On the Connectivity of Two-Hop Networks in Finite Domains

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***Abstract*— In this paper, we study the direct (1-hop) and 2-hop connectivity, between two fixed terminal nodes, amongst N uniformly distributed relay nodes, confined to a finite convex geometry. We develop closed form expressions for average connection probability for special cases of circular and rectangular geometries and consider the case where communication paths are subject to Rayleigh fading. Our analysis shows that connectivity is defined by domain geometry, the mid-point location of the chord separating the terminal nodes and their spatial separation, yielding an interesting property of rotational invariance on the terminal node positions about their mid-point. We show that 2-hop connectivity yields appreciable connectivity gain, over the 1-hop case, with largest gains realised at the domain boundaries. We show that, under a condition of low relay node intensity, a simpler approximation can be employed to evaluate this connectivity metric in a finite domain. The analysis provides an insight into 2-hop connectivity in bounded wireless network domains and has application in many fields including wireless sensor networks, device-to-device communication in 5G networks, which are often modelled as having finite extent.**

I. Introduction

Wireless multi-hop networks in various forms, e.g. ad-hoc networks, wireless sensor networks, vehicle to vehicle (V2V) networks, machine to machine (M2M) networks and millimetre wave communications, to mention a few, have been the subject of intense research in the last few decades [1]–[5]. More recently, research relating to device-to-device (D2D) communications has gained rapid traction, in the context of cellular communication systems, driven by the promise of solving a number of issues: spectral efficiency, energy efficiency and insufficient capacity for example [6], [7]. Looking ahead, [8] predicts that few-hop networking is likely to play a key role in future wireless networks; optimised towards as few hops as possible in order to minimise end-to-end delivery latency and system complexity.

The connectivity between any two devices, wishing to communicate directly, is governed by their spatial separation, channel fading and a number of other system constants: wavelength, SNR, etc. By allowing the two nodes to communicate via an intermediate node, i.e. forming a 2-hop path, it is possible to increase the communication range and/or connection probability. However, since the intermediate nodes are randomly distributed within the domain, we need to consider their spatial distribution and the distributions of other system variables in order to accurately analyse connectivity performance. Further, in the finite domain, the effects of the domain boundary need to be borne in mind [9] and tools taken from the world of stochastic geometry are often called upon to aid analysis [10]. It is worth noting that D2D clusters, within large non-homogeneous wireless networks [11], can be modelled as being contained within finite geometric regions (e.g. a circle).

In this paper, we consider the general case of direct (1-hop) and 2-hop connectivity in finite geometrically bounded wireless networks subject to Rayleigh fading and we assume negligible inter-node interference. The consideration of interference is clearly of concern in many applications; however, for this first foray into 2-hop network analysis, we maintain generality by assuming a suitable MAC protocol is employed to ensure interference is prevented.

Previous works that have considered few-hop connectivity in finite domains include [12], which analyses k-hop connectivity using the unit disk model (UDM), and [13], which reports lower and upper bounds for connection probability for 2-hop connectivity in the unit square. The main contribution of this paper, relative to the state-of-the-art, is the generation of closed form analytic expressions for the direct and 2-hop connection probability between two fixed nodes in circular and rectangular network domains, subject to Rayleigh fading.

The remainder of the paper is organised as follows. Section II provides a system model and details assumptions and constraints. Section III details an analysis of the connection probability between two terminal nodes, at fixed locations within finite networks, by considering the special cases of circular and rectangular boundaries. Section IV analyses the asymptotic behaviour of connection probability, as the domain size increases, which approaches the unbounded case in the $\mathbb{R}^2$ plane. We also consider the case of sparse node density in that section. Section V takes the results of III and provides further analysis of average connectivity from a fixed location in the network and considers the improvement that 2-hop connectivity yields. Much of the analytical work is validated by Monte-Carlo simulations. Conclusions are detailed in section VI.

II. Network Definitions and System Model

The network model considered in this paper is general, but it will become clear that it can be easily applied to a number of scenarios e.g. a D2D cluster where relaying may be employed to maintain low device transmit powers, and thus reduce interference to other cellular devices. However, the model is not limited to any specific case, and thus we abstract our discussion in what follows.

We consider the connectivity between two terminal nodes, node-1 and node-2, at fixed locations, within a finite convex domain $\mathcal{V} \subset \mathbb{R}^2$. Also contained within $\mathcal{V}$, are N identical relay nodes, node-k where $k \in \{3, 4, \ldots, N+2\}$, that are uniformly spatially distributed according to a homogeneous binomial point process (BPP), Φ_B; the node density outside of $\mathcal{V}$ is zero. We denote the location of node-i by the vector $\mathbf{p}_i$ where $i \in I$ and I is the index set $\{1, 2, \ldots, N+2\}$. Hereafter, we use $\mathbf{p}_i$ to represent both node-i and its location interchangeably.

