Distributed stochastic MPC of linear systems with parameter uncertainty and disturbances

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Abstract: In this paper, we propose a distributed stochastic model predictive control (DSMPC) algorithm for a team of linear subsystems sharing coupled probabilistic constraints. Each subsystem is subject to both parameter uncertainty and stochastic disturbances. To handle the probabilistic constraints, we first decompose the state trajectory into a nominal part and an uncertain part. The latter one is further divided into two parts: one is bounded by probabilistic tubes that are calculated offline by making full use of the probabilistic information on disturbances, whereas the other is bounded by polytopic tubes whose scaling is optimized online and whose facets' orientations are chosen offline. Under the update strategy that only one subsystem is permitted to optimize at each time step, probabilistic constraints are transformed into linear constraints, and the original optimization problem is then formulated as a convex problem. In addition, this new algorithm does not rely on instantaneous inter-subsystem exchanges of data during a time step, and therefore may have a relatively low susceptibility to communication delay. By constructing a decoupled terminal set for each subsystem, the proposed algorithm guarantees recursive feasibility with respect to both local and coupled probabilistic constraints and ensures stability in closed-loop operation. Finally, numerical simulations illustrate the efficacy of the theoretical results.

Key Words: Stochastic systems, Probabilistic constraints, Model predictive control (MPC), Distributed control

1 Introduction

Distributed model predictive control (DMPC) has been widely used in practical applications mainly due to its structural flexibility and capability to handle constraints, such as formation control of multi-agent systems. Over the past years, many important results of the deterministic systems have been proposed, and the guaranteed stability and the online computation burden of DMPC have been particularly addressed [1, 2]. However, for systems with parameter uncertainty and stochastic disturbances, the research of DMPC is still in an embryonic stage, and lots of difficult but rather important problems still remain to be solved. For example, how to explicitly deal with parameter uncertainty and stochastic disturbances in a less conservative method; how to design the terminal set for each subsystem to formulate a tractable optimization problem with a finite number of constraints.

To the best of our knowledge, all results on stochastic MPC (SMPC) accounting for both parameter uncertainty and stochastic disturbances are only developed for a single system [3, 4], and the results of DMPC are restricted to a group of subsystems subject to a single uncertainty source. For the case of a team of subsystems with stochastic disturbances, a distributed robust MPC (DRMPC) algorithm was proposed in [5], which can guarantee the recursive feasibility and stability of the overall system by using the constraint-tightening method in [6]. It required that all subsystems must be optimized and communicate with each other during each sampling period, further resulting in excessive communication. [7] improved the result in [5] and achieved a significant reduction in communication cost by using the strategy that only one subsystem is optimized at each time. Recently a cooperative DSMPC method based on output feedback was proposed in [8] which can guarantee the recursive feasibility and the quadratic stability in the presence of disturbances. By using the strategy in [7] and the probabilistic information on disturbances, a set of deterministic constraints are constructed to ensure the satisfaction of coupled probabilistic constraints. For the case of parameter uncertainty, [9] proposed a DSMPC algorithm based on generalized polynomial chaos expansions (GPCEs), which had no restrictions on the distributions of the uncertainty. However, this approach may involve too many constraints if one requires to provide high-precision approximation.

Motivated by the discussion above, this paper considers a more challenging problem of DSMPC for a group of linear subsystems subject to parameter uncertainty and stochastic disturbances as well as coupled probabilistic constraints. To make the optimization problem tractable in a distributed manner, we adopt the update strategy of [7] that only one subsystem is permitted to optimize at each time step. Following the method of [10], the uncertain part of the state trajectory is decomposed into two parts: one is handled by calculating a sufficient tightening margin offline by using the distribution function on disturbances, whereas the other is handled by using polytopic tubes with facets of fixed orientation and online adjustable tube scalings. The main contributions of the paper are summarized as follows: (i) By using the tube methodology, we transform both local and coupled probabilistic constraints into linear deterministic constraints in a less conservative method, further leading to a convex optimization problem, which can be efficiently solved. (ii) Thanks to the distributed terminal sets designed offline, both the recursive feasibility of the optimization problem and the stability of the overall system are guaranteed. (iii) Exchange

This work is supported by the National Basic Research Program of China (973 Program) under Grant 2012CB720000, the National Natural Science Foundation of China under Grant 61225015 and Grant 61105092, and by the Foundation for Innovative Research Groups of the National Natural Science Foundation of China under Grant 61321002.

of data with other subsystems is only required after one subsystem optimization and therefore greater flexibility in communications is offered.

The remainder of the paper is organized as follows. The problem statement is presented in Section 2. Section 3 proposes a DSMPC algorithm, including probabilistic constraint handling strategy and construction of terminal sets, and analyses the recursive feasibility and closed-loop stability. Section 4 demonstrates the effectiveness of our approach by one numerical simulation. Finally, the conclusion is drawn in Section 5.

Notation. Let $\mathbb{N} \triangleq \{0, 1, \ldots\}$. For $x \in \mathbb{R}^n$, x_i denotes the ith component of x. The inequality sign, absolute value $|\cdot|$ and $\max$ operations apply elementwise to vectors. $Pr\{\cdot\}$ denotes the probability and $\mathbb{E}[\cdot]$ denotes the expectation. Let $\mathcal{B}_\gamma^+ \triangleq \{\alpha \mid ||\alpha||_\infty \leq \gamma, \alpha \geq 0\}$. For sets $\mathcal{A}$ and $\mathcal{B}$, the Cartesian product $\mathcal{A} \times \mathcal{B}$ is the set of all ordered pairs (a, b), where $a \in \mathcal{A}$ and $b \in \mathcal{B}$. $\mathbf{0}$ is the zero vector or zero matrix whose elements are all zero and $\mathbf{1}$ is the vector whose elements are all one.

