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Ring Cracks at the Surface of a Half-Space

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Abstract

This paper uses a distribution of ring dislocations to determine the crack tip stress intensity factors for a ring crack at the surface of a half-space subjected to a point force at the axis of symmetry. Three cases are considered: a short crack (relative to its radius), fully open to its root; a longer crack open at the surface but closed and slipping to the root; and a longer crack open at the surface and slipping to some self-determining point in the closing region. The stress intensity factors and points of closure and stick are determined.

Keywords: Ring dislocations, Axisymmetric, Crack, Stress intensity factor

1. Introduction

A powerful method for solving for crack tip stress intensity factors and slip conditions for cracks in simple geometries, but where there may be at least some crack closure, is to use a combination of the nominal stress field in the crack's absence, together with arrays of strain nuclei in the form of dislocations. This expedient is numerically very efficient and is well established for plane problems [1]. In principle, the same ideas may be applied both to general three dimensional problems (where a cruder discretisation of the crack face is needed [2]) and to axis-symmetric problems. Here we address the latter, using circular dislocation loops. This permits all the quadratures and techniques developed for plane problems to be used but suffers from a drawback associated with the nature of the dislocation solution: straight dislocations (that is, those used in plane elasticity) are of the Volterra kind, because they have a constant Burgers vector and, as a consequence, the solution is independent of the path cut used to form the dislocation, so that we may tacitly assume that this follows the line of the crack, without detailed considerations of how we actually form the dislocation solution. Turning to circular dislocations, we see that the ring edge dislocation with axial Burgers vector (b_z) is also Volterra in nature, but the same is *not* true of the radial dislocation (b_r), although a number of crack solutions have either been sent to journals or actually appeared in the literature in which this need has been ignored. In order to circumvent this problem, some authors resort to using dislocation dipoles [3], but this results in singular integral equations with hypersingular kernels (i.e., kernels with $1/(t-s)^2$ terms), requiring a much more powerful quadrature scheme to be

solved. The purpose of this paper is to solve an illustrative axis-symmetric crack problem using the path-cut dependent approach, by making use of the dislocation kernels for a circular dislocation in a homogeneous elastic half-space where the plane of the ring is parallel with the free surface, and due to Paynter [4, 5]. Supplementary material will be attached giving MATLAB code to enable others to use these results in further problems without the need to transcribe the solutions manually, we hope, thereby, avoiding errors.

2. Bilateral Problem

The basic, illustrative, bilateral problem is shown in Figure 1. A point force, P is applied acting into an elastic half-space whose Poisson's ratio is ν . In the cylindrical coordinate set of Figure 1, the state of stress within the half-space is given by Timoshenko [6], and the tractions arising on any cylindrical cut (an $r = \text{constant}$ surface), $\tilde{\sigma}_{ij}(r, z)$ are given by

$$\tilde{\sigma}_{rr}(r, z) = \frac{P}{2\pi} \left\{ (1 - 2\nu) \left[\frac{1}{r^2} - \frac{z}{r^2(r^2 + z^2)^{1/2}} - \frac{3r^2 z}{(r^2 + z^2)^{5/2}} \right] \right\} \quad (1)$$

$$\tilde{\sigma}_{rz}(r, z) = -\frac{3P}{2\pi} \left\{ \frac{r z^2}{(r^2 + z^2)^{5/2}} \right\}. \quad (2)$$

As we would expect the stress field is self-similar, as there are no length dimensions in the problem, and the stresses decay as $[L^{-2}]$. For a force acting inwards there is a band, making an angle of approximately 83° with respect to the axis of symmetry where the radial stress is tensile, and a band between the angles of 83° and 15°

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Figure 1: Bands of stress given by the bilateral solution for an inwards force.

where the radial stress is compressive, and then the narrow cone around the axis where it again becomes tensile.

Therefore, a short crack (relative to its radius) will be fully open to its root, Figure 2b, and denoted case *I*. A longer crack, Figure 2c, and where the inter-face coefficient of friction is small, will be open at the surface but closed and slipping to the root, case *II*, but, if the coefficient of friction is high enough slip will not penetrate to the root but will stop at some self-determining point in the closed region, case *III*, Figure 2d. Thus, case *I* will give rise to both modes *I* and *II* stress intensities, case *II* will give to a mode *II* stress intensity only, and case *III* will give rise to a wholly bounded state of stress everywhere. This set of cases is not exhaustive, and we explicitly exclude the case where the crack is so deep relative to its radius that its tip penetrates into the near-axis tension region.

3. Formulation

The general principle in obtaining a solution will be to develop expressions for the tractions on the surface of the crack, $N(z)$, $S(z)$, representing the normal and shear components, respectively, as the sum of the bilateral solution together with an integral representation of slip in the form of a distribution of climb and glide dislocations. The dislocations needed are all ‘edge’ in character, with their Burgers vectors lying in a $\theta = \text{constant}$ plane. A $b_r(\xi)$ climb dislocation loop, of radius c and with the dislocation lying at a depth ξ may be formed by making a path cut along the cylinder $r = c$ between the dislocation and the free surface of the half-space and inserting a tube of wall thickness b_r . It models crack opening, and

Figure 2: A radial crack subjected to a point force.

will induce tractions along the same cylinder $\hat{\sigma}_{ri}(z)$, where $i = r, z$, given by

$$\hat{\sigma}_{ri}(z) = G_{ri}^r(z, \xi) b_r(\xi). \quad (3)$$

The functions $G_{ri}^r(z, \xi)$ are extremely complicated in form and are defined in [5], as well as being explicitly provided in the supplementary material and in Appendix A. They are bounded ('regular') when $i = z$ but display a Cauchy singularity when $i = r$.

