

The Limits of SDP Relaxations for General-Valued CSPs

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Abstract—It has been shown that for a general-valued constraint language Γ the following statements are equivalent: (1) any instance of $\text{VCSP}(\Gamma)$ can be solved to optimality using a constant level of the Sherali-Adams LP hierarchy; (2) any instance of $\text{VCSP}(\Gamma)$ can be solved to optimality using the third level of the Sherali-Adams LP hierarchy; (3) the support of Γ satisfies the “bounded width condition”, i.e., it contains weak near-unanimity operations of all arities.

We show that if the support of Γ violates the bounded width condition then not only is $\text{VCSP}(\Gamma)$ not solved by a constant level of the Sherali-Adams LP hierarchy but it is also not solved by $\Omega(n)$ levels of the Lasserre SDP hierarchy (also known as the sum-of-squares SDP hierarchy). For Γ corresponding to linear equations in an Abelian group, this result follows from existing work on inapproximability of Max-CSPs. By a breakthrough result of Lee, Raghavendra, and Steurer [STOC’15], our result implies that for any Γ whose support violates the bounded width condition no SDP relaxation of polynomial-size solves $\text{VCSP}(\Gamma)$.

We establish our result by proving that various reductions preserve exact solvability by the Lasserre SDP hierarchy (up to a constant factor in the level of the hierarchy). Our results hold for general-valued constraint languages, i.e., sets of functions on a fixed finite domain that take on rational or infinite values, and thus also hold in notable special cases of $\{0, \infty\}$ -valued languages (CSPs), $\{0, 1\}$ -valued languages (Min-CSPs/Max-CSPs), and \mathbb{Q} -valued languages (finite-valued CSPs).

I. INTRODUCTION

A. CSPs and exact solvability

Constraint satisfaction problems (CSPs) constitute a broad class of computational problems that involve assigning labels to variables subject to constraints to be satisfied and/or optimised [26]. One line of research focuses on CSPs parametrised by a set of (possibly weighted) relations known as a constraint language [27]. In their influential paper, Feder and Vardi conjectured that for decision CSPs every constraint language gives rise to a class of problems that belongs to P or is NP-complete [19]. While the dichotomy conjecture of Feder and Vardi is still open in its full generality, it has been verified in several important special cases [5], [8], [10], [25], [45] mostly using the so-called algebraic approach [3], [9]. Using concepts from the extensions of the algebraic approach to optimisation

problems [16], the exact solvability of purely optimisation CSPs, known as finite-valued CSPs, has been established [49] (these include Min/Max-CSPs as a special case). Putting together decision and optimisation problems in one framework, the exact complexity of so-called general-valued CSPs has been established [29], [33] (modulo the classification of decision CSPs). A result that proved useful when classifying both finite-valued and general-valued CSPs is an algebraic characterisation of the power of the basic linear programming relaxation for decision CSPs [34] and general-valued CSPs [30].

B. Approximation

Convex relaxations, such as linear programming (LP) and semidefinite programming (SDP), have long been a powerful tool for designing efficient exact and approximation algorithms [53], [54]. In particular, for many combinatorial problems, the introduction of semidefinite programming relaxations allowed for a new structural and computational perspective [1], [21], [28]. The Lasserre SDP hierarchy [38] is a sequence of semidefinite relaxations for certain 0-1 polynomial programs, each one more constrained than the previous one. The k th level of the Lasserre SDP hierarchy requires any set of k variables of the relaxation, which live in a finite-dimensional real vector space, to be consistent in a very strong sense. The k th level of the hierarchy with n variables can be solved in time $n^{O(k)}$. If an integer program has n variables then the n th level of the Lasserre SDP hierarchy is tight, i.e., the only feasible solutions are convex combinations of integral solutions. The Lasserre SDP hierarchy is similar in spirit to the Lovász-Schrijver SDP hierarchy [42] and the Sherali-Adams LP hierarchy [47], but the Lasserre SDP hierarchy is stronger [39].

An important line of research, going back to a seminal work of Yannakakis [55], focuses on proving lower bounds on the size of LP formulations. Chan et al. [13] showed that Sherali-Adams LP relaxations are universal for Max-CSPs in the sense that for every polynomial-size LP relaxation of a Max-CSP instance I there is a constant level of the Sherali-Adams LP hierarchy of I that achieves the same approximation guarantees. This result has been improved to subexponential-size LP relaxations [31]. Moreover, [20] shows that in fact the basic LP relaxation enjoys the same universality property (among super-constant levels of the Sherali-Adams LP hierarchy). For related work on the integrality gaps for the

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Sherali-Adams LP and Lovász-Schrijver SDP hierarchies, we refer the reader to [14], [15], [46] and the references therein.

Recent years have seen some remarkable progress on lower bounds for the Lasserre SDP hierarchy. Using the idea of expansion [6], Schoenebeck showed that certain problems have integrality gaps even after $\Omega(n)$ levels of the Lasserre SDP hierarchy. In particular, Schoenebeck showed, among other things, that $\Omega(n)$ levels of the Lasserre SDP hierarchy cannot prove that certain Max-CSPs (corresponding to equations on the Boolean domain) are unsatisfiable. Tulsiani extended this work to Max-CSPs corresponding to equations over Abelian groups of prime power orders. Finally, Chan extended this to Max-CSPs corresponding to equations over Abelian groups of arbitrary size [12]. In a recent breakthrough, Lee et al. [41] showed that the Lasserre SDP relaxations are universal for Max-CSPs in the sense that for every polynomial-size SDP relaxation of a Max-CSP instance I there is a constant level of the Lasserre SDP hierarchy of I that achieves the same approximation guarantees. One of the many ingredients of the proof in [41] is to view the Lasserre SDP hierarchy as the Sum-of-Squares algorithm [37], which relates to proof complexity [44]. (In fact, Schoenebeck’s above-mentioned result had independently been obtained by Grigoriev [22] using this view.)

C. Contributions

We study the power of the Lasserre SDP hierarchy for exact solvability of general-valued CSPs. We have previously shown that general-valued CSPs that are solved exactly by a constant level of the Sherali-Adams LP hierarchy are precisely those general-valued CSPs that satisfy the “bounded width condition” (BWC), formally defined in Section II. In Theorem 2, we show that general-valued CSPs that are not solved by a constant level of the Sherali-Adams LP hierarchy are also not solved by $\Omega(n)$ levels of the Lasserre SDP hierarchy. As a direct corollary, the results of Lee et al. [41] imply that such general-valued CSPs are not solved by any polynomial-size SDP relaxation. Our main result, Theorem 2, generalises one of the two main results in [48], [50], namely the implication (i) \implies (iii) of [50, Theorem 3], while the method of proof is closely related, as discussed in Section III-D.

In order to prove our result, we will strengthen the proof of the implication (i) \implies (iii) of [50, Theorem 3] (stated below as Theorem 1). The idea is to show that if $\text{supp}(\Gamma)$ violates the BWC, then Γ can *simulate* linear equations in some Abelian group. It suffices to show that linear equations cannot be solved by $\Omega(n)$ levels of the Lasserre SDP hierarchy and that the simulation preserves exact solvability by the Lasserre SDP hierarchy (up to a constant factor in the level of the hierarchy). As discussed before, the former is actually known (in a stronger sense of inapproximability of linear equations) [12], [22], [46], [51] and will be discussed in Section III-D. Our contribution is proving the latter. While the simulation involves only local replacements via gadgets, it needs to be done with care. In particular, we emphasise that the simulation involves steps, such as going to the core and interpretations, which are commonly used in the algebraic approach to CSPs but not

in the literature on convex relaxations and approximability of CSPs [51]. Indeed, the algebraic approach to CSPs gives the right tools for the intuitive (but non-trivial to capture formally) meaning of “simulating equations”.

We remark that our main result is incomparable with results from papers dealing with (in)approximability [12], [46], [51]. On the one hand, our results capture exact solvability rather than approximability. On the other hand, we give a stronger result as our result applies to general-valued CSPs rather than only to Max-CSPs or finite-valued CSPs. General-valued CSPs are more expressive than their special cases Max-CSPs and finite-valued CSPs since general-valued CSPs also include decision CSPs as a special case and thus can use “hard” or “strict” constraints. The results on Max-CSPs [12], [46], [51] were extended by (problem-specific) reductions to some problems (such as Vertex Cover) which are not captured by Max-CSPs but are captured by general-valued CSPs. Our results are not problem specific and apply to *all* general-valued CSPs. In particular, we give a *complete* characterisation of which *general-valued* CSPs are solved exactly by the Lasserre SDP hierarchy.

D. Related work

We now informally describe the bounded width condition (BWC). A set of operations on a fixed finite domain satisfies the BWC if it contains “weak near-unanimity” operations of all possible arities. An operation is called a weak near-unanimity operation if the value of the operation does not change assuming all the arguments but at most one are the same. A simple example is a ternary majority operation, which satisfies $f(x, x, y) = f(x, y, x) = f(y, x, x)$ for all x and y . Polymorphisms [9] are operations that combine satisfying assignments to a CSP instance and produce a new satisfying assignment. We say that a CSP instance I satisfies the BWC if the clone of polymorphisms of I satisfies the BWC.

In an important series of papers [3], [11], [35], [43] it was established that the BWC captures precisely the decision CSPs that are solved by a certain natural local propagation algorithm [19].