2 Problem Statement

Consider a system of N_p dynamically decoupled subsystems with $x_p \in \mathbb{R}^{N_{x,p}}$, $u_p \in \mathbb{R}^{N_{u,p}}$, and $w_p \in \mathbb{R}^{N_{w,p}}$,

$$x_p(k+1) = A_p(k)x_p(k) + B_p(k)u_p(k) + w_p(k),$$
$$\forall k \in \mathbb{N},\ p \in \mathcal{P} \triangleq \{1, 2, \ldots, N_p\}. \quad (1)$$

The disturbances $w_p(k)$, for $k \geq 0$, are independent and identically distributed (i.i.d.) and the elements of $w_p(k)$ are zero-mean with known distributions satisfying $w_p \in W_p \triangleq \{w_p \mid |w_p| \leq \alpha_p\}$, where $\alpha_p = [\alpha_{p,1}\ \alpha_{p,2}\ \cdots\ \alpha_{p,N_{w,p}}]^T > \mathbf{0}$. $A_p(k)$ and $B_p(k)$ are defined by

$$[A_p(k)\ B_p(k)] = [A_p^0\ B_p^0] + \sum_{j=1}^{L} [A_p^{(j)}\ B_p^{(j)}] q_p^{(j)}(k), \quad (2)$$

$$\sum_{j=1}^{L} q_p^{(j)}(k) = 1,\ q_p^{(j)}(k) \geq 0, \quad (3)$$

where $q_p^{(j)}(k) \in \mathbb{R}, k \geq 0$, are i.i.d. random variables such that $\mathbb{E}[\sum_{j=1}^{L}[A_p^{(j)}\ B_p^{(j)}]q_p^{(j)}(k)] = \mathbf{0}$.

The predicted control input sequence is formulated as

$$u_p(k+i|k) = K_p x_p(k+i|k) + c_p(k+i|k),\ i = 0, 1, \ldots,$$
$$c_p(k+i|k) = \mathbf{0},\ i \geq N, \quad (4)$$

where $c_p(k+i|k), i = 0, 1, \ldots, N-1$, are optimization variables for some finite prediction horizon N. $u_p = K_p x_p$ is assumed to stabilize subsystem (1) by choosing K_p offline.

The system is also subject to both local and coupled probabilistic constraints

$$Pr\{\eta_p^T x_p(k) \leq h_p\} \geq l_p,\ \forall p \in \mathcal{P}, \quad (5)$$

$$Pr\{\sum_{p=1}^{N_p} \mu_{cp}^T x_p(k) \leq b_c\} \geq \bar{l}_c,\ \forall c \in \mathcal{C} \triangleq \{1, \ldots, N_c\}, \quad (6)$$

where $\eta_p, \mu_{cp} \in \mathbb{R}^{N_{x,p}}$; h_p,$b_c \in \mathbb{R}$; $l_p \in [0, 1]$ and $\bar{l}_c \in [0, 1]$ are prescribed thresholds. Note that the hard constraints can be treated as a special case of constraints (5)-(6) by setting $l_p = \bar{l}_c = 1$.

Before ending this section, three sets of [7] are introduced to facilitate the problem formulation, which are defined as

$$\mathcal{P}_c \triangleq \{p \in \mathcal{P} \mid \mu_{cp} \neq \mathbf{0}\},$$
$$\mathcal{C}_p \triangleq \{c \in \mathcal{C} \mid \mu_{cp} \neq \mathbf{0}\},$$
$$\mathcal{Q}_p \triangleq (\bigcup_{c \in \mathcal{C}_p} \mathcal{P}_c) \backslash \{p\}.$$

3 DSMPC Algorithm

At time k, only subsystem p_k is permitted to optimize for a new perturbation sequence

$$\mathbf{c}_{p_k}(k) \triangleq [c_{p_k}^T(k)\ c_{p_k}^T(k+1|k)\ \cdots\ c_{p_k}^T(k+N-1|k)]^T.$$

Meanwhile, all other subsystems $p \neq p_k$ adopt the feasible perturbation sequences

$$\tilde{\mathbf{c}}_p(k) \triangleq [c_p^T(k|k-1)\ \cdots\ c_p^T(k+N-2|k-1)\ \mathbf{0}]^T.$$

To construct an implementable optimization problem, a strategy to handle the probabilistic constraints is first investigated.

3.1 Probabilistic Constraint Handling Strategy

To facilitate constraint handling, the state trajectory predicted at time k is split into two parts

$$x_p(k+i|k) = z_p(k+i|k) + e_p(k+i|k), \quad (7)$$
$$z_p(k+i+1|k) = \Psi_p^0 z_p(k+i|k) + B_p^0 c_p(k+i|k), \quad (8)$$
$$e_p(k+i+1|k) = \Psi_p(k+i|k)e_p(k+i|k) + \bar{\Psi}_p(k+i|k)$$
$$\times z_p(k+i|k) + \bar{B}_p(k+i|k)c_p(k+i|k) + w_p(k+i|k) \quad (9)$$

with $\Psi_p^0 = A_p^0 + B_p^0 K_p$, $\Psi_p(k+i|k) = A_p(k+i|k) + B_p(k+i|k)K_p$, $\bar{\Psi}_p(k+i|k) = \Psi_p(k+i|k) - \Psi_p^0$, and $\bar{B}_p(k+i|k) = B_p(k+i|k) - B_p^0$. The constraints (5)-(6) can now be expressed as

$$Pr\{\eta_p^T z_p(k+i|k) + \eta_p^T e_p(k+i|k) \leq h_p\} \geq l_p, \quad (10)$$

$$Pr\{\sum_{p=1}^{N_p}[\mu_{cp}^T z_p(k+i|k) + \mu_{cp}^T e_p(k+i|k)] \leq b_c\} \geq \bar{l}_c. \quad (11)$$

Following [10], $e_p(k+i|k)$ is further split into two parts

$$e_p(k+i|k) = \varepsilon_p(k+i|k) + \zeta_p(k+i|k), \quad (12)$$
$$\varepsilon_p(k+i+1|k) = \Psi_p^0 \varepsilon_p(k+i|k) + w_p(k+i|k), \quad (13)$$
$$\zeta_p(k+i+1|k) = \Psi_p(k+i|k)\zeta_p(k+i|k) + \bar{\Psi}_p(k+i|k)$$
$$\times z_p(k+i|k) + \bar{B}_p(k+i|k)c_p(k+i|k)$$
$$+ \bar{\Psi}_p(k+i|k)\varepsilon_p(k+i|k), \quad (14)$$

with $\varepsilon_p(k) = \mathbf{0}$ and $z_p(k) + \zeta_p(k) = x_p(k)$. After the decomposition, the two components of e_p in (10)-(11) can be treated separately in two steps.