For convenience, we consider a fixed connectivity scale r_0 and, hereafter, normalise all distance variables to this scale. Further, we define r_0 as the hard connection range [14] determined solely by large scale path loss according to the well known modified Friis equation

$$r_0^\eta = \frac{P G_t G_r}{\tau \sigma^2} \left(\frac{\lambda}{4\pi} \right)^2 \tag{1}$$

where η is the path loss exponent, τ is the required SNR for signal detection at the receiving node, P is the transmit power, σ^2 represents the thermal noise variance at the receiving node, G_t and G_r are transmit and receive antenna gains[1] respectively and λ is the wavelength of the RF carrier employed. The receiver performance is assumed to be independent of interference generated by other transmitting nodes in the network.

The direct communication path between any two randomly selected nodes, at locations $\mathbf{p}_i$ and $\mathbf{p}_j$, is considered undirected with reciprocal channel gain and connected with probability $H(r_{ij}) \equiv H_{ij}$ where $r_{ij} = \|\mathbf{p}_i - \mathbf{p}_j\|$. We employ a soft connection model by including a Rayleigh[2] distributed small scale fading component $h_{ij} \sim \mathcal{CN}(0,1)$ which yields a connection probability defined by [15]

$$H_{ij} = e^{-r_{ij}^\eta} \tag{2}$$

In addition to the direct connection between $\mathbf{p}_1$ and $\mathbf{p}_2$, we allow 2-hop connections to be made via an intermediate node $\mathbf{p}_k \in \Phi_B$, as depicted in Fig. 1. For mathematical tractability, we assume that intermediate nodes employ regenerative (decode-and-forward) relaying and the probability of 2-hop connectivity, between $\mathbf{p}_1$ and $\mathbf{p}_2$ via $\mathbf{p}_k$, is given by $H_{1k}H_{2k}$ due to the assumption that the 1-k and 2-k channels are statistically independent[3]. For convenience, we define the average of an observable O over all possible spatial configurations of

[1]With constant gain in azimuth i.e. non directional in the $\mathbb{R}^2$ plane

[2]We defer the treatment of more complex fading models to a future paper

[3]Although this is not necessarily the case when $\mathbf{p}_i$ and $\mathbf{p}_j$ are located within the coherence distance of each other, it is a standard assumption that has been shown on several occasions to yield accurate results [9].

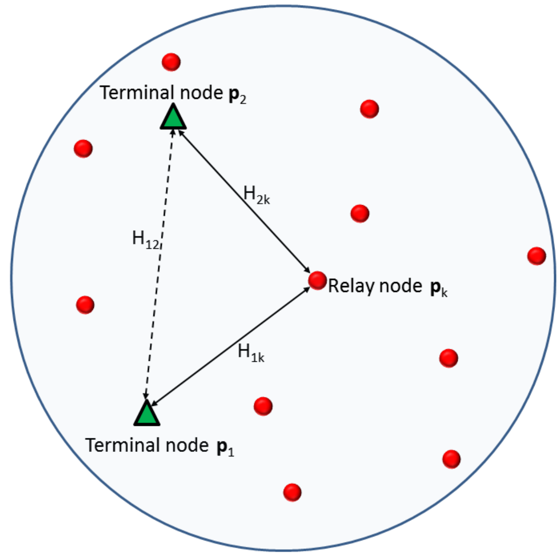

Fig. 1. Direct and 2-hop connectivity between two terminal nodes, $\mathbf{p}_1$ and $\mathbf{p}_2$, in a finite convex domain $\mathcal{V}$ containing N uniformly distributed intermediate relay nodes $\mathbf{p}_k \in \Phi_B$.

u.i.d. relay nodes by [9]

$$\langle O \rangle = \frac{1}{|\mathcal{V}|^N} \int_{\mathcal{V}^N} O(\mathbf{p}_3, \mathbf{p}_4, \ldots, \mathbf{p}_{N+2}) \mathrm{d}\mathbf{p}_3, \mathrm{d}\mathbf{p}_4, \ldots, \mathrm{d}\mathbf{p}_{N+2} \tag{3}$$

III. Connectivity Analysis

The probability that fixed nodes $\mathbf{p}_1$ and $\mathbf{p}_2$ can communicate, either directly or via any intermediate node, is given by the complement of the probability that no direct path and no two-hop paths exist and is denoted by $P_{12}^{(1,2)}$, where the superscript denotes 'via 1-hop and/or 2-hop' and is given by

$$P_{12}^{(1,2)} = 1 - (1 - H_{12}) \prod_{\mathbf{p}_k \in \Phi_B} (1 - H_{1k} H_{2k}) \tag{4}$$

We now proceed to develop closed form analytic expressions for this connection probability for the bounded network and consider two special cases of finite convex boundary: a circle of radius R and a rectangle of dimension $A \times B$. For comparison we also consider an infinite network in the $\mathbb{R}^2$ plane.

