Geometry and Temperature

Enhancing and breaking conformality



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Alla mia famiglia.

Ragàs, a' glò caveda dabòun!

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Geometry and Temperature

Enhancing and breaking conformality

Enrico Marchetto

Conformal field theories have been playing a central role in the latest decades of research in high-energy physics and condensed matter: much of their success is due to the conformal symmetry group, a large set of symmetries which can be exploited to find exact solutions and analytic results. In this Thesis, we explore two different topics connected by a common theme: the modification of conformal symmetry.

In the first part of the Thesis, we study the dynamics of (1,0) superstrings, well known for being the model of heterotic string theory. This dynamics, described by a two-dimensional superconformal field theory on the worldsheet, is ruled by extensions of the Virasoro algebra, called super $\mathcal{W}$-algebras, whose features depend on the G-structures of the internal manifold employed in the string compactification. We discuss the super $\mathcal{W}$-algebras of (1,0) superstrings propagating on manifolds with Spin(7), G_2 and SU(3)-structures. We allow for a non-trivial torsion H and we study how the OPE coefficients of the special holonomy algebras, originally due to Odake and Satashvili–Vafa, are deformed by the scalar torsion classes. This is achieved by comparing an exact computation of the OPEs with a classical one, where a fundamental role is played by the $\mathcal{W}$-symmetries of a (1,0) non-linear σ-model.

In the second part of the Thesis, our focus is not on extending the conformal symmetry group, but on breaking it and exploring the consequences. The perfect motivation is provided by the study of (former) CFTs where a finite temperature T has been introduced. Mathematically, this is achieved in the Euclidean space by replacing the infinite time direction with a circle S^1_β of length $\beta = 1/T$. Since the scaling invariance is now broken, a plethora of one-point functions is allowed to be non-zero, with interesting consequences on the physics. We study how the conformal symmetry group breaks by deriving a set of broken Ward identities. We discuss some non-perturbative implications of this result and novel kinematic constraints on the correlation functions. Finally, we study an interesting setup to bootstrap the thermal one-point functions: the physical principle lies in the periodicity of the observables over the S^1_β circle, while the computations of the errors are made precise thanks to a Tauberian theorem that we prove.

Statement of originality

Chapters discussing the geometry of string compactifications and its relation with the super $\mathcal{W}$-algebras describing the dynamics on the string worldsheet are based on the following article written in collaboration:

◊ Chapter 2: X. de la Ossa, M. Galdeano, E. Marchetto (2024) *Superstrings with torsion and $\mathcal{W}$-algebras*, *in preparation*

Chapters discussing the conformal field theories placed on the thermal geometry $\mathcal{M}_\beta = S^1_\beta \times \mathbb{R}^{d-1}$, their symmetries and their dynamical features are based on the following articles written in collaboration:

◊ Chapter 4: E. Marchetto, A. Miscioscia, E. Pomoni (2023) *Broken (super) conformal Ward identities at finite temperature*, Journal of High Energy Physics **12** (2023) 186 [2306.12417]

◊ Chapter 5: E. Marchetto, A. Miscioscia, E. Pomoni (2023) *Sum rules & Tauberian theorems at finite temperature*, Journal of High Energy Physics **09** (2024) 044 [2312.13030]

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Introduction

The story of high-energy physics in the last decades has been, among many crucial things, a story of symmetries, and we could not tell it without talking about conformal symmetry. This is also going to be the starting point of this Thesis.

Conformal symmetry. The conformal group originally found one of its most important applications in the description of condensed matter systems, for example, models of ferromagnets like the Ising model in two or three dimensions. It is well known that such models exhibit a fascinating phase transition, which happens at a critical value of their coupling. The effective models describing their behaviour at this special point in the moduli space enjoy a high number of symmetries, which become infinite when we consider two-dimensional theories [1]. These symmetries are encoded in a group structure, known as the *conformal group*. In $d \geq 3$ Euclidean dimensions, the group looks like an extension of the usual Poincaré set of symmetries, adding to the already familiar translations and rotations an invariance under scalings and an invariance under inversion of the coordinates, a non-linear symmetry "masked" as a linear one under the name of special conformal transformations, obtained by composing an inversion, translations, and a second inversion in this order. The scaling invariance is probably the most stunning feature, manifesting itself in condensed matter systems as the correlation length diverging to infinity. One of the most interesting aspects of conformal invariance is its power in constraining correlation functions: local operators are forbidden to gain a vacuum expectation value (it would introduce a scale in the theory and break the scaling invariance), and the two-point functions are completely fixed, while three-point functions are fixed only up to an overall coefficient. Although higher-point functions do not enjoy such strict constraints, this is compensated by another outstanding feature of conformal field theories: the radius of convergence of the operator product expansion (OPE) is infinite. In field theory, given two local fields Φ_i and

Φ_j close enough, we can define their product as an infinite sum over all the fields appearing in the theory, each multiplied by an *OPE coefficient*: schematically,

$$\Phi_i(x_i)\Phi_j(x_j) \sim a_{ij}^{k_1}(x_i,x_j)\Phi_{k_1}(x_j) + a_{ij}^{k_2}(x_i,x_j)\Phi_{k_2}(x_j) + \dots \ .$$

The attractiveness of this tool lies in the fact that a given correlation function $\langle \Phi_1(x_1)\dots\Phi_n(x_n)\rangle$ could be broken down into lower-point correlation functions by systematically performing the OPE between the fields inside the bra-ket. However, in a generic field theory, we immediately encounter two obstacles: we do not know, in general, the kinematic structure of the OPE coefficients $a_{ij}^{k_\ell}(x_i,x_j)$, and we do not know the radius of convergence of the OPE. Working with a CFT solves both problems, since conformal invariance fixes $a_{ij}^{k_\ell}(x_i,x_j) = c_{ij}^{k_\ell}|x_i-x_j|^{\Delta_{k_\ell}-\Delta_i-\Delta_j}$, with $c_{ij}^{k_\ell}$ being a number, and the radius of convergence can be shown to be infinite, thanks to the scaling invariance. The strategy to completely solve a CFT becomes the following: first of all, we fix a normalization for the two-point functions of the theory, and then we iteratively perform the OPE inside the correlation functions to fix them up to the OPE coefficients c_{ij}^{k}. Of course, the numbers $c_{ij}^{k_\ell}$ are not arbitrary, but they must comply with precise constraints: an iconic example is given by the crossing symmetry of the four-point function of 4 scalar fields φ_i, which reads schematically as follows

$$\left\langle \prod_{i=1}^{4} \varphi_i(x_i) \right\rangle = \sum_{\Phi} c_{12}^{\Phi} c_{34}^{\Phi}\, \mathcal{G}_{\Phi}^{(s)}(x_1,x_2,x_3,x_4) = \sum_{\Phi'} c_{13}^{\Phi'} c_{24}^{\Phi'}\, \mathcal{G}_{\Phi'}^{(t)}(x_1,x_2,x_3,x_4) \ .$$

In the first sum, denominated s-channel, the OPEs $\varphi_1\times\varphi_2$ and $\varphi_3\times\varphi_4$ were performed, and then the two-point function between two copies of the field Φ (one for each OPE) was evaluated; $\mathcal{G}_{\Phi}^{(s)}$ is called a conformal block and it encodes all the kinematic information. The same procedure can be repeated by taking different OPEs: in the second sum in the equation above, denominated t-channel, the OPEs $\varphi_1\times\varphi_3$ and $\varphi_2\times\varphi_4$ were taken. The crossing symmetry of the four-point function is the requirement that the s-channel and the t-channel are identical. Thus, we say that the crossing symmetry establishes a conformal bootstrap[1] problem, and the problem is formulated as a set of sum rules, where the sum runs over the fields Φ appearing in the $\varphi\times\varphi$ OPE. The conformal bootstrap program was initiated by the seminal papers of Polyakov, Ferrara, Grillo, and Gatto [2, 3] and by

[1]The name derives from the idiom "to pull yourself up by your own bootstraps", which is said whenever you improve your situation without any external help. In physics, it is an attribute for problems which do not require any input apart from internal consistency.

Belavin, Polyakov, and Zamolodchikov in two dimensions [1]; more recent important developments in higher dimensions came from the works [4, 5], and a comprehensive review can be found in [6].

CFTs, physics and geometry. As already anticipated, one of the most prominent physics applications of conformal field theories is the study of statistical and condensed matter systems, where a large amount of degrees of freedom are described through a mean field approach. However, another important motivation to explore conformal invariance comes from *string theory*. This theory is a model of fundamental strings: not only they can translate in the spacetime, but since they are extended objects they have additional degrees of freedom, from which the particle spectrum we are familiar with (scalars, gauge bosons, tensor fields etc.) emerges. Strings propagating in the spacetime span two-dimensional surfaces called *worldsheets*. Similarly to the worldlines spanned by relativistic point particles, the worldsheet should be regarded as the truly physical object; the action describing the string dynamics, the Polyakov action, is defined as a surface integral over the worldsheet. As we review in Chapter 1, at the end of a gauge fixing procedure the Polyakov action describes a two-dimensional conformal field theory, as a consequence of diffeomorphisms and Weyl invariance. In an vast range of string theory applications, conformal invariance goes together with *supersymmetry*, which establishes a symmetry between bosonic and fermionic degrees of freedom on the worldsheet. The combination of conformal symmetry and supersymmetry goes under the name of *superconformal symmetry*, and supersymmetric string theories are called superstring theories. Superstrings are very popular because they get rid of the existence of a tachyon in the particle spectrum thanks to the Gliozzi-Scherk-Olive projection. The massless particles in the spectrum (usually a graviton G_{IJ}, an antysimmetric 2-rank tensor B_{IJ}, a dilaton ϕ, a set of gauge fields $A_I{}^{\alpha}{}_{\beta}$ and additional fluxes) are instead described by an effective field theory in ten dimensions, called *supergravity*, which can be further reduced to a theory in $d < 10$ dimensions by performing a *compactification*. An important example is the type IIB supergravity theory compactified on $\mathrm{AdS}_5 \times S^5$, since it represents the first example of the AdS/CFT correspondence [7–9]. In the context of string theory, CFTs not only appear as two-dimensional worldsheet theories, but also as theories living at the boundary of an asymptotically AdS spacetime, providing a full description of quantum gravity according to the holographic principle. The CFT living at the boundary of type IIB supergravity on $\mathrm{AdS}_5 \times S^5$

is the $\mathcal{N} = 4$ super Yang-Mills theory, which represents a perfect example of superconformal field theory in $d > 2$. Superconformal field theories are well studied independently from their connection to quantum gravity, since they represent laboratories to study the fundamental properties of quantum field theories. In fact, the combination of supersymmetry and conformal symmetry strongly enhances the possibility to produce exact statements, mostly due to the power of representation theory, vastly explored in [10–12]. Non-renormalization theorems are an important example of exact result: they employ superconformal invariance to prove that some three-point functions coefficients do not depend on the theory's couplings [13–16]. Of course, superconformal field theories allow for the formulation of powerful bootstrap problems, which have been at the center of a prolific scientific production in the last decade [17–21].

To summarize, CFTs have proven to be an essential tool to understand statistical systems, to study string theory and quantum gravity and to probe the fundamental features of quantum field theories. We can make a couple of important comments:

- ⋆ in quantum field theory applications, CFTs are always defined on a flat space $\mathbb{R}^d$ (choosing the Euclidean signature);
- ⋆ when we compactify the ten-dimensional effective string theory we choose a specific internal manifold with a number of geometrical features. Since the final effective action is sensitive to them, the worldsheet theory should be as well.

From these considerations we learn that there is a meaningful relationship between CFTs and geometry. The main goal of this Thesis is to explore this relationship proceeding in a double direction:

- ⋆ we would like to understand the consequences of replacing the flat space $\mathbb{R}^d$ with a non-trivial manifold $\mathcal{M}$. In particular, we would like to study the breaking of conformal invariance, and how many useful CFT structures survive the replacement;
- ⋆ we would like to understand the effect on the worldsheet CFT of choosing a compactification with specific geometric characteristics.

Enhancing conformal invariance in string theory. In Part I of this Thesis, we focus on the latter direction of those listed above. We need to choose a string theory setup to conduct our exploration: we opt for the (1,0) superstring [22–25] for both physical and practical reasons. The physical motivation is given by heterotic string theory, a superstring theory where $\mathcal{N} = 1$ supersymmetry is introduced only for left-moving Majorana–Weyl fermions on the worldsheet; the right-moving sector hosts a bosonic string theory and a set of 32 right-moving Majorana–Weyl fermions, which are sections of a gauge bundle with the string background as a base; the gauge bundle sector can couple to a gauge boson $A_I{}^\alpha{}_\beta$. The practical reason lies in the simplicity of the (1,0) superspace formalism, which leads to leaner computations compared to the (1,1) superspace. The worldsheet action is the (1,0) *non-linear σ-model*, studied in detail in [25], and is invariant under infinitesimal superconformal transformations of the worldsheet superfields (we refer to Section 1.2.1 for the details):

$$\delta_\epsilon^{\mathcal{T}} X^i = -\epsilon\, \partial X^i - \frac{1}{2} D\epsilon\, DX^i \ .$$

There exists a known way to enhance conformal invariance in this setup, i.e., it is possible to introduce additional classical symmetries enjoyed by the worldsheet action. A possible class of symmetries is called *$\mathcal{W}$-symmetries*, and their infinitesimal transformations read

$$\delta_\epsilon^{\Phi} X^i = \frac{\epsilon}{(p-1)!} \Phi^i{}_{j_2 \dots j_p}(X) DX^{j_2} \cdots DX^{j_p} \ ,$$

where Φ is a well-defined, nowhere-vanishing, and covariantly constant p-form with support on the string background. These symmetries have been extensively studied in [25–30], sometimes in the context of worldsheet supersymmetry enhancement [24], and have seen recent interest in the works [31,32]. The existence of the form Φ amounts to a reduction of the frame bundle, with structure group $\mathrm{GL}(d, \mathbb{R})$, to a principal sub-bundle with structure group $\mathrm{G} \subset \mathrm{GL}(d, \mathbb{R})$: this reduction defines a G-*structure* on the string background, and Φ is a *characteristic tensor* of the G-structure. The $\mathcal{W}$-symmetries encode this geometrical information at the level of the field theory. Moreover, they carry information about the Neveu–Schwarz 3-flux $H = \mathrm{d}B$: the connection compatible with the G-structure has symbols

$$\Gamma^i_{jk} = \mathring{\Gamma}^i_{jk} + \frac{1}{2} H^i{}_{jk} \ ,$$

where $\mathring{\Gamma}$ are the Levi-Civita symbols. Hence, in this Thesis we will refer to the 3-flux H as *torsion*. It is important that the G-structure admits a totally antisymmetric torsion to accommodate the symbols defined above (we refer to Chapter 2 for more details).

Although $\mathcal{W}$-symmetries are a precious tool for investigating geometry at the level of the σ-model, they can only be used to make classical or semiclassical statements. This obstacle is usually overcome by making very strict additional assumptions on the string background, choosing it to be a sphere, a Lie group, or a coset [33, 34]: the end result is a σ-model where computations can be performed without relying on perturbation theory, producing exact results. In this Thesis, to obtain this goal we employ the idea of *$\mathcal{W}$-algebra*, introduced by Zamolodchikov in the seminal paper [35] and then largely explored to describe condensed matter systems and string theories. $\mathcal{W}$-algebras can be seen as extensions of the Virasoro algebra, where new primary chiral currents are introduced. The construction of $\mathcal{W}$-algebras with one and two additional generators has been explored in the past [36–39]: the procedure returns a fully consistent set of OPEs corresponding to a symmetry enhancement of the Virasoro algebra. The OPE coefficients are functions of a set of free parameters *a priori* detached from any physical interpretation. In string theory, $\mathcal{W}$-algebras have been employed to describe the worldsheet theory associated with a compactification on a special holonomy manifold, in the torsionless limit [40, 41].

The goal of Part I of this Thesis will be to combine the information coming from $\mathcal{W}$-symmetries, classical but sensitive to the features of the compactification, and the information coming from the $\mathcal{W}$-algebra construction, which provides an exact quantum framework. The procedure can be used to perform exact, quantum computations in a general setup, encompassing different G-structures and the possibility of introducing a torsion. This can be seen as a complementary approach with respect to WZW models or similar tools: unlike those, it is possible to work without strict constraints on the string background. The price to pay is the approximation of the OPE coefficients in the final $\mathcal{W}$-algebras, which can be interpreted as correct only in the *large radius limit*, when the string length ℓ_s can be neglected: however, this can constitute a good starting point for a semiclassical approach, aiming at introducing order $O(\ell_s)$ corrections at the level of the worldsheet superconformal field

theory.

Breaking conformal invariance at finite temperature. In Part II of this Thesis, we focus instead on the first direction: replacing the flat ambient manifold $\mathbb{R}^d$ of a CFT with a non-trivial manifold $\mathcal{M}$. There is a perfect physical motivation supporting this direction of research: the study of *finite temperature* theories. According to the laws of statistical physics, the introduction of a finite temperature T in a quantum field theory requires the adoption of a density matrix $\rho = e^{-\beta H}$, where H is the Hamiltonian of the system. The density matrix ρ can be inserted into the correlation functions, modifying the observables and introducing a temperature dependence:

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_\beta = \mathrm{Tr}\left[e^{-\beta H} \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \right] \ .$$

It can be shown [42–44] that for systems in thermal equilibrium, there exists an alternative method to introduce a finite temperature T: we can replace the flat ambient manifold $\mathbb{R}^d$ with a circle compactification of the time direction, obtaining the ambient manifold $\mathcal{M}_\beta = \mathbb{R}^{d-1} \times S^1_\beta$, where β is the length of the circle. Of course, this replacement must be accompanied by a correct field redefinition, which makes bosonic fields periodic over S^1_β and fermionic fields antiperiodic. The result is a theory with the same operator content as the original theory (with the aforementioned *caveat*) and the same OPEs, but with completely different symmetries and observables compared to the original theory. We refer to the theory living on flat space as the "zero temperature" theory. When we consider a zero-temperature CFT, replacing the flat space manifold with the thermal one breaks the conformal symmetry group, introduces new observables, and modifies the already existing ones. It should be highlighted that in this Thesis, the temperature will not be considered as an *external* parameter of the theory to be tuned (usually to reach a certain critical point) but as a fixed geometrical feature. Moreover, it should be noted that models at the critical point (for example, the critical two-dimensional Ising model) should be regarded as zero-temperature CFTs: in fact, they enjoy the full conformal symmetry group. Finite-temperature CFTs should be considered as non-conformal quantum field theories, and they can be explored using an effective field theory approach [45–50].

Is it possible to say anything about the observables and the physics of the model once the conformal group is broken? Can any of the CFT machinery still be employed? In this Thesis, the first step is

to gain a quantitative understanding of how the conformal group breaks. In quantum field theory, (continuous) symmetries are associated with differential equations to which correlation functions must obey, known as *Ward identities*. When a symmetry is broken, we do not expect the original Ward identity to hold, but we expect that it will be corrected by a *breaking term*. Hence, we devote some space to the computation of the broken Ward identities of the conformal symmetry group, and in particular to the exact derivation of the breaking terms. Although they are more difficult to apply and solve, these novel identities still constrain the correlation functions. Another tool that is still at our disposal is the OPE between local operators, which reads exactly as in the zero-temperature CFT: the difference lies in the radius of convergence, which is not infinite anymore but proportional to β. Provided that the operators are close enough, we can still iterate the OPE inside the correlation functions to simplify computations or get a final result. Finally, at zero temperature, conformal symmetry completely fixed the one-point functions to be equal to zero: since the symmetry is now broken, many local operators develop a thermal expectation value. The symmetries are still able to fix the structure of the thermal one-point function up to an overall coefficient $b_{\mathcal{O}}$:

$$\langle \mathcal{O}^{\mu_1 \cdots \mu_J}(x) \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} \left(e^{\mu_1} \cdots e^{\mu_J} - \text{traces} \right) \ .$$

It is natural to wonder if it is possible to set up a bootstrap problem to extract the $b_{\mathcal{O}}$ coefficients: this project has recently been pursued with promising results [45,46,51–53]. In this Thesis, we study a bootstrap problem based on a set of sum rules: while at zero temperature the guiding principle was the crossing symmetry of the four-point function, at finite temperature we consider the *periodicity* of the thermal two-point function around the circle S^1_β, encoded in the *Kubo-Martin-Schwinger (KMS) condition* [54, 55]. The analogy between crossing symmetry and the KMS condition was first drawn in [56]. A novel contribution to the study of this bootstrap problem is presented in this Thesis as a Tauberian theorem for thermal CFTs. Tauberian theorems are important results borrowed from mathematics that have been used proficiently to study conformal bootstrap problems at zero temperature [57–61]. Being able to apply them at finite temperature opens the stage to very interesting results, which will be explored in detail in Chapter 5.

This Thesis is organized as follows:

- ◇ in Chapter 1 we review the string theoretical motivations for studying conformal invariance in two dimensions and $\mathcal{W}$-algebras. Then, we review the construction of generic $\mathcal{W}$-algebras and (1,0) super $\mathcal{W}$-algebras, introducing the superspace formalism. We review the (1,0) non-linear σ-model, the superconformal transformations, and the $\mathcal{W}$-symmetry transformations, together with the definition of G-structure. We present the commutators between superconformal and $\mathcal{W}$-symmetry transformations and discuss a procedure to translate the information of the commutators into classical OPEs between Noether currents;

- ◇ in Chapter 2 we present a strategy to interpolate between the exact OPEs of a $\mathcal{W}$-algebra and the classical OPEs obtained from $\mathcal{W}$-symmetries, deriving a geometrical and physical interpretation of the OPE coefficients in terms of the torsion and the G-structure. We apply the procedure to an $\mathrm{O}(d-n)$-structure as a warm-up, then present new results for the $\mathcal{W}$-algebras associated with strings propagating on manifolds endowed with Spin(7), G_2, and SU(3)-structures;

- ◇ in Chapter 3 we draw conclusions regarding Part I of this Thesis;

- ◇ in Chapter 4 we review how to derive Ward identities through the path integral formalism and present the derivation of new broken Ward identities associated with the symmetries of the conformal group, broken by finite temperature. We derive new non-perturbative results regarding the energy spectrum and the thermal mass of thermal CFTs. Moreover, we apply the broken Ward identities as constraints for thermal correlators, enriching an already existing bootstrap problem for scalar thermal two-point functions and solving it in the $4d$ free scalar theory, recovering the known result in an alternative way;

- ◇ in Chapter 5 we comment on the necessity of formulating a bootstrap problem for thermal one-point functions and formulate one in the form of a set of sum rules. We test the sum rules on control theories and comment on the predictivity of the bootstrap problem. We prove a Tauberian theorem at finite temperature and use it to derive an asymptotic behavior for thermal one-point function coefficients in the large Δ limit. Then, we employ this result to estimate the errors made by truncating the sum rules;

- ◇ in Chapter 6 we draw conclusions regarding Part II of this Thesis;
- ◇ in Appendix A we report some noteworthy technical computations regarding the commutator of two $\mathcal{W}$-transformations; moreover, we report the special holonomy $s\mathcal{W}$-algebras associated with the holonomy groups Spin(7) and SU(3);
- ◇ in Appendix B we present some technical details in the derivation of the sum rules and in the proof of the Tauberian theorem.

Part I

Super $\mathcal{W}$-algebras and torsion

1
$\mathcal{W}$-algebras and $\mathcal{W}$-symmetries

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1.1 $\mathcal{W}$-algebras and symmetry enhancements

1.1.1 From Virasoro to $\mathcal{W}$-algebras

The study of $\mathcal{W}$-algebras, which can roughly be defined as higher-spin extensions of the Virasoro algebra [62], is backed by multiple interesting motivations, the most relevant being understanding critical phenomena in statistical physics and the dynamics of string theory worldsheets. The latter point of view will be the one discussed in the following.

Conformal symmetry and string theory. Free strings propagate in a flat ambient manifold $\mathbb{R}^{1,D-1}$ and they span two-dimensional surfaces called worldsheets. Their dynamics is classically described by the Nambu-Goto action

$$S[x] = \int_\Sigma \frac{\mathrm{d}\Sigma}{4\pi\alpha'} \sqrt{h(x)} \,, \tag{1.1}$$

where Σ represents the worldsheet, $\mathrm{d}\Sigma$ the infinitesimal surface element, $h(x)$ the determinant of the induced metric on the worldsheet and x^I are[1] maps from the worldsheet to the flat ambient manifold $x^I : \Sigma \to \mathbb{R}^{1,D-1}$. It is convenient to trade this action for the Polyakov action, where the metric h_{ab} and the fields x^I are independent degrees of freedom

$$S[x] = \int_\Sigma \frac{\mathrm{d}\Sigma}{4\pi\alpha'} \sqrt{h}\, h_{ab}\, \eta_{IJ}\, \partial^a x^I \partial^b x^J \ . \tag{1.2}$$

This action enjoys a rich amount of symmetry, being classically invariant under worldsheet general coordinate transformations and Weyl rescalings of the worldsheet metric. This large freedom can be used to fix a certain number of components of the metric h, and it turns out that there is enough gauge freedom to fix $h_{ab} = \delta_{ab}$, the flat metric: this choice goes under the name of *flat gauge*. This is the "miracle" of working with one-dimensional fundamental objects, or better, with their two-dimensional world-volumes. We made the choice to adopt the Euclidean signature on the worldsheet Σ. The procedure to fix the flat metric on the worldsheet does not exhaust the full gauge invariance of the theory, and we are left with a residual gauge symmetry being encoded in the *two-dimensional conformal transformations*. If we adopt complex coordinates $(z, \bar{z})$ on the worldsheet such that the flat metric is given by $\mathrm{d}s^2 = 2\mathrm{d}z\mathrm{d}\bar{z}$, the conformal transformations become very simple: they are holomorphic, or anti-holomorphic, change of coordinates

$$z \to f(z) \ , \quad \bar{z} \to \bar{f}(\bar{z}) \ , \tag{1.3}$$

where f is an analytic function and $\bar{f}$ anti-analytic. In these coordinates, the stress-energy tensor $T^{ab}(z, \bar{z})$ splits into a holomorphic component $T(z) = T^{zz}(z)$ and an anti-holomorphic component $\bar{T}(\bar{z}) = T^{\bar{z}\bar{z}}(\bar{z})$. The mixed components vanish due to the *tracelessness condition* $T^a{}_a = 0$, which holds in every classical CFT. Focusing on the holomorphic component, a consequence of the tracelessness is that $T(z)$ must satisfy the constraint

$$\bar{\partial}\, T(z) = 0 \ , \tag{1.4}$$

which makes $T(z)$ a *chiral* current. Since the conformal symmetry is a residual gauge symmetry, the consistency of the quantum theory defined by the action (1.2) would be compromised if it developed

[1] In Part I of this Thesis, we will adopt spacetimes indices I, J, K running from 0 to $D-1$ and worldsheet indices a, b, c running over $1, 2$. Once we will introduce an internal manifold $\mathcal{Y}$ in the context of string compactifications, the internal indices i, j, k will run from 1 to d, where d is the dimension of $\mathcal{Y}$.

an anomaly. The conformal anomaly can be studied by examining the quantum OPE

$$T(z_1)T(z_2) \sim \frac{c/2}{z_{12}^4} + \frac{2}{z_{12}^2}T(z_2) + \frac{1}{z_{12}}\partial T(z_2) + \dots \ , \tag{1.5}$$

where the dots represent regular terms. With a slight abuse of terminology, the OPE (1.5) is called Virasoro algebra. The parameter c, called *central charge*, is a real number encoding the quantum breaking of conformal invariance, since it can also be defined as

$$\langle T^a{}_a \rangle = -\frac{c}{12}R \ , \tag{1.6}$$

where R is the Ricci scalar on the worldsheet. A non-zero central charge forbids the quantum stress-energy tensor to be traceless, and the breaking of the conformal invariance follows. In string theory, this conundrum is solved thanks to the emergence of a ghost action from the gauge-fixing procedure: if the metric h_{ab} is flat, this action reduces to a bc-system with central charge $c_{\text{ghost}} = -26$. Since the stress-energy tensor of the whole system (including Polyakov fields and ghosts) can be written as $T_{\text{tot.}} = T + T_{\text{ghost}}$, conformal invariance is preserved by requiring

$$c_{\text{tot.}} = c + c_{\text{ghost}} = 0 \ . \tag{1.7}$$

Non-trivial string backgrounds. In the flat gauge $h_{ab} = \delta_{ab}$, the action (1.2) reduces to a model of D free scalars x^I, where the index I runs from 0 to $D-1$. The maps x^I are interpreted as the coordinates of the target manifold $\mathcal{M}$ in which the string is propagating, so D is the dimension of the manifold $\mathcal{M}$. A two-dimensional free scalar is associated with a central charge $c = 1$, so the whole Polyakov action carries a central charge $c = D$. The solution of the constraint (1.7) is $D = 26$: to ensure the consistency of the theory, the string worldsheet must be embedded in a 26-dimensional manifold.

It is well known that the action (1.2) can be quantized, returning all the possible string states, each associated with a mass. For our purposes, the most important states are the *massless* states, associated respectively with a graviton $G_{IJ}(x)$, a Kalb-Ramond field $B_{IJ}(x)$ and a dilaton $\phi(x)$. We consider a string which does not freely propagate in the string background, but in a *condensate* generated by the massless string states. To describe the dynamics, we need to couple the Polyakov

fields x^I to the graviton and the Kalb-Ramond field in the flat gauge

$$S[x] = \int_\Sigma \frac{\mathrm{d}\Sigma}{2\pi\alpha'} \left(G_{IJ}(x) + B_{IJ}(x)\right) \partial x^I \bar{\partial} x^J \ , \tag{1.8}$$

where the term including the dilaton was omitted since it contains the two-dimensional Ricci scalar R, which vanishes in flat coordinates. The action (1.8) goes under the name of *non-linear σ-model* and it describes a set of D two-dimensional scalars x^I interacting with themselves. The vertices ruling these interactions can be unveiled only by perturbatively expanding the action (1.8) around a classical background [63–65]; the perturbative parameter is given by

$$\ell_s = \sqrt{2\pi\alpha'} \ , \tag{1.9}$$

which is the *string length scale*. The central charge can be perturbatively expanded as well[2]

$$c = d + \ell_s \, c_1[G, B, \phi] + \frac{1}{2} \, \ell_s^2 \, c_2[G, B, \phi] + \dots \tag{1.10}$$

where the first term maintains the role of fixing the dimension of the string background, while $c_1, c_2, \dots$ are functionals of the fields G_{IJ}, B_{IJ} and ϕ. To ensure conformal symmetry, it is necessary to set each functional equal to zero

$$c_r[G, B, \phi] = 0 \ , \quad r = 1, 2, \dots \tag{1.11}$$

These constraints can be interpreted as *equations of motion* for the fields G_{IJ}, B_{IJ} and ϕ, leading to a perturbative formulation of string theory in the string background $\mathcal{M}$. This formulation is known as supergravity.

Solving the constraints (1.11) is rather challenging even at first order in ℓ_s. Moreover, the action (1.8) can be improved by coupling it with two-dimensional fermions and to additional massless condensates (Ramond-Ramond fluxes, gauge connections, etc.), increasing the level of difficulty. In the literature, this problem is usually tackled by making two main assumptions:

- ◇ the effective theory defined on the string background should be supersymmetric;
- ◇ the background manifold $\mathcal{M}$ should be a direct product of two submanifolds $\mathcal{X}$ and $\mathcal{Y}$

$$\mathcal{M} = \mathcal{X} \times \mathcal{Y} \ , \tag{1.12}$$

[2]Notice that the dependence on the dilaton field ϕ is recovered when this computation is performed [64].

where usually $\mathcal{X}$ is a maximally symmetric space endowed with the Lorentz symmetry group (Minkowski, de Sitter or anti-de Sitter space), while $\mathcal{Y}$ is an *internal* manifold. Although it is not strictly required, $\mathcal{Y}$ is usually considered to be compact.

For the purposes of this Thesis, we focus on the assumption (1.12). The direct product splits the Polyakov fields x^I in a set of maps $x^\mu : \Sigma \to \mathcal{X}$ and in a set of maps $x^i : \Sigma \to \mathcal{Y}$. We also assume that the mixed components of the background fields $G_{\mu i}$ and $B_{\mu i}$ vanish, so that the non-linear σ-model (1.8) can be split in two terms, one describing the propagation on $\mathcal{X}$, and the other describing the propagation on $\mathcal{Y}$. Each action is associated with a central charge, and the full Polyakov central charge will be given by $c = c_{\mathcal{X}} + c_{\mathcal{Y}}$. In the following, we will focus on the action describing the string dynamics on the internal manifold $\mathcal{Y}$.

Construction of $\mathcal{W}$-algebras. We can summarize the most important points of the previous Paragraph:

- ◇ the worldsheet dynamics of a free, bosonic string is described by a two-dimensional conformal field theory, encoded in the Virasoro algebra (1.5). Conformal symmetry is a residual gauge symmetry;
- ◇ despite conformal symmetry being anomalous, the consistency of the theory is ensured by the constraint (1.7);
- ◇ on non-trivial string background, the Polyakov central charge is a functional of the background fields, and it can be expanded in powers of the string length scale ℓ_s. If the string background is split in a direct product (1.12), the Polyakov action splits into two actions;
- ◇ the Polyakov action associated with $\mathcal{Y}$ is associated with a CFT with central charge $c_{\mathcal{Y}}$.

We should expect the two-dimensional CFT associated with the manifold $\mathcal{Y}$ to be sensitive to some of the manifold's geometrical features. It has been known since the early works of Odake, Satashvili and Vafa [40,41] that strings propagating on special holonomy manifolds are described by extensions of the Virasoro algebra [39]. In the following, we study the technology that allows us to extend the two-dimensional Virasoro algebra. There are multiple paths that could be followed. Mathematically

rigorous approaches have been proposed in [66, 67] and reviewed in [68]. These approaches focus on the algebra of Laurent modes and revolve around the notion of Vertex Operator Algebras (VOA). Another mathematically rigorous direction is studied in [69] and developed in [70], where the discussion revolves around the concept of λ-*brackets*. Notice that these references are not exhaustive. The approach of this Thesis will instead be more physically oriented and inspired by [36–38, 71, 72].

Shortly after the seminal work by Belavin, Polyakov and Zamolodchikov on two-dimensional CFTs [1], the project of extending conformal symmetry was started Zamolodchikov [35], who realized that the algebras extending Virasoro are not, in general, Lie algebras, since the OPEs can be non-linear in the generators. Such algebras are known in the literature as $\mathcal{W}$-algebras. To introduce a definition of $\mathcal{W}$-algebra, we need to consider first the definition of Operator Product Algebra. We first provide an abstract definition, in which we will infuse physical meaning right after.

Definition 1 (Operator Product Algebra)**.** *An Operator Product Algebra (OPA) is a $\mathbb{Z}_2$-graded vector space $\mathcal{F}$ with elements $\mathbb{1}, A, B, C, \dots$, an even linear map ∂, and a bilinear map*

$$[\cdot\ \cdot]_p : \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{F}\ , \quad p \in \mathbb{Z} \tag{1.13}$$

such that there exists $\widehat{p} \in \mathbb{Z}$ for which

$$[\cdot\ \cdot]_{p>\widehat{p}} : A \otimes B \longmapsto 0\ , \quad A, B \in \mathcal{F}\ . \tag{1.14}$$

For any triplet $A, B, C \in \mathcal{F}$, the bilinear map satisfies three properties:

- ⋆ *unity:*

$$[\mathbb{1}A]_0 = A\ , \quad [\mathbb{1}A]_{r\neq 0} = 0\ ; \tag{1.15}$$

- ⋆ *commutation:*

$$[BA]_r = (-1)^{|A||B|} \sum_{s\geq r} \frac{(-1)^s}{(s-r)!} \partial^{s-r} [AB]_s\ , \quad \forall r \in \mathbb{Z}\ ; \tag{1.16}$$

- ⋆ *associativity:*

$$[A[BC]_r]_s = (-1)^{|A||B|}[B[AC]_s]_r + \sum_{t>0} \binom{s-1}{t-1} [[AB]_t C]_{r+s-t}\ , \quad \forall r, s \in \mathbb{Z}\ . \tag{1.17}$$

From a physical point of view, we identify the vector space $\mathcal{F}$ with the operator content of the theory, the element $\mathbb{1}$ with the identity operator and $A, B, C \dots$ with the local operators in the field content. The grading $|A|$ returns 0 if the operator is bosonic, while it returns 1 if the operator is fermionic; the linear map ∂ is easily identified as the holomorphic derivative on the two-dimensional space. Finally, given two operators A and B, $[AB]_p$ returns the p-th pole in the OPE between A and B when $p > 0$; the properties (1.15), (1.16) and (1.17) encode the OPE consistency conditions. An OPA is *generated* by a set of operators $\langle \mathbb{1}, A_1, A_2, A_3, \dots \rangle$ if any operator in $\mathcal{F}$ can be built out of the generators $\mathbb{1}, A_1, A_2, A_3, \dots$ using addition, scalar multiplication, action of ∂ and *normal ordering*. The last operation is defined as follows: given $A, B \in \mathcal{F}$, their normal ordering $N(AB)$ reads

$$N(AB)(z) = \oint_z \frac{\mathrm{d}w}{2\pi i} \frac{A(w)B(z)}{w - z} \ . \tag{1.18}$$

Notice that $N(AB) = [AB]_0$. In this Thesis, we will focus on *finitely generated* OPAs, meaning that the set of generators has a finite number of elements.[3]

If the theory has a Virasoro element T, some operators enjoy interesting properties:

- ⋄ an operator A is a *scaling* operator if

$$[TA]_2 = h_A \, A \ , \quad [TA]_1 = \partial A \ . \tag{1.19}$$

  h_A is a real number and is called *weight* of the scaling operator A ;

- ⋄ a scaling operator A is a *quasi-primary* operator if

$$[TA]_3 = 0 \ ; \tag{1.20}$$

- ⋄ a scaling operator A is a *primary* operator if

$$[TA]_r = 0 \quad \forall r \geq 3 \ . \tag{1.21}$$

In the following, the Virasoro element will be identified with the holomorphic chiral component of the stress-energy tensor $T(z)$; with an abuse of language, we will refer to $T(z)$ simply as the stress-energy tensor, since we will never consider the anti-holomorphic component. We can finally give the definition of conformal OPA and $\mathcal{W}$-algebra [72].

[3] Infinitely generated OPAs received particular attention due to their role in the study of asymptotically flat spacetimes and celestial holography [73–75].

Definition 2 (Conformal OPA)**.** *A conformal OPA is an OPA with an operator content $\mathcal{F}$ containing only the identity $\mathbb{1}$, a stress-energy tensor T, and scaling operators.*

Definition 3 ($\mathcal{W}$-algebra)**.** *A $\mathcal{W}$-algebra is a conformal OPA generated by the identity operator and a set of quasi-primary operators.*

Adopting this definition, we can identify the Virasoro algebra as the minimal $\mathcal{W}$-algebra, generated only by $\langle \mathbb{1}, T \rangle$. Breaking the Virasoro OPE (1.5) into its poles

$$[TT]_4 = \frac{c}{2} \ , \quad [TT]_2 = 2\,T \ , \quad [TT]_1 = \partial T \ , \tag{1.22}$$

we confirm that the stress-energy tensor is a quasi-primary operator, hence a good $\mathcal{W}$-algebra generator. In his seminal work [35], Zamolodchikov studied for the first time the extension of the Virasoro algebra obtained by adding to the generators $\langle \mathbb{1}, T \rangle$ a chiral current J_3 of weight $h = 3$, the so-called $\mathcal{W}_3$ algebra, which describes the 3-state critical Potts model [76, 77]. A generalization is available for the $\mathbb{Z}_N$ symmetric lattice model, for $N = 2, 3, \dots$, which at the critical point is described by a two-dimensional CFT with central charge $c_N = 2(N-1)/(N+2)$ [78]: the symmetry algebra describing the critical point is the $\mathcal{W}_N$ algebra, obtained by extending Virasoro with primary chiral currents of weight $h = 3, 4, \dots, N$.

We want to generalize the examples discussed above. We extend the Virasoro algebra with a set of primary chiral currents $J_{h_1}, \dots, J_{h_n}$, with conformal weights $h_1, \dots, h_n$

$$\langle \mathbb{1}, T \rangle \longrightarrow \langle \mathbb{1}, T, J_{h_1}, \dots, J_{h_n} \rangle \ . \tag{1.23}$$

We denote the extended algebra by $\mathcal{W}(2, h_1, \dots, h_n)$, where the first argument is the weight of the stress-energy tensor (the identity operator is implicit). The algebra is well-defined if the OPEs between the generators comply with the conditions (1.15), (1.16), (1.17). By definition, we already know the OPE between the stress-energy tensor and the extra currents

$$T(z_1) J_{h_r}(z_2) \sim \frac{h_r}{z_{12}^2} J_{h_r}(z_2) + \frac{1}{z_{12}} \partial J_{h_r}(z_2) + \dots \ . \tag{1.24}$$

Since the OPEs (1.5) and (1.24) are fixed, the new algebra will be completely characterized by the OPEs between the newly added currents

$$J_{h_r} \times J_{h_s} \ . \tag{1.25}$$

The procedure to construct the algebra consists of writing an Ansatz for the OPEs (1.25) and imposing the consistency conditions. To produce the Ansatz, we consider the operator content of the theory $\mathcal{F}$. By definition of $\mathcal{W}$-algebra, we can split $\mathcal{F}$ as follows [37]

$$\mathcal{F} = \bigoplus_{s,r=0}^{\infty} \partial^s \mathcal{F}_r^{\text{q.p.}} \ , \tag{1.26}$$

where $\mathcal{F}_r^{\text{q.p.}}$ is the set of all quasi-primary operators of weight h_r.

Notice that the normal ordering product (1.18) of two quasi-primary currents J_{h_r} and J_{h_s} in general does not return a quasi-primary operator: however, $N(J_{h_r}J_{h_s})$ can always be *projected* on the space $\mathcal{F}_{h_r+h_s}^{\text{q.p.}}$ of quasi-primaries of weight $h_r + h_s$ [37]. We indicate with $\mathcal{N}(J_{h_r}J_{h_s})$ the quasi-primary projection of the normal-ordering $N(J_{h_r}J_{h_s})$. For example, the normal ordered product $N(TT)$ is not a quasi-primary operator, despite being built out of two copies of the stress-energy tensor T: in fact, $[T, N(TT)]_3 = 3\,\partial T$. The quasi-primary projection of $N(TT)$ on the space $\mathcal{F}_4^{\text{q.p.}}$ is $\mathcal{N}(TT) = N(TT) - \frac{3}{10}\partial\partial T$, and $[T, \mathcal{N}(TT)]_3 = 0$, as we wanted.

If we consider a basis of quasi-primary operators $\{\phi_i\}$ generating the $\mathcal{W}$-algebra, we can identify two important sets of CFT data: the two-point functions coefficients d_{ij} and the three-point functions coefficients C_{ijk}, defined as

$$\langle \phi_i(z_1)\phi_j(z_2)\rangle = \frac{d_{ij}}{z_{12}^{2h_i}} \ , \quad \langle \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle = \frac{C_{ijk}}{z_{12}^{h_{kij}} z_{23}^{h_{ijk}} z_{13}^{h_{kji}}} \ , \tag{1.27}$$

where $h_{ijk} = h_i + h_j - h_k$. The coefficient d_{ij} is symmetric under the exchange of the indices i and j; the coefficient C_{ijk} is invariant under even permutations of the indices and it gains a sign $(-1)^{h_{ijk}}$ under an odd permutation. Then, the OPE between those quasi-primary operators is given by [36–38, 62]

$$\phi_i(z_1)\phi_j(z_2) \sim \sum_{\phi_k \in \mathcal{F}_k^{\text{q.p.}}} \sum_{r=0}^{\infty} C_{ij}^k \, \frac{a_{ijk}^r}{r!} \, \frac{1}{z_{12}^{h_{ijk}-r}} \, \partial^r \phi_k(z_2) \ , \tag{1.28}$$

where the coefficients a_{ijk}^r are determined only by the weights of the quasi-primaries operators involved

$$a_{ijk}^r = \binom{h_{kij} + h_r - 1}{h_r} \binom{2h_k + h_r - 1}{h_r}^{-1} \ . \tag{1.29}$$

The OPE data C_{ij}^k can be determined as the solutions of the following linear system[4] [37]

$$C_{ijk} = C_{ij}^{\ell} d_{\ell k} \ . \tag{1.30}$$

[4]Due to the symmetries of the two-point and three-point functions coefficients, C_{ij}^k enjoys important symmetries as well: we will study them in more detail in Section 1.1.2.

The general OPE (1.28) provides the Ansatz for the OPE (1.25)

$$J_{h_i}(z_1)J_{h_j}(z_2) \sim \sum_{\phi_k \in \mathcal{F}_k^{\text{q.p.}}} \sum_{r=0}^{\infty} C_{ij}^k \, \frac{a_{ijk}^r}{r!} \, \frac{1}{z_{12}^{h_{ijk}-r}} \, \partial^r \phi_k(z_2) \ . \tag{1.31}$$

It should be noted that the currents J_{h_i} and J_{h_j} are primaries operator, hence their OPE could be written only in terms of primaries and their conformal families [1,38]. However, in the following we will adopt the Ansatz (1.31) for its simplicity and its easier implementation in the Mathematica® package `OPEdefs` [72].

Null fields. In some instances, it is possible to relax the OPE consistency conditions (1.15), (1.16) and (1.17). Since correlation functions represent the observables of the theory, the consistency conditions should always be considered to hold up to *null fields*. Any correlator containing a null field insertion vanishes identically: in the following we refine this statement, providing at the same time an operative definition of null fields.

If we consider a null field $\mathfrak{N} \in \mathcal{F}$ and a generic operator $A \in \mathcal{F}$, then $[\mathfrak{N}A]_r$ must be a null field for any values of $r \in \mathbb{Z}$. Let us assume that $B \equiv [\mathfrak{N}A]_r$ is not a null field for a certain value of r: then we could engineer the correlation function $\langle \mathfrak{N}\, A\, B \rangle \neq 0$, proving the statement *per absurdum*. As a consequence, any null field $\mathfrak{N} \in \mathcal{F}$ generates an *ideal* $\mathcal{I}_{\mathfrak{N}} \subset \mathcal{F}$, where every element of $\mathcal{I}_{\mathfrak{N}}$ is a null field. Physical statements are meant to hold up to these ideals.

A necessary condition for a field $\mathfrak{N}$ to be null is

$$[\mathfrak{N}A]_r \neq \mathbb{1} \quad \forall A \in \mathcal{F} \ , \forall r \in \mathbb{Z} \ . \tag{1.32}$$

Usually, this is also regarded as a sufficient condition if all the operators apart from the identity have strictly positive conformal dimensions [72]. Considering a $\mathcal{W}$-algebra finitely generated by the set of operators

$$J_{h_1}, \ldots, J_{h_n} \ , \tag{1.33}$$

we argue that testing the condition (1.32) on the set of generators is sufficient. By definition, the rest of the operators of the theory are constructed out of the generators employing addition, multiplication by scalar, action of ∂ and normal ordering:

- ⋆ *addition and multiplication by scalar*: by linearity

$$[\mathfrak{N}\,(\alpha J_{h_i} + \beta J_{h_j})]_r = \alpha[\mathfrak{N}J_{h_i}]_r + \beta[\mathfrak{N}J_{h_j}]_r \ , \tag{1.34}$$

where α and β are real numbers. Since by hypothesis both terms in the right-hand side of the equation (1.34) are null fields, then we conclude that the left-hand side is a null field as well;

⋆ *action of* ∂: using the properties of the OPE [72]

$$[\mathfrak{N}\,\partial J_{h_i}]_{r+1} = r[\mathfrak{N}J_{h_i}]_r + \partial[\mathfrak{N}J_{h_i}]_{r+1} \ . \tag{1.35}$$

The right-hand side is composed of a null field by hypothesis and by the derivative of a null field by hypothesis, which is still a null field. Hence, the left-hand side is a null field as well;

⋆ *normal ordering*: we consider the generalization of Wick theorem for interacting fields [77]

$$\mathfrak{N}(z_1)N(J_{h_i}J_{h_j})(z_2) = \frac{1}{2\pi i}\oint_{z_2}\frac{\mathrm{d}z_3}{z_3-z_2}\sum_{r>0}\Big(\frac{[\mathfrak{N}J_{h_i}]_r(z_3)J_{h_j}(z_2)}{(z_1-z_3)^r} + (-1)^{|\mathfrak{N}||J_{h_i}|}\frac{J_{h_i}(z_3)[\mathfrak{N}J_{h_j}]_r(z_2)}{(z_1-z_2)^r}\Big) \ . \tag{1.36}$$

Provided that

$$[[\mathfrak{N}J_{h_i}]_r J_{h_j}]_s \neq \mathbb{1} \quad \forall r,s>0 \ , \tag{1.37}$$

the generalized Wick theorem ensures that the OPE $\mathfrak{N}\times N(J_{h_i}J_{h_j})$ contains only null fields.

Iterating these four operations and checks leads to proving the condition (1.32). In conclusion, the condition (1.32), which is difficult to test since the operator content is infinite, can be replaced by the conditions

$$[\mathfrak{N}J_{h_i}]_r \neq \mathbb{1} \quad \forall r>0 \ , \tag{1.38}$$

$$[[\mathfrak{N}J_{h_i}]_r J_{h_j}]_s \neq \mathbb{1} \quad \forall r,s>0 \ , \tag{1.39}$$

where J_{h_i}, J_{h_j} are generators of the $\mathcal{W}$-algebra.

1.1.2 (1,0) superspace and super $\mathcal{W}$-algebras

Although studying the bosonic non-linear σ-model (1.8) allowed us to highlight many important features of the two-dimensional CFTs describing the dynamics on the worldsheet, in this Thesis we will focus on (1,0) superstrings. (1,0) supersymmetry is the minimal amount achievable in two dimensions [22]; it can be classically realized by adding D left-moving fermions to the Polyakov σ-model and imposing invariance under the action of a supercharge. We will study the (1,0) non-linear

σ-model in detail in Section 1.2. In this Section, we focus on the realization of (1,0) supersymmetry at the level of the $\mathcal{W}$-algebra. We start by considering the algebra $\mathcal{W}(2, 3/2)$, obtained by extending Virasoro by adding a primary chiral *supercurrent* G of weight $h = 3/2$. The OPEs read

$$T(z_1)T(z_2) \sim \frac{c/2}{z_{12}^4} + \frac{2}{z_{12}^2}T(z_2) + \frac{1}{z_{12}}\partial T(z_2) + \dots \ , \tag{1.40}$$

$$T(z_1)G(z_2) \sim \frac{3/2}{z_{12}^2}G(z_2) + \frac{1}{z_{12}}\partial G(z_2) + \dots \ , \tag{1.41}$$

$$G(z_1)G_(z_2) \sim \frac{2c/3}{z_{12}^3} + \frac{2}{z_{12}}T(z_2) + \dots \ . \tag{1.42}$$

Similarly to Virasoro being the basis for more complicated $\mathcal{W}$-algebras, $\mathcal{W}(2, 3/2)$ will be the basis for (1,0) super $\mathcal{W}$-algebras. (1,0) supersymmetry can be made manifest by employing the *superspace formalism* [36–38]. In addition to the holomorphic coordinates $(z, \bar{z})$, we consider a Grassmann coordinate θ with conformal weight $h_\theta = -1/2$. The coordinates on the entire superspace will be called $\zeta = (\theta, z, \bar{z})$, while the holomorphic coordinates will be called $Z = (\theta, z)$. We introduce the super derivatives

$$D = \partial_\theta + \theta\partial \ , \quad \bar{\partial} \ , \tag{1.43}$$

and we highlight the property $D^2 = \partial$. Moreover, we introduce two covariant intervals on the super worldsheet [37, 69]

$$Z_{12} \equiv z_1 - z_2 - \theta_1\theta_2 \ , \qquad \theta_{12} \equiv \theta_1 - \theta_2 \ . \tag{1.44}$$

It should be noted that

$$\frac{1}{Z_{12}^n} = \frac{1}{(z_1 - z_2)^n}\left(1 + n\frac{\theta_1\theta_2}{z_1 - z_2}\right) \ . \tag{1.45}$$

Working in the superspace, each operator and its (1,0) supersymmetric partner can be organized in a *super operator*: for example, the stress-energy tensor T and the supercurrent G can be packed into

$$\mathcal{T}(z, \theta) = -\frac{1}{2}G(z) + \theta\, T(z) \ , \tag{1.46}$$

and the entire $\mathcal{W}(2, 3/2)$ algebra can be encoded in a single super OPE

$$\mathcal{T}(Z_1)\mathcal{T}(Z_2) \sim \frac{c/6}{Z_{12}^3} + \frac{3}{2}\frac{\theta_{12}}{Z_{12}^2}\mathcal{T}(Z_2) + \frac{\theta_{12}}{Z_{12}}\partial\mathcal{T}(Z_2) + \frac{1}{2Z_{12}}D\mathcal{T}(Z_2) + \dots \ . \tag{1.47}$$

Notice that we can recover the OPEs (1.40), (1.41) and (1.42) by expanding intervals and super operators in the Grassmann coordinate θ. We can rename this algebra $s\mathcal{W}(3/2)$, where $3/2$ is the

conformal weight of the superfield $\mathcal{T}$.

