$1 + 1$ dimensional cobordism categories and invertible TQFT for Klein surfaces

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A thesis submitted for the degree of

Master of Science by Research

Trinity 2012

For Mum and Dad.

And for Richard, my husband to be.

Abstract

We discuss a method of classifying 2-dimensional invertible topological quantum field theories (TQFTs) whose domain surface categories allow non-orientable cobordisms. These are known as Klein TQFTs. To this end we study the $1+1$ dimensional open-closed unoriented cobordism category $\mathcal{K}$, whose objects are compact 1-manifolds and whose morphisms are compact (not necessarily orientable) cobordisms up to homeomorphism. We are able to compute the fundamental group of its classifying space $\mathrm{B}\mathcal{K}$ and, by way of this result, derive an infinite loop splitting of $\mathrm{B}\mathcal{K}$, a classification of functors $\mathcal{K} \to \mathbb{Z}$, and a classification of 2-dimensional open-closed invertible Klein TQFTs. Analogous results are obtained for the two subcategories of $\mathcal{K}$ whose objects are closed or have boundary respectively, including classifications of both closed and open invertible Klein TQFTs. The results obtained throughout the paper are generalisations of previous results by Tillmann [Til96] and Douglas [Dou00] regarding the $1+1$ dimensional closed and open-closed oriented cobordism categories. Finally we consider how our results should be interpreted in terms of the known classification of 2-dimensional TQFTs in terms of Frobenius algebras.

Acknowledgements

I would like to thank my MSc (Res) supervisors, Prof. Ulrike Tillmann and Dr. Chris Douglas, for their invaluable ideas and support throughout my stay in Oxford. I would also like to thank my parents, John and Linda, for their encouragement. Finally I would like to thank Richard Manthorpe for many helpful discussions, and Martin Palmer for his critical comments and suggestions.

Contents

1 Introduction

1.1 Background

The mid 1980s saw a shift in the nature of the relationship between mathematics and physics. Differential equations and geometry applied in a classical setting were no longer the principal players; in the quantum world topology and algebra began to move to the fore. This was a result of extracting the topological aspects of a quantum field theory, largely attributed in its infancy to Witten, for example in [Wit82]. The move to a formal mathematical theory came in 1989 with Atiyah's axiomatisation of a topological quantum field theory (TQFT) [Ati89]. These ideas have since shown themselves to have applications to both topology (low dimensional manifolds) and physics (string theory) alike.

Roughly speaking, a closed 2-dimensional TQFT is a functor from a category whose objects are closed oriented 1-manifolds (disjoint unions of circles), and whose morphisms are homeomorphism classes of oriented surfaces, to the category of complex vector spaces. It is well known that closed 2-dimensional TQFTs are in one-to-one correspondence with commutative Frobenius algebras. Explicitly, the image of the circle S^1 under such a theory is an algebra of this form and, conversely, for any such algebra A there exists a theory F with $F(S^1) = A$. Proofs of this folk theorem have been given by Abrams [Abr96] and Kock [Koc04].

Atiyah's axiomatisation can be extended in two ways. Firstly, we can drop the assumption that all objects (1-manifolds) in the category are closed, requiring them only to be compact. This leads to the notion of *open-closed* and *open* TQFTs, where for the latter we insist that the objects have boundary. Theories of this form have also been classified. Moore and Segal [MS06] showed that open 2-dimensional TQFTs are equivalent to symmetric Frobenius algebras. In the open-closed case there is a one-to-one correspondence with “knowledgeable Frobenius algebras” (see Lauda and Pfeiffer [LP08] for a definition and proof).

The second way in which we can extend the axiomatisation is by allowing non-orientable surfaces in the category. Field theories of this form are known as *Klein* TQFTs. Turaev and Turner [TT06] proved that closed 2-dimensional Klein TQFTs correspond to finite dimensional commutative Frobenius algebras with additional structure coming from the non-orientable generators of the surface category. Open Klein TQFTs are equivalent to symmetric Frobenius algebras with some extra structure, as shown by Braun [Bra11]. Finally, open-closed Klein theories correspond to “structure algebras” (see Alexeevski and Natanzon [AN06] for a definition and proof).

Here we are interested in invertible 2-dimensional TQFTs, that is to say TQFTs for which the defining functor F is morphism inverting. These play an important role in ordinary quantum field theory, for example as anomalies, and over recent years have proven crucial in the understanding of the subject. For orientable surfaces invertible TQFTs have been classified, in the closed case by Tillmann [Til96] and in the open-closed case by Douglas [Dou00]. Our aim is to extend these results to the corresponding non-orientable cases, and also to give a classification in the open case which has not previously been considered.

Our interest in TQFT lies not only with the functors themselves, but also with the surface categories involved. By studying their classifying spaces one can gain an understanding of the underlying combinatorics of these categories. Tillmann [Til96] and Douglas [Dou00] computed the fundamental groups of the classifying spaces of the oriented closed and open-closed categories respectively. Here we adapt their methods in order to analyse the classifying spaces of the non-orientable cobordism categories. We are able to compute the fundamental group in all three cases, obtaining the desired classification of invertible field theories as a corollary to these results.

It is worth mentioning that the categorical problem at hand is the discrete version of an extended problem, in which one includes diffeomorphism of surfaces in the categorical data. In their recent paper [GTMW09] Galatius, Madsen, Tillmann and Weiss showed that there is a weak equivalence between the classifying space of the extended d-dimensional cobordism category $\mathcal{C}_d$ and the infinite loop space of a certain Thom spectrum. A comprehensive survey of the discrete and extended cobordism categories and the TQFTs they define, along with various examples and further applications, is given by Freed [Fre12].

1.2 List of categories

We provide a list of the main categories of interest (see Table 1) in the hope that this may serve as a reference point for the reader. Formal definitions will be given in Section 2.

Category	**Description**
The open-closed category $\mathcal{K}$	Objects are unoriented compact 1-manifolds (disjoint unions of circles and intervals). A morphism $\Sigma : M_0 \to M_1$ is a (not necessarily orientable) compact cobordism from M_0 to M_1 up to homeomorphism. We call $\partial\Sigma \smallsetminus (M_0 \cup M_1)$ the *free* boundary of Σ. Morphisms are composed by glueing along their common boundary.
The closed object category $\bar{\mathcal{C}}$	The full subcategory of $\mathcal{K}$ on those objects which are closed (disjoint unions of circles).
The closed category $\mathcal{C}$	The subcategory of $\mathcal{K}$ whose objects are disjoint unions of circles, and whose morphisms have no free boundary.
The open object category $\bar{\mathcal{O}}$	The full subcategory of $\mathcal{K}$ on disjoint unions of intervals.
The open category $\mathcal{O}$	The subcategory of $\mathcal{K}$ whose objects are disjoint unions of intervals, and whose morphisms have no closed components.

Table 1: The main categories.

The relationship between the five main categories is as follows.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] & \bar{\mathcal{C}} \arrow[dr, hook] & \\
 & & \mathcal{K} \\
\mathcal{O} \arrow[r, hook] & \bar{\mathcal{O}} \arrow[ur, hook] &
\end{tikzcd}$$

We will also be studying the subcategories of $\mathcal{C}$ listed in Table 2.

Subcategory	**Description**
$\mathcal{C}_0$	The full subcategory on one object, the empty 1-manifold $\emptyset$.
$\mathcal{C}_b$	Contains all objects of $\mathcal{C}$ and those morphisms no connected component of which is a cobordism to $\emptyset$. Known as the *birth category.*
$\mathcal{C}_{>0}$	Contains all objects of $\mathcal{C}$ except $\emptyset$, and those morphisms no component of which is a cobordism to or from $\emptyset$.
$\mathcal{C}_1$	The full subcategory of $\mathcal{C}_b$ on the object S^1.
$\mathcal{C}_1^+$	The subcategory of $\mathcal{C}_1$ whose morphisms are orientable.
$\mathcal{C}_1^-$	The subcategory of $\mathcal{C}_1$ whose morphisms are non-orientable.

Table 2: Subcategories of $\mathcal{C}$.

The relationship between these subcategories is summarised below.

$$\begin{tikzcd}
 & & & \mathcal{C}_0 \arrow[dr, hook] & \\
\mathcal{C}_1^+ \arrow[dr, hook] & & & & \mathcal{C} \\
 & \mathcal{C}_1 \arrow[r, hook] & \mathcal{C}_{>0} \arrow[r, hook] & \mathcal{C}_b \arrow[ur, hook] & \\
\mathcal{C}_1^- \arrow[ur, hook] & & & &
\end{tikzcd}$$

1.3 Outline and statement of the main results

The layout of the paper is as follows. Section 2 is devoted to formally defining the main categories, including a description in terms of generators. We also briefly recall some useful facts regarding classifying spaces of categories, and two important functors will be defined.

In Section 3 we study the closed category $\mathcal{C}$, beginning with an analysis of the subcategories listed in Table 2. We are able to give a complete description of the classifying space of each, with the results obtained being highly useful throughout the course of the paper. We then consider the classifying space $\mathrm{B}\mathcal{C}$ of the whole category $\mathcal{C}$. The main result of the section is that the fundamental group $\pi_1(\mathrm{B}\mathcal{C})$ is isomorphic to $\mathbb{Z}$. We derive various consequences of this result, including an infinite loop splitting of $\mathrm{B}\mathcal{C}$, a classification of functors $\mathcal{C} \to \mathbb{Z}$, and a classification of invertible closed 2-dimensional Klein TQFTs up to isomorphism.

Main Theorem. $\pi_1(\mathrm{B}\mathcal{C}) = \mathbb{Z}$.

Corollary. There is a simply connected infinite loop space X such that $\mathrm{B}\mathcal{C} \simeq X \times S^1$.

Corollary. The category $\mathrm{InvTFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$ whose objects are invertible closed 2-dimensional Klein TQFTs and whose morphisms are certain natural transformations is equivalent to the discrete category $\mathbb{C}^*$ of non-zero complex numbers.

We conclude the section by describing the Frobenius algebras which correspond to these invertible field theories.

In Section 4 we consider the open case. The category $\bar{\mathcal{O}}$ is more complicated than $\mathcal{C}$, and so we do not compute the fundamental group of its classifying space directly. Instead we show that there is a canonical splitting $\mathrm{B}\bar{\mathcal{O}} \simeq \mathrm{B}\mathcal{O} \times \mathrm{B}\mathcal{C}_0$, thereby obtaining results for $\bar{\mathcal{O}}$ as corollaries to those we derive for $\mathcal{O}$.

Main Theorem.

(i) $\pi_1(\mathrm{B}\mathcal{O}) = \mathbb{Z}$.

(ii) $\pi_1(\mathrm{B}\bar{\mathcal{O}})$ is free abelian of infinite rank.

Corollary.

(i) There is a simply connected infinite loop space Y such that $\mathrm{B}\mathcal{O} \simeq Y \times S^1$.

(ii) $\mathrm{B}\bar{\mathcal{O}} \simeq Y \times T^\infty$.

Corollary. The category $\mathrm{InvTFT}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_*$ whose objects are invertible open 2-dimensional Klein TQFTs and whose morphisms are certain natural transformations is equivalent to the discrete category $\mathrm{Seq}(\mathbb{C}^*)$ of infinite sequences of non-zero complex numbers.

Again we conclude with a description of the corresponding Frobenius algebras.

Section 5 is concerned with the most general case: the open-closed category $\mathcal{K}$. We begin by showing that the fundamental group of the classifying space of the subcategory $\bar{\mathcal{C}}$ is isomorphic to $\mathbb{Z}\times\mathbb{Z}$. Using this result, along with those from Sections 3 and 4, we are able to compute the fundmental group $\pi_1(\mathrm{B}\mathcal{K})$ with very little extra effort. Analogous to the closed and open cases, we consequently derive an infinite loop splitting of $\mathrm{B}\mathcal{K}$, a classification of functors $\mathcal{K} \to \mathbb{Z}$, and a classification of invertible open-closed 2-dimensional Klein TQFTs up to isomorphism. We conclude with a description of the structure algebras which correspond to the these field theories.

Main Theorem. $\pi_1(\mathrm{B}\mathcal{K}) = \mathbb{Z}$.

Corollary. There is a simply connected infinite loop space Z such that $\mathrm{B}\mathcal{K} \simeq Z \times S^1$.

Corollary. The category $\mathrm{InvTFT}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_*$ whose objects are invertible open-closed 2-dimensional Klein TQFTs and whose morphisms are certain natural transformations is equivalent to the discrete category $\mathbb{C}^*$ of non-zero complex numbers.

Finally, the appendix contains some general categorical notions and results which will be essential for our classification of invertible Klein TQFTs in all three cases.

Remark. Tillmann [Til96] carried out the above computations and classifications for the closed oriented category $\mathscr{S}$, showing in particular that $\pi_1(\mathrm{B}\mathscr{S}) = \mathbb{Z}$. Initially it may seem surprising that $\pi_1(\mathrm{B}\mathcal{C})$ and $\pi_1(\mathrm{B}\mathscr{S})$ are isomorphic; in particular the absence of 2-torsion in the non-orientable case is somewhat unexpected. However, we will see that the former group is in fact isomorphic to a double cover of the latter, and that this arises in a very natural way via the interaction between the crosscaps and handles of a surface. On the other hand, the free boundary components of a cobordism do not interact in any natural way with the other features of its underlying manifold. It is therefore rather surprising that $\pi_1(\mathrm{B}\mathcal{K}) = \mathbb{Z}$. Douglas [Dou00] showed that the fundamental group of the open-closed oriented category $\mathcal{S}$ is also isomorphic to $\mathbb{Z}$, which in the same way is a surprising result.

2 Definitions

The open-closed category $\mathcal{K}$

The objects of $\mathcal{K}$ are compact unoriented 1-manifolds. Morphisms are (not necessarily orientable) cobordisms up to homeomorphism relative to the boundary. Explicitly, a morphism $(\Sigma, \sigma_0, \sigma_1) : M_0 \to M_1$ in $\mathcal{K}$ is a compact 2-manifold Σ with maps $\sigma_0 : M_0 \to \partial\Sigma$ and $\sigma_1 : M_1 \to \partial\Sigma$ which are homeomorphisms onto their images and satisfy $\sigma_0(M_0) \cap \sigma_1(M_1) = \emptyset$. The morphism $(\Sigma, \sigma_0, \sigma_1)$ will often be abbreviated as Σ. We call $\sigma_0(M_0)$, $\sigma_1(M_1)$ and $\partial\Sigma \smallsetminus (\sigma_0(M_0) \cup \sigma_1(M_1))$ the *source boundary*, the *target boundary* and the *free boundary* respectively. We say two cobordisms $\Sigma, \Sigma' : M_0 \to M_1$ are homeomorphic relative to the boundary (and therefore define the same morphism in $\mathcal{K}$) if there exists a homeomorphism $\Psi : \Sigma \to \Sigma'$ such that the following diagram commutes.

$$\begin{tikzcd} & M_0 \arrow[dl, bend right, "\sigma_0"'] \arrow[dr, bend left, "\sigma_0'"] & \\ \Sigma \arrow[rr, "\Psi"] & & \Sigma' \\ & M_1 \arrow[ul, bend left, "\sigma_1"] \arrow[ur, bend right, "\sigma_1'"'] & \end{tikzcd}$$

Composition of $\Sigma : M_0 \to M_1$ and $\Sigma' : M_1 \to M_2$ is given by glueing the two cobordisms via $\sigma_0' \circ \sigma_1^{-1}$, and has boundary maps σ_0 and σ_1'. The identity morphism from M_0 to itself is given by the cylinder $M_0 \times I$. We will show that $\mathcal{K}$ is a symmetric strict monoidal category under disjoint union of manifolds.

It is convenient to identify $\mathcal{K}$ with a skeleton. Let M be a 1-manifold in $\mathcal{K}$ with m closed components and n components with boundary. For each closed component M_i ($1 \leq i \leq m$) we choose a homeomorphism $f_i : M_i \to S^1$, where S^1 denotes the standard (unoriented) circle. Similarly for each component N_i ($1 \leq i \leq n$) with boundary we choose a homeomorphism $g_i : N_i \to I$, where $I = [0,1]$ denotes the standard (unoriented) interval. Denote the mapping cylinders of f_i and g_i by C_{f_i} and C_{g_i} respectively. Then the union of mapping cylinders $(\coprod_{i=1}^{m} C_{f_i}) \amalg (\coprod_{i=1}^{n} C_{g_i})$ defines an isomorphism $M \to (\coprod_{i=1}^{m} S^1) \amalg (\coprod_{i=1}^{n} I)$ in $\mathcal{K}$. Therefore, the full subcategory on disjoint unions of copies of a single unoriented circle S^1, and a single unoriented interval I, is a skeleton for $\mathcal{K}$. We identify these objects with the product $\mathbb{N} \times \mathbb{N}$, where $(0,0)$ represents the empty 1-manifold and (m,n) represents m ordered copies of S^1 and n ordered copies of I, and we shall often refer to an object of $\mathcal{K}$ in this manner.

For each morphism in $\mathcal{K}$, we choose a representative cobordism whose source and target boundary components are identified with copies of S^1 and I, and whose boundary maps are either inclusions or reflections on each component of an object (m,n). Note that the boundary maps are essential in determining the homeomorphism class of a cobordism in $\mathcal{K}$. For example, consider the cylinder as a morphism $S^1 \to S^1$. If both boundary maps are defined to be inclusions, then this is just the identity $S^1 \times I$. However, if we define one boundary map to be an inclusion, and the other to be a reflection, we obtain an entirely different morphism. Although the underlying manifolds of the two cobordisms are homeomorphic, we cannot find a homeomorphism which commutes with their boundary maps. The latter cobordism can be thought of as the mapping cylinder of the reflection $r : S^1 \to S^1$ and, with this in mind, we denote it by $C_r^{S^1}$. Similarly, the disc as a morphism $I \to I$ comes in two guises; the identity, and the mapping cylinder of the reflection $r : I \to I$. We shall often refer to the latter as the *bow tie*, denoting it by C_r^I.

There is an orientable symmetric strict monoidal subcategory $\mathcal{S}$ of $\mathcal{K}$ consisting of oriented 1-manifolds and oriented cobordisms whose boundary maps are orientation preserving.[1] A generators and relations description of $\mathcal{S}$ has been given by Lauda and Pfeiffer [LP08]. The fourteen cobordisms in Figure 2.1 form a generating set for $\mathcal{S}$ under the operations of composition and disjoint union. In the figure, source boundary components are on the left, and target boundaries are on the right. For those cobordisms with free boundary, the thickened lines are intended to distinguish source and target boundary components from free components. For example, the bottom right hand cobordism is a morphism from S^1 to I.

