HyperKähler and Quaternionic Kähler Geometry

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Thesis submitted Hilary Term, 1990, in support of application to supplicate for the degree of D. Phil.

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Abstract

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A quaternion-Hermitian manifold, of dimension at least 12, with closed fundamental 4-form is shown to be quaternionic Kähler. A similar result is proved for 8-manifolds.

HyperKähler metrics are constructed on the fundamental quaternionic line bundle (with the zero-section removed) of a quaternionic Kähler manifold (indefinite if the scalar curvature is negative). This construction is compatible with the quaternionic Kähler and hyperKähler quotient constructions and allows quaternionic Kähler geometry to be subsumed into the theory of hyperKähler manifolds. It is shown that the hyperKähler metrics that arise admit a certain type of $SU(2)$-action, possess functions which are Kähler potentials for each of the complex structures simultaneously and determine quaternionic Kähler structures via a variant of the moment map construction. Quaternionic Kähler metrics are also constructed on the fundamental quaternionic line bundle and a twistor space analogy leads to a construction of hyperKähler metrics with circle actions on complex line bundles over Kähler-Einstein (complex) contact manifolds.

Nilpotent orbits in a complex semi-simple Lie algebra, with the hyperKähler metrics defined by Kronheimer, are shown to give rise to quaternionic Kähler metrics and various examples of these metrics are identified. It is shown that any quaternionic Kähler manifold with positive scalar curvature and sufficiently large isometry group may be embedded in one of these manifolds. The twistor space structure of the projectivised nilpotent orbits is studied.

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Chapter 1

INTRODUCTION AND DEFINITIONS

A hyperKähler manifold is a Riemannian $4n$-manifold with a family of almost complex structures which behave under composition like the multiplicative, pure-imaginary, unit quaternions and which are covariantly constant with respect to the Levi-Civita connection. If we only require that these almost complex structures exist locally and that the Levi-Civita connection preserves this family as a whole, then we obtain a quaternionic Kähler structure, at least if $n \geqslant 2$. Thus hyperKähler manifolds are a special case of quaternionic Kähler manifolds. However, note that quaternionic Kähler manifolds need not be Kähler.

HyperKähler and quaternionic Kähler manifolds are special classes of Einstein manifolds. From this viewpoint it is natural to try and construct Einstein metrics on bundles over these manifolds. Einstein metrics on certain sphere bundles over homogeneous manifolds, for example, were constructed by Jensen (1973) and more recently Page & Pope (1986) discussed the Einstein equations on quaternionic line bundles. We look for special solutions of these equations which carry information about a quaternionic Kähler base manifold. In Chapter 2 we construct hyperKähler and quaternionic Kähler metrics on the $\mathbb{Z}/2$-quotient $\mathcal{U}(M)$ of the natural quaternionic line bundle associated to a quaternionic Kähler manifold M. These constructions are generalisations of fibrations

$$\mathbb{H} \setminus 0 \longrightarrow \mathbb{H}^{n+1} \setminus 0 \longrightarrow \mathbb{H}\mathrm{P}(n)$$

$$\mathbb{H} \setminus 0 \longrightarrow \mathbb{H}\mathrm{P}(n+1) \setminus \mathbb{H}\mathrm{P}(n) \longrightarrow \mathbb{H}\mathrm{P}(n).$$

In four-dimensional geometry, hyperKähler manifolds are already well-known under a different name: they are Ricci-flat self-dual spaces. Quaternionic Kähler manifolds are then a higher dimensional generalisation of self-dual Einstein manifolds. These 4-manifolds possess twistor spaces which are complex 3-folds encoding the conformal structure of the original manifolds in holomorphic data. Salamon (1982) showed that higher dimensional twistor spaces are associated to any quaternionic Kähler manifold and the inverse of this twistor construction has been studied by LeBrun (1989) and Pedersen & Poon (1989).

The manifold $\mathcal{U}(M)$ with its hyperKähler metric may be regarded as a generalised twistor space, the geometry of M now being encoded in triholomorphic data. In Chapter 3 we discuss the inverse construction which produces a quaternionic Kähler manifold from a hyperKähler manifold admitting a certain type of $SU(2)$ action. The construction is a variation of the Marsden-Weinstein symplectic reduction procedure: this produces a symplectic manifold of dimension $k - 2\dim G$ from a symplectic k-manifold N and an action of a Lie group G on N which preserves the symplectic structure. This reduction also applies to Kähler manifolds, and Hitchin et al. (1987) showed how it could be generalised to hyperKähler manifolds (this time the dimension is reduced by four times $\dim G$). Galicki & Lawson (1988) produced a quaternionic Kähler version of this moment map construction and in Chapter 3 we see that quaternionic Kähler and hyperKähler quotients are compatible with our construction of $\mathcal{U}(M)$, that is, the associated bundle $\mathcal{U}$ of the quaternionic Kähler quotient of M by G may be obtained as the hyperKähler quotient of $\mathcal{U}(M)$. Thus quaternionic Kähler $4n$-manifolds may be studied in terms of special types of hyperKähler $(4n+4)$-manifolds. Since the product of two hyperKähler manifolds is again hyperKähler, this enables us to construct a quaternionic Kähler manifold $\mathcal{J}(M_1, M_2)$ of dimension $4 + \dim M_1 + \dim M_2$ from two quaternionic Kähler manifolds M_1, M_2. An interpretation of the quaternionic Kähler quotient

construction in terms of twistor spaces will be discussed in Chapter 4. Moment maps for circle actions on hyperKähler manifolds are intimately related to Kähler potentials. In Chapter 3 we see that the particular type of $SU(2)$-action required above gives rise to a function which is simultaneously a Kähler potential for each complex structure.

HyperKähler metrics are essentially distinguished from more general quaternionic Kähler ones by having zero scalar curvature. The hyperKähler metrics on $\mathcal{U}(M)$ are a special case of the quaternionic Kähler metrics and arise as a limit as the scalar curvature tends to zero. In this setting, the underlying quaternionic structure of $\mathcal{U}(M)$ contains Einstein metrics with both zero and non-zero Ricci tensors, a phenomenon which in four-dimensions only occurs when the underlying space is conformally flat (Brinkman, 1925). When M is $\mathrm{Gr}_2(\mathbb{C}^n)$, M may be regarded as a quaternionic Kähler quotient of $\mathbb{HP}(n-1)$ by $U(1)$, and so $\mathcal{U}(\mathrm{Gr}_2(\mathbb{C}^n))$ is a hyperKähler quotient of flat space $\mathbb{H}^n$ by the circle. It is well-known that this particular hyperKähler quotient may be deformed so as to produce the hyperKähler metric on $T^*\mathbb{CP}(n-1)$ constructed by Calabi (1979). Thus the hyperKähler metric on $\mathcal{U}(\mathrm{Gr}_2(\mathbb{C}^n))$ also arises as a limit of complete hyperKähler metrics.

The compact homogeneous quaternionic Kähler manifolds were studied by Wolf (1965). The associated bundles for these manifolds may be interpreted as orbits of highest roots in the Lie algebra of the complexified isometry group. In Chapter 4, we study the nilpotent orbits of a complex, semi-simple Lie group and see that these are always associated bundles $\mathcal{U}(M)$. We show that the quaternionic Kähler manifolds lie in a certain subvariety of the Grassmannian of three-planes in the real Lie algebra. In general, these manifolds are not complete, but various examples may be described in terms of finite quotients of Wolf spaces. The twistor spaces of these manifolds are just the projectivised nilpotent orbits and the twistor space of any quaternionic Kähler manifold, with positive scalar curvature

and sufficiently large isometry group will arise in this way.

The holonomy groups of irreducible Riemannian manifolds were classified by Berger (1955) and Alekseevskiĭ (1968). For spaces which are not locally symmetric, they obtained the families $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)\,Sp(1)$, $Sp(n)$ and two exceptional groups G_2 and $Spin(7)$ (which can only occur in dimensions 7 and 8, respectively). Simons (1962) observed that these were most of the simply-connected groups which act transitively on the finite-dimensional spheres. The group $SO(n)$ corresponds to generic geometry; $U(n)$ and $SU(n)$ give Kähler and special Kähler manifolds; and $Sp(n)\,Sp(1)$ and $Sp(n)$ correspond to quaternionic Kähler and hyperKähler geometries, respectively. Complete, non-symmetric metrics with holonomy either G_2 or $Spin(7)$ have now been constructed by Bryant & Salamon (1989). Each of the above geometries is given by invariant parallel differential forms: $U(n)$ holonomy is given by a Kähler 2-form; special Kähler manifolds possess a parallel complex volume form in addition to the Kähler form; quaternionic Kähler manifolds have a fundamental 4-form; hyperKähler metrics are given by a quaternion-valued 2-form; holonomy G_2 is determined by a 3-form and $Spin(7)$ metrics have an invariant parallel 4-form. In the cases of G_2 and $Spin(7)$, there are simple criteria in terms of exterior derivatives for these forms to define structures with the required holonomy. For hyperKähler structures, Hitchin (1987) shows that it is sufficient for the quaternionic 2-form to be closed. In Chapter 5, we show that for a manifold of dimension at least 12, the holonomy is contained in $Sp(n)\,Sp(1)$ if and only if the fundamental 4-form is closed. Thus to construct a quaternionic Kähler metric on a manifold of dimension at least 12, it is sufficient to find a 4-form of the correct algebraic type and show that this 4-form is closed. This result forms the basis of many of our constructions, but its proof bears little relation to the applications and so it is delayed to a self-contained part of Chapter 5.

The reader is referred to Kobayashi & Nomizu (1963) and Griffiths & Harris

(1978) for standard terminology from differential and algebraic geometry. Manifolds will be taken to be smooth (C^∞), second countable and connected, unless otherwise stated.

1.1 HyperKähler Manifolds

An *almost complex structure* on a manifold M is an endomorphism $J \in \mathrm{End}(TM)$ of the tangent bundle such that $J^2 = -1$. This makes each T_xM into a complex vector space and forces M to be even-dimensional.

Suppose M admits two anti-commuting, almost complex structures I and J, so we have $I^2 = -1 = J^2$ and $IJ = -JI$. If we define $K = JI$ then $K^2 = -1$, so K is also an almost complex structure. In fact, M has a family $\mathcal{I}$ of almost complex structures parameterised by the space S^2 of unit imaginary quaternions in $\mathrm{Im}\,\mathbb{H}$, since, if $ai + bj + ck \in S^2$, then

$$(aI + bJ + cK)^2 = -(a^2 + b^2 + c^2) = -1.$$

We may define a left action of $\mathbb{H}$ on T_xM by

$$(\alpha + \beta i + \gamma j + \delta k) \cdot X = \alpha X + \beta IX + \gamma JX + \delta KX,$$

for $X \in T_xM$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. So T_xM is a left $\mathbb{H}$-module and, in particular, its real dimension is a multiple of 4. If, in addition, M has a pseudo-Riemannian metric g which is preserved by I and J, then for each $A \in \mathcal{I}$ we may define a 2-form ω_A on M by

$$\omega_A(X, Y) = g(X, AY),$$

for each $X, Y \in T_xM$.

Definition 1.1.1. A (pseudo-) Riemannian manifold M is *(pseudo-) hyperKähler* if it has a pair of anti-commuting almost complex structures I and J which preserve the metric and are such that

$$d\omega_I = d\omega_J = d\omega_K = 0, \tag{1.1.1}$$

where ω_A is defined above and $K = IJ$.

Note that ω_A is non-degenerate, so (1.1.1) is just the requirement that ω_A is a symplectic form for each $A \in \mathcal{I}$. We may put these three conditions into one, by defining a quaternion-valued 2-form $\omega \in \Omega^2(M, \mathbb{H})$

$$\omega = \omega_I i + \omega_J j + \omega_K k$$

and requiring that $d\omega = 0$. Note that this definition only depends on $\mathcal{I}$ and so could be studied in the language of bunches of forms, see Gindikin (1982a,b, 1986).

The basic example of a hyperKähler manifold is flat space $\mathbb{H}^n$, or more generally $\mathbb{H}^{n*}/\Gamma = (\mathbb{H}^n \setminus \{0\})/\Gamma$, for Γ a finite subgroup of $SU(2)$. An important case for us will be the quotient $\mathbb{H}^*/\mathbb{Z}_2 = \mathbb{R}_{>0} \times \mathbb{R}\mathrm{P}(3)$. Examples of compact hyperKähler manifolds are provided by K3 surfaces, which are complex surfaces with first betti number $b_1 = 0$ and first Chern class $c_1 = 0$ (see Calabi, 1980). Compact examples in higher dimensions have been constructed by Fujiki (1983), Mukai (1984) and Beauville (1983). Closely related to K3 surfaces are the Enriques surfaces; these are complex surfaces with $b_1 = 0$ and $2c_1 = 0$ (see Hitchin, 1974). However, in general, Enriques surfaces are only locally hyperKähler. Calabi (1979) gives examples of hyperKähler structures on cotangent bundles of certain spaces, for example $T^*\mathbb{C}\mathrm{P}(n)$, and hyperKähler metrics have been shown to exist on various moduli spaces of solutions of the Yang-Mills equations. These will be discussed later, but

one important case is the result of Kronheimer (1988, 1989b) that both the nilpotent and semi-simple orbits of a semi-simple complex Lie algebra are hyperKähler. The Calabi metric on $T^*\mathbb{C}\mathrm{P}(n)$ is then an example of a hyperKähler metric on a semi-simple orbit in $\mathfrak{su}(n+1,\mathbb{C})$. Note that any Ricci-flat, self-dual 4-manifold is locally hyperKähler (see Besse, 1987; the definition of self-duality will be given later in the chapter).

If N is a complex manifold, the complex structure on N gives rise to an almost complex structure $J \in \mathrm{End}(TN)$ (see Wells, 1979). An almost complex structure J on N is said to be *integrable* if it comes from some complex structure on N. The *torsion* of an almost complex structure J is defined by

$$N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\}$$

for each $X, Y \in TN$. The relationship between the torsion and integrability of J is given by the following theorem, proofs of which may also be found in Hörmander (1966) and Kohn (1963).

Theorem 1.1.2. (Newlander & Nirenberg, 1957) *An almost complex structure J on N is integrable if and only if its torsion vanishes.* □

The definition of a hyperKähler manifold should be compared with that for a Kähler manifold. In the Kähler case, we have only one almost complex structure I and require that $d\omega_I = 0$ and that I is integrable. The integrability requirement does not enter the definition of hyperKähler manifold as we have the following lemma.

Lemma 1.1.3. (Hitchin, 1987) *On a pseudo-Riemannian manifold M with almost complex structures I, J and K as above, I, J and K are integrable if $d\omega = 0$, i.e. if M is pseudo-hyperKähler.* □

In future, when we talk about a manifold with almost complex structures I, J and K we will require that they satisfy the relations

$$I^2 = -1 = J^2 \qquad IJ = -JI = K.$$

1.2 Other Quaternionic Geometries

A $4n$-dimensional manifold M is said to be *almost quaternionic* if there is a subbundle $\mathcal{G}$ of $\mathrm{End}(TM)$ such that for each $x \in M$ there is a neighbourhood U over which $\mathcal{G}|_U$ has a basis $\{I, J, K\}$ of almost complex structures with $K = IJ = -JI$. Note that this is only a local basis. If M is Riemannian, the metric g is said to be compatible with $\mathcal{G}$ if

$$g(AX, AY) = g(X, Y),$$

for each $X, Y \in T_xM$ and $A \in \mathcal{G}_x$ such that $A^2 = -1$. We can construct a compatible metric g from any Riemannian metric g' by defining

$$g(X, Y) = \tfrac{1}{4}(g'(X, Y) + g'(IX, IY) + g'(JX, JY) + g'(KX, KY)).$$

An almost quaternionic manifold with a compatible metric is called a *quaternion-Hermitian manifold.*

On a quaternion-Hermitian manifold $(M, \mathcal{G}, g)$ we have an isometric embedding $\mathcal{G} \to \Lambda^2 T^*M$, given by $A_x \mapsto (\omega_A)_x$. If I, J, K is a local basis of $\mathcal{G}$ as above, we define the *fundamental 4-form* of M by

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K.$$

This is a non-degenerate form which is well-defined globally.

If $\dim M$ is at least 8, we are now in a position to say what is meant by a quaternionic Kähler structure on M. The definition for 4-manifolds will be given later.

Definition 1.2.1. If $n > 1$, a $4n$-dimensional quaternion-Hermitian manifold is *quaternionic Kähler* if $\nabla\Omega = 0$, where ∇ is the Levi-Civita connection.

This immediately implies that Ω is closed and, since

$$\nabla\Omega = 2(\nabla\omega_I \wedge \omega_I + \nabla\omega_J \wedge \omega_J + \nabla\omega_K \wedge \omega_K),$$

the condition $\nabla\Omega = 0$ implies that the Riemannian connection preserves the subbundle $\mathcal{G}$ of $\Lambda^2 T^*M$. The following result will be used in constructions of quaternionic Kähler metrics in later chapters. However, its proof can be safely delayed until Chapter 5 since it is does not depend on those results.

Theorem 1.2.2. *If M is a quaternion-Hermitian manifold of dimension at least 12, then $d\Omega$ determines $\nabla\Omega$. In particular, $d\Omega = 0$ implies $\nabla\Omega = 0$ and that M is quaternionic Kähler.*

A quaternion-Hermitian 8-manifold is quaternionic Kähler if and only if the fundamental 4-form Ω is closed and the algebraic ideal generated by the subbundle $\mathcal{G}$ of $\Lambda^2 T^*M$ is a differential ideal.

It is not known whether there exist compact quaternion-Hermitian 8-manifolds for which $d\Omega$ vanishes but $\nabla\Omega$ is non-zero.

Let the group $Sp(n)$ be defined by

$$Sp(n) = \{\, A \in M_n(\mathbb{H}) : \bar{A}^t A = 1 \,\},$$

where $M_n(\mathbb{H})$ denotes the quaternionic $n \times n$-matrices and $\bar{A}^t$ is the conjugate transpose of A. We let $(A, q) \in Sp(n) \times Sp(1)$ act on $\xi \in \mathbb{H}^n$ by

$$(A, q) \cdot \xi = A\xi\bar{q}.$$

Now $(-A,-q)$ acts as (A,q), so we have an induced action of $Sp(n)\,Sp(1) = Sp(n)\times_{\mathbb{Z}_2} Sp(1)$ on $\mathbb{H}^n$ which exhibits $Sp(n)\,Sp(1)$ as a subgroup of $SO(4n)$.

The definition of a quaternionic Kähler manifold is equivalent to the requirement that the linear holonomy group (see Kobayashi & Nomizu, 1963) is contained in the subgroup $Sp(n)\,Sp(1)$ of $SO(4n)$. Alekseevskiĭ (1968) shows that this implies M is an Einstein manifold. In particular, if we regard the curvature tensor R of M as a self-adjoint endomorphism of $\Lambda^2 T^*M$, then

$$R|_{\mathcal{G}} = \lambda\,\mathrm{id}_{\mathcal{G}}, \tag{1.2.1}$$

where λ is a positive multiple of the scalar curvature κ of M (see also Ishihara, 1974).

If M is 4-dimensional, then $\nabla\Omega = 0$ is automatically satisfied, since Ω is just three times the volume form. However, there is an extension of the definition of quaternionic Kähler manifold. Suppose M is an oriented, Riemannian 4-manifold with volume form ν. The Hodge star operator $*\colon \Lambda^r T^*M \to \Lambda^{4-r} T^*M$ is defined by $\alpha\wedge\beta = g(\alpha,\beta)\nu$, for $\alpha,\beta\in\Lambda^r T^*M$. For $r=2$, $*^2 = 1$ and $\Lambda^2 T^*M = \Lambda^2_+ \oplus \Lambda^2_-$, where $\Lambda^2_\pm$ are the ±1-eigenspaces of $*$. The two eigenspaces are the bundles of self-dual and anti-self-dual 2-forms respectively. With respect to this decomposition of $\Lambda^2 T^*M$, the Riemann curvature tensor R is $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ and Singer & Thorpe (1969) give the following decomposition into irreducible components

$$R \longmapsto (\operatorname{Tr} A, B, A - \tfrac{1}{3}\operatorname{Tr} A, C - \tfrac{1}{3}\operatorname{Tr} C).$$

Here $\operatorname{Tr} A = \operatorname{Tr} C = \kappa/4$, where κ is the scalar curvature, B is the traceless Ricci tensor and, putting $W_+ = A - \frac{1}{3}\operatorname{Tr} A$, $W_- = C - \frac{1}{3}\operatorname{Tr} C$, $W = W_+ + W_-$ is the

Weyl tensor. Now M is said to be *Einstein* if $B \equiv 0$ and it is said to be *self-dual* if $W \equiv W_+$, that is if $W_- \equiv 0$. As in (1.2.1), these two conditions imply that

$$R|_{\mathcal{G}} = \frac{\kappa}{12}\mathrm{id}_{\mathcal{G}},$$

where $\mathcal{G} = \Lambda^2_-$.

Definition 1.2.3. A 4-dimensional, oriented, Riemannian manifold M is *quaternionic Kähler* if M is Einstein and self-dual.

Further motivation for this definition comes from the following result of Marchiafava (1990). Call a submanifold N of a quaternion-Hermitian manifold M *quaternionic*, if for each $x \in N$, T_xN is an $\mathbb{H}$-submodule of T_xM. Marchiafava shows that a four-dimensional quaternionic submanifold of a quaternionic Kähler manifold is self-dual and Einstein with respect to the induced metric.

Note that in both the 4- and $4n$-dimensional cases, quaternionic Kähler manifolds are not necessarily Kähler. Even if a quaternionic Kähler manifold does possess a Kähler structure, this need not be compatible with the quaternionic structure, as in the case of $\mathrm{Gr}_2(\mathbb{C}^n)$. Because quaternionic Kähler manifolds are Einstein of dimension strictly bigger than 2, their scalar curvature is necessarily constant (see Kobayashi & Nomizu, 1963). If M is a quaternionic Kähler manifold with zero scalar curvature, then the restriction of the Riemann tensor to $\mathcal{G}$ is zero and so $\mathcal{G}$ is flat. If M is also simply connected then this implies that $\mathcal{G}$ is trivial and that M is hyperKähler. Conversely, if M is hyperKähler then its scalar curvature vanishes and its linear holonomy group is a subgroup of $Sp(n) \leqslant Sp(n)\,Sp(1)$. In view of this discussion, from now on we will usually take the expression 'quaternionic Kähler' to imply non-zero scalar curvature.

The model example of a quaternionic Kähler manifold is quaternionic projective space $\mathbb{H}\mathrm{P}(n)$. The compact homogeneous quaternionic Kähler manifolds were studied by Wolf (1965). He obtains five exceptional spaces, which we list in Chapter 2, and three infinite families: $\mathbb{H}\mathrm{P}(n)$; the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ of 2-planes in $\mathbb{C}^n$ and the Grassmannian $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^n)$ of oriented 4-planes in $\mathbb{R}^n$. The non-compact duals of these spaces are also quaternionic Kähler. Non-compact, non-symmetric examples were obtained by Alekseevskiĭ (1970, 1975) and quaternionic Kähler orbifolds have been constructed by Galicki & Lawson (1988). If $\mathbb{H}\mathrm{P}(n)$ is embedded in $\mathrm{Gr}_2(\mathbb{C}^{2n+2})$ by regarding $\mathbb{C}^{2n+2}$ as $\mathbb{H}^{n+1}$, then the normal bundle is $T\,\mathbb{H}\mathrm{P}(n)$ and this inherits an incomplete quaternionic Kähler metric from that of the Grassmannian.

Moving away from the Riemannian category, there are the following two notions corresponding to hyperKähler and quaternionic Kähler. A *hypercomplex* manifold is a manifold with *complex* structures I, J and K such that $IJ = -JI = K$. Even though we have not fixed a compatible metric, there is a natural connection.

Theorem 1.2.4. (Obata, 1956) *If X is a $4n$-dimensional manifold with a pair I, J of integrable, anti-commuting almost complex structures, then there is a unique torsion-free connection ∇ such that*

$$\nabla I \equiv 0 \equiv \nabla J.$$

□

On a hyperKähler manifold this connection is just the Levi-Civita connection. The following definition, which will appear naturally in Chapter 5, is due independently to Bérard-Bergery (see Besse, 1987) and Salamon (1986): a *quaternionic* manifold is a manifold M with a bundle $\mathcal{G}$ together and a compatible torsion-free connection. This implies that locally there are sections of $\mathcal{G}$ which are integrable

complex structures. Quaternionic Kähler manifolds are quaternionic, the relevant connection being the Levi-Civita connection.

Chapter 2
BUNDLE CONSTRUCTIONS

The basic examples of hyperKähler and quaternionic Kähler manifolds are $\mathbb{H}^{n+1}$ and $\mathbb{H}\mathrm{P}(n)$, with their flat and symmetric metrics respectively. The twistor space of $\mathbb{H}\mathrm{P}(n)$ is $\mathbb{C}\mathrm{P}(2n+1)$. These spaces are clearly intimately related from the algebraic viewpoint. In this chapter we begin our exploration of the links between their natural metrics. We show how the hyperKähler metric of $\mathbb{H}^{n+1}$ can be recovered either from $\mathbb{H}\mathrm{P}(n)$ or from $\mathbb{C}\mathrm{P}(2n+1)$. Also, we see how $\mathbb{H}\mathrm{P}(n)$ defines the quaternionic Kähler metric on $\mathbb{H}^{n+1}$ induced from the identification $\mathbb{H}^{n+1} \cong \mathbb{H}\mathrm{P}(n+1) \setminus \mathbb{H}\mathrm{P}(n)$ in such a way that the picture can be generalised. If M is any quaternionic Kähler $4n$-manifold with positive scalar curvature, then Salamon (1982) showed that there is a Kähler manifold Z naturally associated to M. We define a $(4n+4)$-manifold $\mathcal{U}(M)$ and obtain the following mappings.

$$\begin{array}{ccc} \mathbb{H}^{n+1} \setminus 0 & & \\ \downarrow & \searrow & \\ & & \mathbb{C}\mathrm{P}(2n+1) \\ & \swarrow & \\ \mathbb{H}\mathrm{P}(n) & & \end{array} \qquad \begin{array}{ccc} \mathcal{U}(M) & & \\ \downarrow & \searrow & \\ & & Z \\ & \swarrow & \\ M & & \end{array}$$

We construct hyperKähler and quaternionic Kähler metrics on $\mathcal{U}(M)$. The twistor space Z determines the quaternionic Kähler structure of M, so it is also possible to construct the metrics on $\mathcal{U}(M)$ directly from Z. In the second part of this chapter we show that hyperKähler metrics can be constructed on complex line bundles over a larger class of complex-contact manifolds. We then describe $\mathcal{U}(M)$ for the compact

homogeneous quaternionic Kähler manifolds and show that the construction for projective spaces is as claimed above.

In Section 4 we forget the symplectic forms and look at the integrability of the almost complex structures on $\mathcal{U}(M^4)$, obtaining a hypercomplex structure precisely when M is self-dual. Integrability considerations over Kähler-Einstein manifolds show that the metrics we construct over twistor spaces (and their generalisations) are degenerate versions of the Ricci-flat, Kähler metrics constructed by Calabi (1979) on the canonical bundle. Calabi's construction produces metrics with holonomy $SU(n+1)$ from Einstein metrics with holonomy $U(n)$. Our construction over a quaternionic Kähler base is analogous, creating holonomy $Sp(n+1)$ from holonomy $Sp(n)\,Sp(1)$. The construction on complex line bundles cannot be described quite so cleanly, but working in the category of Einstein manifolds, it produces $Sp(n+1)$-holonomy from metrics with holonomy $U(2n+1)$ and structure group $(1 \times Sp(n))\,U(1)$. Note that $U(2n+1)$ is the smallest irreducible holonomy group consistent with such a structure group.

All the metrics in this chapter will be constructed on associated bundles. To fix our conventions we recall how these are defined. Let M be an oriented Riemannian manifold. A frame u on M at a point x is an orientation-preserving linear isometry $u\colon \mathbb{R}^n \to T_xM$. These maps form a principal $SO(n)$-bundle which we denote by $SO(M)$. If M has holonomy group $G \leqslant SO(n)$ then the bundle $SO(M)$ may be reduced to a principal G-bundle F (see Kobayashi & Nomizu, 1963). To any G-module W we associate a bundle $\mathcal{W}$ defined by

$$\mathcal{W} = F \times_G W = \frac{F \times W}{G},$$

where $g \in G$ acts on $(f,w) \in F \times W$ by $(f,w)\cdot g = (f\cdot g, g^{-1}\cdot w)$. This is a vector bundle over M with fibre W. We usually write W for $\mathcal{W}$ and not distinguish between the bundle and the representation.

2.1 Metrics Over a Quaternionic Kähler Base

If M is a quaternionic Kähler manifold of dimension $4n$, then its holonomy group is a subgroup of $Sp(n)\,Sp(1)$ and the reduced frame bundle F consists of frames $u\colon \mathbb{H}^n \to T_xM$ which are compatible with the quaternionic structure. Locally, F can be lifted to a principal $Sp(n)\times Sp(1)$-bundle $\tilde{F}$ which double covers F and this enables the construction of bundles associated to representations of $Sp(n)\times Sp(1)$. These bundles exists globally if either $\tilde{F}$ exists globally or $(-1,-1)\in Sp(n)\times Sp(1)$ acts as the identity.

The obstruction to the global existence of the double cover $\tilde{F}$ is a cohomology class $\varepsilon \in H^2(M,\mathbb{Z}_2)$ defined by Marchiafava & Romani (1976). If M is $4n$-dimensional with n odd, then ε vanishes if and only if the second Stiefel-Witney class w_2 vanishes. The class w_2 is the obstruction to lifting $SO(M)$ to a principal $Spin(4n)$-bundle. If w_2 vanishes, M is said to be a spin manifold. Salamon (1982) noted that quaternionic Kähler $8n$-manifolds are automatically spin and proved that the only complete quaternionic Kähler manifolds with positive scalar curvature for which ε vanishes are the quaternionic projective spaces $\mathbb{H}\mathrm{P}(n)$.

There are two basic modules we consider: E is $\mathbb{H}^n$ with

$$A\cdot\xi = A\xi,$$

where $A\in Sp(n)$ and $\xi\in\mathbb{H}^n$; and H is $\mathbb{H}$ with

$$q\cdot y = y\bar{q},$$

where $q\in Sp(1)$ and $y\in\mathbb{H}$. If M is a 4-manifold then $E=V_+$ and $H=V_-$ are just the positive and negative spin bundles. From the realisation of $Sp(n)\,Sp(1)$ as a subgroup of $SO(4n)$, we have that the complexified tangent bundle $T_{\mathbb{C}}M$ is $E\otimes_{\mathbb{C}}H$.

The bundle F carries a canonical 1-form θ with values in $\mathbb{H}^n$ given by

$$\theta_u(v) = u^{-1}(\pi_* v),$$

where $\pi\colon F \to M$ is the projection map, $u \in F$ and $v \in T_u F$. The Levi-Civita connection on M induces a connection $\omega \in \Omega^1(F, \mathfrak{sp}(n) \oplus \mathfrak{sp}(1))$ on F which is torsion-free. The first structure equation then gives

$$d\theta = -\omega \bar{\wedge} \theta.$$

Here $\omega \bar{\wedge} \theta$ is the 2-form whose value on $X, Y \in T_x F$ is

$$(\omega \bar{\wedge} \theta)(X, Y) = \tfrac{1}{2}(\omega(X) \cdot \theta(Y) - \omega(Y) \cdot \theta(X)),$$

where $\cdot$ denotes the action of $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ on $\mathbb{H}^n$. Corresponding to this Lie algebra splitting we write $\omega = \omega_+ + \omega_-$.

Lemma 2.1.1. *If $\wedge$ denotes the usual exterior product on quaternion-valued forms, then*

$$d\theta = -\omega_+ \bar{\wedge} \theta - \omega_- \bar{\wedge} \theta = -\omega_+ \wedge \theta - \theta \wedge \omega_-. \tag{2.1.1}$$

PROOF. The Lie algebra $\mathfrak{sp}(n)$ consists of quaternion-valued $n \times n$-matrices A such that $A + \bar{A}^t = 0$. If $A \in \mathfrak{sp}(n)$, $q \in \mathfrak{sp}(1) \cong \operatorname{Im}\mathbb{H}$ and $\xi \in \mathbb{H}^n$ then differentiating the $Sp(n)\,Sp(1)$-action on $\mathbb{H}^n$ gives

$$(A + q) \cdot \xi = A\xi + \xi\bar{q} = A\xi - \xi q.$$

It is now clear that $\omega_+ \bar{\wedge} \theta = \omega_+ \wedge \theta$. The algebra $\Omega(F, \mathbb{H}^n)$ is $\Omega(F) \otimes \mathbb{H}^n$, so we can write $\omega_- = \sum_i \alpha_i \otimes q_i$ and $\theta = \sum_j \beta_j \otimes \xi_j$ with $\alpha_i, \beta_j \in \Omega^1(F)$ and $q_i \in \operatorname{Im}\mathbb{H}$, $\xi_j \in \mathbb{H}^n$, then

$$\begin{aligned}\omega_- \bar{\wedge} \theta &= \sum_{i,j} \alpha_i \wedge \beta_j \otimes q_i \cdot \xi_j = -\sum_{i,j} \alpha_i \wedge \beta_j \otimes \xi_j q_i \\ &= \sum_{i,j} \beta_j \wedge \alpha_i \otimes \xi_j q_i = \theta \wedge \omega_-,\end{aligned}$$

as required. □

The uniqueness of the Levi-Civita connection implies that (2.1.1) determines $\omega = \omega_+ + \omega_-$ as an element of $\Omega^1(F, \mathfrak{sp}(n) \oplus \mathfrak{sp}(1))$.