Step One: Handle ε_p using the distribution function of w_p. Since the evolution of ε_p in (13) is described by a linear time-invariant system with disturbance, as in [8] and [12], the conditions which ensure the satisfaction of probabilistic constraints (10)-(11) are given below.

Theorem 1 *For subsystem p, the satisfaction of (10)-(11) is guaranteed, if* $\mathbf{c}_p(k)$ *and* $z_p(k)$ *satisfy the recursively feasible constraints, for* $i = 1, 2, \ldots$,

$$\eta_p^T H_p^i \mathbf{c}_p(k) + \eta_p^T (\Psi_p^0)^i z_p(k) + \eta_p^T \zeta_p(k+i|k) \le h_p - \beta_p^i, \quad (15)$$

$$\mu_{cp}^T H_p^i \mathbf{c}_p(k) + \mu_{cp}^T (\Psi_p^0)^i z_p(k) + \mu_{cp}^T \zeta_p(k+i|k) + \sum_{q\in\mathcal{P}_c\setminus\{p\}} [\mu_{cq}^T H_q^i \mathbf{c}_q^*(k) + \mu_{cq}^T (\Psi_q^0)^i z_q^*(k) + \mu_{cq}^T \zeta_q(k+i|k)] \le b_c - \kappa_c^i, \ \forall c \in \mathcal{C}_p, \quad (16)$$

where $H_p^i = [(\Psi_p^0)^{i-1} B_p^0 \ \ (\Psi_p^0)^{i-2} B_p^0 \ \cdots \ B_p^0 \ \mathbf{0} \ \cdots \ \mathbf{0}]$. β_p^i *and* κ_c^i*, respectively, are the maximum elements of the ith column of*

$$\begin{bmatrix} \gamma_p^1 & \gamma_p^2 & \gamma_p^3 & \cdots \\ 0 & \gamma_p^1 + a_p^1 & \gamma_p^2 + a_p^2 & \cdots \\ 0 & 0 & \gamma_p^1 + a_p^1 + a_p^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (17)$$

$$\begin{bmatrix} \xi_c^1 & \xi_c^2 & \xi_c^3 & \cdots \\ 0 & \xi_c^1 + d_c^1 & \xi_c^2 + d_c^2 & \cdots \\ 0 & 0 & \xi_c^1 + d_c^1 + d_c^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (18)$$

with

$$a_p^i = \max_{w_p\in W_p} \eta_p^T (\Psi_p^0)^i w_p = |\eta_p^T (\Psi_p^0)^i| \alpha_p,$$

$$d_c^i = \max_{w_p\in W_p} \sum_{p=1}^{N_p} \mu_{cp}^T (\Psi_p^0)^i w_p = \sum_{p=1}^{N_p} |\mu_{cp}^T (\Psi_p^0)^i| \alpha_p.$$

γ_p^i *and* ξ_c^i *are the minimum values such that, for* $i = 1, 2, \ldots$,

$$Pr\{\eta_p^T (\Psi_p^0)^{i-1} w_p(k) + \eta_p^T (\Psi_p^0)^{i-2} w_p(k+1|k) + \ldots + \eta_p^T w_p(k+i-1|k) \le \gamma_p^i\} = l_p, \quad (19)$$

$$Pr\{\sum_{p=1}^{N_p} [\mu_{cp}^T (\Psi_p^0)^{i-1} w_p(k) + \mu_{cp}^T (\Psi_p^0)^{i-2} w_p(k+1|k) + \ldots + \mu_{cp}^T w_p(k+i-1|k)] \le \xi_c^i\} = \bar{l}_c. \quad (20)$$

Proof. By iteration of (8) and (13), the predictions at time k are made as follows

$$z_p(k+i|k) = (\Psi_p^0)^i z_p(k) + H_p^i \mathbf{c}_p(k), \quad (21)$$

$$\varepsilon_p(k+i|k) = (\Psi_p^0)^{i-1} w_p(k) + (\Psi_p^0)^{i-2} w_p(k+1|k) + \ldots + w_p(k+i-1|k). \quad (22)$$

Substituting them into (10)-(11) and combining the definitions of γ_p^i and ξ_c^i in (19)-(20), we conclude that if β_p^i and κ_c^i in (15)-(16) are replaced by γ_p^i and ξ_c^i in the first row of the matrices (17)-(18) respectively, then (15)-(16) can guarantee that (10)-(11) are satisfied at time k. However, since γ_p^i and ξ_c^i are computed based on the information available at time k, this cannot guarantee the future feasibility.