In a similar way, a second edge dislocation may be formed by (for example) making the same path cut on the same surface as above, but now sliding the outer wall with respect to the inner wall by a constant amount b_z . This generates a glide dislocation which is of the Volterra kind and will be used to represent crack slip displacement. It will induce tractions along the same cylinder $\hat{\sigma}_{zi}(z)$, where $i = r, z$, given by

$$\hat{\sigma}_{zi}(z) = G_{zi}^z(z, \xi) b_z(\xi). \quad (4)$$

The functions $G_{zi}^z(z, \xi)$ are again extremely complicated in form and are also defined in [5], as well as being explicitly provided in the supplementary material and in Appendix A. This time they are bounded ('regular') when $i = r$ but display a Cauchy singularity when $i = z$. The resulting tractions on the plane of the crack surface are given by

$$N(z) = \tilde{\sigma}_{rr}(z) + \int_{\text{opening}} G_{rr}^r(z, \xi) B_r(\xi) d\xi + \int_{\text{slip}} G_{rr}^z(z, \xi) B_z(\xi) d\xi \quad (5)$$

$$S(z) = \tilde{\sigma}_{rz}(z) + \int_{\text{opening}} G_{rz}^r(z, \xi) B_r(\xi) + \int_{\text{slip}} G_{rz}^z(z, \xi) B_z(\xi) d\xi \quad (6)$$

where $B_i(\xi) = db_i/d\xi$, $i = r, z$. Glide dislocations are installed over the whole of the length of the crack where slipping occurs, including the open portion, and climb dislocations are installed over that part of the crack which is open. Equations (5) and (6) form the basis of the solution and integral equations may be generated in slightly different forms for the three cases of response described in §2.

3.1. Case I

This is the most straightforward problem to consider, and is hence dealt with first, Figure 2b. The crack is of depth a , which is sufficiently small (relative to c) for it to be open to the root. The boundary conditions for the crack faces to be traction-free are

$$N(z) = 0 \quad (7)$$

$$S(z) = 0 \quad 0 \leq z \leq a \quad (8)$$

which, with eqs. (5) and (6) define the two simultaneous Cauchy integral equations

$$\int_0^a \left[G_{rr}^r(z, \xi) B_r(\xi) + G_{rr}^z(z, \xi) B_z(\xi) \right] d\xi = -\tilde{\sigma}_{rr}(z) \quad 0 \leq z \leq a \quad (9)$$

$$\int_0^a \left[G_{rz}^r(z, \xi) B_r(\xi) + G_{rz}^z(z, \xi) B_z(\xi) \right] d\xi = -\tilde{\sigma}_{rz}(z) \quad 0 \leq z \leq a. \quad (10)$$

These equations possess the property that the range of the integrals is the same as the range of imposition of the right hand side, so that they constitute two well-posed integral equations with generalized Cauchy kernels. They must be solved numerically using a standard numerical quadrature devised by Erdogan, Gupta and Cook [7]. First, we put them in standard form over the intervals $[-1, 1]$ utilizing the substitutions

$$s = \frac{2\xi}{a} - 1, \quad t = \frac{2z}{a} - 1 \quad 0 \leq z, \xi \leq a \quad (11)$$

which give

$$\int_{-1}^1 \left[G_{rr}^r(t, s) B_r(s) + G_{rr}^z(t, s) B_z(s) \right] ds = -\tilde{\sigma}_{rr}(t) \quad -1 \leq t \leq 1 \quad (12)$$

$$\int_{-1}^1 \left[G_{rz}^r(t, s) B_r(s) + G_{rz}^z(t, s) B_z(s) \right] ds = -\tilde{\sigma}_{rz}(t) \quad -1 \leq t \leq 1. \quad (13)$$

The next step is to consider the general form of the solution required, and here we note that, at the crack root ($t = 1$), the displacement gradient must be square root singular in nature, while at the surface it must be bounded. The quadratures available do not give us the option of letting the gradients take a finite value, so we instead choose it to be square root bounded. This is slightly over restrictive – it means that the crack faces must be parallel, for example – but is known to have only a very tiny effect on the crack root [1], and so we assume fundamental functions $B_i(s)$ as shown below, which leaves the unknown dislocation densities being represented by the functions $\phi_i(s)$:

$$B_i(s) = \phi_i(s) \sqrt{\frac{1+s}{1-s}} \quad i = r, z \quad (14)$$

$$(15)$$

and eqs. (12) and (13) become, in normalised form

$$\sum_{i=1}^N \left\{ W_i \left[G_{rr}^r(t_k, s_i) \phi_r(s_i) + G_{rr}^z(t_k, s_i) \phi_z(s_i) \right] \right\} = -\frac{2}{\pi a} \tilde{\sigma}_{rr}(t_k) \quad k = 1, \dots, N \quad (16)$$

$$\sum_{i=1}^N \left\{ W_i \left[G_{rz}^r(t_k, s_i) \phi_r(s_i) + G_{rz}^z(t_k, s_i) \phi_z(s_i) \right] \right\} = -\frac{2}{\pi a} \tilde{\sigma}_{rz}(t_k) \quad k = 1, \dots, N \quad (17)$$

where the integration points s_i , collocation points t_k and weights W_i for the quadrature are given in Appendix B.

