \mathbb{1}_{(0,t]}(s),
$$

which is an element of $L^2([0,\infty))$.

On the other hand, for every $s \in (0,t)$, there exists some i such that

$$
u_t^{(n)}(s) = \frac{\int_0^{(i+1)2^{-n}t} u_t(r)\mathrm{d}r - \int_0^{i2^{-n}t} u_t(r)\mathrm{d}r}{2^{-n}t}
$$

which converges to $u_t(s)$ pointwise as n tends to infinity due to the continuity of $u_t(s)$ over $(0,t)$. Now we may apply the dominated convergence theorem and conclude that $u_t^{(n)} \to u_t$ in $L^2([0,\infty))$ as $n \to \infty$.

For fixed $t \in [0,1]$, set

$$B_t^{(n)}(\omega) = \begin{cases} \sum_{i=0}^{2^n-1} F_i^{t,(n)} \left(\omega_{(i+1)2^{-n}t} - \omega_{i2^{-n}t}\right), & 0 < t \leq 1, \\ 0, & t = 0 \end{cases} \tag{4.3}$$

for all $\omega \in \boldsymbol{W}$. Let $\mathscr{G} = (\mathscr{G}_n)_{n\in\mathbb{N}}$, where

$$\mathscr{G}_n = \sigma\left(\omega_{i2^{-n}t}, 0 \leq i \leq 2^n\right)$$

is the σ-algebra generated by $\omega_{i2^{-n}t}$'s, $0 \leq i \leq 2^n$. Then $(B_t^{(n)})_{n\in\mathbb{N}}$ is a discrete martingale with respect to the filtration $\mathscr{G}$. This was observed by Decreusefond and Üstünel in [10].

The proof of this claim relies on the fact that for a standard Brownian motion ω_t and any $0 \leq t_0 < t_1 < \cdots < t_n$,

$$\mathbb{E}\left[\omega_{t_i} \middle| \omega_{t_0}, \omega_{t_1}, \cdots, \omega_{t_{i-1}}, \omega_{t_{i+1}}, \cdots, \omega_{t_n}\right] = \frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}}\omega_{t_{i-1}} + \frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}}\omega_{t_{i+1}}. \tag{4.4}$$

To verify (4.4), one only needs to spot that for each i and n,

$$X_i := \omega_{t_i} - \frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}}\omega_{t_{i-1}} - \frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}}\omega_{t_{i+1}}$$

is independent of $\sigma(\omega_{t_0}, \omega_{t_1}, \cdots, \omega_{t_{i-1}}, \omega_{t_{i+1}}, \cdots, \omega_{t_n})$. Indeed, for any $0 \leq j < i \leq n$,

$$\begin{aligned} \mathbb{E}\left[X_i \omega_{t_j}\right] &= \mathbb{E}\left[\omega_{t_i}\omega_{t_j}\right] - \frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}}\mathbb{E}\left[\omega_{t_{i-1}}\omega_{t_j}\right] - \frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}}\mathbb{E}\left[\omega_{t_{i+1}}\omega_{t_j}\right] \\ &= t_j - \frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}}t_j - \frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}}t_j \\ &= 0. \end{aligned}$$

As both X_i and ω_{t_j} are Gaussian, they are independent. One may verify X_i and ω_{t_j}

are independent via similar computation when $0 < i < j \leq n$. Thus ω_{t_i} is independent of all linear combinations of

$$\omega_{t_0}, \omega_{t_1}, \cdots, \omega_{t_{i-1}}, \omega_{t_{i+1}}, \cdots, \omega_{t_n}$$

and hence the σ-algebra

$$\mathscr{F}_i := \sigma(\omega_{t_0}, \omega_{t_1}, \cdots, \omega_{t_{i-1}}, \omega_{t_{i+1}}, \cdots, \omega_{t_n}).$$

Therefore, we get that

$$\begin{aligned}
&\mathbb{E}\left[\omega_{t_i} | \mathscr{F}_i\right] \\
&= \mathbb{E}\left[X_i + \frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}} \omega_{t_{i-1}} + \frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}} \omega_{t_{i+1}} | \mathscr{F}_i\right] \\
&= \frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}} \omega_{t_{i-1}} + \frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}} \omega_{t_{i+1}}.
\end{aligned}$$

For each $1 \leq i \leq 2^n - 1$, if i is odd, then we may write $i = 2k+1$, $0 \leq k \leq 2^{n-1} - 1$, and thus by (4.4),

$$\begin{aligned}
\mathbb{E}\left[\omega_{(i+1)2^{-n}t} - \omega_{i2^{-n}t} | \mathscr{G}_{n-1}\right] &= \mathbb{E}\left[\omega_{(k+1)2^{-n+1}t} - \omega_{(2k+1)2^{-n}t} | \mathscr{G}_{n-1}\right] \\
&= \frac{1}{2} \omega_{(k+1)2^{-n+1}t} - \frac{1}{2} \omega_{k2^{-n+1}t}.
\end{aligned}$$

If i is even, write $i = 2k$ for $0 \leq k \leq 2^{n-1} - 1$, then it holds that

$$\begin{aligned}
\mathbb{E}\left[\omega_{(i+1)2^{-n}t} - \omega_{i2^{-n}t} | \mathscr{G}_{n-1}\right] &= \mathbb{E}\left[\omega_{(2k+1)2^{-n}t} - \omega_{k2^{-n+1}t} | \mathscr{G}_{n-1}\right] \\
&= \frac{1}{2} \omega_{(k+1)2^{-n+1}t} - \frac{1}{2} \omega_{k2^{-n+1}t}.
\end{aligned}$$

Therefore, by the definition of $F_i^{t,(n)}$, we conclude that

$$\mathbb{E}\left[B_t^{(n)} \middle| \mathscr{G}_{n-1}\right] = \sum_{i=0}^{2^n - 1} F_i^{t,(n)} \mathbb{E}\left[\omega_{(i+1)2^{-n}t} - \omega_{i2^{-n}t} \middle| \mathscr{G}_{n-1}\right]$$

$$
\begin{aligned}
&= \sum_{k=0}^{2^{n-1}-1} F_{2k+1}^{t,(n)} \left(\frac{1}{2}\omega_{(k+1)2^{-n+1}t} - \frac{1}{2}\omega_{k2^{-n+1}t} \right) \\
&\quad + \sum_{k=0}^{2^{n-1}-1} F_{2k}^{t,(n)} \left(\frac{1}{2}\omega_{(k+1)2^{-n+1}t} - \frac{1}{2}\omega_{k2^{-n+1}t} \right) \\
&= \sum_{k=0}^{2^{n-1}-1} \frac{2^{-n+1}}{t} \left(\omega_{(k+1)2^{-n+1}t} - \omega_{k2^{-n+1}t} \right) \\
&\quad \cdot \left(\int_{k2^{-n+1}t}^{(2k+1)2^{-n}t} u_t(s)ds + \int_{(2k+1)2^{-n}t}^{(k+1)2^{-n+1}t} u_t(s)ds \right) \\
&= B_t^{(n-1)}.
\end{aligned}
$$

For $p \in (1, \infty)$, because the increments of ω_t over different time intervals are independent, and $B_t^{(n)}$ is contained in the first Wiener chaos, by (2.9) in Chapter 2 with $N = 1$, we have that

$$
\begin{aligned}
\left\| B_t^{(n)} \right\|_p &\leq 2\sqrt{p-1} \left\| B_t^{(n)} \right\|_2 \\
&= 2\sqrt{p-1} \left[\sum_{i=1}^{2^n-1} \left(\frac{2^n}{t} \right)^2 \left(\int_{(i-1)2^{-n}t}^{i2^{-n}t} u_t(s)\mathrm{d}s \right)^2 \mathbb{E}\left[\left(\omega_{(i+1)2^{-n}} - \omega_{i2^{-n}t} \right)^2 \right] \right]^{\frac{1}{2}} \\
&= 2\sqrt{p-1} \left[\sum_{i=1}^{2^n-1} \frac{2^n}{t} \left(\int_{(i-1)2^{-n}t}^{i2^{-n}t} u_t(s)\mathrm{d}s \right)^2 \right]^{\frac{1}{2}} \\
&\leq 2\sqrt{p-1} \left(\sum_{i=1}^{2^n-1} \int_{(i-1)2^{-n}t}^{i2^{-n}t} u_t^2(s)\mathrm{d}s \right)^{\frac{1}{2}} \\
&= 2\sqrt{p-1}t^H,
\end{aligned}
$$

where the right-hand side is a uniform bound for all n, and hence

$$
\sup_{n\in\mathbb{N}} \mathbb{E}[|B_t^{(n)}|^p] < \infty.
$$

It thus follows from the martingale convergence theorem that for each fixed t, the sequence $(B_t^{(n)})_{n\in\mathbb{N}}$ converges to B_t a.s. and in $L^p(\boldsymbol{W})$ as n tends to infinity. Furthermore, B_t is a Gaussian random variable with mean zero and covariance given by

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left[B_s^{(n)}B_t^{(n)}\right] &= \lim_{n\to\infty}\mathbb{E}\left[\left(\int_0^\infty u_t^{(n)}(r)\mathrm{d}\omega_r\right)\left(\int_0^\infty u_s^{(n)}(r)\mathrm{d}\omega_r\right)\right]\\ &= \lim_{n\to\infty}\int_0^\infty u_t^{(n)}(r)u_s^{(n)}(r)\mathrm{d}r\\ &= \int_0^{s\wedge t}K(t,r)K(s,r)\mathrm{d}r\\ &= R(s,t)\end{aligned}$$

for any $s,t>0$. In particular, the variance of B_t is given by

$$\lim_{n\to\infty}\mathbb{E}[|B_t^{(n)}|^2] = t^{2H}$$

for every $t>0$.

Now by the definition of Malliavin derivative, for $t>0$ and each $n\in\mathbb{N}$,

$$DB_t^{(n)}(s) = \int_0^s u_t^{(n)}(v)\mathrm{d}v,$$

and all higher-order derivatives of $B_t^{(n)}$ vanish.

We have already proved that

$$B_t^{(n)}\to B_t$$

in $L^p(\boldsymbol{W})$ and

$$u_t^{(n)}\to u_t$$

in $L^2([0,\infty))$, so for any $r\in\mathbb{N}$ and $p\in(1,\infty)$, as

$$\begin{aligned}\|B_t^{(n)}-B_t^{(m)}\|_{\mathbb{D}_r^p} &= \left(\mathbb{E}\left[|B_t^{(n)}-B_t^{(m)}|^p\right]+\mathbb{E}\left[\left|\|DB_t^{(n)}-DB_t^{(m)}\|_{\mathcal{H}}\right|^p\right]\right)^{1/p}\\ &= \left(\mathbb{E}\left[|B_t^{(n)}-B_t^{(m)}|^p\right]+\mathbb{E}\left[\left|\|u_t^{(n)}-u_t^{(m)}\|_{L^2([0,\infty))}\right|^p\right]\right)^{1/p},\end{aligned}$$

we obtain that $(B_t^{(n)})_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{D}_r^p$. By the completeness of $\mathbb{D}_r^p$, this sequence tends to a limit random variable in $\mathbb{D}_r^p$ as n tends to infinity. Now by the definition of $\|\cdot\|_{\mathbb{D}_r^p}$, this convergence implies the convergence in $L^p(\boldsymbol{W})$, and by the uniqueness of limit, this random variable must coincide with B_t. Moreover,

$$DB_t(s) = \int_0^{s\wedge t} K(t,u)\mathrm{d}u,$$

where $DB_t \in \mathcal{H}$ is the Malliavin derivative of B_t with respect to standard Brownian motion, and all of its higher-order Malliavin derivatives equal to zero. □

Therefore, according to the above lemma, we see that for each fixed time t, $B_t : \boldsymbol{W} \to \mathbb{R}$ is a smooth random variable.

Remark 4.1.1. *Indeed, according to Malliavin (see Theorem 2.3.3, Chapter IV, Part II, [49]), for each fixed* t*,* $B_t \in \mathbb{D}_r^p$*, there exists some* $\widetilde{B}_t$ *which is a* (p,r)*-redefinition of* B_t*, i.e. for all* $\varepsilon > 0$*, it is possible to find some subset* $O_\varepsilon \subset \boldsymbol{W}$ *whose* (p,r)*-capacity satisfies*

$$c_{p,r}(O_\varepsilon) < \varepsilon,$$

and $\widetilde{B_t}$ *is a continuous function on* $\boldsymbol{W}\backslash O_\varepsilon$*. This result is Lusin's theorem in the setting of capacities.*

4.2 Several technical facts

In this section, we shall prove several technical facts about fBMs which will be used in the sequel.

The first one is the following inequality, which is similar to the result due to Fukushima in [21].

Lemma 4.2.1. *Let* $(B_t)_{t\geq 0}$ *be a fractional Brownian motion with Hurst parameter*

H. Then for all $H \in (0,1)$,

$$(c_{p,r}(|B_t - B_s| > \eta))^p \leq 2\left(\sum_{l=0}^{r}\left(\frac{\eta}{p(t-s)^H}\right)^{lp}\right) e^{-\frac{\eta^2}{2(t-s)^{2H}}}$$

for any $r \in \mathbb{N}$, $1 < p < \infty$, $\eta > 0$, and $0 \leq s < t$.

Proof. Denote

$$M_{s,t} = B_t - B_s,$$

where $0 \leq s < t$. Then by the definition of Malliavin derivative, we obtain that

$$DM_{s,t}(u) = \int_0^u K(t,r)\mathbb{1}_{[0,t]}(r) - K(s,r)\mathbb{1}_{[0,s]}(r)\mathrm{d}r \in \mathcal{H}$$

and higher-order derivatives of $M_{s,t}$ are all equal to zero.

We first show that for all $\alpha \geq 0$,

$$e^{\frac{\alpha}{p}M_{s,t}} \in \mathbb{D}_r^p,$$

and

$$D^l e^{\frac{\alpha}{p}M_{s,t}} = \left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}} DM_{s,t} \otimes \cdots \otimes DM_{s,t} \in L^p(\boldsymbol{W}; \mathcal{H}^{\otimes l})$$

for all $1 \leq l \leq r$.

Set $f(x) = e^{\frac{\alpha}{p}x}$. For each $N \in \mathbb{N}$, let $\psi_N \in C_c^\infty(\mathbb{R})$ be a cut-off function taking values in $[0,1]$ such that

$$\psi_N(x) = \begin{cases} 1, & |x| \leq N, \\ 0, & |x| \geq N+1, \end{cases}$$

and

$$\sup_{x,N} |\psi_N^{(k)}(x)| = C < \infty$$

for all $1 \le k \le r$. Set

$$f_N(x) = f(x)\psi_N(x).$$

For convenience, write $F_N = f_N(M_{s,t})$, then $F_N \in \mathcal{S}$ as $f_N \in C_0^\infty(\mathbb{R})$, and by the chain rule for Malliavin derivatives, we have

$$D^l F_N = f_N^{(l)}(M_{s,t}) DM_{s,t} \otimes \cdots \otimes DM_{s,t}$$

for $1 \le l \le r$. We note here that $F_N \to e^{\frac{\alpha}{p}M_{s,t}}$ as N tends to infinity in $L^p(\boldsymbol{W})$.

Hence,

$$\begin{aligned}
&\mathbb{E}\left[\left|\left\|D^l F_N - \left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}} DM_{s,t} \otimes \cdots \otimes DM_{s,t}\right\|_{\mathcal{H}^{\otimes l}}\right|^p\right] \\
&= \mathbb{E}\left[\left|f_N^{(l)}(M_{s,t}) - \left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}}\right|^p \left\|DM_{s,t}\right\|_{\mathcal{H}}^{lp}\right] \\
&= \mathbb{E}\left[\left|\sum_{j=0}^{l} \binom{l}{j} f^{(j)}(M_{s,t})\psi_N^{(l-j)}(M_{s,t}) - \left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}}\right|^p\right] \left\|DM_{s,t}\right\|_{\mathcal{H}}^{lp} \\
&= \mathbb{E}\left[\left|\sum_{j=0}^{l-1} \binom{l}{j} f^{(j)}(M_{s,t})\psi_N^{(l-j)}(M_{s,t}) + \left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}}\psi_N(M_{s,t}) - \left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}}\right|^p\right] \\
&\quad \cdot \left\|DM_{s,t}\right\|_{\mathcal{H}}^{lp} \\
&\le l^{p-1}\mathbb{E}\left[\sum_{j=0}^{l-1} \left|\binom{l}{j} f^{(j)}(M_{s,t})\psi_N^{(l-j)}(M_{s,t})\right|^p + \left|\left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}}\left(\psi_N(M_{s,t}) - 1\right)\right|^p\right] \\
&\quad \cdot \left\|DM_{s,t}\right\|_{\mathcal{H}}^{lp} \\
&\le l^{p-1}\mathbb{E}\left[\sum_{j=0}^{l-1} \left|\binom{l}{j}\left(\frac{\alpha}{p}\right)^j e^{\frac{\alpha}{p}M_{s,t}} C \cdot \mathbb{1}_{\{|M_{s,t}| \ge N\}}\right|^p + \left|\left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}} \mathbb{1}_{\{|M_{s,t}| \ge N\}}\right|^p\right] \\
&\quad \cdot \left\|DM_{s,t}\right\|_{\mathcal{H}}^{lp},
\end{aligned}$$

which tends to zero as $N \to \infty$ by the dominated convergence theorem. According to the previous estimate, we deduce that

$$D^l F_N \to \left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}} DM_{s,t} \otimes \cdots \otimes DM_{s,t}$$

in $L^p(\boldsymbol{W}; \mathcal{H}^{\otimes l})$ as $N \to \infty$.

Since D^l is closable, together with the definition of $\mathbb{D}_l^p$, we conclude that

$$D^l F = \left(\frac{\alpha}{p}\right)^l e^{\frac{\alpha}{p}M_{s,t}} DM_{s,t} \otimes \cdots \otimes DM_{s,t}$$

for each $1 \leq l \leq r$ and $e^{\frac{\alpha}{p}M_{s,t}} \in \mathbb{D}_r^p$.

By Chebyshev inequality for (p, r)-capacity, it follows that

$$\begin{aligned}
&\left(c_{p,r}\left(M_{s,t} - \frac{\alpha}{2}(t-s)^{2H} > \beta\right)\right)^p \\
&= \left(c_{p,r}\left(\frac{\alpha}{p}M_{s,t} - \frac{\alpha^2}{2p}(t-s)^{2H} > \frac{\alpha\beta}{p}\right)\right)^p \\
&= \left(c_{p,r}\left(\exp\left(\frac{\alpha}{p}M_{s,t}\right) > \exp\left(\frac{\alpha^2}{2p}(t-s)^{2H} + \frac{\alpha\beta}{p}\right)\right)\right)^p \\
&\leq \exp\left(-\frac{\alpha^2}{2}(t-s)^{2H} - \alpha\beta\right) \left\|e^{\frac{\alpha}{p}M_{s,t}}\right\|_{\mathbb{D}_r^p}^p,
\end{aligned} \tag{4.5}$$

for any $\alpha, \beta > 0$. It is clear that

$$\begin{aligned}
\langle DM_{s,t}, DM_{s,t}\rangle_{\mathcal{H}} &= \int_0^\infty \left[K(t,u)\mathbb{1}_{[0,t]}(u) - K(s,u)\mathbb{1}_{[0,s]}(u)\right]^2 \mathrm{d}u \\
&= R(t,t) - 2R(s,t) + R(s,s) \\
&= (t-s)^{2H}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle D^l e^{\frac{\alpha}{p}M_{s,t}}, D^l e^{\frac{\alpha}{p}M_{s,t}}\rangle_{\mathcal{H}^{\otimes l}} &= \left(\frac{\alpha}{p}\right)^{2l} e^{\frac{2\alpha}{p}M_{s,t}} (\langle DM_{s,t}, DM_{s,t}\rangle_{\mathcal{H}})^l \\
&= \left(\frac{\alpha}{p}\right)^{2l} e^{\frac{2\alpha}{p}M_{s,t}} (t-s)^{2lH},
\end{aligned}$$

which implies that

$$\mathbb{E}\left[\left|\|D^l e^{\frac{\alpha}{p}M_{s,t}}\|_{\mathcal{H}^{\otimes l}}\right|^p\right] = \mathbb{E}\left[\left(\frac{\alpha}{p}\right)^{lp} e^{\alpha M_{s,t}}(t-s)^{lHp}\right]$$
$$= \left(\frac{\alpha}{p}\right)^{lp}(t-s)^{lHp}e^{\frac{\alpha^2}{2}(t-s)^{2H}},$$

where we have used that $M_{s,t} \sim N(0,(t-s)^{2H})$. Hence,

$$\|e^{\frac{\alpha}{p}M_{s,t}}\|^p_{\mathbb{D}^p_r} = \mathbb{E}\left[\left|e^{\frac{\alpha}{p}M_{s,t}}\right|^p\right] + \sum_{l=1}^{r}\mathbb{E}\left[\left|\|D^l e^{\frac{\alpha}{p}M_{s,t}}\|_{\mathcal{H}^{\otimes l}}\right|^p\right]$$
$$= \mathbb{E}\left[e^{\alpha M_{s,t}}\right] + \sum_{l=1}^{r}\left(\frac{\alpha}{p}\right)^{lp}(t-s)^{lHp}e^{\frac{\alpha^2}{2}(t-s)^{2H}}$$
$$= \left(\sum_{l=0}^{r}\left(\frac{\alpha}{p}\right)^{lp}(t-s)^{lHp}\right)e^{\frac{\alpha^2}{2}(t-s)^{2H}}.$$

Now by (4.5), we obtain that

$$\left(c_{p,r}\left(M_{s,t} - \frac{\alpha}{2}(t-s)^{2H} > \beta\right)\right)^p \leq \left(\sum_{l=0}^{r}\left(\frac{\alpha}{p}\right)^{lp}(t-s)^{lHp}\right)e^{-\alpha\beta}.$$

For any positive η, optimise the above inequality by setting

$$\alpha = \frac{\eta}{(t-s)^{2H}}$$

and

$$\beta = \frac{\eta}{2},$$

and then we arrive at

$$\left(c_{p,r}\left(M_{s,t} > \eta\right)\right)^p \leq \left(\sum_{l=0}^{r}\left(\frac{\eta}{p(t-s)^{2H}}\right)^{lp}(t-s)^{lHp}\right)e^{-\frac{\eta^2}{2(t-s)^{2H}}}$$
$$= \sum_{l=0}^{r}\left(\frac{\eta}{p(t-s)^{H}}\right)^{lp}e^{-\frac{\eta^2}{2(t-s)^{2H}}}.$$

By replacing B with $-B$, we may conclude that

$$(c_{p,r}\left(|M_{s,t}| > \eta\right))^p \le 2 \left(\sum_{l=0}^{r} \left(\frac{\eta}{p(t-s)^H} \right)^{lp} \right) e^{-\frac{\eta^2}{2(t-s)^{2H}}}.$$

□

The second technical lemma establishes a comparison result between the classical Wiener capacity and the Wiener measure. Before we state the result, let us first introduce the following notation.

Let $0 \le u < r < s < t \le T$. Set

$$X = \frac{B_t - B_s}{(t-s)^H},$$

and

$$Y = \frac{B_r - B_u}{(r-u)^H},$$

so that $X, Y \sim N(0,1)$. Then by definition,

$$\begin{aligned}\mathbb{E}[XY] &= (t-s)^{-H}(r-u)^{-H}\left(R(t,r) - R(t,u) - R(s,r) + R(s,u)\right) \\ &= \frac{1}{2(t-s)^H(u-r)^H}\left[(t-u)^{2H} - (t-r)^{2H} \right. \\ &\quad \left. - \left((s-u)^{2H} - (s-r)^{2H}\right)\right],\end{aligned} \tag{4.6}$$

which is non-negative when $H \in \left[\frac{1}{2}, 1\right)$, and non-positive when $H \in \left(0, \frac{1}{2}\right]$. We also need the following simple observation. By (4.1), we compute that

$$\langle DX, DY \rangle_{\mathcal{H}} = (t-s)^{-H}(r-u)^{-H}\langle DB_t - DB_s, DB_r - DB_u \rangle_{\mathcal{H}} \tag{4.7}$$

$$= (t-s)^{-H}(r-u)^{-H}\int_0^\infty \left(K(t,v)\mathbb{1}_{[0,t]}(v) - K(s,v)\mathbb{1}_{[0,s]}(v)\right) \tag{4.8}$$

$$\cdot \left(K(r,v)\mathbb{1}_{[0,r]}(v) - K(u,v)\mathbb{1}_{[0,u]}(v)\right) \mathrm{d}v \tag{4.9}$$

$$= \mathbb{E}[XY]. \tag{4.10}$$

Next technical lemma contains results similar to Proposition 1 in Fukushima [21] and Proposition 2 in Takeda [67].

