Geometrical Changes: Change and Motion in Aristotle's Philosophy of Geometry

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It is often said that Aristotle takes geometrical objects to be absolutely unmovable and unchangeable. However, Greek geometrical practice does appeal to motion and change, and geometers seem to consider their objects apt to be manipulated. In this paper, I examine if and how Aristotle's philosophy of geometry can account for the geometers' practices and way of talking. First, I illustrate three different ways in which Greek geometry appeals to change. Second, I examine Aristotle's ontology of geometrical objects and argue that although the truth-makers of geometrical statements are in fact unmovable because they are properties of sensible objects, geometers 'separate them in thought' and treat them as substances apt to be modified. Finally, I examine whether allowing for the possibility of manipulating these semi-fictional geometrical individuals creates problems for the applicability of geometry. I find that it does not, insofar as one accepts that geometry is not meant to track physical change but merely to study the instantaneous geometrical configuration of sensible bodies, and is thus only applicable at the instant.

Aristotle often mentions that geometry studies objects that are 'separable in thought from motion' (*Physics* II.2), 'immovable' (*Metaphysics* E.1), or 'without motion' (*Metaphysics* A.8). These remarks are usually taken at face value, as implying that geometrical objects are absolutely unmovable and unchangeable (for example, Katz 2019, p. 507; Mendell 2004). However, the way in which Greek geometers act and speak suggests that they consider their objects apt to be manipulated, changed, and maybe even moved around—even if not subject to decay.

This way of speaking seems to cause problems for philosophers. Plato, for example, acknowledges that geometers 'talk as if they were doing something … they talk about making the square, applying and adding' but describes this way of talking as 'quite ridiculous', although 'necessary' (μάλα γελοίως τε καὶ ἀναγκαίως)

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(cf. Burnyeat 2000, pp. 33 ff.). The reason, he claims, is that geometry 'is the knowledge of the eternally real' (*Republic* VII, 527a6–b). While he acknowledges that geometers need to use physical and visual prompts such as diagrams, he claims that geometry is ultimately about separate, eternal and unchangeable objects. It follows that the constructive talk of geometers is misleading and should not be taken seriously, for they cannot act on those objects. At the very best, geometers can manipulate and modify diagrams and models that are nothing but heuristic tools.

Aristotle's position towards the constructive character of Greek geometry must be different from Plato's. Aristotle does not think that geometry is about separate objects, but about physical objects (although 'considered not *qua* physical'); thus he cannot explain away the geometers' talk on the same grounds as Plato.

Note that Aristotle considers geometrical methodology as a model for scientific reasoning, and frequently refers to it to clarify his theory of demonstration. So we would expect him to be able to account for common geometrical practices in his philosophy of geometry, rather than dismissing them. The goal of this paper is to understand if and how Aristotle's philosophy of geometry can account for the way in which Greek geometry appeals to motion and change. Can his remarks about the absence of change in geometry be interpreted in such a way as to be compatible with the actual practice of geometry of his time?

I

Appeals to change and motion are very frequent in Greek geometrical practice, and essential to it. We can distinguish three main cases.

1. *Diagrams*

Greek geometry makes extensive use of diagrams, and motion is required to draw them. This basic appeal to motion seems easy to dismiss, for example by claiming that diagrams are not the objects of geometry, but representations of these objects. However, note that diagrams are not just an illustration of the text of the proof, but provide information that is necessary to its validity. As Netz (1999) proves, the text alone is often insufficient to yield a valid argument: both deductive and diagrammatic steps are involved in geometrical proofs. Interestingly, the diagram is reliable, not because of the way it looks (diagrams might not even be accurate), but because it

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encodes information about its construction. The diagram does not represent merely the complete configuration under examination, but also its construction (Netz 1999, pp. 54, 181). The underlying assumption seems to be that the corresponding geometrical object is constructed in the same way.

2. *Definitions*

Some geometrical objects are defined by appealing to motion, others as changeable. For example, the solids of revolution are presented in *Elements* XI as the result of the revolution of a given plane figure: 'when a semicircle with fixed diameter is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a sphere' (Definition 14). Similarly, curves are usually defined either stereometrically or kinematically; kinematic definitions appeal to the motion of a point, while stereometric definitions take curves to be divisions of solids, thus implying that it is possible to divide (and thus modify) the solid. In both cases, the object is defined by referring to its construction. Other objects are themselves defined as changeable: for example, lines are by definition extendable. This is granted by Euclid's second postulate, which gives licence 'to produce a finite straight line continuously in a straight line'.