The metrics we construct will be defined on the bundle H which is the $Sp(n) \times Sp(1)$-quotient of $\tilde{F} \times \mathbb{H}$. The forms θ, $\omega_\pm$ can be pulled back to $\tilde{F} \times \mathbb{H}$, where they will also be denoted by θ and $\omega_\pm$ respectively. There is a natural quaternion-valued function x on $\tilde{F} \times \mathbb{H}$ which is the pull-back of the identity map on $\mathbb{H}$. We define a quaternion-valued 1-form α on $\tilde{F} \times \mathbb{H}$ by

$$\alpha = dx - x\omega_-.$$

Proposition 2.1.2. *The forms $x\bar{\theta}^t \wedge \theta\bar{x}$ and $\alpha \wedge \bar{\alpha}$ on $\tilde{F} \times \mathbb{H}$ are pull-backs of forms on the $Sp(n) \times Sp(1)$-quotient H.*

PROOF. We use the following well-known Lemma.

Lemma 2.1.3. *If a Lie group G acts smoothly on a manifold N in such a way that the quotient N/G is a manifold, then the projection $p\colon N \to N/G$ induces a natural isomorphism*

$$\Omega^*(N/G) \xrightarrow{\cong} \Omega^*_{\mathrm{bas}}(N)$$

between forms on N/G and basic forms on N, that is forms $\mu \in \Omega^*(N)$ such that

$$g^*\mu = \mu \qquad \text{and} \qquad X \lrcorner \mu = 0,$$

for all $g \in G$ and for all $X \in TN$ such that $p_*X = 0$.

In our case G is $Sp(n) \times Sp(1)$ and the action on $\Omega^*(\tilde{F} \times \mathbb{H})$ will be denoted by $R^*_{A,q}$. From the definitions of θ and x, we have

$$R^*_{A,q}\theta = \bar{A}^t \theta q, \tag{2.1.2}$$

and

$$R^*_{A,q}x = xq.$$

If $\mu \in \Omega^p(X, \mathbb{H})$, $\nu \in \Omega^q(X, \mathbb{H})$ are two quaternion valued forms on a space X, then

$$\overline{\mu \wedge \nu} = (-1)^{pq} \bar{\nu} \wedge \bar{\mu}. \tag{2.1.3}$$

Therefore,

$$\begin{aligned} R^*_{A,q}(x\bar{\theta}^t \wedge \theta\bar{x}) &= xq\overline{\left(\bar{A}^t\theta q\right)}^t \wedge \bar{A}^t\theta q\overline{(xq)} \\ &= xq\bar{q}\bar{\theta}^t A \wedge \bar{A}^t\theta q\bar{q}\bar{x} \\ &= x\bar{\theta}^t \wedge \theta\bar{x}. \end{aligned}$$

Consider the following commutative diagram in which all the maps are projections:

$$\begin{array}{ccc} \tilde{F} \times \mathbb{H} & \xrightarrow{\varpi} & H \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi^H} \\ \tilde{F} & \xrightarrow{\pi} & M \end{array}$$

If $X \in T(\tilde{F}\times\mathbb{H})$ then $\theta(X)$ is zero if and only if $\pi_* p_{1*} X = 0$. But $\pi_* p_{1*} X = \pi_*^H p_* X$, which is zero if $p_* X = 0$, so $x\bar{\theta}^t \wedge \theta\bar{x}$ descends to H.

From equations (2.1.1) and (2.1.2)

$$R^*_{A,q} d\theta = -\left(R^*_{A,q}\omega_+\right) \wedge \left(\bar{A}^t\theta q\right) - \left(\bar{A}^t\theta q\right) \wedge \left(R^*_{A,q}\omega_-\right).$$

Since $R^*_{A,q}$ commutes with d, we also have

$$R^*_{A,q} d\theta = d\left(\bar{A}^t\theta q\right) = -\left(\bar{A}^t\omega_+ A\right) \wedge \left(\bar{A}^t\theta q\right) - \left(\bar{A}^t\theta q\right) \wedge \left(\bar{q}\omega_- q\right).$$

From the uniqueness of ω, we obtain

$$R^*_{A,q}\omega_+ = \bar{A}^t\omega_+ A,$$

$$R^*_{A,q}\omega_- = \bar{q}\omega_- q.$$

Thus for α, we have

$$R^*_{A,q}\alpha = \alpha q, \tag{2.1.4}$$

so $\alpha \wedge \bar{\alpha}$ is $Sp(n)\times Sp(1)$-invariant.

If $X \in T_{(u,y)}(\tilde{F}\times\mathbb{H})$ is such that $p_* X = 0$, then X is tangent to the $Sp(n)\times Sp(1)$-action and we may write $X = (u\cdot(P,Q), yQ)$ for some $(P,Q) \in \mathfrak{sp}(n)\oplus\mathfrak{sp}(1)$. By definition, $\omega_+(X) = P$ and $\omega_-(X) = Q$, so

$$\alpha(X) = dx(X) - (x\omega_-)(X) = (yQ) - y\,Q = 0,$$

as required. □

The function $r^2 = x\bar{x}$ also descends to H, so we may consider the 2-form

$$\upsilon = f(r^2)\alpha \wedge \bar{\alpha} + g(r^2)x\bar{\theta}^t \wedge \theta\bar{x},$$

where f and g are arbitrary real-valued functions. From (2.1.3), we see that υ takes values in $\mathrm{Im}\,\mathbb{H}$ and so defines three real 2-forms. In general, if $\mu = \mu_0 + \mu_1 i + \mu_2 j + \mu_3 k \in \Omega^1(X, \mathbb{H})$ then

$$\bar{\mu} \wedge \mu = (\mu_0 \wedge \mu_1 - \mu_2 \wedge \mu_3)i + (\mu_0 \wedge \mu_2 - \mu_3 \wedge \mu_1)j + (\mu_0 \wedge \mu_3 - \mu_1 \wedge \mu_2)k.$$

Thus, the components of υ are good candidates for symplectic forms on H.

Proposition 2.1.4. *If f and $r^2 g$ are nowhere zero and υ is the form defined above, then the i, j and k components of υ are non-degenerate, real 2-forms on H.*

PROOF. The projection $\pi\colon \tilde{F} \to M$ has an $Sp(n) \times Sp(1)$-invariant distribution of horizontal subspaces given by $\ker\omega$. This defines a distribution $\mathcal{D}$ on $\tilde{F} \times \mathbb{H}$ such that for each $(u, y) \in \tilde{F} \times \mathbb{H}$

$$T_{(u,y)}(\tilde{F} \times \mathbb{H}) = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \oplus \mathcal{D}_{u,y} \oplus \mathbb{H}. \tag{2.1.5}$$

Thus $\mathcal{D}_{u,y}$ is isomorphic to $\ker\omega_u \leqslant T_u\tilde{F}$ and is $Sp(n) \times Sp(1)$-invariant. So $\mathcal{D}$ descends to a distribution $\mathcal{H}$ of horizontals on H such that at $a \in H$

$$T_a H \cong \mathcal{H}_a \oplus \mathbb{H} \cong \mathcal{D}_{u,y} \oplus \mathbb{H}$$

whenever $(u, y) \in p^{-1}(a)$. Here $\mathbb{H}$ is the space of vertical vectors, that is vectors tangent to the fibre of $H \to M$ over $\pi^H(a)$.

From the commutative diagram above, $x\bar{\theta}^t \wedge \theta\bar{x}$ vanishes on verticals. It is non-degenerate on $\mathcal{H}$ since $\bar{\theta}^t \wedge \theta$ is non-degenerate on $\ker\omega$. The form $\alpha = dx - x\omega_-$ vanishes on $\mathcal{D}$, hence $\alpha \wedge \bar{\alpha}$ vanishes on $\mathcal{H}$. It is sufficient to show that $\alpha \wedge \bar{\alpha}$ is non-degenerate on the verticals $\mathbb{H}$; but on $\mathbb{H}$, $\alpha \wedge \bar{\alpha}$ is just $dx \wedge d\bar{x}$, which is clearly non-degenerate. □

We can define a pseudo-Riemannian metric g_1 on H by

$$g_1 = \mathrm{Re}(f(r^2)\alpha \otimes \bar{\alpha} + g(r^2)r^2\bar{\theta}^t \otimes \theta).$$

Away from the zero section.
This metric is positive definite if f and g are both strictly positive everywhere and the almost complex structures defined by g_1 and v are independent of f and g. If we put

$$\begin{aligned}\Upsilon &= v \wedge v \\ &= f^2\alpha \wedge \bar{\alpha} \wedge \alpha \wedge \bar{\alpha} \\ &\quad + fg(\alpha \wedge \bar{\alpha} \wedge x\bar{\theta}^t \wedge \theta\bar{x} + x\bar{\theta}^t \wedge \theta\bar{x} \wedge \alpha \wedge \bar{\alpha}) \\ &\quad + g^2r^4\bar{\theta}^t \wedge \theta \wedge \bar{\theta}^t \wedge \theta,\end{aligned}$$

then g_1 is pseudo-hyperKähler if $dv = 0$, and by Theorem 1.2.2, if $\dim M \geqslant 8$, g_1 is pseudo-quaternionic Kähler if $d\Upsilon = 0$. Let $\mathcal{G}_1$ be the subbundle of $\Lambda^2 T^*\mathcal{U}(M)$ generate by the i, j and k components of v. If M is four-dimensional, we obtain pseudo-quaternionic Kähler metrics if, in addition, the algebraic ideal generated by $\mathcal{G}_1$ is a differential ideal. In order to compute these exterior derivatives we introduce the curvature forms $\Omega_\pm$ of $\omega_\pm$:

$$\Omega_+ = d\omega_+ + \omega_+ \wedge \omega_+, \qquad \Omega_- = d\omega_- + \omega_- \wedge \omega_-.$$

The base manifold M is Einstein, or in 4-dimensions Einstein and self-dual, if and only if

$$\Omega_- = c\bar{\theta}^t \wedge \theta, \tag{2.1.6}$$

where c is a constant positive multiple of the scalar curvature κ. For example, in 4-dimensions, c is $\kappa/48$.

Theorem 2.1.5. *The 4-form Υ is closed if and only if*

$$f = \frac{q}{c(pr^2+q)^2} \qquad \text{and} \qquad g = \frac{1}{pr^2+q},$$

for some p, $q \in \mathbb{R}$ and the algebraic ideal generated by $\mathcal{G}_1$ is a differential ideal if and only if

$$f = \frac{1}{c(1+\gamma r^2)} g,$$

for some real constant γ. Thus,

$$g_1 = \frac{q}{c(pr^2+q)^2}\,\mathrm{Re}(\alpha \otimes \bar{\alpha}) + \frac{r^2}{pr^2+q}\,\mathrm{Re}(\bar{\theta}^t \otimes \theta)$$

is a pseudo-quaternionic Kähler metric on $H \setminus 0$ away from the (possibly empty) submanifold $pr^2+q=0$ and we have a pseudo-hyperKähler metric if $p=0$.

PROOF. We check the hyperKähler property first. From (2.1.6),

$$d\alpha = -\alpha \wedge \omega_- - x\Omega_- = -\alpha \wedge \omega_- - cx\bar{\theta}^t \wedge \theta,$$

so

$$d(\alpha \wedge \bar{\alpha}) = -c(x\bar{\theta}^t \wedge \theta \wedge \bar{\alpha} + \alpha \wedge \bar{\theta}^t \wedge \theta\bar{x}).$$

Also, by (2.1.1),

$$d(\bar{\theta}^t \wedge \theta) = -\omega_- \wedge \bar{\theta}^t \wedge \theta + \bar{\theta}^t \wedge \theta \wedge \omega_-,$$

so

$$d(x\bar{\theta}^t \wedge \theta\bar{x}) = \alpha \wedge \bar{\theta}^t \wedge \theta\bar{x} + x\bar{\theta}^t \wedge \theta \wedge \bar{\alpha}.$$

This gives $cp\,dv = d(\alpha \wedge \bar{\alpha} + cx\bar{\theta}^t \wedge \theta\bar{x}) = 0$ and that g_1 is pseudo-hyperKähler in this case.

We now return to Υ. For greater generality, consider

$$\Xi = A(\alpha \wedge \bar{\alpha} \wedge \alpha \wedge \bar{\alpha}) + B(\alpha \wedge \bar{\alpha} \wedge x\bar{\theta}^t \wedge \theta\bar{x} + x\bar{\theta}^t \wedge \theta\bar{x} \wedge \alpha \wedge \bar{\alpha}) + Cr^4(\bar{\theta}^t \wedge \theta \wedge \bar{\theta}^t \wedge \theta),$$

where A, B and C are real-valued functions of r^2. In addition to the derivatives calculated above, we have

$$d(\bar{\theta}^t \wedge \theta \wedge \bar{\theta}^t \wedge \theta) = d(\pi^{H^*}\Omega) = 0$$

and

$$dr^2 = xd\bar{x} + dx\,\bar{x} = x\bar{\alpha} + \alpha\bar{x}.$$

Using this second equation to rearrange terms, we calculate that

$$\begin{aligned} d\Xi = &\left(-\frac{3c}{r^2}A + \frac{3}{r^2}B + B'\right) dr^2 \wedge (\alpha \wedge \bar{\alpha} \wedge x\bar{\theta}^t \wedge \theta\bar{x} + x\bar{\theta}^t \wedge \theta\bar{x} \wedge \alpha \wedge \bar{\alpha}) \\ &+ \left(-\frac{2c}{r^2}B + \frac{2}{r^2}C + C'\right) r^4 dr^2 \wedge \bar{\theta}^t \wedge \theta \wedge \bar{\theta}^t \wedge \theta, \end{aligned}$$

where $'$ denotes differentiation with respect to r^2. (Note that $dr^2 \wedge \alpha \wedge \bar{\alpha} \wedge \alpha \wedge \bar{\alpha}$ is zero since it is a 5-form that can only be non-zero on a 4-dimensional distribution.) Thus $d\Xi$ vanishes if and only if

$$-\frac{3c}{r^2}A + \frac{3}{r^2}B + B' = 0$$

and

$$-\frac{2c}{r^2}B + \frac{2}{r^2}C + C' = 0.$$

Putting $A = f^2$, $B = fg$ and $C = g^2$ these equations imply that $d\Upsilon = 0$ if and only if

$$-\frac{3c}{r^2}f^2 + \frac{3}{r^2}fg + f'g + fg' = 0$$

and

$$g\left(-\frac{c}{r^2}f+\frac{g}{r^2}+g'\right)=0.$$

If $g(r^2)\neq 0$ then the second equation gives

$$f(r^2)=\frac{1}{c}(r^2g'+g).$$

Substituting in the first equation implies

$$g''g=2g'^2$$

and that

$$g(r^2)=(pr^2+q)^{-1},$$

giving

$$f(r^2)=\frac{q}{c}(pr^2+q)^{-2}.$$

To find when the algebraic ideal generated by $\mathcal{G}_1$ is a differential ideal, we calculate dv:

$$dv=\left(\frac{g}{r^2}-\frac{cf}{r^2}\right)(x\bar{\theta}^t\wedge\theta\bar{x}\wedge x\bar{\alpha}+\alpha\bar{x}\wedge x\bar{\theta}^t\wedge\theta\bar{x})$$
$$+f'dr^2\wedge\alpha\wedge\bar{\alpha}+g'dr^2\wedge x\bar{\theta}^t\wedge\theta\bar{x}.$$

Because

$$dr^2\wedge\alpha\wedge\bar{\alpha}=-\alpha\bar{x}\wedge\alpha\wedge\bar{\alpha}-\alpha\wedge\bar{\alpha}\wedge x\bar{\alpha},$$

we see that the components of dv lie in $\mathcal{G}_1\wedge T^*\mathcal{U}(M)\subset\Lambda^3T^*\mathcal{U}(M)$ if and only if

$$\frac{f'}{f}=\frac{g'}{g}-\frac{1}{r^2}\left(1-\frac{cf}{g}\right).$$

Solving this differential equation give the desired result. □

Instead of υ we can consider the 2-form

$$\frac{1}{r^2}\left(a(r^2)\, x\bar{\alpha}\wedge\alpha\bar{x} + b(r^2)\, x\bar{\theta}^t\wedge\theta\bar{x}\right),$$

where a and b are arbitrary functions of r^2. This corresponds formally to inverting the fibre of H by mapping r^2 to $1/r^2$. In exactly the same way as above we obtain pseudo-quaternionic Kähler metrics.

Theorem 2.1.6. *On H there are pseudo-quaternionic Kähler metrics*

$$g_2 = \frac{p}{c(pr^2+q)^2}\operatorname{Re}\bar{\alpha}\otimes\alpha + \frac{1}{pr^2+q}\operatorname{Re}\bar{\theta}^t\otimes\theta$$

whose 4-forms are given by

$$\begin{aligned}&\frac{p^2}{c^2(pr^2+q)^4}(\bar{\alpha}\wedge\alpha\wedge\bar{\alpha}\wedge\alpha)\\ &\quad+\frac{p}{c(pr^2+q)^3}(\bar{\alpha}\wedge\alpha\wedge\bar{\theta}^t\wedge\theta+\bar{\theta}^t\wedge\theta\wedge\bar{\alpha}\wedge\alpha)\\ &\quad+\frac{1}{(pr^2+q)^2}(\bar{\theta}^t\wedge\theta\wedge\bar{\theta}^t\wedge\theta).\end{aligned}$$

When $q = 0$ these metrics are pseudo-hyperKähler away from the zero-section. □

Note that these quaternionic Kähler metrics are defined across the zero-section, so the original g_1 metrics can be extended at infinity by adjoining a copy of M, but they will still degenerate on the zero-section. We shall see in Section 3 that if the base manifold is $\mathbb{H}\mathrm{P}(n)$, then the total space of $H\setminus 0$ is $\mathbb{H}^{n+1^*}$ ($\mathbb{H}^{n+1}$ with the origin removed). The hyperKähler metric is just the usual flat metric, which may be completed by adjoining a point, and the quaternionic Kähler metrics are

induced by the inclusion $\mathbb{H}^{n+1^*} \subset \mathbb{H}\mathrm{P}(n+1)$. These quaternionic Kähler metrics are completed by adjoining a copy of $\mathbb{H}\mathrm{P}(n)$ at infinity to give $\mathbb{H}\mathrm{P}(n+1)$.

So far we have only produced metrics on the bundle H which, as remarked above, may only exist locally over M. Global statements may be obtained by considering the bundle

$$\mathcal{U}(M) = F \times_{Sp(n)\,Sp(1)} (\mathbb{H}^*/\mathbb{Z}_2).$$

Locally, this is just the quotient of H by $\mathbb{Z}_2 = \{\pm 1\}$. The 2-form v and the metrics g_1 and g_2 are invariant by this action and so descend to $\mathcal{U}(M)$. The above theorems thus hold on $\mathcal{U}(M)$. In particular, fixing attention on g_1 and taking $p, q \geqslant 0$, we have:

Theorem 2.1.7. *If M is a quaternionic Kähler manifold with non-zero scalar curvature, then the associated bundle $\mathcal{U}(M)$ is pseudo-hyperKähler. Also $\mathcal{U}(M)$ admits a family of pseudo-quaternionic Kähler metrics which have the pseudo-hyperKähler structure as a scalar curvature tends to zero limit. All these metrics share the same underlying quaternionic structure and they are positive definite if M has positive scalar curvature.* □

That hyperKähler metrics may be obtained as the scalar curvature tends to zero limit of families of quaternionic Kähler metrics is suggested by the result of Alekseevskiĭ (1968). He shows that the curvature tensor of a quaternionic Kähler manifold decomposes as $\lambda R_{\mathbb{H}\mathrm{P}(n)} + R_0$, where $R_{\mathbb{H}\mathrm{P}(n)}$ and R_0 have the symmetries of the curvature tensors of $\mathbb{H}\mathrm{P}(n)$ and a hyperKähler manifold respectively. Galicki (1986) showed, using results of Bagger & Witten (1983), that the Calabi metric on $T^*\mathbb{C}\mathrm{P}(n)$ could be obtained as a limit of the symmetric metric on the Wolf space $\mathrm{Gr}_2(\mathbb{C}^{n+2})$. The families of metrics we produced above have the additional property that the topology does not change in the limit.

Corollary 2.1.8. *If M is a self-dual, Einstein, spin 4-manifold with scalar curvature κ and $V_- \setminus 0$ is its negative spin bundle with the zero section removed. Then $V_- \setminus 0$ is hyperKähler if $\kappa > 0$ and $V_- \setminus 0$ is pseudo-hyperKähler if $\kappa < 0$. Also $V_- \setminus 0$ admits pseudo-quaternionic Kähler metrics in the same quaternionic structure which are positive-definite if $\kappa > 0$.*

□

The bundle V_- is naturally associated to the conformal structure on M^4. In fact we may write the fibre of $\mathcal{U}(M)$ as

$$\frac{CO^+(4)}{SU(2)_-} = \frac{Sp(1)\,GL(1,\mathbb{H})}{Sp(1)_-} = \mathbb{H}^*/\mathbb{Z}_2 = \mathbb{RP}(3) \times \mathbb{R}_{>0}.$$

The last two descriptions of the fibre of $\mathcal{U}(M)$ apply in all dimensions. From the definition, $\mathcal{U}(M)$ is a principal $\mathbb{H}^*/\mathbb{Z}_2$-bundle, with action induced from the action of $q \in \mathbb{H}^*$ on $\xi \in \mathbb{H}$ given by $\xi \cdot q = \bar{q}\xi$. The action permutes the complex structures $\mathcal{U}(M)$, thus showing that they are all equivalent. This is not true in general where the cohomology class of I, say, may be different from those for the other structures, as in case of the moduli space for solutions of the self-duality equations over a Riemann surface (Hitchin, 1987), or all three cohomology classes may differ, as for the asymptotically locally Euclidean (ALE) 4-manifolds (Kronheimer, 1989a). Note that $\mathbb{RP}(3) = (Sp(n)\,Sp(1))/\,Sp(n)$ parameterises those $Sp(n)$-structures which are compatible with a given quaternionic Kähler structure on $\mathbb{R}^{4n}$. This is exactly analogous to the way that $SO(2n)/\,U(n)$ parameterises all almost complex structures compatible with a given metric and orientation on $\mathbb{R}^{2n}$. Quaternionic Kähler manifolds have a twistor operator D defined to be the composition

$$H \xrightarrow{\nabla} H \otimes T^*M = H \otimes EH = ES^2H \oplus H \longrightarrow ES^2H,$$

where the last map is projection. From Salamon (1986) we see that a section s of $\mathcal{U}(M)$ is a hyperKähler submanifold only if it is a solution of the twistor equation $Ds = 0$.

Since the constructions of metrics in this section only use properties of the representations of the complexified group $Sp(n, \mathbb{C})\, Sp(1, \mathbb{C})$, they may be generalised to the case of a pseudo-quaternionic Kähler base manifold M. If the metric on M has signature $(4p, 4q)$ then the pseudo-hyperKähler metrics of the last theorem will have signature $(4p + 4, 4q)$ if the scalar curvature of M is positive or $(4p, 4q + 4)$ if it is negative.

2.2 Metrics Over Kähler-Einstein Contact Manifolds

The twistor space Z of a quaternionic Kähler manifold is the $\mathbb{C}\mathrm{P}(1)$-bundle consisting of almost complex structures compatible with the quaternionic Kähler structure on M. It may be obtained from $\mathcal{U}(M)$ by fixing one of the complex structures and then projectivising, so

$$Z = \mathbb{P}(\mathcal{U}(M)),$$

which is locally $\mathbb{P}(H)$. This induces a complex structure on Z and exhibits $\mathcal{U}(M)$ as the total space of a holomorphic line bundle on Z with the zero section removed. Salamon (1982) shows that Z is Kähler-Einstein manifold and that it has a complex contact structure.

Definition 2.2.1. If N is a complex $(2n + 1)$-manifold, then a *complex contact structure* on N is a line bundle $\mathcal{L}$ together with a holomorphic 1-form φ taking values in $\mathcal{L}$, such that $\varphi \wedge (d\varphi)^n$ is nowhere zero.

These structures on Z may be derived from the hyperKähler metric on $\mathcal{U}(M)$ as follows. Fix one of the complex structures I on $\mathcal{U}(M)$, then there is a subgroup $U(1)$

of $Sp(1)$ which acts preserving I and permuting J and K. The moment map associated to this $U(1)$-action is just the function r^2, and so we obtain the twistor space as the Kähler quotient of $\mathcal{U}(M)$. The norm of the vector field generated by the circle action is constant on the level sets of the moment map and this can be used to show that the twistor space is also Einstein (cf. Futaki, 1988). The complex contact structure is given by the following result.

Proposition 2.2.2. *Let N be a hyperKähler manifold, with complex structures I, J, K. Suppose $U(1)$ acts on N preserving I and permuting J and K, then the Kähler quotient of N by $U(1)$ is a Kähler contact manifold.*

PROOF. This proof is essentially due to LeBrun (1983). Let X be the vector field generated by the $U(1)$-action. Without loss of generality we may assume $L_X J = K$ and $L_X K = -J$. Let P be the symplectic quotient of N^{4n} with respect to the Kähler form ω_I associated to I. Hitchin et al. (1987) show that this quotient manifold is actually Kähler, so it only remains to show that it is a complex contact manifold.

If μ is the moment map associated to the Kähler quotient, then $S := \mu^{-1}(x)$ is a principal $U(1)$-bundle over P. Define a line bundle L over P by

$$L = S \times_{U(1)} \mathbb{C},$$

where $g \in U(1)$ acts by $g \cdot (s, z) = (g \cdot s, gz)$. Let ω_J, ω_K be the Kähler forms associated to J and K respectively and define $\omega = \omega_J + i\omega_K$. We define the contact form φ on P to be $\varphi = X \lrcorner \omega$. If $g \in U(1)$, then $g^*\varphi = g\varphi$, so φ takes values in L and if $\sigma\colon P \to S$ is a local section, then

$$\sigma^*(\varphi \wedge (d\varphi)^{n-1}) = \sigma^*((X \lrcorner \omega) \wedge (L_X \omega)^{n-1}) = i^{n-1}\sigma^*(X \lrcorner \omega^n),$$

which is does not vanish since X is vertical for the projection and $(\omega \wedge \bar{\omega})^n$ is a non-zero multiple of the volume form of N. □

Note that $\mathcal{U}(M)$ is dual to the contact bundle and this leads us to construct hyperKähler metrics over other Kähler-Einstein contact manifolds. Twistor spaces also have a real structure σ. This is an antiholomorphic involution and is naturally defined by the $Sp(1)$-action on $\mathcal{U}(M)$. Suppose the $U(1)$ preserving I is given by $e^{i\theta}$. Let j be an element of $Sp(1)$ such that on $\mathcal{U}(M)$, $je^{i\theta} = e^{-i\theta}j$. Then j descends to the real structure σ on Z. Theorems indicating when a general Kähler-Einstein contact manifold is a twistor space will be discussed in Chapter 4.

Let N be a complex $(2n+1)$-manifold with a complex contact structure $\{\mathcal{L}, \varphi\}$. We have an exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow T'N \xrightarrow{\varphi} \mathcal{L} \longrightarrow 0,$$

where $T'N$ is the holomorphic tangent bundle of N and $\mathcal{D}$ is the distribution $\ker \varphi$. Kobayashi (1959) shows that the structure group of N may be reduced to $U(1) \times (Sp(n) \otimes U(1))$, since $T'N \cong \mathcal{D} \oplus \mathcal{L}$ and $d\varphi$ behaves like a symplectic form on $\mathcal{D}$. The definition of the contact structure shows that $(\mathcal{L}^*)^{\otimes n+1}$ is isomorphic to the canonical bundle K of N. This implies that $\mathcal{L}^n = \Lambda^{2n}\mathcal{D}$ and we may restrict to frames (l, D): $\mathbb{C} \oplus \mathbb{H}^n \to T'N$ such that $l^n = \det D$. Thus the structure group actually reduces to $(1 \times Sp(n)) \otimes U(1)$.

Let F be the principal $(1 \times Sp(n)) \otimes U(1)$-bundle of frames $u : \mathbb{C} \oplus \mathbb{H}^n \to T_x N$ compatible with the complex contact structure of N. The tangent bundle TN is $(\mathbb{C} \oplus E) \otimes V$ as an $(1 \times Sp(n)) \otimes U(1)$-module, where E is the basic representation of $Sp(n)$, as before, and V is the $U(1)$-module $\mathbb{C}$ with $w \cdot v = \bar{w}v$, for $w \in U(1)$ and $v \in \mathbb{C}$. The contact line bundle is now just the bundle associated to V and the contact form is projection.

If N has a Kähler structure compatible with the complex contact structure, then the holonomy group of N is a subgroup of $U(2n+1)$. If $\theta \in \Omega^1(F, \mathbb{C} \oplus \mathbb{H}^n)$ is

the canonical 1-form, then the Levi-Civita connection descends to give a 1-form ω in $\Omega^1(F, \mathfrak{u}(2n+1))$ such that $d\theta = -\omega \wedge \theta$. Corresponding to the Lie algebra splitting $\mathfrak{u}(2n+1) = \mathfrak{u}(1) \oplus \mathfrak{su}(2n+1)$, we write $\omega = \omega_+ + \omega_-$. Locally, we can work on a line bundle $\mathcal{L}^{-1/2}$ such that $\mathcal{L}^{-1/2} \otimes \mathcal{L}^{-1/2} \cong \mathcal{L}^*$. This may be constructed by choosing a double cover S^1 of $U(1)$ and taking the bundle associated to the standard representation of S^1 on $W \cong \mathbb{C}$. Let z be the pull-back of the identity map on W and let $r^2 = z\bar{z}$. As before, define a 1-form α by

$$\alpha = dz - z\omega_+.$$

Lemma 2.2.3. *If N is also an Einstein manifold, then there is an almost Kähler structure on $\mathcal{L}^{-1/2} \setminus 0$ with metric*

$$g = \mathrm{Re}(\alpha \otimes \bar{\alpha} + \lambda r^2 \bar{\theta}^t \otimes \theta)$$

and 2-form

$$\nu = \alpha \wedge \bar{\alpha} + \lambda r^2 \bar{\theta}^t \wedge \theta,$$

where λ is some positive constant multiple of the scalar curvature of N.

PROOF. For $A \in Sp(n)$ and $\zeta \in S^1$, we have $R^*_{A,\zeta}\theta = \bar{A}^t\theta\zeta^2$ and $R^*_{A,\zeta}z = \zeta z$. Also, ω_+ is invariant under this group action. Exactly as for the bundle H, we see that ν and g are well-defined on $\mathcal{L}^{-1/2}$.

It is now sufficient to show that ν is closed, as it is clearly non-degenerate away from the zero section. From the definition of α,

$$d\alpha = -dz\,\omega_+ - z\,d\omega_+ = -\alpha\omega_+ - z\Omega_+,$$

where $\Omega_+ = d\omega_+ + \omega_+ \wedge \omega_+$. Since N is Kähler-Einstein, we have $\Omega_+ = \lambda\bar{\theta}^t \wedge \theta$ and that

$$d(\alpha \wedge \bar{\alpha}) = -\Omega_+ \wedge dr^2 = -\lambda\bar{\theta}^t \wedge \theta \wedge dr^2 = -\lambda d(r^2\bar{\theta}^t \wedge \theta).$$

Thus ν is closed. □

The almost complex structure so defined on $\mathcal{L}^{-1/2} \setminus 0$ is integrable, as will be seen from Hitchin's lemma, but a direct proof will also be presented when we discuss the relation with the Calabi metric later in this chapter.

The canonical 1-form θ on F splits as $\theta = \theta_+ + \theta_-$ corresponding to the splitting of $\Omega^1(F, \mathbb{C} \oplus \mathbb{H}^n)$. With respect to the complex structure defined by g and ν, we have that $z^2\bar{\theta}_+$ is a well-defined holomorphic 1-form on $\mathcal{L}^{-1/2}$. This is essentially the contact form on N and $d(z^2\bar{\theta}_+)$ provides the rest of the hyperKähler structure on $\mathcal{L}^{-1/2} \setminus 0$ and also its $\mathbb{Z}/2$-quotient $\mathcal{L}^* \setminus 0$.

Proposition 2.2.4. *Let N be a $(2n+1)$-dimensional Kähler-Einstein manifold with a compatible complex contact structure and positive scalar curvature. Then*

$$g = \mathrm{Re}(\alpha \otimes \bar{\alpha} + \lambda r^2 \bar{\theta}^t \otimes \theta)$$

is a hyperKähler metric on $\mathcal{L}^* \setminus 0$, where $\mathcal{L}$ is the contact line bundle over N.

PROOF. It remains to show that $d(z^2\bar{\theta}_+)$ and ν are compatible. We will assume that the scalar curvature of N has been normalised so that $\lambda = 1$. The compatibility requirement is that there is a $\mathbb{H}^{n+1}$-valued 1-form $\mu + \mu' j$ such that $(\mu + \mu' j) \wedge \overline{(\mu + \mu' j)} = \mu \wedge \bar{\mu} + \mu' \wedge \bar{\mu'} + 2\mu' \wedge \mu j$ is the 2-form $\nu + d(z^2\bar{\theta}_+)j$.