A super operator is said to be super quasi-primary if both its components are quasi-primary operators; similarly, a super operator is said to be super primary is both is components are primary operators. The super algebra $s\mathcal{W}(3/2)$ can be extended by including super primary currents $\mathcal{J}_{h_1}, \dots, \mathcal{J}_{h_n}$, whose OPE with the super stress-energy tensor reads

$$\mathcal{T}(Z_1)\mathcal{J}_{h_r}(Z_2) \sim \frac{h_r\,\theta_{12}}{Z_{12}^2}\mathcal{J}_{h_r}(Z_2) + \frac{\theta_{12}}{Z_{12}}\partial\mathcal{J}_{h_r}(Z_2) + \frac{1}{2\,Z_{12}}D\mathcal{J}_{h_r}(Z_2) + \dots \,. \tag{1.48}$$

The extended super algebra will be called $s\mathcal{W}(3/2, h_1, \dots, h_n)$. In the following, we will drop the suffix "super" from super algebras, super operators and super OPE, leaving it implicit.

We need to produce an Ansatz for the OPE $\mathcal{J}_{h_r} \times \mathcal{J}_{h_s}$. Given a set of quasi primary operators $\{\mathcal{O}_i\}$, we introduce the two-point and the three-point functions coefficients d_{ij} and C_{ijk} [37]

$$\langle \mathcal{O}_i(Z_1)\mathcal{O}_j(Z_2)\rangle = \frac{d_{ij}}{Z_{12}^{2h_i}}\ , \tag{1.49}$$

$$\langle \mathcal{O}_i(Z_1)\mathcal{O}_j(Z_2)\mathcal{O}_k(Z_3)\rangle = \begin{cases} \dfrac{C_{ijk}}{Z_{12}^{h_{kij}}\,Z_{23}^{h_{ijk}}\,Z_{13}^{h_{jki}}} & \text{if } h_{ijk} \in \mathbb{Z} \\ (-1)^{2h_i+1}\Theta_{123}\,\dfrac{C_{ijk}}{Z_{12}^{h_{kij}}\,Z_{23}^{h_{ijk}}\,Z_{13}^{h_{jki}}} & \text{if } h_{ijk} \in \mathbb{Z} + 1/2 \end{cases}\ , \tag{1.50}$$

where $\Theta_{123} = \theta_1 Z_{23} - \theta_2 Z_{13} + \theta_3 Z_{12} + \theta_1\theta_2\theta_3$. The coefficients C_{ijk} are invariant under an even permutation of the indices, while they pick a sign $(-1)^F$ under an odd permutation: the exponent F is determined by

$$F = \begin{cases} h_{ijk} + 4h_ih_j & \text{if } h_{ijk} \in \mathbb{Z} \\ h_{ijk} + 4(h_i + 1/2)(h_j + 1/2) + 1/2 & \text{if } h_{ijk} \in \mathbb{Z} + 1/2 \end{cases}\ . \tag{1.51}$$

The operator content of the theory $\mathcal{F}$ can be split as follows

$$\mathcal{F} = \bigoplus_{s,r=0}^{\infty} D^s\mathcal{F}_r^{\text{q.p.}}\ . \tag{1.52}$$

Combining all the elements, the Ansatz is [37]

$$\mathcal{J}_{h_i}(Z_1)\mathcal{J}_{h_j}(Z_2) \sim \sum_{\mathcal{O}_k \in \mathcal{F}_k^{\text{q.p.}}} \sum_{r=0}^{\infty} C_{ij}^k\,A_{ijk}^r\,\frac{1}{Z_{12}^{h_{ijk}-r/2}}\,D^r\mathcal{O}_k(Z_2)\ , \tag{1.53}$$

where the coefficients A^k_{ijk} and C^k_{ij} can be exactly computed. If $h_{ijk} \in \mathbb{Z}$, we have

$$A^r_{ijk} = \begin{cases} \frac{1}{\nu!}\binom{h_{kij}+\nu-1}{\nu}\binom{2h_k+\nu-1}{\nu}^{-1} & \text{if } r = 2\nu \\ \\ \frac{1}{\nu!}\binom{h_{kij}+\nu}{\nu+1}\binom{2h_k+\nu}{\nu+1}^{-1} & \text{if } r = 2\nu+1 \end{cases} \tag{1.54}$$

and the couplings C^k_{ij} are the solutions of the linear system

$$C^\ell_{ij} d_{\ell k} = C_{ijk} \ ; \tag{1.55}$$

if instead $h_{ijk} \in \mathbb{Z} + 1/2$, we have

$$A^r_{ijk} = \begin{cases} \frac{1}{\nu!}\binom{h_{kij}+\nu-1/2}{\nu}\binom{2h_k+\nu-1}{\nu}^{-1} & \text{if } r = 2\nu \\ \\ \frac{1}{(\nu+1)!}\binom{h_{kij}+\nu-1/2}{\nu}\binom{2h_k+\nu}{\nu+1}^{-1} & \text{if } r = 2\nu+1 \end{cases} \tag{1.56}$$

and the couplings C^k_{ij} are determined by

$$C^\ell_{ij} d_{\ell k} = (-1)^{2h_j+1} C_{ijk} \ . \tag{1.57}$$

Symmetry of the C^k_{ij} couplings. The OPE coefficients C^k_{ij}, sometimes referred to as *couplings*, are solutions of the linear systems (1.55) and (1.57): since d_{ij} and C_{ijk} enjoy symmetries under the permutation of the indices, it is natural to investigate the symmetries of the coupling C^k_{ij} under permutation of its indices as well. If we normalize the basis of quasi-primary generators $\{\mathcal{O}_i\}$ in such a way that the d_{ij} tensor is diagonal, the coefficients of the two-point functions take the form

$$d_{ij} = C^{\mathbb{1}}_{ii}\, \delta_{ij} \ . \tag{1.58}$$

This hypothesis leads to the following identities

$$C_{ijk} = C^k_{ij} C^{\mathbb{1}}_{kk} \quad \text{if } h_{ijk} \in \mathbb{Z} \ , \qquad C_{ijk} = (-1)^{2h_j+1} C^k_{ij} C^{\mathbb{1}}_{kk} \quad \text{if } h_{ijk} \in \mathbb{Z} + 1/2 \ . \tag{1.59}$$

The symmetries of the three-point functions coefficient C_{ijk} can be explicited as follows, where the exponent F is given by (1.51)

$$C_{ijk} = C_{jki} = C_{kij} = (-1)^F C_{kji} = (-1)^F C_{ikj} = (-1)^F C_{jik} \ . \tag{1.60}$$

Combining the identities (1.59) with the identities (1.60), we can extract two lists of constraints, reported in Table 1.1. In the following, these symmetries will be useful to simplify the Ansatz (1.53).

Coupling	$h_{ijk} \in \mathbb{Z}$	$h_{ijk} \in \mathbb{Z} + 1/2$
C^i_{jk}	$\big(C^{\mathbb{1}}_{kk}/C^{\mathbb{1}}_{ii}\big)C^k_{ij}$	$(-1)^{2(h_j-h_k)}\big(C^{\mathbb{1}}_{kk}/C^{\mathbb{1}}_{ii}\big)C^k_{ij}$
C^j_{ki}	$\big(C^{\mathbb{1}}_{kk}/C^{\mathbb{1}}_{jj}\big)C^k_{ij}$	$(-1)^{2(h_j-h_i)}\big(C^{\mathbb{1}}_{kk}/C^{\mathbb{1}}_{jj}\big)C^k_{ij}$
C^i_{kj}	$(-1)^F\big(C^{\mathbb{1}}_{kk}/C^{\mathbb{1}}_{ii}\big)C^k_{ij}$	$(-1)^F\big(C^{\mathbb{1}}_{kk}/C^{\mathbb{1}}_{ii}\big)C^k_{ij}$
C^j_{ik}	$(-1)^F\big(C^{\mathbb{1}}_{kk}/C^{\mathbb{1}}_{jj}\big)C^k_{ij}$	$(-1)^F(-1)^{2(h_j-h_k)}\big(C^{\mathbb{1}}_{kk}/C^{\mathbb{1}}_{jj}\big)C^k_{ij}$
C^k_{ji}	$(-1)^F C^k_{ij}$	$(-1)^F(-1)^{2(h_j-h_i)}C^k_{ij}$

Table 1.1: *Given three quasi-primary operators $\mathcal{O}_i, \mathcal{O}_j$ and $\mathcal{O}_k$, the couplings appearing in the first column can be rewritten in terms of a single coupling C^k_{ij}, the two-point functions coefficients and an overall signs which depend on the conformal weights of the operators. The couplings in the first column should be identified with the ones in the second column if h_{ijk} is an integer number; otherwise, they should be identified with the ones in the third column.*

1.2 $\mathcal{W}$-symmetries and (1,0) non-linear σ-model

In this Section, we review the $\mathcal{W}$-symmetries, a fundamental tool to set a bridge between the geometrical features of the internal manifold $\mathcal{Y}$ introduced in Section 1.1.1 and the dynamics on the worldsheet. As anticipated, we will work with (1,0) superstrings in the superspace formalism. From the physical point of view, this model is employed to describe the dynamics of a *heterotic superstring* propagating in a non-trivial string background. The main feature of this model is the possibility of realizing an effective Yang-Mills theory in the string background without relying on Dp-branes[5] but introducing a gauge bundle whose sections λ^α are introduced in the σ-model as Majorana–Weyl two-dimensional fermions.

The classical (1,0) non-linear σ-model. Let $\mathcal{Y}$ be a d-dimensional Riemannian manifold endowed with a metric G_{ij} and a Kalb-Ramond field B_{ij}. Let us also consider a gauge bundle $\mathcal{V} \hookrightarrow \mathcal{Y}$ of dimension q on $\mathcal{Y}$ with connection $A_i{}^\alpha{}_\beta$; the metric on the fibres of the bundle can be set to be constant without loss of generality [25].[6] We will work with the minimal amount of supersymmetry $(1,0)$ and we will denote the (super) worldsheet as $\mathbb{\Sigma}$. The worldsheet is parametrized by the coordinates $\zeta = (z, \bar{z}, \theta)$. The measure on $\mathbb{\Sigma}$ will be indicated as $\mathrm{d}^{2|1}\zeta = \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}\theta$. The Polyakov fields

[5] Notice that heterotic string theory cannot accommodate for Dp-branes.

[6] In the following, the indices $i, j, k, \ldots$ run from 1 to d; the indices $\alpha, \beta, \gamma, \ldots$ run from 1 to q.

can be packed in two sets of superfields

$$X^i = x^i + i\theta\psi^i \ , \qquad \Lambda^\alpha = \lambda^\alpha + \theta f^\alpha \ . \tag{1.61}$$

The X^i superfields are bosonic: the components x^i have already been introduced in Section 1.1.1 and they describe the coordinates of the internal manifold $\mathcal{Y}$, while ψ^i is a set of left-moving Majorana–Weyl fermions. The Λ^α superfield is fermionic: the fields λ^α are right-moving Majorana-Weyl fermions and they represent sections of the gauge bundle $\mathcal{V}$; f^α are a set of auxiliary fields. The action of the classical two-dimensional $(1,0)$ non-linear σ-model reads [23, 25]

$$S[X,\Lambda] = \int_{\mathbb{\Sigma}} \frac{\mathrm{d}^{2|1}\zeta}{4\pi\alpha'} \Big[M_{ij}(X)\bar{\partial}X^i DX^j + \mathrm{tr}\,(\Lambda D_A \Lambda) \Big] \ , \tag{1.62}$$

where $M_{ij}(X) = G_{ij}(X) + B_{ij}(X)$ and $D_A\Lambda^\alpha = D\Lambda^\alpha + \widehat{A}^\alpha{}_\beta(X)\Lambda^\beta$, with $\widehat{A}^\alpha{}_\beta(X) = A_i{}^\alpha{}_\beta(X)DX^i$. The trace is performed over the gauge bundle indices. It is instructive to integrate out the Grassmann variable θ and the auxiliary fields f because we are required to introduce the *gauge field strength* $F = \mathrm{d}A + A \wedge A$ and the *Neveu–Schwarz 3-flux* (NS 3-flux) $H = \mathrm{d}B$. The action now reads as follows

$$S[x,\psi,\lambda] = \int_\Sigma \frac{\mathrm{d}z\mathrm{d}\bar{z}}{4\pi\alpha'} \Big[M_{ij}(x)\bar{\partial}x^i\partial x^j + G_{ij}(x)\psi^i\bar{\nabla}\psi^j - \mathrm{tr}\,(\lambda\nabla_A\lambda) \Big] \ , \tag{1.63}$$

where $\bar{\nabla}\psi^i = \bar{\partial}\psi^i + \Gamma(x)^i{}_{k\ell}\,\bar{\partial}x^\ell\psi^k$, $\nabla_A\lambda^\alpha = \partial\lambda^\alpha - \widetilde{F}(x)^\alpha{}_\beta\lambda^\beta$ and

$$\Gamma(x)^i{}_{jk} = \mathring{\Gamma}(x)^i{}_{jk} + \frac{1}{2}\,H(x)^i{}_{jk} \ , \qquad \widetilde{F}(x)^\alpha{}_\beta = \frac{1}{2}F(x)_{ij}{}^\alpha{}_\beta\psi^i\psi^j \ . \tag{1.64}$$

$\mathring{\Gamma}(x)$ represents the Christoffel symbols associated with the target metric $G_{ij}(x)$. Most importantly, we should notice that the torsion tensor $\mathrm{Tor}^i{}_{jk}$ associated with the connection with indices $\Gamma^i{}_{jk}$ is given by

$$\mathrm{Tor}^i{}_{jk} = 2\,\Gamma^i{}_{[jk]} = H^i{}_{jk} \ . \tag{1.65}$$

Since the torsion tensor can be identified with the NS 3-flux H, in the rest of the Thesis the NS 3-flux will be called "torsion", to emphasize its geometric role.

To conclude this Section, we explicitly list the differential forms conventions we will adopt in the following. The indices of a *hatted* p-form $\Phi_{i_1\cdots i_p}$ are saturated with p copies of the superfield DX^i

$$\widehat{\Phi} = \frac{1}{p!}\,\Phi_{i_1\cdots i_p}(X)\,DX^{i_1}\cdots DX^{i_p} \ . \tag{1.66}$$

In this case, the form takes value on the worldsheet via the pull-back $X^i : \mathbb{\Sigma} \to \mathcal{Y}$. The indices of a *tilded* p-form $\Phi_{i_1 \cdots i_p}$ are contracted instead with with p copies of the field ψ^i

$$\widetilde{\Phi} = \frac{1}{p!}\, \Phi_{i_1 \cdots i_p}(x)\, \psi^{i_1} \cdots \psi^{i_p} \ . \tag{1.67}$$

In this case, the form takes value on the worldsheet via the pull-back $x^i : \Sigma \to \mathcal{M}$.

1.2.1 Symmetries and classical currents

In what follows we study the chiral symmetries of the action (1.62). In particular, we will focus on the holomorphic sector and largely ignore the anti-holomorphic sector. Given a continuous symmetry of the action (1.62), its infinitesimal variation is given by

$$\delta S = 2 \int_{\mathbb{\Sigma}} \frac{\mathrm{d}^{2|1}\zeta}{4\pi\alpha'} \Big\{ \Big[-G_{ij}\big(D\bar{\partial}X^j + \Gamma^j{}_{k\ell} DX^k \bar{\partial} X^\ell\big) + \mathrm{tr}(\Lambda \widehat{F}_i \Lambda) \Big]\, \delta X^i + \mathrm{tr}(D_A\Lambda\, \delta_A \Lambda) \Big\} \ , \tag{1.68}$$

where we introduced a gauge covariant variation of the fields Λ^α

$$\delta_A \Lambda^\alpha = \delta\Lambda^\alpha + A_i{}^\alpha{}_\beta\, \delta X^i\, \Lambda^\beta \ . \tag{1.69}$$

Classically, a continuous, chiral holomorphic symmetry, whose infinitesimal transformations are parametrized by the parameter $\epsilon(Z)$, allows the variation of the action to be written in the form

$$\delta S = \int_{\mathbb{\Sigma}} \frac{\mathrm{d}^{2|1}\zeta}{4\pi\alpha'} \epsilon(Z)\, \bar{\partial}\mathcal{J}(Z) \ . \tag{1.70}$$

Thus, up to equations of motion, $\bar{\partial}\mathcal{J}(Z) = 0$ classically. We can associate a charge to the chiral current $\mathcal{J}(Z)$, acting on worldsheet operators as follows

$$\boldsymbol{\mathcal{Q}}_{\mathcal{J}}\, \mathcal{O}(\zeta) = \frac{1}{2\pi i} \oint_\zeta \mathrm{d}Z\, \epsilon(Z) \mathcal{J}(Z) \mathcal{O}(\zeta) \ , \tag{1.71}$$

where we introduced the measure $\mathrm{d}Z = \mathrm{d}z\mathrm{d}\theta$. The integral must be understood as the usual integration on the Grassmann variable θ, plus an anti-clockwise integration over a contour circling the insertion of the $\mathcal{O}(\zeta)$ operator.

Superconformal symmetry. The first chiral symmetry we want to consider is superconformal symmetry. The action on the fields reads

$$\delta^{\mathcal{T}}_\epsilon X^i = -\epsilon\, \partial X^i - \frac{1}{2} D\epsilon\, DX^i \ , \qquad \delta^{\mathcal{T}}_{\epsilon,A} \Lambda = -\epsilon\, \partial_A \Lambda - \frac{1}{2} D\epsilon\, D_A \Lambda \ . \tag{1.72}$$

The chiral current is the *classical* stress-energy tensor

$$\mathcal{T}_{\text{cl.}} = -\frac{1}{2}\left(G_{ij}\partial X^i DX^j - \widehat{H}\right) \ , \tag{1.73}$$

where there is an explicit dependence on the torsion H. The expression (1.73) should be thought as a classical realization of the stress-tensor operator introduced in Section 1.1.2. We recall that the stress-energy tensor can be decomposed as in equation (1.45)

$$\mathcal{T}_{\text{cl.}}(Z) = -\frac{1}{2}G_{\text{cl.}}(z) + \theta\, T_{\text{cl.}}(z) \ , \tag{1.74}$$

and each component can be associated with a classical realization

$$G_{\text{cl.}} = i\left(G_{ij}\partial x^i\psi^j + \widetilde{H}\right) \ , \quad T_{\text{cl.}} = -\frac{1}{2}G_{ij}\left[\partial x^i\partial x^j - (\partial\psi^i + \Gamma^i{}_{k\ell}\,\partial x^k\psi^\ell)\psi^j\right] \ . \tag{1.75}$$

As a comment, it should be noted that the torsion associated with (1,0) superstrings receives perturbative corrections in the string length scale ℓ_s, introduced in equation (1.9): the first correction is

$$H = \mathrm{d}B + \frac{\ell_s^2}{8\pi}\big(\mathrm{CS}_3(A) - \mathrm{CS}_3(\Theta)\big) + \dots \ , \tag{1.76}$$

where $\mathrm{CS}_3(\cdot)$ is the Chern-Simons 3-form, A is the gauge bundle connection, and Θ is the spin connection on the tangent bundle $T\mathcal{Y}$ with symbols[7]

$$\Gamma^{-i}{}_{jk} = \mathring{\Gamma}^i{}_{jk} - \frac{1}{2}\,(\mathrm{d}B)^i{}_{jk} \ . \tag{1.77}$$

In [25] the authors showed that the stress-energy tensor is still a chiral current up to the first correction in ℓ_s, i.e. including the Chern-Simons corrections (1.76) to the torsion. In formula,

$$\bar\partial\,\mathcal{T}_{\text{cl.}} = o(\ell_s^2) \ . \tag{1.78}$$

G-structures and holonomy. When considering a specific d-dimensional Riemannian manifold $\mathcal{Y}$, some of its geometric features are encoded in a G-*structure*.[8] A G-structure is defined as a reduction of the frame bundle $F\mathcal{Y}$ with structure group $\mathrm{GL}(d,\mathbb{R})$ to a principal sub-bundle with structure group $\mathrm{G} \subset \mathrm{GL}(d,\mathbb{R})$, where G is a Lie group. In other words, it is a choice of local frames connected on overlaps via local G-transformations.

[7] Notice the different sign with respect to the symbols (1.64).

[8] For a mathematically detailed description of G-structures, we refer to [79–81].

Given the existence of a G-invariant non-zero tensor of $\mathbb{R}^d$, this defines a well-defined, nowhere vanishing *characteristic tensor field* on any d-dimensional manifold $\mathcal{Y}$ endowed with a G-structure. From this point of view, Riemannian manifolds are manifolds endowed with an $\mathrm{O}(d)$-structure, and the Riemannian metric is the associated characteristic tensor.

In this Thesis, we will focus on G-structures whose characteristic tensors are represented by nowhere vanishing, globally defined p-forms $\Phi_{i_1\dots i_p}$. There is a physical motivation for this choice. In fact, the existence of a specific G-structure is often associated with the preservation of a certain amount of supersymmetry in the string background, which requires the existence of one (or more than one) globally defined, nowhere wanishing spinor η. The existence of this spinor requires the ambient manifold $\mathcal{Y}$ to be a spin manifold. Out of the spinor η, it is possible to build a set of bilinears $b_{i_1\dots i_p} = \eta^\dagger\, \gamma_{i_1\dots i_p}\, \eta$, where $\gamma_{i_1\dots i_p}$ is an element of the Clifford algebra in d dimensions. It can be argued that the set of these bilinears $\{b_1, \dots, b_n\}$ can be identified with the the set of p-forms $\Phi_{i_1\dots i_p}$ mentioned above [82].

Another important idea is the *holonomy group*. Let $\mathcal{Y}$ be a d dimensional Riemannian manifold and let us consider the Levi-Civita connection ∇, defined by the Christoffel symbols $\mathring{\Gamma}(x)^i{}_{kj}$. Let $p \in \mathcal{Y}$ be a point and let us consider the set of closed loops $\ell_p(t)$ at p, such that $0 \leq t \leq 1$ and $\ell(0) = \ell(1) = p$. The parallel transport defined via the connection ∇ along the closed loop $\ell(t)$ induces a linear transformation

$$P_\ell : T_p\mathcal{Y} \longrightarrow T_p\mathcal{Y} \ . \tag{1.79}$$

The set of transformations P_ℓ has a group structure and is called *holonomy group* (at p) Hol. Given a tensor field $\mathcal{S}$ on $\mathcal{Y}$, we have

$$\nabla\mathcal{S} = 0 \Longleftrightarrow h \cdot \mathcal{S} = \mathcal{S} \ , \quad \forall h \in \mathrm{Hol} \ . \tag{1.80}$$

In this case, we say that the manifold $\mathcal{Y}$ is a *special holonomy manifold*. It is useful to compare this notion with the one of manifold endowed with a certain G-structure. If there exists a G-structure whose characteristic tensor is represented by the p-form Φ_p on the manifold $\mathcal{Y}$, then Φ_p sits in the singlet representation of the group G: however, this does not automatically imply neither of the two equivalent conditions (1.80). Hence, it is perfectly allowed to consider a G-structure manifold which

Holonomy group	Manifold
$\mathrm{U}(d/2)$	Kähler manifold
$\mathrm{SU}(d/2)$	Calabi–Yau manifold
$\mathrm{Sp}(d/4)$	Hyper-Kähler manifold
$\mathrm{Sp}(d/4)\cdot\mathrm{Sp}(1)$	Quaternionic-Kähler manifold
G_2	G_2-manifold
$\mathrm{Spin}(7)$	$\mathrm{Spin}(7)$-manifold

Table 1.2: *Berger's classification of holonomy groups for Riemannian manifolds. In this Thesis, particular attention will be given to the groups* $\mathrm{SU}(d/2)$, G_2 and $\mathrm{Spin}(7)$.

is *not* a special holonomy manifold as well. The possible groups G that can be holonomy groups have been classified by Berger in the case of Riemannian manifolds [83]: we report them in Table 1.2.

$\mathcal{W}$-symmetry. The geometrical information provided by the G-structures is encoded at the level of the non-linear σ-model (1.62) as a continuous, *non-linear* symmetry, which in the literature goes under the name of *$\mathcal{W}$-symmetry* [25–32]. Let us consider the σ-model (1.62), and let us suppose that the ambient manifold $\mathcal{Y}$ is endowed with a certain G-structure associated with the existence of one (or more) well-defined, nowhere vanishing p-form $\Phi_{i_1\dots i_p}$. Then, the field transformations encoding the $\mathcal{W}$-symmetry are [25]

$$\delta^{\Phi}_{\epsilon} X^i = \frac{\epsilon}{(p-1)!}\Phi^i{}_{j_2\dots j_p}(X) DX^{j_2}\cdots DX^{j_p} \ , \tag{1.81}$$

$$\delta^{\Phi}_{\epsilon,A}\Lambda^{\alpha} = \frac{2\,\epsilon}{(p-2)!}\Upsilon^{\alpha}{}_{\beta j_1\dots j_{p-2}}(X) DX^{j_1}\cdots DX^{j_{p-2}} D_A\Lambda^{\beta} \ . \tag{1.82}$$

The infinitesimal parameter $\epsilon(Z)$ has conformal weight $h_\epsilon = \frac{1-p}{2}$ and it has odd/even Grassmann parity if p is an even/odd integer. The definition of the transformation (1.82) requires the introduction of an $\mathrm{End}(\mathcal{V})$-valued differential $(p-2)$-form $\Upsilon^{\alpha}{}_{\beta j_1\dots j_{p-2}}(X)$, which at this stage is completely arbitrary. The transformations define a symmetry if and only if the following conditions are met [25]

$$\nabla^{+}_{j}\Phi_{i_1\dots i_p} = 0 \ , \quad F_{j[i_1}\Phi^{j}{}_{i_2\dots i_p]} = 0 \ , \quad \Upsilon_{(\alpha\beta)i_1\dots i_{p-2}} = 0 \ , \tag{1.83}$$

where the covariant derivative ∇_i^+ is associated to a connection with the symbols introduced in (1.64). It is important to notice that if $H = 0$, we have $\nabla\,\Phi = 0$ and the condition (1.80) is satisfied! Hence, in the absence of torsion the classical invariance of the σ-model under the action of $\mathcal{W}$-symmetry requires the ambient manifold $\mathcal{Y}$ to be a special holonomy manifold. This scenario does not realize if $H \neq 0$, since in that case $\nabla_i\,\Phi_{j_1\dots j_p} = \frac{1}{2}H^i{}_{j_1 k}\,\Phi^k{}_{j_2\dots j_p}$ and so $\mathrm{Hol} \not\subseteq \mathrm{G}$.

When the conditions (1.83) are satisfied, we can associate to the $\mathcal{W}$-symmetry the classical chiral current [25]

$$\mathcal{J}^{\Phi}_{\mathrm{cl.}} = (-1)^{p-1}\,\widehat{\Phi}\ . \tag{1.84}$$

This current should be seen as the classical limit of a new operator extending the super Virasoro algebra on the worldsheet. The components of the chiral current (1.84) are

$$\mathcal{J}^{\Phi}_{\mathrm{cl.}}(Z) = (-i)^{p+2}\big(P^{\Phi}_{\mathrm{cl.}}(z) - \theta\,K^{\Phi}_{\mathrm{cl.}}(z)\big)\ . \tag{1.85}$$

It should be noticed that in the definition of the new current (1.84) Υ does not play any role. In particular, the definition looks completely independent from the structures defined on the gauge bundle: this suggests that Υ does not provide additional information compared to Φ, even though they both appear in the $\mathcal{W}$-symmetry transformations (1.81) and (1.82). The role of Υ has been studied in relation to worldsheet supersymmetry enhancement [24], and some comments on the $\mathcal{W}$-symmetry transformation of the gauge bundle sections have recently appeared [31]. In what follows, we will neglect the gauge bundle sector, leaving it as the object of a future exploration.

Similarly to the case of the stress-energy tensor, it was shown in [25] that the $\mathcal{W}$-symmetry current (1.84) is chiral up to the first correction in ℓ_s, provided that an instantonic condition holds for the spin connection Θ

$$R^-(\Theta)_{j[i_1}\,\Phi^j{}_{i_2\dots i_p]} = 0\ . \tag{1.86}$$

The two instantonic conditions can rewritten in a index-less notation

$$i_\Phi(F(A)) = 0\ , \qquad i_\Phi(R^-(\Theta)) = 0\ , \tag{1.87}$$

where, given a p-form Φ, the *interior product* acts as follows

$$i_\Phi(\cdot) : \Psi \longmapsto i_\Phi(\Psi) = \Phi^i \wedge \Psi_i \ , \tag{1.88}$$

where Ψ is a differential form and

$$\Phi_i = \frac{1}{(p-1)!}\Phi_{ij_2\dots j_p}\mathrm{d}x^{j_2}\dots\mathrm{d}x^{j_p} \ . \tag{1.89}$$

If the conditions (1.87) hold, the connections A and Θ are said to be *quasi-instantons*.

In what follows, we will encounter $\mathcal{W}$-symmetries associated with p-forms obtained as a wedge product of two forms, each satisfying the conditions (1.83). In this situation, the current admits a description in terms of the currents of each form. Let Φ be a p-form and Ψ a q-form complying to the conditions (1.83). We consider their wedge product $\Phi \wedge \Psi$: this automatically satisfies the properties required by the $\mathcal{W}$-symmetry. The chiral current is

$$\mathcal{J}^{\Phi\wedge\Psi}_{\text{cl.}}(Z) = (-1)^{p+q-1}\,\widehat{\Phi\wedge\Psi}(Z) = -\mathcal{J}^{\Phi}_{\text{cl.}}(Z)\mathcal{J}^{\Psi}_{\text{cl.}}(Z) \ , \tag{1.90}$$

and its components are

$$P^{\Phi\wedge\Psi}_{\text{cl.}}(z) = P^{\Phi}_{\text{cl.}}(z)P^{\Psi}_{\text{cl.}}(z) \ , \qquad K^{\Phi\wedge\Psi}_{\text{cl.}}(z) = K^{\Phi}_{\text{cl.}}(z)P^{\Psi}_{\text{cl.}}(z) + (-1)^p\, P^{\Phi}_{\text{cl.}}(z)K^{\Psi}_{\text{cl.}}(z) \ . \tag{1.91}$$

1.2.2 Commutators of classical symmetries

In Section 1.2.1 we introduced two symmetries for the classical non-linear σ-model (1.62). The Noether currents associated with superconformal and $\mathcal{W}$-symmetry should be regarded as classical realizations of the quantum operators discussed in Section 1.1.2. Although the information encoded at the classical level can only be partial if compared to the full quantum $\mathcal{W}$-algebra, the classical currents $\mathcal{T}_{\text{cl.}}$ and $\mathcal{J}^{\Phi}_{\text{cl.}}$ store important geometrical features of the string background since they depend on H and on the G-structure(s) Φ. For this reason, it is important to work out a connection between classical superconformal and $\mathcal{W}$-symmetries and the $\mathcal{W}$-algebras defined in Section 1.1. In the following, we present the commutators associated with these symmetries. The algebra of classical $\mathcal{W}$-symmetries for σ-models without bundle terms has been studied case by case in terms of Poisson brackets in [27–30, 84], see also [85].

$[\delta^{\mathcal{T}}, \delta^{\mathcal{T}}]$**: two superconformal transformations.** The commutator of two superconformal transformations with parameters $\epsilon_1(Z)$ and $\epsilon_2(Z)$ returns another superconformal transformation with parameter $\epsilon_3(Z)$

$$[\delta^{\mathcal{T}}_{\epsilon_1}, \delta^{\mathcal{T}}_{\epsilon_2}] X^i = \delta^{\mathcal{T}}_{\epsilon_3} X^i \ , \tag{1.92}$$

where the new infinitesimal parameter is given by

$$\epsilon_3 = \epsilon_1 \partial \epsilon_2 - \partial \epsilon_1 \, \epsilon_2 + \frac{1}{2} D\epsilon_1 D\epsilon_2 \ . \tag{1.93}$$

$[\delta^{\mathcal{T}}, \delta^{\Phi}]$**: superconformal and $\mathcal{W}$-transformations.** The commutator between a superconformal transformation and a $\mathcal{W}$-transformation associated with a p-form Φ returns a $\mathcal{W}$-transformation associated again with the p-form Φ

$$[\delta^{\mathcal{T}}_{\epsilon_1}, \delta^{\Phi}_{\epsilon_2}] \, X^i = \delta^{\Phi}_{\epsilon_3} X^i \ , \tag{1.94}$$

where the new infinitesimal parameter is given by

$$\epsilon_3 = \epsilon_1 \partial \epsilon_2 - \frac{1}{2}(p-1) \partial \epsilon_1 \, \epsilon_2 + \frac{1}{2} D\epsilon_1 D\epsilon_2 \ . \tag{1.95}$$

$[\delta^{\Phi}, \delta^{\Psi}]$**: two $\mathcal{W}$-transformations.** Finally, we consider the case where two $\mathcal{W}$-symmetries participate in the commutator. If the symmetries are associated respectively with a p-form Φ and a q-form Ψ, then the commutator is the sum of three different symmetry transformations [30]

$$[\delta^{\Phi}_{\epsilon_1}, \delta^{\Psi}_{\epsilon_2}] X^i = \delta^{U}_{\epsilon_U} X^i + \delta^{N}_{\epsilon_N} X^i + \delta^{\mathcal{T}V}_{\epsilon_{\mathcal{T}V}} \, X^i \ . \tag{1.96}$$

In the following, we provide some details on the transformations $\delta^{U}_{\epsilon_U}$, $\delta^{N}_{\epsilon_N}$ and $\delta^{\mathcal{T}V}_{\epsilon_{\mathcal{T}V}}$:

- $\diamond$ the transformation $\delta^{U}_{\epsilon_U}$ is a $\mathcal{W}$-transformation associated with the $(p+q-2)$-form

$$U = \frac{1}{c_U} i_{\Phi}(\Psi) = \frac{1}{c_U} \Phi^i \wedge \Psi_i \ , \tag{1.97}$$

  where c_U is a constant parameter that can be tuned. The infinitesimal parameter of the transformation is given by

$$\epsilon_U = c_U \frac{(-1)^{pq}}{p+q-2} \left[(q-1) D\epsilon_1 \, \epsilon_2 + (-1)^p \, (p-1) \epsilon_1 \, D\epsilon_2 \right] \ ; \tag{1.98}$$

◇ the transformation $\delta^N_{\epsilon_N}$ is a $\mathcal{W}$-transformation associated with the $(p+q-1)$-form

$$N = \frac{1}{c_N}\left(H_{jk}\wedge\Phi^j\wedge\Psi^k + (-1)^{p+1}\,\frac{2\,c_V}{d-(p+q-4)}\,H\wedge V\right)\ , \tag{1.99}$$

where the torsion H and the $(p+q-4)$-form V appear

$$V = \frac{1}{c_V}\Phi^{ij}\wedge\Psi_{ij}\ . \tag{1.100}$$

In the formula (1.99), d stands for the dimension of the ambient manifold $\mathcal{Y}$, on which the forms Φ and Ψ have support. Similarly to the previous transformation, the constants c_V and c_N can be tuned. The infinitesimal parameter reads

$$\epsilon_N = c_N\,(-1)^{(p-1)(q-1)}\epsilon_1\epsilon_2\ ; \tag{1.101}$$

◇ finally, the last transformation appearing in (1.96) can be written as follows

$$\delta^{\mathcal{T}V}_{\epsilon_{\mathcal{T}V}}X^i = -\left(\epsilon_{\mathcal{T}V}\partial X^i + \frac{1}{2}(-1)^{p+q}D\epsilon_{\mathcal{T}V}DX^i\right)\widehat{V} + \epsilon_{\mathcal{T}V}\left(-\mathcal{T}_{\text{cl.}}\widehat{V}^i + \frac{1}{2}\,DX^iD\widehat{V}\right)\ , \tag{1.102}$$

where the infinitesimal parameter reads

$$\epsilon_{\mathcal{T}V} = (-1)^{(p-1)q+1}\,\frac{2\,c_V}{d-(p+q-4)}\,\epsilon_1\epsilon_2\ . \tag{1.103}$$

The current associated with the transformation (1.102) can be rewritten as a composite current

$$\mathcal{J}^{\mathcal{T}V}_{\text{cl.}}(Z) = -\mathcal{T}_{\text{cl.}}(Z)\,\mathcal{J}^V_{\text{cl.}}(Z)\ ; \tag{1.104}$$

if $V = \mathbb{1}$, the transformation (1.102) is simply a superconformal transformation, and the associated current is $\mathcal{J}^{\mathcal{T}\mathbb{1}}_{\text{cl.}} = \mathcal{T}_{\text{cl.}}$.

Some technical details regarding these transformations can be found in Appendix A.1.

1.2.3 From classical commutators to OPEs

The natural step following the computation of the commutators (1.92), (1.94) and (1.96) is to translate the information they carry in an OPE language. The result will be a set of *classical* OPEs, which will be compared to the $\mathcal{W}$-algebra OPEs in Chapter 2.

Let us consider two chiral currents $\mathcal{J}_1(Z)$ and $\mathcal{J}_2(Z)$ and the charges $\boldsymbol{\mathcal{Q}}_1$ and $\boldsymbol{\mathcal{Q}}_2$, defined as in (1.71)

$$\boldsymbol{\mathcal{Q}}_1\,\mathcal{O}(\zeta)=\frac{1}{2\pi i}\oint_\zeta \mathrm{d}Z\,\epsilon_1(Z)\mathcal{J}_1(Z)\mathcal{O}(\zeta)\ ,\quad \boldsymbol{\mathcal{Q}}_2\,\mathcal{O}(\zeta)=\frac{1}{2\pi i}\oint_\zeta \mathrm{d}Z\,\epsilon_2(Z)\mathcal{J}_2(Z)\mathcal{O}(\zeta)\ . \tag{1.105}$$

The commutator of the two charges can be written in terms of the currents as follows [77]

$$[\boldsymbol{\mathcal{Q}}_1,\boldsymbol{\mathcal{Q}}_2]\,\mathcal{O}(\zeta)=-(-1)^{|\epsilon_1||\mathcal{J}_2|}\oint_\zeta\frac{\mathrm{d}Z_2}{2\pi i}\oint_{\zeta_2}\frac{\mathrm{d}Z_1}{2\pi i}\,\epsilon_1(Z_1)\epsilon_2(Z_2)\,\mathcal{J}_1(Z_1)\mathcal{J}_2(Z_2)\,\mathcal{O}(\zeta)\ , \tag{1.106}$$

where the prefactor is sensitive to the parity of the infinitesimal parameters and of the currents. Given that the action of a charge on a local operator is given by the associated infinitesimal symmetry transformation, in formula $\boldsymbol{\mathcal{Q}}\,\mathcal{O}=\delta\mathcal{O}$, the left-hand side can be rewritten as follows

$$[\boldsymbol{\mathcal{Q}}_1,\boldsymbol{\mathcal{Q}}_2]\,\mathcal{O}(\zeta)=[\delta_1,\delta_2]\,\mathcal{O}(\zeta)\ . \tag{1.107}$$

Now the key idea is to consider the commutator (1.107) acting on the field $X^i(\zeta)$. Thanks to the discussion in Section 1.2.2, we know that *classically* the commutator can be traded for another trasformation, or a linear combination of transformations

$$[\boldsymbol{\mathcal{Q}}_1,\boldsymbol{\mathcal{Q}}_2]\,X^i(\zeta)=[\delta_1,\delta_2]\,X^i(\zeta)\overset{\text{cl.}}{=}\sum_r\delta_r X^i(\zeta)=\sum_r\boldsymbol{\mathcal{Q}}_r\,X^i(\zeta)\ , \tag{1.108}$$

where we highlighted that this statement works at the classical level by employing the notation "$\overset{\text{cl.}}{=}$". Considering that the infinitesimal parameter ϵ_r participating in the action of the charges $\boldsymbol{\mathcal{Q}}_r$ is a functional of the infinitesimal parameters $\epsilon_1(Z)$ and $\epsilon_2(Z)$, we write

$$\sum_r\boldsymbol{\mathcal{Q}}_r\,X^i(\zeta)=\oint_\zeta\frac{\mathrm{d}Z}{2\pi i}\sum_r\epsilon_r[\epsilon_1(Z),\epsilon_2(Z)]\,\mathcal{J}_r(Z)\,X^i(\zeta)\ , \tag{1.109}$$

and we obtain the following *master equation*

$$\begin{aligned}\oint_\zeta\frac{\mathrm{d}Z_2}{2\pi i}\oint_{\zeta_2}\frac{\mathrm{d}Z_1}{2\pi i}\,\epsilon_1(Z_1)\epsilon_2(Z_2)\,\mathcal{J}_1(Z_1)\mathcal{J}_2(Z_2)\,X^i(\zeta)&\\ \overset{\text{cl.}}{=}-(-1)^{|\epsilon_1||\mathcal{J}_2|}\oint_\zeta\frac{\mathrm{d}Z_2}{2\pi i}\sum_r\epsilon_r[\epsilon_1(Z_2),\epsilon_2(Z_2)]\,\mathcal{J}_r(Z_2)\,X^i(\zeta)\ .&\end{aligned} \tag{1.110}$$

The connection with the OPE can be made explicit working on the right hand side of the equation (1.110). To this end, we need to manipulate the functional $\epsilon_r[\epsilon_1(Z),\epsilon_2(Z)]$, using integration by parts to reproduce the $\epsilon_1(Z_1)\epsilon_2(Z_2)$ structure on the left-hand side of the equation (1.110). In the following, we study the three classical OPEs associated with the commutators studied in Section 1.2.2.

$\mathcal{T}_{\text{cl.}} \times \mathcal{T}_{\text{cl.}}$ **OPE: Witt algebra.** Let us illustrate how to use the master equation (1.110) working out the case where $\mathcal{J}_1 = \mathcal{T}_{\text{cl.}}$ and $\mathcal{J}_2 = \mathcal{T}_{\text{cl.}}$. The equation (1.110) reads, in this case,

$$\begin{aligned}\oint_\zeta \frac{\mathrm{d}Z_2}{2\pi i} \oint_{\zeta_2} \frac{\mathrm{d}Z_1}{2\pi i}\, \epsilon_1(Z_1)\epsilon_2(Z_2)\, \mathcal{T}_{\text{cl.}}(Z_1)\mathcal{T}_{\text{cl.}}(Z_2)\, X^i(\zeta) \\ \overset{\text{cl.}}{=} -\oint_\zeta \frac{\mathrm{d}Z_2}{2\pi i} \Big(\epsilon_1(Z_2)\partial\epsilon_2(Z_2) - \partial\epsilon_1(Z_2)\epsilon_2(Z_2) + \frac{1}{2} D\epsilon_1(Z_2) D\epsilon_2(Z_2)\Big)\, \mathcal{T}_{\text{cl.}}(Z_2)\, X^i(\zeta)\ , \end{aligned} \tag{1.111}$$

where we plugged the expression (1.93) in place of the functional $\epsilon_{\mathcal{T}}[\epsilon_1(Z_2), \epsilon_2(Z_2)]$. The double-integral structure appearing in the left-hand side of equation (1.111) can be reproduced in the right-hand side by manipulating the infinitesimal parameter $\epsilon_1(Z_2)$ and its derivatives. We introduce the Dirac δ-distribution on the super worldsheet [36, 69]

$$\delta(Z_1 - Z_2) = \frac{1}{2\pi i} \frac{\theta_{12}}{Z_{12}}\ , \tag{1.112}$$

and we compute the following derivatives

$$\partial_2\, \delta(Z_1 - Z_2) = \frac{1}{2\pi i} \frac{\theta_{12}}{Z_{12}^2}\ , \quad D_2\, \delta(Z_1 - Z_2) = -\frac{1}{2\pi i} \frac{1}{Z_{12}}\ . \tag{1.113}$$

The infinitesimal parameter $\epsilon_1(Z_2)$ can be rewritten as

$$\epsilon_1(Z_2) = \oint_{\zeta_2} \mathrm{d}Z_1\, \delta(Z_1 - Z_2)\, \epsilon_1(Z_1) = \oint_{\zeta_2} \frac{\mathrm{d}Z_1}{2\pi i}\, \frac{\theta_{12}}{Z_{12}}\, \epsilon_1(Z_1)\ , \tag{1.114}$$

and its derivatives as follows, using the equations (1.113)

$$\partial_2\epsilon_1(Z_2) = \oint_{\zeta_2} \frac{\mathrm{d}Z_1}{2\pi i}\, \frac{\theta_{12}}{Z_{12}^2}\, \epsilon_1(Z_1)\ , \quad D_2\epsilon_1(Z_2) = \oint_{\zeta_2} \frac{\mathrm{d}Z_1}{2\pi i}\, \frac{1}{Z_{12}}\, \epsilon_1(Z_1)\ . \tag{1.115}$$

We have all the tools needed to reshape the right-hand side of the equation (1.111). The first step is to integrate by parts to remove all the derivatives from the parameter $\epsilon_2(\zeta_2)$. Focusing on the right-hand side, we have

$$\begin{aligned}\ldots \overset{\text{cl.}}{=} \oint_\zeta \frac{\mathrm{d}Z_2}{2\pi i} \Big(\frac{3}{2}\partial\epsilon_1(Z_2)\epsilon_2(Z_2)\mathcal{T}_{\text{cl.}}(Z_2) + \epsilon_1(Z_2)\epsilon_2(Z_2)\partial\mathcal{T}_{\text{cl.}}(Z_2) \\ + \frac{1}{2} D\epsilon_1(Z_2)\epsilon_2(Z_2) D\mathcal{T}_{\text{cl.}}(Z_2)\Big) X^i(\zeta)\ . \end{aligned} \tag{1.116}$$

The second step requires to substitute $\epsilon_1(Z_2)$ with the expression (1.114) and its derivatives with the expressions (1.115)

$$\ldots \overset{\text{cl.}}{=} \oint_\zeta \frac{\mathrm{d}Z_2}{2\pi i} \oint_{\zeta_2} \frac{\mathrm{d}Z_1}{2\pi i} \epsilon_1(Z_1)\epsilon_2(Z_2) \Big(\frac{3}{2}\frac{\theta_{12}}{Z_{12}^2}\mathcal{T}_{\text{cl.}}(Z_2) + \frac{\theta_{12}}{Z_{12}}\partial\mathcal{T}_{\text{cl.}}(Z_2) + \frac{D\mathcal{T}_{\text{cl.}}(Z_2)}{2Z_{12}}\Big) X^i(\zeta)\ . \tag{1.117}$$

By comparing the expression (1.117) with the left-hand side of the expression (1.111), we extract the classical OPE between the Noether current $\mathcal{T}_{\text{cl.}}$ and itself

$$\mathcal{T}_{\text{cl.}}(Z_1)\mathcal{T}_{\text{cl.}}(Z_2) \sim \frac{3}{2}\frac{\theta_{12}}{Z_{12}^2}\mathcal{T}_{\text{cl.}}(Z_2) + \frac{\theta_{12}}{Z_{12}}\partial\mathcal{T}_{\text{cl.}}(Z_2) + \frac{1}{2Z_{12}}D\mathcal{T}_{\text{cl.}}(Z_2) + \dots \ . \tag{1.118}$$

We immediately notice that this classical OPE reproduces correctly every term in the Virasoro algebra (1.47), excluding the $1/z_{12}^3$ pole, proportional to the central charge c. In fact, the information encoded in (1.118) is the same as the one encoded in the classical commutator (1.92), hence we cannot access the central extension of the algebra through this procedure.

$\mathcal{T}_{\text{cl.}} \times \mathcal{J}^{\Phi}_{\text{cl.}}$ OPE: primaries. Following the same procedure used to derive the classical OPE $\mathcal{T}_{\text{cl.}} \times \mathcal{T}_{\text{cl.}}$, we convert the classical commutator (1.94) into a classical OPE. If we consider Φ to be a p-form, we have

$$\mathcal{T}_{\text{cl.}}(Z_1)\mathcal{J}^{\Phi}_{\text{cl.}}(Z_2) \sim \frac{p}{2}\frac{\theta_{12}}{Z_{12}^2}\mathcal{J}^{\Phi}_{\text{cl.}}(Z_2) + \frac{\theta_{12}}{Z_{12}}\partial\mathcal{J}^{\Phi}_{\text{cl.}}(Z_2) + \frac{1}{2\,Z_{12}}D\mathcal{J}^{\Phi}_{\text{cl.}}(Z_2) + \dots \ . \tag{1.119}$$

This classical OPE should be compared with the quantum OPE (1.48): the weight $h_{\mathcal{J}_\Phi} = p/2$ of the $\mathcal{W}$-symmetry current $\mathcal{J}^{\Phi}_{\text{cl.}}$ depends on the degree of the associated G-structure Φ.

$\mathcal{J}^{\Phi}_{\text{cl.}} \times \mathcal{J}^{\Psi}_{\text{cl.}}$ OPE: symmetry enhancement. Finally, we consider the commutator (1.96). Let Φ be a p-form and Ψ a q-form, then the procedure returns the classical OPE

$$\begin{aligned}\mathcal{J}^{\Phi}_{\text{cl.}}(Z_1)\mathcal{J}^{\Psi}_{\text{cl.}}(Z_2) \sim\ & (-1)^{p+1}\,c_U\,\frac{\mathcal{J}^{U}_{\text{cl.}}(Z_2)}{Z_{12}} + (-1)^{p+1}\,c_U\left(\frac{p-1}{p+q-2}\right)\frac{\theta_{12}}{Z_{12}}D\mathcal{J}^{U}_{\text{cl.}}(Z_2) \\ & + (-1)^{p}\,c_N\,\frac{\theta_{12}}{Z_{12}}\mathcal{J}^{N}_{\text{cl.}}(Z_2) + c_V\left(\frac{2}{d-(p+q-4)}\right)\frac{\theta_{12}}{Z_{12}}\,\mathcal{T}_{\text{cl.}}(Z_2)\mathcal{J}^{V}_{\text{cl.}}(Z_2) + \dots \ .\end{aligned} \tag{1.120}$$

Notice that the last term in the right-hand side of the classical OPE (1.120) is simply a product of $\mathcal{T}_{\text{cl.}}$ and $\mathcal{J}^{V}_{\text{cl.}}$, as prescribed by the equation (1.104), and it should not be regarded as an operator product. This classical OPE is extremely interesting: although it is not able to capture all the terms encoded in the quantum OPE, the forms U, V, N and the classical stress-energy tensor $\mathcal{T}_{\text{cl.}}$ carry geometrical information about the G-structure and, most importantly, the torsion H. The classical OPE $\mathcal{J}^{\Phi}_{\text{cl.}} \times \mathcal{J}^{\Psi}_{\text{cl.}}$ is meant to be compared with the outcome of the Ansatz (1.53): this comparison will be the focus of Chapter 2.