3. *Proofs*

The most widespread and interesting appeals to motion and change happen in the course of geometrical proofs.[1] We can distinguish three cases:

1 New objects are introduced. The Postulates give permission to draw additional lines and circles. These can be combined in such a way as to construct higher-dimensional objects, or made to intersect so as to individuate points.
2 Objects can be modified, for example, divided or extended.
3 Some proofs seem to rely on the possibility of moving geometrical objects around. Euclid appeals to an intuitive idea of congruence according to which two objects are congruent if and only if they would

[1] This is particularly evident in the tradition of problems, which require the geometer to construct a geometrical object. Problems start with the instruction to perform an action, and are solved when the action is concluded and it has been proved that it achieved the desired effect. But constructive steps are essential in theorems too.

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coincide when superposed. This intuition comes to light in a couple of proofs where locomotion is directly appealed to. See for example *El.* I.4, which proves the side-angle-side condition of congruence for triangles by claiming that 'if the triangle ABC be applied (ἐφαρμοζομένου) to the triangle DEF, and if the point A be placed on the point D, and the straight line AB on DE, then the point B will also coincide with E, because AB is equal to DE'.

II

What is the problem with change, that might warrant its ban from geometry? To begin with, it is important to stress that Aristotle shares with Plato (and the other participants in the debate) a realist conception of mathematical truth, which is taken to be eternal and necessary, and independent of what geometers do or know. Thus, Aristotle surely cannot hold that adding objects to the configuration of a proof amounts to bringing them into existence.

The distinction between actuality and potentiality allows him to solve this problem. In *Metaph*. Θ.9, 1051a21–31, Aristotle examines a geometrical proof and argues that geometrical truths are discovered by bringing into actuality (ἐνεργείᾳ) what was present only potentially (ἐνυπάρχει δυνάμει). This means that adding an element to a geometrical configuration does not bring it into *existence*, but brings into *actuality* what was previously merely potentially present. Crucially, Aristotle believes that what exists potentially does exist, and can be the truthmaker of propositions.[2] Thus bringing such items into actuality does not affect the truth-value of geometrical theorems. This passage confirms both that geometrical objects are merely actualized and not brought into existence, and that the operations of manipulation and construction are legitimate tools and methods when doing geometry. The two things are not in contradiction.

However, Aristotle's observations about the unchangeability of geometrical objects cannot just amount to denying their coming-to-be and passing away. Aristotle argues that the geometer

[2] Saying that something exists potentially is not identical to saying that it is possible that it exists. Potential existence is a mode of being that grounds such a modal claim, but cannot be reduced to it. For example, Aristotle claims that parts exist potentially in the whole. This surely means that they can be actualized and thus come into actual and autonomous existence, but there is a sense in which parts already existed before being actualized; saying that they exist potentially grasps this mode of being.

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'separates' (χωρίζει) geometrical objects, 'for in thought they are separable from motion' (χωριστὰ γὰρ τῇ νοήσει κινήσεώς ἐστι, *Ph*. II.2, 193b33–4). He maintains that 'some parts of mathematics deal with things which are immovable (ἀκίνητα)' (*Metaph*. E.1, 1026a14–5),[3] and that 'mathematical objects, except those of astronomy, are without motion' (τὰ γὰρ μαθηματικὰ τῶν ὄντων ἄνευ κινήσεώς ἐστιν ἔξω τῶν περὶ τὴν ἀστρολογίαν, *Metaph*. A.8, 989b32–33). In these passages, Aristotle uses the language of *kinesis* and not that of *genesis*: he is not primarily concerned with denying that geometrical objects come into being, but with denying that they move or change.[4]

To understand why Aristotle is so concerned with stressing that geometry does not deal with *kinesis*, we should turn to his ontology of geometry. Aristotle argues that geometers study sensible objects, although considered 'not *qua* sensible' (*Ph*. II.2; *Metaph*. M.3). The *qua* operator works as a property-filter (Lear 1982), which allows geometers to ignore all the features that are irrelevant to their inquiry—in particular, the sensible and material aspects of physical bodies. Notice that sensible properties are characterized as those that belong to movable things (*Metaph*. Δ.14, 1020b9–11, 17–18), and that nature is defined as the 'inner principle of change and rest' (*Ph*. II.1, 192b21–2). Ignoring the nature and sensible features of the physical body under consideration requires us to ignore its motion too (Pfeiffer 2018, p. 38).