In our main result, Theorem 2, the BWC is required to hold, as in [48], [50], for the support of the fractional polymorphisms [16] of the general-valued CSPs. Intuitively, fractional polymorphisms of a general-valued CSP instance I are probability distributions over polymorphisms of I with some desirable properties. (A formal definition is given in Section II.) This is a natural requirement since polymorphisms do not capture the complexity of general-valued CSPs but the fractional polymorphisms do so [16], [29].

The BWC was also shown [4], [17] to capture precisely the Max-CSPs that can be robustly approximated, as conjectured in [23]. This work is similar to ours but different. In particular, Dalmau and Krokhin showed that various reductions preserve robust approximability of equations, and thus showing that Max-CSPs not satisfying the BWC cannot be robustly approximated, assuming $P \neq NP$ and relying on Håstad’s inapproximability results for linear equations [24]. (Barto and Kozik [4] then showed that Max-CSPs satisfying the BWC can be robustly

approximated.) However, note that linear equations *can* be solved exactly using Gaussian elimination and thus this result is not applicable in our setting. Our result, on the other hand, shows that various reductions preserve exact solvability of equations by a *particular* algorithm (the Lasserre SDP hierarchy) independently of P vs. NP. Moreover, the pp-definitions and pp-interpretations used in [4], [17] were required to be equality-free. We prove that our reductions are well-behaved without this assumption.

Using different techniques (definability in counting logics), Dawar and Wang have recently obtained a similar result in the special case of \mathbb{Q} -valued languages, i.e., for finite-valued CSPs [18].

II. PRELIMINARIES

A. General-valued CSPs

We first describe the framework of general-valued constraint satisfaction problems (VCSPs). Let $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ denote the set of rational numbers extended with positive infinity. Throughout the paper, let D be a fixed finite set of size at least two, also called a *domain*; we call the elements of D *labels*. We denote by $[n]$ the set $\{1, \dots, n\}$.

Definition 1. An r -ary weighted relation over D is a mapping $\phi : D^r \rightarrow \overline{\mathbb{Q}}$. We write $\text{ar}(\phi) = r$ for the arity of ϕ .

A weighted relation $\phi : D^r \rightarrow \{0, \infty\}$ can be seen as the (ordinary) relation $\{\mathbf{x} \in D^r \mid \phi(\mathbf{x}) = 0\}$. We will use both viewpoints interchangeably.

For any r -ary weighted relation ϕ , we denote by $\text{Feas}(\phi) = \{\mathbf{x} \in D^r \mid \phi(\mathbf{x}) < \infty\}$ the underlying r -ary *feasibility relation*, and by $\text{Opt}(\phi) = \{\mathbf{x} \in \text{Feas}(\phi) \mid \forall \mathbf{y} \in D^r : \phi(\mathbf{x}) \leq \phi(\mathbf{y})\}$ the r -ary *optimality relation*, which contains the tuples on which ϕ is minimised.

Definition 2. Let $V = \{x_1, \dots, x_n\}$ be a set of variables. A valued constraint over V is an expression of the form $\phi(\mathbf{x})$ where ϕ is a weighted relation and $\mathbf{x} \in V^{\text{ar}(\phi)}$. The tuple \mathbf{x} is called the *scope* of the constraint.

Definition 3. An instance I of the valued constraint satisfaction problem (VCSP) is specified by a finite set $V = \{x_1, \dots, x_n\}$ of variables, a finite set D of labels, and an objective function ϕ_I expressed as follows:

$$\phi_I(x_1, \dots, x_n) = \sum_{i=1}^q \phi_i(\mathbf{x}_i),$$

where each $\phi_i(\mathbf{x}_i)$, $1 \leq i \leq q$, is a valued constraint. Each constraint may appear multiple times in I . An assignment to I is a map $\sigma : V \rightarrow D$. The goal is to find an assignment that minimises the objective function.

For a VCSP instance I , we write $\text{Val}_{\text{VCSP}}(I, \sigma)$ for $\phi_I(\sigma(x_1), \dots, \sigma(x_n))$, and $\text{Opt}_{\text{VCSP}}(I)$ for the minimum of $\text{Val}_{\text{VCSP}}(I, \sigma)$ over all assignments σ .

An assignment σ with $\text{Val}_{\text{VCSP}}(I, \sigma) < \infty$ is called *satisfying*.

An assignment σ with $\text{Val}_{\text{VCSP}}(I, \sigma) = \text{Opt}_{\text{VCSP}}(I)$ is called *optimal*.

A VCSP instance I is called *satisfiable* if there is a satisfying assignment to I . Constraint satisfaction problems (CSPs) are a special case of VCSPs with (unweighted) relations with the goal to determine the existence of a satisfying assignment.

A *general-valued constraint language* (or just a *constraint language* for short) over D is a set of weighted relations over D . We denote by $\text{VCSP}(\Gamma)$ the class of all VCSP instances in which the weighted relations are all contained in Γ . A constraint language Γ is called *crisp* if Γ contains only (unweighted) relations. For a crisp language Γ , $\text{VCSP}(\Gamma)$ is equivalent to the well-studied (decision) CSP(Γ) [26]. We remark that for $\{0, 1\}$ -valued constraint languages, $\text{VCSP}(\Gamma)$ is also known as Min-CSP(Γ) or Max-CSP(Γ) (since for exact solvability these are equivalent).

For a constraint language Γ , let $\text{ar}(\Gamma)$ denote $\max\{\text{ar}(\phi) \mid \phi \in \Gamma\}$.

B. Fractional polymorphisms and cores

We next define fractional polymorphisms, which are algebraic properties known to capture the computational complexity of the underlying class of VCSPs. We also introduce the important notion of cores.

Given an r -tuple $\mathbf{x} \in D^r$, we denote its i th entry by $\mathbf{x}[i]$ for $1 \leq i \leq r$. A mapping $f : D^m \rightarrow D$ is called an m -ary *operation* on D ; f is *idempotent* if $f(x, \dots, x) = x$. We apply an m -ary operation f to m r -tuples $\mathbf{x}_1, \dots, \mathbf{x}_m \in D^r$ coordinatewise, that is, $f(\mathbf{x}_1, \dots, \mathbf{x}_m) = (f(\mathbf{x}_1[1], \dots, \mathbf{x}_m[1]), \dots, f(\mathbf{x}_1[r], \dots, \mathbf{x}_m[r]))$.

Definition 4. Let ϕ be a weighted relation on D and let f be an m -ary operation on D . We call f a *polymorphism* of ϕ if, for any $\mathbf{x}_1, \dots, \mathbf{x}_m \in \text{Feas}(\phi)$, we have that $f(\mathbf{x}_1, \dots, \mathbf{x}_m) \in \text{Feas}(\phi)$.

For a constraint language Γ , we denote by $\text{Pol}(\Gamma)$ the set of all operations which are polymorphisms of all $\phi \in \Gamma$. We write $\text{Pol}(\phi)$ for $\text{Pol}(\{\phi\})$.

The following notions are known to capture the complexity of general-valued constraint languages [16], [33] and will also be important in this paper. A probability distribution ω over the set of m -ary operations on D is called an m -ary *fractional operation*. We define $\text{supp}(\omega)$ to be the set of operations assigned positive probability by ω .

Definition 5. Let ϕ be a weighted relation on D and let ω be an m -ary fractional operation on D . We call ω a *fractional polymorphism* of ϕ if $\text{supp}(\omega) \subseteq \text{Pol}(\phi)$ and for any $\mathbf{x}_1, \dots, \mathbf{x}_m \in \text{Feas}(\phi)$, we have

$$\mathbb{E}_{f \sim \omega} [\phi(f(\mathbf{x}_1, \dots, \mathbf{x}_m))] \leq \text{avg}\{\phi(\mathbf{x}_1), \dots, \phi(\mathbf{x}_m)\}.$$

For a general-valued constraint language Γ , we denote by $\text{fPol}(\Gamma)$ the set of all fractional operations which are fractional polymorphisms of all weighted relations $\phi \in \Gamma$. We write $\text{fPol}(\phi)$ for $\text{fPol}(\{\phi\})$.

Definition 6. Let Γ be a general-valued constraint language on D . We define

$$\text{supp}(\Gamma) = \bigcup_{\omega \in \text{fPol}(\Gamma)} \text{supp}(\omega).$$

Definition 7. Let Γ be a general-valued constraint language with domain D and let $S \subseteq D$. The sub-language $\Gamma[S]$ of Γ induced by S is the constraint language defined on domain S and containing the restriction of every weighted relation $\phi \in \Gamma$ onto S .

Definition 8. A general-valued constraint language Γ is a core if all unary operations in $\text{supp}(\Gamma)$ are bijections. A general-valued constraint language Γ' is a core of Γ if Γ' is a core and $\Gamma' = \Gamma[f(D)]$ for some unary $f \in \text{supp}(\Gamma)$.

III. LOWER BOUNDS ON LP AND SDP RELAXATIONS

Every VCSP instance has a natural LP relaxation known as the *basic LP relaxation* (BLP). The power of BLP for exact solvability of $\text{CSP}(\Gamma)$, where Γ is a crisp constraint language, has been characterised (in terms of the polymorphisms of Γ) in [34]. The power of BLP for exact solvability of $\text{VCSP}(\Gamma)$, where Γ is a general-valued constraint language, has been characterised (in terms of the fractional polymorphisms of Γ) in [30].