2

G-structures, torsion and $s\mathcal{W}$-algebras

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2.1 Connecting G-structures and torsion to $\mathcal{W}$-algebras

The starting point of this Chapter is the string theory defined at the classical level by the σ-model (1.62), having as a target space a d-dimensional Riemannian manifold $\mathcal{Y}$, endowed with a G-structure with characteristic p-forms $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively of degree $p_1, \ldots, p_n$. There are two directions we can follow:

- ◇ the dynamics on the worldsheet is described by a $\mathcal{W}$-algebra extending the Virasoro algebra. The $\mathcal{W}$-algebra is generated by the identity, the stress-energy tensor and a set of primary operators

$$\langle \mathbb{1}, \mathcal{T}, \mathcal{J}_{h_1}, \ldots, \mathcal{J}_{h_n} \rangle \ . \tag{2.1}$$

The $\mathcal{W}$-algebra is completely determined by the Ansatz (1.53)

$$\mathcal{J}_{h_i}(Z_1)\mathcal{J}_{h_j}(Z_2) \sim \sum_{\mathcal{O}_k \in \mathcal{F}_k^{\text{q.p.}}} \sum_{r=0}^{\infty} C_{ij}^k \, A_{ijk}^r \, \frac{1}{Z_{12}^{h_{ijk}-r/2}} \, D^r \mathcal{O}_k(Z_2) \tag{2.2}$$

and the OPE consistency conditions (1.15), (1.16) and (1.17);

◇ the σ-model enjoys superconformal and $\mathcal{W}$-symmetries associated with a set of classical Noether currents

$$\mathcal{T}_{\text{cl.}}, \; \mathcal{J}_{\text{cl.}}^{\Phi_1}, \; \dots, \; \mathcal{J}_{\text{cl.}}^{\Phi_n} \; . \tag{2.3}$$

These currents carry information about the G-structures and the torsion H, and they are tied by a set of classical OPEs

$$\begin{aligned} \mathcal{J}_{\text{cl.}}^{\Phi_1}(Z_1)\mathcal{J}_{\text{cl.}}^{\Phi_2}(Z_2) \sim{}& (-1)^{p_1+1} \, c_U \, \frac{\mathcal{J}_{\text{cl.}}^U(Z_2)}{Z_{12}} + (-1)^{p_1+1} \, c_U \left(\frac{p_1 - 1}{p_1 + p_2 - 2} \right) \frac{\theta_{12}}{Z_{12}} D\mathcal{J}_{\text{cl.}}^U(Z_2) \\ & + (-1)^{p_1} \, c_N \, \frac{\theta_{12}}{Z_{12}} \mathcal{J}_{\text{cl.}}^N(Z_2) + c_V \left(\frac{2}{d - (p_1 + p_2 - 4)} \right) \frac{\theta_{12}}{Z_{12}} \, \mathcal{T}_{\text{cl.}}(Z_2)\mathcal{J}_{\text{cl.}}^V(Z_2) + \dots \; . \end{aligned} \tag{2.4}$$

In the following, we will assume that the currents (2.3) are the classical limit of the generators (2.1) [25, 39]: under this assumption, the generators of the $\mathcal{W}$-algebra read

$$\left\langle \mathbb{1}, \mathcal{T}, \mathcal{J}_{\frac{p_1}{2}}^{\Phi_1}, \dots, \mathcal{J}_{\frac{p_n}{2}}^{\Phi_n} \right\rangle \; . \tag{2.5}$$

In the following, we will omit the weight of the generator $\mathcal{J}_{\frac{p_i}{2}}^{\Phi_i} \to \mathcal{J}^{\Phi_i}$ to make the notation less cluttered. Since symmetry arguments (see Table 1.1) and the associativity condition (1.17) will fix the OPE coefficients C_{ij}^k up to a small number of free parameters, the idea is to compare the two OPEs (2.2) and (2.4) in order to write those free parameters as a function of the torsion H. To check the result, we will employ the torsionless limit $H \to 0$ and recover the $\mathcal{W}$-algebras associated to special holonomy manifold, i.e. the case where Hol $\subseteq$ G. The list of cases we will study, together with the associated special holonomy $s\mathcal{W}$-algebra, are listed in Table 2.1. The special holonomy algebras can be found in Appendix A.2. The explicit $\mathcal{W}$-algebras reported in this Chapter have been obtained with the help of the Mathematica® package `OPEdefs`, developed by Kris Thielemans [72].

2.1.1 Warm up: vector fields

We test the procedure described at the beginning of this Chapter by studying a d-dimensional Riemannian manifold $\mathcal{Y}$ endowed with a set of linearly independent, globally defined and nowhere

Special holonomy group	Manifold	Algebra
$\mathrm{SU}(d/2)$	Calabi–Yau manifold	$\mathrm{Od}(d/2)$ [40]
G_2	G_2-manifold	$\mathrm{SV}_{\mathrm{G}_2}$ [41, 86]
Spin(7)	Spin(7)-manifold	$\mathrm{SV}_{\mathrm{Spin}(7)}$ [41, 86]

Table 2.1: *Special holonomy groups and the manifolds they are associated with, together with the $\mathcal{W}$-algebras ruling the dynamics on the worldsheet.*

vanishing vector fields $\xi^i_1, \dots, \xi^i_n$. These vectors are naturally associated to a set of 1-forms $\sigma_1, \dots, \sigma_n$, defined via the isomorphism $\sigma_i = G_{ij}\xi^j$, where G_{ij} is the metric on $\mathcal{Y}$. These 1-forms are the characteristic tensors of an $\mathrm{O}(d-n)$-structure [82] and are associated with the classical currents $\mathcal{J}^{\sigma_I}_{\mathrm{cl.}} \equiv \mathcal{J}^{\sigma}_{\mathrm{cl.},I}$, where $I = 1, \dots, n$. If we consider explicitly the condition (1.83), which ensures that the currents $\mathcal{J}^{\sigma}_{\mathrm{cl.},I}$ are chiral [87], this reads

$$\nabla^+_i \sigma_{I,j} = 0 \ , \qquad \forall I = 1, \dots, n \ . \tag{2.6}$$

Recalling that the torsion tensor $\mathrm{Tor} = H$ involved in the equation (2.6) is totally antisymmetric, it can be deduced that the vector fields ξ_I must satisfy the Killing equation and be the infinitesimal generators of a set of isometries [87].

Construction of the $\mathcal{W}$-algebra. The construction of the algebra is relatively simple, due to the low weights of the currents involved in the Virasoro extension. Following the prescription (2.5), the generators of the algebra are

$$\langle \mathbb{1}, \mathcal{T}, \mathcal{J}^{\sigma}_1, \dots, \mathcal{J}^{\sigma}_n \rangle \ , \tag{2.7}$$

where each current $\mathcal{J}^{\sigma}_I$ has a weight $h_I = 1/2$ and is associated with a 1-form σ_I. Virasoro extends to the algebra $s\mathcal{W}(3/2, 1/2, \dots 1/2)$. By dimensional analysis, the Ansatz (2.2) specializes to the $s\mathcal{W}(3/2, 1/2, \dots 1/2)$ case as follows

$$\mathcal{J}^{\sigma}_I(Z_1)\mathcal{J}^{\sigma}_J(Z_2) \sim \frac{C_{IJ}}{Z_{12}} + C^K_{IJ}\frac{\theta_{12}}{Z_{12}}\mathcal{J}^{\sigma}_K(Z_2) + \dots \ . \tag{2.8}$$

Geometrical interpretation of the couplings. In order to produce an Ansatz for the classical OPE (2.4), we need to compute the forms U, N and V first:

◇ U is a 0-form given by the scalar product between 1-forms

$$U_{IJ} = \frac{1}{c_U}\sigma_I \cdot \sigma_J \ ; \tag{2.9}$$

◇ V cannot be defined, since it can be constructed only for forms of degree 2 or superior. Whenever V appears, we will set the term equal to zero;

◇ N is a 1-form given by

$$N_{IJ} = \frac{1}{c_N}\left(H_{ijk}\sigma_I^i\sigma_J^j\right)\mathrm{d}x^k = \frac{1}{c_N} i_{\sigma_J}\left(i_{\sigma_I}\left(H\right)\right) \ . \tag{2.10}$$

Plugging the forms above in the classical OPE (2.4), we obtain

$$\mathcal{J}^{\sigma}_{\mathrm{cl.},I}(Z_1)\mathcal{J}^{\sigma}_{\mathrm{cl.},J}(Z_2) \sim \frac{\sigma_I \cdot \sigma_J}{Z_{12}} - \frac{\theta_{12}}{Z_{12}}\mathcal{J}^{i_{\sigma_J}\left(i_{\sigma_I}(H)\right)}_{\mathrm{cl.}}(Z_2) + \dots \ . \tag{2.11}$$

We can now compare the OPEs (2.8) and (2.11) to give a geometrical interpretation to the couplings C_{IJ} and C_{IJ}^K. Comparing the poles $1/Z_{12}$, we find that

$$C_{IJ} = \sigma_I \cdot \sigma_J + O(\ell_s) \ , \tag{2.12}$$

where we took into account possible perturbative corrections in ℓ_s to the classical OPE (2.11) , obtained by performing loop computations.[1] We can now compare the θ_{12}/Z_{12} poles and we find

$$C_{IJ}^K\,\sigma_K^k = -H_{ij}{}^k\sigma_I^i\sigma_J^j + O(\ell_s) \ . \tag{2.13}$$

Multiplying both sides of the equation (2.13) by $G_{k\ell}\,\sigma_L^\ell$ and applying the result (2.12), we obtain

$$f_{IJL} \equiv C_{IJ}^K C_{KL} = -H_{ij\ell}\sigma_I^i\sigma_J^j\sigma_L^\ell + O(\ell_s) \ , \tag{2.14}$$

where we introduced the *Cartan 3-form* f_{IJK}. In summary, we found an interpretation of the couplings C_{IJ} and C_{IJ}^K of the $s\mathcal{W}(3/2, 1/2, \dots 1/2)$ algebra in terms of the characteristic 1-forms σ_I of the $\mathrm{O}(d-n)$-structure and the torsion H. This concludes the problem and illustrates the procedure we will employ for the next G-structures.

[1] See [88, 89] for efforts in this direction in a similar context.

2.2 Applications to specific G-structures

In Section 2.1.1 we connected the free parameters of the $\mathcal{W}$-algebra to the torsion in a straightforward way. In this Section, for each G-structure we consider its *intrinsic* torsion, which can be decomposed into *torsion classes* [90–96]. Each torsion class sits in a representation of the group G associated with the chosen G-structure: to introduce them, we will mostly follow the presentation of [95].
If the ambient manifold $\mathcal{Y}$ is endowed with a certain G-structure defined by a p-form Φ, tensors and spinors can be decomposed according to the representations of the group G. In particular, we will focus on the decomposition of the $(p+1)$-form $\mathrm{d}\Phi$. In fact, if we consider a connection ∇^{Tor} with generic torsion tensor Tor, the condition (1.83) can be rewritten as

$$\nabla^{\mathrm{Tor}}\Phi = 0 \Longrightarrow \mathrm{d}\Phi = i_{\mathrm{Tor}}(\Phi) \ . \tag{2.15}$$

Even though we would like to impose $\mathrm{Tor} = H$, this is not straightforward, since it would assume the existence of a *totally antisymmetric torsion*, which is not allowed in general. The hypothesis that needs to be satisfied for a totally antisymmetric torsion to exist is formulated in terms of the torsion classes. Moreover, notice that a vanishing torsion, associated with special holonomy, requires $\mathrm{d}\Phi = 0$.

If we consider the intrinsic torsion Tor associated to a G-structure on a Riemannian manifold $\mathcal{Y}$, we have that [91, 95]

$$\mathrm{Tor} \in \Lambda^1(T\mathcal{Y}) \otimes \mathfrak{g}^{\perp} \tag{2.16}$$

where $\Lambda^1(T\mathcal{Y})$ is the space of 1-forms defined on $\mathcal{Y}$ and $\mathfrak{g}^{\perp}$ is defined as the orthogonal complement of the Lie algebra $\mathfrak{g}$ (of the group G) in $\mathfrak{so}(d)$, i.e. $\mathfrak{g} \oplus \mathfrak{g}^{\perp} = \mathfrak{so}(d)$. Computing the tensor product (2.16), the intrinsic torsion can be decomposed into irreducible G-*modules*[2]

$$\mathrm{Tor} \in \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n \ . \tag{2.17}$$

The specific component of Tor in the module $\mathcal{W}_i$ will be called i-th *torsion class* and it will be indicated with W_i (or with an alternative name more established in the literature). The condition

[2] An R-*module* (or simply a module) is defined as a vector space where the field of scalars is replaced by a ring R. A G-module can be seen as a similar structure, where the vector space is replaced by an Abelian group M and the ring multiplication is replaced by the action ρ of G on M, such that $\rho : \mathrm{G} \times \mathrm{M} \to \mathrm{M}$.

to achieve special holonomy can be translated in terms of torsion classes as follows

$$W_i = 0 \ , \quad \forall i = 1, \dots, n \ \Leftrightarrow \ \text{Hol} \subseteq \text{G} \ . \tag{2.18}$$

2.2.1 Spin(7)-structures and the $s\mathcal{W}\,(3/2, 2)$ algebra

To immediately apply the discussion above on torsion classes, we study Spin(7)-structures and the algebra associated with them. This is a perfect laboratory due to the simplicity of the G-structure, since it only requires the existence of one 4-form Ψ on the ambient manifold $\mathcal{Y}$.

Spin(7)-structures. A Spin(7)-structure is determined by the existence of a well defined, nowhere-vanishing 4-form Ψ, called *Cayley form*. In the following, the Cayley form will have support on an eight-dimensional Riemannian manifold $\mathcal{Y}$. At any given point on the manifold $\mathcal{Y}$, the Cayley form can be written as [95]

$$\begin{aligned} \Psi = {} & \mathrm{d}x^{1234} + \mathrm{d}x^{1256} + \mathrm{d}x^{1278} + \mathrm{d}x^{3456} + \mathrm{d}x^{3478} + \mathrm{d}x^{5678} + \mathrm{d}x^{1357} \\ & - \mathrm{d}x^{1368} - \mathrm{d}x^{1458} - \mathrm{d}x^{1467} - \mathrm{d}x^{2358} - \mathrm{d}x^{2367} - \mathrm{d}x^{2457} + \mathrm{d}x^{2468} \ , \end{aligned} \tag{2.19}$$

where $\mathrm{d}x^i$ define a local frame and $\mathrm{d}x^{ijk\ell} \equiv \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^\ell$. The Cayley form Ψ can be used to determine a metric and an orientation on $\mathcal{Y}$ with respect to which Ψ is self-dual [95]

$$\Psi = *\Psi \ , \tag{2.20}$$

where the Hodge star operator $*$ acts on a generic p-form $\alpha = \frac{1}{k!}\alpha_{i_1 \dots i_p}\mathrm{d}x^{i_1} \dots \mathrm{d}x^{i_p}$ as follows

$$*\alpha = \frac{\sqrt{G}}{(d-k)!\,k!}\alpha^{i_1 \dots i_p}\epsilon_{i_1 \dots i_p j_1 \dots j_{d-p}}\mathrm{d}x^{j_1} \dots \mathrm{d}x^{j_{d-p}} \ . \tag{2.21}$$

We now study the torsion classes associated with the Spin(7)-structures in detail. The Lie algebra $\mathfrak{so}(8)$ decomposes as follows

$$\underset{\mathbf{28}}{\mathbf{so}(8)} = \underset{\mathbf{21}}{\mathbf{spin}(7)} \oplus \mathfrak{spin}(7)^\perp \ , \tag{2.22}$$

where **28** is the adjoint representation of SO(8) and **21** is the adjoint representation of Spin(7). This hints at the following decomposition of the **28** representation

$$\mathbf{28} \to \mathbf{7} + \mathbf{21} \ , \tag{2.23}$$

Considering that a 1-form $\sigma \in \Lambda^1(T\mathcal{Y})$ sits in the fundamental representation **8** of SO(8), the formula (2.16) states that the intrinsic torsion must sit in the representations **8** and **48** of Spin(7)

$$\mathbf{8} \times \mathbf{7} = \mathbf{8} + \mathbf{48} \ . \tag{2.24}$$

Thanks to this analysis, we conclude that [90, 95]

$$\mathrm{Tor} \in \mathcal{W}_1 \oplus \mathcal{W}_2 \ . \tag{2.25}$$

The torsion classes W_1 and W_2, called in the literature τ_1 and τ_3, appear in the decomposition of $\mathrm{d}\Psi$ into Spin(7) representations [96]

$$\mathrm{d}\Psi = i_{\mathrm{Tor}}(\Psi) = \tau_1 \wedge \Psi + *\tau_3 \ , \quad \tau_1 \in \Lambda^1_{\mathbf{8}}(T\mathcal{Y}) \ , \tau_3 \in \Lambda^3_{\mathbf{48}}(T\mathcal{Y}) \ . \tag{2.26}$$

Interestingly, on a manifold with Spin(7)-structure there *always* exists a connection admitting a totally antisymmetric torsion [93], so we do not have to impose any constraints on the torsion classes. Such torsion, compatible with (2.26), is given by

$$H = -\frac{1}{6}\, \tau_1 \lrcorner\, \Psi - \tau_3 \ , \tag{2.27}$$

where we introduced the notation $\alpha \lrcorner \beta$ to contract a p-form α into a $(p+n)$-form β

$$\alpha \lrcorner \beta = \frac{1}{p!n!} \alpha^{i_1 \dots i_p} \beta_{i_1 \dots i_p j_1 \dots j_n}\, \mathrm{d}x^{j_1} \dots \mathrm{d}x^{j_n} \ . \tag{2.28}$$

Construction of the $\mathcal{W}$-algebra. Following the prescription (2.5), the generators of the algebra will be

$$\left\langle \mathbb{1}, \mathcal{T}, \mathcal{J}^{\Psi} \right\rangle \ , \tag{2.29}$$

where the current $\mathcal{J}^{\Psi}$ has weight $h_{\Psi} = 2$. Virasoro extends to the algebra $s\mathcal{W}(3/2, 2)$ algebra, derived for the first time in [38]. To formulate an Ansatz for the OPE $\mathcal{J}^{\Psi} \times \mathcal{J}^{\Psi}$, we need to introduce the following quasi-primary projections of the normal ordered operators $N(D\mathcal{T}\mathcal{T})$ and $N(\mathcal{J}^{\Psi}\mathcal{T})$ (see Section 1.1.1 for more details)

$$\mathcal{N}(\mathcal{T}D\mathcal{T}) = N(D\mathcal{T}\mathcal{T}) - \frac{3}{8}\partial\partial\mathcal{T} \ , \tag{2.30}$$

$$\mathcal{N}(\mathcal{T}\mathcal{J}^{\Psi}) = N(\mathcal{J}^{\Psi}\mathcal{T}) + \frac{3}{10} D\partial\mathcal{J}^{\Psi} \ . \tag{2.31}$$

$\mathcal{J}^1_{\rm cl.}$	$\mathcal{J}^2_{\rm cl.}$	$\mathcal{J}^U_{\rm cl.}$	$\mathcal{J}^N_{\rm cl.}$	$\mathcal{J}^V_{\rm cl.}$	c_U	c_N	c_V
$\mathcal{J}^\Psi_{\rm cl.}$	$\mathcal{J}^\Psi_{\rm cl.}$	-	-	$\mathcal{J}^\Psi_{\rm cl.}$	-	-	4!

Table 2.2: *Currents appearing in the classical OPE $\mathcal{J}^1_{cl.} \times \mathcal{J}^2_{cl.}$, in the case of a* Spin(7)*-structure. The torsion classes τ_1 and τ_3 do not explicitly appear in the coefficients c_U, c_N and c_V.*

The most general Ansatz for the OPE $\mathcal{J}^\Psi \times \mathcal{J}^\Psi$ is[3]

$$\begin{aligned}\mathcal{J}^\Psi(Z_1)\mathcal{J}^\Psi(Z_2) \sim{}& \frac{C^{\mathbb{1}}_{\Psi\Psi}}{Z_{12}^4} + C^\Psi_{\Psi\Psi}\left(\frac{1}{Z_{12}^2}\mathcal{J}^\Psi + \frac{\theta_{12}}{2\,Z_{12}^2}D\mathcal{J}^\Psi + \frac{1}{2\,Z_{12}}\partial\mathcal{J}^\Psi + \frac{3}{10}\frac{\theta_{12}}{Z_{12}}D\partial\mathcal{J}^\Psi\right)\\ &+ C^{\mathcal{T}}_{\Psi\Psi}\left(\frac{\theta_{12}}{Z_{12}^3}\mathcal{T} + \frac{1}{Z_{12}^2}D\mathcal{T} + \frac{2}{3}\frac{\theta_{12}}{Z_{12}^2}\partial\mathcal{T} + \frac{1}{4\,Z_{12}}D\partial\mathcal{T} + \frac{\theta_{12}}{4\,Z_{12}}\partial\partial\mathcal{T}\right)\\ &+ C^{\mathcal{T}D\mathcal{T}}_{\Psi\Psi}\frac{\theta_{12}}{Z_{12}}\mathcal{N}(\mathcal{T}D\mathcal{T}) + C^{\mathcal{T}\Psi}_{\Psi\Psi}\frac{\theta_{12}}{Z_{12}}\mathcal{N}(\mathcal{T}\mathcal{J}^\Psi) + \dots\ .\end{aligned} \tag{2.32}$$

The coupling symmetries reported in Table 1.1 do not provide any further constraints. The OPE consistency conditions fix the couplings to be functions of the central charge c

$$C^{\mathcal{T}}_{\Psi\Psi} = \frac{12}{c}C^{\mathbb{1}}_{\Psi\Psi}\ , \quad C^{\mathcal{T}D\mathcal{T}}_{\Psi\Psi} = \frac{216}{c\,(21+4c)}C^{\mathbb{1}}_{\Psi\Psi}\ , \quad C^{\mathcal{T}\Psi}_{\Psi\Psi} = \frac{54}{6+5c}C^{\Psi}_{\Psi\Psi}\ , \tag{2.33}$$

and

$$\left(C^\Psi_{\Psi\Psi}\right)^2 = -\frac{8(5c+6)^2}{c\,(c-15)(4c+21)}C^{\mathbb{1}}_{\Psi\Psi}\ . \tag{2.34}$$

Since the coupling $C^{\mathbb{1}}_{\Psi\Psi}$ only depends on the normalization of the Ψ form, we conclude that the $s\mathcal{W}(3/2, 2)$ algebra represents a *family* of algebras with each family component associated with a value of the central charge c. The special locus $c = 12$ corresponds to the $\mathrm{SV}_{\mathrm{Spin}(7)}$ algebra [41], describing Spin(7) special holonomy. The $\mathrm{SV}_{\mathrm{Spin}(7)}$ is reported in Appendix A.2.

Geometrical interpretation of the couplings. The Spin(7) case provides a good example to examine the difference between the classical and the quantum OPE: in fact, the former is drastically simpler than the latter, and reads

$$\mathcal{J}^\Psi_{\rm cl.}(Z_1)\mathcal{J}^\Psi_{\rm cl.}(Z_2) \sim 12\,\frac{\theta_{12}}{Z_{12}}\,\mathcal{T}_{\rm cl.}\mathcal{J}^\Psi_{\rm cl.} + \dots\ . \tag{2.35}$$

The currents $\mathcal{J}^U_{\rm cl.}, \mathcal{J}^V_{\rm cl.}$ and $\mathcal{J}^N_{\rm cl.}$ are listed in Table 2.2, together with the coefficients c_U, c_V and c_N. Their explicit computations can be found in Appendix A.3.1. If we identify the product of classical

[3]The Z_2 dependence of the operators in the right-hand side has been omitted to unclutter the expression. It will always be omitted in the rest of this Section.

currents $\mathcal{T}_{\text{cl.}}\mathcal{J}^{\Psi}_{\text{cl.}}$ as the classical limit of the quasi-primary normal ordered operator $\mathcal{N}(\mathcal{T}\mathcal{J}^{\Psi})$, the comparison between the expression (2.32) and (2.35) is straightforward and leads to

$$C^{\mathcal{T}\Psi}_{\Psi\Psi} = 12 + O(\ell_s) \ . \tag{2.36}$$

Since the consistency of the OPE requires the constraints (2.33) and (2.34) to be satisfied, we have

$$C^{\Psi}_{\Psi\Psi} = \frac{2}{9}(6+5c) + O(\ell_s) \ , \tag{2.37}$$

$$C^{\mathbb{1}}_{\Psi\Psi} = -\frac{c\,(c-15)(4c+21)}{162} + O(\ell_s) \ , \tag{2.38}$$

$$C^{\mathcal{T}}_{\Psi\Psi} = -\frac{2}{27}(c-15)(4c+21) + O(\ell_s) \ , \tag{2.39}$$

$$C^{\mathcal{T}D\mathcal{T}}_{\Psi\Psi} = -\frac{4}{3}(c-15) + O(\ell_s) \ . \tag{2.40}$$

The central charge c retains its role of free parameter of the family of algebras. In this case, the outcome of the comparison between the classical and the quantum OPE was to fix a normalization for the two-point function $\langle \mathcal{J}^{\Psi}\mathcal{J}^{\Psi} \rangle$. The role of the torsion is relegated to loop contributions of order $O(\ell_s)$. Finally, we can check that setting $c = 12$ and taking the $H \to 0$ limit returns the special holonomy algebra: the predicted couplings are

$$C^{\mathcal{T}\Psi}_{\Psi\Psi} = 12 \ , \quad C^{\Psi}_{\Psi\Psi} = \frac{44}{3} \ , \quad C^{\mathbb{1}}_{\Psi\Psi} = \frac{46}{3} \ , \quad C^{\mathcal{T}}_{\Psi\Psi} = \frac{46}{3} \ , \quad C^{\mathcal{T}D\mathcal{T}}_{\Psi\Psi} = 4 \ , \tag{2.41}$$

and we find a perfect match with the $\mathrm{SV}_{\mathrm{Spin}(7)}$ algebra.

2.2.2 G_2-structures and the $s\mathcal{W}\left(3/2, 3/2, 2\right)$ algebra: an *incipit*

Together with Spin(7), G_2 is the only exceptional group belonging to Berger's list, reported in Table 1.2. In this Thesis, we will not present a prediction for the couplings, but we will limit ourselves to comment on the classical OPEs and their relation with the torsion classes. The complete results will appear in a future work [97].

G_2-structures. A G_2-structure is determined by the existence of a well defined, nowhere-vanishing 3-form φ, called *associative form*. The associative form will have support on a seven-dimensional Riemannian manifold $\mathcal{Y}$. Similarly to the Cayley form (2.19), at any given point on the manifold $\mathcal{Y}$ it can be written as follows [92]

$$\varphi = \mathrm{d}x^{246} - \mathrm{d}x^{235} - \mathrm{d}x^{145} - \mathrm{d}x^{136} + \mathrm{d}x^{127} + \mathrm{d}x^{347} + \mathrm{d}x^{567} \ . \tag{2.42}$$

The associative form φ determines a metric and an orientation on $\mathcal{Y}$, and we can construct the Hodge dual 4-form $\psi = *\varphi$, called the *co-associative* form.

We analyze the torsion classes of a G_2-structure. The decomposition $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{g}_2^{\perp}$ hints at decomposing the adjoint representation of SO(7) as follows

$$\mathbf{21} \to \mathbf{7} + \mathbf{14} \ , \tag{2.43}$$

where $\mathbf{14}$ is the adjoint representation of G_2. The intrinsic torsion must sit in the representation

$$\mathbf{7} \times \mathbf{7} = \mathbf{1} + \mathbf{7} + \mathbf{14} + \mathbf{27} \ , \tag{2.44}$$

corresponding to the decomposition

$$\mathrm{Tor} \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \ , \tag{2.45}$$

leading to the existence of four torsion classes, known in the literature as τ_0, τ_1, τ_2 and τ_3. We gain access to them by decomposing $\mathrm{d}\varphi$ and $\mathrm{d}\psi$ into G_2 representations [96]

$$\mathrm{d}\varphi = i_{\mathrm{Tor}}(\varphi) = \tau_0 \, \psi + 3 \, \tau_1 \wedge \varphi + *\tau_3 \ , \tag{2.46}$$

$$\mathrm{d}\psi = i_{\mathrm{Tor}}(\psi) = 4 \, \tau_1 \wedge \psi + *\tau_2 \ , \tag{2.47}$$

where the torsion classes are elements of the following set of forms, each in a specific representation

$$\tau_0 \in \Lambda^0_{\mathbf{1}}(T\mathcal{Y}) \ , \quad \tau_1 \in \Lambda^1_{\mathbf{7}}(T\mathcal{Y}) \ , \quad \tau_2 \in \Lambda^2_{\mathbf{14}}(T\mathcal{Y}) \ , \quad \tau_3 \in \Lambda^3_{\mathbf{27}}(T\mathcal{Y}) \ . \tag{2.48}$$

On a manifold with a G_2-structure, the torsion class τ_2 represents an *obstruction* to the existence of a connection with totally antisymmetric torsion [98]. Hence, we set $\tau_2 = 0$, and the unique totally antisymmetric torsion is given by

$$H = \frac{1}{6} \, \tau_0 \, \varphi - \tau_1 \lrcorner \, \psi - \tau_3 \ . \tag{2.49}$$

Classical OPEs. Employing the information reported in Table 2.3, we can produce the following classical OPEs

$$\mathcal{J}^{\varphi}_{\mathrm{cl.}}(Z_1) \mathcal{J}^{\varphi}_{\mathrm{cl.}}(Z_2) \sim \frac{6}{Z_{12}} \mathcal{J}^{\psi}_{\mathrm{cl.}} + \frac{3 \, \theta_{12}}{Z_{12}} D\mathcal{J}^{\psi}_{\mathrm{cl.}} + \dots \ , \tag{2.50}$$

$$\mathcal{J}^{\varphi}_{\mathrm{cl.}}(Z_1) \mathcal{J}^{\psi}_{\mathrm{cl.}}(Z_2) \sim 6 \, \frac{\theta_{12}}{Z_{12}} \, \mathcal{T}_{\mathrm{cl.}} \mathcal{J}^{\varphi}_{\mathrm{cl.}} + \dots \ , \tag{2.51}$$

$\mathcal{J}^1_{\text{cl.}}$	$\mathcal{J}^2_{\text{cl.}}$	$\mathcal{J}^U_{\text{cl.}}$	$\mathcal{J}^N_{\text{cl.}}$	$\mathcal{J}^V_{\text{cl.}}$	c_U	c_N	c_V
$\mathcal{J}^\varphi_{\text{cl.}}$	$\mathcal{J}^\varphi_{\text{cl.}}$	$\mathcal{J}^\psi_{\text{cl.}}$	-	-	6	-	-
$\mathcal{J}^\varphi_{\text{cl.}}$	$\mathcal{J}^\psi_{\text{cl.}}$	-	-	$\mathcal{J}^\varphi_{\text{cl.}}$	-	-	12
$\mathcal{J}^\psi_{\text{cl.}}$	$\mathcal{J}^\psi_{\text{cl.}}$	-	$-\mathcal{J}^\varphi_{\text{cl.}}\mathcal{J}^\psi_{\text{cl.}}$	$\mathcal{J}^\psi_{\text{cl.}}$	-	$\frac{2}{3}\,\tau_0$	12

Table 2.3: *Currents appearing in the classical OPE $\mathcal{J}^1_{cl.} \times \mathcal{J}^2_{cl.}$, in the case of a G_2-structure. The scalar torsion class τ_0 explicitly appears in one of the c_N coefficients: it provides a non-trivial geometrical information that can be used to interpret one of the free parameters of the algebra $s\mathcal{W}(3/2, 3/2, 2)$.*

$$\mathcal{J}^\psi_{\text{cl.}}(Z_1)\mathcal{J}^\psi_{\text{cl.}}(Z_2) \sim -\frac{2\,\tau_0}{3}\frac{\theta_{12}}{Z_{12}}\mathcal{J}^\varphi_{\text{cl.}}\mathcal{J}^\psi_{\text{cl.}} + 8\,\frac{\theta_{12}}{Z_{12}}\,\mathcal{T}_{\text{cl.}}\mathcal{J}^\psi_{\text{cl.}} + \dots\ . \tag{2.52}$$

The computations to produce the Table 2.3 can be found in Appendix A.3.2. The $\mathcal{W}$-algebra describing the worldsheet dynamics on a manifold endowed with a G_2 structure is $s\mathcal{W}(3/2, 3/2, 2)$, computed for the first time in [36]. The algebra depends on two free parameters: by looking at the classical OPE, we can anticipate that the coupling $C^{\varphi\psi}_{\psi\psi}$ will be fixed to be

$$C^{\varphi\psi}_{\psi\psi} = -\frac{2}{3}\,\tau_0 + O(\ell_s)\ . \tag{2.53}$$

This is a very interesting direction since it promises to provide a first general description of the deformations of the special holonomy algebra $\mathrm{SV}_{\mathrm{G}_2}$ induced by the torsion. Efforts in this direction have already been made in the framework of WZW models on 7-manifolds $\mathcal{Y} = S^3 \times \mathcal{M}_4$ [99]: in particular, it is interesting to notice that a non-zero torsion class τ_0 on $\mathcal{Y}$ allows the 3-dimensional spacetime component to be *curved*, opening to further insights into dS_3 or AdS_3 supergravity solutions.

2.2.3 $\mathrm{SU}(3)$-structures and the $\mathrm{Od}^\varepsilon(3)$ algebra

The case of SU(3)-structures with torsion is extremely interesting, due to its connection with Calabi–Yau manifolds in the torsionless limit, but it also poses a challenge. Following the examples set by the other G-structures studied in the previous Sections, the algebra describing the string dynamics on a manifold endowed with an SU(3)-structure should belong to the family $s\mathcal{W}(3/2, 3/2, 3/2, 1)$, generated by the stress-energy tensor, two primaries of weight $h = 3/2$ and a primary of weight $h = 1$. Up to our best knowledge, a full classification of this algebra is missing in the literature, although two special loci associated with the central charges $c = 9$ and $c = 1$ are known and studied [100]. The full OPEs are not available: imposing the associativity of the algebra on the Ansatz (2.2) is a hard task,

due to the high number of non-linear constraints. Although a full study of the $s\mathcal{W}(3/2, 3/2, 3/2, 1)$ algebra could surely be possible in the future, in this Thesis we focus instead on an *infinitesimal deformation* of the Od(3) algebra, associated with special holonomy manifolds. We call this new algebra $\mathrm{Od}^{\varepsilon}(3)$, where ε is the infinitesimal parameter regulating the deformation, and it should be physically interpreted as the algebra ruling the worldsheet dynamics in a string background with an infinitesimal amount of totally antisymmetric torsion, i.e. the Neveu–Schwarz 3-flux H.

SU(3)-structures. An SU(3)-structure is determined by a complex 3-form Ω and a real 2-form ω, both well-defined and nowhere-vanishing. These forms will have support on a six-dimensional Riemannian manifold $\mathcal{Y}$; moreover, they are not independent from each other since they must satisfy the structure equations

$$\Omega \wedge \omega = 0 \ , \qquad \frac{1}{6}\, \omega \wedge \omega \wedge \omega = \frac{i}{||\Omega||^2}\, \Omega \wedge \overline{\Omega} \ , \tag{2.54}$$

where $||\Omega||^2 = \overline{\Omega} \lrcorner \Omega$. The forms ω and Ω define a metric and an orientation on $\mathcal{Y}$. By employing this metric, the 2-form ω defines an *almost complex structure* J on $\mathcal{Y}$

$$J^i{}_j = G^{ik}\omega_{kj} \ . \tag{2.55}$$

With respect to J, ω is a (1,1)-form, Ω a (3,0)-form and $\overline{\Omega}$ a (0,3)-form.

Let us move to the study of the torsion classes. The Lie algebra $\mathfrak{so}(6)$ decomposes according to $\mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathfrak{su}(3)^{\perp}$, and the adjoint representation of SO(6) decomposes as [95]

$$\mathbf{15} \to \mathbf{1} + \mathbf{3} + \overline{\mathbf{3}} + \mathbf{8} \ , \tag{2.56}$$

where $\mathbf{8}$ is the adjoint representation of SU(3). The fundamental representation $\mathbf{6}$ of SO(6) decomposes instead as

$$\mathbf{6} \to \mathbf{3} + \overline{\mathbf{3}} \ , \tag{2.57}$$

so the intrinsic torsion must sit in the representation

$$(\mathbf{3} + \overline{\mathbf{3}}) \times (\mathbf{1} + \mathbf{3} + \overline{\mathbf{3}}) = (\mathbf{1} + \overline{\mathbf{1}}) + (\mathbf{8} + \mathbf{8}) + (\mathbf{6} + \overline{\mathbf{6}}) + (\mathbf{3} + \overline{\mathbf{3}}) + (\mathbf{3} + \overline{\mathbf{3}}) \ , \tag{2.58}$$

corresponding to five SU(3)-modules

$$\mathrm{Tor} \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \ . \tag{2.59}$$

We deduce that the intrinsic torsion associated with an SU(3)-structure is described by five torsion classes, known in the literature as W_0, W_1, W_2, W_3 and ϑ. The exterior derivatives of ω and Ω satisfy the following equations [101, 102]

$$\mathrm{d}\omega = i_{\mathrm{Tor}}(\omega) = -\frac{12}{||\Omega||^2}\,\mathrm{Im}(W_0\,\overline{\Omega}) + W_1 \wedge \omega + W_3 \ , \tag{2.60}$$

$$\mathrm{d}\Omega = i_{\mathrm{Tor}}(\Omega) = W_0\,\omega \wedge \omega + W_2 \wedge \omega + \bar{\vartheta} \wedge \Omega \ , \tag{2.61}$$

where W_0 is a complex function, W_1 is a real 1-form, ϑ is a (1,0)-form, W_2 is a complex primitive (1,1)-form, and W_3 is a real primitive 3-form of type $(2,1)+(1,2)$. On the manifold $\mathcal{Y}$, a connection compatible with the SU(3)-structure and equipped with a totally antisymmetric torsion exists if and only if the *Nijenhuis tensor* is a 3-form [98]. The Nijenhuis tensor is defined as a tensor $N_J : T\mathcal{Y} \otimes T\mathcal{Y} \to T\mathcal{Y}$, such that, given two vector fields $X, Y \in T\mathcal{Y}$, it returns

$$N_J[X,Y] = [X,Y] + J\left([JX,Y] + [X,JY]\right) - [JX,JY] \ , \tag{2.62}$$

where $[\cdot,\cdot]$ is the Lie-bracket between two vector fields, and $J(\cdot)$ is the endomorphism defined by the almost complex structure (2.55). This tensor can be turned into a 3-rank tensor by defining $N'_J[X,Y,Z] \equiv G(N_J[X,Y],Z)$, with G being the metric on $\mathcal{Y}$. N'_J is a a 3-form if and only if the torsion class W_2 vanishes [92, 98], and the intrinsic totally antisymmetric torsion reads

$$H = -J(\mathrm{d}\omega) + N'_J \ . \tag{2.63}$$

In the following, we will work in a basis composed of ω and the real and imaginary parts of Ω

$$\Omega_+ = \mathrm{Re}\,\Omega \ , \quad \Omega_- = \mathrm{Im}\,\Omega \ . \tag{2.64}$$

Similarly, the complex function W_0 will be decomposed into two real torsion classes

$$W_0^+ = \mathrm{Re}\,W_0 \ , \quad W_0^- = \mathrm{Im}\,W_0 \ . \tag{2.65}$$

Infinitesimal deformation of the Od(3) **algebra.** Following the prescription (2.5), the generators of the algebra will be

$$\left\langle \mathbb{1}, \mathcal{T}, \mathcal{J}^{\omega}, \mathcal{J}^{\Omega_+}, \mathcal{J}^{\Omega_-} \right\rangle \ , \tag{2.66}$$

where the current $\mathcal{J}^{\omega}$ has weight $h_{\omega} = 1$ and the currents $\mathcal{J}^{\Omega_\pm}$ have weight $h_{\Omega_\pm} = 3/2$. In this case, Virasoro extends to the algebra $s\mathcal{W}(3/2, 3/2, 3/2, 1)$. As anticipated, the high weights and the number

of generators makes the explicit construction this algebra a challenging task.

To formulate the Ansatz, we need to introduce the following quasi-primary projections of normal ordered products: notice how their structures depend on the couplings, many to be determined

$$\mathcal{N}(\mathcal{T}\mathcal{J}^{\omega}) = N(\mathcal{J}^{\omega}\mathcal{T}) + \frac{1}{6}D\partial\mathcal{J}^{\omega} \ , \tag{2.67}$$

$$\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\omega}) = N(\mathcal{J}^{\omega}\mathcal{J}^{\omega}) - \frac{1}{3}C^{\Omega_+}_{\omega\omega}D\mathcal{J}^{\Omega_+} - \frac{1}{3}C^{\Omega_-}_{\omega\omega}D\mathcal{J}^{\Omega_-} - \frac{1}{3}C^{\mathcal{T}}_{\omega\omega}D\mathcal{T} \ , \tag{2.68}$$

$$\begin{aligned}\mathcal{N}(\mathcal{J}^{\Omega_\pm}\mathcal{J}^{\omega}) = N(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_\pm}) + \frac{1}{6}C^{\omega}_{\omega\Omega_\pm}D\partial\mathcal{J}^{\omega} + \frac{1}{4}C^{\omega\omega}_{\omega\Omega_\pm}D\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\omega}) \\ + \frac{1}{3}C^{\mathcal{T}}_{\omega\Omega_\pm}\partial\mathcal{T} + \frac{1}{3}C^{\Omega_+}_{\omega\Omega_\pm}\partial\mathcal{J}^{\Omega_+} + \frac{1}{3}C^{\Omega_-}_{\omega\Omega_\pm}\partial\mathcal{J}^{\Omega_-} \ .\end{aligned} \tag{2.69}$$

The symmetry arguments and the normalization reported in Section 1.1.2 allow us to set the following couplings to zero

$$\begin{aligned} &C^{\omega}_{\omega\omega} = 0 \ , && C^{\omega}_{\Omega_\pm\Omega_\pm} = 0 \ , && C^{\mathcal{T}\omega}_{\Omega_\pm\Omega_\pm} = 0 \ , && C^{\Omega_\pm\omega}_{\Omega_\pm\Omega_\pm} = 0 \ , \\ &C^{\Omega_+}_{\omega\Omega_+} = 0 \ , && C^{\Omega_-}_{\omega\Omega_-} = 0 \ , && C^{\mathbb{1}}_{\omega\Omega_\pm} = 0 \ , && C^{\mathbb{1}}_{\Omega_+\Omega_-} = 0 \ . \end{aligned} \tag{2.70}$$

Moreover, we normalize the non-zero two-point functions to be the same employed in the Od(3) algebra (see Appendix A.2)

$$C^{\mathbb{1}}_{\omega\omega} = -3 \ , \quad C^{\mathbb{1}}_{\Omega_\pm\Omega_\pm} = 4 \ . \tag{2.71}$$

Additional symmetries can be employed to further constrain the couplings: adopting the normalization (2.71)

$$C^{\omega}_{\omega\Omega_\pm} = \frac{4C^{\Omega_\pm}_{\omega\omega}}{3} \ , \quad C^{\Omega_+}_{\Omega_+\Omega_-} = C^{\Omega_-}_{\Omega_+\Omega_+} \ , \quad C^{\Omega_-}_{\Omega_+\Omega_-} = C^{\Omega_+}_{\Omega_-\Omega_-} \ , \quad C^{\Omega_-}_{\omega\Omega_+} = -\frac{3C^{\omega}_{\Omega_+\Omega_-}}{4} \ , \quad C^{\Omega_+}_{\omega\Omega_-} = \frac{3C^{\omega}_{\Omega_+\Omega_-}}{4} \ . \tag{2.72}$$

The most general Ansatz satisfying the symmetry constraints listed above is the following

$$\mathcal{J}^{\omega}(Z_1)\mathcal{J}^{\omega}(Z_2) \sim -\frac{3}{Z_{12}^2} + \frac{\theta_{12}}{Z_{12}}\left(C^{\mathcal{T}}_{\omega\omega}\mathcal{T} + C^{\Omega_+}_{\omega\omega}\mathcal{J}^{\Omega_+} + C^{\Omega_-}_{\omega\omega}\mathcal{J}^{\Omega_-}\right) + \dots \ , \tag{2.73}$$

$$\begin{aligned}\mathcal{J}^{\omega}(Z_1)\mathcal{J}^{\Omega_+}(Z_2) \sim \frac{4}{3}C^{\Omega_+}_{\omega\omega}\left(\frac{\theta_{12}}{Z_{12}^2}\mathcal{J}^{\omega} + \frac{1}{2\,Z_{12}}D\mathcal{J}^{\omega} + \frac{\theta_{12}}{2\,Z_{12}}\partial\mathcal{J}^{\omega}\right) \\ + C^{\mathcal{T}}_{\omega\Omega_+}\left(\frac{1}{Z_{12}}\mathcal{T} + \frac{\theta_{12}}{3\,Z_{12}}D\mathcal{T}\right) - \frac{3}{4}C^{\omega}_{\Omega_+\Omega_-}\left(\frac{1}{Z_{12}}\mathcal{J}^{\Omega_-} + \frac{\theta_{12}}{3\,Z_{12}}D\mathcal{J}^{\Omega_-}\right) \\ + C^{\omega\omega}_{\omega\Omega_+}\frac{\theta_{12}}{Z_{12}}\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\omega}) + \dots \ ,\end{aligned} \tag{2.74}$$

$$\mathcal{J}^{\omega}(Z_1)\mathcal{J}^{\Omega_-}(Z_2) \sim \frac{4}{3}\,C^{\Omega_-}_{\omega\omega}\left(\frac{\theta_{12}}{Z_{12}^2}\mathcal{J}^{\omega}+\frac{1}{2\,Z_{12}}D\mathcal{J}^{\omega}+\frac{\theta_{12}}{2\,Z_{12}}\partial\mathcal{J}^{\omega}\right)$$
$$+\,C^{\mathcal{T}}_{\omega\Omega_-}\left(\frac{1}{Z_{12}}\mathcal{T}+\frac{\theta_{12}}{3\,Z_{12}}D\mathcal{T}\right)+\frac{3}{4}\,C^{\omega}_{\Omega_+\Omega_-}\left(\frac{1}{Z_{12}}\mathcal{J}^{\Omega_+}+\frac{\theta_{12}}{3\,Z_{12}}D\mathcal{J}^{\Omega_+}\right)$$
$$+\,C^{\omega\omega}_{\omega\Omega_-}\frac{\theta_{12}}{Z_{12}}\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\omega})+\dots\ , \tag{2.75}$$

$$\mathcal{J}^{\Omega_\pm}(Z_1)\mathcal{J}^{\Omega_\pm}(Z_2) \sim \frac{4}{Z_{12}^3}+C^{\mathcal{T}}_{\Omega_\pm\Omega_\pm}\left(\frac{\theta_{12}}{Z_{12}^2}\mathcal{T}+\frac{1}{3\,Z_{12}}D\mathcal{T}+\frac{2}{3}\frac{\theta_{12}}{Z_{12}}\partial\mathcal{T}\right)$$
$$+\,C^{\Omega_+}_{\Omega_\pm\Omega_\pm}\left(\frac{\theta_{12}}{Z_{12}^2}\mathcal{J}^{\Omega_+}+\frac{1}{3\,Z_{12}}D\mathcal{J}^{\Omega_+}+\frac{2}{3}\frac{\theta_{12}}{Z_{12}}\partial\mathcal{J}^{\Omega_+}\right)$$
$$+\,C^{\Omega_-}_{\Omega_\pm\Omega_\pm}\left(\frac{\theta_{12}}{Z_{12}^2}\mathcal{J}^{\Omega_-}+\frac{1}{3\,Z_{12}}D\mathcal{J}^{\Omega_-}+\frac{2}{3}\frac{\theta_{12}}{Z_{12}}\partial\mathcal{J}^{\Omega_-}\right)$$
$$+\,C^{\omega\omega}_{\Omega_\pm\Omega_\pm}\left(\frac{1}{Z_{12}}\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\omega})+\frac{\theta_{12}}{2\,Z_{12}}D\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\omega})\right)+\dots\ , \tag{2.76}$$

$$\mathcal{J}^{\Omega_+}(Z_1)\mathcal{J}^{\Omega_-}(Z_2) \sim C^{\omega}_{\Omega_+\Omega_-}\left(\frac{\mathcal{J}^{\omega}}{Z_{12}^2}+\frac{\theta_{12}}{2\,Z_{12}^2}D\mathcal{J}^{\omega}+\frac{1}{2\,Z_{12}}\partial\mathcal{J}^{\omega}+\frac{\theta_{12}}{3\,Z_{12}}D\partial\mathcal{J}^{\omega}\right)$$
$$+\,C^{\mathcal{T}}_{\Omega_+\Omega_-}\left(\frac{\theta_{12}}{Z_{12}^2}\mathcal{T}+\frac{1}{3\,Z_{12}}D\mathcal{T}+\frac{2}{3}\frac{\theta_{12}}{Z_{12}}\partial\mathcal{T}\right)$$
$$+\,C^{\Omega_-}_{\Omega_+\Omega_+}\left(\frac{\theta_{12}}{Z_{12}^2}\mathcal{J}^{\Omega_+}+\frac{1}{3\,Z_{12}}D\mathcal{J}^{\Omega_+}+\frac{2}{3}\frac{\theta_{12}}{Z_{12}}\partial\mathcal{J}^{\Omega_+}\right)$$
$$+\,C^{\Omega_+}_{\Omega_-\Omega_-}\left(\frac{\theta_{12}}{Z_{12}^2}\mathcal{J}^{\Omega_-}+\frac{1}{3\,Z_{12}}D\mathcal{J}^{\Omega_-}+\frac{2}{3}\frac{\theta_{12}}{Z_{12}}\partial\mathcal{J}^{\Omega_-}\right)$$
$$+\,C^{\omega\omega}_{\Omega_+\Omega_-}\left(\frac{1}{Z_{12}}\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\omega})+\frac{\theta_{12}}{2\,Z_{12}}D\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\omega})\right)$$
$$+\,\frac{\theta_{12}}{Z_{12}}\left(C^{\mathcal{T}\omega}_{\Omega_+\Omega_-}\mathcal{N}(\mathcal{T}\mathcal{J}^{\omega})+C^{\Omega_+\omega}_{\Omega_+\Omega_-}\mathcal{N}(\mathcal{J}^{\Omega_+}\mathcal{J}^{\omega})+C^{\Omega_-\omega}_{\Omega_+\Omega_-}\mathcal{N}(\mathcal{J}^{\Omega_-}\mathcal{J}^{\omega})\right)+\dots\ . \tag{2.77}$$

To avoid the challenge of closing the complicated algebra reported above, we work as anticipated at a perturbative level, constructing an *infinitesimal deformation* of the Od(3) algebra. The construction proceeds as follows. The Virasoro algebra and the $\mathcal{T}\times\mathcal{J}^{\omega,\Omega_\pm}$ OPEs are considered to be fixed and they will be kept equal to the OPEs (1.118) and (1.119). Let us consider instead a generic coupling C_{ij}^{k} appearing explicitly in the Ansatz written above, and the same coupling in the Od(3) algebra $C_{\mathrm{Od},ij}{}^{k}$, where special holonomy is achieved. In the special holonomy locus, $H\to 0$ and $c=9$. We introduce an infinitesimal parameter ε: assuming that C_{ij}^{k} is an analytic function of ε, we write

$$C_{ij}^{k}=C_{\mathrm{Od},ij}{}^{k}+\varepsilon\,\delta C_{ij}^{k}+\frac{1}{2}\varepsilon^2\delta\delta C_{ij}^{k}+O(\varepsilon^3)\ . \tag{2.78}$$

If we plug the expansion (2.78) in the OPEs of the Ansatz listed above, we produce an infinitesimal deformation of the Od(3) algebra, which we will call $\mathrm{Od}^{\varepsilon}(3)$. The algebra $\mathrm{Od}^{\varepsilon}(3)$ will be considered consistent if it complies with the conditions (1.15), (1.16) and (1.17) up to terms of order $O(\varepsilon^3)$.

We present the final result. Interestingly, every coupling is either equal to zero or a function of the two deformed couplings[4]

$$C_{\omega\omega}^{\Omega_+} = \varepsilon\,\delta C_{\omega\omega}^{\Omega_+} + \frac{1}{2}\varepsilon^2\delta\delta C_{\omega\omega}^{\Omega_+} + O(\varepsilon^3)\ , \quad C_{\omega\omega}^{\Omega_-} = \varepsilon\,\delta C_{\omega\omega}^{\Omega_-} + \frac{1}{2}\varepsilon^2\delta\delta C_{\omega\omega}^{\Omega_-} + O(\varepsilon^3)\ . \tag{2.79}$$

To unclutter the formulas, we introduce a lighter notation

$$C_{\omega\omega}^{\Omega_+} = \varepsilon\, v_+ + \frac{1}{2}\varepsilon^2 u_+ + O(\varepsilon^3)\ , \quad C_{\omega\omega}^{\Omega_-} = \varepsilon\, v_- + \frac{1}{2}\varepsilon^2 u_- + O(\varepsilon^3)\ . \tag{2.80}$$

In the following, we list all the couplings constrained by imposing associativity on the OPE. We start with the central charge, which exhibits a correction of order $O(\varepsilon^2)$

$$c = 9 - \frac{3}{4}\varepsilon^2\left(v_+^2 + v_-^2\right) + O(\varepsilon^3)\ . \tag{2.81}$$

Differently from the Spin(7) case, the central charge is not a free parameter of the $\mathrm{Od}^{\varepsilon}(3)$ algebra. However, we conjecture that c should be a free parameter of the full $s\mathcal{W}(3/2, 3/2, 3/2, 1)$ algebra: in particular, the equation (2.81) fails to capture the existence of another known special locus in the $s\mathcal{W}(3/2, 3/2, 3/2, 1)$ family, associated with $c = 1$ [100].