Now

$$\nu = \alpha \wedge \bar{\alpha} + r^2\bar{\theta}_+ \wedge \theta_+ + r^2\bar{\theta}^t_- \wedge \theta_-$$

and

$$d(z^2\bar{\theta}_+) = 2z\, dz \wedge \bar{\theta}_+ + z^2 d\bar{\theta}_+.$$

Since, θ_+ is the contact form, our assumption of compatibility of the complex contact and Kähler structures shows that $d\theta_+$ and $-\bar{\theta}^t_- \wedge \theta_-$ are compatible on the distribution $\mathcal{D}$. Thus, $z^2 d\bar{\theta}_+$ and $r^2\bar{\theta}^t_- \wedge \theta_-$ are compatible. Also, taking $\mu + \mu' j = z\bar{\theta}_+ + \alpha j$ we see that $2z\alpha \wedge \bar{\theta}_+$ and $\alpha \wedge \bar{\alpha} + r^2\bar{\theta}_+ \wedge \theta_+$ are compatible. □

If N *is* the twistor space of a quaternionic Kähler manifold M then this metric agrees with one previously constructed on $\mathcal{U}(M) \cong \mathcal{L}^* \setminus 0$. This can be seen as follows. If we apply the proposition to $\mathbb{C}\mathrm{P}(1)$ with its standard structure, then a computation analogous to the one below for $\mathbb{H}\mathrm{P}(n)$ shows that we obtain the flat metric on $\mathbb{C}^{2^*}$. The metric constructed over N now consists of this flat metric in the (θ_+, α)-direction and a multiple of the metric on M in the remaining directions, which is the way that the metric of $\mathcal{U}(M)$ decomposes with respect to the fibre and horizontal directions.

If we just regard $\mathcal{L}^* \setminus 0$ as a complex symplectic manifold then it is the symplectification of the complex contact manifold N in the sense of Arnol'd (1978). The construction of hyperKähler metrics in the first section may then be regarded as a quaternionic enhancement.

2.3 Examples

The model examples of quaternionic Kähler manifolds are $\mathbb{H}\mathrm{P}(n)$. For these spaces it is possible to compute the metrics on $\mathcal{U}(\mathbb{H}\mathrm{P}(n))$ explicitly. This is largely because the frame bundle F may be readily computed.

Lemma 2.3.1. *The double cover of the reduced frame bundle of* $\mathbb{H}\mathrm{P}(n)$ *may be identified with* $Sp(n+1)$.

PROOF. Recall that

$$\mathbb{H}\mathrm{P}(n) \cong \frac{Sp(n+1)}{Sp(n) \times Sp(1)},$$

so $Sp(n+1)$ is naturally a principal $Sp(n) \times Sp(1)$-bundle over $\mathbb{H}\mathrm{P}(n)$; we denote the projection by π. Explicitly, let e be the vector $(0, \ldots, 0, 1)^t \in \mathbb{H}^{n+1}$ and for $A \in Sp(n+1)$ put

$$\pi(A) = [Ae] = [a_{1n}, \ldots, a_{nn}],$$

where $A = (a_{ij})$. Let $(B, q) \in Sp(n) \times Sp(1)$ act on A via right multiplication by $\begin{pmatrix} B & 0 \\ 0 & q \end{pmatrix}$. Then the tangent space of $\mathbb{H}\mathrm{P}(n)$ at $\pi(A)$ is

$$T_{\pi(a)}\,\mathbb{H}\mathrm{P}(n) = AT_e\bar{A}^t,$$

where

$$T_e = T_{[e]}\,\mathbb{H}\mathrm{P}(n) = \left\{ \begin{pmatrix} 0 & \xi \\ -\bar{\xi}^t & 0 \end{pmatrix} : \xi \in \mathbb{H}^n \right\}.$$

Now A defines an orientation preserving linear isometry $\mathbb{H}^n \to T_{\pi(a)}\,\mathbb{H}\mathrm{P}(n)$ by

$$\xi \mapsto A \cdot \xi = A \begin{pmatrix} 0 & \xi \\ -\bar{\xi}^t & 0 \end{pmatrix} \bar{A}^t.$$

This map from $Sp(n+1)$ to the frame bundle of $\mathbb{H}\mathrm{P}(n)$ has kernel $\{\pm 1\}$ and so the lemma is proved. □

For $\mathbb{H}\mathrm{P}(n)$, we have that the bundle $H \setminus 0$ equals

$$\tilde{F} \times_{Sp(n)\times Sp(1)} \mathbb{H}^* = \frac{Sp(n+1)}{Sp(n)} \times_{Sp(1)} \mathbb{H}^* = \frac{Sp(n+1)}{Sp(n)} \times \mathbb{R}_{>0}.$$

But $Sp(n+1)/\,Sp(n)$ is just the $(4n+3)$-sphere S^{4n+3}. So, topologically, $H \setminus 0$ is $\mathbb{H}^{n+1^*}$.

Proposition 2.3.2. *The hyperKähler metric on $H \setminus 0$ over $\mathbb{H}\mathrm{P}(n)$ is the flat metric on $\mathbb{H}^{n+1^*}$.*

PROOF. If $A \in Sp(n+1)$ then

$$T_A\,Sp(n+1) \cong \left\{ v = A \begin{pmatrix} M & \xi \\ -\bar{\xi}^t & \nu \end{pmatrix} \bar{A}^t : \xi \in \mathbb{H}^n,\ M \in \mathfrak{sp}(n),\ \nu \in \mathrm{Im}\,\mathbb{H} \right\}.$$

Thus, if $v \in T_A\, Sp(n+1)$ then $\theta_A(v) = A^{-1} \cdot (\pi_* v) = \xi$. For any $v, v' \in T_A\, Sp(n+1)$ we have $d\theta_A(v,v') = v\theta_A(v') - v'\theta_A(v) - \theta_A[v,v']$. We may extend v, v' to vector fields in such a way as to make $v\theta_A(v') - v'\theta_A(v)$ vanish. Then, in components,

$$d\theta_A(v,v') = d\theta_A\left(A \cdot \begin{pmatrix} M & \xi \\ -\bar{\xi}^t & \nu \end{pmatrix}, A \cdot \begin{pmatrix} M' & \xi' \\ -\bar{\xi'}^t & \nu' \end{pmatrix}\right) = -(M\xi' + \xi\nu' - M'\xi - \xi'\nu).$$

From (2.1.1), we see that

$$\omega_{+A}(v) = M, \qquad \omega_{-A}(v) = \nu.$$

Let $s\colon H \setminus 0 \to Sp(n+1) \times \mathbb{H}^*$ be a local section such that the component of s in $\mathbb{H}^*$ is real. If (v,z) is tangent to s at (A,y), then

$$\alpha(v,z) = z - y\nu \qquad \text{and} \qquad (\theta\bar{x})(v,z) = \xi y.$$

So the hyperKähler metric g on $H \setminus 0$ is

$$g_{\varpi(A,y)}(\varpi_*(v,z), \varpi_*(v,z)) = z^2 + y^2(|\nu|^2 + c|\xi|^2),$$

where c is the constant such that $\Omega_- = d\omega_- + \omega_- \wedge \omega_- = c\bar{\theta}^t \wedge \theta$. But $\Omega_-(v,v') = \bar{\xi}^t\xi' - \bar{\xi'}^t\xi = (\bar{\theta}^t \wedge \theta)(v,v')$, so $c = 1$. The final bracket in the expression for g is just the standard metric on S^{4n+3}, since the inclusion $\mathfrak{sp}(n+1) \subset \mathfrak{so}(4n+4)$ shows that it agrees with the expression for the $SO(4n+4)$-invariant metric. □

Note that it was not sufficient in the above proof to check that we had an $Sp(n+1)$-invariant metric, since S^{4n+3} admits several families of such metrics some of which are even Einstein (see Besse, 1987).

We may also consider other homogeneous base spaces. Wolf (1965) and Alekseevskiĭ (1968) classified the compact homogeneous quaternionic Kähler manifolds, obtaining precisely one for each compact simple Lie group:

$$\mathbb{HP}(n) = \frac{Sp(n+1)}{Sp(n)\times Sp(1)}, \qquad \mathrm{Gr}_2(\mathbb{C}^n) = \frac{SU(n)}{S(U(n-2)\times U(2))},$$
$$\widetilde{\mathrm{Gr}}_4(\mathbb{R}^n) = \frac{SO(n)}{SO(n-4)\times SO(4)}, \qquad \frac{G_2}{SO(4)},$$
$$\frac{F_4}{Sp(3)\,Sp(1)}, \qquad \frac{E_6}{SU(6)\,Sp(1)},$$
$$\frac{E_7}{Spin(12)\,Sp(1)} \quad \text{and} \quad \frac{E_8}{E_7\,Sp(1)}.$$

These are constructed as follows. Let G be a compact, centreless, simple Lie group with Lie algebra $\mathfrak{g}$. For each root α define

$$\mathfrak{g}_\alpha = \{\, X \in \mathfrak{g}^{\mathbb{C}} : [H,X] = \alpha(H)X \text{ for all } H \in \mathfrak{h} \,\},$$

where $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Choose a highest root ρ. Then the subalgebra generated by $\mathfrak{g}_\rho$ and $\mathfrak{g}_{-\rho}$ is isomorphic to $\mathfrak{sp}(1,\mathbb{C})$. This has normaliser $\mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{sp}(1,\mathbb{C})$, where, if $\langle\cdot,\cdot\rangle$ denotes he Killing form on $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{l}$ is defined by

$$\mathfrak{l} = \mathfrak{g} \cap \Big(\{\, H \in \mathfrak{h} : \rho(H) = 0 \,\} \oplus \sum_{\substack{\alpha>0 \\ \langle\alpha,\rho\rangle=0}} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \Big).$$

The Wolf space associated to G is then $M = G/(L\cdot Sp(1))$, where L and $Sp(1)$ have Lie algebras $\mathfrak{l}$ and $\mathfrak{sp}(1)$, respectively. Wolf also shows that the twistor space of M is $Z = G/(L\cdot S^1) \cong G^{\mathbb{C}}/U$, where the Lie algebra of U is

$$\mathfrak{u} = \mathfrak{h} \oplus \sum_{\langle\alpha,\rho\rangle\geqslant 0} \mathfrak{g}_\alpha.$$

The twistor space may also be described as follows. Let E_ρ be a non-zero element of $\mathfrak{g}_\rho$ and consider the adjoint action of $G^{\mathbb{C}}$ on the linear span $[E_\rho]$ of E_ρ. If $H \in \mathfrak{h}$ then $[H, E_\rho] = \rho(H)E_\rho$ and for $X \in \mathfrak{g}_\alpha$ we have

$$[X, E_\rho]\begin{cases} \in \mathfrak{g}_{\alpha+\rho} & \text{if } \alpha+\rho \text{ is a root,} \\ = 0 & \text{otherwise.} \end{cases}$$

If $\langle \alpha, \rho \rangle \geqslant 0$, then $\alpha+\rho$ is not a root, since ρ is the highest root. Whereas, if $\langle \alpha, \rho \rangle < 0$, then $\alpha+\rho$ is a root. So the Lie algebra of the stabiliser of $[E_\rho]$ is $\mathfrak{h} \oplus \sum_{\langle \alpha, \rho \rangle \geqslant 0} \mathfrak{g}_\alpha = \mathfrak{u}$ and the twistor space of M may be identified with the projectivised orbit of E_ρ. The orbit $\mathcal{O}$ of E_ρ itself is $B = G^{\mathbb{C}}/U_1$ where

$$\mathfrak{u}_1 = \{ X \in \mathfrak{u} : \langle X, H_\rho \rangle = 0 \},$$

(Here H_ρ is the element of $\mathfrak{h}$ defined by $\langle H_\rho, H \rangle = \rho(H)$ for all $H \in \mathfrak{h}$.) The contact line bundle on Z is $\mathcal{L} = B^*$ and the contact form φ is given by $(\varphi(\pi_* X))(b) = \langle E_\rho, X \rangle$, where $\pi \colon B \to Z$ is the projection, $b \in \pi^{-1}(1)$ and $X \in \langle H_\rho \rangle \oplus \sum_{\langle \alpha, \rho \rangle < 0} \mathfrak{g}_\alpha$ which is isomorphic to the tangent space of B. Thus the nilpotent orbit $\mathcal{O}$ is the dual of the contact line bundle and hence is the associated bundle $\mathcal{U}(M)$.

Recall that the Wolf space M is the quotient of the real group G by the normaliser $N(Sp(1))$ of $Sp(1)$. Thus M is the G-orbit of real subalgebras $\mathfrak{sp}(1)$ of $\mathfrak{g}$ such that the nilpotent elements of $\mathfrak{sp}(1) \otimes \mathbb{C}$ lie in the highest root orbit $\mathcal{O}$. The fibre of $\mathcal{U}$ over a point of M consists of the nilpotent elements in $\mathfrak{sp}(1) \otimes \mathbb{C}$. Let σ be the real structure on $\mathfrak{g}^{\mathbb{C}}$ corresponding to $\mathfrak{g}$. This is a conjugate-linear Lie endomorphism of $\mathfrak{g}^{\mathbb{C}}$ with $\sigma^2 = 1$ and whose $+1$-eigenspace is $\mathfrak{g}$. If $X \in \mathcal{O}$ then the real $\mathfrak{sp}(1)$ corresponding to X has complexification spanned by X, σX and $[X, \sigma X]$. The $\mathbb{H}^*$-action on $\mathcal{U}(M)$ is essentially given by the following lemma.

Lemma 2.3.3. *If Y, H are such that $\langle Y, -\sigma Y, H\rangle$ is an $\mathfrak{sl}(2,\mathbb{C})$-triplet, i.e.*

$$[Y,-\sigma Y] = H, \qquad [H,Y] = 2Y \quad \text{and} \quad [H,-\sigma Y] = -2(-\sigma Y),$$

then there is an $\mathbb{H}^*$-action on the nilpotent elements of $\langle Y, \sigma Y, H\rangle$ given by

$$(a+bj)\cdot Y = a^2Y + b^2\sigma Y + ab[Y,\sigma Y], \qquad \text{for } a+bj \in Sp(1),$$
$$\lambda \cdot Y = \lambda^2 Y, \qquad \text{for } \lambda \in \mathbb{R}.$$

PROOF. We need to verify that this is an action. It sufficient to check we have an $Sp(1)$-action. Let $a+bj$ and $c+dj$ be elements of $Sp(1)$.

$$\begin{aligned}(c+dj)\cdot((a+bj)\cdot Y) &= (c^2a^2 + d^2\bar{b}^2)Y + (c^2b^2 + d^2\bar{a}^2)\sigma Y \\ &\quad + (c^2ab - d^2\bar{a}\bar{b})[Y,\sigma Y] + cd(|a|^4 - |b|^4)[Y,\sigma Y] \\ &\quad - cda\bar{b}(|a|^2 + |b|^2)[[Y,\sigma Y],Y] \\ &\quad - cd\bar{a}b(|a|^2 + |b|^2)[[Y,\sigma Y],\sigma Y] \\ &= (ca - d\bar{b})^2Y + (cb + d\bar{a})^2\sigma Y + (ca - d\bar{b})(cb + d\bar{a})[Y,\sigma Y] \\ &= ((c+dj)(a+bj))\cdot Y\end{aligned}$$

□

To get an $\mathbb{H}^*$-action on the orbit we need to show that there is a positive real number λ such that $Y = \lambda X$ satisfies the hypotheses of the lemma. It is sufficient to show that $[X,[X,\sigma X]]$ is a positive multiple of X. Now $\langle X, \sigma X, [X,\sigma X]\rangle$ is isomorphic to $\mathfrak{sp}(1,\mathbb{C})$ with its standard real structure, so we only need to work

in $\mathfrak{sp}(1,\mathbb{C}) \cong \mathfrak{su}(2,\mathbb{C})$. Write X as a 2×2 complex matrix, then $\sigma X = -\bar{X}^t$. Let $\{e_1, e_2\}$ be an orthonormal basis for $\mathbb{C}^2$. Since X is a highest root vector, $X = a \otimes \bar{b} = a_i \bar{b}_j e_i e_j$, for some $a, b \in \mathbb{C}^2$ such that $\sum a_i \bar{b}_j = 0$. Now $[X, \sigma X] = (\|a\|^2 b_i \bar{b}_j - \|b\|^2 a_i \bar{a}_j) e_i e_j$. So $[X, [X, \sigma X]]$ is $2\|a\|^2\|b\|^2 X$, as required.

From the last description of M, we can regard $\mathcal{O}$ as the set of Lie algebra homomorphisms from $\mathfrak{sp}(1)$ to $\mathfrak{g}$ such that the nilpotent elements in $\mathfrak{sp}(1) \otimes \mathbb{C}$ map into $\mathcal{O}$ when the homomorphism is extended complex linearly. The $\mathbb{H}^*$-action above can then be viewed as the action of $\mathrm{Aut}(\mathfrak{sp}(1)) \cong \mathbb{H}^*/\mathbb{Z}_2$ via composition. Thus the hyperKähler structures constructed by Kronheimer (1988) give the same metric on the highest root orbit. His metrics on the rest of the nilpotent variety will be studied in a later chapter.

Wolf also shows that any compact, homogeneous, complex contact manifold arises as the twistor space of one of the quaternionic Kähler manifolds above. So there are no compact homogeneous examples of our second construction which are not already covered by the quaternionic Kähler case. An alternative description of these twistor spaces in terms of nilpotent Lie algebras has been given by Burstall (1989).

2.4 Integrability

Recall that for an oriented, Riemannian four-manifold M, the bundle H is just the negative spin bundle V_-. Much of the discussion in the first section of this chapter still applies; in particular, $V_- \setminus 0$ carries a family of almost complex structures. When the base manifold is Einstein and self-dual we automatically get the integrability of these structures from the fact that all three symplectic forms are closed (Hitchin, 1987). However, if we are only concerned with when the structures are integrable these conditions are stronger than necessary.

Theorem 2.4.1. *If M is an oriented, 4-dimensional, Riemannian, spin manifold, then the almost complex structures I, J and K on $V_- \setminus 0$ are integrable if and only if M is self-dual.*

PROOF. Write V for $V_- \setminus 0$. We first fix attention on one almost complex structure I. Let $T^{1,0}$, $T^{0,1}$ denote the $+i$, $-i$ eigenspaces of I on $TV \otimes \mathbb{C}$, respectively, and write $\Lambda^{p,q}$ for $\Lambda^p T^{1,0\,*} \otimes \Lambda^q T^{0,1\,*}$. The Newlander-Nirenberg Theorem says that I is integrable if and only if its torsion tensor N vanishes identically. It is well known (see Kobayashi & Nomizu, 1963) that N is zero if and only if for each $\beta \in \Lambda^{1,0}$, $(d\beta)^{0,2} = 0$.

Fix $a \in V$ and choose a section $s\colon V \to \tilde{F} \times \mathbb{H}$ in a neighbourhood of a. Write

$$s^*\alpha = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k,$$

$$s^*(\theta\bar{x}) = \theta_0 + \theta_1 i + \theta_2 j + \theta_3 k,$$

where $\alpha_0, \ldots, \alpha_3, \theta_0, \ldots, \theta_3 \in \Omega^1(V, \mathbb{R})$. This gives an orthogonal basis $\{\alpha_0, \ldots, \alpha_3, \theta_0, \ldots, \theta_3\}$ for T_a^*V with respect to which

$$\eta_I = -2(\alpha_0 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3 - c(\theta_0 \wedge \theta_1 - \theta_2 \wedge \theta_3))$$

and

$$g = \sum_{i=0}^{3} (\alpha_i \otimes \alpha_i + c\theta_i \otimes \theta_i).$$

A typical element of $\Lambda^{1,0}$ is $\beta - iI\beta$, so $\alpha_0 + i\alpha_1$, $\alpha_2 + i\alpha_3$, $\theta_0 - i\theta_1$, $\theta_2 + i\theta_3$ is a basis for $\Lambda_a^{1,0}V$. Recall that

$$d\alpha = -\alpha \wedge \omega_- - x\Omega_-$$

and

$$d(\theta\bar{x}) = -\omega_+ \wedge \theta\bar{x} - \theta \wedge \omega_-\bar{x} + \theta \wedge d\bar{x}.$$

We may choose s such that $s_*(T_aV) = \mathcal{D}_{s(a)} \oplus \mathbb{H}$, where $\mathcal{D}$ is the distribution of horizontals given in (2.1.5). For such a choice $(s^*d\alpha)_a = (-s^*(x\Omega_-))_a$ and $(s^*d(\theta\bar{x}))_a = 0$, since $\omega_\pm$ and dx vanish on $\mathcal{D}_{s(a)}$ and ω_- and θ vanish on $\mathbb{H}$. Also, we still have enough freedom left in the choice of s to arrange that $x(s(a))$ is real. Thus, if we write $s^*(\Omega_-) = \Omega_1 i + \Omega_2 j + \Omega_3 k$, then the remark at the start of this proof gives that I is integrable if and only if

$$\begin{aligned} \Omega_1^{0,2} &= 0, \\ (\Omega_2 + i\Omega_3)^{0,2} &= 0. \end{aligned} \tag{2.4.1}$$

Since I preserves the horizontal subspace $\mathcal{H}_a$ of T_aV, a choice of a section $\sigma : M \to V$ in neighbourhood of $b = \pi^H(a) \in M$ such that $\sigma_* T_bM = \mathcal{H}_a$, gives a decomposition under I of

$$(\Lambda^2 T_b^* M)_{\mathbb{C}} = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}.$$

We have $\sigma^*\alpha_i = 0$ and that $\sigma^*\theta_0, \ldots, \sigma^*\theta_3$ is a basis for T_b^*M. In $\Lambda^{1,1}$ we obtain the distinguished linear subspace $\langle \sigma^*\eta_I \rangle$ spanned by $\sigma^*\eta_I$. Write $\Lambda_0^{1,1}$ for the orthogonal complement of this subspace in $\Lambda^{1,1}$. Now $\sigma^*(\theta_0 \wedge \theta_1 \wedge \theta_2 \wedge \theta_3)$ is a positive multiple of the volume form on M at b, so a direct computation is all that is required to verify the well-known correspondence

$$\begin{aligned} \Lambda_0^{1,1} &= \Lambda^2_{+\mathbb{C}}, \\ \Lambda^{0,2} \oplus \Lambda^{2,0} \oplus \langle \sigma^*\eta_I \rangle &= \Lambda^2_{-\mathbb{C}}, \end{aligned}$$

at b. Thus, as equations (2.4.1) only involve the (0,2)-components of Ω_-, they only place restrictions on Ω_-^-, the component of Ω_- lying in Λ^2_-. (Here we are regarding

Ω_- as a 2-form on M, which is possible since $\Omega_-(X,Y) = 0$ unless X and Y are both horizontal.)

If we write θ_i for $\sigma^*\theta_i$, then

$$\Omega_-^- = a(\theta_0 \wedge \theta_1 - \theta_2 \wedge \theta_3) + b(\theta_0 \wedge \theta_2 - \theta_3 \wedge \theta_1) + c(\theta_0 \wedge \theta_3 - \theta_1 \wedge \theta_2) = (a,b,c),$$

for some $a, b, c \in \mathrm{Im}\,\mathbb{H}$. Write $a = a_1 i + a_2 j + a_3 k$, etc. Now, with respect to the decomposition given by I, $\Lambda^{0,2}$ is the span of $(\theta_0 \wedge \theta_2 - \theta_3 \wedge \theta_1) - i(\theta_0 \wedge \theta_3 - \theta_1 \wedge \theta_2)$. So (2.4.1) is equivalent to

$$b = -ic.$$

Similarly, integrability of J and K are equivalent to $c = -ja$ and $a = -kb$, respectively. So I, J, K are integrable if and only if

$$\Omega_-^- = -\{(\theta_0 \wedge \theta_1 - \theta_2 \wedge \theta_3)i + (\theta_0 \wedge \theta_2 - \theta_3 \wedge \theta_1)j + (\theta_0 \wedge \theta_3 - \theta_1 \wedge \theta_2)k\}a,$$

for some real number a, which is precisely the condition that M be self-dual. □

Fixing a complex structure on $V_- \setminus 0$ we can form the twistor space and obtain the theorem of Atiyah et al. (1978a).

Theorem 2.4.2. (Atiyah, Hitchin & Singer) *If M is an oriented, Riemannian, 4-manifold then the natural almost complex structure on the twistor space is integrable if and only if M is self-dual.* □

When M is self-dual and spin, the complex structures I, J and K define a hypercomplex structure on $V_- \setminus 0$. Salamon (1986) shows that this is also true in higher dimensions if we replace the self-duality condition by a requirement that M be quaternionic and if we replace $V_- \setminus 0$ by the appropriate rank 4 bundle. Even though we have not defined a compatible metric, Obata's theorem gives us a natural connection and therefore geodesics.

We now briefly return to the metrics constructed over Kähler-Einstein contact manifolds earlier in this chapter. If we drop the assumption of possessing a complex contact structure and allow N to be any Kähler-Einstein $2n$-manifold we may perform the construction of Lemma 2.2.3 on the canonical bundle K of N and obtain an almost Kähler metric. This metric is actually Kähler-Einstein, has zero scalar curvature and is a degenerate case of a family of metrics described by Calabi (1979).

Proposition 2.4.3. *Away from the zero-section, the canonical bundle of a Kähler-Einstein manifold with positive scalar curvature admits a Kähler-Einstein metric which is Ricci flat.*

PROOF. Put V to be the $U(n)$-representation $\Lambda^n\mathbb{C}^n$, where $\mathbb{C}^n$ is the standard $U(n)$-module. Then for $A \in U(n)$, $R_A^*\theta = \bar{A}^t\theta$ and $R_A^*dz = (\det A)^{-1}dz$. As in Lemma 2.2.3, we have

$$g = \mathrm{Re}(\alpha \otimes \bar{\alpha} + \lambda r^2 \bar{\theta}^t \otimes \theta)$$

and a closed, non-degenerate 2-form

$$\nu = \alpha \wedge \bar{\alpha} + \lambda r^2 \bar{\theta}^t \wedge \theta,$$

where λ is some positive constant multiple of the scalar curvature of N. We need to check that the almost complex structure is integrable and that the Ricci tensor is zero.

The integrability computation is exactly as above in the case of H, and shows that the almost complex structure is integrable if and only if $\Omega_+^{0,2}$ vanishes. However, N is Einstein and $\Omega_+ = \lambda\bar{\theta}^t \wedge \theta$ which is of type $(1,1)$.

The Ricci tensor ρ of K is given by $\rho = -id'd'' \log\det g_{\alpha\bar{\beta}}$, where $g_{\alpha\bar{\beta}} = g(v_\alpha, v_{\bar{\beta}})$ for some local holomorphic basis $\{v_\alpha\}$. But choosing a local section s, we have $s^*\alpha = \alpha_0 + i\alpha_1$ and $s^*(\theta\bar{z}) = \theta_0 + i\theta_1$, giving a local basis $\alpha_0 + i\alpha_1, \theta_0 - i\theta_1$ of holomorphic 1-forms for which $\log\det g_{\alpha\bar{\beta}}$ is constant. □

Chapter 3

QUOTIENT CONSTRUCTIONS

Suppose M is a symplectic manifold of dimension $2n$ and a compact Lie group G of rank k acts smoothly, isometrically and freely on M. Then M/G is a manifold, but it has dimension $2n-k$ and so, in general, it cannot be symplectic. If the G-action preserves the symplectic structure then Marsden & Weinstein (1974) showed that under certain mild restrictions on M or G, there is a moment map $\mu\colon M \to \mathfrak{g}^*$ such that $\mu^{-1}(0)/G$ is a symplectic manifold of dimension $2n-2k$. This construction has been generalised to hyperKähler manifolds by Hitchin et al. (1987) and to quaternionic Kähler manifolds by Galicki & Lawson (1988). In this chapter we discuss these constructions, give some examples and show that these reductions are compatible with the hyperKähler metric construction of the previous chapter. In particular, if M_G is the quaternionic Kähler quotient of a quaternionic Kähler manifold M by G, Z_G is its twistor space and $\rightsquigarrow$ denotes an appropriate quotient by G, we have the following diagram.

$$\begin{array}{ccc} \mathcal{U}(M) & \rightsquigarrow & \mathcal{U}(M_G) \\ \downarrow & & \downarrow \\ Z & \rightsquigarrow & Z_G \\ \downarrow & & \downarrow \\ M & \rightsquigarrow & M_G \end{array}$$

In Chapter 2 we constructed various hyperKähler metrics from different types of base space. In this chapter we show how the structure of the quaternionic Kähler

base spaces may be recovered from the hyperKähler structures. The construction is a moment map construction based on the observation that the total spaces admit a special type of $Sp(1)$-action. Finally we return to the $Sp(1)$-actions and show that these hyperKähler metrics are determined by special Kähler potentials.

3.1 HyperKähler Quotients

Let M be a $4n$-dimensional hyperKähler manifold, with Riemannian metric g, complex structures I, J and K, and symplectic form $\eta = \eta_I i + \eta_J j + \eta_K k$. If X is a vector field on M, let L_X denote the Lie derivative with respect to X. We say that X is *Killing* if $L_X g = 0$, *triholomorphic* if $L_X I = L_X J = L_X K = 0$, or *Hamiltonian* if $L_X \eta = 0$. Note that any two of these conditions imply the third one.

Now suppose that G is a compact Lie group acting freely and smoothly on M preserving g and η. For any vector field X generated by this action we have $L_X g = 0 = L_X \eta$. Now η is closed, so

$$0 = L_X \eta = X \lrcorner d\eta + d(X \lrcorner \eta) = d(X \lrcorner \eta).$$

If the first cohomology group $H^1(M, \mathbb{R})$ vanishes then there is a function $\mu_X \in \Omega^0(M, \operatorname{Im}\mathbb{H})$ such that

$$X \lrcorner \eta = d\mu_X.$$

These functions can be combined into a single map $\mu \colon M \to \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H}$ defined by

$$\langle \mu(m), X \rangle = \mu_X(m), \tag{3.1.1}$$

for $m \in M$. Note that μ_X is only defined upto an additive constant, but if G is compact, then (Guillemin & Sternberg, 1984) there is no obstruction to choosing

these constants so that μ is equivariant with respect to the actions of G on M and $\mathfrak{g}^*$. In this case, μ (or μ_X) is called a *moment map* for the action of G on M.

Given a moment map μ, suppose $a \in \mathfrak{g}^* \otimes \mathrm{Im}\,\mathbb{H}$ is invariant under the (coadjoint) action of G. Then, if we set $M_a = \mu^{-1}(a)$, M_a is a G-invariant submanifold of M of dimension $4n - 3k$, where k is the rank of G. Now G acts freely and isometrically on M_a, so $\hat{M}_a = M_a/G$ is a Riemannian manifold of dimension $4n - 4k$. Since G also preserves η, it can be shown that η and the almost complex structures on M define a hyperKähler structure on $\hat{M}_a$. Details of these assertions may be found in Hitchin, Karlhede, Lindström & Roček (1987).

This construction may be generalised to pseudo-hyperKähler manifolds. We require that if we consider the distribution $\mathcal{A}$ of vectors tangent to the action of G on M, then the restriction of the pseudo-metric g to $\mathcal{A}$ is non-degenerate. This ensures that g gives a G-invariant distribution of horizontals in TM_a for $M_a \to M_a/G$ (see Lang, 1984). With this additional hypothesis, the proofs cited above show that M_a/G is pseudo-hyperKähler. Hitchin (1988) has also studied the case where we have no metric, just a hypersymplectic structure, and obtains a further generalisation of the quotient construction.

The first example of a hyperKähler quotient is that of $U(1)$ acting on flat space $\mathbb{H}^n$ via

$$e^{i\theta} \cdot (a + jb) = e^{i\theta} a + j e^{-i\theta} b.$$

The value of the moment map for this action at $q \in \mathbb{H}^n$ is $\bar{q}^t i q$ and we obtain a hyperKähler quotient for each $a \in \mathrm{Im}\,\mathbb{H}$. If a is non-zero, there is an $\alpha \in \mathbb{H}$ such that $\bar{\alpha} i \alpha = a$ and the $U(1)$-equivariant map $q \mapsto q\alpha^{-1}$ takes the level set associated to a to that associated to i. Thus the hyperKähler quotient associated to a is homothetic

that associated to i. Writing $q = a + jb$, the moment map equation $\bar{q}^t i q = i$ becomes

$$|a|^2 - |b|^2 = 1,$$

$$b^t a = 0.$$

The hyperKähler quotient is topologically $T^*\mathbb{C}\mathrm{P}(n-1)$, since a defines a point of $\mathbb{C}\mathrm{P}(n-1)$ and $b \otimes \bar{a}$ defines an element of $T^*_{[a]}\,\mathbb{C}\mathrm{P}(n-1) \cong (\mathrm{Hom}(\langle a\rangle, \langle a\rangle^{\perp}))^*$. Hitchin (1986) shows that the metric obtained on $T^*\mathbb{C}\mathrm{P}(n-1)$ coincides with the one constructed by Calabi (1979, 1980).

The 0-level set of the moment map is

$$|a|^2 = |b|^2 \qquad \text{and} \qquad b^t a = 0.$$

The map $(a, b) \mapsto a \otimes \bar{b}$ is well-defined on the quotient space. The image in $M_n(\mathbb{C}) \cong \mathfrak{u}(n, \mathbb{C})$ is the highest root orbit, together with 0, and the action of $Sp(1)$ on $\mathbb{H}^n$ on the right descends to the action described in Chapter 2. So the hyperKähler quotient is the associated bundle $\mathcal{U}(\mathrm{Gr}_2(\mathbb{C}^n))$ with its hyperKähler metric and we may view this metric as a singular limit of the Calabi metric on $T^*\mathbb{C}\mathrm{P}(n-1)$. Note that Galicki (1986) obtains this Calabi metric as a limit of the quaternionic Kähler metrics on $\mathrm{Gr}_2(\mathbb{C}^{n+1})$.