The Sherali-Adams LP hierarchy [47] gives a systematic way of strengthening the BLP relaxation. BLP being the first level, the k th level of the Sherali-Adams LP hierarchy adds to the BLP linear constraints satisfied by the integral solutions and involving at most k variables. One can think of the variables of the k th level as probability distributions over assignments to at most k variables of the original instance.

The Lasserre SDP hierarchy [38] is a significant strengthening of the Sherali-Adams LP hierarchy: real-valued variables are replaced by vectors from a finite-dimensional real vector space. Intuitively, the norms of these vectors again induce probability distributions over assignments to at most k variables of the original instance (for the k th level of the Lasserre SDP hierarchy). Since these distributions have to come from inner products of vectors, this is a tighter relaxation. In particular, it is known that the k th level of the Lasserre SDP hierarchy is at least as tight as the k th level of the Sherali-Adams LP hierarchy [39].

It is well known that for a problem with n variables, the n th levels of both of these two hierarchies are exact, i.e., the solutions to the n th levels are precisely the convex combinations of the integral solutions. However, it is not clear how to solve the n th levels in polynomial time. In general, the k th level of both hierarchies can be solved in time $n^{O(k)}$ for a problem with n variables. In particular, this is polynomial for a fixed k .

In this section, we will define the Sherali-Adams LP and the Lasserre SDP hierarchies and state known and new results regarding their power and limitations for exact solvability of general-valued CSPs.

A. Sherali-Adams LP Hierarchy

Let I be an instance of the VCSP with $\phi_I(x_1, \dots, x_n) = \sum_{i=1}^q \phi_i(\mathbf{x}_i)$, $X_i \subseteq V = \{x_1, \dots, x_n\}$ and $\phi_i: D^{\text{ar}(\phi_i)} \rightarrow \overline{\mathbb{Q}}$. We will use the notational convention to denote by X_i the set of variables occurring in the scope \mathbf{x}_i .

A *null constraint* on a set $X \subseteq V$ is a constraint with a weighted relation identical to 0. It is sometimes convenient to add null constraints to a VCSP instance as placeholders, to ensure that they have scopes where required, even if these relations may not necessarily be members of the corresponding constraint language Γ . In order to obtain an equivalent instance that is formally in $\text{VCSP}(\Gamma)$, the null constraints can simply be dropped, as they are always satisfied and do not influence the value of the objective function.

Let k be an integer. The k th level of the Sherali-Adams LP hierarchy [47], henceforth called the $\text{SA}(k)$ -relaxation of I , is given by the following linear program. Ensure that for every non-empty $X \subseteq V$ with $|X| \leq k$ there is some constraint $\phi_i(\mathbf{x}_i)$ with $X_i = X$, possibly by adding null constraints. The variables of the $\text{SA}(k)$ -relaxation, given in Figure 1, are $\lambda_i(\sigma)$ for every $i \in [q]$ and assignment $\sigma: X_i \rightarrow D$. We slightly abuse notation by writing $\sigma \in \text{Feas}(\phi_i)$ for $\sigma: X_i \rightarrow D$ such that $\sigma(\mathbf{x}_i) \in \text{Feas}(\phi_i)$.

We write $\text{Opt}_{\text{LP}}(I, k)$ for the optimal value of an LP-solution to the $\text{SA}(k)$ -relaxation of I .

Definition 9. Let Γ be a general-valued constraint language. We say that $\text{VCSP}(\Gamma)$ is solved by the k th level of the Sherali-Adams LP hierarchy if for every instance I of $\text{VCSP}(\Gamma)$ we have $\text{Opt}_{\text{VCSP}}(I) = \text{Opt}_{\text{LP}}(I, k)$.

We now describe the main result from [48], [50], which captures the power of Sherali-Adams LP relaxations for exact optimisation of VCSPs.

An m -ary idempotent operation $f: D^m \rightarrow D$ is called a *weak near-unanimity* (WNU) operation if, for all $x, y \in D$,

$$f(y, x, x, \dots, x) = f(x, y, x, x, \dots, x) = \dots = f(x, x, \dots, x, y). \quad (W)$$

Definition 10. A set of operations satisfies the bounded width condition (BWC) if it contains a (not necessarily idempotent) m -ary operation satisfying the identities (W), for every $m \geq 3$.

Theorem 1 ([48], [50]). Let Γ be a general-valued constraint language of finite size. The following are equivalent:

- (i) $\text{VCSP}(\Gamma)$ is solved by a constant level of the Sherali-Adams LP hierarchy.
- (ii) $\text{VCSP}(\Gamma)$ is solved by the third level of the Sherali Adams LP hierarchy.
- (iii) $\text{supp}(\Gamma)$ satisfies the BWC.

We remark that while it is not clear from the definition that condition (iii) of Theorem 1 is decidable, it is known to be equivalent to a decidable condition [32], see also [50, Proposition 3]. Moreover, [50, Section 3.6] discusses how to obtain a solution to an instance I of $\text{VCSP}(\Gamma)$ from the optimal value of the $\text{SA}(3)$ -relaxation of I . Finally, Theorem 1 says that

$$\begin{aligned}
& \text{minimise } \sum_{i=1}^q \sum_{\sigma \in \text{Feas}(\phi_i)} \lambda_i(\sigma) \phi_i(\sigma(\mathbf{x}_i)) \\
& \text{subject to} \\
& \lambda_i(\sigma) \geq 0 \quad \forall i \in [q], \sigma: X_i \rightarrow D \quad (\text{S1}) \\
& \lambda_i(\sigma) = 0 \quad \forall i \in [q], \sigma: X_i \rightarrow D, \sigma(\mathbf{x}_i) \notin \text{Feas}(\phi_i) \quad (\text{S2}) \\
& \sum_{\sigma: X_i \rightarrow D} \lambda_i(\sigma) = 1 \quad \forall i \in [q] \quad (\text{S3}) \\
& \sum_{\substack{\sigma: X_i \rightarrow D \\ \sigma|_{X_j} = \tau}} \lambda_i(\sigma) = \lambda_j(\tau) \quad \forall i, j \in [q]: X_j \subseteq X_i, |X_j| \leq k, \tau: X_j \rightarrow D \quad (\text{S4})
\end{aligned}$$

Fig. 1. The k th level of the Sherali-Adams LP hierarchy, $\text{SA}(k)$.

if $\text{supp}(\Gamma)$ violates the BWC then $\text{VCSP}(\Gamma)$ requires more than a constant level of the Sherali-Adams LP hierarchy for exact solvability. [50] actually shows that assuming the BWC is violated then $\Omega(\sqrt{n})$ levels are required for exact solvability of n -variable instances of $\text{VCSP}(\Gamma)$.

B. Lasserre SDP Hierarchy

Let I be an instance of the VCSP with $\phi_I(x_1, \dots, x_n) = \sum_{i=1}^q \phi_i(\mathbf{x}_i)$, $X_i \subseteq V = \{x_1, \dots, x_n\}$ and $\phi_i: D^{\text{ar}(\phi_i)} \rightarrow \mathbb{Q}$. For $\sigma_i: X_i \rightarrow D$ and $\sigma_j: X_j \rightarrow D$, if $\sigma_i|_{X_i \cap X_j} = \sigma_j|_{X_i \cap X_j}$ then we write $\sigma_i \circ \sigma_j: (X_i \cup X_j) \rightarrow D$ for the assignment defined by $\sigma_i \circ \sigma_j(x) = \sigma_i(x)$ for $x \in X_i$ and $\sigma_i \circ \sigma_j(x) = \sigma_j(x)$ otherwise.

Let k be an integer with $k \geq \max_i(\text{ar}(\phi_i))$.¹ The k th level of the Lasserre SDP hierarchy [36], [37], henceforth called the Lasserre(k)-relaxation of I , is given by the following semidefinite program (we follow the presentation from [51]). Ensure that for every non-empty $X \subseteq V$ with $|X| \leq k$ there is some constraint $\phi_i(\mathbf{x}_i)$ with $X_i = X$, possibly by adding null constraints. The vector variables of the Lasserre(k)-relaxation, given in Figure 2, are $\lambda_0 \in \mathbb{R}^t$ and $\lambda_i(\sigma) \in \mathbb{R}^t$ for every $i \in [q]$ and assignment $\sigma: X_i \rightarrow D$. Here t is the dimension of the real vector space.²

For any fixed k and any t polynomial in the size of I , the Lasserre(k)-relaxation of I is of polynomial size in terms of I and can be solved in polynomial time [52].³ Note that k may not necessarily be constant but it could depend on n , the number of variables of I .

¹It also makes sense to consider relaxations with $k < \max_i(\text{ar}(\phi_i))$, in particular for positive (algorithmic) results, such as the implication (iii) \Rightarrow (ii) in Theorem 1. For our main (impossibility) result, we will be interested in k which is linear in the number of variables of I .

²Typically, $t = (nd)^{O(k)}$ for an instance with n variables over a domain of size d .

³SDPs can be solved only approximately; i.e., for any ϵ there is an algorithm that given an SDP returns vectors for which the objective function is at most ϵ away from the optimum value and the running time is polynomial in the input size and $\log(1/\epsilon)$ [52]. However, for any language Γ of finite size there is $\epsilon = \epsilon(\Gamma)$ such that solving the SDP up to an additive error of ϵ suffices for exact solvability. For instance, take ϵ such that $\epsilon < \min_{\phi \in \Gamma} \min_{\mathbf{x}, \mathbf{y} \in \text{Feas}(\phi), \phi(\mathbf{x}) \neq \phi(\mathbf{y})} |\phi(\mathbf{x}) - \phi(\mathbf{y})|$. Since this paper deals with impossibility results these matters are not relevant but we mention it here for completeness.

