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Pseudo-exponentiation on algebraically closed fields of characteristic zero

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Abstract

We construct and study structures imitating the field of complex numbers with exponentiation. We give a natural, albeit non first-order, axiomatisation for the corresponding class of structures and prove that the class has a unique model in every uncountable cardinality. This gives grounds to conjecture that the unique model of cardinality continuum is isomorphic to the field of complex numbers with exponentiation.

1. Introduction

We construct and study here structures imitating $\mathbb{C}_{\exp} = (\mathbb{C}, +, \times, \exp)$, the complex numbers equipped with the field operations and exponentiation. The idea and the aims of the project were described in [4].

The version of the structures, the *strongly exponentially–algebraically closed fields with pseudo-exponentiation*, that we study here is very close to $\mathbb{C}_{\exp}$, and one of the main results is the statement that there is exactly one, up to isomorphism, strongly exponentially–algebraically closed field with pseudo-exponentiation of a given uncountable cardinality, and we give precise and simple conditions under which $\mathbb{C}_{\exp}$ is the one of cardinality

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continuum. In fact the conditions can be interpreted as two conjectures about the complex exponentiation, the first being the well known Schanuel conjecture and the second the conjecture stating that certain systems of exponential equations over complex numbers do have solutions in the complex numbers.

The definition of the above-mentioned systems of exponential equations is given in natural, albeit rather technical, terms (normality and freeness) which roughly speaking amount to saying that the system is *not overdetermined*. We prove that these definitions are first order, in fact that certain properties of exponential varieties are of finite character, which we hope to use for a further analysis of the fields with pseudo-exponentiation.

We also prove an elimination-of-quantifiers result in $L_{\omega_1,\omega}$ for the fields with pseudo-exponentiation and give a, we hope useful, criterion for elementary extensions in the class.

2. Definitions and notation

We start with a class of structures $\mathbf{F} = \langle F, L\rangle$ where F is a field of characteristic 0 in the language L consisting of a binary operation $+$, unary operations $\frac{1}{m}\cdot$, for every positive integer m, a binary relation E and a collection of n-ary predicates $V(x_1, \dots, x_n)$ for each algebraic subvariety $V \subseteq F^n$, defined and irreducible over $\mathbb{Q}$;
These are interpreted in $\mathbf{F}$ as follows:
$+$ is the usual addition in the field F;
$\frac{1}{m}\cdot$ multiplies the argument by the corresponding rational number;
n-ary predicates $V(x_1, \dots, x_n)$ correspond to algebraic subvarieties $V \subseteq F^n$;
the binary relation $E(x, y)$ is the graph of a function $\mathrm{ex} : F \to F$.

Definition. We let $\mathcal{E}$ to be the class of L-structures $\mathbf{F}$ defined by the (first-order) axioms stating that F is an algebraically closed field of characteristic zero and $E(x, y)$ the graph of a surjective map

$$\mathrm{ex} : F \to F^\times = F \setminus \{0\}$$

satisfying the homomorphism condition

$$\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2).$$

Definition. Let L^- be the language L without predicate E. Let *sub*$\mathcal{E}$ be the class of L-structures A such that for some $F \in \mathcal{E}$

(i) $A \subseteq F$ as L^--structures;
(ii) $E(A) \subseteq E(F)$;
(iii) the domain of the partial mapping ex_A is closed under addition and multiplication by rationals.

The following lemma provides a description of *sub*$\mathcal{E}$ in algebraic terms.

Lemma 2.1. *Suppose A is a divisible subgroup of the additive group of an algebraically closed field F, $A_0 \subseteq A$ a divisible subgroup and*

$$\mathrm{ex}_A : A_0 \to F^\times$$

a homomorphism into the multiplicative group such that $\mathrm{ex}_A(A_0) \subseteq A$. *Let* E *be the graph of* ex_A *and let, for any* L*-name* V *for an algebraic variety over* $\mathbb{Q}$*, us interpret* V *on* A *as induced by the embedding* $A \subseteq F$.

Suppose also that the ranks of the abelian groups satisfy the inequality $\mathrm{rk}\, A_0 < \mathrm{rk}\, F$. *Then* A *viewed as an* L*-structure is in sub*$\mathcal{E}$.

Proof. By definition it is enough to be able to extend ex_A to a surjective homomorphism:

$$\mathrm{ex} : F \to F^{\times}.$$

By standard theory of abelian groups,

$$F \cong A_0 \times B \text{ and } F^{\times} \cong \mathrm{ex}(A_0) \times B',$$

where B and B' are divisible groups in, respectively, additive and multiplicative representations. Considering the ranks one gets easily from the assumption that

$$\mathrm{rk}\, B = \mathrm{rk}\, B' \geq \aleph_0.$$

Since B is torsion free, it follows that there is a surjective homomorphism from B onto B', and hence we can extend ex_A in a surjective way. □

Notation. We write $X \subseteq_{fin} Y$ to say that X is a finite subset of Y. We also often write XY instead of $X \cup Y$.

Notation. For $A \in sub\mathcal{E}$,

$$\mathrm{D}_A = \{x \in A : \exists y \in A\ E(x, y)\}.$$

For $X \subseteq_{fin} A$

$$\mathrm{ex}_A(X) = \{y \in A : \exists x \in X\ E(x, y)\};$$

tr.d.(X) is the transcendence degree of X over $\mathbb{Q}$ and
lin.d.(X) the dimension of the vector space $\mathrm{span}_{\mathbb{Q}}(X)$ generated by X over $\mathbb{Q}$;

Definition. The predimension of $X \subseteq_{fin} A$ is

$$\delta_A(X) = \mathrm{tr.d.}(\mathrm{span}_{\mathbb{Q}} X \cup \mathrm{ex}_A \mathrm{span}_{\mathbb{Q}} X) - \mathrm{lin.d.}(\mathrm{span}_{\mathbb{Q}} X).$$

Remark. If $X \subseteq_{fin} A \subseteq B \in sub\mathcal{E}$ then obviously

$$\delta_A(X) \leq \delta_B(X).$$

We usually omit the subscript in $\delta_A(X)$ when A is fixed.

Definition. For $X, X' \subseteq_{fin} A$

$$\delta(X/X') = \delta(XX') - \delta(X').$$

For infinite $Z \subseteq A$ and $k \in \mathbb{Z}$, $\delta(X/Z) \geq k$ by definition means that for any $Y \subseteq_{fin} Z$ there is $Y \subseteq_{fin} Y' \subseteq Z$ such that $\delta(X/Y') \geq k$, and $\delta(X/Z) = k$ means $\delta(X/Z) \geq k$ and not $\delta(X/Z) \geq k + 1$.

Remark. Letting $\text{tr.d.}(X/X') = \text{tr.d.}(XX') - \text{tr.d.}(X')$ and $\text{lin.d.}(X/X') = \text{lin.d.}(XX') - \text{lin.d.}(X')$ and assuming $X = \text{span}_{\mathbb{Q}} X$ and $X' = \text{span}_{\mathbb{Q}} X'$ we have

$$\delta(X/X') = \text{tr.d.}(X \cup \text{ex}X/X' \cup \text{ex}X') - \text{lin.d.}(X/X').$$

Notation. $sub\mathcal{E}^0$ is the subclass of $sub\mathcal{E}$ consisting of all $A \in sub\mathcal{E}$ satisfying the condition

$$\delta_A(X) \geq 0 \text{ for all } X \subseteq_{fin} \mathrm{D}_A.$$

$$\mathcal{E}^0 = \mathcal{E} \cap sub\mathcal{E}^0.$$

Notation. For W an algebraic variety, $\bar{b} = \langle b_1, \ldots, b_l \rangle$, let

$$W(\bar{b}) = \{\langle x_{l+1}, \ldots, x_{n+l} \rangle : \langle b_1, \ldots, b_l, x_{l+1}, \ldots, x_{n+l} \rangle \in W\}.$$

Lemma 2.2. *If $X = \{x_1, \ldots, x_n\} \subseteq A$, $\bar{x} = \langle x_1, \ldots, x_n \rangle$, then:* $\text{tr.d.}(X) = \dim V$, *where $V \subseteq F^n$ is the minimal algebraic variety over $\mathbb{Q}$ containing $\bar{x}$;*
$\text{lin.d.}(X) = \dim L$, *where L is the minimal linear subspace of F^n containing $\bar{x}$ and given by homogeneous linear equations over $\mathbb{Q}$.*

Proof. Immediate from definitions. □

Lemma 2.3. *Let $A \in sub\mathcal{E}$. Then for $X \subseteq_{fin} A$ and $Z \subseteq A$ there is an $Y \subseteq_{fin} Z$, such that if $Y \subseteq Y' \subseteq Z$, then $\delta(X/Y') = \delta(X/Z)$.*

Proof. Choose $Y \subseteq_{fin} Z$ such that $\text{tr.d.}(X \cup \text{ex}X/Y \cup \text{ex}Y) = \text{tr.d.}(X \cup \text{ex}X/Z \cup \text{ex}Z)$ and $\text{lin.d.}(X/Y) = \text{lin.d.}(X/Z)$. This choice is possible, since $\text{tr.d.}(X/Y)$ and $\text{lin.d.}(X/Y)$ are non-increasing functions of Y. □

Remark. The condition $\text{lin.d.}(X/Z) = \text{lin.d.}(X/Y)$ for $Y \subseteq Z$ is satisfied iff $\text{span}_{\mathbb{Q}}(X) \cap Z \subseteq Y$ and correspondingly for the transcendence degree.

Definition. For $A, B \in sub\mathcal{E}$, we say that A is **strongly embedded in** B, writing $A \leq B$, if $A \subseteq B$ as L-structures and the following two conditions hold:

(S1) $\delta_A(Y/Z) \leq \delta_B(Y/Z)$ for any $Y, Z \subseteq_{fin} \mathrm{D}_A$; and
(S2) $\delta_B(X/\mathrm{D}_A) \geq 0$, for all $X \subseteq_{fin} \mathrm{D}_B$.

Lemma 2.4. *Condition* (S1) *is satisfied if the following condition holds:*
(S1A) *any algebraically independent subset of* $\text{ex}_B(A) \setminus A$ *is algebraically independent over A.*

Proof. The inequality in (S1) is equivalent by definition to

$$\text{tr.d.}(Y \cup \text{ex}_A Y/Z \cup \text{ex}_A Z) \leq \text{tr.d.}(Y \cup \text{ex}_B Y/Z \cup \text{ex}_B Z).$$

W.l.o.g., $Z \subseteq Y$. We can equivalently replace $\text{ex}_B Y$ on the right by $\text{ex}_B Y' \cup \text{ex}_A Y$ such that $\text{ex}_B Y' \subseteq \text{ex}_B Y \setminus A$, $\text{ex}_B Y'$ is a transcendence basis of $\text{ex}_B Y \setminus A$ and $\text{ex}_B(Y' \cap Z)$ is a basis of $\text{ex}_B Z \setminus A$. Then

$$\begin{aligned}\text{tr.d.}(Y \cup \text{ex}_B Y/Z \cup \text{ex}_B Z) &= \text{tr.d.}(Y \cup \text{ex}_B Y' \cup \text{ex}_A Y/Z \cup \text{ex}_B(Y' \cap Z) \cup \text{ex}_A Z) \\ &= \text{tr.d.}(Y \cup \text{ex}_A Y/Z \cup \text{ex}_A Z) + |\text{ex}_B(Y' \setminus Z)|. \quad \square\end{aligned}$$

We also use a relativized notion of a strong embedding:

Definition. For subsets $U \subseteq V \subseteq C$ and $C \in sub\mathcal{E}$, we say that U is **strongly embedded into** V **relative to** C, writing

$$U \leq_C V,$$

meaning that $\delta_C(X/U \cap \mathrm{D}_C) \geq 0$ for any $X \subseteq V \cap \mathrm{D}_C$.

Notice that this definition agrees with the absolute strong embedding when $U = A$, $V = B$ and $A, B \subseteq C$ with the property that $\mathrm{D}_A = A \cap \mathrm{D}_C$.

Lemma 2.5. *For any structure A of the class $sub\mathcal{E}$ and $X, Y, Z \subseteq_{fin} A$:*

(i) *If* $\mathrm{span}_{\mathbb{Q}}(X') = \mathrm{span}_{\mathbb{Q}}(X)$ *then* $\delta(X') = \delta(X)$.
(ii) *If* $\mathrm{span}_{\mathbb{Q}}(X'Y) = \mathrm{span}_{\mathbb{Q}}(XY)$ *then* $\delta(X/Y) = \delta(X'/Y)$.
(iii) *If* $\mathrm{span}_{\mathbb{Q}}(Y) = \mathrm{span}_{\mathbb{Q}}(Y')$ *then* $\delta(X/Y) = \delta(X/Y')$.
(iv) $\delta(XY/Z) = \delta(X/YZ) + \delta(Y/Z)$.

Proof. Immediate from definitions. □

Lemma 2.6. (i) *For $A, B, C \in sub\mathcal{E}$,*
if $A \leq B$ and $B \leq C$, then $A \leq C$.
(ii) *For $C \in sub\mathcal{E}$ and its subsets $A \subseteq B \subseteq C$,*
if $A \leq_C B$ and $B \leq_C C$, then $A \leq_C C$.

