

Asymptotic Theory for Kernel Estimators under Moderate Deviations from a Unit Root, with an Application to the Asymptotic Size of Nonparametric Tests

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Abstract

We provide new asymptotic theory for kernel density estimators, when these are applied to autoregressive processes exhibiting moderate deviations from a unit root. This fills a gap in the existing literature, which has to date considered only nearly integrated and stationary autoregressive processes. These results have applications to nonparametric predictive regression models. In particular, we show that the null rejection probability of a nonparametric t test is controlled uniformly in the degree of persistence of the regressor. This provides a rigorous justification for the validity of the usual nonparametric inferential procedures, even in cases where regressors may be highly persistent.

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1 Introduction

Consider the predictive regression model

$$y_t = m(x_{t-1}) + u_t \quad (1.1)$$

where m is an unknown function and u_t is a martingale difference sequence. x_t is a time series with an unknown – but possibly very high – degree of persistence, which we shall parametrise as

$$x_t = \rho x_{t-1} + v_t, \quad (1.2)$$

for $\rho \in \mathcal{P} := [-1 + \delta, 1]$, where v_t is weakly dependent.

In this setting, parametric estimators of m are known to have a limiting distribution that is non-Gaussian, and which depends on the proximity of ρ to unity. The difficulties that this poses for inference has spawned a large literature (see e.g. Cavanagh, Elliott, and Stock, 1995; Campbell and Yogo, 2006; Jansson and Moreira, 2006; Magdalinos and Phillips, 2009; Phillips and Lee, 2013; and Elliott, Müller, and Watson, 2015). In contrast, nonparametric estimators of m have been shown to be asymptotically normal even when regressors are nearly integrated (see, in particular, Wang and Phillips, 2009a,b). Because this is also true when x_t is stationary, it has been recently argued by Kasparis, Andreou, and Phillips (2015, hereafter KAP) that valid inferences on m may be drawn simply by referring a nonparametric t statistic to normal critical values. That is, so far as *nonparametric* inferences are concerned, it is not necessary to make any adjustments when ρ is close to unity. KAP provide some simulation evidence in support of this claim.

The primary motivation for the present work is to provide a rigorous proof of the asymptotic validity of the nonparametric t test, in the setting of the model (1.1)–(1.2), thereby putting KAP’s thesis on a surer footing. What do we mean by ‘asymptotic validity’ in this context? For a test of

$$H_0 : m(x) = \theta \quad \text{against} \quad H_1 : m(x) \neq \theta$$

(for a chosen $x \in \mathbb{R}$), the null rejection probability of the t test needs to be controlled uniformly over all parameters left unrestricted by H_0 : in particular, over all $\rho \in \mathcal{P}$. (For a discussion of this issue in more general contexts, see for example Romano, 2004, Mikusheva, 2007, and Andrews, Cheng, and Guggenberger, 2011.) More precisely, the nonparametric t test is said to be asymptotically of size α if

$$\limsup_{n \rightarrow \infty} \sup_{\rho \in \mathcal{P}} \mathbb{P}_\rho \{ |\hat{t}_n(x)| \geq z_{1-\alpha/2} \} \leq \alpha, \quad (1.3)$$

where the ‘ ρ ’ subscript on \mathbb{P}_ρ indicates the dependence of this probability on the value of ρ in (1.2), z_τ denotes τ th quantile of the standard normal distribution, and

$$\hat{t}_n(x) = s_n(x)^{-1} [\hat{m}_n(x) - \theta]$$

denotes the nonparametric t statistic for $\hat{m}_n(x)$, the local level (Nadaraya–Watson) estimator of m at x , and $s_n^2(x)$ an estimate of its asymptotic variance (see Section 2.2 below for precise definitions).

Existing limit theory for nonparametric regression estimators establishes that $\hat{t}_n(x)$ is asymptotically normal when ρ is either fixed and in the stationary region ($\rho < 1$), or is local to unity in the sense that $\rho = 1 + c/n$ (see, e.g., Wu and Mielniczuk, 2002; Wang and Phillips, 2009a,b; and KAP). As we shall argue in Section 2, these results are sufficient only to establish what might be termed the ‘pointwise asymptotic validity’ of the t test, i.e. that

$$\limsup_{n \rightarrow \infty} \mathbb{P}_\rho\{|\hat{t}_n(x)| \geq z_{1-\alpha/2}\} \leq \alpha, \quad \forall \rho \in \mathcal{P}.$$

To prove (1.3), we additionally need to show that $\hat{t}_n(x) \rightsquigarrow N[0, 1]$ when x_t exhibits ‘moderate deviations from a unit root’, in the sense that $\rho_n \rightarrow 1$ but $n(1 - \rho_n) \rightarrow \infty$; we refer to these as *mildly integrated processes* (see Giraitis and Phillips, 2006; Phillips and Magdalinos, 2007).

Accordingly, Section 3 of this paper provides new asymptotic theory for sums of integrable transformations of mildly integrated processes – i.e. for kernel density estimators applied to such processes. This fills a significant gap in the existing technical literature, and allows for a successful proof of (1.3). The development of this theory relies on an interesting combination of arguments appropriate to stationary and local-to-unity processes. The dependence of mildly integrated processes is sufficiently weak that kernel density estimators converge not to the local time of some limiting process, but to the standard normal density. In particular, we have

$$\frac{d_n}{nh_n} \sum_{t=1}^n f\left(\frac{x_t - d_na}{h_n}\right) \xrightarrow{p} \varphi(a) \int_{\mathbb{R}} f,$$

where f is an integrable function, $d_n := \text{var}(x_n)$, $\varphi(a) := (2\pi)^{-1/2}e^{-a^2/2}$, and $h_n = o(1)$ is a bandwidth sequence. In this respect, mildly integrated processes are more akin to stationary processes, except for the noted normality of the limiting density. On the other hand, they also share the diminished recurrence and slower rates of convergence characteristic of local-to-unity processes.

The rest of this paper is organised as follows. We begin by outlining a simplified version of the inferential problem studied by KAP (Sections 2.1–2.2). We then provide an explanation of how the asymptotic validity of the t test – in the sense of (1.3) above – may be established with the aid of new results on integrable transformations of mildly integrated processes (Section 2.3). These results are developed in Section 3. Proofs of the main results appear in Appendices A–C. Proofs of technical results that are either conceptually straightforward, or closely related to those that have already appeared in the literature, are given in the Online Supplement to this article, available at Cambridge Journals Online (journals.cambridge.org/ect).

Notation. All limits are taken as $n \rightarrow \infty$ unless otherwise stated. \mathbb{R} denotes the real

numbers. For sequences $\{a_n\}, \{b_n\}$: $a_n \asymp b_n$ denotes $\lim_{n \rightarrow \infty} a_n/b_n = c \in \mathbb{R} \setminus \{0\}$, and $a_n \sim b_n$ denotes $\lim_{n \rightarrow \infty} a_n/b_n = 1$. For positive sequences: $a_n \lesssim b_n$ denotes $\limsup_{n \rightarrow \infty} a_n/b_n < \infty$ – equivalently, $a_n = O(b_n)$. For random sequences $\{x_n\}, \{y_n\}$: $x_n \lesssim_p y_n$ denotes $x_n = O_p(y_n)$. \rightsquigarrow denotes weak convergence in the sense of van der Vaart and Wellner (1996), and $\rightsquigarrow_{\text{fdd}}$ the convergence of finite-dimensional distributions. For $x \geq 0$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x .

2 Nonparametric predictive regression

2.1 Data generating process

As outlined above, the data generating process (DGP) is the same as that studied by KAP. We have the following nonlinear predictive regression model

$$y_t = m(x_{t-1}) + u_t.$$

where m and the series $\{x_t, u_t\}$ are assumed to satisfy the following

Assumption DGP.

DGP1 m is Lipschitz continuous.

DGP2 $\{\varepsilon_t\}$ is a scalar i.i.d. sequence; ε_0 has a characteristic function $\psi_\varepsilon(\lambda) := \mathbb{E}e^{i\lambda\varepsilon_0}$ that is integrable, and a probability density f_ε that is Lipschitz continuous and everywhere nonzero; $\mathbb{E}\varepsilon_0 = 0$ and $\mathbb{E}\varepsilon_0^2 = 1$.

DGP3 $\{x_t\}$ and $\{v_t\}$ are generated according to

$$x_t = \rho x_{t-1} + v_t \qquad v_t := \sum_{k=0}^{\infty} \phi_k \varepsilon_{t-k}, \qquad (2.1)$$

with $x_0 = 0$; $\rho \in \mathbb{P} := [-1 + \delta, 1]$ for some $\delta > 0$; $\phi_0 \neq 0$; $\sum_{k=0}^{\infty} |\phi_k| < \infty$; and $\phi := \sum_{k=0}^{\infty} \phi_k \neq 0$.

DGP4 $\{u_t\}$ is a martingale difference sequence with respect to $\mathcal{G}_t := \sigma(\{x_s, u_s\}_{s \leq t})$, with $\mathbb{E}[u_t^2 \mid \mathcal{G}_{t-1}] = \sigma_u^2$ a.s. constant, and $\sup_t \mathbb{E}[|u_t|^4 \mid \mathcal{G}_{t-1}] < \infty$ a.s.

Remark 2.1. (a) Our assumptions closely correspond to those of KAP. In particular, DGP3 is cognate with their Assumptions 2.3 and 2.4, with the key difference that we do not restrict $\{x_t\}$ to the local-to-unity region, in which $\rho = 1 + \frac{c}{n}$ for some fixed $c \in \mathbb{R}$. We instead allow ρ to range over the entirety of $\mathbb{P} = [-1 + \delta, 1]$. On the other hand, $\sum_{k=0}^{\infty} |\phi_k| < \infty$ implies that $\{v_t\}$ is a short-memory process, and so excludes the long-memory and anti-persistent cases that are also considered in KAP. While it is likely that our results could be extended to cover these cases, we have excluded these to keep this paper to a manageable length.

(b) Owing to the initialisation $x_0 = 0$, the regressor process is nonstationary, regardless of the value of ρ . However, (2.1) has a stationary solution when $\rho < 1$, which corresponds to the weak limit of x_n as $n \rightarrow \infty$. The assumption of a fixed initialisation is made only for convenience; our results below would still hold provided x_0 is stochastically bounded (and adapted to \mathcal{G}_0).

(c) The assumption that f_ε is Lipschitz is used only in the stationary region, i.e. when $\rho < 1$, to facilitate the direct application of results from Wu, Huang, and Huang (2010). Strict positivity of f_ε is also assumed merely for convenience, to ensure that the stationary solution to (2.1) has a density that is strictly positive at every $x \in \mathbb{R}$, thereby avoiding any inadvertent attempts to estimate $m(x)$ at points of zero density. (Aside from ensuring such points are avoided, this assumption is *not* needed for Proposition 2.1 below.)

2.2 Estimation and inference

KAP develop two nonparametric tests for the ‘predictability’ of y_t by x_{t-1} , each of which involve taking either the average or the maximum of a finite collection of nonparametric t statistics, evaluated at selected points in the domain of the regressor. Critical values for these tests are derived from the normal distribution, which is justified if each of the t statistics are asymptotically normal (and asymptotically independent). In what follows, we consider a simplified version of their testing problem, which involves testing hypotheses about the value of m at a single $x \in \mathbb{R}$, by comparing a t statistic to normal critical values. The asymptotic validity of this simplified procedure is of interest in its own right, and has direct implications for the validity of the predictability tests developed by KAP.¹

Following KAP, an estimate of the regression function m , at a chosen $x \in \mathbb{R}$, is provided by the local level (Nadaraya-Watson) regression estimator,

$$\hat{m}_n(x; h) := \frac{\sum_{t=1}^n K_h(x_t - x) y_{t+1}}{\sum_{t=1}^n K_h(x_t - x)},$$

where $K : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth probability density, $h > 0$ denotes the bandwidth, and $K_h(x) := h^{-1}K(h^{-1}x)$. For the purposes of developing the asymptotics of \hat{m}_n , we shall suppose $h = h_n$, for $\{h_n\}$ a bandwidth sequence satisfying

Assumption SM (smoothing).

SM1 K is non-negative, bounded and Lipschitz, with $\int_{\mathbb{R}} |x| K(x) dx < \infty$ and $\int_{\mathbb{R}} K = 1$;

SM2 $h_n = o(1)$ and $n^{1/2}h_n \rightarrow \infty$.

Remark 2.2. The maximum rate at which h_n may shrink to zero, while still ensuring the consistency of \hat{m}_n , will be determined by the values of ρ for which $\{x_t\}$ is least recurrent – i.e. when $\rho = 1$. This accounts for the requirement that $n^{1/2}h_n \rightarrow \infty$ in SM2. This could be relaxed if h_n were chosen so as to adapt to the (unknown) recurrence of $\{x_t\}$.

¹These implications are fully developed in an earlier version of this paper (arXiv:1509.05017v3).

For each $x \in \mathbb{R}$, a test of

$$H_0 : m(x) = \theta \quad \text{against} \quad H_1 : m(x) \neq \theta \quad (2.2)$$

may then be based on the nonparametric t -statistic

$$\hat{t}_n(x) := s_n(x)^{-1}[\hat{m}(x; h_n) - \theta],$$

where

$$s_n^2(x) := \frac{\hat{\sigma}_u^2(x) \int_{\mathbb{R}} K^2}{h_n \sum_{t=1}^n K_{h_n}(x_t - x)} \quad \hat{\sigma}_u^2(x) := \frac{\sum_{t=1}^n K_{h_n}(x_t - x)[y_{t+1} - \hat{m}_n(x)]^2}{\sum_{t=1}^n K_{h_n}(x_t - x)}.$$

As in KAP, critical values for the test are provided by the quantiles of a standard normal distribution: so that for a test having nominal size α , H_0 would be rejected if $|\hat{t}_n(x)| > z_{1-\alpha/2}$, where z_τ denotes the τ th quantile of the standard normal distribution.