Lemma 4.2.2. *For all $H \in (0,1)$ and each $N \in \mathbb{N}$, let $0 \le t_0 < t_1 < \cdots < t_N$ with $|t_i - t_{i-1}| = L$, $1 \le i \le N$. Take $-\infty < a_i < b_i < \infty$, $c_i > 0$, $1 \le i \le N$. Then it holds that*

$$\left(c_{p,r}\left(\bigcap_{i=1}^{N}\{a_i < X_i < b_i\}\right)\right)^p \le \left(\sum_{l=0}^{r} N^{lp} C_H^{lp/2}\left(\frac{M_r}{c}\right)^{lp}\right) \cdot P\left(\bigcap_{i=1}^{N}\{a_i - c_i < X_i < b_i + c_i\}\right)$$

for all $r \in \mathbb{N}$ and $p \in (1,\infty)$, where

$$X_i = \frac{B_{t_i} - B_{t_{i-1}}}{L^H} \sim N(0,1), \tag{4.11}$$

$$c = \min_{1 \le i \le N} c_i,$$

M_r is a constant depending only on r, and

$$C_H = \max\left\{2^{2H-1} - 1, 1\right\} \le 1$$

is some constant depending only on H.

Proof. The proof is a modification of Takeda's argument in [67]. For $i = 1, 2, \cdots, N$, let $f_i \in C_c^\infty(\mathbb{R})$ be the cut-off functions valued in $[0,1]$ such that

$$f_i(x) = \begin{cases} 1, & x \in (a_i, b_i), \\ 0, & x \in (-\infty, a_i - c_i) \cup (b_i + c_i, \infty), \end{cases}$$

and

$$\left|\frac{d^l f_i}{dx^l}\right| \le \frac{M_r}{c_i^l}$$

for all $l \le r$, where $M_r \ge 1$ is a constant depending on r. Set

$$F(x_{1,} \cdots, x_N) = \prod_{i=1}^{N} f_i(x_i),$$

then according to the above conditions, we have that

$$\left|\partial^l_{n_1,\cdots,n_l} F(x_{1,} \cdots, x_N)\right| \le \left(\frac{M_r}{c}\right)^l \mathbb{1}_{\prod_{i=1}^{N}(a_i - c_i, b_i + c_i)}(x_1, \cdots, x_N) \tag{4.12}$$

for each $l \le r$, where $c = \min_{1\le i\le N} c_i$. Here, $\partial^l_{n_1,\cdots,n_l} F$ denotes the l-th partial derivative of F in the $n_1, \cdots, n_l$-th components, where $1 \le n_i \le N$ for $1 \le i \le l$ and we allow $n_i = n_j$ for $i \ne j$.

For simplicity, write

$$Y = F(X_1, \cdots, X_N),$$

where X_i's are defined as in (4.11). Then $Y \in \mathbb{D}^p_l$. Moreover, since all Malliavin derivatives of X_i of order higher than 2 vanish, it holds that

$$D^l Y = \sum_{1\le n_1,\cdots,n_l \le N} \partial^l_{n_1,\cdots,n_l} F(X_1, \cdots, X_N) DX_{n_1} \otimes \cdots \otimes DX_{n_l}.$$

Furthermore, $D^l F \in \mathcal{H}^{\otimes l}$ and

$$\|D^l Y\|^2_{\mathcal{H}^{\otimes l}} = \sum_{\substack{1\le n_1,\cdots,n_l\le N\\ 1\le m_1,\cdots,m_l\le N}} \left(\partial^l_{n_1,\cdots,n_l} F(X_1, \cdots, X_N) \partial^l_{m_1,\cdots,m_l} F(X_1, \cdots, X_N) \cdot \prod_{i=1}^{l} \langle DX_{n_i}, DX_{m_i}\rangle_{\mathcal{H}} \right). \tag{4.13}$$

Our next step is to find an upper bound for $|\langle DX_j, DX_k\rangle_{\mathcal{H}}|$ for all $1 \le j, k \le N$.

When $1 \leq j = k \leq N$, $\langle DX_j, DX_k \rangle_{\mathcal{H}} = 1$; when $1 \leq j < k \leq N$, by (4.6) and (4.10),

$$\begin{aligned} \langle DX_j, DX_k \rangle_{\mathcal{H}} &= \mathbb{E}[X_j X_k] \\ &= \frac{1}{2} \left[(k-j+1)^{2H} + (k-j-1)^{2H} - 2\,(k-j)^{2H} \right]. \end{aligned}$$

Set

$$g(x) = \frac{1}{2} \left[(x+1)^{2H} + (x-1)^{2H} - 2x^{2H} \right].$$

Observe that when $H < \frac{1}{2}$, x^{2H} is concave, so $g(x) \leq 0$, and similarly when $H > \frac{1}{2}$, $g(x) \geq 0$. The derivative of g is given by

$$g'(x) = H \left[\left((x+1)^{2H-1} - x^{2H-1} \right) - \left(x^{2H-1} - (x-1)^{2H-1} \right) \right].$$

Using the fact that the function x^{2H-1} is convex if $H \in \left(0, \frac{1}{2}\right)$, we deduce that when $H \in \left(0, \frac{1}{2}\right)$, $g'(x) \geq 0$. As $k - j \in \{1, 2, \cdots, N-1\}$, it follows that

$$|\langle DX_j, DX_k \rangle_{\mathcal{H}}| \leq 2^{2H-1} - 1.$$

When $H \in \left(\frac{1}{2}, 1\right)$, $g'(x) \leq 0$ and thus

$$|\langle DX_j, DX_k \rangle_{\mathcal{H}}| \leq 2^{2H-1} - 1.$$

When $H = \frac{1}{2}$, $\langle DX_j, DX_k \rangle_{\mathcal{H}} = 0$ by independence. Set

$$C_H = \max \left\{ 2^{2H-1} - 1, 1 \right\},$$

then

$$|\langle DX_j, DX_k \rangle_{\mathcal{H}}| \leq C_H$$

for all $1 \leq j, k \leq N$. Moreover, as H takes values in $(0,1)$, $C_H \leq 1$.

Therefore, by (4.13), together with (4.12), it follows that

$$\|D^l Y\|^2_{\mathcal{H}^{\otimes l}} \leq N^{2l} \left(\frac{M_r}{c}\right)^{2l} \mathbb{1}_{\prod_{i=1}^N (a_i - c_i, b_i + c_i)}(X_1, \cdots, X_N) C_H^l$$

for all $l \leq r$. Hence

$$\left|\|D^l Y\|_{\mathcal{H}^{\otimes l}}\right|^p \leq N^{lp} C_H^{lp/2} \left(\frac{M_r}{c}\right)^{lp} \mathbb{1}_{\prod_{i=1}^N (a_i - c_i, b_i + c_i)}(X_1, \cdots, X_N).$$

By the definition of (p,r)-capacity,

$$\begin{aligned}
&\left(c_{p,r}\left(\bigcap_{i=1}^N \{a_i < X_i < b_i\}\right)\right)^p \\
&\leq \|Y\|^p_{\mathbb{D}^p_r} \\
&= \mathbb{E}\left[|Y|^p\right] + \sum_{l=1}^r \mathbb{E}\left[\left|\|D^l Y\|_{\mathcal{H}^{\otimes l}}\right|^p\right] \\
&\leq P\left(\bigcap_{i=1}^N \{a_i - c_i < X_i < b_i + c_i\}\right) \\
&\quad + \sum_{l=1}^r \left(N^{lp} C_H^{lp/2} \left(\frac{M_r}{c}\right)^{lp}\right) P\left(\bigcap_{i=1}^N \{a_i - c_i < X_i < b_i + c_i\}\right) \\
&= \left(\sum_{l=0}^r N^{lp} C_H^{lp/2} \left(\frac{M_r}{c}\right)^{lp}\right) P\left(\bigcap_{i=1}^N \{a_i - c_i < X_i < b_i + c_i\}\right).
\end{aligned}$$

□

Throughout this chapter, we always use the notation $X_\cdot$ to denote normalised increments of fBMs, though it may refer to increments over time intervals of different lengths, it always has the standard Gaussian distribution.

Before we state the third lemma, let us first mention one important tool we used in the proof of next lemma, which is Slepian's lemma. Here, we adopt the version provided in Theorem 3.11, Chapter 3, [39].

Proposition 4.2.1. *Let $X = (X_1, \cdots, X_N)$ and $Y = (Y_1, \cdots, Y_N)$ be two Gaussian random variables in $\mathbb{R}^N$. Assume that*

$$\begin{aligned} \mathbb{E}[X_iX_j] &\le \mathbb{E}[Y_iY_j], && \text{if } (i,j) \in A, \\ \mathbb{E}[X_iX_j] &\ge \mathbb{E}[Y_iY_j], && \text{if } (i,j) \in B, \\ \mathbb{E}[X_iX_j] &= \mathbb{E}[Y_iY_j], && \text{if } (i,j) \notin A \cup B, \end{aligned}$$

where A and B are subsets of $\{1, \cdots, N\} \times \{1, \cdots, N\}$. Let f be a function on $\mathbb{R}^N$ such that its second derivatives in the sense of distributions satisfy

$$D_{ij} f \ge 0, \quad \text{if } (i,j) \in A,$$

$$D_{ij} f \le 0, \quad \text{if } (i,j) \in B.$$

Then

$$\mathbb{E}[f(X)] \le \mathbb{E}[f(Y)].$$

We shall omit the proof of this result, and one may refer to [39] for a proof.

The third lemma we include here is a $(2,1)$-capacity estimate on the supremum process corresponding to the fBM with Hurst parameter $H \in (0,1)$.

Lemma 4.2.3. *Let $0 \le s < t$. For $H \in (0,1)$ and $\eta > 0$,*

$$c_{2,1}\left(\sup_{s\le u\le t} (B_u - B_s) > \eta\right) \le C_{s,t,\eta,H} \cdot \exp\left(-\frac{\eta^2}{4\left(\gamma_H (t-s)^{2H} + (t-s)\right)}\right), \quad (4.14)$$

and

$$c_{2,1}\left(\sup_{s\le u\le t} |B_u - B_s| > \eta\right) \le \sqrt{2} C_{s,t,\eta,H} \cdot \exp\left(-\frac{\eta^2}{4\left(\gamma_H (t-s)^{2H} + (t-s)\right)}\right), \quad (4.15)$$

$$c_{2,1}\left(\sup_{s\le u\le t} |B_t - B_u| > \eta\right) \le \sqrt{2} C_{s,t,\eta,H} \cdot \exp\left(-\frac{\eta^2}{4\left(\gamma_H (t-s)^{2H} + (t-s)\right)}\right), \quad (4.16)$$

where

$$\gamma_H = \begin{cases} 1, & H \le \frac{1}{2}, \\ \frac{3}{2}, & H > \frac{1}{2}, \end{cases} \tag{4.17}$$

and

$$C_{s,t,\eta,H} = \sqrt{\frac{\eta^2 (t-s)^{2H}}{2\left(\gamma_H (t-s)^{2H} + (t-s)\right)^2} + 2}.$$

Proof. We shall follow the same ideas as for the proof of Proposition 2 and 3 in [21], while we have to overcome several difficulties arising from the fact that the distribution of supremum processes is not known for fBMs.

When $H = \frac{1}{2}$, the above inequality is already covered by the result due to Fukushima in [21].

From now on, we assume that $H \neq \frac{1}{2}$. We prove (4.14) and (4.15) first. For simplicity, define

$$M^*_{s,t} = \sup_{s \le u \le t} (B_u - B_s)$$

for any $0 \le s < t$. Following Fukushima's notation in [21], for $s < t_1 < \cdots < t_n \le t$, let us define

$$B_{s;t_1,\cdots,t_n} = (B_{t_1} - B_s, \cdots, B_{t_n} - B_s),$$

and let

$$g(x_1, \cdots, x_n) = x_1 \vee \cdots \vee x_n,$$

and define

$$M_{s;t_1,\cdots,t_n} = g(B_{s;t_1,\cdots,t_n}) = \max_{1 \le i \le n} (B_{t_i} - B_s).$$

Then we proceed in 4 steps.

Step 1. In this step, only the law of fBMs will be involved, so the argument is indeed applicable to various Gaussian processes. As t_i's are fixed in the first two steps, we simplify our notations by writing

$$B^{(n)}_{s,t} = B_{s;t_1,\cdots,t_n}$$

and

$$M^{(n)}_{s,t} = M_{s;t_1,\cdots,t_n}$$

for the moment. In this step, we establish an upper bound for $\mathbb{E}\left[e^{\alpha M^{(n)}_{s,t}}\right]$, where $\alpha > 0$.

Consider the following correlation:

$$\mathbb{E}\left[(B_{t_i} - B_s)\left(B_{t_j} - B_s\right)\right] = \mathbb{E}\left[(B_{t_i} - B_s)^2 + (B_{t_i} - B_s)\left(B_{t_j} - B_{t_i}\right)\right]. \tag{4.18}$$

When $H < \frac{1}{2}$, for any $1 \leq i \leq j \leq n$, the increments of $(B_t)_{t\geq 0}$ over different time intervals are negatively correlated, which leads to

$$\begin{aligned}
\mathbb{E}\left[(B_{t_i} - B_s)\left(B_{t_j} - B_s\right)\right] &\leq \mathbb{E}\left[(B_{t_i} - B_s)^2\right] \\
&= (t_i - s)^{2H} \\
&\leq (t - s)^{2H}.
\end{aligned} \tag{4.19}$$

When $H > \frac{1}{2}$, we seek for an upper bound of

$$\mathbb{E}\left[(B_{t_i} - B_s)\left(B_{t_j} - B_{t_i}\right)\right].$$

We compute that

$$\begin{aligned}
\mathbb{E}\left[(B_{t_i} - B_s)\left(B_{t_j} - B_{t_i}\right)\right] &= \frac{1}{2}\left[(t_j - s)^{2H} - (t_j - t_i)^{2H} - (t_i - s)^{2H}\right] \\
&\leq \frac{1}{2}(t - s)^{2H},
\end{aligned}$$

where $0 \le s < t_i < t_j \le t$. Combining with (4.18), we have

$$\mathbb{E}\left[(B_{t_i} - B_s)\left(B_{t_j} - B_s\right)\right] \le \frac{3}{2}(t-s)^{2H}.$$

Therefore, for all $H \in (0,1)$,

$$\mathbb{E}\left[(B_{t_i} - B_s)\left(B_{t_j} - B_s\right)\right] \le \gamma_H (t-s)^{2H},$$

where γ_H is defined as in (4.17).

For convenience, set

$$Z_i = B_{t_i} - B_s \sim N(0, (t_i - s)^{2H}),$$

and by the above estimate, correlations between any two Z_i's are bounded by $\gamma_H(t-s)^{2H}$. We want to apply Proposition 4.2.1 to overcome the difficulties in finding the distribution of the supremum process of $(B_t)_{t\ge 0}$, so we take a random variable

$$\xi_{s,t} \sim N(0, \gamma_H (t-s)^{2H}),$$

which is independent of the standard Brownian motion $(\omega_t)_{t\ge 0}$ on $(\boldsymbol{W}, \mathcal{H}, P)$, so that $\xi_{s,t}$ and $\omega_{t_i} - \omega_s$ are independent for all $i \in \{1, 2, \cdots, n\}$. Define

$$Y_i = \omega_{t_i} - \omega_s + \xi_{s,t},$$

for all $1 \le i \le n$ and let

$$N_{s,t}^{(n)} = \max_{1\le i\le n} Y_i = \max_{1\le i\le n} (\omega_{t_i} - \omega_s) + \xi_{s,t}.$$

Then by independence,

$$\begin{aligned}\mathbb{E}\left[Y_iY_j\right] &= \mathbb{E}\left[(\omega_{t_i}-\omega_s+\xi_{s,t})(\omega_{t_j}-\omega_s+\xi_{s,t})\right]\\ &= \mathbb{E}\left[(\omega_{t_i}-\omega_s)(\omega_{t_j}-\omega_s)\right]+\mathbb{E}\left[\xi_{s,t}^2\right]\\ &= t_i-s+\gamma_H(t-s)^{2H}\\ &\geq \gamma_H(t-s)^{2H},\end{aligned}$$

for $1\leq i\leq j\leq n$, and hence by (4.19),

$$\mathbb{E}[Z_iZ_j]\leq \mathbb{E}[Y_iY_j]$$

for all $1\leq i,j\leq n$. Since both exponential function and maximum function are convex, their composition is also convex, and hence according to Proposition 4.2.1, Slepian's lemma, we obtain that

$$\mathbb{E}\left[e^{\alpha M_{s,t}^{(n)}}\right]=\mathbb{E}\left[e^{\alpha\max_{1\leq i\leq n}Z_i}\right]\leq \mathbb{E}\left[e^{\alpha\max_{1\leq i\leq n}Y_i}\right]=\mathbb{E}\left[e^{\alpha N_{s,t}^{(n)}}\right],$$

for all $\alpha>0$. Due to independence and the fact that

$$\max_{1\leq i\leq n}\left(\omega_{t_i}-\omega_s\right)\leq \sup_{s\leq u\leq t}\left(\omega_u-\omega_s\right),$$

we deduce that

$$\begin{aligned}\mathbb{E}\left[e^{\alpha N_{s,t}^{(n)}}\right] &= \mathbb{E}\left[e^{\alpha\xi_{s,t}}\right]\mathbb{E}\left[\exp\left(\alpha\max_{1\leq i\leq n}\left(\omega_{t_i}-\omega_s\right)\right)\right]\\ &\leq \exp\left(\frac{\alpha^2}{2}\gamma_H(t-s)^{2H}\right)\mathbb{E}\left[\exp\left(\alpha\sup_{s\leq u\leq t}\left(\omega_u-\omega_s\right)\right)\right].\end{aligned}$$

Using the distribution of the supremum of standard Brownian motion, we obtain that

$$\mathbb{E}\left[e^{\alpha M_{s,t}^{(n)}}\right]=\mathbb{E}\left[\exp\left(\alpha M_{s;t_1,\cdots,t_n}\right)\right]$$

$$\leq 2\exp\left(\frac{\alpha^2}{2}\left[\gamma_H(t-s)^{2H}+(t-s)\right]\right) \tag{4.20}$$

for all $\alpha > 0$ and $0 \leq s < t$.

Step 2. The difference between our method and classical approaches will be demonstrated in this step since we use only the capacities induced by standard Brownian motion.

In this step, we show that $e^{\frac{\alpha}{2}M_{s,t}^{(n)}} \in \mathbb{D}_1^2$ and

$$De^{\frac{\alpha}{2}M_{s,t}^{(n)}} = \frac{\alpha}{2}\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right)DM_{s,t}^{(n)}.$$

We observe that g is Lipschitz, so by Proposition 1.2.4, Chapter 1, [58],

$$M_{s,t}^{(n)} = g(B_{s,t}^{(n)}) \in \mathbb{D}_1^2,$$

and the chain rule applies, which is

$$\begin{aligned} DM_{s,t}^{(n)}(u) &= \sum_{i=1}^{n} \mathbb{1}_{\{M_{s,t}^{(n)}=B_{t_i}-B_s\}}(B_{s,t}^{(n)})D(B_{t_i}-B_s) \\ &= \sum_{i=1}^{n} \mathbb{1}_{\{M_{s,t}^{(n)}=B_{t_i}-B_s\}}(B_{s,t}^{(n)})\left[K(t_i,u)\mathbb{1}_{[0,t_i]}(u)-K(s,u)\mathbb{1}_{[0,s]}(u)\right]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \langle DM_{s,t}^{(n)}, DM_{s,t}^{(n)}\rangle_{\mathcal{H}} &= \sum_{i=1}^{n} \mathbb{1}_{\{M_{s,t}^{(n)}=B_{t_i}-B_s\}}(B_{s,t}^{(n)}) \\ &\quad \cdot \int_0^\infty \left[K(t_i,u)\mathbb{1}_{[0,t_i]}(u)-K(s,u)\mathbb{1}_{[0,s]}(u)\right]^2 \mathrm{d}u \\ &= \sum_{i=1}^{n} \mathbb{1}_{\{M_{s,t}^{(n)}=B_{t_i}-B_s\}}(B_{s,t}^{(n)})(t_i-s)^{2H} \\ &\leq (t-s)^{2H}\sum_{i=1}^{n} \mathbb{1}_{\{M_{s,t}^{(n)}=B_{t_i}-B_s\}}. \end{aligned} \tag{4.21}$$

Similar to the argument in Lemma 4.2.1, we set $f(x) = e^{\frac{\alpha}{2}x}$, and $\psi_N(x)$ as in Lemma 4.2.1, and set $f_N = f \cdot \psi_N$. For simplicity, denote $F = e^{\frac{\alpha}{2}M_{s,t}^{(n)}}$, and $F_N = f_N(M_{s,t}^{(n)})$. Then since $f_N \in C_c^\infty(\mathbb{R})$, the chain rule applies, and

$$DF_N = f_N'(M_{s,t}^{(n)})DM_{s,t}^{(n)}.$$

Similarly, we have that

$$\begin{aligned}
&\mathbb{E}\left[\|DF_N - \frac{\alpha}{2}e^{\frac{\alpha}{2}M_{s,t}^{(n)}}DM_{s,t}^{(n)}\|_{\mathcal{H}}^2\right] \\
&= \int_{\boldsymbol{W}} \left| f'(M_{s,t}^{(n)})\psi_N(M_{s,t}^{(n)}) + f(M_{s,t}^{(n)})\psi_N'(M_{s,t}^{(n)}) - \frac{\alpha}{2}e^{\frac{\alpha}{2}M_{s,t}^{(n)}} \right|^2 \|DM_{s,t}^{(n)}\|_{\mathcal{H}}^2 \mathrm{d}P \\
&\leq \int_{\boldsymbol{W}} \left| \frac{\alpha}{2}e^{\frac{\alpha}{2}M_{s,t}^{(n)}}\left(\psi_N(M_{s,t}^{(n)}) - 1\right) + e^{\frac{\alpha}{2}M_{s,t}^{(n)}}\psi_N'(M_{s,t}^{(n)}) \right|^2 \\
&\quad \cdot \left(\sum_{i=1}^n \mathbb{1}_{\{M_{s,t}^{(n)} = B_{t_i} - B_s\}}\right)(t-s)^{2H}\mathrm{d}P \\
&\leq 2\mathbb{E}\left[\left|\frac{\alpha}{2}e^{\frac{\alpha}{2}M_{s,t}^{(n)}} \cdot \mathbb{1}_{\{|M_{s,t}^{(n)}| \geq N\}}\right|^2 + \left|e^{\frac{\alpha}{2}M_{s,t}^{(n)}} C \cdot \mathbb{1}_{\{|M_{s,t}^{(n)}| \geq N\}}\right|^2\right](t-s)^{2H},
\end{aligned}$$

which tends to zero as $N \to \infty$, where C is defined as in Lemma 4.2.1. Therefore, since $F_N \to F$ in $L^2(\boldsymbol{W})$,

$$DF_N \to \frac{\alpha}{2}e^{\frac{\alpha}{2}M_{s,t}^{(n)}}DM_{s,t}^{(n)}$$

in $L^2(\boldsymbol{W};\mathcal{H})$ and D is closable from $L^2(\boldsymbol{W})$ to $L^2(\boldsymbol{W};\mathcal{H})$, so it follows that

$$DF = \frac{\alpha}{2}e^{\frac{\alpha}{2}M_{s,t}^{(n)}}DM_{s,t}^{(n)} \tag{4.22}$$

and $F \in \mathbb{D}_1^2$.