To ensure the recursive feasibility, consider the prediction of (13) at time $k+j$

$$\begin{aligned} &\varepsilon_p(k+j+i|k+j) \\ &= (\Psi_p^0)^i \varepsilon_p(k+j) + (\Psi_p^0)^{i-1} w_p(k+j) + \ldots \\ &\quad + w_p(k+j+i-1|k+j). \end{aligned} \quad (23)$$

Note that, at time $k+j$, $\varepsilon_p(k+j)$ cannot be treated as a random variable. Therefore, when computing the distribution of $\varepsilon_p(k+j+i|k+j)$, we must consider the worst-case value for $\varepsilon_p(k+j)$ over the class of allowable uncertainty

$$\max_{w_p\in W_p} \eta_p^T (\Psi_p^0)^i \varepsilon_p(k+j) = \sum_{t=i}^{i+j-1} a_p^t, \quad (24)$$

$$\max_{w_p\in W_p} \sum_{p=1}^{N_p} \mu_{cp}^T (\Psi_p^0)^i \varepsilon_p(k+j) = \sum_{t=i}^{i+j-1} d_c^t. \quad (25)$$

The ith elements of the $(j+1)$th rows of (17)-(18) are the predicted values of $\eta_p^T \varepsilon_p(k+j+i|k+j)$ and $\sum_{p=1}^{N_p} \mu_{cp}^T \varepsilon_p(k+j+i|k+j)$, which are obtained by considering the worst-case values for $\eta_p^T (\Psi_p^0)^i \varepsilon_p(k+j)$ and $\sum_{p=1}^{N_p} \mu_{cp}^T (\Psi_p^0)^i \varepsilon_p(k+j)$.

Therefore, the satisfaction of probabilistic constraints (5)-(6) and the recursive feasibility are guaranteed, if β_p^i and κ_c^i are selected as the maximum elements of the ith columns of (17)-(18), respectively. □

In (16), note that $\mathbf{c}_q(k)$ and $z_q(k)$, $q \in \mathcal{Q}_p$, are not affected by the optimization variables, so they can be treated as fixed values, denoted by $\mathbf{c}_q^*(k)$ and $z_q^*(k)$.

Lemma 1 *[12] The sequences* $\{\beta_p^1, \beta_p^2, \ldots\}$ *and* $\{\kappa_c^1, \kappa_c^2, \ldots\}$ *are monotonically non-decreasing and converge to* $\bar{\beta}_p$ *and* $\bar{\kappa}_c$ *respectively. Furthermore, the arbitrarily tight upper bounds on* $\bar{\beta}_p$ *and* $\bar{\kappa}_c$ *are given as*

$$\bar{\beta}_p \le \gamma_p^1 + \sum_{j=1}^{v_p-1} a_p^j + \tfrac{\rho_p^{v_p}}{1-\rho_p} \parallel \eta_p \parallel_{S_p}, \ \forall p \in \mathcal{P}, \quad (26)$$

and

$$\bar{\kappa}_c \le \xi_c^1 + \sum_{i=1}^{\mu_c-1} d_c^i + \sum_{p=1}^{N_p} \tfrac{\rho_p^{\mu_c}}{1-\rho_p} \parallel \mu_{cp} \parallel_{S_p}, \ \forall c \in \mathcal{C}, \quad (27)$$

for any integers $v_p > 1$ *and* $\mu_c > 1$*. The parameters* ρ_p *and* S_p *are defined as the solution of a semidefinite program (see [12]).*

Step Two: Handle ζ_p using polytopic tubes. In this step, a polytopic tube with bounded facets of fixed orientation is constructed online to bound ζ_p [11]. The cross-section of the tube is parameterized in the form

$$\{\zeta_p(k+i|k) \mid V_p \zeta_p(k+i|k) \le \theta_p(k+i|k)\}, \ i \ge 0, \quad (28)$$

where $V_p \in \mathbb{R}^{N_{v,p}\times N_{x,p}}$ is chosen offline and $\theta_p \in \mathbb{R}^{N_{v,p}}$ is optimized online.

Theorem 2 *For any* $\zeta_p(k+i|k) \in \{\zeta \mid V_p\zeta \le \theta_p(k+i|k)\}$*, a necessary and sufficient condition for* $V_p\zeta_p(k+i+1|k) \le \theta_p(k+i+1|k)$ *is that for* $i = 0, 1, \ldots$*, there exist nonnegative matrices* $M_p^{(j)} \ge 0, j = 1, 2, \ldots, L$*, satisfying*

$$M_p^{(j)} V_p = V_p(\Psi_p^0 + \Psi_p^{(j)}), \quad (29)$$

$$M_p^{(j)} \theta_p(k+i|k) + V_p \Psi_p^{(j)} z_p(k+i|k) + V_p B_p^{(j)} c_p(k+i|k) + d_p^{(j)}(i) \le \theta_p(k+i+1|k), \quad (30)$$

where $d_p^{(j)}(i) = \max_{w_p\in W_p} V_p \Psi_p^{(j)} \varepsilon_p(k+i|k)$.

Proof. From (14) and Proposition 3.31 in [13], it is easy to obtain (29)-(30). □

Remark 3.1 *For each vertex $j = 1, 2, \ldots, L$, the sequence $\{d_p^{(j)}(0), d_p^{(j)}(1), \ldots\}$ converges to a limit $\bar{d}_p^{(j)}$. The arbitrarily close upper bound on $\bar{d}_p^{(j)}$ can be computed by*

$$\bar{d}_p^{(j)} \leq \sum_{l=0}^{\tau-1} b_p^{(j)}(l) + \frac{\rho_p^\tau}{1-\rho_p}\|(V_p\Psi_p^{(j)})^T\|_{S_p}, \quad (31)$$

where $b_p^{(j)}(l) = \max_{w_p \in W_p} V_p\Psi_p^{(j)}(\Psi_p^0)^l w_p$ and τ is any given integer.

Theorem 3 reformulates (15)-(16) as linear constraints by using the definition of tube cross-section (28).