2.3 Asymptotic validity of the t test

The purpose of this section is to show that the testing procedure described above has the correct size asymptotically, in the sense that the nominal and actual size of the test approximately agree in large samples.

To that end, recall that the size of a test of is commonly defined as its maximum rejection probability over all values of the model parameters consistent with the null hypothesis (see e.g. Lehmann and Romano, 2005, p. 57). In the present setting, H_0 restricts only the value of m (at x), leaving the nuisance parameter $\rho \in \mathcal{P}$ entirely unrestricted.² Thus the t test for H_0 has size α asymptotically if

$$\limsup_{n \rightarrow \infty} \sup_{\rho \in \mathcal{P}} \mathbb{P}_\rho\{|\hat{t}_n(x)| \geq z_{1-\alpha/2}\} = \alpha. \quad (2.3)$$

where the ' ρ ' subscript on \mathbb{P}_ρ makes explicit the dependence of this probability of the value of ρ in (2.1). It is known from previous work – e.g. from Lemma 2 in KAP – that

$$\hat{t}_n(x) \rightsquigarrow N[0, 1] \quad (2.4)$$

for every *fixed* $\rho \in \mathcal{P}$, and indeed when $\rho = 1 + c/n$. But while this result is highly suggestive, it is insufficient to establish (2.3).

What would be sufficient for (2.3)? Since there must be a sequence $\{\rho_n^*\} \subset \mathcal{P}$ such

²We might also regard other aspects of the model, such as the distributions of ϵ_t and u_t , as (infinite-dimensional) nuisance parameters. The size of the t test would then be more properly computed by taking the maximum rejection probability over the parameter space for these distributions (as well as over $\rho \in \mathcal{P}$). Our results could be extended in this direction, but we have refrained from doing so here in order to keep the paper to a reasonable length.

that

$$\limsup_{n \rightarrow \infty} \sup_{\rho \in \mathcal{P}} \mathbb{P}_\rho\{|\hat{t}_n(x)| \geq z_{1-\alpha/2}\} = \lim_{n \rightarrow \infty} \mathbb{P}_{\rho_n^*}\{|\hat{t}_n(x)| \geq z_{1-\alpha/2}\},$$

(2.3) will follow once we have shown that (2.4) holds for the drifting sequence $\rho = \rho_n^*$. Rather than try to characterise $\{\rho_n^*\}$ and show that (2.4) holds for that specific sequence, Proposition 2.1 below establishes that (2.4) holds for *every* drifting sequence $\{\rho_n\} \subset \mathcal{P}$. This immediately implies (2.3), and carries the further implication that the t -test is asymptotically similar, in the sense that

$$\liminf_{n \rightarrow \infty} \inf_{\rho \in \mathcal{P}} \mathbb{P}_{m,\rho}\{|\hat{t}_n(x)| \geq z_{1-\alpha/2}\} = \alpha$$

holds additionally. (For a further discussion, see Andrews et al., 2011.)

Our main result on the asymptotic size of the t test may now be stated. We shall additionally assume $h_n = o(n^{-1/3})$, so as to ensure that the bias in \hat{m}_n is asymptotically negligible.³

Proposition 2.1. *Suppose DGP and SM hold, and that additionally $h_n = o(n^{-1/3})$. Then under H_0*

$$\hat{t}_n(x) \rightsquigarrow N[0, 1] \tag{2.5}$$

along every $\{\rho_n\} \subset \mathcal{P}$, and the nonparametric t test of (2.2) is asymptotically similar.

The proof of Proposition 2.1 appears in Appendix A. The problem reduces to one of proving that

$$v_n(x) := \frac{h_n^{1/2} \sum_{t=1}^n K_{h_n}(x_t - x) u_{t+1}}{\sigma_u [\sum_{t=1}^n K_{h_n}(x_t - x) \int K^2]^{1/2}} \rightsquigarrow N[0, 1] \tag{2.6}$$

when $\rho = \rho_n$, for all drifting sequences $\{\rho_n\} \subset \mathcal{P}$. Define the following classes of sequences:

- *stationary*: $\rho_n \rightarrow \rho$ for some $\rho \in [-1 + \delta, 1)$, and $\rho_n < 1$ for all n ;
- *mildly integrated*: $\rho_n \rightarrow 1$ but $n(\rho_n - 1) \rightarrow -\infty$, and $\rho_n < 1$ for all n ; and
- *local to unity*: $\rho_n \rightarrow 1$, and $n(\rho_n - 1) \rightarrow c$ for some $c \leq 0$;

and let \mathcal{P} denote the collection of all such sequences $\{\rho_n\}$. Though \mathcal{P} is evidently a strict subset of all sequences in \mathcal{P} , by an argument given in the proof of Proposition 2.1, the convergence (2.6) must hold for *all* sequences in \mathcal{P} if it holds for all those in \mathcal{P} (here we adapt the proof of Lemma 2.1 in Andrews and Cheng, 2012).

It then remains to prove that (2.6) holds for stationary, mildly integrated, and local-to-unity sequences $\{\rho_n\}$. In all cases, the numerator of (2.6) is a martingale, and so is in principle amenable to the application of existing martingale central limit theory. The main difficulty is to show that the conditional variance $\sigma_u^2 \sum_{t=1}^n K_{h_n}^2(x_t - x)$ converges

³If DGP1 were strengthened such that the *second* derivatives of m were uniformly bounded, then it would be possible to relax this requirement to $h_n = o(n^{-1/6})$: see e.g. Wang and Phillips (2009b, Rem. C; 2011).

weakly to an a.s. nonzero limit upon standardisation. This follows by an application of Theorem 3.2 below. Convergence results of this kind are available in the literature when $\{\rho_n\}$ is stationary or local to unity, but the proof of this convergence when $\{\rho_n\}$ is mildly integrated requires some genuinely new limit theory for kernel density estimators, which is the principal contribution of the following section.

3 Density estimation: a unified limit theory

Our remaining objective is thus to provide some new results on the asymptotics of functionals of the form $\sum_{t=1}^n f_{h_n}(x_t - x)$ – where f is an integrable function and $f_h(x) := h^{-1}f(h^{-1}x)$ – in the case where $\{x_t\}$ is mildly integrated, i.e. when $\rho_n \rightarrow 1$ but $n(\rho_n - 1) \rightarrow -\infty$. We shall do this by means of an extension to Theorem 2.1 in Wang and Phillips (2009a, hereafter WP), which is stated as Theorem 3.1 below. An application of this result to mildly integrated processes, in conjunction with existing results for local-to-unity and stationary processes, gives the asymptotics of $\sum_{t=1}^n f_{h_n}(x_t - x)$ for all three classes of processes considered in the preceding section, which are collected in Theorem 3.2 below.

3.1 A general framework

In order to provide our extension of Theorem 2.1 in Wang and Phillips (2009a, hereafter WP), we first restate their assumptions, some of which will also be needed here. Let $\{\tilde{x}_{n,t}\}_{t=1}^n$ be a triangular array, $\{\tilde{\mathcal{F}}_{n,t}\}_{t=1}^n$ a collection of σ -fields such that each $\tilde{x}_{n,t}$ is $\tilde{\mathcal{F}}_{n,t}$ -measurable, $f : \mathbb{R} \rightarrow \mathbb{R}$, and define

$$\Omega_n(\eta) := \{(s, t) \mid \eta n \leq s \leq (1 - \eta)n, s + \eta n \leq t \leq n\}$$

for $\eta \in (0, 1)$. Let L^p denote the class of Lebesgue p -integrable functions on \mathbb{R} .

Assumption WP (Ass. 2.1–2.3 in Wang and Phillips, 2009a).

WP1 $f \in L^1 \cap L^2$.

WP2 *There exists a stochastic process $X(r)$ on $[0, 1]$ having continuous local time $\mathcal{L}_X(r, a)$ such that $\tilde{x}_{n, \lfloor nr \rfloor} \rightsquigarrow X(r)$ in $\ell_\infty([0, 1])$.*

WP3 *There exists an $n_0 \in \mathbb{N}$ such that for all $0 \leq s < t \leq n$ and $n \geq n_0$,⁴ there are constants $\{d_{n,s,t}\}$ such that*

(a) *for some $m_0 > 0$ and $C > 0$, $\inf_{(s,t) \in \Omega_n(\eta)} d_{n,s,t} \geq \eta^{m_0}/C$ as $n \rightarrow \infty$, and*

- i. $\lim_{\eta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=(1-\eta)n}^n d_{n,0,t}^{-1} = 0$,
- ii. $\lim_{\eta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \max_{0 \leq s \leq (1-\eta)n} \sum_{t=s+1}^{s+\eta n} d_{n,s,t}^{-1} = 0$,
- iii. $\limsup_{n \rightarrow \infty} \frac{1}{n} \max_{0 \leq s \leq n-1} \sum_{t=s+1}^n d_{n,s,t}^{-1} < \infty$;

⁴Note that WP have $n_0 = 1$ in their statement of this condition, but it is clearly sufficient for their result that this condition hold only for n sufficiently large.

- (b) *conditional on $\tilde{\mathcal{F}}_{n,s}$, $(\tilde{x}_{n,t} - \tilde{x}_{n,s})/d_{n,s,t}$ has a density $h_{n,s,t}(x)$ which is uniformly bounded (in n , s and t) by a constant $K < \infty$, and*

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \sup_{(s,t) \in \Omega_n(\delta^{1/2m_0})} \sup_{|u| \leq \delta} |h_{n,s,t}(u) - h_{n,s,t}(0)| = 0. \quad (3.1)$$

It is evident from Jeganathan (2004) that WP2 may be weakened to finite dimensional convergence (i.e. $\tilde{x}_{n,[nr]} \rightsquigarrow_{\text{fdd}} X(r)$) if $\{\tilde{x}_{n,[nr]}\}$ satisfies the following weak asymptotic ‘equicontinuity in probability’ condition: that for every $\epsilon > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{|r_1 - r_2| \leq \delta} \mathbb{P}\{|\tilde{x}_{n,[nr_1]} - \tilde{x}_{n,[nr_2]}| > \epsilon\} = 0. \quad (3.2)$$

This is considerably weaker than asymptotic equicontinuity (tightness), which would require control over $\sup_{|r_1 - r_2| \leq \delta} |\tilde{x}_{n,[nr_1]} - \tilde{x}_{n,[nr_2]}|$ (and which is of course implied by WP2). However, as discussed further in Remark 3.3 below, when $\{\tilde{x}_{n,t}\}$ is derived from a mildly integrated process, even such an apparently weak requirement as (3.2) fails to hold: though the finite-dimensional limit of $\tilde{x}_{n,[nr]}$ exists, it is not separable. However, it is possible in this case to verify the following strictly weaker condition, which turns out to be sufficient for the purposes of Theorem 3.1 below.

Assumption WP (continued).

WP2' *There exists a stochastic process $\tilde{\mu} : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}_+$, which is continuous a.s. with $\int_{\mathbb{R}} \tilde{\mu}(r, x) dx < \infty$ for all $r \in [0, 1]$, such that for every bounded and Lipschitz $g : \mathbb{R} \rightarrow \mathbb{R}$,*

$$\frac{1}{n} \sum_{t=1}^{[nr]} g(\tilde{x}_{n,t} - a) \rightsquigarrow_{\text{fdd}} \int_{\mathbb{R}} g(x - a) \tilde{\mu}(r, x) dx, \quad (3.3)$$

over $(r, a) \in [0, 1] \times \mathbb{R}$.

Replacing WP2 by WP2', we thus have the following extension of WP's Theorem 2.1. The proof appears in Appendix B.

Theorem 3.1. *Suppose WP1, WP2' and WP3 hold. Then if $\tilde{c}_n \rightarrow \infty$ and $\tilde{c}_n/n \rightarrow 0$*

$$\frac{\tilde{c}_n}{n} \sum_{t=1}^{[nr]} f[\tilde{c}_n(\tilde{x}_{n,t} - a)] \rightsquigarrow_{\text{fdd}} \tilde{\mu}(r, a) \int_{\mathbb{R}} f$$

over $(r, a) \in [0, 1] \times \mathbb{R}$.

3.2 Application to mildly integrated processes

Theorem 3.1 is broad enough to cover the entire class of regressor processes contemplated in DGP, even when $\rho = \rho_n$ varies with n . Indeed, it is the manner in which ρ_n approaches unity (if at all) that determines the density $\tilde{\mu}$ appearing in (3.3). In accordance with the

division of the sequences $\{\rho_n\} \in \mathcal{P}$ given in Section 2.3 above, define

$$\mu(r, a; \{\rho_n\}) := \begin{cases} r\sigma_\rho\nu_\rho(\sigma_\rho a) & \text{if } \{\rho_n\} \text{ is stationary} \\ r\varphi(a) & \text{if } \{\rho_n\} \text{ is mildly integrated} \\ \mathcal{L}_c(r, a) & \text{if } \{\rho_n\} \text{ is local to unity} \end{cases} \quad (3.4)$$

where ν_ρ is the density corresponding to the stationary solution to (2.1), which has variance σ_ρ^2 ; φ is the standard normal density; and $\mathcal{L}_c(r, a)$ is the local time density (at time $r \in [0, 1]$ and point $a \in \mathbb{R}$) associated with the normalised Ornstein–Uhlenbeck process,

$$J_c(r) := \left(\int_0^1 e^{2(1-s)c} ds \right)^{-1/2} \int_0^r e^{(r-s)c} dW(s), \quad (3.5)$$

for W a standard Brownian motion on $[0, 1]$.