Step 3. In this step, we find an upper bound for $\mathbb{E}\left[e^{\alpha M_{s,t}^*}\right]$ for any $\alpha > 0$, then we prove that $e^{\alpha M_{s,t}^*} \in \mathbb{D}_1^2$ and find an upper bound for $\|e^{\alpha M_{s,t}^*}\|_{\mathbb{D}_1^2}$. As $M_{s;t_1,\cdots,t_n}$ increases to $M_{s,t}^*$ when we refine the partition and let n tend to infinity, the monotone convergence theorem and (4.20) imply that

$$\mathbb{E}\left[e^{\alpha M^*_{s,t}}\right] \le 2\exp\left(\frac{\alpha^2}{2}\left[\gamma_H(t-s)^{2H}+(t-s)\right]\right).$$

We have already proved that $e^{\alpha M_{s;t_1,\cdots,t_n}} \in \mathbb{D}^2_1$ in the previous step, and by (4.21) and (4.22),

$$\langle D\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right), D\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right)\rangle_{\mathcal{H}} \le \frac{\alpha^2}{4}(t-s)^{2H}\exp\left(\alpha M_{s;t_1,\cdots,t_n}\right) \cdot \sum_{i=1}^n \mathbb{1}_{\{M_{s;t_1,\cdots,t_n}=B_{t_i}-B_s\}}.$$

Therefore, by (4.20), we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\langle D\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right), D\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right)\rangle_{\mathcal{H}}\right]\\
&\le \frac{\alpha^2}{4}(t-s)^{2H}\sum_{i=1}^n\int_{\{M_{s;t_1,\cdots,t_n}=B_{t_i}-B_s\}}\exp\left(\alpha M_{s;t_1,\cdots,t_n}\right)\mathrm{d}P\\
&= \frac{\alpha^2}{4}(t-s)^{2H}\mathbb{E}\left[\exp\left(\alpha M_{s;t_1,\cdots,t_n}\right)\right]\\
&\le \frac{\alpha^2}{2}(t-s)^{2H}\exp\left(\frac{\alpha^2}{2}\left[\gamma_H(t-s)^{2H}+(t-s)\right]\right),
\end{aligned}$$

which implies that

$$\sup_n \mathbb{E}\left[\|D\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right)\|^2_{\mathcal{H}}\right] < \infty.$$

Applying Lemma 1.2.3 in [58], we deduce that $e^{\frac{\alpha}{2}M^*_{s,t}} \in \mathbb{D}^2_1$ and

$$D\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right) \to De^{\frac{\alpha}{2}M^*_{s,t}}$$

weakly in $L^2(\boldsymbol{W};\mathcal{H})$. As a consequence, we have

$$\begin{aligned}
&\mathbb{E}\left[\langle De^{\frac{\alpha}{2}M^*_{s,t}}, De^{\frac{\alpha}{2}M^*_{s,t}}\rangle_{\mathcal{H}}\right]\\
&\le \liminf_{n\to\infty}\mathbb{E}\left[\langle D\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right), D\exp\left(\frac{\alpha}{2}M_{s;t_1,\cdots,t_n}\right)\rangle_{\mathcal{H}}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\alpha^2}{4}(t-s)^{2H}\mathbb{E}\left[e^{\alpha M^*_{s,t}}\right] \\
&\leq \frac{\alpha^2}{2}(t-s)^{2H}\exp\left(\frac{\alpha^2}{2}\left[\gamma_H(t-s)^{2H}+(t-s)\right]\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\|e^{\frac{\alpha}{2}M^*_{s,t}}\right\|^2_{\mathbb{D}^2_1} &= \mathbb{E}\left[e^{\alpha M^*_{s,t}}\right] + \mathbb{E}\left[\langle De^{\frac{\alpha}{2}M^*_{s,t}}, De^{\frac{\alpha}{2}M^*_{s,t}}\rangle_{\mathcal{H}}\right] \\
&\leq \left(\frac{\alpha^2}{2}(t-s)^{2H}+2\right)\exp\left(\frac{\alpha^2}{2}\left[\gamma_H(t-s)^{2H}+(t-s)\right]\right).
\end{aligned}
\tag{4.23}
$$

Step 4. By Chebyshev's inequality for capacities and (4.23), we thus have

$$
\begin{aligned}
&\left(c_{2,1}\left(M^*_{s,t}-\frac{\alpha}{2}(t-s)^{2H}>\beta\right)\right)^2 \\
&= \left(c_{2,1}\left(\frac{\alpha}{2}M^*_{s,t}-\frac{\alpha^2}{4}(t-s)^{2H}>\frac{\alpha\beta}{2}\right)\right)^2 \\
&= \left(c_{2,1}\left(\exp\left(\frac{\alpha}{2}M^*_{s,t}\right)>\exp\left(\frac{\alpha\beta}{2}+\frac{\alpha^2}{4}(t-s)^{2H}\right)\right)\right)^2 \\
&\leq \exp\left(-\alpha\beta-\frac{\alpha^2}{2}(t-s)^{2H}\right)\left\|e^{\frac{\alpha}{2}M^*_{s,t}}\right\|^2_{\mathbb{D}^2_1} \\
&\leq \left(\frac{\alpha^2}{2}(t-s)^{2H}+2\right)\exp\left(-\alpha\beta+\frac{\alpha^2}{2}\left[(\gamma_H-1)(t-s)^{2H}+(t-s)\right]\right)
\end{aligned}
\tag{4.24}
$$

for any positive constants α and β.

Notice that the exponential function is the dominating part in the last term of (4.24), so we optimise the above quantity by minimising the exponent, and setting

$$
\alpha = \frac{\eta}{\gamma_H(t-s)^{2H}+(t-s)},
$$

and

$$
\beta = \eta - \frac{\alpha}{2}(t-s)^{2H}.
$$

Therefore, we get that

$$\left(c_{2,1}\left(M^*_{s,t} > \eta\right)\right)^2 \leq C^2_{s,t,\eta,H} \exp\left(-\frac{\eta^2}{2\left[\gamma_H (t-s)^{2H} + (t-s)\right]}\right),$$

where

$$C_{s,t,\eta,H} = \sqrt{\frac{\eta^2 (t-s)^{2H}}{2\left[\gamma_H (t-s)^{2H} + (t-s)\right]^2} + 2}.$$

Moreover, by replacing B with $-B$, it follows that

$$\left(c_{2,1}\left(\sup_{s\leq u\leq t} |B_u - B_s| > \eta\right)\right)^2 \leq 2C^2_{s,t,\eta,H} \exp\left(-\frac{\eta^2}{2\left[\gamma_H (t-s)^{2H} + (t-s)\right]}\right).$$

Finally, (4.16) may be established directly following the same argument with slight modification in the definition of $M^*_{s,t}$. □

Remark 4.2.1. *The results in the previous lemma can be considered as a version of the maximal inequality for fBMs, but with respect to the classical Wiener capacity. For a similar result when $H = \frac{1}{2}$, one may refer to Fukushima [21] for the case when $p = 2$ and $r = 1$, or Takeda [67] for any $r \in \mathbb{N}$ and $p \in (1, \infty)$. Though we establish these inequalities for all $H \in (0,1)$, when considering a sufficiently small time interval $[s,t]$, the result looks weaker when $H > \frac{1}{2}$ due to the appearance of $(t-s)$ in the exponent. In fact, when $H > \frac{1}{2}$, $(t-s)$ will be the dominating part rather than $(t-s)^{2H}$. However, the factor $(t-s)$ appears necessary for small time intervals.*

4.3 Modulus of continuity

In this part, we shall show the result on the modulus of continuity for fBMs with respect to the (p,r)-capacity defined on the classical Wiener space. We shall adopt the arguments in Fukushima's work [21] and use the method from the original proof by Lévy [40], who proved the modulus of continuity for standard Brownian motion in probability sense.

Theorem 4.3.1. *Let $(B_t)_{t\geq 0}$ be an fBM with Hurst parameter H. Then it holds that*

$$\limsup_{\delta\downarrow 0} \frac{1}{\sqrt{2\delta^{2H}\log(1/\delta)}} \max_{\substack{0\leq s<t\leq 1\\ t-s\leq\delta}} |B_t - B_s| \leq 1, \quad q.s. \tag{4.25}$$

when $H \in (0,1)$ and

$$\limsup_{\delta\downarrow 0} \frac{1}{\sqrt{2\delta^{2H}\log(1/\delta)}} \max_{\substack{0\leq s<t\leq 1\\ t-s\leq\delta}} |B_t - B_s| \geq 1, \quad q.s. \tag{4.26}$$

when $H \in \left(0, \frac{1}{2}\right]$.

Proof. Let us prove (4.26) first. For any $r \in \mathbb{N}$ and $p \in (1,\infty)$, we want to show that

$$c_{p,r}\left(\limsup_{\delta\downarrow 0} \frac{1}{g(\delta)} \max_{\substack{0\leq s<t\leq 1\\ t-s\leq\delta}} |B_t - B_s| < 1\right) = 0,$$

where

$$g(\delta) = \sqrt{2\delta^{2H}\log(1/\delta)}.$$

By Lemma 4.2.2, we have

$$\begin{aligned}
&c_{p,r}\left(\max_{1\leq j\leq 2^n} \left|B_{\frac{j}{2^n}} - B_{\frac{j-1}{2^n}}\right| \leq (1-\theta)g(2^{-n})\right)\\
&= c_{p,r}\left(\max_{1\leq j\leq 2^n} 2^{nH}\left|B_{\frac{j}{2^n}} - B_{\frac{j-1}{2^n}}\right| \leq (1-\theta)\sqrt{2\log(2^n)}\right)\\
&\leq \left(\sum_{k=0}^{r}(2n)^{kp}C_H^{kp/2}\left(\frac{M_r}{c}\right)^{kp}\right)^{1/p}\\
&\quad\cdot\left(P\left(\max_{1\leq j\leq 2^n} 2^{nH}\left|B_{\frac{j}{2^n}} - B_{\frac{j-1}{2^n}}\right| \leq (1-\theta)\sqrt{2\log(2^n)} + c\right)\right)^{1/p}
\end{aligned}$$

for $\theta \in (0,1)$, where c is some small constant such that $c < \theta\sqrt{2\log 2}$. Set

$$X_j = 2^{nH}\left(B_{\frac{j}{2^n}} - B_{\frac{j-1}{2^n}}\right),$$

then $X_j \sim N(0,1)$ and when $H \leq \frac{1}{2}$,

$$\mathbb{E}[X_j X_k] \leq 0$$

for $j \neq k$. Take a sequence of independent standard Gaussian random variables Y_j's so that

$$\mathbb{E}[X_j X_k] \leq 0 = \mathbb{E}[Y_j Y_k]$$

for $j \neq k$. Let

$$c' = \frac{c}{\sqrt{2\log 2}}$$

so that $\theta - c' > 0$, and hence $0 < 1 - \theta + c' < 1$. Then Proposition 4.2.1 implies that

$$\begin{aligned}
& P\left(\bigcap_{1\leq j\leq 2^n}\left\{|X_j| \leq (1-\theta+c')\sqrt{2\log(2^n)}\right\}\right) \\
&\leq P\left(\bigcap_{1\leq j\leq 2^n}\left\{|X_j| \leq (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right\}\right) \\
&\leq P\left(\bigcap_{1\leq j\leq 2^n}\left\{X_j \leq (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right\}\right) \\
&\leq P\left(\bigcap_{1\leq j\leq 2^n}\left\{Y_j \leq (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right\}\right) \\
&= \prod_{1\leq j\leq 2^n} P\left(Y_j \leq (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right) \\
&= \left(1 - P\left(Y_j > (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right)\right)^{2^n} \\
&\leq \exp\left(-\xi 2^n\right),
\end{aligned}$$

where

$$\xi = P\left(Y_j > (1-\theta+c')^{1/2}\sqrt{2\log(2^n)}\right)$$
$$\geq \frac{(1-\theta+c')^{1/2}\sqrt{2\log(2^n)}}{1+2(1-\theta+c')\log(2^n)}\exp\left(-(1-\theta+c')\log(2^n)\right)$$
$$\geq C2^{-n(1-\theta+c')}$$

for n sufficiently large, hence it follows that

$$P\left(\bigcap_{1\leq j\leq 2^n}\left\{|X_j| \leq (1-\theta+c')\sqrt{2\log(2^n)}\right\}\right) \leq \exp\left(-C2^{n(\theta-c')}\right).$$

The right-hand side is a term of a convergent series, and hence by the first Borel-Cantelli lemma for (p,r)-capacity, (4.26) follows immediately.

For the upper bound, we first notice that

$$g(k2^{-n}) = (k2^{-n})^H\sqrt{2\log(\frac{2^n}{k})}.$$

For any $\varepsilon > 0$, applying Lemma 4.2.1 with

$$\eta = (1+\varepsilon)g(k2^{-n}),$$

we get that

$$I_n^p = \left(c_{p,r}\left(\max_{\substack{0<k=j-i\leq 2^{n\theta}\\ 0\leq i<j\leq 2^n}}\frac{|B_{j2^{-n}}-B_{i2^{-n}}|}{g(k2^{-n})}\geq 1+\varepsilon\right)\right)^p$$
$$\leq M_{p,r}\sum_{\substack{0<k=j-i\leq 2^{n\theta}\\ 0\leq i<j\leq 2^n}}\left(c_{p,r}\left(\frac{|B_{j2^{-n}}-B_{i2^{-n}}|}{g(k2^{-n})}\geq 1+\varepsilon\right)\right)^p$$

$$
\begin{aligned}
&\leq M_{p,r}2^n \sum_{1\leq k\leq 2^{n\theta}} \left(2\sum_{l=0}^{r}\left(\frac{(1+\varepsilon)g(k2^{-n})}{p(k2^{-n})^H}\right)^{lp}\right)(k2^{-n})^{(1+\varepsilon)^2}\\
&= M_{p,r}2^n \sum_{1\leq k\leq 2^{n\theta}} \left(2\sum_{l=0}^{r}\left(\frac{(1+\varepsilon)}{p}\sqrt{2\log\left(\frac{2^n}{k}\right)}\right)^{lp}\right)(k2^{-n})^{(1+\varepsilon)^2}\\
&\leq M_{p,r}2^{n(1+\theta)} \left(2\sum_{l=0}^{r}\left(\frac{(1+\varepsilon)}{p}\sqrt{2n\log 2}\right)^{lp}\right)2^{-n(1-\theta)(1+\varepsilon)^2},
\end{aligned}
$$

where the first inequality follows from the subadditivity property of capacities.

Now we only need to pick up a suitable θ such that $\sum_n I_n < \infty$. To this end, we want

$$1+\theta < (1-\theta)(1+\varepsilon)^2.$$

In fact, any

$$\theta \in \left(0, \frac{(1+\varepsilon)^2-1}{(1+\varepsilon)^2+1}\right)$$

will do. The proof is complete by applying the first Borel-Cantelli lemma for (p,r)-capacity and letting $\varepsilon \to 0$. □

The upper bound (4.25) indicates the following result:

Corollary 4.3.1. *$(B_t)_{t\geq 0}$ is α-Hölder-continuous for $\alpha < H$ quasi-surely with respect to the classical Wiener capacity.*

Remark 4.3.1. *We regard $(B_t)_{t\geq 0}$ as a family of measurable functions on $(\boldsymbol{W}, \mathscr{F})$ with parameter $t \geq 0$, where $\mathscr{F}$ is the completion of $\mathscr{B}(\boldsymbol{W})$ with respect to the Wiener measure P. What we proved previously is that apart from on a slim set, $t \mapsto B_t(\omega)$ is continuous. Therefore, we can modify $(B_t)_{t\geq 0}$ on the slim set K by for example setting $B_t(\omega) = 0$ for all $\omega \in K$ such that the modified process is continuous, and $K \in \mathscr{F}$ has measure $P(K) = 0$ as $c_{p,r}$ is increasing in p and r. From now on, we always refer $(B_t)_{t\geq 0}$ to its continuous modification.*

4.4 Non-differentiability

In this part, we will generalise a very standard result and prove that fBM sample paths are nowhere differentiable, based on the argument in [17] (see also Theorem 9.18, Section 2.9, Chapter 2, [33]), [21] and [67].

Theorem 4.4.1. *Let $H \in (0,1)$. Then for a fractional Brownian motion $(B_t)_{t\geq 0}$ with Hurst parameter H,*

$$\limsup_{h\downarrow 0} \frac{|B_{t+h} - B_t|}{h} = \infty \quad \textit{for all } t \in [0,1] \quad \textit{q.s.}$$

Proof. Let

$$A = \left\{ \limsup_{h\downarrow 0} \frac{|B_{t+h} - B_t|}{h} < \infty \text{ for some } t \in [0,1] \right\}.$$

The goal is to show that A is a slim set. If $\omega \in A$, then there exists a $t \in [0,1]$, positive integers M and k, such that

$$|B_{t+h}(\omega) - B_t(\omega)| \leq Mh$$

for all $0 \leq h \leq \frac{1}{k}$. Therefore, we may consider

$$A^t_{k,M} = \left\{ \sup_{h\in[0,\frac{1}{k}]} \frac{|B_{t+h} - B_t|}{h} \leq M \right\},$$

where M and k are positive integers. Then

$$A = \bigcup_{t\in[0,1]} \bigcup_{M=1}^{\infty} \bigcup_{k=1}^{\infty} A^t_{k,M}.$$

By the sub-additivity property of (p,r)-capacity, it remains to show that

$$c_{p,r}\Big(\bigcup_{t\in[0,1]} A^t_{k,M} \Big) = 0$$

for all $r \in \mathbb{N}$ and $1 < p < \infty$.

Fix r, p, k and M. For $H \in (0,1)$, take N to be the smallest integer such that

$$\frac{N(1-H)}{p} > 1,$$

and divide $[0,1]$ into n subintervals with $n \geq (N+1)k$. Then for all $t \in [\frac{i-1}{n}, \frac{i}{n}]$, $1 \leq i \leq n$,

$$\frac{i+N}{n} - t \leq \frac{1}{k},$$

which indicates that for $1 \leq j \leq N$,

$$\frac{i+j-1}{n} - t \leq \frac{i+j}{n} - t \leq \frac{i+N}{n} - t \leq \frac{1}{k}. \tag{4.27}$$

Now if $\omega \in A^t_{k,M}$ with $t \in [\frac{i-1}{n}, \frac{i}{n}]$, then for each $1 \leq j \leq N$, by (4.27),

$$\begin{aligned} \left|B_{\frac{i+j}{n}}(\omega) - B_{\frac{i+j-1}{n}}(\omega)\right| &\leq \left|B_{t+(\frac{i+j}{n}-t)}(\omega) - B_t(\omega)\right| + \left|B_t(\omega) - B_{t+(\frac{i+j-1}{n}-t)}(\omega)\right| \\ &\leq \left[\left(\frac{i+j}{n} - t\right) + \left(\frac{i+j-1}{n} - t\right)\right] M \\ &\leq \frac{(2j+1)M}{n}. \end{aligned}$$

Therefore, if we define

$$C_{i,n} = \bigcap_{j=1}^{N} \left\{ n^H \left| B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}} \right| \leq \frac{(2j+1)M}{n^{1-H}} \right\}, \quad 1 \leq i \leq n$$

for each $n \geq (N+1)k$, then

$$\bigcup_{t \in [0,1]} A^t_{k,M} \subset \bigcup_{i=1}^{n} C_{i,n}.$$

Therefore, it suffices to prove that $\sum_{i=1}^{n} c_{p,r}(C_{i,n}) \to 0$ as $n \to \infty$.

To this end, we apply Lemma 4.2.2 to bound $c_{p,r}(C_{i,n})$ from above. For each fixed

i, set

$$X_j = n^H \left(B_{\frac{i+j}{n}} - B_{\frac{i+j-1}{n}} \right)$$

and

$$\alpha_j = \frac{(2j+1)M}{n^{1-H}}$$

for all $1 \leq j \leq N$. By Lemma 4.2.2 with $L = \frac{1}{n}$, it follows that

$$\begin{aligned} \sum_{i=1}^{n} c_{p,r}(C_{i,n}) \leq & \left(\sum_{l=0}^{r} \left(N^{lp} C_H^{lp/2} \left(\frac{M_r}{c} \right)^{lp} \right) \right)^{1/p} \\ & \cdot \sum_{i=1}^{n} \left(P \left(\bigcap_{j=1}^{N} \{ -\alpha_j - c \leq X_j \leq \alpha_j + c \} \right) \right)^{1/p}, \end{aligned} \tag{4.28}$$

where $c > 0$ is a constant, M_r and C_H are given as in Lemma 4.2.2. Note that $(X_1, \cdots, X_N)$ is a centred Gaussian random variable with covariance matrix Σ, determined by

$$\mathbb{E}[X_j X_k] = \frac{1}{2} \left[(k-j+1)^{2H} + (k-j-1)^{2H} \right] - (k-j)^{2H},$$

depending only on j and k. Σ is an $N \times N$ positive definite matrix independent of n. Therefore, the right-hand side of (4.28) may be computed explicitly as

$$\begin{aligned} & P \left(\bigcap_{j=1}^{N} \{ |X_j| \leq \alpha_j + c \} \right) \\ = & 2^N \int_0^{\alpha_N + c} \cdots \int_0^{\alpha_1 + c} \frac{1}{\sqrt{2\pi |\Sigma|}} \exp \left(-\frac{1}{2} \boldsymbol{x}^T \Sigma^{-1} \boldsymbol{x} \right) \mathrm{d}x_1 \cdots \mathrm{d}x_N \\ \leq & 2^N \frac{1}{\sqrt{2\pi |\Sigma|}} \prod_{j=1}^{N} (\alpha_j + c) \\ = & O(n^{-N(1-H)}), \end{aligned}$$

hence it follows that

$$\sum_{i=1}^{n}\left(P\left(\bigcap_{j=1}^{N}\{-\alpha_j - c \le X_j \le \alpha_j + c\}\right)\right)^{1/p} \le O\left(n \cdot n^{-N(1-H)/p}\right) \to 0$$

as $n \to \infty$, which completes the proof. $\square$

4.5 Law of the iterated logarithm

In this section, we establish the result on the law of the iterated logarithm for fBMs with Hurst parameter $H \in \left(0, \frac{1}{2}\right]$ with respect to the $(2,1)$-capacity defined on the classical Wiener space, using the method from [21].

Theorem 4.5.1. *Let $(B_t)_{t\ge 0}$ be a one-dimensional fBM on $(\boldsymbol{W}, \mathcal{H}, P)$ with Hurst parameter $H \in \left(0, \frac{1}{2}\right]$. Then it holds that*

$$c_{2,1}\left(\limsup_{t\downarrow 0}\frac{B_t}{\sqrt{2t^{2H}\log\log(1/t)}} > 1\right) = 0.$$

Proof. When $H = \frac{1}{2}$, the problem will be reduced to the case of standard Brownian motion, which will become the same as in [21] and [67].