Theorem 3 *For any $\zeta_p(k+i|k) \in \{\zeta \mid V_p\zeta \leq \theta_p(k+i|k)\}$, (15)-(16) for $i = 1, 2, \ldots$ are satisfied if and only if there exist nonnegative matrices $M_p \geq 0$ and $M_{cp} \geq 0$ satisfying*

$$M_pV_p = \eta_p^T, \quad (32)$$

$$M_p\theta_p(k+i|k) + \beta_p^i + \eta_p^T H_p^i \mathbf{c}_p(k) + \eta_p^T(\Psi_p^0)^i z_p(k) \leq h_p, \quad (33)$$

$$M_{cp}V_p = \mu_{cp}^T, \quad (34)$$

$$M_{cp}\theta_p(k+i|k) + \kappa_c^i + \mu_{cp}^T H_p^i \mathbf{c}_p(k) + \mu_{cp}^T(\Psi_p^0)^i z_p(k) + \sum_{q\in\mathcal{P}_c\setminus\{p\}} [\mu_{cq}^T H_q^i \mathbf{c}_q^*(k) + \mu_{cq}^T(\Psi_q^0)^i z_q^*(k) + M_{cq}\theta_q^*(k+i|k)] \leq b_c, \ \forall c \in \mathcal{C}. \quad (35)$$

In (35), $\theta_q(k+i|k)$ can also be regarded as a fixed value. To relax constraints (30), (33), and (35) applied online, $M_p^{(j)}$, M_p, and M_{cp} can be calculated offline as the solutions of the linear programmings [11]

$$(M_p^{(j)})_i = arg \min_{\mathbf{m}^T}\{\mathbf{1}^T\mathbf{m} \mid \mathbf{m}^TV_p = (V_p)_i(\Psi_p^0 + \Psi_p^{(j)}), \ \mathbf{m} \geq \mathbf{0}\}, \quad (36)$$

$$M_p = arg \min_{\mathbf{m}^T}\{\mathbf{1}^T\mathbf{m} \mid \mathbf{m}^TV_p = \eta_p^T, \ \mathbf{m} \geq \mathbf{0}\}, \quad (37)$$

$$M_{cp} = arg \min_{\mathbf{m}^T}\{\mathbf{1}^T\mathbf{m} \mid \mathbf{m}^TV_p = \mu_{cp}^T, \ \mathbf{m} \geq \mathbf{0}\}, \quad (38)$$

where $(M)_i$ is the ith row of matrix M.

3.2 Construction of Terminal Sets

To use an infinite prediction horizon cost but keep a finite number of constraints, $z_p(k+N_1|k)$ and $\theta_p(k+N_1|k)$, $N_1 > N$, will be bounded to lie in some terminal set. By requiring $z_p(k+N|k) \in \chi_p^0$, where χ_p^0 is a robust positively invariant set for $z_p(k+1) = \Psi_p^0 z_p(k) + (\Psi_p^0)^N w_p(k)$, $z_p(k+N_1|k)$ will enter a smaller set $\chi_p^l \triangleq (\Psi_p^0)^l\chi_p^0, l = N_1 - N$. Based on it, Theorem 4 designs a terminal set for $\theta_p(k+N_1|k)$.

Theorem 4 *Define $\bar{v}_{p,l}^{(j)} = \max_{z_p\in\chi_p^l}\{V_p\Psi_p^{(j)}z_p\}$, $\bar{f}_{p,l} = \max_{z_p\in\chi_p^l}\{\eta_p^Tz_p\}$, and $\bar{f}_{cp,l} = \max_{z_p\in\chi_p^l}\{\mu_{cp}^Tz_p\}$. If $\bar{\theta}_p$ satisfies*

$$\bar{\theta}_p \geq \frac{\max_j(\|\bar{v}_{p,l}^{(j)}\|_\infty + \|\bar{d}_p^{(j)}\|_\infty)}{1 - \max_j \|M_p^{(j)}\|_\infty}, \quad (39)$$

$$\bar{\theta}_p \leq \frac{h_p - \bar{\beta}_p - \|\bar{f}_{p,l}\|_\infty}{\|M_p\|_\infty}, \quad (40)$$

$$\sum_{p=1}^{N_p}\|M_{cp}\|_\infty\bar{\theta}_p \leq b_c - \bar{\kappa}_c - \sum_{p=1}^{N_p}\|\bar{f}_{cp,l}\|_\infty, \quad (41)$$

then $\chi_p^l \times \mathcal{B}_{\bar{\theta}_p}^+$ is an invariant set with respect to

$$z_p(k+i+1|k) = \Psi_p^0 z_p(k+i|k), \quad (42)$$

$$\theta_p(k+i+1|k) = \max_j\{M_p^{(j)}\theta_p(k+i|k) + d_p^{(j)}(i) + V_p\Psi_p^{(j)}z_p(k+i|k)\}, \ i \geq N_1, \quad (43)$$

and constraints (33) and (35) are satisfied in $\chi_p^l \times \mathcal{B}_{\bar{\theta}_p}^+$.

Due to space limitations, the proof of Theorem 4 is omitted here. In Theorem 4, (39) ensures that the set $\chi_p^l \times \mathcal{B}_{\bar{\theta}_p}^+$ is invariant for the dynamics given in (42)-(43) and (40)-(41) are conditions of $\bar{\theta}_p$ corresponding to the local constraint (33) and the coupled constraint (35) respectively.

3.3 Optimization Problem Formulation and Distributed Implementation Algorithm

Define the cost function in terms of the nominal state as

$$J_p^k = \sum_{i=0}^{\infty}[\|z_p(k+i|k)\|_{Q_p}^2 + \|K_pz_p(k+i|k) + c_p(k+i|k)\|_{R_p}^2], \quad (44)$$

where the weighting matrices Q_p and R_p are symmetric positive-definite. Lemma 2 states that (44) can be rewritten as a quadratic function of the optimization variables.