We write $\text{Val}_{\text{SDP}}(I, \lambda, k)$ for the value of the SDP-solution λ to the Lasserre(k)-relaxation of I , and $\text{Opt}_{\text{SDP}}(I, k)$ for its optimal value.

Definition 11. Let Γ be a general-valued constraint language. We say that $\text{VCSP}(\Gamma)$ is solved by the k th level of the Lasserre SDP hierarchy if for every instance I of $\text{VCSP}(\Gamma)$ we have $\text{Opt}_{\text{VCSP}}(I) = \text{Opt}_{\text{SDP}}(I, k)$.

We say that an instance I of $\text{VCSP}(\Gamma)$ is a *gap instance* for the k th level of the Lasserre SDP hierarchy if $\text{Opt}_{\text{SDP}}(I, k) < \text{Opt}_{\text{VCSP}}(I)$.

Definition 12. Let Γ be a general-valued constraint language. We say that $\text{VCSP}(\Gamma)$ requires linear levels of the Lasserre SDP hierarchy if there is a constant $0 < c < 1$ such that for sufficiently large n there is an n -variable gap instance I_n of $\text{VCSP}(\Gamma)$ for Lasserre($\lfloor cn \rfloor$).

C. Main Results

We are now ready to state our main results.

Theorem 2. Let Γ be a general-valued constraint language of finite size. The following are equivalent:

- (i) $\text{VCSP}(\Gamma)$ requires linear levels of the Lasserre SDP hierarchy.
- (ii) Γ can simulate linear equations.
- (iii) $\text{supp}(\Gamma)$ violates the BWC.

Theorems 1 and 2 give the following.

Corollary 1. Let Γ be a general-valued constraint language. Then, either $\text{VCSP}(\Gamma)$ is solved by the third level of the Sherali-Adams LP relaxation, or $\text{VCSP}(\Gamma)$ requires linear levels of the Lasserre SDP relaxation.

Proof. Either $\text{supp}(\Gamma)$ satisfies the BWC, in which case $\text{VCSP}(\Gamma)$ is solved by the third level of the Sherali-Adams LP relaxation by Theorem 1, or $\text{supp}(\Gamma)$ violates the BWC, in which case $\text{VCSP}(\Gamma)$ requires linear levels of the Lasserre SDP hierarchy by Theorem 2. \square

A constraint language Γ is called *finite-valued* [49] if for every $\phi \in \Gamma$ it holds $\phi(\mathbf{x}) < \infty$ for every \mathbf{x} . In this special

$$\begin{aligned}
& \text{minimise } \sum_{i=1}^q \sum_{\sigma \in \text{Feas}(\phi_i)} \|\lambda_i(\sigma)\|^2 \phi_i(\sigma(\mathbf{x}_i)) \\
& \text{subject to} \\
& \quad \|\lambda_0\| = 1 \tag{L1} \\
& \quad \langle \lambda_i(\sigma_i), \lambda_j(\sigma_j) \rangle \geq 0 \quad \forall i, j \in [q], \sigma_i: X_i \rightarrow D, \sigma_j: X_j \rightarrow D \tag{L2} \\
& \quad \|\lambda_i(\sigma)\|^2 = 0 \quad \forall i \in [q], \sigma: X_i \rightarrow D, \sigma(\mathbf{x}_i) \notin \text{Feas}(\phi_i) \tag{L3} \\
& \quad \sum_{a \in D} \|\lambda_i(a)\|^2 = 1 \quad \forall i \text{ with } |X_i| = 1 \tag{L4} \\
& \quad \langle \lambda_i(\sigma_i), \lambda_j(\sigma_j) \rangle = 0 \quad \forall i, j \in [q], \sigma_i: X_i \rightarrow D, \sigma_j: X_j \rightarrow D \tag{L5} \\
& \quad \langle \lambda_i(\sigma_i), \lambda_j(\sigma_j) \rangle = \langle \lambda_{i'}(\sigma_{i'}), \lambda_{j'}(\sigma_{j'}) \rangle \quad \forall i, j, i', j' \in [q], X_i \cup X_j = X_{i'} \cup X_{j'} \tag{L6} \\
& \quad \sigma_i|_{X_i \cap X_j} \neq \sigma_j|_{X_i \cap X_j} \\
& \quad \sigma_i: X_i \rightarrow D, \sigma_j: X_j \rightarrow D, \sigma_{i'}: X_{i'} \rightarrow D, \sigma_{j'}: X_{j'} \rightarrow D \\
& \quad \sigma_i \circ \sigma_j = \sigma_{i'} \circ \sigma_{j'}
\end{aligned}$$

Fig. 2. The k th level of the Lasserre SDP hierarchy, $\text{Lasserre}(k)$.

case, we get the following result, which was independently obtained (using a different proof) in [18].

Corollary 2. *Let Γ be a finite-valued constraint language. Then, either $\text{VCSP}(\Gamma)$ is solved by the first level of the Sherali-Adams LP relaxation, or $\text{VCSP}(\Gamma)$ requires linear levels of the Lasserre SDP relaxation.*

Proof. Let D be the domain of Γ . If $\text{VCSP}(\Gamma)$ is not solved by the first level of the Sherali-Adams LP relaxation then [49, Theorem 3.4] shows that there are distinct $a, b \in D$ such that Γ can simulate a weighted relation ϕ with $\text{argmin } \phi = \{(a, b), (b, a)\}$. This implies that the BWC is violated and Theorem 2 proves the claim. \square

Finally, using the recent work of Lee et al. [41], Theorem 2 gives the following.

Corollary 3. *Let Γ be a general-valued constraint language. Then, either $\text{VCSP}(\Gamma)$ is solved by the third level of the Sherali-Adams LP relaxation, or $\text{VCSP}(\Gamma)$ is not solved by any polynomial-size SDP relaxation.*

Lee et al. [40], [41] give some very strong results on approximation-preserving reductions between SDP relaxations. The results in [40], [41] are formulated for the sum-of-squares SDP hierarchy, which is equivalent to the Lasserre SDP hierarchy: the k th level of the sums-of-squares SDP hierarchy is the same as the $(k/2)$ th level of the Lasserre SDP hierarchy. We will only need a special case of one of their results. We remark that [40], [41] deals with maximisation problems but for exact solvability we can equivalently turn to minimisation problems.

Proof. If $\text{supp}(\Gamma)$ violates the BWC then, by Theorem 2, we have that $\text{VCSP}(\Gamma)$ requires linear levels of the Lasserre SDP hierarchy. [40, Theorem 6.4] gives a general reduction turning lower bounds on the number of levels of the Lasserre

SDP hierarchy to lower bounds on the size arbitrary SDP relaxations.⁴ In particular, with the right parameters (e.g., setting the number of variables, N , and the function giving the number of levels of the Lasserre SDP hierarchy, $d(n)$, so that $n = \Theta(\log N)$ and $d(n) = \Theta(\frac{\log N}{\log \log N})$ – see also the end of Section 6 in [40]), [40, Theorem 6.4] gives that if $\text{VCSP}(\Gamma)$ requires linear levels of the Lasserre SDP hierarchy then no polynomial-size SDP relaxation solves $\text{VCSP}(\Gamma)$. \square

D. Proof of Theorem 2

Let Γ be a general-valued constraint language of finite size. If $\text{supp}(\Gamma)$ violates the BWC then we aim to prove that $\text{VCSP}(\Gamma)$ requires linear levels of the Lasserre SDP hierarchy.

We will follow the approach used to prove the implication (i) \implies (iii) of Theorem 1. This is based on the idea that if $\text{supp}(\Gamma)$ violates the BWC, then Γ can simulate linear equations in some Abelian group. In order to establish the implications (iii) \implies (ii) \implies (i) of Theorem 2, it suffices to show that linear equations require linear levels of the Lasserre SDP hierarchy and that the simulation preserves exact solvability by the Lasserre SDP hierarchy (up to a constant factor in the level of the hierarchy). Our contribution is proving the latter. The former is known [22], [46], [51], as we will now discuss.

Let \mathcal{G} be an Abelian group over a finite set G and let $r \geq 1$ be an integer. Denote by $E_{\mathcal{G}, r}$ the crisp constraint language over domain G with, for every $a \in G$, and $1 \leq m \leq r$, a relation $R_a^m = \{(x_1, \dots, x_m) \in G^m \mid x_1 + \dots + x_m = a\}$.

Theorem 3 ([12]). *Let \mathcal{G} be a finite non-trivial Abelian group. Then, $\text{VCSP}(E_{\mathcal{G}, 3})$ requires linear levels of the Lasserre SDP hierarchy.*

⁴We note that [40, Theorem 6.4] is stated only for Boolean Max-CSPs and proved using [40, Theorem 3.8]. However, a generalisation to non-Boolean domains follows from [40, Theorem 7.2]. We thank Prasad Raghavendra and David Steurer for clarifying this.