Our main result on the finite-dimensional convergence of density estimators, when applied to a series $\{x_t\}$ satisfying DGP, may be stated as follows. Let $\{h_n\}$ denote a deterministic, nonzero bandwidth sequence, define $d_n := \text{var}(x_n)^{1/2}$, and recall $f_h(x) := h^{-1}f(h^{-1}x)$. The proof appears in Appendix B.⁵

Theorem 3.2. *Suppose DGP holds with $\rho = \rho_n$ for some $\{\rho_n\} \in \mathcal{P}$, and $f \in L^1 \cap L^2$. Then if $h_n = o(d_n)$ and $nd_n^{-1}h_n \rightarrow \infty$,*

$$\frac{d_n}{n} \sum_{t=1}^{\lfloor nr \rfloor} f_{h_n}(x_t - d_n a) \rightsquigarrow_{\text{fdd}} \mu(r, a; \{\rho_n\}) \int_{\mathbb{R}} f,$$

over $(r, a) \in [0, 1] \times \mathbb{R}$.

Remark 3.1. $d_n \rightarrow \infty$ whenever $\{\rho_n\}$ is mildly integrated or local to unity, and so the arguments given in the proof of Theorem 3.2 also imply that, in these cases,

$$\frac{d_n}{n} \sum_{t=1}^{\lfloor nr \rfloor} f_{h_n}(x_t - x) \rightsquigarrow \mu(r, 0; \{\rho_n\}) \int_{\mathbb{R}} f$$

for each $x \in \mathbb{R}$.

Remark 3.2. The stationary and local-to-unity cases are covered by the results of Wu and Mielniczuk (2002), Wang and Phillips (2009b) and Wu et al. (2010). The proof under mild integration is new to the literature, though the arguments employed are a combination of those appropriate to the stationary and local-to-unity cases. Our strategy is to use a kind of law of large numbers to establish (3.3) for the scale-normalised array

$$\tilde{x}_{n,t} := \text{var}(x_n)^{-1/2} x_t = d_n^{-1} x_t, \quad (3.6)$$

⁵In an earlier version of this paper (arXiv:1509.05017v3) we showed that the finite dimensional convergence in Theorem 3.2 may be strengthened to weak convergence.

(see Proposition B.1 in Appendix B), whence it follows that $\{\tilde{x}_{n,t}\}$ satisfies WP2'. Since WP1 and WP3 also hold, it is then possible to invoke Theorem 3.1.

Remark 3.3. The tripartite classification in (3.4) is reflected in the different possible finite-dimensional limits $X(r; \{\rho_n\})$ of the standardised regressor process $X_n(r) := d_n^{-1}x_{\lfloor nr \rfloor}$. Under both stationarity and mild integration, the relatively weak dependence between $X_n(r_1)$ and $X_n(r_2)$ vanishes in the limit, and so X has the property that $X(r_1)$ and $X(r_2)$ are independent for every $r_1 \neq r_2$. This explains why even such an apparently mild equicontinuity requirement as (3.2) is unavailing for the purposes of proving Theorem 3.2.

Under mild integration, $d_n \rightarrow \infty$ and an invariance principle operates to ensure that the marginals of $X(r)$ are standard normal; whereas in the stationary case, d_n is bounded and the marginals have density ν_ρ , which depends on the distribution of $\{\varepsilon_t\}$. The limiting process X under mild integration thus corresponds to a continuous-time, standard normal white noise process. (A rigorous basis for these assertions is provided by Proposition B.1(ii) in Appendix B, and the proof thereof.)

The strong dependence between $X_n(r_1)$ and $X_n(r_2)$ that is a characteristic of local-to-unity processes ensures that, in this case, X_n converges weakly to the diffusion J_c (see (3.5) above). As $c \rightarrow -\infty$, the finite-dimensional distributions of J_c converge to those of standard normal white noise process: and in this sense there is continuity, in the limit, at the boundary demarcating mildly integrated and local-to-unity processes.

4 Conclusion

This paper has established the asymptotic size of the nonparametric t test in a predictive regression, when the regressor is possibly highly persistent. Our work on this problem has necessitated the development of some new limit theory for kernel density estimators, when these are applied to mildly integrated processes. These new results have allowed us to give a unified treatment of kernel density and regression estimators that encompasses stationary, mildly integrated and local-to-unity processes.

A notable implication of our results is that conventional nonparametric inferences, using normal critical values, remain valid regardless of the degree of persistence of the regressor. This may be counted a significant advantage of kernel nonparametric estimators over their parametric counterparts, which partially compensates for their lower rates of convergence and – in the case of integrated regressors – their limited applicability to models with multiple regressors.

5 References

- ABADIR, K. M., W. DISTASO, L. GIRAITIS, AND H. L. KOUL (2014): “Asymptotic normality for weighted sums of linear processes,” *Econometric Theory*, 30, 252–84.
- ANDREWS, D. W. K. AND X. CHENG (2012): “Estimation and inference with weak, semi-strong, and strong identification,” *Econometrica*, 80, 2153–2211.

- ANDREWS, D. W. K., X. CHENG, AND P. GUGGENBERGER (2011): “Generic results for establishing the asymptotic size of confidence sets and tests,” Cowles Foundation Discussion Paper 1813, Yale University.
- CAMPBELL, J. Y. AND M. YOGO (2006): “Efficient tests of stock return predictability,” *Journal of Financial Economics*, 81, 27–60.
- CAVANAGH, C. L., G. ELLIOTT, AND J. H. STOCK (1995): “Inference in models with nearly integrated regressors,” *Econometric Theory*, 11, 1131–1147.
- ELLIOTT, G., U. K. MÜLLER, AND M. W. WATSON (2015): “Nearly optimal tests when a nuisance parameter is present under the null hypothesis,” *Econometrica*, 83, 771–811.
- FELLER, W. (1971): *An Introduction to Probability Theory and its Applications*, vol. II, New York, USA: Wiley.
- GIRAITIS, L. AND P. C. B. PHILLIPS (2006): “Uniform limit theory for stationary autoregression,” *Journal of Time Series Analysis*, 27, 51–60.
- HALL, P. AND C. C. HEYDE (1980): *Martingale Limit Theory and Its Application*, New York (USA): Academic Press.
- JANSSON, M. AND M. J. MOREIRA (2006): “Optimal inference in regression models with nearly integrated regressors,” *Econometrica*, 74, 681–714.
- JEGANATHAN, P. (2004): “Convergence of functionals of sums of r.v.s to local times of fractional stable motions,” *Annals of Probability*, 32, 1771–95.
- KASPARIS, I., E. ANDREOU, AND P. C. B. PHILLIPS (2015): “Nonparametric predictive regression,” *Journal of Econometrics*, 185, 468–94.
- LEHMANN, E. L. AND J. P. ROMANO (2005): *Testing Statistical Hypotheses*, Springer, 3rd ed.
- MAGDALINOS, T. AND P. C. B. PHILLIPS (2009): “Econometric inference in the vicinity of unity,” CoFie Working Paper 7, Singapore Management University.
- MIKUSHEVA, A. (2007): “Uniform inference in autoregressive models,” *Econometrica*, 75, 1411–1452.
- PHILLIPS, P. C. B. AND J. H. LEE (2013): “Predictive regression under various degrees of persistence and robust long-horizon regression,” *Journal of Econometrics*, 177, 250–264.
- PHILLIPS, P. C. B. AND T. MAGDALINOS (2007): “Limit theory for moderate deviations from a unit root,” *Journal of Econometrics*, 136, 115–130.
- RAY, D. (1963): “Sojourn times of diffusion processes,” *Illinois Journal of Mathematics*, 7, 615–30.
- ROMANO, J. P. (2004): “On non-parametric testing, the uniform behaviour of the t-test, and related problems,” *Scandinavian Journal of Statistics*, 31, 567–584.
- VAN DER VAART, A. W. AND J. A. WELLNER (1996): *Weak Convergence and Empirical Processes: with Applications to Statistics*, New York (USA): Springer.

- WANG, Q. (2014): “Martingale limit theorems revisited and non-linear cointegrating regression,” *Econometric Theory*, 30, 509–35.
- WANG, Q. AND P. C. B. PHILLIPS (2009a): “Asymptotic theory for local time density estimation and nonparametric cointegrating regression,” *Econometric Theory*, 25, 710–38.
- (2009b): “Structural nonparametric cointegrating regression,” *Econometrica*, 77, 1901–1948.
- (2011): “Asymptotic theory for zero energy functionals with nonparametric regression applications,” *Econometric Theory*, 27, 235–259.
- WU, W. B., Y. HUANG, AND Y. HUANG (2010): “Kernel estimation for time series: an asymptotic theory,” *Stochastic Processes and their Applications*, 120, 2412–2431.
- WU, W. B. AND J. MIELNICZUK (2002): “Kernel density estimation for linear processes,” *Annals of Statistics*, 30, 1441–59.

A Proof of Proposition 2.1

Throughout the Appendices (excepting Section B.1), Assumptions DGP and SM are always maintained, even when not explicitly referenced.

Notation. For $p \in (1, \infty)$ and a function $f : \mathbb{R} \rightarrow \mathbb{R}$, define $\|f\|_p := (\int |f(x)|^p dx)^{1/p}$ and $\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)|$; for a random variable X , $\|X\|_p := (\mathbb{E}|X|^p)^{1/p}$, and $\|X\|_\infty$ denotes the essential supremum of X . C, C_1 , etc., denote generic constants which may take on different values even at different places in the same proof.

We shall need the following auxiliary results, the proofs of which appear in Section S.1 of the Online Supplement. Recall the definition of \mathcal{P} , and the classification of the sequences $\{\rho_n\} \in \mathcal{P}$ given in Section 2.3. Let

$$e_n := e_n(\{\rho_n\}) := nd_n^{-1}$$

where $d_n := \text{var}(x_n)^{1/2}$ as was defined in (3.6).

Lemma A.1. *Suppose $\{\rho_n\} \in \mathcal{P}$. Then*

$$n^{1/2} \lesssim e_n(\{\rho_n\}) \lesssim n$$

The next lemma is a direct consequence of Theorem 3.2, and is the principal implication of that theorem needed for the proof Proposition 2.1. For $\{\rho_n\} \in \mathcal{P}$, define

$$\tau(x) := \tau(x, \{\rho_n\}) := \begin{cases} \sigma_\rho \nu_\rho(x) & \text{if } \{\rho_n\} \text{ is stationary} \\ \varphi(0) & \text{if } \{\rho_n\} \text{ is mildly integrated} \\ \mathcal{L}_c(1, 0) & \text{if } \{\rho_n\} \text{ is local to unity,} \end{cases}$$

where ν_ρ denotes the density of the stationary solution to (2.1) (for $\rho < 1$), and σ_ρ^2 its variance.

Lemma A.2. *Suppose $\{\rho_n\} \in \mathcal{P}$. Then if $\alpha \geq 1$ and $\beta = 0$, or $\alpha = 1$ and $\beta \in [0, 1]$,*

$$\frac{1}{e_n} \sum_{t=1}^n \frac{1}{h_n} K^\alpha \left(\frac{x_t - x}{h_n} \right) \left| \frac{x_t - x}{h_n} \right|^\beta \rightsquigarrow \tau(x) \int_{\mathbb{R}} K^\alpha(u) |u|^\beta du,$$

where $\tau(x) > 0$ a.s.

Lemma A.3. *For every $x \in \mathbb{R}$, $\hat{\sigma}_u^2(x) = \sigma_u^2 + o_p(1)$.*

Proof of Proposition 2.1. Suppose that we:

- (i) show that (2.5) holds for every $\{\rho_n\} \in \mathcal{P}$; and then
- (ii) deduce from (i) that (2.5) holds for all $\{\rho_n\} \subset \mathcal{P}$.

The proofs of (i) and (ii) are given immediately below. Now by definition of the limit supremum, there must exist a $\{\rho_n^*\} \subset \mathcal{P}$ such that

$$\limsup_{n \rightarrow \infty} \sup_{\rho \in \mathcal{P}} \mathbb{P}_\rho \{ |\hat{t}_n(x)| \geq z_{1-\alpha/2} \} = \lim_{n \rightarrow \infty} \mathbb{P}_{\rho_n^*} \{ |\hat{t}_n(x)| \geq z_{1-\alpha/2} \}.$$

It follows from (ii) that

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\rho_n^*} \{ |\hat{t}_n(x)| \geq z_{1-\alpha/2} \} = \mathbb{P} \{ |N[0, 1]| \geq z_{1-\alpha/2} \} = \alpha$$

whence the t test has asymptotic size α . Asymptotic similarity of the t test follows by an analogous argument.