Lemma 2 *Let W_p be the solution of*

$$W_p - \begin{bmatrix} \Psi_p^0 & B_p^0E_p \\ \mathbf{0} & T_p \end{bmatrix}^T W_p \begin{bmatrix} \Psi_p^0 & B_p^0E_p \\ \mathbf{0} & T_p \end{bmatrix} = \bar{Q}_p, \quad (45)$$

where T_p is a matrix with the identity matrix on its super-diagonal blocks and zeros elsewhere, $E_p = \begin{bmatrix} I & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$, $\bar{Q}_p = \begin{bmatrix} Q_p + K_p^TR_pK_p & K_p^TR_pE_p \\ E_p^TR_pK_p & E_p^TR_pE_p \end{bmatrix}$, then we have

$$J_p^k = \left\|\begin{bmatrix} z_p(k) \\ \mathbf{c}_p(k) \end{bmatrix}\right\|_{W_p}^2. \quad (46)$$

Proof. Pre- and post- multiplication of (45) by $[z_p^T(k+i|k) \ c_p^T(k+i|k)]$ and $[z_p^T(k+i|k) \ c_p^T(k+i|k)]^T$ and then summing over $i = 0, 1, \ldots$ will lead to (46). □

The MPC optimization problem $\mathbf{P}_p(x_p(k); Z_p^*(k))$ can now be formulated as follows.

$$\min_{\mathbf{c}_p(k), z_p(k), \theta_p(k|k), \ldots, \theta_p(k+N_1|k)} J_p^k = \left\|\begin{bmatrix} z_p(k) \\ \mathbf{c}_p(k) \end{bmatrix}\right\|_{W_p}^2$$

s.t.

$$V_p(x_p(k|k) - z_p(k|k)) \leq \theta_p(k|k), \ x_p(k|k) = x_p(k) \quad (47)$$

$$\forall i \in \{0, 1, \ldots, N_1 - 1\}, \ \forall j \in \{1, 2, \ldots, L\}:$$

$$z_p(k+i+1|k) = \Psi_p^0 z_p(k+i|k) + B_p^0 c_p(k+i|k) \quad (48)$$

$$M_p^{(j)}\theta_p(k+i|k) + V_p\Psi_p^{(j)}z_p(k+i|k) + V_pB_p^{(j)}c_p(k+i|k) + d_p^{(j)}(i) \leq \theta_p(k+i+1|k) \quad (49)$$

$\forall i \in \{1, 2, \ldots, N_1 - 1\}$:

$$M_p\theta_p(k+i|k) + \beta_p^i + \eta_p^T H_p^i \mathbf{c}_p(k) + \eta_p^T(\Psi_p^0)^i z_p(k) \leq h_p \quad (50)$$

$$M_{cp}\theta_p(k+i|k) + \kappa_c^i + \mu_{cp}^T H_p^i \mathbf{c}_p(k) + \mu_{cp}^T(\Psi_p^0)^i z_p(k) + \sum_{q\in\mathcal{P}_c\setminus\{p\}} [\mu_{cq}^T H_q^i \mathbf{c}_q^*(k) + \mu_{cq}^T(\Psi_q^0)^i z_q^{i*}(k) + M_{cq}\theta_q^*(k+i|k)] \leq b_c,\ \forall c \in \mathcal{C}_p \quad (51)$$

$$z_p(k+N|k) \in \chi_p^0 \quad (52)$$

$$\|\theta_p(k+N_1|k)\|_\infty \leq \bar{\theta}_p. \quad (53)$$

$\mathbf{c}_q^*(k)$, $z_q^{i*}(k)$, and $\theta_q^*(k+i|k)$ in (51) can be constructed by iteration of (54)-(57) with the optimal solutions $\mathbf{c}_q^*(\hat{k}_q)$, $z_q^{i*}(\hat{k}_q) = z_q^*(\hat{k}_q)$, and $\theta_q^*(\hat{k}_q + i|\hat{k}_q), i = 0, 1, \ldots, N_1$, of subsystems $q \in \mathcal{P}_c \setminus \{p\}$ at their last update step $\hat{k}_q$,

$$\mathbf{c}_q^*(k) = T_q \mathbf{c}_q^*(k-1), \quad (54)$$

$$z_q^{i*}(k) = z_q^{i+1*}(k|k-1) + sgn(\mu_{cq}^T(\Psi_q^0)^i)^T \cdot \alpha_q, \quad (55)$$

$$\theta_q^*(k+i|k) = \theta_q^*(k+i|k-1),\ i = 0, 1, \ldots, N_1 - 1, \quad (56)$$

$$\theta_q^*(k+N_1|k) = \max_j\{M_q^{(j)}\theta_q^*(k-1+N_1|k-1) + V_q\Psi_q^{(j)} z_q^*(k-1+N_1|k-1) + d_q^{(j)}(N_1)\}, \quad (57)$$

where for $x, y \in \mathbb{R}^n$, $sgn(x) = [sign(x_1) \cdots\ sign(x_n)]$ and $x \cdot y = [x_1y_1\ x_2y_2\ \cdots\ x_ny_n]^T$.

The distributed algorithm can now be summarized below.

Algorithm 1.

1) Initialization: Wait for a feasible solution and information $Z_p^*(0)$ for each $p \in \mathcal{P}$. If a solution cannot be found, stop.
2) All subsystems p: Apply $u_p^*(k) = K_p x_p(k) + c_p^*(k)$.
3) All subsystems p: Update k and sample states $x_p(k)$.
4) Subsystem $p = p_k$:
   a) Solve $\mathbf{P}_p^D(x_p(k); Z_p^*(k))$ to obtain $\{\mathbf{c}_p(k), z_p(k), \theta_p(k|k), \theta_p(k+1|k), \ldots, \theta_p(k+N_1|k)\}$.
   b) Transmit it to subsystems $q \in \mathcal{Q}_p$.
   c) Set $\hat{k}_p = k$.

   Subsystems $p \neq p_k$: Construct $\mathbf{c}_p(k) = T_p^{k-\hat{k}_p}\mathbf{c}_p(\hat{k}_p)$.
5) Go to step 2).

Different from [8], here the exchange of information is only required when a subsystem is optimized for a new solution.