Thus, we have a set of $2N$ equations for $2N$ unknowns. Once the vector $[\phi_r(s_i), \phi_z(s_i)]$ is known the crack tip stress intensities may be found from the following relations

$$K_I = \sqrt{2\pi a} \frac{2\mu}{(\kappa + 1)} \phi_r(1) \quad (18)$$

$$K_{II} = \sqrt{2\pi a} \frac{2\mu}{(\kappa + 1)} \phi_z(1), \quad (19)$$

where μ is the modulus of rigidity (shear modulus) and κ Kolosov's constant ($\kappa = 3 - 4\nu$ for plane strain).

3.2. Case II

The solution described above will be valid for sufficiently small a/c . As this ratio is increased the mode I crack tip stress intensity factor will decrease until it reaches zero at a unique value of a/c , which corresponds to crack-tip closure. In this section we look at what happens for larger values of a/c , and denote the closure point by b , Figure 2c. Notice that, in this problem, point a is specified but point b is an output of the problem. The boundary conditions on the tractions are now rather different. In the open part we require, as before,

$$N(z) = 0 \quad (20)$$

$$S(z) = 0 \quad 0 \leq z \leq b \quad (21)$$

whereas, in the closed, slipping region, the shearing traction is limited by friction:

$$N(z) < 0 \quad (22)$$

$$S(z) = f N(z) \quad b \leq z \leq a. \quad (23)$$

These two sets of conditions may be combined into one by making use of the Heaviside step function, $H(\cdot)$, giving

$$N(z) = 0 \quad 0 \leq z \leq b \quad (24)$$

$$S(z) - f H(z - b) N(z) = 0 \quad 0 \leq z \leq a. \quad (25)$$

A further complication is that the climb dislocations are non-zero only over the interval $0 \leq z \leq b$, whereas

the glide dislocations must be distributed over the whole interval $0 \leq z \leq a$:

$$N(z) = \tilde{\sigma}_{rr}(z) + \int_0^b G_{rr}^r(z, \xi) B_r(\xi) d\xi + \int_0^a G_{rr}^z(z, \xi) B_z(\xi) d\xi \quad (26)$$

$$S(z) = \tilde{\sigma}_{rz}(z) + \int_0^b G_{rz}^r(z, \xi) B_r(\xi) d\xi + \int_0^a G_{rz}^z(z, \xi) B_z(\xi) d\xi \quad (27)$$

or

$$\int_0^b G_{rr}^r(z, \xi) B_r(\xi) d\xi + \int_0^a G_{rr}^z(z, \xi) B_z(\xi) d\xi = -\tilde{\sigma}_{rr}(z) \quad (28)$$

$$\int_0^b B_r(\xi) \left[G_{rz}^r(z, \xi) - f H(z - b) G_{rr}^r(z, \xi) \right] d\xi + \int_0^a B_z(\xi) \left[G_{rz}^z(z, \xi) - f H(z - b) G_{rr}^z(z, \xi) \right] d\xi = -\left[\tilde{\sigma}_{rz}(z) - f H(z - b) \tilde{\sigma}_{rr}(z) \right]. \quad (29)$$

Notice that, as required, the region of imposition of the integral equations (eqs. (24) and (25)) matches those integrals in eqs. (28) and (29) which are Cauchy in character.

Since there are two regions of imposition for the integral equations, two sets of quadrature points are needed. Again, the equation must be solved numerically, using Gauss-Chebyshev quadrature [7]. Putting them in standard form over the intervals $[-1, 1]$ using the substitutions

$$u = \frac{2\xi}{b} - 1, \quad v = \frac{2z}{b} - 1 \quad 0 \leq z, \xi \leq b \quad (30)$$

$$s = \frac{2\xi}{a} - 1, \quad t = \frac{2z}{a} - 1 \quad 0 \leq z, \xi \leq a \quad (31)$$

gives

$$\int_{-1}^1 G_{rr}^r(v, u) B_r(u) du + \int_{-1}^1 G_{rr}^z(v, s) B_z(s) ds = -\tilde{\sigma}_{rr}(v) \quad -1 \leq v \leq 1 \quad (32)$$

$$\int_{-1}^1 B_r(u) \left[G_{rz}^r(t, u) - f H(\zeta) G_{rr}^r(t, u) \right] du + \int_{-1}^1 B_z(s) \left[G_{rz}^z(t, s) - f H(\zeta) G_{rr}^z(t, s) \right] ds = -\left[\tilde{\sigma}_{rz}(t) + f H(\zeta) \tilde{\sigma}_{rr}(s) \right] \quad -1 \leq t \leq 1 \quad (33)$$

where $\zeta = (t + 1) - 2b/a$.