(i) Let $x \in \mathbb{R}$ and $\{\rho_n\} \in \mathcal{P}$. In view of Lemma A.3, straightforward calculations yield that under H_0

$$\hat{t}_n(x) = [v_n(x) + b_n(x)](1 + o_p(1)) \tag{A.1}$$

where

$$v_n(x) = \frac{h_n^{1/2} \sum_{t=1}^n K_{h_n}(x_t - x) u_{t+1}}{\sigma_u \left[\sum_{t=1}^n K_{h_n}(x_t - x) \int K^2 \right]^{1/2}}$$

is as defined in (2.6), and

$$b_n(x) := \frac{h_n^{1/2} \sum_{t=1}^n K_{h_n}(x_t - x) [m(x_t) - m(x)]}{\sigma_u \left(\int_{\mathbb{R}} K^2 \sum_{t=1}^n K_{h_n}(x_t - x) \right)^{1/2}} =: \frac{b_{n,1}(x)}{b_{n,2}(x)}. \tag{A.2}$$

By Lemma A.2 and the fact that $|m(x_t) - m(x)| \leq C|x_t - x|$ (by DGP1),

$$b_{n,1}(x) \leq h_n^{3/2} \sum_{t=1}^n \frac{1}{h_n} K \left(\frac{x_t - x}{h_n} \right) \left| \frac{x_t - x}{h_n} \right| \lesssim_p h_n^{3/2} e_n, \tag{A.3}$$

and by Lemma A.2,

$$e_n^{-1/2}b_{n,2}(x) \rightsquigarrow \sigma_u \left(\tau(x) \int_{\mathbb{R}} K^2 \right)^{1/2}, \quad (\text{A.4})$$

which is strictly positive a.s. Together (A.2)–(A.4) yield

$$|b_n(x)| \lesssim_p h_n^{3/2} e_n^{1/2} = o(1) \quad (\text{A.5})$$

since $h_n = o(n^{-1/3})$ by assumption, and $e_n \lesssim n$ by Lemma A.1.

The limiting distribution of $v_n(x)$ may be obtained via an application of an appropriate martingale CLT. Consider the closely related quantity

$$M_n := \left(\frac{h_n}{e_n} \right)^{1/2} \sum_{t=1}^n K_{h_n}(x_t - x) u_{t+1}. \quad (\text{A.6})$$

Under DGP4, the summands $\{K_{h_n}(x_{t-1} - x)u_t\}$ are adapted to $\mathcal{G}_t = \sigma(\{x_s, u_s\}_{s \leq t})$, with

$$\mathbb{E}[K_{h_n}(x_{t-1} - x)u_t \mid \mathcal{F}_{t-1}] = K_{h_n}(x_{t-1} - x) \cdot \mathbb{E}[u_t \mid \mathcal{F}_{t-1}] = 0.$$

Hence M_n is the row sum of a martingale difference array, with conditional variance

$$\langle M_n \rangle = \frac{\sigma_u^2}{e_n h_n} \sum_{t=1}^n K^2 \left(\frac{x_t - x}{h_n} \right) \rightsquigarrow \sigma_u^2 \tau(x) \int_{\mathbb{R}} K^2, \quad (\text{A.7})$$

by Lemma A.2. Furthermore, the (standardised) summands in (A.6) satisfy a conditional Lyapunov condition, since

$$\begin{aligned} & \sum_{t=1}^n \mathbb{E} \left[\left\{ \left(\frac{h_n}{e_n} \right)^{1/2} K_{h_n}(x_t - x) u_{t+1} \right\}^4 \mid \mathcal{G}_t \right] \\ &= \frac{1}{(e_n h_n)^2} \sum_{t=1}^n K^4 \left(\frac{x_t - x}{h_n} \right) \mathbb{E}[|u_{t+1}|^4 \mid \mathcal{G}_t] \\ &\leq \frac{C}{e_n h_n} \cdot \frac{1}{e_n h_n} \sum_{t=1}^n K^4 \left(\frac{x_t - x}{h_n} \right) \\ &\lesssim_p \frac{1}{e_n h_n} = o(1) \end{aligned} \quad (\text{A.8})$$

by DGP4, Lemma A.2 and the fact that $e_n h_n \gtrsim n^{1/2} h_n \rightarrow \infty$ (by SM2 and Lemma A.1).

When $\{\rho_n\}$ is stationary or mildly integrated, the r.h.s. of (A.7) is non-random, and so the asymptotic normality of (A.6) follows from Theorem 3.2 in Hall and Heyde (1980): the relevant conditions having been verified by (A.7) and (A.8). Thus in both cases

$$M_n := \left(\frac{h_n}{e_n} \right)^{1/2} \sum_{t=1}^n K_{h_n}(x_t - x) u_{t+1} \rightsquigarrow \sigma_u \left(\tau(x) \int_{\mathbb{R}} K^2 \right)^{1/2} \cdot \xi, \quad (\text{A.9})$$

where $\tau(x) > 0$ is a constant.

When $\{\rho_n\}$ is local to unity, $\tau(x) = \mathcal{L}_c(1, 0)$ is a random local time density, and we must instead apply Theorem 2.1 of Wang (2014). This requires that we additionally verify the stronger conditions of that theorem. Under DGP, it is easy to see that $\{(\varepsilon_t, u_t), \mathcal{G}_t\}$ satisfy his Assumption 1. That his Assumption 2 is satisfied follows from

$$\max_{t \leq n} \left| \left(\frac{h_n}{e_n} \right)^{1/2} K_{h_n}(x_t - x) \right| \leq \frac{1}{(e_n h_n)^{1/2}} \sup_{x \in \mathbb{R}} |K(x)| = o(1)$$

and

$$\begin{aligned} & \left(\frac{h_n}{n e_n} \right)^{1/2} \sum_{t=1}^n K_{h_n}(x_t - x) |\mathbb{E}_t \varepsilon_{t+1} u_{t+1}| \\ & \leq \sigma_u \left(\frac{h_n}{n e_n} \right)^{1/2} \sum_{t=1}^n K_{h_n}(x_t - x) \lesssim_p \left(\frac{h_n^{3/2} e_n}{n} \right)^{1/2} \lesssim \left(\frac{e_n}{n} \right)^{1/2} = o(1), \end{aligned}$$

which follows by the Cauchy-Schwarz inequality and Lemma A.2. Finally, for the purposes of verifying his Assumption 3, we note that

$$\frac{1}{e_n h_n} \sum_{t=1}^n K^2 \left(\frac{x_t - x}{h_n} \right) u_{t+1}^2 = \langle M_n \rangle + o_p(1) \rightsquigarrow \tau(x) \int_{\mathbb{R}} K^2 = \mathcal{L}_c(1, 0) \int_{\mathbb{R}} K^2 \quad (\text{A.10})$$

by Theorem 2.23 in Hall and Heyde (1980) and (A.7). \mathcal{L}_c is the local time density of the process J_c given in (3.5), and is thus a functional of the standard Brownian motion W that emerges as the weak limit of $n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} \varepsilon_t \rightsquigarrow W(r)$. This convergence holds jointly with (A.10), and so Wang's Assumption 3 is satisfied. It therefore follows by Theorem 2.1 of Wang (2014) that

$$M_n := \left(\frac{h_n}{e_n} \right)^{1/2} \sum_{t=1}^n K_{h_n}(x_t - x) u_{t+1} \rightsquigarrow \sigma_u \left(\mathcal{L}_c(1, 0) \int_{\mathbb{R}} K^2 \right)^{1/2} \cdot \xi \quad (\text{A.11})$$

holds jointly with (A.7), where $\xi \sim N[0, 1]$ is independent of $\mathcal{L}_c(1, 0)$.

Finally, deduce from (A.1), (A.5), (A.7), (A.9) and (A.11) that

$$\hat{t}_n(x) = v_n(x) + o_p(1) = \frac{M_n}{\langle M_n \rangle} + o_p(1) \rightsquigarrow \frac{\sigma_u \left(\tau(x) \int_{\mathbb{R}} K^2 \right)^{1/2} \cdot \xi}{\sigma_u \left(\tau(x) \int_{\mathbb{R}} K^2 \right)^{1/2}} = \xi \sim N[0, 1]$$

for all $\{\rho_n\} \in \mathcal{P}$.

(ii). The argument here largely follows the proof of Lemma 2.1 in Andrews and Cheng (2012). Let f be an arbitrary bounded and Lipschitz function. It follows from part (i) of the proof that

$$\mathbb{E}_{\rho_n} f(\hat{t}_n) \rightarrow \mathbb{E} f(\xi) \quad (\text{A.12})$$

for every $\{\rho_n\} \in \mathcal{P}$, where $\xi \sim N[0, 1]$ and where \mathbb{E}_{ρ_n} is indexed by the true parameters ρ_n . We need to show that the preceding holds for every $\{\rho_n\} \subset \mathbf{P}$: i.e. that it holds for *all* sequences, not merely those in \mathcal{P} . To that end, let $\{\rho_n\} \subset \mathbf{P}$ be given. To prove (A.12), it suffices to show that for every subsequence $\{p_n\}$ of $\{n\}$, there exists a further subsequence $\{w_n\}$ of $\{p_n\}$ such that

$$\mathbb{E}_{\rho_{w_n}} f(\hat{t}_{w_n}) \rightarrow \mathbb{E} f(\xi). \quad (\text{A.13})$$

Let $\{p_n\}$ be an arbitrary subsequence of $\{n\}$, and $c_n := n(\rho_n - 1)$. By a compactification of \mathbb{R} , $\{(\rho_{p_n}, c_{p_n})\}$ has an accumulation point $(\bar{\rho}, \bar{c}) \in \mathbf{P} \times [-\infty, 0]$. Now let $\{w_n\}$ be a subsequence of $\{p_n\}$, chosen as follows. If

- (i) $\bar{\rho} < 1$: choose $\{w_n\}$ such that $\rho_{w_n} \rightarrow \bar{\rho}$ and $\rho_{w_n} < 1$, for all $n \in \mathbb{N}$;
- (ii) $\bar{\rho} = 1$ and either:
 - (a) $\bar{c} \in (-\infty, 0]$: choose $\{w_n\}$ such that $c_{w_n} \rightarrow \bar{c}$; or
 - (b) $\bar{c} = -\infty$: choose $\{w_n\}$ such that $(\rho_{w_n}, c_{w_n}) \rightarrow (1, -\infty)$.

Note that in case (ii)(b),

$$w_n^{-1} c_{w_n} = \rho_{w_n} - 1 \rightarrow 0 \quad (\text{A.14})$$

as $n \rightarrow \infty$.

Corresponding to the three cases above, construct a new sequence $\{\rho'_n\}$ as follows.

- (i) $\rho'_n = \rho_{w_k}$ for $w_k \leq n < w_{k+1}$: then $\rho'_n \rightarrow \bar{\rho} < 1$, whence $\{\rho'_n\}$ is a stationary sequence.
- (ii) $\rho'_n = 1 + n^{-1} c_{w_k}$ for $w_k \leq n < w_{k+1}$. Then by construction,

$$c'_n := n(\rho'_n - 1) = c_{w_k} \quad \text{for } w_k \leq n < w_{k+1},$$

and hence in case:

- (a) $\lim_{n \rightarrow \infty} c'_n = \lim_{k \rightarrow \infty} c_{w_k} = \bar{c} \in (-\infty, 0]$, so $\{\rho'_n\}$ is a local-to-unity sequence;
- (b) $\lim_{n \rightarrow \infty} c'_n = \lim_{k \rightarrow \infty} c_{w_k} = -\infty$, and for $w_k \leq n < w_{k+1}$,

$$|\rho'_n - 1| = n^{-1} |c'_n| = n^{-1} |c_{w_k}| \leq w_k^{-1} |c_{w_k}| \rightarrow 0$$

as $k \rightarrow \infty$, by (A.14). Thus $\rho'_n \rightarrow 1$ and $\{\rho'_n\}$ is a mildly integrated sequence.

It follows that $\{\rho'_n\} \in \mathcal{P}$ in all cases, and thus (A.12) holds for $\{\rho'_n\}$ by part (i) of the proof. Since by construction $\rho'_{w_n} = \rho_{w_n}$ for all $n \in \mathbb{N}$, we finally have

$$\mathbb{E} f(\xi) = \lim_{n \rightarrow \infty} \mathbb{E}_{\rho'_n} f(\hat{t}_n) = \lim_{n \rightarrow \infty} \mathbb{E}_{\rho'_{w_n}} f(\hat{t}_{w_n}) = \lim_{n \rightarrow \infty} \mathbb{E}_{\rho_{w_n}} f(\hat{t}_{w_n})$$

and thus (A.13) holds. \square

B Proofs of Theorems 3.1 and 3.2

B.1 Proof of Theorem 3.1

Similarly to the proof of Theorem 2.1 in Wang and Phillips (2009a), define

$$L_n(r, a) := \frac{\tilde{c}_n}{n} \sum_{k=1}^{\lfloor nr \rfloor} f[\tilde{c}_n(\tilde{x}_{k,n} - a)]$$

$$L_{n,\epsilon}(r, a) := \frac{\tilde{c}_n}{n} \sum_{k=1}^{\lfloor nr \rfloor} \int_{\mathbb{R}} f[\tilde{c}_n(\tilde{x}_{k,n} - a + z\epsilon)] \varphi(z) dz,$$

and set $\varphi_\epsilon(x) := \epsilon^{-1} \varphi(\epsilon^{-1}x)$. It follows from Lemma 7 in Jeganathan (2004) that, for each $\epsilon > 0$ fixed,

$$\lim_{n \rightarrow \infty} \left| L_{n,\epsilon}(r, a) - \frac{1}{n} \sum_{k=1}^{\lfloor nr \rfloor} \varphi_\epsilon(\tilde{x}_{k,n} - a) \int_{\mathbb{R}} f \right| = 0.$$

Furthermore, the arguments used by Wang and Phillips (2009a) to prove that

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{E}|L_n(r, a) - L_{n,\epsilon}(r, a)| = 0,$$

for each $a \in \mathbb{R}$, which corresponds to (5.1) in that paper, require only their Assumptions 2.1 and 2.3, both of which are maintained here (as WP1 and WP3 respectively). Finally, by WP2',

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{\lfloor nr \rfloor} \varphi_\epsilon(\tilde{x}_{k,n} - a) &\rightsquigarrow_{\text{fdd}} \int_{\mathbb{R}} \varphi_\epsilon(x - a) \tilde{\mu}(r, x) dx \\ &= \int_{\mathbb{R}} \varphi(x) \tilde{\mu}(r, \epsilon x + a) dx = \tilde{\mu}(r, a) + o_p(1) \end{aligned}$$

over $(r, a) \in [0, 1] \times \mathbb{R}$ as $n \rightarrow \infty$ and then $\epsilon \rightarrow 0$, since $\tilde{\mu}$ is continuous a.s. \square

B.2 Proof of Theorem 3.2

We separately consider $\{\rho_n\} \in \mathcal{P}$ that are local to unity, mildly integrated, and stationary.