Proof. (i) Let $X \subseteq_{fin} \mathrm{D}_C$ and let $Z \subseteq_{fin} \mathrm{D}_A$ be large enough that $\delta_C(X/Z) = \delta_C(X/\mathrm{D}_A)$. We need to prove that $\delta_C(X/Z) \geq 0$. Choose $Y \subseteq_{fin} \mathrm{D}_B$ so that $\mathrm{span}_{\mathbb{Q}}(YZ) = \mathrm{D}_B \cap \mathrm{span}_{\mathbb{Q}}(XZ)$. Then $\mathrm{lin.d.}(X/YZ) = \mathrm{lin.d.}(X/\mathrm{D}_B)$.

From the definition of δ_C it follows that $\delta_C(X/YZ) \geq \delta_C(X/\mathrm{D}_B) \geq 0$. Also $\delta_C(Y/Z) \geq \delta_B(Y/Z) \geq 0$ by (S1) and (S2). Hence
$\delta_C(XY/Z) = \delta_C(X/YZ) + \delta_C(Y/Z) \geq 0$. Now notice that $\delta_C(X/Z) = \delta_C(XY/Z)$ by definition.

(ii) We may assume that $A = \mathrm{span}_{\mathbb{Q}} A$ and $B = \mathrm{span}_{\mathbb{Q}} B$ and then apply the same arguments as in (i). □

Definition. Let $A \in sub\mathcal{E}^0$ and $X \subseteq_{fin} \mathrm{D}_A$. The **dimension of** X in A is

$$\partial_A(X) = \min\{\delta_A(X') : X \subseteq X' \subseteq_{fin} X \cup \mathrm{D}_A\}.$$

Lemma 2.7. *Let $A \in sub\mathcal{E}^0$.*

(i) *If $X \subseteq X' \subseteq_{fin} \mathrm{D}_A$ are such that $\delta_A(X') = \partial_A(X)$, then $X' \leq_A A$.*
(ii) *Given $X \subseteq_{fin} \mathrm{D}_A$ there exists $X' \subseteq_{fin} \mathrm{D}_A$ satisfying* (i).

Proof. Immediate from definitions. □

Lemma 2.8. *Let $A, B \in sub\mathcal{E}$, $A \leq B$ and $X \subseteq_{fin} A$. Then*

$$\partial_A(X) = \partial_B(X).$$

Proof. Let $Y \subseteq_{fin} \mathrm{D}_B$ be such that

$$\delta_B(XY) = \partial_B(X).$$

Let Y_0 be a $\mathbb{Q}$-linear basis of Y over A and $X_1 \subseteq_{fin} A$ a superset of X such that

$$\mathrm{span}_Q(X_1) = \mathrm{span}_Q(XY) \cap A.$$

Then $\mathrm{lin.d.}(Y_0/A) = \mathrm{lin.d.}(Y_0/X_1)$. On the other hand, it is obvious that $\mathrm{tr.d.}(Y_0 \cup \mathrm{ex}_B Y_0/A \cup \mathrm{ex}_B A) \leq \mathrm{tr.d.}(Y_0 \cup \mathrm{ex}_B Y_0/X_1 \cup \mathrm{ex}_B X_1)$. It follows that

$$\delta_B(Y_0/X_1) \geq \delta_B(Y_0/A) \geq 0.$$

Also

$$\mathrm{span}_Q(XY) = \mathrm{span}_Q(X_1 Y_0).$$

Hence

$$\delta_B(XY) = \delta_B(X_1 Y_0) = \delta_B(X_1) + \delta_B(Y_0/X_1).$$

By the above proof, $\delta_B(XY) \geq \delta_B(X_1)$. By definitions $\delta_B(X_1) \geq \delta_A(X_1)$ and $\delta_A(X_1) \geq \partial_A(X)$. Thus $\partial_B(X) \geq \partial_A(X)$, and the converse is obvious. □

Lemma 2.9. *Suppose* $A \in sub\mathcal{E}^0$, $B \in sub\mathcal{E}$,
$A \subseteq B$ *as* L*-structures*, $\mathrm{D}_B = \mathrm{D}_A + \mathrm{span}_Q(X)$,
the condition (S1A) *of* Lemma 2.4 *is satisfied and*
$\delta_B(X'/\mathrm{D}_A) \geq 0$ *for all* $X' \subseteq_{fin} \mathrm{span}_Q X$.
Then $B \in sub\mathcal{E}^0$ *and* $A \leq B$.

Proof. We may assume that X is $\mathbb{Q}$-linearly independent over D_A. Let $Z \subseteq \mathrm{D}_B$, $Z = \{z_1, \ldots, z_n\}$ and $z_i = x_i + y_i$ for some $x_i \in \mathrm{span}_Q(X)$, $y_i \in \mathrm{D}_A$. Let $\{x_1, \ldots, x_k\}$ be a $\mathbb{Q}$-linear basis of $\{x_1, \ldots, x_n\}$. Then, using Lemma 2.5, for $\delta = \delta_B$ we have

$\delta(Z) = \delta(x_1 + y_1, \ldots, x_k + y_k, y'_{k+1}, \ldots, y'_n)$, for $y'_{k+1}, \ldots, y'_n$ appropriate $\mathbb{Q}$-linear combinations of $y_1, \ldots, y_n$.

We rewrite as follows:

$$\delta(Z) = \delta(\{x_1 + y_1, \ldots, x_k + y_k\}/\{y'_{k+1}, \ldots, y'_n\}) + \delta(y'_{k+1}, \ldots, y'_n).$$

By assumption, $\delta(y'_{k+1}, \ldots, y'_n) \geq 0$. On the other hand,

$$\delta(\{x_1 + y_1, \ldots, x_k + y_k\}/\{y'_{k+1}, \ldots, y'_n\}) \geq \delta(\{x_1, \ldots, x_k\}/\mathrm{D}_A) \geq 0$$

since

$$\begin{aligned} &\mathrm{tr.d.}(x_1 + y_1, \ldots, x_k + y_k, \mathrm{ex}(x_1 + y_1), \ldots, \mathrm{ex}(x_k + y_k)/y'_{k+1}, \ldots, y'_n, \\ &\quad \mathrm{ex}(y'_{k+1}), \ldots, \mathrm{ex}(y'_n)) \\ &\quad \geq \mathrm{tr.d.}(\{x_1 + y_1, \ldots, x_k + y_k, \mathrm{ex}(x_1 + y_1), \ldots, \mathrm{ex}(x_k + y_k)\}/\mathrm{D}_A \cup \mathrm{ex}_A A) \\ &\quad \geq \mathrm{tr.d.}(\{x_1, \ldots, x_k, \mathrm{ex}(x_1), \ldots, \mathrm{ex}(x_k)\}/\mathrm{D}_A \cup \mathrm{ex}_A A) \end{aligned}$$

and

$$\mathrm{lin.d.}(\{x_1 + y_1, \ldots x_k + y_k\}/\{y'_{k+1}, \ldots, y'_n\}) = k = \mathrm{lin.d.}(\{x_1, \ldots, x_k\}/\mathrm{D}_A).$$

Thus

$$\delta(Z) \geq 0.$$

The same argument shows that

$$\delta(Z/\mathrm{D}_A) \geq 0.$$

This proves (S2) of the definition of strong embedding. Lemma 2.4 completes the proof. □

Definition. Let $A \in sub\mathcal{E}$. Write

$$\ker_{|A} = \{a \in A : \mathrm{ex}(a) = 1\}.$$

A is said to be **with standard kernel** if

$$\ker_{|A} = \omega \cdot \mathbb{Z}$$

for some transcendental $\omega \in A$.

A is said to be **with full kernel** if for $\ker = \ker_{|A} = \{a \in A : \mathrm{ex}(a) = 1\}$ the group $A/\ker$ is isomorphic to a multiplicative subgroup of an algebraically closed field containing all torsion points of the field.

Proposition 2.10. *There is an $A \in sub\mathcal{E}^0$ with standard full kernel.*

Proof. Let F be an algebraically closed field and $\omega \in F$ a transcendental element. Consider the subgroup $A_0 = \omega \cdot \mathbb{Q}$ of the additive group F and define $H = A_0/\ker$ for ker the standard kernel with generator ω. Then H considered as a multiplicative group is characterized by the property that it is a torsion group such that any equation of the form $x^n = h$, for any h, has exactly n solutions in the group. In other words, H is isomorphic to the torsion subgroup of the algebraically closed field F. Define ex_A as the canonical homomorphism $A_0 \to H \subseteq F^\times$ corresponding to this isomorphism and $A = A_0 + \mathrm{span}_{\mathbb{Q}} H$. Now we can view A as an L-structure from $sub\mathcal{E}$, by Lemma 2.1.

Since ω is transcendental, $A_0 \cap \mathrm{span}_{\mathbb{Q}} H = \emptyset$, $\mathrm{D}_A = A_0$ and $\delta(X) = 0$ for any $X \subseteq_{fin} A_0$. It follows that $A \in sub\mathcal{E}^0$. □

Lemma 2.11. *Suppose $A \in sub\mathcal{E}^0$ and A is with full kernel. Then there is $F \in \mathcal{E}^0$ and an embedding of A into F such that $A \leq F$ and* $\ker_{|F} = \ker_{|A}$.

Proof. Choose an algebraically closed field F of characteristic zero such that $A \subseteq F$ and $\mathrm{tr.d.}(F/A) \geq \mathrm{card}\, A + \aleph_0$. We want to define $\mathrm{ex} : F \to F^\times$ extending ex_A so that $F \in \mathcal{E}^0$.

Fix a well-ordering of F. Let

$$D_0 = \mathrm{D}_A, \quad \mathrm{ex}_0 = \mathrm{ex}_A : D_0 \to A \text{ and } A_0 = A.$$

Proceed by induction defining D_α, A_α and a homomorphism

$$\mathrm{ex}_\alpha : D_\alpha \to F^\times \text{ with } \mathrm{D}_{A_\alpha} = D_\alpha \text{ and } \mathrm{ex}_\alpha(D_\alpha) \subseteq A_\alpha$$

as follows:

If α is even, choose the first element $a \in F \setminus D_\alpha$ and define $\mathrm{ex}_{\alpha+1}(a)$ to be any element in $F^\times \setminus \mathrm{acl}(A_\alpha)$. Put $D_{\alpha+1} = D_\alpha + \mathbb{Q} \cdot a$ and extend $\mathrm{ex}_{\alpha+1}$ to $D_{\alpha+1}$ as a group homomorphism.
If α is odd, choose the first element $b \in F^\times \setminus \mathrm{ex}_\alpha(D_\alpha)$ and an $a \in F \setminus \mathrm{acl}(A_\alpha)$, put $\mathrm{ex}_{\alpha+1}(a) = b$, $D_{\alpha+1} = D_\alpha + \mathbb{Q} \cdot a$ and again extend $\mathrm{ex}_{\alpha+1}$ to $D_{\alpha+1}$ as a group homomorphism.

Define in both cases

$$A_{\alpha+1} = \mathrm{span}_{\mathbb{Q}}(A_\alpha \cup D_{\alpha+1} \cup \mathrm{ex}_{\alpha+1}(D_{\alpha+1}))$$

with E on the set defined by $\mathrm{ex}_{\alpha+1}$.

On any step it follows from Lemma 2.9 that $A_{\alpha+1} \in sub\mathcal{E}^0$ and $A_\alpha \leq A_{\alpha+1}$. Also $D_{\alpha+1}$ is divisible and, since A is with full kernel, in $\mathrm{ex}(A_{\alpha+1})$ any equation of the form $x^n = b$ has exactly n solutions.

Finally,

$$\ker_{|A_{\alpha+1}} = \ker_{|A_\alpha},$$

since if $\mathrm{ex}(qa + a') = 1$ for a generating $D_{\alpha+1}$ over D_α as above, some rational $q = \frac{m}{n}$ and $a' \in D_\alpha$, then $b^m = g^n$ for $b = \mathrm{ex}(a)$, $g = \mathrm{ex}(-a')$. Since all the roots of degree m of g^n are in $\mathrm{ex}(D_\alpha)$ it would contradict $b \notin \mathrm{ex}(D_\alpha)$ unless $q = 0$. $\square$

Notation. Let $sub\mathcal{E}^0_{\mathrm{st}}$ be the subclass of $sub\mathcal{E}^0$ consisting of the structures with standard full kernel.

Let

$$\mathcal{E}^0_{\mathrm{st}} = sub\mathcal{E}^0_{\mathrm{st}} \cap \mathcal{E}.$$

3. Normality and freeness

In this section we consider the class of structures with standard kernel $\omega\mathbb{Z}$, which we denote just as ker. We extend the language L by naming ω.

Definition. We say that an algebraic variety $V \subseteq F^{2n}$ is **ex-defined over** some $C \subseteq F$ if V can be defined with parameters in the field $\mathbb{Q}(C + \ker + \mathrm{ex}C)$.

We let $\tilde{C} = \mathbb{Q}(C + \ker + \mathrm{ex}C)$.

We say that the variety V ex-definable over C is **ex-irreducible over** C if the ideal of the polynomials in $x_1, \ldots, x_n, y_1, \ldots, y_n$ over $\tilde{C}$ vanishing on V is prime.

Definition. For an algebraic variety $V \subseteq F^{2n}$, written in variables $x_1, \ldots, x_n, y_1, \ldots, y_n$, define $\mathrm{pr}_x V$ to be the Zariski closure of the projection of V onto the first n coordinates. Correspondingly, $\mathrm{pr}_y V$ is the Zariski closure of the projection onto the last n coordinates.

Remark. If the variety V is ex-definable and ex-irreducible over some $C \subseteq F$, then so are the projections.