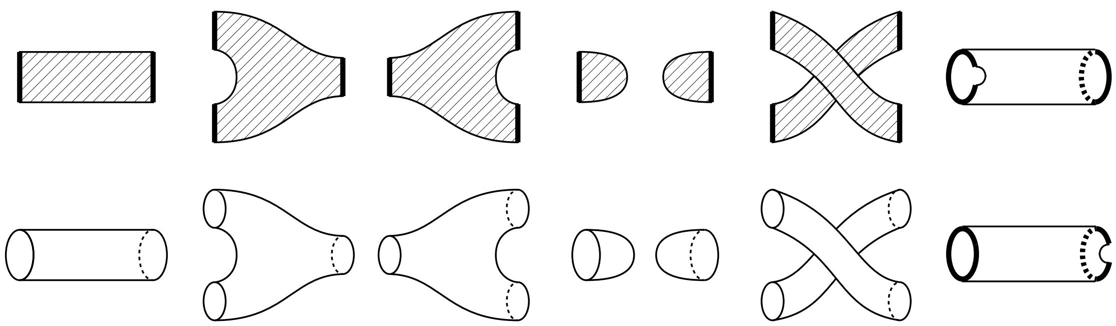

Figure 2.1: Generators of $\mathcal{S}$.

By moving crosscaps and taking out factors of $C_r^{S^1}$ and C_r^I, we can decompose any cobordism in $\mathcal{K}$ as an element of $\mathcal{S}$ composed with a combination of the four additional cobordisms in Figure 2.2. In the figure, a dotted circle surrounding a cross represents a crosscap attached to a surface. From left to right, the cobordisms are: the twice punctured projective plane, the Möbius strip, the cylinder $C_r^{S^1}$, and the disc C_r^I.

[1]We identify $\mathcal{S}$ with its skeleton, the full subcategory on unions of copies of one fixed oriented circle and one fixed oriented interval. In this way we view $\mathcal{S}$ as a subcategory of $\mathcal{K}$ by forgetting orientations.

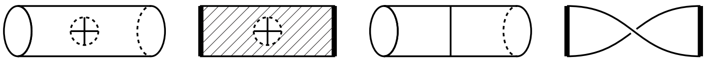

Figure 2.2: Additional generators of $\mathcal{K}$.

It follows that the eighteen cobordisms of Figures 2.1 and 2.2 form a generating set for $\mathcal{K}$ under composition and disjoint union. An example of the decomposition of a general morphism $(\Sigma, \sigma_0, \sigma_1) : (2, 0) \to (2, 1)$ in $\mathcal{K}$ into generators is shown in Figure 2.3. The manifold Σ has one handle, one crosscap, and two free boundary components. Of the free components, one is an interval, and the other is a circle. We often refer to a free boundary circle as a *window*. The boundary maps σ_0 and σ_1 are $i \amalg i : S^1 \amalg S^1 \to \partial\Sigma$ and $r \amalg i \amalg i : S^1 \amalg S^1 \amalg I \to \partial\Sigma$ respectively, where i denotes the inclusion and r denotes the reflection.

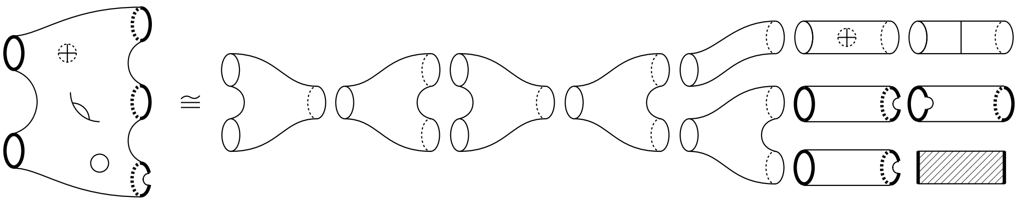

Figure 2.3: Decomposition of a general morphism in $\mathcal{K}$ into generators.

We denote the classifying space of $\mathcal{K}$ by $\mathrm{B}\mathcal{K}$, recalling that the classifying space $\mathrm{B}\mathscr{C}$ of any category $\mathscr{C}$ is by definition the geometric realisation of its nerve. A functor $\mathrm{F} : \mathscr{C} \to \mathscr{C}'$ induces a map $\mathrm{B}\mathscr{C} \to \mathrm{B}\mathscr{C}'$. When the meaning is clear we will simply denote this map by F, otherwise we use BF. We briefly recall the following two useful facts regarding classifying spaces. Firstly, for product categories we have $\mathrm{B}(\mathscr{C} \times \mathscr{C}') \simeq \mathrm{B}\mathscr{C} \times \mathrm{B}\mathscr{C}'$. Secondly, a natural transformation between two functors $\mathrm{F}_0, \mathrm{F}_1 : \mathscr{C} \to \mathscr{C}'$ induces a homotopy $\mathrm{BF}_0 \to \mathrm{BF}_1$ [Seg68]. It follows that a pair of adjoint functors $\mathrm{F}_0 : \mathscr{C} \to \mathscr{C}'$ and $\mathrm{F}_1 : \mathscr{C}' \to \mathscr{C}$ induce a homotopy equivalence $\mathrm{B}\mathscr{C} \simeq \mathrm{B}\mathscr{C}'$. In particular, equivalent categories have homotopy equivalent classifying spaces, which justifies our identification of $\mathcal{K}$ with its skeleton.

The closed category $\mathcal{C}$

We define the closed category $\mathcal{C}$ to be the following subcategory of $\mathcal{K}$. The objects of $\mathcal{C}$ are all closed 1-manifolds in $\mathcal{K}$, and morphisms $\Sigma : M_0 \to M_1$ must have boundary maps satisfying the additional condition $\sigma_0(M_0) \cup \sigma_1(M_1) = \partial\Sigma$. In other words, we do not allow cobordisms in $\mathcal{C}$ to have any free boundary. We will show that $\mathcal{C}$ is also symmetric strict monoidal under disjoint union. Once again we identify $\mathcal{C}$ with its skeleton, the full subcategory on disjoint unions of copies of a single unoriented S^1. We identify these objects with the natural numbers $\mathbb{N}$, where 0 represents the empty 1-manifold and n represents the union of n ordered circles. The classifying space of $\mathcal{C}$ will be denoted by $\mathrm{B}\mathcal{C}$.

It is well known that any connected closed surface is homeomorphic to one and only one of the following: the sphere S^2, a connect sum of g tori for $g \geq 1$, or a connect sum of k real projective planes for $k \geq 1$. The Euler characteristic χ of such a surface can be computed by $2-2g$ in the orientable case, and $2-k$ in the non-orientable case. Removing n discs from the surface reduces its Euler characteristic by n. In general, for spaces X_1 and X_2 whose union is X, we have $\chi(X) = \chi(X_1)+\chi(X_2)-\chi(X_1 \cap X_2)$. Since the Euler characteristic of the circle is zero, it immediately follows that $\chi : \mathcal{C} \to \mathbb{Z}$ is a functor, where χ sends all objects of $\mathcal{C}$ to the only object of $\mathbb{Z}$, and sends a morphism to its Euler characteristic.

The open object category $\bar{\mathcal{O}}$

The open category $\bar{\mathcal{O}}$ is the full (symmetric strict monoidal) subcategory of $\mathcal{K}$ on disjoint unions of intervals. Again we identify the objects of $\bar{\mathcal{O}}$ with $\mathbb{N}$, and denote its classifying space by $\mathrm{B}\bar{\mathcal{O}}$.

Since the Euler characteristic of the unit interval is 1, χ is not a functor on $\bar{\mathcal{O}}$, and nor is it a functor on $\mathcal{K}$. However, we define a functor $\Theta : \mathcal{K} \to \mathbb{Z}$ which is constant on objects, and sends a morphism $\Sigma : (m,n) \to (p,q)$ to $\Theta(\Sigma) := p + q - m - \chi(\Sigma)$.

Proposition 2.1. *$\Theta : \mathcal{K} \to \mathbb{Z}$ is a functor.*

Proof. Let $\Sigma_1 : (p,q) \to (r,s)$ and $\Sigma_2 : (m,n) \to (p,q)$ be morphisms in $\mathcal{K}$. Then

$$\begin{aligned}\Theta(\Sigma_1 \circ \Sigma_2) &= r + s - m - \chi(\Sigma_1 \circ \Sigma_2)\\ &= r + s - m - (\chi(\Sigma_1) + \chi(\Sigma_2) - q)\\ &= r + s - p - \chi(\Sigma_1) + p + q - m - \chi(\Sigma_2)\\ &= \Theta(\Sigma_1) + \Theta(\Sigma_2).\end{aligned}$$ □

We can restrict the functor Θ to $\bar{\mathcal{O}}$ in the obvious way. The functors χ and Θ will be crucial in computing the fundamental groups of the three categories we have just defined.

3 The closed category $\mathcal{C}$

3.1 Analysis of subcategories

Before looking at the classifying space of the whole category $\mathcal{C}$, we will first study some important subcategories. This will motivate the techniques used in calculating $\pi_1(\mathrm{B}\mathcal{C})$. Moreover, we are able to give a complete description of the classifying space of each subcategory.

Definition 3.1. We define the category $\mathcal{C}_0$ to be the full subcategory of $\mathcal{C}$ on one object, the empty 1-manifold $\emptyset$.

Proposition 3.2. *The classifying space $B\mathcal{C}_0$ is homotopy equivalent to the infinite dimensional torus T^∞.*

Proof. Morphisms in $\mathcal{C}_0$ are closed surfaces, and are completely determined by the genus or number of crosscaps of each of their components. We first consider morphisms whose every component is orientable. These are in one-to-one correspondence with the monoid $\mathbb{N}^{\mathbb{N}}$, where a sequence $(n_0, n_1, n_2, \ldots)$ with finitely many non-zero entries represents the morphism comprising n_0 spheres, n_1 tori, n_2 double tori, and so on. Composition of endomorphisms of zero is simply given by taking the disjoint union, so we see that composition of orientable morphisms in $\mathcal{C}_0$ corresponds to addition in $\mathbb{N}^{\mathbb{N}}$.

Similarly, the surfaces in $\mathcal{C}_0$ consisting of only non-orientable components are in one-to-one correspondence with the monoid $\mathbb{N}^{\mathbb{N}_{>0}}$. Here a sequence $(m_1, m_2, m_3, \ldots)$ represents the morphism comprising m_1 projective planes, m_2 Klein bottles, m_3 connect sums of three projective planes, etc.

A general morphism in $\mathcal{C}_0$ is just the union of an element of $\mathbb{N}^{\mathbb{N}}$ and an element of $\mathbb{N}^{\mathbb{N}_{>0}}$, with composition corresponding to component-wise addition in each monoid. We can therefore identify the whole category $\mathcal{C}_0$ with the monoid $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}_{>0}}$, and hence $\mathrm{B}\mathcal{C}_0 = \mathrm{B}(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}_{>0}})$.

Let $\mathscr{G}(M)$ denote the group completion of a monoid M, that is the target of a universal homomorphism from M to a group, and let Ω denote the loop space functor. The canonical homomorphism $M \to \mathscr{G}(M)$ induces a natural transformation $\Omega\mathrm{B}(M) \to \mathscr{G}(M)$ which is a weak equivalence when M satisfies certain properties as defined by Quillen [Qui94]. In particular, this map is a weak equivalence when M is abelian. It follows that $\mathrm{B}(M) \simeq \mathrm{B}\mathscr{G}(M)$ for abelian monoids M. Applying this result to $\mathcal{C}_0$ we see that

$$\mathrm{B}\mathcal{C}_0 = \mathrm{B}\left(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}_{>0}}\right) \simeq \mathrm{B}\left(\mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^{\mathbb{N}_{>0}}\right) \simeq T^{\infty}$$

as required, where we have used the fact that $\mathrm{B}\mathbb{Z} \simeq S^1$. □

Corollary 3.3.

$\chi : B\mathcal{C}_0 \to B\mathbb{Z}$ corresponds to the $((2, 0, -2, -4, -6\ldots), (1, 0, -1, -2, -3\ldots))$*-fold cover of S^1.*

Proof. In our identification of $\mathcal{C}_0$ with $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}_{>0}}$, the i-th generator of $\mathbb{N}^{\mathbb{N}}$ has Euler characteristic $2-2i$, and the j-th generator of $\mathbb{N}^{\mathbb{N}_{>0}}$ has Euler characteristic $2-j$. $\square$

Definition 3.4. We define the category $\mathcal{C}_b$ to be the subcategory of $\mathcal{C}$ containing all objects of $\mathcal{C}$, and those morphisms no connected component of which is a cobordism to zero.

For example, a disc as a morphism $0 \to 1$ would be in $\mathcal{C}_b$, but a disc as a morphism $1 \to 0$ would not. In order to study the classifying space of $\mathcal{C}_b$ we must first look at an even smaller subcategory.

Definition 3.5. Let $\mathcal{C}_1$ be the full subcategory of $\mathcal{C}_b$ on one object, the circle S^1.

Morphisms in $\mathcal{C}_1$ are connected surfaces with one source and one target boundary circle, and are determined by their genus or number of crosscaps and their boundary maps. Note that composition in $\mathcal{C}_1$ is abelian. The aim of the following discussion is to show that $\mathcal{C}_1$ can be identified with the monoid $\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim}$ where the relation is generated by $(g, k, 0) \sim (0, 2g+k, 1)$ if and only if k is non-zero. We will then compute the group completion of this monoid, and hence identify $\mathrm{B}\mathcal{C}_1$.

First we consider the boundary maps of a morphism in $\mathcal{C}_1$. Recall from Section 2 that for each morphism we choose a representative cobordism whose boundary circles are identified with S^1, and whose boundary maps are either inclusions or reflections.

Definition 3.6. Let $(\Sigma, \sigma_0, \sigma_1)$ be a cobordism of the chosen form representing a morphism in $\mathcal{C}_1$. If σ_0 and σ_1 are both inclusions, or both reflections, then we say that they have the *same direction*. In this case we say that Σ is of *type 0*. If one of σ_0

and σ_1 is an inclusion, and the other is a reflection, then we say that the boundary maps have *opposite directions*. In this case we say that Σ is of *type 1*.

For orientable surfaces,[2] the type of a cobordism carries through to a well-defined notion of the type of a morphism. To see this, we make the following observations. Firstly, for a cobordism of genus g and type 0, the two possible choices (boundary maps are both inclusions or boundary maps are both reflections) lie in the same homeomorphism class. Similarly, for a cobordism of genus g and type 1 the two possible choices define the same morphism. Finally, there is no homeomorphism between an orientable cobordism of genus g and type 0, and an orientable cobordism of genus g and type 1, which commutes with all boundary maps. Thus the type of an orientable morphism is well-defined. For example, the cylinder of type 1 in $\mathcal{C}_1$ is the morphism $C_r^{S^1}$, whilst the cylinder of type 0 is the identity.

Now consider the subcategory $\mathcal{C}_1^+$ of $\mathcal{C}_1$ whose morphisms are all orientable. Morphisms in $\mathcal{C}_1^+$ are determined by their genera and their type. In terms of genera, composition in $\mathcal{C}_1^+$ corresponds to addition. If two morphisms in $\mathcal{C}_1^+$ have the same type, then their composition will always have type 0. However, composing two morphisms of different types gives a morphism of type 1. It follows that the type of a morphism lies in the group $\mathbb{Z}_2$. We therefore identify $\mathcal{C}_1^+$ with the monoid $\mathbb{N} \times \mathbb{Z}_2$, where the element (g, ϵ) stands for the unique morphism in $\mathcal{C}_1^+$ of genus g and type $\epsilon \in \{0, 1\}$.

Similarly, we can consider the subcategory $\mathcal{C}_1^-$ of $\mathcal{C}_1$ whose morphisms are non-orientable cobordisms along with the cylinder. For non-orientable surfaces, the type of a cobordism does not carry through to a well-defined notion of morphism type. This is precisely because a cobordism from S^1 to S^1 with $k \geq 1$ crosscaps and type 0 lies in the same homeomorphism class as a cobordism from S^1 to S^1 with

[2]Note that by orientable we do not mean oriented, and so no assumption is made about the direction of boundary maps.

k crosscaps and type 1. To see this, let $(\Sigma, \sigma_0, \sigma_1)$ be the punctured Möbius band as a cobordism from S^1 to S^1 of type 1. This is depicted in the left of Figure 3.1, where the boundary circle of the Möbius band represents the image of σ_0, and the boundary of the removed disc represents the image of σ_1. The arrows denote the directions of the embeddings σ_0 and σ_1. The surface on the right of Figure 3.1 is the punctured Möbius band Σ' of type 0. The required homeomorphism $\Sigma \to \Sigma'$ is then given by the crosscap slide, that is by pushing the image of σ_1 once round the Möbius strip, and this commutes with all boundary maps.

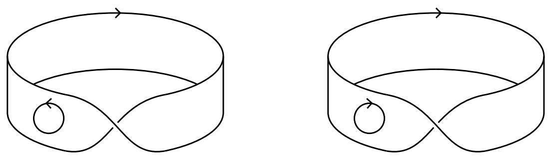

Figure 3.1: Möbius bands of type 1 and of type 0.

Thus for each $k \geq 1$ there is only one morphism with k crosscaps in $\mathcal{C}_1^-$. In terms of crosscaps, composition in $\mathcal{C}_1^-$ corresponds to addition. It follows that $\mathcal{C}_1^-$ is identifiable with the monoid $\mathbb{N} \cup C_r^{S^1}$, where $C_r^{S^1}$ acts as an identity for all elements other than itself and the cylinder $S^1 \times I$. For compatibility with the notation defined for $\mathcal{C}_1^+$, we write this as $\mathcal{C}_1^- = \frac{\mathbb{N}\times\mathbb{Z}_2}{\sim}$ where the relation is $(k, 0) \sim (k, 1)$ for non-zero k.

Moving back to $\mathcal{C}_1$, we now consider composing an orientable morphism Σ of genus g with a non-orientable morphism Σ' with $k \geq 1$ crosscaps. The composition $\Sigma \circ \Sigma'$ is a non-orientable surface with g handles and k crosscaps. Since the connect sum of a projective plane and a torus is homeomorphic to the connect sum of three projective planes, it follows that $\Sigma \circ \Sigma'$ is the unique morphism in $\mathcal{C}_1$ with $2g + k$ crosscaps.

Piecing together these results, we see that $\mathcal{C}_1$ can be identified with the monoid $\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2$ modulo the relations discussed above, where the class (g, k, ϵ) is represented by a cobordism with g handles, k crosscaps, and type $\epsilon \in \{0, 1\}$. One checks that the single relation $(g, k, 0) \sim (0, 2g + k, 1) \iff k \neq 0$ generates all others, and hence $\mathcal{C}_1 = \frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim}$ where the relation is $(g, k, 0) \sim (0, 2g + k, 1)$ for non-zero k.

Proposition 3.7. *The classifying space of $\mathcal{C}_1$ is homotopy equivalent to the circle S^1.*

By our earlier discussion on group completion, the proposition immediately follows from the following theorem.