A. Connectivity in a Finite Domain $\mathcal{V}$

Since the individual pairwise connection probabilities in (4) are functions of node locations, we define $\langle P_{12}^{(1,2)} \rangle$ as the connection probability (4) averaged over all locations of intermediate relay nodes over the domain. Since relay node positions are i.i.d. then $\langle P_{12}^{(1,2)} \rangle$ can be simply expressed as

$$\langle P_{12}^{(1,2)} \rangle = 1 - (1 - H_{12})(1 - \langle H_{12}^{(2)} \rangle)^N \tag{5}$$

where $\langle H_{12}^{(2)} \rangle$ denotes the average 2-hop connection probability between $\mathbf{p}_1$ and $\mathbf{p}_2$ as defined by

$$\langle H_{12}^{(2)}\rangle = \frac{1}{|\mathcal{V}|}\int_{\mathcal{V}} H_{1k}H_{2k}\mathrm{d}\mathbf{p}_k \tag{6}$$

We now solve (6) for the two special cases of bounded network.

1) Circular Domain: We employ a polar coordinate system, relative to the centre of the circle of radius R. The Euclidean distance between $\mathbf{p}_i = r_i\angle\phi_i$ and $\mathbf{p}_j = r_j\angle\phi_j$ is given by

$$r_{ij} = \sqrt{r_i^2 + r_j^2 - 2r_ir_j\cos(\phi_j-\phi_i)} \tag{7}$$

and (6) becomes

$$\langle H_{12}^{(2)}\rangle = \frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} r_k e^{-(r_{1k}^\eta + r_{2k}^\eta)}\mathrm{d}\phi_k\mathrm{d}r_k \tag{8}$$

For mathematical tractability, we consider the case $\eta = 2$ and (8) becomes

$$\langle H_{12}^{(2)}\rangle = \frac{2}{\pi R^2}e^{-(r_1^2+r_2^2)}\int_0^R r_k e^{-2r_k^2} \times \int_0^\pi e^{-2|\overline{\mathbf{p}}_{12}|r_k\cos\phi_k}\mathrm{d}\phi_k\mathrm{d}r_k \tag{9}$$

where $\overline{\mathbf{p}}_{12} = (\mathbf{p}_1+\mathbf{p}_2)/2$ is the mid-point between $\mathbf{p}_1$ and $\mathbf{p}_2$. Evaluation of the inner integral of (9) yields

$$\langle H_{12}^{(2)}\rangle = \frac{1}{2R^2}e^{-(r_1^2+r_2^2)}\int_0^R r_k e^{-2r_k^2} I_0(2|\overline{\mathbf{p}}_{12}|r_k)\mathrm{d}r_k \tag{10}$$

where $I_0(a) = \frac{1}{\pi}\int_0^\pi e^{a\cos\theta}\mathrm{d}\theta$ is the zeroth order modified Bessel function of the first kind. It can be shown that evaluating the remaining integral yields the following closed form expression for average 2-hop connection probability in a circular domain

$$\langle H_{12}^{(2)}\rangle = \frac{1}{2R^2}e^{-\frac{1}{2}\|\mathbf{p}_2-\mathbf{p}_1\|^2}\Big[1 - \mathrm{Q}_1\left(2|\overline{\mathbf{p}}_{12}|, 2R\right)\Big] \tag{11}$$

where $\mathrm{Q}_1(a,b) = \int_b^\infty x e^{-\frac{(x^2+a^2)}{2}} I_o(ax)\mathrm{d}x$ is the first order Marcum-Q-function. Substituting (11) into (5) yields the expected connection probability, between terminal nodes $\mathbf{p}_1$ and $\mathbf{p}_2$, employing direct and 2-hop connectivity.

$\langle H_{12}^{(2)}\rangle$ is clearly governed by the product of two monotonic functions of Euclidean distance, between the terminal nodes, and the position of their mid-point, within the domain $\mathcal{V}$, respectively. Intuitively, the closer the nodes to the domain centre, the higher the probability of connecting to a proximity relay node. Fig. 2 serves to illustrate the behaviour of (11). Note that when nodes lie on the same radial axis (top right panel), the contour of the graph is such that fixing one node and moving another yields improved connectivity up to a maximum before meeting the fixed node, then connectivity reduces. This is further illustrated in Fig. 3, where we consider the case $|\mathbf{p}_1| = 0.5$ and the maximum 2-hop connectivity is observed at approximately $|\mathbf{p}_2| = 0.25$. The other (bottom) panels do not exhibit this global maximum since the nodes always move further apart. By equating the first derivative of (11) to 0, with respect to r_1, it can be shown that this maximum occurs when the following equality is satisfied for $\Delta\phi < \pi/2$