For Abelian groups of prime power orders, Tulsiani showed that there is a constant $0 < c < 1$ such that for every large enough n there is an instance I_n of $\text{VCSP}(E_{G,r})$ on n variables with $\text{Opt}_{\text{VCSP}}(I_n) = \infty$ and $\text{Opt}_{\text{SDP}}(I_n, \lfloor cn \rfloor) = 0$; i.e., I_n is a gap instance for Lasserre($\lfloor cn \rfloor$) [51, Theorem 4.2].⁵ This work was based on the result of Schoenebeck who showed it for Boolean domains [46], thus rediscovering the work of Grigoriev [22]. A generalisation to all Abelian groups was then established by Chan in [12, Appendix D]. We remark that the results in [12], [46], [51] actually prove something much stronger: $\Omega(n)$ levels of the Lasserre SDP hierarchy not only cannot distinguish unsatisfiable instances of $\text{VCSP}(E_{G,3})$ from satisfiable ones, but also cannot distinguish unsatisfiable instance from those in which a certain (large) fraction of the constraints is guaranteed to be satisfiable. (Moreover, the quantitative statements are optimal [12], [51].)

The following notion of reduction is key in this paper.

Definition 13. Let Γ and Δ be two general-valued constraint languages of finite size. We write $\Delta \leq_L \Gamma$ if there is a polynomial-time reduction from $\text{VCSP}(\Delta)$ to $\text{VCSP}(\Gamma)$ with the following property: there is a constant $c \geq 1$ depending only on Γ and Δ such that for any $k \geq 1$, if Lasserre(k) solves $\text{VCSP}(\Gamma)$ then Lasserre(ck) solves $\text{VCSP}(\Delta)$.

By Definition 13, \leq_L reductions compose. Let $\Delta \leq_L \Gamma$. By Definitions 12 and 13, if $\text{VCSP}(\Delta)$ requires linear levels of the Lasserre SDP hierarchy then so does $\text{VCSP}(\Gamma)$. An analogous notion of reduction for the Sherali-Adams LP hierarchy, \leq_{SA} , was used in [48], [50].

We now describe the various types of gadget constructions needed to establish Theorem 2.

Definition 14. We say that an m -ary weighted relation ϕ is expressible over a general-valued constraint language Γ if there exists an instance I of $\text{VCSP}(\Gamma)$ with variables $x_1, \dots, x_m, v_1, \dots, v_p$ such that

$$\phi(x_1, \dots, x_m) = \min_{v_1, \dots, v_p} \phi_I(x_1, \dots, x_m, v_1, \dots, v_p).$$

For a fixed set D , let ϕ_{\equiv}^D denote the binary equality relation $\{(x, x) \mid x \in D\}$. We denote by $\langle \Gamma \rangle$ the set of weighted relations obtained by taking the closure of $\Gamma \cup \{\phi_{\equiv}^D\}$, where D is the domain of Γ , under expressibility and the Feas and Opt operations.

Definition 15. Let Γ and Δ be general-valued constraint languages on domain D and D' , respectively. We say that Δ has an interpretation in Γ with parameters (d, S, h) if there exists a $d \in \mathbb{N}$, a set $S \subseteq D^d$, and a surjective map $h : S \rightarrow D'$ such that $\langle \Gamma \rangle$ contains the following weighted relations:

- $\phi_S : D^d \rightarrow \overline{\mathbb{Q}}$ defined by $\phi_S(\mathbf{x}) = 0$ if $\mathbf{x} \in S$ and $\phi_S(\mathbf{x}) = \infty$ otherwise;
- $h^{-1}(\phi_{\equiv}^{D'})$; and
- $h^{-1}(\phi_i)$, for every weighted relation $\phi_i \in \Delta$,

⁵We note that [51] uses different terminology from ours: $\text{Max-CSP}(P)$ for a k -ary predicate P applied to literals rather than variables.

where $h^{-1}(\phi_i)$, for an m -ary weighted relation ϕ_i , is the dm -ary weighted relation on D defined by $h^{-1}(\phi_i)(\mathbf{x}_1, \dots, \mathbf{x}_m) = \phi_i(h(\mathbf{x}_1), \dots, h(\mathbf{x}_m))$, for all $\mathbf{x}_1, \dots, \mathbf{x}_m \in S$.

Remark 1. A weighted relation being expressible over $\Gamma \cup \{\phi_{\equiv}^D\}$ is the analogue of a relation being definable by a *primitive positive* (pp) formula (using existential quantification and conjunction) over a relational structure with equality. Indeed, when Γ is crisp, the two notions coincide. Also, for a crisp Γ the notion of an interpretation coincides with the notion of a *pp-interpretation* for relational structures [7].

We can now give a formal definition of the notion of *simulation* used in the statement of Theorem 2. Let $\mathcal{C}_D = \{\{(a)\} \mid a \in D\}$ be the set of constant unary relations on the set D .

Definition 16. Let Γ' be a core of a general-valued constraint language Γ on domain $D' \subseteq D$. We say that Γ can simulate a general-valued constraint language Δ if Δ has an interpretation in $\Gamma' \cup \mathcal{C}_{D'}$.

The following theorem is the main technical contribution of the paper. It shows that a general-valued constraint language can be augmented with various additional weighted relations while preserving exact solvability in the Lasserre SDP hierarchy up to a constant factor in the level of the hierarchy. It is a strengthening of Theorem [50, Theorem 16], which showed that the same additional weighted relations preserve exact solvability in the Sherali-Adams LP hierarchy.

Theorem 4. Let Γ be a general-valued constraint language of finite size on domain D . The following holds:

- 1) If ϕ is expressible in Γ , then $\Gamma \cup \{\phi\} \leq_L \Gamma$.
- 2) $\Gamma \cup \{\phi_{\equiv}^D\} \leq_L \Gamma$.
- 3) If Γ interprets the general-valued constraint language Δ of finite size, then $\Delta \leq_L \Gamma$.
- 4) If $\phi \in \Gamma$, then $\Gamma \cup \{\text{Opt}(\phi)\} \leq_L \Gamma$ and $\Gamma \cup \{\text{Feas}(\phi)\} \leq_L \Gamma$.
- 5) If Γ' is a core of Γ on domain $D' \subseteq D$, then $\Gamma' \cup \mathcal{C}_{D'} \leq_L \Gamma$.

Proof. The proof is to a large extent based on a technical lemma, Lemma 4, which is stated and proved in Section IV. This lemma shows that, subject to some consistency conditions, a polynomial-time reduction between two constraint languages Δ and Γ that is based on locally replacing valued constraints with weighted relations in Δ by gadgets expressed in Γ can be turned into an \leq_L -reduction. The same approach was used in [50, Theorem 16] for constructing \leq_{SA} -reductions for (1–3), and (5). In these cases, it therefore essentially suffices to replace the applications of [50, Lemma 9] by applications of Lemma 4 in the proofs of [50, Lemmas 10–12, and 15]. For case (3), we remark that our definition differs slightly from that of [50] in that we incorporate applications of the operations Opt and Feas in the definition of $\langle \Gamma \rangle$. To accommodate for this in the proof, it suffices to add an application of (4). For case (5), the proof in [50, Lemmas 15] also refers to [50, Lemma 7] which also hold for \leq_L -reductions by Lemma 1 below, and cases (1) and (4).

The remaining two reductions in (4) are shown in a more straightforward way for \leq_{SA} -reductions in [50, Lemmas 13 and 14]. Here, we argue that the proof of [50, Lemmas 13] goes through for \leq_{L} -reductions as well, which shows that $\Gamma \cup \{\text{Opt}(\phi)\} \leq_{\text{L}} \Gamma$. We omit the analogous argument for the reduction $\Gamma \cup \{\text{Feas}(\phi)\} \leq_{\text{L}} \Gamma$. An instance I of $\text{VCSP}(\Gamma \cup \{\text{Opt}(\phi)\})$ is transformed into an instance of $\text{VCSP}(\Gamma)$ by replacing all occurrences of $\text{Opt}(\phi)$ by multiple copies of ϕ . In the proof of [50, Lemmas 13], it is then shown that if I is a gap instance for the $\text{SA}(k)$ -relaxation, and λ is an optimal solution to this relaxation, then λ is also a solution to the $\text{SA}(k)$ -relaxation of J . Moreover, λ attains a better value than $\text{Opt}_{\text{VCSP}}(J)$, hence J is also a gap instance.⁶ This argument goes through also if we take I to be a gap instance for the Lasserre(k)-relaxation, and λ an optimal solution to this relaxation. The exact same solution λ then also shows that J is a gap instance for the Lasserre(k)-relaxation. \square

In order to finish the proof of Theorem 2, we need few additional results. The following result follows, as described in the proof of [50, Theorem 15], from [2], [32].

Theorem 5 ([50, Theorem 15]). *Let Δ be a crisp constraint language of finite size that contains all constant unary relations. If $\text{Pol}(\Delta)$ violates the BWC, then there exists a finite non-trivial Abelian group \mathcal{G} such that Δ interprets $E_{\mathcal{G},r}$, for every $r \geq 1$.*

The following two lemmas, together with cases (1) and (4) of Theorem 4, extend [50, Lemma 7 and Lemma 8] from \leq_{SA} -reductions to \leq_{L} -reductions.