Definition. For $V \subseteq F^{2n}$ ex-definable over C, we say that $\mathrm{pr}_x V$ is **free of additive dependencies over** C if no $\bar{a} \in \mathrm{pr}_x V$ generic over $\tilde{C}$ satisfies $m_1 \cdot a_1 + \cdots + m_n \cdot a_n = c$ for a $c \in \mathrm{span}_{\mathbb{Q}}(C + \ker)$ and a non-zero tuple of integers $m_1, \ldots, m_n$.

$\mathrm{pr}_x V$ is said to be **absolutely free of additive dependencies over** C if $\mathrm{pr}_x V$ is free of additive dependencies over $\mathrm{acl}(\tilde{C})$.

We say that $\mathrm{pr}_y V$ is **free of multiplicative dependencies over** C if no $\bar{b} \in \mathrm{pr}_y V$ generic over $\tilde{C}$ satisfies $b_1^{m_1} \cdot \ldots \cdot b_n^{m_n} = r$ for an $r \in \mathrm{ex}(\mathrm{span}_{\mathbb{Q}}(C + \ker))$.

$\mathrm{pr}_y V$ is said to be **absolutely free of multiplicative dependencies over** C if no $\bar{b} \in \mathrm{pr}_y V$ generic over $\tilde{C}$ satisfies $b_1^{m_1} \cdot \ldots \cdot b_n^{m_n} = r$ for an $r \in \mathrm{acl}(\tilde{C})$.

V is said to be **free** if both $\mathrm{pr}_x V$ is free of additive dependencies and $\mathrm{pr}_y V$ is free of multiplicative dependencies over C.

Notation. $G_n(F) = F^n \times (F^\times)^n$ is an algebraic group, the product of n copies of the additive group F and n copies of the multiplicative group $F^\times$.

Given $m \in \mathbb{Z}$ denote as $[m] : G_n(F) \to G_n(F)$ the homomorphism mapping given by $x \mapsto mx$ on the first n coordinates and $y \mapsto y^m$ on the last n ones.

More generally, given an integer $(k \times n)$-matrix

$$M = \{m_{i,j}\}_{1 \le i \le k;\ 1 \le j \le n},$$

we denote by

$$[M] : G_n(F) \to G_k(F)$$

the homomorphism mapping given by $\langle x_1, \ldots, x_n \rangle \mapsto \langle x'_1, \ldots, x'_k \rangle$, with $x'_i = m_{i,1}x_1 + \cdots + m_{i,n}x_n$ on the first n coordinates and $\langle y_1, \ldots, y_n \rangle \mapsto \langle y'_1, \ldots, y'_k \rangle$ with $y'_i = y_1^{m_{i,1}} \cdot \ldots \cdot y_n^{m_{i,n}}$ on the last n ones.

Definition. $V \subseteq G_n(F)$ is said to be **ex-normal over** C if in some extensions of the field there are $\langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle \in V$ such that for any $k \le n$ independent integer vectors $m_i = \langle m_{i,1}, \ldots, m_{i,n} \rangle$, $i = 1, \ldots, k$, and

$$a'_i = m_{i,1}a_1 + \cdots + m_{i,n}a_n, \quad b'_i = b_1^{m_{i,1}} \cdot \ldots \cdot b_n^{m_{i,n}},$$

the following inequality holds:

$$\mathrm{tr.d.}(\langle a'_1, \ldots, a'_k, b'_1, \ldots, b'_k \rangle / \tilde{C}) \ge k. \tag{1}$$

Equivalently, the varieties

$$V'_{1,\ldots,k} = \mathrm{locus}_C(a'_1, \ldots, a'_k, b'_1, \ldots, b'_k)$$

satisfy the inequality

$$\dim V'_{1,\ldots,k} \ge k. \tag{2}$$

Notice that the varieties $V'_{1,\ldots,k}$ are just the images of V under the corresponding regular homomorphisms $[M] : G_n(F) \to G_k(F)$. We denote the image of the variety under $[M]$ by V^M.

If $W = \mathrm{pr}_y V$ then we write W^M for $\mathrm{pr}_y(V^M)$. Obviously, this W^M is equal to the image of W under the above multiplicative homomorphism $(F^\times)^n \to (F^\times)^k$ determined by M.

Definition. Given an n-tuple $\bar{a} \in F^n$ and a subset $C \subseteq F$, we define **the ex-locus of** $\bar{a}$ **over** C to be the smallest algebraic variety $V \subseteq F^{2n}$ ex-defined over C and containing the $2n$-tuple $\langle a_1, \dots, a_n, \mathrm{ex}a_1, \dots, \mathrm{ex}a_n \rangle$.

Remark. The ex-locus of a tuple over C is ex-irreducible over C.

Lemma 3.1. *Let $\tilde{C} \subseteq A$, $\omega \in C \leq_A A \in sub\mathcal{E}^0$, and let $\bar{a} = \langle a_1, \dots, a_n \rangle$ be a string of elements of A linearly independent over C and $b_i = \mathrm{ex}a_i$ be defined for all $i = 1, \dots, n$. Then the ex-locus V of $\bar{a}$ over C is ex-normal and free. If $\tilde{C}$ is an algebraically closed subfield of F, then V is absolutely free.*

Proof. The inequalities (1) in the definition of ex-normality under the assumptions of the lemma are equivalent to

$$\delta(a'_1, \dots, a'_k/C) \geq 0$$

and the latter follow from the fact that $C \leq_A A$.

An additive dependence for $\mathrm{pr}_x V$ would mean by the definition of V a linear dependence of $\bar{a}$ over C, which does not hold by the assumptions.

A multiplicative dependence for $\mathrm{pr}_y V$ is equivalent to $\mathrm{ex}\bar{a}$ being multiplicatively dependent over the subgroup generated by $\mathrm{ex}C$, which is equivalent under the assumptions to $\bar{a}$ being linearly dependent over C. $\square$

Theorem 3.2. *Let $V(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_k) \subseteq F^{2n+k}$ be an algebraic variety over some C. Then the following sets are quantifier-free definable in the language of fields:*

$$\{\langle a_1, \dots, a_k \rangle \in F^k : V(x_1, \dots, x_n, y_1, \dots, y_n, a_1, \dots, a_k) \text{ is irreducible}\}; \tag{3}$$

$$\{\langle a_1, \dots, a_k \rangle \in F^k : \mathrm{pr}_y V(y_1, \dots, y_n, a_1, \dots, a_k) \text{ is absolutely free of multiplicative dependencies}\}; \tag{4}$$

$$\{\langle a_1, \dots, a_k \rangle \in F^k : V(x_1, \dots, x_n, y_1, \dots, y_n, a_1, \dots, a_k) \text{ is ex-normal}\}. \tag{5}$$

Proof. Throughout the proof we let $W = \mathrm{pr}_{yz} V$, the variety in the variables $y_1, \dots, y_n$, $z_1, \dots, z_k$. We let $a = \langle a_1, \dots, a_k \rangle$ and denote by $W(a)$ the variety in the variables $y_1, \dots, y_n$ obtained from W by letting $z = a$.

For (3) the fact is well known and widely used.

To prove the statement for (4) it is enough to prove that, for $W = \mathrm{pr}_y V$, there is a finite set $\mu(W)$ of basic tori (algebraic subgroups of $(F^\times)^n$) of codimension 1 such that, given $W(a) \subseteq F^n$ which is not free of multiplicative dependencies, there are $Q \in \mu(W)$ and an $e \in (F^\times)^n$ with $W(a) \subseteq Qe$, the shift of Q by e. This statement is a special case of Corollary 3 of [5]. For (5) we will need a stronger version of the same Corollary 3 which is obtained by simply combining the former with the 'function field' version of Proposition 1 of [5]:

Fact. *Let $P \subseteq (F^\times)^n$ be a basic torus and $W(a) \subseteq P$ an algebraic variety. Then there is a finite collection $\pi_P(W)$ of basic subtori of P (depending on W but not on a) such that given a torus $T \subseteq P$, for any connected infinite atypical component X of $W(a) \cap T$, there exists $Q \in \pi_P(W)$ and $c \in P$ such that $X \subseteq Q \cdot c$ and X is typical in $W(a) \cap T$ with respect to $Q \cdot c$.*

Here a component X of $W(a) \cap T$ is said to be **atypical** (with respect to P) if

$$\dim X > \dim W(a) + \dim T - \dim P$$

and **typical** if

$$\dim X = \dim W(a) + \dim T - \dim P.$$

The last statement will be proved through the following sequence of lemmas.

Definition. Given a basic torus $T \subseteq (F^\times)^n$ there is a uniquely determined algebraic (group) variety $(F^\times)^n/T$ and the corresponding regular surjective homomorphism

$$(F^\times)^n \to (F^\times)^n/T.$$

We write $W(a)/T$ for the image of $W(a)$ under the homomorphism. Also, since T is uniquely determined by any of its cosets, we use the notation also when T is a non-basic torus, i.e. a coset of an algebraic subgroup of $(F^\times)^n$.

Let $T \subseteq P$ be tori, $W(a) \subseteq P$. We say that $W(a)/T$ is an **atypical image with respect to** P if

$$\dim W(a)/T < \min\{\dim P/T, \dim W(a)\}.$$

Easy dimension calculations show, for irreducible $W(a) \subseteq P$ with an atypical image $W(a)/T$, that for any generic $w \in W(a)$ it holds that

$$\dim W(a) \cap Tw > 0 \tag{6}$$

and

$$\dim W(a) \cap Tw > \dim W(a) - \dim P/T. \tag{7}$$

Proposition 3.3.[1] *Given $W(a) \subseteq P = (F^*)^n$, an irreducible algebraic variety, for any basic torus $T \subseteq P$ with atypical image $W(a)/T$ with respect to P, there is $Q \in \pi_P(W)$ such that*

$$\dim W(a)/Q = \dim W(a)/T - \dim Q/(Q \cap T)$$

and

$$\dim W(a)/T = \dim W(a)/(Q \cap T).$$

Proof. Let $w \in W(a)$ be generic and $X \subseteq W(a) \cap T \cdot w$ be a component of the intersection of maximal dimension. Then by the additive formula

$$\dim W(a)/T = \dim W(a) - \dim X \tag{8}$$

and $\dim X = \dim W(a) \cap T \cdot w > 0$. We may assume $w \in X$. By the Fact above there is $Q \in \pi_P(W)$ such that (i) $X \subseteq Q \cdot w$ and (ii) X is a typical component of the intersection

[1] I am grateful to Kitty Holland for detecting a serious error in the formulation of the Proposition in the previous version of the paper. The present version is quite similar to her result in [2], the proof of which is based on the same Section 5 of [5].

$(W(a) \cap Qw) \cap Tw$ with respect to Qw. By (i) and the maximality of $\dim X$, we have $\dim W(a)/T = \dim W(a)/(Q \cap T)$. And (ii) means that, given a connected component $Y \supseteq X$ of the variety $W(a) \cap Qw$, we have

$$\dim X = \dim Y + \dim Q \cap T - \dim Q. \tag{9}$$

But Y is a component of a generic fibre of the mapping $W(a) \to W(a)/Q$ and, by the classical theorem on the dimension of fibres ([3], Chapter 1, s.6, Thm 7),

$$\dim Y = \dim W(a) \cap Qw = \dim W(a) - \dim W(a)/Q. \tag{10}$$

Combining (8), (9) and (10) we get the required equality on $\dim W(a)/Q$. □

In the case of $P = (F^{\times})^n$ we write $\pi(W)$ instead of $\pi_P(W)$.

Lemma 3.4. *If the variety $V(a) \subseteq F^{2n}$ is not ex-normal then either* $\dim V(a) < n$ *or, for $W = \mathrm{pr}_y V$, there is $Q \in \pi(W)$ defined by a matrix q on $l =$* codim *Q independent integer n-rows as $Q = \{y \in (F^{\times})^n : y^q = 1\}$ such that*

$$\dim V(a)^q < l.$$

Proof. Suppose $\dim V(a) \geq n$, and $V(a)$ is not ex-normal, which is witnessed by M, a matrix of $k < n$ independent integer n-rows, as

$$\dim V(a)^M < k. \tag{11}$$

By definition, on x-coordinates the mapping $x \to Mx$ is a linear surjective mapping $F^n \to F^k$, and on y-coordinates $y \to y^M$ is a surjective homomorphism $(F^{\times})^n \to (F^{\times})^k$. Denote the kernel of the second one as T; thus the latter mapping in the notation above is $P \to P/T$ and $W(a)^M = W(a)/T$. Notice also that the dimension of the kernel of $[M]$ on x-coordinates is equal to $\dim T$, since both are equal to the co-rank of the matrix M.

Claim 1. $W(a)/T$ is an atypical image.

Suppose not. Then, in the case of $\dim P/T \leq \dim W(a)$, we have by definition that $\dim W(a)/T = \dim P/T$ and $\dim P/T = k$, a contradiction. In the case of $\dim W(a) < \dim P/T$ we have $\dim W(a)/T = \dim W(a)$. It follows that the mapping $W(a) \to W(a)^M$ is finite; thus the fibres of the mapping $V(a) \to V(a)^M$ are at most of dimension $\dim T$ and hence $\dim V(a)^M \geq \dim V(a) - \dim T \geq n - \dim T = \dim P/T$, which contradicts the assumptions again. The claim is proved.