This is not the end of the story, though. From a metaphysical point of view, what geometers study are geometrical properties and features of sensible objects (Katz 2019). These properties do not exist on their own, as separate objects: there is no Form of Triangularity, according to Aristotle, but only sensible things to which triangularity belongs as a property. This is important because triangularity and all other geometrical properties, being properties, are not the kind of things that can be the subjects of change. If a sensible object changes from being triangular to being square, it is not triangularity that changes: it is the object, which loses the property of triangularity

[3] He seems to have in mind geometry and arithmetic, as opposed to subordinate sciences such as astronomy.

[4] '*Kinesis*' is at times used as an umbrella term to cover all changes that are compatible with the persistence of the subject of change (as opposed to radical changes), and at times with the more restricted sense of motion. This is why in the paper I speak both of motion and of change.

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and acquires the property of squareness. So there is a strong sense in which the objects of geometry, that is, the things that make geometrical statements true, are unchangeable: they are properties and not substances.[5]

However, geometers talk of their objects as objects: not of triangularity but of triangles. In the Aristotelian *kosmos*, strictly speaking, there are no triangles: there is no individual sensible object whose essence is completely exhausted by its triangularity. Rather, there are triangular *things*, that is, objects that are triangular, to whom triangularity belongs as a property.[6] This is why in addition to applying the *qua* operator to filter out the irrelevant features of sensible triangular objects, Aristotle also claims that geometers 'separate and posit what is not separate' (*Metaph.* M.3, 1078a21-22, τὸ μὴ κεχωρισμένον θείη χωρίσας). This step involves a certain degree of fiction, which has to do with the mode of existence of the object under consideration: even if the object of geometry really is a property of a sensible object, geometers treat it as a substance (cf. Corkum 2012). This means that they treat it as if it were the kind of thing that could itself be the ultimate subject of properties (and indeed of contrary properties at different times), even if in fact it is not.

I submit that when geometers 'separate' in thought, they obtain semi-fictional objects which are not completely unmovable and unchangeable, but are apt to be modified according to the geometrical conventions and rules of construction. I call them 'semi-fictional' (rather than simply 'fictional') because the fiction only regards the mode of existence of the item. The properties that are fictionally attributed the status of substances do in fact exist, as properties of sensibles.

Note that this gives rise to an ambiguity in the use of the expression *ta mathematika*: Aristotle could mean either the things in virtue of which geometrical statements are true or the objects to which these statements refer. I take the former to be properties of sensible bodies and as such unchangeable, while the latter are semi-fictional objects that can undergo modifications in thought. The former are

[5] Note that being unchangeable in this sense does not amount to always being, nor to being eternal. In my account the stability of mathematical truth is granted by allowing for potential objects to be truthmakers, not by the fact that geometrical objects are properties of sensibles.

[6] It might be even a necessary or essential property. For example, it is essential to the celestial spheres to be spherical. But their essence is not exhausted by their sphericity: their essence is to be celestial spheres.

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the truthmakers of geometrical statements, while the latter are the referent of the singular terms that appear in those statements.[7]

III

Is this theory coherent? If geometers study sensible bodies (although not *qua* sensible), is it legitimate to 'separate in thought' and subject the resulting semi-fictional objects to certain sets of modifications? A worry arises with respect to the applicability of geometry. If geometry is supposed to study sensible objects (although not *qua* sensible), but proceeds by subjecting its semi-fictional objects to diachronic modifications that *do not track* the changes of the sensible bodies, can geometry really be applicable to the sensible world? Note that geometry does in fact ignore the motions and changes of the sensible bodies that instantiate geometrical properties: the 'semi-fictional' geometrical objects that result from the separation of such properties do not have natural motions, they do not grow or decay. The modifications and changes that are allowed in geometry are very specific and codified, and differ from the motions that sensible objects would undergo. How can Aristotle be sure that this operation does not lead to any falsity?

