Distributed stochastic MPC of linear systems with parameter uncertainty and disturbances

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Abstract: In this paper, we propose a distributed stochastic model predictive control (DSMPC) algorithm for a team of linear subsystems sharing coupled probabilistic constraints. Each subsystem is subject to both parameter uncertainty and stochastic disturbances. To handle the probabilistic constraints, we first decompose the state trajectory into a nominal part and an uncertain part. The latter one is further divided into two parts: one is bounded by probabilistic tubes that are calculated offline by making full use of the probabilistic information on disturbances, whereas the other is bounded by polytopic tubes whose scaling is optimized online and whose facets’ orientations are chosen offline. Under the update strategy that only one subsystem is permitted to optimize at each time step, probabilistic constraints are transformed into linear constraints, and the original optimization problem is then formulated as a convex problem. In addition, this new algorithm does not rely on instantaneous inter-subsystem exchanges of data during a time step, and therefore may have a relatively low susceptibility to communication delay. By constructing a decoupled terminal set for each subsystem, the proposed algorithm guarantees recursive feasibility with respect to both local and coupled probabilistic constraints and ensures stability in closed-loop operation. Finally, numerical simulations illustrate the efficacy of the theoretical results.

Key Words: Stochastic systems, Probabilistic constraints, Model predictive control (MPC), Distributed control

1 Introduction

Distributed model predictive control (DMPC) has been widely used in practical applications mainly due to its structural flexibility and capability to handle constraints, such as formation control of multi-agent systems. Over the past years, many important results of the deterministic systems have been proposed, and the guaranteed stability and the online computation burden of DMPC have been particularly addressed [1, 2]. However, for systems with parameter uncertainty and stochastic disturbances, the research of DMPC is still in an embryonic stage, and lots of difficult but rather important problems still remain to be solved. For example, how to explicitly deal with parameter uncertainty and stochastic disturbances in a less conservative method; how to design the terminal set for each subsystem to formulate a tractable optimization problem with a finite number of constraints.

To the best of our knowledge, all results on stochastic MPC (SMPC) accounting for both parameter uncertainty and stochastic disturbances are only developed for a single system [3, 4], and the results of DMPC are restricted to a group of subsystems subject to a single uncertainty source. For the case of a team of subsystems with stochastic disturbances, a distributed robust MPC (DRMPC) algorithm was proposed in [5], which can guarantee the recursive feasibility and stability of the overall system by using the constraint-tightening method in [6]. It required that all subsystems must be optimized and communicate with each other during each sampling period, further resulting in excessive communication. [7] improved the result in [5] and achieved a significant reduction in communication cost by using the strategy that only one subsystem is optimized at each time. Recently a cooperative DSMPC method based on output feedback was proposed in [8] which can guarantee the recursive feasibility and the quadratic stability in the presence of disturbances. By using the strategy in [7] and the probabilistic information on disturbances, a set of deterministic constraints are constructed to ensure the satisfaction of coupled probabilistic constraints. For the case of parameter uncertainty, [9] proposed a DSMPC algorithm based on generalized polynomial chaos expansions (GPCEs), which had no restrictions on the distributions of the uncertainty. However, this approach may involve too many constraints if one requires to provide high-precision approximation.

Motivated by the discussion above, this paper considers a more challenging problem of DSMPC for a group of linear subsystems subject to parameter uncertainty and stochastic disturbances as well as coupled probabilistic constraints. To make the optimization problem tractable in a distributed manner, we adopt the update strategy of [7] that only one subsystem is permitted to optimize at each time step. Following the method of [10], the uncertain part of the state trajectory is decomposed into two parts: one is handled by calculating a sufficient tightening margin offline by using the distribution function on disturbances, whereas the other is handled by using polytopic tubes with facets of fixed orientation and online adjustable tube scalings. The main contributions of the paper are summarized as follows: (i) By using the tube methodology, we transform both local and coupled probabilistic constraints into linear deterministic constraints in a less conservative method, further leading to a convex optimization problem, which can be efficiently solved. (ii) Thanks to the distributed terminal sets designed offline, both the recursive feasibility of the optimization problem and the stability of the overall system are guaranteed. (iii) Exchange
of data with other subsystems is only required after one sub-

The remainder of the paper is organized as follows. The

problem statement is presented in Section 2. Section 3

proposes a DSMPC algorithm, including probabilistic con-

straint handling strategy and construction of terminal sets,

and analyses the recursive feasibility and closed-loop sta-

bility. Section 4 demonstrates the effectiveness of our

approach by one numerical simulation. Finally, the conclu-

sion is drawn in Section 5.