The twistor space of $\mathrm{Gr}_2(\mathbb{C}^n)$ is the flag manifold

$$F_{1,2,n} = \frac{U(n)}{U(n-2) \times U(1) \times U(1)}.$$

This clearly fibres over $\mathbb{C}\mathrm{P}(n-1)$, and, away from the zero sections, $T^*\mathbb{C}\mathrm{P}(n-1)$ may be regarded as $\mathcal{O}(-2)$. Letting $\Longrightarrow$ denote a degenerate limit of a family of metrics, and using the subscripts ${}_{qK}$ and ${}_{hK}$ to indicate whether the space in question is

to be regarded as a quaternionic Kähler or hyperKähler manifold respectively, we have the following diagram (away from zero sections).

$$\begin{array}{ccccccc} \mathcal{U}(\mathrm{Gr}_2(\mathbb{C}^n))_{qK} & \Longrightarrow & \mathcal{U}(\mathrm{Gr}_2(\mathbb{C}^n))_{hK} & \Longleftarrow & T^*\mathbb{C}\mathrm{P}(n-1) & \Longleftarrow & \mathrm{Gr}_2(\mathbb{C}^{n+1}) \\ & & & \searrow \quad \swarrow & & & \\ & & & F_{1,2,n} & & & \\ & & & \swarrow \quad \searrow & & & \\ & & \mathrm{Gr}_2(\mathbb{C}^n) & & \mathbb{C}\mathrm{P}(n-1) & & \end{array}$$

More generally, we can consider the bundle $\mathcal{U}(M)$ associated to a quaternionic Kähler manifold M with non-zero scalar curvature.

Lemma 3.1.1. *If X is a Killing vector field on M such that $L_X\Omega = 0$, where Ω is the fundamental 4-form of M, then X may be lifted to a Hamiltonian Killing vector field $\tilde{X}$ on $\mathcal{U}(M)$.*

PROOF. Let ϕ be an isometry of M such that $\phi^*\Omega = \Omega$. Then ϕ_* defines a map $\phi_*\colon F_x \to F_{\phi(x)}$ which commutes with the $Sp(n)\,Sp(1)$-action on the reduced frame bundle F, since

$$(\phi_* u(\xi)) \cdot (A, q) = \phi_* u(A\xi\bar{q}) = \phi_*(u(\xi) \cdot (A, q)),$$

for $u \in F$, $\xi \in \mathbb{H}$ and $(A, q) \in Sp(n) \times Sp(1)$. Similarly, the pull-back of ϕ_* to $F \times (\mathbb{H}^*/\mathbb{Z}_2)$ commutes with the $Sp(n)\,Sp(1)$-action, so we have an induced map $\phi_*\colon \mathcal{U}(M) \to \mathcal{U}(M)$.

Now ϕ^* preserves θ, for if $u \in F$, $v \in T_uF$ then

$$\phi^*\theta_u(v) = \theta_{\phi_* u}(\phi_{**}v) = (\phi_* u)^{-1}\,(\pi_*\phi_{**}v)$$

and we have that $\pi\phi_* = \phi\pi$, so $\phi^*\theta_u(v) = u^{-1}{\phi_*}^{-1}(\phi_*\pi_* v) = \theta_u(v)$. Also, ϕ^* commutes with d and the connection forms $\omega_\pm$ are uniquely determined by

$d\theta = -\omega_+ \wedge \theta - \theta \wedge \omega_-$, so $\phi^*\omega_- = \omega_-$. The induced map ϕ_* acts trivially on $\mathbb{H}$, and so preserves x and dx. Thus ϕ^* is an isometry on $\mathcal{U}(M)$ which preserves the 2-form υ.

Let ϕ_t ba a local 1-parameter group of local isometries generating X. Since $L_X\Omega = 0$, we may assume that $\phi_t^*\Omega = \Omega$. As above, we can lift ϕ_t to a local 1-parameter group ϕ_{t*} of local isometries on $\mathcal{U}(M)$. The vector field $\tilde{X}$ generated by ϕ_{t*} is thus a Hamiltonian Killing field on $\mathcal{U}(M)$. Note that $\pi_*^H\tilde{X} = X$. □

Suppose a compact Lie group G acts on M freely and isometrically and that this action preserves Ω. The lemma above then tells us that the action of G may be lifted to $\mathcal{U}(M)$ and that this action is isometric and triholomorphic. Since $\pi_*^H\tilde{X} = X$ for each vector field generated by the action, the induced action preserves the fibres of $\pi^H\colon \mathcal{U}(M) \to M$, and is thus free.

Let Y be a Killing field generated by the action of G on M and let X denote the lift of Y to $\mathcal{U}(M)$. The proof of the lemma yields an $Sp(n)\,Sp(1)$-invariant vector field, which we again denote by X, on F and $\varpi_*X = X$. If we define μ_X on $F \times \mathbb{H}^*$ by

$$\mu_X = -X \lrcorner (x\omega_-\bar{x}), \tag{3.1.2}$$

then μ_X is $Sp(n) \times Sp(1) \times \mathbb{Z}_2$-invariant and so is well-defined on $\mathcal{U}(M)$.

Proposition 3.1.2. *Let $\mu\colon \mathcal{U}(M) \to \mathfrak{g}^* \otimes \mathrm{Im}\,\mathbb{H}$ be defined from* (3.1.2) *as in* (3.1.1). *Then μ is a moment map for the induced action of G on $\mathcal{U}(M)$.*

PROOF. First we check that $d\mu_X = X \lrcorner \upsilon$. Now, since M is Einstein we have

$$\begin{aligned} d\mu_X &= -(L_X - X \lrcorner d)(x\omega_-\bar{x}) \\ &= -L_X(x\omega_-\bar{x}) + X \lrcorner (dx \wedge \omega_-\bar{x} + xc\bar{\theta}^t \wedge \theta\bar{x} - x\omega_- \wedge \omega_-\bar{x} - x\omega_- \wedge d\bar{x}). \end{aligned}$$

But $X \lrcorner dx = 0$ and $L_X x = 0$, so

$$d\mu_X = -x(L_X \omega_-)\bar{x} + X \lrcorner (\alpha \wedge \bar{\alpha} + cx\bar{\theta}^t \wedge \theta \bar{x}).$$

Let ϕ_t be a 1-parameter group of isometries preserving Ω and generating Y. Then ϕ_{t*} generates X and it preserves ω_-, so $L_X \omega_- = 0$. Hence, $d\mu_X = X \lrcorner v$. The equivariance of μ follows directly from the definition of μ_X. □

Define α_Y to be $Y \lrcorner g$, where g is the metric on M. Then $\nabla \alpha_Y$ is an element of $\Lambda^2 T^*M$, since Y is a Killing field (see Kobayashi & Nomizu, 1963). We have a subbundle $\mathfrak{sp}(1)$ of $\Lambda^2 T^*M$ whose fibre is the Lie algebra $\mathfrak{sp}(1) \cong \mathrm{Im}\,\mathbb{H}$; this bundle is just the isometric embedding of $\mathcal{G}$ in $\Lambda^2 T^*M$. Let $(\nabla \alpha_Y)^{\mathfrak{sp}(1)}$ denote the orthogonal projection of $\nabla \alpha_Y$ onto $\mathfrak{sp}(1)$. If we choose a frame $u \in F$, then $(\nabla \alpha_Y)^{\mathfrak{sp}(1)}_{\pi(u)} = \omega_-(X)_u$, where X is the lift of Y to TF, as above. Thus, μ_X vanishes at a point $a \in \mathcal{U}(M)$ if and only if $(\nabla \alpha_Y)^{\mathfrak{sp}(1)}_{\pi^H(a)} = 0$. So, if μ_X vanishes at a, then μ_X vanishes on the whole fibre through a and if X is a lifted Killing vector field, then X is horizontal over each point of the fibre through a.

3.2 Quaternionic Kähler Quotients

Suppose M is a quaternionic Kähler manifold with fundamental 4-form Ω. A Killing vector field X on M is said to be *quaternionic* if $L_X \Omega = 0$. Let X be such a Killing field, I, J and K a local basis for $\mathcal{G}$ and η_I, η_J and η_K their corresponding 2-forms. Then locally we can define a 1-form Θ_X with values in $\mathcal{G}$ by

$$\Theta_X = (X \lrcorner \eta_I) \otimes \eta_I + (X \lrcorner \eta_J) \otimes \eta_J + (X \lrcorner \eta_K) \otimes \eta_K.$$

This definition does not depend on the choice of local basis for $\mathcal{G}$ and so Θ_X is globally defined. If $\alpha_X = X \lrcorner g$, as above, then the map $\Omega^1(M) \to \Omega^1(\mathcal{G})$ given by $\alpha_X \mapsto \Theta_X$ is injective so we may regard Θ_X as an element of $\Omega^1(M)$.

The Riemannian connection ∇ on M preserves $\mathcal{G}$, so we can define a sequence

$$\Omega^0(\mathcal{G}) \xrightarrow{d^\nabla} \Omega^1(\mathcal{G}) \xrightarrow{d^\nabla} \Omega^2(\mathcal{G}) \xrightarrow{d^\nabla} \cdots,$$

where $d^\nabla(\alpha \otimes \sigma) = d\alpha \otimes \sigma + (-1)^p \alpha \otimes \nabla\sigma$ for $\alpha \in \Omega^p(M)$ and $\sigma \in \Omega^0(\mathcal{G})$. It is well-known that if $\mathcal{R}$ is the curvature of $\mathcal{G}$ then

$$d^\nabla \circ d^\nabla f = \mathcal{R}(f),$$

for each $f \in \Omega^0(\mathcal{G})$. This is used to show the following:

Theorem 3.2.1. (Galicki & Lawson, 1988) *if M has non-zero scalar curvature, then for each quaternionic Killing vector field X on M there is a unique $f_X \in \Omega^0(\mathcal{G})$ such that $\nabla f_X = \Theta_X$.* □

Remark. If we regard f_X as a 2-form, then the condition $\nabla f_X = \Theta_X$ is equivalent to $df_X = X \lrcorner \Omega$. The map $X \mapsto X \lrcorner \Omega$ is just the inclusion of $\Lambda^1 T^*M$ in $\Lambda^3 T^*M$ which can be seen in the representation theory of Chapter 5.

We assume from now on that M has non-zero scalar curvature.

The map f_X is actually determined by the function $\mu_{\tilde{X}}$ introduced in (3.1.2). In the notation of the previous section, let

$$f_X = \frac{1}{\lambda}(\nabla \alpha_Y)^{\mathfrak{sp}(1)},$$

where λ is the same constant positive multiple of the scalar curvature of M as in the introduction.

Lemma 3.2.2. *The function f_X defined above satisfies $\nabla f_X = \Theta_X$.*

PROOF. Fix $m \in M$ and let η_1, η_2, η_3 be a local basis of $\mathcal{G}$ regarded as a subbundle of $\Lambda^2 T^*M$, such that

$$\nabla \eta_i|_m = 0, \qquad \text{for } i = 1, 2, 3.$$

Let $\mathbf{a}\colon V^{\otimes p} \to \Lambda^p V$ be the alternation map, then $d\eta_i|_m = \mathbf{a}(\nabla \eta_i|_m) = 0$; so at m,

$$L_X \eta_i = d(X \lrcorner \eta_i) + X \lrcorner d\eta_i = \mathbf{a}(\nabla(X \lrcorner \eta_i)) = \mathbf{a}(\nabla \mathbf{c}(\alpha_X, \eta_i)).$$

Here $\mathbf{c}\colon \Lambda^p \otimes \Lambda^q \to \Lambda^{p-1} \otimes \Lambda^{q-1}$ is contraction with the metric over each of the first two indices. Now $\mathbf{c}$ commutes with ∇, so

$$L_X \eta_i = \mathbf{a}(\mathbf{c}(\alpha_X, \nabla \eta_i) + \mathbf{c}(\nabla \alpha_X, \eta_i)) = \mathbf{a}(\mathbf{c}(\nabla \alpha_X, \eta_i)),$$

by our choice of η. Now, M is irreducible, so $\nabla \alpha_X \otimes \eta_i$ is an element of $(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) \otimes \mathfrak{sp}(1) \cong (\mathfrak{sp}(n) \otimes \mathfrak{sp}(1)) \oplus S^2(S^2\mathbb{C}^2) \oplus \mathfrak{sp}(1)$. But $\mathfrak{sp}(n) \otimes \mathfrak{sp}(1)$ is an irreducible $Sp(n) \times Sp(1)$-module and $\mathbf{ac}(\mathfrak{sp}(n) \otimes \mathfrak{sp}(1))$ is a proper submodule, so by Schur's Lemma it is $\{0\}$. Thus $\mathbf{ac}(\nabla \alpha_X, \eta_i)$ is an element of $\mathfrak{sp}(1)$ and only involves the $\mathfrak{sp}(1)$-component of $\nabla \alpha_X$. Hence

$$L_X \eta_i = [(\nabla \alpha_X)^{\mathfrak{sp}(1)}, \eta_i],$$

and the function f_X agrees with the explicit form given in Galicki & Lawson (1988, equation (2.11)), that is,

$$f_X|_m = \sum_{i=1}^{3} f_i \otimes \eta_i|_m = \frac{1}{\lambda} \sum_i r_i \otimes \eta_i|_m,$$

where $L_X \eta_j = \sum_{i,k} \varepsilon_{ijk} r_i \eta_k$. □

Thus f_X is just the projection of $\mu_{\tilde{X}}$ to M.

Let G be a compact Lie group acting freely and smoothly on M. Suppose that this action preserves the metric and the 4-form Ω.

Definition 3.2.3. (Galicki, 1987) The *moment map* $\Phi\colon M \to \mathfrak{g}^* \otimes \mathcal{G}$ associated to the G-action on M is defined by $\langle \Phi(m), X\rangle = f_X(m)$, for each $m \in M$ and each vector field X generated by the action of G.

From the theorem above we have that Φ is G-equivariant. Unlike the hyperKähler case, there is only one natural G-invariant submanifold to consider, namely $M_0 = \Phi^{-1}(0)$, where 0 denotes the zero section. If we now define $M_G = M_0/G$, then Galicki & Lawson (1982) have shown that M_G inherits a quaternionic Kähler structure from M.

Examples of quaternionic Kähler quotients of $\mathbb{H}\mathrm{P}(n)$ may be found in Galicki (1987). In particular, the action of $U(1)$ on $\mathbb{H}^{n+1}$ given in the previous section descends to $\mathbb{H}\mathrm{P}(n)$. The moment map at $[q_0, \ldots, q_n] \in \mathbb{H}\mathrm{P}(n)$ is $\bar{q}^t i q$. Write $q = a + jb$, then there is an $A \in U(n+1)$ such that $Aa = (1, 0, \ldots, 0)^t$. Now, if q is a zero of the moment map, then so is $Aq = Aa + j\bar{A}b$ and in particular, $b^t \bar{A}^t A a = b^t a = 0$, so $\bar{A}b = (0, b_1, \ldots, b_n)^t$ and we still have the freedom to choose A so that $\bar{A}b = (0, 1, 0, \ldots, 0)^t$. Thus, $U(n+1)$ acts transitively on the zero set of the moment map. The stabiliser of a point is $SU(2) \times U(n-1)$, so the quaternionic Kähler quotient is

$$\frac{U(n+1)}{SU(2) \times U(n-1) \times U(1)} = \mathrm{Gr}_2(\mathbb{C}^{n+1}).$$

Recall that the corresponding hyperKähler quotient of $\mathbb{H}^{n+1}$ constructed the manifold $\mathcal{U}(\mathrm{Gr}_2(\mathbb{C}^{n+1}))$ which is the associated bundle of the quaternionic Kähler quotient. In the following section we show that this is true for all quaternionic Kähler quotients.

3.3 Commutativity of Constructions

In this section, when we perform the hyperKähler quotient of $\mathcal{U}(M)$ we use the zero level set of the moment map.

Theorem 3.3.1. *Suppose M is a quaternionic Kähler manifold with non-zero scalar curvature. If a Lie group G acts isometrically, freely and quaternionically on M then G induces a free, triholomorphic, isometric action on the associated bundle $\mathcal{U}(M)$. The pseudo-hyperKähler quotient of $\mathcal{U}(M)$ by this G-action is precisely the associated bundle to the quaternionic Kähler quotient of M by G.*

PROOF. Let M_0, M_G be as above and let $\pi\colon M_0 \to M_G$ and $i\colon M_0 \to M$ be the projection and inclusion maps, respectively. Let F be the reduced frame bundle of M: this is a principal $Sp(n)\,Sp(1)$-bundle, where $\dim M = 4n$. If $X_1, \ldots, X_k$ is a basis of quaternionic Killing vector fields for the action of G on M, and I, J, K is a basis for $\mathcal{G}$ in a neighbourhood U of a point $m \in M$, then $\{ IX_i, JX_i, KX_i : i = 1, \ldots, k \}$ span the orthogonal complement $(TM_0)^{\perp}|_U$ of TM_0 in $TM|_{M_0}$ (see Galicki & Lawson, 1988). Also, $X_1, \ldots, X_k$ span the vertical tangent space for the projection π and its orthogonal complement in TM_0 is horizontal for the Riemannian submersion $\pi\colon M_0 \to M_G$. Define E to be the subbundle of $TM|_{M_0}$ whose fibre at m is the span of $\{ X_i, IX_i, JX_i, KX_i : i = 1, \ldots, k \}$ and consider the subbundle P of i^*F of frames which restrict to frames on E. This is a principal bundle over M_0 with fibre

$$(Sp(n-k) \times Sp(k))\,Sp(1) = (Sp(n-k) \times Sp(k)) \times_{\mathbb{Z}_2} Sp(1).$$

We have a natural projection $\rho\colon P \to F_G$ to the reduced frame bundle of M_G obtained by restricting frames to the horizontal directions. Note that ρ is just the quotient map obtained from the free group action of $G \times Sp(k)$ on P.

Let $\hat{\mathcal{U}}(M)$ denote the pseudo-hyperKähler quotient of $\mathcal{U}(M)$ by the induced G-action, so $\hat{\mathcal{U}}(M) = \mu^{-1}(0)/G = \mathcal{U}(M)_0/G$. This quotient may be performed, since on $\mu^{-1}(0)$ each vector field induced by the G-action is horizontal, so the restriction of the metric to these vector fields is non-degenerate. The discussion in earlier sections of this chapter shows that $\mathcal{U}(M)_0 = i^*\mathcal{U}(M)$.

We first show that $\mathcal{U}(M_G) = \hat{\mathcal{U}}(M)$ topologically. We have that

$$\begin{aligned}\pi^*\mathcal{U}(M_G) &= \pi^*F_G \times_{Sp(n-k)\,Sp(1)} (\mathbb{H}^*/\mathbb{Z}_2) \\ &= (P \times_{Sp(n-k)\,Sp(1)} (\mathbb{H}^*/\mathbb{Z}_2))/\,Sp(k) \\ &= P \times_{(Sp(n-k)\times Sp(k))\,Sp(1)} (\mathbb{H}^*/\mathbb{Z}_2),\end{aligned}$$

where $Sp(k)$ acts trivially on $\mathbb{H}^*$. Let ϕ be the composition

$$P \times (\mathbb{H}^*/\mathbb{Z}_2) \longrightarrow i^*F \times (\mathbb{H}^*/\mathbb{Z}_2) \longrightarrow \mathcal{U}(M)_0,$$

where the first map is inclusion and the second is the quotient by $Sp(n)\,Sp(1)$. It is sufficient to show that ϕ is the quotient map obtained from the action of $(Sp(n-k) \times Sp(k))\,Sp(1)$ on $P \times (\mathbb{H}^*/\mathbb{Z}_2)$, but this is easily checked directly. Thus $\mathcal{U}(M_G) = \hat{\mathcal{U}}(M)$ topologically.

Now it is sufficient to prove that $\mathcal{U}(M_G)$ and $\hat{\mathcal{U}}(M)$ are locally isometric. If U is a sufficiently small open subset of M_0, then we may choose a double cover $i^*\tilde{F}|_U$ of $i^*F|U$ and this gives a double cover $\tilde{P}$ of P. Now $G \times Sp(k)$ acts freely on $\tilde{P}$, so we may take the quotient and obtain a double cover $\tilde{F}_G|_{\pi(U)}$ of $F_G|_{\pi(U)}$. Let $\sigma\colon U \to \tilde{P}$ be a section and identify $\pi^*TM_G|_U$ with the horizontal space. We may define a map $\tau_\sigma\colon \pi^*\tilde{F}_G|_U \to \tilde{F}|_U$ by

$$\tau_\sigma(u)_m = \begin{pmatrix} u_0 \\ \\ v_0 \end{pmatrix} \cdot \left(\begin{pmatrix} A & 0 \\ \\ 0 & 1 \end{pmatrix}, q \right).$$

Here $\sigma_m = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ with u_0 in the double cover of a frame on $\pi^* TM_G|_m$ and $u = u_0 \cdot (A, q)$, $A \in Sp(n-k)$, $q \in Sp(1)$ and 1 denotes the identity matrix in $Sp(k)$. If we extend τ_σ by the identity to $\mathbb{H}^*$ then, as τ_σ commutes with the $Sp(n-k) \times Sp(1)$-action, we get an induced map $\tau_\sigma : \pi^* H^*(M_G) \to H^*(M_0)$ over U. It may easily be checked that this map is injective. Let $\hat{}$ over a form denote the version on $H^*(M_G)$ or $\tilde{F}_G$, e.g. $\hat{\theta}$ is the canonical 1-form of $\tilde{F}_G$. From the following commutative diagram

$$\begin{array}{ccccccc}
\tilde{F}_G & \xleftarrow{\pi_*} & \pi^*\tilde{F}_G & \xrightarrow{\tau_\sigma} & i^*\tilde{F} & \xrightarrow{i_*} & \tilde{F} \\
\Big\downarrow{\scriptstyle \pi_G} & & \Big\downarrow{\scriptstyle \pi_0} & & \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow \\
M_G & \xleftarrow[\pi]{} & M_0 & = & M_0 & \xrightarrow[i]{} & M
\end{array}$$

we have

$$\pi^* \hat{\theta}_v(w) = \hat{\theta}_{\pi_* v}(\pi_{**} w) = (\pi_* v)^{-1}(\pi_{G*}\pi_{**} w) = (\pi_* v)^{-1}(\pi_* \pi_{0*} w)$$

for $v \in \pi^* \tilde{F}_G|_U$, $w \in T_v(\pi^* F_G)$. Also, in the notation of the definition of τ_σ,

$$\begin{aligned}
{\tau_\sigma}^* i^* \theta_v(w) = \theta_{\tau_\sigma v}(\tau_{\sigma *} w) &= \left(\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \cdot \left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, q \right) \right)^{-1} (\pi_{1*}\tau_{\sigma *} w) \\
&= (u_0 \cdot (A, q))^{-1}(\pi_{0*} w) = v^{-1}(\pi_{0*} w).
\end{aligned}$$

But π_* is an isomorphism on each fibre, so $\pi^* \hat{\theta}_v(w) = {\tau_\sigma}^* i^* \theta_v(w)$. Hence, ${\tau_\sigma}^* i^* \eta = \pi^* \hat{\eta}$ and τ_σ is a local isometry. □

Taking M to be flat space gives the following: if G acts isometrically and triholomorphically on $\mathbb{H}^{n+1}$, then the hyperKähler quotient $\mu^{-1}(x)/G$ projects to a quaternionic Kähler quotient of $\mathbb{HP}(n)$, provided $x \neq 0$.

3.4 Examples

In the first two sections of this chapter we have already seen how the hyperKähler metric on $\mathcal{U}(\mathrm{Gr}_2(\mathbb{C}^n))$ arises as the hyperKähler quotient of flat space $\mathbb{H}^n$ by $U(1)$. Galicki (1987) also shows that the space $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{4n})$ may be obtained as a quaternionic Kähler quotient of $\mathbb{H}\mathrm{P}(n-1)$ by $Sp(1)$ and so we obtain $\mathcal{U}(\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{4n}))$ as the hyperKähler quotient of $\mathbb{H}^n$ by the diagonal action of $Sp(1)$. In the case of $\mathcal{U}(\mathrm{Gr}_2(\mathbb{C}^n))$, we obtained hyperKähler deformations of the metric. This procedure does not work for $\mathcal{U}(\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{4n}))$ because the centre of $\mathfrak{sp}(1)$ is trivial.

HyperKähler quotients of flat space have been used by Kronheimer (1989a), extending work of Gibbons & Hawking (1978) and Hitchin (1979), to construct and classify hyperKähler ALE 4-manifolds. Here ALE stands for 'asymptotically locally Euclidean.' These manifolds have one end and at infinity it resembles $\mathbb{R}^4/\Gamma$, for some finite subgroup Γ of $SU(2)$. Briefly, the construction is as follows. Let V be the regular representation of Γ and define M to be $((\mathrm{End}\, V) \otimes_{\mathbb{C}} \mathbb{H})^{\Gamma}$, the space of Γ-invariant elements in $(\mathrm{End}\, V) \otimes_{\mathbb{C}} \mathbb{H}$, where Γ acts on $\mathbb{H}$ in the standard way. Let G be the group of unitary transformations of V which commute with the action of Γ. Then the ALE manifolds are obtained as the hyperKähler quotient of M by G. Note that G is in general a product of unitary groups with non-trivial centre Z, so we obtain a space X_ζ for each $\zeta \in \mathrm{Im}\,\mathbb{H} \otimes \mathfrak{Z}^*$. The three real components of ζ then correspond to the cohomology classes of the symplectic forms on X_ζ. Note that when $\zeta = 0$ we obtain the singular space $\mathbb{C}^2/\Gamma$ with its flat metric.

As a contrasting example of quotients of flat space with an irreducible group action, consider the nth-symmetric power $S^n = S^n V$ of the basic $SU(2)$-module $V \cong \mathbb{C}^2$, where $SU(2)$ acts by left multiplication. If $\{x, y\}$ is an orthonormal basis for V,

then $\{x^n, n^{1/2}x^{n-1}y, \binom{n}{2}^{1/2}x^{n-2}y^2, \ldots, y^n\}$ is an orthonormal basis for S^n, where

$$x^{n-1}y = \frac{1}{n}(x \otimes \cdots \otimes x \otimes y + x \otimes \cdots \otimes x \otimes y \otimes x + \cdots + y \otimes x \otimes \cdots \otimes x),$$

etc.

A *structure map* on a $\mathbb{C}G$-module A is a conjugate-linear automorphism j of A such that j commutes with the G-action and either $\mathrm{j}^2 = 1$ or $\mathrm{j}^2 = -1$. In the first case j is a *real* structure map and A is the complexification an $\mathbb{R}G$-module. In the second case j is a *quaternionic* structure map and A is isomorphic to an $\mathbb{H}G$-module.

The basic $SU(2)$-module V has a quaternionic structure map j given by

$$\mathrm{j}(ax + by) = -\bar{b}x + \bar{a}y.$$

This gives a structure map j on S^n which is real if n is even and quaternionic if n is odd.

We restrict our attention to the quaternionic modules $S^{2m+1} \cong \mathbb{H}^m$. There is a natural $SU(2)$-invariant homomorphism $\Omega\colon S^{2m+1} \otimes S^{2m+1} \to S^0 \cong \mathbb{C}$ which is the restriction of the linear map $\Omega\colon V^{2m+1} \otimes V^{2m+1} \to S^0$ given on decomposables by

$$\Omega(u \otimes v \otimes \ldots, u' \otimes v' \otimes \ldots) = (u \wedge u') \otimes (v \wedge v') \otimes \ldots.$$

If we set

$$\begin{aligned}
\eta_I(X,Y) &= \frac{i}{2}(\Omega(X,\mathrm{j}Y) - \Omega(Y,\mathrm{j}X)),\\
\eta_J(X,Y) &= \operatorname{Re}\Omega(X,Y),\\
\eta_K(X,Y) &= \operatorname{Im}\Omega(X,Y),
\end{aligned}$$

for $X, Y \in TS^{2m+1} \cong S^{2m+1}$, then η_I, η_J and η_K are symplectic forms on S^{2m+1} corresponding to the complex structures given by left-multiplication by i, j and k

on $\mathbb{H}^m$, respectively. Similarly, we have an $SU(2)$-invariant linear map $P\colon S^{2m+1} \otimes S^{2m+1} \to S^2$ which is the restriction of the map given on decomposables by

$$P(u\otimes v\otimes w\otimes\ldots, u'\otimes v'\otimes w'\otimes\ldots) = \frac{1}{2}(u\otimes u'+u'\otimes u)\otimes(v\wedge v')\otimes(w\wedge w')\otimes\cdots.$$

Let

$$\mu_I(\phi) = imP(\phi, \mathrm{j}\phi),$$
$$\mu_J(\phi) = m\,\mathrm{Re}\,P(\phi,\phi),$$
$$\mu_K(\phi) = m\,\mathrm{Im}\,P(\phi,\phi),$$

for $\phi \in S^{2m+1}$. Now $\mu = \mu_1 i + \mu_2 j + \mu_3 k$ is a map from S^{2m+1} to $S^2 \otimes \mathrm{Im}\,\mathbb{H}$ and $S^2 \cong \mathfrak{su}(2)^* \otimes \mathbb{C}$ as $SU(2)$-modules. To show μ is a moment map one checks that each μ_i is $\mathfrak{su}(2)$-invariant, which is automatic, and that

$$\langle d\mu_i(\phi)(\xi), a\rangle = (\Omega_i)_\phi(\xi, a_\phi),$$

where $\phi \in S^{2m+1}$, $\xi \in T_\phi S^{2m+1}$ and $a \in \mathfrak{su}(2)$; $\langle\cdot,\cdot\rangle$ is the natural pairing between $\mathfrak{su}(2)^*$ and $\mathfrak{su}(2)$ and $\phi \to a_\phi$ is the vector field on S^{2m+1} induced by a.

For S^1, S^3 and S^5, 0 is the only zero of μ (see Salamon, 1984, for S^5). For S^7 ($m=3$), $SU(2)$ does not act freely on $\mathbb{H}^4$, but the action descends to $\mathbb{H}\mathrm{P}(3)$. The map μ then becomes a moment map on $\mathbb{H}\mathrm{P}(3)$ and the corresponding quaternionic Kähler quotient is a finite number of points. Thus the connected components of the hyperKähler quotient $\mu^{-1}(0)\setminus\{0\}/\,SU(2)$ are copies of $\mathbb{H}^*/\mathbb{Z}_2$. For S^9 ($m=4$), we have that $Q = \mu^{-1}(0)/\,SU(2)$ is an 8-dimensional manifold with singularities and a natural hyperKähler structure at the regular points, which is the associated bundle of some (possibly singular) self-dual Einstein 4-manifold. With respect to the basis

$\{x^9, x^8y, \ldots, y^9\}$ of S^9, the equations $\mu(\phi) = \mu(A, B, C, D, E, F, G, H, L, M) = 0$ take the form

$$\begin{aligned}
252AL - 56BH + 21CG - 12DF + 5E^2 &= 0,\\
2268AM - 196BL + 35CH - 18DG + 2EF &= 0,\\
252BM - 56CL + 21DH - 12EG + 5F^2 &= 0,\\
252(A\bar{B} + L\bar{M}) + 56(B\bar{C} + H\bar{L}) + 21(C\bar{D} + G\bar{H})&\\
+12(D\bar{E} + F\bar{G}) + 10E\bar{F} &= 0,\\
2268(A\bar{A} - M\bar{M}) + 196(B\bar{B} - L\bar{L}) + 35(C\bar{C} - H\bar{H})&\\
+18(D\bar{D} - G\bar{G}) + 2(E\bar{E} - F\bar{F}) &= 0.
\end{aligned}$$

Notice that these equations are invariant under $\phi \to \lambda\phi$ for $\lambda \in \mathbb{R}$. The problem of the determination of this manifold is essentially a question in classical invariant theory (see Mumford & Fogarty, 1982). In the second half on the nineteenth century the invariants of S^n for $n \leqslant 6$ were classified and Shioda (1967) dealt with the case of the binary octavic (S^8). As far as we know the classification in the cases of the binary septimic and nonic have not been completed.