The other couplings are constrained as follows

$$\begin{aligned}
C_{\omega\omega}^{\mathcal{T}} &= -2 - \frac{1}{6}\varepsilon^2\left(v_+^2 + v_-^2\right) + O(\varepsilon^3)\ , & C_{\omega\Omega_+}^{\mathcal{T}} &= O(\varepsilon^3)\ ,\\
C_{\omega\Omega_+}^{\omega\omega} &= -\frac{1}{2}C_{\omega\omega}^{\Omega_-} + O(\varepsilon^3)\ , & C_{\omega\Omega_-}^{\mathcal{T}} &= O(\varepsilon^3)\ ,\\
C_{\omega\Omega_-}^{\omega\omega} &= \frac{1}{2}C_{\omega\omega}^{\Omega_+} + O(\varepsilon^3)\ , & C_{\Omega_\pm\Omega_\pm}^{\mathcal{T}} &= 4 + \frac{1}{3}\varepsilon^2(v_+^2 + v_-^2) + O(\varepsilon^3)\ ,\\
C_{\Omega_+\Omega_+}^{\Omega_+} &= -\frac{3}{2}C_{\omega\omega}^{\Omega_+} + O(\varepsilon^3)\ , & C_{\Omega_-\Omega_-}^{\Omega_+} &= -\frac{1}{2}C_{\omega\omega}^{\Omega_+} + O(\varepsilon^3)\ ,\\
C_{\Omega_+\Omega_+}^{\Omega_-} &= -\frac{1}{2}C_{\omega\omega}^{\Omega_-} + O(\varepsilon^3)\ , & C_{\Omega_-\Omega_-}^{\Omega_-} &= -\frac{3}{2}C_{\omega\omega}^{\Omega_-} + O(\varepsilon^3)\ ,\\
C_{\Omega_+\Omega_+}^{\omega\omega} &= -2 + \frac{1}{48}\varepsilon^2\left(-v_+^2 + 3v_-^2\right) + O(\varepsilon^3)\ , & C_{\Omega_-\Omega_-}^{\omega\omega} &= -2 + \frac{1}{48}\varepsilon^2\left(3v_+^2 - v_-^2\right) + O(\varepsilon^3)\ ,
\end{aligned}$$

[4]The couplings $\delta\delta C_{\Omega_+\Omega_-}^{\Omega_\pm\omega}$ are not constrained as well: however, they do not carry any physical meaning, as we argue in the following.

$$C^{\omega}_{\Omega_+\Omega_-} = 4 - \frac{1}{6}\varepsilon^2\left(v_+^2 + v_-^2\right) + O(\varepsilon^3)\ , \qquad C^{\mathcal{T}}_{\Omega_+\Omega_-} = O(\varepsilon^3)\ ,$$

$$C^{\omega\omega}_{\Omega_+\Omega_-} = -\frac{1}{12}\varepsilon^2 v_+ v_- + O(\varepsilon^3)\ , \qquad C^{\mathcal{T}\omega}_{\Omega_+\Omega_-} = 4 + \frac{13}{24}\varepsilon^2\left(v_+^2 + v_-^2\right) + O(\varepsilon^3)\ ,$$

$$C^{\Omega_+\omega}_{\Omega_+\Omega_-} = -2\varepsilon v_+ + \frac{1}{2}\varepsilon^2\delta\delta C^{\Omega_+\omega}_{\Omega_+\Omega_-} + O(\varepsilon^3)\ , \qquad C^{\Omega_-\omega}_{\Omega_+\Omega_-} = -2\varepsilon v_- + \frac{1}{2}\varepsilon^2\delta\delta C^{\Omega_-\omega}_{\Omega_+\Omega_-} + O(\varepsilon^3)\ . \quad (2.82)$$

The $\mathrm{Od}^{\varepsilon}(3)$ closes up to the ideals generated by two null fields $\mathfrak{N}_1$ and $\mathfrak{N}_2$, which can be expressed as expansions in powers of ε

$$\begin{aligned}\mathfrak{N}_1 = {} & N(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_-}) - \partial\mathcal{J}^{\Omega_+} + \frac{1}{12}\varepsilon\left[-2\,v_-\left(N(\mathcal{T}\mathcal{J}^{\omega}) + D\partial\mathcal{J}^{\omega}\right) - 3\,v_+\left(N(\mathcal{J}^{\omega}D\mathcal{J}^{\omega}) - \frac{2}{3}\partial\mathcal{T}\right)\right] \\ & + \frac{1}{96}\varepsilon^2\Big[-8\,u_-\left(N(\mathcal{T}\mathcal{J}^{\omega}) + D\partial\mathcal{J}^{\omega}\right) - 12\,u_+\left(N(\mathcal{J}^{\omega}D\mathcal{J}^{\omega}) - \frac{2}{3}\partial\mathcal{T}\right) \\ & \qquad + \left(v_-^2 - 3v_+^2\right)N(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_-}) + \left(3v_-^2 - v_+^2\right)\partial\mathcal{J}^{\Omega_+} + 8\,v_+v_-N(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_+})\Big] + O(\varepsilon^3)\ ,\end{aligned} \quad (2.83)$$

$$\begin{aligned}\mathfrak{N}_2 = {} & N(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_+}) + \partial\mathcal{J}^{\Omega_-} + \frac{1}{12}\varepsilon\left[-2\,v_+\left(N(\mathcal{T}\mathcal{J}^{\omega}) + D\partial\mathcal{J}^{\omega}\right) + 3\,v_-\left(N(\mathcal{J}^{\omega}D\mathcal{J}^{\omega}) - \frac{2}{3}\partial\mathcal{T}\right)\right] \\ & + \frac{1}{96}\varepsilon^2\Big[-8\,u_+\left(N(\mathcal{T}\mathcal{J}^{\omega}) + D\partial\mathcal{J}^{\omega}\right) + 12\,u_-\left(N(\mathcal{J}^{\omega}D\mathcal{J}^{\omega}) - \frac{2}{3}\partial\mathcal{T}\right) \\ & \qquad + \left(v_+^2 - 3v_-^2\right)N(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_+}) + \left(v_-^2 - 3v_+^2\right)\partial\mathcal{J}^{\Omega_-} + 8\,v_+v_-N(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_-})\Big] + O(\varepsilon^3)\ .\end{aligned} \quad (2.84)$$

The null fields $\mathfrak{N}_1$ and $\mathfrak{N}_2$ satisfy the conditions (1.38) and (1.39) up to terms of order $O(\varepsilon^3)$. In the limit $\varepsilon \to 0$, they reduce to the superfields

$$\mathfrak{N}_1 \overset{\varepsilon\to 0}{\longrightarrow} N(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_-}) - \partial\mathcal{J}^{\Omega_+}\ , \quad \mathfrak{N}_2 \overset{\varepsilon\to 0}{\longrightarrow} N(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_+}) + \partial\mathcal{J}^{\Omega_-}\ , \quad (2.85)$$

whose fermionic components f_1 and f_2 read exactly as the Od(3) null fields [40, 103]

$$if_1 = N(P^{\omega}P^{\Omega_-}) - \partial P^{\Omega_+}\ , \quad if_2 = N(P^{\omega}P^{\Omega_+}) + \partial P^{\Omega_-}\ . \quad (2.86)$$

Since the limits (2.85) can be rewritten up to terms of order $O(\varepsilon)$ as

$$\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_-}) = N(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_-}) - \partial\mathcal{J}^{\Omega_+} + O(\varepsilon)\ , \quad \mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_+}) = N(\mathcal{J}^{\omega}\mathcal{J}^{\Omega_+}) + \partial\mathcal{J}^{\Omega_-} + O(\varepsilon)\ , \quad (2.87)$$

we conclude that the parameters $\delta\delta C^{\Omega_\pm\omega}_{\Omega_+\Omega_-}$ are arbitrary and can be set to zero, since they are the coefficients of the null fields $\mathfrak{N}_1$ and $\mathfrak{N}_2$ written up to terms of order $O(\varepsilon)$. This completely fixes the entire $\mathrm{Od}^{\varepsilon}(3)$ up to the free parameters $v_\pm, u_\pm$.

$\mathcal{J}^1_{\rm cl.}$	$\mathcal{J}^2_{\rm cl.}$	$\mathcal{J}^U_{\rm cl.}$	$\mathcal{J}^N_{\rm cl.}$	$\mathcal{J}^V_{\rm cl.}$	c_U	c_N	c_V
$\mathcal{J}^\omega_{\rm cl.}$	$\mathcal{J}^\omega_{\rm cl.}$	-	$\mathcal{J}^{\Omega_+}_{\rm cl.}$, $\mathcal{J}^{\Omega_-}_{\rm cl.}$	$-\mathbb{1}$	-	$\frac{32}{\|\|\Omega\|\|^2}\, W_0^+$, $\frac{32}{\|\|\Omega\|\|^2}\, W_0^-$	6
$\mathcal{J}^\omega_{\rm cl.}$	$\mathcal{J}^{\Omega_\pm}_{\rm cl.}$	$\mathcal{J}^{\Omega_\mp}_{\rm cl.}$	$-\mathcal{J}^\omega_{\rm cl.}\mathcal{J}^\omega_{\rm cl.}$	-	± 3	$\pm 2\, W_0^\mp$	-
$\mathcal{J}^{\Omega_\pm}_{\rm cl.}$	$\mathcal{J}^{\Omega_\pm}_{\rm cl.}$	$-\mathcal{J}^\omega_{\rm cl.}\mathcal{J}^\omega_{\rm cl.}$	-	-	$\frac{\|\|\Omega\|\|^2}{4}$	-	-
$\mathcal{J}^{\Omega_+}_{\rm cl.}$	$\mathcal{J}^{\Omega_-}_{\rm cl.}$	-	-	$\mathcal{J}^\omega_{\rm cl.}$	-	-	$\|\|\Omega\|\|^2$

Table 2.4: *Currents appearing in the classical OPE $\mathcal{J}^1_{cl.} \times \mathcal{J}^2_{cl.}$, in the case of an* SU(3)*-structure. Geometrical information is encoded in the norm of the 3-forms $||\Omega||$ and in the scalar torsion classes W_0^+ and W_0^-. The norm $||\Omega||$ can be tuned to be equal to $||\Omega|| = 2\sqrt{2}$.*

Geometrical interpretation of the couplings. In Table 2.4 we report the data needed to build the classical OPEs. The details can be found in Appendix A.3.3. We normalize $||\Omega||^2 = 8$: the classical OPEs read

$$\mathcal{J}^\omega_{\rm cl.}(Z_1)\mathcal{J}^\omega_{\rm cl.}(Z_2) \sim 4\, W_0^+\, \frac{\theta_{12}}{Z_{12}}\mathcal{J}^{\Omega_+}_{\rm cl.} + 4\, W_0^-\, \frac{\theta_{12}}{Z_{12}}\mathcal{J}^{\Omega_-}_{\rm cl.} - 2\, \frac{\theta_{12}}{Z_{12}}\, \mathcal{T}_{\rm cl.} + \dots \ , \tag{2.88}$$

$$\mathcal{J}^\omega_{\rm cl.}(Z_1)\mathcal{J}^{\Omega_\pm}_{\rm cl.}(Z_2) \sim \mp 3\left(\frac{1}{Z_{12}}\, \mathcal{J}^{\Omega_\mp}_{\rm cl.} + \frac{\theta_{12}}{3\, Z_{12}} D\mathcal{J}^{\Omega_\mp}_{\rm cl.}\right) \mp 2\, W_0^\mp\, \frac{\theta_{12}}{Z_{12}}\mathcal{J}^\omega_{\rm cl.}\mathcal{J}^\omega_{\rm cl.} + \dots \ , \tag{2.89}$$

$$\mathcal{J}^{\Omega_\pm}_{\rm cl.}(Z_1)\mathcal{J}^{\Omega_\pm}_{\rm cl.}(Z_2) \sim -2\left(\frac{1}{Z_{12}}\mathcal{J}^\omega_{\rm cl.}\mathcal{J}^\omega_{\rm cl.} + \frac{\theta_{12}}{2\, Z_{12}} D\left(\mathcal{J}^\omega_{\rm cl.}\mathcal{J}^\omega_{\rm cl.}\right)\right) + \dots \ , \tag{2.90}$$

$$\mathcal{J}^{\Omega_+}_{\rm cl.}(Z_1)\mathcal{J}^{\Omega_-}_{\rm cl.}(Z_2) \sim 4\, \frac{\theta_{12}}{Z_{12}}\, \mathcal{T}_{\rm cl.}\mathcal{J}^\omega_{\rm cl.} + \dots \ . \tag{2.91}$$

We can identify the products of classical currents $\mathcal{J}^\omega_{\rm cl.}\mathcal{J}^\omega_{\rm cl.}$ and $\mathcal{T}_{\rm cl.}\mathcal{J}^\omega_{\rm cl.}$ as the classical limit respectively of the quasi-primary normal ordered operators $\mathcal{N}(\mathcal{J}^\omega\mathcal{J}^\omega)$ and $\mathcal{N}(\mathcal{T}\mathcal{J}^\omega)$. By comparing the couplings of the classical OPEs with the list of couplings (2.82), we immediately realize that the comparison is effective only up to order $O(\varepsilon)$. Then, we can extract the prediction

$$C^{\Omega_\pm}_{\omega\omega} = \varepsilon\, v_\pm + O(\varepsilon^2) = 4\, W_0^\pm + O(\ell_s) \ . \tag{2.92}$$

From the equation (2.92) we infer that W_0 is of order $O(\varepsilon)$, so the worldsheet theory described by the algebra $\mathrm{Od}^\varepsilon(3)$ must be associated with an *infinitesimal torsion*, ruled by the same parameter ε. This scenario can be realized as follows. Let us consider the σ-model (1.62) and slightly modify the string background, in particular the Kalb-Ramond field by adding a contribution $b(\varepsilon, X)$

$$S[X,\Lambda] = \int_\Sigma \frac{\mathrm{d}^{2|1}\zeta}{4\pi\alpha'}\, \left[(G_{ij}(X) + h_{ij}(\varepsilon,X)) + (B_{ij}(X) + b_{ij}(\varepsilon,X))\right] \bar\partial X^i DX^j + \dots \tag{2.93}$$

in such a way that B is a closed form and b is an analytic function of ε, equal to zero at the 0th order

$$b(\varepsilon, X) = \varepsilon\, \delta b(X) + \frac{1}{2}\varepsilon^2\, \delta\delta b(X) + O(\varepsilon^3)\ . \tag{2.94}$$

Then, the torsion reads

$$\begin{aligned} H &= \mathrm{d}\,(B + b(\varepsilon)) + \frac{\ell_s^2}{8\pi}\big(\mathrm{CS}_3(A) - \mathrm{CS}_3(\Theta)\big) + \dots \\ &= \varepsilon\, \mathrm{d}\, \delta b(X) + O(\varepsilon^2, \ell_s)\ . \end{aligned} \tag{2.95}$$

Recalling the decomposition into torsion classes, we can define a W_0 scalar torsion class for H and a w_0 scalar torsion class for the "small" torsion $\mathrm{d}\,\delta b(X)$: we obtain

$$W_0^{\pm} = \varepsilon\, w_0^{\pm} + O(\varepsilon^2, \ell_s)\ . \tag{2.96}$$

We can plug the equation (2.96) in the prediction (2.92), obtaining the final result

$$v_{\pm} = 4\, w_0^{\pm} + O(\ell_s) \tag{2.97}$$

This outcome is extremely interesting since all the couplings (2.82) only depend on the free parameters $v_{\pm}$, without any reference to the parameters $u_{\pm}$ (which probably appears at higher orders in ε). For example, the central charge of the special holonomy algebra is corrected by the infinitesimal torsion as follows

$$c = 9 - 12\, \varepsilon^2 ||w_0||^2 + O(\varepsilon^3, \ell_s)\ . \tag{2.98}$$

To conclude, we notice how the limit $H \to 0$ converts into $w_0^{\pm} \to 0$ in the case we are considering: in this torsionless limit, the algebra flows back to the Od(3) algebra, and the geometry to a special holonomy manifold.

3
Conclusions

In Chapters 1 and 2 we discussed the construction of $s\mathcal{W}$-algebras and their role in the study of (1,0) superstring compactifications on manifolds endowed with specific G-structures. In Chapter 1 we laid down the physical motivations for our problem and the required technology to describe and build $\mathcal{W}$-algebras and super $\mathcal{W}$-algebras. To define the classical limit of the additional symmetry generators appearing in the extensions of Virasoro, we studied the Noether currents of the $\mathcal{W}$-symmetries of a (1,0) non-linear σ-model, employed as the classical action describing the dynamics of (1,0) superstrings. Then, we computed the commutators of superconformal and $\mathcal{W}$-symmetry transformations, following the steps of [30] and adopting a new notation; moreover, we encoded the information of the commutators in a set of classical OPEs. In Chapter 2 we employed all the technology previously introduced to study the worldsheet $s\mathcal{W}$-algebras of (1,0) superstrings propagating on internal manifolds with chiral isometries, with Spin(7)-structures and SU(3)-structures, commenting on future developments involving G_2-structures as well. The element of novelty with respect to previous works in the literature is the introduction of a generic torsion H, a direction which, to our best knowledge, was started only recently in the work [99]. From a geometrical point of view, the torsion represents an obstacle for a manifold endowed with a given G-structure to be promoted to a G special holonomy manifold. In this Thesis, this obstruction is mirrored at the level of the worldsheet by a *deformation* of the special holonomy super $\mathcal{W}$-algebras induced by the scalar torsion classes, i.e. τ_0 for G_2-structures and W_0 for SU(3)-structures. We identified these deformations by comparing a fully quantum computation with a classical one: by constructing a specific $\mathcal{W}$-algebra from first principles, we find consistent OPEs at the quantum level, but the couplings depend on free parameters whose interpretation is not tied to any specific physical mean-

ing; by studying the classical OPEs between $\mathcal{W}$-symmetries' Noether currents, we cannot find the complete OPE (all the central extensions go missing, for instance), but we tie some couplings to the G-structures and to the torsion by computing explicitly the forms U, V, and N (see equations (1.97), (1.100), (1.99)). Putting together these two incomplete perspectives, we obtained super $\mathcal{W}$-algebras whose OPEs are consistent and whose couplings hold physical meaning. It would be interesting to explore the similarities between this strategy and the λ-bracket formalism [70, 104], at the basis of another approach to the interpolation between $\mathcal{W}$-symmetries and special holonomy $\mathcal{W}$-algebras. It would also be significant to study the role of 1-form (for instance, W_1 and τ_1) and 3-form (W_3, τ_3) torsion classes: given their index structure, they likely appear in the deformation of the $s\mathcal{W}$-algebras combined together through scalar products, for example, $||W_1||^2 = G_{ij}W_1^iW_2^j$. Hence, we conjecture that they should appear in loop corrections, at order $O(\ell_s)$ or higher. A perturbative computation could be performed to shed more light on this.

The main results of the first part of this Thesis are the classical OPEs for Spin(7), G_2, and SU(3)-structures, and the computation of the $\mathrm{Od}^{\varepsilon}(3)$ algebra, completed with a geometrical interpretation of its couplings. It would be interesting to complete the study of SU(3)-structures, fully constructing the $s\mathcal{W}(3/2, 3/2, 3/2, 1)$ algebra and finding an exact correspondence between the couplings of the quantum and classical OPEs.

The operative procedure described in Chapter 2 can be applied to any G-structure and any combination of generators. In particular, it can be a starting point to investigate, from a CFT point of view, more complicated geometrical constructions: for instance, adding a chiral current of weight $h_\sigma = 1/2$ to the generators of a $s\mathcal{W}(3/2, 3/2, 2)$ algebra with torsion should lead to a CFT description of the *almost contact structure* machinery on G_2-manifolds [102]. Furthermore, a solid connection of the OPE coefficients of the worldsheet algebras to the string background geometry and fields could help us improve our understanding of the relationship between string dualities and string compactifications, interpreting switching between different equivalent string backgrounds as performing an automorphism on the worldsheet algebra.

In recent years, a lot of progress has been made in the context of the AdS/CFT correspondence in order to tie together the features of the worldsheet theories with NS flux to those of the boundary

CFTs [105–109]. It would be interesting to study the possible holographic applications of the worldsheet super $\mathcal{W}$-algebras with torsion and to expand the results already available for WZW models to more general non-linear σ-models.

Finally, a possible direction to pursue in the future involves the gauge bundle, introduced for completeness in the (1,0) non-linear σ-model. Almost nothing is known concerning the structures on the gauge bundle and their relationship with the worldsheet theory. It would be important to study superconformal and $\mathcal{W}$-symmetry commutators applied to the superfield Λ^α, and explore the general constraints that the existence of a G-structure in the background imposes on the gauge bundle.

Part II

Conformal field theories at finite temperature

4

Operators, symmetries and the thermal geometry

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4.1 Broken symmetries at finite temperature

The study of the characteristics and the features of conformal field theories on the thermal geometry $\mathcal{M}_\beta = S^1_\beta \times \mathbb{R}^{d-1}$ starts from considering a generic CFT on the flat space $\mathbb{R}^d$. It is possible to associate with this theory a given operator content, represented by *local operators* $\{\mathcal{O}_i\}$. The observables of this theory can be constructed out of n-point correlation functions

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle \ . \tag{4.1}$$

It is well known that one-point, two-point and three-point functions of a generic CFT living on the flat spacetime $\mathbb{R}^d$ are completely fixed, up to overall constants, by the symmetries composing the global conformal group: translations, dilatations, rotations, and special conformal transformations. At the quantum level, these symmetries are associated with a *Ward identity*, representing a specific

differential equation that correlation functions must satisfy. In the flat spacetime $\mathbb{R}^d$, these Ward identities are easily solvable and they return known results [110–113]. In particular, it is well known that the Ward identities associated with translations and dilatations impose the following requirements on the correlation functions

$$\sum_{r=1}^{n} \langle \mathcal{O}_1(x_1) \dots \partial_\mu \mathcal{O}_r(x_r) \dots \mathcal{O}_n(x_n) \rangle = 0 \ , \tag{4.2}$$

$$\sum_{r=1}^{n} \langle \mathcal{O}_1(x_1) \dots (x_r^\mu \partial_{r,\mu} + \Delta_r) \, \mathcal{O}_r(x_r) \dots \mathcal{O}_n(x_n) \rangle = 0 \ . \tag{4.3}$$

When (4.2) and (4.3) are applied to one-point functions, they force those observables to be trivially equal to zero, unless the operator under study is the identity

$$\langle \mathbb{1} \rangle = 1 \ , \qquad \langle \mathcal{O}(x) \rangle = 0 \ . \tag{4.4}$$

It is natural to wonder about the fate of these constraints when we consider theories living not in the flat space but on a generic manifold $\mathcal{M}$. It is possible to provide a quantitative answer to this question by computing the Ward identities (4.2) and (4.3) in the new ambient manifold, and then solve them. In general, we expect the Ward identities to develop new terms compared to their structure in the flat space: we refer to such terms as *breaking terms*, and the respective Ward identity as a *broken* Ward identity. Physically, this states that the symmetries are not preserved in general on a given ambient manifold $\mathcal{M}$. For instance, we expect dilatations to be broken because of the insertion of a mass scale in the theory: a finite temperature setup represents an easy example, where the temperature $1/\beta = T$ is the only mass scale.

What is a "CFT at finite temperature"? The discussion above can naturally lead to a crucial question: if the features of a given ambient manifold $\mathcal{M}$, and in particular of the thermal geometry $\mathcal{M}_\beta = S^1_\beta \times \mathbb{R}^{d-1}$, can lead to the breaking of the symmetries composing the conformal group, what is a CFT at finite temperature (sometimes also referred to as "thermal CFT" in the literature)? To provide the reader with an answer, we refer to Figure 4.1. The Figure represents a schematic space of theories, each one associated with a value of the temperature T and of a generic coupling g. A generic point in the (g, T) plane is associated with a QFT, and the conformal group is broken. On the $T = 0$ axis, a blue point identifies a fixed point $g^\star$ of the RG flow, associated with a CFT at zero

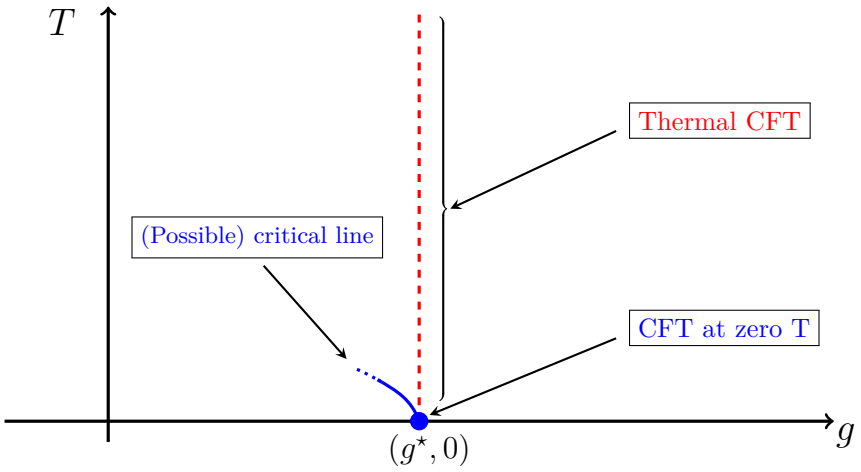


Figure 4.1: *A schematic representation of the (g, T) plane, where T is the temperature and g is a coupling constant. The zero temperature CFT sits on the critical point $(g^\star, 0)$, represented by a blue dot. The thermal CFTs sit on the vertical dashed red line above the critical point; each point of the line represents a thermal CFT at a different temperature. Theories sitting on the blue line represent critical points in the neighbourhood of the zero-temperature fixed point.*

temperature, i.e. a standard CFT on the Euclidean space $\mathbb{R}^d$. Turning on a finite temperature could trigger an RG flow involving the coupling g and it could identify a locus $g = g(T)$ originating from the CFT at zero temperature. This locus, represented with a blue line in Figure 4.1, is a *critical line* of theories which enjoy conformal symmetry. These theories are extremely interesting both from a theoretical and experimental point of view [114–116].

We identify as CFTs at finite temperature, or thermal CFTs, the theories sitting on the vertical, dashed red line. Since they sit outside of the critical line, these theories should be regarded as QFTs that do not enjoy the full conformal symmetry, as it will be proven in the following by computing the broken Ward identities. These theories recover conformality in the limit $T \to 0$, and they inherit important structures from the zero temperature CFT they are associated with: most importantly, they inherit its *local* properties, among which we find

- ◇ the operator content $\{\mathcal{O}_i\}$;
- ◇ the operators' conformal dimensions $\{\Delta_i\}$;
- ◇ the structure of the OPEs.

These theories are the subject of the Chapters 4 and 5.

Broken Ward identities. The procedure to derive the Ward identity (broken or unbroken) associated with a given classical symmetry starts from the correlation function

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_{\mathcal{M}} \ , \tag{4.5}$$

where $\langle \dots \rangle_{\mathcal{M}}$ indicates the correlation function on a generic manifold $\mathcal{M}$, and where it is understood that the operators satisfy specific boundary conditions intrinsically tied to the features of $\mathcal{M}$.[1] The derivation of zero temperature Ward identities, i.e. Ward identities in the ambient flat spacetime manifold, is well known in the literature (see for example [77,117]). We start from a theory described by a formal action S and we define an infinitesimal symmetry transformation $\phi'(x) = \phi(x) - i\omega_a \boldsymbol{G}_a \phi(x)$, where

$$i\omega_a \boldsymbol{G}_a \phi(x) = \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \phi - \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a} \ . \tag{4.6}$$

Here $\mathcal{F}$ is a function that parameterizes the variation of the field ϕ under the symmetry generated by $\boldsymbol{G}_a$. Here and in the following of this Section, the index a stands for a generic collection of indices tied to the representation in which the charge $\boldsymbol{G}_a$ sits. The correlation function (4.5) can be represented as a path integration over the fundamental fields

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_{\mathcal{M}} = \frac{1}{\mathcal{Z}} \int [\mathrm{d}\phi] \ \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) e^{-S[\phi]} \ ; \tag{4.7}$$

since the fundamental fields ϕ are integration variables, we can apply the transformation (4.6) as a functional change of variables

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_{\mathcal{M}} = \frac{1}{\mathcal{Z}} \int [\mathrm{d}\phi'] \ \left(\mathcal{O}_1(x_1) + \delta\mathcal{O}_1(x_1)\right) \dots \left(\mathcal{O}_n(x_n) + \delta\mathcal{O}_n(x_n)\right) e^{-S'[\phi']} \ , \tag{4.8}$$

where $\delta\mathcal{O}_i(x_i)$ denotes the effect of the transformation (4.6) on the i-th local operator. If the theory is assumed to be invariant under rigid symmetry transformations, the transformed action on a generic manifold $\mathcal{M}$ can be written as

$$S'[\phi'] = S[\phi] + \int_{\mathcal{M}} \mathrm{d}^d x \sqrt{G} \ \nabla_\mu \omega_a(x) J_a^\mu(x) \ , \tag{4.9}$$

where G is the determinant of the metric on the manifold $\mathcal{M}$, and J_a^μ are the current operators associated with the conservation of the symmetry generated by $\boldsymbol{G}_a$. In a Lagrangian theory, the

[1] As a clarifying example, we can consider operators on the thermal manifold $\mathcal{M}_\beta = S^1_\beta \times \mathbb{R}^{d-1}$: the geometrical features require bosonic operators to be periodic and fermionic operators to be antiperiodic over the thermal circle S^1_β.

current operators admit an explicit realization where the canonical stress-energy tensor $T^{\mu\nu}_{\text{can.}}$ appears

$$J^{\mu}_{a} = \frac{\delta x^{\mu}}{\delta \omega_a}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\partial_\nu \phi \frac{\delta x^{\nu}}{\delta \omega_a} + \sum_{\phi} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\frac{\delta \mathcal{F}}{\delta \omega_a} = T^{\mu\nu}_{\text{can.}}\frac{\delta x_\nu}{\delta \omega_a} + \sum_{\phi} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\frac{\delta \mathcal{F}}{\delta \omega_a} \ . \tag{4.10}$$

By focusing solely on the currents associated with the symmetries of the conformal group, we can reshape them in terms of the symmetric, traceless stress-energy tensor $T^{\mu\nu}$,[2] which is a well-defined operator in any local CFT, even if not Lagrangian. The explicit realizations of the canonical currents associated with the symmetries of the conformal group are listed in Table 4.2.

By inserting the transformed action in the expression (4.8) and expanding at first order in the infinitesimal parameters ω_a we get

$$\begin{aligned}\sum_{i=1}^{n} \langle \mathcal{O}_1(x_i) \dots \delta\mathcal{O}_i(x_i) \dots \mathcal{O}_n(x_n)\rangle_{\mathcal{M}} &= \langle \delta S \ \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n)\rangle_{\mathcal{M}} \\ &= \int_{\mathcal{M}} \mathrm{d}^d x \sqrt{G}\ \nabla_\mu \omega_a(x)\, \langle J^{\mu}_{a}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_{\mathcal{M}} \ .\end{aligned} \tag{4.11}$$

Recalling that $\delta\mathcal{O}(x) = -i\omega_a(x)\boldsymbol{G}_a\mathcal{O}(x)$ and taking the functional derivatives in $\omega_b(y)$ both on the left and on the right side of the equation (4.11), the expression reads

$$\begin{aligned}-i\sum_{i=1}^{n}\delta(x_i - y)\,\langle \mathcal{O}_1(x_1)\dots \boldsymbol{G}_a\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\rangle_{\mathcal{M}} = \\ = \int_{\mathcal{M}} \mathrm{d}^d x\sqrt{G}\ \partial_\mu\delta(x-y)\ \ \langle J^{\mu}_{a}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_{\mathcal{M}} \ .\end{aligned} \tag{4.12}$$

The derivative in the x coordinates can be replaced with a derivative in the y coordinates up to a sign. We, therefore, get the *unintegrated* broken Ward identities

$$i\sum_{i=1}^{n}\delta(x_i - y)\,\langle \mathcal{O}_1(x_1)\dots \boldsymbol{G}_a\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\rangle_{\mathcal{M}} = \nabla^{y}_{\mu}\,\langle J^{\mu}_{a}(y)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_{\mathcal{M}} \ . \tag{4.13}$$

It is crucial to notice that the current on the right-hand side of the equation (4.13) iteratively acts on all the operators in the correlation function. For example, if we consider the dilatation broken Ward identity for a one-point function on the thermal manifold $\mathcal{M}_\beta = S^1 \times \mathbb{R}^{d-1}$, the equation (4.13) specializes to

$$i\delta(x-y)\,\langle \boldsymbol{D}\,\mathcal{O}(x)\rangle_\beta = \frac{\partial}{\partial y^\mu}\left[(y-x)_\nu\,\langle T^{\mu\nu}(y)\mathcal{O}(x)\rangle_\beta\right] \ , \tag{4.14}$$

[2]The current associated with spacetime translations is $a_\nu T^{\mu\nu}_{\text{can}}$, where $T^{\mu\nu}_{\text{can}}$ is the canonical stress-energy tensor; however, as in the case of a CFT at zero temperature [77], it can be shown that the broken Ward identities are not altered by replacing the canonical stress-energy tensor with the traceless, symmetric one.

where here and in the following we use $\langle\,\cdot\,\rangle_\beta = \langle\,\cdot\,\rangle_{\mathcal{M}_\beta}$ to indicate thermal correlators, i.e. correlators computed on the thermal geometry. The kinematic term $(y-x)_\nu$ in the equation (4.14) can be easily understood considering the fact that we are integrating on the boundary of an infinite-dimensional ball, having as a center the position of the operator $\mathcal{O}$. The un-integrated broken Ward identities, even if very explicit, can be simplified if we specify the manifold $\mathcal{M}$; as anticipated we are mainly interested in the thermal case, therefore from now on we will focus on this specific choice.

Thermally broken Ward identities. In the thermal case $\mathcal{M}_\beta = S^\beta_1 \times \mathbb{R}^{d-1}$, after integrating both the right and the left-hand sides of (4.11), we get

$$i\sum_{i=1}^{n}\langle\mathcal{O}_1(x_1)\dots\boldsymbol{G}_a\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\rangle_\beta = \int \mathrm{d}^{d-1}y\int_0^\beta \mathrm{d}y_0\;\frac{\partial}{\partial y^\mu}\,\langle J^\mu_a(y)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_\beta\;. \tag{4.15}$$

By replacing in the equation (4.15) the thermal manifold $\mathcal{M}_\beta$ with the flat space $\mathbb{R}^d$, turning the volume integral into a boundary integral sets the right-hand side to zero (with the correct boundary conditions on the fundamental fields) and we recover the well known (un-broken) Ward identities

$$\sum_{i=1}^{n}\langle\mathcal{O}_1(x_1)\dots\boldsymbol{G}_a\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\rangle = 0\;. \tag{4.16}$$

However, on a generic manifold $\mathcal{M}$ the integration over the boundary $\partial\mathcal{M}$ does not return a trivial result, hence the right-hand side of the equation (4.15) can be interpreted as a *breaking term* correcting the Ward identity. In order to allow for explicit computations, we will parameterize the thermal circle S^1_β with a coordinate $\tau\in[0,\beta)$, i.e. we impose[3]

$$\tau \equiv \tau+\beta\;. \tag{4.17}$$

We have to further impose boundary conditions on the fields: periodic and anti-periodic boundary conditions are assigned to bosons and fermions respectively. Now we can evaluate the breaking term on the thermal manifold. We can rewrite the right-hand side of the equation (4.15) as the sum of two contributions: the first one is given by

$$\int_{\mathbb{R}^{d-1}}\mathrm{d}^{d-1}y\;\left\langle\left[J^0_a(\beta,\vec{y})-J^0_a(0,\vec{y})\right]\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle_\beta\;, \tag{4.18}$$

[3]In the following, the indices μ,ν,ρ will run over all the directions of the thermal manifold (from 0 to $d-1$), while the indices i,j,k will run over the spatial directions of the component $\mathbb{R}^{d-1}$ (from 1 to $d-1$). We work in Euclidean signature, so there is no difference between raised and lowered indices.

and the second one by the integral over the boundary parallel to S^1_β

$$\lim_{R\to\infty}\int_0^\beta \mathrm{d}\tau \int_{S_1^{d-2}} \mathrm{d}\Omega\ R^{d-2} n_i \left\langle J_a^i(\tau,R,\Omega)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle_\beta \ , \tag{4.19}$$

where n_i is the unit vector fixing the orientation of the boundary and $\mathrm{d}\Omega$ is the measure on the $(d-2)$-sphere of unitary radius. The term that appears in equation (4.19) can be discussed by invoking the *clustering property*

$$\lim_{R\to\infty}\left\langle J_a^i(\tau,R,\Omega)\mathcal{O}(x_1)\dots\mathcal{O}_n(x_n)\right\rangle_\beta = \lim_{R\to\infty}\left\langle J_a^i(\tau,R,\Omega)\right\rangle_\beta \left\langle\mathcal{O}(x_1)\dots\mathcal{O}_n(x_n)\right\rangle_\beta \ . \tag{4.20}$$

To study the clustered correlator we can use translational invariance on the thermal manifold: in fact, no point in the spatial component $\mathbb{R}^{d-1}$ nor on the thermal circle S^1_β should be regarded as preferred. Since the invariance under space-time translations is unbroken, the Ward identity (4.2) is not modified and every one-point function is still required to be a constant. Consequently, the term (4.20) is constant in R, τ, and Ω and therefore its integral over the thermal manifold is divergent unless the one-point function of the current is exactly equal to zero. Therefore, equation (4.20) encodes a possible source of *infrared divergences* in the correlation functions. The divergent term, if present, will be proportional to

$$\left\langle\mathcal{O}(x_1)\dots\mathcal{O}_n(x_n)\right\rangle_\beta \times \mathrm{Vol}\left[\mathcal{M}_\beta\right] \ . \tag{4.21}$$

This suggests that the breaking term appearing in the broken Ward identities should be renormalized by subtracting all the possible infrared divergences: the immediate consequence is that the boundary term (4.19) is either zero or infrared divergent and therefore it does not contribute to the breaking term. We find that this procedure is consistent and makes explicit some comments already present in literature [117].

Let us now take into account the other boundary term in equation (4.18). If we define the object

$$\Gamma_a^\beta(\vec{y}) = J_a^0(\beta,\vec{y}) - J_a^0(0,\vec{y}) \ , \tag{4.22}$$

we can rewrite the equation (4.18) as

$$\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}y \left\langle\Gamma_a^\beta(\vec{y})\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\right\rangle_\beta \ . \tag{4.23}$$

Symmetry	Infinitesimal transformations: $i\omega_a \boldsymbol{G}_a \mathcal{O}_i(x_i)$
Translations	$a^\mu \frac{\partial}{\partial x_i^\mu} \mathcal{O}_i(x_i)$
Dilatations	$b\left(x_i^\mu \frac{\partial}{\partial x_i^\mu} + \Delta_i\right) \mathcal{O}_i(x_i)$
Rotations	$c^{\mu\nu}\left(-x_{i,\mu}\frac{\partial}{\partial x_i^\nu} + x_{i,\nu}\frac{\partial}{\partial x_i^\mu} + i\boldsymbol{S}_{\mu\nu}\right) \mathcal{O}_i(x_i)$
Special conf. tr.	$d^\mu\left(-x_i^2\frac{\partial}{\partial x_i^\mu} + 2x_{i,\mu}x_i^\nu\frac{\partial}{\partial x_i^\nu} + 2x_{i,\mu}\Delta_i - 2ix_i^\nu \boldsymbol{S}_{\mu\nu}\right) \mathcal{O}_i(x_i)$

Table 4.1: *Symmetries of the global conformal group and their actions on the local operator $\mathcal{O}_i(x_i)$ living in an irreducible representation of the Lorentz group* [77]*. a^μ and d^μ are infinitesimal vectors, b is an infinitesimal scalar and $c^{\mu\nu}$ an antisymmetric tensor; $\boldsymbol{S}_{\mu\nu}$ represents the spin operator.*

From this definition it is already clear that translations and spatial rotations are unbroken because of the periodicity of their respective Noether currents $J^\mu(\beta, \vec{x}) = J^\mu(0, \vec{x})$. However, dilatations, rotations involving the time direction, and special conformal transformations are broken since their respective currents are not periodic. The final form of the broken Ward identities is then given by

$$i\sum_{i=1}^{n} \langle \mathcal{O}_1(x_1) \dots \boldsymbol{G}_a \mathcal{O}_i(x_i) \dots \ \mathcal{O}_n(x_n)\rangle_\beta = \int \mathrm{d}^{d-1}y \ \left\langle \Gamma_a^\beta(\vec{y}) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n)\right\rangle_\beta \ . \tag{4.24}$$

From a physical point of view, the equation (4.24) reveals that the simple structure of the zero-temperature Ward identities (see (4.2) and (4.3), for instance) is due to the specific geometrical features of the flat ambient manifold $\mathbb{R}^d$; moreover, symmetries can be unbroken on non-trivial manifolds (like the thermal manifold), but the symmetry current must comply to specific requirements (like periodicity, in the thermal case). It is interesting to notice that the thermally broken Ward identities (4.24) provide an example of dialogue between conformal field theories and geometry, the common trait between the two Parts of this Thesis.

4.1.1 Breaking conformality at finite temperature

By plugging in the general broken Ward identity (4.24) the specific breaking terms associated with the symmetries of the conformal group (whose associated operators are listed in Table 4.1), it is possible to produce specific constraints on the thermal correlation functions of a generic CFT at finite temperature. In Table 4.2 we list for every symmetry in the Conformal Group the current and the associated breaking term $\Gamma_a^\beta(x)$, which can be computed by plugging the explicit expressions for the currents $J_a^\mu(x)$ in equation (4.22). In the following, for the sake of notation clarity, we will

employ the leaner notation

$$\mathcal{O}_{1\cdots n}(\boldsymbol{x}) \equiv \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\ . \tag{4.25}$$

Symmetry	Currents: $\omega_a J^\mu_a$	Breaking terms: $\Gamma^\beta_a(0,\vec{x})$
Translations	$a_\nu T^{\mu\nu}$	0
Dilatations	$b\, x_\nu T^{\mu\nu}$	$\beta\, T^{00}(0,\vec{x})$
Spatial rot.	$-c^j_i x_j T^{\mu i} + c^j_i x_j T^{\mu i}$	0
Boosts	$-c^0_i \tau T^{\mu i} + c^0_i \tau T^{\mu i}$	$\beta\, T^{0i}(0,\vec{x})$
S.c.t. on S^1_β	$d\left(-x^2\delta_{0\nu} + 2\,\tau x_\nu\right) T^{\mu\nu}$	$\beta\left[(\beta-2\tau)\, T^{00}(0,\vec{x}) + 2(x_i-y_i)T^{0i}(0,\vec{x})\right]$
S.c.t. on $\mathbb{R}^{d-1}$	$d^i\left(-x^2\delta_{i\nu} + 2\, x_i x_\nu\right) T^{\mu\nu}$	$\beta\left[(\beta+2\tau)\, T^{0j}(0,\vec{x}) + 2(x_i-y_i)T^{ij}(0,\vec{x})\right]$

Table 4.2: *Explicit* $\Gamma^\beta_a(x)$ *terms for the symmetries of the global conformal group. The* $(\tau,\vec{y})$ *coordinates locates the operator to which the breaking term is applied. Notice that the thermal contributions explicitly break the manifest* $\mathrm{SO}(d)$ *invariance, but preserve the manifest* $\mathrm{SO}(d-1)$ *invariance.*

Translations. The Ward identity associated with translations in every direction is left unbroken by the introduction of a finite temperature. The constraints are simply

$$\sum_{r=1}^n \frac{\partial}{\partial\tau_r}\left\langle\mathcal{O}_{1\cdots n}(\boldsymbol{x})\right\rangle_\beta = 0\ , \qquad \sum_{r=1}^n \frac{\partial}{\partial x^i_r}\left\langle\mathcal{O}_{1\cdots n}(\boldsymbol{x})\right\rangle_\beta = 0\ . \tag{4.26}$$

Dilatations. The Ward identity associated with dilatations is broken by the introduction of a finite temperature. In fact, the theory ceases to be scale-less: the effect is captured by the broken Ward identity

$$\sum_{r=1}^n\left(\tau_r\frac{\partial}{\partial\tau_r} + x^i_r\frac{\partial}{\partial x^i_r} + \Delta_r\right)\left\langle\mathcal{O}_{1\cdots n}(\boldsymbol{x})\right\rangle_\beta = \beta\int \mathrm{d}^{d-1}y\,\left\langle T^{00}(0,\vec{y})\mathcal{O}_{1\cdots n}(\boldsymbol{x})\right\rangle_\beta\ , \tag{4.27}$$

where Δ_r indicates the classical conformal dimension of the r-th operator.

Rotations. From the list of breaking terms in Table 4.2, we can notice that only boosts are associated with a broken Ward identity. Spatial rotations are associated with

$$\sum_{r=1}^n\left(x_{r,i}\frac{\partial}{\partial x^j_r} - x_{r,j}\frac{\partial}{\partial x^i_r} - i\boldsymbol{S}^r_{ij}\right)\left\langle\mathcal{O}_{1\cdots n}(\boldsymbol{x})\right\rangle_\beta = 0\ , \tag{4.28}$$

where $\boldsymbol{S}^r_{\mu\nu}$ denotes the application of the spin operator to the r-th operator. Boosts are instead associated with the following broken Ward identity

$$\sum_{r=1}^{n}\left(\tau_r\frac{\partial}{\partial x_r^i}-x_{r,i}\frac{\partial}{\partial\tau_r}+i\boldsymbol{S}^r_{i0}\right)\langle\mathcal{O}_{1\cdots n}(\boldsymbol{x})\rangle_\beta=\beta\int \mathrm{d}^{d-1}y\,\left\langle T^{0i}(0,\vec{y})\mathcal{O}_{1\cdots n}(\boldsymbol{x})\right\rangle_\beta\ . \tag{4.29}$$

Special conformal transformations. We conclude the list with the broken Ward identities associated with the special conformal transformations. On the S^1_β component, the identity reads

$$\begin{aligned}&\sum_{r=1}^{n}\left[\left(\tau_r^2-x_r^ix_r^i\right)\frac{\partial}{\partial\tau_r}+2\,\tau_r x_r^i\frac{\partial}{\partial x_r^i}+2\,\tau_r\Delta_r+2i\,x_r^i\boldsymbol{S}^r_{i0}\right]\langle\mathcal{O}_{1\cdots n}(\boldsymbol{x})\rangle_\beta=\\&\quad=\sum_{r=1}^{n}\beta\left(\beta-2\tau_i\right)\int \mathrm{d}^{d-1}y\left(\left\langle T^{00}(0,\vec{y})\mathcal{O}_{1\cdots n}(\boldsymbol{x})\right\rangle_\beta+(x_i-y_i)\left\langle T^{0i}(0,\vec{y})\mathcal{O}_{1\cdots n}(\boldsymbol{x})\right\rangle_\beta\right)\ ,\end{aligned} \tag{4.30}$$

while on the $\mathbb{R}^{d-1}$ spacetime component we have

$$\begin{aligned}&\sum_{r=1}^{n}\left[\left(\tau_r^2+x_r^jx_r^j\right)\frac{\partial}{\partial x_r^i}-2\,x_{r,i}x_r^\mu\frac{\partial}{\partial x_r^\mu}-2\,x_{r,i}\Delta_r+2\,i\tau_r\boldsymbol{S}^r_{i0}+2i\,x_r^j\boldsymbol{S}^r_{ij}\right]\langle\mathcal{O}_{1\cdots n}(\boldsymbol{x})\rangle_\beta=\\&\quad=\sum_{r=1}^{n}\beta\left(\beta+2\tau_i\right)\int \mathrm{d}^{d-1}y\left(\left\langle T^{00}(0,\vec{y})\mathcal{O}_{1\cdots n}(\boldsymbol{x})\right\rangle_\beta+(x_i-y_i)\left\langle T^{0i}(0,\vec{y})\mathcal{O}_{1\cdots n}(\boldsymbol{x})\right\rangle_\beta\right)\ .\end{aligned} \tag{4.31}$$

4.1.2 Thermal Hamiltonian and momentum

Although we introduced broken Ward identities to constrain the thermal correlation functions, it is important to notice that they are *non-perturbative* results, hence they can provide non-perturbative insights on thermal CFTs. In particular, it can be observed that the newly found broken Ward identities (4.27) and (4.29) assume a very interesting physical interpretation when we introduce the *Hamiltonian* and the *spatial momentum*[4]

$$H=-\int_{\mathbb{R}^{d-1}}\mathrm{d}^{d-1}x\ \left(T^{00}(\tau,\vec{x})+E_0\right)\ ,\qquad P^i=i\int_{\mathbb{R}^{d-1}}\mathrm{d}^{d-1}x\ T^{0i}(\tau,\vec{x})\ , \tag{4.32}$$

where E_0 is the energy of a thermal bath. By using the translational invariance of one-point functions, equation (4.27) can be converted into an *operatorial* equation

$$\boldsymbol{H}=-\frac{1}{\beta}\boldsymbol{D}-E_0\ , \tag{4.33}$$

where $\boldsymbol{H}$ is the Hamiltonian operator of the theory at finite temperature. Recalling Figure 4.1, in general this theory should be thought of as a $(d-1)$-dimensional, interacting QFT at finite

[4]The minus sign is because we are using the convention for Euclidean theories, where $T^{00}_{\text{Lor.}}=i^2\,T^{00}_{\text{Euc.}}=-T^{00}_{\text{Euc.}}$. The same for the i in front of the definition of the momentum, $T^{0i}_{\text{Lor.}}=i\,T^{0i}_{\text{Euc.}}$.

temperature [45, 46]. This means that the energies of the $(d-1)$-dimensional, interacting theory can be written in terms of the conformal spectrum of the d-dimensional conformal field theory at zero temperature. In particular, we can choose a basis of $\boldsymbol{D}$ eigenstates $\mathcal{O}_i\,|0\rangle$ (where $\mathcal{O}_i$ is a scaling operator): since the equation (4.33) sets an equivalence between a zero temperature operator $\boldsymbol{D}$ (acting on zero temperature states built with the vacuum $|0\rangle$) and a finite temperature operator $\boldsymbol{H}$ (acting on finite temperature states built with a vacuum $|0\rangle_\beta$), we know that the eigenstates of $\boldsymbol{D}$ must be eigenstates of $\boldsymbol{H}$ as well, so for each zero temperature state $\mathcal{O}_i\,|0\rangle$ there must be a finite temperature state $|i\rangle_\beta$ equal to that. With these important considerations, it is possible to write down the matrix elements of the Hamiltonian as

$$\langle i|\boldsymbol{H}|j\rangle_\beta = -\left(\frac{\Delta_i}{\beta}+E_0\right)\delta_{ij}\ , \tag{4.34}$$

where $\{\Delta_i\}$ is the spectrum of the dilatation operator when applied to local operators (inserted in the origin). Those eigenvalues are zero temperature data and they correspond to the conformal dimensions of the operator content of the zero-temperature CFT. Equation (4.34) connects non-perturbatively zero temperature data to the Hamiltonian at finite temperature, i.e. on the vertical red dashed axis in Fig. 4.1. From the expression above one can read the energy spectrum of the theory at finite temperature

$$E_n = -\left(\frac{\Delta_n}{\beta}+E_0\right)\ , \tag{4.35}$$

where the Δ_n are ordered from the lowest to the highest one. The unitarity of the original CFT implies that the minimum value of Δ_n is given by 0: this corresponds to the identity operator. As already anticipated, E_0 is the energy of the thermal vacuum, which takes into account the Casimir energy of the system.[5] Equation (4.35) is perfectly in agreement with a trivial dimensional analysis: since the only scale in the theory is the temperature, i.e. β, then $E \propto 1/\beta$.[6] The non-triviality of equation (4.35) comes from the explicit coefficient in front of the temperature.

The energy spectrum of a thermal CFT (4.35) allows to indulge in extremely interesting speculations. In particular, it is possible to hint at a connection between thermal CFTs, associated with

[5] The Casimir energy of a thermal system can be understood as a contribution given by the one-point function of the stress-energy tensor: in fact, while at zero temperature $\langle T^{00}\rangle = 0$, we anticipated that at finite temperature $\langle T^{00}\rangle_\beta = -\left(1-\frac{1}{d}\right)\frac{b_T}{\beta^d}$, which returns the energy density $\mathcal{E}_0^{\text{gas}} = -\frac{d-1}{d}\frac{b_T}{\beta^{d-1}}$ [45, 118, 119].

[6] E_0 is an exception since $E_0 = \mathcal{E}_0^{\text{gas}} \times \text{Vol}\,[\mathcal{M}_\beta]$.

the geometry $\mathcal{M}_\beta = S^1_\beta \times \mathbb{R}^{d-1}$, and CFTs at finite volume, associated instead with the geometry $\mathcal{M}_\beta = \mathbb{R} \times S^{d-1}_R$. In two dimensions, Hamiltonian and momentum of finite volume theories are well known in the literature [120–122] and can be checked explicitly, by exploiting the conformal map between the plane $\mathbb{R}^2 \simeq \mathbb{C}$ and the cylinder $\mathbb{R} \times S^1_R$. Generalizations to higher dimensions are also provided in the literature together with interesting applications [45, 123, 124]. In this perspective, the equation (4.33) is curiously and intriguingly similar (the only differences concern the Euclidean/Lorentzian conventions) to the Hamiltonian at finite volume, hinting at a possible relation between finite temperature and finite volume systems. This fact is well understood in two dimensions, since it is a consequence of the modular invariance [77, 122, 125], but it is not completely clear in higher dimensions. As already proposed in the literature [126, 127], we speculate that there could be a possibility to *generalize the modular invariance* in higher dimensions. One way is to consider the conformal field theory on $S^\beta_1 \times S^R_{d-1}$ in which we choose the time direction to be defined by $0 \leq \tau \leq \beta$ and the spatial direction to be $0 \leq |\vec{x}| \leq R$. This geometry is related to $\mathbb{R}^d$ via conformal maps; the thermal CFT can be derived by sending $R \to \infty$ while the finite volume theory is reached in the limit $\beta \to \infty$. Therefore, the geometry $S^\beta_1 \times S^R_{d-1}$ contains information both on the thermal and on the finite volume theory.