2 Problem Statement

Consider a system of $N_p$ dynamically decoupled subsys-

tems with $x_p \in \mathbb{R}^{N_{x,p}}$, $u_p \in \mathbb{R}^{N_{u,p}}$, and $w_p \in \mathbb{R}^{N_{w,p}}$.

$$x_p(k+1) = A_p(k)x_p(k) + B_p(k)u_p(k) + w_p(k), \quad \forall k \in \mathbb{N}, \; p \in \mathcal{P} \triangleq \{1, 2, \ldots, N_p\}. \quad (1)$$

The disturbances $w_p(k)$, for $k \geq 0$, are independent and

identically distributed (i.i.d.) and the elements of $w_p(k)$ are

zero-mean with known distributions satisfying $w_p \in \mathcal{W}_p \triangleq \{w_p \mid |w_p| \leq \alpha_p\}$, where $\alpha_p = [\alpha_{p,1}, \alpha_{p,2}, \ldots, \alpha_{p,N_{w,p}}]^T > 0$. $A_p(k)$ and $B_p(k)$ are defined by

$$[A_p(k) B_p(k)] = [A_0^p B_0^p] + \sum_{j=1}^L [A_p^{(j)} B_p^{(j)}] q_p^{(j)}(k), \quad (2)$$

$$\sum_{j=1}^L q_p^{(j)}(k) = 1, \quad q_p^{(j)}(k) \geq 0, \quad (3)$$

where $q_p^{(j)}(k) \in \mathbb{R}, \; k \geq 0$, are i.i.d. random variables such that $\mathbb{E} \left[\sum_{j=1}^L [A_p^{(j)} B_p^{(j)}] q_p^{(j)}(k)\right] = 0$.

The predicted control input sequence is formulated as

$$u_p(k+i) = K_p x_p(k+i) + e_p(k+i), \quad i = 0, 1, \ldots, e_p(k+i) = 0, i \geq N, \quad (4)$$

where $e_p(k+i), i = 0, 1, \ldots, N - 1$, are optimization variables for some finite prediction horizon $N$. $u_p = K_p x_p$ is assumed to stabilize subsystem (1) by choosing $K_p$ offline.

The system is also subject to both local and coupled prob-

abilistic constraints

$$Pr\{\eta_p^T x_p(k) \leq h_p\} \geq l_p, \forall p \in \mathcal{P}, \quad (5)$$

$$Pr\{\sum_{p=1}^{N_p} \mu_p^T x_p(k) \leq b_c\} \geq \tilde{l}_c, \forall c \in \mathcal{C} \triangleq \{1, 2, \ldots, N_c\}, \quad (6)$$

where $\eta_p, \mu_p \in \mathbb{R}^{N_{x,p}}; h_p, b_c \in \mathbb{R}; l_p \in [0, 1]$ and $\tilde{l}_c \in [0, 1]$ are prescribed thresholds. Note that the hard constraints can be treated as a special case of constraints (5)-(6) by setting $l_p = \tilde{l}_c = 1$.

Before ending this section, three sets of [7] are introduced to facilitate the problem formulation, which are defined as

$$\mathcal{P}_c \triangleq \{p \in \mathcal{P} \mid \mu_p \neq 0\},$$

$$\mathcal{C}_p \triangleq \{c \in \mathcal{C} \mid \mu_p \neq 0\},$$

$$\mathcal{Q}_p \triangleq (\bigcup_{c \in \mathcal{C}_p} \mathcal{P}_c) \setminus \{p\}.$$ 

3 DSMPC Algorithm

At time $k$, only subsystem $p_k$ is permitted to optimize for a new perturbation sequence

$$e_{p_k}(k) \triangleq [e_{p_k}(k) \; e_{p_k}^T(k+1) \; \cdots e_{p_k}^T(k+N-1|k)].^T$$