Lemma 1. *Let Γ be a general-valued constraint language over domain D and let F be a set of operations over D . If $\text{supp}(\Gamma) \cap F = \emptyset$, then there exists a crisp constraint language $\Delta \subseteq \langle \Gamma \rangle$ such that $\text{Pol}(\Delta) \cap F = \emptyset$. Moreover, if Γ and F are finite then so is Δ .*

Proof. By [50, Lemma 2], for each $f \in F \cap \text{Pol}(\Gamma)$, there is an instance I_f of $\text{VCSP}(\Gamma)$ such that $f \notin \text{Pol}(\text{Opt}(\phi_{I_f}))$. Let $\Delta = \{\text{Opt}(\phi_{I_f}) \mid f \in F\} \cup \{\text{Feas}(\phi) \mid \phi \in \Gamma\} \subseteq \langle \Gamma \rangle$. For $f \in F \cap \text{Pol}(\Gamma)$, we have $f \notin \text{Pol}(\text{Opt}(\phi_{I_f})) \supseteq \text{Pol}(\Delta)$. For $f \in F \setminus \text{Pol}(\Gamma)$, we have $f \notin \text{Pol}(\phi)$, for some $\phi \in \Gamma$, so $f \notin \text{Pol}(\Delta)$. It follows that $\text{Pol}(\Delta) \cap F = \emptyset$. \square

Lemma 2. *Let Γ be a general-valued constraint language of finite size. If $\text{supp}(\Gamma)$ violates the BWC, then there is a crisp constraint language $\Delta \subseteq \langle \Gamma \rangle$ of finite size such that $\text{Pol}(\Delta)$ violates the BWC.*

Proof. Since $\text{supp}(\Gamma)$ violates the BWC, there exists an $m \geq 3$ such that $\text{supp}(\Gamma)$ does not contain any m -ary WNU. Let F be the (finite) set of all m -ary WNUs. The result follows by applying Lemma 1 to Γ and F . \square

We are now ready to prove Theorem 2.

Proof of Theorem 2. Theorem 1 gives the implication (i) \implies (iii) by contraposition: if $\text{supp}(\Gamma)$ satisfies the BWC then, by

⁶We remark that the reviewers of [50] noticed that the proof of [50, Lemma 13] omits one case. For completeness, we discuss this in Appendix A.

Theorem 1, $\text{VCSP}(\Gamma)$ is solved by any constant level k of the Sherali-Adams LP hierarchy with $k \geq 3$, and thus also by the k th level of the Lasserre SDP hierarchy for $k \geq \text{ar}(\Gamma)$.

Now, suppose that $\text{supp}(\Gamma)$ violates the BWC. Let Γ' be a core of Γ on a domain $D' \subseteq D$ and let $\Gamma_c = \Gamma' \cup \mathcal{C}_{D'}$. By [50, Lemma 5], $\text{supp}(\Gamma_c)$ also violates the BWC. By Lemma 2, there exists a finite crisp constraint language Δ such that Δ has an interpretation in Γ_c and $\text{Pol}(\Delta)$ violates the BWC. Since $\mathcal{C}_D \subseteq \Gamma_c$, we may assume, without loss of generality, that $\mathcal{C}_D \subseteq \Delta$. By Theorem 5, there exists a finite non-trivial Abelian group \mathcal{G} and an interpretation of $E_{\mathcal{G},3}$ in Δ . It is easy to see that interpretations compose, and hence, $E_{\mathcal{G},3}$ has an interpretation in Γ_c . Therefore, Γ can simulate $E_{\mathcal{G},3}$ which gives the implication (iii) \implies (ii).

Finally, by Theorem 3, $\text{VCSP}(E_{\mathcal{G},3})$ requires linear levels of the Lasserre SDP hierarchy. By Theorem 4(3) and (5), we have $E_{\mathcal{G},3} \leq_{\text{L}} \Gamma_c \leq_{\text{L}} \Gamma$. Consequently, $\text{VCSP}(\Gamma)$ requires linear levels of the Lasserre SDP hierarchy as well. This gives the implication (ii) \implies (i). \square

IV. AN \leq_{L} -REDUCTION SCHEME

In this section, we will prove Lemma 4, which is a key technique used to establish cases (1)–(3) and (5) of Theorem 4. It is an analogue of [50, Lemma 9], which does the same for the \leq_{SA} -reductions. In order to generalise [50, Lemma 9] to the \leq_{L} -reductions we will use Lemma 3 proved below.

The following observation will be used throughout this section: since the set of vectors $\{\lambda_i(\tau) \mid \tau \in D^{X_i}\}$ for a feasible solution λ is orthogonal by (L5), it follows that $\|\sum_{\tau \in T} \lambda_i(\tau)\|^2 = \sum_{\tau \in T} \langle \lambda_i(\tau), \lambda_i(\tau) \rangle$ for any subset $T \subseteq D^{X_i}$.

Lemma 3. *Every feasible solution λ to the Lasserre(k)-relaxation satisfies, in addition to (L1)–(L6):*

$$\sum_{\tau: \tau|_{X_j} = \sigma} \lambda_i(\tau) = \lambda_j(\sigma) \quad \forall i, j \in [q], X_j \subseteq X_i, |X_i| \leq k, \sigma: X_j \rightarrow D. \quad (\text{L7})$$

Proof. Consider the norm of the vector $\sum_{\tau: \tau|_{X_j} = \sigma} \lambda_i(\tau) - \lambda_j(\sigma)$.

$$\begin{aligned} & \left\| \sum_{\tau: \tau|_{X_j} = \sigma} \lambda_i(\tau) - \lambda_j(\sigma) \right\|^2 \\ &= \left\| \sum_{\tau: \tau|_{X_j} = \sigma} \lambda_i(\tau) \right\|^2 - 2 \left\langle \sum_{\tau: \tau|_{X_j} = \sigma} \lambda_i(\tau), \lambda_j(\sigma) \right\rangle + \|\lambda_j(\sigma)\|^2 \\ &= \left\| \sum_{\tau: \tau|_{X_j} = \sigma} \lambda_i(\tau) \right\|^2 - 2 \sum_{\tau: X_i \rightarrow D} \langle \lambda_i(\tau), \lambda_j(\sigma) \rangle + \|\lambda_j(\sigma)\|^2 \\ &= \left\| \sum_{\tau: \tau|_{X_j} = \sigma} \lambda_i(\tau) \right\|^2 - 2 \sum_{\tau: X_i \rightarrow D} \langle \lambda_i(\tau), \lambda_i(\tau) \rangle + \|\lambda_j(\sigma)\|^2 \\ &= -\left\| \sum_{\tau: \tau|_{X_j} = \sigma} \lambda_i(\tau) \right\|^2 + \|\lambda_j(\sigma)\|^2, \end{aligned}$$

where the next to last equality follows from (L6) since $X_j \subseteq X_i$ and $\sigma = \tau|_{X_j}$. We see that the equality in the lemma is equivalent to:

$$\left\| \sum_{\tau: \tau|_{X_j} = \sigma} \lambda_i(\tau) \right\|^2 = \|\lambda_j(\sigma)\|^2. \quad (1)$$

We finish the proof by induction on $|X_i \setminus X_j| \geq 1$. There are two base cases:

- (i) If $|X_i \setminus X_j| = 1$ and $X_j = \emptyset$, then (1) follows immediately from (L1) and (L4).
- (ii) If $|X_i \setminus X_j| = 1$ and $X_j \neq \emptyset$, then let $X_r = \{x\} = X_i \setminus X_j$ be a scope on the single variable x , and, for $a \in D$, let σ_a be the assignment $\sigma_a(x) = a$. Now, (1) follows from:

$$\begin{aligned} \left\| \sum_{\tau: \tau|_{X_j} = \sigma} \lambda_i(\tau) \right\|^2 &= \sum_{a \in D} \langle \lambda_i(\sigma_a \circ \sigma), \lambda_i(\sigma_a \circ \sigma) \rangle \\ &\stackrel{(L6)}{=} \sum_{a \in D} \langle \lambda_r(\sigma_a), \lambda_j(\sigma) \rangle \\ &= \left\langle \sum_{a \in D} \lambda_r(\sigma_a), \lambda_j(\sigma) \right\rangle \\ &\stackrel{(i)}{=} \langle \lambda_0, \lambda_j(\sigma) \rangle, \\ &\stackrel{(L6)}{=} \langle \lambda_j(\sigma), \lambda_j(\sigma) \rangle. \end{aligned}$$

Finally, assume that $|X_i \setminus X_j| > 1$ and that $x \in X_i \setminus X_j$. Let r be an index such that $X_r = X_j \cup \{x\}$, and, for $a \in D$, let σ_a be the assignment $\sigma_a(x) = a$. Then,

$$\begin{aligned} \sum_{\tau: \tau|_{X_j} = \sigma} \lambda_i(\tau) &= \sum_{a \in D} \sum_{\tau: \tau|_{X_r} = \sigma \circ \sigma_a} \lambda_i(\tau) \\ &= \sum_{a \in D} \lambda_r(\sigma \circ \sigma_a) \\ &= \lambda_j(\sigma), \end{aligned}$$

where the last two equalities follow by induction. \square

For a solution λ to the Lasserre(k)-relaxation of I with the objective function $\sum_{i=1}^q \phi(\mathbf{x}_i)$, we denote by $\text{supp}(\lambda_i)$ the positive support of λ_i , i.e., $\text{supp}(\lambda_i) = \{\sigma: X_i \rightarrow D \mid \|\lambda_i(\sigma)\|^2 > 0\}$.

We extend the convention of denoting the set of variables in \mathbf{x}_i by X_i to tuples \mathbf{y}'_i , whose sets are denoted by Y'_i .

The following technical lemma is the basis for the reductions.