For the general form of the solution, the behaviour of the climb and glide dislocations are distinct. At the surface, both climb and glide solutions must be bounded. At

the closure point b ($v = 1$), the climb solution must be bounded, since this point is a cusp. At the crack root ($z = a$, $t = 1$), the glide solution must be square root singular. Thus, we choose

$$B_r(u) = \phi_r(u) \sqrt{1 - u^2} \quad (34)$$

$$B_z(s) = \phi_z(s) \sqrt{\frac{1+s}{1-s}} \quad (35)$$

and eqs. (32) and (33) become, in normalised form,

$$\sum_{i=1}^N \left\{ X_i b \phi_r(u_i) G_{rr}^r(v_k, u_i) + W_i a \phi_z(s_i) G_{rr}^z(v_k, s_i) \right\} = -\frac{2}{\pi} \tilde{\sigma}_{rr}(v_k) \quad k = 1, \dots, N+1 \quad (36)$$

$$\sum_{i=1}^N \left\{ X_i b \left[G_{rz}^r(t_k, u_i) - f H(\zeta_k) G_{rr}^r(t_k, u_i) \right] \phi_r(u_i) \right\} + \sum_{i=1}^N \left\{ W_i a \left[G_{rz}^z(t_k, s_i) - f H(\zeta_k) G_{rr}^z(t_k, s_i) \right] \phi_z(s_i) \right\} = -\frac{2}{\pi} \left[\tilde{\sigma}_{rz}(t_k) - f H(\zeta_k) \tilde{\sigma}_{rr}(t_k) \right] \quad k = 1, \dots, N \quad (37)$$

where $\zeta_k = (t_k + 1) - 2b/a$. The integration points (s_i , u_i), collocation points (t_k , v_k) and weights (W_i , X_i) for the quadrature are given in Appendix B.

Equations (36) and (37) form a system of $2N+1$ equations and $2N+1$ unknowns. These are the N values of $\phi_r(u_i)$, N values of $\phi_z(s_i)$ and the closure point b . Once the vector $[\phi_r(u_i), \phi_z(s_i)]$ is known the crack tip stress intensity may be found from the following relation

$$K_{II} = \sqrt{2\pi a} \frac{2\mu}{(\kappa+1)} \phi_z(1). \quad (38)$$

The value of the mode II crack tip stress intensity will depend on the coefficient of friction. It will reduce in magnitude as the friction is increased. When it vanishes, there is no longer a mode II stress intensity factor, and the solution is as detailed in the following section.

3.3. Case III

The problem we look at now is that of a relatively deep crack (but not so deep that it approaches the region of radial tension, Figure 1), and the general format of the problem is depicted in Figure 2d. We denote the closure point by b and the stick point by a' (Figure 2d). In this problem, point a does not appear in the solution and points b and a' are outputs of the problem. The boundary conditions are similar to those applied in case II . In the open region we require, as in case I ,

$$N(z) = 0 \quad (39)$$

$$S(z) = 0 \quad 0 \leq z \leq b \quad (40)$$

and in the closed, slipping region, the shearing traction is limited by friction:

$$N(z) < 0 \quad (41)$$

$$S(z) = f N(z) \quad b \leq z \leq a'. \quad (42)$$

Combining the sets of conditions into one, we have

$$N(z) = 0 \quad 0 \leq z \leq b \quad (43)$$

$$S(z) - f H(z-b) N(z) = 0 \quad 0 \leq z \leq a'. \quad (44)$$

The same procedure used for case II can be applied here, giving

$$N(z) = \tilde{\sigma}_{rr}(z) + \int_0^b G_{rr}^r(z, \xi) B_r(\xi) d\xi + \int_0^{a'} G_{rr}^z(z, \xi) B_z(\xi) d\xi \quad (45)$$

$$S(z) = \tilde{\sigma}_{rz}(z) + \int_0^b G_{rz}^r(z, \xi) B_r(\xi) d\xi + \int_0^{a'} G_{rz}^z(z, \xi) B_z(\xi) d\xi, \quad (46)$$

or

$$\int_0^b G_{rr}^r(z, \xi) B_r(\xi) d\xi + \int_0^{a'} G_{rr}^z(z, \xi) B_z(\xi) d\xi = -\tilde{\sigma}_{rr}(z) \quad (47)$$

$$\int_0^b B_r(\xi) \left[G_{rz}^r(z, \xi) - f H(z-b) G_{rr}^r(z, \xi) \right] d\xi + \int_0^{a'} B_z(\xi) \left[G_{rz}^z(z, \xi) - f H(z-b) G_{rr}^z(z, \xi) \right] d\xi = -\left[\tilde{\sigma}_{rz}(z) - f H(z-b) \tilde{\sigma}_{rr}(z) \right]. \quad (48)$$

Equations (47) and (48) need to be solved numerically as before. Normalising the integrals over the intervals $[-1, 1]$ using the substitutions

$$u = \frac{2\xi}{b} - 1, \quad v = \frac{2z}{b} - 1 \quad 0 \leq z, \xi \leq b \quad (49)$$

$$s = \frac{2\xi}{a'} - 1, \quad t = \frac{2z}{a'} - 1 \quad 0 \leq z, \xi \leq a' \quad (50)$$

give

$$\int_{-1}^1 G_{rr}^r(v, u) B_r(u) du + \int_{-1}^1 G_{rr}^z(v, s) B_z(s) ds = -\tilde{\sigma}_{rr}(v) \quad -1 \leq v \leq 1 \quad (51)$$

$$\int_{-1}^1 B_r(u) \left[G_{rz}^r(t, u) - f H(\zeta) G_{rr}^r(t, u) \right] du + \int_{-1}^1 B_z(s) \left[G_{rz}^z(t, s) - f H(\zeta) G_{rr}^z(t, s) \right] ds = -\left[\tilde{\sigma}_{rz}(t) + f H(\zeta) \tilde{\sigma}_{rr}(s) \right] \quad -1 \leq t \leq 1 \quad (52)$$

where $\zeta = (t + 1) - 2b/a'$.