An exact definition for the thermal mass. Similarly to the dilatation operator, the boost operator $\boldsymbol{L}_i$ is related to momentum operator via equation (4.29). Explicitly

$$\boldsymbol{P}_i = \frac{i}{\beta} \boldsymbol{L}_i \ , \tag{4.36}$$

where $\boldsymbol{P}_i$ is the spatial momentum operator of the theory at finite temperature. The physical interpretation of the equation (4.36) is that the boost operator generates the spatial translations. Interestingly, the equations (4.33) and (4.36) connect thermal observables with zero-temperature observables. This simple observation seems to suggest that some finite temperature quantities could be recovered by considering zero-temperature CFT data, through the, previously derived, broken Ward identities.[7] One very important finite temperature quantity is the *thermal mass* of the theory: this is not a physical mass, but it can be defined as the exponential factor that rules the behavior

[7] This observation can be justified by employing the real time formalism to describe finite temperature systems. For more details, we refer to Section 5.1.

of the two-point functions at spatial infinity, i.e. in the limit $|\vec{x}| \to \infty$

$$\langle \phi(\tau, \vec{x}) \phi(0,0) \rangle_\beta \simeq \langle \phi \rangle_\beta^2 + O\left(e^{-m_{\rm th}|\vec{x}|}\right) \ , \tag{4.37}$$

where $m_{\rm th}$ is the thermal mass. When $\vec{x} = (x, 0, 0, \ldots)$, we have

$$\langle \phi(0)\phi(\tau, x) \rangle_\beta = \left\langle \phi(0) e^{i\boldsymbol{H}\tau - \boldsymbol{P}_1 x} \phi(0) \right\rangle_\beta \ , \tag{4.38}$$

where $\boldsymbol{H}$ and $\boldsymbol{P}_1$ are the operators defined in the equation (4.32). Observe that the definition of the thermal mass is simply the first non-zero eigenvalue of $\boldsymbol{P}_1$, which is a well-defined differential operator. Similarly to the previous case involving the thermal Hamiltonian, thanks to the identity established by the equation (4.36) it is possible to choose a basis of scaling operators $|\mathcal{O}_{\Delta,J}\rangle = \mathcal{O}_{\Delta,J}|0\rangle$: then, the thermal mass is given by

$$m_{\rm th} = \frac{i}{\beta} \min_{\Delta,J} \langle \mathcal{O}_{\Delta,J} | \boldsymbol{L}_1 | \mathcal{O}_{\Delta,J} \rangle = \frac{i}{\beta} \min_{\Delta,J} \lim_{y \to \infty} |y|^{2\Delta} \langle \mathcal{O}_{\Delta,J}(y) \boldsymbol{S}_{01} \mathcal{O}_{\Delta,J}(0) \rangle \ , \tag{4.39}$$

where it is understood that the minimum excludes the zero eigenvalue and the correlator is computed at zero temperature. It should be noted how the thermal observable $m_{\rm th}$ is completely determined by a zero temperature computation, where the only crucial ingredient is the knowledge of the zero temperature CFT's spectrum.

Higher dimensional Cardy formula. As we already highlighted, the violation of dilatation invariance carries important physical information about the theory. The associated broken Ward identity provides a quantitative description of the thermal symmetry breaking, and we were able to extract physical predictions by reformulating the identity as the operatorial equation (4.33).

An alternative derivation of the dilatation broken Ward identity is available. Let us start from the simple dimensional analysis of a generic thermal correlator

$$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle_\beta = \frac{1}{\beta^{a_0}} \frac{1}{x_1^{a_1}} \cdots \frac{1}{x_n^{a_n}} f\left(\frac{x_1}{\beta}, \ldots, \frac{x_n}{\beta}\right) \ , \tag{4.40}$$

where f is an unknown function. Dimensional analysis fixes

$$\sum_{i=0}^{n} a_i = \sum_{j=1}^{n} \Delta_j \ . \tag{4.41}$$

The above expression for the correlator can be further simplified by using $\mathrm{SO}(d-1)$ invariance: however, this is not necessary for our purpose. We can simply observe that the correlation function is invariant under the transformation

$$\mathcal{O}_i \to \lambda^{-\Delta_i}\mathcal{O}_i \ , \qquad x_i \to \lambda\, x_i \ , \qquad \beta \to \lambda\, \beta \ . \tag{4.42}$$

The first two transformations are simply dilatations, while the last one is a non-physical scaling of the temperature.[8] Nevertheless, the fact that the dilatation invariance is compensated by a scaling of the temperature implies that

$$\left(\boldsymbol{D} + \beta\frac{\partial}{\partial\beta}\right)\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_\beta = 0 \ . \tag{4.43}$$

Observe that the dilatation breaking term in Table 4.2 can be derived from this equation. To see this, we can consider that a scaling of the temperature is equivalent to a scaling of the metric component G_{00}, which is "dual" to T^{00} in the sense that the stress-energy tensor is the source of G_{00} [56]. Let us consider the case of a one-point function for simplicity: then

$$\frac{\partial}{\partial\beta}\langle\mathcal{O}\rangle_\beta = -\frac{1}{\beta}\int \mathrm{d}^{d-1}x \int_0^\beta \mathrm{d}\tau\ \langle T^{00}(\tau,\vec{x})\mathcal{O}(0)\rangle_\beta = -\int \mathrm{d}^{d-1}x\ \langle T^{00}(0,\vec{x})\mathcal{O}(0)\rangle_\beta \ , \tag{4.44}$$

where the last step is justified by the fact that the stress-energy tensor component T^{00} generates the Hamiltonian, and therefore it commutes with it, proving that the integrand does not depend on τ. Finally, knowing that $\mathcal{O} \sim \beta^{-\Delta_\mathcal{O}}$ (see Section 4.2.1 for more details), we obtain

$$\Delta_\mathcal{O}\langle\mathcal{O}\rangle_\beta = \beta\int \mathrm{d}^{d-1}x\ \langle T^{00}(0,\vec{x})\mathcal{O}(0)\rangle_\beta \ . \tag{4.45}$$

This proves that the broken Ward identities (4.27) and (4.44) are equivalent, and it is possible to set an identity between the two breaking terms

$$\beta\frac{\partial}{\partial\beta}\left\langle\mathcal{O}(0)\right\rangle_\beta = -\beta\int \mathrm{d}^{d-1}x\left\langle T^{00}(0,\vec{x})\mathcal{O}(0)\right\rangle_\beta \ . \tag{4.46}$$

This formula already appeared in literature [56] and it has a very concrete simple interpretation. The temperature is related to the radius of the thermal circle, or equivalently to the value of the metric component G_{00}. Since we can think of G_{00} as the source of T^{00}, then a small change in the temperature is equivalent to the insertion of the T^{00} component of the stress-energy tensor. This

[8] This transformation is not physical because the temperature defines the theory in the theory space parameterized by the couplings and the temperature (cfr. Fig. 4.1).

physical interpretation is extremely useful, in particular considering the specific case in which the equation (4.46) specializes for $\mathcal{O} = T^{00}$

$$\beta \frac{\partial}{\partial \beta} \langle T^{00}(0) \rangle_\beta = -\beta \int \mathrm{d}^{d-1} x \left\langle T^{00}(x) T^{00}(0) \right\rangle_\beta \ . \tag{4.47}$$

It is natural to think (4.47) as the differential equation generalizing the *Cardy formula* in every spacetime dimension [56]. In two dimensions, the Cardy formula establishes a connection between the entropy S of a system and its central charge c, where the latter can also be seen as the coefficient of the $\langle TT \rangle$ two-point function. The formula (4.47) has a similar role: it connects the thermal two-point function of the stress-energy tensor to the thermal one-point function of the same operator. It can be shown that the thermal one-point function $\left\langle T^{00} \right\rangle_\beta$ can be used to compute the entropy of the system [45]: however, its explicit computation depends on an overall coefficient which is not known in general, apart from exceptional cases, leaving the generalization of the Cardy formula in higher dimensions an interesting problem to be addressed in the future [46, 59, 128–132]. In Chapter 5, we will argue that it is possible to set up a bootstrap problem to compute $\left\langle T^{00} \right\rangle_\beta$, along with other thermal one-point functions.

4.2 Constraining thermal correlation functions

After having explored some important dynamical non-perturbative consequences of the broken Ward identities in Section 4.1.2, we can now focus on the goal discussed at the beginning of this Chapter and explore the kinematic constraints that thermal correlation functions need to satisfy. This program is non-trivial to pursue, and the answers will inevitably be partial. However, the combination of the OPE with the broken Ward identities promises to shed more light on the hidden thermal structures.

4.2.1 Thermal one-point functions

The original, intriguing question posed at the beginning of this Part was about the consequences that a different ambient manifold $\mathcal{M}$ would have on the one-point functions of local operators. We know that in a flat space, such observables are trivial as a consequence of the unbroken translations and dilatations Ward identity. The thermal geometry is a perfect laboratory for this problem since it is easy enough to allow for an explicit derivation of the Ward identities' breaking terms, and at the same time, it allows for a wide class of local operators to develop a non-zero one-point function.

Remarkably, a general result exists [45]. If $\mathcal{O}^{\mu_1\cdots\mu_J}$ is a local operator sitting in a traceless, symmetric representation of the group SO(d) with even spin J, then it can develop the thermal one-point function

$$\langle \mathcal{O}^{\mu_1\cdots\mu_J}(x)\rangle_\beta = \frac{b_\mathcal{O}}{\beta^{\Delta_\mathcal{O}}}\left(e^{\mu_1}\cdots e^{\mu_J} - \text{traces}\right) \ , \tag{4.48}$$

where e^μ is a unit vector in the compactified direction of S^1_β. Since invariance under translations is preserved on the thermal manifold, the one-point function (4.48) is constant. The only dynamical and theory-dependent information is encoded in the coefficient $b_\mathcal{O}$. The authors of [45] got the result from a pure symmetry-driven argument. From the point of view adopted in this Thesis, symmetries' constraints are encoded in the broken and unbroken Ward identities and, even if these differential equations are in general complicated, they should be compatible with the general result (4.48). In the following, we will test this statement for scalar, vector and rank-two tensor operators.

Let us consider a generic thermal one-point function $\langle\mathcal{O}(x)\rangle_\beta$, where $\mathcal{O}(x)$ is a local operator. The unbroken Ward identities (4.26) are of fundamental importance: in the case of one-point functions, they specialize to

$$\frac{\partial}{\partial\tau}\langle\mathcal{O}(\tau,\vec{x})\rangle_\beta = 0 \ , \qquad \frac{\partial}{\partial x^i}\langle\mathcal{O}(\tau,\vec{x})\rangle_\beta = 0 \ . \tag{4.49}$$

Thanks to these equations, we conclude that a thermal one-point function (of a local operator) is a constant all over the thermal manifold. Once we acknowledge this, the other Ward identities can be significantly simplified. First of all, the unbroken Ward identity describing spatial rotations simply becomes

$$\langle \boldsymbol{S}_{ij}\mathcal{O}(x)\rangle_\beta = 0 \ ; \tag{4.50}$$

dilatations and boosts are encoded into simplified broken Ward identities

$$\Delta_\mathcal{O}\langle\mathcal{O}(x)\rangle_\beta = \beta\int \mathrm{d}^{d-1}y\ \langle T^{00}(0,\vec{y})\mathcal{O}(x)\rangle_\beta \ , \tag{4.51}$$

$$i\langle \boldsymbol{S}_{i0}\mathcal{O}(x)\rangle_\beta = \beta\int \mathrm{d}^{d-1}y\ \langle T^{0i}(0,\vec{y})\mathcal{O}(x)\rangle_\beta \ . \tag{4.52}$$

The broken Ward identity associated with special conformal transformations in the S^1_β direction

simplifies to

$$2\,\tau\Delta_{\mathcal{O}}\,\langle\mathcal{O}(\tau,\vec{x})\rangle_\beta + 2i\,x^i\,\langle \boldsymbol{S}_{i0}\mathcal{O}(\tau,\vec{x})\rangle_\beta = \beta\,(\beta-2\tau)\int \mathrm{d}^{d-1}y\,\big\langle T^{00}(0,\vec{y})\mathcal{O}(\tau,\vec{x})\big\rangle_\beta + \\ + \beta\int \mathrm{d}^{d-1}y\;(x_i-y_i)\,\big\langle T^{0i}(0,\vec{y})\mathcal{O}(\tau,\vec{x})\big\rangle_\beta\ , \quad (4.53)$$

while the one on the $\mathbb{R}^{d-1}$ component is

$$-\,2\,x_i\Delta_{\mathcal{O}}\,\langle\mathcal{O}(\tau,\vec{x})\rangle_\beta + 2i\,\tau\,\langle \boldsymbol{S}_{i0}\mathcal{O}(\tau,\vec{x})\rangle_\beta = \beta\,(\beta+2\tau)\int \mathrm{d}^{d-1}y\,\big\langle T^{0i}(0,\vec{y})\mathcal{O}(\tau,\vec{x})\big\rangle_\beta + \\ + \beta\int \mathrm{d}^{d-1}y\;(x_j-y_j)\,\big\langle T^{ij}(0,\vec{y})\mathcal{O}(\tau,\vec{x})\big\rangle_\beta\ . \quad (4.54)$$

To summarize, we conclude that the unbroken and broken Ward identities of the global conformal group constrain the thermal one-point functions to be kinematically constant. From the point of view of their tensorial structure, equation (4.50) prescribe them to be SO($d-1$)-invariant tensors. All the other features can be extracted in principle from the equations (4.51) and (4.52). As anticipated, we will now focus on the thermal one-point functions of scalar, vector, and rank-two tensor local operators.

Scalar operators. Scalar operators are particularly easy to study since $\langle \boldsymbol{S}_{0i}\mathcal{O}(x)\rangle_\beta = \langle \boldsymbol{S}_{ij}\mathcal{O}(x)\rangle_\beta = 0$ identically. The rotation broken Ward identity (4.52) simply becomes

$$\int \mathrm{d}^{d-1}y\;\big\langle T^{0i}(0,\vec{y})\mathcal{O}(x)\big\rangle_\beta = 0\ , \quad (4.55)$$

which is satisfied, since for SO($d-1$) invariance the integrand has to be odd in one spatial coordinate. The only meaningful constraint on the thermal one-point function of a scalar operator $\mathcal{O}(x)$ is given by the broken Ward identity (4.51)

$$\langle\mathcal{O}\rangle_\beta = \frac{\beta}{\Delta_{\mathcal{O}}}\int \mathrm{d}^{d-1}y\,\big\langle T^{00}(0,\vec{y})\mathcal{O}(x)\big\rangle_\beta\ . \quad (4.56)$$

Vector operators. When we consider a local vector operator $\mathcal{O}^\mu(x)$, the Ward identities associated with rotations are not trivial anymore. We need to evaluate explicitly the functions $\langle \boldsymbol{S}_{0i}\mathcal{O}^\mu(x)\rangle_\beta$ and $\langle \boldsymbol{S}_{ij}\mathcal{O}^\mu(x)\rangle_\beta$. First of all, we specialize the generators $\boldsymbol{S}_{ij}$ and $\boldsymbol{S}_{0i}$ to the vector representation of SO(d)

$$(\boldsymbol{S}_{ij})^\mu{}_\nu = i\left(\delta_i^\mu\delta_{j\nu}-\delta_j^\mu\delta_{i\nu}\right)\ , \qquad (\boldsymbol{S}_{0i})^\mu{}_\nu = i\,(\delta_0^\mu\delta_{i\nu}-\delta_i^\mu\delta_{0\nu})\ . \quad (4.57)$$

We can now explicitly write down the unbroken and broken Ward identities

$$\langle \boldsymbol{S}_{ij}\mathcal{O}^{\mu}(x)\rangle_{\beta} = i\left(\delta_i^{\mu}\langle\mathcal{O}_j(x)\rangle_{\beta} - \delta_j^{\mu}\langle\mathcal{O}_i(x)\rangle_{\beta}\right) = 0 \ , \tag{4.58}$$

$$\langle \boldsymbol{S}_{0i}\mathcal{O}^{\mu}(x)\rangle_{\beta} = i\left(\delta_0^{\mu}\langle\mathcal{O}_i(x)\rangle_{\beta} - \delta_i^{\mu}\langle\mathcal{O}_0(x)\rangle_{\beta}\right) = -i\beta\int \mathrm{d}^{d-1}y\,\big\langle T^{0i}(0,\vec{y})\mathcal{O}^{\mu}(x)\big\rangle_{\beta} \ . \tag{4.59}$$

Let us start by solving the equation (4.58). If we set $\mu = 0$, then the identity is trivially satisfied; if we set $\mu = k$, then we can multiply both sides by δ^i_k, obtaining

$$(d-2)\langle\mathcal{O}_j(x)\rangle_{\beta} = 0 \ . \tag{4.60}$$

This equation is always satisfied in $d = 2$, hence it doesn't provide any additional constraint; in $d > 2$, it returns

$$\langle\mathcal{O}_i(x)\rangle_{\beta} = 0 \tag{4.61}$$

which is also a solution of the unbroken Ward identity (4.58). This was to be expected for symmetry reasons. Moreover, it simplifies the equation (4.59)

$$\delta_i^{\mu}\big\langle\mathcal{O}^0(x)\big\rangle_{\beta} = \beta\int \mathrm{d}^{d-1}y\,\big\langle T^{0i}(0,\vec{y})\mathcal{O}^{\mu}(x)\big\rangle_{\beta} \ . \tag{4.62}$$

By setting $\mu = 0$ in the equation (4.62), we get an equation satisfied for symmetry reasons; instead, by setting $\mu = j$ and multiplying by δ^i_j we get an explicit expression for the 0th component

$$\big\langle\mathcal{O}^0(x)\big\rangle_{\beta} = \frac{\beta}{d-1}\int \mathrm{d}^{d-1}y\,\big\langle T^{0i}(0,\vec{y})\mathcal{O}^{i}(x)\big\rangle_{\beta} \ . \tag{4.63}$$

The dilatations broken Ward identity gives

$$\langle\mathcal{O}^{\mu}(x)\rangle_{\beta} = \frac{\beta}{\Delta_{\mathcal{O}}}\int \mathrm{d}^{d-1}y\,\big\langle T^{00}(0,\vec{y})\mathcal{O}^{\mu}(x)\big\rangle_{\beta} \ . \tag{4.64}$$

If we consider $\mu = i$ we get an identity for symmetry reasons, while if we consider $\mu = 0$, we get a second constraint on the 0th component

$$\big\langle\mathcal{O}^0(x)\big\rangle_{\beta} = \frac{\beta}{\Delta_{\mathcal{O}}}\int \mathrm{d}^{d-1}y\,\big\langle T^{00}(0,\vec{y})\mathcal{O}^{0}(x)\big\rangle_{\beta} \ . \tag{4.65}$$

In conclusion, the most general statement we can produce only by applying the unbroken and broken Ward identities is

$$\big\langle\mathcal{O}^i(x)\big\rangle_{\beta} = 0 \ , \tag{4.66}$$

$$\big\langle\mathcal{O}^0(x)\big\rangle_{\beta} = \frac{\beta}{\Delta_{\mathcal{O}}}\int \mathrm{d}^{d-1}y\,\big\langle T^{00}(0,\vec{y})\mathcal{O}^{0}(x)\big\rangle_{\beta} = \frac{\beta}{d-1}\int \mathrm{d}^{d-1}y\,\big\langle T^{0i}(0,\vec{y})\mathcal{O}^{i}(x)\big\rangle_{\beta} \ . \tag{4.67}$$

In Section 4.2.3, we will show $\big\langle\mathcal{O}^0(x)\big\rangle_{\beta} = 0$ if $\mathcal{O}^{\mu}$ lives in the OPE of two scalar local operators.

Rank-two tensor operators. We focus on a local 2-rank tensor operator $\mathcal{O}^{\mu\nu}(x)$. The unbroken Ward identity associated with spatial rotations can be written as

$$\langle \boldsymbol{S}_{ij}\mathcal{O}_{\mu\rho}(x)\rangle_\beta = i\left(\delta_{\mu i}\langle \mathcal{O}_{j\rho}(x)\rangle_\beta - \delta_{\mu j}\langle \mathcal{O}_{i\rho}(x)\rangle_\beta + \delta_{i\rho}\langle \mathcal{O}_{\mu j}(x)\rangle_\beta - \delta_{j\rho}\langle \mathcal{O}_{\mu i}(x)\rangle_\beta\right) = 0 \ . \tag{4.68}$$

If we set $\mu = \rho = 0$, we get a trivial identity, so we simply set

$$\langle \mathcal{O}_{00}(x)\rangle_\beta = a \ , \tag{4.69}$$

where a is a constant. By setting $\mu = k$ and $\rho = 0$, we get an equation similar to (4.60)

$$(d-2)\langle \mathcal{O}_{j0}(x)\rangle_\beta = 0 \ , \tag{4.70}$$

which again is trivially solved if $d = 2$, while in higher dimensions it produces the following constraint

$$\langle \mathcal{O}_{i0}(x)\rangle_\beta = 0 \ . \tag{4.71}$$

We can get $\langle \mathcal{O}_{0j}(x)\rangle_\beta = 0$ in a similar way. Finally, we want to constrain $\langle \mathcal{O}_{ij}(x)\rangle_\beta$ by choosing $\mu = k$ and $\rho = \ell$

$$\langle \boldsymbol{S}_{ij}\mathcal{O}_{k\ell}(x)\rangle_\beta = i\left(\delta_{ki}\langle \mathcal{O}_{j\ell}(x)\rangle_\beta - \delta_{kj}\langle \mathcal{O}_{i\ell}(x)\rangle_\beta + \delta_{i\ell}\langle \mathcal{O}_{kj}(x)\rangle_\beta - \delta_{j\ell}\langle \mathcal{O}_{ki}(x)\rangle_\beta\right) = 0 \ . \tag{4.72}$$

This equation is trivially satisfied in $d = 2$ since the only existing rotation is the one generated by $\boldsymbol{S}_{01}$. In higher dimension, the solution is given by the only $\mathrm{SO}(d-1)$-invariant tensor with two indices

$$\langle \mathcal{O}_{ij}(x)\rangle_\beta = b\,\delta_{ij} \ , \tag{4.73}$$

where b is a constant.

We conclude that the thermal one-point function $\langle \mathcal{O}^{\mu\nu}(x)\rangle_\beta$ is completely determined by the constants a and b. We can use the other Ward identities to set constraints on these numbers. In particular, by using the dilatation broken Ward identity, we get

$$a = \langle \mathcal{O}^{00}\rangle_\beta = \frac{\beta}{\Delta_\mathcal{O}}\int \mathrm{d}^{d-1}y\,\langle T^{00}(0,\vec{y})\mathcal{O}^{00}(x)\rangle_\beta \ , \tag{4.74}$$

$$b = \frac{\beta\delta_{ij}}{d-1}\langle \mathcal{O}^{ij}\rangle_\beta = \delta_{ij}\frac{\beta}{(d-1)\Delta_\mathcal{O}}\int \mathrm{d}^{d-1}y\,\langle T^{00}(0,\vec{y})\mathcal{O}^{ij}(x)\rangle \ . \tag{4.75}$$

The broken Ward identity associated with boosts establishes a relationship between the two constants: the identity reads

$$\delta_{\mu 0}\langle \mathcal{O}_{i\rho}(x)\rangle_\beta - \delta_{\mu i}\langle \mathcal{O}_{0\rho}(x)\rangle_\beta + \delta_{0\rho}\langle \mathcal{O}_{\mu i}(x)\rangle_\beta - \delta_{i\rho}\langle \mathcal{O}_{\mu 0}(x)\rangle_\beta = -\int \mathrm{d}^{d-1}y\,\langle T^{0i}(0,\vec{y})\mathcal{O}_{\mu\rho}(x)\rangle_\beta \ . \tag{4.76}$$

By choosing $\mu = 0$ and $\rho = i$, we extract the only non-trivial identity

$$b = a - \frac{1}{d-1} \int \mathrm{d}^{d-1} y \left\langle T^{0i}(0, \vec{y}) \mathcal{O}^{0i}(x) \right\rangle_\beta . \tag{4.77}$$

In Section 4.2.3, we will show that, if $\mathcal{O}^{\mu\nu}$ lives in the OPE of two scalar operators, a and b must comply to an additional constraint that sets the thermal one-point function to be traceless.

4.2.2 Thermal scalar two-point functions

The natural step following the analysis of the thermal one-point functions is to consider thermal two-point functions. Two-point functions, differently from one-point functions, retain an explicit dependence on the coordinates. In this case, simply applying the broken Ward identities return constraints that are too complicated to solve without additional conditions. Thus, the goal in this case will be to formulate a *bootstrap problem* for the two-point function of two *scalar* local operators $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$. Working with scalars, the set of broken and unbroken Ward identities significantly simplifies, motivating our choice. We employ the compact notation

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle_\beta \equiv f_\beta \left(\tau_1, \tau_2, \vec{x}_1, \vec{x}_2\right) \ . \tag{4.78}$$

The easiest constraint is given by the unbroken translations Ward identity

$$\frac{\partial}{\partial \tau_1} f_\beta \left(\tau_1, \tau_2, \vec{x}_1, \vec{x}_2\right) + \frac{\partial}{\partial \tau_2} f_\beta \left(\tau_1, \tau_2, \vec{x}_1, \vec{x}_2\right) = 0 \ , \tag{4.79}$$

$$\frac{\partial}{\partial x_1^i} f_\beta \left(\tau_1, \tau_2, \vec{x}_1, \vec{x}_2\right) + \frac{\partial}{\partial x_2^i} f_\beta \left(\tau_1, \tau_2, \vec{x}_1, \vec{x}_2\right) = 0 \ . \tag{4.80}$$

These equations are solved in full generality by

$$f_\beta \left(\tau_1, \tau_2, \vec{x}_1, \vec{x}_2\right) = f_\beta \left(\tau_1 - \tau_2, \vec{x}_1 - \vec{x}_2\right) = f_\beta \left(\tau_{12}, \vec{x}_{12}\right) \ . \tag{4.81}$$

We can now consider the constraints coming from applying the broken and unbroken rotations Ward identities. Since we are focusing on scalar local operators, the equations simplify considerably. The unbroken Ward identity (4.28) reads

$$\left(x_{12,i} \frac{\partial}{\partial x_{12}^j} - x_{12,j} \frac{\partial}{\partial x_{12}^i} \right) f_\beta \left(\tau_{12}, \vec{x}_{12}\right) = 0 \tag{4.82}$$

and it is solved by

$$f_\beta \left(\tau_{12}, \vec{x}_{12}\right) = f_\beta \left(\tau_{12}, |\vec{x}_{12}|\right) \ ; \tag{4.83}$$

The broken Ward identity (4.29) instead reads

$$\left(\tau_{12}\frac{\partial}{\partial x_{12}^i}-x_{12,i}\frac{\partial}{\partial\tau_{12}}\right)f_\beta\left(\tau_{12},|\vec{x}_{12}|\right)=\beta\int \mathrm{d}^{d-1}y\left\langle T^{0i}(0,\vec{y})\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\right\rangle_\beta\ , \tag{4.84}$$

which represents a new non-trivial constraint on the two-point function. Interestingly, the right-hand side of equation (4.84) depends only on the difference τ_{12}. This observation allows us to highlight that the distance between two operators on the thermal circle can be measured in two different ways, corresponding to the two arcs identified by the two points τ_1 and τ_2. The independence of the correlation function on the choice of the arc is a consequence of the KMS condition. When introducing a third operator, such as the stress-energy tensor in equation (4.84), the aforementioned choice generates two possibilities: one where the stress-energy tensor is between the two operators, i.e. the distance is measured on the arc including the third operator and another where it is not. However, the KMS condition ensures the equivalence of these two choices. The dilatations broken Ward identity provides an additional non-trivial constraint

$$\left(\tau_{12}\frac{\partial}{\partial\tau_{12}}+x_{12}^i\frac{\partial}{\partial x_{12}^i}+\Delta_1+\Delta_2\right)f_\beta\left(\tau_{12},|\vec{x}_{12}|\right)=\beta\int \mathrm{d}^{d-1}y\left\langle T^{00}(0,\vec{y})\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\right\rangle_\beta\ . \tag{4.85}$$

We complete the exposition of the complete set of constraints on the thermal scalar two-point functions coming from the Ward identities with the special conformal transformations. The equations (4.30) and (4.31) in this case specialize to complicated equations, however the breaking terms of the special conformal transformations can be written in terms of breaking terms of dilatations and rotations. Therefore, also considering $\beta-\tau=\tau$, we do not expect any additional constraints coming from special conformal transformations, and we will neglect them in what follows.

In addition to the novel constraints provided by the broken Ward identities, we recap the other important conditions already known and discussed in the literature [45, 53] to formulate a complete *bootstrap problem* for $f_\beta(\tau,|\vec{x}|)$.

KMS condition. The periodicity of correlations functions for finite temperature theories is known in the literature as Kubo-Martin-Schwinger (KMS) condition [45, 53–55], which reads

$$f_\beta(\tau,|\vec{x}|)=f_\beta(\tau+\beta,|\vec{x}|)\ . \tag{4.86}$$

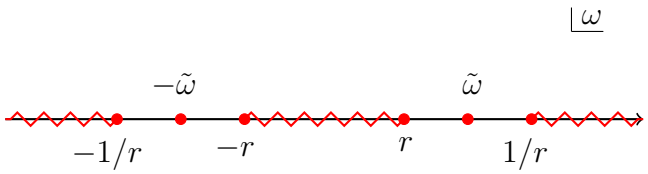


Figure 4.2: *Scheme of the analyticity condition of the two-point function* $f_\beta(\tau, |\vec{x}|) = f_\beta(r/\omega, r\omega)$ *in the* ω *plane. As shown in [45], the two-point function must be analytic everywhere outside the poles and the cuts drawn in the* ω *plane.*

Analyticity. Let us define in the complex plane the variables $\omega = \text{Arg}(\tau + i|\vec{x}|)$, and $r = \text{Abs}(\tau + i|\vec{x}|)$. In these coordinates, the two-point function reads

$$f_\beta(\tau, |\vec{x}|) = f_\beta\left(\frac{r}{\omega}, r\omega\right). \tag{4.87}$$

Then, the analyticity consistency condition states that the two-point function can only have cuts starting in $\omega = \pm r$ and $\omega = \pm r^{-1}$; moreover, it has poles in $\omega = 0, \pm\widetilde{\omega}$, where $\widetilde{\omega} = \sqrt{(\tau + i|\vec{x}|)/(\tau - i|\vec{x}|)}$. Outside of these poles and cuts, the two-point function must be an analytic function [45, 53]. This consistency condition is illustrated in Figure 4.2.

Regge limit. Let us define in the complex plane the variables $z = \tau + i|\vec{x}|$ and $\bar{z} = \tau - i|\vec{x}|$. With this set of coordinates, we can introduce the variables $\rho = \sqrt{z\bar{z}}$ and $\eta = \frac{z+\bar{z}}{2\sqrt{z\bar{z}}}$. In these coordinates, the two-point function reads $f_\beta(\rho, \eta)$. The Regge limit is

$$\lim_{|\eta|\to\infty} f_\beta(\rho, \eta) \ , \quad \rho \text{ fixed} \ . \tag{4.88}$$

The two-point function must be polynomially bounded in the Regge limit [45, 53].[9]

Large distance limit. As mentioned before, it is believed that the compactification on the thermal manifold produces a massive theory with a mass gap $m_{\text{th}} > 0$.[10] Assuming this *folk-theorem*, the two-point function $f_\beta(\tau, |\vec{x}|)$ approaches spatial infinity as [45, 133]

$$f_\beta(\tau, |\vec{x}|) \sim \langle \mathcal{O}_1 \rangle_\beta \langle \mathcal{O}_2 \rangle_\beta + O\left(e^{-m_{\text{th}}|\vec{x}|}\right) \ . \tag{4.89}$$

[9] Analiticity in the spin is a consequence of polynomial boundedness in the $|\eta| \to \infty$ limit, since this implies that the two-point function does not grow faster than $\eta^{J_{\text{Regge.}}}$. This is the analogue of the Regge limit at zero temperature.

[10] By dimensional analysis this mass has to be proportional to β^{-1}. Moreover, notice that, as already suggested in [45, 46], the existence of a thermal mass is not always guaranteed: in some cases (e.g. bosonic free theory) symmetries preserve the theory from a mass gap. This could also be the case if the theory spontaneously breaks a symmetry at finite temperature as in the case of [115, 116], however, we are not aware of any example in integer spacetime dimension. Nevertheless, in a generic interacting CFT at finite temperature the existence of such a mass gap is expected from the fact that thermal fluctuations make correlations between local operators negligible at large distances.

Zero temperature limit. Since a thermal two-point function can be defined on the thermal manifold with an arbitrary value of β, we expect $f_\beta(\tau, |\vec{x}|)$ to be a continuous function in β; moreover, the physical expectation is for it to be smooth and to have a well-defined limit for $\beta \to \infty$

$$\lim_{\beta\to\infty} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle_\beta = \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \ . \tag{4.90}$$

Consistency with the OPE. It is a well known fact that the radius of convergence of the OPE between two local operators is infinite in a CFT: this fact can be proven thanks to the state-operator correspondence and the invariance under dilatations. On a generic ambient manifold, locality ensures that the OPE does not change; however, the radius of convergence is sensitive to the global properties of the manifold and to the mass scales of the theories, and it is unknown for a generic QFT. A crucial feature of the thermal geometry is the exact knowledge of the OPE convergence radius, which is equal to β, the length of the thermal circle [45]. This radius makes sure that two local scalar operators $\mathcal{O}_1$ and $\mathcal{O}_2$ on the thermal geometry lie in a sphere whose interior is flat. If the two operators have conformal dimensions Δ_1 and Δ_2 respectively, their OPE reads

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \sim \sum_{\mathcal{O}\in\mathcal{O}_1\times\mathcal{O}_2} \frac{f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}}{c_\mathcal{O}} \left(\tau^2+|\vec{x}|^2\right)^{\frac{1}{2}(\Delta_\mathcal{O}-\Delta_1-\Delta_2-J)} x_{\mu_1}\dots x_{\mu_J}\mathcal{O}^{\mu_1\dots\mu_J}(x_2) \ , \tag{4.91}$$

where $x^\mu = x_1^\mu - x_2^\mu$, $\Delta_\mathcal{O}$ is the conformal dimensions of $\mathcal{O}$, $f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}$ is the three-point function OPE constant at zero temperature, and $c_\mathcal{O}$ is the normalization coefficient appearing in the two-point function at zero temperature

$$\langle \mathcal{O}^{\mu_1\dots\mu_J}(x)\mathcal{O}_{\nu_1\dots\nu_J}(0)\rangle = c_\mathcal{O} \frac{I^{(\mu_1\dots\mu_J)}_{(\nu_1\dots\nu_J)}}{|x|^{2\Delta_\mathcal{O}}} \ , \qquad I^\mu_\nu = \delta^\mu_\nu - \frac{2x_\nu x^\mu}{x^2} \ . \tag{4.92}$$

The two-point function $f_\beta(\tau, |\vec{x}|)$ is consistent with the OPE if, for $|\vec{x}|^2+\tau^2 < \beta^2$,

$$f_\beta(\tau,|\vec{x}|) = \sum_{\mathcal{O}\in\mathcal{O}_1\times\mathcal{O}_2} \frac{f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}}{c_\mathcal{O}} \left(\tau^2+|\vec{x}|^2\right)^{\frac{1}{2}(\Delta_\mathcal{O}-\Delta_1-\Delta_2-J)} x_{\mu_1}\dots x_{\mu_J} \langle \mathcal{O}^{\mu_1\dots\mu_J}\rangle_\beta \ . \tag{4.93}$$

where, as already noticed, all the conformal dimensions (and the anomalous dimensions) are inherited from zero temperature [53]. Working in the OPE regime, it is possible to apply the constraints derived from the broken and unbroken Ward identities to the expression (4.93).

A bootstrap problem for $f(\tau, |\vec{x}|)$. The bootstrap problem for the thermal scalar two-point function $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle_\beta$ consists of finding a function of the coordinates x_1, x_2 and the inverse

of the temperature β such that it satisfies all the conditions previously listed in this Section. First of all, the unbroken Ward identities set the dependence on the coordinates to be like in equation (4.83). Then, the function:

1. must be periodic over the thermal circle (see equation (4.86));
2. in the ω plane, must be analytic everywhere apart from the poles and cuts illustrated in Figure 4.2;
3. must be polynomially bounded in the Regge limit; the zero temperature limit must return the same two-point function, but computed on the flat space; the large distance limit must return a clustered structure like in equation (4.89);
4. must be compatible with the OPE expansion in the OPE regime, i.e. $\tau^2 + \vec{x}^2 < \beta^2$;
5. must be symmetric with respect to $\tau \to -\tau$. Following [45], if the CFT at zero temperature is not parity invariant under the transformation $\tau \to -\tau$, then this transformation has to be performed together with a reflection of one (or more) direction in the spatial component $\mathbb{R}^{d-1}$;[11]
6. must satisfy the constraints (4.84) and (4.85), imposed by the broken symmetries of the global conformal group.

If the bootstrap problem set above returns a unique solution or not is an unsolved problem; in particular, the minimal amount of conditions to impose to uniquely determine the two-point functions is not known. Let us suppose the existence of two different solutions f^1 and f^2 to the problem. Imagine first that the two solutions differ by the contribution of a finite number of local operators in the OPE regime: then, the large distance limit of the difference $\Delta f = f_1 - f_2$ is[12]

$$\lim_{r\to\infty} |\Delta f(\tau, r)| = \infty \ , \tag{4.94}$$

and therefore if f_1 satisfies the clustering property, expressed in equation (4.89), then f_2 does not satisfy it (and *vice-versa*), so only one among f_1 and f_2 is a solution to the bootstrap problem. If

[11] Both realizations of the parity invariance enhance the unbroken Lorentz symmetry subgroup preserved at finite temperature from $\mathrm{SO}(d-1)$ to $\mathrm{O}(d-1)$.

[12] The limit holds only for even spin operators; however it is sufficient because the one point function of an odd spin operator is zero and their contribution to the two-point function thereof.

f_1 and f_2 differ for an overall factor then the limit (4.90) differs for the same overall factor, which can be identified with the normalization of the two-point function at zero temperature. If we fix this normalization before solving the bootstrap problem, then it fixes the residue of the poles of the two-point function at finite temperature. Two solutions f_1 and f_2 that do not differ by an overall factor must have the same poles with the same residues since the poles are given by the zero temperature contribution and the limit $\beta \to \infty$ defines the coefficients in front.

It is not trivial to find a way to discriminate between two solutions complying with all the conditions above but differing for one or more branch cuts. This problem was already pointed out in [53] where the authors proved unicity assuming some specific conditions,[13] by using the Regge boundedness of the thermal correlator and the subtracted thermal dispersion relations.

4.2.3 Thermal free scalar theory

Although in the previous Section it was argued that finding a unique solution to the bootstrap problem is a highly non-trivial task, in this Section we argue that it is possible to accomplish this goal for the theory of a *free scalar φ in four dimensions* and at finite temperature. Although the thermal two-point function in this theory is a well known result [42–44], the novelty is represented by the possibility to derive it from *first principles*, all collected in the bootstrap problem previously described. As an anticipation, the peculiarity of this theory relies in the simplicity of the OPE expansion between two fundamental scalars.

Combining broken Ward identities and OPE. The constraints set up by the broken symmetries of the global conformal group are difficult to solve by themselves since the structure of the thermal breaking term requires the knowledge of a thermal $(n+1)$-point function to study a thermal n-point function. However, in this Section we show how the breaking term splits into lower-point functions in the OPE regime, allowing for an explicit derivation of the solution to the bootstrap problem and additional constraints on the thermal one-point functions. We are going to focus on (broken) dilatations and rotations.

[13] The necessary condition to argue for the unicity proposed in [53] is $\mathrm{Disc}[\Delta f(z,\bar{z})] = 0$, $0 \leq \bar{z} = \tau - ir \leq 1$ and $1 \leq z = \tau + ir$.

Let us start by focusing on the new constraint coming from the dilatations broken Ward identity. We study the thermal scalar two-point function

$$f_\beta(\tau,|\vec{x}|) = \langle\varphi(x)\varphi(0)\rangle_\beta \ , \tag{4.95}$$

where both scalar operators now have conformal dimension Δ_φ. By introducing the radial coordinate $r = |\vec{x}|$, the dilatations broken Ward identity (4.85) reads

$$\left(\tau\frac{\partial}{\partial\tau} + r\frac{\partial}{\partial r} + 2\Delta_\varphi\right) f_\beta(\tau,r) = \beta\int \mathrm{d}^{d-1}y \left\langle T^{00}(0,\vec{y})\varphi(x)\varphi(0)\right\rangle_\beta \ . \tag{4.96}$$

If we consider the local operators φ to be in the OPE regime, the broken Ward identity can be rewritten as

$$\begin{aligned}&\left(\tau\frac{\partial}{\partial\tau} + r\frac{\partial}{\partial r} + 2\Delta_\varphi\right) f_\beta(\tau,r) = \\ &\quad = \beta\sum_{\mathcal{O}\in\varphi\times\varphi} \frac{f_{\varphi\varphi\mathcal{O}}}{c_\mathcal{O}} \left(\tau^2+r^2\right)^{\frac{1}{2}(\Delta_\mathcal{O}-2\Delta_\varphi-J)} x_{\mu_1}\dots x_{\mu_J}\int \mathrm{d}^{d-1}y \left\langle T^{00}(0,\vec{y})\mathcal{O}^{\mu_1\dots\mu_J}(0)\right\rangle_\beta \ .\end{aligned} \tag{4.97}$$

The right-hand side can be rewritten by plugging the equation (4.56), which is the dilatations broken Ward identity applied to thermal one-point functions

$$\begin{aligned}&\left(\tau\frac{\partial}{\partial\tau} + r\frac{\partial}{\partial r} + 2\Delta_\varphi\right) f_\beta(\tau,r) = \\ &\qquad = \sum_{\mathcal{O}\in\varphi\times\varphi} \frac{f_{\varphi\varphi\mathcal{O}}\Delta_\mathcal{O}}{c_\mathcal{O}} \left(\tau^2+r^2\right)^{\frac{1}{2}(\Delta_\mathcal{O}-2\Delta_\varphi-J)} x_{\mu_1}\dots x_{\mu_J}\left\langle\mathcal{O}^{\mu_1\dots\mu_J}\right\rangle_\beta \ .\end{aligned} \tag{4.98}$$

The left-hand side of the equation (4.98) respects the OPE regime as well, so it can be expanded in a similar fashion. We get

$$\sum_{\mathcal{O}\in\varphi\times\varphi}\left(\tau\frac{\partial}{\partial\tau} + r\frac{\partial}{\partial r} + 2\Delta_\varphi - \Delta_\mathcal{O}\right) f_\beta^\mathcal{O}(\tau,r) = 0 \ , \tag{4.99}$$

where

$$f_\beta^\mathcal{O}(\tau,r) = \frac{f_{\varphi\varphi\mathcal{O}}}{c_\mathcal{O}} \left(\tau^2+r^2\right)^{\frac{1}{2}(\Delta_\mathcal{O}-2\Delta_\varphi-J)} x_{\mu_1}\dots x_{\mu_J}\left\langle\mathcal{O}^{\mu_1\dots\mu_J}\right\rangle_\beta \ . \tag{4.100}$$

Since $\langle\mathcal{O}^{\mu_1\dots\mu_J}\rangle_\beta \propto \beta^{-\Delta_\mathcal{O}}$ by dimensional analysis, we can interpret the sum over the operators in the OPE appearing in the equation (4.100) as a sum over powers of β.[14] Hence, each term of the

[14] This requires additional care in the case of a degenerate spectrum. In this case, we can either distinguish the two operators from the spin or from other global symmetries; in the first case, the contribution of any operator is always captured by equation (4.108) by the linearity of the differential equation (4.100), in the second case the one-point function is zero because the operator is not in the trivial representation of the global symmetry's group.

series constitutes an independent differential equation

$$\left(\tau\frac{\partial}{\partial\tau}+r\frac{\partial}{\partial r}+2\Delta_\varphi\right)f_\beta^{\mathcal{O}}(r,\tau)=\Delta_{\mathcal{O}}f_\beta^{\mathcal{O}}\ . \tag{4.101}$$

If we consider the identity operator $\mathbb{1}\in\mathcal{O}_1\times\mathcal{O}_2$, then $\Delta_{\mathbb{1}}=0$ and the equation (4.101) returns the zero temperature result. For all the other cases we can also write down the boosts broken Ward identity

$$\left(\tau r\frac{\partial}{\partial r}-r^2\frac{\partial}{\partial\tau}\right)f_\beta(\tau,r)=x_i\int \mathrm{d}^{d-1}y\left\langle T^{0i}(0,\vec{y})\varphi(x)\varphi(0)\right\rangle_\beta\ . \tag{4.102}$$

This equation can be expanded in OPE as well; we expand the right-hand side first

$$\begin{aligned}&\left(\tau r\frac{\partial}{\partial r}-r^2\frac{\partial}{\partial\tau}\right)f_\beta(\tau,r)=\\&=\beta\sum_{\mathcal{O}\in\varphi\times\varphi}\frac{f_{\varphi\varphi\mathcal{O}}}{c_{\mathcal{O}}}\left(\tau^2+r^2\right)^{\frac{1}{2}(\Delta_{\mathcal{O}}-2\Delta_\varphi-J)}x_{\mu_1}\dots x_{\mu_J}x_i\int \mathrm{d}^{d-1}y\left\langle T^{0i}(0,\vec{y})\mathcal{O}^{\mu_1\dots\mu_J}(0)\right\rangle_\beta\ ,\end{aligned} \tag{4.103}$$

and then we apply the broken Ward identity (4.52) to rewrite the integral as

$$\begin{aligned}&\left(\tau r\frac{\partial}{\partial r}-r^2\frac{\partial}{\partial\tau}\right)f_\beta(\tau,r)=\\&\qquad=i\beta\sum_{\mathcal{O}\in\varphi\times\varphi}\frac{f_{\varphi\varphi\mathcal{O}}}{c_{\mathcal{O}}}\left(\tau^2+r^2\right)^{\frac{1}{2}(\Delta_{\mathcal{O}}-2\Delta_\varphi-J)}x_{\mu_1}\dots x_{\mu_J}x_i\left\langle \boldsymbol{S}_{i0}\mathcal{O}^{\mu_1\dots\mu_J}(0)\right\rangle_\beta\ .\end{aligned} \tag{4.104}$$

The equation (4.101) together with the equation (4.104) can be interpreted as a system of partial differential equations. In general, we have that the first differential equation, i.e. (4.101), fixes

$$f_\beta^{\mathcal{O}}(\tau,r)\propto\left(r^2+\tau^2\right)^{\frac{1}{2}(\Delta_{\mathcal{O}}-2\Delta_\varphi-J)}\tau^J g_{\mathcal{O}}\left(\frac{r}{\tau}\right)\ , \tag{4.105}$$

where the function $g_{\mathcal{O}}(r/\tau)$ remains unfixed. The next step is to plug this solution in the rotation broken Ward identities; order by order in the OPE expansion equation (4.104) reads

$$\frac{r}{\tau^2}\tau^J\left[-J\,r\,\tau\,g_{\mathcal{O}}\left(\frac{r}{\tau}\right)+\left(r^2+\tau^2\right)g'_{\mathcal{O}}\left(\frac{r}{\tau}\right)\right]=x_{\mu_1}\dots x_{\mu_J}x_i\left\langle \boldsymbol{S}_{i0}\mathcal{O}^{\mu_1\dots\mu_J}\right\rangle_\beta\ . \tag{4.106}$$

For scalars, i.e. $J=0$, we get $g_{\mathcal{O}}(r/\tau)=\text{const.}$, meaning that the one-point function of a scalar $\langle\mathcal{O}\rangle_\beta$ does not have any kinematical structure, as expected. For vectors, i.e. $J=1$, by imposing the reality condition on the solution,[15] we get $\langle\mathcal{O}^\mu\rangle=0$ and only a term proportional to an odd power of τ can contribute to the OPE. However, this contribution is zero because of the symmetry

[15]The differential equation is also solved by a second function proportional to a hypergeometric function: however, this function is not real for all the possible values of r and τ.

$\tau \to -\tau$, which forbids odd functions of τ to appear.[16] For 2-rank tensors, i.e. $J = 2$, we can recall the results of Section (4.2.1) to see that the solution to the two differential equations is given by

$$f_\beta^{\mathcal{O}}(\tau, r) \propto \left(\tau^2 + r^2\right)^{\frac{1}{2}(\Delta_{\mathcal{O}} - 2\Delta_\varphi)} C_2^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2 + r^2}}\right) , \tag{4.107}$$

where $C_J^{(\nu)}(x)$ is a *Gegenbauer polynomial*. This result can be extended to all values of J by observing that the first differential equation (4.101) is solved by the Ansatz

$$f_\beta^{\mathcal{O}}(\tau, r) = \frac{\mathcal{C}_{J,\nu}^{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} \left(\tau^2 + r^2\right)^{\frac{1}{2}(\Delta_{\mathcal{O}} - 2\Delta_\varphi)} C_J^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2 + r^2}}\right) , \tag{4.108}$$

where the $\beta^{-\Delta_{\mathcal{O}}}$ factor is introduced via dimensional analysis and $\mathcal{C}_{J,\nu}^{\mathcal{O}}$ is an overall coefficient to be fixed. We can fix it by imposing consistency of the thermal two-point function with the OPE. By comparing the expressions (4.100) and (4.108) we identify

$$\frac{\mathcal{C}_{J,\nu}^{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} C_J^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2 + r^2}}\right) = \frac{f_{\varphi\varphi\mathcal{O}}}{c_{\mathcal{O}}} \left(\tau^2 + r^2\right)^{-\frac{J}{2}} x_{\mu_1} \dots x_{\mu_J} \left\langle \mathcal{O}^{\mu_1 \dots \mu_J} \right\rangle_\beta . \tag{4.109}$$

If $\nu = \frac{d-2}{2}$, the following identity holds [45]

$$C_J^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2 + r^2}}\right) = \frac{2^J (\nu)_J}{J!} \left(\tau^2 + r^2\right)^{-\frac{J}{2}} x_{\mu_1} \dots x_{\mu_J} \left(e^{\mu_1} \dots e^{\mu_J} - \text{traces}\right) , \tag{4.110}$$

leading to

$$\frac{\mathcal{C}_{J,\nu}^{\mathcal{O}}}{b_{\mathcal{O}}} \frac{2^J (\nu)_J}{J!} \left[\frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} \left(e^{\mu_1} \dots e^{\mu_J} - \text{traces}\right)\right] = \frac{f_{\varphi\varphi\mathcal{O}}}{c_{\mathcal{O}}} \left\langle \mathcal{O}^{\mu_1 \dots \mu_J} \right\rangle_\beta , \tag{4.111}$$

from which we can both recover the formula (4.48) and fix the overall coefficient $\mathcal{C}_{J,\nu}^{\mathcal{O}}$, reaching the result

$$f_\beta^{\mathcal{O}}(\tau, r) = \frac{J!}{2^J (\nu)_J} \frac{1}{\beta^{\Delta_{\mathcal{O}}}} \frac{f_{\varphi\varphi\mathcal{O}} b_{\mathcal{O}}}{c_{\mathcal{O}}} \left(\tau^2 + r^2\right)^{\frac{1}{2}(\Delta_{\mathcal{O}} - 2\Delta_\varphi)} C_J^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2 + r^2}}\right) , \tag{4.112}$$

which immediately leads to the solution[17]

$$f_\beta(\tau, r) = \sum_{\mathcal{O} \in \varphi \times \varphi} \frac{J!}{2^J (\nu)_J} \frac{1}{\beta^{\Delta_{\mathcal{O}}}} \frac{f_{\varphi\varphi\mathcal{O}} b_{\mathcal{O}}}{c_{\mathcal{O}}} \left(\tau^2 + r^2\right)^{\frac{1}{2}(\Delta_{\mathcal{O}} - 2\Delta_\varphi)} C_J^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2 + r^2}}\right) . \tag{4.113}$$

Furthermore, observe that the OPE decomposition of the two-point function implies that odd spin operators' one-point functions must be zero; this can be easily checked by considering the fact that

[16] The use of $\tau \to -\tau$ is necessary to get the correct result. This corresponds to the use of the full $\mathrm{O}(d-1)$ symmetry group: $\mathrm{SO}(d-1)$ is indeed not sufficient to exclude odd spin operators to have a non-zero thermal one-point function.