Meanwhile, all other subsystems $p \neq p_k$ adopt the feasible perturbation sequences

$$\tilde{e}_p(k) \triangleq [e_{p}^T(k)|k-1) \; \cdots e_{p}^T(k+N-2|k-1) 0].^T.$$ 

To construct an implementable optimization problem, a s-

trategy to handle the probabilistic constraints is first investi-

3.1 Probabilistic Constraint Handling Strategy

To facilitate constraint handling, the state trajectory predicted at time $k$ is split into two parts

$$x_p(k+i) = z_p(k+i) + e_p(k+i), \quad (7)$$

$$z_p(k+i+1) = \Psi_p z_p(k+i) + B_0^p e_p(k+i), \quad (8)$$

$$e_p(k+i+1) = \Psi_p e_p(k+i) + \Psi_p(k+i)\psi_p(k+i) + \Psi_p(k+i)\psi_p(k+i) + \Psi_p(k+i)\psi_p(k+i) + \Psi_p(k+i)\psi_p(k+i) + \Psi_p(k+i)\psi_p(k+i), \quad (9)$$

with $\Psi_p = A_0^p + B_0^p K_p, \Psi_p(k+i) = A_p(k+i) + B_p(k+i) K_p, \Psi_p(k+i) = \Psi_p(k+i) - \Psi_p^0$, and $\tilde{B}_p(k+i) = B_p(k+i) - B_0^p$. The constraints (5)-(6) can now be expressed as

$$Pr\{\eta_p^T z_p(k+i) + \mu_p^T e_p(k+i) \leq h_p\} \geq l_p, \quad (10)$$

$$Pr\{\sum_{p=1}^{N_p} [\mu_p^T z_p(k+i) + \mu_p^T e_p(k+i)] \leq b_c\} \geq \tilde{l}_c. \quad (11)$$

Following [10], $e_p(k+i)$ is further split into two parts

$$e_p(k+i) = e_p(k+i) + c_p(k+i), \quad (12)$$

$$e_p(k+i+1) = \Psi_p e_p(k+i) + w_p(k+i), \quad (13)$$

$$c_p(k+i+1) = \Psi_p(k+i) c_p(k+i) + \Psi_p(k+i) c_p(k+i) + \Psi_p(k+i) c_p(k+i) + \Psi_p(k+i) c_p(k+i) + \Psi_p(k+i) c_p(k+i), \quad (14)$$

with $c_p(k) = 0$ and $z_p(k) + c_p(k) = x_p(k)$. After the decomposition, the two components of $e_p$ in (10)-(11) can be treated separately in two steps.

Step One: Handle $e_p$ using the distribution function of $w_p$. Since the evolution of $e_p$ in (13) is described by a linear time-invariant system with disturbance, as in [8] and [12], the conditions which ensure the satisfaction of probabilistic constraints (10)-(11) are given below.
Theorem 1  For subsystem $p$, the satisfaction of (10)-(11) is guaranteed, if $c_p(k)$ and $z_p(k)$ satisfy the recursively feasible constraints, for $i = 1, 2, \ldots,$

$$
\eta_p^T H_p^i c_p(k) + \eta_p^T (\Psi_p^0)^i z_p(k) + \eta_p^T q_p(k + i|k) \leq h_p - \beta_p^i,
$$

(15)

where $H_p^i = [(\Psi_p^0)^{-1} B_p^0 (\Psi_p^0)^{-2} B_p^0 \ldots B_p^0 0 \ldots 0]$. $\beta_p^i$ and $\kappa_p^i$, respectively, are the maximum elements of the $i$th column of

$$
\begin{bmatrix}
\gamma_p^1 & \gamma_p^2 & \ldots & \\
0 & \gamma_p^2 + a_p & \ldots & \\
0 & 0 & \gamma_p^3 + a_p + a_p & \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & \xi_c^1 + d_c^1 & \ldots \\
0 & 0 & \xi_c^1 + d_c^1 + d_c^2 & \ldots \\
& & & \\
\end{bmatrix},
$$