One can try to obtain the exceptional Wolf spaces as quaternionic Kähler quotients of other Wolf spaces. The examples of the quotients of $\mathbb{H}\mathrm{P}(n)$ by $U(1)$ and $SU(2)$ suggest looking at isotropy irreducible spaces of the form $G/(H \times K)$ where G is the isometry group of the original Wolf space G/L, H is the isometry group of the quaternionic Kähler quotient of G/L by K. We are interested in examples where H is an exceptional group. Using the tables in Wolf (1968) and computing the dimensions, we see that the only possible example of this form arises from $E_6/(G_2 \times SU(3))$. Note that $SU(3)$ does not act freely on the Wolf

space $E_6/(SU(6)\,Sp(1))$. This may be seen as follows, using the notation of Buccella et al. (1989). Let e_i, $i = 0, \ldots, 7$ be the standard basis for $\mathbb{O}$ and let e'_i be a basis for a second copy $\mathbb{O}'$ of $\mathbb{O}$. Then

$$\mathfrak{su}(6,\mathbb{C}) = \Lambda^2\mathbb{C}^3 \oplus \mathfrak{su}(2)_S \oplus (S_0^2\mathbb{C}^3 \oplus \Lambda^2\mathbb{C}^3 \otimes e'_7) \otimes e_q \oplus (S_0^2\mathbb{C}^3 \otimes e'_7) \hookrightarrow \mathfrak{e}_6^{\mathbb{C}}$$
$$\mathfrak{e}_6^{\mathbb{C}} = \Lambda^2\mathbb{C}^3 \oplus \mathfrak{g}_2^{\mathbb{C}} \oplus (S_0^2\mathbb{C}^3 \otimes e_i) \oplus (\Lambda^2\mathbb{C}^3 \otimes e'_7 \otimes e_i) \oplus (S_0^2\mathbb{C}^3 \otimes e'_7),$$

where i runs from 1 to 7 and q takes the values 3, 6 and 7. Now $\mathfrak{su}(3,\mathbb{C}) = \Lambda^2\mathbb{C}^3 \oplus S_0^2\mathbb{C}^3 \otimes e'_7$, which is contained in $\mathfrak{su}(6,\mathbb{C})$. However, this does not imply that the quaternionic Kähler quotient does not exist, since it is only necessary that $SU(3)$ acts freely on the zero set of the moment map. The group G_2 commutes with $SU(3)$ and so acts on the quaternionic Kähler quotient; this leads us to conjecture that the quaternionic Kähler quotient of $E_6/(SU(6)SU(2))$ by $SU(3)$ is the space $G_2/SO(4)$.

The growing importance of hyperKähler quotients arises from their use in an infinite-dimensional setting, first discovered by Atiyah & Bott (1982). If P is a principal G-bundle over a symplectic manifold (M^{2m}, ω), for some compact semi-simple group G, then the space $\mathcal{A}$ of G-connections may be identified with

$$\mathcal{A} = \Omega^1(\mathfrak{g}_P),$$

where $\mathfrak{g}_P = P \times_G \mathfrak{g}$ is the associated bundle. We obtain a symplectic structure on $\mathcal{A}$ via $(A, B) = \int_M \mathrm{Tr}(A \wedge B) \wedge \omega^{m-1}$ for $A, B \in \mathcal{A}$. The group $\mathcal{G} = \mathrm{Aut}\, P$ acts on $\mathcal{A}$ preserving this symplectic structure. The moment map for this action maybe identified with $\mu(A) = F_A \wedge \omega^{m-1}$, where $F_A = dA + \frac{1}{2}[A, A]$ is the curvature of the

connection A. If M is a hyperKähler 4-manifold, the symplectic forms span $\Lambda^2_+ T^*M$ and we obtain a formal hyperKähler quotient of $\mathcal{A}$ by $\mathcal{G}$

$$\frac{\mu^{-1}(0)}{\mathcal{G}} = \frac{\{\, A : F_A \in \Omega^2_+(\mathfrak{g}_P)\,\}}{\mathcal{G}},$$

which is the moduli space of self-dual solutions of the Yang-Mills equations. When M is $\mathbb{R}^4$, this space may also be obtained as a *finite*-dimensional hyperKähler quotient via the ADHM construction (Atiyah et al., 1978b) which describes the moduli space in terms of matrix algebras. The remark at the end of Section 3.3 shows that these moduli spaces are associated to quaternionic Kähler manifolds.

3.5 Quaternionic Kähler Metrics

Recall that the hyperKähler metric constructed on the associated bundle $\mathcal{U}(M)$ of a quaternionic Kähler manifold M admits a homothetic $\mathbb{H}^*$-action. The subgroup $Sp(1)$ of $\mathbb{H}^*$ acts isometrically, but it permutes the complex structures on $\mathcal{U}(M)$, so the quotient constructions earlier in this chapter do not apply to this group action. However, if we fix one complex structure I, then there is a subgroup $U(1) \leqslant Sp(1)$ which preserves I and we have a moment map for this circle action. Upto a constant, this is the function r^2 and fibrewise the level sets correspond to the spheres in $\mathbb{H}$. These are not only $U(1)$-invariant, but they are preserved by the $Sp(1)$-action. The $Sp(1)$-quotient of a level set is just the original quaternionic manifold M.

This discussion generalises as follows.

Theorem 3.5.1. *Suppose N is a hyperKähler manifold which admits an isometric $Sp(1)$-action such that*

i) *there is a finite subgroup* Γ *of* $Sp(1)$ *such that* $Sp(1)/\Gamma$ *acts freely on* N,

ii) $Sp(1)$ *induces a transitive action on the 2-sphere of complex structures on* N, *and*

iii) *if* X_I *is the vector field generated by the circle subgroup of* $Sp(1)$ *preserving* I, *then the (real) linear span of* IX_I *in* TN *is independent of the choice of complex structure* I.

Choose a subgroup $U(1) \leqslant Sp(1)$ *preserving a complex structure* I. *Let* $\mu\colon N \to \mathbb{R}$ *be a moment map for this* $U(1)$ *with respect to the Kähler structure defined by* I, *then* $\mu^{-1}(x)$ *is* $Sp(1)$*-invariant and* $\mu^{-1}(x)/\,Sp(1)$ *is a quaternionic Kähler manifold.*

PROOF. Fix a complex structure I preserved by a subgroup $U(1) \leqslant Sp(1)$ and let ω_I denote the Kähler form associated to I. Let X be the vector field generated by the $U(1)$-action and let Y be a vector field generated by the $Sp(1)$-action. Then Y is orthogonal to IX, since Y arises from a circle group preserving a complex structure A and Y is orthogonal to $AY = IX$. Now,

$$Y \lrcorner d\mu = Y \lrcorner (X \lrcorner \omega_I) = -g(Y, IX) = 0,$$

so μ is $Sp(1)$-invariant.

We now give two different proofs that the quotient $\mu^{-1}(x)/\,Sp(1)$ is quaternionic Kähler. The first shows that we have a closed quaternionic 4-form on the quotient and then uses Theorem 1.2.2 to deduce that it is quaternionic Kähler. This is only valid if the quotient manifold has dimension at least 12. The second proof applies in all dimensions and is more in the spirit of Galicki & Lawson's proof (1988) of the quaternionic Kähler quotient construction.

Let J and K be complex structures such that $IJ = K = -JI$ and let ω_J, ω_K be the corresponding Kähler forms. Define a 4-form by $\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$. Let $i\colon \mu^{-1}(x) \hookrightarrow N$ and $\pi\colon \mu^{-1}(x) \to \mu^{-1}(x)/\,Sp(1)$ be the inclusion and projection respectively. If n is a zero of μ, then we can write $T_nN = \mathcal{V}_n \oplus \mathcal{H}_n$ where $\mathcal{V}$ is the quaternionic span of X and $\mathcal{H}$ is the orthogonal complement. Note that this splitting is quaternionic. The hypotheses of the theorem imply that $\mathcal{V}$ contains all vectors tangent to the $Sp(1)$-action and that $\mathcal{H}$ is an $Sp(1)$-invariant distribution of horizontals for the projection π. The restriction $i^*\Omega$ of Ω vanishes on $\mathcal{V}$ and is $Sp(1)$-invariant, so $i^*\Omega$ is the pull-back of a 4-form Ω' on the quotient. Since $\mathcal{H}$ is quaternionic and $\pi^*\Omega'$ is just the restriction of Ω on $\mathcal{H}$, we see that Ω' is of the correct algebraic type to define a quaternionic structure on the quotient. Now $i^*\Omega = \pi^*\Omega'$ and π^* is injective, so $d\Omega = 0$ implies that Ω' is closed. This concludes the proof if the quotient is at least 12-dimensional.

For the second proof we need to work with the covariant derivative ∇' on the quotient. As above, for $n \in \mu^{-1}(x)$ split T_nN as $\mathcal{V}_n \oplus \mathcal{H}_n$, write i and π for the inclusion and projection maps, and let $\cdot^{\mathcal{H}}$ denote the horizontal component of a tangent vector. Then if $Z \in \mathcal{H}_n$ is the pull-back of a tangent vector Z on the quotient, we have

$$\nabla' Z = \pi_*((\nabla Z)^{\mathcal{H}}),$$

where ∇ is the Riemannian connection on N. Note that if ω_A is one of the Kähler forms on N, then if $Z \in \mathcal{H}$ and Y is any tangent vector, we have $\omega_A(Y, Z) = \omega_A(Y^{\mathcal{H}}, Z)$. Choosing a local section s of π we obtain a local 2-form $s^*\omega_A$ on the quotient and there we obtain a rank 3 vector bundle $\mathcal{G}$ generated by all such local 2-forms. Except in 4-dimensions, it is sufficient to show that this bundle is preserved by ∇'. If ω_A is such a local 2-form, then $\pi^*\omega_A = (f_I\omega_I + f_J\omega_J + f_K\omega_K)^{\mathcal{H}}$ for some

functions f_I, f_J, f_K. Now

$$\begin{aligned}\pi^*\nabla'\omega_A &= (\nabla((\pi^*\omega_A)^{\mathcal{H}}))^{\mathcal{H}} \\ &= (\nabla(f_I\omega_I + f_J\omega_J + f_K\omega_K)^{\mathcal{H}})^{\mathcal{H}} \\ &= (df_I \otimes \omega_I + df_J \otimes \omega_J + df_K \otimes \omega_K)^{\mathcal{H}},\end{aligned}$$

since by the remark above, $(\nabla(\omega_I)^{\mathcal{H}})^{\mathcal{H}} = (\nabla\omega_I)^{\mathcal{H}} = 0$. Thus $\nabla'\omega_A$ is in $\mathcal{G}$, as required.

In 4-dimensions we need to calculate the curvature. If s is a local section, then $s^*\omega_I$ defines a complex structure if and only if it is a holomorphic section with respect to I. As in the integrability proof in the previous chapter, this implies that the curvature tensor lies in the complement of $\Lambda^{0,2} \otimes \Lambda^{0,2}$. This holds for each complex structure on N and forces the curvature to be self-dual. The Einstein condition follows from an immersion computation and the formulae of O'Neill (1966) applied to the Riemannian submersion $\mu^{-1}(x) \to \mu^{-1}(x)/\,Sp(1)$. □

The above theorem generalises to the pseudo-Riemannian category in the same way as we remarked that the hyperKäher quotient construction could be generalised.

The hypotheses on the group action in the above theorem, imply that the action is essentially determined by a $U(1)$-action and the quaternionic structure of N. This is a quaternionic analogue of having a complexified $U(1)$-action, which is what is used in the ordinary Kähler quotient construction. Note that an $Sp(1)$-action also determines much of the hyperKähler structure of N.

Lemma 3.5.2. *Let (N, g) be a Riemannian manifold with symplectic form ω. Suppose N admits an isometric $SU(2)$-action such that the distribution $\mathcal{A}$ spanned by $SU(2) \cdot I$ is 3-dimensional and the orthogonal complement $I^{\perp}$ in $\mathcal{A}$ is preserved by the almost complex structure I defined by ω and g. Then N is locally hyperKähler.*

PROOF. Locally there is a $g \in SU(2)$ such that $g^*\omega$ is orthogonal to ω. Define ω_I to be ω and put $\omega_J = g^*\omega$. Define J by $g(X, JY) = \omega_J(X, Y)$ for tangent vectors X and Y. Then, the I-invariance of the decomposition of $\mathcal{A}$ implies that $IJ \in SU(2)\cdot I$ and hence that $(IJ)^2 = -1$, so $IJ = -JI$. Define K to be IJ and define ω_K in the same way as ω_J. Then ω_I, ω_J and ω_K are closed 2-forms and so define a local hyperKähler structure. □

The product of two quaternionic Kähler manifolds need not be quaternionic, but the above theorem suggests a type of quaternionic *join* construction. Let M_1, M_2 be quaternionic Kähler manifolds with positive scalar curvature. Consider the bundles $\mathcal{U}(M_1)$, $\mathcal{U}(M_2)$ with their hyperKähler metrics. The product $\mathcal{U}(M_1) \times \mathcal{U}(M_2)$ is hyperKähler and $\mathbb{H}^*$ acts diagonally. The $Sp(1)$-action so defined satisfies the hypotheses of the theorem and we obtain a quaternionic Kähler manifold $\mathcal{J}(M_1, M_2)$ of dimension $\dim M_1 + \dim M_2 + 4$. Topologically this is the $\mathbb{Z}_2$-quotient of the quaternionic projectivisation of the bundle which is locally $\pi_1^* H(M_1) \oplus \pi_2^* H(M_2) \to M_1 \times M_2$, where π_i is projection onto the ith-factor. Thus if M_2 is a point $*$, $\mathcal{J}(M_1, *)$ is topologically $\mathcal{U}(M_1)$.

Proposition 3.5.3. *Let M be a quaternionic Kähler manifold with positive scalar curvature. The quaternionic Kähler metric on $\mathcal{J}(M, *)$ agrees with the metric g_1 constructed on $\mathcal{U}(M)$ in Chapter 2.*

PROOF. First note that the join of two points $\mathcal{J}(*, *)$ is the $\mathbb{Z}_2$-quotient of $\mathbb{HP}(1)$ with two points removed with its standard symmetric metric $g^{\mathbb{HP}(1)}$. In inhomogeneous coordinates,

$$g^{\mathbb{HP}(1)} = \mathrm{Re}\left(\frac{d\bar{z} \otimes dz}{1 + \|z\|^2} - \frac{\bar{z}dz \otimes d\bar{z}z}{(1 + \|z\|^2)^2}\right) = \mathrm{Re}\left(\frac{d\bar{z} \otimes dz}{(1 + \|z\|^2)^2}\right).$$

Let g^M denote the metric on M, which we will assume to be normalised so that the constant c appearing in the definition of g_1 is 1. Let $g^{\mathbb{H}}_{(i)}$ denote various copies of the standard flat metric on $\mathbb{H}$. Then the metric on $\mathcal{J}(M,*)$ is

$$\frac{1}{r^2}\left(r_1^2\pi^{H^*}g^M + g^{\mathbb{H}}_{(1)} + g^{\mathbb{H}}_{(2)}\right)^{\mathcal{H}},$$

where $r^2 = r_1^2 + r_2^2$; r_1^2, r_2^2 are the radius functions on $\mathcal{U}(M)$ and $\mathcal{U}(*)$ respectively; and $\mathcal{H}$ denotes projection to an appropriate horizontal subspace. Now $\pi^{H^*}g^M$ is horizontal and $\frac{1}{r^2}\left(g^{\mathbb{H}}_{(1)} + g^{\mathbb{H}}_{(2)}\right)^{\mathcal{H}}$ is the metric on $\mathcal{J}(*,*)$. Fix $r_2^2 = 1$ and $r_1^2 = \|z\|^2$, then the metric on $\mathcal{J}(M,*)$ is

$$\frac{\|z\|^2}{1+\|z\|^2}\pi^{H^*}g^M + \frac{1}{(1+\|z\|^2)^2}g^{\mathbb{H}},$$

which is one of the g_1 metrics. □

If M is $\mathbb{H}\mathrm{P}(n)$ then $\mathcal{J}(M,*)$ is the quotient of $\mathbb{H}^{n+2}$ by $\mathbb{H}^*$ and we just obtain the standard quaternionic Kähler metric on an open subset of $\mathbb{H}\mathrm{P}(n+1)$. Thus, the quaternionic Kähler metrics on $\mathcal{U}(\mathbb{H}\mathrm{P}(n))$ arise from the standard inclusion $\mathbb{H}^{n+1} \hookrightarrow \mathbb{H}\mathrm{P}(n+1)$. Other examples of joins of Wolf spaces will be discussed in the next chapter. In particular, we will see that the join of $\mathbb{C}\mathrm{P}(2)$ with a point is not locally symmetric.

3.6 Potentials

Let N be a hyperKähler manifold admitting a circle action preserving I and permuting J and K. Hitchin et al. (1987) showed that if μ is the moment associated to the circle action and the Kähler structure I, then μ is a Kähler potential for J and for K. This raises the possibility of N admitting a Kähler potential for all the complex structures simultaneously.

Definition 3.6.1. If N is a hyperKähler manifold, a function $\mu\colon N \to \mathbb{R}$ is a *hyperKähler potential* if for each complex structure A compatible with the hyperKähler structure of N, μ satisfies

$$i\partial_A\bar{\partial}_A\mu = \omega_A,$$

where ω_A is the Kähler form of A.

The result quoted above suggests that the existence of a hyperKähler potential may be linked with the existence of certain group actions.

Proposition 3.6.2. *Suppose N is a hyperKähler manifold. If N has a hyperKähler potential then N admits a local $Sp(1)$-action which permutes the complex structures and, in the notation of the previous section, is such that the vector field IX_I is independent of I.*

Conversely, if N admits such an $Sp(1)$-action, then N has a hyperKähler potential. [If N is compact, the same is true even if we only require that $Sp(1)$ permutes the complex structures.]

PROOF. Let μ be a hyperKähler potential on N. For a complex structure I compatible with the hyperKähler structure of N, define a vector field X_I by $d\mu = X_I \lrcorner \omega_I = -(IX_I)\lrcorner g$, where g is the metric on N. Thus the vector field $X = IX_I$ is independent of the choice of complex structure I. It is now sufficient to show that the bracket $[IX, JX]$ a non-zero multiple of KX. Firstly, if A and B are vector fields then

$$0 = -d^2\mu(A,B) = d(X\lrcorner g)(A,B) = Ag(X,B) - Bg(X,A) - g(X,[A,B]).$$

Taking $A = IX$ and $B = JX$ yields $g(X, [IX, JX]) = 0$, so $[IX, JX]$ is orthogonal to X. If C is also a vector field then

$$0 = d\omega_I(A, B, C) = A\omega_I(B, C) + B\omega_I(C, A) + C\omega_I(A, B) \\ - \omega_I(A, [B, C]) - \omega_I(B, [C, A]) - \omega_I(C, [A, B]).$$

Fix a point n of N and let $Y \in T_nN$ be orthogonal to the quaternionic span of X. Extend Y locally so that it commutes with IX. Take $A = Y$, $B = IX$ and $C = JX$ in the above formula. This gives

$$0 = g(Y, I[IX, JX]) + g(X, [JX, Y]) + g(JX, I[Y, IX]) = g(Y, I[IX, JX]),$$

so $[IX, JX]$ is in the span of IX, JX and KX. Now μ is a Kähler potential for I, so $d(Id\mu) = \omega_I$ for each I. Thus $L_{IX}\omega_J = d((KX) \lrcorner g) = d(K(-X \lrcorner g)) = d(Kd\mu) = \omega_K$ and similarly for cyclic permutations of I, J and K. Also, $L_{IX}\omega_I = d(X \lrcorner g) = d^2\mu = 0$. Thus

$$\begin{aligned} L_{[IX,JX]}\omega_I &= [L_{IX}, L_{JX}]\omega_I \\ &= (L_{IX}L_{JX} - L_{JX}L_{IX})\omega_I \\ &= -L_{IX}\omega_K - 0 \\ &= \omega_J, \end{aligned}$$

showing that $[IX, JX]$ is non-zero and that the $Sp(1)$-action, resulting from integration, permutes the complex structures.

Conversely, if N admits a permuting $Sp(1)$-action with IX_I fixed, let μ_I be the moment map associated to I, then $d\mu_I = X_I \lrcorner \omega_I = -(IX_I) \lrcorner g$. Hitchin et al. (1987) show that μ_I is a Kähler potential for J and K. Now let μ_J be the moment map

associated to J. Then since JX_J is independent of J, we have $d\mu_J = -(JX_J)\lrcorner g = d\mu_I$. Now μ_J is a Kähler potential for I, so applying ∂_I to this equation shows that $i\partial_I d\mu_I = \omega_I$ and that μ_I is a hyperKähler potential.

[If N compact and $Sp(1)$ permutes the complex structures, then μ_I and μ_J are both Kähler potentials for K. So $\partial_K\bar{\partial}_K(\mu_I - \mu_J) = 0$ and since N is compact this implies that μ_I and μ_J only differ by a constant. In particular, $d\mu_I = d\mu_J$ and μ_I is a hyperKähler potential.] □

Note that on the associated bundle of a quaternionic Kähler manifold the hyperKähler potential is the function r^2. The hyperKähler potential determines the metric as follows. The first part of the proposition is well-known.

Proposition 3.6.3. 1) *If N is a Kähler manifold with metric g, Riemannian connection ∇, complex structure I and Kähler form ω_I, then for a function μ on N*

$$\frac{1}{2}(\nabla^2_{X,Y}\mu + \nabla^2_{IX,IY}\mu) = g(X,Y) \qquad \textit{for all } X, Y$$

if and only if

$$i\partial_I\bar{\partial}_I\mu = \omega_I.$$

2) *If N is a hyperKähler manifold, then for a function μ*

$$\nabla^2\mu = g \qquad \textit{if and only if} \qquad \mu \textit{ is a hyperKähler potential.}$$

PROOF. 1) Let X, Y be two vector fields. Now ∇ is torsion-free and I is Kähler,

so

$$\begin{aligned}
2(i\partial_I\bar{\partial}_I\mu)(X,Y) &= -id(d\mu + iId\mu)(X,Y) \\
&= d(Id\mu)(X,Y) \\
&= X(Id\mu)(Y) - Y(Id\mu)(X) - Id\mu[X,Y] \\
&= X((IY)\mu) - Y((IX)\mu) - (I[X,Y])\mu \\
&= X((IY)\mu) - Y((IX)\mu) - I(\nabla_X Y - \nabla_Y X)\mu \\
&= X((IY)\mu) - \nabla_X(IY)\mu - Y((IX)\mu) + \nabla_Y(IX)\mu \\
&= \nabla^2_{X,IY}\mu - \nabla^2_{Y,IX}\mu
\end{aligned}$$

and the result follows.

2) This follows from part (1) by fairly general principles. Let $h(X,Y) = \nabla^2_{X,Y}\mu$; we need only show that $g = h$ if and only it $g(X,Y) = \frac{1}{2}(h(X,Y) + h(IX,IY))$ for all vector fields X, Y and each compatible complex structure I. If $g = h$ then this is just the fact that I preserves the metric. If the second condition holds then $h(IX,IY) = h(JX,JY)$. Put $X' = IX$, $Y' = IY$ and suppose $IJ = K = -JI$, then $h(X',Y') = h(KX',KY')$, as required. □

Remark. If N is a hyperKähler manifold with hyperKähler potential μ, then the vector field X dual to $d\mu$ is an infinitesimal quaternionic transformation. In the notation of Proposition 3.6.2, X is the vector field IX_I, so we have a local $\mathbb{H}^*$-action. The system of equations

$$\nabla d\mu = \lambda g,$$

for some constant λ, is over determined and Bochner techniques show that the $\mathbb{H}^*$-orbits are flat and totally geodesic.

Pursuing an idea of Lebrun (1990), suppose M is a quaternionic Kähler manifold with Levi-Civita connection ∇ and that M admits a hyperKähler metric in the

same quaternionic class. The hyperKähler metric trivialises the bundle H and so gives a solution h of the twistor equation. Quaternionic invariance of this equation, implies that $e = \nabla h$ is a section of E. From e and h one can construct infinitesimal vector fields for a local $\mathbb{H}^*$-action. We expect such metrics to be locally isometric to an associated bundle $\mathcal{U}(M')$ for some quaternionic Kähler manifold M'. If M is four-dimensional, the above discussion shows that it is flat.

The classification of hyperKähler 4-manifolds admitting a permuting $Sp(1)$-action was carried out by Gibbons & Pope (1979) and completed by Atiyah & Hitchin (1988). The metrics obtained (upto finite quotients) are the flat metric on $\mathbb{H}$, the TAUB-NUT metric and the hyperKähler metric on the moduli space of charge 2 monopoles. Of these, the only one that can possess a hyperKähler potential is the flat metric.

Maciocia (1989) explicitly constructs the hyperKähler potential for the moduli space $\mathcal{M}_{k,r}$ of framed $SU(r)$-instantons of charge k over $\mathbb{R}^4$. If A is such a connection, then the hyperKähler potential is given by the 'second moment'

$$m_2(a) = \frac{1}{16\pi^2} \int_{\mathbb{R}^4} \|x\|^2 \operatorname{Tr} F_A^2,$$

where F_A is the curvature of A.

Corollary 3.6.4. *The manifold $m_2^{-1}(x)/\,SO(3) \subset \mathcal{M}_{k,r}/\,SO(3)$, for any non-degenerate point $x \in \operatorname{Im} m_2$, is quaternionic Kähler.* □

Chapter 4

ISOMETRY GROUPS AND QUATERNIONIC GEOMETRY

In this chapter we will be concerned with quaternionic Kähler manifolds of positive scalar curvature which admit sufficiently large isometry groups. We start by showing that the twistor spaces of these manifolds may be mapped into the projectivised nilpotent variety of the complexified isometry group. We also obtain a map of the quaternionic Kähler manifold into the Grassmannian of oriented three-planes in the Lie algebra of the isometry group. This leads us to study the nilpotent variety and the Grassmannian from the point of view of quaternionic geometry.

In the second section we try to see as much of the twistor space structure of the projectivised nilpotent orbits as possible and obtain quaternionic manifolds via the inverse twistor construction. The following section studies the same situation from the point of view of the nilpotent orbits themselves using the gradient flows of certain functionals. The nilpotent orbits are already known to be hyperKähler with an action of $\mathbb{H}^*$ permuting the complex structures (Kronheimer, 1988) and in Section 3 we see that the $\mathbb{H}^*$-quotient is a quaternionic Kähler manifold and identify points of this manifold with elements of the Grassmannian encountered in the first section. The chapter ends with various examples of quaternionic Kähler manifolds derived from nilpotent orbits.

4.1 Embeddings of Twistor Spaces

Let M be a quaternionic Kähler manifold with positive scalar curvature and let G be the identity component of its isometry group. For simplicity, we will assume that G is compact and semi-simple. Compactness holds if, for example, M is complete, since the positivity of the scalar curvature now implies that M is compact (Myers, 1935). We have already described the twistor space Z of M as the complex projectivisation of either $\mathcal{U}(M)$ or, locally, H. However, since the vector bundle $\mathcal{G}$ generated by local compatible almost complex structures on M is just the symmetric product S^2H we see that Z is also the sphere bundle of $\mathcal{G}$. Explicitly, a unit vector $x \in H$ is identified with the symmetric product $ix \vee jx \in S^2H$. The Kähler structure on Z now arises as a combination of the natural structure on the fibres $S^2 \cong \mathbb{C}\mathrm{P}(1)$ and the almost complex structure on T_xM defined by the point of $Z = S(\mathcal{G})$.

The isometry group G lifts to Z as a real form of the identity component of the group $G^{\mathbb{C}}$ of holomorphic automorphisms of the contact structure on Z (see Nitta & Takeuchi, 1987).

Theorem 4.1.1. (Salamon, 1982) *If M is a $4n$-dimensional pseudo-quaternionic Kähler manifold with non-zero scalar curvature and Z is the twistor space, then*

i) *the fibres of Z are complex projective lines with normal bundle $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$,*

ii) *Z admits a real structure, that is, a fixed-point free, antiholomorphic involution,*

iii) *Z has a complex contact structure, the contact line bundle $\mathcal{L}$ is positive and the contact distribution is transverse to the fibres, that is, the contact form $\theta \in \Omega^1(\mathcal{L})$ gives an isomorphism $T\,\mathbb{C}\mathrm{P}(1) \cong \mathcal{L}|_{\mathbb{C}\mathrm{P}(1)}$,*

iv) *the space of holomorphic sections* $H^0(\mathcal{L})$ *is isomorphic to the Lie algebra* $\mathfrak{g}^{\mathbb{C}}$. □

For part (iv), see also Poon & Salamon (1989) and Kobayashi (1972).

If Z is compact then Kodaira's Theorem (see Griffiths & Harris, 1978) shows that for sufficiently large r the (meromorphic) map

$$\Phi^{(r)}\colon Z \to \mathbb{P}(H^0(\mathcal{L}^r)^*)$$

induced by evaluation, is a holomorphic embedding. Let $\Phi = \Phi^{(1)}$ be the meromorphic map

$$\Phi\colon Z \to \mathbb{P}(H^0(\mathcal{L})^*) \cong \mathbb{P}(\mathfrak{g}^{\mathbb{C}*}).$$

From Chapter 3, we see that this map is the complex moment map associated to the action of G on $\mathcal{U}(M)$, which should be regarded as the symplectification of Z. So Φ is associated to both the algebraic geometry and the group theory of Z. A *base point* of Φ is a pole of Φ, that is, a point at which all holomorphic sections of $\mathcal{L}$ vanish.

Proposition 4.1.2. *Suppose* Φ *has no base points and that the action of* G *on* M *is* full *in the sense that there is no open set in* M *on which* G *fixes a compatible complex structure. Then the elements of* $\Phi(Z)$ *are nilpotent.*

PROOF. Let z be a point of Z, identify $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}*}$ via an invariant inner product, and choose a representative $e \in \mathfrak{g}^{\mathbb{C}} \setminus \{0\}$ such that $\Phi(z) = [e]$. We claim that e is nilpotent if for every G-invariant polynomial p we have $p(e) = 0$. To see this choose an embedding of $\mathfrak{g}^{\mathbb{C}}$ in $\mathfrak{gl}(m, \mathbb{C})$ (this exists since G is compact). Then put e in Jordan form and consider the polynomials $p_n(x) = \mathrm{Tr}(x^n)$. These are invariant

polynomials for $GL(m,\mathbb{C})$ and hence for G, so $\mathrm{Tr}(e^n) = 0$ for all n and all symmetric polynomials in the eigenvalues of e vanish. Hence zero is the only eigenvalue of e and e is nilpotent.

Suppose there is a non-trivial, G-invariant polynomial $p \in S^k H^0(\mathcal{L}) \cong S^k\mathfrak{g}$ of least degree such that $p(e) \neq 0$. Let s_k be the map

$$s_k : S^k H^0(\mathcal{L}) \to H^0(\mathcal{L}^k).$$

Now $p(e) = (s_k p)(e^{\otimes k})$ is non-zero, so we have a G-invariant section $\sigma = s_k p$ of $H^0(\mathcal{L}^k)$. Restricting to a fibre $\mathbb{C}\mathrm{P}(1)$ of $Z \to M$ we have that $\sigma|_{\mathbb{C}\mathrm{P}(1)}$ is a holomorphic section of $\mathcal{L}^k|_{\mathbb{C}\mathrm{P}(1)} = \mathcal{O}(2k)$, so σ must have a zero on $\mathbb{C}\mathrm{P}(1)$. Furthermore, for a generic $\mathbb{C}\mathrm{P}(1)$ this zero must have multiplicity one, otherwise there is an invariant polynomial of smaller degree. Using the action of G, we now obtain a holomorphic, invariant section of Z on an open set, which is not possible by our fullness hypothesis. So σ is identically zero and $p(e) = 0$. Thus e is a zero of each G-invariant polynomial and so must be nilpotent. □

Since G acts on Z preserving the Kähler structure we have a real moment map $\mu : Z \to \mathfrak{g}^*$. We may extend this linearly to a map $\mathcal{G} \to \mathfrak{g}^*$; this is just the map $A \mapsto (\nabla A)^{\mathfrak{sp}(1)}$ discussed in Chapter 3, because the fibre of $\mathcal{G}$ is isomorphic to $\mathfrak{sp}(1)$. So for each point x in M we have a map $\Psi : \mathcal{G}_x \cong \mathfrak{sp}(1) \to \mathfrak{g}$. In Chapter 2 we saw that for a Wolf space this map is a homomorphism, but in general this map does not preserve the Lie algebra structure. The maps Φ and Ψ are related as follows: let e_1 be a unit vector in $\mathcal{G}_x$ and extend to an oriented, orthonormal basis e_1, e_2, e_3, then $\Phi(e_1) = [\Psi(e_2) + i\Psi(e_3)] \in \mathbb{P}(\mathfrak{g}^{\mathbb{C}})$. Now $e_2 + ie_3$ is isotropic in $\mathcal{G}_x \otimes \mathbb{C}$, so the above proposition shows that $V = \Psi(\mathcal{G}_x)$ is such that the image of the isotropic elements, under the complex linear extension of Ψ, are nilpotent. Let $\mathcal{F}$ denote the set of all oriented 3-planes $V \in \widetilde{\mathrm{Gr}}_3(\mathfrak{g})$ whose isotropic elements are nilpotent.

Proposition 4.1.3. *If the action of G is full and Φ has no base points, then $\Psi(\mathcal{G}_x)$ lies in the variety $\mathcal{F}$.*

PROOF. It only remains to show that $\Psi(\mathcal{G}_x)$ is three-dimensional. The hypothesis that Φ has no base points implies that $\Psi(\mathcal{G}_x)$ is at least one-dimensional. If the image is not three-dimensional, then there exist orthogonal unit vectors e_1 and e_2 in $\mathcal{G}_x$ with $\Psi(e_1) \neq 0$ and $\Psi(e_2) \in \langle \Psi(e_1) \rangle$. Thus $\Psi(e_1 \pm ie_2)$ are complex multiples of $\Psi(e_1) \in \mathfrak{g}$. It is now sufficient to show that $\mathfrak{g}$ contains no (non-zero) nilpotent elements.