The integral equations in eqs. (51) and (52) for case *III* are the same as those found in case *II* (eqs. (32) and (33)), with a' in place of a . The difference in cases lies in the general form of the solution. For case *III*, both the climb and glide dislocations must be bounded at both ends of the interval. Therefore,

$$B_r(u) = \phi_r(u) \sqrt{1 - u^2} \quad (53)$$

$$B_z(s) = \phi_z(s) \sqrt{1 - s^2} \quad (54)$$

and eqs. (51) and (52) become, in normalised form,

$$\sum_{i=1}^N \left\{ X_i b \phi_r(u_i) G_{rr}^r(v_k, u_i) + W_i a' \phi_z(s_i) G_{rr}^z(v_k, s_i) \right\} = -\frac{2}{\pi} \tilde{\sigma}_{rr}(v_k) \quad k = 1, \dots, N + 1 \quad (55)$$

$$\sum_{i=1}^N \left\{ X_i b \left[G_{rz}^r(t_k, u_i) - f H(\zeta) G_{rr}^r(t_k, u_i) \right] \phi_r(u_i) \right\} + \sum_{i=1}^N \left\{ W_i a' \left[G_{rz}^z(t_k, s_i) - f H(\zeta) G_{rr}^z(t_k, s_i) \right] \phi_z(s_i) \right\} = -\frac{2}{\pi} \left[\tilde{\sigma}_{rz}(t_k) - f H(\zeta) \tilde{\sigma}_{rr}(t_k) \right] \quad k = 1, \dots, N + 1 \quad (56)$$

where $\zeta_k = (t_k + 1) - 2b/a'$. The integration points (s_i, u_i) , collocation points (t_k, v_k) and weights (W_i, X_i) for the quadrature are given in Appendix B.

Equations (55) and (56) form a system of $2N + 2$ equations and $2N + 2$ unknowns: N values of $\phi_r(u_i)$, N values of $\phi_z(s_i)$, the closure point b and the stick point a' . For this case, there are no mode *I* or *II* stress intensity factors.

4. Results

Each of the three classes of problem was coded up using the numerical processor MATLAB. Setting N to 80 makes the solution converge for all cases, i.e. the changes in the stress intensity factors become negligible when N is increased beyond 80. The results are for $\nu = 0.3$. In

problems where a length dimension is an unknown of the problem (b in case *II* and both b, a' in case *III*), there are additional collocation equations which need to be satisfied and which enable the values of b/c and a'/c to be found. In practice, we guess values of the lengths to be found and omit the central equations from the $N + 1$ generated. The column vector of ϕ is found. The omitted equation is then evaluated and the lengths needed are adjusted to minimise the residue.

Consider case *I* first, the crack fully open to the root, where the results are portrayed in Figure 4. We choose as a non-dimensionalising constant the crack tip stress intensity factor for a crack of length $2a$ embedded in a half-plane, and choose as a reference stress the radial stress at the surface (eq. (1)), so that

$$K_0 = \frac{P(1 - 2\nu)}{2\pi c^2} \sqrt{\pi a}. \quad (57)$$

This means that, as expected, when $a/c \rightarrow 0$ we recover the standard result for a plane surface breaking crack under uniform tension ($K_I/K_0 = 1.1215$). As the crack is made longer the normalised mode *I* stress intensity falls, and goes to zero when $a/c \rightarrow 0.207$. This contrasts with the depth at which the radial stress becomes zero (eq. (1)) of $a/c = 0.122$. The same non-dimensionalising constant is also used for mode *II* loading. Of course the mode *II* stress intensity vanishes for short cracks and increases as the crack length is increased.

Figure 3 shows what happens when we make the crack longer than those just considered (case *II*). At the left of the figure ($a/c < 0.207$) the mode *II* stress intensity for the open crack is re-plotted, and then the right hand part of the figure shows what happens as the crack is made longer, and hence remains closed, with no mode *I* intensification. First, look at the response with no friction. The K_{II} value, in normalised form, first increases in magnitude, until $a/c \sim 1.1$, beyond which it decreases. We do not attempt to investigate what happens for very large values of a/c , when the crack tip approaches the tension region. Secondly, consider the curves for increasing values of the coefficient of friction. The crack faces are closed along part of their length, including the crack tip, and some of the shear traction is sustained by friction, with the unsatisfied part contributing to a stress intensity of lower magnitude, as expected. The case *II* solution remains valid until the coefficient of friction reaches the value 0.625. At this point K_{II} falls to zero and, if we attempt to obtain a solution for long cracks slipping at higher coefficients of friction we find that the crack slip displacement changes sign, and this is inconsistent with our assumption that the sign of slip is constant along the whole of the closed portion of the crack. Thus, the solution for a short crack with an interfacial coefficient of friction of 0.7 is valid only for cracks which penetrate a very short distance into the compression region, and the mode *II* stress intensity vanishes when $a/c \rightarrow \sim 0.3$.

Figure 3: Non-dimensional mode II stress intensity factors for cases I and II.

Figure 4: Non-dimensional mode I and II stress intensity factors for case I.