Theorem 3.8. *The group completion of $\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim}$ is $\mathbb{Z}$.*

Proof. Recall that the group completion of an abelian monoid M is given by the Grothendieck construction $\frac{M\times M}{\approx}$ where the relation is $(x, y) \approx (x', y')$ if and only if there exists $k \in M$ such that $x + y' + k = y + x' + k$ in M. Here we denote the relation on $M \times M$ by $\approx$ to distinguish it from the relation $\sim$ on $\mathbb{N} \times \mathbb{N} \times \mathbb{Z}_2$.

Therefore, the group completion of $\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim}$ is given by $\frac{(\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim})\times(\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim})}{\approx}$ where $((a, b, \epsilon), (c, d, \eta)) \approx ((a', b', \epsilon'), (c', d', \eta'))$ if and only if there exists $(x, y, \zeta) \in \frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim}$ such that $(a, b, \epsilon) + (c', d', \eta') + (x, y, \zeta) \sim (c, d, \eta) + (a', b', \epsilon') + (x, y, \zeta)$ in $\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim}$.

We will show that the following two relations hold.

$$\begin{array}{lll} \text{(i)} & ((a, b, 0), (c, d, \eta)) \approx ((a, b, 1), (c, d, \eta)) & \text{for all } a, b, c, d, \eta \\ \text{(ii)} & ((a, b, 0), (c, d, 0)) \approx ((0, 2a + b, 0), (c, d, 0)) & \text{for all } a, b, c, d \end{array}$$

It will follow that $\mathscr{G}(\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim})$ is isomorphic to $\frac{(\{0\}\times\mathbb{N}\times\{0\})\times(\{0\}\times\mathbb{N}\times\{0\})}{\approx}$. The relation $\approx$ can then be written as $((0, b, 0), (0, d, 0)) \approx ((0, b', 0), (0, d', 0))$ if and only if there exists $(x, y, \zeta) \in \frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim}$ such that $(x, b+d'+y, \zeta) \sim (x, d+b'+y, \zeta)$, which is the case if and only if $b + d' + y = d + b' + y$ in $\mathbb{N}$. It will follow that $\mathscr{G}(\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim})$ is isomorphic to the group completion $\mathscr{G}(\mathbb{N}) = \mathbb{Z}$.

For the first equivalence (i), take $(x, y, \zeta) = (0, 1, 0) \in \frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim}$. Then $(a, b, 0) + (c, d, \eta) + (0, 1, 0) = (a + c, b + d + 1, \eta) \sim (c + a, d + b + 1, \eta + 1) = (c, d, \eta) + (a, b, 1) + (0, 1, 0)$, where for the second step we have used the fact that $b+d+1 > 0$. Equivalence (ii) follows in a similar fashion, again by taking (x, y, ζ) to be $(0, 1, 0)$. $\square$

Corollary 3.9. *$\chi : B\mathcal{C}_1 \to B\mathbb{Z}$ is a homotopy equivalence.*

Proof. Let $\bar{\chi} : \mathscr{G}(\mathcal{C}_1) \to \mathbb{Z}$ denote the unique map corresponding to χ from the universal property of the group completion. From the proof of Theorem 3.8 we see that a generator for $\mathscr{G}(\mathcal{C}_1) = \mathscr{G}(\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\sim})$ is the twice punctured projective plane. This has Euler characteristic -1, and so $\bar{\chi} : \mathbb{Z} = \mathscr{G}(\mathcal{C}_1) \to \mathbb{Z}$ is multiplication by -1. Hence $\mathrm{B}\bar{\chi}$ is a homotopy equivalence, and therefore so is $\mathrm{B}\chi$. □

We can compute the classifying spaces of the subcategories $\mathcal{C}_1^+$ and $\mathcal{C}_1^-$ via these same methods. Understanding their group completions allows us to describe the maps on classifying spaces induced by the inclusions of each subcategory into $\mathcal{C}_1$.

Proposition 3.10. *The classifying space $B\mathcal{C}_1^+$ is homotopy equivalent to $S^1 \times \mathbb{R}P^\infty$. The map $B\mathcal{C}_1^+ \to B\mathcal{C}_1$ induced by the inclusion $\mathcal{C}_1^+ \hookrightarrow \mathcal{C}_1$ is homotopic to $(z, x) \mapsto 2z$.*

Proof. Recall that we identified $\mathcal{C}_1^+$ with the monoid $\mathbb{N} \times \mathbb{Z}_2$. This group completes to $\mathbb{Z} \times \mathbb{Z}_2$, generated by the twice punctured torus of type 0, and the cylinder $C_r^{S^1}$. The first statement follows since $\mathrm{B}\mathbb{Z}_2 \simeq \mathbb{R}\mathrm{P}^\infty$. For the second statement, note that Equations (ii) and (i) in the proof of Theorem 3.8 yield the following equivalences in $\mathscr{G}(\mathcal{C}_1)$: the twice punctured torus is equivalent to the twice punctured Klein bottle, and $C_r^{S^1}$ is equivalent to the identity. □

Proposition 3.11. *The classifying space $B\mathcal{C}_1^-$ is homotopy equivalent to S^1. The map $B\mathcal{C}_1^- \to B\mathcal{C}_1$ induced by the inclusion $\mathcal{C}_1^- \hookrightarrow \mathcal{C}_1$ is a homotopy equivalence.*

Proof. We identified $\mathcal{C}_1^-$ with the monoid $\frac{\mathbb{N}\times\mathbb{Z}_2}{\sim}$ where the relation is $(k, 0) \sim (k, 1)$ for non-zero k. Via the same methods as in the proof of Theorem 3.8, one can show that the group completion of this monoid is $\mathbb{Z}$, generated by the twice punctured projective plane. Since this also generates $\mathscr{G}(\mathcal{C}_1)$, the inclusion $\mathcal{C}_1^- \hookrightarrow \mathcal{C}_1$ induces a homotopy equivalence. □

The category $\mathcal{C}_1^-$ will be essential to our understanding of the classifying space $\mathrm{B}\mathcal{C}_b$.

Definition 3.12. We define the subcategory ${}^i\mathcal{C}_1^-$ of $\mathcal{C}_1^-$ by ${}^i\mathcal{C}_1^- \coloneqq \mathcal{C}_1^- \setminus \{C_r^{S^1}\} = \mathbb{N}$.

This is equivalent to saying that ${}^i\mathcal{C}_1^-$ is the subcategory of $\mathcal{C}_1^-$ whose morphisms have inclusions for boundary maps, hence the notation. Note that the classifying space $\mathrm{B}({}^i\mathcal{C}_1^-)$ is homotopy equivalent to S^1.

Proposition 3.13. *The classifying space $B\mathcal{C}_b$ is homotopy equivalent to the circle S^1.*

The proposition immediately follows from the following theorem.

Theorem 3.14. *The inclusion ${}^i\mathcal{C}_1^- \hookrightarrow \mathcal{C}_b$ has a left adjoint Φ.*

Proof. We construct the functor $\Phi : \mathcal{C}_b \to {}^i\mathcal{C}_1^-$ by adapting a functor defined by Tillmann [Til96, Theorem 4] of the same name. On objects, Φ is the constant map $n \mapsto 1$. For a morphism $\Sigma : n \to m$ with c components, k crosscaps, and total genus g, we define

$$\Phi(\Sigma) \coloneqq m - n - \chi(\Sigma) = 2g + k + 2m - 2c$$

which is to say $\Phi(\Sigma)$ is the unique morphism in ${}^i\mathcal{C}_1^-$ with precisely $2g + k + 2m - 2c$ crosscaps. Note that this is well-defined since $m \geq c$ for any cobordism in $\mathcal{C}_b$. Let $\Sigma_1 : m \to l$ and $\Sigma_2 : n \to m$ be morphisms in $\mathcal{C}_b$. Then

$$\begin{aligned}\Phi(\Sigma_1 \circ \Sigma_2) &= l - n - \chi(\Sigma_1 \circ \Sigma_2) \\ &= l - m - \chi(\Sigma_1) + m - n - \chi(\Sigma_2) \\ &= \Phi(\Sigma_1) + \Phi(\Sigma_2)\end{aligned}$$

and so Φ is functorial. Note that $\Phi \circ i = \mathrm{Id}_{{}^i\mathcal{C}_1^-}$. Therefore, to prove that Φ is left adjoint to the inclusion $i : {}^i\mathcal{C}_1^- \hookrightarrow \mathcal{C}_b$ it suffices to define a natural transformation $\tau : \mathrm{Id}_{\mathcal{C}_b} \to i \circ \Phi$. For the object n in $\mathcal{C}_b$, let $\tau_n : n \to 1$ be the pair of pants surface with n legs and one crosscap. For a morphism $\Sigma_{g,k,c} : n \to m$ with c components, k crosscaps, and total genus g, we need to check that the following diagram commutes.

$$\begin{tikzcd} n \arrow[r, "\tau_n"] \arrow[d, "\Sigma_{g,k,c}"'] & 1 \arrow[d, "2g+k+2m-2c"] \\ m \arrow[r, "\tau_m"'] & 1 \end{tikzcd}$$

Now, $(2g+k+2m-2c) \circ \tau_n$ is the unique connected non-orientable morphism $n \to 1$ in $\mathcal{C}_b$ with precisely $2g+k+2m-2c+1$ crosscaps. On the other hand, the genus of $\Sigma_{g,k,c}$ increases by precisely $m-c$ on composition with τ_m, and the resulting surface is connected. Therefore $\tau_m \circ \Sigma_{g,k,c}$ is non-orientable with $2(g+m-c)+k+1 = 2g+k+2m-2c+1$ crosscaps. Note that, by ensuring that both compositions are non-orientable, we need not worry about the direction of boundary maps. Therefore the diagram commutes as required. $\square$

Corollary 3.15. *$\chi : B\mathcal{C}_b \to B\mathbb{Z}$ is a homotopy equivalence.*

Proof. ${}^{i}\mathcal{C}_1^- = \mathbb{N}$ is generated by the twice punctured projective plane, which has Euler characteristic -1. It follows that $\chi : {}^{i}\mathcal{C}_1^- = \mathbb{N} \to \mathbb{Z}$ is multiplication by -1, and therefore $\chi : \mathrm{B}({}^{i}\mathcal{C}_1^-) \to \mathrm{B}\mathbb{Z}$ is a homotopy equivalence. Since $\mathrm{B}({}^{i}\mathcal{C}_1^-) \simeq \mathrm{B}\mathcal{C}_b$ induced by the inclusion, we see that χ is a homotopy equivalence on $\mathrm{B}\mathcal{C}_b$ also. $\square$

Definition 3.16. Let $\mathcal{C}_{>0}$ be the subcategory of $\mathcal{C}$ which has all objects except the empty 1-manifold, and morphisms no component of which is a cobordism to or from the empty 1-manifold.

Note that $\mathcal{C}_b$ contains $\mathcal{C}_{>0}$ as a subcategory, and we can restrict the functor Φ to $\mathcal{C}_{>0}$. Via the same arguments as in the proof of Theorem 3.14 we obtain the following.

Proposition 3.17. *The classifying space of $\mathcal{C}_{>0}$ is homotopy equivalent to the circle S^1, and $\chi : B\mathcal{C}_{>0} \to B\mathbb{Z}$ is a homotopy equivalence.* $\square$

3.2 Fundamental group

Theorem 3.18. *$\pi_1(B\mathcal{C}) = \mathbb{Z}$ and $\chi_* : \pi_1(B\mathcal{C}) \to \pi_1(B\mathbb{Z})$ is an isomorphism.*

Proof. Let $\mathfrak{C} = \mathcal{C}[\mathcal{C}^{-1}]$ denote the *localisation* of $\mathcal{C}$, that is the groupoid obtained from $\mathcal{C}$ by adjoining inverses of all morphisms (see [GZ67] for formal construction). Since $\mathcal{C}$ is connected, the fundamental group of its classifying space is canonically isomorphic to the group of automorphisms of any object in $\mathfrak{C}$ [Qui73]. In particular, it is isomorphic to the group $\mathfrak{C}_1$ of automorphisms of the circle.

It seems like a good starting point would be $\mathrm{End}_{\mathcal{C}}(S^1)$, the monoid of endomorphisms of the circle in $\mathcal{C}$. It is important to note that its localisation $\mathscr{G}(\mathrm{End}_{\mathcal{C}}(S^1))$ is most likely not isomorphic to $\mathfrak{C}_1$. The canonical map $\mathscr{G}(\mathrm{End}_{\mathcal{C}}(S^1)) \to \mathfrak{C}_1$ is not necessarily injective, since localising all of $\mathcal{C}$ may yield more equivalence relations among automorphisms of S^1. However, and this will be essential for the remainder of the proof, the map is surjective, and any relations that hold in $\mathscr{G}(\mathrm{End}_{\mathcal{C}}(S^1))$ must also hold in $\mathfrak{C}_1$. The argument for surjectivity goes as follows.

Definition 3.19. A category $\mathscr{C}$ is *strongly connected* if for any pair of objects x, y in $\mathscr{C}$ there exists morphisms $x \to y$ and $y \to x$.

Notice in particular that $\mathcal{C}$ is strongly connected.

Proposition 3.20. *If $\mathscr{C}$ is a strongly connected category, then the canonical map $\mathscr{G}(\mathrm{End}_{\mathscr{C}}(x)) \to \mathrm{Aut}_{\mathscr{C}[\mathscr{C}^{-1}]}(x)$ is surjective for any object x in $\mathscr{C}$.*

Proof. Consider a general automorphism F of x in $\mathscr{C}[\mathscr{C}^{-1}]$, which is a composition of morphisms in $\mathscr{C}$ and their inverses. This is represented by the top line of the diagram below, where t, u, v, w are other objects in $\mathscr{C}$.

$$\begin{tikzcd}
x \arrow[r] & t \arrow[d, bend right] \arrow[d, bend left] & u \arrow[l] \arrow[r] & v \arrow[d, bend right] \arrow[d, bend left] & w \arrow[l] \arrow[r] & x \\
 & x & x \arrow[u, bend left] \arrow[u, bend right] & x & x \arrow[u, bend left] \arrow[u, bend right] &
\end{tikzcd}$$

We move from left to right along the diagram. Since $\mathscr{C}$ is strongly connected, when we reach the object t we can find a morphism $t \to x$ in $\mathscr{C}$, represented by the first downwards arrow. We then map back to t via the inverse of this morphism. Similarly, when we reach u we can find a morphism $x \to u$, and map down to x via its inverse. Continuing in this manner, we can decompose F as a composition of endomorphisms of x in $\mathscr{C}$ and their inverses, that is to say as elements of $\mathscr{G}(\mathrm{End}_{\mathscr{C}}(x))$. $\square$

By proposition 3.20, to calculate $\mathfrak{C}_1$ it suffices to find all relations among elements of the image of $\mathscr{G}(\mathrm{End}_{\mathcal{C}}(S^1))$ in $\mathfrak{C}$.

Disregarding closed components, an element Σ in $\mathrm{End}_{\mathcal{C}}(S^1)$ consists of either a connected surface with two boundary circles (that is, an element of $\mathcal{C}_1$) or of two connected surfaces each with one boundary circle (see Figure 3.2). Each component has some arbitrary genus $g \in \mathbb{N}$ or number $k \in \mathbb{N}$ of crosscaps. Connected endomorphisms as in the left of Figure 3.2 also have an associated type which lies in the group $\mathbb{Z}_2$, as discussed in Section 3.1. Via reflection in a suitable horizontal plane, one sees that the disc with boundary map i is homeomorphic (relative to the boundary) to the disc with boundary map r, and so we need not worry about the direction of boundary maps for disconnected endomorphisms as in the right of Figure 3.2.

The closed components of Σ lie in the monoid $\mathcal{C}_0 = \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}_{>0}}$ from the previous section. It follows that $\mathrm{End}_{\mathcal{C}}(S^1)$ is isomorphic to

$$\Big(\big(\tfrac{\mathbb{N}\times\mathbb{N}\times\mathbb{Z}_2}{\approx}\big) \amalg \Big(\big(\tfrac{\mathbb{N}\times\mathbb{N}}{\sim}\big) \times \big(\tfrac{\mathbb{N}\times\mathbb{N}}{\sim}\big)\Big)\Big) \times \big(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}_{>0}}\big)$$

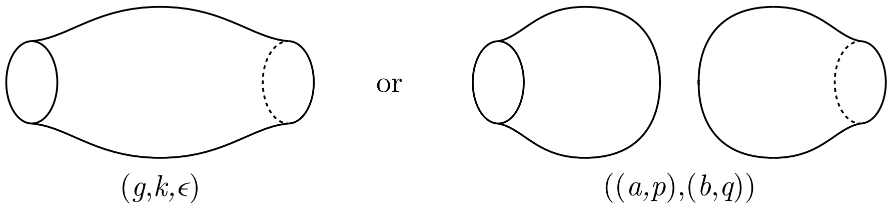

Figure 3.2: Two types of morphisms in $\mathrm{End}_{\mathcal{C}}(S^1)$.

where $\approx$ is the relation $(g,k,0) \approx (0, 2g+k, 1)$ for $k > 0$ and $\sim$ is the relation $(g,k) \sim (0, 2g+k)$ for $k > 0$. Elements of the form $((g,k,\epsilon); n_{00}, n_{10}, n_{01}, \ldots)$ represent morphisms comprising a connected surface as in the left of Figure 3.2, along with n_{00} spheres, n_{10} tori, n_{01} projective planes, n_{20} double tori, n_{02} Klein bottles, and so on. Elements of the form $((a,p),(b,q); n_{00}, n_{10}, n_{01}, \ldots)$ represent morphisms comprising a disconnected surface as in the right of Figure 3.2, along with n_{00} spheres, n_{10} tori etc. Addition in the monoid is non-commutative and defined by the geometry, for example:

$$\begin{aligned} &((a,p),(b,q); n'_{00}, n'_{10}, \ldots) \circ ((g,k,\epsilon); n_{00}, n_{10}, \ldots) \\ &\qquad\qquad = ((g+a, k+p), (b,q); n_{00}+n'_{00}, n_{10}+n'_{10}, \ldots) \end{aligned}$$

all modulo the above relations, and it is straightforward to check that this is well defined.

Since $\mathcal{C}_1 \subset \mathrm{End}_{\mathcal{C}}(S^1)$, relations which hold in $\mathscr{G}(\mathcal{C}_1)$ must also hold in $\mathscr{G}(\mathrm{End}_{\mathcal{C}}(S^1))$, and hence also in $\mathfrak{C}$. In particular the cylinder $C_r^{S^1}$ is equivalent to the identity 1_{S^1} (Theorem 3.8) and so the direction of boundary maps becomes irrelevant upon localisation. Since we will be working in $\mathfrak{C}$ from now on, we will henceforth suppress all mention of boundary maps. In this spirit we adapt our notation, denoting by $((g,k); n_{00}, n_{10}, \ldots)) \in \mathrm{Im}\,(\mathrm{End}_{\mathcal{C}}(S^1) \to \mathscr{G}(\mathrm{End}_{\mathcal{C}}(S^1)))$ a morphism comprising a connected surface as in the left of Figure 3.2 with g handles and k crosscaps, along with closed components.