I believe that a cryptic passage in *De Caelo* (1.10, 279b32–280a11) can help us find a solution. Aristotle is arguing against a non-literal interpretation of Plato's *Timaeus*, according to which the talk of generation found in the dialogue is only 'for the sake of instruction'. The proponents of such a view draw a parallel with the geometers' use of constructions. Aristotle rejects the parallelism, arguing that 'in the production of geometrical figures, if everything involved is supposed to exist simultaneously, the same result is achieved … In geometrical figures no element is separate in time (οὐδὲν τῷ χρόνῳ κεχώρισται)'.

The fact that 'no element is separate in time' is presented as the reason why it is legitimate to use constructions to discover and prove geometrical theorems, even if those constructions involve artificial

[7] From a metaphysical point of view, this surprisingly implies that the truth of geometrical theorems does not require the existence of the referents of the singular terms that appear in them. Note that some contemporary analyses of truthmaking accept that the ontological commitments of a theory might be restricted to the entities that we need as truthmakers for that theory, which might not be the referents of the singular terms that appear in the theory. See, for example, Cameron (2008).

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motions and changes that are different from the natural motion and development of sensible substances. I believe that this is the key to understanding why Aristotle puts so much emphasis on geometry's indifference to change. In short, I take his point to be that geometry is not supposed to study physical bodies as they move and evolve in time, but *at an instant*.

Aristotle maintains that change requires time; at an instant there can be no change nor rest (*Ph*. VI.3, 234a24–b9; VI.6, 237a14–15). I wish to suggest that 'separating from change' or 'removing change' does not mean that geometry studies special objects that are unchangeable, but rather that it considers only the instantaneous configuration of sensible bodies. Such configuration is static, not because geometrical objects cannot undergo motions, but because no change can happen at an instant. I take this to be a restriction on the applicability of geometry to sensible reality—not a statement about the features and characteristics of the geometrical semi-fictional objects.

These semi-fictional objects are the result of separation: geometrical features that belong to a certain sensible body at one instant are considered in their own right and posited as separate. I believe that such objects can be modified and manipulated in thought; the only condition is that these manipulations do nothing more than actualize features that were already potentially present in the instantaneous configuration that we are examining. If this condition is respected, then the artificial modification of the semi-fictional objects widens our understanding of the instantaneous state of the world that we are considering, without introducing changes in the world.

Take the case of a spherical lump of clay that is moulded into a cube. The lump loses the property of sphericity and acquires the property of being a cube. It is the lump which undergoes a change, while its geometrical features cannot be said to change: being properties, they can be lost or acquired by various objects, but they do not undergo change themselves. When we do geometry, we consider the lump not *qua* changeable. According to my reading, this means that geometry can only study the initial and final instantaneous configurations, but not the transition from one to the other. When studying, for instance, the initial configuration, in which the lump is spherical, geometers separate the sphericity and posit it in thought as a substance; having done this, they can reason in a narrative and diachronic way, manipulating and modifying this semi-fictional sphere.

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These modifications and manipulations, which must be operated in accordance with geometrical postulates and rules of construction, do not track the physical changes undergone by the spherical lump of clay—but they were never meant to. The goal of geometrical manipulations is to bring to light (by bringing into actuality) the geometrical properties of the configuration that is instantiated in the sensible world at the instant at which the examination starts. The geometrical reasoning develops by using changes and motions, and does so in a diachronic way. However, the special kinds of changes that are allowed in geometry are such that there is no contradiction if all the stages of the change are supposed to exist at the same time. This allows geometers to apply the results of their diachronic reasoning to the initial state of the physical system.

This is possible, I believe, because geometrical truths are indifferent to the distinction between actual and potential existence. A certain geometrical theorem can be true in virtue of features that are merely potential in a certain configuration, and that might be brought into actuality thanks to geometrical modifications and constructions. Actualizing these features results in the production of knowledge that is still relative to the initial state of the system. Geometry is applicable at the instant and not diachronically, even if its practice appeals to constructions and changes.[8]

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