The rest of our proof will be similar to the argument in [21]. Let

$$h(t) = \sqrt{2t^{2H}\log\log(1/t)}, \quad \forall t > 0.$$

Fix $\theta, \delta \in (0,1)$, and set $\eta = (1+\delta)h(\theta^n)$, $s = 0$, $t = \theta^n$ in Lemma 4.2.3, then it follows that

$$\begin{aligned}&\left(c_{2,1}\left(\sup_{0\le u\le\theta^n} B_u > (1+\delta)h(\theta^n)\right)\right)^2\\ &\le \left(\left(\frac{\theta^{2nH}}{\theta^{2nH}+\theta^n}\right)^2 (1+\delta^2)\log\log(\theta^{-n}) + 2\right)\end{aligned}$$

$$
\begin{aligned}
&\cdot \exp\left(-\frac{\theta^{2nH}}{\theta^{2nH}+\theta^n}(1+\delta)^2 \log\log(\theta^{-n})\right)\\
&\le \big((1+\delta^2)\log\log(\theta^{-n})+2\big)\left(n\log(\theta^{-1})\right)^{-\frac{\theta^{2nH}}{\theta^{2nH}+\theta^n}(1+\delta)^2}\\
&= C_1(\log n + C_2) n^{-\frac{\theta^{2nH}}{\theta^{2nH}+\theta^n}(1+\delta)^2}. \qquad (4.29)
\end{aligned}
$$

For each θ and δ, as $H < \frac{1}{2}$ and $\theta < 1$, there exists some N_0 such that for all $n \ge N_0$,

$$
\frac{\theta^{2nH}}{\theta^{2nH}+\theta^n}(1+\delta)^2 > 1,
$$

so the right-hand side of (4.29) is a term of a convergent series, and thus by the first Borel-Cantelli lemma for capacities,

$$
\sup_{0\le u\le\theta^n} B_u \le (1+\delta)h(\theta^n) \quad \text{eventually}
$$

under $(2,1)$-capacity. The rest of proof remains the same as in probability case. □

Theorem 4.5.2. *Let $(B_t)_{t\ge0}$ be a one-dimensional fBM on $(\boldsymbol{W},\mathcal{H},P)$ with Hurst parameter $H \in \left(0,\frac{1}{2}\right]$. Then it holds that*

$$
c_{2,1}\left(\limsup_{t\downarrow 0}\frac{B_t}{\sqrt{2t^{2H}\log\log(1/t)}} < 1\right) = 0.
$$

Proof. When $H=\frac{1}{2}$, the problem is reduced to the Brownian motion case, so we only need to consider the case when $H \in \left(0,\frac{1}{2}\right)$. Denote

$$
h(t) = \sqrt{2t^{2H}\log\log(1/t)}.
$$

Let $\theta \in (0,1)$, and define

$$
G_n = \left\{B_{\theta^n} - B_{\theta^{n+1}} < (1-\theta^H)h(\theta^n)\right\}.
$$

We want to prove that

$$c_{2,1}\left(\liminf_{n\to\infty} G_n\right) = 0,$$

from which we may deduce that for sufficiently large n,

$$B_{\theta^n} - B_{\theta^{n+1}} > (1-\theta^H)h(\theta^n)$$

apart from on a $(2,1)$-capacity zero set.

Write

$$X_n = \frac{B_{\theta^n} - B_{\theta^{n+1}}}{(\theta^n - \theta^{n+1})^H} \sim N(0,1),$$

then by definition,

$$G_n = \left\{ X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})} \right\}.$$

For any integer $l \leq N$, take a decreasing sequence of real numbers $\{a_i\}_{i=1}^{\infty}$ such that $a_i \downarrow -\infty$ as $i \to \infty$, due to the continuity of capacities (2.7), we have that

$$\begin{aligned}
\left(c_{2,1}\left(\bigcap_{n=l}^{N} G_n\right)\right)^2 &= \left(c_{2,1}\left(\bigcap_{n=l}^{N}\left\{X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}\right\}\right)\right)^2 \\
&= \left(c_{2,1}\left(\bigcup_{i=1}^{\infty}\bigcap_{n=l}^{N}\left\{a_i < X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}\right\}\right)\right)^2 \\
&= \lim_{i\to\infty}\left(c_{2,1}\left(\bigcap_{n=l}^{N}\left\{a_i < X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})}\right\}\right)\right)^2.
\end{aligned}$$

Then we may apply Lemma 4.2.2 to control the intersection capacity with the intersection probability as the following:

$$\left(c_{2,1}\left(\bigcap_{n=l}^{N} G_n\right)\right)^2 \leq \lim_{i\to\infty}\left(1 + (N-l)^2 C_H \left(\frac{M_r}{c}\right)^2\right)$$

$$
\begin{aligned}
&\cdot P\left(\bigcap_{n=l}^{N}\left\{a_i - c_n < X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})} + c_n\right\}\right)\\
&\leq \left(1+(N-l)^2 C_H \left(\frac{M_r}{c}\right)^2\right)\\
&\cdot P\left(\bigcap_{n=l}^{N}\left\{X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})} + c_n\right\}\right). \qquad (4.30)
\end{aligned}
$$

When $H \in \left(0, \frac{1}{2}\right)$, the increments of fBMs over distinct time intervals are negatively correlated, i.e. $\mathbb{E}[X_n X_m] \leq 0$. For all $l \leq n, m \leq N$, we may take a sequence of independent standard Gaussian random variables $\{Y_n\}$, and apply Slepian's lemma (what we use here is a corollary of Proposition 4.2.1, see Corollary 3.12, Chapter 3, [39]) to the intersection probability in the last line in (4.30) to obtain that

$$
\begin{aligned}
&P\left(\bigcap_{n=l}^{N}\left\{X_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})} + c_n\right\}\right)\\
&\leq P\left(\bigcap_{n=l}^{N}\left\{Y_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})} + c_n\right\}\right)\\
&= \prod_{n=l}^{N} P\left(Y_n < \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})} + c_n\right)\\
&= \prod_{n=l}^{N}\left(1 - P\left(Y_n \geq \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})} + c_n\right)\right)\\
&\leq \exp\left(-\sum_{n=l}^{N} P\left(Y_n \geq \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})} + c_n\right)\right),
\end{aligned}
$$

where the last inequality follows from the fact that $1 - x \leq e^{-x}$.

We proceed by picking up suitable c_n's such that the left-hand side of (4.30) vanishes as N tends to infinity. Notice that for each $n \in \{l, \cdots, N\}$, it holds that

$$
\begin{aligned}
&P\left(Y_n \geq \alpha\sqrt{2\log\log(\theta^{-n})}\right)\\
&= \frac{1}{\sqrt{2\pi}}\int_{\alpha\sqrt{2\log\log(\theta^{-n})}}^{\infty} e^{-\frac{x^2}{2}}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{1}{\sqrt{2\pi}} \frac{\alpha\sqrt{2\log\log(\theta^{-n})}}{1+2\alpha^2\log\log(\theta^{-n})} \exp\left(-\alpha^2 \log\log(\theta^{-n})\right) \\
&\geq \frac{1}{\sqrt{2\pi}} \frac{1}{C_1\sqrt{2\alpha^2\log\log(\theta^{-n})}} \cdot \frac{1}{n^{\alpha^2}(\log(\theta^{-1}))^{\alpha^2}} \\
&\geq \frac{C_2}{n^{\alpha^2}\sqrt{\log n}},
\end{aligned} \tag{4.31}
$$

where C_1 and C_2 are positive constants. Choose suitable C and small β such that $\log x < Cx^\beta$ for large x, and set c_n to be small enough such that the quantity

$$
\alpha = \frac{c_n}{\sqrt{2\log\log(\theta^{-n})}} + \frac{1-\theta^H}{(1-\theta)^H}
$$

satisfies

$$
\gamma = \alpha^2 + \frac{\beta}{2} < 1.
$$

By taking α equal to the above value in (4.31), we conclude that

$$
\begin{aligned}
\sum_{n=l}^{N} P\left(Y_n \geq \frac{1-\theta^H}{(1-\theta)^H}\sqrt{2\log\log(\theta^{-n})} + c_n\right) &\geq \sum_{n=l}^{N} \frac{C_3}{n^\gamma} \\
&\geq C_3(N^{1-\gamma} - l^{1-\gamma}),
\end{aligned}
$$

where C_3 is a positive constant. Therefore,

$$
\left(c_{2,1}\left(\bigcap_{n=l}^{\infty} G_n\right)\right)^2 \leq \left(c_{2,1}\left(\bigcap_{n=l}^{N} G_n\right)\right)^2 \leq C'(N-l)^2 C_H e^{-C_3(N^{1-\gamma}-l^{1-\gamma})},
$$

where C' is some positive constant, and

$$
C_H = \max\left\{2^{2H-1}-1, 1\right\} \leq 1
$$

as in Lemma 4.2.2. Since the right-hand side of the above inequality vanishes as N

tends to infinity, we arrive at

$$c_{2,1}\left(\liminf_{n\to\infty} G_n\right) = 0.$$

□

4.6 Self-intersection

Recall that $\boldsymbol{W}_0^d$ consists of all $\mathbb{R}^d$-valued continuous paths, starting at the origin, and $(\boldsymbol{W}_0^d, \mathcal{H}, P)$ is the corresponding classical Wiener space. In this section, a d-dimensional fBM is defined to be the functional on $(\boldsymbol{W}_0^d, \mathcal{H}, P)$ given by the integral

$$B_t = \int_0^t K(t,s)d\omega(s), \tag{4.32}$$

where $\omega \in \boldsymbol{W}_0^d$ is a d-dimensional Brownian motion. By definition, a d-dimensional fBM is d copies of independent one-dimensional fBM defined as in (2.11) due to the definition of multi-dimensional Brownian motion. Like in the one-dimensional case, we take a suitable modification of B_t such that it is quasi-surely continuous with respect to the classical Wiener capacity.

In this section, we will study the self-avoiding property of d-dimensional fBMs and establish a result with respect to $(2,1)$-capacity on $(\boldsymbol{W}_0^d, \mathcal{H}, P)$, following the idea by Kakutani [32] together with several techniques from Fukushima [21] and Takeda [67] to deal with capacities.

Theorem 4.6.1. *Let $B = (B_t)_{t\geq 0}$ be the d-dimensional fBM defined in (4.32) with Hurst parameter H. When $H \leq \frac{1}{2}$ and $d > \frac{2}{H} + 2$, B has no double point under $(2,1)$-capacity on the classical Wiener space; when $H \geq \frac{1}{2}$ and $d > 6$, B has no double point under $(2,1)$-capacity.*

Proof. When $H = \frac{1}{2}$, the above result is proved in Fukushima [21] and Takeda [67].

It suffices to show that for any two disjoint intervals $I = (s_0, s_1)$ and $J = (t_0, t_1)$ with $s_0 < s_1 < t_0 < t_1$,

$$c_{2,1}\left(B_s = B_t, \text{ for some } s \in I \text{ and some } t \in J\right) = 0. \tag{4.33}$$

By the self-similarity property of fBMs, we only need to establish the above equality (4.33) for $0 \le s_0 < s_1 < t_0 < t_1 \le 1$. Denote the set

$$A = \left\{\omega \in \boldsymbol{W}_0^d : B_s(\omega) = B_t(\omega), \text{ for some } s \in I \text{ and some } t \in J\right\}.$$

Then for any $\eta > 0$, we have

$$\begin{aligned} A \subset & \bigcap_{i=1}^{d} \left\{\left|B_{s_1}^i - B_{t_0}^i\right| < 2\eta\right\} \cup \bigcup_{i=1}^{d} \left\{\sup_{s\in I}\left|B_{s_1}^i - B_s^i\right| > \eta\right\} \\ & \cup \bigcup_{i=1}^{d} \left\{\sup_{t\in J}\left|B_t^i - B_{t_0}^i\right| > \eta\right\}, \end{aligned}$$

where B^i is the i-th component of B. It thus follows from the sub-additivity property of capacities that

$$\begin{aligned} c_{2,1}(A) \le & c_{2,1}\left(\bigcap_{i=1}^{d} \left\{\left|B_{s_1}^i - B_{t_0}^i\right| < 2\eta\right\}\right) \\ & + \sum_{i=1}^{d} c_{2,1}\left(\sup_{s\in I}\left|B_{s_1}^i - B_s^i\right| > \eta\right) \\ & + \sum_{i=1}^{d} c_{2,1}\left(\sup_{t\in J}\left|B_t^i - B_{t_0}^i\right| > \eta\right). \end{aligned}$$

Applying Lemma 4.2.2 with $c = c_i = \eta$, $i = 1, 2, \cdots, d$, we obtain that

$$c_{2,1}\left(\bigcap_{i=1}^{d} \left\{-2\eta < B_{s_1}^i - B_{t_0}^i < 2\eta\right\}\right) \le \left(1 + d^2 C_H \left(\frac{M}{\eta}\right)^2\right)$$

$$\cdot P\left(\bigcap_{i=1}^{d}\left\{\left|B_{s_1}^i - B_{t_0}^i\right| < 3\eta\right\}\right),$$

where $C_H = \max\left\{2^{2H-1}-1, 1\right\} \leq 1$, and M is some positive constant.

Therefore,

$$\begin{aligned}
&c_{2,1}\left(\bigcap_{i=1}^{d}\left\{\left|B_{s_1}^i - B_{t_0}^i\right| < 2\eta\right\}\right) \\
&\leq \left(1 + d^2\left(\frac{M}{\eta}\right)^2\right)\prod_{i=1}^{d} P\left(\left|B_{s_1}^i - B_{t_0}^i\right| < 3\eta\right) \\
&= \left(1 + d^2\left(\frac{M}{\eta}\right)^2\right)\left(\frac{1}{\sqrt{2\pi(t_0-s_1)^{2H}}}\int_{-3\eta}^{3\eta}\exp\left(-\frac{x^2}{2(t_0-s_1)^{2H}}\right)\mathrm{d}x\right)^d \\
&\leq \left(1 + d^2\left(\frac{M}{\eta}\right)^2\right)\left(\frac{6\eta}{\sqrt{2\pi\left(d(I,J)\right)^{2H}}}\right)^d,
\end{aligned}$$

where $d(I,J) = t_0 - s_1$ denotes the distance between these two intervals. Also, applying Lemma 4.2.3, we deduce that

$$\begin{aligned}
c_{2,1}\left(\sup_{s\in I}\left|B_{s_1}^i - B_s^i\right| > \eta\right) &\leq \sqrt{\frac{\eta^2(s_1-s_0)^{2H}}{\left[\gamma_H(s_1-s_0)^{2H} + (s_1-s_0)\right]^2} + 4} \\
&\quad\cdot \exp\left(-\frac{\eta^2}{4[\gamma_H(s_1-s_0)^{2H} + (s_1-s_0)]}\right) \\
&= \sqrt{\frac{\eta^2|I|^{2H}}{\left(\gamma_H|I|^{2H} + |I|\right)^2} + 4}\cdot\exp\left(-\frac{\eta^2}{4\left(\gamma_H|I|^{2H} + |I|\right)}\right),
\end{aligned}$$

where $|I| = s_1 - s_0$ denotes the length of I. Accordingly,

$$c_{2,1}\left(\sup_{t\in J}\left|B_t^i - B_{t_0}^i\right| > \eta\right) \leq \sqrt{\frac{\eta^2|J|^{2H}}{\left(\gamma_H|J|^{2H} + |J|\right)^2} + 4}\cdot\exp\left(-\frac{\eta^2}{4\left(\gamma_H|J|^{2H} + |J|\right)}\right),$$

with $|J| = t_1 - t_0$, the length of interval J.

Divide I and J into k subintervals evenly, i.e. $I = \bigcup_{m=1}^{k} I_m$, $J = \bigcup_{l=1}^{k} J_l$, I_m and J_l are disjoint for all $1 \leq m, l \leq k$ and $|I_m| = |I|/k$, $|J_l| = |J|/k$. By sub-additivity and what has been proved above,

$$\begin{aligned}
c_{2,1}(A) \leq & \sum_{m=1}^{k}\sum_{l=1}^{k} c_{2,1}\Big(B_s = B_t, \text{ for some } s \in I_m \text{ and some } t \in J_l\Big) \\
\leq & \sum_{m=1}^{k}\sum_{l=1}^{k} \left[\left(1 + d^2\left(\frac{M}{\eta}\right)^2\right)\left(\frac{6\eta}{\sqrt{2\pi\left(d(I_m, J_l)\right)^{2H}}}\right)^d\right. \\
& + d\sqrt{\frac{\eta^2|I_m|^{2H}}{\left(\gamma_H|I_m|^{2H} + |I_m|\right)^2} + 4\exp\left(-\frac{\eta^2}{4\left(\gamma_H|I_m|^{2H} + |I_m|\right)}\right)} \\
& \left. + d\sqrt{\frac{\eta^2|J_l|^{2H}}{\left(\gamma_H|J_l|^{2H} + |J_l|\right)^2} + 4\exp\left(-\frac{\eta^2}{4\left(\gamma_H|J_l|^{2H} + |J_l|\right)}\right)}\right] \\
\leq & k^2\left[\left(1 + d^2\left(\frac{M}{\eta}\right)^2\right)\left(\frac{6\eta}{\sqrt{2\pi\left(d(I, J)\right)^{2H}}}\right)^d\right. \\
& + d\sqrt{\frac{\eta^2 k^{2H}}{\gamma_H^2|I|^{2H}} + 4\exp\left(-\frac{\eta^2}{4\left(\gamma_H|I|^{2H}k^{-2H} + |I|k^{-1}\right)}\right)} \\
& \left. + d\sqrt{\frac{\eta^2 k^{2H}}{\gamma_H^2|J|^{2H}} + 4\exp\left(-\frac{\eta^2}{4\left(\gamma_H|J|^{2H}k^{-2H} + |J|k^{-1}\right)}\right)}\right].
\end{aligned}$$

Now set $\eta = k^{-\sigma}$, then according to the previous estimate, when k is sufficiently large and $H < \frac{1}{2}$, it holds that

$$c_{2,1}(A) \leq C_1\left(k^{2-\sigma(d-2)} + k^{(H-\sigma)+2}e^{-Ck^{2(H-\sigma)}}\right), \tag{4.34}$$

where C_1 is some constant.

Notice that when

$$\frac{2}{d-2} < \sigma < H,$$

the expression on the right-hand side of (4.34) vanishes as k tends to infinity. This implies that if such a σ exists, then B has no double point under $(2,1)$-capacity, which only requires $\frac{2}{d-2} < H$, i.e. $d > \frac{2}{H} + 2$.

On the other hand, when $H > \frac{1}{2}$, by setting $\eta = k^{-\sigma}$, we get that

$$c_{2,1}(A) \leq C_2 \left(k^{2-\sigma(d-2)} + k^{(H-\sigma)+2} e^{-Ck^{1-2\sigma}} \right)$$

when k is sufficiently large, where C_2 is a constant. Therefore, in order to guarantee that the right-hand side vanishes as k tends to infinity, we require

$$\frac{2}{d-2} < \sigma < \frac{1}{2},$$

which forces $d > 6$. □

Remark 4.6.1. *For d-dimensional Brownian motion, the absence of double points under* $(2,1)$*-capacity was firstly proved by Fukushima in [21]. According to Lyons [43], the critical dimension for such a property is* $d = 6$*. Due to the lack of tools such as potential theory for the case of fBMs, the critical dimension of self-avoiding property for fBMs remains an open question, even in probability setting.*

Chapter 5

Large deviation principles for fractional Brownian motions under the classical Wiener capacity

In this chapter, we establish a version of large deviation principles (LDPs) for fBMs in the context of classical Wiener capacity.

To this end, we shall first prove that fBMs as Wiener functionals are quasi-surely defined on the classical Wiener space. Then we study the large deviations of the corresponding scaling mappings, and determine the rate function. We only prove these results for the case when the Hurst parameter $H \geq \frac{1}{2}$ due to technical reasons.

In this chapter, in order to simplify our computations a bit, we shall consider fractional Brownian motions over finite time interval $[0,1]$, and set $\boldsymbol{W} = C_0([0,1])$, the space of all continuous function on $[0,1]$ with $\omega(0) = 0$. The norm $\|\cdot\|$ on $\boldsymbol{W}$ is the supremum norm, i.e.

$$\|\omega\| = \sup_{t\in[0,1]} |\omega(t)|.$$

We recall that fBMs $B = (B_t)_{t\in[0,1]}$ can be realised as Wiener functionals on the classical Wiener space due to the following integral representation:

$$B_t(\omega) = \int_0^t K(t,s)\mathrm{d}\omega(s), \tag{5.1}$$

where $\{\omega(s) : s \geq 0\}$ is the coordinate mapping process on $\boldsymbol{W}$ (hence a standard Brownian motion). B_t is defined almost surely, and the stochastic integral on the right-hand side is understood in the sense of Itô. K is a kernel given by

$$K(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u, \quad s < t$$

when $H > \frac{1}{2}$, and c_H is some constant depending only on H. In addition, $s \mapsto \int_0^{s\wedge t} K(t,u)\mathrm{d}u$ is not in $\boldsymbol{W}^*$, but only in $\mathcal{H}$.

In this chapter, our strategy of the proof can be described as the following:

1. For each fixed $t > 0$, we define $B_t^{(m)}$ to be a finite linear combination of elements in the classical Wiener space, and show that the sequence of random variables $\left(B_t^{(m)}\right)_{m\in\mathbb{N}}$ converges quasi-surely (with respect to the capacities induced by standard Brownian motion). The main difficulty here is that the kernel $K(t,s)$ is singular in s, and will explode when s approaches 0, so it becomes quite difficult to control its increments in s and estimate the integral of K over small time intervals near time $s = 0$. However, we notice that K as a function of t, behaves more regularly. Therefore, we attempt to control the difference

$$K(t, s+\alpha) - K(t,s)$$

by the difference of K when t varies.

2. Then we may define a family of mappings $\{X^{(n)}\}_{n\in\mathbb{N}}$ on the classical Wiener

space by linear interpolation:

$$X^{(n)}(\omega)(t) := B_{\frac{k-1}{2^n}}(\omega)+2^n\left(t-\frac{k-1}{2^n}\right)\left(B_{\frac{k}{2^n}}(\omega)-B_{\frac{k-1}{2^n}}(\omega)\right), \quad \forall \frac{k-1}{2^n} < t \leq \frac{k}{2^n}.$$

For each n, $X^{(n)}$ is quasi-surely defined. We will show that this sequence $\{X^{(n)}\}_{n\in\mathbb{N}}$ converges to some mapping X. As any countable union of capacity zero sets still has zero capacity, we conclude that the limit random variable X is quasi-surely defined, and we shall also see that this convergence is exponentially fast in the proof. Now as the large deviation principles may be established for $X^{(n)}$'s, using the result of exponentially good approximations from the LDP theory, we deduce the LDP for the limit mapping X.

5.1 FBMs as Wiener functionals

Let $B = (B_t)_{t\in[0,1]}$ be a fractional Brownian motion with Hurst parameter $H \geq \frac{1}{2}$ defined by (5.1), which can be realised as a Wiener functional on the classical Wiener space. According to the result in Chapter 4 (see also the transfer principle, Proposition 5.2.1, page 288, [58]), for each fixed t, $B_t \in \mathbb{D}_r^p$, and its first order Malliavin derivative is given by

$$DB_t(s) = \int_0^{s\wedge t} K(t,u)\mathrm{d}u.$$

The higher-order derivatives of B_t vanish.

Recall that in the previous chapter, to prove that for each fixed t, B_t is a smooth random variable in the sense of Malliavin, we construct a sequence of random variables $(B_t^{(m)})_{m\in\mathbb{N}}$. These random variables are approximations of B_t in the Sobolev space $\mathbb{D}_r^p$, and are given by

$$B_t^{(m)}(\omega) := \sum_{i=0}^{2^m-1} \frac{2^m}{t} \int_{\frac{i}{2^m}t}^{\frac{i+1}{2^m}t} K(t,r)\mathrm{d}r \left(\omega_{\frac{i+1}{2^m}t} - \omega_{\frac{i}{2^m}t}\right), \tag{5.2}$$

and $B_0^{(m)} = 0$, where $m = 1, 2, \cdots$.

Obviously $B_t^{(m)} \in \mathbb{D}_r^p$ and by direct computation,

$$DB_t^{(m)}(s) = u_t^{(m)}(s) = \sum_{i=0}^{2^m-1} \frac{2^m}{t} \int_{\frac{i}{2^m}t}^{\frac{i+1}{2^m}t} K(t,r)\mathrm{d}r \mathbb{1}_{\left(\frac{i}{2^m}t, \frac{i+1}{2^m}t\right]}(s). \tag{5.3}$$

Its higher-order Malliavin derivatives vanish identically.

Theorem 5.1.1. *For all $r \in \mathbb{N}$, $1 < p < \infty$ and $t \in [0,1]$, $\left(B_t^{(m)}\right)_{m\in\mathbb{N}}$ converges (p,r)-quasi-surely to some limit, denoted by B_t too, which is also the limit of $\left(B_t^{(m)}\right)_{m\in\mathbb{N}}$ in $\mathbb{D}_r^p$.*

Proof. The proof is quite technical and will be divided into several steps. Let us begin with the simple fact that

$$\left\{\omega : \left(B_t^{(m)}(\omega)\right)_{m\geq 1} \text{ is not Cauchy}\right\} \subset \limsup_{m\to\infty} \left\{\omega : \left|B_t^{(m+1)}(\omega) - B_t^{(m)}(\omega)\right| > \frac{1}{2^{m\delta}}\right\}$$

for all $\delta > 0$. Therefore, by the first Borel-Cantelli lemma for capacities, as long as we can prove that

$$\sum_{m=1}^{\infty} c_{p,r}\left(\left|B_t^{(m+1)} - B_t^{(m)}\right| > \frac{1}{2^{m\delta}}\right) < \infty$$

for all $p \in (1,\infty)$ and $r \in \mathbb{N}$, then the required result follows. Since $c_{p,r}$ is increasing in p and r, it suffices to prove that the above infinite sum is finite for $p > 2$ and all $r \in \mathbb{N}$. Therefore, we shall assume that $p > 2$ in the sequel.