$\{\rho_n\}$ local to unity. Proposition 7.1 in Wang and Phillips (2009b), together with the arguments used to prove their Proposition 7.2, establish that $\{\tilde{x}_{n,t}\}$ satisfies WP2 and WP3. Thus, in this case, the result follows by Theorem 3.1.

$\{\rho_n\}$ mildly integrated. In this case, we shall need the following two results, the proofs of which are given in Appendix C. Recall the definition of $\tilde{x}_{n,t} := d_n^{-1}x_t$ given in (3.6) above.

Proposition B.1. *Suppose g is bounded and Lipschitz, and $\{\rho_n\}$ is mildly integrated. Then*

- (i) $\frac{1}{n} \sum_{t=1}^{\lfloor nr \rfloor} g(\tilde{x}_{n,t}) = \frac{1}{n} \sum_{t=1}^{\lfloor nr \rfloor} \mathbb{E}g(\tilde{x}_{n,t}) + o_p(1)$; and
- (ii) $\frac{1}{n} \sum_{t=1}^{\lfloor nr \rfloor} \mathbb{E}g(\tilde{x}_{n,t}) \rightarrow r \int_{\mathbb{R}} g(x) \varphi(x) dx$.

Proposition B.2. *Suppose $\{\rho_n\}$ is mildly integrated. Then $\tilde{x}_{n,t}$ satisfies WP3 with $\tilde{\mathcal{F}}_{n,t} := \sigma(\{\varepsilon_s\}_{s \leq t})$.*

It follows immediately from Proposition B.1 that for every g bounded and Lipschitz,

$$\frac{1}{n} \sum_{t=1}^{\lfloor nr \rfloor} g(\tilde{x}_{n,t} - a) = \frac{1}{n} \sum_{t=1}^{\lfloor nr \rfloor} \mathbb{E}g(\tilde{x}_{n,t} - a) + o_p(1) \xrightarrow{p} r \int_{\mathbb{R}} g(x - a) \varphi(x) dx$$

for each $(r, a) \in [0, 1] \times \mathbb{R}$. Thus WP2' holds with $\tilde{\mu}(r, a) = r\varphi(a)$. By Proposition B.2, $\{\tilde{x}_{n,t}\}$ satisfies WP3, whence the result follows by Theorem 3.1.

$\{\rho_n\}$ stationary. Since $d_n \lesssim 1$ in this case, it follows from Theorem 1 in Wu et al. (2010), with minor modifications, that

$$\frac{d_n}{n} \sum_{t=1}^{\lfloor nr \rfloor} f_{h_n}(x_t - d_n a) = \frac{d_n}{n} \sum_{t=1}^{\lfloor nr \rfloor} \mathbb{E}f_{h_n}(x_t - d_n a) + o_p(1).$$

It remains to determine the limit of the r.h.s. To that end, let $\nu_{\rho,t}$ and $\psi_{\rho,t}$ respectively denote the probability density and characteristic function of x_t , and ν_ρ and ψ_ρ those of the stationary solution to (2.1), for $\rho < 1$. Let $t_n \in \mathbb{N}$ with $t_n \leq n$ and $t_n \rightarrow \infty$. Since $\rho_n \rightarrow \rho < 1$ is bounded away from unity, we have

$$x_{t_n} = \sum_{s=0}^{t_n-1} \rho_n^s v_{t_n-s} \stackrel{d}{=} \sum_{s=0}^{t_n-1} \rho_n^s v_{-s} \xrightarrow{p} \sum_{s=0}^{\infty} \rho^s v_{-s} \quad (\text{B.1})$$

where the r.h.s. has density ν_ρ . Deduce $\psi_{\rho_n,t_n}(\lambda) \rightarrow \psi_\rho(\lambda)$ for each $\lambda \in \mathbb{R}$, whence

$$\begin{aligned} \|\nu_{\rho_n,t_n} - \nu_\rho\|_\infty &\leq \int_{\{|\lambda| \leq A\}} |\psi_{\rho_n,t_n}(\lambda) - \psi_\rho(\lambda)| d\lambda + \int_{\{|\lambda| > A\}} [|\psi_{\rho_n,t_n}(\lambda)| + |\psi_\rho(\lambda)|] d\lambda \\ &\rightarrow 0, \end{aligned} \quad (\text{B.2})$$

as $n \rightarrow \infty$ and then $A \rightarrow \infty$, where we have used $|\psi_{\rho_n,t_n}(\lambda)| \vee |\psi_{\rho_n}(\lambda)| \leq |\psi_\varepsilon(\phi_0 \lambda)|$ to control the integral over $\{|\lambda| > A\}$.

Since the convergence in (B.1) also holds in mean square, taking $t_n = n$ yields $d_n = \text{var}(x_n)^{1/2} \rightarrow \sigma_\rho$, the standard deviation associated to the density ν_ρ . Thus by (B.2)

$$\begin{aligned} \mathbb{E}f_{h_n}(x_{t_n} - d_n a) &= \int_{\mathbb{R}} f(x) \nu_{\rho_n,t_n}(d_n a + h_n x) dx \\ &= \int_{\mathbb{R}} f(x) \nu_\rho(d_n a + h_n x) dx + o(1) \rightarrow \nu_\rho(\sigma_\rho a) \int_{\mathbb{R}} f, \end{aligned}$$

and hence

$$\frac{d_n}{n} \sum_{t=\lfloor n\delta \rfloor + 1}^{\lfloor nr \rfloor} \mathbb{E} f_{h_n}(x_t - d_n a) \rightarrow (r - \delta) \sigma_\rho \nu_\rho(\sigma_\rho a) \int_{\mathbb{R}} f \rightarrow r \sigma_\rho \nu_\rho(\sigma_\rho a) \int_{\mathbb{R}} f$$

as $n \rightarrow \infty$ and then $\delta \rightarrow 0$, while

$$\begin{aligned} \left| \frac{d_n}{n} \sum_{t=1}^{\lfloor n\delta \rfloor} \mathbb{E} f_{h_n}(x_t - d_n a) \right| \\ \leq \delta \cdot d_n \max_{1 \leq t \leq \lfloor n\delta \rfloor} |\mathbb{E} f_{h_n}(x_t - d_n a)| \leq C\delta \max_{1 \leq t \leq \lfloor n\delta \rfloor} \|\nu_{\rho_n, t}\|_\infty \|f\|_1 \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ and then $\delta \rightarrow 0$, since

$$\|\nu_{\rho_n, t}\|_\infty \leq \int_{\mathbb{R}} |\psi_{\rho_n, t}(\lambda)| d\lambda \leq \int_{\mathbb{R}} |\psi_\varepsilon(\phi_0 \lambda)| d\lambda < \infty. \quad \square$$

C Proofs of Propositions B.1 and B.2

C.1 Preliminaries

Under DGP, we may write $x_t = \sum_{k=0}^{\infty} a_{t,k} \varepsilon_{t-k}$, where

$$a_{t,k} := a_{t,k}(\rho) := \sum_{l=0}^{k \wedge (t-1)} \rho^l \phi_{k-l}. \quad (\text{C.1})$$

Observe that this quantity does not depend on t for $0 \leq k \leq t-1$, and we will accordingly write $a_k := a_{t,k}$ in this case. We shall make frequent use of the decomposition

$$x_t = \sum_{k=0}^{\infty} a_{t,k} \varepsilon_{t-k} = \sum_{k=t-s+1}^{\infty} a_{t,k} \varepsilon_{t-k} + \sum_{k=0}^{t-s} a_k \varepsilon_{t-k} =: x'_{s-1, t} + x_{s, t}, \quad (\text{C.2})$$

for $s \in \{1, \dots, t\}$: note that $x'_{s-1, t}$ and $x_{s, t}$ are independent.

We shall also need the following lemma, the proof of which appears in Section S.2 of the Online Supplement. Recall that $d_n^2 = \text{var}(x_n)$ and $\phi = \sum_{k=0}^{\infty} \phi_k$.

Lemma C.1. *Suppose $\{\rho_n\}$ is mildly integrated and $\epsilon > 0$. Then*

- (i) $\rho_n^n \rightarrow 0$;
- (ii) $d_n^2 \sim \phi^2(1 - \rho_n^2)^{-1}$; and
- (iii) for any sequence $\{t_n\}$ with $n\epsilon \leq t_n \leq n$,

$$\text{var}(x_{t_n}) \sim \text{var}(x_{1, t_n}) \sim d_n^2.$$

C.2 Proof of Proposition B.1

We first state and prove the following auxiliary lemma, which is the key ingredient in the proof of the first part of Proposition B.1. For a function g bounded and Lipschitz, let $\|g\|_{\text{Lip}} := \sup_{x \neq y} |g(x) - g(y)|/|x - y|$.

Lemma C.2. *For any g bounded and Lipschitz,*

$$\mathbb{E} \left| \sum_{t=1}^n [g(x_t) - \mathbb{E}g(x_t)] \right| \leq \|g\|_{\text{Lip}} \sum_{k=0}^{\infty} \left(\sum_{t=1}^n a_{t,k}^2 \right)^{1/2} \leq \|g\|_{\text{Lip}} n^{1/2} \frac{\sum_{k=0}^{\infty} |\phi_k|}{1 - |\rho|}, \quad (\text{C.3})$$

where the second inequality holds if $|\rho| < 1$.

Proof. Let $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot \mid \mathcal{G}_t]$. We decompose

$$g(x_t) - \mathbb{E}g(x_t) = \sum_{k=0}^{\infty} [\mathbb{E}_{t-k}g(x_t) - \mathbb{E}_{(t-1)-k}g(x_t)]$$

where the sum on the r.h.s. converges a.s., since $\mathbb{E}_{t-k}g(x_t) \rightarrow \mathbb{E}g(x_t)$ a.s. as $k \rightarrow \infty$, by the reverse martingale convergence theorem. Therefore we may write

$$\sum_{t=1}^n [g(x_t) - \mathbb{E}g(x_t)] = \sum_{k=0}^{\infty} \sum_{t=1}^n [\mathbb{E}_{t-k}g(x_t) - \mathbb{E}_{(t-1)-k}g(x_t)] =: \sum_{k=0}^{\infty} M_{n,k}. \quad (\text{C.4})$$

Clearly, by the orthogonality of martingale differences,

$$\mathbb{E}M_{n,k}^2 = \sum_{t=1}^n \mathbb{E}[\mathbb{E}_{t-k}g(x_t) - \mathbb{E}_{(t-1)-k}g(x_t)]^2. \quad (\text{C.5})$$

We have

$$\begin{aligned} x_t &= \sum_{k=0}^{\infty} a_{t,k} \varepsilon_{t-k} = \sum_{s=0}^{k-1} a_{t,s} \varepsilon_{t-s} + a_{t,k} \varepsilon_{t-k} + \sum_{s=k+1}^{\infty} a_{t,s} \varepsilon_{t-s} \\ &\stackrel{=}{=} \sum_{s=0}^{k-1} a_{t,s} \varepsilon_{t-s} + a_{t,k} \varepsilon^* + \sum_{s=k+1}^{\infty} a_{t,s} \varepsilon_{t-s} =: x_t^* \end{aligned}$$

where ‘ $=_d$ ’ denotes equality in distribution, and $\varepsilon^* =_d \varepsilon_0$ is defined to be independent of $\{\varepsilon_t\}$, and hence also of \mathcal{G}_{t-k} . Thus $\mathbb{E}_{(t-1)-k}g(x_t) = \mathbb{E}_{t-k}g(x_t^*)$, whence

$$|\mathbb{E}_{t-k}g(x_t) - \mathbb{E}_{(t-1)-k}g(x_t)| = |\mathbb{E}_{t-k}[g(x_t) - g(x_t^*)]| \leq \|g\|_{\text{Lip}} |a_{t,k}| \cdot |\mathbb{E}_{t-k}[\varepsilon_{t-k} - \varepsilon^*]|.$$

Hence, by (C.5) and Jensen’s inequality, and recalling that $\sigma_{\varepsilon}^2 = 1$,

$$\mathbb{E}M_{n,k}^2 \leq 2\|g\|_{\text{Lip}}^2 \sum_{t=1}^n a_{t,k}^2,$$

which together with (C.4) yields the first inequality in (C.3).