As above, we may assume that G is a subgroup of $SU(n)$ for some n. Now if $A \in \mathfrak{su}(n)$ is nilpotent, we have $\operatorname{Tr} A^m = 0$, for all m. But $A = -\bar{A}^t$, so $0 = \operatorname{Tr} A^2 = -\operatorname{Tr} A\bar{A}^t = -\sum_{i,j} |a_{ij}|^2$ and A is identically zero. □

Thus, $\mathcal{F} \subset \widetilde{\mathrm{Gr}}_3(\mathfrak{g})$ and the nilpotent variety in $\mathfrak{g}^{\mathbb{C}}$ contain local models for M and $\mathcal{U}(M)$, respectively. In the next two sections we will study the quaternionic geometry of these two varieties.

To give an example of $\mathcal{F}$, consider the case where $\mathfrak{g}$ is $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Let $e_1, e_2, e_3, f_1, f_2, f_3$ be oriented orthogonal bases for each of the $\mathfrak{so}(3)$-factors such that

$$[e_1, e_2] = -2e_3, \qquad [f_1, f_2] = -2f_3, \qquad \text{etc.}$$

The elements of $\mathcal{F}$ are now the three planes $\langle \lambda e_1 + \mu f_1, \lambda e_2 + \mu f_2, \lambda e_3 + \mu f_3 \rangle$, for $(\lambda, \mu) = (\cos\theta, \sin\theta) \in S^1$, together with those obtained after applying an element of $SO(3)$ to either the e_i or the f_i. These three planes are images of homomorphisms only if λ/μ is 0, 1 or ∞. The elements of $\mathcal{F}$ with $\lambda/\mu \in (0, \infty)$ and $\lambda > 0$ are in bijective correspondence with $S^4/\mathbb{Z}_2$ with two points removed.

4.2 Projective Nilpotent Orbits

From the previous section one would hope to be able to show that projective nilpotent orbits are twistor spaces of quaternionic Kähler manifolds. We saw in Chapter 2 that this was indeed the case for the highest root orbit, since this is the associated bundle of the Wolf space with isometry group G.

Let $\mathcal{O}$ be a nilpotent orbit of a complex semi-simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$. There is a contact structure on $\mathbb{P}(\mathcal{O})$ derived form the complex symplectic structure ω of $\mathcal{O}$ described by Kirillov, Kostant & Souriau (Kirillov, 1972). If X is an element of $\mathcal{O}$, then $\omega([X,A],[X,B]) = \langle X,[A,B]\rangle$. The contact form θ is the contraction of ω with the vector field generating the $\mathbb{R}$-action, so $\theta([X,A]) = \omega([X,-H],[X,A]) = -\langle X,[H,A]\rangle = \langle [X,H],A\rangle = -2\langle X,A\rangle$. Let σ be the real structure on $\mathfrak{g}^{\mathbb{C}}$ whose fixed point set is $\mathfrak{g}$. Then $\sigma^*\theta = \bar{\theta}$, so the natural real structure σ on $\mathbb{P}(\mathcal{O})$ respects the contact structure.

Given $X \in \mathcal{O}$, the Jacobson-Morosov Theorem (Carter, 1985, Kostant, 1959) states that there are elements Y and H of $\mathfrak{g}^{\mathbb{C}}$ such that $\langle X,Y,H\rangle$ is an $\mathfrak{sl}(2,\mathbb{C})$-triplet, that is X, Y and H satisfy

$$[X,Y] = H, \qquad [H,X] = 2X \quad \text{and} \quad [H,Y] = -2Y.$$

This shows that $\mathbb{R}$ acts on $\mathcal{O}$ via scaling and that we have a Lie algebra homomorphism $\rho\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}^{\mathbb{C}}$ such that $\rho\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X$, given by

$$\rho\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X, \qquad \rho\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = Y \quad \text{and} \quad \rho\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H.$$

Without loss of generality, ρ is the complexification of a real homomorphism $\mathfrak{su}(2) \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of a compact real form G of $G^{\mathbb{C}}$ (see Kronheimer, 1988).

Let $x \in \mathbb{P}(\mathcal{O})$ be the point defined by X and let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{sl}(2,\mathbb{C})$. Then x lies in $\mathbb{P}(\rho\mathcal{N}) = \mathbb{C}\mathrm{P}(1)$. Let ν be the normal bundle of this $\mathbb{C}\mathrm{P}(1)$ in $\mathbb{P}(\mathcal{O})$. If $\mathbb{P}(\mathcal{O})$ is a twistor space then we expect ν to be a direct sum of $\mathcal{O}(1)$'s. Now $\mathfrak{g}^{\mathbb{C}}$ splits under the action of $\mathfrak{sl}(2,\mathbb{C})$ into a direct sum of $SU(2)$-modules

$$\mathfrak{g}^{\mathbb{C}} = S^2 \oplus \bigoplus_{k\geqslant 0} A_k S^k,$$

where the first S^2 is the span of X, Y and H, A_k are trivial as $SU(2)$-modules and juxtaposition denotes tensor product. The tangent space to $\mathcal{O}$ at X is

$$T_X\mathcal{O} = (\operatorname{ad} X)\mathfrak{g}^{\mathbb{C}},$$

so the normal bundle at x is given by

$$\nu_x \cong (\operatorname{ad} X) \bigoplus_{k\geqslant 1} A_k S^k.$$

Restricting to the circle subgroup defined by H, we have a further splitting of ν into $U(1)$-bundles. Since $(\operatorname{ad} X)S^k$ is just S^k without the $(-k)$-weight space, a summand $(\operatorname{ad} X)S^k$ of ν consists of line bundles whose Chern classes are $k, k-2, \ldots, 2-k$. Thus the Chern class of $(\operatorname{ad} X)S^k$ is $k = \dim(\operatorname{ad} X)S^k$ and we have

$$c_1(\nu) = \operatorname{rank}\nu.$$

Let ev_x be the evaluation map $H^0(\nu) \to \nu_x$. Consider the map $A \mapsto [X, A]$. This is a map from $\mathfrak{g}^{\mathbb{C}}$ to the tangent space of the nilpotent orbit at X and gives a well-defined element of the tangent space $T_x\mathbb{P}(\mathcal{O})$ of the projective nilpotent orbit at $x = [X]$. We denote this tangent vector by $x(A)$. The composition

$$\mathfrak{g}^{\mathbb{C}} \longrightarrow T_x\mathbb{P}(\mathcal{O}) \longrightarrow T_x\mathbb{P}(\mathcal{O})/T_x\,\mathbb{C}\mathrm{P}(1) = \nu_x,$$

where the first map is $A \mapsto x(A)$ and the second is projection, is clearly surjective. Letting x range over $\mathbb{C}\mathrm{P}(1)$ gives a map from the complex Lie algebra to $H^0(T\mathbb{P}(\mathcal{O}))$. Now the composition

$$\mathfrak{g}^{\mathbb{C}} \longrightarrow H^0(T\mathbb{P}(\mathcal{O})) \longrightarrow H^0(\nu) \xrightarrow{\mathrm{ev}_x} \nu_x$$

agrees with the map given above and so ev_x must be surjective. Any holomorphic bundle over $\mathbb{C}\mathrm{P}(1)$ splits holomorphically as a direct sum of line bundles $\mathcal{O}(k)$ (see Griffiths & Harris, 1978), so

$$\nu = \bigoplus_k W_k \mathcal{O}(k),$$

for some complex vector spaces W_k. Let μ be the non-negative part of ν, so that $\mu = \bigoplus_{k \geqslant 0} W_k \mathcal{O}(k)$. Then we have the exact sequence

$$0 \longrightarrow \mu \longrightarrow \nu \xrightarrow{\pi} \nu/\mu \longrightarrow 0.$$

If s is a holomorphic section of ν, then $\pi s = 0$ so $ev_x(s) = s(x) \in \mu_x$. But ev_x is surjective so π is zero and $\nu = \bigoplus_{k \geqslant 0} W_k \mathcal{O}(k)$. In particular, we have

$$H^1(\nu) = 0.$$

Let $h^i(\cdot)$ be $\dim H^i(\cdot)$. By the Riemann-Roch Theorem, we have that

$$\begin{aligned} h^0(\nu) - h^1(\nu) = \chi^{\mathrm{hol}}(\nu) &= \mathrm{ch}(\nu) \cdot \mathrm{Td}(\mathbb{C}\mathrm{P}(1)) \\ &= c_1(\nu) + \mathrm{rank}\, \nu \\ &= 2\,\mathrm{rank}\, \nu, \end{aligned}$$

since $\mathrm{ch}(\nu)$ is $\operatorname{rank}\nu$ in dimension 0 and $c_1(\nu)$ in dimension 1 and $\mathrm{Td}(\mathbb{C}\mathrm{P}(1))$ is 1 in both dimensions 0 and 1. Thus,

$$h^0(\nu) = 2\operatorname{rank}\nu.$$

If we identify the nilpotent orbit $\mathcal{N}$ in $\mathfrak{sl}(2,\mathbb{C})$ with its image $\rho(\mathcal{N})$, then the map $(x, A) \mapsto x(A)$ gives the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}\mathrm{P}(1) \times \bigoplus_{k\geqslant 1} A_k S^k & \longrightarrow & \nu \\ & \searrow \quad \swarrow & \\ & \mathbb{C}\mathrm{P}(1) & \end{array}$$

This map is complex linear and hence holomorphic and the splitting $\bigoplus_{k\geqslant 1} A_k S^k$ induces a holomorphic splitting of ν. So, holomorphically, we may work with one summand $\nu' = (\operatorname{ad} X)S^k$ of ν. The above cohomological arguments also apply to ν', in particular

$$c_1(\nu') = k = \operatorname{rank}\nu'$$

$$h^0(\nu') = 2k.$$

Now the map

$$S^k = H^0(\mathbb{C}\mathrm{P}(1) \times S^k) \longrightarrow H^0(\nu') \tag{4.2.1}$$

is equivariant and we have a surjection

$$\mathbb{C}\mathrm{P}(1) \times S^k \longrightarrow \nu'. \tag{4.2.2}$$

Suppose $\nu' = B_0 + B_1\mathcal{O}(1) + \cdots$, where the B_i are $SU(2)$-modules. Then any $S^r \subset B_0$ is in the image of S^k under (4.2.1). But $S^r \subset \nu'$, so $r < k$, since $\operatorname{rank}\nu' = k$

and $\dim S^r = r+1$. Now Schur's Lemma implies that the map $S^k \to S^r$ is zero and so the map $S^k \to B_0$ is also trivial. Surjectivity of (4.2.2) now forces $B_0 = \{0\}$ and

$$\nu' = \bigoplus_{t \geqslant 1} B_t \mathcal{O}(t).$$

But the equation $c_1(\nu') = \operatorname{rank} \nu'$ says $\sum_{t \geqslant 1} (\dim B_t)t = \sum_{t \geqslant 1} \dim B_t$, so $\nu' = B_1 \mathcal{O}(1)$ and

$$\nu = B\mathcal{O}(1),$$

for some $SU(2)$-module B.

Write B_1 as $\bigoplus_i S^{l_i}$. Then

$$H^0(\nu') = B_1 \otimes S^1 \cong \bigoplus_i (S^{l_i+1} + S^{l_i-1})$$

and $h^0(\nu') = 2\sum_i l_i + 1$. Also, $S^k \subset H^0(\nu')$ so one of $S^k + S^{k-2}$ or $S^{k+2} + S^k$ must also be contained in $H^0(\nu')$. A dimension count shows that $H^0(\nu') = S^k \oplus S^{k-2}$ and

$$H^0(\nu) = \bigoplus_{k \geqslant 1} A_k (S^k \oplus S^{k-2}).$$

Thus we have proved the following.

Proposition 4.2.1. *If $\mathfrak{g}^{\mathbb{C}}$ splits as the direct sum $S^2 \oplus \bigoplus_{k \geqslant 0} A_k S^k$, under the action of $\mathfrak{sl}(2, \mathbb{C})$. Then the normal bundle of $\mathbb{C}\mathrm{P}(1) = \mathbb{P}(\rho\mathcal{N})$ in $\mathbb{P}(\mathcal{O})$ is*

$$\left(\bigoplus_{k \geqslant 1} A_k S^{k-1} \right) \otimes \mathcal{O}(1).$$

□

Remark. This decomposition may also be obtained as follows. Let V be the complexification of a real oriented three-plane whose isotropic elements lie in $\mathcal{O}$.

Then $T\mathbb{P}(\mathfrak{g}^{\mathbb{C}})/T\mathbb{P}(V)$ is a direct sum of $\mathcal{O}(1)$'s. If $\mathbb{C}\mathrm{P}(1)$ is the projectivisation of the isotropic elements of V, then the map $\mathbb{C}\mathrm{P}(1) \to \mathbb{P}(V) \cong \mathbb{C}\mathrm{P}(2)$ has degree 2 and $\mu = T\mathbb{P}(\mathfrak{g}^{\mathbb{C}})/T\mathbb{P}(V)|_{\mathbb{C}\mathrm{P}(1)} \cong V^{\perp} \otimes \mathcal{O}(2)$. Now the normal bundle ν of $\mathbb{C}\mathrm{P}(1)$ is a non-negative subbundle of μ, so $\nu = A\mathcal{O} + B\mathcal{O}(1) + C\mathcal{O}(2)$, for some vector spaces A, B and C. An $\mathcal{O}(2)$-summand gives a non-zero holomorphic section s of $\nu(-2)$ and hence of $\mu(-2)$. This is a constant vector v in $V^{\perp}$ which also lies in $T_x\mathcal{O}$ for each $x \in \mathbb{C}\mathrm{P}(1)$. Thus, if $c_1(\nu) = \operatorname{rank}\nu$, we have that ν is a direct sum of $\mathcal{O}(1)$'s if and only if $\bigcap_{[X]\in\mathbb{C}\mathrm{P}(1)} \operatorname{ad}(X)\mathfrak{g}^{\mathbb{C}} = \{0\}$. If V arises from a homomorphism, then this second condition is clearly satisfied.

This information about $\mathbb{P}(\mathcal{O})$ is nearly sufficient to prove directly that it is the twistor space of a quaternionic Kähler manifold.

Theorem 4.2.2. *The manifold $G/N(Sp(1))$ is a submanifold of a pseudo-quaternionic Kähler manifold whose twistor space is an open subset of a projectivised nilpotent orbit $\mathbb{P}(\mathcal{O})$.*

Note that if $\mathcal{O}$ is a highest root orbit and G is simple and centreless, then the manifold $G/N(Sp(1))$ is just the Wolf space discussed in Chapter 2. This result should be contrasted with the classification of homogeneous Kähler manifolds, which may be described as $G/N(U(1))$ for some circle subgroup $U(1)$ of G.

PROOF. Starting with a real homomorphism $\rho\colon \mathfrak{su}(2) \to \mathfrak{g}$ such that $\rho(\mathcal{N}) \subset \mathcal{O}$, we obtain a $\mathbb{C}\mathrm{P}(1) \subset \mathbb{P}(\mathcal{O})$ with normal bundle $\nu \cong B\mathcal{O}(1)$. In particular, $H^1(\nu)$ vanishes, so Kodaira's Theorem provides us with a maximal family $\mathcal{M}$ of deformations of dimension $h^0(\nu) = 2\operatorname{rank}\nu$ (Kodaira, 1962). Let $\mathcal{T}$ be the incidence manifold for $\mathbb{P}(\mathcal{O}) \times \mathcal{M}$ and consider the following diagram, where each map is the canonical

projection.

$$\begin{array}{ccc} & \mathcal{T} & \\ q \swarrow & & \searrow p \\ \mathbb{P}(\mathcal{O}) & & \mathcal{M} \end{array}$$

Now for points of $\mathbb{C}\mathrm{P}(1)$ we have

$$\begin{aligned} \operatorname{rank} q &= \dim \mathcal{T} - h^0(\nu(-1)) \\ &= 2 \operatorname{rank} \nu + 1 - \operatorname{rank} \nu \\ &= \operatorname{rank} \nu + 1 \\ &= \dim \mathbb{P}(\mathcal{O}). \end{aligned}$$

By Riemann-Roch, this computation is equivalent to $h^1(\nu(-1)) = 0$. A result of Grothendieck (Hartshorne, 1977) asserts that the function $x \mapsto h^1(\nu(-1))_x$ is lower semi-continuous and so $h^1(\nu(-1)) = 0$ in a neighbourhood of $\mathbb{C}\mathrm{P}(1)$. Thus q has rank $\dim \mathbb{P}(\mathcal{O})$ in a neighbourhood of the orbit of $\mathbb{C}\mathrm{P}(1)$ under G. This orbit is an S^2-bundle over the manifold $G/N(Sp(1))$ where $N(Sp(1))$ is the normaliser of subgroup $Sp(1)$ of G obtained from $\operatorname{Im} \rho$ via exponentiation. The real and contact structures on the $\mathbb{C}\mathrm{P}(1)$'s in this orbit extend to the deformation space. Our result now follows from a theorem of LeBrun (1989) which states that for a complex contact manifold of dimension $2n + 1 \geqslant 5$ with fixed-point-free anti-holomorphic involution σ respecting the contact structure, the set of σ-invariant rational curves with normal bundle $2n\mathcal{O}(1)$ is naturally a pseudo-quaternionic Kähler manifold of dimension $4n$. (A non-Riemannian version of LeBrun's result was discussed by Pedersen & Poon, 1989.) □

The above theorem would show that the whole of $\mathbb{P}(\mathcal{O})$ is a twistor space if we could prove that there is an appropriate real $\mathbb{C}\mathrm{P}(1)$ through each point. The Jacobson-Morosov Theorem already shows that there each point lies in a $\mathbb{C}\mathrm{P}(1)$ with correct normal bundle, but in general this need not be σ-invariant. Note that the

manifold $\mathcal{M}$ is a complex Riemannian manifold with quaternionic structure (i.e. a manifold with structure group $Sp(n,\mathbb{C})\,Sp(1,\mathbb{C})$ and a compatible torsion-free connection) associated to the whole projectivised orbit.

4.3 Trajectories and Quaternionic Kähler Metrics

The manifold $G/N(Sp(1))$ is contained in the variety $\mathcal{F} \subset \widetilde{\mathrm{Gr}}_3(\mathfrak{g})$ introduced in the first part of this chapter. The proof at the end of the previous section shows that $G/N(Sp(1))$ lies in a quaternionic Kähler manifold which is contained in $\mathcal{F}$. Given an invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, there is a natural functional ψ, defined on $\widetilde{\mathrm{Gr}}_3(\mathfrak{g})$ as follows. Let V be an oriented 3-plane in $\mathfrak{g}$ and let v_1, v_2, v_3 be an oriented basis for V. Then

$$\psi(V) = \psi(v_1, v_2, v_3) = -\frac{1}{\ell}\langle v_1, [v_2, v_3]\rangle,$$

where

$$\begin{aligned}\ell^2 = {} & \|v_1\|^2\|v_2\|^2\|v_3\|^2 \\ & - \langle v_1, v_2\rangle^2\|v_3\|^2 - \langle v_2, v_3\rangle^2\|v_1\|^2 - \langle v_3, v_1\rangle^2\|v_2\|^2 \\ & + 2\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle.\end{aligned}$$

Note that if e_1, e_2, e_3 is an orthonormal basis for V then $\psi(V) = -\langle e_1, [e_2, e_3]\rangle$. The gradient flow equations for ψ, $\dot{V} = \nabla\psi(V)$, are

$$\begin{aligned}\dot{e}_1 &= -[e_2, e_3] - \psi e_1, \\ \dot{e}_2 &= -[e_3, e_1] - \psi e_2, \\ \dot{e}_3 &= -[e_1, e_2] - \psi e_3.\end{aligned}$$

The right-hand sides of these equations are just the components of the Lie brackets which are orthogonal to V.

The non-zero critical points of ψ correspond to images of homomorphisms $\rho\colon \mathfrak{su}(2) \to \mathfrak{g}$, so these critical sets are the manifolds $G/N(Sp(1))$. Dynkin (1952) defines the index j_ρ of ρ to be the constant such that $\langle \rho(x), \rho(y)\rangle = j_\rho \langle x, y\rangle$. So, at a homomorphism ρ, $\psi(\rho(\mathfrak{sp}(1))) = \sqrt{2}/\sqrt{j_\rho}$. Results of Dynkin now show that $0 < \psi(\rho(\mathfrak{sp}(1))) \leqslant \sqrt{2}$ and that $\frac{1}{\sqrt{2}}\psi$ is a calibration on $\widetilde{\mathrm{Gr}}_3(\mathfrak{g})$ with the absolute maxima of ψ occurring at the Wolf space with isometry group G.

Returning to the example of the first section, where $\mathfrak{g} = \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, the trajectories of ψ are obtained by varying the S^1-parameter (λ, μ). The critical points correspond to the homomorphisms from $\mathfrak{sp}(1)$ to each $\mathfrak{so}(3)$ factor of $\mathfrak{so}(4)$ and to the diagonal copy of $\mathfrak{so}(3)$ in $\mathfrak{so}(4)$. In the first two cases the value of ψ is $\sqrt{2}$ and in the diagonal case ψ is 1. The space of trajectories from the diagonal homomorphism to one $\mathfrak{so}(3)$-factor is thus $\mathbb{R}\mathrm{P}(3)$.

Lemma 4.3.1. *The gradient flow of ψ preserves the variety $\mathcal{F}$ and the nilpotent orbits associated to elements of $\mathcal{F}$.*

PROOF. Given an oriented, orthonormal basis e_1, e_2, e_3 for an element $V \in \mathcal{F}$, let

$$e_1' = (1 - t\psi)e_1 - t[e_2, e_3], \quad \text{etc.},$$

be a path parameterised by $t \in \mathbb{R}$. Upto order t^2, e_1', e_2', e_3' are orthonormal, so we need to show that $e_1' + ie_2'$ is nilpotent to first order.

We claim that $X \in \mathfrak{g}^{\mathbb{C}} \leqslant \mathfrak{gl}(m, \mathbb{C})$ is nilpotent if and only if $X = [X, Y]$ for some $Y \in \mathfrak{g}^{\mathbb{C}}$. The forward implication is part of the Jacobson-Morosov Theorem. For the converse, let A be a Jordan block of X, so X is conjugate to $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Now

$$\left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right] = \begin{pmatrix} [A,\alpha] & * \\ * & [B,\delta] \end{pmatrix},$$

and $X = [X,Y]$ implies $A = [A,\alpha]$. Since A is a Jordan block it has minimum polynomial $(x-\lambda I)^n$. But $0 = \mathrm{Tr}[A,\alpha] = \mathrm{Tr}\, A = n\lambda$, so $\lambda = 0$ and A is nilpotent.

Now $V \in \mathcal{F}$, so $e_1 + ie_2$ is nilpotent and there exists Y such that $e_1 + ie_2 = [e_1 + ie_2, Y]$. We have

$$e_1' + ie_2' = (1 - t\psi)(e_1 + ie_2) + t[e_1 + ie_2, ie_3]$$

$$\begin{aligned} \text{and } [e_1' + ie_2', Y] &= (1 - t\psi)[e_1 + ie_2, Y] + t[[e_1 + ie_2, ie_3], Y] \\ &= (1 - t\psi)[e_1 + ie_2, Y] + t[[e_1 + ie_2, Y], ie_3] - t[e_1 + ie_2, [ie_3, Y]], \end{aligned}$$

so

$$e_1' + ie_2' = [e_1' + ie_2', Y + t[ie_3, Y]] \qquad \text{to order } t^2,$$

as required. Note that, to first order, $Y + t[ie_3, Y]$ lies in the $G^{\mathbb{C}}$-orbit of Y. □

From the lower semi-continuity of h^1, we may use the flow of ψ to extend the pseudo-quaternionic Kähler manifold constructed at the end of the previous section. We shall see that the twistor space is the whole of $\mathbb{P}(\mathcal{O})$ from the following argument in the Lie algebra. The proof will not depend on the discussion of $\mathcal{F}$, but we shall see that the variety $\mathcal{F}$ does provide a good description of the quaternionic geometry.

The analogue of ψ on $\mathfrak{g}^3$ is the Chern-Simons functional $\varphi\colon \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by

$$\varphi(A) = \varphi(A_1, A_2, A_3) = \sum_{i=1}^{3} \langle A_i, A_i \rangle + \langle A_1, [A_2, A_3] \rangle.$$

The gradient flow equation $\dot{A} = -\nabla\varphi(A)$ is

$$\begin{aligned} \dot{A}_1 &= -2A_1 - [A_2, A_3], \\ \dot{A}_2 &= -2A_2 - [A_3, A_1], \\ \dot{A}_3 &= -2A_3 - [A_1, A_2]. \end{aligned}$$

Note that these trajectories project to (unparameterised) trajectories of ψ on $\widetilde{\mathrm{Gr}}_3(\mathfrak{g})$, since the right-hand sides project to the component of the Lie bracket that is orthogonal to the three-plane V_A spanned by A_1, A_2, A_3. Let $M(\rho)$ denote the space of solutions of these equations such that $\lim_{t\to\infty} A(t) = 0$ and $\lim_{t\to-\infty} A(t)$ is conjugate to ρ under the adjoint action of the real group G. Here A has been identified with the linear map $\mathfrak{su}(2) \to \mathfrak{g}$ given by

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \mapsto A_1, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto A_2 \quad \text{and} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \mapsto A_3.$$

Lemma 4.3.2. *If A is a solution of $\dot{A} = -\nabla\varphi(A)$ such that $\lim_{t\to-\infty} A(t)$ is a homomorphism $\mathfrak{su}(2) \to \mathfrak{g}$ (for example, if A is a point of $M(\rho)$) then A_1, A_2, A_3 is a conformal basis for V_A.*

PROOF. $\frac{d}{dt}\langle A_1, A_2\rangle = -4\langle A_1, A_2\rangle$, so $\langle A_1, A_2\rangle = Ce^{-4t}$ for some constant C. At a homomorphism $-2A_2 = [A_3, A_1]$, so $2\langle A_1, A_2\rangle = -\langle A_1, [A_3, A_1]\rangle = 0$. Letting $t \to -\infty$ shows that $C = 0$ and that $\langle A_1, A_2\rangle = 0$ for all t.

Now $\frac{d}{dt}\|A_1\|^2 = -4\|A_1\|^2 - 2\langle A_1, [A_2, A_3]\rangle$, so

$$\|A_1\|^2 = c_1 e^{-4t} - 2e^{-4t}\int_0^t e^{4s}\langle A_1, [A_2, A_3]\rangle\, ds.$$

But the integrand is the same for each A_i and as $t \to -\infty$ the norms of the A_i approach the same non-zero constant, so $\|A_1\| = \|A_2\| = \|A_3\|$ for all t. □

Kronheimer (1988) shows that $M(\rho)$ is diffeomorphic to the nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^{\mathbb{C}}$ containing X. He also interprets $M(\rho)$ as a moduli space of framed G-instantons with $SU(2)$-action on the 4-manifold $SU(2) \times \mathbb{R}$ and shows that $M(\rho)$ is a hyperKähler manifold. The space $M(\rho)$ also admits an action of $\mathbb{H}^*/\mathbb{Z}_2 \cong \mathrm{Aut}(\mathfrak{su}(2))$ induced by composition and this action permutes the complex structures.

To obtain an identification between $M(\rho)$ and $\mathcal{O}$, introduce a fourth $\mathfrak{g}$-valued function $A_0(t)$ and replace the gradient flow equations above by

$$\dot{A}_1 = -2A_1 - [A_0, A_1] - [A_2, A_3],$$
$$\dot{A}_2 = -2A_2 - [A_0, A_2] - [A_3, A_1],$$
$$\dot{A}_3 = -2A_3 - [A_0, A_3] - [A_1, A_2].$$

Introducing $\mathfrak{g}^{\mathbb{C}}$-valued functions given by $2\alpha = A_0 + iA_1$ and $2\beta = A_2 + iA_3$, we obtain one real equation and one complex equation

$$(\dot{\alpha} + \dot{\alpha}^*) + 2(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0,$$
$$\dot{\beta} + 2\beta + 2[\alpha, \beta] = 0.$$

These equations are invariant under the action of the real gauge group $\mathbf{G}$ of all smooth maps $g\colon \mathbb{R} \to G$, given by $A_0 \mapsto \mathrm{Ad}(g)(A_0) - \dot{g}g^{-1}$ and $A_i \mapsto \mathrm{Ad}(g)(A_i)$ for $i = 1, 2, 3$. The complex equation is invariant under the complex gauge group $\mathbf{G}^{\mathbb{C}}$ of smooth maps $g\colon \mathbb{R} \to G^{\mathbb{C}}$ which acts via

$$\alpha \mapsto \mathrm{Ad}(g)\alpha - \tfrac{1}{2}\dot{g}g^{-1} \qquad \text{and} \qquad \beta \mapsto \mathrm{Ad}(g)\beta.$$

One now makes the following definitions.

Definition 4.3.3. (Kronheimer) A *complex trajectory* associated to ρ is a pair of smooth functions $\alpha, \beta\colon \mathbb{R} \to \mathfrak{g}^{\mathbb{C}}$ such that

i) α and β satisfy the complex equation,

ii) as $t \to \infty$ we have $\alpha(t) \to 0$ and $\beta(t) \to 0$,

iii) as $t \to -\infty$ we have $2\alpha(t) \to \mathrm{Ad}(g)(H)$ and $\beta(t) \to \mathrm{Ad}(g)(Y)$ for some g in the compact group G,

iv) α and β approach their limits as $t \to \pm\infty$ with exponential decay.

Two complex trajectories are *equivalent* if there is an element g of the complex gauge group $\mathbf{G}^{\mathbb{C}}$ taking one to the other and such that $g(t) \to 1$ as $t \to \infty$.

Proposition 4.3.4. (Kronheimer) *The equivalence classes of the complex trajectories associated with ρ are parameterised by the nilpotent orbit $\mathcal{O}$.*

Explicitly, Kronheimer shows that if two trajectories agree outside a compact subset of $\mathbb{R}$ then they are equivalent and that each complex trajectory is equivalent to a trajectory (α, β) such that

$$(\alpha(t), \beta(t)) = \begin{cases} (\frac{1}{2}H, Y) & \text{if } t \in (-\infty, 0], \\ (0, e^{-2t}\varepsilon) & \text{if } t \in [1, \infty), \end{cases} \tag{4.3.1}$$

where $\varepsilon \in \mathcal{O}$ is uniquely determined by the trajectory. The $U(1)$ in $Sp(1)$ which preserves the natural complex structure I on the space of equivalence classes of complex trajectories, acts on (α, β) as multiplication by $(0, e^{2i\theta})$.

Lemma 4.3.5. *The action of $U(1)$ on the trajectories given in* (4.3.1) *is given by $e^{i\theta} \cdot \varepsilon = e^{2i\theta}\varepsilon$.*

PROOF. We have $e^{i\theta} \cdot (\alpha, \beta) = (\alpha, e^{2i\theta}\beta)$, so it is sufficient to find a gauge transformation f such that

$$f(t) = \begin{cases} g & \text{if } t \in (-\infty, 0], \\ 1 & \text{if } t \in [1, \infty), \end{cases}$$

where $g \in G^{\mathbb{C}}$ is such that $\langle e^{-2i\theta}X, e^{2i\theta}Y, H\rangle = \operatorname{Ad} g\langle X, Y, H\rangle$. Since these two $\mathfrak{sl}(2, \mathbb{C})$-triplets have H in common, a result of Dynkin's shows that they are conjugate under the action of $G^{\mathbb{C}}$. In our case, we may take g to be $\rho \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$.

□

Let X_I denote the vector field generated by this $U(1)$-action, then IX_I is vector field generated by the $\mathbb{R}$-action

$$\lambda \cdot \varepsilon = \lambda^2 \varepsilon,$$

which is independent of the choice of complex structure I on $M(\rho)$. The results of the previous chapter now show the following.

Propoition 4.3.6. *The quotient of $\mathcal{O} \cong M(\rho)$ by $\mathbb{H}^*$ is a quaternionic Kähler manifold.* □

The above $\mathbb{R}$-action can be seen directly on $M(\rho)$ as reparameterization of the trajectories. If we map $t \mapsto t - \tau$ we obtain the trajectory

$$(\alpha(t), \beta(t)) = \begin{cases} (\frac{1}{2}H, Y) & \text{if } t \in (-\infty, \tau], \\ (0, e^{-2\tau} e^{2t} \varepsilon) & \text{if } t \in [1+\tau, \infty), \end{cases}$$

which agrees with (4.3.1) with ε replaced by $e^{2\tau}\varepsilon$ outside the compact interval $[\min\{0,\tau\}, \max\{1, 1+\tau\}]$.

The real structure on a twistor space $\mathbb{P}(\mathcal{O})$ is given by the action of j on $\mathcal{U}(M)$. Using the notation of the Lemma, the action of $j \in \mathbb{H}^*$ is given by $j \cdot (\alpha, \beta) = (\sigma\alpha, \sigma\beta)$ and using the gauge transformation f with $g = \rho(j) = \rho \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ shows that $j \cdot \varepsilon = \sigma\varepsilon$, where σ is the real structure on $\mathfrak{g}^{\mathbb{C}}$ defining $\mathfrak{g}$.

Given two homomorphisms ρ_+ and ρ_- of $\mathfrak{sp}(1)$ into $\mathfrak{g}$, let $M(\rho_-, \rho_+)$ denote the space of trajectories A of φ such that $\lim_{t\to\infty} A(t) = \rho_+$ and $\lim_{t\to-\infty} A(t)$ is conjugate to ρ_- under the action of G.