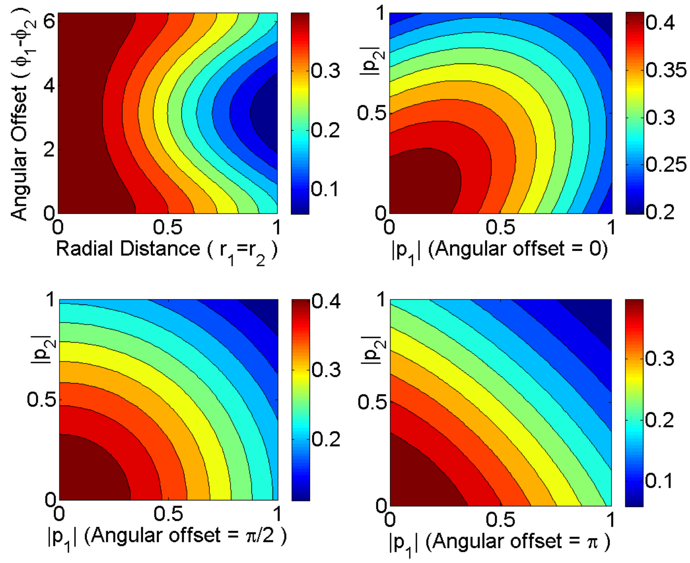

Fig. 2. $\langle H_{12}^{(2)}\rangle$ as a function of $\mathbf{p}_1$ and $\mathbf{p}_2$ within a circular domain of unity radius. Upper-left panel shows effect of variable angular offset and variable radial offset with $r_i = r_j$ and remaining panels show effect of fixed angular offsets $(0, \pi/2, \pi)$ with variable radial offsets of nodes.

$$\frac{|\mathbf{p}_2|\cos(\Delta\phi) - |\mathbf{p}_1|}{|\mathbf{p}_2|\cos(\Delta\phi) + |\mathbf{p}_1|} = \frac{\mathrm{Q}_2\left(2|\overline{\mathbf{p}}_{12}|, 2R\right) - \mathrm{Q}_1\left(2|\overline{\mathbf{p}}_{12}|, 2R\right)}{1 - \mathrm{Q}_1\left(2|\overline{\mathbf{p}}_{12}|, 2R\right)} \tag{12}$$

where $\mathrm{Q}_2(a,b) = \frac{1}{a}\int_b^\infty x^2 e^{-\frac{(x^2+a^2)}{2}} I_1(ax)\mathrm{d}x$ is the second order Marcum-Q-function, $I_1(a) = \frac{1}{\pi}\int_0^\pi e^{a\cos\theta}\cos\theta\mathrm{d}\theta$ is the first order modified Bessel function , and $\Delta\phi = \phi_1 - \phi_2$.

2) Rectangular Domain: We employ the Cartesian coordinate system, $\mathbf{p}_i = (x_i, y_i)$, relative to centre of the rectangle of dimensions $A\times B$. Again we let $\eta = 2$ and (6) becomes

$$\langle H_{12}^{(2)}\rangle = \frac{1}{AB}\int_{-\frac{B}{2}}^{\frac{B}{2}}\int_{-\frac{A}{2}}^{\frac{A}{2}} e^{-(\|\mathbf{p}_k-\mathbf{p}_1\|^2+\|\mathbf{p}_k-\mathbf{p}_2\|^2)}\mathrm{d}x_k\mathrm{d}y_k \tag{13}$$

Expanding and simplifying the exponent in (13) yields

$$\langle H_{12}^{(2)}\rangle = \frac{1}{AB}e^{-(x_1^2+x_2^2+y_1^2+y_2^2)}\int_{-\frac{B}{2}}^{\frac{B}{2}} e^{-2[y_k^2-(y_1+y_2)y_k]}\mathrm{d}y_k \times \int_{-\frac{A}{2}}^{\frac{A}{2}} e^{-2[x_k^2-(x_1+x_2)x_k]}\mathrm{d}x_k \tag{14}$$

Evaluating the integrals and simplifying further yields

$$\langle H_{12}^{(2)}\rangle = \frac{\pi}{2AB}e^{-\frac{1}{2}\|\mathbf{p}_2-\mathbf{p}_1\|^2}\mathrm{g}(A,\overline{x}_{12})\mathrm{g}(B,\overline{y}_{12}) \tag{15}$$

where $(\overline{x}_{12}, \overline{y}_{12})$ is the coordinate of the midpoint between $\mathbf{p}_1$ and $\mathbf{p}_2$. The function $\mathrm{g}(\cdot,\cdot)$ is defined by

$$\mathrm{g}(a,b) = \frac{1}{2}\left[\mathrm{erf}\left(\sqrt{2}(a/2-b)\right) + \mathrm{erf}\left(\sqrt{2}(a/2+b)\right)\right] \tag{16}$$