For the second inequality, we note from (C.1) that

$$\max_{1 \leq t \leq n} |a_{t,k}| \leq \sum_{l=0}^{n-1} |\rho|^l |\phi_{k-l}|,$$

with the convention that $\phi_{-l} := 0$ for $l < 0$. Hence if $|\rho| < 1$,

$$\sum_{k=0}^{\infty} \left(\sum_{t=1}^n a_{t,k}^2 \right)^{1/2} \leq n^{1/2} \sum_{k=0}^{\infty} \max_{1 \leq t \leq n} |a_{t,k}| \leq n^{1/2} \sum_{l=0}^{n-1} |\rho|^l \sum_{k=0}^{\infty} |\phi_{k-l}| \leq n^{1/2} \frac{\sum_{k=0}^{\infty} |\phi_k|}{1 - |\rho|}. \quad \square$$

Proof of Proposition B.1(i). We take $r = 1$ for simplicity; the proof for fixed $r \in [0, 1)$ is analogous. When $\rho \in (0, 1)$, applying Lemma C.2 to the *unstandardised* process $\{x_t\}$ gives the bound

$$\mathbb{E} \left| \sum_{t=1}^n [g(x_t) - \mathbb{E}g(x_t)] \right| \leq \|g\|_{\text{Lip}} n^{1/2} \frac{\sum_{k=0}^{\infty} |\phi_k|}{1 - \rho}. \quad (\text{C.6})$$

It follows that replacing x_t by the rescaled process $\tilde{x}_{n,t} = d_n^{-1} x_t$ in (C.6) gives

$$\begin{aligned} \mathbb{E} \left| \frac{1}{n} \sum_{t=1}^n [g(\tilde{x}_{n,t}) - \mathbb{E}g(\tilde{x}_{n,t})] \right| &\lesssim \frac{1}{n} \cdot \frac{n^{1/2}}{d_n(1 - \rho_n)} \\ &\asymp \frac{1}{n^{1/2}} \cdot \frac{(1 - \rho_n^2)^{1/2}}{1 - \rho_n} \asymp \frac{1}{[n(1 - \rho_n)]^{1/2}} = o(1), \end{aligned}$$

where we have used Lemma C.1. \square

Proof of Proposition B.1(ii). Let $\epsilon > 0$. It is proved below that along every sequence $\{t_n\} \subset [n\epsilon, n]$,

$$\tilde{x}_{n,t_n} \rightsquigarrow N[0, 1], \quad (\text{C.7})$$

whence $\mathbb{E}g(\tilde{x}_{n,t_n}) \rightarrow \int_{\mathbb{R}} g(x) \varphi(x) dx$, since g is bounded. Then by the preceding and the boundedness of g ,

$$\left| \frac{1}{n} \sum_{t=1}^{\lfloor nr \rfloor} \left[\mathbb{E}g(\tilde{x}_{n,t}) - \int g \varphi \right] \right| \leq \epsilon \|g\|_{\infty} + \sup_{t \in [n\epsilon, n]} \left| \mathbb{E}g(\tilde{x}_{n,t}) - \int g \varphi \right| \rightarrow \epsilon \|g\|_{\infty}.$$

Since ϵ was arbitrary, the result follows.

It remains to prove (C.7). It follows from Lemma C.1 that $\text{var}(\tilde{x}_{n,t_n}) \rightarrow 1$. Moreover, we may write $\tilde{x}_{n,t_n} = \sum_{k=-\infty}^n \delta_{n,k} \varepsilon_k$, where

$$\delta_{n,k} = \begin{cases} d_n^{-1} a_{t_n,k} & \text{if } k \leq t_n, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\max_{k \leq n} |\delta_{n,k}| \leq d_n^{-1} \max_{k \leq t_n} |a_{t_n,k}| \leq d_n^{-1} \sum_{i=0}^{\infty} |\phi_i| = o(1).$$

(C.7) therefore follows from Lemma 2.1(i) in Abadir, Distaso, Giraitis, and Koul (2014). \square

C.3 Proof of Proposition B.2

We shall need the following results, proofs of which appear in Section S.2 of the Online Supplement. For $\{\rho_n\}$ mildly integrated, define $k_n := k_n(\{\rho_n\})$ to be the largest integer for which

$$k_n(\{\rho_n\}) \leq [(1 - \rho_n)^{-1} \wedge n]/2; \quad (\text{C.8})$$

observe (by Lemma C.1) that $k_n \asymp d_n^2$. Recall the definition of $a_k = a_k(\rho_n)$ given immediately after (C.1) above.

Lemma C.3. *Suppose $\{\rho_n\}$ is mildly integrated. Then there exist $k_0, n_0 \in \mathbb{N}$ with k_0 even, such that*

- (i) $\rho_n^k, \rho_n^{-k} \in [C_1, C_2]$ for some $C_1, C_2 \in (0, \infty)$ for all $n \geq n_0$ and $0 \leq k \leq 2k_n$; and
- (ii) for some $\underline{a}, \bar{a} \in (0, \infty)$, $|a_0| \geq \underline{a}$ and for all $n \geq n_0$,

$$\underline{a} \leq \min_{k_0/2 \leq k \leq 2k_n} |a_k| \leq \max_{0 \leq k \leq n} |a_k| \leq \bar{a}. \quad (\text{C.9})$$

Lemma C.4. *Let $\{\vartheta_k\}_{k \in \mathbb{N}}$ have $\sigma_{\vartheta}^2 := \sum_{k=1}^{\infty} \vartheta_k^2 > 0$, and $\{\varepsilon_t\}_{t \in \mathbb{Z}}$ be as in DGP2. There exists a bounded function $G(A, \sigma^2, \psi_\epsilon)$, not otherwise depending on $\{\vartheta_k\}$, such that $\sigma^2 \mapsto G(A; \sigma^2, \psi_\epsilon)$ is decreasing in σ ,*

$$\int_{\{|\lambda| \geq A\}} \left| \mathbb{E} \left(i\lambda \sum_{k=1}^{\infty} \vartheta_k \varepsilon_k \right) \right| d\lambda \leq G(A; \sigma_{\vartheta}^2, \psi_\epsilon) \leq C\sigma_{\vartheta}^{-1}, \quad \forall A \geq 0 \quad (\text{C.10})$$

for some $C < \infty$ depending only on $\|\psi_\epsilon\|_1$, and $\lim_{A \rightarrow \infty} G(A; \sigma_{\vartheta}^2, \psi_\epsilon) = 0$.

Lemma C.5. *Let $\{\rho_n\}$ be mildly integrated and $\eta \in (0, 1]$. Then*

$$\frac{1}{n} \int_1^{\eta n} \frac{1}{(1 - \rho_n^u)^{1/2}} du = \eta + o(1).$$

Proof of Proposition B.2. We take

$$d_{n,s,t} := (1 - \rho_n^{2(t-s)})^{1/2}.$$

Since $\rho_n \rightarrow 1$ with $\rho_n < 1$, we assume throughout that $\rho_n \in (\frac{1}{2}, 1)$.

We first consider part (a) of WP3. For (a)(i), we have

$$\frac{1}{n} \sum_{t=(1-\eta)n}^n d_{n,0,t}^{-1} = \frac{1}{n} \sum_{t=(1-\eta)n}^n \frac{1}{(1 - \rho_n^{2t})^{1/2}} \leq \frac{1}{n} \cdot \frac{\eta n}{(1 - \rho_n^{2(1-\eta)n})^{1/2}} \rightarrow \eta$$

by Lemma C.1. For (a)(ii), we note that

$$\begin{aligned} \frac{1}{n} \max_{0 \leq s \leq (1-\eta)n} \sum_{t=s+1}^{s+\eta n} d_{n,s,t}^{-1} &= \frac{1}{n} \sum_{k=1}^{\eta n} \frac{1}{(1 - \rho_n^{2k})^{1/2}} \leq \frac{1}{n} \sum_{k=1}^{\eta n} \frac{1}{(1 - \rho_n^k)^{1/2}} \\ &\leq \frac{1}{n} \left\{ \frac{1}{(1 - \rho_n)^{1/2}} + \int_1^{\eta n} \frac{1}{(1 - \rho_n^u)^{1/2}} du \right\} \rightarrow \eta, \end{aligned}$$

where the final convergence follows by Lemma C.5. Finally, for (a)(iii), essentially the preceding with $\eta = 1$ yields

$$\frac{1}{n} \max_{0 \leq s \leq n-1} \sum_{t=s+1}^n d_{n,s,t}^{-1} = \frac{1}{n} \sum_{k=1}^n \frac{1}{(1 - \rho_n^{2k})^{1/2}} \rightarrow 1.$$

Thus part (a) of WP3 is satisfied.

We next turn to part (b) of WP3. By the Fourier inversion formula and Lemma C.4, uniform boundedness of $\{h_{n,s,t}\}$ will follow if the variance of $(\tilde{x}_{n,t} - \tilde{x}_{n,s})/d_{n,s,t}$, conditional on $\tilde{\mathcal{F}}_{n,s} := \sigma(\{\varepsilon_r\}_{r \leq s})$, is bounded away from zero. As in (C.2) above, we have

$$x_t = \sum_{k=0}^{\infty} a_{t,k} \varepsilon_{t-k} = \sum_{k=t-s}^{\infty} a_{t,k} \varepsilon_{t-k} + \sum_{k=0}^{t-s-1} a_k \varepsilon_{t-k} =: x'_{s,t} + x_{s+1,t}.$$

Since $x_{s+1,t}$ is independent of x_s and $x'_{s,t}$, both of which are $\tilde{\mathcal{F}}_{n,s}$ -measurable, we have

$$\text{var} \left(\frac{\tilde{x}_{n,t} - \tilde{x}_{n,s}}{d_{n,s,t}} \mid \tilde{\mathcal{F}}_{n,s} \right) = \text{var} \left(\frac{x_{n,t} - x_{n,s}}{d_{n,s,t} d_n} \mid \tilde{\mathcal{F}}_{n,s} \right) = \text{var} \left(\frac{x_{s+1,t}}{d_{n,s,t} d_n} \right).$$

Further, taking $r = t - s$ we have

$$x_{s+1,t} = \sum_{k=0}^{t-s-1} a_k \varepsilon_{t-k} =_d \sum_{k=0}^{r-1} a_k \varepsilon_{t-k} = x_{1,r}.$$

Since $d_{n,s,t} = d_{n,0,t-s} = d_{n,0,r}$, it follows that

$$\text{var} \left(\frac{x_{s+1,t}}{d_{n,s,t} d_n} \right) = \frac{\text{var}(x_{1,r})}{d_{n,0,r}^2 d_n^2} \geq C \frac{1 - \rho_n^2}{1 - \rho_n^{2r}} \text{var}(x_{1,r}) =: C g_{n,r}$$

by Lemma C.1, for some $C > 0$ (depending on ϕ), for all n sufficiently large.

We thus need to show that $\inf_{n \geq n_0} \inf_{1 \leq r \leq n} g_{n,r} > 0$ for some $n_0 \in \mathbb{N}$. To that end, we note that for k_0 as in Lemma C.3 and k_n as in (C.8),

$$\text{var}(x_{1,r}) = \sum_{k=0}^r a_k^2 \geq \underline{a}^2 \cdot \begin{cases} 1 & \text{if } 1 \leq r \leq k_0 \\ r/2 & \text{if } k_0 + 1 \leq r \leq k_n \\ k_n/2 & \text{if } k_n + 1 \leq r \leq n \end{cases} \quad (\text{C.11})$$

for n sufficiently large. We also note the inequality

$$\frac{1-x^2}{1-x^{2r}} = \frac{1}{\sum_{l=0}^{r-1} x^{2l}} \geq \frac{1}{r}, \quad \forall r \in \mathbb{N}, x \in (0, 1).$$

Considering each of the three cases in (C.11) in turn, we have:

(i) $1 \leq r \leq k_0$: then

$$g_{n,r} \geq \frac{1-\rho_n^2}{1-\rho_n^{2r}} \cdot \underline{a}^2 \geq \frac{1}{r} \underline{a}^2 \geq \frac{1}{2k_0} \underline{a}^2;$$

(ii) $k_0 + 1 \leq r \leq k_n$: then

$$g_{n,r} \geq \frac{1-\rho_n^2}{1-\rho_n^{2r}} \cdot \frac{r}{2} \cdot \underline{a}^2 \geq \frac{1}{2} \underline{a}^2 \geq \frac{1}{2} \underline{a}^2;$$

(iii) $k_n + 1 \leq r \leq n$: then for some $C \in (0, \infty)$,

$$g_{n,r} \geq \frac{1-\rho_n^2}{1-\rho_n^{2r}} \cdot \frac{k_n}{2} \cdot \underline{a}^2 \geq \frac{C}{(1-\rho_n^{2r})} \underline{a}^2 \geq \frac{C}{2} \underline{a}^2,$$

where the second inequality follows from $k_n \asymp (1-\rho_n)^{-1} \asymp (1-\rho_n^2)^{-1}$, and the third inequality from Lemma C.3.

Thus $\inf_{1 \leq r \leq n} g_{n,r}$ is bounded away from zero for n sufficiently large, whence $\{h_{n,s,t}\}$ is uniformly bounded.

Finally, in view of the definition of $\Omega_n(\eta)$, (3.1) only concerns those s and t for which $(1-\delta)n \geq t-s = r = r_n \geq n\delta$ for some $\delta \in (0, 1)$. For such r_n , we have $d_{n,0,r_n} = (1-\rho_n^{2r_n})^{1/2} \rightarrow 1$ by Lemma C.1, and so arguments given in the proof of Proposition B.1(ii) yield

$$z_n := \frac{x_{1,r_n}}{d_n \cdot d_{n,0,r_n}} = (1 + o_p(1)) \cdot d_n^{-1} x_{1,r_n} \rightsquigarrow N[0, 1].$$

Letting ψ_{z_n} denote the characteristic function of z_n , arguments given in the proof of Corollary 2.2 in Wang and Phillips (2009a) then imply that (3.1) holds if the sequence $\{\psi_{z_n}\}$ is uniformly integrable. But this is immediate from Lemma C.4 and the fact that $\text{var}(z_n) \rightarrow 1$, which itself follows from Lemma C.1. \square

Online Supplementary Material for ‘Asymptotic Theory for Kernel Estimators under Moderate Deviations from a Unit Root’ by J. A. Duffy

Throughout the following, Assumptions DGP and SM are always maintained, even when not explicitly referenced. Section S.1 provides the proofs of Lemmas A.1–A.3, and Section S.2 provides the proofs of Lemmas C.1 and C.3–C.5.