By Proposition 3.3 there is $Q \in \pi(W)$ with $\dim W(a)/Q = \dim W(a)/T - \dim Q/(Q \cap T)$ and $\dim W(a)/(Q \cap T) = \dim W(a)/T$.

Claim 2. W.l.o.g., we may assume that $Q \supseteq T$.

Indeed, the basic torus $Q \cap T$ is given by a system of $k' = \mathrm{codim}\, Q \cap T \geq k$ independent equations $y^{M'} = 1$.

By definition, M' defines on x-coordinates a linear surjective mapping $[M'] : F^n \to F^{k'}$, with $\ker[M'] \subseteq \ker[M]$, so $[M]$ can be obtained as the composition of $[M']$ with another linear mapping with fibres of dimension $k' - k$. Thus, for any $b \in W(a)$, letting

$$V(b^\frown a) = \{c \in F^n : c^\frown b \in V(a)\},$$

a variety on x-coordinates, we have after applying the mappings $[M']$ and $[M]$,

$$\dim V(b^\frown a)^{M'} \leq \dim V(b^\frown a)^M + (k' - k).$$

On the other hand, by the addition formula,

$$\dim V(a)^{M'} = \dim W(a)^{M'} + \min_{b \in W(a)} \dim V(b^\frown a)^{M'}.$$

Since $\dim W(a)^{M'} = \dim W(a)/(Q \cap T) = \dim W(a)/T = \dim W(a)^M$, we have

$$\begin{aligned}\dim V(a)^{M'} &\leq \dim W(a)^M + \min_{b \in W(a)} \dim V(b^\frown a)^M + (k' - k)\\ &= \dim V(a)^M + k' - k < k'.\end{aligned}$$

In other words, we can replace T by $Q \cap T$, and so M by M', and still witness the failure of ex-normality. The claim is proved.

Let now the above basic torus $Q \supseteq T$ be given by $l = \operatorname{codim} Q \leq k$ equations of the form $y^q = 1$, and the matrix q induce the surjective mapping

$$[q] : F^n \times (F^*)^n \to F^l \times (F^*)^l.$$

Since $Q \supseteq T$ we have

$$\dim V(b^\frown a)^q \leq \dim V(b^\frown a)^M,$$

while for y-coordinates we have

$$\dim W(a)^q = \dim W(a)^M - (k - l),$$

by the definition of Q.

Again, the addition formula and the last two formulas yield

$$\dim V(a)^q = \dim W(a)^q + \min_{b \in W(a)} \dim V(b^\frown a)^q \leq \dim V(a)^M + l - k.$$

It follows by (11) that

$$\dim V(a)^q < l. \quad \square$$

End of the Proof of the Theorem. The statement for (5) follows immediately from the lemma, as the condition

$$\dim V(a) \geq n \ \ \& \bigwedge_{Q \in \pi(W)} \dim V(a)^q \geq \operatorname{codim} Q$$

is quantifier-free definable in L. $\square$

Remark. Theorem 3.2 will not be used in the proof of the main result of this paper since the further constructions and proofs are carried out in $L_{\omega_1,\omega}$-terms. Still we hope that with some extra work the theorem can provide a finer description of the fields with pseudo-exponentiation.

4. Exponentially–algebraically closed structures

Definition. Let $V \subseteq G_n(F)$ be an algebraic subvariety ex-defined and ex-irreducible over some $C \subseteq F$. With any such V we associate a sequence $\{V^{\frac{1}{l}} : l \in \mathbb{N}\}$ of algebraic varieties which are ex-definable and ex-irreducible over C and satisfy the following:
$V^1 = V$; and for any $l, m \in \mathbb{N}$ the mapping $[m]$ maps $V^{\frac{1}{lm}}$ onto $V^{\frac{1}{l}}$.

Such a sequence is said to be **a sequence associated with** V over C.
Also, with any $\langle a_1, \ldots, a_n, b_1, \ldots, b_n\rangle \in V$ as above we associate a sequence

$$\{\langle a_1, \ldots, a_n, b_1, \ldots, b_n\rangle^{\frac{1}{l}} : l \in \mathbb{N}\}$$

such that for any $l, m \in \mathbb{N}$ the mapping $[m]$ maps $\langle a_1, \ldots, a_n, b_1, \ldots, b_n\rangle^{\frac{1}{lm}}$ onto $\langle a_1, \ldots, a_n, b_1, \ldots, b_n\rangle^{\frac{1}{l}}$.

Let $V' \subseteq V \subseteq G_n(F)$ be varieties over C, V irreducible over C, $\{V^{\frac{1}{l}} : l \in \mathbb{N}\}$ a sequence associated with V. Then the pair

$$\tau = (V \setminus V', \{V^{\frac{1}{l}} : l \in \mathbb{N}\})$$

is said to be **an [almost finite]** n**-type over** C. A **finite** n**-type** over C is given by an algebraic set of the form $V \setminus V'$, with $V, V' \subseteq F^{2n}$ ex-definable algebraic varieties.

A tuple $\bar{a} = \langle a_1, \ldots, a_n\rangle \in F^n$ is said to realize the type $V \setminus V'$ if

$$\langle a_1, \ldots, a_n, \mathrm{ex}(a_1), \ldots, \mathrm{ex}(a_n)\rangle \in V \setminus V'.$$

The tuple $\bar{a}$ is said to realize the type τ above if $\bar{a}$ realizes $V \setminus V'$ and

$$\left\langle \frac{1}{l}a_1, \ldots, \frac{1}{l}a_n, \mathrm{ex}\left(\frac{1}{l}a_1\right), \ldots, \mathrm{ex}\left(\frac{1}{l}a_n\right)\right\rangle \in V^{\frac{1}{l}}$$

for all $l \in \mathbb{N}$.

We say that $\bar{a}$ realizes τ **generically over** C if V is the ex-locus of $\bar{a}$ over C.

The **complete ex-locus of** $\bar{a}$ **over** C is the type $(V, \{V^{\frac{1}{l}} : l \in \mathbb{N}\}$, where $V^{\frac{1}{l}}$ are the ex-loci of $\langle \frac{1}{l}a_1, \ldots, \frac{1}{l}a_n\rangle$ over C.

We say that $C \subseteq F$ is **finitary** if there are $n \geq 0$, substructures $E_1, \ldots, E_n \subseteq F$, such that $\mathrm{ex}E_1, \ldots, \mathrm{ex}E_n$ are algebraically closed subfields of F, and a finite set A, such that

$$C = \mathrm{span}_{\mathbb{Q}}(A) \cup E_1 \cup \cdots \cup E_n.$$

Below we use the notation $\mathbf{F}$ for an L-structure on the field F.

A crucial tool for the study of types and their realizations in this section will be the following reformulation of the main result of [6]:

Theorem 4.1. *Let* $\mathbf{F} \in \mathcal{E}^0$, $C \subseteq F$, V *an algebraic variety in* $2n$ *variables* $x_1, \ldots, x_n, y_1, \ldots, y_n$, *ex-defined, ex-irreducible over* C *and free, and* $\{V^{\frac{1}{n}} : n \in \mathbb{N}\}$ *a sequence associated with* V *over* C. *Suppose also that* C *is finitary. Then there exists a* *positive integer m such that for any* $l \in \mathbb{N}$, $V^{\frac{1}{lm}}$ *is the unique variety which is irreducible over* $\tilde{C}$ *and satisfies*

$$(V^{\frac{1}{lm}})^l = V^{\frac{1}{m}}\ (x_i \mapsto l \cdot x_i \text{ and } y_i \mapsto y_i^l).$$

Proof. This is based on Theorem 1 of [6].

We may assume that V is ex-defined over finite $A \subseteq C$, $\omega \in A$ as in the definition of finitary C. Obviously, for any m, the field of definition of $V^{\frac{1}{m}}$ is a subfield of $\mathrm{acl}(A \cup \mathrm{ex}A)$, which is of finite transcendence degree. Hence, the statement of the theorem holds for C if and only if it holds for C_0 instead of C, where

$$C_0 = C \cap \ln(\mathrm{acl}(A \cup \mathrm{ex}A)) = \mathrm{span}_{\mathbb{Q}}(A) \cup E_1^0 \cup \dots \cup E_n^0$$

and E_i^0 are substructures such that $\mathrm{ex}(E_i^0) = L_i$ are algebraically closed subfields of F.

Let $\langle a_1, \dots, a_n \rangle$ be a generic over $\tilde{C}$ point in $\mathrm{pr}_x V$ and $\{a_1, \dots, a_r\} = A \cup \{a_1, \dots, a_n\} \cup \mathrm{ex}(A \cup \{a_1, \dots, a_n\})$.

Let $\hat{P} = \mathbb{Q}(a_1, \dots, a_r, \sqrt{1}, L_1, \dots, L_n)$, the field generated by elements $a_1, \dots, a_r$, all the roots of unity and subfields $L_1, \dots, L_n$. (The roots of unity can be omitted if $n > 0$.)

Choose $\langle b_1^{\frac{1}{m}}, \dots, b_n^{\frac{1}{m}} \rangle$ to be generic in $\mathrm{pr}_y V^{\frac{1}{m}}$ over $\hat{P}$ and $(b_i^{\frac{1}{mk}})^k = b_i^{\frac{1}{m}}$. It follows from the freeness assumptions that $b_1, \dots, b_n$ are multiplicatively independent over the group $gp(a_1, \dots, a_r)$ generated by the a_i's. The statement of the theorem follows with this notation directly from Theorem 1 of [6]. □

Definition. A structure $\mathbf{F}$ in $\mathcal{E}_{\mathrm{st}}^0$ is said to be **exponentially–algebraically closed** (e.a.c.) if for any $\mathbf{F}' \in \mathcal{E}_{\mathrm{st}}^0$, such that $\mathbf{F} \le \mathbf{F}'$, any finite quantifier-free type over F which is realized in $\mathbf{F}'$ has a realization in $\mathbf{F}$.
The class of exponentially–algebraically closed structures is denoted as $\mathcal{EC}_{\mathrm{st}}$.

Remark. It follows from Lemma 2.11 that in the definition of $\mathcal{EC}_{\mathrm{st}}$ we can equivalently assume that $\mathbf{F}'$ ranges in $sub\mathcal{E}_{\mathrm{st}}^0$.

Lemma 4.2. *Suppose* $\mathbf{F} \in \mathcal{EC}_{\mathrm{st}}$, $C \subseteq F$ *is finitary and*

$$\tau = (V \setminus V', \{V^{\frac{1}{l}} : l \in \mathbb{N}\})$$

is a type ex-definable over C*. Assume also that* V *is ex-normal over* C *and absolutely free. Then there is an* a *in* $\mathbf{F}$ *realizing* τ*. Moreover, in some extension* $\mathbf{F}' \ge \mathbf{F}$*,* a *can be chosen to realize* τ *generically over* C*.*

Proof. Under the assumptions of the lemma, by Theorem 4.1, after the transformation $x_i \mapsto \frac{1}{m} x_i$ of variables, we may assume that τ is just $V \setminus V'$. Take $a^\frown b$ in an algebraically closed extension F' of the field F, generic in V over F. Choose in F' a sequence $\{(a^\frown b)^{\frac{1}{l}} : l \in \mathbb{N}\}$ associated with $a^\frown b$. This gives us uniquely determined values of $\frac{1}{l} a_i$ and $b_i^{\frac{1}{l}}$ for coordinates a_i of a and b_i of b.

Let $A = F + \mathrm{span}_{\mathbb{Q}}\{(a^\frown b)^{\frac{1}{l}} : l \in \mathbb{N}\}$ and define ex_A with domain $\mathrm{D}_A = D = F + \mathrm{span}_{\mathbb{Q}}(a_1, \dots, a_n)$ as

$$\mathrm{ex}_A\left(f+\sum_i \frac{m_i}{l}a_i\right)=\mathrm{ex}(f)\cdot\prod_i (b_i^{\frac{1}{l}})^{m_i},$$

for any integers m_i, $l \neq 0$ and element $f \in F$. The definition is consistent since $\mathrm{pr}_x V$ is free of additive dependencies over F. Evidently the formula defines a homomorphism; thus $A \in sub\mathcal{E}$.

The kernel of the homomorphism ex_A coincides with that of ex on F, since $\mathrm{pr}_y V$ has no multiplicative dependencies over F. Thus A has a standard full kernel. Notice that, by ex-normality, $\delta(m_1 a, \ldots, m_k a/F) \geq 0$ for any independent integer vectors $m_i = \langle m_{i,1}, \ldots, m_{i,n}\rangle$, $i = 1, \ldots, k$.