(17)

with

$$
a_p = \max_{w_p \in W_p} \eta_p^T (\Psi_p^0)^i w_p = \eta_p^T (\Psi_p^0)^i |a_p|,
$$

$$
d_c = \max_{w_p \in W_p} \sum_{p=1}^{N_p} \mu_p^T (\Psi_p^0)^i w_p = \sum_{p=1}^{N_p} |\mu_p^T (\Psi_p^0)^i| |a_p|.
$$

$\gamma_p^i$ and $\xi_c^i$ are the minimum values such that, for $i = 1, 2, \ldots,$

$$
Pr[\eta_p^T (\Psi_p^0)^i w_p(k) + \eta_p^T (\Psi_p^0)^i-2 w_p(k + 1|k) + \ldots + \eta_p^T w_p(k + i - 1|k) \leq \gamma_p^i] = l_p,
$$

(19)

$$
Pr[\sum_{p=1}^{N_p} |\mu_p^T (\Psi_p^0)^i w_p(k) + \mu_p^T (\Psi_p^0)^i-2 w_p(k + 1|k) + \ldots + \mu_p^T w_p(k + i - 1|k) \leq \xi_c^i] = l_c.
$$

(20)

Proof. By iteration of (8) and (13), the predictions at time $k$ are made as follows

$$
z_p(k + i|k) = (\Psi_p^0)^i z_p(k) + H_p^i c_p(k),
$$

(21)

$$
\varepsilon_p(k + i|k) = (\Psi_p^0)^i-1 w_p(k) + (\Psi_p^0)^i-2 w_p(k + 1|k) + \ldots + w_p(k + i - 1|k).
$$

(22)

Substituting them into (10)-(11) and combining the definitions of $\gamma_p^i$ and $\xi_c^i$ in (19)-(20), we conclude that if $\beta_p^i$ and $\kappa_p^i$ in (15)-(16) are replaced by $\gamma_p^i$ and $\xi_c^i$ in the first row of the matrices (17)-(18) respectively, then (15)-(16) can guarantee that (10)-(11) are satisfied at time $k$. However, since $\gamma_p^i$ and $\xi_c^i$ are computed based on the information available at time $k$, this cannot guarantee the future feasibility.

To ensure the recursive feasibility, consider the prediction of (13) at time $k + j$

$$
\varepsilon_p(k + j + i|k) = (\Psi_p^0)^i \varepsilon_p(k + j) + (\Psi_p^0)^i-1 w_p(k + j) + \ldots + w_p(k + j + i - 1|k).
$$

(23)

Note that, at time $k + j$ , $\varepsilon_p(k + j)$ cannot be treated as a random variable. Therefore, when computing the distribution of $\varepsilon_p(k + j + i|k + j)$, we must consider the worst-case value for $\varepsilon_p(k + j)$ over the class of allowable uncertainty

$$
\max_{w_p \in W_p} \eta_p^T (\Psi_p^0)^i \varepsilon_p(k + j) = \sum_{t=1}^{i+j-1} a_{p,t}^i,
$$

(24)

$$
\max_{w_p \in W_p} \sum_{t=1}^{N_p} \mu_p^T (\Psi_p^0)^i \varepsilon_p(k + j) = \sum_{t=1}^{i+j-1} d_{c,t}^i.
$$

(25)

The $i$th elements of the $(j + 1)$th rows of (17)-(18) are the predicted values of $\eta_p^T \varepsilon_p(k + j + i|k + j)$ and $\sum_{p=1}^{N_p} \mu_p^T \varepsilon_p(k + j + i|k + j)$, which are obtained by considering the worst-case values for $\eta_p^T (\Psi_p^0)^i \varepsilon_p(k + j)$ and $\sum_{p=1}^{N_p} \mu_p^T (\Psi_p^0)^i \varepsilon_p(k + j)$.

Therefore, the satisfaction of probabilistic constraints (5)-(6) and the recursive feasibility are guaranteed, if $\beta_p^i$ and $\kappa_p^i$ are selected as the maximum elements of the $i$th columns of (17)-(18), respectively.

In (16), note that $c_p(k)$ and $z_p(k)$, $q \in Q_p$, are not affected by the optimization variables, so they can be treated as fixed values, denoted by $c_p^c(k)$ and $z_p^c(k)$.