Since $\mathcal{C}$ is connected, conjugation $c_\alpha : \mathfrak{C}_1 \to \mathfrak{C}_0$ defined by $\beta \mapsto \alpha^{-1}\beta\alpha$ is an isomorphism for any $\alpha : 0 \to 1$. In particular it is injective, and we can use this to obtain relations on $\mathfrak{C}_1$ by finding elements with homeomorphic images in $\mathfrak{C}_0$. Let α be the disc as a morphism $0 \to 1$. A representative for the inverse of α in $\mathfrak{C}$ is the disc $: 1 \to 0$ union the inverse of a sphere. The images of elements of $\mathfrak{C}_1$ under c_α are:

$$\pm((g,k); n_{00}, n_{10}, n_{01}, \ldots) \mapsto \pm(n_{00} - 1, n_{10}, n_{01}, \ldots, n_{gk} + 1, \ldots)$$
$$\pm((a,p),(b,q); n_{00}, n_{10}, n_{01}, \ldots) \mapsto \pm(n_{00} - 1, n_{10}, n_{01}, \ldots, n_{ap} + 1, \ldots, n_{bq} + 1, \ldots)$$

where $g, k, a, b, p, q, n_{i0}, n_{0j}$ are non-negative and $\pm$ represents an element of $\mathrm{End}_{\mathcal{C}}(S^1)$ or its inverse respectively. Injectivity of c_α thus forces the following identifications in $\mathfrak{C}_1$:

(i) $\pm\,((g,k); n_{00}, n_{10}, \ldots)$
$$= \pm\left(\left(g + \sum_i i n_{i0}, k + \sum_j j n_{0j}\right); \sum_i n_{i0} + \sum_j n_{0j}, 0, 0, \ldots\right)$$
$$= \pm\left(\left(0, 2g + k + 2\sum_i i n_{i0} + \sum_j j n_{0j}\right); \sum_i n_{i0} + \sum_j n_{0j}, 0, 0, \ldots\right)$$

(ii) $\pm\,((a,p),(b,q); n_{00}, n_{10}, \ldots)$
$$= \pm\left(\left(a + b + \sum_i i n_{i0}, p + q + \sum_j j n_{0j}\right); 1 + \sum_i n_{i0} + \sum_j n_{0j}, 0, 0, \ldots\right)$$
$$= \pm\left(\left(0, 2a + 2b + p + q + 2\sum_i i n_{i0} + \sum_j j n_{0j}\right); 1 + \sum_i n_{i0} + \sum_j n_{0j}, 0, 0, \ldots\right).$$

Here we use '=' rather than the usual '$\sim$' to mean equivalent, in order to avoid confusion with the relation $\sim$ defined earlier. For the second step in each equivalence we have used the fact that the twice punctured torus and twice punctured Klein bottle are equivalent in $\mathscr{G}(\mathcal{C}_1)$ (Theorem 3.8) and therefore also in $\mathfrak{C}_1$. We see that every morphism in $\mathfrak{C}_1$ is equivalent to an element of $\mathcal{C}_1^-$ union a collection of spheres, or its inverse, or a composition of such things. The next step is to eliminate the spheres, meaning that every element of $\mathfrak{C}_1$ is equivalent to an element of $\mathscr{G}(\mathcal{C}_1^-)$.

Proposition 3.21. $((0,-1);0,0,\ldots) = ((0,1);1,0,0,\ldots)$ *in* $\mathfrak{C}$.

Here we have adopted the notation $((0,-k);0,0,\ldots) = -((0,k);0,0,\ldots)$ for the inverse of the twice punctured connect sum of k projective planes. Geometrically speaking, the proposition states that the inverse of the twice punctured projective plane is a twice punctured projective plane union a sphere.

Proof. We show that $((0,-2);0,0,\ldots) = ((0,0);1,0,0,\ldots)$ in $\mathfrak{C}$, in other words the inverse of the twice punctured Klein bottle is a cylinder union a sphere. The result then follows by composing each side of the equation with $((0,1);0,0,\ldots)$, the twice punctured projective plane. It suffices to show that the inverse of the twice punctured torus is a cylinder union a sphere, for which we follow Tillmann [Til96, Theorem 7].

Let $\mathfrak{C}_n$ denote the group of automorphisms of the object n in $\mathfrak{C}$. We recall that conjugation $c_\alpha : \mathfrak{C}_m \to \mathfrak{C}_n$ via $\beta \to \alpha^{-1}\beta\alpha$ is a group isomorphism for any $\alpha : n \to m$. Let α be the union of a cylinder and a disc as a morphism $1 \to 2$. We consider the following two representatives for the inverse of α in $\mathfrak{C}$: let $\beta_1 : 2 \to 1$ be the pair of pants surface, and let $\beta_2 : 2 \to 1$ be the union of a disc, a cylinder and the inverse of a sphere. In order for conjugation to be well defined, we must have $\beta_1\gamma\alpha = \beta_2\gamma\alpha$ in $\mathfrak{C}$ for any morphism $\gamma : 2 \to 2$. Taking γ to be the union of the pair of pants surface and a disc, we see that the twice punctured torus union a sphere is the identity (see Figure 3.3). □

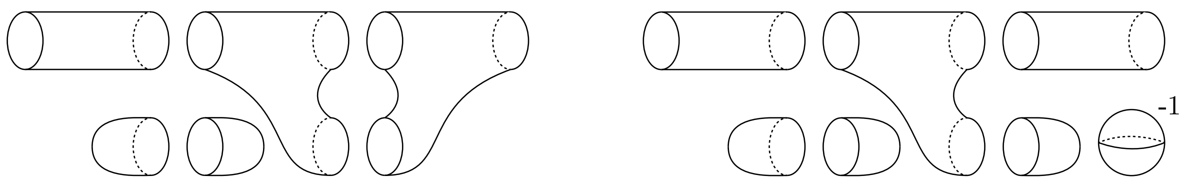

Figure 3.3: $c_\alpha(\gamma) = \beta_1\gamma\alpha$ and $c_\alpha(\gamma) = \beta_2\gamma\alpha$.

Corollary 3.22. $((0,-k);0,0,\ldots) = ((0,k);k,0,0,\ldots)$ *in* $\mathfrak{C}$.

Corollary 3.23. $((0,k);n_{00},0,0,\ldots) = ((0,k-2n_{00});0,0,\ldots)$ *in* $\mathfrak{C}$.

Proof. The first corollary is immediate from the proposition. The second corollary is a consequence of the first. Explicitly, we have

$$\begin{aligned}((0,k);n_{00},0,0,\ldots) &= ((0,k-n_{00});0,0,\ldots)\circ((0,n_{00});n_{00},0,0,\ldots)\\ &= ((0,k-n_{00});0,0,\ldots)\circ((0,-n_{00});0,0,\ldots)\\ &= ((0,k-2n_{00});0,0,\ldots).\end{aligned}$$ □

Applying Corollary 3.23 to the equivalences (i) and (ii) we obtain

$$\begin{aligned}&(\mathrm{i})' \pm((g,k);n_{00},n_{10},\ldots)\\ &\quad = \pm\left(\left(0,2g+k+2\sum_i(i-1)n_{i0}+\sum_j(j-2)n_{0j}\right);0,0,\ldots\right)\\ &(\mathrm{ii})' \pm((a,p),(b,q);n_{00},n_{10},\ldots)\\ &\quad = \pm\left(\left(0,2(a+b-1)+(p+q)+2\sum_i(i-1)n_{i0}+\sum_j(j-2)n_{0j}\right);0,0,\ldots\right)\end{aligned}$$

and so every element of $\mathfrak{C}_1$ is of the desired form. We see that $\mathfrak{C}_1 = \mathbb{Z}$ modulo relations. It remains to show that all further relations are trivial.

The functor $\chi : \mathcal{C} \to \mathbb{Z}$ is morphism inverting and hence factors through $\mathfrak{C}$ (see for example [GZ67]). Let $\bar{\chi}$ denote the unique functor $\mathfrak{C} \to \mathbb{Z}$ corresponding to χ, so that the following diagram commutes, where $\psi : \mathcal{C} \to \mathfrak{C}$ is the canonical projection.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\psi"] \arrow[dr, "\chi"'] & & \mathfrak{C} \arrow[dl, "\bar{\chi}"] \\ & \mathbb{Z} & \end{tikzcd}$$

Since $\bar{\chi}((0,\pm k);0,0,\ldots) = \mp k$ we see that $\bar{\chi}|_{\mathfrak{C}_1} : \mathfrak{C}_1 \to \mathbb{Z}$ is an isomorphism. Hence all further relations on $\mathfrak{C}_1$ are trivial, and therefore $\pi_1(\mathrm{B}\mathcal{C}) = \mathfrak{C}_1 = \mathbb{Z}$.

The final part of the theorem follows since the twice punctured projective plane generates $\pi_1(\mathrm{B}\mathcal{C})$, and has Euler characteristic -1. □

Corollary 3.24. *The classifying space $B\mathfrak{C}$ is homotopy equivalent to the circle S^1.*

Proof. For any connected category $\mathscr{C}$ and for any object $x \in \mathscr{C}$, the inclusion functor $\mathrm{Aut}_{\mathscr{C}[\mathscr{C}^{-1}]}(x) \hookrightarrow \mathscr{C}[\mathscr{C}^{-1}]$ is an equivalence of categories [Qui73]. The result thus immediately follows from Theorem 3.18. $\square$

Remark. The remarks of Section 1.3 can now be made precise. The fundamental group of the oriented category $\mathscr{S}$ is generated by the twice punctured torus [Til96]. In $\mathfrak{C}$ this is equivalent to the twice punctured Klein bottle, with the equivalence induced (upon categorical localisation) by the natural interaction between the crosscaps and handles of surfaces. Thus the map $\pi_1(\mathrm{B}\mathscr{S}) \to \pi_1(\mathrm{B}\mathcal{C})$ induced by the inclusion $\mathscr{S} \to \mathcal{C}$ is multiplication by 2. Identifying $\pi_1(\mathrm{B}\mathscr{S})$ with its image in $\pi_1(\mathrm{B}\mathcal{C})$ we see that the isomorphism $\pi_1(\mathrm{B}\mathcal{C}) \to \pi_1(\mathrm{B}\mathscr{S})$ induced by sending the twice punctured projective plane to the twice punctured torus is a double covering of groups. However, the result is somewhat unexpected in terms of boundary maps. The morphism $C_r^{S^1}$ plays no part in the fundamental group of $\mathcal{C}$, despite the fact that one might expect a $\mathbb{Z}_2$ factor to appear.

3.3 Homotopy type

Theorem 3.25. *$\mathcal{C}$ is a connected symmetric strict monoidal category. Hence its classifying space $B\mathcal{C}$ has the homotopy type of an infinite loop space. Furthermore, there is a simply connected infinite loop space X such that $B\mathcal{C}$ is homotopy equivalent to $X \times S^1$.*

Proof. The product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is given by disjoint union, with the empty 1-manifold as the identity object. Let $\mathrm{t} : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the twist functor. Then there is a natural transformation $\tau : \otimes \to \otimes \circ \mathrm{t}$ which is simply given by crossing cylinders, and so clearly $\tau^2 = \mathrm{Id}$.

The category associated to any abelian monoid is also symmetric strict monoidal. In this case the only object in the category is the identity object, and the product on morphisms is given by addition in the monoid.

It is known that the classifying space of a connected symmetric strict monoidal category has the homotopy type of an infinite loop space [May74], [Seg74]. Moreover, a symmetric monoidal functor induces a map of infinite loop spaces on the classifying spaces of two such categories. Since the Euler characteristic is additive with respect to disjoint union, it follows that the induced map $\chi : \mathrm{B}\mathcal{C} \to \mathrm{B}\mathbb{Z}$ is a map of infinite loop spaces.

The composition $\frac{\mathbb{N}\times\mathbb{Z}_2}{\sim} = \mathcal{C}_1^- \hookrightarrow \mathcal{C} \xrightarrow{\chi} \mathbb{Z}$ is $(n,\epsilon) \mapsto -n$, in other words it is the canonical map $\frac{\mathbb{N}\times\mathbb{Z}_2}{\sim} \to \mathscr{G}\left(\frac{\mathbb{N}\times\mathbb{Z}_2}{\sim}\right) = \mathbb{Z}$ composed with an isomorphism. Therefore the induced map on classifying spaces is a homotopy equivalence. It follows that χ has a section, that is a map $s : \mathrm{B}\mathbb{Z} \to \mathrm{B}\mathcal{C}$ with $\chi \circ s \simeq \mathrm{Id}_{\mathrm{B}\mathbb{Z}}$.

Let X be the homotopy fibre of $\chi : \mathrm{B}\mathcal{C} \to \mathrm{B}\mathbb{Z}$, and denote the map $X \to \mathrm{B}\mathcal{C}$ by ϕ. Note that X is an infinite loop space, since χ is a map of infinite loop spaces. We show that X is simply connected, and that the map $\Psi : X \times \mathrm{B}\mathbb{Z} \to \mathrm{B}\mathcal{C}$ defined by $(x, y) \mapsto \phi(x) \cdot s(y)$ is a weak equivalence, where the product $\cdot$ on $\mathrm{B}\mathcal{C}$ is the loop product.

Since χ has a section, the long exact sequence in homotopy

$$\cdots \to \pi_n X \xrightarrow{\phi_*} \pi_n(\mathrm{B}\mathcal{C}) \xrightarrow{\chi_*} \pi_n S^1 \to \pi_{n-1} X \to \cdots$$

splits as short exact sequences $0 \to \pi_n X \xrightarrow{\phi_*} \pi_n(\mathrm{B}\mathcal{C}) \xrightarrow{\chi_*} \pi_n S^1 \to 0$. Considering this short exact sequence in dimensions $n = 0$ and $n = 1$, and applying Theorem 3.18 in the latter case, we see that X is simply connected. We also see that the lower dimensional homotopy groups are abelian. We therefore have split short exact sequences of abelian groups in every dimension, and hence isomorphisms $\pi_n(X \times S^1) \to \pi_n(\mathrm{B}\mathcal{C})$ given by

$[x, y] \mapsto \phi_*[x]s_*[y]$. But these are precisely the maps induced by Ψ. Hence Ψ is a weak equivalence, and therefore a homotopy equivalence as required. $\square$

3.4 Applications to TQFT

3.4.1 Classification of functors $\mathcal{C} \to \mathbb{Z}$

Any functor $F : \mathcal{C} \to \mathbb{Z}$ is morphism inverting and hence factors through the localised category $\mathfrak{C}$. F is determined by its image of:

- A generator of $\mathfrak{C}_n = \mathbb{Z}$ for any object n in $\mathcal{C}$
- A set of connecting morphisms $p_k : n \to k$, one for each object k in $\mathcal{C}$.

To see this, let $\bar{F}$ denote the functor $\mathfrak{C} \to \mathbb{Z}$ corresponding to F, and note that for any morphism $\Sigma : k \to k'$ in $\mathcal{C}$ we can write $F(\Sigma) = F(p_{k'}) + \bar{F}(\Gamma) + F(p_k)^{-1}$ where $\Gamma = p_{k'}^{-1} \circ \Sigma \circ p_k \in \mathfrak{C}_n$.

In particular we take $n = 0$, and p_k (for $k \geq 1$) to be the union of k discs each as a cobordism from $\emptyset$ to S^1. We define p_0 to be the identity $1_0 = \emptyset$.

Proposition 3.26. *The projective plane P^2 generates $\mathfrak{C}_0$.*

Proof. Conjugation by p_1 defines an isomorphism $\mathfrak{C}_1 \to \mathfrak{C}_0$. Since morphisms in $\mathfrak{C}_1$ are completely determined by their Euler characteristic (that is, two morphisms are equivalent if and only if they have the same image under $\bar{\chi}$) this must also be true in $\mathfrak{C}_0$. The projective plane has Euler characteristic 1, and so it is a generator. $\square$

Geometrically, we see that in $\mathfrak{C}_0$:

- the sphere is equivalent to two projective planes
- a closed orientable surface of genus g is equivalent to $2 - 2g$ projective planes
- a closed non-orientable surface with k crosscaps is equivalent to $2 - k$ projective planes.

The following observation will be useful: the proof of Proposition 3.26 can be adapted for the automorphism group of any object in $\mathfrak{C}$, and so all automorphisms are completely determined by their Euler characteristic. Let $\Sigma \in \mathfrak{C}_n$ be any morphism and let $\Gamma \in \mathfrak{C}_n$ be a morphism with $\chi(\Gamma) = 1$. Then Σ is equivalent to the composition $\Gamma \circ \ldots \circ \Gamma$ ($\bar{\chi}(\Sigma)$ times).

Proposition 3.27. *Any functor $F : \mathcal{C} \to \mathbb{Z}$ is determined by where it sends the projective plane P^2 and the connecting morphisms p_k. Conversely, for any set of integers $\{b_0, b_1, b_2, \ldots\}$ there exists a functor sending the projective plane to b_0, and sending p_k to b_k for all $k \geq 1$.*

Proof. The first statement follows from the above discussion and Proposition 3.26. For the second statement, we build a functor $F : \mathcal{C} \to \mathbb{Z}$ as follows. Let $\Sigma : k \to k'$ be a morphism in $\mathcal{C}$. Any such functor F must satisfy $F(\Sigma) = F(p_{k'}) + \bar{F}(\amalg_\alpha P^2) + F(p_k)^{-1}$ where $\alpha = \bar{\chi}(p_{k'}^{-1} \circ \Sigma \circ p_k) \in \mathbb{Z}$. Hence we define

$$F(\Sigma) := \begin{cases} b_{k'} - b_k - b_0(k' - k - \chi(\Sigma)) & \text{if } k, k' \geq 1 \\ b_{k'} - b_0(k' - \chi(\Sigma)) & \text{if } k = 0,\ k' \geq 1 \\ -b_k - b_0(-k - \chi(\Sigma)) & \text{if } k \geq 1,\ k' = 0 \\ b_0 \chi(\Sigma) & \text{if } k = k' = 0. \end{cases}$$

Note that $F(P^2) = b_0$ and $F(p_k) = b_k$. Let $\Sigma' : k' \to k''$ be another morphism in $\mathcal{C}$, and suppose $k, k', k'' \geq 1$. Then we have

$$\begin{aligned} F(\Sigma' \circ \Sigma) &= b_{k''} - b_k - b_0(k'' - k - \chi(\Sigma' \circ \Sigma)) \\ &= b_{k''} - b_{k'} + b_{k'} - b_k - b_0(k'' - k' + k' - k - \chi(\Sigma' \circ \Sigma)) \\ &= b_{k''} - b_{k'} - b_0(k'' - k' - \chi(\Sigma')) + b_{k'} - b_k - b_0(k' - k - \chi(\Sigma)) \\ &= F(\Sigma') + F(\Sigma). \end{aligned}$$

One checks the remaining seven cases ($k, k' \geq 1, k'' = 0$ etc) to see that F is indeed a functor. $\square$

For a sequence of rational numbers $(a_i)_{i\in\mathbb{N}}$ such that $a_0 \in \mathbb{Z}$ and $ka_k \in \mathbb{Z}$ for all k, we define a functor $\Phi_{(a_i)} : \mathcal{C} \to \mathbb{Z}$ which is constant on objects, and sends a morphism $\Sigma : n \to m$ to

$$\Phi_{(a_i)}(\Sigma) \coloneqq a_m m - a_n n + a_0(m - n - \chi(\Sigma)).$$

One checks that $\Phi_{(a_i)}$ is functorial. Furthermore, since $\Phi_{(a_i)}$ sends the projective plane to $-a_0$, and sends p_k to $a_k k$, we immediately see that the functors are all distinct. For example, the familiar functor χ can be uniquely written as $\Phi_{(-1,1,1,\ldots)}$. Given a set of integers $\{b_0, b_1, b_2, \ldots\}$, if we set $a_0 \coloneqq -b_0$ and $a_k \coloneqq \frac{b_k}{k}$ for $k \geq 1$, then $\Phi_{(a_i)}$ is the functor which sends the projective plane to b_0, and sends p_k to b_k for $k \geq 1$.