Proposition 4.3.7. i) *A trajectory of ψ on $\mathcal{F}$ between the images of homomorphisms V_{ρ_-} and V_{ρ_+} may be lifted to a trajectory in $M(\rho_-, \rho_+)$ and this lifting is unique upto the action of $\mathbb{H}^*$.*

ii) *For a trajectory* $A \in M(\rho_-)$, *the three-plane* $\lim_{t\to\infty} V_{A(t)}$ *is a well-defined element of* $\mathcal{F}$. *Moreover, if* $\lim_{t\to-\infty} A(t) = \rho_-$ *the projection of this trajectory may be extended to a trajectory between* V_{ρ_-} *and* V_{ρ_+}, *for some homomorphism* $\rho_+ : \mathfrak{sp}(1) \to \mathfrak{g}$ *and every point on this extended trajectory arises for some such* φ*-trajectory* A.

PROOF. To show the existence of a lifting, let $e_i(t)$ be a path of orthonormal bases for a ψ-trajectory $V(t)$ in $\mathcal{F}$ and define

$$A_i(t) = g(t)e_i(f(t)), \qquad \text{for } i = 1, 2, 3,$$

where g and f are arbitrary, differentiable, real-valued functions to be determined. For A to be a trajectory of φ, we require

$$\dot{A}_1 = \dot{g}g^{-1}A_1 - \dot{f}\psi A_1 - \dot{f}g^{-1}[A_2, A_3] = -2A_1 - [A_2, A_3], \quad \text{etc.}$$

If A_1, A_2, A_3 do not span the image of a homomorphism, then A_1 and $[A_2, A_3]$ are linearly independent, so

$$\dot{f} = g,$$
$$\dot{g}g^{-1} - \psi\dot{f} = -2.$$

Since ψ is bounded on the trajectory, this has the solution

$$g(t) = \frac{e^{-2t}}{k - \int_0^t \psi(\tau)e^{-2\tau}\,d\tau},$$
$$f(t) = \int_0^t \frac{e^{-2\sigma}}{k - \int_0^\sigma \psi(\tau)e^{-2\tau}\,d\tau}\,d\sigma + c,$$

for some constants k and c. Moreover, for an appropriate choice of k, $\lim_{t\to\infty} g(t)$ is non-zero.

To study the uniqueness of this lifting we look at reparameterizations of φ-trajectories. Let A be a φ-trajectory from ρ_- to ρ_+ and define $\tilde{A}(t)$ to be the path $g(t)h(t)A(f(t))$, where again f and g are real-valued, but now h takes values in the group $SO(3)$. For $\tilde{A}$ to be a φ-trajectory, we have that

$$((g\dot{h} + \dot{g}h) - 2gh\dot{f} + 2gh)A_1 - gh\dot{f}[A_2, A_3] + g^2[hA_2, hA_3] = 0, \quad \text{etc.}$$

Now $h \in SO(3)$ implies that $[hA_2, hA_3] = h[A_2, A_3]$, so if we are not at a homomorphism $\dot{f} = g$ and

$$gh^{-1}\dot{h} = (-\dot{g} + 2g\dot{f} - 2g)\,\mathrm{id}\,.$$

If $\dot{h}$ is non-zero then h must be proportional to the identity matrix, but this implies that $h \equiv 1$. So h is always a constant matrix and $\dot{g} - 2g\dot{f} + 2g = 0$. This equation is just the one obtained above, but with ψ replaced by 2. This now gives

$$g(t) = \frac{1}{1 + \lambda e^{2t}},$$
$$f(t) = -\tfrac{1}{2}\log|e^{-2t} + \lambda| + c,$$

where λ and c are constants.

Fix $h = 1$ and $c = 0$. If $\lambda = 0$, we just obtain the original trajectory. For $\lambda > 0$, $\tilde{A}(t) \to \rho_-$ as $t \to -\infty$. As $t \to \infty$, $\tilde{A}(t) \to 0$ but $f(t) \to -\frac{1}{2}\log\lambda$, so $V_{\tilde{A}(t)} \to V_{A(-\frac{1}{2}\log\lambda)}$.

If $\lambda < 0$, then g and f have singularities at $t = t_0 = -\frac{1}{2}\log(-\lambda)$ and the trajectory splits into two parts. For $t < t_0$, the limit as $t \to -\infty$ is still ρ_-, and in $\mathcal{F}$, as $t \nearrow t_0$, the three-plane $V_{\tilde{A}}$ approaches V_{ρ_+}. For $t > t_0$, we have an orientation reversed version of the case $\lambda > 0$.

Fix $\lambda = -1$, so that $t_0 = 0$. We claim that as $t \nearrow 0$, $\|\tilde{A}_i\|$ approaches a non-zero limit. Recall that

$$\tilde{A}_i(t) = \frac{1}{1-e^{2t}} A_i(-\tfrac{1}{2}\log(e^{-2t}-1)), \qquad \text{for } i = 1,2,3 \text{ and } t < 0.$$

Suppose $A_1(t) = e^{-2t}(\gamma + \varepsilon(t))$, where $\varepsilon(t) \to 0$ as $t \to \infty$ and γ is non-zero. Then, as $t \to 0$, we have $\tilde{A}_1 \to \gamma$, as required.

To show that A_1 has the right behaviour at infinity, recall that

$$\|A_1(t)\|^2 = Ce^{-4t} - 2e^{-4t}\int_T^t e^{4\tau}\phi(\tau)\,d\tau,$$

where $\phi = \langle A_1, [A_2, A_3]\rangle$ and $C = e^{4T}\|A_1(T)\|^2$. Now $\dot{\phi} = -6\phi - (\langle A_1, A_2\rangle^2 + \langle A_2, A_3\rangle^2 + \langle A_3, A_1\rangle^2) \leqslant -6\phi$, so $0 \leqslant \phi(t) \leqslant e^{-6t}k$, where $k = e^{6T}\phi(T)$ for $t > T$ and

$$Ce^{-4t} \geqslant \|A_1(t)\|^2 \geqslant (C-k)e^{-4t} + e^{-6t}k,$$

so we need only ensure that $C > k$, that is $e^{2T}\phi(T) < \|A_1\|^2(T)$. But $\phi \leqslant 2\|A_1\|^3$, so it is sufficient to have $2e^{2T}\|A_1(T)\| < 1$ and this is possible, since $\|A_1(T)\|$ is bounded as $T \to -\infty$.

It now only remains to show that if A is a φ-trajectory from ρ_- to zero, then the limit three-plane lies in $\mathcal{F}$. Fix attention on the isotropic element $\beta = A_2 + iA_3$. Let $\alpha = iA_1$. Now Kronheimer shows that there is a complex gauge transformation g such that on $[1,\infty)$, $g \cdot (\alpha, \beta) = (0, e^{-2t}\varepsilon)$, for some ε in the nilpotent orbit given by ρ_-. In particular, $e^{2t}\beta = \mathrm{Ad}(g^{-1})\varepsilon$, so we obtain a nilpotent element in the limit. □

The manifold $M(\rho_-, \rho_+)$ is a transverse slice to the orbit given by ρ_+ through the orbit associated to ρ_- (Slodowy, 1980) and Kronheimer shows this is a hyper-Kähler manifold. In $\mathcal{F}$, this slice is just seen as all the points on trajectories to V_{ρ_+} from the real orbit of V_{ρ_-}, together with the points of the V_{ρ_-}-orbit.

4.4 Examples

The classification of nilpotent orbits in the complexification of a classical Lie algebra is just the classification of the Jordan forms of matrices which arise and so the orbits are classified by various partitions of n, for some fixed integer n. The dimension of an orbit may be calculated directly from the partitions (Springer & Steinberg, 1970) and, using the results of Dynkin (1952) and Bala & Carter (1974), for the first few semi-simple Lie algebras, we obtain the nilpotent orbits of the dimensions given in Table 1 (see also Carter, 1985). Unless otherwise stated, by 'dimension' we mean 'real dimension.'

Lie algebra	Dimension	Real Dimensions of Nilpotent Orbits
$\mathfrak{so}(3) = \mathfrak{su}(2) = \mathfrak{sp}(1)$	3	4
$\mathfrak{so}(4) = \mathfrak{sp}(1) + \mathfrak{sp}(1)$	6	8, 4, 4
$\mathfrak{su}(3)$	8	12, 8
$\mathfrak{so}(5) = \mathfrak{sp}(2)$	10	16, 12, 8
$\mathfrak{g}_2$	14	24, 20, 16, 12
$\mathfrak{so}(6) = \mathfrak{su}(4)$	15	24, 20, 16, 12
$\mathfrak{sp}(3)$	21	36, 32, 28, 28, 24, 20, 12
$\mathfrak{so}(7)$	21	36, 32, 28, 24, 20, 16
$\mathfrak{su}(5)$	24	40, 36, 32, 28, 24, 16
$\mathfrak{sp}(4)$	36	64, 60, 56, 56, 52, 48, 48, 44, 40, 40, 36, 28, 16

Table 1.

Dimensions of nilpotent orbits of some semi-simple Lie algebras.

Four-dimensional Models. If we regard nilpotent orbits as models for associated bundles of quaternionic Kähler manifolds, then in 4-dimensions we have the highest root orbits of $\mathfrak{su}(3)$ and $\mathfrak{sp}(2)$ and the regular orbit in $\mathfrak{so}(4)$. The first two are just the associated bundles of the Wolf spaces $\mathbb{C}\mathrm{P}(2)$ and $\mathbb{H}\mathrm{P}(1)$, respectively. Hitchin (1979) showed that these were the only two complete, self-dual Einstein manifolds with positive scalar curvature. The regular orbit (the unique nilpotent orbit of highest dimension) in $\mathfrak{so}(4)$ is just the product of the highest root orbits of each $\mathfrak{sp}(1)$ factor, and so it is the associated bundle for the quaternionic join of two points, which is an incomplete manifold. This orbit is a finite quotient of the highest root orbit of $\mathfrak{sp}(2)$. More generally, we may describe the regular orbit of $\mathfrak{sp}(p_1) \oplus \cdots \oplus \mathfrak{sp}(p_r)$ as a finite quotient of an open subset of the highest root orbit of $\mathfrak{sp}(n)$, where $\sum_{i=1}^{r} p_i = n$. The highest root orbit in $\mathfrak{sp}(n, \mathbb{C}) \cong S^2\mathbb{C}^{2n}$ consists of those elements v^2, where $v \in \mathbb{C}^{2n}$. Write $v = v_{p_1} + \cdots + v_{p_r}$ corresponding to the splitting $\mathbb{C}^{2n} = \mathbb{C}^{2p_1} + \cdots + \mathbb{C}^{2p_r}$, then the map to $\mathfrak{sp}(p_1) \oplus \cdots \oplus \mathfrak{sp}(p_r)$ is given by

$$v^2 \mapsto v_{p_1}^2 + \cdots + v_{p_r}^2.$$

Restricting to the open set consisting of v such that no component v_{p_i} is zero gives the regular orbit as a $\mathbb{Z}_2^{r-1}$-quotient and shows that the quaternionic Kähler manifold obtained is

$$\mathbb{H}\mathrm{P}(n-1) \setminus \bigcup_i \mathbb{H}\mathrm{P}(n-p_i-1) \Big/ \mathbb{Z}_2^{r-1}.$$

This space is just the quaternionic join $\mathcal{J}(\mathbb{H}\mathrm{P}(p_1), \ldots, \mathbb{H}\mathrm{P}(p_r))$. Poon (1986) showed that the projectivised regular orbit of $\mathfrak{so}(4, \mathbb{C})$ could be desingularised in such a way as to give a self-dual structure on the connected sum $\mathbb{C}\mathrm{P}(2) \# \mathbb{C}\mathrm{P}(2)$. Donaldson & Friedman (1989) extended this result to define self-dual metrics on other connected

sums of self-dual manifolds. This example suggests that one should be able to extend their result to construct quaternionic metrics on the connected sums of quaternionic manifolds.

Eight-dimensional Models. We have the following. Firstly, the highest root orbits for $\mathfrak{g}_2$, $\mathfrak{su}(4)$ and $\mathfrak{sp}(3)$ and the regular orbits of $\mathfrak{sp}(1)\oplus\mathfrak{sp}(1)\oplus\mathfrak{sp}(1)$ and $\mathfrak{su}(3)\oplus\mathfrak{sp}(1)$ giving the Wolf spaces $G_2/SO(4)$, $\mathrm{Gr}_2(\mathbb{C}^4)$ and $\mathbb{H}\mathrm{P}(2)$ and the incomplete spaces $\mathbb{H}\mathrm{P}(2)\setminus 3\,\mathbb{H}\mathrm{P}(1)/\mathbb{Z}_2\times\mathbb{Z}_2$ and $\mathcal{U}(\mathbb{C}\mathrm{P}(2))$. Next there is the subregular orbit (the unique orbit of codimension 4 in the nilpotent variety) of $\mathfrak{sp}(2,\mathbb{C})\cong\mathfrak{so}(5,\mathbb{C})$. Because of the isomorphism between these algebras, the description of this orbit leads to two different generalisations.

Proposition 4.4.1. i) *In* $\mathfrak{sp}(n,\mathbb{C})$, *one of the nilpotent orbits of dimension* $8n-4$ *is double covered by an open subset of the highest root orbit in* $\mathfrak{su}(2n,\mathbb{C})$. *The quaternionic Kähler manifold obtained from this* $Sp(n,\mathbb{C})$*-orbit is* $\mathrm{Gr}_2(\mathbb{C}^{2n})\setminus\mathbb{H}\mathrm{P}(n-1)/\mathbb{Z}_2$.

ii) *In* $\mathfrak{so}(q,\mathbb{C})$, *one of the nilpotent orbits of dimension* $4q-8$ *is double covered by an open subset of the highest root orbit in* $\mathfrak{so}(q+1,\mathbb{C})$. *The quaternionic Kähler manifold obtained from this* $SO(q,\mathbb{C})$*-orbit is* $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{q+1})\setminus\widetilde{\mathrm{Gr}}_4(\mathbb{R}^q)/\mathbb{Z}_2$.

PROOF. i) The Lie algebra $\mathfrak{su}(m,\mathbb{C})$ consists of the trace-free elements of $\mathrm{End}(\mathbb{C}^m)$ and the highest root orbits contains those elements of the form $a\otimes\bar{b}$, for some $a,b\in\mathbb{C}^m$. If we identify $\mathbb{C}^{2n}$ and $\mathbb{H}^n$, then there is a map from the highest root orbit of $\mathfrak{su}(2n,\mathbb{C})$ to $\mathfrak{sp}(n,\mathbb{C})\cong S^2\mathbb{C}^{2n}$ given by $a\otimes\bar{b}\mapsto a\vee jb$. Generically, this map has kernel $\mathbb{Z}_2$. Its image consists of all elements of the form $e\vee f$, where $e,f\in\mathbb{C}^{2n}$ and, if ω is the symplectic form, then $\omega(e,f)=0$. One of the consequences of the results of Springer & Steinberg (1970) is that the nilpotent orbits of $\mathfrak{sp}(n,\mathbb{C})$ are obtained as the intersection of $\mathfrak{sp}(n,\mathbb{C})$ with the nilpotent orbits of $\mathfrak{sl}(2n,\mathbb{C})$. Thus

studying the Jordan forms of elements shows that the image splits as a disjoint union of two orbits, one is the highest root orbit of $\mathfrak{sp}(n,\mathbb{C})$ and the other is an orbit of $\mathfrak{sp}(n,\mathbb{C})$ of dimension $8n-4$.

On the level of quaternionic Kähler manifolds, the $\mathbb{Z}_2$-action is the action of j on $\mathrm{Gr}_2(\mathbb{C}^{2n})$, where $\mathbb{C}^{2n} \cong \mathbb{H}^n$; so the fixed point set is $\mathbb{H}\mathrm{P}(n-1)$ and the incomplete manifold obtained is $\mathrm{Gr}_2(\mathbb{C}^{2n}) \setminus \mathbb{H}\mathrm{P}(n-1)/\mathbb{Z}_2$.

ii) The Lie algebra of $\mathfrak{so}(q)$ is $\Lambda^2\mathbb{C}^q$, so there is map from $\mathfrak{so}(q+1)$ to $\mathfrak{so}(q)$ induced by projection. In particular,

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 & i \\ 0 & 0 & \dots & 0 & -i & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ -1 & i & \dots & 0 & 0 & 0 \\ -i & -1 & \dots & 0 & 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & -i \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & i & \dots & 0 & 0 \end{pmatrix}.$$

The first matrix is an element of the highest root orbit of $\mathfrak{so}(q+1)$ and its image is a nilpotent matrix in $\mathfrak{so}(q)$ which lies in an orbit of dimension $4q-8$ (using the results of Springer & Steinberg, 1970). Away from the highest root orbit of $\mathfrak{so}(q)$ this map has kernel $\mathbb{Z}_2$. The quaternionic manifold obtained is $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{q+1}) \setminus \widetilde{\mathrm{Gr}}_4(\mathbb{R}^q)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $\mathbb{R}^{q+1} = \mathbb{R}^q \oplus \mathbb{R}$ as $(1,-1)$. Note that the nilpotent elements obtained are those which may be written in the form $a \wedge b$, with $g(a,a) = 0 = g(a,b)$, since the highest root orbit of $\mathfrak{so}(q+1,\mathbb{C})$ consists of elements $c \wedge d$, with $g(c,c) = g(d,d) = 0 = g(c,d)$. □

Our final model for a quaternionic Kähler 8-manifold is the regular orbit of $SU(3)$. Now $\mathfrak{su}(3)$ is a subalgebra of $\mathfrak{g}_2$: let $(0,1)$ be the highest root of $\mathfrak{g}_2^{\mathbb{C}}$ and suppose $(1,0)$ is a short root such that the long roots of $\mathfrak{g}_2$ are $\pm(0,1), \pm(3,2)$,

$\pm(3,1)$, and the short roots are $\pm(1,0)$, $\pm(1,1)$, $\pm(2,1)$; then $\mathfrak{su}(3,\mathbb{C})$ is the subalgebra generated by the long roots. The Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ decomposes under the action of $SU(3)$ as

$$\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{su}(3,\mathbb{C}) \oplus \Lambda^{1,0}\mathbb{C}^3 \oplus \Lambda^{0,1}\mathbb{C}^3.$$

The space $G_2/SU(3)$ is 3-symmetric; the symmetry σ comes from the centre $\mathbb{Z}_3$ of $SU(3)$ and acts on this decomposition of $\mathfrak{g}_2^{\mathbb{C}}$ as $(1, e^{2\pi i/3}, e^{-2\pi i/3})$. The projection of the highest root orbit of $\mathfrak{g}_2^{\mathbb{C}}$ contains elements of both the nilpotent orbits of $\mathfrak{su}(3,\mathbb{C})$ and so counting dimensions, the image must be the whole nilpotent variety of $\mathfrak{su}(3,\mathbb{C})$. To show that both orbits meet the image, we argue as follows. We have a subgroup $SO(4) = Sp(1)_+ Sp(1)_-$ of G_2 and Salamon (1987) shows that the Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ decomposes under $SO(4)$ as

$$\mathfrak{g}_2^{\mathbb{C}} = S^2V_+ \oplus S^2V_- \oplus V_+S^3V_-.$$

If we take $\mathfrak{sp}(1)_+$ to be the span of $E_{(0,1)}, H_{(0,1)}, E_{-(0,1)}$, where H_α, E_λ is a Cartan basis for $\mathfrak{g}_2^{\mathbb{C}}$, then $\mathfrak{sp}(1)_-$ is spanned by $E_{(2,1)}, H_{(2,1)}, E_{-(2,1)}$ and the remaining roots span $V_+S^3V_-$. Now, for definiteness, identify $E_{(0,1)}$ and $E_{(3,1)}$ with the matrices $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This gives $[E_{(0,1)}, E_{(3,1)}] = E_{(3,2)}$ is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and so $E_{(0,1)} + E_{(3,2)}$ is a highest root vector of $\mathfrak{su}(3)$ and hence of $\mathfrak{g}_2$. Consider the orbit of this element under $Sp(1)_-$. The first component lies in $\mathfrak{sp}(1)_+$ so it is $Sp(1)_-$-invariant. The other component lies in the $Sp(1)_-$-module S^3V_- spanned by $E_{(3,2)}, E_{(1,1)}, E_{(-1,0)}, E_{(-3,-1)}$. In particular, there is an element in the $Sp(1)_-$-orbit of $E_{(0,1)} + E_{(3,2)}$ which projects to $E_{(0,1)} + aE_{(3,2)} + bE_{(-3,-1)}$ with both a and b non-zero. Now this element lies in the regular orbit of $\mathfrak{su}(3,\mathbb{C})$ and is the projection of a highest root element of $\mathfrak{g}_2^{\mathbb{C}}$, as required.

Quaternionically, we obtain the following. Let $1, i, j, k, e, ie, je, ke$ be a basis of $\mathbb{O}$, then the Wolf space $G_2/SO(4)$ is the space of quaternionic lines in $\mathbb{O}$. The action of $\sigma \in \mathbb{Z}_3$ on $\mathbb{O}$ is by right multiplication by $e^{2\pi e/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}e$. This is an element of G_2, since

$$(\sigma i)(\sigma j) = \frac{k}{4} - \frac{\sqrt{3}}{4}(ie)j - \frac{\sqrt{3}}{4}i(je) + \frac{3}{4}(ie)(je) = -\frac{k}{2} + \frac{\sqrt{3}}{2}(ke) = \sigma(k)$$

$$\text{and} \qquad \sigma(i)\sigma(e) = \left(-\frac{i}{2} + \frac{\sqrt{3}}{2}ie\right)e = -\frac{ie}{2} - \frac{\sqrt{3}}{2}i = \sigma(ie).$$

The fixed point set of σ consists of those quaternionic lines which are spanned by 1, e and a complex line in $\mathbb{C}^3 = \langle i, j, k, ie, je, ke\rangle$, which is isomorphic to $\mathbb{C}\mathrm{P}(2)$.

Proposition 4.4.2. *There is an open set in the highest root orbit of* $\mathfrak{g}_2^{\mathbb{C}}$ *which is a three-fold cover of the regular orbit in* $\mathfrak{su}(3,\mathbb{C})$. *The quaternionic Kähler manifold associated to this* $\mathfrak{su}(3)$*-orbit is* $G_2 \setminus \mathbb{C}\mathrm{P}(2)/\mathbb{Z}_3$. □

Thus we have the following models for eight-manifolds:

$$\mathbb{H}\mathrm{P}(2), \qquad \mathrm{Gr}_2(\mathbb{C}^4), \qquad \frac{G_2}{SO(4)}, \qquad \mathcal{U}(\mathbb{C}\mathrm{P}(2)),$$

$$\frac{\mathbb{H}\mathrm{P}(2) \setminus 3\,\mathbb{H}\mathrm{P}(1)}{\mathbb{Z}_2 \times \mathbb{Z}_2}, \qquad \frac{\mathrm{Gr}_2(\mathbb{C}^4) \setminus \mathbb{H}\mathrm{P}(1)}{\mathbb{Z}_2} \quad \text{and} \quad \frac{G_2/SO(4) \setminus \mathbb{C}\mathrm{P}(2)}{\mathbb{Z}_3}.$$

These are exactly the models for quaternionic Kähler 8-manifolds with positive scalar curvature obtained by Poon & Salamon (1989), who showed that the only such complete 8-manifolds are the Wolf spaces.

Higher-dimensional Quotients. The hyperKähler quotients of $\mathbb{H}\mathrm{P}(n-1)$ by $U(1)$ and $Sp(1)$ described by Galicki may be understood in terms of Lie algebras

as follows. The normaliser of $U(1)$ in $Sp(n)$ is $U(n)$. Let v be an element of $\mathbb{H}^n$. Corresponding to $\mathbb{H}^n = \Lambda^{1,0}\mathbb{C}^n \oplus \Lambda^{0,1}\mathbb{C}^n = \mathbb{C}^n \oplus \overline{\mathbb{C}^n}$, write $v = a + \bar{b}$. Now

$$\mathfrak{sp}(n,\mathbb{C}) = S^2\mathbb{C}^{2n} = \mathfrak{u}(n,\mathbb{C}) \oplus \mathfrak{m} = \Lambda^{1,1} \oplus (S^{2,0} \oplus S^{0,2}),$$

and, correspondingly, we have

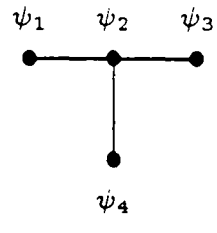
$$v^2 = a \otimes \bar{b} + (a^2 + \bar{b}^2).$$

The orbit of $a \otimes \bar{b} \in \mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u}(n,\mathbb{C})$ is $4(n-1)$-dimensional and these orbits are parameterised by $\mathrm{Tr}(a \otimes \bar{b}) = g(a,\bar{b})$. So the image of $v^2 \mapsto a \otimes \bar{b}$ is $(4n-2)$-dimensional. Restricting to nilpotent elements in the image is equivalent to imposing the condition $g(a,\bar{b}) = 0$ and we obtain the highest root orbit of $\mathfrak{su}(n,\mathbb{C}) \subset \mathfrak{u}(n,\mathbb{C})$. Thus, $g(a,\bar{b})$, which is the component of v^2 in $\mathfrak{u}(1,\mathbb{C}) \subset \mathfrak{u}(n,\mathbb{C})$, gives the complex moment map. Similarly, the normaliser of $Sp(1)$ in $Sp(n)$ is $SO(n)$. The Lie algebra splits as

$$\mathfrak{sp}(n) = \mathfrak{so}(n,\mathbb{C}) \oplus \mathfrak{sp}(1,\mathbb{C}) \otimes (S^2_0\mathbb{R} + \mathbb{R})$$

and the projection to $\mathfrak{sp}(1,\mathbb{C})$ gives the complex moment map. This enables us to give another description of the orbit of $\mathfrak{so}(q,\mathbb{C})$ discussed above. Consider $\mathbb{C}^{q+2} = \mathbb{C}^q + \mathbb{C}^2$. Then, as before, the map from the highest root orbit of $\mathfrak{so}(q+2,\mathbb{C})$ to $\mathfrak{so}(q,\mathbb{C})$ induced by projection onto $\mathbb{C}^q$, contains the highest root orbit and an orbit of dimension $4q-8$. A typical element of the highest root orbit of $\mathfrak{so}(q+2,\mathbb{C})$ is $a \wedge b = (a_q, a_2) \wedge (b_q, b_2)$, where $g(a,a) = g(b,b) = 0 = g(a,b)$. Regarding projection to $\Lambda^2\mathbb{C}^2 \cong \mathfrak{so}(2,\mathbb{C})$ as a complex moment map for the action of $U(1) = SO(2)$, any zero of this map in the highest root orbit may be put in the form $(a_q, a_2) \wedge (b_q, 0)$ and so the $U(1)$-quotient consists precisely of the two orbits above. Thus, the quaternionic Kähler quotient of $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{q+2})$ by $U(1)$ acting on the $\mathbb{R}^2$-factor as $SO(2)$ is $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{q+1})/\mathbb{Z}_2$.

The Algebra $\mathfrak{g}_2$. One might hope to describe the 16-dimensional orbit of $\mathfrak{g}_2^{\mathbb{C}}$ in terms of the highest root orbit of $\mathfrak{so}(7)$. Recall that $SO(8)/G_2$ is a 3-symmetric space. Let $\psi_1, \ldots, \psi_4$ be a set of simple roots for $\mathfrak{so}(8)$.



Let h_{ψ_i}, e_λ be the corresponding Weyl basis for $\mathfrak{so}(8,\mathbb{C})$. The description of the inclusion of $\mathfrak{g}_2$ in $\mathfrak{so}(7)$ given by Gray & Wolf (1968) is as follows. Define

$$a_\pm = e_{\pm\psi_1} + e_{\pm\psi_3} + e_{\pm\psi_4}$$

$$b_\pm = e_{\pm(\psi_1+\psi_2)} + e_{\pm(\psi_3+\psi_2)} + e_{\pm(\psi_4+\psi_2)}$$

$$c_\pm = e_{\pm(\psi_1+\psi_3+\psi_2)} + e_{\pm(\psi_3+\psi_4+\psi_2)} + e_{\pm(\psi_4+\psi_1+\psi_2)}$$

and if $\alpha = \alpha_1+\alpha_2+\alpha_3$, let $\varepsilon = e^{2\pi i/3}$, $\alpha' = \alpha_1+\varepsilon\alpha_2+\varepsilon^2\alpha_3$ and $\alpha'' = \alpha_1+\varepsilon^2\alpha_2+\varepsilon\alpha_3$. Then

$$\mathfrak{g}_2^{\mathbb{C}} = \langle h_{\psi_1} + h_{\psi_3} + h_{\psi_4}, h_{\psi_2}, e_{\pm\psi_2}, e_{\pm(\psi_1+\psi_2+\psi_3+\psi_4)}, e_{\pm(\psi_1+2\psi_2+\psi_3+\psi_4)}, a_\pm, b_\pm, c_\pm \rangle.$$

Also, if we define

$$\mathfrak{g}_2(\varepsilon) = \langle h_{\psi_1} + \varepsilon^2 h_{\psi_3} + \varepsilon h_{\psi_4}, a''_\pm, b''_\pm, c''_\pm \rangle$$

$$\mathfrak{g}_2(\varepsilon^2) = \langle h_{\psi_1} + \varepsilon h_{\psi_3} + \varepsilon^2 h_{\psi_4}, a'_\pm, b'_\pm, c'_\pm \rangle$$

then

$$\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathfrak{g}_2(\varepsilon) \oplus \mathfrak{g}_2(\varepsilon^2),$$

and

$$\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathbb{R}^7,$$

where

$$\mathbb{R}^7 = \langle \tfrac{1}{2}(h_{\psi_3} + h_{\psi_4}), \tfrac{1}{\sqrt{2}}(e_{\pm\psi_3} + e_{\pm\psi_4}), \\ \tfrac{1}{\sqrt{2}}(e_{\pm(\psi_3+\psi_2)} + e_{\pm(\psi_4+\psi_2)}), \tfrac{1}{\sqrt{2}}(e_{\pm(\psi_1+\psi_4+\psi_2)} + e_{\pm(\psi_1+\psi_3+\psi_2)})\rangle.$$

The inclusion $\mathfrak{g}_2 \hookrightarrow \mathfrak{so}(7)$ induces a projection $\mathfrak{so}(7) \to \mathfrak{g}_2$ via the dual map and the Killing forms. Levasseur & Smith (1988) show that the image of highest root orbit contains the orbit of short roots (which contains e_{ψ_2}), which is the 16 dimensional orbit we are interested in, and that this map is a bijection on the closures of these orbits. They also show that the 20-dimensional orbit arises in the image of the highest root orbit of $\mathfrak{so}(8)$.

The above links between nilpotent orbits in low dimensions are summarised in Diagram 1.

Singularities in the Nilpotent Variety. The nilpotent orbits of a simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ may be partially ordered by $\mathcal{O}_1 \prec \mathcal{O}_2$ if and only if $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$. Diagrams showing this partial order for the exceptional groups and for some of the smaller classical groups may be found in Carter (1985). The nilpotent variety of $\mathfrak{g}^{\mathbb{C}}$ contains a unique open dense orbit, the regular orbit, and a unique orbit of codimension 4, the subregular orbit. If ρ and ρ' are homomorphisms $\mathfrak{su}(2) \to \mathfrak{g}$ corresponding to the regular and subregular orbits respectively, then Slodowy (1980) defines a slice S transverse to the subregular orbit by taking S to be the intersection of $S(\rho')$ with the nilpotent variety. Here $S(\rho') = \rho'\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathfrak{z}\rho'\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, where $\mathfrak{z}$ denotes the centraliser in $\mathfrak{g}^{\mathbb{C}}$. The transverse slice S is biholomorphic to $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$ (that is, Γ is either cyclic, dihedral of order $4r$ for some r,

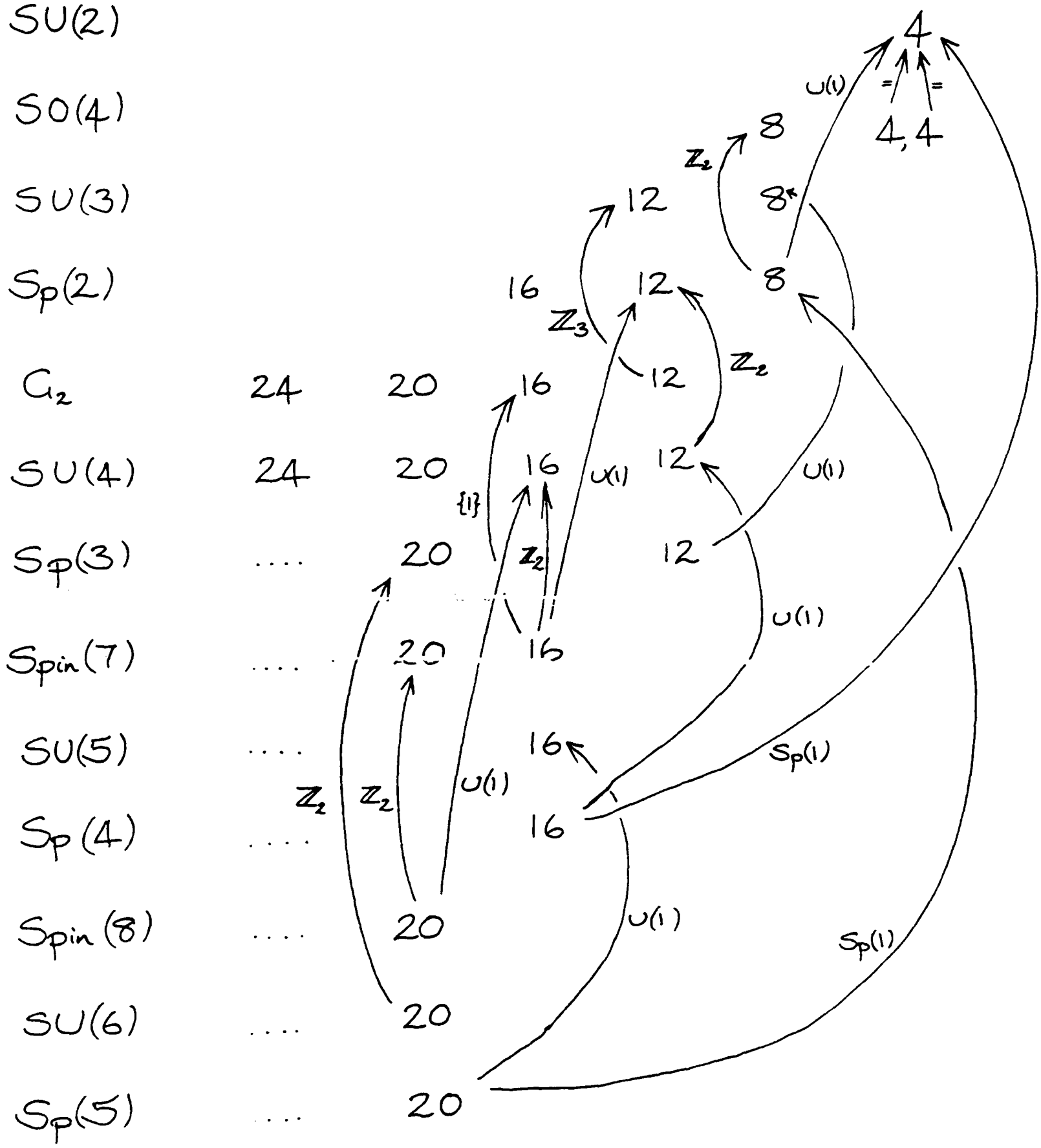


Diagram 1

HyperKähler Quotients of Small Nilpotent Orbits

Arrows are labelled by the group and the quotient is the closure of the indicated orbit.

binary tetrahedral, binary octahedral or binary icosahedral). This was conjectured by Grothendieck and proved by Brieskorn (1970). Kronheimer (1988) shows that this slice is actually hypercomplex. This result was extended by Kraft & Procesi (1982) in the case of classical groups.