Figure 5: Non-dimensional closure and stick points as a function of the coefficient of friction for case III.

The next question which arises is 'What happens when we have a crack which is longer than $0.207c$, and where the coefficient of friction is greater than 0.625 ? These are case *III* problems, in which the crack does not slip all the way to the root, but sticks at some self-determining depth. The depth at which this occurs (a'/c), is shown in Figure 5 as a function of the coefficient of friction. If the coefficient of friction is less than 0.625 the crack will slip to the root. Note that, because of the coupling inherent in the kernels, glide dislocations modify the contact pressure between the crack faces, that is, they induce a change in the direct traction $\sigma_{rr}(z)$, so that we would expect the point of crack closure, b/c , to change. In fact, there is practically no change in this value for realistic coefficients of friction, Figure 5.

5. Conclusions

The crack tip stress intensity factors have been found for a surface-breaking ring crack loaded by a point force. This problem shows the interesting properties that there are regions in which the crack tip suffers both opening and shear loading modes of stress intensity (case *I*), regions in which the crack tip is closed but it suffers shear mode stress intensity only (case *II*), and regions in which the crack is closed and the crack root does not slip (case *III*). The transition from cases *I* to *II* was found to occur at a unique value of crack length a/c , and is independent of the applied load. The transition to case *III*, however, was found to be dependent both on the crack length (only occurs for short cracks) and on the coefficient of friction.

One of the prime motivations for carrying out this calculation was to show that the ring dislocation is a viable quantity to use as a kernel in studying axi-symmetric cracks problems, and also to bring the kernel functions into the public domain. They are attached to this paper as supplementary material suitable for use with MATLAB.

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Appendix A. State of stress induced by circular edge dislocation loops

This section presents the influence functions due to circular edge dislocation loops. An axial edge dislocation may be formed, for example, by making a path cut along the cylinder $r = c$ and sliding the outer wall with respect to the inner wall by a constant amount b_z . This generates a glide dislocation which is of the Volterra kind and *independent* of the choice of path cut. A $b_r(\xi)$ radial dislocation loop, of radius c and with the dislocation lying at a depth ξ may be formed by making a path cut along the cylinder $r = c$ between the dislocation and the free surface of the half-space and inserting a tube of wall thickness b_r . The radial dislocation, however, is not of the Volterra kind, since the Burgers vector changes direction along the path cut (the Burgers vector is not constant) and, therefore, it *depends* on the chosen path cut.

Appendix A.1. Axial dislocation

Consider a glide axial dislocation loop of radius a put at a position (r, z) in a cylindrical coordinate system and being observed at a depth d , with a Burgers vector component b_z . The stress fields at a position (r, z) are given by

$$\sigma_{ri}^z(r, z) = G_{rj}^z(r, z, d) b_z(d) \quad i = r, z. \quad (\text{A.1})$$

The influence functions $G_{ri}^z(\rho, \zeta, \delta)$ ($i = r, z$) for the glide dislocation in a half-space are given as [5]:

$$G_{rr}^z(\rho, \zeta, \delta) = \frac{2\mu}{c(\kappa+1)} \left[-J_{1,0;1} + I_{1,0;1} + (\zeta - \delta) J_{1,0;2} + (\zeta - 3\delta) I_{1,0;2} + \frac{(\kappa-1)}{2\rho} J_{1,1;0} - \frac{(\kappa-1)}{2\rho} I_{1,1;0} - \frac{(\zeta - \delta)}{\rho} J_{1,1;1} + \frac{(\zeta \kappa - \delta)}{\rho} I_{1,1;1} + \frac{(2\zeta\delta)}{\rho} I_{1,1;2} - 2\zeta\delta I_{1,0;3} \right] \quad (\text{A.2})$$

$$G_{rz}^z(\rho, \zeta, \delta) = \frac{2\mu}{c(\kappa+1)} \left[-(\zeta - \delta) J_{1,1;2} + (\zeta - \delta) I_{1,1;2} - 2\zeta\delta I_{1,1;3} \right] \quad (\text{A.3})$$

where ρ , ζ and δ are the normalised coordinates, given as

$$\rho = r/c, \quad \zeta = z/c, \quad \delta = d/c, \quad (\text{A.4})$$

μ is the modulus of rigidity and κ is the Kolosov's constant.

Appendix A.2. Radial dislocation

The radial dislocation is not of Volterra kind and, thus, is path-cut dependent. There are four possible path-cuts to create a climb dislocation of Burgers vector b_r , defined by Paynter & Hills [5] and shown in Figure A.1. These result in significantly different solutions for the influence functions and should match the path in which the integrals are being evaluated.