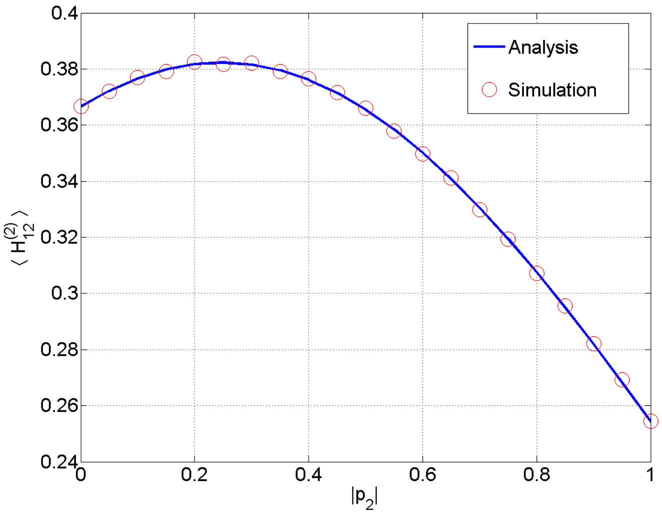

Fig. 3. $\langle H_{12}^{(2)} \rangle$ as a function of $|\mathbf{p}_2|$ where $|\mathbf{p}_1| = 0.5$ and $\mathbf{p}_1$ and $\mathbf{p}_2$ lie on the same radial axis of a circle of radius $R = 1$. The red circles represent results obtained by Monte-Carlo simulation.

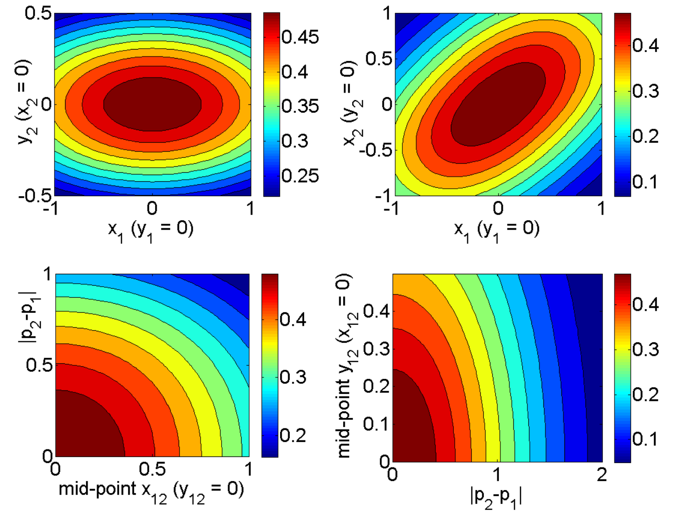

Fig. 4. $\langle H_{12}^{(2)} \rangle$ as a function of $\mathbf{p}_1$ and $\mathbf{p}_2$ within a rectangular domain of dimension 2×1. Upper-left panel shows excursion of $\mathbf{p}_1$ and $\mathbf{p}_2$ about $y_1 = 0$ and $x_2 = 0$ axes respectively and upper-right panel shows excursion of $\mathbf{p}_1$ and $\mathbf{p}_2$ both located on the central horizontal axes $y_1 = y_2 = 0$. Lower panels shows $\langle H_{12}^{(2)} \rangle$ as a function of position of mid-point $\overline{\mathbf{p}}_{12}$, along central horizontal axis (lower left plot) and along central vertical axis (lower right plot) and separation $|\mathbf{p}_2 - \mathbf{p}_1|$.

where $\operatorname{erf}(a) = \frac{2}{\pi}\int_0^a e^{-x^2} \mathrm{d}x$ is the error function. Substituting (15) and (16) into (5) yields the expected connection probability, between terminal nodes $\mathbf{p}_1$ and $\mathbf{p}_2$, employing direct and 2-hop connectivity.

Again, equation (15) shows dependence on Euclidean distance between the terminal nodes and the position of their mid-point within the domain and its behaviour is illustrated in Fig. 4. Note that when the terminal nodes lie on the same axis (upper right figure) we observe that the maximum in connectivity does not occur when the nodes are colocated; with the exception of when they are both at the centre of the domain. Further analysis of global maximum is omitted for brevity.