Lemma 1 [12]. The sequences $\{\beta_p^i, \beta_p^2, \ldots\}$ and $\{\kappa_p^i, \kappa_p^2, \ldots\}$ are monotonically non-decreasing and converge to $\beta_p$ and $\kappa_p$, respectively. Furthermore, the arbitrarily tight upper bounds on $\beta_p$ and $\kappa_p$ are given as

$$
\beta_p \leq \gamma_p^1 + \sum_{i=1}^{v_p-1} a_{p,i}^1 + \frac{v_p}{\mu_p} \|\eta_p\|_{S_p}, \forall p \in P,
$$

(26)

and

$$
\kappa_p \leq \xi_c^1 + \sum_{i=1}^{\mu_p-1} d_{c,i}^1 + \sum_{i=1}^{N_p} \frac{\mu_p}{\mu_p} \|\mu_p\|_{S_p}, \forall c \in C,\forall p \in P.
$$

(27)

Step Two: Handle $q_p$ using polytopic tubes. In this step, a polytopic tube with bounded facets of fixed orientation is constructed online to bound $q_p$. The cross-section of the tube is parameterized in the form

$$
\{q_p(k + i|k) \mid V_p q_p(k + i|k) \leq \theta_p(k + i|k)\}, i \geq 0,
$$

(28)

where $V_p \in \mathbb{R}^{N_p \times N_p}$ is chosen offline and $\theta_p \in \mathbb{R}^{N_p \times p}$ is optimized online.

Theorem 2 For any $q_p(k + i|k) \in \{q \mid V_p q \leq \theta_p(k + i|k)\}$, a necessary and sufficient condition for $V_p q_p(k + i|k + 1) \leq \theta_p(k + i|k + 1)$ is that for $i = 0, 1, \ldots$, there exist nonnegative matrices $M_p^{(i)} \succeq 0, i = 1, 2, \ldots, L$, satisfying

$$
M_p^{(i)} V_p = V_p (\Psi_p^0 + \Psi_p^{(i)}),
$$

(29)

$$
M_p^{(i)} \theta_p(k + i|k) + V_p w_p^{(i)} z_p(k + i|k) + V_p B_p^{(i)} c_p(k + i|k)
$$

(30)

where

$$
d_{c,i}^{(i)} = \max_{w_p \in W_p} V_p w_p^{(i)} \varepsilon_p(k + i|k).
$$

Proof. From (14) and Proposition 3.31 in [13], it is easy to obtain (29)-(30).
Remark 3.1 For each vertex $j = 1, 2, \ldots, L$, the sequence \{d_p^{(j)}(0), d_p^{(j)}(1), \ldots\} converges to a limit $d_p^{(j)}$. The arbitrarily close upper bound on $d_p^{(j)}$ can be computed by
\[
\bar{d}_p^{(j)} \leq \sum_{l=0}^{\bar{\tau}_p-1} b_p^{(j)}(l) + \frac{\rho_p^{(j)}}{\tau_p} \| (V_p \Psi_p^{(j)})^T \|_{sp},
\]
where $b_p^{(j)}(l) = \max_{w_p \in W_p} V_p \Psi_p^{(j)} (\Psi_p^{(j)})^+ w_p$ and $\tau_p$ is any given integer.

Theorem 3 reformulates (15)-(16) as linear constraints by using the architecture of tube closed-section (28).

Theorem 3 For any $\zeta_p(k + i|k) \in \{ \zeta \mid V_p \zeta \leq \theta_p(k + i|k) \}$, (15)-(16) for $i = 1, 2, \ldots$ are satisfied if and only if there exist nonnegative matrices $M_p \geq 0$ and $M_{cp} \geq 0$ satisfying
\[
M_p V_p = \eta_p^T,
\]
\[
M_p \theta_p(k + i|k) + \beta_p + \eta_p^T H_p e_p(k) + \eta_p^T (\Psi_p^T)^* z_p(k) \leq h_p,
\]
\[
M_{cp} V_p = \mu_{cp}^T,
\]
\[
M_{cp} \theta_p(k + i|k) + \kappa_p + \mu_{cp}^T H_p e_p(k) + \mu_{cp}^T (\Psi_p^T)^* z_p(k) + \sum_{q \in \mathcal{P}_s \setminus \{ p \}} [\mu_{cp}^T H_q e_q^* + \mu_{cp}^T (\Psi_q^T)^* z_q(k) + M_{cp} \theta_q(k + i|k)] \leq b_c, \forall c \in \mathcal{C}.
\]