Appendix A.2.1. Outer cut (b_r^o)

An 'outside ring' path cut is made by inserting a ring of material at the depth $z = d$, from $r = c$ to infinity, displacing the materials by an amount b_r (Figure A.1a). The angle parameter α designates the angle of the cut with respect to the r axis. For the outer cut, $\alpha = 0$. The influence functions are given as [5]:

$$^oG_{rz}^r(\rho, \zeta, \delta, \alpha) = \frac{2\mu}{c(\kappa+1)} \left[J_{0,1;1} - I_{0,1;1} - (\zeta - \delta) J_{0,1;2} + (\zeta + \delta) I_{0,1;2} - 2\zeta\delta I_{0,1;3} \right] \quad (\text{A.5})$$

Figure A.1: Path cuts for a radial climb dislocation.

$$^oG_{rz}^r(\rho, \zeta, \delta, \alpha) = \frac{2\mu}{c(\kappa+1)} \left[J_{0,1;1} - I_{0,1;1} - (\zeta - \delta) J_{0,1;2} + (\zeta + \delta) I_{0,1;2} - 2\zeta\delta I_{0,1;3} \right] \quad (\text{A.6})$$

Appendix A.2.2. Inner cut (b_r^i)

An 'inside disk' path cut (Figure A.1b) can be formed by inserting a disk of material at a depth $z = d$, from $r = 0$ to $r = c$, displacing the material by the same amount b_r^i (thickness of the disk). For the inner cut, $\alpha = 0$.

$$^iG_{rr}^r(\rho, \zeta, \delta, \alpha) = \frac{2\mu}{c(\kappa+1)} \left[2J_{2,0;1}(\alpha) + 2I_{2,0;1}(|\alpha|) - (\zeta - \delta) J_{2,0;2} + (\zeta + 3\delta) I_{2,0;2} - \frac{(\kappa+1)}{2\rho} J_{2,1;0}(\alpha) - \frac{(\kappa+1)}{2\rho} I_{2,1;0}(|\alpha|) + \frac{(\zeta - \delta)}{\rho} J_{2,1;1} - \frac{(\zeta + \kappa\delta)}{\rho} I_{2,1;1} - \frac{(2\zeta\delta)}{\rho} I_{2,1;2} + 2\zeta\delta I_{2,0;3} \right] \quad (\text{A.7})$$

$${}^i G_{rz}^r(\rho, \zeta, \delta, \alpha) = \frac{2\mu}{c(\kappa+1)} \left[-J_{2,1;1} + I_{2,1;1} + (\zeta - \delta) J_{2,1;2} + (\zeta + \delta) I_{2,1;2} + 2\zeta \delta I_{2,1;3} \right] \quad (\text{A.8})$$

Appendix A.2.3. Cylindrical cuts ($b_r^{c,e}$)

The two other path cuts involve cylindrical cuts. The ‘exterior’ path cut is made by inserting a cylinder of material of wall thickness b_r^e at a position $(r, z) = (c, d)$ extending along the z axis from $z = d$ to $z = 0$, i.e. towards the ‘exterior’ of the half-space (Figure A.1c). The ‘core’ path cut is made by inserting a cylinder of material of wall thickness b_r^c at a position $(r, z) = (c, d)$ extending along the z axis from $z = d$ to infinity, i.e. towards the ‘core’ of the half-space (Figure A.1d). These are dependent of radial dislocations with inside and outside path-cuts (b_r^i and b_r^o) and the corrective Kelvin and Boussinesq terms (b_r^k and b_r^b). The radial dislocations are also dependent of the angle argument α . For an ‘exterior’ cut, $\alpha = -\pi/2$ and for a ‘core’ cut, $\alpha = \pi/2$. From Paynter & Hills [5]:

$${}^c G_{rj}^j(\rho, \zeta, \delta) = \frac{(\kappa-1)}{(\kappa+1)} {}^i G_{rj}^j(\rho, \zeta, \delta, \pi/2) + \frac{2}{(\kappa+1)} {}^o G_{rj}^j(\rho, \zeta, \delta, \pi/2) - \frac{2\mu(3-\kappa)}{c(\kappa+1)} {}^k G_{rj}^j(\rho, \zeta, \delta, \pi/2) \quad j = r, z \quad (\text{A.9})$$

$${}^e G_{rj}^j(\rho, \zeta, \delta) = \frac{(\kappa-1)}{(\kappa+1)} {}^i G_{rj}^j(\rho, \zeta, \delta, -\pi/2) + \frac{2}{(\kappa+1)} {}^o G_{rj}^j(\rho, \zeta, \delta, -\pi/2) - \frac{2\mu(\kappa-1)}{c(\kappa+1)} \left[{}^k G_{rj}^j(\rho, \zeta, \delta, -\pi/2) - {}^b G_{rj}^j(\rho, \zeta, -\pi/2) \right] \quad j = r, z \quad (\text{A.10})$$

where ${}^c G_{rj}^j$, ${}^e G_{rj}^j$, ${}^i G_{rj}^j$, ${}^o G_{rj}^j$, ${}^k G_{rj}^j$ and ${}^b G_{rj}^j$ are, respectively, the influence functions for a radial dislocation with core, exterior, inner and outer cuts, and the Kelvin and Boussinesq corrective terms. These corrective terms are given as:

Kelvin term (b_r^k)

$${}^k G_{rr}^r(\rho, \zeta, \delta, \alpha) = \frac{c}{(\kappa+1)} \left[-\frac{(3-\kappa)}{2} J_{1,0;0}(\alpha) + \frac{(3\kappa-1)}{2} I_{1,0;0}(|\alpha|) + (\zeta - \delta) J_{1,0;1} + (\kappa\zeta - 3\delta) I_{1,0;1} - \frac{(\zeta - \delta)}{\rho} J_{1,1;0} - \frac{(\zeta - \delta)\kappa}{\rho} I_{1,1;0} + \frac{(2\zeta\delta)}{\rho} I_{1,1;1} + \frac{(1-\kappa^2)}{2\rho} I_{1,1;-1}(|\alpha|) - 2\zeta\delta I_{1,0;2} \right] \quad (\text{A.11})$$

$${}^k G_{rz}^r(\rho, \zeta, \delta, \alpha) = \frac{c}{(\kappa+1)} \left[-\frac{(\kappa-1)}{2} J_{1,1;0} + \frac{(\kappa-1)}{2} I_{1,1;0} - (\zeta - \delta) J_{1,1;1} + (\kappa\zeta - \delta) I_{1,1;1} - 2\zeta\delta I_{1,1;2} \right] \quad (\text{A.12})$$