Step 1. In this step, we convert our problem from estimating the classical Wiener capacity to estimating L^2-norm of Gaussian random variables.

By Chebyshev's inequality, we have that for fixed $t > 0$,

$$c_{p,r}\left(\left|B_t^{(m+1)} - B_t^{(m)}\right| > \lambda\right) = c_{p,r}\left(\left|B_t^{(m+1)} - B_t^{(m)}\right|^2 > \lambda^2\right)$$
$$\leq \lambda^{-2}\left\|\left(B_t^{(m+1)} - B_t^{(m)}\right)^2\right\|_{\mathbb{D}_r^p} \tag{5.4}$$

for any $\lambda > 0$ and $m \in \mathbb{N}$. We notice that $\left(B_t^{(m+1)} - B_t^{(m)}\right)^2$ is a polynomial Wiener functional of degree 2, so Proposition 2.3.1 applies. Furthermore, by direct computation,

$$D\left(B_t^{(m+1)} - B_t^{(m)}\right) = u_t^{(m+1)} - u_t^{(m)},$$

and for all $l \geq 3$,

$$D^l\left(\left(B_t^{(m+1)} - B_t^{(m)}\right)^2\right) = 0.$$

Therefore, by Proposition 2.3.1,

$$\begin{aligned}
\left\|\left\|D^l\left(\left(B_t^{(m+1)} - B_t^{(m)}\right)^2\right)\right\|_{\mathcal{H}^{\otimes l}}\right\|_p &\leq 3(p-1)\left\|\left\|D^l\left(\left(B_t^{(m+1)} - B_t^{(m)}\right)^2\right)\right\|_{\mathcal{H}^{\otimes l}}\right\|_2 \\
&\leq 3(p-1)2^{\frac{l}{2}}\left\|\left(B_t^{(m+1)} - B_t^{(m)}\right)^2\right\|_2 \\
&= 3(p-1)2^{\frac{l}{2}}\left\|B_t^{(m+1)} - B_t^{(m)}\right\|_4^2 \\
&\leq 3(p-1)2^{\frac{l}{2}}\left(2\sqrt{3}\left\|B_t^{(m+1)} - B_t^{(m)}\right\|_2\right)^2 \\
&= 36(p-1)2^{\frac{l}{2}}\left\|B_t^{(m+1)} - B_t^{(m)}\right\|_2^2
\end{aligned}$$

for all $p > 2$ and $0 \leq l \leq r$, and thus we deduce that

$$\begin{aligned}
\left\|\left(B_t^{(m+1)} - B_t^{(m)}\right)^2\right\|_{\mathbb{D}_r^p} &\leq \sum_{l=0}^{r}\left\|\left\|D^l\left(\left(B_t^{(m+1)} - B_t^{(m)}\right)^2\right)\right\|_{\mathcal{H}^{\otimes l}}\right\|_p \\
&\leq 36(r+1)(p-1)2^{\frac{r}{2}}\left\|B_t^{(m+1)} - B_t^{(m)}\right\|_2^2
\end{aligned}$$

$$= C_{r,p} \left\| B_t^{(m+1)} - B_t^{(m)} \right\|_2^2, \tag{5.5}$$

where

$$C_{r,p} = 36(r+1)(p-1)2^{\frac{r}{2}}$$

is a constant depending only on r and p.

The L^2-norm on the right-hand side of the above inequality can be handled as the following. By definition (5.2),

$$\begin{aligned} B_t^{(m+1)}(\omega) - B_t^{(m)}(\omega) = & \sum_{i=0}^{2^{m+1}-1} \frac{2^{m+1}}{t} \int_{\frac{i}{2^{m+1}}t}^{\frac{i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \left(\omega_{\frac{i+1}{2^{m+1}}t} - \omega_{\frac{i}{2^{m+1}}t}\right) \\ & - \sum_{i=0}^{2^m-1} \frac{2^m}{t} \int_{\frac{i}{2^m}t}^{\frac{i+1}{2^m}t} K(t,r)\mathrm{d}r \left(\omega_{\frac{i+1}{2^m}t} - \omega_{\frac{i}{2^m}t}\right). \end{aligned}$$

The integral term in $B_t^{(m)}(\omega)$ may be split into two parts, i.e. for each $i \in \{0, 1, \cdots, 2^m - 1\}$,

$$\begin{aligned} & \frac{2^m}{t} \int_{\frac{i}{2^m}t}^{\frac{i+1}{2^m}t} K(t,r)\mathrm{d}r \left(\omega_{\frac{i+1}{2^m}t} - \omega_{\frac{i}{2^m}t}\right) \\ = & \frac{2^m}{t} \left(\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r)\mathrm{d}r + \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \right) \left(\omega_{\frac{i+1}{2^m}t} - \omega_{\frac{i}{2^m}t}\right) \\ = & \frac{2^m}{t} \left[\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r)\mathrm{d}r \left(\omega_{\frac{2i+2}{2^{m+1}}t} - \omega_{\frac{2i}{2^{m+1}}t}\right) \right. \\ & \left. + \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \left(\omega_{\frac{2i+2}{2^{m+1}}t} - \omega_{\frac{2i}{2^{m+1}}t}\right) \right]. \end{aligned}$$

We observe that the contribution from the interval $\left(\frac{i}{2^m}t, \frac{i+1}{2^m}t\right]$ in $B_t^{(m+1)}(\omega)$ is

$$\begin{aligned} \frac{2^{m+1}}{t} & \left[\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r)\mathrm{d}r \left(\omega_{\frac{2i+2}{2^{m+1}}t} - \omega_{\frac{2i+1}{2^{m+1}}t}\right) \right. \\ & \left. + \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \left(\omega_{\frac{2i+1}{2^{m+1}}t} - \omega_{\frac{2i}{2^{m+1}}t}\right) \right]. \end{aligned}$$

Therefore, it follows that

$$
\begin{aligned}
& B_t^{(m+1)}(\omega) - B_t^{(m)}(\omega) \\
&= \frac{2^m}{t} \sum_{i=0}^{2^m-1} \left[\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r)\mathrm{d}r \left(\omega_{\frac{2i+2}{2^{m+1}}t} - 2\omega_{\frac{2i+1}{2^{m+1}}t} + \omega_{\frac{2i}{2^{m+1}}t} \right) \right. \\
&\quad \left. + \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \left(2\omega_{\frac{2i+1}{2^{m+1}}t} - \omega_{\frac{2i}{2^{m+1}}t} - \omega_{\frac{2i+2}{2^{m+1}}t} \right) \right] \\
&= \frac{2^m}{t} \sum_{i=0}^{2^m-1} \left(\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r)\mathrm{d}r - \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \right) \\
&\quad \cdot \left(\omega_{\frac{2i+2}{2^{m+1}}t} - 2\omega_{\frac{2i+1}{2^{m+1}}t} + \omega_{\frac{2i}{2^{m+1}}t} \right). \qquad (5.6)
\end{aligned}
$$

Since a standard Brownian motion has independent increments, we have

$$
\begin{aligned}
& \left\| B_t^{(m+1)} - B_t^{(m)} \right\|_2^2 \\
&= \mathbb{E} \left[\sum_{i=0}^{2^m-1} \left(\frac{2^m}{t} \left(\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r)\mathrm{d}r - \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \right) \right)^2 \right. \\
&\quad \left. \cdot \left(\omega_{\frac{2i+2}{2^{m+1}}t} - 2\omega_{\frac{2i+1}{2^{m+1}}t} + \omega_{\frac{2i}{2^{m+1}}t} \right)^2 \right] \\
&= \left(\frac{2^m}{t} \right)^2 \sum_{i=0}^{2^m-1} \left(\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r)\mathrm{d}r - \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \right)^2 \\
&\quad \cdot \mathbb{E} \left[\left(\omega_{\frac{2i+2}{2^{m+1}}t} - 2\omega_{\frac{2i+1}{2^{m+1}}t} + \omega_{\frac{2i}{2^{m+1}}t} \right)^2 \right] \\
&= \left(\frac{2^m}{t} \right)^2 \sum_{i=0}^{2^m-1} \left(\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r)\mathrm{d}r - \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \right)^2 \\
&\quad \cdot \mathbb{E} \left[\left(\omega_{\frac{2i+2}{2^{m+1}}t} - \omega_{\frac{2i+1}{2^{m+1}}t} \right)^2 + \left(\omega_{\frac{2i+1}{2^{m+1}}t} - \omega_{\frac{2i}{2^{m+1}}t} \right)^2 \right] \\
&= \left(\frac{2^m}{t} \right)^2 \sum_{i=0}^{2^m-1} \left(\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r)\mathrm{d}r - \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \right)^2 \left(\frac{t}{2^m} \right) \\
&= \left(\frac{2^m}{t} \right) \sum_{i=0}^{2^m-1} \left(\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r)\mathrm{d}r - \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \right)^2. \qquad (5.7)
\end{aligned}
$$

Step 2. In this step, we further simplify our problem using a rather simple observation.

By change of variables, for each $i \in \{0, \cdots, 2^m - 1\}$, define

$$\begin{aligned} M_i &:= \int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r)\mathrm{d}r - \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \\ &= \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K\left(t, s + \frac{t}{2^{m+1}}\right)\mathrm{d}s - \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} K(t,r)\mathrm{d}r \\ &= \int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t} \left[K\left(t, s + \frac{t}{2^{m+1}}\right) - K(t,s)\right]\mathrm{d}s. \end{aligned}$$

Using the definition of kernel K, we observe that for all $\alpha \in (0, t-s)$,

$$\begin{aligned} K(t, s+\alpha) &= c_H (s+\alpha)^{\frac{1}{2}-H} \int_{s+\alpha}^{t} (u - s - \alpha)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u \\ &= c_H (s+\alpha)^{\frac{1}{2}-H} \int_{s}^{t-\alpha} (v-s)^{H-\frac{3}{2}} (v+\alpha)^{H-\frac{1}{2}} \mathrm{d}v \\ &= c_H (s+\alpha)^{\frac{1}{2}-H} \int_{s}^{t-\alpha} (v-s)^{H-\frac{3}{2}} v^{H-\frac{1}{2}} \left(\frac{v+\alpha}{v}\right)^{H-\frac{1}{2}} \mathrm{d}v \\ &\le c_H (s+\alpha)^{\frac{1}{2}-H} \left(\frac{s+\alpha}{s}\right)^{H-\frac{1}{2}} \int_{s}^{t-\alpha} (v-s)^{H-\frac{3}{2}} v^{H-\frac{1}{2}} \mathrm{d}v \\ &= c_H s^{\frac{1}{2}-H} \int_{s}^{t-\alpha} (v-s)^{H-\frac{3}{2}} v^{H-\frac{1}{2}} \mathrm{d}v \\ &= K(t-\alpha, s), \end{aligned}$$

and hence

$$K(t, s+\alpha) - K(t,s) \le K(t-\alpha, s) - K(t,s). \tag{5.8}$$

On the other hand, for every $\alpha \in (0, r)$,

$$\begin{aligned} K(t, r-\alpha) &= c_H (r-\alpha)^{\frac{1}{2}-H} \int_{r-\alpha}^{t} (u - r + \alpha)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u \\ &= c_H (r-\alpha)^{\frac{1}{2}-H} \int_{r}^{t+\alpha} (v-r)^{H-\frac{3}{2}} (v-\alpha)^{H-\frac{1}{2}} \mathrm{d}v \end{aligned}$$

$$\leq c_H(r-\alpha)^{\frac{1}{2}-H}\int_r^{t+\alpha}(v-r)^{H-\frac{3}{2}}v^{H-\frac{1}{2}}\mathrm{d}v$$
$$=\left(\frac{r}{r-\alpha}\right)^{H-\frac{1}{2}}K(t+\alpha,r).$$

By setting $r=s+\alpha$, we deduce that

$$\begin{aligned}K(t,s+\alpha)-K(t,s)&=K(t,s+\alpha)-K(t,s+\alpha-\alpha)\\&\geq K(t,s+\alpha)-\left(\frac{s+\alpha}{s}\right)^{H-\frac{1}{2}}K(t+\alpha,s+\alpha).\end{aligned}\tag{5.9}$$

Now let

$$\alpha=\frac{t}{2^{m+1}}$$

in (5.8) and (5.9), and for all $s<t-\frac{t}{2^{m+1}}$, set

$$\begin{aligned}L_i=&\int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t}K\left(t,s+\frac{t}{2^{m+1}}\right)\\&-\left(\frac{s+\frac{t}{2^{m+1}}}{s}\right)^{H-\frac{1}{2}}K\left(t+\frac{t}{2^{m+1}},s+\frac{t}{2^{m+1}}\right)\mathrm{d}s\\=&\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t}K(t,r)-\left(\frac{r}{r-\frac{t}{2^{m+1}}}\right)^{H-\frac{1}{2}}K\left(t+\frac{t}{2^{m+1}},r\right)\mathrm{d}r,\end{aligned}$$

and

$$U_i:=\int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t}K\left(t-\frac{t}{2^{m+1}},s\right)-K(t,s)\mathrm{d}s.$$

Then

$$L_i\leq M_i\leq U_i$$

for each i, and it thus follows that

$$M_i^2\leq L_i^2\vee U_i^2,$$

which implies that

$$\left\|B_t^{(m+1)} - B_t^{(m)}\right\|_2^2 = \left(\frac{2^m}{t}\right)\sum_{i=0}^{2^m-1} M_i^2$$
$$\leq \left[\left(\frac{2^m}{t}\right)\sum_{i=0}^{2^m-1} L_i^2\right] \vee \left[\left(\frac{2^m}{t}\right)\sum_{i=0}^{2^m-1} U_i^2\right]. \tag{5.10}$$

Step 3. In this step, we find upper bounds for L_i^2 and U_i^2, respectively. We first find a control of

$$\left(\frac{2^m}{t}\right)\sum_{i=0}^{2^m-1} L_i^2 = \left(\frac{2^m}{t}\right)\sum_{i=0}^{2^m-1}\left[\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} K(t,r) - \left(\frac{r}{r-\frac{t}{2^{m+1}}}\right)^{H-\frac{1}{2}} K(t+\frac{t}{2^{m+1}}, r)\mathrm{d}r\right]^2.$$

For all $r \in \left(\frac{t}{2^{m+1}}, t\right)$, consider the function

$$f_r(x) = \left(\frac{r}{r-x}\right)^{H-\frac{1}{2}} K(t+x, r), \quad 0 \leq x < r.$$

Then

$$f_r(0) = K(t,r)$$

and

$$f_r\left(\frac{t}{2^{m+1}}\right) = r^{H-\frac{1}{2}}\left(r - \frac{t}{2^{m+1}}\right)^{\frac{1}{2}-H} K\left(t+\frac{t}{2^{m+1}}, r\right).$$

Therefore, we may write

$$L_i = \int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t}\int_{\frac{t}{2^{m+1}}}^{0} f_r'(x)\mathrm{d}x\mathrm{d}r.$$

We may compute the derivative of f_r, which is

$$f_r'(x) = r^{H-\frac{1}{2}}\left[\left(H-\frac{1}{2}\right)(r-x)^{-\frac{1}{2}-H}K(t+x,r) + (r-x)^{\frac{1}{2}-H}\partial_1 K(t+x,r)\right]$$

$$
\begin{aligned}
&= \left(H - \frac{1}{2}\right) r^{H-\frac{1}{2}} (r-x)^{-\frac{1}{2}-H} K(t+x,r) \\
&\quad + c_H r^{H-\frac{1}{2}} (r-x)^{\frac{1}{2}-H} \left(\frac{t+x}{r}\right)^{H-\frac{1}{2}} (t+x-r)^{H-\frac{3}{2}} \\
&= \left(H - \frac{1}{2}\right) r^{H-\frac{1}{2}} (r-x)^{-\frac{1}{2}-H} K(t+x,r) \\
&\quad + c_H (r-x)^{\frac{1}{2}-H} (t+x)^{H-\frac{1}{2}} (t+x-r)^{H-\frac{3}{2}},
\end{aligned}
$$

where $\partial_1 K(t,s)$ denotes the partial derivative of K with respect to the first variable.

Denote

$$
g_r(x) = \left(H - \frac{1}{2}\right) r^{H-\frac{1}{2}} (r-x)^{-\frac{1}{2}-H} K(t+x,r)
$$

and

$$
h_r(x) = c_H (r-x)^{\frac{1}{2}-H} (t+x)^{H-\frac{1}{2}} (t+x-r)^{H-\frac{3}{2}}.
$$

Then for all $x < r$, $g_r(x) \geq 0$ and $h_r(x) \geq 0$. By Cauchy-Schwarz inequality, we obtain that

$$
\begin{aligned}
L_i^2 &= \left(- \int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} \int_0^{\frac{t}{2^{m+1}}} f_r'(x) \mathrm{d}x \mathrm{d}r \right)^2 \\
&= \left(\int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} \left(\int_0^{\frac{t}{2^{m+1}}} g_r(x) \mathrm{d}x + \int_0^{\frac{t}{2^{m+1}}} h_r(x) \mathrm{d}x \right) \mathrm{d}r \right)^2 \\
&\leq \frac{t}{2^{m+1}} \int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} \left(\int_0^{\frac{t}{2^{m+1}}} g_r(x) \mathrm{d}x + \int_0^{\frac{t}{2^{m+1}}} h_r(x) \mathrm{d}x \right)^2 \mathrm{d}r \\
&\leq \frac{t}{2^m} \int_{\frac{2i+1}{2^{m+1}}t}^{\frac{2i+2}{2^{m+1}}t} \left(\int_0^{\frac{t}{2^{m+1}}} g_r(x) \mathrm{d}x \right)^2 + \left(\int_0^{\frac{t}{2^{m+1}}} h_r(x) \mathrm{d}x \right)^2 \mathrm{d}r,
\end{aligned}
$$

and hence

$$
\left(\frac{2^m}{t}\right) \sum_{i=0}^{2^m-1} L_i^2 \leq \int_{\frac{t}{2^{m+1}}}^{t} \left(\int_0^{\frac{t}{2^{m+1}}} g_r(x) \mathrm{d}x \right)^2 + \left(\int_0^{\frac{t}{2^{m+1}}} h_r(x) \mathrm{d}x \right)^2 \mathrm{d}r. \tag{5.11}
$$

We control the integral of g_r first. When $H > \frac{1}{2}$, since $t \leq 1$,

$$\begin{aligned}
K(t,s)s^{H-\frac{1}{2}} &= c_H \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u \\
&\leq c_H t^{H-\frac{1}{2}} \int_s^t (u-s)^{H-\frac{3}{2}} \mathrm{d}u \\
&= c_H t^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \\
&\leq c_H t^{2H-1} \\
&\leq c_H,
\end{aligned}$$

i.e.

$$K(t,s) \leq c_H s^{\frac{1}{2}-H}.$$

Therefore, we deduce that

$$\begin{aligned}
0 \leq \int_0^{\frac{t}{2^{m+1}}} g_r(x)\mathrm{d}x &= \left(H - \frac{1}{2}\right) \int_0^{\frac{t}{2^{m+1}}} r^{H-\frac{1}{2}} (r-x)^{-\frac{1}{2}-H} K(t+x, r)\mathrm{d}x \\
&\leq c_H \left(H - \frac{1}{2}\right) \int_0^{\frac{t}{2^{m+1}}} (r-x)^{-\frac{1}{2}-H} \mathrm{d}x \\
&= c_H \left[\left(r - \frac{t}{2^{m+1}}\right)^{\frac{1}{2}-H} - r^{\frac{1}{2}-H} \right].
\end{aligned}$$

As

$$\left(r - \frac{t}{2^{m+1}}\right)^{\frac{1}{2}-H} \geq r^{\frac{1}{2}-H},$$

it follows that

$$\begin{aligned}
\left[\left(r - \frac{t}{2^{m+1}}\right)^{\frac{1}{2}-H} - r^{\frac{1}{2}-H} \right]^2 &= \left(r - \frac{t}{2^{m+1}}\right)^{1-2H} - 2\left(r - \frac{t}{2^{m+1}}\right)^{\frac{1}{2}-H} r^{\frac{1}{2}-H} \\
&\quad + r^{1-2H} \\
&\leq \left(r - \frac{t}{2^{m+1}}\right)^{1-2H} - r^{1-2H}.
\end{aligned}$$

Consequently,

$$\left(\int_0^{\frac{t}{2^{m+1}}} g_r(x)\mathrm{d}x\right)^2 \leq C_1\left[\left(r-\frac{t}{2^{m+1}}\right)^{1-2H} - r^{1-2H}\right], \tag{5.12}$$

where C_1 is a constant depending only on H.

As for h_r, due to change of variables,

$$\begin{aligned}
0 \leq \int_0^{\frac{t}{2^{m+1}}} h_r(x)\mathrm{d}x &= c_H \int_0^{\frac{t}{2^{m+1}}} (r-x)^{\frac{1}{2}-H}\,(t+x)^{H-\frac{1}{2}}\,(t+x-r)^{H-\frac{3}{2}}\mathrm{d}x \\
&\leq c_H\,(2t)^{H-\frac{1}{2}} \int_0^{\frac{t}{2^{m+1}}} (r-x)^{\frac{1}{2}-H}(t+x-r)^{H-\frac{3}{2}}\mathrm{d}x \\
&= c_H\,(2t)^{H-\frac{1}{2}} \int_{r-\frac{t}{2^{m+1}}}^{r} y^{\frac{1}{2}-H}(t-y)^{H-\frac{3}{2}}\mathrm{d}y \\
&\leq c_H\,(2t)^{H-\frac{1}{2}} \left(r-\frac{t}{2^{m+1}}\right)^{\frac{1}{2}-H} \int_{r-\frac{t}{2^{m+1}}}^{r} (t-y)^{H-\frac{3}{2}}\mathrm{d}y \\
&\leq \frac{2c_H}{H-\frac{1}{2}} \left(r-\frac{t}{2^{m+1}}\right)^{\frac{1}{2}-H} \\
&\quad \cdot \left[\left(t+\frac{t}{2^{m+1}}-r\right)^{H-\frac{1}{2}} - (t-r)^{H-\frac{1}{2}}\right] \\
&\leq \frac{2c_H}{H-\frac{1}{2}} \left(r-\frac{t}{2^{m+1}}\right)^{\frac{1}{2}-H} \left(\frac{t}{2^{m+1}}\right)^{H-\frac{1}{2}},
\end{aligned}$$

which implies that

$$\left(\int_0^{\frac{t}{2^{m+1}}} h_r(x)dx\right)^2 \leq C_2\left(\frac{t}{2^{m+1}}\right)^{2H-1}\left(r-\frac{t}{2^{m+1}}\right)^{1-2H}, \tag{5.13}$$

with C_2 some constant only depending on the value of H. Using (5.11), we get that

$$\begin{aligned}
\left(\frac{2^m}{t}\right)\sum_{i=0}^{2^m-1} L_i^2 \leq C_1 &\int_{\frac{t}{2^{m+1}}}^{t} \left(r-\frac{t}{2^{m+1}}\right)^{1-2H} - r^{1-2H}\mathrm{d}r \\
&+ C_2\left(\frac{t}{2^{m+1}}\right)^{2H-1}\int_{\frac{t}{2^{m+1}}}^{t}\left(r-\frac{t}{2^{m+1}}\right)^{1-2H}\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&= C_1 \frac{1}{2-2H}\left[\left(t-\frac{t}{2^{m+1}}\right)^{2-2H} - t^{2-2H} + \left(\frac{t}{2^{m+1}}\right)^{2-2H}\right] \\
&\quad + C_2\left(\frac{t}{2^{m+1}}\right)^{2H-1} \frac{1}{2-2H}\left(t-\frac{t}{2^{m+1}}\right)^{2-2H} \\
&\leq C_1 \frac{1}{1-H}\left(\frac{t}{2^{m+1}}\right)^{2-2H} + C_2 \frac{1}{2-2H}\left(\frac{t}{2^{m+1}}\right)^{2H-1} \\
&= c_1\left(\frac{t}{2^{m+1}}\right)^{2-2H} + c_2\left(\frac{t}{2^{m+1}}\right)^{2H-1}, \qquad (5.14)
\end{aligned}
$$

where c_1 and c_2 are two positive constants.