S.1 Proofs of auxiliary lemmas from Appendix A

Proof of Lemma A.1. Since $d_n^2 = \text{var}(x_n)$ is bounded away from zero in all cases, it suffices to prove that $d_n \lesssim n^{1/2}$ when $\{\rho_n\} \in \mathcal{P}$ is mildly integrated or local to unity. To that end, recall from (C.2) that

$$x_n = \sum_{k=1}^{n-1} a_k \varepsilon_{t-k} + \sum_{k=n}^{\infty} a_{t,k} \varepsilon_{t-k}$$

where $a_{t,k} = \sum_{l=0}^{k \wedge (t-1)} \rho^l \phi_{k-l}$. Hence

$$\begin{aligned} \text{var}(x_n) &= \sum_{k=1}^{n-1} a_{t,k}^2 + \sum_{k=n}^{\infty} a_{t,k}^2 \leq \sum_{k=1}^{n-1} \left(\sum_{l=0}^k \rho^l \phi_{k-l} \right)^2 + \sum_{k=n}^{\infty} \left(\sum_{l=0}^{n-1} \rho^l \phi_{k-l} \right)^2 \\ &\leq n \left(\sum_{i=0}^{\infty} |\phi_i| \right)^2 + \sum_{k=n}^{\infty} \left(\sum_{l=0}^{n-1} |\phi_{k-l}| \right)^2 \end{aligned}$$

For the second r.h.s. term, we have

$$\begin{aligned} \sum_{k=n}^{\infty} \left(\sum_{l=0}^{n-1} |\phi_{k-l}| \right)^2 &\lesssim \sum_{k=n}^{\infty} \sum_{l=0}^{n-1} |\phi_{k-l}| = \left(\sum_{k=n}^{2n} + \sum_{k=2n+1}^{\infty} \right) \sum_{l=0}^{n-1} |\phi_{k-l}| \\ &\leq \sum_{k=0}^n \sum_{l=k}^{\infty} |\phi_l| + n \sum_{k=n}^{\infty} |\phi_k| = o(n). \quad \square \end{aligned}$$

Proof of Lemma A.2. As noted in the text, the stated convergence follows immediately from Theorem 3.2: see also Remark 3.1. Regarding the strict positivity of $\tau(x)$: when $\{\rho_n\}$ is local to unity, this follows from Ray’s (1963) theorem; when $\{\rho_n\}$ is mildly integrated this is immediate from φ being the standard normal density; and when $\{\rho_n\}$ is stationary, this follows from the density f_ε of ε_t having been assumed strictly positive (see DGP2). \square

Proof of Lemma A.3. We first show that $\hat{m}_n(x) = m(x) + o_p(1)$. To that end, decompose

$$\hat{m}_n(x) - m(x) = \frac{A_{n,1} + A_{n,2}}{\frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x)}$$

where:

$$\begin{aligned}
 |A_{n,1}| &:= \frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) |m(x_t) - m(x)| \\
 &\leq \frac{Ch_n}{e_n} \sum_{t=1}^n \frac{1}{h_n} K\left(\frac{x_t - x}{h_n}\right) \left| \frac{x_t - x}{h_n} \right| \\
 &\lesssim_p h_n
 \end{aligned} \tag{S.1}$$

by Lemma A.2 and the Lipschitz continuity of m ; and

$$A_{n,2} := \frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) u_{t+1} = o_p(1)$$

where the claimed negligibility follows since $A_{n,2}$ is a martingale with variance

$$\begin{aligned}
 \mathbb{E} A_{n,2}^2 &= \frac{1}{e_n^2 h_n^2} \sum_{t=1}^n \mathbb{E} K^2\left(\frac{x_t - x}{h_n}\right) u_{t+1}^2 \\
 &= \frac{1}{e_n h_n} \cdot \frac{\sigma^2}{e_n} \sum_{t=1}^n \mathbb{E} \frac{1}{h_n} \sum_{t=1}^n K^2\left(\frac{x_t - x}{h_n}\right) \lesssim_p \frac{1}{e_n h_n} = o(1)
 \end{aligned}$$

by Lemma A.2 and $n^{1/2}h_n \rightarrow \infty$ (see SM2). Since by Lemma A.2

$$\frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) \rightsquigarrow \tau(x)$$

which is a.s. positive, we have $\hat{m}_n(x) = m(x) + o_p(1)$ as claimed.

The remainder of the proof follows similar lines to the proof of Theorem 3.2 in Wang and Phillips (2009b). Recalling

$$\hat{\sigma}_u^2(x) = \frac{\frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) [y_{t+1} - \hat{m}_n(x)]^2}{\frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x)}$$

we decompose the numerator as

$$\begin{aligned}
 &\frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) [y_{t+1} - \hat{m}_n(x)]^2 \\
 &= \frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) u_{t+1}^2 + \frac{2}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) [m(x_t) - \hat{m}_n(x)] u_{t+1} \\
 &\quad + \frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) [m(x_t) - \hat{m}_n(x)]^2 \\
 &=: B_{n,1} + 2B_{n,2} + B_{n,3}.
 \end{aligned}$$

Letting $\zeta_t := u_t^2 - \sigma_u^2$, we claim that

$$\begin{aligned} B_{n,1} &= \frac{\sigma_u^2}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) + \frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) \zeta_{t+1} \\ &\rightsquigarrow \sigma_u^2 \tau(x). \end{aligned} \tag{S.2}$$

The convergence of the first r.h.s. term in (S.2) follows from Lemma A.2. Regarding the second r.h.s. term, we note that since $\zeta_{t+1} := u_{t+1}^2 - \sigma_u^2$ is a martingale difference under DGP4, this term is a martingale with conditional variance

$$\frac{1}{e_n h_n} \cdot \frac{1}{e_n} \sum_{t=1}^n \frac{1}{h_n} K^2\left(\frac{x_t - x}{h_n}\right) \mathbb{E}[\zeta_{t+1}^2 \mid \mathcal{G}_t] \lesssim_p \frac{1}{e_n h_n} = o(1)$$

by Lemma A.2 and $\sup_t \mathbb{E}[\zeta_{t+1}^2 \mid \mathcal{G}_t] < \infty$ a.s. (under DGP4). It follows by Corollary 3.1 of Hall and Heyde (1980) that, indeed,

$$\frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) \zeta_{t+1} \xrightarrow{p} 0.$$

Next, we have

$$\begin{aligned} B_{n,3} &\leq C \frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x) \{[m(x_t) - m(x)]^2 + [\hat{m}_n(x) - m(x)]^2\} \\ &= O_p(h_n^2) + o_p(1) \\ &= o_p(1) \end{aligned}$$

by an analogous argument as was used to prove (S.1), and $\hat{m}_n(x) = m(x) + o_p(1)$. Finally

$$B_{n,2} \leq (B_{n,1})^{1/2} (B_{n,3})^{1/2},$$

by the Cauchy-Schwarz inequality; whence by Lemma A.2 and the preceding,

$$\hat{\sigma}_u^2(x) = \frac{B_{n,1} + B_{n,2} + B_{n,3}}{\frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - x)} \rightsquigarrow \frac{\sigma_u^2 \tau(x)}{\tau(x)} = \sigma_u^2. \quad \square$$

S.2 Proofs of auxiliary lemmas from Appendix C

Proof of Lemma C.1. Letting $c_n := n(\rho_n - 1) \rightarrow -\infty$, we note that for every $M < \infty$, we may take n sufficiently large such that $c_n < -M$, whence

$$\rho_n^{n\epsilon} = \left(1 + \frac{c_n}{n}\right)^{n\epsilon} \leq \left(1 - \frac{M}{n}\right)^{n\epsilon} \rightarrow e^{-M\epsilon} \rightarrow 0$$

as $n \rightarrow \infty$ and then $M \rightarrow \infty$. Thus (i) holds.

Now taking $s = 1$ in (C.2), we have

$$x_t = \sum_{k=0}^{t-1} a_k \varepsilon_{t-k} + \sum_{k=t}^{\infty} a_{t,k} \varepsilon_{t-k} = x_{1,t} + x'_{0,t}$$

where $x_{1,t}$ and $x'_{0,t}$ are independent, with variances $\varsigma_{1,t}^2 := \text{var}(x_{1,t})$ and $\varsigma_{2,t}^2 := \text{var}(x'_{0,t})$ respectively. Let $\{t_n\} \subseteq [n\epsilon, n]$ be as in the statement of part (iii) of the lemma. We shall prove below that

$$(1 - \rho_n^2) \text{var}(x_{t_n}) = (1 - \rho_n^2)(\varsigma_{1,t_n}^2 + \varsigma_{2,t_n}^2) = (1 - \rho_n^2)\varsigma_{1,t_n}^2 + o(1) \rightarrow \phi^2,$$

from which both parts (ii) and (iii) of the lemma immediately follow.

Some tedious algebra (verified immediately below this proof) yields

$$\varsigma_{1,t_n}^2 = \sum_{k=0}^{t_n-1} \left(\sum_{l=0}^k \rho_n^{k-l} \phi_l \right)^2 = \sum_{i=0}^{t_n-1} \phi_i^2 \sum_{k=0}^{t_n-i-1} \rho_n^{2k} + 2 \sum_{i=0}^{t_n-1} \sum_{j=i+1}^{t_n-1} \phi_i \phi_j \sum_{k=0}^{t_n-j-1} \rho_n^{2k+(j-i)} \quad (\text{S.3})$$

whence, since $\rho_n \in (0, 1)$,

$$(1 - \rho_n^2) \varsigma_{1,t_n}^2 = \sum_{i=0}^{t_n-1} \phi_i^2 (1 - \rho_n^{2(t_n-i)}) + 2 \sum_{i=0}^{t_n-1} \sum_{j=i+1}^{t_n-1} \phi_i \phi_j (1 - \rho_n^{2(t_n-j)+(j-i)})$$

Since $\rho_n^{2(t_n-i)} \leq \rho_n^{2(\lfloor n\epsilon \rfloor - i)} \rightarrow 0$ as $n \rightarrow \infty$ for each *fixed* $i \in \mathbb{N}$ by part (i), and $\sum_{i=0}^{\infty} |\phi_i| < \infty$, it follows that

$$(1 - \rho_n^2) \varsigma_{1,t_n}^2 \rightarrow \sum_{i=0}^{\infty} \phi_i^2 + 2 \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \phi_i \phi_j = \phi^2.$$

Regarding ς_{2,t_n}^2 , we note that since $|\rho_n| \leq 1$ and $C_\phi := \sum_{i=0}^{\infty} |\phi_i| < \infty$

$$\varsigma_{2,t_n}^2 = \sum_{k=t_n}^{\infty} \left(\sum_{l=0}^{t_n-1} \rho_n^l \phi_{k-l} \right)^2 \leq C_\phi \sum_{k=t_n}^{\infty} \sum_{l=0}^{t_n-1} \rho_n^l |\phi_{k-l}| \leq C_\phi \sum_{l=0}^{t_n-1} \rho_n^l \tilde{\phi}_{t_n-l},$$

where $\tilde{\phi}_j := \sum_{i=j}^{\infty} |\phi_i|$. Further,

$$\begin{aligned} \sum_{l=0}^{t_n-1} \rho_n^l \tilde{\phi}_{t_n-l} &= \left(\sum_{l=0}^{\lfloor t_n/2 \rfloor - 1} + \sum_{l=\lfloor t_n/2 \rfloor}^{t_n-1} \right) \rho_n^l \tilde{\phi}_{t_n-l} \\ &\leq \left(\tilde{\phi}_{\lfloor t_n/2 \rfloor - 1} + C_\phi \rho_n^{\lfloor t_n/2 \rfloor} \right) \sum_{l=0}^{\lfloor t_n/2 \rfloor - 1} \rho_n^l = o[(1 - \rho_n^2)^{-1}], \end{aligned}$$

since $\tilde{\phi}_{\lfloor t_n/2 \rfloor} \rightarrow 0$ and $\rho_n^{\lfloor t_n/2 \rfloor} \rightarrow 0$ by part (i), and

$$\sum_{l=0}^{\lfloor t_n/2 \rfloor} \rho_n^l \leq (1 - \rho_n)^{-1} \asymp (1 - \rho_n^2)^{-1},$$

whence $\varsigma_{2,t_n}^2 = o[(1 - \rho_n^2)^{-1}]$. □

Verification of (S.3). Dropping the n subscript from t_n and ρ_n for simplicity, and setting $m := t - 1$, we have

$$\begin{aligned} \sum_{k=0}^m \left(\sum_{l=0}^k \rho^{k-l} \phi_l \right)^2 &= \sum_{k=0}^m \sum_{i=0}^k \sum_{j=0}^k \rho^{2k-i-j} \phi_i \phi_j \\ &= \sum_{i=0}^m \sum_{j=0}^m \phi_i \phi_j \sum_{k=i \vee j}^m \rho^{2k-i-j} \\ &= \sum_{i=0}^m \phi_i^2 \sum_{k=i}^m \rho^{2(k-i)} + 2 \sum_{i=0}^m \sum_{j=i+1}^m \phi_i \phi_j \sum_{k=j}^m \rho^{2(k-j)+(j-i)} \\ &= \sum_{i=0}^m \phi_i^2 \sum_{k=0}^{m-i} \rho^{2k} + 2 \sum_{i=0}^m \sum_{j=i+1}^m \phi_i \phi_j \sum_{k=0}^{m-j} \rho^{2k+(j-i)}. \end{aligned} \quad \square$$

Proof of Lemma C.3. When $\{\rho_n\}$ is mildly integrated, $\rho_n \in (0, 1)$ and the upper bound in (C.9) follows trivially from $|a_k(\rho_n)| \leq \sum_{i=0}^\infty |\phi_i|$. Further, for any $0 \leq k \leq 2k_n$,

$$\rho_n^{2k_n} \leq \rho_n^k \leq \rho_n^{-k} \leq \rho_n^{-2k_n}.$$

Noting that $\rho^{(1-\rho)^{-1}} \rightarrow e^{-1}$ as $\rho \rightarrow 1$, and $2k_n \sim (1 - \rho_n)^{-1}$, it follows that $(\rho_n^{2k_n}, \rho_n^{-2k_n}) \rightarrow (e^{-1}, e)$ as $n \rightarrow \infty$. Thus there exists an $n_0 \in \mathbb{N}$ and $C_1, C_2 \in (0, \infty)$ such that $\rho_n^k, \rho_n^{-k} \in [C_1, C_2]$ for all $n \geq n_0$ and $0 \leq k \leq 2k_n$.