3.4 Main Results

Theorem 5 *If there exists a feasible solution for subsystem $p \in \mathcal{P}$ to the optimization problem $\mathbf{P}_p(x_p(k_0); Z_p^*(k_0))$ at the initial time, and the system is controlled by Algorithm 1, then the MPC optimization problem is recursively feasible. Moreover, the MPC control law converges to the unconstrained optimal control law as $k \to \infty$.*

Proof. At time k_0, the feasible solution for subsystem $p \in \mathcal{P}$ is assumed to be

$$\{\mathbf{c}_p(k_0), z_p(k_0), \theta_p(k_0|k_0), \theta_p(k_0+1|k_0), \ldots, \theta_p(k_0+N_1|k_0)\}.$$

Next, we will show that

$$\{T_p\mathbf{c}_p(k_0), z_p(k_0+1|k_0) + w_p(k_0), \theta_p(k_0+1|k_0), \ldots, \theta_p(k_0+N_1|k_0), \max_j\{M_p^{(j)}\theta_p(k_0+N_1|k_0) + V_p\Psi_p^{(j)} z_p(k_0+N_1|k_0) + d_p^{(j)}(N_1)\}\} \quad (58)$$

is a feasible solution at time $k_0 + 1$.

From the fact that $\zeta_p(k_0+1|k_0) = x_p(k_0+1) - z_p(k_0+1|k_0) - \varepsilon_p(k_0+1|k_0)$, (47) is satisfied at time k_0+1. The constraints (48) and (49)-(50) for $i = 0, 1, \ldots, N_1 - 2$ are satisfied directly at time $k_0 + 1$. Since $d_p^{(j)}(N_1) \geq V_p\Psi_p^{(j)}(\Psi_p^0)^{N_1-1}w_p(k_0) + d_p^{(j)}(N_1-1)$ and $\beta_p^{N_1} \geq \beta_p^{N_1-1} + \eta_p^T(\Psi_p^0)^{N_1-1}w_p(k_0)$, (49)-(50) also hold for $i = N_1 - 1$. With similar argument, from $\kappa_c^{i+1} = \kappa_c^i + d_c^i$, (51) is satisfied at time k_0+1. From the definitions of χ_p^0 and $\mathcal{B}_{\bar{\theta}_p}^+$, we can verify that the terminal constraints (52)-(53) are satisfied at time k_0+1. Hence the candidate solution (58) is a feasible solution to $\mathbf{P}_p(x_p(k_0+1); Z_p^*(k_0+1))$. The recursive feasibility of $\mathbf{P}_p(x_p(k); Z_p^*(k))$ follows immediately by recursion.

The recursive feasibility implies that the sequence $\{c_p(0), c_p(1), \ldots\}$ is in l^2, and therefore $c_p(k)$ tends to zero as $k \to \infty$. As a consequence, the MPC control law converges asymptotically to the unconstrained optimal control law $u_p(k) = K_p x_p(k)$ which can stabilize the subsystem (1) by assumption, and meanwhile the state of each subsystem converges asymptotically to a set of states in which probabilistic constraints are satisfied by applying the unconstrained optimal control law.

□

4 Numerical Example

The system of three subsystems is defined by

$$A_p^0 = \begin{bmatrix} 1.6 & 1 \\ -0.5 & 1.2 \end{bmatrix},\ B_p^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$A_p^{(1)} = \begin{bmatrix} 0.05 & -0.15 \\ -0.01 & 0.01 \end{bmatrix},\ B_p^{(1)} = \begin{bmatrix} -0.05 \\ 0.05 \end{bmatrix},$$

$$A_p^{(2)} = -A_p^{(1)},\ B_p^{(2)} = -B_p^{(1)}.$$

$q_p^{(1)}(k)$ is a random variable uniformly distributed over $[0,\ 1]$ and $q_p^{(2)}(k) = 1 - q_p^{(1)}(k)$. Each element of w_p is truncated from a normal distribution with mean zero and variance $\frac{1}{10^2}$ and satisfies $\|w_p\|_\infty \leq 0.12$. The constraint parameters are given by

$$\eta_{p,1} = [0.8\ -0.2]^T,\ h_{p,1} = 2.1,\ l_{p,1} = 0.8;$$

$$\eta_{p,2} = [-0.8\ -0.2]^T,\ h_{p,2} = 2.5,\ l_{p,2} = 0.8;$$

$$\mu_{1p} = [0.8\ -0.2]^T,\ b_1 = 5.5,\ \bar{l}_1 = 0.8.$$

Choose $Q_p = [1\ 0.3]^T[1\ 0.3]$, $R_p = 0.1$, $N = 4$, $N_1 = 7$, and $N_{v,p} = 8$. The update sequence adopted is $\{1, 2, 3, 1, 2, 3, \ldots\}$.

For the initial condition $x_1(0) = [3.5\ -3.5]^T$, $x_2(0) = [3.4\ -3.45]^T$, and $x_3(0) = [3.5\ -3.25]^T$, Fig. 1-Fig. 4 respectively show the evolution of $x_p(k)$ for each subsystem and $\sum_{p=1}^3 x_p(k)$ by applying Algorithm 1 and the unconstrained optimal control for 100 Monte Carlo simulations.

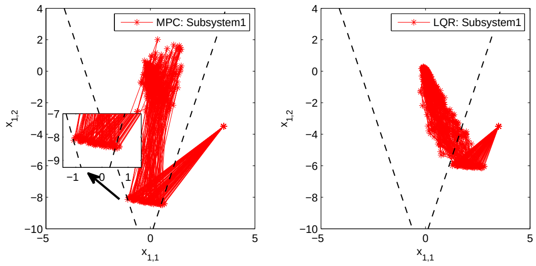

Fig. 1: Closed-loop response of subsystem 1 under Algorithm 1 (left) and unconstrained optimal control (right) for 100 realizations of uncertainty.