Boussinesq term (b_r^b)

Kelvin term with depth set to zero ($\delta = 0$).

$${}^b G_{rr}^r(\rho, \zeta) = c \left[-J_{1,0;0}(\alpha) + \zeta J_{1,0;1} - \frac{\zeta}{\rho} J_{1,1;0} + \frac{(\kappa-1)}{2\rho} J_{1,1;-1}(\alpha) \right] \quad (\text{A.13})$$

$${}^b G_{rz}^r(\rho, \zeta) = c \left[-\zeta J_{1,1;1} \right] \quad (\text{A.14})$$

Appendix A.3. Lipschitz-Hankel integrals

In the influence functions, the terms $J_{n,p;q}$ and $I_{n,p;q}$ represent Lipschitz-Hankel integrals. The standard definition for these functions is as an integral of the product of Bessel functions ($J_i(\cdot)$), an exponential term and a power term. Using normalised coordinate variables, it is given as [8]

$$P_{\mu,\nu;\lambda}(\rho, \zeta) = \int_0^\infty J_\mu(t) J_\nu(\rho t) e^{-\zeta t} t^\lambda dt. \quad (\text{A.15})$$

In the kernels, the follow definition is applied:

$$J_{n,p;q} = P_{n,p;q}(\rho, \zeta - \delta) \quad (\text{A.16})$$

$$I_{n,p;q} = P_{n,p;q}(\rho, -\zeta - \delta) \quad (\text{A.17})$$

The Lipschitz-Hankel integrals needed in the kernels are given in [4, 5].

Appendix B. Gaussian Quadrature for SIEs with Cauchy Kernel

A general singular integral equation of the first kind with a simple Cauchy kernel can be written in standard form (with the crack lying along $[-1, 1]$) as

$$F(t) = \int_{-1}^1 \frac{w(s) \phi(s)}{t - s} ds, \quad |t| < 1 \quad (\text{B.1})$$

where $F(t)$ is the normalised traction component on the line of the crack, $B(s) = w(s) \phi(s)$ is the dislocation density, $\phi(s)$ is a smooth continuous function and $w(s)$ is a weight function of the form

$$w(s) = (1 + s)^\alpha (1 - s)^\beta. \quad (\text{B.2})$$

It can be proved that the exponents α and β must be either $-1/2$ or $+1/2$ [9], resulting in four classes of SIEs, depending on the behaviour of the $B(s)$ at the end points. Solutions were presented by Erdogan, Gupta and Cook [7]. In those cases, the weight function is associated with the Chebyshev polynomials of the first kind T_N . The solution for eq. (B.1) is given as

$$F(t_k) = \sum_{i=1}^N W_i \frac{\phi(s_i)}{t_k - s_i}, \quad k = 1, \dots, N + n \quad (\text{B.3})$$

where s_i are the integration points, t_k the collocation points and W_i the weights. These are all dependent on the behaviour of the dislocation density function at the end points. Since the function's behaviour is dictated by $w(s)$, there are also four sets of s_i , t_k and W_i . These are shown in Table B.1.

For the cracks solved in this paper, the behaviour of $w(s)$ is either bounded at both ends, which corresponds to case *C* in Table B.1, or bounded at $s = -1$ and singular at $s = 1$, which corresponds to case *D* in Table B.1.

Table B.1: Gauss-Chebyshev quadrature formulae for Cauchy kernels

Case	$w(s)$	s_i	t_k	W_i	n
<i>A</i>	$\frac{1}{\sqrt{1-s^2}}$	$\cos\left(\pi \frac{2i-1}{2N}\right)$	$\cos\left(\pi \frac{k}{N}\right)$	$\frac{1}{N}$	-1
<i>B</i>	$\sqrt{\frac{1-s}{1+s}}$	$\cos\left(\pi \frac{2i}{2N+1}\right)$	$\cos\left(\pi \frac{2k-1}{2N+1}\right)$	$\frac{2(1-s_i)}{2N+1}$	0
<i>C</i>	$\sqrt{\frac{1+s}{1-s}}$	$\cos\left(\pi \frac{2i-1}{2N+1}\right)$	$\cos\left(\pi \frac{2k}{2N+1}\right)$	$\frac{2(1+s_i)}{2N+1}$	0
<i>D</i>	$\sqrt{1+s^2}$	$\cos\left(\pi \frac{i}{N+1}\right)$	$\cos\left(\pi \frac{2k-1}{2(N+1)}\right)$	$\frac{1-s_i^2}{2N+1}$	+1

Ring Cracks at the Surface of a Half-Space

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Highlights

- A ring crack in a half-space is analysed
- Ring dislocations are used to determine SIFs
- A short crack presents modes *I* and *II* stress intensification
- A longer crack may present only mode *II* stress intensification
- This depends on the interfacial coefficient of friction