[17] The procedure highlighted in this short Section shows how it is possible to recover the results already presented in [45] by employing only the broken Ward identities and the OPE. It must be noticed that strictly speaking, the formula (4.48) was not recovered in full generality, but only for those operators living in the OPE between two identical scalar operators; moreover, the differential equation (4.106) has to be solved spin by spin.

$C_J^{(\nu)}(\tau/\sqrt{r^2+\tau^2}) = C_J^{(\nu)}(-\tau/\sqrt{r^2+\tau^2})$ only for even spin operator and therefore $\tau \to -\tau$ is a symmetry only if the one-point functions of odd spin operator are zero.

Computation of $\langle\varphi(x)\varphi(0)\rangle_\beta$. The result (4.113) is the starting point to solve the bootstrap problem for the correlation function $\langle\varphi(x)\varphi(0)\rangle_\beta$. Although it is compatible by construction with the OPE, the equation (4.113) is not valid outside the radius of convergence; moreover, it does not provide any information on the one-point functions coefficients $b_\mathcal{O}$. The key information is the particularly simple structure of the $\varphi \times \varphi$ OPE

$$\varphi(x)\varphi(0) \sim \mathbb{1} + a\,\varphi^2(0) + \underbrace{b_\mu \varphi(0)\partial^\mu\varphi(0)}_{\sim V^\mu(0)} + \underbrace{c_{\mu\nu}\varphi(0)\partial^\mu\partial^\nu\varphi(0)}_{\sim T^{\mu\nu}(0)} + \dots \ , \tag{4.114}$$

where the specific expressions of the operators are not important for our purpose. We can extract from the OPE a formula for the conformal dimension of the operator $\mathcal{O} \in \varphi \times \varphi$ (excluding the identity operator $\mathbb{1}$) as a function of its spin J. Recalling that $\Delta_\varphi = \frac{d-2}{2}$,

$$\Delta_\mathcal{O} = d - 2 + J \ . \tag{4.115}$$

Using this formula, we can turn the sum over the operators in the equation (4.114) into a sum over the spins J

$$f_\beta(\tau, r) = \frac{1}{(\tau^2+r^2)^{\frac{d-2}{2}}} + \sum_{J=0}^{\infty} k_J \frac{1}{\beta^{d-2+J}} \frac{J!}{2^J(\nu)_J} \left(\tau^2+r^2\right)^{\frac{J}{2}} C_J^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2+r^2}}\right) \ , \tag{4.116}$$

where we grouped all the dynamical information into the coefficients

$$k_J = \frac{f_{\varphi\varphi\mathcal{O}} b_\mathcal{O}}{c_\mathcal{O}} \ . \tag{4.117}$$

Since the equation (4.116) is missing manifest periodicity over the thermal circle, it should be established by constraining the coefficients k_J. Since the coefficients k_J are functions of the $b_\mathcal{O}$ one-point function data, imposing manifest periodicity to the two-point scalar function will provide information about the one-point functions of the scalar free theory at the same time [45].

From now on, we will set $d = 4$. This is another important simplification, since the equation (4.116) now reads

$$f_\beta^{d=4}(\tau, r) = \frac{1}{\tau^2+r^2} + \sum_{J=0}^{\infty} \frac{1}{\beta^{2+J}} \frac{k_J}{2^J} \left(\tau^2+r^2\right)^{\frac{J}{2}} U_J\left(\frac{\tau}{\sqrt{\tau^2+r^2}}\right), \tag{4.118}$$

where we used the identity

$$C_J^{(1)}(x) = U_J(x)\,, \tag{4.119}$$

with $U_J(x)$ being the Jth *Čebyšëv polynomial of the second kind*. These polynomials admit a closed formula

$$U_J(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{J+1} - \left(x-\sqrt{x^2-1}\right)^{J+1}}{2\sqrt{x^2-1}} \tag{4.120}$$

which can be plugged into the $d=4$ solution

$$f_\beta^{d=4}(\tau,r) = \frac{1}{\tau^2+r^2} + \frac{1}{2\beta^2 r}\sum_{J=0}^{\infty} k_J \left(\frac{i}{2\beta}\right)^J \left[(r-i\tau)^{J+1} + (-1)^J (r+i\tau)^{J+1}\right]. \tag{4.121}$$

The structure of the solution can be further improved by noticing that the zero temperature contribution can be rewritten as

$$\frac{1}{\tau^2+r^2} = \frac{1}{2\,\beta^2 r}\left(-\frac{1}{4}\right)\left(\frac{i}{2\beta}\right)^{-2}\left[(r-i\tau)^{-1} + (-1)^{-2}(r+i\tau)^{-1}\right] \tag{4.122}$$

and by adding the following factor, identically equal to 0

$$0 = \frac{1}{2\,\beta^2 r} k_{-1}\left(\frac{i}{2\beta}\right)^{-1}\left[(r-i\tau)^0 + (-1)^{-1}(r+i\tau)^0\right]. \tag{4.123}$$

After having defined $k_{-2} = -1/4$, the solution reads

$$f_\beta^{d=4}(\tau,r) = \frac{1}{2\,\beta^2 r}\sum_{J=-2}^{\infty} k_J \left(\frac{i}{2\beta}\right)^J \left[(r-i\tau)^{J+1} + (-1)^J (r+i\tau)^{J+1}\right]. \tag{4.124}$$

The expression can be reshaped by relabeling $J = \ell - 2$ and imposing the parity $\tau \to -\tau$ to leave the two-point function invariant, forcing the sum to run only over even integers $\ell = 2n$: this can be achieved by setting $k_{\text{odd}} = 0$

$$f_\beta^{d=4}(\tau,r) = \frac{\pi}{2\beta r}\sum_{n=0}^{\infty} \frac{k_{2n-2}}{\pi^{2n}}\left(-\frac{1}{4}\right)^{n-1}\left\{\left[\frac{\pi}{\beta}(r-i\tau)\right]^{2n-1} + \left[\frac{\pi}{\beta}(r+i\tau)\right]^{2n-1}\right\}. \tag{4.125}$$

By introducing the adimensional complex variables

$$w = \frac{\pi}{\beta}(r-i\tau)\,, \qquad \overline{w} = \frac{\pi}{\beta}(r+i\tau)\,, \tag{4.126}$$

we conclude that the two-point function can be reduced to

$$f_\beta^{d=4}(w,\overline{w}) = \left(\frac{\pi}{\beta}\right)^2 \frac{g(w) + g(\overline{w})}{w+\overline{w}}\,, \tag{4.127}$$

with the sums over n converging for $0 < |w| < \pi$. $g(w)$ is an odd function of w, and similarly $g(\overline{w})$ is an odd function of $\overline{w}$. We can now solve the bootstrap problem posed in Section 4.2.2 for the function (4.127). The function $g(z)$ is periodic and it has a single pole in $z = 0$, plus all the poles that come from the periodicity of the function. The residue of the pole is fixed by $k_{-2} = -1/4$ (and by periodicity, this is the residue of all the other poles as well). A function that satisfies all the analytic properties required is the hyperbolic cotangent

$$g(w) = \coth(w) \ , \tag{4.128}$$

and therefore the candidate solution is given by

$$\widetilde{f}_{\beta}^{d=4}(w,\overline{w}) = \left(\frac{\pi}{\beta}\right)^2 \frac{\coth(w) + \coth(\overline{w})}{w+\overline{w}} \ . \tag{4.129}$$

Let us assume that another solution to the bootstrap problem exists: since the equation (4.129) already encodes all the analytic properties that can be read from (4.125), then the new solution must have the following form

$$\widehat{f}_{\beta}^{d=4}(w,\overline{w}) = \left(\frac{\pi}{\beta}\right)^2 \frac{(\coth(w) + \delta f(w)) + (\coth(\overline{w}) + \delta f(\overline{w}))}{w+\overline{w}} \ , \tag{4.130}$$

where $\delta f(w)$ is a holomorphic, regular, and periodic function. Thanks to its periodicity, $\delta f(w)$ (and similarly its complex conjugate) can be expanded in an infinite series of hyperbolic sines and cosines

$$\delta f(w) = \sum_n \Big(a_n \sinh(nw) + b_n \cosh(nw) \Big). \tag{4.131}$$

We can read from the OPE (4.125) that the solution must be odd in w, hence we set all the coefficients b_n equal to 0 and we are left with

$$\widehat{f}_{\beta}^{d=4}(w,\overline{w}) = \left(\frac{\pi}{\beta}\right)^2 \frac{\coth(w) + \coth(\overline{w})}{w+\overline{w}} + \sum_n a_n \left(\frac{\sinh(nw)}{w+\overline{w}} + \frac{\sinh(n\overline{w})}{w+\overline{w}} \right) \ . \tag{4.132}$$

We can now test the solution candidate (4.132) in the Regge limit. Recalling the prescription provided in Section 4.2.2, we switch to Regge coordinates

$$w = \frac{\rho}{\eta - \sqrt{\eta^2-1}} \ , \quad \overline{w} = \rho\left(\eta - \sqrt{\eta^2-1}\right) \ , \tag{4.133}$$

and the Regge limits of the terms composing the solution candidate (4.132) are

$$\begin{aligned} &\frac{\coth(w)}{w+\overline{w}} \xrightarrow{|\eta|\to\infty} \frac{1}{2\eta\rho} \ , && \frac{\coth(\overline{w})}{w+\overline{w}} \xrightarrow{|\eta|\to\infty} \frac{1}{\rho^2} \ , \\ &\frac{\sinh(nw)}{w+\overline{w}} \xrightarrow{|\eta|\to\infty} \frac{e^{2n\eta\rho}}{4\eta\rho} \ , && \frac{\sinh(n\overline{w})}{w+\overline{w}} \xrightarrow{|\eta|\to\infty} \frac{n}{4\eta^2} \ , \end{aligned} \tag{4.134}$$

where the check was performed for $n > 0$[18] and a fixed value of ρ. We immediately notice that the corrections given by the hyperbolic sines are not allowed, since they are not polynomially bound in the Regge limit. We deduce that $\delta f(w) = 0$ and $\widehat{f}_\beta^{d=4}(w,\overline{w}) = \widetilde{f}_\beta^{d=4}(w,\overline{w})$. Therefore the two-point function of a free scalar theory reads

$$f_\beta^{d=4}(\tau, r) = \frac{\pi}{2\beta r}\left[\coth\left(\frac{\pi}{\beta}(r+i\tau)\right) + \coth\left(\frac{\pi}{\beta}(r-i\tau)\right)\right] \ . \tag{4.135}$$

The two-point function above satisfies all the bootstrap conditions listed in Section 4.2.2 and agrees with the known result [134].

The procedure presented in this Section can in principle be extended to other theories; however the possibility of reading a factorization of the form (4.127) from the OPE is not guaranteed. In fact, the split of the numerator between an holomorphic $g(w)$ and an antiholomorphic $g(\overline{w})$ functions in equation (4.129) descends from the structure of the Čebyšëv polynomials, which are equal to the Gegenbauer polynomials only in $d = 4$ dimensions. In general, formulating an Ansatz from the OPE can be very complicated, and we are not able to prove that the solution is unique. For instance, in the case of the thermal two-point functions in a generalized free field theory, the OPE reads

$$\begin{aligned} f_\beta^{\mathrm{GFF}}(\tau, r) = \frac{1}{(\tau^2+r^2)^{\Delta_\varphi}} &+ \frac{1}{2r}\sum_{J=0}^{\infty}\sum_{n=0}^{\infty}\frac{f_{\varphi\varphi(J,n)}b_{(J,n)}}{c_{(J,n)}}\left(\frac{1}{\beta}\right)^{2\Delta_\varphi+J+n}\left(\frac{i}{2}\right)^J(r-i\tau)^{J+\frac{n}{2}+1}(r+i\tau)^{\frac{n}{2}} \\ &+ \frac{1}{2r}\sum_{J=0}^{\infty}\sum_{n=0}^{\infty}\frac{f_{\varphi\varphi(J,n)}b_{(J,n)}}{c_{(J,n)}}\left(\frac{1}{\beta}\right)^{2\Delta_\varphi+J+n}\left(\frac{-i}{2}\right)^J(r+i\tau)^{J+\frac{n}{2}+1}(r-i\tau)^{\frac{n}{2}} \ . \end{aligned} \tag{4.136}$$

The complicated structure of the equation above makes it very difficult to guess a factorization similar to (4.127), and consequently, it is difficult to prove the unicity of the solution too. This problem surely deserves further future investigations.

One-point functions from the two-point function. We can now study the final solution (4.135) in the OPE regime again to extract all the one-point functions CFT data b_J, spin by spin. This procedure is well known in the literature [45, 134, 135] and we present it both for completeness and as a check of the thermal OPE structure underlying the thermal two-point function.

[18] If $n < 0$, we can reduce ourselves to the case already considered knowing that $\sinh(-x) = -\sinh(x)$.

In order to extract the b_J coefficients, we need to expand the equation (4.135) in the OPE regime. This can be achieved thanks to the series expansion of the hyperbolic cotangent

$$\coth(z) = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} z^{2n-1} \ , \quad 0 < |z| < \pi \ , \tag{4.137}$$

where B_{2n} is the $2n$-th Bernoulli number. Notice that we highlighted the domain of convergence: if we apply the equation (4.137) to the solution (4.135), we can immediately see that the domain of convergence corresponds to the OPE regime. The solution now reads

$$f_\beta^{d=4}(\tau, r) = \frac{\pi}{2\beta r} \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} \left\{ \left[\frac{\pi}{\beta}(r - i\tau) \right]^{2n-1} + \left[\frac{\pi}{\beta}(r + i\tau) \right]^{2n-1} \right\} \ , \tag{4.138}$$

and it can be compared to the equation (4.125). The two expressions coincide if

$$\frac{2^{2n} B_{2n}}{(2n)!} = \frac{k_{2n-2}}{\pi^{2n}} \left(-\frac{1}{4} \right)^{n-1} \ , \quad n \geq 0 \ , \tag{4.139}$$

which is consistent with the choice $k_{-2} = -\frac{1}{4}$ made previously. For $n \geq 1$, the Bernoulli numbers enjoy a relationship with the Riemann ζ function

$$B_{2n} = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n) \ , \tag{4.140}$$

so that the expression above can be rewritten as

$$k_{2n-2} = 2^{2n-1} \zeta(2n) \ . \tag{4.141}$$

Recalling the change of variable $2n = \ell = J + 2$ made after the equation (4.124), we can reintroduce the dependence on the spin J

$$k_J = \frac{f_{\varphi\varphi\mathcal{O}_J} b_J}{c_{\mathcal{O}_J}} = 2^{J+1} \zeta(J + 2) \ , \quad J \geq 2 \ , \quad J \in 2\mathbb{Z} \ . \tag{4.142}$$

Recall that the condition $k_{\text{odd}} = 0$ had already been imposed. This completely solves the bootstrap problem for the thermal one-point functions of the free scalar theory in $d = 4$

$$b_{\mathbb{1}} = 1 \ , \qquad b_J = 0 \quad J \in 2\mathbb{Z} + 1 \ , \qquad b_J = 2^{J+1} \zeta(J + 2) \frac{c_{\mathcal{O}_J}}{f_{\varphi\varphi\mathcal{O}_J}} \quad J \in 2\mathbb{Z} \ . \tag{4.143}$$

5

A bootstrap problem at finite temperature

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5.1 From the KMS condition to thermal sum rules

Why should we set up a bootstrap problem at finite temperature? This Chapter will be dedicated to setting up and studying a bootstrap problem at finite temperature. The coefficients we are interested in bootstrapping are the one-point functions coefficients $b_{\mathcal{O}}$. Recalling the thermal OPE between two local scalar operators $\mathcal{O}_1$ and $\mathcal{O}_2$

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \sim \sum_{\mathcal{O}\in\mathcal{O}_1\times\mathcal{O}_2} \frac{f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}}{c_{\mathcal{O}}} \left(\tau^2+|\vec{x}|^2\right)^{\frac{1}{2}(\Delta_{\mathcal{O}}-\Delta_1-\Delta_2-J)} x_{\mu_1}\dots x_{\mu_J}\mathcal{O}^{\mu_1\dots\mu_J}(x_2)\ , \tag{5.1}$$

we will consider the CFT data of the zero temperature theory ($f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}$, $c_{\mathcal{O}}$ and the conformal dimensions) as an *input*. If we want to compute the two-point function $\langle\mathcal{O}_1\mathcal{O}_2\rangle_\beta$ using the OPE,

the only missing information is the one encoded in the one-point thermal functions

$$\langle \mathcal{O}^{\mu_1 \dots \mu_J} \rangle_\beta = \frac{b_\mathcal{O}}{\beta^{\Delta_\mathcal{O}}} \left(e^{\mu_1} \cdots e^{\mu_J} - \text{traces} \right) \ . \tag{5.2}$$

However, it should be noted that the coefficients $b_\mathcal{O}$ are, in principle, functions of the CFT data at zero temperature. This can be made manifest if we adopt the *real time formalism*. In this formalism, thermal correlation functions are realized by inserting in the zero temperature correlators a density matrix $\rho = e^{-\beta H}$, where H is the Hamiltonian operator

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_\beta = \text{Tr} \left[e^{-\beta H} \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \right] \ . \tag{5.3}$$

Notice that the ambient space manifold did not change in this case, and the correlators are still computed on the flat space. The great advantage of this approach lies in the Hilbert space of the thermal theory being the same as the zero temperature theory. If we choose to span the Hilbert space with a basis made by local, scaling operators $\mathcal{O}_i \left| 0 \right\rangle = \left| i \right\rangle$, where the vacuum state $\left| 0 \right\rangle$ is the zero-temperature vacuum, then the thermal correlation function (5.3) will schematically be proportional to

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_\beta \sim \frac{1}{\mathcal{Z}} \sum_{i,j} \left\langle 0 \left| \mathcal{O}_i(y) e^{-\beta H} \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \mathcal{O}_j(0) \right| 0 \right\rangle \ . \tag{5.4}$$

Interestingly, if we consider thermal one-point functions, we obtain

$$\langle \mathcal{O}(x) \rangle_\beta \sim \frac{1}{\mathcal{Z}} \sum_{i,j} \left\langle \mathcal{O}_i(y) e^{-\beta H} \mathcal{O}(x) \mathcal{O}_j(0) \right\rangle \ . \tag{5.5}$$

Although it seems that the thermal one-point function could be computed by evaluating a weighted sum of zero-temperature three-point functions, this operation is ambiguous, since the Hamiltonian operator is not well-defined for zero-temperature CFTs. A possible solution to this problem could be to consider a *finite volume* CFT, substituting the flat space with the geometry $\mathcal{M}_R = \mathbb{R} \times S^{d-1}_R$. Radial quantization provides a well-defined Hilbert space via state-operator correspondence and a well-defined Hamiltonian

$$\boldsymbol{H} = \frac{\boldsymbol{D}}{R} \ . \tag{5.6}$$

In this case, the thermal one-point functions can be schematically written as

$$\langle \mathcal{O}(x) \rangle_\beta \sim \lim_{R \to \infty} \frac{\sum_{i,j} e^{-\beta \frac{\Delta_i}{R}} \langle \mathcal{O}_i(y) \mathcal{O}(x) \mathcal{O}_j(0) \rangle_R}{\sum_i e^{-\beta \frac{\Delta_i}{R}} \langle \mathcal{O}_i(y) \mathcal{O}_i(0) \rangle_R} \ , \tag{5.7}$$

where the limit is needed to recover a flat space component $\mathbb{R}^{d-1}$. The equation (5.7) provides a legitimate way to compute the thermal one-point functions from zero temperature data, but the explicit evaluation turns out to be very complicated [45]. These considerations corroborate the importance of setting up a bootstrap problem for the coefficients $b_{\mathcal{O}}$, which means that they should be considered as *new data* of the theory. In this optic, working on the thermal geometry is extremely convenient, since the periodicity condition imposed on the thermal two-point functions is the keystone to set up a bootstrap problem to solve.

Sum rules. In Chapter 4, we considered the KMS condition (4.86) as one of the consistency requirements for a thermal scalar two-point function. In general, this condition holds for any thermal correlation function since it encodes the *periodicity* of the observables along the thermal circle. In Section 4.2.3 we already had the opportunity to notice that the sum (4.125), constructed out of the thermal OPE, returned a periodic function if and only if the coefficients k_J were given by (4.142). This idea can be generalized to any thermal CFT: given a generic thermal two-point function between identical scalar operators ϕ, the thermal two-point function is given in the OPE regime by

$$\langle\phi(\tau,r)\phi(0,0)\rangle_\beta = \sum_{\mathcal{O}\in\phi\times\phi} \frac{J!}{2^J(\nu)_J}\frac{1}{\beta^{\Delta_\mathcal{O}}}\frac{f_{\phi\phi\mathcal{O}}b_\mathcal{O}}{c_\mathcal{O}}\left(\tau^2+r^2\right)^{\frac{1}{2}(\Delta_\mathcal{O}-2\Delta_\phi)} C_J^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2+r^2}}\right) , \tag{5.8}$$

where Δ and J are the conformal dimension and the spin of the operator $\mathcal{O}\in\phi\times\phi$, $C_J^{(\nu)}$ are the Gegenbauer polynomials and

$$(\nu)_J = \frac{\Gamma(\nu+J)}{\Gamma(\nu)} , \qquad \nu = \frac{d-2}{2} . \tag{5.9}$$

We can define an OPE coefficient $a_\mathcal{O}$ encoding all the constant, overall factors

$$a_\mathcal{O} = \frac{f_{\phi\phi\mathcal{O}}b_\mathcal{O}}{c_\mathcal{O}}\frac{J!}{2^J(\nu)_J} . \tag{5.10}$$

As it has been already stated, the OPE (5.8) converges only in a radius $\tau^2+r^2<\beta^2$ on the thermal manifold. Knowing this, it is convenient to combine the KMS condition with the parity (or time reversal) transformation $\tau\to-\tau$. This is to ensure that the OPE expansion around $\tau=0$ is convergent and well-defined on both sides of the equation. Indeed

$$\langle\phi(\beta/2+\tau,r)\phi(0)\rangle_\beta = \langle\phi(\beta/2-\tau,r)\phi(0)\rangle_\beta . \tag{5.11}$$

The equation (5.11) had already been identified by El-Showk and Papadodimas [56] as the crossing equation analogue for thermal CFTs. In this Section, we explicitly explore this idea by extracting *explicit sum rules* out of the condition (5.11), making the analogy proposed by the aforementioned authors closer. The details of the derivation of the sum rules are explicitly given in Appendix B.1. Schematically, the idea is the following: we expand both the left and the right side of (5.11) by using the OPE, then we further expand the result in powers of τ and r. This can be achieved by using the definition of the Gegenbauer polynomials (B.1) and the binomial theorem, applied twice. It is easy to see from equation (5.11) that the even powers of the τ variable identically cancel, while the terms proportional to odd powers have to be set equal to zero and return non-trivial equations. Imposing this order by order in the r variable, we obtain explicit sum rules for the combination $b_{\mathcal{O}} f_{\mathcal{O}\phi\phi}$

$$\sum_{\mathcal{O}\in\phi\times\phi} b_{\mathcal{O}} f_{\mathcal{O}\phi\phi}\, F_{\ell,n}(h,J) = 0 \ , \qquad n\in\mathbb{N} \ , \quad \ell\in 2\mathbb{N}+1 \ , \tag{5.12}$$

where we introduced the *twist* variable

$$h = \Delta - J \ , \tag{5.13}$$

we defined the function

$$F_{\ell,n}(h,J) = \frac{1}{2^{h+J}} \binom{\frac{h-2\Delta_\phi}{2}}{n} \binom{h+J-2\Delta_\phi-2n}{\ell} {}_3F_2\left[\begin{matrix} \frac{1-J}{2}, -\frac{J}{2}, \frac{h}{2}-\Delta_\phi+1 \\ \frac{h}{2}-\Delta_\phi-n+1, -J-\nu+1 \end{matrix} \middle| 1 \right] \ , \tag{5.14}$$

and we adopted the normalization $c_{\mathcal{O}} = 1$. As expected, the function $F_{\ell,n}(h,J)$ depends both on the conformal dimensions Δ (via the h twist variable) and the spins J of the operators appearing in the OPE; moreover, it also depends on two positive integer numbers ℓ and n. Adopting a point of view closer to the zero-temperature analytical (and numerical) conformal bootstrap (see [6, 136–141] and references therein), the above formula corresponds to setting to zero the $(2\ell+1)$-th derivative in τ and the n-th derivative in r of $f(\tau,r)$ when computing in $\tau=\beta/2$ and $r=0$ [45]

$$\frac{\partial^{2\ell+1}}{\partial\tau^{2\ell+1}} \frac{\partial^n}{\partial r^n} \langle \phi(\tau,r)\phi(0,0)\rangle_\beta \Big|_{\tau=\frac{\beta}{2}, r=0} = 0 \ . \tag{5.15}$$

In this way, it is clear that the sum rules in equation (5.12) are the incarnation of an expansion around the fixed point of the relation (5.11).

A comment on free scalar theory. Let us comment on the simplest example, with ϕ being a fundamental free scalar field. In this particular scenario, the upper argument of the first binomial

in the equation (5.14) is fixed to zero. Indeed, apart from the case of the identity operator $\mathbb{1}$, the conformal dimensions of the operators $\mathcal{O} \in \phi \times \phi$ are given by $\Delta = 2\Delta_\phi + J$, which leads to $h - 2\Delta_\phi = 0$. This implies that all the equations for $n \neq 0$ are trivially satisfied. This was to be expected since the operators $\mathcal{O} \in \phi \times \phi$ can be labeled only by their conformal dimensions, which are in a bijection with the spins. Therefore it is natural to think that setting the spatial coordinates to zero from the beginning (which is equivalent to set $n = 0$, as it can be inferred from equation (5.15)) is enough to solve for the OPE coefficients. This theory can be explicitly solved, as we highlighted in Section 4.2.2, where we extracted all the thermal OPE coefficients out of the thermal two-point function. However, an explicit, closed expression for the thermal two-point function is a very rare occurrence: in those cases, the setup provided by the sum rules (5.12) can be a powerful tool to approach the problem if not analytically, at least numerically.

5.1.1 Sum rules on a thermal circle

Although the sum rules (5.12) already represent a well-defined bootstrap problem for the coefficients $b_\mathcal{O}$, they can be considerably simplified by considering the two scalar fields ϕ placed on the *same thermal circle*, i.e. separated only by a distance τ. Indeed, when a two-point function of a d-dimensional theory is restricted to zero spatial coordinates the thermal OPE takes the form

$$\langle \phi(\tau,0)\phi(0,0)\rangle_\beta = \sum_\Delta \frac{a_\Delta}{\beta^\Delta} \tau^{\Delta - 2\Delta_\phi} \ , \qquad a_\Delta \equiv \sum_{\mathcal{O}\in\phi\times\phi}^{\Delta \text{ fixed}} a_\mathcal{O}\, C_J^{(\nu)}(1) \ , \tag{5.16}$$

where $a_\mathcal{O}$ is given in (5.10) and the last sum is performed over operators in the OPE $\phi \times \phi$ of dimension Δ (but different spin J). In this case, it is possible to reduce the general sum rules (5.12) to a simpler form[1]

$$\frac{\Gamma(2\Delta_\phi + \ell)}{\Gamma(2\Delta_\phi)} = \sum_\Delta \frac{a_\Delta}{2^\Delta} \frac{\Gamma(\Delta - 2\Delta_\phi + 1)}{\Gamma(\Delta - 2\Delta_\phi - \ell + 1)} \ , \qquad \ell \in 2\mathbb{N} + 1 \ . \tag{5.17}$$

The sum above is performed over all the possible conformal dimensions appearing in the OPE between $\phi \times \phi$ and, due to the reduction, it is not sensitive to the spin J of the operators anymore. These equations are, in principle, non-trivial constraints for the OPE coefficients a_Δ. It is important to notice that constraining the sum rules on a single thermal circle introduces a *spin degeneracy*:

[1] See Appendix B.1 for the derivation of this formula from the more general result (5.14). Alternatively one can directly start with the correlator at zero spatial coordinates and use the binomial theorem, as in the proof of equation (5.12) in Appendix B.1.

the coefficient a_Δ is a weighted sum over all the operators $\mathcal{O}$ with conformal dimension Δ, whatever their spin is. The weight is provided by the Gegenbauer factor $C_J^{(\nu)}(1)$. In deriving the equation (5.17) we used the fact that in the OPE the identity operator $\mathbb{1}$ appears. The conformal dimension of the identity is zero (and so its spin), and its one-point function and structure constant can be set to $b_{\mathbb{1}} = f_{\phi\phi\mathbb{1}} = 1$. We can then isolate the contribution of the identity, producing the left side of equation (5.17). This information is important to define a normalization, otherwise the coefficients a_Δ could only be defined up to an overall rescaling (this is clear from equation (5.12)). Let us comment that equations (5.12) and (5.17) can be straightforwardly generalized to the case of unequal external operators. The structure of the equations is the same but the identity will not be included in the OPE.

Bootstrapping the $b_\mathcal{O}$ coefficients requires solving the sum rules (5.17), so we need to check if they behave correctly in controlled conditions. In the following, we provide some tests, performed for two-dimensional thermal CFTs and generalized free scalar theories. In all these models, the thermal scalar two-point function can be computed *analytically*, and it can be used to extract the $b_\mathcal{O}$ coefficients, in a similar fashion to the one adopted to extract the coefficients (4.143) of the $4d$ free scalar theory.

Testing the sum rules with generalized free fields. Generalized free scalar fields (GFF) represent an important class of theories with interesting applications to holography. The feature we are interested in is the OPE of the fundamental scalar ϕ, of conformal dimension Δ_ϕ, with itself: schematically

$$\phi \times \phi \sim \mathbb{1} + [\phi\phi]_{p,q} \ , \tag{5.18}$$

where the conformal dimensions of the operators in the OPE are $\Delta = 2\Delta_\phi + p + 2q$, $p, q \in \mathbb{N}$ (up to the identity $\mathbb{1}$, whose dimension is zero). The compact notation $[\phi\phi]_{p,q}$ refers to the *double-twist operators*: more explicitly, they can be written as

$$[\phi\phi]_{p,q} = \phi \,\Box^q \partial_{\mu_1} \dots \partial_{\mu_p} \phi \tag{5.19}$$

These theories are well studied and solved for any spacetime dimensions [45, 53] thanks to the possibility of employing the *method of images*. This method consists of writing the thermal propagator $G_\beta^{\phi\phi}(\tau,\vec{x})$ between two scalar fields ϕ as an infinite sum over zero temperature propagators $G^{\phi\phi}(\tau,\vec{x})$ [42–44]

$$G_\beta^{\phi\phi}(\tau,\vec{x}) = \sum_{n=-\infty}^{\infty} G^{\phi\phi}(\tau+n\beta,\vec{x}) = \sum_{n=-\infty}^{\infty} \frac{1}{((\tau+n\beta)^2+|\vec{x}|^2)^{\Delta_\phi}} \ . \tag{5.20}$$

This prescription can be employed as a quick derivation of the result (4.135) by noticing that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n\beta)^2+|\vec{x}|^2} = \frac{\pi}{2\beta|\vec{x}|}\left[\coth\left(\frac{\pi}{\beta}(|\vec{x}|+i\tau)\right)+\coth\left(\frac{\pi}{\beta}(|\vec{x}|-i\tau)\right)\right] \ . \tag{5.21}$$

When we constrain the two scalar fields to sit on the same thermal circle, the method of images (5.20) considerably simplifies, and it can return a closed result for generic values of Δ_ϕ

$$\langle\phi(\tau)\phi(0)\rangle_\beta = \sum_{m=-\infty}^{\infty} \frac{1}{(\tau+m\beta)^{2\Delta_\phi}} = \frac{1}{\beta^{2\Delta_\phi}}\left[\zeta_H\left(2\Delta_\phi,\frac{\tau}{\beta}\right)+\zeta_H\left(2\Delta_\phi,1-\frac{\tau}{\beta}\right)\right] \ , \tag{5.22}$$

where we introduced the *Hurwitz ζ-function*

$$\zeta_H(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \ . \tag{5.23}$$

There are cases in which the solution above simplifies even more. If we consider the GFF theory with $\Delta_\phi = 1$,

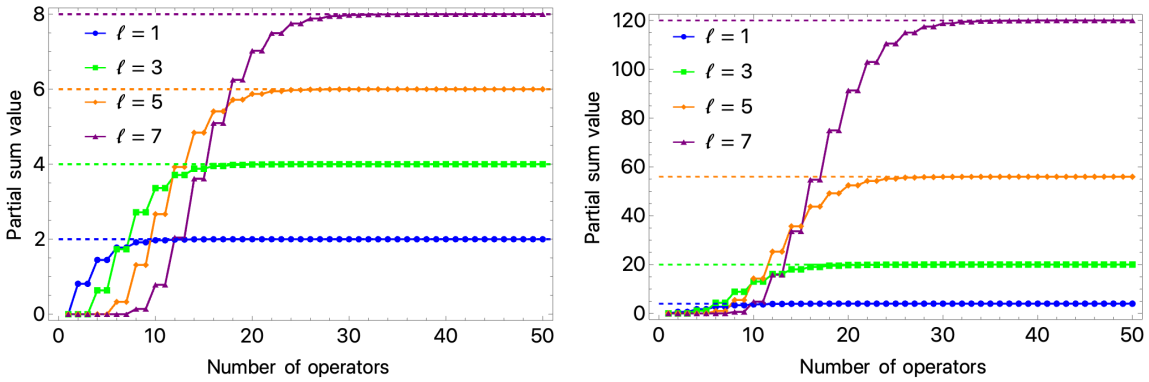
$$\langle\phi(\tau)\phi(0)\rangle_\beta = \frac{\pi^2}{\beta^2}\csc^2\left(\frac{\pi\tau}{\beta}\right) \ . \tag{5.24}$$

Another special case is given by the GFF with $\Delta_\phi = 2$. In this scenario, the solution (5.22) reduces to

$$\langle\phi(\tau)\phi(0)\rangle_\beta = \frac{\pi^4}{3\beta^4}\left[2+\cos\left(\frac{2\pi\tau}{\beta}\right)\right]\csc^4\left(\frac{\pi\tau}{\beta}\right) \ . \tag{5.25}$$

GFF theories represent an optimal playground to test the equation (5.17) since the OPE coefficients a_Δ can be easily extracted from the exact solutions (5.24) and (5.25) by comparing their expasions in powers of τ with the OPE (5.16). After extracting the coefficients and plugging them in the sum rules (5.17), the numerical tests for GFF theories with $\Delta_\phi = 1$ and $\Delta_\phi = 2$ are presented in Figs. 5.1a and 5.1b. From these checks, we learn that

- ◇ the sum rules (5.17) work numerically: the right-hand side converges to the left-hand side, represented by an horizontal, dashed line;



(a) GFF $\Delta_\phi = 1$ (b) GFF $\Delta_\phi = 2$

Figure 5.1: *Comparing the left (straight, dashed lines) and right (continuous lines) sides of equation* (5.17) *for different values of* ℓ*. The bigger* ℓ *is, the more operators need to be summed for the right-hand side to converge to the correct value, namely the left-hand side. The left panel corresponds to a numerical test for a GFF with* $\Delta_\phi = 1$*. The right panel corresponds to a numerical test for a GFF with* $\Delta_\phi = 2$*. In both cases, it is convenient to multiply both sides of the equation* (5.17) *by a factor* $1/\ell!$ *for presentation purposes.*

- ◇ the parameter ℓ has an important role. Although the sum rules (5.17) are supposed to work for any odd value of the integer ℓ, by looking at the Figures 5.1a and 5.1b we realize that a larger values of ℓ requires to consider a larger number of operators in the OPE to achieve a satisfying convergence;

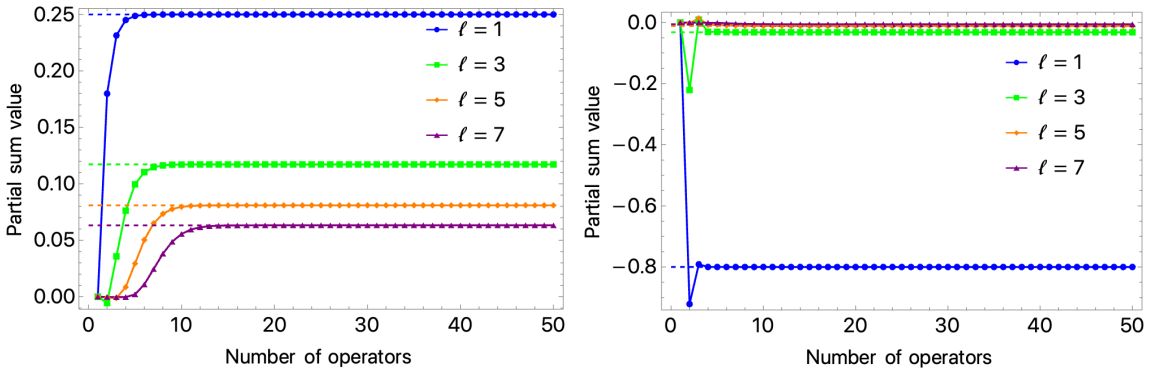
- ◇ we notice that the contribution of "heavy" operators, i.e. operators with a large conformal dimension Δ, decreases until it becomes negligible.

Testing the sum rules with two-dimensional models. Another simple test for the sum rules can be performed by considering two-dimensional correlators between Virasoro primary fields. The two-point functions can be computed by considering the conformal map between the plane $\mathbb{R}^2$ and the cylinder $\mathbb{R} \times S^1_\beta$ [77, 122, 142]

$$\langle \phi(\tau,\sigma)\phi(0,0)\rangle_\beta = \left(\frac{\pi}{\beta}\right)^{2h_\phi+2\bar{h}_\phi} \operatorname{csch}^{2h_\phi}\left[\frac{\pi}{\beta}(\sigma+i\tau)\right] \operatorname{csch}^{2\bar{h}_\phi}\left[\frac{\pi}{\beta}(\sigma-i\tau)\right] , \tag{5.26}$$

where $(h_\phi, \bar{h}_\phi)$ are the conformal weights of the Virasoro primary ϕ of conformal dimension $\Delta = h_\phi + \bar{h}_\phi$. When reduced to zero spatial coordinates the correlator above is given by

$$\langle \phi(\tau,0)\phi(0,0)\rangle_\beta = \left(\frac{\pi}{\beta}\right)^{2\Delta} \csc^{2\Delta}\left(\frac{\pi\tau}{\beta}\right) . \tag{5.27}$$



(a) Critical Ising model

(b) Lee-Yang model

Figure 5.2: *Comparing left (straight, dashed lines) and right (continuous lines) side of equation* (5.17) *for different values of* ℓ*. Observe that similarly to Figs. 5.1a and 5.1b, the bigger* ℓ *is, the more operators we need to sum for convergence. The left panel corresponds to a numerical test for the* $\langle\sigma(\tau)\sigma(0)\rangle_\beta$ *correlator in the two-dimensional critical Ising model. The right panel corresponds to a numerical test for the* $\langle\phi(\tau)\phi(0)\rangle_\beta$ *correlator in the Lee-Yang model, i.e. the non-unitary minimal model* $\mathcal{M}_{2,5}$*. In both cases, it is convenient to multiply both sides of equation* (5.17) *by a factor* $1/\ell!$ *for presentation purposes.*

Therefore we can test the equation (5.17) for any dimension Δ. The numerical tests are presented in Figs. 5.2a and 5.2b: similarly to 5.1a and 5.1b, they compare the left and right-hand sides of the equation (5.17) for different values of ℓ. The considered examples are $\langle\sigma(\tau)\sigma(0)\rangle_\beta$, where the field σ is the Virasoro primary of conformal weights $(1/16, 1/16)$ in the two-dimensional critical Ising model, and $\langle\phi(\tau)\phi(0)\rangle_\beta$, where ϕ is the only Virasoro primary field in the Lee-Yang minimal model. The Lee-Yang model (also known as the Lee-Yang singularity [77]) is the simplest conformal field theory since it is the minimal model with only one Virasoro primary, of conformal weights $(-1/5, -1/5)$. Despite its simplicity, the Lee-Yang model is not unitary: interestingly, the sum rules work for this theory as well, remarking that the only assumption we used to derive equation (5.17) was the KMS condition.

5.1.2 Sum rules as a tool for bootstrap

Truncating the sum rules. In the previous Section we derived thermal sum rules and we tested them on a set of theories where explicit thermal two-point functions are available. The next step is to investigate the potential of the equation (5.17) as an efficient bootstrap problem. The question is: given a generic thermal two-point function between identical scalars ϕ, and known the operator content of the OPE $\phi \times \phi$ together with the conformal dimensions, is it possible to use it to extract

the coefficients $a_{\mathcal{O}}$?

First of all, if we consider the spin degeneracy introduced in equation (5.16), we immediately realize that we can only be predictive for the coefficients a_Δ: the coefficients $a_{\mathcal{O}}$ are available if and only if we can solve the degeneracy. Furthermore, the coefficients a_Δ are infinite in number, so the sum rules become predictive if and only if we can *truncate* the sum to a certain conformal dimension Δ_H, and neglect the tail of heavy operators with conformal dimension $\Delta > \Delta_H$. Truncating the sum rules returns a linear equation where the variables are the coefficients a_Δ with $\Delta < \Delta_H$; this equation is also associated to a certain value of ℓ. If we desire to determine a number n of coefficients a_Δ, we will need to consider a number n of equations and so a number n of values for ℓ. However, by looking at the numerical tests 5.1a, 5.1b, 5.2a, 5.2b, higher values of ℓ require more operators to be considered in the truncation for it to be a good approximation of the whole, infinite sum.

Due to these technical issues, in this Thesis we will not present a new solution to the bootstrap problem set up in this Chapter, even if current work in progress is being done in this direction. In the following, we will show how to solve the problem in a class of toy models called "large-gap theories", for illustrative purposes. In Section 5.2 instead, we will tackle the important issue of estimating the tail of heavy operators mentioned above, studying the asymptotic behaviour of a_Δ when $\Delta \to \infty$ and predicting the *error* introduced by the truncation procedure for a given value of Δ_H and ℓ.

Solving the bootstrap problem for "large-gap theories". We can define as "large-gap theories" any theory where the OPE between two identical scalars decomposes in a light operators' sector, made by a *finite* number of operators, and in a heavy operators' sector. The conformal dimensions of the two sectors are separated by a large gap. The OPE can be truncated so that only the light operators contribute and the heavy ones can be neglected. In this scenario, we can approximately solve the bootstrap problem. In what follows, we present a solution for one and two non-trivial (i.e. different from the identity) light operators, estimating their one-point functions. As we already commented, when the sum rules in equation (5.17) are truncated, an approximation error is

inevitably introduced. The error associated with this approximation will be estimated in Section 5.2.

The simplest case is when the OPE between two fields ϕ contains one light operator in addition to the identity operator $\mathbb{1}$, namely $\mathcal{O}_1$: schematically,

$$\phi(\tau)\phi(0) \sim \mathbb{1} + \mathcal{O}_1 + \text{heavy operators} \ . \tag{5.28}$$

If we assume that all the heavy operators' contributions can be neglected, and we truncate the sum rules (5.17) to $\mathcal{O}_1$, then the only equation we need is

$$2\Delta_\phi \simeq \frac{a_{\Delta_1}}{2^{\Delta_1}} \left(\Delta_1 - 2\Delta_\phi\right) \ , \tag{5.29}$$

where we chose $\ell = 1$. Recalling the definition (5.10), the equation can be solved for the one-point function thermal OPE coefficient $b_{\mathcal{O}_1}$

$$b_{\mathcal{O}_1} \simeq 2^{1+\Delta_1} \frac{c_{\mathcal{O}_1}}{f_{\phi\phi\mathcal{O}_1}} \frac{\Delta_\phi}{\Delta_1 - 2\Delta_\phi} \ . \tag{5.30}$$

The next simple case is the one in which only the identity plus two light operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are considered, while all the others are heavy enough to be negligible. We are therefore assuming an OPE which schematically looks like

$$\phi(\tau)\phi(0) \sim \mathbb{1} + \mathcal{O}_1 + \mathcal{O}_2 + \text{heavy operators} \ , \tag{5.31}$$

from which we can extract two equations of the form (5.17) corresponding to $\ell = 1$ and $\ell = 3$. In particular, the two equations to solve are

$$2\Delta_\phi \simeq \frac{a_{\Delta_1}}{2^{\Delta_1}} \left(\Delta_1 - 2\Delta_\phi\right) + \frac{a_{\Delta_2}}{2^{\Delta_2}} \left(\Delta_2 - 2\Delta_\phi\right) \ , \tag{5.32}$$

and

$$4\Delta_\phi(1+\Delta_\phi)(1+2\Delta_\phi) \simeq \frac{a_{\Delta_1}}{2^{\Delta_1}} \frac{\Gamma(\Delta_1 - 2\Delta_\phi + 1)}{\Gamma(\Delta_1 - 2\Delta_\phi - 2)} + \frac{a_{\Delta_2}}{2^{\Delta_2}} \frac{\Gamma(\Delta_2 - 2\Delta_\phi + 1)}{\Gamma(\Delta_2 - 2\Delta_\phi - 2)} \ . \tag{5.33}$$

Solving the two equations for the thermal OPE coefficients $b_{\mathcal{O}_1}$ and $b_{\mathcal{O}_2}$, we get

$$b_{\mathcal{O}_1} \simeq 2^{1+\Delta_1} \frac{c_{\mathcal{O}_1}}{f_{\phi\phi\mathcal{O}_1}} \frac{\Delta_\phi}{\Delta_1 - 2\Delta_\phi} \left[\frac{\Delta_2(\Delta_2 - 4\Delta_\phi - 3)}{(\Delta_2 - \Delta_1)(\Delta_1 + \Delta_2 - 4\Delta_\phi - 3)}\right] \ , \tag{5.34}$$

$$b_{\mathcal{O}_2} \simeq 2^{1+\Delta_2} \frac{c_{\mathcal{O}_2}}{f_{\phi\phi\mathcal{O}_2}} \frac{\Delta_\phi}{\Delta_2 - 2\Delta_\phi} \left[\frac{\Delta_1(\Delta_1 - 4\Delta_\phi - 3)}{(\Delta_1 - \Delta_2)(\Delta_1 + \Delta_2 - 4\Delta_\phi - 3)}\right] \ . \tag{5.35}$$

Including more light operators is straightforward and the linear system can be solved. The case where two light operators have the same conformal dimension is different. In this case, equation (5.17) is not sufficient to solve for one-point functions and we need to consider the full set of equations in (5.12).

To conclude, it is interesting to notice that the sum rules (5.17) can also provide information about the zero temperature CFT data, such as conformal dimensions $\Delta_{\mathcal{O}}$ and the structure constants $f_{\phi\phi\mathcal{O}}$; if we consider (5.14), the sum rules also contain information on the spins $J_{\mathcal{O}}$. If we work with "large-gap theories" and we assume that the contribution from heavy operators can be neglected in the right-hand side of the equation (5.17), it is possible to constrain the conformal spectrum for the light operators' sector. To show this, let us consider a theory with a light sector composed of N light operators $\mathcal{O}_1, \dots \mathcal{O}_N$: the sum rule (5.17) can be approximated by

$$\sum_{i=1}^{N} a_{\Delta_i} F(\Delta_i, 2p-1) \simeq 0 \ , \qquad F(\Delta_i, 2p-1) = \frac{\Gamma(\Delta_i - 2\Delta_\phi + 1)}{2^{\Delta_i}\Gamma(\Delta_i - 2\Delta_\phi - 2p + 2)} \ , \tag{5.36}$$

where $p = 1, \dots, N$; the identity contribution has been moved to the left side of this equation, i.e. $\mathcal{O}_1 = \mathbb{1}$. As already stated, truncating the sum rules produces an error whose estimation will be discusses in Section 5.2. The equation (5.36) is a set of N linear homogeneous equations in N variables. We require the system to have a physical solution, i.e. a set of non-trivial thermal OPE coefficients $a_{\Delta_1}, \dots, a_{\Delta_N}$: if we consider $F(\Delta_i, 2p-1)$ as a square matrix with columns labeled by $i = 1, \dots, N$ and rows labeled by $p = 1, \dots, N$, then we must require its determinant to be zero

$$\det\left[F(\Delta_i, 2p-1)\right] = 0 \ . \tag{5.37}$$

This is a non-trivial equation for the conformal dimensions $\Delta_1, \dots, \Delta_N$ of the light operators. As a concrete example, we consider the case of two light operators $\mathcal{O}_1 = \mathbb{1}$ and $\mathcal{O}_2$: the equation (5.37) implies that the light conformal dimensions are $\Delta_{\mathbb{1}} = 0$, and $\Delta_2 \in \{2\Delta_\phi, 4\Delta_\phi + 3\}$.

To our best knowledge, this approach has never been explored before in the context of thermal CFTs, but similar comments are present in the literature for zero temperature CFTs when considering the crossing equation instead of equation (5.17). Interestingly, this strengthens the analogy between the KMS condition and the crossing symmetry , proposed by [56] and stressed in [45, 51]. The idea of

using a truncated OPE goes under the name of *Gliozzi method*, introduced in [143], and it is used in different cases, including non-unitary theories [144–148].

5.2 Heavy operators, errors and Tauberian theorems

In the previous Section, we introduced a specific bootstrap problem for thermal CFTs and we derived a set of sum rules as a tool to solve the problem. After having performed a few successful tests as a proof of concept, we employed the truncation of the sum rules as a procedure to extract dynamic information out of them. This Section is dedicated to complete that discussion, providing

- ◇ an asymptotic behaviour for the OPE coefficients a_Δ when $\Delta \to \infty$;
- ◇ an estimation of the error that is made every time the sum rules are truncated.

These goals can be achieved by invoking the *Tauberian theory*. Tauberian theory is a mathematical theory that found application in physics first in [149], and more recently in [57, 59–61, 150–156]. Notably, its application to one-dimensional CFTs is rigorously detailed in [58]. Although we will provide a summary of the notions of Tauberian theory useful to derive the results of this Thesis, we will not be exhaustive: we suggest the reader to consult [157–160] for a detailed treatment of the topic.