Theorem 3.28. *Every functor $\mathcal{C} \to \mathbb{Z}$ is of the form $\Phi_{(a_i)}$ for some unique sequence $(a_i)_{i\in\mathbb{N}}$, where $a_0 \in \mathbb{Z}$ and $ka_k \in \mathbb{Z}$ for all k. Its isomorphism class and homotopy class are both determined by a_0, and it is strict monoidal if and only if $a_1 = a_2 = a_3 = \ldots$.*

Proof. The first statement follows from the above discussion. By the homotopy class of $\Phi_{(a_i)}$ we mean the homotopy class of the induced map on classifying spaces. Recall that the twice punctured projective plane generates $\pi_1(\mathrm{B}\mathcal{C})$, and note that it is mapped to a_0 under $\Phi_{(a_i)}$. Now suppose we have a natural transformation $\tau : \Phi_{(a_i)} \to \Phi_{(a'_i)}$. Then $\mathrm{B}\Phi_{(a_i)}$ and $\mathrm{B}\Phi_{(a'_i)}$ are homotopic, and hence $a_0 = a'_0$. Conversely, we define a natural transformation $\tau : \Phi_{(a_0,a_i)} \to \Phi_{(a_0,a'_i)}$ (where $i \geq 1$) by $\tau_k \coloneqq k(a'_k - a_k) \in \mathbb{Z}$ for each object $k \in \mathcal{C}$. Then for any morphism $\Sigma : n \to m$ in $\mathcal{C}$, the following diagram commutes as required.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\tau_n = n(a'_n - a_n)"] \arrow[d, "a_m m - a_n n + a_0(m-n-\chi(\Sigma))"'] & \mathbb{Z} \arrow[d, "a'_m m - a'_n n + a_0(m-n-\chi(\Sigma))"] \\
\mathbb{Z} \arrow[r, "\tau_m = m(a'_m - a_m)"'] & \mathbb{Z}
\end{tikzcd}$$

Finally, $\Phi_{(a_i)}$ is clearly monoidal on objects. On morphisms, since it is always monoidal on $\mathfrak{C}_0$, it is monoidal if and only if $\Phi_{(a_i)}(p_{k+k'}) = \Phi_{(a_i)}(p_k) + \Phi_{(a_i)}(p_{k'})$

for all k, k'. This is the case if and only if $\Phi_{(a_i)}(p_k) = k\Phi_{(a_i)}(p_1)$, in other words if and only if $ka_k = ka_1$ for all k. $\square$

3.4.2 Classification of invertible closed Klein TQFTs

Let $\mathrm{Vect}_{\mathbb{C}}$ denote the symmetric monoidal catgeory of finite-dimensional complex vector spaces and $\mathbb{C}$-linear maps under tensor product.

Definition 3.29. We define a 2-dimensional closed Klein topological quantum field theory (KTQFT) to be a based symmetric monoidal[3] functor $(F, F_2) : \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$. We say that (F, F_2) is *invertible* if F is a morphism inverting functor.

Note that F must send the object 0 in $\mathcal{C}$ to $\mathbb{C}$, the unit for tensor product.

Definition 3.30. We define the category $\mathrm{TFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$ of 2-dimensional closed KTQFTs by

$$\mathrm{TFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_* \coloneqq \mathrm{SymmMon}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$$

where $\mathrm{SymmMon}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$ denotes the category of based symmetric monoidal functors $\mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$ and based monoidal natural transformations as defined in Section A.1 of the appendix. Recall that any such natural transformation τ must satisfy $\tau_0 = 1_{\mathbb{C}}$.

Definition 3.31. We define the category $\mathrm{InvTFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$ of invertible 2-dimensional closed KTQFTs by

$$\mathrm{InvTFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_* \coloneqq \mathrm{InvSymmMon}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$$

where $\mathrm{InvSymmMon}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$ denotes the full subcategory of $\mathrm{SymmMon}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$ whose objects are morphism inverting functors.

[3] By a based symmetric monoidal functor we mean a symmetric strong monoidal functor in the sense of Mac Lane [ML98] which is strict with respect to units. For a precise definition, along with some elementary properties, we refer the reader to Section A.1 of the appendix.

By Corollary A.13 we have an equivalence of categories

$$\mathrm{InvSymmMon}[\mathcal{C}, \mathrm{Vect}^s_{\mathbb{C}}]_* \to \mathrm{InvTFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$$

where $\mathrm{Vect}^s_{\mathbb{C}}$ denotes the skeleton of $\mathrm{Vect}_{\mathbb{C}}$ described in Section A.2.1 equipped with the symmetric monoidal structure defined in Proposition A.9. There are two important facts to recall regarding this skeleton. Firstly, the objects of $\mathrm{Vect}^s_{\mathbb{C}}$ are $\{\mathbb{C}^n | n \in \mathbb{N}_{>0}\}$ and the unit is $\mathbb{C}$. Secondly, the monoidal structure on $\mathrm{Vect}^s_{\mathbb{C}}$ when restricted to the full subcategory on the single object $\mathbb{C}$ is strict. Furthermore, if we remove the zero map and consider just the invertible linear maps $\mathbb{C} \to \mathbb{C}$, then this monoidal structure coincides with the monoidal structure on $\mathbb{C}^*$ thought of as an abelian group.

Lemma 3.32. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and let $F : \mathscr{C} \to \mathscr{D}$ be a morphism inverting functor. Suppose $\mathscr{C}$ is connected and $\mathscr{D}$ is skeletal. Then F factors through $\mathrm{Aut}_{\mathscr{D}}(F(a))$ for any object $a \in \mathscr{C}$.*

Proof. $\mathscr{D}$ is skeletal and F is morphism inverting, so $F(\mathscr{C})$ is contained in a union of automorphism groups in $\mathscr{D}$. But $\mathscr{C}$ is connected and so $F(\mathscr{C})$ is also connected. Hence $F(\mathscr{C})$ is contained in the single automorphism group $\mathrm{Aut}_{\mathscr{D}}(F(a))$ for any $a \in \mathscr{C}$. □

In particular, if $\mathscr{C}$ and $\mathscr{D}$ are symmetric monoidal categories and F is a based symmetric monoidal functor, then F factors through the automorphism group of the unit in $\mathscr{D}$. We thus have an equivalence of categories

$$\mathrm{InvSymmMon}[\mathcal{C}, \mathbb{C}]_* \to \mathrm{InvTFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*.$$

Any object (F, F_2) in $\mathrm{InvSymmMon}[\mathcal{C}, \mathbb{C}]_*$ can be viewed as a symmetric monoidal functor from $\mathcal{C}$ to the abelian group $\mathbb{C}^*$. Hence, via the same methods as were used to classify functors $\mathcal{C} \to \mathbb{Z}$ (Theorem 3.28) we obtain the following theorem.

Theorem 3.33. *Every functor $\mathcal{C} \to \mathbb{C}^*$ is of the form $F^{\boldsymbol{\mu}}$ for some unique sequence $\boldsymbol{\mu} = (\mu_i)_{i\in\mathbb{N}}$ of non-zero complex numbers. $F^{\boldsymbol{\mu}}$ sends all objects of $\mathcal{C}$ to the only object of $\mathbb{C}^*$, and sends a morphism $\Sigma : n \to m$ in $\mathcal{C}$ to*

$$F^{\boldsymbol{\mu}}(\Sigma) := \mu_m^m \mu_n^{-n} \mu_0^{-(m-n-\chi(\Sigma))}. \qquad \square$$

Note that $\mu_0 = F^{\boldsymbol{\mu}}(P^2)$ and $\mu_k^k = F^{\boldsymbol{\mu}}(p_k)$ for $k \geq 1$. We now consider how $F^{\boldsymbol{\mu}}$ might be given the structure of a symmetric monoidal functor $(F^{\boldsymbol{\mu}}, F_2^{\boldsymbol{\mu}})$.

By considering the lower two diagrams of Definition A.2, and recalling that $\mathcal{C}$ and $\mathbb{C}^*$ are both strict monoidal categories, we see that the isomorphisms $F_2^{\boldsymbol{\mu}} \in \mathbb{C}^*$ must satisfy

$$F_2^{\boldsymbol{\mu}}(n,0) = F_2^{\boldsymbol{\mu}}(0,n) = 1 \text{ for all } n \in \mathbb{N}. \tag{$\dagger$}$$

The $F_2^{\boldsymbol{\mu}}$ must also be natural, and so the following diagram must commute for all objects n, n', m, m' and all morphisms $\Sigma : n \to m$ and $\Sigma' : n' \to m'$ in $\mathcal{C}$.

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "F_2^{\boldsymbol{\mu}}"] \arrow[d, "\mu_m^m\mu_n^{-n}\mu_0^{-(m-n-\chi(\Sigma))}\mu_{m'}^{m'}\mu_{n'}^{-n'}\mu_0^{-(m'-n'-\chi(\Sigma'))}"'] & \mathbb{C}^* \arrow[d, "\mu_{m+m'}^{m+m'}\mu_{n+n'}^{-(n+n')}\mu_0^{-(m+m'-(n+n')-\chi(\Sigma\amalg\Sigma'))}"] \\ \mathbb{C}^* \arrow[r, "F_2^{\boldsymbol{\mu}}"'] & \mathbb{C}^* \end{tikzcd}$$

Hence we have

$$\mu_{m+m'}^{m+m'}\mu_{n+n'}^{-(n+n')}F_2^{\boldsymbol{\mu}}(n,n') = F_2^{\boldsymbol{\mu}}(m,m')\mu_m^m\mu_n^{-n}\mu_{m'}^{m'}\mu_{n'}^{-n'} \text{ for all } n,n',m,m' \in \mathbb{N}.$$

By choosing $m = m' = 0$ and using Equation ($\dagger$) we see that

$$F_2^{\boldsymbol{\mu}}(n,n') = \mu_n^{-n}\mu_{n'}^{-n'}\mu_{n+n'}^{n+n'}$$

and one checks that this also satisfies the symmetry axiom (the upper right hand diagram of Definition A.2). Therefore every functor $\mathcal{C} \to \mathbb{C}^*$ can be uniquely given the structure of a symmetric monoidal functor, and we have completely identified the objects of $\mathrm{InvSymmMon}[\mathcal{C}, \mathbb{C}]_*$.

Remark. The functor $(F^{\boldsymbol{\mu}}, F_2^{\boldsymbol{\mu}})$ is symmetric strict monoidal if and only if all the μ_i are equal for $i \geq 1$.

We now describe the morphisms in $\mathrm{InvSymmMon}[\mathcal{C}, \mathbb{C}]_*$. Suppose we have a based natural transformation $\tau : F^{\boldsymbol{\mu}} \to F^{\boldsymbol{\mu}'}$. Then, by considering the projective plane P^2 as a morphism $0 \to 0$, naturality of τ and the fact that $\tau_0 = 1$ give $\mu_0 = \mu_0'$. So suppose this is the case. For a general morphism $\Sigma : n \to m$ in $\mathcal{C}$, the following diagram must commute.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tau_n"] \arrow[d, "\mu_m^m \mu_n^{-n} \mu_0^{-(m-n-\chi(\Sigma))}"'] & \mathbb{C} \arrow[d, "(\mu_m')^m (\mu_n')^{-n} \mu_0^{-(m-n-\chi(\Sigma))}"] \\ \mathbb{C} \arrow[r, "\tau_m"'] & \mathbb{C} \end{tikzcd}$$

Hence $(\mu_m')^m (\mu_n')^{-n} \tau_n = \tau_m \mu_m^m \mu_n^{-n}$ for all $n, m \in \mathbb{N}$. Choosing $m = 0$ we deduce

$$\tau_n = (\mu_n' \mu_n^{-1})^n.$$

Finally, one checks that τ defines a based monoidal natural transformation $(F^{\boldsymbol{\mu}}, F_2^{\boldsymbol{\mu}}) \to (F^{\boldsymbol{\mu}'}, F_2^{\boldsymbol{\mu}'})$; in other words the diagram of Definition A.4 automatically commutes. Hence there is precisely one morphism between any two objects in $\mathrm{InvSymmMon}[\mathcal{C}, \mathbb{C}]_*$ which are indexed by the same μ_0, and this morphism is an isomorphism. In particular there are no non-trivial automorphisms. Furthermore, there are no morphisms connecting isomorphism classes of objects. Hence, taking a skeleton, we obtain a discrete category with one object for each $\mu_0 \in \mathbb{C}^*$. We have proved the following theorem.

Theorem 3.34. *The category* $\mathrm{InvTFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$ *of invertible closed* 2*-dimensional KTQFTs and based monoidal natural transformations is equivalent to the discrete category* $\mathbb{C}^*$ *of non-zero complex numbers.* □

Remark. We have also shown that $\mathrm{InvTFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$ is equivalent to the category $\mathrm{Inv}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$ of based morphism inverting functors $\mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$ and based natural transformations.

Remark. Given symmetric monoidal categories $\mathscr{C}$ and $\mathscr{D}$, there are two possible definitions that one can make for the category of morphism inverting (and in our case based symmetric monoidal) functors $\mathscr{C} \to \mathscr{D}$. The first is the one that we chose: the category $\mathrm{InvSymmMon}[\mathscr{C}, \mathscr{D}]_*$. We made this choice so that $\mathrm{InvTFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$ would be full as a subcategory of $\mathrm{TFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$. But one could equally well consider the category $\mathrm{SymmMon}[\mathscr{C}, \mathscr{D}^*]_*$ where $\mathscr{D}^* \subset \mathscr{D}$ denotes the subcategory consisting of all objects and those morphisms which are isomorphisms. Although these two categories consist of essentially the same objects (any morphism inverting functor $\mathscr{C} \to \mathscr{D}$ must factor through $\mathscr{D}^*$) they are not in general equivalent, since a natural transformation in $\mathrm{SymmMon}[\mathscr{C}, \mathscr{D}^*]_*$ is required to be an isomorphism, whereas one in $\mathrm{InvSymmMon}[\mathscr{C}, \mathscr{D}]_*$ is not. However, it follows from Theorem 3.34 that any morphism in $\mathrm{InvTFT}[\mathcal{C}, \mathrm{Vect}_{\mathbb{C}}]_*$ is in fact an isomorphism, and so in the case of KTQFTs the two definitions coincide.

Remark. It is known that any 2-dimensional closed KTQFT (F, F_2) corresponds uniquely to a commutative Frobenius algebra A with the following additional structure [TT06].

- An involutive automorphism $\mathbf{x} \mapsto \mathbf{x}^*$ which preserves the pairing on A, that is $(\mathbf{x}^*)^* = \mathbf{x}$, $(\mathbf{xy})^* = \mathbf{x}^*\mathbf{y}^*$ and $\langle \mathbf{x}^*, \mathbf{y}^* \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.
- An element $U \in A$ satisfying:
    - (i) $(\mathbf{a}U)^* = \mathbf{a}U$ for all $\mathbf{a} \in A$
    - (ii) $U^2 = \sum \alpha_{ij} a_i a_j^*$ where $\{a_i\}$ is a basis for A and the copairing $\mathbb{C} \to A \bigotimes A$ is given by $1 \mapsto \sum_{ij} \alpha_{ij} a_i \otimes a_j$.

In terms of cobordisms, the multiplication on the vector space $F(S^1) = A$ corresponds to the pair of pants surface as a morphism $S^1 \amalg S^1 \to S^1$, the unit is given by the image of 1 under the linear map $F(p_1) : \mathbb{C} \to A$, the pairing corresponds to the composition of the pair of pants with the disc as a morphism $S^1 \to \emptyset$, the involution is given by $F(C_r^{S^1})$, and the element U is the image of 1 under the linear map $\mathbb{C} \to A$ corresponding to the Möbius band as a morphism $\emptyset \to S^1$. For the invertible field theory defined by $F^{\mu_0} := F^{\boldsymbol{\mu}}$ with $\mu_i = 1$ for all $i \geq 1$ we observe the following.

Corollary 3.35. *The Frobenius algebra corresponding to the invertible closed KTQFT defined by F^{μ_0} is $\mathbb{C}$ with its usual algebra structure, and pairing given by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \mu_0^2 \boldsymbol{x}\boldsymbol{y}$. The involutive automorphism is the identity, and the element U is equal to μ_0^{-1}.* □

4 The open object category $\bar{\mathcal{O}}$

4.1 A splitting of the classifying space

We would like to compute the fundamental group of the classifying space $\mathrm{B}\bar{\mathcal{O}}$ via the same methods used in the closed case. If we were to attempt this directly, that is by looking at $\mathrm{Aut}_{\bar{\mathcal{O}}[\bar{\mathcal{O}}^{-1}]}(I)$, a slight problem would arise in the final stages. This is because there is no clear way in which to eliminate the closed components of a morphism via equivalences, as we did in Section 3.2. This points towards a component of infinite rank in the fundamental group of $\mathrm{B}\bar{\mathcal{O}}$, generated by all closed surfaces. This can be explained as follows.

Definition 4.1. We define the open category $\mathcal{O}$ to be the subcategory of $\bar{\mathcal{O}}$ containing all objects, and those morphisms which do not have any closed components.