Proposition 4.4.3. *If $\mathfrak{g}$ is a classical simple Lie algebra, then for a minimal degeneration (that is, if the two orbits considered are adjacent in the partial order) the singularity obtained is either of the form described above (i.e. $\mathbb{C}^2/\Gamma$), or two copies of $\mathbb{C}^2/\mathbb{Z}_k$, for some k, or the transverse slice is smoothly equivalent to a highest root orbit for some simple, complex Lie algebra.* □

This result was also proved in the case of G_2 by Kraft (1988) and indicates that a quaternionic description of these singularities should be possible.

Bundles Over Real Orbits. The second section of this chapter gives an explicit description of the normal bundle of $G/N(Sp(1))$ in the quaternionic Kähler manifold associated to the corresponding nilpotent orbit. One could try and construct quaternionic Kähler metrics on these bundles, perhaps using the type of techniques seen in Chapter 2. For the regular orbit in $\mathfrak{su}(3)$, the bundle is S^2 and the tangent space of this bundle splits as $S^4 + S^2$ under the action of $Sp(1)$. The closure of this bundle in $\mathcal{F}$ contains the Wolf space $\mathbb{C}\mathrm{P}(2)$ and we have already seen that adding this space to a triple cover of the bundle gives the quaternionic Kähler manifold $G_2/SO(4)$.

Similarly, the orbit of short roots in $\mathfrak{g}_2^{\mathbb{C}}$ is a bundle over $G_2/SO(4)$ whose tangent space splits as $2S^3 + 2S^1$. This time the closure contains another copy of $G_2/SO(4)$ (this comes from the orbit of the long roots) and the completed manifold is $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$. Regarding this manifold as $\widetilde{\mathrm{Gr}}_3(\mathbb{R}^7)$, it is natural to conjecture that

$\frac{1}{\sqrt{2}}\psi$ coincides with the standard G_2-calibration on $\widetilde{\mathrm{Gr}}_3(\mathbb{R}^7)$ (see the discussion of exceptional geometries in the next chapter for the definition of this calibration). The two copies of $G_2/SO(4)$ would then consist of the associative and anti-associative 3-planes in $\mathbb{R}^7 \cong \mathrm{Im}\,\mathbb{O}$.

One might hope to use this type of procedure to construct other complete quaternionic Kähler manifolds which may not be Wolf spaces. For example, we have not yet given a description of the quaternionic Kähler manifold associated to the regular orbit of $\mathfrak{sp}(2,\mathbb{C})$. This is the bundle S^4 over $SO(5)/SO(3)$ and the closure is obtained by adjoining $\mathrm{Gr}_2(\mathbb{C}^4)/\mathbb{Z}_2$. It is natural to ask whether there is a complete 12-dimensional quaternionic Kähler manifold with an open subset covering this manifold.

Chapter 5
DIFFERENTIAL FORMS

This chapter is more in the nature of an appendix. The main purpose is to show that a quaternion-Hermitian manifold, of dimension at least 12, with a closed fundamental 4-form is quaternionic Kähler. (This result was announced in Swann, 1989.) The proof uses various techniques from representation theory which are discussed in the first section of this chapter whilst decomposing part of the exterior algebra of a quaternion-Hermitian manifold. Any notions not defined there may be found in Bröcker & tom Dieck (1985). The main result is then proved in the second section and the last section is devoted to discussion of related topics in exceptional Riemannian geometry.

5.1 Representation Theory and Exterior Algebras

Recall that a quaternion-Hermitian manifold M is a $4n$-manifold with structure group $Sp(n)\,Sp(1)$. As remarked in Chapter 2, we may associate a bundle over M to each representation of $Sp(n)\,Sp(1)$. In particular, $T_{\mathbb{C}}M$ is the bundle $E \otimes H$, where E and H are the basic representations of $Sp(n)$ and $Sp(1)$, respectively, described in Chapter 2. Now $Sp(n)\,Sp(1)$ is a subgroup of $SO(4n)$, in fact, Gray (1969) shows that it is a maximal subgroup if $n > 1$ (if $n = 1$, then $Sp(1)\,Sp(1) = SO(4)$). Thus, by restriction, any $SO(4n)$-module decomposes as a direct sum of irreducible $Sp(n)\,Sp(1)$-modules. In particular, $\Lambda^p_{\mathbb{C}}T^*M$ decomposes under the action of $Sp(n)\,Sp(1)$. The $Sp(n)\,Sp(1)$-modules are the $Sp(n) \times Sp(1)$-modules in

which $(-1,-1)$-acts as the identity. Any irreducible $Sp(n) \times Sp(1)$-module is just a tensor product of an irreducible $Sp(n)$-module and an irreducible $Sp(1)$-module. The irreducible representations of $Sp(1) \cong SU(2)$ are the symmetric powers $S^r H$ of the basic $Sp(1)$-module H.

The irreducible $Sp(n)$-modules are determined by their dominant weights $\lambda = (\lambda_1, \ldots, \lambda_n)$, where λ_i are positive integers with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ (see Bröcker & tom Dieck, 1985). We write $V^{(\cdots)}$ for the irreducible $Sp(n)$-module of dominant weight $(\ldots)$. In particular, we have

$$V^{(r0\ldots)} \cong S^r E,$$

$$V^{(1\ldots10\ldots)} \cong \Lambda_0^r E,$$

where $(1\ldots10\ldots)$ has r ones and $\Lambda_0^r E$ is the $Sp(n)$-invariant complement to the subspace $\{\omega\}\Lambda^{r-2}E$ of $\Lambda^r E$, where ω is the symplectic form preserved by $Sp(n)$. We will also write K for $V^{(210\ldots)}$ which is a submodule of $E \otimes \Lambda_0^2 E \cong K \oplus E$.

The Weyl group $\mathbf{W}$ of $Sp(n)$ is $\mathbb{Z}_2 \wr S_n$ and this acts on weights λ by permuting entries and flipping signs of entries. The weights of $Sp(n)$ can be partially ordered as follows: for each weight γ there is an element w of $\mathbf{W}$ such that $\lambda = w\gamma$ has entries λ_i satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$; now $\gamma \preccurlyeq \gamma'$ if for the corresponding λ and λ' we have $\lambda_i \leqslant \lambda'_i$ for all i. To calculate the character of the $Sp(n)$-module of dominant weight λ we use the following result.

Proposition 5.1.1. (Kostant) *Let W be an irreducible representation of a compact Lie group G with dominant weight λ and suppose $\lambda' \prec \lambda$. Then the multiplicity $m(\lambda', \lambda)$ of λ' as a weight of W is given by*

$$m(\lambda', \lambda) = -\sum_{w \in \mathbf{W} \setminus \{1\}} \det(w)\, m(\lambda' + \varrho - w(\varrho), \lambda),$$

where ϱ is half the sum of the positive roots. □

For $Sp(n)$ the element ϱ is $(n, n-1, \ldots, 1)$. The results of some of these calculations will be given in Table 2.

Theorem 5.1.2. *On a quaternion-Hermitian manifold M of dimension $4n$ we have*

$$T^*_{\mathbb{C}}M \cong E \otimes H,$$
$$\Lambda^2_{\mathbb{C}}T^*M \cong \Lambda^2_0 E \otimes S^2H + S^2H + S^2E.$$

If M has dimension at least 8, then

$$\Lambda^3_{\mathbb{C}}T^*M \cong (\Lambda^3_0 E + E) \otimes S^3H + (V^{(21)} + E) \otimes H,$$
$$\Lambda^4_{\mathbb{C}}T^*M \cong (\Lambda^4_0 E + \Lambda^2_0 E + \mathbb{R}) \otimes S^4H + (V^{(211)} + S^2E + \Lambda^2_0 E) \otimes S^2H + V^{(22)} + \Lambda^2_0 E + \mathbb{R}.$$

Also,

$$\Lambda^5_{\mathbb{C}}T^*M \cong \begin{cases} (\Lambda^5_0 E + \Lambda^3_0 E + E) \otimes S^5H & \\ \quad +(V^{(2111)} + V^{(21)} + \Lambda^3_0 E + E) \otimes S^3H & \\ \quad +(V^{(221)} + V^{(21)} + \Lambda^3_0 E + E) \otimes H & \text{if } \dim M \geqslant 12, \\ & \\ E \otimes S^3H + (V^{(21)} + E) \otimes H & \text{if } \dim M = 8. \end{cases}$$

(Here any module of length greater than n is to be regarded as the zero-module $\{0\}$ and $\mathbb{R}$ denotes $\mathbb{C}$ together with a real structure.)

PROOF. The decompositions upto Λ^4 have been given by Salamon (1989), so we will concentrate on the decomposition of Λ^5.

We will take $n \geqslant 5$ to avoid special cases, but the results of the corresponding calculations in those other cases will be included in Table 2.

We compute with characters. During this proof, m runs over $\{1, 2, \ldots, n\}$ and $\alpha, \beta, \gamma, \delta, \epsilon$ run over $\{\pm 1, \pm 2, \ldots, \pm n\}$. Let $t = \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_1}, \ldots, e^{-i\theta_n})$, $t' = \operatorname{diag}(e^{i\theta}, e^{-i\theta})$ be diagonal matrices representing typical elements of the maximal tori of $Sp(n)$ and $Sp(1)$ respectively. Let χ_V denote the character of V. Then

$$\chi_{\Lambda^p_{\mathbb{C}} T^*M}(t \otimes t') = \sigma_p(e^{\pm i\theta_m \pm i\theta}),$$

where σ_p is the pth elementary symmetric polynomial (see Adams, 1967). This immediately gives the decomposition of Bonan (1982)

$$\Lambda^r_{\mathbb{C}} T^*M \cong \Lambda^r ES^r H \oplus \bigoplus_{1 \leqslant s \leqslant \left[\frac{r}{2}\right]} V^r_s S^{r-2s} H,$$

where V^r_s is are $Sp(n)$-modules. In particular, this gives

$$\Lambda^5_{\mathbb{C}} T^*M \cong \Lambda^5 ES^5 H + V_1 S^3 H + V_2 H.$$

If we write $\theta_{-a} = -\theta_a$, then explicitly we have

$$\begin{aligned}
\chi_{\Lambda^5_{\mathbb{C}} T^*M}(t \otimes t') = &\sum_{\alpha<\beta<\gamma<\delta<\epsilon} e^{i(\theta_\alpha+\theta_\beta+\theta_\gamma+\theta_\delta+\theta_\epsilon)} \\
&\qquad \times \left\{e^{5i\theta} + e^{-5i\theta} + 5(e^{3i\theta} + e^{-3i\theta}) + 10(e^{i\theta} + e^{-i\theta})\right\} \\
&+ \sum_{\substack{\gamma<\delta<\epsilon \\ \gamma,\delta,\epsilon \neq \beta}} e^{i(2\theta_\beta+\theta_\gamma+\theta_\delta+\theta_\epsilon)} \left\{e^{3i\theta} + e^{-3i\theta} + 3(e^{i\theta} + e^{-i\theta})\right\} \\
&+ \sum_{\substack{\beta<\gamma \\ \beta,\gamma \neq \epsilon}} e^{i(2\theta_\beta+2\theta_\gamma+\theta_\epsilon)} \left\{e^{i\theta} + e^{-i\theta}\right\}.
\end{aligned}$$

The character of S^rH is $\chi_{S^rH}(t) = \sum_{j=0}^{r} e^{i(r-2j)\theta}$, so

$$\chi_{V_1}(t \otimes t') = \sum_{\substack{\gamma<\delta<\epsilon \\ \gamma,\delta,\epsilon\neq\beta}} e^{i(2\theta_\beta+\theta_\gamma+\theta_\delta+\theta_\epsilon)} + 4 \sum_{\alpha<\beta<\gamma<\delta<\epsilon} e^{i(\theta_\alpha+\theta_\beta+\theta_\gamma+\theta_\delta+\theta_\epsilon)}$$

$$\chi_{V_2}(t \otimes t') = \sum_{\substack{\beta<\gamma \\ \beta,\gamma\neq\epsilon}} e^{i(2\theta_\beta+2\theta_\gamma+\theta_\epsilon)} + 2 \sum_{\substack{\gamma<\delta<\epsilon \\ \gamma,\delta,\epsilon\neq\beta}} e^{i(2\theta_\beta+\theta_\gamma+\theta_\delta+\theta_\epsilon)}$$

$$+ 5 \sum_{\alpha<\beta<\gamma<\delta<\epsilon} e^{i(\theta_\alpha+\theta_\beta+\theta_\gamma+\theta_\delta+\theta_\epsilon)}$$

We introduce the following notation

$$(a_1\ a_2\ a_3\ \ldots) = \sum_{\substack{\alpha_1,\alpha_2,\alpha_3,\ldots \text{ distinct} \\ \alpha_j<\alpha_k \text{ if } a_j=a_k \text{ and } j<k}} e^{i(\pm a_1\theta_{\alpha_1} \pm a_2\theta_{\alpha_2} \pm a_3\theta_{\alpha_3} \pm \cdots)}.$$

The multiplicities with which $(a_1\ a_2\ a_3\ \ldots)$ occur in the above expressions for χ_{V_i} are the multiplicities with which the weight $(a_1, a_2, a_3, \ldots, 0, \ldots, 0)$ occurs in the modules V_i. Rearranging those expressions gives

$$\sum_{\alpha<\beta<\gamma<\delta<\epsilon} e^{i(\theta_\alpha+\theta_\beta+\theta_\gamma+\theta_\delta+\theta_\epsilon)} = (11111) + (n-3)(111) + \frac{1}{2}(n-1)(n-2)(1)$$

$$\sum_{\substack{\gamma<\delta<\epsilon \\ \gamma,\delta,\epsilon\neq\beta}} e^{i(2\theta_\beta+\theta_\gamma+\theta_\delta+\theta_\epsilon)} = (2111) + 3(111) + (n-2)(21) + (n-1)(1)$$

$$\sum_{\substack{\beta<\gamma \\ \beta,\gamma\neq\epsilon}} e^{i(2\theta_\beta+2\theta_\gamma+\theta_\epsilon)} = (221) + (21) + (n-1)(1).$$

So we obtain the multiplicities given in Table 2 and the decompositions claimed above. □

		Weight					
		(221)	(2111)	(11111)	(21)	(111)	(1)
V_1	$n \geqslant 5$		1	4	$n-2$	$4n-9$	$(n-1)(2n-3)$
	$n=4$		1		2	7	15
	$n=3$				1	3	6
	$n=2$						1
V_2	$n \geqslant 5$	1	2	5	$2n-3$	$5n-9$	$\frac{1}{2}(n-1)(5n-4)$
	$n=4$	1	2		5	11	24
	$n=3$	1			3	6	11
	$n=2$				1		3
$V^{(221)}$	$n \geqslant 5$	1	2	5	$2(n-2)$	$5n-12$	$\frac{5}{2}(n-1)(n-2)$
	$n=4$	1	2		4	8	15
	$n=3$	1			2	3	5
$V^{(2111)}$	$n \geqslant 5$		1	4	$n-3$	$4(n-3)$	$2(n-3)(n-1)$
	$n=4$		1		1	4	5
K	$n \geqslant 3$				1	2	$2(n-1)$
	$n=2$				1		2
$\Lambda_0^3 E$	$n \geqslant 3$					1	$n-2$
E	$n \geqslant 1$						1

Table 2.
Multiplicities of weights in $Sp(n)$-representations.

If M is a quaternion-Hermitian 8-manifold then the Hodge $*$-operator splits $\Lambda^4_{\mathbb{C}} T^*M$ into its ± 1-eigenspaces $\Lambda^4_{\pm}$. These decompose as

$$\Lambda^4_+ \cong \mathbb{R} + S^4 H + \Lambda^2_0 E \otimes S^2 H + V^{(22)},$$

$$\Lambda^4_- \cong \Lambda^2_0 E + S^2 E \otimes S^2 H.$$

5.2 The Fundamental 4-Form

For a quaternion-Hermitian manifold M^{4n} $(n > 1)$, $SO(4n)$ acts on the fundamental 4-form Ω with stabiliser the maximal subgroup $Sp(n)\,Sp(1)$. Differentiation gives a map $\mathfrak{so}(4n) \to \Lambda^4$ whose kernel is $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$. We thus have an inclusion of the orthogonal complement of $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ in $\mathfrak{so}(4n)$,

$$(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))^{\perp_{\mathfrak{so}(4n)}} \hookrightarrow \Lambda^4 T^*M.$$

Proposition 5.2.1. *If ∇ is the Levi-Civita connection and $X \in T_xM$, then*

$$\nabla_X \Omega \in (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))^{\perp} \cong \Lambda_0^2 ES^2H.$$

PROOF. Let $\tilde{\nabla}$ be an $Sp(n)\,Sp(1)$-connection, not necessarily torsion-free. If we let $\xi_X = \nabla_X - \tilde{\nabla}_X$, then ξ_X is a tensor which is zero on functions and such that $\xi_X \Omega = \nabla_X \Omega$. But ξ_X is in $\mathfrak{so}(4n)$, since the holonomy group of M is a subgroup of $SO(4n)$, so $\nabla_X \Omega$ lies in $(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))^{\perp}$.

Now

$$\begin{aligned}\mathfrak{so}(4n) \cong \Lambda^2 T^*M &\cong S^2E \oplus S^2H \oplus \Lambda_0^2 ES^2H \\ &\cong \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \oplus \Lambda_0^2 ES^2H.\end{aligned}$$

So $(\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))^{\perp} \cong \Lambda_0^2 ES^2H$. □

From this it follows that $\nabla\Omega \in T^* \otimes \Lambda_0^2 ES^2H$.

Proposition 5.2.2. *On a quaternion-Hermitian manifold of dimension at least 8,*

$$\nabla\Omega \in (K + \Lambda^3_0 E + E) \otimes (S^3H + H).$$

(On an 8-manifold, $\Lambda^3_0 E = \{0\}$.) □

Notice that each of these six terms appears in the decomposition of $\Lambda^5 T^*M$ when $\dim M \geqslant 12$, but that KS^3H is absent from this decomposition for an 8-manifold.

Since ∇ is the Levi-Civita connection, $d\Omega = \mathbf{a}\nabla\Omega$, where $\mathbf{a}$ is the alternation map $T^* \otimes \Lambda^4 T^* \to \Lambda^5 T^*$. Now, by Schur's Lemma, when $\dim M \geqslant 12$, to show that $d\Omega$ determines $\nabla\Omega$ it is sufficient to check that $\mathbf{a}$ is non-zero on each of the six components above.

Theorem 5.2.3. *If M is a quaternion-Hermitian manifold of dimension at least 12, then $d\Omega$ determines $\nabla\Omega$. In particular, $d\Omega = 0$ implies $\nabla\Omega = 0$ and that M is quaternionic Kähler.*

PROOF. Let $e_1, \ldots, e_n, \tilde{e_1}, \ldots, \tilde{e_n}$ be a basis for E, $h, \tilde{h}$ a basis of H such that $\tilde{e_i} = je_i$ and $\tilde{h} = jh$. Consider

$$\alpha_{12} = e_1 h \wedge e_2 h \wedge \sum_{i=1}^{n} e_i \tilde{h} \wedge \tilde{e_i}\tilde{h} - e_1 \tilde{h} \wedge e_2 \tilde{h} \wedge \sum_{i=1}^{n} e_i h \wedge \tilde{e_i} h.$$

The symplectic form on E is given by $\sum_{i=1}^{n} e_i \tilde{e_i} - \tilde{e_i} e_i$, so contracting with the symplectic form on the E-components of the third and fourth terms gives

$$e_1 \wedge e_2 (hh\tilde{h}\tilde{h} - \tilde{h}\tilde{h}hh) \in \Lambda^2_0 E \otimes \Lambda^2 S^2 H \cong \Lambda^2_0 E S^2 H.$$

So $\alpha_{12} \in \Lambda_0^2 ES^2H$. Now

$$\mathbf{a}(e_3 h \otimes \alpha_{12}) = (e_3 e_1 e_2 - e_3 e_2 e_1)h\tilde{h}h + (e_2 e_3 e_1 - e_1 e_3 e_2)\tilde{h}hh + e_1 \wedge e_2 \wedge e_3 hh\tilde{h}$$

having applied the contraction above and contracting the third and fifth H-components. But this has non-zero components in KS^3H, KH, $\Lambda_0^3 ES^3H$ and $\Lambda_0^3 EH$.

To get components in $E(S^3H + H)$, consider

$$\mathbf{a}\left(\sum_{i=1}^{n} \tilde{e}_i h \otimes \sum_{j=1}^{n} \alpha_{j2}\right),$$

where α_{j2} is defined in the same way as α_{12}, but with e_1 replaced by e_j.

Thus $\mathbf{a}$ is non-zero on all six components, as required. □

Because all the representations above were complex we see that the above theorem also holds for pseudo-quaternion-Hermitian manifolds.

Corollary 5.2.4. *If M is a pseudo-quaternion-Hermitian manifold of dimension at least 12, then $d\Omega$ determines $\nabla\Omega$.* □

A natural question after the above results is whether there are any complete 8-dimensional (pseudo-) quaternion-Hermitian manifolds for which $d\Omega$ vanishes but $\nabla\Omega$ is non-zero. These conditions imply that $\nabla\Omega$ lies in KS^3H. This question has an analogue in symplectic geometry where one looks for manifolds which are symplectic but not Kähler. Compact examples of such symplectic manifolds were first obtained by Thurston (1976) whose methods were generalised by Cordero et al. (1985a, b, 1986) and by Fernández & Gray (1985). Simply-connected examples have been produced by McDuff (1984).

The decomposition of $\nabla\Omega$ in Proposition 5.2.2 can be used to study various types of quaternion-Hermitian manifolds. For example, the components of $\nabla\Omega$ in

$(K + \Lambda^3_0 E) \otimes (S^3H + H)$ vanish if and only if the algebraic ideal generated by the subbundle $\mathcal{G}$ of $\Lambda^2 T^*M$ is a differential ideal. This is because the covariant derivative of a section α of $\mathcal{G}$ lies in

$$\mathcal{G} \wedge T^*M \cong ES^3H \oplus EH$$

modulo terms arising from $\nabla\Omega \otimes \alpha$. Combining this with the theorem above we have.

Proposition 5.2.5. *A pseudo-quaternion-Hermitian 8-manifold is pseudo-quaternionic Kähler if and only if the fundamental 4-form is closed and the algebraic ideal generated by $\mathcal{G}$ is a differential ideal.* □

Note that the discussion in Chapter 2 gives examples of complete quaternion-Hermitian metrics (on H) for which the differential ideal condition is satisfied but which are not quaternionic Kähler. A variation of the above argument shows that the components of $\nabla\Omega$ involving S^3H all vanish if and only if M is quaternionic in the sense defined in Chapter 1. The metrics of Chapter 2 satisfying the differential ideal condition, all lie in the same quaternionic structure and so have $\nabla\Omega$ in the module EH.

5.3 Relationship to Exceptional Geometries

From the point of view of holonomy classification there are two exceptional Riemannian geometries which occur on non-homogeneous manifolds. The first of these has holonomy in $G_2 \subset SO(7)$. Let $e^1, \ldots, e^7$ be an orthonormal dual basis for $\mathbb{R}^7$. The group G_2 is then characterised as the stabiliser in $GL(7, \mathbb{R})$ of the 3-form

$$\varphi = e^1e^2e^5 - e^3e^4e^5 + e^1e^3e^6 - e^4e^2e^6 + e^1e^4e^7 - e^2e^3e^7 + e^5e^6e^7.$$

The other group that occurs is $Spin(7) \subset SO(8)$. Again this is may be characterised as follows: if we extend the dual basis e^i to $\mathbb{R}^8$ then $Spin(7)$ is the stabiliser in $GL(8,\mathbb{R})$ of the 4-form

$$\begin{aligned}\psi &= e^1e^2e^5e^8 - e^3e^4e^5e^8 + e^1e^3e^6e^8 - e^4e^2e^6e^8 + e^1e^4e^7e^8 - e^2e^3e^7e^8 + e^5e^6e^7e^8 \\ &\quad + e^1e^2e^3e^4 - e^1e^2e^6e^7 + e^3e^4e^6e^7 - e^1e^3e^7e^5 + e^4e^2e^7e^5 - e^1e^4e^5e^6 + e^2e^3e^5e^6 \\ &= \varphi \wedge e^8 + *_7\varphi.\end{aligned}$$

For comparison, the form preserved by $Sp(2)$ is

$$\begin{aligned}\Omega &= \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K \\ &= 3(e^1e^2e^3e^4 + e^5e^6e^7e^8) + e^1e^2e^5e^6 + e^1e^2e^7e^8 + e^3e^4e^5e^6 + e^3e^4e^7e^8 + e^1e^3e^5e^7 \\ &\quad + e^1e^3e^8e^6 + e^4e^2e^5e^7 + e^4e^2e^8e^6 + e^1e^4e^5e^8 + e^1e^4e^6e^7 + e^2e^3e^5e^8 + e^2e^3e^6e^7.\end{aligned}$$

If the sign in front of $\omega_K \wedge \omega_K$ is changed, then the new 4-form obtained is ψ but with $e^8 \mapsto e^7 \mapsto e^6 \mapsto e^5 \mapsto e^4 \mapsto e^3 \mapsto e^2 \mapsto e^1 \mapsto -e^8$. This fact was first observed by Bryant & Harvey (1989) and reflects the inclusion $Sp(2) \subset Spin(7)$.

The forms φ, ψ and Ω are non-degenerate in the sense that they cannot be written as a smaller sum of indecomposables than the expressions above. Thus, 7 indecomposable 3-forms are required for φ and 14 indecomposable 4-forms are required for ψ and Ω. In fact, they are maximally non-degenerate, in that no other forms of the same degree can require more indecomposable summands. This can be shown as follows. Let $M(r,n)$ be an upper bound for the number of indecomposables required for an element of $\Lambda^r\mathbb{R}^n$. If $\alpha \in \Lambda^r\mathbb{R}^n$ and $e^1 \in \Lambda^1\mathbb{R}^n$ then, upto constants,

$$\alpha = e^1 \wedge (e^1 \lrcorner \alpha) + e^1 \lrcorner (e^1 \wedge \alpha).$$

So $M(r,n) \leqslant M(r-1,n-1)+M(n-r-1,n-1)$, since $*(e^1 \wedge \alpha) \in \Lambda^{n-r-1}\mathbb{R}^{n-1}$. Also, by Hodge $*$-duality, $M(r,n) = M(n-r,n)$. Putting $M(0,n) = M(1,n) = 1$ and $M(2,n) = \left[\frac{n}{2}\right]$ we obtain the following bounds

$n\backslash r$	0	1	2	3	4
1	1	1			
2	1	1	1		
3	1	1	1	1	
4	1	1	2	1	1
5	1	1	2	2	1
6	1	1	3	4	3
7	1	1	3	7	7
8	1	1	4	10	14

as claimed.

Classification of manifolds with structure group either G_2 or $Spin(7)$ has been carried out by Fernández (1986) and Fernández & Gray (1982). Various metrics with exceptional holonomy have been produced by Bryant (1984, 1987), Salamon (1987) and Bryant & Salamon (1989). We can reproduce one of their $Spin(7)$ metrics using the techniques of Chapter 2. Recall that in the proof of the existence of quaternionic Kähler manifolds we derived the equations for closure of the 4-form

$$\Xi = A(\alpha \wedge \bar{\alpha} \wedge \alpha \wedge \bar{\alpha}) + B(\alpha \wedge \bar{\alpha} \wedge x\bar{\theta}^t \wedge \theta\bar{x} + x\bar{\theta}^t \wedge \theta\bar{x} \wedge \alpha \wedge \bar{\alpha}) + Cr^4(\bar{\theta}^t \wedge \theta \wedge \bar{\theta}^t \wedge \theta).$$

We showed that this was closed if and only if

$$-\frac{3c}{r^2}A + \frac{3}{r^2}B + B' = 0$$

and

$$-\frac{2c}{r^2}B + \frac{2}{r^2}C + C' = 0.$$

If we put $A = f^2$, $B = 3fg$ and $C = g^2$ then Ξ is the type of 4-form defining a $Spin(7)$-structure. Bryant now tells us that if we show Ξ is closed then we have a metric with holonomy in $Spin(7)$.

Theorem 5.3.1. *The bundle $H/\mathbb{Z}_2$ over a self-dual Einstein 4-manifold carries a metric of holonomy $Spin(7)$ with 4-form*

$$\begin{aligned}(r^2)^{-8/5}\Bigg(&\frac{q^2}{25c^2}(pr^2+q)^{-4/5}(\alpha\wedge\bar{\alpha}\wedge\alpha\wedge\bar{\alpha})\\ &-\frac{3q}{5c}(pr^2+q)^{1/5}(\alpha\wedge\bar{\alpha}\wedge x\bar{\theta}^t\wedge\theta\bar{x}+x\bar{\theta}^t\wedge\theta\bar{x}\wedge\alpha\wedge\bar{\alpha})\\ &+(pr^2+q)^{6/5}(x\bar{\theta}^t\wedge\theta\bar{x}\wedge x\bar{\theta}^t\wedge\theta\bar{x})\Bigg).\end{aligned}$$

There is a similar result that shows that the holonomy is contained in G_2 if $d\varphi = 0 = d^*\varphi$, which is used to construct metrics with holonomy G_2 on Λ^2_- of a self-dual Einstein 4-manifold.

A common feature of the metrics obtained via these bundle constructions is that they admit an $\mathbb{R}$-action. Let the closed differential form associated to the geometry in question be α. Then if X is the vector field generated by this action, we have $L_X\alpha = \alpha$. This implies

$$\alpha = L_X\alpha = X\lrcorner d\alpha + d(X\lrcorner\alpha) = d(X\lrcorner\alpha),$$

and so α is exact.

The 4-form $\frac{1}{3}\Omega$ on a quaternionic Kähler manifold M^{4n} is a example of a *calibration*, that is, for any oriented orthonormal set $e_1,\ldots,e_4$ of T_xM we have

$|\frac{1}{3}\Omega(e_1, e_2, e_3, e_4)| \leqslant 1$ (see Harvey & Lawson, 1982, and Harvey, 1989). Dadok et al. (1988) have classified all self-dual, constant coefficient calibrations on $\mathbb{R}^8$ and a natural question that arises is which submanifolds they determine. Submanifolds of quaternionic Kähler manifolds are rare; Gray (1969) showed that a quaternionic submanifold of any quaternionic Kähler manifold is necessarily totally geodesic. However, one can look for submanifolds M of a quaternionic Kähler manifold for which the pull-back of the fundamental 4-form has the correct algebraic type (at least if $\dim M \geqslant 12$). For example, the existence of such quaternionic Kähler deformations (M_t, Ω_t) of a flat subspace $M_0 = \mathbb{H}^n$ of $\mathbb{H}^m$ can in principle be tackled by Cartan-Kähler theory. Observe that in the decomposition

$$\Lambda^4 T^*M = \mathbb{R} \oplus \Lambda^2_0 E S^2 H \oplus \Lambda^2_0 E \oplus S^2 E S^2 H \oplus W,$$

the first four summands constitute the $\mathfrak{gl}(4n, \mathbb{R})$-orbit of Ω, so the first order equations of deformation are determined by the condition that $\dot{\Omega} = \frac{d\Omega}{dt}\big|_{t=0}$ have zero component in W.

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