B. Connectivity in the $\mathbb{R}^2$ plane

For comparison, we also consider a network of infinite extent on the $\mathbb{R}^2$ plane, where relay nodes are uniformly distributed according to a spatial Poisson point process (PPP), Φ_U, of intensity ρ. The average connection probability for the infinite network will be given by

$$\langle P_{12}^{(1,2)} \rangle = 1 - (1 - H_{12})\mathbb{E}_{\mathbf{p}_k}\left[\prod_{\mathbf{p}_k \in \Phi_U}(1 - H_{1k}H_{2k})\right] \quad (17)$$

Employing the probability generating functional for Φ_U to solve the expectation, gives

$$\langle P_{12}^{(1,2)} \rangle = 1 - (1 - H_{12})\exp\left(- \rho\int_{\mathbb{R}^2} H_{1k}H_{2k}\mathrm{d}\mathbf{p}_k\right) \quad (18)$$

We solve (18) by considering the case $\eta = 2$, as before, and employing the Cartesian coordinate system to evaluate the integral. It can be shown that

$$\langle P_{12}^{(1,2)} \rangle = 1 - (1 - H_{12})\exp\left[-\frac{\pi\rho}{2}e^{-\frac{1}{2}\|\mathbf{p}_2 - \mathbf{p}_1\|^2}\right] \quad (19)$$

which, owing to the translation invariance property of Φ_U, is solely a function of the Euclidean distance between the terminal nodes $\mathbf{p}_1$ and $\mathbf{p}_2$ and node intensity, i.e. independent of absolute position within the network.

IV. Asymptotic Behaviour of $\langle P_{12}^{(1,2)} \rangle$

We now consider the behaviour of connection probability $P_{12}^{(1,2)}$ as the domain tends to infinite extent, keeping node intensity constant, and also consider the scenario where node intensity is considered sparse. In this way, the number of relay nodes N grows with the size of the domain, but the intensity $\rho = N/|V|$ remains constant. Firstly, the results (11) and (15) suggests that $\langle H_{12}^{(2)} \rangle$ can be expressed in the general form

$$\langle H_{12}^{(2)} \rangle = \frac{\pi}{2|\mathcal{V}|}e^{-\frac{1}{2}\|\mathbf{p}_2 - \mathbf{p}_1\|^2} f_{\mathcal{V}}(\overline{\mathbf{p}}_{12}, \hat{\mathcal{V}}) \quad (20)$$

where $0 \leq f_{\mathcal{V}}(\overline{\mathbf{p}}_{12}, \hat{\mathcal{V}}) \leq 1$ is a domain specific function of the midpoint location, $\overline{\mathbf{p}}_{12}$, of terminal nodes within the domain, and a set of parameter(s), $\hat{\mathcal{V}}$, which define the domain geometry i.e. $\hat{\mathcal{V}} \in \{R, \{A, B\}\}$ for our special cases of circular and rectangular domains. At this point we can only conjecture that this general form applies to all convex domain geometries. One interesting observation from (20) is the property of rotational invariance on the terminal nodes about $\overline{\mathbf{p}}_{12}$ for a fixed spatial separation. For any two nodes located at the extremes of the diameter of a circle, centred at $\overline{\mathbf{p}}_{12}$, then the connection probability is unchanged as the diameter, and the nodes on it, rotates.

A. $\hat{\mathcal{V}} \to \infty$

In order to consider the domain scaling to infinite extent we rewrite equation (5) in the form

$$\langle P_{12}^{(1,2)} \rangle = 1 - (1 - H_{12}) \exp\left[N \ln(1 - \langle H_{12}^{(2)} \rangle)\right] \qquad (21)$$

and taking a series expansion of the $\ln(\cdot)$ terms gives

$$\langle P_{12}^{(1,2)} \rangle = 1 - (1 - H_{12}) \exp\left[N\big(-\langle H_{12}^{(2)} \rangle - \frac{\langle H_{12}^{(2)} \rangle^2}{2} - \dots\big)\right] \qquad (22)$$

By substitution of (20) into (22) and taking the limit $\hat{\mathcal{V}} \to \infty$ it can be trivially shown that

$$\langle P_{12}^{(1,2)} \rangle \sim 1 - (1 - H_{12}) \exp\left[-\frac{\pi\rho}{2} e^{-\frac{1}{2}\|\mathbf{p}_2 - \mathbf{p}_1\|^2}\right] \qquad (23)$$

which ties up nicely with (19).

B. Sparse Network

Here we consider the condition of network 'sparsity' and show how connection probability can be approximated by simpler expressions. We define a sparse network as meeting the condition $\rho \ll |\mathcal{V}|$ or equivalently $N \ll |\mathcal{V}|^2$. Substituting (20) into (22) and applying our sparsity condition, the higher order terms of the series expansion become negligible, relative to the first order term, and (22) reduces to

$$\langle P_{12}^{(1,2)} \rangle \approx 1 - (1 - H_{12}) \exp\left[-\frac{\pi\rho}{2} e^{-\frac{1}{2}\|\mathbf{p}_2 - \mathbf{p}_1\|^2} f_{\mathcal{V}}(\overline{\mathbf{p}}_{12}, \hat{\mathcal{V}})\right] \qquad (24)$$

We now consider the behaviour where the midpoint $\overline{\mathbf{p}}_{12}$ approaches the domain centre and boundary and also the asymptotic behaviour as the domain grows in size.