Consider the category $\mathcal{C}_0 = \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}_{>0}}$, the first subcategory of $\mathcal{C}$ defined in Section 3.1. We can view $\mathcal{C}_0$ as a subcategory of $\bar{\mathcal{O}}$; it is the subcategory of closed endomorphisms of the empty 1-manifold 0. We define a functor $\Pi : \mathcal{O} \times \mathcal{C}_0 \to \bar{\mathcal{O}}$ as follows. On objects, Π is the inclusion of the first factor, that is $\Pi((n, 0)) = n$. On morphisms we define $\Pi((\Sigma, \Sigma')) = \Sigma \amalg \Sigma'$. It is clear that Π is functorial since composition in $\mathcal{C}_0$

is just disjoint union. It is also easy to see that Π is an isomorphism of categories, since every morphism in $\bar{\mathcal{O}}$ can be uniquely decomposed into a union of its closed components and its components with boundary. We therefore obtain a splitting $\mathrm{B}\bar{\mathcal{O}} \simeq \mathrm{B}\mathcal{O} \times \mathrm{B}\mathcal{C}_0 \simeq \mathrm{B}\mathcal{O} \times T^\infty$ (where the final equivalence follows by Proposition 3.2). Hence we study the open category $\mathcal{O}$, obtaining results for $\bar{\mathcal{O}}$ as corollaries.

4.2 Fundamental group

Throughout this section Θ will denote the restriction to $\mathcal{O}$ of the functor $\Theta : \mathcal{K} \to \mathbb{Z}$ defined in Section 2.

Theorem 4.2. *$\pi_1(B\mathcal{O}) = \mathbb{Z}$, and $\Theta_* : \pi_1(B\mathcal{O}) \to \pi_1(B\mathbb{Z})$ is an isomorphism.*

Corollary 4.3. *$\pi_1(B\bar{\mathcal{O}})$ is free abelian of infinite rank.*

Proof of Theorem 4.2. Let $\mathfrak{O} := \mathcal{O}[\mathcal{O}^{-1}]$ denote the localisation of $\mathcal{O}$, and for each object $n \in \mathcal{O}$ define $\mathfrak{O}_n := \mathrm{Aut}_{\mathfrak{O}}(n)$. The category $\mathcal{O}$ is strongly connected, so we proceed by looking at $\mathrm{End}_{\mathcal{O}}(I)$.

Disregarding components with entirely free boundary, elements of this monoid take one of three forms depicted in Figure 4.1. For each cobordism in the picture, the thickened lines represent the source and target boundaries of the morphism. The thin lines represent free boundary. The morphism on the left is the disc as an endomorphism of the interval I, and the centre morphism is two discs. The morphism on the right is the cylinder as an endomorphism of I, and we often refer to this surface as the *whistle*. Each component has some genus g, number of crosscaps k, and number of windows w. Components of a morphism which have entirely free boundary are discs, also with genus, crosscaps and windows. We will refer to the standard disc D^2 with entirely free boundary as the *free* disc.

Remark. Once again we need not mention boundary maps. In the localised category $\mathfrak{O}$, conjugation $c_\alpha : \mathfrak{O}_1 \to \mathfrak{O}_0$ is an isomorphism for any $\alpha : 0 \to 1$. Let α be the disc as a morphism $0 \to 1$, and note that an inverse for α is the disc: $1 \to 0$ union the inverse of a free disc. Under c_α, the disc C_r^I and the identity 1_I both map to $\emptyset$. It follows that C_r^I and 1_I are equivalent in $\mathfrak{O}$.

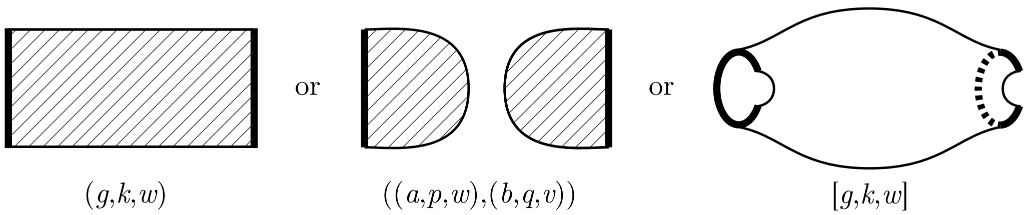

Figure 4.1: Three types of morphisms in $\mathrm{End}_{\mathcal{O}}(I)$.

From the above discussion we can write the monoid $\mathrm{End}_{\mathcal{O}}(I)$ as

$$\Big(\big(\tfrac{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}{\sim}\big) \amalg \Big(\big(\tfrac{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}{\sim}\big) \times \big(\tfrac{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}{\sim}\big)\Big) \amalg \big(\tfrac{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}{\sim}\big)\Big) \times \big(\mathbb{N}^{\mathbb{N}\times\mathbb{N}} \times \mathbb{N}^{\mathbb{N}_{>0}\times\mathbb{N}}\big)$$

where the relation on each component is $(g, k, w) \sim (0, 2g + k, w)$ for non-zero k. Addition in the monoid is non-commutative and defined by the geometry. Elements are written in one of the following ways:

- $((g, k, w); n_{000}, n_{100}, n_{010}, \ldots, n_{001}, n_{101}, n_{011}, \ldots, \ldots)$ represents a connected morphism as in the left of Figure 4.1 along with n_{000} free discs, n_{100} punctured tori, n_{010} Möbius bands,…,n_{001} annuli, n_{101} twice punctured tori, n_{011} twice punctured projective planes, and so on (n_{ijt} is the number of discs with i handles, j crosscaps, and t windows).

- $((a, p, w), (b, q, v); n_{000}, n_{100}, n_{010}, \ldots, n_{001}, n_{101}, n_{011}, \ldots, \ldots)$ represents a disconnected morphism as in the centre of Figure 4.1 along with n_{000} free discs, n_{100} punctured tori, and so on.

- $([g,k,w];n_{000},n_{100},n_{010},\ldots,n_{001},n_{101},n_{011},\ldots,\ldots)$ represents a morphism as in the right of Figure 4.1 along with n_{000} free discs etc, where we use square brackets to distinguish the whistle from the disc in the marked component.

We consider the images of these cobordisms under $c_\alpha : \mathfrak{O}_1 \to \mathfrak{O}_0$, where α and α^{-1} are defined as in the previous remark. We have:

$$\begin{aligned}
\pm((g,k,w);n_{000},n_{100},n_{010},\ldots) &\mapsto \pm(n_{000}-1,n_{100},\ldots,n_{gkw}+1,\ldots)\\
\pm((a,p,w),(b,q,v);n_{000},n_{100},n_{010},\ldots) &\mapsto \pm(n_{000}-1,n_{100},\ldots,n_{apw}+1,\ldots,\\
&\qquad\qquad n_{bqv}+1,\ldots)\\
\pm([g,k,w];n_{000},n_{100},n_{010},\ldots) &\mapsto \pm(n_{000}-1,n_{100},\ldots,n_{gk(w+1)}+1,\ldots)
\end{aligned}$$

and so injectivity of c_α forces the following identifications in $\mathfrak{O}_1$:

(i) $\pm((g,k,w);n_{000},n_{100},n_{010},\ldots)$

$$= \pm\left(\left(g+\sum_{i,t} in_{i0t}, k+\sum_{j,t} jn_{0jt}, w+\sum_{i,t} tn_{i0t}+\sum_{j,t} tn_{0jt}\right);\right.$$
$$\left.\sum_{i,t} n_{i0t}+\sum_{j,t} n_{0jt},0,0,\ldots\right)$$

(ii) $\pm((a,p,w),(b,q,v);n_{000},n_{100},n_{010},\ldots)$

$$= \pm\left(\left(a+b+\sum_{i,t} in_{i0t}, p+q+\sum_{j.t} jn_{0jt}, w+v+\sum_{i,t} tn_{i0t}+\sum_{j,t} tn_{0jt}\right);\right.$$
$$\left.1+\sum_{i,t} n_{i0t}+\sum_{j,t} n_{0jt},0,0,\ldots\right)$$

(iii) $\pm([g,k,w];n_{000},n_{100},n_{010},\ldots)$

$$= \pm\left(\left(g+\sum_{i,t} in_{i0t}, k+\sum_{j,t} jn_{0jt}, w+1+\sum_{i,t} tn_{i0t}+\sum_{j,t} tn_{0jt}\right);\right.$$
$$\left.\sum_{i,t} n_{i0t}+\sum_{j,t} n_{0jt},0,0,\ldots\right).$$

We now show that the genus and free discs can be eliminated from each of the above, meaning that every element of $\mathfrak{O}_1$ can be written in the form $\pm((0,K,W);0,0,\ldots)$ for some $K,W \in \mathbb{N}$. This requires the following lemma and proposition.

Lemma 4.4. *In* $\mathfrak{O}$*,* $((g,k,w);n_{000},0,0,\ldots) = ((0,2g+k,w);n_{000},0,0,\ldots)$ *for all* k*.*

Proof. Let ${}^{i}\mathcal{O}_1$ denote the subcategory of $\mathcal{O}$ with one object, the interval I, and morphism set consisting of connected surfaces of the form $((g,k,w);0,0,\ldots)$, as in the left of Figure 4.1, whose boundary maps are inclusions. We identify ${}^{i}\mathcal{O}_1$ with the monoid $(\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}{\sim})$ where the relation is $((g,k,w);0,0,\ldots) \sim ((0,2g+k,w);0,0,\ldots)$ for non-zero k. Via the methods of Theorem 3.8 one checks that $\mathscr{G}({}^{i}\mathcal{O}_1) = \mathbb{Z}\times\mathbb{Z}$, where localising induces the required equivalence. $\square$

For the following arguments it is convenient to adopt $\pm$ notation on each entry in the marked component. For example, $((0,-1,0);0,0,\ldots)$ denotes an inverse Möbius band, and $((0,1,-1);0,0,\ldots)$ denotes the composition of a Möbius band with an inverse annulus.

Proposition 4.5. $((0,0,-1);0,0,\ldots) = ((0,0,0);1,0,0,\ldots)$ *in* $\mathfrak{O}$*.*

Proof. We consider conjugation $c_\alpha : \mathfrak{O}_2 \to \mathfrak{O}_1$ where α is the union of two discs as a morphism $1 \to 2$ as in Figure 4.2. The following are both inverses for α. Let β_1 be the disc as a morphism $2 \to 1$, and let $\beta_2 : 2 \to 1$ be the union of two discs and an inverse free disc (see Figure 4.2). Let $\gamma : 2 \to 2$ be two discs as in the figure. Since we must have $\beta_1\gamma\alpha = \beta_2\gamma\alpha$ in $\mathfrak{O}$, we see that the annulus union a free disc is equivalent to the identity, both as morphisms $1 \to 1$. $\square$

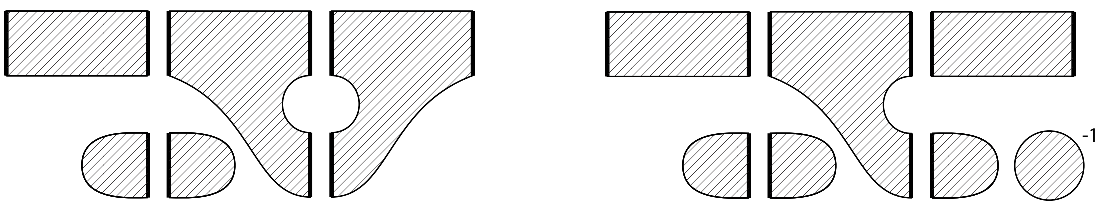

Figure 4.2: $c_\alpha(\gamma) = \beta_1\gamma\alpha$ and $c_\alpha(\gamma) = \beta_2\gamma\alpha$.

Corollary 4.6. $((0,0,-w);0,0,\ldots) = ((0,0,0);w,0,0,\ldots)$ *in* $\mathfrak{O}$*.*

Corollary 4.7. $((0,k,w);n_{000},0,0,\dots) = ((0,k,w-n_{000});0,0,\dots)$ *in* $\mathfrak{O}$. $\square$

Using Lemma 4.4 and Corollary 4.7 we can rewrite relations (i), (ii) and (iii) as follows.

(i)$'$ $\pm\,((g,k,w);n_{000},n_{100},n_{010},\dots)$

$$= \pm\left(\left(0, 2g+k+2\sum_{i,t} in_{i0t} + \sum_{j,t} jn_{0jt},\right.\right.$$
$$\left.\left. w + \sum_{i,t}(t-1)n_{i0t} + \sum_{j,t}(t-1)n_{0jt}\right);0,0,\dots\right)$$

(ii)$'$ $\pm\,((a,p,w),(b,q,v);n_{000},n_{100},n_{010},\dots)$

$$= \pm\left(\left(0, 2(a+b)+(p+q)+2\sum_{i,t} in_{i0t} + \sum_{j.t} jn_{0jt},\right.\right.$$
$$\left.\left. (w+v-1) + \sum_{i,t}(t-1)n_{i0t} + \sum_{j,t}(t-1)n_{0jt}\right);0,0,\dots\right)$$

(iii)$'$ $\pm\,([g,k,w];n_{000},n_{100},n_{010},\dots)$

$$= \pm\left(\left(0, 2g+k+2\sum_{i,t} in_{i0t} + \sum_{j,t} jn_{0jt},\right.\right.$$
$$\left.\left. w + 1 + \sum_{i,t}(t-1)n_{i0t} + \sum_{j,t}(t-1)n_{0jt}\right);0,0\dots\right)$$

We see that $\pi_1(\mathrm{B}\mathcal{O}) = \mathbb{Z}\times\mathbb{Z}$ modulo further relations (which may be trivial).

The functor $\Theta : \mathcal{O} \to \mathbb{Z}$ is morphism inverting and hence factors through $\mathfrak{O}$. We denote the corresponding functor $\mathfrak{O} \to \mathbb{Z}$ by $\bar{\Theta}$. Since $\bar{\Theta}|_{\mathfrak{O}_1}$ is surjective (for example $\bar{\Theta}((0,0,\pm w);0,0,\dots) = \pm w$) it follows that $\pi_1(\mathrm{B}\mathcal{O})$ cannot be smaller than $\mathbb{Z}$. This reduces us to three possible cases: $\pi_1(\mathrm{B}\mathcal{O})$ is either isomorphic to $\mathbb{Z}\times\mathbb{Z}$, or to $\mathbb{Z}\times\mathbb{Z}_a$ for some $a>1$, or simply to $\mathbb{Z}$.

It remains to show that the first two cases result in a contradiction. We do this by showing that every functor $F : \mathcal{O} \to H$, where H is a group, must satisfy a certain property.

Proposition 4.8. $((0,2,0);0,0,\ldots) = ((0,0,2);0,0,\ldots)$ *in* $\mathfrak{O}$.

Proof. Using equation (iii)$''$ above, it suffices to show that $([1,0,0];0,0,\ldots) = ([0,0,2];0,0,\ldots)$ in $\mathfrak{O}$. Once again we consider the conjugation map $c_\alpha : \mathfrak{O}_2 \to \mathfrak{O}_1$, this time taking $\alpha : 1 \to 2$ to be the whistle union a disc. Let $\beta_1 : 2 \to 1$ be the pair of pants surface, and let $\beta_2 : 2 \to 1$ be the punctured whistle, union a disc, union the inverse of a free disc. Each of these morphisms is depicted in Figure 4.3. Both $\beta_1\alpha$ and $\beta_2\alpha$ are homeomorphic to the twice punctured whistle. Composing by α^{-1} on the right, we see that $\beta_1 = \beta_2$ in $\mathfrak{O}$. Therefore $\beta_1\gamma\alpha = \beta_2\gamma\alpha$ for any $\gamma : 2 \to 2$. We choose γ to be the pair of pants surface union a disc as in Figure 4.3.

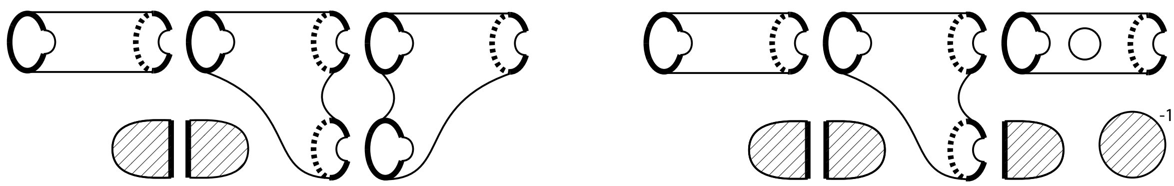

Figure 4.3: $c_\alpha(\gamma) = \beta_1\gamma\alpha$ and $c_\alpha(\gamma) = \beta_2\gamma\alpha$.

The surface $\beta_1\gamma\alpha$ is homeomorphic to a thrice punctured whistle of genus 1 union a free disc. The surface $\beta_2\gamma\alpha$ is the four-punctured whistle. Composing each with an inverse twice punctured whistle, we see that the whistle of genus 1 is equivalent to the punctured whistle union an inverse free disc. The result then follows by Proposition 4.5. $\square$

Corollary 4.9. *Let H be a group. Every functor $F : \mathcal{O} \to H$ satisfies* $F(((0,2,0);0,0,\ldots)) = F(((0,0,2);0,0,\ldots))$ *in* H.

Proof. Any such functor F is morphism inverting, and hence factors through $\mathfrak{O}$ via the functor $\bar{F}$. By Proposition 4.8 we must have $F(((0,2,0);0,0,\ldots)) = \bar{F}(((0,2,0);0,0,\ldots)) = \bar{F}(((0,0,2);0,0,\ldots)) = F(((0,0,2);0,0,\ldots))$. $\square$

To complete the proof we show that $\pi_1(\mathrm{B}\mathcal{O}) = \mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}_a$ implies the existence of a functor contradicting Corollary 4.9. In order to do this we need one more result. The following is a slight adaptation of a proposition by Douglas [Dou00, Proposition 1.3].