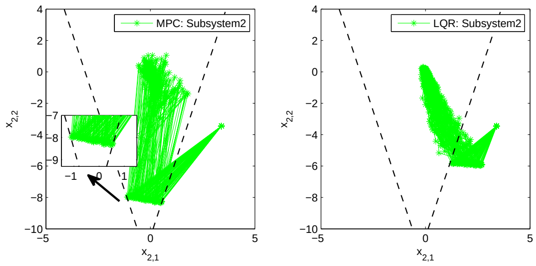

Fig. 2: Closed-loop response of subsystem 2 under Algorithm 1 (left) and unconstrained optimal control (right) for 100 realizations of uncertainty.

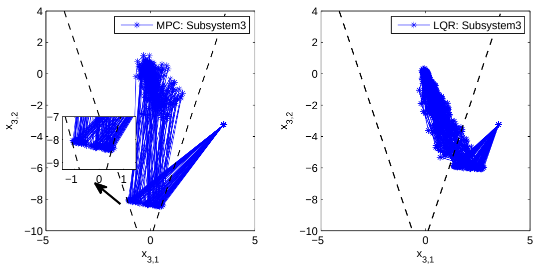

Fig. 3: Closed-loop response of subsystem 3 under Algorithm 1 (left) and unconstrained optimal control (right) for 100 realizations of uncertainty.

Under the unconstrained optimal control, the violation probabilities of both local and coupled probabilistic constraints are 100% at time $k = 1$. While under Algorithm 1, 5.1%, 1%, and 2.5% of the same uncertainty realizations violate the local constraints of subsystems 1, 2, and 3 at time $k = 1$, and the coupled constraint is violated with a probability of 5.3%.

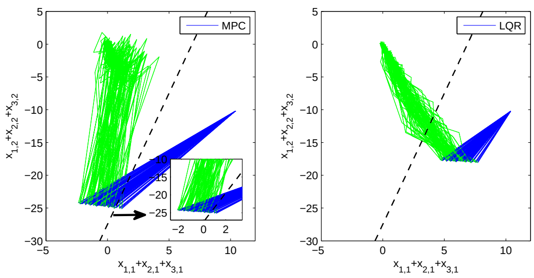

Fig. 4: Evolution of coupled state $\sum_{p=1}^{3} x_p(k)$ under Algorithm 1 (left) and unconstrained optimal control (right) for 100 realizations of uncertainty of each subsystem.

5 Conclusion

By using the tube methodology, a DSMPC algorithm is proposed for multiple subsystems subject to parameter uncertainty and disturbances as well as coupled probabilistic constraints. we divide the uncertain component of the state trajectory into two parts and deal with them separately in two steps: one part with respect to disturbances is first constrained to lie in probabilistic tubes which are derived offline from the distribution of disturbances; the other part is then constrained to lie in polytopic tubes with facets of fixed orientation determined offline, and the cross-section of the tube is determined online. By applying the update strategy that only one subsystem is optimized at each time step, the coupled probabilistic constraints are satisfied in a distributed manner. The formulated MPC optimization problem for each subsystem is a convex optimization, which can be effectively solved by many available methods. The proposed algorithm guarantees the satisfaction of probabilistic constraints, and ensures the recursive feasibility and the stability. A numerical example is given to illustrate the efficacy of the algorithm.

References

[1] J. Liu, D. de la Peña, and P. Christofides, Distributed model predictive control of nonlinear systems subject to asynchronous and delayed measurements, *Automatica*, 46(1): 52–61, 2010.

[2] J. Maestre, and R. Negenborn, *Distributed model predictive control made easy*. Springer, 2014.

[3] M. Cannon, B. Kouvaritakis, and X. Wu, Probabilistic constrained MPC for multiplicative and additive stochastic uncertainty, *IEEE Transactions on Automatic Control*, 54(7): 1626–1632, 2009.

[4] J. Li, D. Li, Y. Xi, and P. Zhang, Stochastic output-feedback model predictive control for systems with multiplicative and additive uncertainty, in *Proceedings of 34th Chinese Control Conference*, 2015: 4179–4184.

[5] A. Richards, and J. How, Robust distributed model predictive control, *International Journal of Control*, 80(9): 1517–1531, 2007.

[6] D. Mayne, M. Seron, and S. Raković, Robust model predictive control of constrained linear systems with bounded disturbances, *Automatica*, 41(2): 219–224, 2005.

[7] P. Trodden, and A. Richards, Distributed model predictive control of linear systems with persistent disturbances, *International Journal of Control*, 83(8): 1653–1663, 2010.

[8] L. Dai, Y. Xia, Y. Gao, B. Kouvaritakis, and M. Cannon, Cooperative distributed stochastic MPC of linear systems with state estimation and coupled probabilistic constraints, *Automatica*, 61: 89–96, 2015.

[9] L. Dai, Y. Xia, and Y. Gao, Distributed MPC of linear systems with stochastic parametric uncertainties and coupled probabilistic constraints, *SIAM Journal on Control and Optimization*, 53(6): 3411–3431, 2015.

[10] Q. Cheng, M. Cannon, B. Kouvaritakis, and M. Evans, Stochastic MPC for systems with both multiplicative and additive disturbances, in *Proceedings of 19th IFAC World Congress*, 2014: 2291–2296.

[11] J. Fleming, B. Kouvaritakis, and M. Cannon, Regions of attraction and recursive feasibility in robust MPC, in *Proceedings of 21st Mediterranean Conference on Control & Automation*, 2013: 801–806.

[12] B. Kouvaritakis, M. Cannon, S. Raković, and Q. Cheng, Explicit use of probabilistic distributions in linear predictive control, *Automatica*, 46(10): 1719–1724, 2010.

[13] F. Blanchini, and S. Miani, *Set-theoretic methods in control*. Boston: Springer, 2007.