1) Domain centre: In the limit $|\overline{\mathbf{p}}_{12}| \to 0$ we get

$$1 - \mathrm{Q}_1\left(2|\overline{\mathbf{p}}_{12}|, 2R)\right]\big) \sim 1 - e^{-2R^2} \qquad (25)$$

and

$$\mathrm{g}(A, \overline{x}_{12})\mathrm{g}(B, \overline{y}_{12}) \sim \operatorname{erf}\left(\frac{A}{\sqrt{2}}\right) \operatorname{erf}\left(\frac{B}{\sqrt{2}}\right) \qquad (26)$$

By considering the asymptotic expansion for erf, then in the limit, $A, B \to \infty$, (26) becomes

$$\mathrm{g}(A, \overline{x}_{12})\mathrm{g}(B, \overline{y}_{12}) \sim \left(1 - \sqrt{\frac{2}{\pi}}\frac{1}{A}e^{\frac{-A^2}{2}}\right)\left(1 - \sqrt{\frac{2}{\pi}}\frac{1}{B}e^{\frac{-B^2}{2}}\right) \qquad (27)$$

We see that, as the area of the network domains grow large, (25) and (27) rapidly converge to unity and the connection probability tends to the limiting case (19).

2) Domain boundary: As $\overline{\mathbf{p}}_{12}$ approaches the domain boundary (one of the horizontal edges in the case of the rectangular boundary)

$$\mathrm{Q}_1\left(2|\overline{\mathbf{p}}_{12}|, 2R\right) \sim \frac{1}{2}\left[1 + e^{-4R^2} I_0(4R^2)\right] \qquad (28)$$

and, by considering the asymptotic expansion for $I_0(4R^2)$, in the limit $R \to \infty$, (28) reduces to

$$\mathrm{Q}_1\left(2|\overline{\mathbf{p}}_{12}|, 2R\right) \sim \frac{1}{2}\left[1 + \frac{1}{\sqrt{8\pi}}\frac{1}{R}\right] \qquad (29)$$

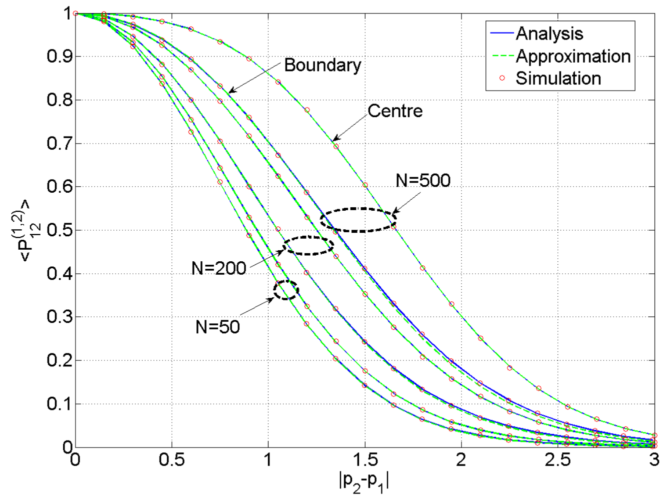

Fig. 5. Connectivity performance (analysis, approximation and Monte-Carlo simulation) at boundary and centre of circular domain of radius $R = 10$ and total number of nodes $N \in \{50, 200, 500\}$. The boundary and central performances are the left and right curves respectively for each node density.

For the rectangular domain, as $\overline{\mathbf{p}}_{12} \to (\frac{A}{2}, 0)$, we get

$$\mathrm{g}(A, \overline{x}_{12})\mathrm{g}(B, \overline{y}_{12}) \sim \frac{1}{2}\operatorname{erf}\left(\sqrt{2}A\right)\operatorname{erf}\left(\frac{B}{\sqrt{2}}\right) \qquad (30)$$

which, in the limit, $A, B \to \infty$, becomes

$$\mathrm{g}(A, \overline{x}_{12})\mathrm{g}(B, \overline{y}_{12}) \sim \frac{1}{2}\left(1 - \frac{1}{\sqrt{2\pi}}\frac{1}{A}e^{-2A^2}\right) \times \left(1 - \sqrt{\frac{2}{\pi}}\frac{1}{B}e^{\frac{-B^2}{2}}\right) \qquad (31)$$

As the area of the network domains grow large, (29) and (31) converge to $\frac{1}{2}$ and the connection probability can be approximated by

$$\langle P_{12}^{(1,2)} \rangle \sim 1 - (1 - H_{12}) e^{-\left(\frac{\pi\rho}{4} e^{-\frac{1}{2}\|\mathbf{p}_2 - \mathbf{p}_1\|^2}\right)} \qquad (32)$$

Provided the sparsity condition $N \ll |\mathcal{V}|^2$ is met, (19) and (32) provide good approximations for connection probability at the domain centre and boundary respectively. This is illustrated in Fig. 5 for a circular domain of radius 10. It should be noted that as the node separation increases, the midpoint necessarily moves away from the boundary of the circle since nodes cannot exist outside of it (the simulation results account for this). Even in this case, the results hold well.