Lemma 4. *Let Δ and Δ' be general-valued constraint languages of finite size over domains D and D' , respectively.*

Let $(I, i) \mapsto J_i$ be a map that to each instance I of $\text{VCSP}(\Delta)$ with variables V and objective function $\sum_{i=1}^q \phi_i(\mathbf{x}_i)$, and index $i \in [q]$, associates an instance J_i of $\text{VCSP}(\Delta')$ with variables Y_i and objective function ϕ_{J_i} . Let J be the $\text{VCSP}(\Delta')$ instance with variables $V' = \bigcup_{i=1}^q Y_i$ and objective function $\sum_{i=1}^q \phi_{J_i}$.

Suppose that the following holds:

- (a) *For every satisfying and optimal assignment α of J , there exists a satisfying assignment σ^α of I such that*

$$\text{Val}_{\text{VCSP}}(I, \sigma^\alpha) \leq \text{Val}_{\text{VCSP}}(J, \alpha).$$

Furthermore, suppose that for any $k \geq \text{ar}(\Delta)$, and any feasible solution λ of the Lasserre(k)-relaxation of I , the following properties hold:

- (b) *For $i \in [q]$, and $\sigma: X_i \rightarrow D$ with positive support in λ , there exists a satisfying assignment α_i^σ of J_i such that*

$$\phi_i(\sigma(\mathbf{x}_i)) \geq \text{Val}_{\text{VCSP}}(J_i, \alpha_i^\sigma);$$

- (c) *for $i, r \in [q]$, any $X \subseteq V$ with $X_i \cup X_r \subseteq X$, and $\sigma: X \rightarrow D$ with positive support in λ ,*

$$\alpha_i^{\sigma_i} |_{Y_i \cap Y_r} = \alpha_r^{\sigma_r} |_{Y_i \cap Y_r},$$

where $\sigma_i = \sigma|_{X_i}$ and $\sigma_r = \sigma|_{X_r}$.

Then, $I \mapsto J$ is a many-one reduction from $\text{VCSP}(\Delta)$ to $\text{VCSP}(\Delta')$ that verifies $\Delta \leq_L \Delta'$.

Proof. First, we show that $\text{Opt}_{\text{VCSP}}(I) = \text{Opt}_{\text{VCSP}}(J)$. From condition (a), if J is satisfiable, then so is I and $\text{Opt}_{\text{VCSP}}(I) \leq \text{Opt}_{\text{VCSP}}(J)$. Conversely, if I is satisfiable, and σ is an optimal assignment to I , then the Lasserre($2k$) solution λ , where $k \geq \text{ar}(\Delta)$, that assigns a fixed unit vector to $\sigma|_X$ for every $X \subseteq V$ with $|X| \leq 2k$ is feasible. Let $\sigma_i = \sigma|_{X_i}$. By (b), there exist satisfying assignments $\alpha_i^{\sigma_i}$ of J_i , for all $i \in [q]$, such that $\text{Opt}_{\text{VCSP}}(I) \geq \text{Opt}_{\text{SDP}}(I, 2k) \geq \sum_{i \in [q]} \text{Val}_{\text{VCSP}}(J_i, \alpha_i^{\sigma_i})$. Define an assignment $\alpha: V' \rightarrow D'$ by letting $\alpha(y) = \alpha_i^{\sigma_i}(y)$ for an arbitrary i such that $y \in Y_i$. We claim that $\alpha|_{Y_i} = \alpha_i^{\sigma_i}$, for all $i \in [q]$. From this it follows that α is a satisfying assignment to J such that $\sum_{i \in [q]} \text{Val}_{\text{VCSP}}(J_i, \alpha_i^{\sigma_i}) = \text{Val}_{\text{VCSP}}(J, \alpha) \geq \text{Opt}_{\text{VCSP}}(J)$, and hence that $\text{Opt}_{\text{VCSP}}(I) \geq \text{Opt}_{\text{VCSP}}(J)$. Indeed, let $y \in V'$ and assume that $y \in Y_i$ and $y \in Y_r$. Let $X = X_i \cup X_r$. Then, since $k \geq \text{ar}(\Delta)$ and $\|\lambda(\sigma|_X)\|^2 > 0$, it follows from (c) that $\alpha_i^{\sigma_i}(y) = \alpha_r^{\sigma_r}(y)$.

Let k' be arbitrary and let $k = \max\{k', \text{ar}(\Delta')\} \cdot \text{ar}(\Delta)$. Assume that I is a gap instance for the Lasserre($2k$)-relaxation of $\text{VCSP}(\Delta)$, and let λ be a feasible solution such that $\text{Val}_{\text{SDP}}(I, \lambda, 2k) < \text{Opt}_{\text{VCSP}}(I)$ (where $\text{Opt}_{\text{VCSP}}(I)$ may be ∞ , i.e. I may be unsatisfiable). We show that there is a feasible solution κ to the Lasserre(k')-relaxation of J such that $\text{Val}_{\text{SDP}}(J, \kappa, k') \leq \text{Val}_{\text{SDP}}(I, \lambda, 2k)$.⁷ Then, by condition (a), we have $\text{Opt}_{\text{VCSP}}(I) \leq \text{Opt}_{\text{VCSP}}(J)$. Hence, $\text{Val}_{\text{SDP}}(J, \kappa, k') \leq \text{Val}_{\text{SDP}}(I, \lambda, 2k) < \text{Opt}_{\text{VCSP}}(I) \leq \text{Opt}_{\text{VCSP}}(J)$, so J is a gap instance for the Lasserre(k')-relaxation of $\text{VCSP}(\Delta')$. Since k' was chosen arbitrarily, we have $\Delta \leq_L \Delta'$.

To this end, augment I with null constraints on $X_{q+1}, \dots, X_{q'}$ so that for every at most $2k$ -subset $X \subseteq V$, there exists an $i \in [q']$ such that $X_i = X$. Rewrite the objective function of J as $\sum_{j=1}^p \phi'_j(\mathbf{y}'_j)$, $\phi'_j \in \Delta'$, where, by possibly first adding extra null constraints to J , we will assume that for every at most k' -subset $Y \subseteq V'$, there exists a $j \in [p]$ such that $Y'_j = Y$. For each $i \in [q]$, let C_i be the set of indices $j \in [p]$ corresponding to the valued constraints in the instance J_i .

⁷We remark here that the vectors in the feasible solution κ will live in the same space \mathbb{R}^t as those of λ . This is not a problem as long as t is chosen sufficiently large enough for both of the relaxations.

Extend α_i^σ to indices $i \in [q'] \setminus [q]$ as follows. For $X \subseteq V$, let $Y_X = \bigcup_{j \in [q]: X_j \subseteq X} Y_j$. For $\sigma \in \text{supp}(\lambda_i)$, and any $r, s \in [q]$ such that $X_r \cup X_s \subseteq X_i$ and $y \in Y_r \cap Y_s$, by (c), it holds that $\alpha_r^{\sigma_r}(y) = \alpha_s^{\sigma_s}(y)$. Therefore, we can uniquely define $\alpha_i^\sigma: Y_{X_i} \rightarrow D'$ by letting $\alpha_i^\sigma(y) = \alpha_r^{\sigma_r}(y)$ for any choice of $r \in [q]$ with $X_r \subseteq X_i$ and $y \in Y_r$. This definition is consistent with α_i^σ for $i \in [q]$ in the sense that (c) now holds for all $i, r \in [q']$.

For $m \geq 1$, let $X_{(\leq m)} = \{X = \bigcup_{i \in S} X_i \mid S \subseteq [q], |X| \leq m\}$, and for $Y \subseteq V'$ with $|Y| \leq k'$, let $X_{(\leq m)}(Y) = \{X \in X_{(\leq m)} \mid Y \subseteq Y_X\}$.

Let $j \in [p]$ be arbitrary and let $X = \bigcup_{i \in S} X_i \in X_{(\leq n)}(Y'_j)$, for some $S \subseteq [q]$, where $n = |V|$. For each $y \in Y'_j$, let $i(y) \in S$ be an index such that $y \in Y_{i(y)}$ and let $X' = \bigcup_{y \in Y'_j} X_{i(y)}$. Then, $Y'_j \subseteq Y_{X'}$, $X' \subseteq X$, and $|X'| \leq \max\{k', \text{ar}(\Delta')\} \cdot \text{ar}(\Delta) \leq k$, so $X' \in X_{(\leq k)}(Y'_j)$. In other words,

$$\text{for all } X \in X_{(\leq n)}(Y'_j), \text{ there exists } i \in [q'] \text{ such that } X_i \subseteq X \text{ and } X_i \in X_{(\leq k)}(Y'_j). \quad (2)$$

In particular (2) shows that for every j there exists $i \in [q']$ such that $X_i \in X_{(\leq 2k)}(Y'_j)$, since $\bigcup_{i \in [q]} X_i \in X_{(\leq n)}(Y'_j)$ for all j .