5.2.1 A Tauberian theorem at finite temperature

A heuristic derivation of the theorem. We start our discussion on the Tauberian theorem by going through a *heuristic* derivation, mostly based on physical assumptions. This will provide a proof-of-concept which will be later supported by a mathematically sound derivation. We start by considering a scalar, local operator $\phi(\tau, \vec{x})$ and its thermal two-point function with itself

$$\langle \phi(\tau, r)\phi(0,0)\rangle_\beta \ . \tag{5.38}$$

We consider the same two-point function in the limit of zero spatial coordinates

$$\langle \phi(\tau)\phi(0)\rangle_\beta \equiv \langle \phi(\tau,0)\phi(0,0)\rangle_\beta \ . \tag{5.39}$$

In the same limit, the thermal OPE expansion of the two-point function takes the form (5.16)

$$\langle \phi(\tau)\phi(0)\rangle_\beta = \tau^{-2\Delta_\phi} \sum_\Delta \frac{a_\Delta}{\beta^\Delta} \tau^\Delta = \tau^{-2\Delta_\phi} \int_0^\infty \mathrm{d}\Delta \ \rho(\Delta) x^\Delta \ , \tag{5.40}$$

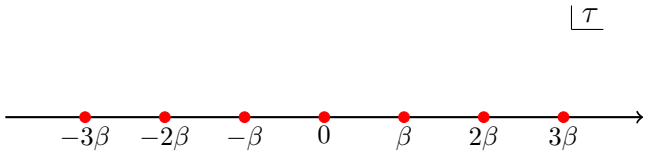


Figure 5.3: *Location of the poles of the function $\langle\phi(\tau)\phi(0)\rangle_\beta$ on the τ real axis. Although they do not appear in this picture, the presence of possible branch cuts is allowed.*

where we introduced the dimensionless ratio $x = \tau/\beta$ and the *OPE spectral density* $\rho(\Delta)$

$$\rho(\Delta) = \sum_{\tilde{\Delta}} a_{\tilde{\Delta}}\ \delta(\Delta - \tilde{\Delta})\ , \tag{5.41}$$

where every $\tilde{\Delta}$ is the conformal dimension of an operator $\widetilde{\mathcal{O}}$ (or a set of operators sharing the same conformal dimension but with different spins) in the OPE $\phi \times \phi$. As already commented in Chapter 4, the thermal OPE does not converge everywhere, but only in the open interval $0 < x < 1$ [45]: this is due to the presence of an infinite number of poles on the real τ axis (see Fig.5.3). We can easily convince ourselves of this by observing that the KMS condition implies that

$$\langle\phi(\tau)\phi(0)\rangle_\beta \overset{\tau\to k\beta}{\sim} (k\beta - \tau)^{-2\Delta_\phi}\ , \tag{5.42}$$

for any $k \in \mathbb{Z}$. Hence, recalling the equation (5.40) we can write

$$\int_0^\infty \mathrm{d}\Delta\ \rho(\Delta) x^\Delta \overset{x\to 1}{\sim} (1-x)^{-2\Delta_\phi}\ . \tag{5.43}$$

Now, we make an Ansatz: the behavior of $\rho(\Delta)$ in the limit $\Delta \to \infty$ should be ruled by a *power-law*

$$\rho(\Delta) \overset{\Delta\to\infty}{\sim} A\Delta^{-\alpha+1}\ . \tag{5.44}$$

The Ansatz will be justified in the following. Plugging it in the two-point function (5.40) we find

$$\langle\phi(\tau)\phi(0)\rangle_\beta \overset{\tau\to\beta}{\sim} A\,\Gamma(2-\alpha)\tau^{-2\Delta_\phi}\left(1 - \frac{\tau}{\beta}\right)^{\alpha-2}\ , \tag{5.45}$$

from which we can read

$$\alpha = -2\Delta_\phi + 2\ , \qquad A = \frac{1}{\Gamma(2\Delta_\phi)}\ . \tag{5.46}$$

Therefore, we find that the behavior of $\rho(\Delta)$ in the limit $\Delta \to \infty$ is

$$\rho(\Delta) \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)}\ . \tag{5.47}$$

The physical interpretation of this result is in terms of the density, in the $\phi \times \phi$ OPE, of heavy operators (captured by the limit $\Delta \to \infty$). Let us observe that the formula (5.47) should not be

regarded as physical: indeed, the conformal spectrum could be discrete, and therefore the density could, in general, be a sum of δ-functions. Nevertheless, the formula (5.47) acquires a correct physical meaning when integrated from 0 to a certain conformal dimension $\Delta \to \infty$

$$\int_0^{\Delta} \rho(\tilde{\Delta})\, \mathrm{d}\tilde{\Delta} \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi}}{\Gamma(2\Delta_\phi + 1)} \ . \tag{5.48}$$

A precise derivation of the theorem. In order to make the heuristic derivation mathematically sound, the Ansatz (5.44) must be proved in its "integrated version", shown in the equation (5.48). The proof, whose details can be found in the Appendix B.2, is based on the following assumptions:

- ⋄ the function $\rho(\Delta)$ must show the behavious

$$\int_0^{\infty} \mathrm{d}\Delta\ \rho(\Delta)\, x^{\Delta} \overset{x\to 1}{\sim} (1-x)^{-\alpha} \ , \quad \alpha > 0 \ ; \tag{5.49}$$

  and the coefficient α must be bigger than 1

$$\alpha > 1 \ . \tag{5.50}$$

- ⋄ the function $\rho(\Delta)$ must be a real function bounded from below, which means that there exists a real positive number C such that

$$\rho(\Delta) \geq -C \ ; \tag{5.51}$$

By looking at the equation (5.43), the assumption (5.50) can be reformulated as follows

$$\Delta_\phi > \frac{1}{2} \ . \tag{5.52}$$

This is physically satisfied by the unitarity bound for scalars

$$\Delta_\phi \geq \frac{d-2}{2} \ , \tag{5.53}$$

which means that the only unitary, physically relevant theory that does not satisfy this bound (in $d > 2$) is the free scalar theory in three dimensions (which saturates the bound). Regarding two-dimensional theories, the conformal dimensions of scalars do not have to be greater than 1/2, since these theories can be solved analytically by using the conformal map between the plane and the cylinder. Acknowledging these special cases, the bound (5.52) allows for a very wide range of theories to satisfy the hypothesis (5.50).

Recalling the definition (5.41) of $\rho(\Delta)$, the reality condition is inherited by the coefficients a_Δ, which will be considered to be real numbers in the following. This assumption can be physically satisfied by working with unitary theories.

The boundedness hypothesis. The most delicate hypothesis is the assumption (5.51), which also represents one of the novelties of this work. Tauberian theorems have been employed in the bootstrap literature with success, and their regimes of validity have been discussed in detail. In particular, an interesting problematic situation was proposed in [58], where the authors observed that a possible behaviour of the integral of the $\rho(\Delta)$ function, without any other assumption, can be an *oscillation* between two power-law behaviours. In [57] and [58] the key assumption to avoid this situation was the hypothesis that the weighted density $\rho(\Delta)$ is *non-negative*. However, in our scenario we cannot make this assumption: recalling the definition of our density (5.41), the coefficients a_Δ are allowed to be negative. The lack of positivity of the thermal OPE coefficients has been known to be one of the main obstacles in the thermal conformal bootstrap program. In this Thesis, the positivity condition was relaxed to the hypothesis (5.51), and in the Appendix B.2 it is shown that the Tauberian theorem can still be proven. The problem we need to address is how to justify the boundedness from below of the thermal OPE coefficients.

To conclude the precise derivation of the Tauberian theorem, we will prove that there exists a cutoff conformal dimension $\widehat{\Delta}$ such that, for $\Delta > \widehat{\Delta}$, the thermal OPE coefficients all share the same sign (say positive[2]). Since each thermal OPE coefficient is finite (to ensure the convergence of the OPE expansion), the boundedness from below will come as a *corollary* of the same-sign lemma, with the bound from below given by the lowest OPE coefficient among the operators with conformal dimension $0 < \Delta < \widehat{\Delta}$. A pictorial representation of the behavior of the coefficients a_Δ as a function of Δ is shown in Fig.(5.4). The proof of the sign stability for large Δ is given in the Appendix B.3. In summary, we use the Euclidean inversion formula combined with the KMS condition to compute a coefficient a_Δ in terms of all the other OPE coefficients. We then compare a_Δ and $a_{\Delta+\delta\Delta}$, where $\delta\Delta$ is the minimal distance between two adjacent operators in the OPE spectrum. The result for

[2]In the case in which the sign is negative, the same procedure can be adopted by flipping the sign of the density.

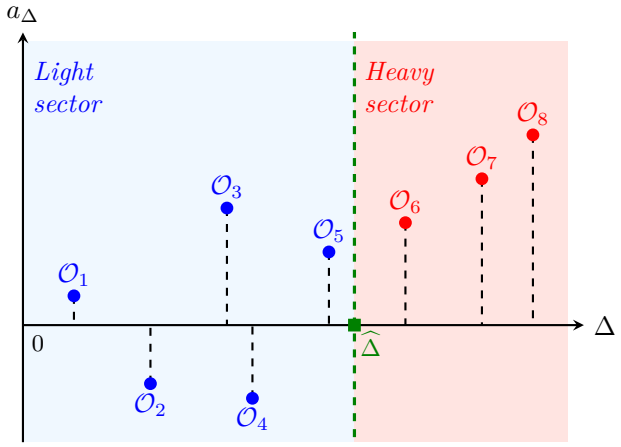


Figure 5.4: *Schematic depiction of the expected behavior of the OPE coefficients a_Δ. The cutoff conformal dimension $\widehat{\Delta}$ is represented by a green square and dashed line, and it divides the set of all the operators in the OPE $\phi \times \phi$ into two sectors. In the "light sector", represented in blue, we have a finite number of operators with $\Delta < \widehat{\Delta}$, and the coefficients a_Δ are allowed to be both positive and negative. In the "heavy sector", represented in red, we have an infinite number of operators with $\Delta > \widehat{\Delta}$; in this Thesis, it is argued that the sign of the coefficients a_Δ is always the same (positive in the picture). An OPE spectrum with a behaviour similar to this pictorial representation ensures that the Tauberian theorem holds.*

$\Delta \gg 1$ is

$$a_{\Delta+\delta\Delta} = a_\Delta \left[1 + O\left(\frac{1}{\Delta}\right)\right] , \quad \Delta \gg 1 . \tag{5.54}$$

Then if a_Δ is not going to zero, there will be a cutoff $\widehat{\Delta}$ such that the correction $O\left(1/\widehat{\Delta}\right)$ becomes negligible compared to $a_{\widehat{\Delta}}$. As a consequence, all the heavy contributions $a_{\Delta > \widehat{\Delta}}$ will share the same sign. This conclusion can be explicitly checked and it is correct in all the known examples, up to our knowledge. It should be noticed that this does not imply that all the coefficients a_Δ are positive, but rather that all the OPE coefficients of the operators in the heavy sector share the same sign.

5.2.2 An asymptotic behaviour at large Δ

The main purpose of the previous Section was to introduce a new tool, the Tauberian theorem. Summarizing, the thermal two-point function can be written as

$$\langle \phi(\tau)\phi(0)\rangle_\beta = \tau^{-2\Delta_\phi} \sum_\Delta \frac{a_\Delta}{\beta^\Delta} \tau^\Delta = \tau^{-2\Delta_\phi} \int_0^\infty \mathrm{d}\Delta \; \rho(\Delta) x^\Delta , \tag{5.55}$$

where the *exact* expression of the OPE spectral density is

$$\rho(\Delta) = \sum_{\tilde{\Delta}} a_{\tilde{\Delta}} \; \delta(\Delta - \tilde{\Delta}) . \tag{5.56}$$

If the hypothesis (5.49), (5.51) and (5.50) are satisfied, the Tauberian theorem states that

$$\int_0^{\Delta} \rho(\tilde{\Delta})\ \mathrm{d}\tilde{\Delta} \ \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi}}{\Gamma(2\Delta_\phi+1)}\ , \tag{5.57}$$

and provides an *approximated* expression of the OPE spectral density in the large Δ limit

$$\rho(\Delta) \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\ . \tag{5.58}$$

Using these tools, the first goal we want to achieve is to obtain an asymptotic behaviour for the OPE coefficients a_Δ at large Δ. Let us consider the OPE coefficient a_{Δ_H} associated with the conformal dimension Δ_H. Considering the exact expression of the OPE spectral density (5.56), we can write the coefficient as an integration of the spectral density from $\Delta_H-\delta\Delta$ to Δ_H, where $\delta\Delta$ parameterizes the distance between Δ_H and the conformal dimension of the operators in the OPE with dimension just smaller than Δ_H. We assume that $\delta\Delta \ll \Delta_H$. The integration is engineered in such a way that only the δ-function multiplying the coefficient a_{Δ_H} is picked up

$$a_{\Delta_H} = \int_{\Delta_H-\delta\Delta}^{\Delta_H} \rho(\Delta)\mathrm{d}\Delta\ . \tag{5.59}$$

Now we consider Δ_H to be large and we perform the same integration, but using the Tauberian theorem (5.57)

$$a_{\Delta_H} = \left(\int_0^{\Delta_H} \mathrm{d}\Delta - \int_0^{\Delta_H-\delta\Delta} \mathrm{d}\Delta\right) \rho(\Delta) \overset{\Delta_H\to\infty}{\sim} \frac{\Delta_H^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\,\delta\Delta\ . \tag{5.60}$$

Therefore,

$$a_{\Delta_H} \overset{\Delta_H\to\infty}{\sim} \frac{\Delta_H^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\delta\Delta\ . \tag{5.61}$$

We stress that in this derivation Δ_H is the conformal dimension of a heavy operator and $\delta\Delta$ is the difference between Δ_H and the conformal dimension of the nearest neighbourhood operator of smaller conformal dimension.[3] The noteworthy feature of the asymptotic (5.61) is that is a prediction for *any thermal CFT*, including (and especially) any thermal CFT where the thermal two-point function $\langle\phi\phi\rangle_\beta$ is unknown. In particular, it holds for holographic thermal CFTs, where interesting applications connected to black branes and black holes solutions are being explored by employing large Δ asymptotic behaviours [161].

[3]We do not claim this result for any Δ_H and arbitrary $\delta\Delta$. To claim such a result one should prove a finite-size version of the Tauberian theorem, which is usually stronger than the Tauberian theorem presented here.

A comment on the subtraction procedure. It was possible to derive the result (5.61) thanks to the passage (5.60), where we employed a subtraction prescription. This passage is delicate and it requires an additional assumption. First of all, we explain why the passage (5.60) is not completely rigorous. The key point is to recall that the Tauberian theorem is an approximated result, and it can accommodate for sub-leading corrections. In the absence of any further assumptions, the Tauberian theorem proved in this Thesis admits the following correction

$$\int_0^{\Delta} \rho(\tilde{\Delta}) \mathrm{d}\tilde{\Delta} \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi}}{\Gamma(2\Delta_\phi+1)} \left[1 + O\left(\frac{1}{\log\Delta}\right)\right] \ . \tag{5.62}$$

In the literature, this is known as the *real Tauberian theorem*. When extracting the OPE coefficients following the prescription

$$a_\Delta = \left(\int_0^{\Delta} \mathrm{d}\tilde{\Delta} - \int_0^{\Delta-\delta\Delta} \mathrm{d}\tilde{\Delta}\right) \rho(\tilde{\Delta}) \ , \tag{5.63}$$

the subtraction between the two leading terms gives

$$a_\Delta \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)} \delta\Delta \ . \tag{5.64}$$

Possible problems could arise due to the $O(1/\log\Delta)$ correction in the equation (5.62): in the most general case, this leads to

$$a_\Delta \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)} \delta\Delta \left[1 + O\left(\frac{\Delta}{\log\Delta}\right)\right] \ , \tag{5.65}$$

In this case, the prediction on a_Δ is spoiled since in the limit $\Delta \to \infty$ the error dominates over the leading term! Therefore, this clarifies why additional conditions are needed to produce correct predictions. The situation slightly improves if we employ the *complex Tauberian theorem* [160]: in this case, the correction $O(1/\log\Delta)$ in the equation (5.62) is traded for one of order $O\left(1/\Delta\right)$. The subtraction procedure (5.63) returns

$$a_\Delta \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)} \delta\Delta \Big[1 + O\left(1\right)\Big] \ . \tag{5.66}$$

This is a step forward since the error cannot spoil the power-law behaviour; however, it is still not possible to fix the overall numerical coefficient. A solution to this problem, and the final assumption we will make, is to consider a Tauberian theorem with a sub-leading correction of the form

$$\int_0^{\Delta} \rho(\tilde{\Delta}) \ \mathrm{d}\tilde{\Delta} \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi}}{\Gamma(2\Delta_\phi+1)} \left[1 + \frac{C}{\Delta} + \dots\right] \ , \qquad C = \text{const.} \ . \tag{5.67}$$

The theorem (5.67) is stronger than the complex Tauberian theorem: in fact, the correction is not only of order $O\left({}^{1}/_{\Delta}\right)$ but it can be written asymptotically as ${}^{C}/_{\Delta}$. A simple example that satisfies the complex Tauberian theorem, but not its stronger version (5.67), could be a theory where

$$\int_0^{\Delta} \rho(\tilde{\Delta})\, \mathrm{d}\tilde{\Delta} \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi}}{\Gamma(2\Delta_\phi+1)} \left[1+\frac{\cos\Delta}{\Delta}+\dots\right] . \tag{5.68}$$

Although $\cos\Delta = O(1)$, its oscillating behaviour makes the error important when we want to compute the OPE coefficients by performing the subtraction (5.63). Let us remark that the scenario (5.68) is just an illustrative example and we do not know if a physical theory exhibiting such behaviour exists. On the contrary, we can easily come up with a physically motivated example where the stronger version of the Tauberian theorem (5.67) is satisfied. Let us consider a two-dimensional correlator between Virasoro primaries of conformal dimension $\Delta = 2$. The leading Tauberian behaviour is given by

$$\int_0^{\Delta} \rho(\tilde{\Delta})\, \mathrm{d}\tilde{\Delta} \overset{\Delta\to\infty}{\sim} \frac{\Delta^4}{24} \left[1+\dots\right] . \tag{5.69}$$

Since the thermal two-point functions between Virasoro primaries of two-dimensional models are always known analytically, we can extract the one-point function coefficients a_n and compute their partial sums $\sum_{n=0}^{\Delta} a_n$. The comparison between the Tauberian prediction and the exact computation is plotted in Figure 5.5. As can be seen from the plot, the lines do not exactly coincide and this is a hint of the presence of a subleading correction. To find it, we subtracted the Tauberian prediction from the exact results and we fit the result. Interestingly enough, the error is well approximated by a cubic polynomial

$$\int_0^{\Delta} \rho(\tilde{\Delta})\, \mathrm{d}\tilde{\Delta} \overset{\Delta\to\infty}{\sim} \frac{\Delta^4}{24} \left[1+\frac{72}{\Delta}+\dots\right] . \tag{5.70}$$

Assuming the stronger Tauberian theorem (5.67) to hold completely justifies the subtraction procedure (5.63), since the result is

$$a_\Delta \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)} \delta\Delta \left[1-\left(\Delta_\phi-\frac{1}{2}\right)\frac{\delta\Delta}{\Delta}+\dots\right] \tag{5.71}$$

and the error on the prediction becomes unequivocably subleading in the large Δ limit. From a physical point of view, this result can be justified by asking the approximated spectral density $\rho(\Delta)$ to be *analytic* in the large Δ limit. It would be interesting to see if this assumption could be formally connected to the analyticity of the thermal OPE coefficients, as the result (5.71) seems to suggest.

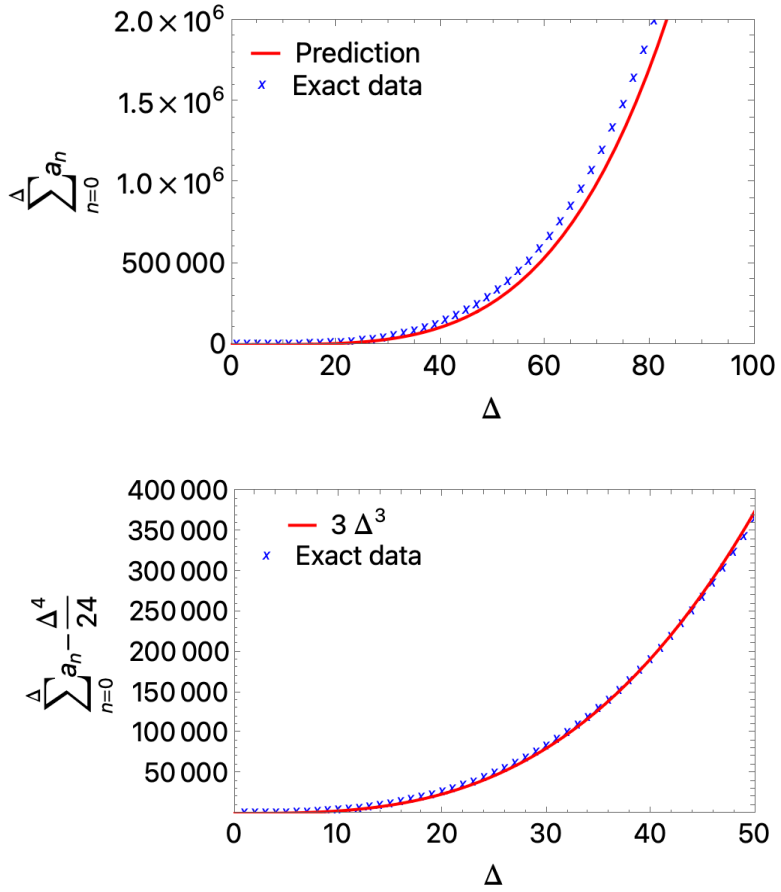


Figure 5.5: *In the first panel from above, we plot the behaviour of the partial sums over the exact OPE coefficients (indicated by blue dots) together with the Tauberian prediction (indicated by a straight red line). In the second panel from above, we report the comparison between the computed error between the exact partial sums and the Tauberian prediction compared with a naive fit.*

5.2.2.1 Tauberian predictions and the $3d$ critical $\mathrm{O}(N)$ model at large N

In this Section, we want to test the sum rules (5.17) and the Tauberian prediction (5.61) on an interacting, higher-dimensional theory: the $3d$ critical $\mathrm{O}(N)$ model at large N. This theory has been studied in the literature both at zero temperature (see e.g. [162–168] and citation thereof) and at finite temperature [52, 115, 116, 169]. We start by reviewing a few features of this model. The action of the theory is given by

$$S = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2 x \left[\frac{1}{2} \left(\partial_\mu \phi_i\right)^2 + \frac{\lambda}{4!} \left(\phi_i \phi_i\right)^2 \right] \ . \tag{5.72}$$

It is a standard procedure in the literature to apply the *Hubbard-Stratonovich transformation* to the $(\phi_i\phi_i)^2$ coupling. The master formula is given by the following functional integration

$$\exp\left(-\frac{a}{2}\mathcal{F}[\phi]\mathcal{F}[\phi]\right) = \int [\mathrm{d}\Sigma] \exp\left(\frac{a}{2}\,\Sigma^2 - a\,\Sigma\,\mathcal{F}[\phi]\right) \ , \tag{5.73}$$

where a is a constant coupling, $\mathcal{F}[\phi]$ a functional of the fundamental fields ϕ_i and Σ an auxiliary field. If we consider $\mathcal{F}[\phi] = \phi_i\phi_i$, then we can write

$$\exp\left(-\frac{\lambda}{4!}(\phi_i\phi_i)^2\right) = \int [\mathrm{d}\Sigma] \exp\left(\frac{\lambda}{24}\,\Sigma^2 - \frac{\lambda}{12}\,\Sigma\,\mathcal{F}[\phi]\right) \ . \tag{5.74}$$

If we redefine the auxiliary field as $\sigma \equiv \frac{\lambda}{6}\,\Sigma$, we end up with the transformed action

$$S = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^2x \left[\frac{1}{2}\left(\partial_\mu\phi_i\right)^2 + \frac{1}{2}\sigma\left(\phi_i\phi_i\right) - \frac{3\,\sigma^2}{2\lambda}\right] \ . \tag{5.75}$$

The critical point is reached for $\lambda \to \infty$. The thermal mass of this theory (see equation (4.37) and the annexed discussion) is given by the thermal one-point function of σ, which can be computed as in [45, 170]

$$\langle\sigma\rangle_\beta = m_{\mathrm{th}}^2 = \frac{4}{\beta^2}\log^2\left(\frac{1+\sqrt{5}}{2}\right) \ . \tag{5.76}$$

The thermal two-point function $\langle\phi_i\phi_j\rangle_\beta$ was also studied in [45] and can be obtained by inverse Fourier transforming the propagator of the field ϕ:

$$\langle\phi_i(\tau,r)\phi_j(0,0)\rangle_\beta = \delta_{ij}\sum_{m=-\infty}^{\infty}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{e^{-i\vec{k}\cdot\vec{x}-i\omega_m\tau}}{\omega_n^2+\vec{k}^2+m_{\mathrm{th}}^2} = \delta_{ij}\sum_{m=-\infty}^{\infty}\frac{e^{-m_{\mathrm{th}}\sqrt{(\tau+m\beta)^2+r^2}}}{\sqrt{(\tau+m\beta)^2+r^2}} \ . \tag{5.77}$$

Interestingly, reducing the expression (5.77) on the same thermal circle, i.e. imposing $r \to 0$, the thermal two-point function achieves a closed form

$$\begin{aligned}\langle\phi_i(\tau)\phi_j(0)\rangle_\beta &= \delta_{ij}\sum_{m=-\infty}^{\infty}\frac{e^{-m_{\mathrm{th}}|\tau+m\beta|}}{|\tau+m\beta|} \\ &= \delta_{ij}\frac{e^{m_{\mathrm{th}}(\tau-\beta)}}{\beta}\Phi\left(e^{-m_{\mathrm{th}}\beta},1,\frac{\beta-\tau}{\beta}\right) + \delta_{ij}\frac{e^{-m_{\mathrm{th}}\tau}}{\beta}\Phi\left(e^{-m_{\mathrm{th}}\beta},1,\frac{\tau}{\beta}\right) \ ,\end{aligned} \tag{5.78}$$

for $\tau \in (0,\beta)$,[4] where Φ is the *Lerch transcendent*

$$\Phi(z,s,\alpha) = \sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s} \ . \tag{5.79}$$

[4]This breaks explicitly parity: $\tau \to -\tau$. However, it is easy to recover it by imposing it explicitly, adding by hand the second term in the right-hand side of the equation (5.78).

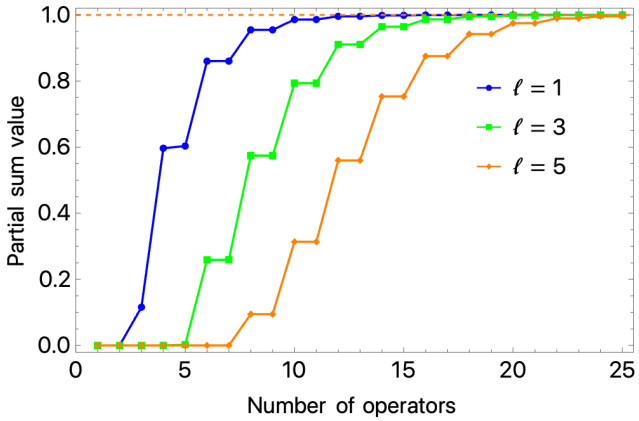


Figure 5.6: *Comparing left (straight, dashed lines) and right (continuous lines) side of equation* (5.17) *for different values of* ℓ *for the two-point function of fundamental scalars in the* 3*d critical* O(N) *model at large* N*. The operators participating in the sum rules are powers of the Hubbard-Stratonovich field* σ^m *and the double-twists of the fundamental field* $[\phi_i\phi_i]_{p,q}$*. It was convenient to multiply both sides of equation* (5.17) *by a factor* $1/\ell!$ *for presentation purposes.*

The Lerch transcendent enjoys a representation in terms of hypergeometric functions

$$\langle\phi_i(\tau)\phi_j(0)\rangle_\beta = \delta_{ij}\frac{e^{-m_{\rm th}(\beta-\tau)}}{\beta-\tau}\,{}_2F_1\left[\begin{matrix}1,\frac{\beta-\tau}{\beta}\\1+\frac{\beta-\tau}{\beta}\end{matrix}\middle|e^{-m_{\rm th}\beta}\right]+\delta_{ij}\frac{e^{-m_{\rm th}\tau}}{\tau}\,{}_2F_1\left[\begin{matrix}1,\frac{\tau}{\beta}\\1+\frac{\tau}{\beta}\end{matrix}\middle|e^{-m_{\rm th}\beta}\right]\,. \quad (5.80)$$

The equation (5.80) allows for the extraction the exact OPE coefficients, by expanding the right-hand side in powers of τ. Let us notice that the conformal dimensions of the fundamental fields are

$$\Delta_{\phi_i}=\frac{1}{2}+O\left(\frac{1}{N}\right)\,,\qquad \Delta_\sigma=2+O\left(\frac{1}{N}\right)\,, \quad (5.81)$$

therefore we expect only operators with integer conformal dimensions in the $\phi_i\times\phi_j$ OPE, since operators constructed out of an odd number of copies of ϕ_i are forbidden by $\mathbb{Z}_2$ symmetry. This makes sure that the OPE expansion of the two-point function (5.80) is equivalent to its Taylor expansion. Similarly to what was done in Section 5.1.1, the sum rules (5.17) can be numerically checked and the result is plotted in Figure 5.6

We are now interested in checking the Tauberian prediction for the coefficients a_Δ. In particular, recalling the equation (5.48) and considering the conformal dimensions (5.81) we expect the partial sums over the OPE coefficients to satisfy

$$\int_0^\Delta \rho(\tilde\Delta)\mathrm{d}\tilde\Delta\overset{\Delta\to\infty}{\sim}\Delta\left[1+O\left(\frac{\log\Delta}{N}\right)\right]\sim\Delta\,,\qquad N\to\infty\,. \quad (5.82)$$

Considering that the distance between two adjacent operators at large Δ is $\delta\Delta = 2$,[5] we predict

$$a_\Delta \overset{\Delta\to\infty}{\sim} 2 \ . \tag{5.83}$$

The predictions (5.82) and (5.83) are explicitly checked in Figure 5.7, where we compared them with the exact partial sums and the exact values of the coefficients a_Δ. In particular, the second plot from above in Figure 5.7 should be compared with the pictorial representation of the expected behaviour of the a_Δ coefficients, illustrated in Figure 5.4. Notice how the exact coefficients a_Δ are badly represented by the Tauberian prediction for $\Delta \lesssim 10$, but they are well fitted by the red line once the conformal dimension Δ becomes larger. Moreover, notice that they achieve sign stability and stop oscillating for $\Delta \gtrsim 10$, as described by the formula (5.54).

5.2.3 Error estimation for the truncated sum rules

Now that the Tauberian theorem has been proven and tested, we can employ it to complete the analysis of the bootstrap problem we set up in Section 5.1.1. We recall the sum rules for the coefficients a_Δ

$$\frac{\Gamma\left(2\Delta_\phi + \ell\right)}{\Gamma\left(2\Delta_\phi\right)} = \sum_\Delta a_\Delta F(\Delta, \ell) \ , \tag{5.84}$$

where the right-hand side was rewritten in a more concise way by introducing the function

$$F(\Delta, \ell) \equiv \frac{\Gamma\left(\Delta - 2\Delta_\phi + 1\right)}{2^\Delta \Gamma\left(\Delta - 2\Delta_\phi - \ell + 1\right)} \ . \tag{5.85}$$

The truncation procedure splits the right hand side in two terms after selecting a conformal dimension Δ_H

$$\frac{\Gamma\left(2\Delta_\phi + \ell\right)}{\Gamma\left(2\Delta_\phi\right)} = \sum_{\Delta<\Delta_H} a_\Delta F(\Delta, \ell) + \sum_{\Delta\geq\Delta_H} a_\Delta F(\Delta, \ell) \ . \tag{5.86}$$

The bootstrap problem becomes predictive if we are allowed to neglect the heavy operators' tail and we end up with a linear equation depending on a value of ℓ and a finite set of parameters a_Δ

$$\frac{\Gamma\left(2\Delta_\phi + \ell\right)}{\Gamma\left(2\Delta_\phi\right)} \sim \sum_{\Delta<\Delta_H} a_\Delta F(\Delta, \ell) \ . \tag{5.87}$$

This procedure is plausible and the numerical tests in Figures 5.1a, 5.1b, 5.2a, 5.2b, and 5.6 confirm this intuition. Now we provide an estimation of the error introduced going from the equation (5.86)

[5] We consider the number $\delta\Delta = 2$ because it is the distance between two adjacent double-twist operators, or between two adjacent operators constructed out of powers of the Hubbard-Stratonovich field.

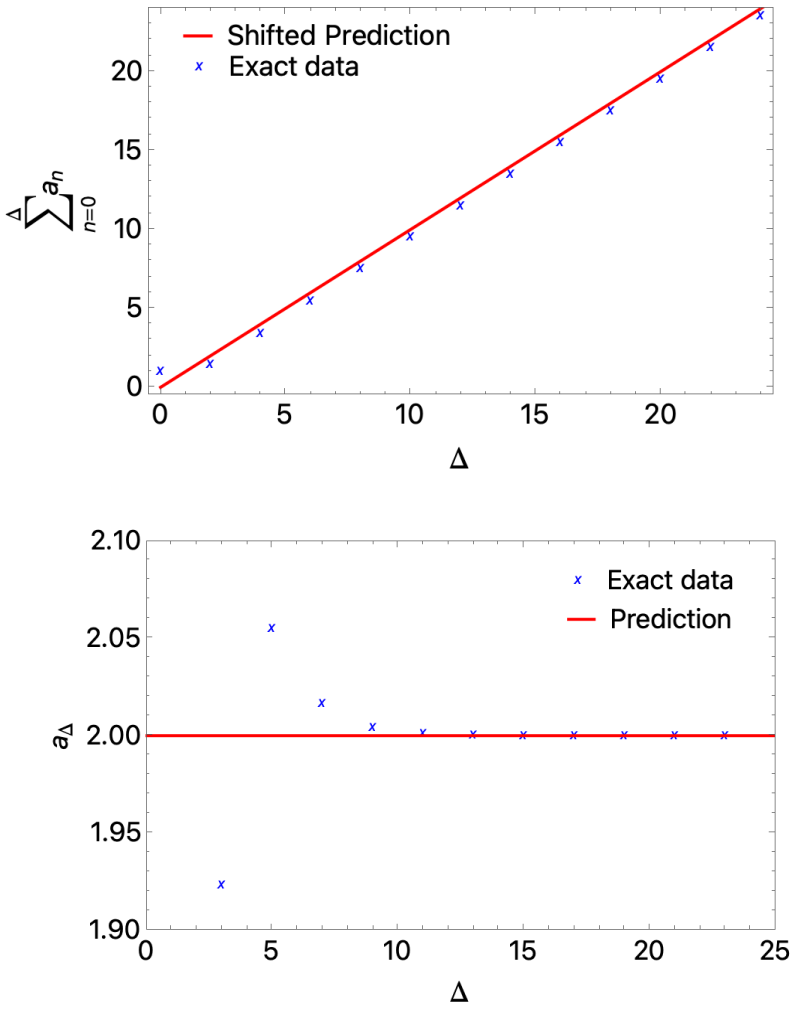


Figure 5.7: *In the first panel from above, we plot the behaviour of the partial sums over the exact OPE coefficients (indicated by blue dots) together with the Tauberian prediction (indicated by a straight red line). Since the Tauberian theorem only predicts the leading behaviour of the sum, it returns a linear behaviour in* Δ*. To properly compare the exact data with the prediction, the latter needs to be vertically shifted by a constant* $k = 2$*: this is not predicted by the theorem, which can only return the slope of the linear behaviour. In the second panel from above, we report the comparison between the exact OPE coefficients (represented by blue dots) and the Tauberian prediction* (5.83) *(represented by a constant, straight red line).*

to the equation (5.87). We consider again the right-hand side of the equation (5.86), assuming Δ_H to be large: employing the Tauberian prediction (5.61), we can rewrite as follows

$$\ldots = \sum_{\Delta<\Delta_H} a_\Delta F(\Delta,\ell) + \sum_{\Delta\geq\Delta_H} \left(\frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\delta\Delta\right) F(\Delta,\ell) \tag{5.88}$$

By assuming that $\Delta_H \geq 2\Delta_\phi + \ell - 1$, the second term is an infinite sum of positive terms. Hence, we can bound the correction due to the tail of heavy operators. Let us start by bounding $\delta\Delta$. The

reasoning is the following: among all the possible heavy operators appearing in the OPE $\phi \times \phi$, we surely have *double-twist operators* $[\phi\phi]_{p,q}$; among the double-twist operators, we have those with $q = 0$, i.e. where no Laplacian is applied. The distance $\delta\Delta$ between two double-twist operators in the class $[\phi\phi]_{p,0}$ can be estimated by noticing that their conformal dimensions can be written as (taking into account the anomalous dimensions as discussed in [136])

$$\Delta_{[\phi\phi]_{p,0}} = 2\Delta_\phi + p + O\left(\frac{1}{p}\right) \ , \tag{5.89}$$

which implies that $\delta\Delta \leq \delta p + O(1/p)$; since we are considering the large Δ limit, this is achieved for this particular class of operators in the large spin limit $p \to \infty$. Therefore, given that $\delta p = 2$ for two adjacent double-twist operators in the class $[\phi\phi]_{p,0}$,[6] we have $\delta\Delta \leq 2$. The inequality symbol ensures the possibility of having other operators in between the two double-twist operators. Hence, we find the upper bound

$$\sum_{\Delta \geq \Delta_H} \left(\frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)} \delta\Delta\right) F(\Delta, \ell) \leq 2 \sum_{\Delta \geq \Delta_H} \left(\frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)}\right) F(\Delta, \ell) \ . \tag{5.90}$$

Finally, we can estimate the the right-hand side of the inequality (5.90) by turning the sum into an integration

$$2\int_{\Delta_H}^{\infty} \mathrm{d}\Delta \left(\frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)}\right) F(\Delta, \ell) \overset{\Delta_H \gg 1}{\sim} \frac{1}{\Gamma(2\Delta_\phi)\log 2} \frac{\Delta_H^{2\Delta_\phi + \ell - 1}}{2^{\Delta_H}} = O\left(\frac{\Delta_H^{2\Delta_\phi + \ell - 1}}{2^{\Delta_H}}\right) \ . \tag{5.91}$$

We conclude that the sum rule (5.86) can be rewritten by bounding the sum over the heavy operators as follows

$$\frac{\Gamma\left(2\Delta_\phi + \ell\right)}{\Gamma\left(2\Delta_\phi\right)} = \sum_{\Delta < \Delta_H} a_\Delta F(\Delta, \ell) + O\left(\frac{\Delta_H^{2\Delta_\phi + \ell - 1}}{2^{\Delta_H}}\right) \ . \tag{5.92}$$

Notice that the estimated error grows with ℓ: it was already noticed that higher values of ℓ make the convergence of the sum rules more difficult to achieve. Moreover, the error decreases in the large Δ_H limit, as predicted.

[6] The one-point functions of odd spin operators are set to be zero by symmetry arguments following [45].

6
Conclusions

In Chapters 4 and 5, we studied thermal CFTs (in the sense provided by Figure 4.1) from many points of view. First of all, we derived a set of *broken* Ward identities for the symmetries composing the conformal group. Although some symmetries are preserved on the thermal manifold, others are broken with important consequences for physics. In particular, we exactly derived the corrective terms modifying the flat-space Ward identities. We derived non-perturbative results from these identities, and in particular, we made explicit predictions on the *energy spectrum* of a generic thermal CFT and on the *thermal mass*. Moreover, we commented on the connection between the broken Ward identity associated with dilatations and the extension of the Cardy formula in dimensions higher than two. We then explored how to employ these identities to constrain the kinematics of thermal correlators, and we formulated a *kinematic bootstrap problem* for the thermal two-point function between two identical scalars, providing the example of the $4d$ free scalar theory at finite temperature as a proof of concept. A different bootstrap problem was set up in Chapter 5, with the ultimate goal of computing the thermal one-point function coefficients $b_{\mathcal{O}}$, defined by the equation (4.48). To use an analogy, since they play a crucial role in the thermal OPE between two local operators, the information they provide is as important as the knowledge of the three-point function coefficients in zero-temperature CFT. The bootstrap problem is defined by a set of *sum rules*, and the unknowns are numerical coefficients a_Δ defined as in equation (5.16). We tested the sum rules on some theories where the control over the kinematics of the thermal two-point functions is complete (GFFs, two-dimensional theories, $3d$ O(N) model at large N). Facing the problem of truncating the sum rules to achieve predictivity, we estimated the error produced by doing so, relying on the technology provided by a *Tauberian theorem* that we proved. The main prediction that can be

extracted from the theorem is the asymptotic behaviour of the coefficients a_Δ at large Δ: with this information, the error cited above was estimated.

There are multiple possible paths to explore ahead. First of all, referring to Figure 4.1, it would be interesting to combine the broken Ward identities we derived with a *perturbative* study of the space of theories around the critical value of the coupling $g^\star$. For this purpose, it might be convenient to combine the broken Ward identities with the *Källén-Lehmann* expansion, in addition to the OPE. Secondly, a very clear goal consists of solving *numerically* the bootstrap problem set by the sum rules (5.17). The asymptotic behaviour of large Δ coefficients could be employed to precisely re-sum the tail of large conformal dimension operators in the $\phi \times \phi$ OPE (in analogy with [171]), avoiding a naive truncation of the sum rules. Reinforcement learning techniques recently developed [172–174] could contribute as well in order to achieve a well-approximated result.

Another crucial direction to pursue is making use of what has been learned about thermal CFTs to study *holographic* thermal CFTs, gaining new insights into quantum gravity in asymptotically Anti-de Sitter spaces. This purpose, which has recently garnered a lot of interest [53, 161, 175–177], would likely require further exploration of the broken Ward identities machinery in *momentum space* [178], rather than in position space. In particular, in [177], it was pointed out that thermal two-point functions in holographic theories can be written in terms of the *quasi-normal modes* of AdS black holes: a very significant result would be connecting quasi-normal modes with the thermal one-point function coefficients. Furthermore, studying black holes through holographic thermal CFTs would require a modification of the thermal manifold $\mathcal{M}_\beta = S^1_\beta \times \mathbb{R}^{d-1}$, replacing it with $\mathcal{M}_\beta = S^1_\beta \times S^{d-1}_R$. This would require the extension of our procedures to more general ambient manifolds; a similar direction has been pursued in [179, 180]. Moreover, it would be important to reach a better understanding of the *phase transition* that is expected to occur at large N for a certain critical value of the ratio β/R. Expectation values of Polyakov loops are regarded as the order parameters of this transition, hence it is important to study line defects in thermal CFT setups. Some works have recently started exploring this direction [48, 181].

A
Part I Appendices

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A.1 The $[\delta^\Phi, \delta^\Psi]$ commutator: some details

The δ^U, δ^N and $\delta^{\mathcal{T}V}$ transformations. By performing a tedious but straightforward computation, it can be shown that the commutator $[\delta^\Phi, \delta^\Psi]X^i$ reads explicitly

$$[\delta^\Phi_{\epsilon_1}, \delta^\Psi_{\epsilon_2}]X^i = \epsilon_2 D\epsilon_1 \widehat{\Phi^j\, \Psi^i{}_j} - \epsilon_1 D\epsilon_2 \widehat{\Psi^j\, \Phi^i{}_j} + (-1)^{p-1}\epsilon_2\epsilon_1 \left(\widehat{[\Phi,\Psi]^i_{\mathrm{F.N.}}} + \widehat{i_{\partial X}(\Phi)^j \Psi^i{}_j} + (-1)^{pq}\, \widehat{i_{\partial X}(\Psi)^j\, \Phi^i{}_j}\right) , \quad \text{(A.1)}$$

where we introduced the *Frölicher–Nijenhuis bracket*

$$[\Phi,\Psi]^i_{\mathrm{F.N.}} = \mathrm{d}\Phi^j \wedge \Psi^i{}_j - (-1)^p\, \Phi^j \wedge \partial_j \Psi^i + (-1)^{pq} \left(\mathrm{d}\Psi^j \wedge \Phi^i{}_j - (-1)^q\, \Psi^j \wedge \partial_j \Phi^i\right) . \quad \text{(A.2)}$$

We are interested in rewriting the right-hand side of the commutator in terms of a sum of symmetry transformations. The following lemma, first appeared in [30], helps in this direction.

Lemma 1 (Introduction of U and V forms)**.** *Let Φ be a p-form and Ψ a q-form with support on a d-dimensional Riemannian manifold $\mathcal{Y}$ with metric G_{ij}, and define*

$$U = \frac{1}{c_U}\,\Phi^i \wedge \Psi_i \ , \quad V = \frac{1}{c_V}\,\Phi^{ij} \wedge \Psi_{ij} \ . \tag{A.3}$$

Then, for $d \neq p+q-4$ and $p+q \neq 2$, we have the following identities

$$\Phi^j \wedge \Psi^i{}_j = \alpha_{pq}\, U^i + \beta_{pq}\, \mathrm{d}x^i \wedge V \ , \tag{A.4}$$

$$\Phi^k{}_{(i} \wedge \Psi_{j)k} = \beta_{pq} \left(G_{ij} V - G_{k(i}\, \mathrm{d}x^k \wedge V_{j)} \right) \ , \tag{A.5}$$

where

$$\alpha_{pq} = c_U\, (-1)^p\, \frac{q-1}{p+q-2} \ , \quad \beta_{pq} = -c_V\, \frac{1}{d-(p+q-4)} \ . \tag{A.6}$$

Let us comment the cases where $p+q=2$ or $p+q-4=d$.

- ⋆ $p+q=2$: this scenario happens if and only if $p=q=1$, i.e. when Φ and Ψ are both 1-forms. In this case, $V=0$ and U is a constant, therefore all the identities are empty and one defines $\alpha_{11}=0$ *ad hoc*;
- ⋆ $p+q-4=d$: although this scenario never happens in the work discussed in this Thesis, this corresponds to a case where the degree of the form V equals the dimension of the ambient manifold $\mathcal{Y}$. We will not comment further into this in the following.

Making use of Lemma 1 and of the fact that the forms Φ and Ψ are covariantly constant (and so are U and V), we can rewrite the commutator as in (1.96). To conclude the argument, we only need to prove that the form N is covariantly constant. Let us recall the definition of the N form

$$N = \frac{1}{c_N} \left(H_{jk} \wedge \Phi^j \wedge \Psi^k + 2\,(-1)^p\, \beta_{pq}\, H \wedge V \right) \ , \tag{A.7}$$

where β_{pq} is given in equation (A.6). We want to prove that

$$\nabla_i^+ \left(H_{jk} \wedge \Phi^j \wedge \Psi^k \right) = 2\,(-1)^{p+1}\, \beta_{pq} \nabla_i^+ \left(H \wedge V \right) \ . \tag{A.8}$$

Since Φ and Ψ are covariantly constant by hypothesis, we have

$$\nabla_i^+ \left(H_{jk} \wedge \Phi^j \wedge \Psi^k \right) = \left(\nabla_i^+ H_{jk\ell} \right) \mathrm{d}x^\ell \wedge \Phi^j \wedge \Psi^k \ . \tag{A.9}$$

By using the identity

$$\nabla_i^+ H = -\frac{1}{2}\left[(\mathrm{d}H)_i + R^+_{jk\ell i}\mathrm{d}x^{jk\ell}\right] \ , \tag{A.10}$$

where $R^+_{jk\ell i}$ is the Riemann tensor constructed out of the connection symbols (1.64), we can rewrite the right-hand side of (A.9) as

$$\begin{aligned}\nabla_i^+\left(H_{jk}\wedge\Phi^j\wedge\Psi^k\right) &= -\frac{1}{2}\left[(\mathrm{d}H)_{ijk\ell} + 6\,R^+_{[jk\ell]i}\right]\mathrm{d}x^\ell\wedge\Phi^j\wedge\Psi^k \\ &= -\frac{1}{2}\left[(\mathrm{d}H)_{ijk\ell} + 2\left(R^+_{jk\ell i} + R^+_{k\ell ji} + R^+_{\ell jki}\right)\right]\mathrm{d}x^\ell\wedge\Phi^j\wedge\Psi^k \ .\end{aligned} \tag{A.11}$$

We can use the identity

$$[\nabla_i^+,\nabla_j^+]\Phi = 0 \Rightarrow R^+_{ijk\ell}\,\mathrm{d}x^j\wedge\Phi^i = 0 \tag{A.12}$$

to set the last two terms in the right-hand side of the equation (A.11) equal to zero

$$\nabla_i^+\left(H_{jk}\wedge\Phi^j\wedge\Psi^k\right) = -\frac{1}{2}\left[(\mathrm{d}H)_{ijk\ell} + 2\,R^+_{jk\ell i}\right]\mathrm{d}x^\ell\wedge\Phi^j\wedge\Psi^k \ . \tag{A.13}$$

We notice that the identity (A.12) can be reshaped as follows

$$0 = R^+{}_{ijk\ell}\,\mathrm{d}x^j\wedge\Phi^i{}_m\wedge\mathrm{d}x^m = R^+_{jk\ell i}\Phi^j - R^+_{jm\ell i}\mathrm{d}x^m\wedge\Phi^j{}_k \ , \tag{A.14}$$

so we can rewrite the right-hand side of the equation (A.13)

$$\nabla_i^+\left(H_{jk}\wedge\Phi^j\wedge\Psi^k\right) = -\frac{1}{2}(\mathrm{d}H)_{ijk\ell}\,\mathrm{d}x^\ell\wedge\Phi^j\wedge\Psi^k + \ R^+_{jmi\ell}\,\mathrm{d}x^{\ell m}\wedge\Phi^j{}_k\wedge\Psi^k \ . \tag{A.15}$$

Using Lemma 1, and in particular the identity (A.4), we obtain the following expression

$$\begin{aligned}\nabla_i^+\left(H_{jk}\wedge\Phi^j\wedge\Psi^k\right) = -\frac{1}{2}(\mathrm{d}H)_{ijk\ell}\,\mathrm{d}x^\ell\wedge\Phi^j\wedge\Psi^k + (-1)^p\beta_{pq}\,R^+_{jmi\ell}\,\mathrm{d}x^{\ell mj}\wedge V \\ + (-1)^{q+1}\alpha_{qp}\,R^+_{jmi\ell}\,\mathrm{d}x^{\ell m}\wedge U^j \ .\end{aligned} \tag{A.16}$$

We first focus on the term proportional to the form U. Since the form U is covariantly constant by construction, we recall the identity (A.12)

$$R^+_{jmi\ell}\,\mathrm{d}x^m\wedge U^i = 0 \ , \tag{A.17}$$

and we immediately conclude that the term in the right-hand side of the equation (A.16) vanishes. Focusing instead on the term proportional to $\mathrm{d}H$, recalling that $\mathrm{d}H = O(\ell_s)$ (see equation (1.76)), we can reshape the right-hand side of equation (A.16) up to terms of order $O(\ell_s)$

$$\nabla_i^+\left(H_{jk}\wedge\Phi^j\wedge\Psi^k\right) = (-1)^p\beta_{pq}\left[(\mathrm{d}H)_i + R^+_{j\ell mi}\,\mathrm{d}x^{j\ell m}\right]\wedge V + O(\ell_s) \ . \tag{A.18}$$

Using again the identity (A.10), we find that

$$\nabla_i^+ \left(H_{jk} \wedge \Phi^j \wedge \Psi^k\right) = 2(-1)^{p+1}\beta_{pq}\left(\nabla_i^+ H\right) \wedge V = 2\,(-1)^{p+1}\beta_{pq}\nabla_i^+ \left(H \wedge V\right) + O(\ell_s)\ , \tag{A.19}$$

which proves that the form N is covariantly constant up to the first correction of order $O(\ell_s)$

$$\nabla_i^+ N = O(\ell_s)\ . \tag{A.20}$$

A.2 Special holonomy $s\mathcal{W}$-algebras

A.2.1 $\mathrm{SV}_{\mathrm{Spin}(7)}$ and $\mathrm{Spin}(7)$ special holonomy

The $\mathrm{SV}_{\mathrm{Spin}(7)}$ algebra first appeared in [41] and it describes the symmetry algebra on the worldsheet for superstrings propagating on a background with Spin(7) special holonomy

$$\mathcal{T}(Z_1)\mathcal{T}(Z_2) \sim \frac{2}{Z_{12}^3} + \frac{3}{2}\frac{\theta_{12}}{Z_{12}^2}\mathcal{T} + \frac{\theta_{12}}{Z_{12}}\,\partial\mathcal{T} + \frac{1}{2\,Z_{12}}\,D\mathcal{T} + \dots\ , \tag{A.21}$$

$$\mathcal{T}(Z_1)\mathcal{J}^\Psi(Z_2) \sim 2\frac{\theta_{12}}{Z_{12}^2}\mathcal{J}^\Psi + \frac{\theta_{12}}{Z_{12}}\,\partial\mathcal{J}^\Psi + \frac{1}{2\,Z_{12}}\,D\mathcal{J}^\Psi + \dots\ , \tag{A.22}$$

$$\begin{aligned}\mathcal{J}^\Psi(Z_1)\mathcal{J}^\Psi(Z_2) \sim\ & \frac{46}{3}\frac{1}{Z_{12}^4} + \frac{44}{3}\left(\frac{1}{Z_{12}^2}\mathcal{J}^\Psi + \frac{\theta_{12}}{2\,Z_{12}^2}D\mathcal{J}^\Psi + \frac{1}{2\,Z_{12}}\partial\mathcal{J}^\Psi + \frac{3}{10}\frac{\theta_{12}}{Z_{12}}D\partial\mathcal{J}^\Psi\right)\\ &+ \frac{46}{3}\left(\frac{\theta_{12}}{Z_{12}^3}\mathcal{T} + \frac{1}{Z_{12}^2}D\mathcal{T} + \frac{2}{3}\frac{\theta_{12}}{Z_{12}^2}\partial\mathcal{T} + \frac{1}{4\,Z_{12}}D\partial\mathcal{T} + \frac{\theta_{12}}{4\,Z_{12}}\partial\partial\mathcal{T}\right)\\ &+ 4\,\frac{\theta_{12}}{Z_{12}}\mathcal{N}(\mathcal{T}D\mathcal{T}) + 12\,\frac{\theta_{12}}{Z_{12}}\mathcal{N}(\mathcal{T}\mathcal{J}^\Psi) + \dots\ .\end{aligned} \tag{A.23}$$

The OPEs appearing in the original paper can be recovered from the OPEs (A.21), (A.22) and (A.23) by performing the field redefinition

$$\mathcal{J}^\Psi = -\mathcal{J}^\Psi_{\mathrm{SV}} - \frac{1}{3}D\mathcal{T} \tag{A.24}$$

and breaking the superspace expression into the singular components. $\mathcal{J}^\Psi_{\mathrm{SV}}$ encodes the operators appearing in the original formulation of this algebra.