Thus $\mathbf{F} \subseteq A$ satisfy the assumptions of Lemma 2.9 (with $\mathrm{ex}_A(F) \setminus F = \emptyset$) and hence $A \in sub\mathcal{E}^0_{\mathrm{st}}$, $\mathbf{F} \leq A$. By the choice of a the tuple realizes τ. Since $\mathbf{F} \in \mathcal{EC}_{\mathrm{st}}$ there is a realization of the type in $\mathbf{F}$. □

Proposition 4.3. *A structure $\mathbf{F} \in \mathcal{E}^0_{\mathrm{st}}$ is in $\mathcal{EC}_{\mathrm{st}}$ iff for any irreducible ex-normal free V over F there is a realization of the finite type given by V in $\mathbf{F}$.*

First we prove:

Lemma 4.4. *Given an irreducible free ex-normal $V \subseteq F^{2n}$ and non-empty $V' \subseteq V$ there is a free ex-normal $V^* \subseteq F^{2n+2m}$ such that $\langle a_1, \ldots, a_n, b_1, \ldots, b_n\rangle \in F^{2n}$ realizes $V \setminus V'$ iff there is $\langle a_{n+1}, \ldots, a_{n+m}, b_{n+1}, \ldots, b_{n+m}\rangle \in F^{2m}$ such that $\langle a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}, b_1, \ldots, b_n, b_{n+1}, \ldots, b_{n+m}\rangle$ realizes V^*.*

Proof. Let $g(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be a polynomial in the annihilator of V', but not zero on V. We may assume that for no positive integer k and a non-zero integer tuple $\langle m_1, \ldots, m_n\rangle$,

$$g(x_1, \ldots, x_n, y_1, \ldots, y_n)^k \cdot y_1^{m_1} \cdot \ldots \cdot y_n^{m_n}$$

is constant on V, since otherwise g does not vanish on V'. Add new variables x_{n+1}, y_{n+1} together with the new identity

$$g(x_1, \ldots, x_n, y_1, \ldots, y_n) \cdot y_{n+1} = 1.$$

Denote the resulting variety in F^{2n+2} by V^g. By construction, V^g is irreducible and its projection onto the first $2n$ coordinates is equal to

$$V \setminus \{\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle : g(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0\}.$$

By our assumptions V^g is free of multiplicative dependencies and obviously free of additive dependencies. It is also ex-normal since we do not impose any condition on x_{n+1}.

Repeating the construction with all the polynomials in the basis of the annihilator of V' we come to V^* as required. □

Proof of the Proposition. The left-to-right implication follows from Lemma 4.2. Indeed, since the field F is algebraically closed, V is absolutely free. On the other hand, we can obviously choose $C \subseteq F$ finite such that V is ex-definable over C.

To get the inverse, assume that a is a tuple in some $\mathbf{F}' \geq \mathbf{F}$ and we need to realize an almost finite type $(V \setminus V', \{V^{\frac{1}{l}} : l \in \mathbb{N}\})$, where $\tau = (V, \{V^{\frac{1}{l}} : l \in \mathbb{N}\})$ is the ex-locus of

a over F. It is enough to solve the problem for a $\mathbb{Q}$-linear basis a_0 of a over F, so we may assume that a is $\mathbb{Q}$-linearly independent over F. Thus V is free ex-normal, by Lemma 3.1, and ex-irreducible, because F is algebraically closed. So we may assume that τ is a finite type. By Lemma 4.4 we reduce the type $V \setminus V'$ to a type of the form V and V is ex-normal and free. By the assumptions of the Proposition the type is realized in $\mathbf{F}$. $\square$

Corollary 4.5. *The structure $\mathbb{C}_{\exp}$ on complex numbers is in $\mathcal{EC}_{\text{st}}$ iff it satisfies the Schanuel conjecture and for any ex-normal free algebraic variety $V \subseteq \mathbb{C}^{2n}$ there is $a \in \mathbb{C}^n$ such that $a^\frown \exp(a) \in V$.*

Proof. By definition, $\mathbb{C}_{\exp} \in \mathcal{E}^0_{\text{st}}$ iff the Schanuel conjecture holds. The rest is Proposition 4.3

Corollary 4.6. *There is a collection* EC *of first-order formulas such that for any $\mathbf{F} \in \mathcal{E}^0_{\text{st}}$*

$$\mathbf{F} \models EC \textit{ iff } \mathbf{F} \in \mathcal{EC}_{\text{st}}.$$

Proof. For each algebraic variety $V \subseteq F^{2n+k}$ over $\mathbb{Q}$ in variables $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_k$, by Theorem 3.2 there exists a first-order quantifier-free formula $\Psi_V(z_1, \ldots, z_k)$, in the language of fields, such that for any $a_1, \ldots, a_k \in F$,

$$\mathbf{F} \models \Psi_V(a_1, \ldots, a_k) \text{ iff } V(a_1, \ldots, a_k) \text{ is irreducible, ex-normal and free.}$$

It follows that the statement

for any $a_1, \ldots, a_k$, if $V(a_1, \ldots, a_k)$ is irreducible, ex-normal and free, then there is $\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle \in V(a_1, \ldots, a_k)$ such that $\mathrm{ex}(x_1) = y_1, \ldots, \mathrm{ex}(x_n) = y_n$

is first order (in fact, an ∀∃-sentence).

Since any variety over F has the form $V(a_1, \ldots, a_k)$ for some V and $a_1, \ldots, a_k$ as above, we can write down the condition given in Proposition 4.3 by an infinite collection of first-order formulas. $\square$

We are going to weaken the assumptions in Lemma 4.2. We assume below that $\mathbf{F} \in \mathcal{EC}_{\text{st}}$.

Lemma 4.7. *Let C be a finitary subset in $\mathbf{F}$, $\bar{a}$ a finite string in some $\mathbf{F}' \geq \mathbf{F}$, $\{V^{\frac{1}{l}} : l \in \mathbb{N}\})$ be the complete ex-locus of $\bar{a}$ over C and τ the type $(V \setminus V', \{V^{\frac{1}{l}} : l \in \mathbb{N}\})$ for V' some proper subvariety of V, ex-defined over C. Then there is an $m \in \mathbb{N}$ such that τ is equivalent to a finite type $(V \setminus V', \{V^{\frac{1}{l}} : l \leq m\})$ and there is a realization of τ in $\mathbf{F}$.*

Proof. Passing to a linear basis of $\bar{a}$ over $C \cup \{\omega\}$ we may assume that $\bar{a}$ is linearly independent over $C \cup \{\omega\}$. Hence V is free. By Theorem 4.1 the infinite part of τ, the system of equations saying that $\frac{1}{l}\bar{x}^\frown \mathrm{ex}(\frac{1}{l}\bar{x}) \in V^{\frac{1}{l}}, l \in \mathbb{N}$, is equivalent to a finite subsystem. Thus τ is equivalent to a finite type. By the assumption for $\mathbf{F}$ the finite type is realized in $\mathbf{F}$. $\square$

Proposition 4.8. *Let $C \leq_F \mathbf{F}$ be finitary and V in coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ be ex-definable irreducible over C and ex-normal. Then for any sequence $\{V^{\frac{1}{l}} : l \in \mathbb{N}\}$ and $V' \subsetneq V$ over C there is in $\mathbf{F}$ a realization $\bar{a}$ of the type*

$$\tau = (V \setminus V', \{V^{\frac{1}{l}} : l \in \mathbb{N}\})$$

Moreover, in some extension $\mathbf{F}' \geq \mathbf{F}$ *there is a generic realization of* τ.

Proof. If V is absolutely free, then we can apply Lemma 4.2 and get the statement. So we assume that V is not absolutely free and prove the statement by induction on n.

For $n = 1$, from the fact that V is not absolutely free we get that $\dim \mathrm{pr}_x V = 0$ or $\dim \mathrm{pr}_y V = 0$. Suppose that the first takes place. Then by ex-normality, if $a \in \mathrm{pr}_x V$ and $b \notin \mathrm{acl}(\tilde{C} \cup \{a\})$ then $\langle a, b\rangle \in V$. Choose any a in $\mathrm{pr}_x V(F) \subseteq \mathrm{acl}(\tilde{C})$. Then, since $\delta(a/C) \geq 0$ and a is additively independent of C, we have that $\mathrm{ex}(a)$ is algebraically independent of $\tilde{C}$; thus $\langle a, \mathrm{ex}(a)\rangle \in V$. For the same reason, $\langle \frac{1}{l}a, \mathrm{ex}(\frac{1}{l}a)\rangle$ satisfies $V^{\frac{1}{l}}$ and $\langle a, \mathrm{ex}(a)\rangle \notin V'$.

The case $\mathrm{pr}_y V = 0$ can be dealt with symmetrically.

Consider now the general n assuming that the statement holds for smaller values of n and that $\mathrm{pr}_x V$ is not absolutely free of additive dependencies.

Let $\mathbf{F} \leq \mathbf{F}'$ with F' of infinite transcendence degree over F. Applying a transformation $[M] : G_n(F') \to G_n(F')$ induced by an appropriate rational matrix to V we may assume that for any $\bar{a} = \langle a_1, \ldots, a_n\rangle \in \mathrm{pr}_x V$ in F', generic over $\tilde{C}$, the elements $a_1, \ldots, a_k$ are linearly independent over $\mathrm{acl}(\tilde{C})$ and $a_{k+1}, \ldots, a_n \in \mathrm{acl}(\tilde{C})$. Choose such a tuple $\bar{a}$ in $\mathbf{F}'$. By genericity, $\bar{a} \notin V'$.

Write $W = \mathrm{pr}_y V$ and $W_{k+1,\ldots,n}$ the variety induced by W on $\{k+1, \ldots, n\}$-coordinates. It follows from ex-normality and the fact that $\mathrm{tr.d.}(a_{k+1}, \ldots, a_n/\tilde{C}) = 0$ that $\dim W_{k+1,\ldots,n} = n - k$.

Since $\delta(a_{k+1}, \ldots, a_n/C) \geq 0$, we have

$$\mathrm{tr.d.}(\mathrm{ex}(a_{k+1}), \ldots, \mathrm{ex}(a_n)/\tilde{C}) \geq \mathrm{lin.d.}(a_{k+1}, \ldots, a_n/C) = n - k,$$

which implies that $\langle a_{k+1}, \ldots, a_n, \mathrm{ex}(a_{k+1}), \ldots, \mathrm{ex}(a_n)\rangle$ is generic in $V_{k+1,\ldots,n}$ over $\tilde{C}$. For the same reason,

$$\left\langle \frac{1}{l}a_{k+1}, \ldots, \frac{1}{l}a_n, \mathrm{ex}\left(\frac{1}{l}a_{k+1}\right), \ldots, \mathrm{ex}\left(\frac{1}{l}a_n\right)\right\rangle \in V^{\frac{1}{l}}_{k+1,\ldots,n}.$$

Finally notice that the type $\tau(a_{k+1}, \ldots, a_n)$ over $C \cup \{a_{k+1}, \ldots, a_n\}$, corresponding to the first k coordinates in τ when the rest are replaced by the highlighted elements, satisfies the assumptions of Lemma 4.2. Thus it has a realization $\langle a_1, \ldots, a_k\rangle$ in $\mathbf{F}$. This completes the construction of a realization $\langle a_1, \ldots, a_n\rangle$ of τ.

The case when $\mathrm{pr}_y V$ is not absolutely free of multiplicative dependencies can be treated symmetrically. □

Now we study the rank notion ∂_F for F e.a.c.

Lemma 4.9. *For* $\mathbf{F} \in \mathcal{EC}_{\mathrm{st}}$, *given* $A \subseteq_{fin} F$ *and* $\mathbf{F}' \in \mathcal{E}^0_{\mathrm{st}}$ *such that* $F \leq F'$,

$$\partial_F(A) = \partial_{F'}(A).$$

Proof. This is just a special case of Lemma 2.8. □

Lemma 4.10. *Given* $\mathbf{F} \in \mathcal{EC}_{\rm st}$, *for any* $A \subseteq_{fin} F$ *and any* $a, b \in F$:

(i) $\partial(A) \leq \partial(aA) \leq \partial(A) + 1$;
(ii) $\partial(abA) = \partial(aA) = \partial(A)$ *implies* $\partial(bA) = \partial(A)$;
(iii) $\partial(abA) = \partial(aA) \& \partial(A) < \partial(bA)$ *implies* $\partial(abA) = \partial(bA)$;
(iv) $\partial(aA) = \partial(A) = \partial(bA)$ *implies* $\partial(abA) = \partial(A)$;
(v) $\partial(aA) = \partial(A)$ *implies* $\partial(bA) = \partial(abA)$.

Proof. (i) follows immediately from the definitions of δ and ∂. (ii) and (iii) are immediate from (i).
(iv) Let $B' \supseteq aA$, $B'' \supseteq bA$ be such that $\delta(B') = \partial(aA)$ and $\delta(B'') = \partial(bA)$. Let $B = \text{span}_{\mathbb{Q}}(B') \cap \text{span}_{\mathbb{Q}}(B'')$.
Notice that $\delta(B' \cup B'') \leq \delta(B'')$. Indeed by Lemma 2.5(iii),
$\delta(B' \cup B'') = \delta(B'/B'') + \delta(B'')$
$= [\text{tr.d.}(B' \cup \text{ex}(B')/B'' \cup \text{ex}(B'')) - \text{lin.d.}(B'/B'')] + \delta(B'')$.
By the modularity of linear dimension, $\text{lin.d.}(B'/B'') = \text{lin.d.}(B'/B)$. Also, by properties of algebraic dependence,
$\text{tr.d.}(B' \cup \text{ex}(B')/B'' \cup \text{ex}(B'')) \leq \text{tr.d.}(B' \cup \text{ex}(B')/B \cup \text{ex}(B))$. Hence
$\delta(B'/B'') \leq \delta(B'/B)$. The latter is less than or equal to zero by the choice of A, B' and B.

Now, since $abA \subseteq B' \cup B''$ and $\delta(B' \cup B'') \leq \delta(B'') = \partial(A)$, we have $\partial(abA) = \partial(A)$.

(v) is immediate from (iv). □

Notation. For finite $A \subseteq \mathbf{F}$,

$$\text{cl}_{\mathbf{F}}(A) = \{b \in F : \partial(Ab) = \partial(A)\}.$$

For infinite A,

$$\text{cl}_{\mathbf{F}}(A) = \bigcup_{X \subseteq_{fin} A} \text{cl}_{\mathbf{F}}(X).$$

$\text{cl}_{\mathbf{F}}(A)$ will be called the ∂-closure of A in $\mathbf{F}$.