Next, we move onto the estimate of U_i's. By the definition of U_i's and Cauchy-Schwarz inequality, we have that

$$
\begin{aligned}
&\left(\frac{2^m}{t}\right)\sum_{i=0}^{2^m-1} U_i^2 \\
&\leq \left(\frac{2^m}{t}\right)\sum_{i=0}^{2^m-1}\left(\frac{t}{2^{m+1}}\right)\int_{\frac{2i}{2^{m+1}}t}^{\frac{2i+1}{2^{m+1}}t}\left(K\left(t-\frac{t}{2^{m+1}}, s\right) - K(t,s)\right)^2 \mathrm{d}s \\
&= \frac{1}{2}\int_0^{t-\frac{t}{2^{m+1}}}\left(K\left(t-\frac{t}{2^{m+1}}, s\right) - K(t,s)\right)^2 \mathrm{d}s \\
&\leq \frac{1}{2}\int_0^{t-\frac{t}{2^{m+1}}} K^2\left(t-\frac{t}{2^{m+1}}, s\right)\mathrm{d}s - \int_0^{t-\frac{t}{2^{m+1}}} K\left(t-\frac{t}{2^{m+1}}, s\right)K(t,s)\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t K^2(t,s)\mathrm{d}s \\
&= \frac{1}{2}\left(\frac{t}{2^{m+1}}\right)^{2H}. \qquad (5.15)
\end{aligned}
$$

Step 4. We complete our proof using the above estimates in this step.

It follows from (5.10), (5.14) and (5.15) that

$$
\begin{aligned}
\left\|B_t^{(m+1)} - B_t^{(m)}\right\|_2^2 &\leq \left[c_1\left(\frac{t}{2^{m+1}}\right)^{2-2H} + c_2\left(\frac{t}{2^{m+1}}\right)^{2H-1}\right] \vee \left[\frac{1}{2}\left(\frac{t}{2^{m+1}}\right)^{2H}\right] \\
&\leq c_3\left(\frac{t}{2^{m+1}}\right)^{2-2H} + c_4\left(\frac{t}{2^{m+1}}\right)^{2H-1},
\end{aligned}
$$

where c_3 and c_4 are some constants.

Therefore, for any $\lambda > 0$, it holds that

$$c_{p,r}\left(\left|B_t^{(m+1)} - B_t^{(m)}\right| > \lambda\right) \le C_{r,p}\lambda^{-2}\left\|B_t^{(m+1)} - B_t^{(m)}\right\|_2^2$$
$$\le C_{r,p}\lambda^{-2}\left[c_3\left(\frac{t}{2^{m+1}}\right)^{2-2H} + c_4\left(\frac{t}{2^{m+1}}\right)^{2H-1}\right].$$

Set $\lambda = 2^{-m\delta}$, then as $t \le 1$,

$$c_{p,r}\left(\left|B_t^{(m+1)} - B_t^{(m)}\right| > \frac{1}{2^{m\delta}}\right) \le C_{r,p,H}\left(\frac{1}{2^{2m(1-H-\delta)}} + \frac{1}{2^{2m(H-\frac{1}{2}-\delta)}}\right),$$

where $C_{r,p,H}$ is a suitable constant depending on r, p and H only.

Hence, if we choose δ small enough such that

$$0 < \delta < (1-H) \wedge \left(H - \frac{1}{2}\right),$$

then

$$\sum_{m=1}^{\infty} c_{p,r}\left(\left|B_t^{(m+1)} - B_t^{(m)}\right| > \frac{1}{2^{m\delta}}\right) < \infty,$$

which implies that $(B_t^{(m)})_{m\ge 0}$ converges quasi-surely to some random variable, denoted by $\widetilde{B}_t$, as m tends to infinity by the first Borel-Cantelli lemma.

In Chapter 4, we have already shown that $(B_t^{(m)})_{m\ge 0}$ converges in $\mathbb{D}_r^p$ to B_t for all $r \in \mathbb{N}$ and $1 < p < \infty$, and the Malliavin derivative of the limit random variable is given by

$$DB_t(s) = K(t,s)\mathbb{1}_{[0,t]}(s),$$

with all higher-order Malliavin derivatives of B_t equal to zero. Now we can easily prove that there exists a subsequence $(B_t^{(m_k)})_{k\ge 0}$ converging quasi-surely by choosing this sequence to be such that (for example by applying Hölder's inequality)

$$\left\| \left(B_t^{(m_{k+1})} - B_t^{(m_k)} \right)^2 \right\|_{\mathbb{D}_r^p} \leq \frac{1}{2^{k+1}},$$

and then applying the first Borel-Cantelli lemma as before. If for $\omega \in \boldsymbol{W}$, there are infinitely many k's such that

$$\left| B_t^{(m_{k+1})}(\omega) - B_t^{(m_k)}(\omega) \right| > 1,$$

then $(B_t^{(m_k)}(\omega))_{k\geq 0}$ is not Cauchy. Therefore, by Chebyshev's inequality,

$$\begin{aligned}
&\sum_{k=0}^{\infty} c_{p,r} \left\{ \omega : \left| B_t^{(m_{k+1})}(\omega) - B_t^{(m_k)}(\omega) \right| > 1 \right\} \\
&= \sum_{k=0}^{\infty} c_{p,r} \left\{ \omega : \left(B_t^{(m_{k+1})}(\omega) - B_t^{(m_k)}(\omega) \right)^2 > 1 \right\} \\
&\leq \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \\
&< \infty,
\end{aligned}$$

and hence by the first Borel-Cantelli lemma,

$$c_{p,r} \left\{ \left| B_t^{(m_{k+1})} - B_t^{(m_k)} \right| > 1 \text{ infinitely often} \right\} = 0.$$

As a consequence, $(B_t^{(m_k)})_{k\geq 0}$ converges to B_t apart from on a slim set, and the uniqueness of limit forces its limit to be $\widetilde{B}_t$, which implies $B_t = \widetilde{B}_t$ q.s. □

Consequently, we will have the following corollary.

Corollary 5.1.1. *Let $r \in \mathbb{N}$, $1 < p < \infty$, and let $H \in \left[\frac{1}{2}, 1\right)$. For each $t \in [0,1]$, there is a modification of B_t, given by the integral representation (5.1) of the fractional Brownian motion with Hurst parameter H, that is defined (p,r)-quasi-surely.*

From now on, we work with the modification of B_t which is defined as a quasi-sure limit of the approximations $B_t^{(m)}$.

5.2 Exponential tightness

For each fixed t, B_t is quasi-surely defined (with $B_0(\omega) = 0$ for all $\omega \in \boldsymbol{W}$), so for each integer $m \in \mathbb{N}$, we may define a map $X^{(m)} : \boldsymbol{W} \to \boldsymbol{W}$ by

$$X^{(m)}(\omega)(t) := B_{t_{k-1}^m}(\omega) + 2^m \left(t - t_{k-1}^m\right) \left(B_{t_k^m}(\omega) - B_{t_{k-1}^m}(\omega)\right), \quad \forall t_{k-1}^m \leq t \leq t_k^m,$$

where $t_k^m = \frac{k}{2^m}$.

Then $X^{(n)}$ is quasi-surely defined as it is a linear interpolation of finitely many $B_{t_k^m}$'s. For each m, let $X^{\varepsilon,(m)}$ to be the scaled map, which is defined as

$$X^{\varepsilon,(m)}(\omega) = X^{(m)}(\varepsilon\omega).$$

As B_t is the limit of linear combinations of ω_t's, it follows that

$$X^{\varepsilon,(m)}(\omega) = \varepsilon X^{(m)}(\omega).$$

Our goal is to show that the sequence $\left(X^{(m)}\right)_{m\in\mathbb{N}}$ converges to some X quasi-surely as m tends to infinity, which implies that $\left(X^{\varepsilon,(m)}\right)_{m\in\mathbb{N}}$ converges to X^ε quasi-surely, where the scaled map X^ε is given by

$$X^\varepsilon(\omega) = X(\varepsilon\omega) = \varepsilon X(\omega).$$

Moreover, the fact that the family of scaled maps $\left(X^{\varepsilon,(m)}\right)_{m\in\mathbb{N}}$ converges exponentially fast will be revealed in the proof as well. Since X^ε is quasi-surely defined with exponentially good approximations $\left(X^{\varepsilon,(m)}\right)_{m\in\mathbb{N}}$, we may apply the result from the

LDP theory to conclude the final result.

We will need the following estimate from the rough path analysis, which is contained in [45] (see Proposition 4.1.1, Chapter 4 or equation (4.15) on page 64). Here, we adapt the result to our case and state it as the following:

Proposition 5.2.1. *Let u and w be two continuous paths in a Banach space. Then for any $q > 1$ and $\gamma > q - 1$, there exists a constant $C_{q,\gamma}$ depending only on q and γ such that*

$$\sup_D \sum_l \left|u_{t_{l-1},t_l} - w_{t_{l-1},t_l}\right|^q \le C_{q,\gamma} \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left|u_{t^n_{k-1},t^n_k} - w_{t^n_{k-1},t^n_k}\right|^q,$$

where the supremum is taken over all finite partitions D of $[0,1]$, $t^n_l = \frac{l}{2^n}$ for $n = 1, 2, \cdots$, $l = 0, \cdots, 2^n$, and $u_{s,t} = u_t - u_s$ is the increment of path u.

Together with Proposition 2.3.1, the above estimate allows us to convert our problem from controlling capacities to controlling the L^2-norm of Gaussian processes, which is much easier to work with.

Theorem 5.2.1. *For $r \in \mathbb{N}$ and $1 < p < \infty$, $\left(X^{(m)}\right)_{m\in\mathbb{N}}$ converges (p,r)-quasi-surely to some limit X, and the family of scaled maps $\left(X^{\varepsilon,(m)}\right)_{m\in\mathbb{N}}$ is a family of exponentially good approximations of X^ε under the capacity $c_{p,r}$.*

Proof. Here we use a technique from the theory of rough paths to control the tails of $X^{(m)}$'s, which are Gaussian. Let us first prove that the sequence $\left\{X^{(m)} : m \ge 1\right\}$ converges uniformly quasi-surely. By using the elementary fact that

$$\|u - w\| \le \sup_D \left(\sum_l \left|u_{t_{l-1},t_l} - w_{t_{l-1},t_l}\right|^q\right)^{\frac{1}{q}},$$

for any $u, w \in \boldsymbol{W}$ and for any $q > 1$, where the supremum is taken over all possible finite partition of $[0, 1]$, and $u_{s,t} = u_t - u_s$, together with Proposition 5.2.1, we obtain that

$$\|u-w\|^q \leq C_{q,\gamma} \sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} \left| u_{t^n_{k-1},t^n_k} - w_{t^n_{k-1},t^n_k} \right|^q$$

for $\gamma > q-1$, where $C_{q,\gamma}$ is a constant depending on q and γ, and $t^n_k = \frac{k}{2^n}$.

We will apply the above estimate to $X^{(n)}$ to estimate

$$I_m(\lambda) := c_{p,r}\left(\left\| X^{(m+1)} - X^{(m)} \right\| > \lambda \right), \tag{5.16}$$

where $\lambda > 0$. Since (p,r)-capacity is increasing in p, we shall assume that $p > 2$.

By monotonicity and sub-additivity properties of capacities, we obtain that for $\theta > 0$,

$$\begin{aligned}
I_m(\lambda) &\leq c_{p,r}\left(C_{q,\gamma} \sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} \left| X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k} \right|^q > \lambda^q \right) \\
&= c_{p,r}\left(\sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} \left| X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k} \right|^q > C^{-1}_{q,\gamma} C_{\theta,\gamma} \sum_{n=1}^{\infty} n^{\gamma} \frac{\lambda^q}{2^{n\theta}} \right) \\
&\leq \sum_{n=1}^{\infty} c_{p,r}\left(\sum_{k=1}^{2^n} \left| X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k} \right|^q > C^{-1}_{q,\gamma} C_{\theta,\gamma} \frac{\lambda^q}{2^{n\theta}} \right) \\
&\leq \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} c_{p,r}\left(\left| X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k} \right| > C^{-\frac{1}{q}}_{q,\gamma} C^{\frac{1}{q}}_{\theta,\gamma} \frac{\lambda}{2^{\frac{n(1+\theta)}{q}}} \right),
\end{aligned} \tag{5.17}$$

where

$$C_{\theta,\gamma} = \left(\sum_{n=1}^{\infty} \frac{n^{\gamma}}{2^{n\theta}} \right)^{-1}.$$

We introduce a new parameter N, whose value is to be determined at the end of our proof, and consider

$$c_{p,r}\left(\left| X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k} \right|^{2N} > C^{-\frac{2N}{q}}_{q,\gamma} C^{\frac{2N}{q}}_{\theta,\gamma} \frac{\lambda^{2N}}{2^{\frac{2nN(1+\theta)}{q}}} \right). \tag{5.18}$$

Notice that when $n \leq m$, $X^{(m)}_{t^n_k} = B_{t^m_{2^{m-n}k}}$. Since $t^m_{2^{m-n}k} = t^{m+1}_{2^{m+1-n}k}$, we have that

$X^{(m+1)}_{t^n_{k-1},t^n_k} = X^{(m)}_{t^n_{k-1},t^n_k}$. By Chebyshev's inequality, we obtain that

$$
\begin{aligned}
&c_{p,r}\left(\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N} > C_{q,\gamma,\theta,N}\frac{\lambda^{2N}}{2^{\frac{2nN(1+\theta)}{q}}}\right) \\
&\leq C^{-1}_{q,\gamma,\theta,N}\lambda^{-2N}2^{\frac{2nN(1+\theta)}{q}}\left\|\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right\|_{\mathbb{D}^p_r}, \qquad (5.19)
\end{aligned}
$$

where

$$
C_{q,\gamma,\theta,N} = C_{q,\gamma}^{-\frac{2N}{q}}C_{\theta,\gamma}^{\frac{2N}{q}}
$$

is a constant. Now by Proposition 2.3.1, with the fact that

$$
\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}
$$

is a polynomial functional of degree $2N$, where $N \geq \frac{r}{2}$, we have that

$$
\begin{aligned}
&\left\|\left\|D^l\left(\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right)\right\|_{\mathcal{H}^{\otimes l}}\right\|_p \\
&\leq (2N+1)(p-1)^{\frac{N}{2}}\left\|\left\|D^l\left(\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right)\right\|_{\mathcal{H}^{\otimes l}}\right\|_2,
\end{aligned}
$$

and

$$
\left\|\left\|D^l\left(\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right)\right\|_{\mathcal{H}^{\otimes l}}\right\|_2 \leq (2N)^{\frac{r}{2}}\left\|\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right\|_2
$$

for any $0 \leq l \leq r$. The above two inequalities imply that

$$
\begin{aligned}
\left\|\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right\|_{\mathbb{D}^p_r} &\leq (r+1)(2N+1)(p-1)^{\frac{N}{2}}(2N)^{\frac{r}{2}} \\
&\cdot\left\|\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right\|_2. \qquad (5.20)
\end{aligned}
$$

For $N \in \mathbb{N}$, $f(x) = x^{2N}$ is convex, so by Jensen's inequality,

$$\begin{aligned} f(x+y) &= 2^{2N}\left(\frac{1}{2}x + \frac{1}{2}y\right)^{2N} \\ &\leq 2^{2N}\left(\frac{1}{2}x^{2N} + \frac{1}{2}y^{2N}\right) \\ &= 2^{2N-1}\left(f(x) + f(y)\right), \end{aligned}$$

and hence

$$\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N} \leq 2^{2N-1}\left(\left|X^{(m+1)}_{t^n_{k-1},t^n_k}\right|^{2N} + \left|X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right).$$

Therefore, it suffices to estimate $\left\|\left|X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right\|_2$.

By definition, if $t^m_{l-1} \leq t^n_{k-1} < t^n_k \leq t^m_l$ for some l, then

$$X^{(m)}_{t^n_{k-1},t^n_k} = 2^m\left(t^n_k - t^n_{k-1}\right)\left(B_{t^m_l} - B_{t^m_{l-1}}\right).$$

Hence, by Proposition 2.3.1,

$$\begin{aligned} \left\|\left|X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right\|_2 &= \left\|X^{(m)}_{t^n_{k-1},t^n_k}\right\|^{2N}_{4N} \\ &\leq 2^{2N}(4N-1)^N\left\|X^{(m)}_{t^n_{k-1},t^n_k}\right\|^{2N}_2 \\ &= 2^{2N}(4N-1)^N\mathbb{E}\left[\left(2^m\left(t^n_k - t^n_{k-1}\right)\left(B_{t^m_l} - B_{t^m_{l-1}}\right)\right)^2\right]^N \\ &= 2^{2N}(4N-1)^N 2^{2mN}\frac{1}{2^{2nN}}\mathbb{E}\left[\left(B_{t^m_l} - B_{t^m_{l-1}}\right)^2\right]^N \\ &= 2^{2N}(4N-1)^N 2^{2mN}\frac{1}{2^{2nN}}\frac{1}{2^{2mNH}} \\ &= 2^{2N}(4N-1)^N\frac{2^{2mN(1-H)}}{2^{2nN}}. \end{aligned}$$

It thus implies that

$$\left\|\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right\|_2 \leq 2^{2N-1}\left(\left\|\left|X^{(m+1)}_{t^n_{k-1},t^n_k}\right|^{2N}\right\|_2 + \left\|\left|X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right\|_2\right)$$
$$\leq 2^{4N-1}(4N-1)^N\left(\frac{2^{2(m+1)N(1-H)}}{2^{2nN}} + \frac{2^{2mN(1-H)}}{2^{2nN}}\right)$$
$$= C_{N,H}\frac{2^{2mN(1-H)}}{2^{2nN}}, \tag{5.21}$$

where

$$C_{N,H} = 2^{4N-1}\left(4N-1\right)^N\left(1+2^{2N(1-H)}\right)$$

is a constant depending only on N and H. We may conclude from (5.20) and (5.21) that

$$\left\|\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N}\right\|_{\mathbb{D}^p_r} \leq C_{r,p,N,H}\frac{2^{2mN(1-H)}}{2^{2nN}} \tag{5.22}$$

for $n > m$, where

$$C_{r,p,N,H} = (r+1)(2N+1)(p-1)^{\frac{N}{2}}(2N)^{\frac{r}{2}}C_{N,H} \tag{5.23}$$

depends on r, p, N and H. Plugging (5.22) into (5.19), we obtain that

$$c_{p,r}\left(\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N} > C_{q,\gamma,\theta}\frac{\lambda^{2N}}{2^{\frac{2nN(1+\theta)}{q}}}\right)$$
$$\leq C^{-1}_{q,\gamma,\theta,N}C_{r,p,N,H}\lambda^{-2N}\frac{2^{2mN(1-H)}}{2^{2nN\left(1-\frac{1+\theta}{q}\right)}}. \tag{5.24}$$

Therefore, according to (5.17), (5.18) and (5.24),

$$I_m(\lambda) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{2^n} c_{p,r}\left(\left|X^{(m+1)}_{t^n_{k-1},t^n_k} - X^{(m)}_{t^n_{k-1},t^n_k}\right|^{2N} > C_{q,\gamma,\theta,N}\frac{\lambda^{2N}}{2^{\frac{2nN(1+\theta)}{q}}}\right)$$
$$\leq \sum_{n=m+1}^{\infty}\sum_{k=1}^{2^n} C^{-1}_{q,\gamma,\theta,N}C_{r,p,N,H}\lambda^{-2N}\frac{2^{2mN(1-H)}}{2^{2nN\left(1-\frac{1+\theta}{q}\right)}}$$

$$
\begin{aligned}
&= C_{q,\gamma,\theta,N}^{-1} C_{r,p,N,H} \lambda^{-2N} \sum_{n=m+1}^{\infty} \frac{2^{2mN(1-H)}}{2^{n\left(2N\left(1-\frac{1+\theta}{q}\right)-1\right)}} \\
&= C_{q,\gamma,\theta,N}^{-1} C_{r,p,N,H} \lambda^{-2N} \frac{1}{2^{m\left(2N\left(H-\frac{1+\theta}{q}\right)-1\right)}} \sum_{k=1}^{\infty} \frac{1}{2^{k\left(2N\left(1-\frac{1+\theta}{q}\right)-1\right)}}.
\end{aligned}
$$

Since

$$
2N\left(1-\frac{1+\theta}{q}\right)-1 > 2N\left(H-\frac{1+\theta}{q}\right)-1,
$$

the above series converges as long as

$$
2N\left(H-\frac{1+\theta}{q}\right)-1 > 0,
$$

which means that we need

$$
qH - \frac{q}{2N} - 1 > 0
$$

for some $N \in \mathbb{N}$. Therefore, if we choose

$$
q > \left(H-\frac{1}{2}\right)^{-1}
$$

and

$$
\theta \in \left(0, qH - \frac{q}{2N} - 1\right),
$$

then the above series converges. As a consequence, we thus have

$$
I_m(\lambda) \leq C_{q,\gamma,\theta,N}^{'} C_{r,p,N,H} \frac{\lambda^{-2N}}{2^{m\left(2N\left(H-\frac{1+\theta}{q}\right)-1\right)}} \tag{5.25}
$$

for every $m = 1, 2, \cdots$, where

$$
\begin{aligned}
C_{q,\gamma,\theta,N}^{'} &= C_{q,\gamma,\theta,N}^{-1} \sum_{k=1}^{\infty} \frac{1}{2^{k\left(2N\left(1-\frac{1+\theta}{q}\right)-1\right)}} \\
&= C_{q,\gamma,\theta,N}^{-1} \frac{1}{2^{2N\left(1-\frac{1+\theta}{q}\right)-1}-1}
\end{aligned}
$$

$$\leq C_{q,\gamma,\theta,N}^{-1} \frac{1}{2^{2\left(1-\frac{1+\theta}{q}\right)-1} - 1}$$

$$= C_{q,\gamma,\theta,N}^{-1} C_{q,\theta}$$

with

$$C_{q,\theta} = \left(2^{2\left(1-\frac{1+\theta}{q}\right)-1} - 1\right)^{-1}.$$

From (5.25), we may deduce that

$$I_m(\lambda) \leq C_{q,\gamma,\theta,N}^{-1} C_{q,\theta} C_{r,p,N,H} \frac{\lambda^{-2N}}{2^{m\left(2N\left(H-\frac{1+\theta}{q}\right)-1\right)}}. \tag{5.26}$$

Applying the same argument as in the previous theorem, we see that the problem may be reduced to proving that for some suitable positive $\delta > 0$,

$$\sum_{m=1}^{\infty} I_m\left(\frac{1}{2^{m\delta}}\right) < \infty. \tag{5.27}$$

Then by the first Borel-Cantelli lemma for capacities, we obtain the quasi-sure convergence for $(X^{(m)})_{m\in\mathbb{N}}$. Since

$$I_m\left(\frac{1}{2^{m\delta}}\right) \leq C_{q,\gamma,\theta,N}^{-1} C_{q,\theta} C_{r,p,N,H} \frac{1}{2^{m\left(2N\left(H-\frac{1+\theta}{q}-\delta\right)-1\right)}},$$

so the series in (5.27) converges as long as we choose δ such that

$$\delta < H - \frac{1+\theta}{q} - \frac{1}{2N},$$

which must exist as we have chosen q and θ such that

$$2N\left(H - \frac{1+\theta}{q}\right) - 1 > 0$$

for some $N \in \mathbb{N}$. The convergence of the series in (5.27) implies the convergence of

$\left(X^{(m)}\right)_{m\in\mathbb{N}}$. Denote its limit by X, then X is defined quasi-surely on $\boldsymbol{W}$.