A.2.2 $\mathrm{Od}(3)$ and $\mathrm{SU}(3)$ special holonomy

The Od(3) algebra first appeared in [40] and it describes the symmetry algebra on the worldsheet for superstrings propagating on a background with SU(3) special holonomy. We have a few significant

OPEs, reading

$$\mathcal{T}(Z_1)\mathcal{T}(Z_2) \sim \frac{3/2}{Z_{12}^3} + \frac{3}{2}\frac{\theta_{12}}{Z_{12}^2}\mathcal{T} + \frac{\theta_{12}}{Z_{12}}\,\partial\mathcal{T} + \frac{1}{2\,Z_{12}}\,D\mathcal{T} + \dots\ , \tag{A.25}$$

$$\mathcal{T}(Z_1)\mathcal{J}^{\omega}(Z_2) \sim \frac{\theta_{12}}{Z_{12}^2}\mathcal{J}^{\omega} + \frac{\theta_{12}}{Z_{12}}\,\partial\mathcal{J}^{\omega} + \frac{1}{2\,Z_{12}}\,D\mathcal{J}^{\omega} + \dots\ , \tag{A.26}$$

$$\mathcal{T}(Z_1)\mathcal{J}^{\Omega_\pm}(Z_2) \sim \frac{3}{2}\frac{\theta_{12}}{Z_{12}^2}\mathcal{J}^{\Omega_\pm} + \frac{\theta_{12}}{Z_{12}}\,\partial\mathcal{J}^{\Omega_\pm} + \frac{1}{2\,Z_{12}}\,D\mathcal{J}^{\Omega_\pm} + \dots\ , \tag{A.27}$$

$$\mathcal{J}^{\omega}(Z_1)\mathcal{J}^{\omega}(Z_2) \sim -\frac{3}{Z_{12}^2} - 2\,\frac{\theta_{12}}{Z_{12}}\mathcal{T} + \dots\ , \tag{A.28}$$

$$\mathcal{J}^{\omega}(Z_1)\mathcal{J}^{\Omega_\pm}(Z_2) \sim \mp\frac{3}{Z_{12}}\mathcal{J}^{\Omega_\mp} \mp \frac{\theta_{12}}{Z_{12}}\,D\mathcal{J}^{\Omega_\mp} + \dots\ , \tag{A.29}$$

$$\mathcal{J}^{\Omega_\pm}(Z_1)\mathcal{J}^{\Omega_\pm}(Z_2) \sim \frac{4}{Z_{12}^3} + 4\frac{\theta_{12}}{Z_{12}^2}\mathcal{T} + \frac{4}{3}\frac{1}{Z_{12}}D\mathcal{T} + \frac{8}{3}\frac{\theta_{12}}{Z_{12}}\partial\mathcal{T} - \frac{2}{Z_{12}}\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\omega}) \tag{A.30}$$

$$- \frac{\theta_{12}}{Z_{12}}D\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\omega}) + \dots\ , \tag{A.31}$$

$$\mathcal{J}^{\Omega_+}(Z_1)\mathcal{J}^{\Omega_-}(Z_2) \sim \frac{4}{Z_{12}^2}\mathcal{J}^{\omega} + 2\frac{\theta_{12}}{Z_{12}^2}D\mathcal{J}^{\omega} + \frac{2}{Z_{12}}\partial\mathcal{J}^{\omega} + \frac{4}{3}\frac{\theta_{12}}{Z_{12}}D\partial\mathcal{J}^{\omega} + 4\frac{\theta_{12}}{Z_{12}}\mathcal{N}(\mathcal{T}\mathcal{J}^{\omega}) + \dots\ . \tag{A.32}$$

Interestingly, in the special holonomy case we can hint towards a CFT-geometry dictionary. Let us consider the structure equations (2.54), which in the real basis we adopted read

$$\Omega_\pm \wedge \omega = 0\ , \quad \omega\wedge\omega\wedge\omega + \frac{3}{2}\,\Omega_-\wedge\Omega_+ = 0\ . \tag{A.33}$$

The Od(3) algebra closes up to the ideals $\mathcal{I}_{\mathfrak{N}_1}, \mathcal{I}_{\mathfrak{N}_2}$ generated by the null fields

$$\mathfrak{N}_1 = \mathcal{N}(\mathcal{J}^{\Omega_+}\mathcal{J}^{\omega})\ , \quad \mathfrak{N}_2 = \mathcal{N}(\mathcal{J}^{\Omega_-}\mathcal{J}^{\omega})\ , \tag{A.34}$$

and in the ideals we can easily find the null field

$$\frac{i}{2}[\mathcal{J}^{\Omega_-}, \mathfrak{N}_1]_1 = \frac{i}{2}[\mathcal{J}^{\Omega_+}, \mathfrak{N}_2]_1 = \mathcal{N}(\mathcal{J}^{\omega}\mathcal{N}(\mathcal{J}^{\omega}\mathcal{J}^{\omega})) + \frac{3}{2}\mathcal{N}(\mathcal{J}^{\Omega_-}\mathcal{J}^{\Omega_+}) \in \mathcal{I}_{\mathfrak{N}_{1,2}}\ . \tag{A.35}$$

The null fields (A.34) and (A.35) are in exact correspondence with the structure equation (A.33), and they translate the geometrical statement into a quantum field theory statement, setting to zero any correlators in which they are inserted. This dictionary is lost when we introduce torsion since it can be checked that the fields $\mathcal{N}(\mathcal{J}^{\Omega_\pm}\mathcal{J}^{\omega})$ are not null anymore.

A.3 Classical OPE computations: some details

In order to compute the classical OPEs we considered in Chapter 2, it is necessary to first identify the forms U, V and N built out of the two forms Φ and Ψ associated with the commutator of $\mathcal{W}$-transformations. It is useful to recall the formulas for these forms

$$U = \frac{1}{c_U}\Phi^i \wedge \Psi_i \ , \quad V = \frac{1}{c_V}\Phi^{ij} \wedge \Psi_{ij} \ , \quad N = \frac{1}{c_N}\left(H_{jk} \wedge \Phi^j \wedge \Psi^k + 2\,(-1)^p\,\beta_{pq}\,H \wedge V\right) \ . \tag{A.36}$$

A.3.1 Spin(7)-structures and classical OPEs

$\mathcal{J}^{\Psi}_{\mathbf{cl.}} \times \mathcal{J}^{\Psi}_{\mathbf{cl.}}$ **classical OPE.** There is only one classical OPE to study. The form U trivially vanishes

$$U = \frac{1}{c_U}\Psi^i \wedge \Psi_i \ , \tag{A.37}$$

while the form V can be computed using the identity [95, 182]

$$\Psi^{ijk\ell}\Psi_{mnk\ell} = 12\delta^{[i}_{[m}\delta^{j]}_{n]} + 4\Psi^{ij}{}_{mn} \ , \tag{A.38}$$

leading to

$$V = \frac{1}{c_V}\Psi^{ij} \wedge \Psi_{ij} = \frac{4!}{c_V}\Psi \ . \tag{A.39}$$

The choice $c_V = 4!$ leads to $V = \Psi$. Finally, we prove that the form N vanishes. Let us consider the Hodge dual form of N

$$*N = \frac{1}{c_N} * \left(H_{jk} \wedge \Psi^j \wedge \Psi^k - 12\,H \wedge \Psi\right) \ , \tag{A.40}$$

so we need to show that

$$*\left(H_{jk} \wedge \Psi^j \wedge \Psi^k\right) = 12\,(H \wedge \Psi) \ . \tag{A.41}$$

We start by writing down the Hodge dual

$$\begin{aligned} *\left(H_{jk} \wedge \Psi^j \wedge \Psi^k\right) &= \frac{\sqrt{G}}{7!}\frac{7!}{3!\,3!}H^{i_1}{}_{jk}\,\Psi^{ji_2i_3i_4}\Psi^{ki_5i_6i_7}\epsilon_{i_1i_2i_3i_4i_5i_6i_7i_8}\mathrm{d}x^{i_8} \\ &= -\frac{5}{3!}H^{i_1}{}_{jk}\,\Psi^{ji_2i_3i_4}\delta^k_{[i_1}\Psi_{i_2i_3i_4i_8]}\mathrm{d}x^{i_8} \ , \end{aligned} \tag{A.42}$$

which we rewrote using the identity

$$\frac{\sqrt{G}}{3!}\Psi^{ki_5i_6i_7}\epsilon_{i_1i_2i_3i_4i_5i_6i_7i_8} = -5\,\delta^k_{[i_1}\Psi_{i_2i_3i_4i_8]} \ . \tag{A.43}$$

Using the identity (A.38) together with the identity

$$\Psi^{ijk\ell}\Psi_{mjk\ell} = 42\,\delta^i_m \ , \tag{A.44}$$

we reach the desired result

$$*\left(H_{jk}\wedge\Psi^j\wedge\Psi^k\right) = 12\,H\lrcorner\Psi = 12\,*\left(H\wedge\Psi\right) \ , \tag{A.45}$$

where we used the generic identity for a p-form α and a $(p+k)$-form β

$$\alpha\lrcorner\beta = (-1)^{p(d-p-k)}\,*\left(\alpha\wedge *\beta\right) \ , \tag{A.46}$$

and the self-duality of the Cayley form $\Psi = *\Psi$.

A.3.2 G_2-structures and classical OPEs

$\mathcal{J}^{\varphi}_{\mathbf{cl.}}\times\mathcal{J}^{\varphi}_{\mathbf{cl.}}$ **classical OPE.** Starting from the computation of the form V, we see that it trivially vanishes

$$V = \frac{1}{c_V}\varphi^{ij}\wedge\varphi_{ij} = 0 \ . \tag{A.47}$$

The form N simplifies to

$$N = \frac{1}{c_N}\,H_{jk}\wedge\varphi^j\wedge\varphi^k = 0 \ , \tag{A.48}$$

and it trivially vanishes as well. The form U can be computed using the identity [95]

$$\varphi^{ijk}\varphi_{mnk} = 2\delta^{[i}_{[m}\delta^{j]}_{n]} + \psi^{ij}{}_{mn} \ , \tag{A.49}$$

leading to

$$U = \frac{1}{c_U}\varphi^i\wedge\varphi_i = \frac{6}{c_U}\psi \ , \tag{A.50}$$

The choice $c_U = 6$ leads to $U = \psi$.

$\mathcal{J}^{\varphi}_{\mathbf{cl.}}\times\mathcal{J}^{\psi}_{\mathbf{cl.}}$ **classical OPE.** We can start by proving that the form U vanishes, thanks to the identity [95]

$$\varphi^{ijk}\psi_{\ell mnk} = \varphi^{[i}{}_{[\ell m}\delta^{j]}_{n]} \ , \tag{A.51}$$

so $U = 0$. The computation of the V form needs instead the identity

$$\varphi^{ijk}\psi_{\ell mjk} = 4\,\varphi^i{}_{\ell m} \ , \tag{A.52}$$

leading to

$$V = \frac{1}{c_V}\varphi^{ij} \wedge \psi_{ij} = \frac{12}{c_V}\,\varphi \ , \tag{A.53}$$

so choosing $c_V = 12$ we have $V = \varphi$. Finally, we prove that $N = 0$ with a strategy similar to the one employed in the Spin(7) case. We want to prove that

$$*(H_{jk} \wedge \varphi^j \wedge \psi^k) = -6 * (H \wedge \varphi) \ . \tag{A.54}$$

We write the left-hand side explicitly

$$\begin{aligned} *(H_{jk} \wedge \varphi^j \wedge \psi^k) &= \frac{\sqrt{G}}{6!}\frac{6!}{2!\,3!} H^{i_1}{}_{jk}\,\varphi^{ji_2i_3}\psi^{ki_4i_5i_6}\epsilon_{i_1i_2i_3i_4i_5i_6i_7}\mathrm{d}x^{i_7} \\ &= 2\,H^{i_1}{}_{jk}\,\varphi^{ji_2i_3}\delta^k_{[i_1}\varphi_{i_2i_3i_7]}\mathrm{d}x^{i_7} \ , \end{aligned} \tag{A.55}$$

where we used the identity

$$\frac{\sqrt{G}}{3!}\psi^{ki_4i_5i_6}\epsilon_{i_1i_2i_3i_4i_5i_6i_7} = 4\,\delta^k_{[i_1}\varphi_{i_2i_3i_7]} \ . \tag{A.56}$$

Considering the identity

$$\varphi^{ijk}\varphi_{\ell jk} = 6\,\delta^i_\ell \ , \tag{A.57}$$

together with (A.49) and (A.46), we obtain

$$*(H_{jk} \wedge \varphi^j \wedge \psi^k) = 6\,H \lrcorner \psi = -6 * (H \wedge \varphi) \ , \tag{A.58}$$

concluding the proof.

$\mathcal{J}^\psi_{\mathbf{cl.}} \times \mathcal{J}^\psi_{\mathbf{cl.}}$ **classical OPE.** The form U trivially vanishes: $U = 0$. The form V instead can be computed using the identity

$$\psi^{ijk\ell}\psi_{mnk\ell} = 8\,\delta^{[i}_{[m}\delta^{j]}_{n]} + 2\,\psi^{ij}{}_{mn} \ , \tag{A.59}$$

leading to

$$V = \frac{1}{c_V}\psi^{ij} \wedge \psi_{ij} = \frac{12}{c_V}\psi \ ; \tag{A.60}$$

we can select $c_V = 12$ in order to define $V = \psi$. Regarding the form N, this takes form

$$N = \frac{1}{c_N}\left(H_{jk} \wedge \psi^j \wedge \psi^k - 8\,H \wedge \psi\right) \ . \tag{A.61}$$

we can use the identities (A.52) and (A.56) to write

$$\begin{aligned} *\left(H_{jk}\wedge\psi^j\wedge\psi^k\right) &= 12\,\varphi\lrcorner H \\ &= 12\,\varphi\lrcorner\left(\frac{1}{6}\,\tau_0\,\varphi-\tau_1\lrcorner\psi-\tau_3\right) \\ &= 14\,\tau_0\ , \end{aligned} \tag{A.62}$$

where we used the explicit expression of the torsion (2.49), the norm of the associative form $||\varphi||^2=7$ and the orthogonality between $\varphi\in\Lambda^3_{\mathbf{1}}(T\mathcal{Y})$, $\tau_3\in\Lambda^3_{\mathbf{27}}(T\mathcal{Y})$ and $\tau_1\lrcorner\psi\in\Lambda^3_{\mathbf{7}}(T\mathcal{Y})$ [96]. Since the Hodge dual is a scalar, the form we are computing will be proportional to the volume form $\mathrm{Vol}=\varphi\wedge\psi$ [102]

$$H_{jk}\wedge\psi^j\wedge\psi^k=2\,\tau_0\,\varphi\wedge\psi\ . \tag{A.63}$$

In addition, we have that $H\wedge\psi=\frac{1}{6}\tau_0\,\varphi\wedge\psi$. Hence, the N form takes the shape

$$N=\frac{1}{c_N}\left(2\,\tau_0\,\varphi\wedge\psi-\frac{4}{3}\tau_0\,\varphi\wedge\psi\right)=\frac{1}{c_N}\frac{2}{3}\tau_0\,\varphi\wedge\psi\ . \tag{A.64}$$

Hence, we choose $c_N=\frac{2}{3}\tau_0$, so that $N=\varphi\wedge\psi$.

A.3.3 SU(3)-structures and classical OPEs

$\mathcal{J}^\omega_{\mathbf{cl.}}\times\mathcal{J}^\omega_{\mathbf{cl.}}$ **classical OPE.** The U-form trivially vanishes: $U=0$. The V 0-form is given by the norm of the ω form

$$V=\frac{1}{c_V}\omega^{ij}\wedge\omega_{ij}=\frac{6}{c_V}\ , \tag{A.65}$$

hence we can choose $c_V=6$ to get $V=\mathbb{1}$. With this choice, the form N reduces to

$$N=\frac{1}{c_N}\left(H_{jk}\wedge\omega^j\wedge\omega^k-2\,H\right)\ . \tag{A.66}$$

Making use of the definition of almost complex structure (2.55), we can write

$$H_{jk}\wedge\omega^j\wedge\omega^k=H_{ijk}J^i_{\ \ell}\,J^j_{\ m}\,\mathrm{d}x^{k\ell m}\ , \tag{A.67}$$

and using $\mathrm{d}\omega=i_H(\omega)$, we end up with the identity

$$H_{jk}\wedge\omega^j\wedge\omega^k=-2\,J(\mathrm{d}\omega)\ , \tag{A.68}$$

so the N form becomes

$$N=-\frac{2}{c_N}\left(J(\mathrm{d}\omega)+H\right) \tag{A.69}$$

and using the explicit expression of the torsion (2.63), we get

$$N = -\frac{2}{c_N} N'_J = \frac{1}{c_N} \frac{32}{||\Omega||^2} \left(W_0^+ \Omega_+ + W_0^- \Omega_- \right) \ . \tag{A.70}$$

We have two N forms $N_+ = \Omega_+$ and $N_- = \Omega_-$, if we choose respectively $c_{N_+} = \frac{32}{||\Omega||^2} W_0^+$ and $c_{N_-} = \frac{32}{||\Omega||^2} W_0^-$.

$\mathcal{J}^{\omega}_{\mathbf{cl.}} \times \mathcal{J}^{\Omega_\pm}_{\mathbf{cl.}}$ classical OPE. We start by considering the computation of the U form

$$U_\pm = \frac{1}{c_U} \omega^i \wedge \Omega_{\pm,i} = -\frac{1}{c_U} J^i \wedge \Omega_{\pm,i} \ . \tag{A.71}$$

Focusing on Ω_+ (the case of Ω_- is analogous), using the property of the almost complex structure we get

$$U_+ = -\frac{1}{c_U} J^i{}_j \, \mathrm{d}x^j \wedge \frac{1}{2} \left(\Omega_i + \overline{\Omega}_i \right) = \frac{1}{c_U} \mathrm{d}x^j \wedge \frac{1}{2i} \left(\Omega_j - \overline{\Omega}_j \right) = \frac{1}{c_U} \mathrm{d}x^j \wedge \Omega_{-,j} = \frac{3}{c_U} \Omega_- \ ; \tag{A.72}$$

similarly, we get $U_- = -\frac{3}{c_U} \Omega_-$. Hence, we can choose $c_U = \pm 3$ and obtain $U = \Omega_\mp$. The V form vanishes

$$V = \frac{1}{c_V} \omega^{ij} \wedge \Omega_{\pm,ij} = 0 \ . \tag{A.73}$$

Finally, we compute the N form. The starting point is

$$N = \frac{1}{c_N} H_{jk} \wedge \omega^j \wedge \Omega^k_\pm \ . \tag{A.74}$$

We can plug the explicit expression of the torsion[1]

$$H = -J(W_1 \wedge \omega + W_3) - \frac{4}{||\Omega||^2} \mathrm{Re}(W_0 \, \overline{\Omega}) \tag{A.75}$$

obtaining

$$N = \frac{1}{c_N} \left(-J(W_1 \wedge \omega + W_3) - \frac{4}{||\Omega||^2} \mathrm{Re}(W_0 \, \overline{\Omega}) \right)_{jk} \wedge \omega^j \wedge \Omega^k_\pm \ . \tag{A.76}$$

It can be shown that only the $(3,0)+(0,3)$ component of the torsion contributes to the N form, so we isolate

$$N = -\frac{2}{c_N ||\Omega||^2} \left(W_0 \, \overline{\Omega}_{jk} + \overline{W}_0 \, \Omega_{jk} \right) \wedge \omega^j \wedge \Omega^k_\pm \ . \tag{A.77}$$

We focus on the $(3,0)$ component, and the $(0,3)$ will be analogous. First of all, we can see that (focusing on Ω_+ and on the associated component $N^{(3,0)}_+$, for example)

$$H^{(3,0)}_{jk} \wedge \omega^j \wedge \Omega^k_+ = \frac{1}{2} H^{(3,0)}_{jk} \wedge J^j \wedge \overline{\Omega}^k = -i \, H^{(3,0)}_j \wedge \overline{\Omega}^j \ , \tag{A.78}$$

[1]The formula (A.75) provides an explicit expression for the torsion of the Bismut connection [183, 184].

so we can rewrite the $N_+^{(3,0)}$ form as

$$N_+^{(3,0)} = \frac{2i\,\overline{W}_0}{c_N ||\Omega||^2}\,\Omega_j \wedge \overline{\Omega}^j \ . \tag{A.79}$$

We can use the identity

$$\Omega^i \wedge \overline{\Omega}_i = \frac{1}{2}||\Omega||^2 \omega \wedge \omega \tag{A.80}$$

to rewrite everything as

$$N_+^{(3,0)} = \frac{i}{c_N}\overline{W}_0\,\omega \wedge \omega \ . \tag{A.81}$$

Considering the (0,3) component, we reach the final result (in the case where we consider the form Ω_+)

$$N_+ = N_+^{(3,0)} + N_+^{(0,3)} = \frac{2}{c_N}\mathrm{Im}\,(W_0)\,\omega \wedge \omega = \frac{2}{c_N}W_0^-\omega \wedge \omega \ . \tag{A.82}$$

The case where we consider Ω_- is analogous. We can choose $c_{N_\pm} = \pm 2\,W_0^{\mp}$, so that $N_\pm = \omega \wedge \omega$.

$\mathcal{J}_{\mathbf{cl.}}^{\Omega_\pm} \times \mathcal{J}_{\mathbf{cl.}}^{\Omega_\pm}$ **classical OPE.** The form V vanishes trivially

$$V = \frac{1}{c_V}\Omega_\pm^{ij} \wedge \Omega_{\pm,ij} = 0 \ , \tag{A.83}$$

while the form U requires a quick computation. We focus on Ω_+, since the Ω_- will be analogous:

$$U = \frac{1}{c_U}\Omega_+^i \wedge \Omega_{+,i} = \frac{1}{4\,c_U}\left(\Omega + \overline{\Omega}\right)^i \wedge \left(\Omega + \overline{\Omega}\right)_i = \frac{1}{2\,c_U}\Omega^i \wedge \overline{\Omega}_i \ . \tag{A.84}$$

We can use the identity (A.80) to rewrite U as

$$U = \frac{||\Omega||^2}{4\,c_U}\omega \wedge \omega \ . \tag{A.85}$$

If we select $c_U = \frac{||\Omega||^2}{4}$, we obtain $U = \omega \wedge \omega$. Finally, we compute the form N, which reads

$$N = \frac{1}{c_N}H_{jk} \wedge \Omega_\pm^j \wedge \Omega_\pm^k = 0 \tag{A.86}$$

and vanishes trivially due to the symmetry of the indices.

$\mathcal{J}_{\mathbf{cl.}}^{\Omega_+} \times \mathcal{J}_{\mathbf{cl.}}^{\Omega_-}$ **classical OPE.** We consider the form U: it can be quickly shown that it vanishes

$$U = \frac{1}{c_U}\Omega_+^i \wedge \Omega_{-,i} = -\frac{i}{4\,c_U}(\Omega^i \wedge \overline{\Omega}_i - \overline{\Omega}^i \wedge \Omega_i) = 0 \ . \tag{A.87}$$

Regarding the V form, we can write

$$V = \frac{1}{c_V}\Omega_+^{ij} \wedge \Omega_{-,ij} = -\frac{i}{2\,c_V}\Omega^{ij} \wedge \overline{\Omega}_{ij} \tag{A.88}$$

and we can use the identity

$$\Omega^{ij} \wedge \overline{\Omega}_{ij} = 2i\,||\Omega||^2\omega \tag{A.89}$$

to obtain the final result

$$V = \frac{1}{c_V}\Omega_+^{ij} \wedge \Omega_{-,ij} = \frac{1}{c_V}||\Omega||^2\omega\ . \tag{A.90}$$

Choosing $c_V = ||\Omega||^2$, we have $V = \omega$. Finally, we consider the N form. The explicit expression is

$$N = \frac{1}{c_N}\left(H_{jk} \wedge \Omega_+^j \wedge \Omega_-^k + \frac{1}{2}||\Omega||^2\, H \wedge \omega\right)\ . \tag{A.91}$$

We start by considering the Hodge dual of the first term

$$\begin{aligned} *\left(H_{jk} \wedge \Omega_+^j \wedge \Omega_-^k\right) &= \frac{\sqrt{G}}{5!}\frac{5!}{2!\,2!}H_{jk}{}^{i_1}\Omega_+^{ji_2i_3}\Omega_-^{ki_4i_5}\epsilon_{i_1i_2i_3i_4i_5i_6}\mathrm{d}x^{i_6} \\ &= -H_{jk}{}^{i_1}\Omega_+^{ji_2i_3}\delta^k_{[i_1}\Omega_{+,i_2i_3i_6]}\mathrm{d}x^{i_6} \end{aligned} \tag{A.92}$$

where we used the identity

$$\frac{\sqrt{G}}{2}\Omega_-^{ki_4i_5}\epsilon_{i_1i_2i_3i_4i_5i_6} = -4\delta^k_{[i_1}\Omega_{+,i_2i_3i_6]}\ . \tag{A.93}$$

Further computations lead to

$$*\left(H_{jk} \wedge \Omega_+^j \wedge \Omega_-^k\right) = \frac{1}{2}||\Omega||^2 J(\omega \lrcorner H) = ||\Omega||^2\, W_1\ , \tag{A.94}$$

where we used the fact that $W_1 = \frac{1}{2}J(\omega \lrcorner H)$. We can now undo the Hodge dual to compute the first term of the N form (recalling that for a generic p-form α, $**\alpha = (-1)^{p(d-p)}\alpha$)

$$**\left(H_{jk} \wedge \Omega_+^j \wedge \Omega_-^k\right) = -H_{jk} \wedge \Omega_+^j \wedge \Omega_-^k = ||\Omega||^2 * W_1\ . \tag{A.95}$$

The Hodge dual of W_1 can be computed using the identity (A.46), the identity $*1 = \mathrm{Vol}$ and the definition of the volume form $\mathrm{Vol} = \frac{1}{6}\omega \wedge \omega \wedge \omega$ [101]

$$*W_1 = *(W_1 \wedge 1) = W_1 \lrcorner \mathrm{Vol} = \frac{1}{2}\,(W_1 \lrcorner \omega) \wedge \omega \wedge \omega = -\frac{1}{2}J(W_1) \wedge \omega \wedge \omega\ . \tag{A.96}$$

We conclude that the first term of the N form reads

$$H_{jk} \wedge \Omega_+^j \wedge \Omega_-^k = \frac{1}{2}||\Omega||^2\, J(W_1) \wedge \omega \wedge \omega\ . \tag{A.97}$$

The second term can be found much faster: recalling the explicit expression of the torsion (A.75), only the $J(W_1 \wedge \omega)$ term contributes, hence we have

$$\frac{1}{2}||\Omega||^2 H \wedge \omega = -\frac{1}{2}||\Omega||^2 J(W_1) \wedge \omega \wedge \omega \ . \tag{A.98}$$

Putting all together, we prove that the N form vanishes.

B
Part II Appendices

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B.1 Derivation of the sum rules

In this Appendix, we derive the sum rules in equation (5.12). The starting point is the OPE (5.8) and the general consistency relation (5.11) encoding the KMS condition. As a first step, it is convenient to expand the Gegenbauer polynomials as

$$C_J^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2+r^2}}\right)=\sum_{k=0}^{[J/2]}2^{J-2k}(-1)^k\frac{\Gamma\left(J-k+\nu\right)}{\Gamma(\nu)k!(J-2k)!}\left(\frac{\tau}{\sqrt{r^2+\tau^2}}\right)^{J-2k}\,. \tag{B.1}$$

Plugging the decomposition in the equation (5.8)

$$f(\tau,r)=\sum_{\mathcal{O}\in\phi\times\phi}\sum_{k=0}^{[J/2]}\frac{a_{\mathcal{O}}}{\beta^{\Delta}}(-1)^k\frac{\Gamma\left(J-k+\nu\right)}{\Gamma(\nu)k!(J-2k)!}2^{J-2k}|\tau^2+r^2|^{\frac{h-2\Delta_\phi+2k}{2}}\tau^{J-2k}\,, \tag{B.2}$$

where we adopted the twist variable (5.13). By using the (generalized) binomial theorem, we obtain

$$f(\tau,r)=\sum_{\mathcal{O}\in\phi\times\phi}\sum_{k=0}^{[J/2]}\sum_{n=0}^{\infty}\frac{a_{\mathcal{O}}}{\beta^{\Delta}}(-1)^k\frac{\Gamma\left(J-k+\nu\right)}{\Gamma(\nu)k!(J-2k)!}2^{J-2k}\binom{\frac{h-2\Delta_\phi+2k}{2}}{n}\tau^{\Delta-2\Delta_\phi-2n}r^{2n}\,. \tag{B.3}$$

We now compute the function in $\frac{\beta}{2}\pm\tau$, and we apply the binomial theorem a second time

$$\begin{aligned}f\left(\frac{\beta}{2}\pm\tau,r\right)=\sum_{\mathcal{O}\in\phi\times\phi}\sum_{k=0}^{[J/2]}\sum_{n,\ell=0}^{\infty}\frac{a_{\mathcal{O}}}{\beta^{2\Delta_\phi+2n+\ell}}(-1)^k\frac{\Gamma\left(J-k+\nu\right)}{\Gamma(\nu)k!(J-2k)!}2^{-h-2k+2\Delta_\phi+2n+\ell}\\ \times\binom{\Delta-2\Delta_\phi-2n}{\ell}\binom{\frac{h-2\Delta_\phi+2k}{2}}{n}(\pm\tau)^{\ell}\,r^{2n}\,.\end{aligned} \tag{B.4}$$

It is crucial to notice that the expression is sensitive to the sign difference in $\frac{\beta}{2} \pm \tau$ if and only if ℓ is an odd positive integer. If we impose the KMS condition (5.11), we get

$$\sum_{\mathcal{O}\in\phi\times\phi} \sum_{k=0}^{[J/2]} \sum_{n=0}^{\infty} \frac{a_{\mathcal{O}}}{\beta^{2n}} (-1)^k \frac{\Gamma(J-k+\nu)}{\Gamma(\nu)k!(J-2k)!} 2^{-h-2k+2n} \binom{\Delta - 2\Delta_\phi - 2n}{\ell} \binom{\frac{h-2\Delta_\phi+2k}{2}}{n} r^{2n} = 0 \ , \quad \text{(B.5)}$$

for a fixed $\ell \in 2\mathbb{N}+1$. These equations must be true for any $n \in \mathbb{N}$ and for any $\ell \in 2\mathbb{N}+1$, therefore they reduce to

$$\sum_{\mathcal{O}\in\phi\times\phi} \sum_{k=0}^{[J/2]} a_{\mathcal{O}} (-1)^k \frac{\Gamma(J-k+\nu)}{\Gamma(\nu)k!(J-2k)!} 2^{-h-2k} \binom{\Delta - 2\Delta_\phi - 2n}{\ell} \binom{\frac{h-2\Delta_\phi+2k}{2}}{n} = 0 \ . \quad \text{(B.6)}$$

Once we re-sum over k, the equation can be recast as

$$\sum_{\mathcal{O}\in\phi\times\phi} b_{\mathcal{O}} f_{\mathcal{O}\phi\phi} F_{\ell,n}(h,J) = 0 \ , \quad \text{(B.7)}$$

where we used the definition (5.10) of the $a_{\mathcal{O}}$ coefficients (the normalization constants $c_{\mathcal{O}}$ have been set equal to 1 identically)

$$a_{\mathcal{O}} = \frac{f_{\phi\phi\mathcal{O}} b_{\mathcal{O}}}{c_{\mathcal{O}}} \frac{J!}{2^J (\nu)_J} \ , \quad \text{(B.8)}$$

and we introduced the function

$$F_{\ell,n}(h,J) = \frac{1}{2^{h+J}} \binom{\frac{h-2\Delta_\phi}{2}}{n} \binom{h+J-2\Delta_\phi-2n}{\ell} {}_3F_2 \left[\begin{matrix} \frac{1-J}{2}, -\frac{J}{2}, \frac{h}{2} - \Delta_\phi + 1 \\ \frac{h}{2} - \Delta_\phi - n + 1, -J - \nu + 1 \end{matrix} \middle| 1 \right] . \quad \text{(B.9)}$$

This concludes the derivation of the sum rules (5.12).

Reduction to $r = 0$. Now, we want to specialize the sum rules to the case of two operators lying on the same thermal circle. This limit is achieved by setting $r = 0$. By looking at equation (B.3), we see that the only significative equation is returned by setting $n = 0$. This considerably simplifies the expression (B.9)

$$F_{\ell,0}(h,J) = \frac{1}{2^{h+J}} \binom{h+J-2\Delta_\phi}{\ell} {}_2F_1 \left[\begin{matrix} \frac{1-J}{2}, -\frac{J}{2} \\ -J-\nu+1 \end{matrix} \middle| 1 \right] . \quad \text{(B.10)}$$

This expression of $F_{\ell,0}$ is such that the sum rule (5.12) is equivalent, in the $n = 0$ reduction, to (5.17). To see this we use the fact that, by symmetry arguments, $J \in 2\mathbb{N}$, and $\nu \in \mathbb{N}/2$ by definition: hence, it can be shown that

$$C_J^{(\nu)}(1) = \frac{2^J (\nu)_J}{J!} {}_2F_1 \left[\begin{matrix} \frac{1-J}{2}, -\frac{J}{2} \\ -J-\nu+1 \end{matrix} \middle| 1 \right] . \quad \text{(B.11)}$$

The sum rules (B.6) are now in the form, for a fixed odd integer ℓ

$$\sum_{\mathcal{O}\in\phi\times\phi} a_{\mathcal{O}}\, C_J^{(\nu)}(1)\frac{\Gamma(\Delta-2\Delta_\phi+1)}{2^\Delta\Gamma(\Delta-2\Delta_\phi-\ell+1)}=0\ . \tag{B.12}$$

Since the only spin-dependent term is $a_{\mathcal{O}}\, C_J^{(\nu)}(1)$, we can just define a weighted sum of thermal OPE coefficients

$$a_\Delta \equiv \sum_{\mathcal{O}\in\phi\times\phi}^{\Delta\ \text{fixed}} a_{\mathcal{O}}\, C_J^{(\nu)}(1)\ . \tag{B.13}$$

Finally, by separating the identity $\mathbb{1}$ contribution

$$-\frac{\Gamma(2\Delta_\phi+k)}{\Gamma(2\Delta_\phi)}\ , \tag{B.14}$$

we end up with equation (5.17).

B.2 A proof for the Tauberian theorem

In this Appendix, we sketch the proof for the Tauberian theorem discussed in Section 5.2: our proposal is an adaptation of Karamata's proof presented in Section 7.53 of [159].

Let us start by considering the integral

$$\int_0^\infty \mathrm{d}\Delta\ \rho(\Delta)x^\Delta \overset{x\to1}{\sim} (1-x)^{-\alpha}\ ,\quad \alpha>0\ . \tag{B.15}$$

The first step is to prove that for every polynomial $P(y)$

$$\lim_{x\to1}(1-x)^\alpha\int_0^\infty \mathrm{d}\Delta\ \rho(\Delta)x^\Delta P\left(x^\Delta\right)=\int_0^1 \mathrm{d}t\ P(t)\ . \tag{B.16}$$

It is sufficient to prove the statement for $P(y)=y^n$. After a straightforward manipulation

$$\lim_{x\to1}(1-x)^\alpha\int_0^\infty \mathrm{d}\Delta\ \rho(\Delta)\left(x^{1+n}\right)^\Delta=\lim_{x\to1}\left(\frac{1-x}{1-x^{n+1}}\right)^\alpha\left(1-x^{n+1}\right)^\alpha\int_0^\infty \mathrm{d}\Delta\ \rho(\Delta)\left(x^{1+n}\right)^\Delta\ . \tag{B.17}$$

Recalling the hypothesis (B.15), the integral is diverging as

$$\int_0^\infty \mathrm{d}\Delta\ \rho(\Delta)\left(x^{1+n}\right)^\Delta \overset{x\to1}{\sim} \left(1-x^{n+1}\right)^{-\alpha}\ , \tag{B.18}$$

and therefore

$$\lim_{x\to1}(1-x)^\alpha\int_0^\infty \mathrm{d}\Delta\ \rho(\Delta)\left(x^{1+n}\right)^\Delta=\frac{1}{n+1}=\int_0^1 \mathrm{d}t\ t^n\ , \tag{B.19}$$

as claimed above. In the following, let us consider a function $g(x)$ defined on the closed interval $[0,1]$. We want to approximate $g(x)$ with arbitrary precision from above and from below by using polynomials $p(x)$ and $P(x)$

$$p(x) \leq g(x) \leq P(x) \ , \tag{B.20}$$

where

$$\int_0^1 \mathrm{d}x \, (g(x) - p(x)) \leq \epsilon \ , \qquad \int_0^1 \mathrm{d}x \, (P(x) - g(x)) \leq \epsilon \ , \tag{B.21}$$

for an arbitrarily small $\epsilon > 0$. If $g(x)$ is a continuous function, it is sufficient to invoke Weierstrass's approximation theorem. However, we need this approximation also in the case of a function $g(x)$ that admits discontinuities of the first kind, i.e. for any point $c \in (0,1)$, both $\lim_{x \to c^-} g(x)$ and $\lim_{x \to c^+} g(x)$ exist, but $\lim_{x \to c^-} g(x) \neq \lim_{x \to c^+} g(x)$. The claim is correct also under this relaxed hypothesis: suppose without loss of generality that $\lim_{x \to c^-} g(x) < \lim_{x \to c^+} g(x)$, and let

$$\phi(x) = \begin{cases} g(x) + \epsilon/2 & c - \delta > x \text{ and } x > c \\ \max\left\{ l(x), g(x) + \epsilon/4 \right\} & c - \delta \leq x \leq c \end{cases} \ , \tag{B.22}$$

where $l(x)$ is a linear function defined in such a way that $l(c-\delta) = g(c-\delta) + \epsilon/2$ and $l(c) = \lim_{x \to c^+} g(x) + \epsilon/2$. Then $\phi(x)$ is continuous by construction and $\phi(x) > g(x)$: for small enough δ, we can apply Weierstrass's approximation theorem to $\phi(x)$ and consequently polynomially bound $g(x)$

$$P(x) \geq \phi(x) > g(x) \ . \tag{B.23}$$

A similar construction can be repeated for the polynomial bound from below.

The next step is to prove that for any $g(x)$ with only discontinuities of the first kind we have

$$\lim_{x \to 1} (1-x)^\alpha \int_0^\infty \mathrm{d}\Delta \ \rho(\Delta) x^\Delta g\left(x^\Delta\right) = \int_0^1 \mathrm{d}t \ g(t) \ . \tag{B.24}$$

Now, we need to introduce a crucial hypothesis: we consider the density $\rho(\Delta)$ to be positive definite

$$\rho(\Delta) \geq 0 \ , \quad \text{for any } \Delta \geq 0 \ . \tag{B.25}$$

Using this hypothesis, we can derive the following series of inequalities

$$\begin{aligned} \lim_{x \to 1} (1-x)^\alpha \int_0^\infty \mathrm{d}\Delta \ \rho(\Delta) x^\Delta g\left(x^\Delta\right) &\leq \lim_{x \to 1} (1-x)^\alpha \int_0^\infty \mathrm{d}\Delta \ \rho(\Delta) x^\Delta P\left(x^\Delta\right) \\ &= \int_0^1 \mathrm{d}t \ P\left(t\right) \leq \int_0^1 \mathrm{d}t \ g\left(t\right) + \epsilon \ , \end{aligned} \tag{B.26}$$

which leads to

$$\lim_{x\to 1}(1-x)^{\alpha}\int_0^{\infty}\mathrm{d}\Delta\ \rho(\Delta)x^{\Delta}g\left(x^{\Delta}\right)\leq\int_0^1\mathrm{d}t\ g\left(t\right)\ , \tag{B.27}$$

when $\epsilon\to 0$. Similarly, by using the polynomial bound from below, one can see that

$$\lim_{x\to 1}(1-x)^{\alpha}\int_0^{\infty}\mathrm{d}\Delta\ \rho(\Delta)x^{\Delta}g\left(x^{\Delta}\right)\geq\int_0^1\mathrm{d}t\ g\left(t\right)\ , \tag{B.28}$$

and therefore the equation B.24 is proved.

The final step consists to choose a specific function $g(x)$ with a discontinuity of the first kind

$$g(x)=\begin{cases}0\ , & 0\leq x\leq e^{-1}\\ 1/x\ , & e^{-1}<x\leq 1\end{cases}\ . \tag{B.29}$$

The function is engineered in such a way that

$$\int_0^1 g(x)\mathrm{d}x=\int_{1/e}^1\frac{\mathrm{d}x}{x}=1\ , \tag{B.30}$$

so that we can prove

$$\int_0^{\infty}\mathrm{d}\Delta\ \rho(\Delta)x^{\Delta}g\left(x^{\Delta}\right)\overset{x\to 1}{\sim}(1-x)^{-\alpha}\ . \tag{B.31}$$

With this specific choice and parametrizing $x=e^{-1/\bar{\Delta}}$, we have

$$\int_0^{\bar{\Delta}}\mathrm{d}\Delta\ \rho(\Delta)=\int_0^{\infty}\mathrm{d}\Delta\ \rho(\Delta)e^{-\Delta/\bar{\Delta}}g\left(e^{-\Delta/\bar{\Delta}}\right)\overset{\bar{\Delta}\to\infty}{\sim}\left(1-e^{-1/\bar{\Delta}}\right)^{-\alpha}\sim\bar{\Delta}^{\alpha}\ . \tag{B.32}$$

The last equation is exactly the Tauberian theorem (5.48).

Relaxing the positivity hypothesis. In the scenario considered in this Thesis, the hypothesis (B.25) is not true in general. However, this hypothesis can be relaxed asking $\rho(\Delta)$ to be bounded from below, i.e. there exists a constant C such that

$$\rho(\Delta)\geq -C\ . \tag{B.33}$$

To see that this a sufficient condition for the proof in Appendix B.2 to hold, it is enough to consider the same proof starting from the integral

$$\int_0^{\infty}\mathrm{d}\Delta\ \widetilde{\rho}(\Delta)x^{\Delta}\overset{x\to 1}{\sim}(1-x)^{-\alpha}\ ,\quad \alpha>0\ , \tag{B.34}$$

where $\widetilde{\rho}(\Delta)=\rho(\Delta)+C\geq 0$. If we isolate the contribution from the constant C, it generates a divergence of the type

$$C\int_0^{\infty}\mathrm{d}\Delta\ x^{\Delta}\overset{x\to 1^{-}}{\sim}-\frac{C}{\log x}\ . \tag{B.35}$$

In conclusion, if $\rho(\Delta)$ is bounded from below

$$\int_0^\infty \mathrm{d}\Delta\ \rho(\Delta)x^\Delta \overset{x\to 1^-}{\sim} (1-x)^{-\alpha} + \frac{C}{\log(x)}\ . \tag{B.36}$$

Observe that

$$\frac{C}{\log(x)} \sim \frac{C}{1-x} + \text{regular terms}\ , \tag{B.37}$$

hence this divergence will be subleading with respect to the divergence in (B.34) if and only if $\alpha > 1$. In our scenario, this last condition gets translated into $\Delta_\phi > 1/2$.

B.3 Boundedness of the thermal OPE coefficients

In this Appendix, we prove that the OPE coefficients a_Δ, defined at zero spatial coordinates in equation (5.16), are bounded from below. To see this, we combine the KMS condition with the Euclidean inversion formula [45]. In the following, β will not play any role, so we will set $\beta = 1$ without loss of generality.

We start by packing the OPE coefficients into the function

$$a(\Delta) = \int_0^1 \frac{\mathrm{d}\tau}{\tau}\tau^{2\Delta_\phi-\Delta} f(\tau)\ , \tag{B.38}$$

where $f(\tau) = \langle\phi(\tau)\phi(0)\rangle_\beta$ is the thermal two-point function at zero spatial coordinates. The OPE coefficients a_Δ are encoded in the function $a(\Delta)$ as the residues of first-order poles

$$a(\Delta) = \sum_{\widehat{\Delta}} \frac{a_{\widehat{\Delta}}}{\widehat{\Delta}-\Delta}\ . \tag{B.39}$$

Then, we apply the KMS condition (plus parity) to the two-point function $f(\tau)$

$$a(\Delta) = \int_0^1 \frac{\mathrm{d}\tau}{\tau}\tau^{2\Delta_\phi-\Delta} f(1-\tau)\ . \tag{B.40}$$

The integrand can be expanded using the thermal OPE giving

$$a(\Delta) = \int_0^1 \frac{\mathrm{d}\tau}{\tau}\tau^{2\Delta_\phi-\Delta} \sum_{\tilde{\Delta}} a_{\tilde{\Delta}}(1-\tau)^{\tilde{\Delta}-2\Delta_\phi}\ . \tag{B.41}$$

We can now switch integral and series, making sure to first regularize the integration to allow the operations to exchange

$$a(\Delta) = \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{\mathrm{d}\tau}{\tau}\tau^{2\Delta_\phi-\Delta} \sum_{\tilde{\Delta}} a_{\tilde{\Delta}}(1-\tau)^{\tilde{\Delta}-2\Delta_\phi} = \lim_{\epsilon\to 0}\sum_{\tilde{\Delta}} a_{\tilde{\Delta}}\int_\epsilon^1 \frac{\mathrm{d}\tau}{\tau}\tau^{2\Delta_\phi-\Delta}(1-\tau)^{\tilde{\Delta}-2\Delta_\phi}\ . \tag{B.42}$$

The integration can be explicitly performed

$$a(\Delta) = -\lim_{\epsilon\to 0}\sum_{\tilde{\Delta}} a_{\tilde{\Delta}}\left[\mathrm{B}_\epsilon\left(2\Delta_\phi-\Delta,1-2\Delta_\phi+\tilde{\Delta}\right)-\mathrm{B}\left(2\Delta_\phi-\Delta,1-2\Delta_\phi+\tilde{\Delta}\right)\right]\,, \tag{B.43}$$

where the result is written in terms of the *Euler Beta function*

$$\mathrm{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,. \tag{B.44}$$

Furthermore, we make use of the identity

$$\mathrm{B}_\epsilon(a,b)=\mathrm{B}(a,b)-\frac{\epsilon^a(1-\epsilon)^b}{b}{}_2F_1\left[\begin{matrix}1,a+b\\ b+1\end{matrix}\middle|\,1-\epsilon\right] \tag{B.45}$$

to write the function $a(\Delta)$ as

$$a(\Delta)=\lim_{\epsilon\to 0}\sum_{\tilde{\Delta}} a_{\tilde{\Delta}}\frac{\epsilon^{2\Delta_\phi-\Delta}(1-\epsilon)^{1-2\Delta_\phi+\tilde{\Delta}}}{1-2\Delta_\phi+\tilde{\Delta}}{}_2F_1\left[\begin{matrix}1,\tilde{\Delta}-\Delta+1\\ 2-2\Delta_\phi+\tilde{\Delta}\end{matrix}\middle|\,1-\epsilon\right]\,. \tag{B.46}$$

Having found an explicit expression of $a(\Delta)$, we need to compute its residues to extract a single coefficient $a_{\widehat{\Delta}}$ in terms of all the other OPE coefficients. We can easily project on $a_{\widehat{\Delta}}$ recalling the definition of $a(\Delta)$ in (B.39)

$$a_{\widehat{\Delta}}=-\lim_{\Delta\to\widehat{\Delta}}\lim_{\epsilon\to 0}\sum_{\tilde{\Delta}} a_{\tilde{\Delta}}\left(\widehat{\Delta}-\Delta\right)\frac{\epsilon^{2\Delta_\phi-\Delta}(1-\epsilon)^{1-2\Delta_\phi+\tilde{\Delta}}}{1-2\Delta_\phi+\tilde{\Delta}}{}_2F_1\left[\begin{matrix}1,\tilde{\Delta}-\Delta+1\\ 2-2\Delta_\phi+\tilde{\Delta}\end{matrix}\middle|\,1-\epsilon\right]\,. \tag{B.47}$$

To unclutter the notation, we can define $\Delta=\widehat{\Delta}+\eta$ and relabel $\widehat{\Delta}\to\Delta$

$$a_\Delta=\lim_{\eta\to 0}\lim_{\epsilon\to 0}\sum_{\tilde{\Delta}} a_{\tilde{\Delta}}\frac{\eta\epsilon^{2\Delta_\phi-\Delta-\eta}(1-\epsilon)^{1-2\Delta_\phi+\tilde{\Delta}}}{1-2\Delta_\phi+\tilde{\Delta}}{}_2F_1\left[\begin{matrix}1,\tilde{\Delta}-\Delta-\eta+1\\ 2-2\Delta_\phi+\tilde{\Delta}\end{matrix}\middle|\,1-\epsilon\right]\,. \tag{B.48}$$

The equation (B.48) is a consequence of the KMS condition and it expresses one single OPE coefficient in terms of all the other OPE coefficients. The expression is quite complicated and hard to use. However, we only need to prove that

$$a_{\Delta+\delta\Delta}=a_\Delta\left[1+O\left(\frac{1}{\Delta}\right)\right]\,,\quad \Delta\gg 1\,, \tag{B.49}$$

where it is understood that ${}^{\delta\Delta}/_{\Delta}\ll 1$. The idea is the following: we consider $a_{\Delta+\delta\Delta}$

$$a_{\Delta+\delta\Delta}=\lim_{\eta\to 0}\lim_{\epsilon\to 0}\sum_{\tilde{\Delta}} a_{\tilde{\Delta}}\frac{\eta\epsilon^{2\Delta_\phi-\Delta-\delta\Delta-\eta}(1-\epsilon)^{1-2\Delta_\phi+\tilde{\Delta}}}{1-2\Delta_\phi+\tilde{\Delta}}{}_2F_1\left[\begin{matrix}1,\tilde{\Delta}-\Delta-\delta\Delta-\eta+1\\ 2-2\Delta_\phi+\tilde{\Delta}\end{matrix}\middle|\,1-\epsilon\right]\,, \tag{B.50}$$

and we expand the expression above in powers of $\delta\Delta$. In general, $\delta\Delta$ cannot be considered a small parameter: the best bound we have is $\delta\Delta\lesssim 2$, corresponding to the distance in GFF between the

conformal dimensions of two adjacent double-twist operators. However, every order of the expansion in powers of $\delta\Delta$ is suppressed by a term of order $O(1/\Delta)$. Let us consider the terms in the right-hand side of the equation (B.50) sensitive to $\delta\Delta$: the exponential of ϵ, for $\Delta \gg 1$, can be approximated as

$$\epsilon^{2\Delta_\phi-\Delta-\delta\Delta-\eta} \approx \epsilon^{2\Delta_\phi-\Delta-\eta} \ , \quad {}^{\delta\Delta}/_{\Delta} \ll 1 \ , \Delta \gg 1 \ , \tag{B.51}$$

while the hypergeometric function is expanded as

$$\begin{aligned} {}_2F_1\left[\begin{matrix} 1, \tilde{\Delta}-\Delta-\delta\Delta-\eta+1 \\ 2-2\Delta_\phi+\tilde{\Delta} \end{matrix}\middle| 1-\epsilon\right] &= {}_2F_1\left[\begin{matrix} 1, \tilde{\Delta}-\Delta-\eta+1 \\ 2-2\Delta_\phi+\tilde{\Delta} \end{matrix}\middle| 1-\epsilon\right] + \\ + \delta\Delta \sum_{n=0}^{\infty} \frac{(1)_{n+1}(\tilde{\Delta}-\Delta-\eta+1)_{n+1}}{(2-2\Delta_\phi+\tilde{\Delta})_{n+1}} \frac{(1-\epsilon)^{n+1}}{(n+1)!} \sum_{m=0}^{n} \frac{1}{-m-\tilde{\Delta}+\Delta+\eta-1} &+ O(\delta\Delta^2) \ . \end{aligned} \tag{B.52}$$

It is clear that there are three different cases to study:

1. $\Delta \gg 1$ and $\Delta \gg \tilde{\Delta}$: in this case the first order term is suppressed by factors of order $O\left({}^{1}/_{\Delta}\right)$;

2. $\Delta \gg 1$ and $\Delta \ll \tilde{\Delta}$: in this case the first order term is suppressed by factors of order $O\left({}^{1}/_{\tilde{\Delta}}\right)$, which are consequently also of order $O\left({}^{1}/_{\Delta}\right)$;

3. $\Delta \gg 1$ and $\Delta \sim \tilde{\Delta}$: in this case, the sum over m in the right-hand side of the equation (B.52) does not suppress the function. Instead, we must consider the term

$$\frac{(\tilde{\Delta}-\Delta-\eta+1)_{n+1}}{(2-2\Delta_\phi+\tilde{\Delta})_{n+1}} \sim \tilde{\Delta}^{1-n} \sim \Delta^{1-n} \ . \tag{B.53}$$

   Hence, the first order term is suppressed by factors of order $O(1/\Delta)$.

The proof for higher orders in the expansion in powers of $\delta\Delta$ follows in the same way, such that the OPE coefficients for large Δ satisfy

$$a_{\Delta+\delta\Delta} = a_\Delta \left[1 + O\left(\frac{1}{\Delta}\right)\right] \ , \quad \Delta \gg 1 \ . \tag{B.54}$$

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