We usually omit the subscript $\mathbf{F}$ when no ambiguity can arise.

Proposition 4.11. *The operator* $A \mapsto \text{cl}(A)$ *in* $\mathbf{F} \in \mathcal{EC}_{\rm st}$ *is a closure operator, satisfying for any* $A \subseteq F$:

(i)

$$\text{cl}(A) = \bigcup_{X \subseteq A,\ X \textit{ finite}} \text{cl}(X);$$

(ii)

if $A \subseteq A' \subseteq F$, *then* $\text{cl}(A) \subseteq \text{cl}(A')$;

(iii)

$$\text{cl}(\text{cl}(A)) = \text{cl}(A);$$

(iv)

$$a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A) \Rightarrow b \in \mathrm{cl}(Ab);$$

(v)

$$\mathrm{cl}(A) \le F;$$

(vi) $\mathrm{cl}(A)$ *is an existentially–algebraically closed substructure of* $\mathbf{F}$.

Proof. (i) is immediate from definitions.
(ii) and (iii) follow from Lemma 4.10.
(iv) follows from Lemma 4.10(iii).
(v) Assume first that A is finite. Then there is a finite $A' \supseteq A$, $A' \subseteq F$, such that $\partial(A) = \delta(A')$ and so $A' \le_F F$. Since for every $a \in A'$ by definition $\partial(Aa) = \partial(A)$, we have $A' \subseteq \mathrm{cl}(A)$. The same argument shows in the general case that for any $B \subseteq_{fin} \mathrm{cl}(A)$ there is a finite B', $B \subseteq B' \subseteq \mathrm{cl}(A)$, such that $B' \le_F F$. Notice also that $\mathrm{cl}(A)$ is closed under ex. It follows that $\mathrm{cl}(A) \le F$.
(vi) follows from Proposition 4.3. Indeed, we need to check that given a free ex-normal V in $2n$ variables ex-definable over $\mathrm{cl}(A)$ there is a realization of V in $\mathrm{cl}(A)$. Notice that $\mathrm{cl}(A) \cap F$ is algebraically closed in the field F.

We prove the existence of the realization by induction on $n = \mathrm{lin.d.}(\bar{a}/\mathrm{cl}(A))$.

Let $\bar{a}$ be a realization of V in $\mathbf{F}$ with minimal $\delta(\bar{a}/\mathrm{cl}(A))$. This number is non-negative since $\mathrm{cl}(A) \le F$. If $\delta(\bar{a}/\mathrm{cl}(A)) = 0$, then $\bar{a}$ is a tuple from $\mathrm{cl}(A)$ by definition and we are done. Suppose, towards a contradiction, that $\delta(\bar{a}/\mathrm{cl}(A)) > 0$. If we apply a transformation induced by an integer matrix M of rank $k < n$, then still $\delta(M\bar{a}/\mathrm{cl}(A)) > 0$, since otherwise we see that $M\bar{a}$ is in $\mathrm{cl}(A)$ and the linear dimension of $\bar{a}$ over $M\bar{a}$, and so over $\mathrm{cl}(A)$, is not bigger than $n - k$. We can also assume that $\bar{a}$ is linearly independent over $\mathrm{cl}(A)$ and V is the ex-locus of $\bar{a}$ over $\mathrm{cl}(A)$. Then V is ex-normal and absolutely free.

Let $C \le_F \mathrm{cl}(A)$ be finitary and such that V is ex-defined over C. It follows from $\delta(\bar{a}/C) > 0$ that $\dim \mathrm{pr}_x V > 0$, so we may also assume that $a_n \notin \mathrm{acl}(\tilde{C})$.

Then there exists a $c_n \in \mathrm{acl}(\tilde{C})$ such that every component V' of the subvariety $V \cap \{x_n = c_n\}$ is non-empty and has dimension equal to $\dim V - 1$. Consider such an ex-definable irreducible variety V' over the finitary set $\mathrm{span}_{\mathbb{Q}}(Cc_n)$. This is ex-normal. Indeed, consider a generic over $\widetilde{Cc_n}$ tuple $\langle c_1, \ldots, c_n, b_1, \ldots, b_n\rangle \in V'$ and

$$a'_i = m_{i,1}c_1 + \cdots + m_{i,n}c_n, \quad \text{and } b'_i = b_1^{m_{i,1}} \cdot \ldots \cdot b_n^{m_{i,n}} \quad i = 1, \ldots, k,$$

for some $k \times n$ integer matrix

$$M = \{m_{i,l} : 1 \le i \le k,\ 1 \le l \le n\}$$

of rank $k \le n$. We can also write down in vector form

$$\bar{a}' = M\bar{a} \quad \text{and} \quad \bar{b}' = \bar{b}^M.$$

We need to see that

$$\mathrm{tr.d.}(\bar{a}'^\frown \bar{b}'/\widetilde{Cc_n}) \ge k.$$

It follows from the fact that the tuple was chosen to be generic that the required inequality is equivalent to

$$\dim(V')^M \geq k.$$

But $\dim V' = \dim V - 1$, thus $\dim(V')^M \geq \dim V^M - 1$, so

$$\dim(V')^M - k \geq \dim V^M - k - 1 = \text{tr.d.}(M\bar{a}^\frown \text{ex}(\bar{a})^M/\tilde{C}) - k - 1,$$

and the latter is non-negative because $\delta(M\bar{a}/\text{cl}(A)) > 0$. The ex-normality follows.

Now we use Proposition 4.8 to find a realization $\bar{a}'$ for V' in $\mathbf{F}$, which by definition is of linear dimension at most $n - 1$ over $\text{cl}(A)$, contradicting the minimality. □

5. Strongly exponentially–algebraically closed fields

Definition. A structure $\mathbf{F}$ in $\mathcal{E}^0_{\text{st}}$ is said to be **strongly exponentially–algebraically closed** (s.e.a.c.) if $\mathbf{F} \in \mathcal{EC}_{\text{st}}$, and, for any ex-irreducible free ex-normal V in $2n$ variables ex-defined over a finite $C \subseteq \mathbf{F}$, with $\dim V = n$, there is a generic over C realization of V in $\mathbf{F}$. The class of strongly exponentially–algebraically closed structures is denoted as $\mathcal{EC}^*_{\text{st}}$.

Remark. The definition assumes a 'slight saturatedness' of the exponentially–algebraically closed structure.

Remark. Corollary 4.5 can be obviously amended to a criterion for $\mathbb{C}_{\exp}$ to be s.e.a.c.

Definition. We say that a structure $\mathbf{F} \in \mathcal{E}$ has the **countable closure property** (or c.c.p. for short) if, given a $C \subseteq F$ and an algebraic variety $V \subseteq F^{2n}$ of dimension n which is ex-definable, ex-irreducible, ex-normal and free over C, the set of generic realizations of V over C is at most countable.

We prove below (Lemma 5.12) that $\mathbb{C}_{\exp}$ has the c.c.p.

Our main goal in this final section of the paper is to prove that the class of exponentially–algebraically closed structures with the countable closure property has a unique model in every uncountable cardinality. We show first that the class is definable by an $L_{\omega_1,\omega}$-sentence and the c.c.p. (which can be written as an $L_{\omega_1,\omega}(Q)$-sentence in this case). The author's paper [7] lays out sufficient conditions under which such a class is categorical in all uncountable cardinals. The main theorem of [7] is a contribution to the theory of excellency developed by Shelah and adapted here for algebraic applications. We present the result below with some simplifications sufficient for the purposes of the present paper.

A class $\mathcal{C}$ of L-structures is said to be **quasi-minimal excellent** if the following three assumptions hold:

Assumption I (*Pregeometry*). There is an $L_{\omega_1,\omega}$-definable operator $X \to \text{cl}(X)$ acting on subsets of an $\mathbf{F} \in \mathcal{C}$ and satisfying:

(i) $\text{cl}(X) \in \mathcal{C}$ as a substructure of $\mathbf{F}$;
(ii) $\text{cl}(Y) = \bigcup\{\text{cl}(X) : X \subseteq Y, \; X \text{ finite}\}$;
(iii) $X \to \text{cl}(X)$ is a monotone idempotent operator.

Definition. Let $\mathbf{F}, \mathbf{F}' \in \mathcal{C}$ and $G \subseteq \mathbf{F}$, $G \subseteq \mathbf{F}'$. Then a (partial) mapping, identical on G, $\varphi : \mathbf{F} \to \mathbf{F}'$ is called a G-**monomorphism** if it preserves quantifier-free formulas over G.

Assumption II (*ω-Homogeneity over a Submodel*). Let $G \subseteq \mathbf{F}$, $G \subseteq \mathbf{F}'$, $G \in \mathcal{C}$ or $G = \emptyset$. Then

(i) if X and X' are cl-independent subsets over G in some $\mathbf{F}, \mathbf{F}' \in \mathcal{C}$, respectively, then any bijection $\varphi : X \to X'$ is a G-monomorphism;
(ii) if a partial $\varphi : \mathbf{F} \to \mathbf{F}'$ is a G-monomorphism, Dom $\varphi = X$, with X finite, then for any $y \in \mathbf{F}$ there is an extension φ' of φ with Dom $\varphi' = X \cup \{y\}$;
(iii) if $\varphi : X \cup \{y\} \to X' \cup \{y'\}$ is a monomorphism, then

$$y \in \mathrm{cl}(X) \text{ iff } y' \in \mathrm{cl}(X').$$

Definition. Given $X, C \subseteq \mathbf{F}$ we say that **the type of X over C is defined over** C_0 if any $\varphi : X \to \mathbf{F}'$ which is a C_0-monomorphism is also a C-monomorphism.

Definition. A subset $C \subseteq \mathbf{F}$ will be called **special** if there is a cl-independent $A \subseteq \mathbf{F}$ and $A_1, \ldots, A_k \subseteq A$ such that

$$C = \bigcup_i \mathrm{cl}(A_i).$$

Assumption III. Suppose $C \subseteq \mathbf{F}$ is special and X is a finite subset of $\mathrm{cl}(C)$. Then the type of X over C is defined over a finite subset $C_0 \subseteq C$.

Main Theorem of [7]. *Let $\mathcal{C}$ be quasi-minimal excellent and $\mathcal{C}^\#$ be its subclass consisting of structures satisfying the countable closure property. Suppose also that some $\mathbf{F} \in \mathcal{C}^\#$ contains an infinite* cl-*independent subset A. Then for any uncountable κ there is a unique, up to isomorphism, $\mathbf{F}_\kappa \in \mathcal{C}^\#$ of cardinality κ. Moreover, $\mathbf{F}_\kappa$ is prime over any maximal* cl-*independent subset (basis).*

Now we proceed to check the Assumptions for $\mathcal{C} = \mathcal{EC}^*_{\mathrm{st}}$. It is obvious that Proposition 4.11 implies Assumption I. It remains to prove the other two.

Lemma 5.1. *There is an $L_{\omega_1,\omega}$-sentence* $\mathrm{EC}^*_{\mathrm{st}}$ *such that, given $\mathbf{F} \in \mathcal{E}^0$,*

$$\mathbf{F} \models \mathrm{EC}^*_{\mathrm{st}} \text{ iff } \mathbf{F} \in \mathcal{EC}^*_{\mathrm{st}}.$$

Proof. Follow the proof of Corollary 4.6 and observe that for every $V(\bar{a})$ we can say, by an $L_{\omega_1,\omega}$-formula, that $\bar{x}^\frown\bar{y}$ is generic in $V(\bar{a})$ over $\bar{a}$. □

It follows from Lemma 4.2 that:

Proposition 5.2. *For any $\mathbf{F} \in \mathcal{E}^0_{\mathrm{st}}$ there is an $\mathbf{F}^\sharp \in \mathcal{EC}^*_{\mathrm{st}}$ such that $\mathbf{F} \leq \mathbf{F}^\sharp$.*

Lemma 5.3. *If $\mathbf{F} \in \mathcal{EC}^*_{\mathrm{st}}$ and $A \subseteq F$, then $\mathrm{cl}(A) \leq \mathbf{F}$ and $\mathrm{cl}(A) \in \mathcal{EC}^*_{\mathrm{st}}$.*

Proof. By Proposition 4.11(v) $\mathrm{cl}(A) \leq \mathbf{F}$. By Proposition 4.11(vi) we have $\mathrm{cl}(A) \in \mathcal{EC}_{\mathrm{st}}$. Now given a free ex-normal V over a finite $C \leq_F \mathrm{cl}(A)$ in $2n$ variables with $\dim V = n$, by definition there is a realization $\bar{a}$ of V generic over C in $\mathbf{F}$. But then

$$\delta(\bar{a}/C) = \dim V - n = 0$$

and thus $\bar{a}$ is in $\mathrm{cl}(A)$. So $\mathrm{cl}(A) \in \mathcal{EC}^*_{\mathrm{st}}$. $\square$

Definition. Given $\mathbf{F} \in \mathcal{EC}^*_{\mathrm{st}}$ and subsets $B, C \subseteq F$, we say that B **is** cl**-independent over** C if $\mathrm{cl}(B'C) \neq \mathrm{cl}(BC)$ for any proper subset $B' \subset B$.