In (35), $\theta_q(k + i|k)$ can also be regarded as a fixed value. To relax constraints (30), (33), and (35) applied online, $\theta_p$, $M_p$, and $M_{cp}$ can be calculated offline as the solutions of the linear programing [111]
\[
(M_p^{(j)})_i = \arg \min_{m^T} \{ 1^T m \mid m^T V_p = (V_p \psi (\Psi_p^0 + \Psi_p))_i, m \geq 0 \},
\]
\[
M_p = \arg \min_{m^T} \{ 1^T m \mid m^T V_p = \eta_p^T, m \geq 0 \},
\]
\[
M_{cp} = \arg \min_{m^T} \{ 1^T m \mid m^T V_p = \mu_{cp}^T, m \geq 0 \},
\]
where $(M)$ is the $i$th row of matrix $M$.

3.2 Construction of Terminal Sets
To use an infinite prediction horizon cost but keep a finite number of constraints, $z_p(k + N|k)$ and $\theta_p(k + N|k)$, $N_1 > N$, will be bounded to lie in some terminal set. By requiring $z_p(k + N|k) \in \chi_p^0$, where $\chi_p^0$ is a robust positively invariant set for $z_p(k + 1) = \Psi_p z_p(k) + (\Psi_p^0)^N w_p(k)$, $z_p(k + 1|k)$ will enter a smaller set $\chi_p^0 \triangleq (\Psi_p^0)^{N_1} I = N_1 - N$. Based on it, Theorem 4 designs a terminal set for $\theta_p(k + N|k)$.

Theorem 4 Define $\bar{z}_p^{(j)} = \sum_{z \in \chi_p^0} (V_p \Psi_p^{(j)})^T z_p^{(j)}$, $\bar{f}_p^{(j)} = \max_{z \in \chi_p^0} \{ \eta_p^T z_p^{(j)} \}$, and $f_{cp}^{(j)} = \max_{z \in \chi_p^0} \{ \eta_p^T z_p^{(j)} \}$. If $\bar{\theta}_p$ satisfies
\[
\bar{\theta}_p \geq \frac{\max \{ \bar{z}_p^{(j)} - \eta_p^T M_p^{(j)} \| \infty + \| M_p^{(j)} \| \infty \}}{1 - \max \{ M_p^{(j)} \| \infty \}},
\]
\[
\bar{\theta}_p \leq \frac{h_p - \beta_p - \bar{f}_p^{(j)} \| \infty }{\| M_p^{(j)} \| \infty },
\]
\[
\sum_{p=1}^{N_p} \| M_{cp} \| \infty \bar{\theta}_p \leq b_c - \kappa_c \leq \sum_{p=1}^{N_p} \| f_{cp}^{(j)} \| \infty ,
\]
then $\chi_p^0 \times B_p^+$ is an invariant set with respect to
\[
z_p(k + i + 1|k) = \Psi_p z_p(k + i|k),
\]
\[
\theta_p(k + i + 1|k) = \max_j \{ M_p^{(j)} \theta_p(k + i|k) + d_p^{(j)}(i) + V_p \Psi_p^{(j)} z_p(k + i|k) \}, i \geq N_1,
\]
and constraints (33) and (35) are satisfied in $\chi_p^0 \times B_p^+$.

Due to space limitations, the proof of Theorem 4 is omitted here. In Theorem 4, (39) ensures that the set $\chi_p^0 \times B_p^+$ is invariant for the dynamics given in (42)-(43) and (40)-(41) are conditions of $\theta_p$ corresponding to the local constraint (33) and the coupled constraint (35) respectively.