For $j \in [p]$, $\alpha: Y'_j \rightarrow D'$, and $i \in [q']$ such that $X_i \in X_{(\leq 2k)}(Y'_j)$, define

$$\mu_j^i(\alpha) = \sum_{\sigma: \alpha_i^\sigma|_{Y'_j} = \alpha} \lambda_i(\sigma). \quad (3)$$

Claim: Definition (3) is independent of the choice of $X_i \in X_{(\leq 2k)}(Y'_j)$. That is,

$$\mu_j^r = \mu_j^i \quad \forall r, i \in [q'] \text{ such that } X_r, X_i \in X_{(\leq 2k)}(Y'_j). \quad (4)$$

First, we prove (4) for $X_r \subseteq X_i$ with $X_r \in X_{(\leq k)}(Y'_j)$ and $X_i \in X_{(\leq 2k)}(Y'_j)$. We have

$$\begin{aligned} \mu_j^r(\alpha) &\stackrel{(3)}{=} \sum_{\tau: \alpha_r^\tau|_{Y'_j} = \alpha} \lambda_r(\tau) \\ &\stackrel{(L7)}{=} \sum_{\tau: \alpha_r^\tau|_{Y'_j} = \alpha} \sum_{\sigma: \sigma|_{X_r} = \tau} \lambda_i(\sigma) \\ &= \sum_{\sigma: \alpha_r^{\sigma_r}|_{Y'_j} = \alpha} \lambda_i(\sigma) \\ &\stackrel{(c)}{=} \sum_{\sigma: \alpha_i^\sigma|_{Y'_j} = \alpha} \lambda_i(\sigma) \\ &\stackrel{(3)}{=} \mu_j^i(\alpha), \end{aligned}$$

Next, let $X_r \in X_{(\leq 2k)}(Y'_j)$ and $X_i \in X_{(\leq 2k)}(Y'_j)$ be arbitrary. From (2), it follows that X_r contains a subset $X_s \in X_{(\leq k)}(Y'_j)$ and that X_i contains a subset $X_t \in X_{(\leq k)}(Y'_j)$. Since $|X_s \cup X_t| \leq 2k$, there exists an index u such that $X_u = X_s \cup X_t$. The claim (4) now follows by a repeated application of the first case: $\mu_j^r = \mu_j^s = \mu_j^u = \mu_j^t = \mu_j^i$.

By (4), we can pick an arbitrary $X_i \in X_{(\leq 2k)}(Y'_j)$ and uniquely define $\kappa_j = \mu_j^i$.

We now show that this definition of κ satisfies the equations (L1)–(L6).

- The equation (L1) holds as $\kappa_0 = \sum_\sigma \lambda_i(\sigma) = 1$ for an arbitrary i by (L7).
- The equations (L2) holds by the linearity of the inner product.
- The equations (L3) hold trivially if ϕ'_j is a null constraint. Otherwise, $j \in C_i$ for some $i \in [q]$. This implies that $X_i \in X_{(\leq k)}(Y'_j)$, and by (4) we have $\kappa_j = \mu_j^i$. Then, $\alpha \in \text{supp}(\kappa_j)$ implies that there is a $\sigma \in \text{supp}(\lambda_i)$ such that $\alpha_i^\sigma|_{Y'_j} = \alpha$. By condition (b) and equation (L3) for λ_i , the tuple $\alpha_i^\sigma(\mathbf{y}'_j) \in \text{Feas}(\phi'_j)$, so κ_j satisfies (L3).
- We show that the equations (L4) hold for κ . Let $Y'_j = \{y\}$ be a singleton and let $X_i \in X_{(\leq k)}(Y'_j)$. We have

$$\begin{aligned} &\sum_{a' \in D'} \|\kappa_j(a')\|^2 \\ &\stackrel{(3)}{=} \sum_{a' \in D'} \langle \sum_{\sigma: \alpha_i^\sigma(y) = a'} \lambda_i(\sigma), \sum_{\sigma: \alpha_i^\sigma(y) = a'} \lambda_i(\sigma) \rangle \\ &\stackrel{(L5)}{=} \sum_{a' \in D'} \sum_{\sigma: \alpha_i^\sigma(y) = a'} \langle \lambda_i(\sigma), \lambda_i(\sigma) \rangle \\ &= \sum_{\sigma} \langle \lambda_i(\sigma), \lambda_i(\sigma) \rangle \\ &= \|\sum_{\sigma} \lambda_i(\sigma)\|^2 \\ &\stackrel{(L7)}{=} 1. \end{aligned}$$

- The equations (L5) hold by linearity of the inner product and by the equations (L5) for λ .
- Finally, we show that the equations (L6) hold for κ . Let $Y'_r, Y'_s, Y'_{r'}, Y'_{s'}$ be such that $Y'_r \cup Y'_s = Y'_{r'} \cup Y'_{s'}$ and $|Y'_r|, |Y'_s|, |Y'_{r'}|, |Y'_{s'}| \leq k$. Furthermore, let $\alpha_r: Y'_r \rightarrow D', \alpha_s: Y'_s \rightarrow D', \alpha_{r'}: Y'_{r'} \rightarrow D', \alpha_{s'}: Y'_{s'} \rightarrow D'$ be such that $\alpha_r \circ \alpha_s = \alpha_{r'} \circ \alpha_{s'}$. Let $X_i \in X_{(\leq 2k)}(Y'_r \cup Y'_s)$. Then,

$$\begin{aligned} &\langle \kappa_r(\alpha_r), \kappa_s(\alpha_s) \rangle \\ &\stackrel{(3)}{=} \langle \sum_{\sigma: \alpha_i^\sigma|_{Y'_r} = \alpha_r} \lambda(\sigma), \sum_{\sigma': \alpha_i^{\sigma'}|_{Y'_s} = \alpha_s} \lambda(\sigma') \rangle \\ &= \sum_{\sigma: \alpha_i^\sigma|_{Y'_r} = \alpha_r} \sum_{\sigma': \alpha_i^{\sigma'}|_{Y'_s} = \alpha_s} \langle \lambda(\sigma), \lambda(\sigma') \rangle \\ &\stackrel{(L5)}{=} \sum_{\sigma: \alpha_i^\sigma|_{Y'_r \cup Y'_s} = \alpha_r \circ \alpha_s} \langle \lambda(\sigma), \lambda(\sigma) \rangle \end{aligned}$$

Since $Y'_r \cup Y'_s = Y'_{r'} \cup Y'_{s'}$ and $\sigma_r \circ \sigma_s = \sigma_{r'} \circ \sigma_{s'}$, it follows that the right-hand side is identical for $\langle \kappa_r(\alpha_r), \kappa_s(\alpha_s) \rangle$ and $\langle \kappa_{r'}(\alpha_{r'}), \kappa_{s'}(\alpha_{s'}) \rangle$.

We conclude that κ is a feasible solution to the Lasserre(k')-relaxation of J .

Let $i \in [q]$ and note that by (4), for every $j \in C_i$, we have $\kappa_j = \mu_j^i$. Therefore,

$$\begin{aligned}
& \sum_{j \in C_i} \sum_{\alpha \in \text{Feas}(\phi'_j)} \|\kappa_j(\alpha)\|^2 \phi'_j(\alpha(\mathbf{y}'_j)) \\
&= \sum_{j \in C_i} \sum_{\alpha \in \text{Feas}(\phi'_j)} \sum_{\sigma: \alpha_i^\sigma|_{Y'} = \alpha} \|\lambda_i(\sigma)\|^2 \phi'_j(\alpha(\mathbf{y}'_j)) \\
&= \sum_{\sigma: \alpha_i^\sigma|_{Y'} \in \text{Feas}(\phi'_j)} \|\lambda_i(\sigma)\|^2 \sum_{j \in C_i} \phi'_j(\alpha_i^\sigma(\mathbf{y}'_j)) \\
&\leq \sum_{\sigma \in \text{supp}(\lambda_i)} \|\lambda_i(\sigma)\|^2 \phi_i(\sigma),
\end{aligned} \tag{5}$$

where the inequality follows from assumption (b). Summing inequality (5) over $i \in [q]$ shows that $\text{Val}_{\text{SDP}}(J, \kappa, k') \leq \text{Val}_{\text{SDP}}(I, \lambda, k)$ and the lemma follows. \square

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$\text{Opt}_{\text{VCSP}}(J)$, so again Γ does not have valued relational width (k, ℓ) . Since k and ℓ were chosen arbitrarily, the result follows.

APPENDIX

NOTE ON THE PROOF OF [50, LEMMA 13]

There is a missing case in the last paragraph of the proof of [50, Lemma 13]. In particular, if J is satisfiable then I is not necessarily satisfiable as claimed in the proof of [50, Lemma 13] and thus the case J being satisfiable and I being unsatisfiable needs to be considered as well.

The last paragraph of the proof of [50, Lemma 13] should be replaced with the following paragraph.

If J is unsatisfiable, then $\text{Opt}_{\text{LP}}(J) < \text{Opt}_{\text{VCSP}}(J)$, so Γ does not have valued relational width (k, ℓ) . If J is satisfiable and I is also satisfiable then it was shown above that $\text{Opt}_{\text{VCSP}}(J) = \text{Opt}_{\text{VCSP}}(I) + CN \min(\phi)$, so $\text{Opt}_{\text{LP}}(J) \leq \text{Opt}_{\text{LP}}(I) + CN \min(\phi) < \text{Opt}_{\text{VCSP}}(I) + CN \min(\phi) = \text{Opt}_{\text{VCSP}}(J)$, and again Γ does not have valued relational width (k, ℓ) . Finally, if J is satisfiable and I is unsatisfiable then $\text{Opt}_{\text{VCSP}}(J) > CN \min(\phi) + U$. Since λ is a feasible solution, we have $\text{Opt}_{\text{LP}}(I) \leq U$ from the definition of U . Then, $\text{Opt}_{\text{LP}}(J) \leq \text{Opt}_{\text{LP}}(I) + CN \min(\phi) \leq U + CN \min(\phi) <$