Next, we prove the sequence $\left\{X^{\varepsilon,(m)} : m \geq 1, \varepsilon > 0\right\}$ converges to $\{X^{\varepsilon} : \varepsilon > 0\}$ exponentially fast with respect to the capacity $c_{p,r}$, that is,

$$\lim_{m\to\infty} \limsup_{\varepsilon\to 0} \varepsilon^2 \log c_{p,r} \left\{\omega : \left\|X^{\varepsilon,(m)}(\omega) - X^{\varepsilon}(\omega)\right\| > \lambda\right\} = -\infty$$

for $\lambda > 0$. To this end, we shall use a similar argument as in the proof above. By the sub-additivity of capacity, for $\alpha > 0$,

$$\begin{aligned}
& c_{p,r} \left\{\omega : \left\|X^{\varepsilon,(m)}(\omega) - X^{\varepsilon}(\omega)\right\| > \lambda\right\} \\
\leq\ & c_{p,r} \left\{\omega : \sum_{k=m}^{\infty} \left\|X^{\varepsilon,(k)}(\omega) - X^{\varepsilon,(k+1)}(\omega)\right\| > \lambda\right\} \\
=\ & c_{p,r} \left\{\omega : \sum_{k=m}^{\infty} \left\|X^{\varepsilon,(k)}(\omega) - X^{\varepsilon,(k+1)}(\omega)\right\| > C_\alpha \sum_{k=m}^{\infty} \frac{\lambda}{2^{(k-m)\alpha}}\right\} \\
\leq\ & \sum_{k=m}^{\infty} c_{p,r} \left\{\omega : \left\|X^{(k)}(\omega) - X^{(k+1)}(\omega)\right\| > C_\alpha \varepsilon^{-1} \frac{\lambda}{2^{(k-m)\alpha}}\right\} \\
=\ & \sum_{k=m}^{\infty} I_k \left(\frac{\lambda C_\alpha}{2^{(k-m)\alpha}\varepsilon}\right),
\end{aligned} \tag{5.28}$$

where we have used the notations in (5.16), and

$$C_\alpha = \left(\sum_{i=0}^{\infty} \frac{1}{2^{i\alpha}}\right)^{-1}$$

is some positive constant depending only on α.

Recall that up to now, the only assumption on N is that $N \geq \frac{r}{2}$, and now we shall pick up a suitable N to show that the convergence of $\left(X^{\varepsilon,(m)}\right)_{m\in\mathbb{N}}$ is exponentially fast. By (5.26),

$$I_k \left(\frac{\lambda C_\alpha}{2^{(k-m)\alpha}\varepsilon}\right) \leq C_{q,\gamma,\theta,N}^{-1} C_{r,p,N,H} C_{q,\theta} \frac{1}{2^{k\left(2N\left(H - \frac{1+\theta}{q} - \alpha\right) - 1\right)}} \frac{1}{2^{2Nm\alpha}} \frac{\varepsilon^{2N}}{\lambda^{2N} C_\alpha^{2N}}$$

$$= C_{q,\gamma,\theta,N}^{-1} C_{r,p,N,H} C_{q,\theta} C_{\alpha}^{-2N} \frac{1}{2^{k\beta}} \frac{1}{2^{2Nm\alpha}} \frac{\varepsilon^{2N}}{\lambda^{2N}}, \tag{5.29}$$

where

$$\beta = 2N\left(H - \frac{1+\theta}{q} - \alpha\right) - 1.$$

As $C_{q,\gamma,\theta,N} = C_{q,\gamma}^{-\frac{2N}{q}} C_{\theta,\gamma}^{\frac{2N}{q}}$, where $C_{r,p,N,H}$ is given as in (5.23) with

$$C_{N,H} = 2^{4N-1}\left(4N-1\right)^{N}\left(1 + 2^{2N(1-H)}\right),$$

we have that

$$\begin{aligned}
&C_{q,\gamma,\theta,N}^{-1} C_{r,p,N,H} \\
&= C_{q,\gamma}^{\frac{2N}{q}} C_{\theta,\gamma}^{-\frac{2N}{q}} (r+1)(2N+1)(p-1)^{\frac{N}{2}} (2N)^{\frac{r}{2}} 2^{4N-1}\left(4N-1\right)^{N}\left(1 + 2^{2N(1-H)}\right) \\
&= \frac{1}{2}(r+1)(2N+1)(2N)^{\frac{r}{2}} \left(C_{q,\gamma}^{\frac{2}{q}} C_{\theta,\gamma}^{-\frac{2}{q}} (p-1)^{\frac{1}{2}} 2^4\right)^{N} \left(4N-1\right)^{N}\left(1 + 2^{2N(1-H)}\right) \\
&\leq (r+1)(2N+1)(2N)^{\frac{r}{2}} \left(16 C_{q,\gamma}^{\frac{2}{q}} C_{\theta,\gamma}^{-\frac{2}{q}} (p-1)^{\frac{1}{2}}\right)^{N} \left(4N\right)^{N} 2^{2N(1-H)} \\
&= (r+1)(2N+1)(2N)^{\frac{r}{2}} \left(64 C_{q,\gamma}^{\frac{2}{q}} C_{\theta,\gamma}^{-\frac{2}{q}} (p-1)^{\frac{1}{2}} 2^{2(1-H)}\right)^{N} N^{N} \\
&:= P_r(N) C_{q,\gamma,\theta,p,H}^{N} N^{N},
\end{aligned} \tag{5.30}$$

where

$$P_r(N) = (r+1)(2N+1)(2N)^{\frac{r}{2}}$$

is a polynomial of N depending only on r, and

$$C_{q,\gamma,\theta,p,H} = 64 C_{q,\gamma}^{\frac{2}{q}} C_{\theta,\gamma}^{-\frac{2}{q}} (p-1)^{\frac{1}{2}} 2^{2(1-H)}$$

is a constant. If we set α such that

$$\alpha < H - \frac{1+\theta}{q} - \frac{1}{2N},$$

then $\beta > 0$, together with (5.29) and (5.30), we obtain that

$$\begin{aligned}\sum_{k=m}^{\infty} I_k\left(\frac{\lambda C_\alpha}{2^{(k-m)\alpha}\varepsilon}\right) &\le P_r(N) C^N_{q,\gamma,\theta,p,H} N^N C_{q,\theta} C_\alpha^{-2N} \frac{1}{2^{2Nm\alpha}} \frac{\varepsilon^{2N}}{\lambda^{2N}} \sum_{k=m}^{\infty} \frac{1}{2^{k\beta}} \\ &= P_r(N) C^N_{q,\gamma,\theta,p,H} N^N C_{q,\theta} C_\alpha^{-2N} \lambda^{-2N} \varepsilon^{2N} \sum_{k=0}^{\infty} \frac{1}{2^{k\beta}} \frac{1}{2^{m(2N\alpha+\beta)}} \\ &= P_{r,q,\theta,\beta}(N) C^N_{q,\gamma,\theta,p,H,\alpha} \lambda^{-2N} \varepsilon^{2N} N^N \frac{1}{2^{m\left(2N\left(H-\frac{1+\theta}{q}\right)-1\right)}},\end{aligned}$$

where

$$C_{q,\gamma,\theta,p,H,\alpha} = C_{q,\gamma,\theta,p,H} C_\alpha^{-2}$$

and

$$P_{r,q,\theta,\beta}(N) = C_{q,\theta}\left(\sum_{k=0}^{\infty} \frac{1}{2^{k\beta}}\right) P_r(N).$$

According to (5.28), it holds that

$$\begin{aligned}&\varepsilon^2 \log c_{p,r}\left\{\omega : \left\|X^{\varepsilon,(m)}(\omega) - X^\varepsilon(\omega)\right\| > \lambda\right\} \\ \le\, &\varepsilon^2 \log P_{r,q,\theta,\beta}(N) + \varepsilon^2 N \log C_{q,\gamma,\theta,p,H,\alpha} \\ &+ \varepsilon^2 N \log\left(\frac{\varepsilon^2 N}{\lambda^2}\right) - \varepsilon^2\left(2N\left(H - \frac{1+\theta}{q}\right) - 1\right) m \log 2.\end{aligned}$$

For ε small enough, choose $N = \lfloor \varepsilon^{-2} \rfloor$. Then since $P_{r,q,\theta,\beta}(N)$ is a polynomial of N, it holds that

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log c_{p,r}\left\{\omega : \left\|X^{\varepsilon,(m)}(\omega) - X^\varepsilon(\omega)\right\| > \lambda\right\} \le \log C - 2\left(H - \frac{1+\theta}{q}\right) m \log 2,$$

where $C = C_{q,\gamma,\theta,p,H,\alpha}\lambda^{-2}$ is a constant. Therefore, as $H > \frac{1+\theta}{q}$,

$$\lim_{m\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^2\log c_{p,r}\left\{\omega : \|X^{\varepsilon,m}(\omega) - X^{\varepsilon}(\omega)\| > \lambda\right\} = -\infty,$$

which completes the proof. □

5.3 $c_{p,r}$-LDPs for fBMs

In this section, we will state the main result of this chapter-the large deviation principles for fractional Brownian motions with respect to the classical Wiener capacity, and the rest of this section is devoted to the proof of this result. In order to state the large deviation principles for fBMs, we need to identify their rate functions, which must be the same rate functions in the context of probability as probability is a very special case of capacities. To properly define the rate functions, we need to identify the Cameron-Martin space associated with the fBM whose Hurst parameter is H.

We notice that the integral representation of fBMs

$$B_t(\omega) = \int_0^t K(t,s)\mathrm{d}\omega(s)$$

is defined almost surely, i.e. it defines a measurable mapping $B : (\boldsymbol{W}, \mathscr{B}(\boldsymbol{W})) \to (\boldsymbol{W}, \mathscr{B}(\boldsymbol{W}))$ which takes almost all standard Brownian motion paths to the sample paths of fractional Brownian motion. From the previous section, we see that indeed this mapping is defined except for on a set even smaller than a null set – it is indeed quasi-surely defined, and this mapping is denoted by X in the previous section. We shall keep using X to denote this mapping in the sequel.

By the definition of X, we see that it is a measurable mapping from $(\boldsymbol{W}, \mathscr{B}(\boldsymbol{W}))$ to $(\boldsymbol{W}, \mathscr{B}(\boldsymbol{W}))$. Now let $Q = P \circ X^{-1}$ be the push-forward of the Wiener measure, which is a Gaussian measure on $(\boldsymbol{W}, \mathscr{B}(\boldsymbol{W}))$. Similar to the case of Brownian motion, there is a canonical way to associate this Gaussian measure and $\boldsymbol{W}$ with a separable Hilbert space $\hat{\mathcal{H}}$, which can be embedded into $\boldsymbol{W}$ via a continuous and dense mapping such that this triple $(\boldsymbol{W}, \hat{\mathcal{H}}, Q)$ forms an abstract Wiener space.

In [10], Decreusefond and Üstünel identified the Cameron-Martin space corresponding to the fractional Brownian motion with Hurst parameter equal to H, which is the space consisting of all elements of the form

$$h(t) = \int_0^t K(t,s)\dot{h}(s)\mathrm{d}s,$$

where K is the singular kernel as in the integral representation, and $\dot{h} \in L^2([0,1])$. It turns out this Cameron-Martin space is the one that we are looking for. The inner product structure on this space is defined as

$$\langle h_1, h_2 \rangle_{\hat{\mathcal{H}}} = \int_0^1 \dot{h}_1(s)\dot{h}_2(s)\mathrm{d}s,$$

for all $h_1, h_2 \in \hat{\mathcal{H}}$ such that $h_i(t) = \int_0^t K(t,s)\dot{h}_i(s)\mathrm{d}s$, $i = 1, 2$. Then

$$\int_{\boldsymbol{W}} e^{il(\omega)} Q(\mathrm{d}\omega) = e^{-\frac{\|l\|^2_{\hat{\mathcal{H}}}}{2}}, \quad \forall l \in \boldsymbol{W}^*.$$

Let $\{Q_\varepsilon\}$ be the family of scaled measures, the laws of $\{\varepsilon\omega\}$ under Q. By definition,

$$Q_\varepsilon(A) = Q\{\omega \in \boldsymbol{W} : \varepsilon\omega \in A\} = P\{\omega \in \boldsymbol{W} : \varepsilon X(\omega) \in A\} \tag{5.31}$$

for each $A \in \mathscr{B}(\boldsymbol{W})$. According to Theorem 3.4.12 in [12], the family of scaled measures $\{Q_\varepsilon\}$ satisfies the LDP with the good rate function I given by

$$I(\omega) = \begin{cases} \frac{\|\omega\|^2_{\hat{\mathcal{H}}}}{2}, & \omega \in \hat{\mathcal{H}}, \\ \infty, & \text{otherwise.} \end{cases}$$

Now we are in a position to state the main result of this section.

Theorem 5.3.1. *Let* $r \in \mathbb{N}$, $1 < p < \infty$ *and* $\frac{1}{2} \leq H < 1$. *Let* $X_t^\varepsilon(\omega) = B_t(\varepsilon\omega)$ *for all* ω *expect for a* $c_{p,r}$*-zero subset (for all* $t \in [0,1]$, $\varepsilon > 0$*). Then* $\{X^\varepsilon : \varepsilon > 0\}$ *(which are scaled fBMs with Hurst parameter* H*) satisfies the* $c_{p,r}$*-LDP with the good rate function*

$$I(\omega) = \begin{cases} \frac{\|\omega\|^2_{\hat{\mathcal{H}}}}{2}, & \omega \in \hat{\mathcal{H}}, \\ \infty, & \text{otherwise.} \end{cases} \tag{5.32}$$

Notice that for each m, $X^{(m)}$, which is a Wiener functional on $\boldsymbol{W}$ defined quasi-surely, is a linear interpolation of some Gaussian random variables, so we may consider $F_m : \mathbb{R}^{2^m+1} \to \boldsymbol{W}$ given by

$$F_m(x_0, \cdots x_{2^m})(t) = x_{k-1} + 2^m \left(t - \frac{k-1}{2^m}\right)(x_k - x_{k-1}), \quad \forall t \in \left[\frac{k-1}{2^m}, \frac{k}{2^m}\right], \tag{5.33}$$

which maps a $(2^m + 1)$-dimensional vector to its linear interpolation. Let us apply Varadhan's contraction principle to the family of maps above. As the rate function for the vector-valued Gaussian random variable $\left(B_0, B_{\frac{1}{2^m}}, \cdots, B_{\frac{k}{2^m}}, \cdots, B_1\right)$ is computable, the capacity version of LDPs may be established easily for $X^{(m)}$.

Proposition 5.3.1. *Let* $\boldsymbol{t} = \{0 \leq t_1 < t_2 < \cdots < t_n \leq 1\}$ *be a finite partition of* $[0,1]$. *Define* $T^\varepsilon : \boldsymbol{W} \to \mathbb{R}^n$ *(for* $\varepsilon > 0$*) by* $T^\varepsilon(\omega) = \boldsymbol{B_t}(\varepsilon\omega)$*, where*

$$\boldsymbol{B_t}(\omega) = (B_{t_1}(\omega), \cdots, B_{t_n}(\omega))$$

is a Gaussian vector with covariance matrix $\boldsymbol{\Sigma} = (\sigma_{ij})_{1 \leq i,j \leq n}$ *and* $\sigma_{ij} = R(t_i, t_j)$. *Then* $\{T^\varepsilon : \varepsilon > 0\}$ *satisfies* $c_{p,r}$*-LDP with the good rate function* $I_n : \mathbb{R}^n \to [0, \infty]$ *given by*

$$I_n(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{x}.$$

Proof. According to the definition of $c_{p,r}$-LDP, we need to establish the upper bound and lower bound. Since (p,r)-capacity is increasing in p and r, the lower bound part follows directly from the classical LDPs for Gaussian measures, and thus we have for all open G,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon^2\log c_{p,r}\left\{\omega\in\boldsymbol{W}:T^\varepsilon(\omega)\in G\right\} &\geq \frac{1}{p}\limsup_{\varepsilon\to 0}\varepsilon^2\log P\left\{\omega\in\boldsymbol{W}:T^\varepsilon(\omega)\in G\right\}\\ &\geq -\frac{1}{p}\inf_{y\in G}I_n(y).\end{aligned}$$

For the upper bound part, we first establish the result when $n=1$. Let $a>0$. By Chebyshev's inequality, for any $t\in[0,1]$,

$$\begin{aligned}c_{p,r}\left\{\omega:B_t(\varepsilon\omega)>a\right\} &= c_{p,r}\left\{\omega:e^{\lambda\varepsilon B_t(\omega)}>e^{\lambda a}\right\}\\ &\leq e^{-\lambda a}\left\|e^{\lambda\varepsilon B_t}\right\|_{\mathbb{D}_r^p}\\ &= e^{-\lambda a}\left(\sum_{l=0}^{r}\mathbb{E}\left[\left|\left\|D^l\left(e^{\lambda\varepsilon B_t}\right)\right\|_{\mathcal{H}^{\otimes l}}\right|^p\right]\right)^{\frac{1}{p}}.\end{aligned}$$

Recall that

$$D\left(e^{\lambda\varepsilon B_t}\right)(s)=\lambda\varepsilon e^{\lambda\varepsilon B_t}K(t,s)\mathbb{1}_{(0,t)}(s),$$

so by iteration,

$$D^l\left(e^{\lambda\varepsilon B_t}\right)(s_1,s_2,\cdots,s_l)=(\lambda\varepsilon)^l e^{\lambda\varepsilon B_t}\left(K(t)\mathbb{1}_{(0,t)}\right)^{\otimes l}(s_1,s_2,\cdots,s_l)$$

for all $l\leq r$, where $\left(K(t)\mathbb{1}_{(0,t)}\right)^{\otimes l}$ denotes l-fold tensor product of $K(t,s)\mathbb{1}_{(0,t)}(s)$ with itself.

Therefore,

$$\left\|D^l\left(e^{\lambda\varepsilon B_t}\right)\right\|_{\mathcal{H}^{\otimes l}}^2=(\lambda\varepsilon)^{2l}e^{2\lambda\varepsilon B_t}\left(\int_0^1 K^2(t,s)\mathbb{1}_{(0,t)}(s)\mathrm{d}s\right)^l$$

$$= (\lambda\varepsilon)^{2l} e^{2\lambda\varepsilon B_t} t^{2Hl},$$

and therefore

$$\begin{aligned}\mathbb{E}\left[\left|\left\|D^l\left(e^{\lambda\varepsilon B_t}\right)\right\|_{\mathcal{H}^{\otimes l}}\right|^p\right] &= (\lambda\varepsilon)^{lp} t^{Hlp}\mathbb{E}\left[e^{\lambda\varepsilon p B_t}\right] \\ &= (\lambda\varepsilon)^{lp} t^{Hlp} e^{\frac{(\lambda\varepsilon p)^2 t^{2H}}{2}}.\end{aligned}$$

It thus follows that

$$\begin{aligned}c_{p,r}\left\{\omega : B_t(\varepsilon\omega) > a\right\} &\le e^{-\lambda a}\left(\sum_{l=0}^{r}\mathbb{E}\left[\left|\left\|D^l\left(e^{\lambda\varepsilon B_t}\right)\right\|_{\mathcal{H}^{\otimes l}}\right|^p\right]\right)^{\frac{1}{p}} \\ &\le e^{-\lambda a}\sum_{l=0}^{r}\mathbb{E}\left[\left|\left\|D^l\left(e^{\lambda\varepsilon B_t}\right)\right\|_{\mathcal{H}^{\otimes l}}\right|^p\right]^{\frac{1}{p}} \\ &= e^{\frac{(\lambda\varepsilon)^2 p t^{2H}}{2} - \lambda a}\sum_{l=0}^{r}\left(\lambda\varepsilon t^H\right)^l,\end{aligned}$$

so that

$$\varepsilon^2 \log c_{p,r}\left\{\omega : B_t(\varepsilon\omega) > a\right\} \le \frac{\lambda^2\varepsilon^4 p t^{2H}}{2} - \lambda a\varepsilon^2 + \varepsilon^2\log\left(\sum_{l=0}^{r}\left(\lambda\varepsilon t^H\right)^l\right). \qquad (5.34)$$

Setting

$$\lambda = \frac{a}{p\varepsilon^2 t^{2H}}$$

so that the sum of first two terms in (5.34) attains its minimum, we obtain that

$$\begin{aligned}\varepsilon^2 \log c_{p,r}\left\{\omega : B_t(\varepsilon\omega) > a\right\} &\le -\frac{a^2}{2pt^{2H}} + \varepsilon^2\log\left((r+1)\cdot\max_{0\le l\le r}\left(\frac{a}{\varepsilon p t^H}\right)^l\right) \\ &= -\frac{a^2}{2pt^{2H}} + \varepsilon^2\log(r+1) + \max_{0\le l\le r} l\varepsilon^2\log\left(\frac{a}{\varepsilon p t^H}\right).\end{aligned}$$

It follows that

$$\limsup_{\varepsilon\to 0}\varepsilon^2\log c_{p,r}\left\{\omega : B_t(\varepsilon\omega) > a\right\} \le -\frac{1}{2p}\cdot\frac{a^2}{t^{2H}} = -\frac{1}{p}\inf_{x>a} I_1(x), \tag{5.35}$$

which remains true if we replace $\{\omega : B_t(\varepsilon\omega) > a\}$ with $\{\omega : B_t(\varepsilon\omega) \ge a\}$. We may also deduce the similar results for $\{\omega : B_t(\varepsilon\omega) < b\}$ and $\{\omega : B_t(\varepsilon\omega) \le b\}$ with $b < 0$ by symmetry.

Now let us deal with the case of a finite partition $\boldsymbol{t} = \{0 \le t_1 \le \cdots \le t_n \le 1\}$. Then by linearity $\boldsymbol{B_t}(\varepsilon\omega) = \varepsilon\boldsymbol{B_t}(\omega)$. Introduce an inner product $\langle\cdot,\cdot\rangle_{\boldsymbol{\Sigma}}$ on $\mathbb{R}^n$:

$$\langle\boldsymbol{x},\boldsymbol{y}\rangle_{\boldsymbol{\Sigma}} = \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{y},$$

and denote the corresponding norm by $|\cdot|_{\boldsymbol{\Sigma}}$. Notice that for any $\boldsymbol{x} = (x_1,\cdots,x_n) \in B(\boldsymbol{a},r)$, the open ball in $(\mathbb{R}^n, |\cdot|_{\boldsymbol{\Sigma}})$ with centre $\boldsymbol{a}$ and radius r,

$$\langle\boldsymbol{\lambda},\boldsymbol{a}-\boldsymbol{x}\rangle_{\boldsymbol{\Sigma}} \le |\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}|\boldsymbol{a}-\boldsymbol{x}|_{\boldsymbol{\Sigma}} \le r|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}$$

for all $\boldsymbol{\lambda}\in\mathbb{R}^n$, which implies that

$$B(\boldsymbol{a},r) \subset \left\{\boldsymbol{x} : \langle\boldsymbol{\lambda},\boldsymbol{a}-\boldsymbol{x}\rangle_{\boldsymbol{\Sigma}} \le r|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}\right\}.$$

Based on this observation, we may apply Chebyshev's inequality and get that

$$\begin{aligned} c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega)\in B(\boldsymbol{a},r)\right\} &\le c_{p,r}\left\{\omega : \langle\boldsymbol{\lambda},\boldsymbol{a}-\boldsymbol{B_t}(\varepsilon\omega)\rangle_{\boldsymbol{\Sigma}} \le r|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}\right\} \\ &= c_{p,r}\left\{\omega : e^{\langle\boldsymbol{\lambda},\boldsymbol{B_t}(\varepsilon\omega)\rangle_{\boldsymbol{\Sigma}}} \ge e^{\langle\boldsymbol{\lambda},\boldsymbol{a}\rangle_{\boldsymbol{\Sigma}}-r|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}}\right\} \\ &\le e^{r|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}-\langle\boldsymbol{\lambda},\boldsymbol{a}\rangle_{\boldsymbol{\Sigma}}}\left\|e^{\langle\boldsymbol{\lambda},\boldsymbol{B_t}(\varepsilon\omega)\rangle_{\boldsymbol{\Sigma}}}\right\|_{\mathbb{D}_r^p} \end{aligned} \tag{5.36}$$

for all $\boldsymbol{\lambda} = (\lambda_1,\cdots,\lambda_n)\in\mathbb{R}^n$.