We say that B **is a** cl**-basis of F over** C if B is cl-independent over C and $\mathrm{cl}(BC) = F$. If $C = \emptyset$ then B is just called a cl-basis.

By the properties of cl, any two bases of **F** are of the same cardinality. This cardinality is called the cl**-dimension of F**.

Lemma 5.4. *Suppose that $A \leq \mathbf{F}$ and $B \subseteq_{fin} F$ is* cl*-independent over A. Then $AB \leq \mathbf{F}$.*

Proof. We have by assumption $\partial(B/A) = \delta(B/A)$ and hence $\delta(BD/A) \geq \delta(B/A)$ for any finite $D \subseteq F$. $\square$

Lemma 5.5. *If $\mathbf{F} \in \mathcal{EC}^*_{\mathrm{st}}$, then for every finite $C \subseteq F$ for any $\mathbf{F}' \geq \mathbf{F}$ and finite $A \subseteq \mathrm{cl}_{\mathbf{F}'}(C)$ there is an $A' \subseteq \mathrm{cl}_{\mathbf{F}}(C)$ such that the quantifier-free types of A and A' over C coincide.*

Proof. Extend C to a finite $C' \leq_F \mathbf{F}$. We may replace A by its linear basis over C'; thus we assume w.l.o.g. that A is linearly independent over C'. Since $A \subseteq \mathrm{cl}(C')$, there is a finite extension $B \supseteq A$ in $\mathbf{F}'$, of size n say, such that $\delta(B/C') = 0$ and B is linearly independent over C'. Then the ex-locus V of B over C' is free, ex-normal and $\dim V = n$. By Lemma 4.7 the complete ex-locus is equivalent to its finite part and we may assume that this is just V. By definition it has a generic realization B' in $\mathbf{F}$, and the genericity implies that the quantifier-free type of B' over C' coincides with that of B, $\delta(B'/C') = 0$ and thus $B' \subseteq \mathrm{cl}_{\mathbf{F}}(C)$. $\square$

Notation. We now extend the language L to a language L^* for structures in $\mathcal{EC}^*_{\mathrm{st}}$. Let V be a variety in $2(n+l)$ variables over $\mathbb{Q}$. Given an (ordered) subset X of size l of a field with pseudo-exponentiation, let V_X be the variety obtained by replacing $2l$ of the variables by $X \cup \mathrm{ex}(X)$. Thus V_X is an ex-definable over X variety in $2n$ variables. For any such V we introduce the predicate $E_V(X)$ in variables X saying that

"V_X is ex-irreducible, free over X and there exists a generic over X realization of V_X in $\mathrm{cl}(X)$".

Obviously, this is an $L_{\omega_1,\omega}$-definable expansion of the language; thus the notions of L^*- and L-isomorphisms coincide, which is not necessarily true for monomorphisms, the bijections between subsets preserving the basic relations.

Lemma 5.6. *Given C in* **F** *with $\partial(C) \leq m$ this fact is witnessed by the L^*-quantifier-free type of C in the following sense:*

If C' in $\mathbf{F}'$ satisfies the same L^*-quantifier-free type, then $\partial(C') \leq m$.

Proof. W.l.o.g., we may assume that C is linearly independent. Suppose $\partial(C) = m_0 \leq m$. We have by definition that, for some finite and linearly independent over C finite set $D \subseteq \mathrm{cl}(C)$ of size, say l,

$$\mathrm{tr.d.}(CD\mathrm{ex}(CD)) - (n+l) \leq m_0, \text{ for } n = |C|.$$

In other words, there is an algebraic variety V in F^{2n+2l} irreducible over $\mathbb{Q}$, with $\dim V \leq m+n+l$, and $CD\mathrm{ex}(CD)$ is its generic point. Then

$$\mathbf{F} \models E_V(C) \text{ and } \mathbf{F}' \models E_V(C')$$

and the fact on the right implies

$$\text{tr.d.}(C'D'\mathrm{ex}(C'D')) - (n+l) \leq m_0,$$

for some D' linearly independent over C' (by genericity). Hence $\partial(C') \leq m$. □

It follows, in particular, that the fact that some $B \subseteq \mathrm{cl}(C)$ is witnessed by L^*-quantifier-free formulas as well.

Lemma 5.7. *Suppose $\mathbf{F} \in \mathcal{EC}^*_{\mathrm{st}}$, $C \subseteq \mathbf{F}$ and $C\bar{a} \leq_F \mathbf{F}$. Then the L-quantifier-free type of $\bar{a}$ over C determines the L^*-quantifier-free type of $\bar{a}$ over C; that is, any realization $\bar{b}$ of the L-quantifier-free type of $\bar{a}$, with $C\bar{b} \leq_F \mathbf{F}$, has the same L^*-quantifier-free type over C.*

Proof. We show that if for some V, $\mathbf{F} \models E_V(C\bar{a})$, then $\mathbf{F} \models E_V(C\bar{b})$. Indeed, assuming the first holds, let $\bar{u}$ be a generic realization of $V_{C\bar{a}}$ in $\mathrm{cl}(C\bar{a})$. Since $C\bar{a} \leq_F \mathbf{F}$ we can extend $\bar{u}$ to a $\bar{u}'$ such that $\delta(\bar{u}'/C\bar{a}) = 0$. Without loss of generality we consider the ex-locus of $\bar{u}'$ over $C\bar{a}$ instead of $V_{C\bar{a}}$ and we assume that $\bar{u} = \bar{u}'$. It follows from the assumptions that $V_{C\bar{a}}$ is ex-irreducible, free, ex-normal and also $\dim V_{C\bar{a}} = n$. The same is true for $V_{C\bar{b}}$, as the parameters of the variety are of the same algebraic type. Then $V_{C\bar{b}}$ has a generic realization too. □

Lemma 5.8. *Suppose $\mathbf{F}, \mathbf{F}' \in \mathcal{EC}^*_{\mathrm{st}}$. Then*

$$\mathbf{F} \leq \mathbf{F}' \Leftrightarrow \mathbf{F} \subseteq_{L^*} \mathbf{F}'$$

Proof. $\Rightarrow$. Suppose $\mathbf{F} \leq \mathbf{F}'$ and let $C \subseteq F$ be finite. If $\mathbf{F}' \models E_V(C)$, let $\bar{a}$ be a generic realization of V_C in $\mathrm{cl}_{\mathbf{F}'}(C)$. By Lemma 5.5 there is $\bar{a}'$ in $\mathrm{cl}_{\mathbf{F}}(C)$ realizing V_C generically. Thus $\mathbf{F} \models E_V(C)$. If $\mathbf{F} \models E_V(C)$ then $\mathbf{F}' \models E_V(C)$ follows by definition.
$\Leftarrow$. Suppose $\mathbf{F} \subseteq \mathbf{F}'$ in L^*. Consider $A = \mathrm{cl}_{\mathbf{F}'}(F)$. By Lemma 5.3, $A \leq F'$. We want to show that $F \leq A$, which by transitivity would imply $\mathbf{F} \leq \mathbf{F}'$. So suppose, towards a contradiction, that there is a finite $\bar{a}$ in A such that $\delta(\bar{a}/F) < 0$. We may assume that $\bar{a}$ is linearly independent over F, and let n be the length of the tuple. Then, letting V_X be the ex-locus of $\bar{a}$ over F, some finite $X \leq_F F$, we have that V is free and ex-irreducible over F and

$$\dim V_X - n = \delta(\bar{a}/F) < 0.$$

Then $\mathbf{F}' \models E_V(X)$ and thus $\mathbf{F} \models E_V(X)$. But the latter implies that there is a generic realization $\bar{a}'$ of V_X in $\mathbf{F}$. By definition, $\delta(\bar{a}'/X) = \dim V - n < 0$. This contradicts the fact that $X \leq_F F$. □

Proposition 5.9. *Let $\mathbf{F}_1, \mathbf{F}_2, \mathbf{G} \in \mathcal{EC}^*_{\mathrm{st}}$, $\mathbf{G} \leq \mathbf{F}_1$, $\mathbf{G} \leq \mathbf{F}_2$ and $\mathbf{G}$ be finite-*cl*-dimensional countable or $\mathbf{G} = \emptyset$. Suppose also that $C_1 \subseteq F_1$, $C_2 \subseteq F_2$ are finite subsets and $\varphi : C_1 \to C_2$ is an L^*-monomorphism over $\mathbf{G}$.*

Then:

(i) *For any* $b_1 \in \mathrm{cl}(C_1)$ *there is an* L^*-$\mathbf{G}$*-monomorphism* φ' *extending* φ*, with* $\mathrm{Dom}\,\varphi' \supseteq C_1 \cup \{b_1\}$.

(ii) *If* $\varphi' : C_1 \cup \{b_1\} \to C_2 \cup \{b_2\}$ *is an* L^**-monomorphism extending* φ*, then*

$$b_1 \in \mathrm{cl}(C_1) \text{ iff } b_2 \in \mathrm{cl}(C_2).$$

(iii) *Suppose some* $B_1 \subseteq F_1$, $B_2 \subseteq F_2$ *are* cl*-independent over* GC_1 *and* GC_2*, respectively. Then, given a bijection* $\psi_0 : B_1 \to B_2$*, the mapping* $\varphi \cup \psi_0$ *is an* L^*-$\mathbf{G}$*-monomorphism.*

Proof. Let C_1' be a cl-basis of C_1 over G in $\mathbf{F}_1$, i.e. a cl-independent subset such that $C_1' \subseteq C_1 \subseteq \mathrm{cl}(GC_1')$. We have then $GC_1' \leq F$, by Lemma 5.4.

The image $C_2' = \varphi(C_1')$ is a basis of C_2 in $\mathbf{F}_2$, since cl-dependence is witnessed by basic formulas of L^* (see Lemma 5.6).

Let $C_1 b_1 \subseteq C_1'' \leq_{F_1} \mathrm{cl}(GC_1')$, C_1'' finite. Then

$$0 \leq \delta(C_1''/GC_1') \leq \delta(C_1'/GC_1') = 0;$$

that is, $\delta(C_1''/GC_1') = 0$ and this property of C_1'' is witnessed by $V(C_1)$, the ex-locus of C_1'' over GC_1.

The L^*-quantifier-free formula stating the existence of a generic realization of type $V(C_1)$ guarantees that the corresponding type $V(C_2)$ over GC_2 has a generic realization C_2'' in $\mathbf{F}_2$. It follows that $\delta(C_2''/GC_2') = 0$ and then again $\delta(C_2''D/GC_2') \geq \delta(C_2''/GC_2')$, for every D, since we know already that $GC_2' \leq \mathbf{F}_2$. Thus $GC_2'' \leq_{F_2} \mathbf{F}_2$.

This allows us to extend φ by letting $\varphi'(C_1'') = C_2''$. We have

$$C_1 \subseteq GC_1'' \leq_{F_1} \mathrm{cl}(GC_1) \leq_{F_1} \mathbf{F}_1, \quad C_2 \subseteq GC_2'' \leq_{F_2} \mathrm{cl}(GC_2) \leq_{F_2} \mathbf{F}_2,$$

and φ' preserves all the basic L-formulas. By Lemma 5.7, φ' is an L^*-monomorphism over G.

(ii) Immediate by Lemma 5.6.

(iii) By (i) we may assume that $C_1 \leq_{F_1} \mathbf{F}_1$, $C_2 \leq_{F_2} \mathbf{F}_2$ and φ is an L-monomorphism between C_1 and C_2.

Let

$$C_1^0 = \mathrm{span}_{\mathbb{Q}}(B_1 \cup C_1 \cup \ker) \text{ and } C_2^0 = \mathrm{span}_{\mathbb{Q}}(B_2 \cup C_2 \cup \ker).$$

With a slight abuse of notation we call ψ_0 the mapping which is the extension of the initial $\psi_0 \cup \varphi$ onto the domain C_1^0 by linearity. Any L-quantifier-free formula which holds for a finite subset of C_l^0 ($l = 1, 2$) is a conjunction of polynomial equalities and inequalities in $B_l \cup \mathrm{ex}(B_l)$ with coefficients in $\tilde{C}_l$. By independence, only the inequalities are possible for both values of l. Thus

$$\psi_0 : C_1^0 \to C_2^0$$

is an L-monomorphism.

It is also obvious that $C_l^0 \leq_{F_l} \mathbf{F}_l$, $l = 1, 2$; thus ψ_0 is an L^*-monomorphism. $\square$

Corollary 5.10. *Let* **F** *be a s.e.a.c. structure and* C *a finite subset in* **F**. *Then there are only countably many complete* $L_{\omega_1,\omega}$*-*n*-types over* C *realized in* **F**.

Proof. Extend first C to a finite $\bar{C} \le_F$ **F**.

We claim that the number of complete n-types over $\bar{C}$ realized in **F** is $\aleph_0$, which obviously implies the statement of the corollary.

Let $\bar{b}$ be a finite n-tuple in F such that $\bar{C}\bar{b} \le_F F$. By Proposition 5.9 and Lemma 5.7 the automorphism type of $\bar{b}$ over $\bar{C}$ is determined by its L-quantifier-free type. On the other hand, by definition there is an L-monomorphism between two such tuples if and only if the complete ex-loci of the tuples over $\bar{C}$ coincide. By Theorem 4.1 the complete ex-loci are principal types over $\bar{C}$; thus there are only countably many n-types over $\bar{C}$. □

Now we discuss the countable closure property in the class of s.e.a.c. structures.