Proposition 4.10. *Let H be a group. Let $\mathscr{C}$ be a strongly connected category, and x any object of $\mathscr{C}$. Denote the projection $\mathscr{C} \to \mathscr{C}[\mathscr{C}^{-1}]$ by ψ. For any homomorphism $f : \mathrm{Aut}_{\mathscr{C}[\mathscr{C}^{-1}]}(x) \to H$, there exists functors $F : \mathscr{C} \to H$ and $\bar{F} : \mathscr{C}[\mathscr{C}^{-1}] \to H$ such that $F = \bar{F} \circ \psi$ and $\bar{F}|_{\mathrm{Aut}_{\mathscr{C}[\mathscr{C}^{-1}]}(x)} = f$.*

The proposition is summarised by the following commutative diagram.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "\psi"] \arrow[d, dashed, "F"'] & \mathscr{C}[\mathscr{C}^{-1}] \arrow[dl, dashed, "\bar{F}"'] \\
H & \mathrm{Aut}_{\mathscr{C}[\mathscr{C}^{-1}]}(x) \arrow[l, "f"] \arrow[u, hook]
\end{tikzcd}$$

Proof. We construct the functor $\bar{F}$, and then define F to be the composition $\bar{F} \circ \psi$. Since $\mathscr{C}$ is strongly connected, we may choose a morphism $p_k : x \to k$ for every object k in $\mathscr{C}$. We choose p_x to be the identity 1_x. Define the functor $\bar{F}$ to take all objects of $\mathscr{C}$ to the single object of H, and to send a morphism $g : k \to k'$ to $\bar{F}(g) := f(p_{k'}^{-1} \circ g \circ p_k)$. For a second morphism $g' : k' \to k''$ we have

$$\begin{aligned}
\bar{F}(g' \circ g) &= f(p_{k''}^{-1} \circ (g' \circ g) \circ p_k) \\
&= f(p_{k''}^{-1} \circ g' \circ (p_{k'} \circ p_{k'}^{-1}) \circ g \circ p_k) \\
&= f(p_{k''}^{-1} \circ g' \circ p_{k'}) \circ f(p_{k'}^{-1} \circ g \circ p_k) \\
&= \bar{F}(g') \circ \bar{F}(g)
\end{aligned}$$

and so $\bar{F}$ is a functor. Finally, for a morphism l in $\mathrm{Aut}_{\mathscr{C}[\mathscr{C}^{-1}]}(x)$ we have $\bar{F}(l) = f(1_x \circ l \circ 1_x) = f(l)$ as required. $\square$

We can now complete the proof of Theorem 4.2. Suppose $\mathfrak{O}_1 = \mathbb{Z} \times \mathbb{Z}$, and let $f : \mathfrak{O}_1 \to \mathbb{Z} \times \mathbb{Z}$ be the identity. Then by Proposition 4.10 there exists a functor $F : \mathcal{O} \to \mathbb{Z} \times \mathbb{Z}$ whose corresponding functor $\bar{F}$ restricts to the identity on $\mathfrak{O}_1$. In particular we have $F(((0,2,0);0,0,\ldots)) = (2,0)$ and $F(((0,0,2);0,0,\ldots)) = (0,2)$, contradicting Corollary 4.9. Assuming $\mathfrak{O}_1 = \mathbb{Z} \times \mathbb{Z}_a$ for some $a > 1$, and again considering the identity homomorphism, yields the same contradiction. Therefore we must have $\pi_1(\mathrm{B}\mathcal{O}) = \mathfrak{O}_1 = \mathbb{Z}$.

The final part of the theorem follows since $\pi_1(\mathrm{B}\mathcal{O})$ is generated by the Möbius band (or equivalently the annulus) which maps to 1 under the functor Θ. $\square$

4.3 Homotopy type

Theorem 4.11. *$\mathcal{O}$ is a connected symmetric strict monoidal category. Hence its classifying space $B\mathcal{O}$ has the homotopy type of an infinite loop space. Furthermore, there is a simply connected infinite loop space Y such that $B\mathcal{O}$ is homotopy equivalent to $Y \times S^1$.*

Corollary 4.12. *$\bar{\mathcal{O}}$ is also a connected symmetric strict monoidal category. The classifying space $B\bar{\mathcal{O}}$ is homotopy equivalent to $Y \times T^\infty$.*

Proof of Theorem 4.11. The product $\otimes : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ is given by disjoint union. The natural transformation from the product to the twisted product is given by crossing discs. It follows that $\mathrm{B}\mathcal{O}$ has the homotopy type of an infinite loop space.

Let $\Sigma : n \to m$ and $\Sigma' : n' \to m'$ be morphisms in $\mathcal{O}$. Then

$$\begin{aligned}\Theta(\Sigma \amalg \Sigma') &= m + m' - \chi(\Sigma \amalg \Sigma') \\ &= m - \chi(\Sigma) + m' - \chi(\Sigma') \\ &= \Theta(\Sigma) + \Theta(\Sigma')\end{aligned}$$

and therefore the functor $\Theta : \mathcal{O} \to \mathbb{Z}$ is symmetric monoidal. It follows that $\mathrm{B}\Theta$ is a map of infinite loop spaces, and hence its homotopy fibre Y is an infinite loop space.

Let ${}^{i}\mathcal{O}_1^{-}$ denote the subcategory of ${}^{i}\mathcal{O}_1$ whose morphisms are non-orientable (with the exception of the identity 1_I) and have no windows. The composition $\mathbb{N} = {}^{i}\mathcal{O}_1^{-} \hookrightarrow \mathcal{O} \xrightarrow{\Theta} \mathbb{Z}$ is just the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$, and therefore $\mathrm{B}\Theta$ has a section. By Theorem 4.2 the induced map $\mathrm{B}\Theta_*$ is an isomorphism on π_1, and so one obtains the desired splitting. □

4.4 Applications to TQFT

4.4.1 Classification of functors $\bar{\mathcal{O}} \to \mathbb{Z}$

We classify functors $\mathcal{O} \to \mathbb{Z}$, thereby obtaining a classification of functors $\bar{\mathcal{O}} \to \mathbb{Z}$ as a corollary.

Via similar arguments to those in Section 3.4.1 we see that any functor $F : \mathcal{O} \to \mathbb{Z}$ is determined by its image of:

- A generator of $\mathfrak{O}_0 = \mathbb{Z}$
- A set of connecting morphisms $p_k : 0 \to k$, one for each object k in $\mathcal{O}$.

We take p_k (for $k \geq 1$) to be the union of k discs, each as a cobordism from $\emptyset$ to I. We set p_0 to be the identity $1_0 = \emptyset$. For a generator of $\mathfrak{O}_0$ we choose the free disc D^2. Note that this is indeed a generator since it maps to -1 under the functor Θ.

Proposition 4.13. *Any functor $F : \mathcal{O} \to \mathbb{Z}$ is determined by where it sends the free disc D^2 and the connecting morphisms p_k. Conversely, for any set of integers $\{b_0, b_1, b_2, \ldots\}$ there exists a functor sending the free disc to b_0, and sending p_k to b_k for all $k \geq 1$.*

Proof. The first statement follows from the above discussion. For the second statement, let $\Sigma : k \to k'$ be a morphism in $\mathcal{O}$. Any such functor $F : \mathcal{O} \to \mathbb{Z}$ must satisfy $F(\Sigma) = F(p_{k'}) + \bar{F}(\amalg_\alpha D^2) + F(p_k)^{-1}$ where $\alpha = -\bar{\Theta}(p_{k'}^{-1} \circ \Sigma \circ p_k) = -\Theta(\Sigma) \in \mathbb{Z}$. Hence we define

$$F(\Sigma) := \begin{cases} b_{k'} - b_k - b_0\Theta(\Sigma) & \text{if } k, k' \geq 1 \\ b_{k'} - b_0\Theta(\Sigma) & \text{if } k = 0,\ k' \geq 1 \\ -b_k - b_0\Theta(\Sigma) & \text{if } k \geq 1,\ k' = 0 \\ -b_0\Theta(\Sigma) & \text{if } k = k' = 0. \end{cases}$$

One checks that F is functorial, and that $F(D^2) = b_0$ and $F(p_k) = b_k$ for $k \geq 1$. □

For a sequence of rational numbers $(a_i)_{i\in\mathbb{N}}$ such that $a_0 \in \mathbb{Z}$ and $ka_k \in \mathbb{Z}$ for all k, we define a functor $\Phi_{(a_i)} : \mathcal{O} \to \mathbb{Z}$ which is constant on objects, and sends a morphism $\Sigma : n \to m$ to

$$\Phi_{(a_i)}(\Sigma) := a_m m - a_n n + a_0\Theta(\Sigma).$$

One checks that $\Phi_{(a_i)}$ is functorial. Since $\Phi_{(a_i)}$ sends the free disc to $-a_0$, and sends p_k to $a_k k$, we see that the functors are all distinct. For example, the functor Θ can be uniquely written as $\Phi_{(1,0,0,\ldots)}$. Suppose we are given a set of integers $\{b_0, b_1, b_2, \ldots\}$. If we set $a_0 = -b_0$ and $a_k = \frac{b_k}{k}$ for $k \geq 1$ then $\Phi_{(a_i)}$ is the functor which sends D^2 to b_0, and sends p_k to b_k for $k \geq 1$.

Theorem 4.14. *Every functor $\mathcal{O} \to \mathbb{Z}$ is of the form $\Phi_{(a_i)}$ for some unique sequence $(a_i)_{i\in\mathbb{N}}$, where $a_0 \in \mathbb{Z}$ and $ka_k \in \mathbb{Z}$ for all k. Its isomorphism class and homotopy class are both determined by a_0, and it is strict monoidal if and only if $a_1 = a_2 = a_3 = \ldots$.*

Proof. The first statement follows from the above discussion. For the second statement, note that the functor $\Phi_{(a_i)}$ sends the Möbius band (as a morphism $1 \to 1$) to a_0. Suppose that we have a natural transformation $\tau : \Phi_{(a_i)} \to \Phi_{(a'_i)}$. Then $\mathrm{B}\Phi_{(a_i)}$ and $\mathrm{B}\Phi_{(a'_i)}$ are homotopic and hence $a_0 = a'_0$. Conversely we define a natural transformation $\tau : \Phi_{(a_0,a_i)} \to \Phi_{(a_0,a'_i)}$ (where $i \geq 1$) by $\tau_k := k(a'_k - a_k)$ for each object $k \in \mathcal{O}$. One checks that the appropriate square commutes as required. Finally, $\Phi_{(a_i)}$

is monoidal if and only if it is monoidal on the p_k, in other words if and only if $\Phi_{(a_i)}(p_k)=k\Phi_{(a_i)}(p_1)$. $\square$

We can extend this result to give a classification of functors $\bar{\mathcal{O}} \to \mathbb{Z}$ via the isomorphism $\Pi : \bar{\mathcal{O}} \to \mathcal{O} \times \mathcal{C}_0$ defined in Section 4.1. Any such functor is determined by its image of the free disc D^2, the connecting morphisms p_k, and all closed surfaces. Denote by $\Sigma_{g,0}$ (for $g \geq 0$) the orientable generator of $\mathcal{C}_0$ with g handles, and denote by $\Sigma_{0,k}$ (for $k \geq 1$) the non-orientable generator of $\mathcal{C}_0$ with k crosscaps. We (uniquely) write each morphism Σ in $\bar{\mathcal{O}}$ as $\Sigma = (\Sigma', \Sigma'') \in \mathcal{O} \times \mathcal{C}_0$ via the isomorphism Π. For rational sequences $(a_i)_{i\in\mathbb{N}}, (b_j)_{j\in\mathbb{N}}, (c_t)_{t\in\mathbb{N}_{>0}}$ where $a_0 \in \mathbb{Z}$ and $ka_k, b_j, c_t \in \mathbb{Z}$ for all k, j, t, we define a functor $\Psi_{(a_i),(b_j),(c_t)} : \bar{\mathcal{O}} \to \mathbb{Z}$ which is constant on objects, and sends a morphism Σ to

$$\Psi_{(a_i),(b_j),(c_t)}(\Sigma) := \Phi_{(a_i)}(\Sigma') + \sum_{j\geq 0} b_j n_j(\Sigma'') + \sum_{t\geq 1} c_t m_t(\Sigma'')$$

where $n_j(\Sigma'')$ is the number of copies of $\Sigma_{j,0}$ in Σ'' and $m_t(\Sigma'')$ is the number of copies of $\Sigma_{0,t}$ in Σ''. It is clear that $\Psi_{(a_i),(b_j),(c_t)}$ is functorial since $\Phi_{(a_i)}$ is a functor and composition in $\mathcal{C}_0$ is just disjoint union. Note that $\Psi_{(a_i),(b_j),(c_t)}$ sends D^2 to $-a_0$, sends p_k to $a_k k$, sends $\Sigma_{g,0}$ to b_g, and sends $\Sigma_{0,k}$ to c_k. Furthermore, given integers $\{d_i\}_{i\in\mathbb{N}}$, $\{e_j\}_{j\in\mathbb{N}}$, $\{f_t\}_{t\in\mathbb{N}_{>0}}$, if we set $a_0 = -d_0$, $a_k = \frac{d_k}{k}$ for $k \geq 1$, $b_j = e_j$ and $c_t = f_t$, then $\Psi_{(a_i),(b_j),(c_t)}$ is the functor which sends D^2 to d_0, p_k to d_k for $k \geq 1$, $\Sigma_{g,0}$ to e_g and $\Sigma_{0,k}$ to f_k. We thus obtain the following corollary to Theorem 4.14.

Corollary 4.15. *Every functor $\bar{\mathcal{O}} \to \mathbb{Z}$ is of the form $\Psi_{(a_i),(b_j),(c_t)}$ for some unique triple of sequences $(a_i)_{i\in\mathbb{N}}, (b_j)_{j\in\mathbb{N}}, (c_t)_{t\in\mathbb{N}_{>0}}$ where $a_0 \in \mathbb{Z}$ and $ka_k, b_j, c_t \in \mathbb{Z}$ for all k, j, t. Its isomorphism class and homotopy class are both determined by $a_0, \{b_j\}$ and $\{c_t\}$, and it is strict monoidal if and only if $a_1 = a_2 = a_3 = \ldots$.* $\square$

4.4.2 Classification of invertible open Klein TQFTs

Definition 4.16. We define a 2-dimensional open KTQFT to be a based symmetric monoidal functor $(F, F_2) : \bar{\mathcal{O}} \to \mathrm{Vect}_{\mathbb{C}}$. We say that (F, F_2) is *invertible* if F is a morphism inverting functor.

Definition 4.17. We define the category $\mathrm{TFT}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_*$ of 2-dimensional open KTQFTs by

$$\mathrm{TFT}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_* := \mathrm{SymmMon}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_*.$$

Definition 4.18. We define the category $\mathrm{InvTFT}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_*$ of invertible 2-dimensional open KTQFTs by

$$\mathrm{InvTFT}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_* := \mathrm{InvSymmMon}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_*.$$

By Corollary A.13 and Lemma 3.32 we have an equivalence of categories

$$\mathrm{InvSymmMon}[\bar{\mathcal{O}}, \mathbb{C}]_* \to \mathrm{InvTFT}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_*.$$

As in the closed case we view an object (F, F_2) in $\mathrm{InvSymmMon}[\bar{\mathcal{O}}, \mathbb{C}]_*$ as a symmetric monoidal functor $\bar{\mathcal{O}} \to \mathbb{C}^*$. Via the same methods as were used to prove Corollary 4.15 we obtain the following.

Theorem 4.19. *Every functor $\bar{\mathcal{O}} \to \mathbb{C}^*$ is of the form $F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}$ for some unique triple of sequences $\boldsymbol{\mu} = (\mu_i)_{i\in\mathbb{N}}, \boldsymbol{b} = (b_j)_{j\in\mathbb{N}}, \boldsymbol{c} = (c_t)_{t\in\mathbb{N}_{>0}}$ of non-zero complex numbers. $F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}$ sends all objects of $\bar{\mathcal{O}}$ to the only object of $\mathbb{C}^*$, and sends a morphism $\Sigma : n \to m$ in $\bar{\mathcal{O}}$ to*

$$F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(\Sigma) := \mu_m^m \mu_n^{-n} \mu_0^{-\Theta(\Sigma')} \prod_{j\geq 0} b_j^{n_j(\Sigma'')} \prod_{t\geq 1} c_t^{m_t(\Sigma'')}.$$

where $\Sigma = (\Sigma', \Sigma'') \in \mathcal{O} \times \mathcal{C}_0$, $n_j(\Sigma'')$ is the number of copies of $\Sigma_{j,0}$ in Σ'' and $m_t(\Sigma'')$ is the number of copies of $\Sigma_{0,t}$ in Σ''. □

Note that $\mu_0 = F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(D^2)$, $\mu_k^k = F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(p_k)$ for $k \geq 1$, $b_g = F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(\Sigma_{g,0})$ and $c_k = F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(\Sigma_{0,k})$. We now consider how $F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}$ might be given the structure of a symmetric monoidal functor $(F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}, F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}})$.

By considering the lower two diagrams of Definition A.2, and recalling that $\bar{\mathcal{O}}$ and $\mathbb{C}^*$ are both strict monoidal categories, we see that the isomorphisms $F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}} \in \mathbb{C}^*$ must satisfy

$$F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(n,0) = F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(0,n) = 1 \text{ for all } n \in \mathbb{N}. \tag{‡}$$

The $F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}$ must also be natural, and so the following diagram must commute for all objects n, n', m, m' and all morphisms $\Sigma : n \to m$ and $\Gamma' : n' \to m'$ in $\bar{\mathcal{O}}$

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r, "F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}"] \arrow[d, "F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(\Sigma)F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(\Gamma)"'] & \mathbb{C}^* \arrow[d, "F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(\Sigma\amalg\Gamma)"] \\
\mathbb{C}^* \arrow[r, "F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}"'] & \mathbb{C}^*
\end{tikzcd}$$

where

$$F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(\Sigma)F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(\Gamma) = \mu_m^m \mu_n^{-n} \mu_{m'}^{m'} \mu_{n'}^{-n'} \mu_0^{-\Theta(\Sigma')-\Theta(\Gamma')} \prod_{j\geq 0} b_j^{n_j(\Sigma'')+n_j(\Gamma'')} \prod_{t\geq 1} c_t^{m_t(\Sigma'')+m_t(\Gamma'')}$$

and

$$F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(\Sigma \amalg \Gamma) = \mu_{m+m'}^{m+m'} \mu_{n+n'}^{-(n+n')} \mu_0^{-\Theta(\Sigma'\amalg\Gamma')} \prod_{j\geq 0} b_j^{n_j(\Sigma''\amalg\Gamma'')} \prod_{t\geq 1} c_t^{m_t(\Sigma''\amalg\Gamma'')}.$$

Noting that $n_j(\Sigma'' \amalg \Gamma'') = n_j(\Sigma'') + n_j(\Gamma'')$ and $m_t(\Sigma'' \amalg \Gamma'') = m_t(\Sigma'') + m_t(\Gamma'')$, we have

$$\mu_{m+m'}^{m+m'} \mu_{n+n'}^{-(n+n')} F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(n,n') = F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(m,m') \mu_m^m \mu_n^{-n} \mu_{m'}^{m'} \mu_{n'}^{-n'} \text{ for all } n, n', m, m' \in \mathbb{N}.$$

By choosing $m = m' = 0$ and using Equation (‡) we see that

$$F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}(n,n') = \mu_n^{-n} \mu_{n'}^{-n'} \mu_{n+n'}^{n+n'}$$

and one checks that this also satisfies the symmetry axiom (the upper right hand diagram of Definition A.2). Therefore every functor $\bar{\mathcal{O}} \to \mathbb{C}^*$ can be uniquely given the structure of a symmetric monoidal functor, and we have completely identified the objects of $\mathrm{InvSymmMon}[\bar{\mathcal{O}}, \mathbb{C}]_*$.

Remark. The functor $(F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}, F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}})$ is symmetric strict monoidal if and only if all the μ_i are equal for $i \geq 1$.