3.3 Optimization Problem Formulation and Distributed Implementation Algorithm
Define the cost function in terms of the nominal state as
\[
J^k_p = \sum_{i=0}^{\bar{\tau}_p} \| z_p(k + i|k) \|_{\tilde{Q}_p} + \| K_p \|_{\tilde{R}_p},
\]

where $z_p(k + i|k)$ is any $m$-dimensional vector and $Q_p$ and $R_p$ are symmetric positive-definite. Lemma 2 states that (44) can be rewritten as a quadratic function of the optimization variables.

Lemma 2 Let $W_p$ be the solution of
\[
W_p = \left[ \begin{array}{ccc}
\Psi_p^0 & B_p^0 E_p^T & T_p^T \\
0 & 0 & T_p^T \\
T_p & 0 & 0
\end{array} \right]^T \Psi_p^0 W_p \left[ \begin{array}{ccc}
\Psi_p^0 & B_p^0 E_p^T & T_p^T \\
0 & 0 & T_p^T \\
T_p & 0 & 0
\end{array} \right] = \tilde{Q}_p,
\]

where $T_p$ is a matrix with the identity matrix on its super-diagonal blocks and zeros elsewhere, $E_p = \left[ I \ 0 \ \cdots \ 0 \right]$, $\tilde{Q}_p = \left[ Q_p + K_p^T R_p K_p \ K_p^T R_p E_p^T \ E_p^T R_p K_p \ E_p^T R_p E_p \right]$, then we have
\[
J^k_p = \left\| \begin{array}{c}
z_p(k) \\
\eta_p^T \bar{c}_p
\end{array} \right\|_{W_p}^2.
\]

Proof. Pre- and post- multiplication of (44) by $[z_p^T(k + i|k) c_p^T(k + i|k)]$ and $[z_p^T(k + i|k) c_p^T(k + i|k)]^T$ and then summing over $i = 0, 1, \ldots$ will lead to (46).

The MPC optimization problem $P_p(x_p(k); Z_p^i(k))$ can now be formulated as follows.

\[
\begin{array}{ll}
\min_{x_p(k), \eta_p^T \bar{c}_p, \ldots, \theta_p(k + N|k)} & J^k_p = \left\| \begin{array}{c}
z_p(k) \\
\eta_p^T \bar{c}_p
\end{array} \right\|_{W_p}^2 \\
\text{s.t.} & V_p(x_p(k)) - z_p(k) \leq \theta_p(k), x_p(k) = x_p(k) \\
& \forall p \in \{ 0, 1, \ldots, N_1 - 1 \} \setminus \{ j \in \{ 1, 2, \ldots, L \} \} \\
& z_p(k + i + 1|k) = \Psi_p z_p(k + i|k) + B_p^0 c_p(k + i|k) \\
& M_p^j \theta_p(k + i|k) + V_p \Psi_p^j z_p(k + i|k) + V_p B_p^j c_p(k + i|k) + d_p^{(j)}(i) \leq \theta_p(k + i + 1|k)
\end{array}
\]
∀i ∈ \{1, 2, \ldots, N_1 - 1\}:
\[
M_p \eta_p(k_i) \leq \theta_p.
\]

Next, we will show that
\[
\{T_p \eta_p(k_0), z_p(k_0 + 1|k_0) + w_p(k_0), \theta_p(k_0 + 1|k_0), \ldots, \theta_p(k_0 + N_1|k_0), \max_j \{M_p^{(j)} \mid \theta_p(k_0 + N_1|k_0) + V_p \Psi_p^{(j)} z_p(k_0 + N_1|k_0) + d_p^{(j)}(N_1)\}\}
\]
is a feasible solution at time k_0 + 1. From the fact that \(z_p(k_0 + 1|k_0) = x_p(k_0 + 1) - z_p(k_0 + 1|k_0) - \zeta_p(k_0 + 1|k_0)\), (47) is satisfied at time k_0 + 1. The constraints (48) and (49)-(50) for \(i = 0, 1, \ldots, N_1 - 2\) are satisfied directly at time k_0 + 1. Since \(d_p^{(j)}(N_1) \geq V_p \Psi_p^{(j)} (\Psi_p^{(j)})^{N_1} - 1 w_p(k_0) + d_p^{(j)}(N_1 - 1)\) and \(\beta_p \geq \beta_p^{(N_1)} \geq d_p^{(j)}(N_1)\), (49)-(50) also hold for \(i = N_1 - 1\). With similar argument, from \(\kappa_p^{N_1} + 1 = \kappa_p + d_p\), (51) is satisfied at time k_0 + 1. From the definitions of \(\chi_p^0\) and \(\beta_p^1\), we can verify that the terminal constraints (52)-(53) are satisfied at time k_0 + 1. Hence the candidate solution (58) is a feasible solution to \(P_\text{p}(x_p(k_0 + 1); Z_p^T(k_0 + 1))\). The recursive feasibility of \(P_\text{p}(x_p(k_0); Z_p^T(k_0))\) follows immediately by recursion.