Lemma 5.11. *For* **F** *exponentially–algebraically closed,* **F** *has the countable closure property iff* $\mathrm{cl}_{\mathbf{F}}(C)$ *is countable for any finite* $C \subseteq F$.

Proof. The right-to-left statement is obvious. We prove the statement on the right assuming that the countable closure property holds.

By Proposition 4.11(v) we may assume that $C \le_F$ **F**. Then $a \in \mathrm{cl}(C)$ iff $\partial(a/C) = 0$ iff there is a tuple $\bar{a}$ extending a such that $\delta(\bar{a}/C) = 0$. We may assume that $\bar{a}$ is linearly independent over C. Then the ex-locus V of $\bar{a}$ over C is of dimension equal to the length of $\bar{a}$, ex-definable, ex-irreducible, ex-normal and free over C. Thus by the assumption of the lemma there are at most countably many choices for such an $\bar{a}$. It follows that $\mathrm{cl}(C)$ is countable. □

Remark. With the use of the quantifier Qx expressing the fact that 'there are uncountably many x such that ...', one can write down the obvious $L_{\omega_1,\omega}(Q)$-sentence, call it $\mathrm{EC}^*_{\mathrm{st,ccp}}$, such that

$$\mathbf{F} \models \mathrm{EC}^*_{\mathrm{st,ccp}} \text{ iff } \mathbf{F} \in \mathcal{EC}^*_{\mathrm{st}} \text{ and has the c.c.p.}$$

Lemma 5.12. $\mathbb{C}_{\exp}$ *has the countable closure property.*

Proof. We need to show that for any finite $B \subseteq \mathbb{C}$ and an algebraic variety $V \subseteq F^{2n}$ of dimension n, which is ex-definable, ex-irreducible, ex-normal and free over B, the set of the of generic realization of V over B is at most countable. Let $\bar{a}$ be a generic realization of V. We claim that $\bar{a}$ is an isolated point in the analytic set

$$S = \{\bar{x} \in \mathbb{C}^n : \bar{x}^\frown \exp(\bar{x}) \in V\}.$$

Suppose not. Then there is a non-constant C^∞-mapping from the real unit interval into a neighbourhood of $\bar{a}$ in S:

$$t \in [0,1] \mapsto \bar{x}(t) \in S, \quad \bar{x}(0) = \bar{a};$$

we denote the mapping as $x_i(t)$ coordinate-wise. Let $y_i(t) = \exp(x_i(t))$. Then the $x_i(t)$ and $y_i(t)$ can be considered as elements of a differential field of germs of functions differentiable near 0, with the differentiation operator $\mathrm{D}f = \frac{\mathrm{d}f}{\mathrm{d}t}$. By definition,

$$\mathrm{D}y_i = y_i \mathrm{D}x_i, \text{ all } i. \tag{12}$$

Also, $\bar{x}(t)^\frown\bar{y}(t) \in V$ and, by the assumptions on V and $\bar{a}$,

$$\text{tr.d.}(\bar{x}(t)^\frown\bar{y}(t)/B) = n. \tag{13}$$

Consider additive dependencies between the differentials $\mathrm{d}x_1(t), \ldots, \mathrm{d}x_n(t)$. If there are some, then by a suitable $\mathbb{Q}$-linear transformation of variables we may assume that $\mathrm{d}x_1(t) \equiv 0, \ldots, \mathrm{d}x_k(t) \equiv 0$ and $\mathrm{d}x_{k+1}(t), \ldots, \mathrm{d}x_n(t)$ are additively independent (in the case where there are no dependencies, $k = 0$). It follows that $x_1(t) \equiv a_1, \ldots, x_k(t) \equiv a_k$ and $y_1(t) \equiv \exp(a_1), \ldots, y_k(t) \equiv \exp(a_k)$. By the ex-normality of V and the fact that $\bar{a}^\frown \exp(\bar{a})$ is generic in V,

$$\text{tr.d.}(\{a_1, \ldots, a_k, \exp(a_1), \ldots, \exp(a_k)\}/B) \geq k.$$

Hence

$$\begin{aligned}
&\text{tr.d.}(\{x_{k+1}(t), \ldots, x_n(t), y_{k+1}(t), \ldots, y_n(t)\}/\mathbb{C})\\
&\quad \leq \text{tr.d.}(\bar{x}(t)^\frown\bar{y}(t)/B \cup \{a_1, \ldots, a_k, \exp(a_1), \ldots, \exp(a_k)\})\\
&\quad = \text{tr.d.}(\bar{x}(t)^\frown\bar{y}(t)/B) - \text{tr.d.}(\{a_1, \ldots, a_k, \exp(a_1), \ldots, \exp(a_k)\}/B)\\
&\quad \leq n - k.
\end{aligned} \tag{14}$$

By the theorem of J. Ax [1], under (12) and (14) $\mathrm{D}x_{k+1}(t), \ldots, \mathrm{D}x_n(t)$ must be additively dependent. The contradiction proves the claim and the lemma. □

Theorem 5.13. *Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be s.e.a.c. structures of infinite* cl*-dimensions. Then the two structures are $L_{\omega_1,\omega}$-equivalent; moreover,*

$$\mathbf{F}_1 \subseteq_{L^*} \mathbf{F}_2 \textit{ iff } \mathbf{F}_1 \leq \mathbf{F}_2 \textit{ iff } \mathbf{F}_1 \preccurlyeq_{L_{\omega_1,\omega}} \mathbf{F}_2,$$

and any $L_{\omega_1,\omega}$-definable subset of F_1 is quantifier-free definable in $L^*_{\omega_1,\omega}$.

Proof. The first 'iff' is Lemma 5.8. It follows that, given a finite $C \subseteq F_1$, the identity embedding $C \subseteq F_2$ is an L^*-monomorphism. Suppose now $C \subseteq C_1 \subseteq F_1$, $C \subseteq C_2 \subseteq F_2$ finite, $\varphi : C_1 \to C_2$ is an L^*-monomorphism fixing C, and $b_i \in F_i$ for $i = 1$ or $i = 2$. By the symmetry of what follows we may assume w.l.o.g. that $i = 1$.

If $b_1 \in \text{cl}(C_1)$, then by Proposition 5.9 we can extend the φ to $C_1 \cup \{b_1\}$. If $b_1 \notin \text{cl}(C_1)$ then, using the fact that $\mathbf{F}_2$ is infinite dimensional, choose $b_2 \in F_2 \setminus \text{cl}(C_2)$. By Proposition 5.9(iii) we can put $\varphi(b_1) = b_2$ and again extend φ to $C_1 \cup \{b_1\}$. Thus by the Ehrenfeucht–Fraisse criterion we obtain that $\mathbf{F}_1$ and $\mathbf{F}_2$ are $L_{\omega_1,\omega}$-equivalent over C, for any finite $C \subseteq F_1$. This by definition means that

$$\mathbf{F}_1 \preccurlyeq_{L_{\omega_1,\omega}} \mathbf{F}_2.$$

On the other hand, the same argument shows that once there is an L^*-monomorphism between $C_1 \subseteq F_1$ and $C_2 \subseteq F_1$, we can play the Ehrenfeucht–Fraisse game extending the monomorphism any finite number of steps, which implies that

$$C_1 \equiv_{L_{\omega_1,\omega}} C_2.$$

The latter means that the type of an n-tuple is equivalent to a quantifier-free type (formula) in $L^*_{\omega_1,\omega}$. But there are only countably many complete $L_{\omega_1,\omega}$-n-formulas realized in $\mathbf{F}_1$, by the Lemma 5.10.

Since an $L_{\omega_1,\omega}$-definable set in $\mathbf{F}_1$ is the union of the subsets definable by complete $L_{\omega_1,\omega}$-formulas, there is a quantifier-free $L^*_{\omega_1,\omega}$-formula defining the set. □

Lemma 5.14. *Suppose A is a* cl-*independent subset of* $\mathbf{F}$, $A_1, \dots A_k \subseteq A$ *and*

$$C = \bigcup_{1\le i\le k} \mathrm{cl}(A_i). \tag{15}$$

Then $C \le_F \mathbf{F}$.

Proof. Notice first that $A \le_F F$, by definition. We may assume that A is finite and $A = \bigcup_i A_i$.

Now, let $c_1 \in C$, that is $c_1 \in \mathrm{cl}(A_i)$, for some i. By definition, there is a finite X_1 such that $\delta(X_1c_1/A_i) = 0$. It follows that $\delta(X_1c_1/A) = 0$ and $X_1 \subseteq \mathrm{cl}(A_i) \subseteq C$.

Applying this observation we get for any finite $\{c_1, \dots, c_m\} \subseteq C$ finite $X_1, \dots, X_m \subseteq C$ such that $\delta(X_i c_i/A) = 0$ for each $i \in \{1, \dots, m\}$ and hence $\delta(X_1 \cup \cdots \cup X_m \cup \{c_1, \dots, c_m\}/A) = 0$.

It follows that

$$A \cup X_1 \cup \cdots \cup X_m \cup \{c_1, \dots, c_m\} \le_F F;$$

thus for any finite subset $C' \subseteq C$ there is a $C'' \le_F F$ such that $C' \subseteq C'' \subseteq C$. This immediately implies that $C \le_F F$. □

Proposition 5.15. *Given* $\mathbf{F} \in \mathcal{EC}^*_{\mathrm{st}}$, ker $\le_F C \le_F \mathbf{F}$, *C finitary and a finite $A \subseteq \mathrm{cl}(C)$, there is a finite subset $C_0 \subseteq C$ such that the complete $L_{\omega_1,\omega}$-type $tp(A/C)$ is isolated by the type $tp(A/C_0)$.*

In other words any $\varphi : A \to \mathbf{F}' \in \mathcal{EC}^*_{\mathrm{st}}$ which is an L^*-monomorphism over C_0 is also a monomorphism over C.

In particular, the statement holds for C satisfying (15) *above.*

Proof. Replacing if needed A by its linear basis over C we assume that A is linearly independent over C. Since $A \subseteq \mathrm{cl}(C)$ there is a finite B, linearly independent over CA, such that $\delta(AB/C) = 0$.

Choose $C_0 \le_F C$ finite with the property $\delta(AB/C_0) = 0$ and containing a generator of the cyclic group ker, the kernel of ex. We also may assume, by Lemma 4.7, that the ex-locus of AB over C is determined uniquely by an irreducible variety V_0 which is ex-defined over a finite C_0 and contains the tuple $AB\mathrm{ex}A\mathrm{ex}B$ as a generic element.

We claim that this C_0 satisfies the requirements.

Indeed, let $\varphi : A \to \mathbf{F}'$ be an L^*-monomorphism over C_0.

The $L_{\omega_1,\omega}$-formula over C_0 stating that "$A' = \varphi(A)$ can be extended by a B' so that $A'B'\mathrm{ex}A'\mathrm{ex}B'$ satisfies V_0 generically over C_0" holds in $\mathbf{F}'$, by Theorem 5.13. This implies $\delta(A'B'/C_0) = 0 = \delta(A'B'/C)$, and thus $CA'B' \le_F \mathbf{F}'$. It follows also that any ex-definable V' over C satisfied by $A'B'\mathrm{ex}A'\mathrm{ex}B'$ must contain V_0, because otherwise $\delta(A'B'/C) < 0$. In other words, we have proved that AB and $A'B'$ have the same L-quantifier-free types over C.

Suppose now that $E_V(\bar{c})$ is a predicate in the language L^* over $\bar{c} \in CA$ satisfied by Y, for some $Y \subseteq \mathrm{cl}_{\mathbf{F}}(CA) = \mathrm{cl}_{\mathbf{F}}(CAB)$. We can without loss of generality assume that Y is linearly independent over CAB. By extending Y we can also assume that $\delta(Y/CAB) = 0$. Let V^* be the ex-locus of Y over CAB and suppose V^* is ex-defined over C_1AB for some finite $C_1 \subseteq C$ such that $C_1AB \leq_F \mathbf{F}$.

By Lemma 3.1, V^* is free and ex-normal.

Let V' be the variety over $C_1A'B'$ obtained from V^* by replacing AB by $A'B'$. Since the property of being free and ex-normal is L-quantifier-free definable, by the above proof V' is free and ex-normal, and for the same reason $\dim V' - n = 0$, for n equal to the number of x-variables in V' or, equivalently, the number of elements in Y.

By Lemma 4.8 and the fact that $\mathbf{F}' \in \mathcal{EC}^*_{\text{st}}$, there is a generic realization Y' of V' in $\mathbf{F}'$. By construction, Y' witnesses the validity of $E_V(\varphi(\bar{c}))$ in $\mathbf{F}'$. This finally proves that φ preserves quantifier-free L^*-formulas and thus, by Theorem 5.13, all $L_{\omega_1,\omega}$-formulas. □

The combined meaning of Propositions 4.11, 5.9 and 5.15 is that $\mathcal{EC}^*_{\text{st}}$ is quasi-minimal excellent. By Proposition 5.2 there is an infinite-dimensional member of this class of cardinality $\aleph_0$, hence with the countable closure property. Thus, we get:

Theorem 5.16 (*Categoricity Theorem*). *For any uncountable cardinal κ there is a unique, up to isomorphism, structure $\mathbf{F} \in \mathcal{EC}^*_{\text{st,ccp}}$ of cardinality κ.*

Moreover, $\mathbf{F}$ is prime over any basis.

In other words the $L_{\omega_1,\omega}(Q)$-sentence $\mathrm{EC}^*_{\text{st,ccp}}$ is categorical in all uncountable cardinalities.

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