Now $a_k(\rho_n) = \rho_n^k \sum_{l=0}^k \rho_n^{-l} \phi_l$, and for any $m \leq k \leq 2k_n$,

$$\sum_{l=0}^k \rho_n^{-l} \phi_l = \sum_{l=0}^m \phi_l - \sum_{l=0}^m (1 - \rho_n^{-l}) \phi_l + \sum_{l=m+1}^k \rho_n^{-l} \phi_l.$$

Therefore, since $|\rho_n^k| \leq 1$,

$$\left| a_k(\rho_n) - \rho_n^k \sum_{l=0}^m \phi_l \right| \leq \sum_{l=0}^m |1 - \rho_n^{-l}| |\phi_l| + \sum_{l=m+1}^k |\phi_l|$$

Let m_0 be chosen such that both

$$\rho_n^k \left| \sum_{l=0}^{m_0} \phi_l \right| \geq C_1 \left| \sum_{l=0}^{m_0} \phi_l \right| \geq \frac{C_1}{2} |\phi| =: 3a$$

for all $n \geq n_0$, and $\sum_{l=m_0+1}^{\infty} |\phi_l| \leq \underline{a}$. Since $\rho_n^{-l} \rightarrow 1$ for each l , there exists an $n_1 \geq n_0$ such that

$$|a_k(\rho_n)| \geq \rho_n^k \left| \sum_{l=0}^{m_0} \phi_l \right| - \sum_{l=0}^{m_0} |1 - \rho_n^{-l}| |\phi_l| - \sum_{l=m_0+1}^k |\phi_l| \geq \underline{a}$$

for all $n \geq n_1$. Taking $k_0 := 2m_0$ and re-designating n_1 as n_0 gives the claimed lower bound in (C.9).

Finally, since $a_0 = \phi_0$ is nonzero by DGP3, replacing \underline{a} by $\underline{a} \wedge |\phi_0|$ yields a lower bound that also applies to $|a_0|$. \square

Proof of Lemma C.4. Since $\psi_\varepsilon \in L^1$, ε_0 has a bounded continuous density. Thus by the Riemann-Lebesgue lemma (Feller, 1971, Lem. XV.3.3) $\limsup_{|\lambda| \rightarrow \infty} |\psi_\varepsilon(\lambda)| = 0$. Further, $\psi_\varepsilon \in L^1$ cannot be periodic, and so $|\psi_\varepsilon(\lambda)| < 1$ for all $\lambda \neq 0$ (Feller, 1971, Lem. XV.1.4); since ψ_ε is necessarily continuous (Feller, 1971, Lem. XV.1.1), it follows that $\sup_{|\lambda| \geq 1} |\psi_\varepsilon(\lambda)| \geq e^{-\gamma_0}$ for some $\gamma_0 \in (0, \infty)$. By the moments theorem for characteristic functions (Feller, 1971, Lem. XV.4.2), we have $\psi_\varepsilon(\lambda) = 1 - \frac{1}{2}\lambda^2(1 + o(1))$ as $\lambda \rightarrow 0$. Thus there exists a $\gamma_1 \in (0, \infty)$ such that $|\psi_\varepsilon(\lambda)| \leq e^{-\gamma_1\lambda^2}$. Taking $\gamma := \gamma_0 \wedge \gamma_1$ thus gives

$$|\psi_\varepsilon(\lambda)| \leq \begin{cases} e^{-\gamma\lambda^2} & \text{if } |\lambda| \in [0, 1], \\ e^{-\gamma} & \text{if } |\lambda| \geq 1. \end{cases} \quad (\text{S.4})$$

Let $\psi_\vartheta(\lambda) := \mathbb{E} \exp(i\lambda \sum_{k=1}^{\infty} \vartheta_k \varepsilon_k) = \prod_{k=1}^{\infty} \psi_\varepsilon(\vartheta_k \lambda)$; we want to control the integral of (the modulus of) this function over $[A, \infty)$. Without loss of generality, assume the coefficients $\{\vartheta_k\}$ are ordered such that $|\vartheta_i| \geq |\vartheta_{i+1}|$. Since

$$\sum_{k=1}^{\infty} \frac{3\sigma_\vartheta^2}{\pi} \cdot k^{-2} = \frac{\sigma_\vartheta^2}{2} = \frac{1}{2} \sum_{k=1}^{\infty} \vartheta_k^2,$$

the set

$$\mathcal{K} := \left\{ k \in \mathbb{N} \mid \vartheta_k^2 \geq \frac{3\sigma_\vartheta^2}{\pi} \cdot k^{-2} \right\}$$

must be nonempty; let k^* denote the smallest element of \mathcal{K} .

We will bound the integral of $|\psi_\vartheta|$ separately over each of the two r.h.s. sets in

$$[A, \infty) = [A, A \vee \vartheta_{k^*}^{-1}] \cup [A \vee \vartheta_{k^*}^{-1}, \infty).$$

We first have

$$\begin{aligned} \int_{\{|\lambda| \in [A, A \vee \vartheta_{k^*}^{-1}]\}} |\psi_\vartheta(\lambda)| \, d\lambda &\leq \int_{\{|\lambda| \in [A, A \vee \vartheta_{k^*}^{-1}]\}} \prod_{k \in \mathcal{K}} |\psi_\varepsilon(\vartheta_k \lambda)| \, d\lambda \\ &\stackrel{(2)}{\leq} \int_{\{|\lambda| \geq A\}} \exp\left(-\gamma\lambda^2 \sum_{k \in \mathcal{K}} \vartheta_k^2\right) \, d\lambda \end{aligned}$$

$$\leq_{(3)} \int_{\{|\lambda| \geq A\}} \exp(-\gamma \lambda^2 \sigma_\vartheta^2/2) d\lambda$$

where $\leq_{(2)}$ follows from (S.4) and

$$|\lambda| \in [A, A \vee \vartheta_{k^*}^{-1}] \implies |\vartheta_{k^*} \lambda| \leq 1 \implies |\vartheta_k \lambda| \leq 1, \quad \forall k \geq k^*;$$

while $\leq_{(3)}$ follows from

$$\sum_{k \in \mathcal{K}} \vartheta_k^2 = \sigma_\vartheta^2 - \sum_{k \notin \mathcal{K}} \vartheta_k^2 \geq \sigma_\vartheta^2 - \frac{3\sigma_\vartheta^2}{\pi} \cdot \sum_{k \notin \mathcal{K}} k^{-2} \geq \frac{\sigma_\vartheta^2}{2}.$$

Next, we have

$$\begin{aligned} \int_{\{|\lambda| \in [A \vee \vartheta_{k^*}^{-1}, \infty)\}} |\psi_\vartheta(\lambda)| d\lambda &\leq \int_{\{|\lambda| \in [A \vee \vartheta_{k^*}^{-1}, \infty)\}} \prod_{k=1}^{k^*} \psi_\epsilon(\vartheta_k \lambda) d\lambda \\ &\leq_{(2)} e^{-\gamma(k^*-1)} \int_{\{|\lambda| \in [A \vee \vartheta_{k^*}^{-1}, \infty)\}} |\psi_\epsilon(\vartheta_{k^*} \lambda)| d\lambda \\ &\leq e^{-\gamma(k^*-1)} \int_{\{|\lambda| \geq A\}} |\psi_\epsilon(\vartheta_{k^*} \lambda)| d\lambda \\ &= e^{-\gamma(k^*-1)} \vartheta_{k^*}^{-1} \int_{\{|\lambda| \geq \vartheta_{k^*} A\}} |\psi_\epsilon(\lambda)| d\lambda \\ &\leq_{(5)} c_0^{-1} \sigma_\vartheta^{-1} e^{-\gamma(k^*-1)} k^* \int_{\{|\lambda| \geq c_0 \sigma_\vartheta A/k^*\}} |\psi_\epsilon(\lambda)| d\lambda, \end{aligned}$$

for $c_0 := (3/\pi)^{1/2}$, where $\leq_{(2)}$ holds trivially if $k^* = 1$, and otherwise follows from

$$|\lambda| \in [A \vee \vartheta_{k^*}^{-1}, \infty) \implies |\vartheta_{k^*} \lambda| \geq 1 \implies |\vartheta_k \lambda| \geq 1, \quad \forall k \leq k^*;$$

while $\leq_{(5)}$ follows from $\vartheta_{k^*}^2 \geq (3\sigma_\vartheta^2/\pi) \cdot (k^*)^{-2}$.

Finally, define

$$\begin{aligned} G(A; \sigma^2, \psi_\epsilon) &:= \int_{\{|\lambda| \geq A\}} \exp(-\gamma \lambda^2 \sigma^2/2) d\lambda + c_0^{-1} \sigma^{-1} \sup_{k \geq 1} e^{-\gamma(k-1)} k \int_{\{|\lambda| \geq c_0 \sigma A/k\}} |\psi_\epsilon(\lambda)| d\lambda, \end{aligned}$$

which clearly satisfies the first inequality in (C.10), and is decreasing in σ^2 ; the second inequality in (C.10) follows by evaluating $G(0; \sigma^2, \psi_\epsilon)$, and noting $\sup_{k \geq 1} e^{-\gamma(k-1)} k < \infty$. It thus remains to show that $G(A; \sigma^2, \psi_\epsilon) \rightarrow 0$ as $A \rightarrow \infty$. To that end, let $\epsilon > 0$ and note that there exists a k' such that

$$e^{-\gamma(k'-1)} k' \int_{\mathbb{R}} |\psi_\epsilon(\lambda)| d\lambda < \epsilon.$$

Since

$$e^{-\gamma(k-1)}k \int_{\{|\lambda| \geq c_0 \sigma A/k\}} |\psi_\epsilon(\lambda)| d\lambda \rightarrow 0$$

as $A \rightarrow \infty$, for each fixed $k \in \{1, \dots, k'\}$, the claim follows. \square

Proof of Lemma C.5. Making the change of variables $u = \rho^x$, we have

$$\int_1^a \frac{1}{(1 - \rho^x)^{1/2}} dx = \frac{1}{-\log \rho} \int_{\rho^a}^{\rho} \frac{1}{(1 - u)^{1/2} u} du = \frac{1}{-\log \rho} \left[-2 \tanh^{-1}\{(1 - u)^{1/2}\} \right]_{\rho^a}^{\rho}.$$

for $\rho \in (0, 1)$, where $\tanh^{-1}(x) := \frac{1}{2} \log\{(1 + x)/(1 - x)\}$ is inverse hyperbolic tangent function. Now set $\rho = \rho_n$, for $\{\rho_n\}$ mildly integrated, and $a = n\eta$: and note that $\rho_n \rightarrow 1$, whereas $\rho_n^{\eta n} \rightarrow 0$ by Lemma C.1. Then

$$\begin{aligned} \frac{1}{n} \int_1^{\eta n} \frac{1}{(1 - \rho_n^x)^{1/2}} dx &= \frac{1}{n} \cdot \frac{1}{-\log \rho_n} \left\{ 2 \tanh^{-1}[(1 - \rho_n^{\eta n})^{1/2}] + o(1) \right\} \\ &\sim \frac{1}{n} \cdot \frac{\log[1 - (1 - \rho_n^{\eta n})^{1/2}]}{\log \rho_n}. \end{aligned}$$

Next, note that by two applications of L'Hôpital's rule

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\log[1 - (1 - x)^{1/2}]}{\log x} &= \lim_{x \rightarrow 0} \frac{\frac{1}{2}(1 - x)^{-1/2}/[1 - (1 - x)^{1/2}]}{1/x} \\ &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{x}{1 - (1 - x)^{1/2}} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{\frac{1}{2}(1 - x)^{-1/2}} = 1, \end{aligned}$$

whence

$$\frac{1}{n} \cdot \frac{\log[1 - (1 - \rho_n^{\eta n})^{1/2}]}{\log \rho_n} \sim \frac{1}{n} \cdot \frac{\log(\rho_n^{\eta n})}{\log \rho_n} = \eta$$

and the result follows. \square