By the Chain rule for Malliavin derivatives,

$$D\left(e^{\langle \boldsymbol{\lambda}, \boldsymbol{B_t}(\varepsilon\omega)\rangle_{\boldsymbol{\Sigma}}}\right)(s) = \varepsilon e^{\langle \boldsymbol{\lambda}, \boldsymbol{B_t}(\varepsilon\omega)\rangle_{\boldsymbol{\Sigma}}}\langle \boldsymbol{\lambda}, D\boldsymbol{B_t}\rangle_{\boldsymbol{\Sigma}}(s),$$

where

$$D\boldsymbol{B_t} = \left(DB_{t_1}, \cdots, DB_{t_n}\right),$$

and by iteration,

$$D^l\left(e^{\langle \boldsymbol{\lambda}, \boldsymbol{B_t}(\varepsilon\omega)\rangle_{\boldsymbol{\Sigma}}}\right)(s_1, \cdots, s_l) = \varepsilon^l e^{\langle \boldsymbol{\lambda}, \boldsymbol{B_t}(\varepsilon\omega)\rangle_{\boldsymbol{\Sigma}}}\langle \boldsymbol{\lambda}, D\boldsymbol{B_t}\rangle_{\boldsymbol{\Sigma}}^{\otimes l}(s_1, \cdots, s_l),$$

where $\langle \boldsymbol{\lambda}, D\boldsymbol{B_t}\rangle_{\boldsymbol{\Sigma}}^{\otimes l}$ denotes the l-fold tensor product of $\langle \boldsymbol{\lambda}, D\boldsymbol{B_t}\rangle_{\boldsymbol{\Sigma}}$ with itself.

Since $\langle DB_{t_i}, DB_{t_j}\rangle_{\mathcal{H}} = \sigma_{ij}$ for all $1 \le i, j \le n$, it follows that

$$\begin{aligned}
\left\|D^l\left(e^{\langle \boldsymbol{\lambda}, \boldsymbol{B_t}(\varepsilon\omega)\rangle_{\boldsymbol{\Sigma}}}\right)\right\|_{\mathcal{H}^{\otimes l}}^2 &= \varepsilon^{2l} e^{2\varepsilon\langle \boldsymbol{\lambda}, \boldsymbol{B_t}(\omega)\rangle_{\boldsymbol{\Sigma}}}\langle\langle \boldsymbol{\lambda}, D\boldsymbol{B_t}\rangle_{\boldsymbol{\Sigma}}, \langle \boldsymbol{\lambda}, D\boldsymbol{B_t}\rangle_{\boldsymbol{\Sigma}}\rangle_{\mathcal{H}}^l \\
&= \left\langle \sum_{i=1}^n \left(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-1}\right)_i DB_{t_i}, \sum_{j=1}^n \left(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-1}\right)_j DB_{t_j}\right\rangle_{\mathcal{H}}^l \\
&\quad \cdot \varepsilon^{2l} e^{2\varepsilon\langle \boldsymbol{\lambda}, \boldsymbol{B_t}(\omega)\rangle_{\boldsymbol{\Sigma}}} \\
&= \left(\sum_{1\le i,j\le n} \left(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-1}\right)_i \left(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-1}\right)_j \langle DB_{t_i}, DB_{t_j}\rangle_{\mathcal{H}}\right)^l \\
&\quad \cdot \varepsilon^{2l} e^{2\varepsilon\langle \boldsymbol{\lambda}, \boldsymbol{B_t}(\omega)\rangle_{\boldsymbol{\Sigma}}} \\
&= \varepsilon^{2l} e^{2\varepsilon\langle \boldsymbol{\lambda}, \boldsymbol{B_t}(\omega)\rangle_{\boldsymbol{\Sigma}}}\left(\sum_{1\le i,j\le n} \left(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-1}\right)_i \left(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-1}\right)_j \sigma_{ij}\right)^l \\
&= \varepsilon^{2l} e^{2\varepsilon\langle \boldsymbol{\lambda}, \boldsymbol{B_t}(\omega)\rangle_{\boldsymbol{\Sigma}}}\left(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\left(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-1}\right)^T\right)^l \\
&= \varepsilon^{2l}|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}^{2l} e^{2\varepsilon\langle \boldsymbol{\lambda}, \boldsymbol{B_t}(\omega)\rangle_{\boldsymbol{\Sigma}}},
\end{aligned}$$

where $\left(\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-1}\right)_i$ denotes the i-th component of $\boldsymbol{\lambda}^T\boldsymbol{\Sigma}^{-1}$. Thus,

$$\begin{aligned}\mathbb{E}\left[\left|\left\|D^l\left(e^{\langle\boldsymbol{\lambda},\boldsymbol{B_t}(\varepsilon\omega)\rangle_{\boldsymbol{\Sigma}}}\right)\right\|_{\mathcal{H}^{\otimes l}}\right|^p\right] &= \varepsilon^{lp}|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}^{lp}\mathbb{E}\left[e^{\varepsilon p\langle\boldsymbol{\lambda},\boldsymbol{B_t}(\omega)\rangle_{\boldsymbol{\Sigma}}}\right]\\ &= \varepsilon^{lp}|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}^{lp}e^{\frac{1}{2}(\varepsilon p)^2\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}\right)^T\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}\right)}\\ &= \varepsilon^{lp}|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}^{lp}e^{\frac{1}{2}(\varepsilon p)^2|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}^2},\end{aligned}$$

which implies that

$$\begin{aligned}\left\|e^{\langle\boldsymbol{\lambda},\boldsymbol{B_t}(\varepsilon\omega)\rangle}\right\|_{\mathbb{D}_r^p} &\leq \sum_{l=0}^{r}\mathbb{E}\left[\left|\left\|D^l\left(e^{\langle\boldsymbol{\lambda},\boldsymbol{B_t}(\varepsilon\omega)\rangle}\right)\right\|_{\mathcal{H}^{\otimes l}}\right|^p\right]^{\frac{1}{p}}\\ &= \sum_{l=0}^{r}\varepsilon^l|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}^l e^{\frac{1}{2}\varepsilon^2 p|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}^2}.\end{aligned}$$

Therefore, by (5.36),

$$c_{p,r}\left\{\omega:\boldsymbol{B_t}(\varepsilon\omega)\in B(\boldsymbol{a},r)\right\}\leq\sum_{l=0}^{r}\left(\varepsilon|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}\right)^l e^{\frac{1}{2}\varepsilon^2 p|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}^2+r|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}-\langle\boldsymbol{\lambda},\boldsymbol{a}\rangle_{\boldsymbol{\Sigma}}}.$$

As a consequence we have,

$$\begin{aligned}\varepsilon^2\log c_{p,r}\left\{\omega:\boldsymbol{B_t}(\varepsilon\omega)\in B(\boldsymbol{a},r)\right\} &\leq \frac{1}{2}\varepsilon^4 p|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}^2+\varepsilon^2 r|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}-\varepsilon^2\langle\boldsymbol{\lambda},\boldsymbol{a}\rangle_{\boldsymbol{\Sigma}}\\ &\quad+\varepsilon^2\log\left(\sum_{l=0}^{r}\left(\varepsilon|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}\right)^l\right).\end{aligned}\tag{5.37}$$

Choosing $\boldsymbol{\lambda}$ such that

$$f(\boldsymbol{\lambda})=\frac{1}{2}\varepsilon^4 p|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}^2+\varepsilon^2 r|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}-\varepsilon^2\langle\boldsymbol{\lambda},\boldsymbol{a}\rangle_{\boldsymbol{\Sigma}}$$

attains its minimum, which happens when $\boldsymbol{\lambda}$ has the same direction as $\boldsymbol{a}$ since the first two terms only depend on the magnitude of $\boldsymbol{\lambda}$, so we may write $\boldsymbol{\lambda}=\boldsymbol{a}|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}|\boldsymbol{a}|_{\boldsymbol{\Sigma}}^{-1}$.

Then the function becomes

$$f(\boldsymbol{\lambda}) = \frac{1}{2}\varepsilon^4 p|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}^2 + \varepsilon^2 r|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}} - \varepsilon^2|\boldsymbol{a}|_{\boldsymbol{\Sigma}}|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}},$$

which is a quadratic function of $|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}}$, we thus deduce that it reaches its minimum when

$$|\boldsymbol{\lambda}|_{\boldsymbol{\Sigma}} = \frac{(|\boldsymbol{a}|_{\boldsymbol{\Sigma}} - r)^+}{\varepsilon^2 p}.$$

Therefore, the minimum is attained at

$$\boldsymbol{\lambda} = \frac{(|\boldsymbol{a}|_{\boldsymbol{\Sigma}} - r)^+}{\varepsilon^2 p|\boldsymbol{a}|_{\boldsymbol{\Sigma}}}\boldsymbol{a}.$$

By setting $\boldsymbol{\lambda}$ to take the above value in (5.37), we obtain that

$$\begin{aligned}
\varepsilon^2 \log c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in B(\boldsymbol{a}, r)\right\} &\leq -\frac{1}{2p}\left((|\boldsymbol{a}|_{\boldsymbol{\Sigma}} - r)^+\right)^2 \\
&\quad + \varepsilon^2 \log\left(\sum_{l=0}^{r}\left(\frac{(|\boldsymbol{a}|_{\boldsymbol{\Sigma}} - r)^+}{\varepsilon p}\right)^l\right) \\
&\leq -\frac{1}{2p}\left((|\boldsymbol{a}|_{\boldsymbol{\Sigma}} - r)^+\right)^2 + \varepsilon^2 \log(r+1) \\
&\quad + \max_{0\leq l\leq r} \varepsilon^2 l \log\left(\frac{(|\boldsymbol{a}|_{\boldsymbol{\Sigma}} - r)^+}{\varepsilon p}\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \varepsilon^2 \log c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in B(\boldsymbol{a}, r)\right\} &\leq -\frac{1}{2p}\left((|\boldsymbol{a}|_{\boldsymbol{\Sigma}} - r)^+\right)^2 \\
&= -\frac{1}{p}\inf_{\boldsymbol{x}\in B(\boldsymbol{a},r)} I_n(\boldsymbol{x}).
\end{aligned}$$

Now for any compact $K \subset (\mathbb{R}^n, |\cdot|_{\boldsymbol{\Sigma}})$ and any $\delta > 0$, there exists a finite open cover $\{B(\boldsymbol{a}_i,\delta)\}_{i\in I}$ in $(\mathbb{R}^n, |\cdot|_{\boldsymbol{\Sigma}})$ of K with $\boldsymbol{a}_i \in K$ and I a finite index set. Therefore,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon^2 \log c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in K\right\} \\
&\leq \limsup_{\varepsilon\to 0} \varepsilon^2 \log c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in \bigcup_{i\in I} B(\boldsymbol{a}_i,\delta)\right\} \\
&\leq \limsup_{\varepsilon\to 0} \varepsilon^2 \log \left(\sum_{i\in I} c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in B(\boldsymbol{a}_i,\delta)\right\}\right) \\
&\leq \limsup_{\varepsilon\to 0} \varepsilon^2 \log |I| + \limsup_{\varepsilon\to 0} \varepsilon^2 \log \left(\max_{i\in I} c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in B(\boldsymbol{a}_i,\delta)\right\}\right) \\
&= \max_{i\in I} \limsup_{\varepsilon\to 0} \varepsilon^2 \log c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in B(\boldsymbol{a}_i,\delta)\right\} \\
&\leq \max_{i\in I} -\frac{1}{p} \inf_{\boldsymbol{x}\in B(\boldsymbol{a}_i,\delta)} I_n(\boldsymbol{x}) \\
&= -\frac{1}{p} \min_{i\in I} \inf_{\boldsymbol{x}\in B(\boldsymbol{a}_i,\delta)} I_n(\boldsymbol{x}) \\
&\leq -\frac{1}{p} \inf_{\boldsymbol{x}\in B(K,\delta)} I_n(\boldsymbol{x}),
\end{aligned}$$

where

$$B(K,\delta) = \left\{\boldsymbol{x} \in \mathbb{R}^n : \inf_{y\in K} |\boldsymbol{x} - \boldsymbol{y}|_{\boldsymbol{\Sigma}} < \delta\right\}.$$

Let $\delta \to 0$, then the upper bound for compact sets is established.

Now for any $F \subset \mathbb{R}^n$ closed under the Euclidean metric, as all norms on $\mathbb{R}^n$ are equivalent, F is also closed in $(\mathbb{R}^n, |\cdot|_{\boldsymbol{\Sigma}})$. For $\rho > 0$, let

$$H_\rho = \{\boldsymbol{x} = (x_1, \cdots, x_n) : |x_i| \leq \rho, \forall 1 \leq i \leq n\}$$

be a hypercube in $\mathbb{R}^n$. Then by sub-additivity property,

$$\begin{aligned}
c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in F\right\} &\leq c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in F \cap H_\rho\right\} + c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in H_\rho^C\right\} \\
&\leq c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in F \cap H_\rho\right\} + \sum_{i=1}^{n} c_{p,r}\left\{\omega : |B_{t_i}(\varepsilon\omega)| > \rho\right\}.
\end{aligned}$$

Therefore, by the result for compact sets and (5.35), as well as Lemma 1.2.15, Chapter 1, [11], we have that

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon^2 \log c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in F\right\} \\
\le \max &\left\{ \limsup_{\varepsilon\to 0} \varepsilon^2 \log c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in F \cap H_\rho\right\}, \right. \\
&\left. \quad \limsup_{\varepsilon\to 0} \varepsilon^2 \log \left(\sum_{i=1}^{n} c_{p,r}\left\{\omega : |B_{t_i}(\varepsilon\omega)| > \rho\right\} \right) \right\} \\
\le \max &\left\{ \limsup_{\varepsilon\to 0} \varepsilon^2 \log c_{p,r}\left\{\omega : \boldsymbol{B_t}(\varepsilon\omega) \in F \cap H_\rho\right\}, \right. \\
&\left. \quad \limsup_{\varepsilon\to 0} \varepsilon^2 \log \left(c_{p,r}\left\{\omega : |B_{t_i}(\varepsilon\omega)| > \rho\right\} \right) \right\} \\
\le \max &\left\{ -\frac{1}{p} \inf_{\boldsymbol{x}\in F\cap H_\rho} I_n(\boldsymbol{x}), -\frac{1}{p} \inf_{x>\rho} I_1(x) \right\}
\end{aligned}
$$

for all $\rho > 0$. The proof is complete by letting $\rho \to \infty$. □

Now we may conclude our proof of the large deviation principles and obtain Theorem 5.3.1.

As $F_m : \mathbb{R}^{2^m+1} \to \boldsymbol{W}$ defined in (5.33) is continuous and by definition $F_m \circ T^\varepsilon = X^{\varepsilon,(m)}$, so by the contraction principle, the family $\left\{X^{\varepsilon,(m)}\right\}$ satisfies the $c_{p,r}$-LDP with the good rate function

$$
J_m(\omega) = \inf_{\boldsymbol{x}:F_m(\boldsymbol{x})=\omega} I_{2^m+1}(\boldsymbol{x}), \quad \omega \in \boldsymbol{W}
$$

where we define $\inf \emptyset = \infty$. When $p = 1$ and $r = 0$, the capacity $c_{p,r}$ coincides with the Wiener measure P, and we would expect that the classical LDPs for fBMs defined on the classical Wiener space hold.

Now define $\hat{F}_m : \boldsymbol{W} \to \boldsymbol{W}$ by

$$
\hat{F}_m(\omega) = \omega\left(\frac{k-1}{2^m}\right) + 2^m \left(t - \frac{k-1}{2^m}\right)\left(\omega\left(\frac{k}{2^m}\right) - \omega\left(\frac{k-1}{2^m}\right)\right), \quad t \in \left[\frac{k-1}{2^m}, \frac{k}{2^m}\right], \tag{5.38}
$$

which is a continuous mapping with respect to the uniform convergence topology. Then by the contraction principle for measures (see Theorem 4.2.1, Chapter 4, [11]), the family $\left\{\mathbb{Q}_\varepsilon \circ \hat{F}_m^{-1}\right\}$ on $(\boldsymbol{W}, \mathscr{B}(\boldsymbol{W}))$ satisfies the LDP with the good rate function

$$\hat{J}_m(\omega) = \inf\left\{I(x) : x \in \boldsymbol{W}, \hat{F}_m(x) = \omega\right\}, \quad \forall \omega \in \boldsymbol{W}, \tag{5.39}$$

where $\inf \emptyset = \infty$.

By (5.31), for each $A \in \mathscr{B}(\boldsymbol{W})$, $\hat{F}_m^{-1}(A) \in \mathscr{B}(\boldsymbol{W})$, and

$$\begin{aligned} Q_\varepsilon \circ \hat{F}_m^{-1}(A) = Q_\varepsilon\left(\hat{F}_m^{-1}(A)\right) \\ &= P\left\{\omega \in \boldsymbol{W} : \varepsilon B(\omega) \in \hat{F}_m^{-1}(A)\right\} \\ &= P\left\{\omega \in \boldsymbol{W} : \hat{F}_m\left(\varepsilon B(\omega)\right) \in A\right\}. \end{aligned}$$

Since $\hat{F}_m(\varepsilon B) = X^{\varepsilon,(m)}$ P-a.s. on $\boldsymbol{W}$, we obtain that

$$Q_\varepsilon \circ \hat{F}_m^{-1}(A) = P\left\{\omega \in \boldsymbol{W} : X^{\varepsilon,(m)}(\omega) \in A\right\}.$$

Therefore, by the uniqueness of rate functions (see Lemma 4.1.4, Chapter 4, [11]), J_m and $\hat{J}_m$ coincides.

As shown in Theorem 5.2.1, $\left\{X^{\varepsilon,(m)}\right\}$ are exponentially good approximations of $\{X^\varepsilon\}$, so it suffices to verify that the function I defined above coincides with the function J given in (2.15) and satisfies all conditions in Proposition 2.5.5.

Let us first check if I satisfies all conditions. We observe that I given in (5.32) is a good rate function by definition.

For any closed $C \subset \boldsymbol{W}$, denote $\eta_m = \inf_{\omega \in C} \hat{J}_m(\omega)$, where $\hat{J}_m = J_m$ is defined as in (5.39). By definition,

$$\eta_m = \inf_{\omega \in \hat{F}_m^{-1}(C)} I(\omega).$$

Suppose that

$$\liminf_{m\to\infty} \eta_m = \eta < \infty,$$

then as I is a good rate function and as lower semi-continuous functions attain their minimums on compact sets, we conclude that I attains its minimum on the closed subset $\hat{F}_m^{-1}(C) \subset \boldsymbol{W}$. Therefore, for each m , there exists some $\omega_m \in \boldsymbol{W}$ such that $\omega_m \in \hat{F}_m^{-1}(C)$ and

$$\eta_m = \inf_{\omega \in \hat{F}_m^{-1}(C)} I(\omega) = I(\omega_m).$$

We notice that for all $\omega \in \boldsymbol{W}$, $\hat{F}_m(\omega) \to \omega$ in $(W, \|\cdot\|)$ as $m \to \infty$. Indeed, by the definition of $\hat{F}_m$ in (5.38),

$$\begin{aligned}
\|\hat{F}_m(\omega) - \omega\| &= \sup_{t\in[0,1]} \left|\hat{F}_m(\omega)(t) - \omega(t)\right| \\
&= \max_{1\le i\le 2^m} \sup_{t\in\left[\frac{i-1}{2^m},\frac{i}{2^m}\right]} \left|\hat{F}_m(\omega)(t) - \omega(t)\right| \\
&= \max_{1\le i\le 2^m} \sup_{t\in\left[\frac{i-1}{2^m},\frac{i}{2^m}\right]} \Bigg|\omega\left(\frac{i-1}{2^m}\right) \\
&\quad +2^m\left(t - \frac{i-1}{2^m}\right)\left(\omega\left(\frac{i}{2^m}\right) - \omega\left(\frac{i-1}{2^m}\right)\right) - \omega(t)\Bigg| \\
&\le \max_{1\le i\le 2^m} \sup_{t\in\left[\frac{i-1}{2^m},\frac{i}{2^m}\right]} \Bigg(\left|\omega\left(\frac{i-1}{2^m}\right) - \omega(t)\right| \\
&\quad +2^m\left(t - \frac{i-1}{2^m}\right)\left(\left|\omega\left(\frac{i}{2^m}\right) - \omega(t)\right| + \left|\omega\left(\frac{i-1}{2^m}\right) - \omega(t)\right|\right)\Bigg) \\
&\le \max_{1\le i\le 2^m} \sup_{t\in\left[\frac{i-1}{2^m},\frac{i}{2^m}\right]} \left(2\left|\omega\left(\frac{i-1}{2^m}\right) - \omega(t)\right| + \left|\omega\left(\frac{i}{2^m}\right) - \omega(t)\right|\right),
\end{aligned}$$

and since ω is uniformly continuous over $[0,1]$, for any $\varepsilon > 0$, there exists some $N > 0$ such that for all $s, t \in [0,1]$ satisfying

$$|s - t| < \frac{1}{2^N},$$

it holds that

$$|\omega(s) - \omega(t)| < \frac{\varepsilon}{3}.$$

Therefore, it follows that $\forall \varepsilon > 0$, there exists some $N > 0$ such that $\|\hat{F}_m(\omega) - \omega\| < \varepsilon$.

Since $\hat{F}_m(\omega_m) \in C$ for all m, for each $\delta > 0$, $\omega_m \in C_\delta$ for large m, where

$$C_\delta = \{\omega : \|\omega - C\| \leq \delta\}.$$

It follows that

$$\inf_{\omega \in C_\delta} I(\omega) \leq I(\omega_m) = \eta_m = \inf_{\omega \in C} \hat{J}_m(\omega)$$

for m sufficiently large, and hence by taking limit infimum on both sides, we have that

$$\inf_{\omega \in C_\delta} I(\omega) \leq \liminf_{m \to \infty} \inf_{\omega \in C} \hat{J}_m(\omega).$$

According to Lemma 4.1.6 (a), Chapter 4, [11],

$$\inf_{\omega \in C} I(\omega) \leq \liminf_{m \to \infty} \inf_{\omega \in C} \hat{J}_m(\omega) \tag{5.40}$$

when letting $\delta \to 0$, and hence the condition (2.16) is fulfilled. The case when $\liminf_{m \to \infty} \eta_m = \infty$ is trivial, so we have verified all conditions in Proposition 2.5.5.

Next, we prove that I coincides with the function J defined as in (2.15) by

$$J(\omega) = \sup_{\lambda > 0} \liminf_{m \to \infty} \inf_{x \in B(\omega, \lambda)} \hat{J}_m(x).$$

For any $\omega \in \boldsymbol{W}$, set $C = \overline{B(\omega, \lambda)}$ in (5.40). It holds that

$$\begin{aligned} \inf_{x \in C} I(x) &\leq \liminf_{m \to \infty} \inf_{x \in C} \hat{J}_m(x) \\ &\leq \liminf_{m \to \infty} \inf_{x \in B(\omega, \lambda)} \hat{J}_m(x) \end{aligned}$$

$$\leq J(\omega).$$

By letting $\lambda \to 0$ and applying Lemma 4.1.6 (a), Chapter 4 in [11], we may conclude that $I(\omega) \leq J(\omega)$ for all ω. For the reverse part, denote $\hat{\omega}_m = \hat{F}_m(\omega)$ for $\omega \in \boldsymbol{W}$, and as shown above, $\hat{\omega}_m \to \omega$ as $m \to \infty$. Therefore, for any $\lambda > 0$, there exists some $M > 0$ such that for all $m \geq M$,

$$\inf_{x \in B(\omega,\lambda)} \hat{J}_m(x) \leq \hat{J}_m(\hat{\omega}_m).$$

By the definition of $\hat{J}_m$, $\hat{J}_m(\hat{\omega}_m) \leq I(\omega)$. By taking limit infimum in m first, then supremum in λ, we obtain that

$$J(\omega) = \sup_{\lambda>0} \liminf_{m\to\infty} \inf_{x\in B(\omega,\lambda)} \hat{J}_m(x) \leq I(\omega),$$

and hence $I = J$.

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