The recursive feasibility implies that the sequence \(\{c_p(0), c_p(1), \ldots\}\) is in \(l^2\), and therefore \(c_p(k)\) tends to zero as \(k \to \infty\). As a consequence, the MPC control law converges asymptotically to the unconstrained optimal control law \(u_{\text{p}}(k) = \bar{K}_p x_p(k)\) which can stabilize the subsystem (1) by assumption, and meanwhile the state of each subsystem converges asymptotically to a set of states in which probabilistic constraints are satisfied by applying the unconstrained optimal control law.

4 Numerical Example

The system of three subsystems is defined by
\[
A_p^0 = \begin{bmatrix} 1.6 & 1 \\ -0.5 & 1.2 \end{bmatrix}, \quad B_p^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]
\[
A_p^{(1)} = \begin{bmatrix} 0.05 & -0.15 \\ -0.01 & 0.01 \end{bmatrix}, \quad B_p^{(1)} = \begin{bmatrix} -0.05 \\ 0.05 \end{bmatrix},
\]
\[
A_p^{(2)} = -A_p^{(1)}, \quad B_p^{(2)} = -B_p^{(1)}.
\]

For each subsystem \(i\), a random variable uniformly distributed over \([0, 1]\) and \(q_p^{(i)}(k) = 1 - q_p^{(i)}(k)\). Each element of \(w_p\) is truncated from a normal distribution with mean zero and variance \(1/\sigma^2\) and satisfies \(||w_p||_\infty \leq 0.12\). The constraint parameters are given by
\[
\eta_p = [0.8 - 0.2]T, \quad h_p = 2.1, \quad l_p = 0.8;
\]
\[
\mu_p = [0.8 - 0.2]T, \quad b_p = 5.5, \quad \tilde{l}_p = 0.8.
\]
Choose \(Q_p = [1, 0.3]^T[1, 0.3],\quad R_p = 0.1, \quad N = 4, \quad N_1 = 7, \quad N_2 = 8\). The update sequence adopted is \(\{1, 2, 3, 1, 2, 3, \ldots\}\).

For the initial condition \(x_1(0) = [3.5 - 3.5]^T, x_2(0) = [3.4 - 3.45]^T, x_3(0) = [3.5 - 3.25]^T\).

Fig. 1-Fig. 4 respectively show the evolution of \(x_p(k)\) for each subsystem and \(\sum_{p=1}^3 x_p(k)\) by applying Algorithm 1 and the unconstrained optimal control for 100 Monte Carlo simulations.
5 Conclusion

By using the tube methodology, a DSMPC algorithm is proposed for multiple subsystems subject to parameter uncertainty and disturbances as well as coupled probabilistic constraints. We divide the uncertain component of the state trajectory into two parts and deal with them separately in two steps: one part with respect to disturbances is first constrained to lie in probabilistic tubes which are derived offline from the distribution of disturbances; the other part is then constrained to lie in polytopic tubes with facets of fixed orientation determined offline, and the cross-section of the tube is determined online. By applying the update strategy that only one subsystem is optimized at each time step, the coupled probabilistic constraints are satisfied in a distributed manner. The formulated MPC optimization problem for each subsystem is a convex optimization, which can be effectively solved by many available methods. The proposed algorithm guarantees the satisfaction of probabilistic constraints, and ensures the recursive feasibility and the stability. A numerical example is given to illustrate the efficacy of the algorithm.

References