V. Averaging over the terminal node $\mathbf{p}_2$

An insightful metric is the average connection probability from a fixed node to any randomly chosen node within the domain. One possible example is a fixed node that acts as the gateway into another fixed network, but is unaware of the position of other nodes to which it needs to communicate. Given we now have closed form expressions for average connection probability, between two fixed nodes, then averaging over all

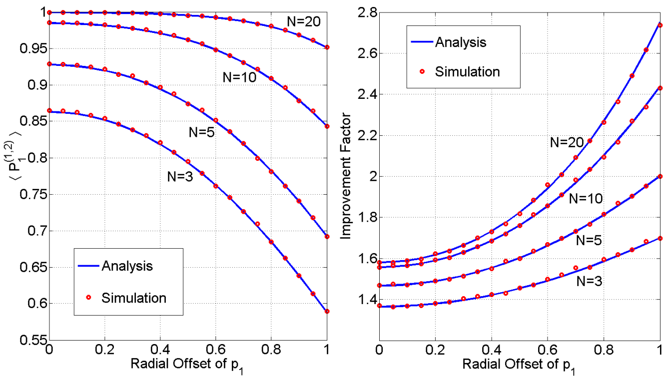

Fig. 6. Left Panel: Average connection probability, via direct link or 2-hop path, $\langle P_1^{(1,2)} \rangle$, in a circular domain of unity radius and varying the position of a fixed node, $\mathbf{p}_1$, along a fixed radial axis. Right Panel: Improvement in average connection probability. Total relay nodes in network $N \in \{3, 5, 10, 20\}$. Results are supported by Monte-Carlo simulation as shown by red circles.

possible positions of one of the terminal nodes[4] is relatively straightforward and given by

$$\langle P_1^{(1,2)} \rangle = \frac{1}{|\mathcal{V}|} \int_{\mathcal{V}} \langle P_{12}^{(1,2)} \rangle \mathrm{d}\mathbf{p}_2 \tag{33}$$

Since this does not yield a nice closed form solution for either special case of domain, we evaluate numerically and illustrate in the left plot of Fig. 6 for the circular domain. The figure shows loss of connectivity performance towards the boundary, which can be attributed to a reduction in connectivity mass [16] as the node moves towards the boundary. Further, in order to gauge the improvement in employing 2-hop connectivity, we can consider the ratio of $\langle P_1^{(1,2)} \rangle / \langle P_1^{(1)} \rangle$, hereafter called the 'improvement factor', where $\langle P_1^{(1)} \rangle$ is the average connection probability for a direct path only from a fixed terminal node location to any randomly chosen destination node. For the case of the circular domain it can be shown that this is given by

$$\langle P_1^{(1)} \rangle = \frac{1}{R^2} \left[1 - \mathrm{Q}_1 \left(\sqrt{2} r_1, \sqrt{2} R \right) \right] \tag{34}$$

where r_1 is the radial offset of the terminal node. Evaluation of the above ratio for the circular domain, of unity radius and varying the number of relay nodes, is shown in the right panel of Fig. 6. The figure clearly shows appreciable gain in performance, particularly towards the domain boundary.

VI. Conclusion and Discussions

In this paper, we have studied the direct and 2-hop connectivity, between two fixed terminal nodes within finite convex geometric boundaries, subject to Rayleigh fading. We derive analytic expressions, in closed form, for average connection probability for circular and rectangular domains which show a dependence on: boundary geometry, spatial separation of nodes and their mid-point location. Boundary effects have been shown to tie up with previous observations [9] and employment of 2-hop connectivity was shown to yield appreciable improvements in connectivity, particularly at the domain boundaries. These general results can be used in the design and simulation of bounded homogeneous random wireless networks, the analysis of D2D connectivity within clusters contained within larger non-homogeneous wireless networks and provides a foundation for further research e.g. evaluation of alternative boundary geometries and the effect of interference in finite networks employing relaying.

[4] Assumed to have uniform spatial distribution within $\mathcal{V}$

Acknowledgement

This work was supported by EPSRC grant number EP/N002350/1 'Spatially Embedded Networks'.

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