We now describe the morphisms in $\mathrm{InvSymmMon}[\bar{\mathcal{O}}, \mathbb{C}]_*$. Suppose we have a based natural transformation $\tau : F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}} \to F^{\boldsymbol{\mu}',\boldsymbol{b}',\boldsymbol{c}'}$. Then, by considering the free disc D^2 as a morphism $0 \to 0$, naturality of τ and the fact that $\tau_0 = 1$ give $\mu_0 = \mu'_0$. Similarly, considering the closed surface $\Sigma_{g,0}$ as a morphism $0 \to 0$ gives $b_g = b'_g$ for all $g \geq 0$, and considering the closed surface $\Sigma_{0,k}$ gives $c_k = c'_k$ for all $k \geq 1$. So suppose this is the case. For a general morphism $\Sigma : n \to m$ in $\bar{\mathcal{O}}$, the following diagram must commute.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tau_n"] \arrow[d, "\mu_m^m \mu_n^{-n} \mu_0^{-\Theta(\Sigma')} \prod_{j\geq 0} b_j^{n_j(\Sigma'')} \prod_{t\geq 1} c_t^{m_t(\Sigma'')}"'] & \mathbb{C} \arrow[d, "(\mu'_m)^m (\mu'_n)^{-n} \mu_0^{-\Theta(\Sigma')} \prod_{j\geq 0} b_j^{n_j(\Sigma'')} \prod_{t\geq 1} c_t^{m_t(\Sigma'')}"] \\ \mathbb{C} \arrow[r, "\tau_m"'] & \mathbb{C} \end{tikzcd}$$

Hence $(\mu'_m)^m(\mu'_n)^{-n}\tau_n = \tau_m \mu_m^m \mu_n^{-n}$ for all $n, m \in \mathbb{N}$. Choosing $m = 0$ we deduce

$$\tau_n = (\mu'_n \mu_n^{-1})^n.$$

Finally, one checks that τ defines a based monoidal natural transformation $(F^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}, F_2^{\boldsymbol{\mu},\boldsymbol{b},\boldsymbol{c}}) \to (F^{\boldsymbol{\mu}',\boldsymbol{b},\boldsymbol{c}}, F_2^{\boldsymbol{\mu}',\boldsymbol{b},\boldsymbol{c}})$; in other words the diagram of Definition A.4 automatically commutes. Hence there is precisely one morphism between any two objects in $\mathrm{InvSymmMon}[\bar{\mathcal{O}}, \mathbb{C}]_*$ which are indexed by the same $\mu_0, \boldsymbol{b}$ and $\boldsymbol{c}$, and this morphism is an isomorphism. In particular there are no non-trivial automorphisms. Furthermore, there are no morphisms connecting isomorphism classes of objects. Hence, taking a skeleton, we obtain a discrete category with one object for each sequence $(\mu_0, \boldsymbol{b}, \boldsymbol{c})$ of non-zero complex numbers. We have proved the following theorem.

Theorem 4.20. *The category* $\mathrm{InvTFT}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_*$ *of invertible open* $2-$*dimensional KTQFTs and based monoidal natural transformations is equivalent to the discrete category* $\mathrm{Seq}(\mathbb{C}^*)$ *of infinite sequences of non-zero complex numbers.* $\square$

Remark. We have also shown that $\mathrm{InvTFT}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_*$ is equivalent to the category $\mathrm{Inv}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_*$ of based morphism inverting functors $\bar{\mathcal{O}} \to \mathrm{Vect}_{\mathbb{C}}$ and based natural transformations.

Remark. We have also shown that $\mathrm{InvTFT}[\bar{\mathcal{O}}, \mathrm{Vect}_{\mathbb{C}}]_* = \mathrm{SymmMon}[\bar{\mathcal{O}}, \mathrm{Vect}^*_{\mathbb{C}}]_*$.

Remark. An open 2-dimensional KTQFT (F, F_2) is equivalent to a symmetric Frobenius algebra B with an involutive anti-automorphism $\mathbf{x} \mapsto \mathbf{x}^*$ preserving the pairing [Bra11]. Here the multiplication on the vector space $B = F(I)$ corresponds to the disc as a morphism $I \amalg I \to I$. The unit corresponds to the disc as a morphism $\emptyset \to I$, the pairing corresponds the composition of the former disc with the disc as a morphism $I \to \emptyset$, and the involution corresponds to the bow tie C^I_r. For the invertible field theory defined by $F^{\mu_0, \boldsymbol{b}, \boldsymbol{c}} := F^{\boldsymbol{\mu}, \boldsymbol{b}, \boldsymbol{c}}$ with $\mu_i = 1$ for all $i \geq 1$ we observe the following.

Corollary 4.21. *The Frobenius algebra corresponding to the invertible open KTQFT defined by $F^{\mu_0, \boldsymbol{b}, \boldsymbol{c}}$ is $\mathbb{C}$ with its usual algebra structure (in particular it is commutative) and pairing given by $\langle \mathbf{x}, \mathbf{y} \rangle = \mu_0 \mathbf{x}\mathbf{y}$. The involutive anti-automorphism is the identity.*

□

5 The open-closed category $\mathcal{K}$

5.1 Analysis of the subcategory $\bar{\mathcal{C}}$

Definition 5.1. Let $\bar{\mathcal{C}}$ be the full subcategory of $\mathcal{K}$ on those objects which are closed.

Note that $\bar{\mathcal{C}}$ contains and is closely related to $\mathcal{C}$, the difference being that in $\bar{\mathcal{C}}$ we allow morphisms to have windows. In this section we compute the fundamental group of the classifying space $\mathrm{B}\bar{\mathcal{C}}$. The results obtained, along with those from previous sections, will enable us to easily compute the fundamental group $\pi_1(\mathrm{B}\mathcal{K})$, and hence classify invertible open-closed Klein TQFTs.

We define a functor $\chi_c : \bar{\mathcal{C}} \to \mathbb{Z} \times \mathbb{Z}$ which is constant on objects, and on a morphism $\Sigma : n \to m$ with w windows acts as follows. First we form the surface $\Sigma^f \in \mathcal{C}$ corresponding to Σ by glueing in w discs, one to 'fill' each window. We then map to $\mathbb{Z} \times \mathbb{Z}$ by χ on the first component, and by remembering the number of windows of Σ on the second component. For example, the image of the thrice punctured Klein bottle as a morphism $1 \to 1$ under χ_c is $(-2, 1)$.

Proposition 5.2. *χ_c is functorial.*

Proof. Let $\Sigma_1 : m \to l$ and $\Sigma_2 : n \to m$ be morphisms in $\bar{\mathcal{C}}$. Denote the number of windows of Σ_i by w_i. Since all objects in $\bar{\mathcal{C}}$ are unions of circles, no extra windows can be created by composition. Therefore the number of windows in $\Sigma_1 \circ \Sigma_2$ is $w_1 + w_2$. For the same reason $(\Sigma_1 \circ \Sigma_2)^f = \Sigma_1^f \circ \Sigma_2^f$, and we see that

$$\begin{aligned}
\chi_c(\Sigma_1 \circ \Sigma_2) &= (\chi((\Sigma_1 \circ \Sigma_2)^f), w_1 + w_2) \\
&= (\chi(\Sigma_1^f \circ \Sigma_2^f), w_1 + w_2) \\
&= (\chi(\Sigma_1^f) + \chi(\Sigma_2^f), w_1 + w_2) \\
&= (\chi(\Sigma_1^f), w_1) + (\chi(\Sigma_2^f), w_2), \\
&= \chi_c(\Sigma_1) + \chi_c(\Sigma_2).
\end{aligned}$$

□

Theorem 5.3. *$\pi_1(B\bar{\mathcal{C}}) = \mathbb{Z} \times \mathbb{Z}$, and $(\chi_c)_* : \pi_1(B\bar{\mathcal{C}}) \to \pi_1(B(\mathbb{Z} \times \mathbb{Z}))$ is an isomorphism.*

Proof. $\bar{\mathcal{C}}$ is strongly connected, so we proceed by looking at $\mathscr{G}(\mathrm{End}_{\bar{\mathcal{C}}}(S^1))$. Let $\bar{\mathfrak{C}} := \bar{\mathcal{C}}[\bar{\mathcal{C}}^{-1}]$ denote the localisation of $\bar{\mathcal{C}}$, and define $\bar{\mathfrak{C}}_n := \mathrm{Aut}_{\bar{\mathfrak{C}}}(n)$. Note that any relations which hold in $\mathfrak{C}$ must also hold in $\bar{\mathfrak{C}}$.

Disregarding closed components and components with entirely free boundary, endomorphisms of S^1 in $\bar{\mathcal{C}}$ take one of the two forms in Figure 3.2, with the additional property that each component has $w \in \mathbb{N}$ windows. Again we do not worry about boundary maps, since $C_r^{S^1} = 1_{S^1}$ in $\mathfrak{C}$ and hence also in $\bar{\mathfrak{C}}$. The monoid of endomorphisms of S^1 in $\bar{\mathcal{C}}$ can therefore be written as

$$\left(\left(\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}{\sim}\right) \amalg \left(\left(\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}{\sim}\right) \times \left(\frac{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}{\sim}\right)\right)\right) \times \left(\mathbb{N}^{\mathbb{N}\times\mathbb{N}} \times \mathbb{N}^{\mathbb{N}_{>0}\times\mathbb{N}}\right)$$

where the relation is $(g,k,w) \sim (0, 2g+k, w)$ for non-zero k on each component. An element of the form $((g,k,w); n_{000}, n_{100}, n_{010}, \ldots, n_{001}, n_{101}, n_{011}, \ldots, \ldots)$ represents a connected surface as in the left of Figure 3.2 with genus g, k crosscaps, and w windows, along with n_{000} spheres, n_{100} tori, n_{010} projective planes, …, n_{001} discs, n_{101} punctured tori, n_{011} Möbius bands, and so on. An element of the form $((a,p,w),(b,q,v); n_{000}, \ldots)$ stands for a disconnected surface as in the right of Figure 3.2, where the left-hand component has w windows and the right-hand component has v windows, along with closed components. Addition is non-commutative and defined by the geometry.

Lemma 5.4. *$((g,k,w); n_{000}, 0, 0, \ldots) = ((0, 2(g-n_{000})+k, w); 0, 0, \ldots)$ in $\bar{\mathfrak{C}}$.*

Proof. We decompose the morphism $((g,k,w); n_{000}, 0, 0, \ldots)$ into an element of $\mathcal{C}$ composed with a w-punctured cylinder. The result then follows by Theorem 3.8 and Corollary 3.23. Explicitly:

$$\begin{aligned}((g,k,w); n_{000}, 0, 0, \ldots) &= ((0,0,w); 0, 0, \ldots) \circ ((g,k,0); n_{000}, 0, 0, \ldots)\\ &= ((0,0,w); 0, 0, \ldots) \circ ((0, 2(g-n_{000})+k, 0); 0, 0, \ldots)\\ &= ((0, 2(g-n_{000})+k, w); 0, 0, \ldots).\end{aligned}$$ ∎

Letting $\alpha : 0 \to 1$ be the disc, injectivity of the conjugation map $c_\alpha : \bar{\mathfrak{C}}_1 \to \bar{\mathfrak{C}}_0$ forces the following identifications in $\bar{\mathfrak{C}}$. For the second step in each equivalence we have used Lemma 5.4.

(i) $\pm\left((g,k,w);n_{000},n_{100},\dots\right)$

$$= \pm\left(\left(g+\sum_{i,t} i n_{i0t}, k+\sum_{j,t} j n_{0jt}, w+\sum_{i,t} t n_{i0t}+\sum_{j,t} t n_{0jt}\right); \sum_{i,t} n_{i0t}+\sum_{j,t} n_{0jt},0,0,\dots\right)$$

$$= \pm\left(\left(0, 2g+k+2\sum_{i,t}(i-1)n_{i0t}+\sum_{j,t}(j-2)n_{0jt}, w+\sum_{i,t} t n_{i0t}+\sum_{j,t} t n_{0jt}\right);0,0,\dots\right)$$

(ii) $\pm\left((a,p,w),(b,q,v);n_{000},n_{100},\dots\right)$

$$= \pm\left(\left(a+b+\sum_{i,t} i n_{i0t}, p+q+\sum_{j,t} j n_{0jt}, w+v+\sum_{i,t} t n_{i0t}+\sum_{j,t} t n_{0jt}\right); 1+\sum_{i,t} n_{i0t}+\sum_{j,t} n_{0jt},0,0,\dots\right)$$

$$= \pm\left(\left(0, 2(a+b-1)+(p+q)+2\sum_{i,t}(i-1)n_{i0t}+\sum_{j,t}(j-2)n_{0jt}, (w+v)+\sum_{i,t} t n_{i0t}+\sum_{j,t} t n_{0jt}\right);0,0,\dots\right)$$

The functor χ_c is morphism inverting. Denote the corresponding functor $\bar{\mathfrak{C}} \to \mathbb{Z}\times\mathbb{Z}$ by $\bar{\chi}_c$. Since $\bar{\chi}_c(\pm((0,k,w);0,0,\dots)) = \pm(-k,w)$ we see that $\bar{\chi}_c|_{\bar{\mathfrak{C}}_1} : \bar{\mathfrak{C}}_1 \to \mathbb{Z}\times\mathbb{Z}$ is an isomorphism, and therefore $\pi_1(\mathrm{B}\bar{\mathcal{C}}) = \mathbb{Z}\times\mathbb{Z}$ as required.

The final part of the theorem follows since $\pi_1(\mathrm{B}\bar{\mathcal{C}})$ is generated by the twice punctured projective plane and the cylinder with one window. Under χ_c these map to $(-1,0)$ and $(0,1)$ respectively. □

5.2 Fundamental group

Let $\mathfrak{K}$ denote the localisation of $\mathcal{K}$, and define $\mathfrak{K}_{(m,n)} := \mathrm{Aut}_{\mathfrak{K}}((m,n))$. Note that any relations which hold in $\bar{\mathfrak{C}}$ or $\mathfrak{O}$ must also hold in $\mathfrak{K}$.

Theorem 5.5. *$\pi_1(B\mathcal{K}) = \mathbb{Z}$, and $\Theta_* : \pi_1(B\mathcal{K}) \to \pi_1(B\mathbb{Z})$ is an isomorphism.*

Proof. We proceed by looking at $\mathscr{G}(\mathrm{End}_{\mathcal{K}}(S^1))$. Endomorphisms of the circle in $\mathcal{K}$ take the same form as those in $\bar{\mathcal{C}}$, so we adopt the notation of Section 5.1. Relations (i) and (ii) of Section 5.1 must hold in $\mathfrak{K}$, and so $\pi_1(\mathrm{B}\mathcal{K}) = \mathfrak{K}_{(1,0)} = \mathbb{Z} \times \mathbb{Z}$ modulo relations.

The functor $\Theta : \mathcal{K} \to \mathbb{Z}$ is morphism inverting and hence factors through $\mathfrak{K}$ via the functor $\bar{\Theta} : \mathfrak{K} \to \mathbb{Z}$. The restriction $\bar{\Theta}|_{\mathfrak{K}_{(1,0)}} : \mathfrak{K}_{(1,0)} \to \mathbb{Z}$ is surjective, and therefore $\mathfrak{K}_{(1,0)}$ is isomorphic to one of $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}_a$ for some $a > 1$, or $\mathbb{Z}$.

Proposition 5.6. *Let H be a group. Every functor $F : \mathcal{K} \to H$ satisfies $F(((0,2,0);0,0,\ldots)) = F(((0,0,2);0,0,\ldots))$ in H.*

Proof. We show that $((0,0,3);0,0,\ldots) = ((0,2,1);0,0,\ldots)$ in $\mathfrak{K}$. The result then follows since any such functor F factors through $\mathfrak{K}$.

We can decompose the triple punctured cylinder $((0,0,3);0,0,\ldots)$ into the following morphisms: the cylinder : $S^1 \to I$, followed by the twice punctured disc : $I \to I$, followed by the cylinder : $I \to S^1$. By Proposition 4.8, we obtain an equivalent morphism by replacing the central disc by a punctured Klein bottle : $I \to I$. Glueing these three cobordisms back together we obtain the thrice punctured Klein bottle : $S^1 \to S^1$ as required. □

Now suppose $\mathfrak{K}_{(1,0)} = \mathbb{Z}\times\mathbb{Z}$, and let $f : \mathfrak{K}_{(1,0)} \to \mathbb{Z}\times\mathbb{Z}$ be the identity. By Proposition 4.10 there exists a functor $F : \mathcal{K} \to \mathbb{Z} \times \mathbb{Z}$ with $F(((0,2,0);0,0,\ldots)) = (2,0) \neq (0,2) = F(((0,0,2);0,0,\ldots))$, contradicting Proposition 5.6. Assuming $\mathfrak{K}_{(1,0)} = \mathbb{Z} \times \mathbb{Z}_a$ for $a > 1$ also results in contradiction. Hence $\pi_1(\mathrm{B}\mathcal{K}) = \mathbb{Z}$, generated by the twice punctured projective plane (or equivalently the punctured cylinder). □

5.3 Homotopy type

Theorem 5.7. *$\mathcal{K}$ is a connected symmetric strict monoidal category. Hence its classifying space $B\mathcal{K}$ has the homotopy type of an infinite loop space. Furthermore, there is a simply connected infinite loop space Z such that $B\mathcal{K}$ is homotopy equivalent to $Z \times S^1$.*

Proof. The product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is given by disjoint union. The natural transformation from the product to the twisted product is given by crossing cylinders and discs. Therefore B$\mathcal{K}$ has the homotopy type of an infinite loop space.

Let $\Sigma : (m, n) \to (p, q)$ and $\Sigma' : (m', n') \to (p', q')$ be morphisms in $\mathcal{K}$. Then

$$\begin{aligned}\Theta(\Sigma \amalg \Sigma') &= p + p' + q + q' - m - m' - \chi(\Sigma \amalg \Sigma') \\ &= p + q - m - \chi(\Sigma) + p' + q' - m' - \chi(\Sigma') \\ &= \Theta(\Sigma) + \Theta(\Sigma')\end{aligned}$$

and therefore the functor $\Theta : \mathcal{K} \to \mathbb{Z}$ is symmetric monoidal. It follows that BΘ is a map of infinite loop spaces, and hence its homotopy fibre Z is an infinite loop space.

The composition $\frac{\mathbb{N}\times\mathbb{Z}_2}{\sim} = \mathcal{C}_1^- \hookrightarrow \mathcal{K} \xrightarrow{\Theta} \mathbb{Z}$ is $(n, \epsilon) \mapsto n$, in other words it the canonical map $\frac{\mathbb{N}\times\mathbb{Z}_2}{\sim} \to \mathscr{G}\left(\frac{\mathbb{N}\times\mathbb{Z}_2}{\sim}\right) = \mathbb{Z}$. Therefore the induced map on classifying spaces is a homotopy equivalence, and so BΘ has a section. By Theorem 5.5 the induced map BΘ_* is an isomorphism on π_1, and so one obtains the desired splitting. □

5.4 Applications to TQFT

5.4.1 Classification of functors $\mathcal{K} \to \mathbb{Z}$

Via similar arguments to those in Section 3.4.1 we see that any functor $F : \mathcal{K} \to \mathbb{Z}$ is determined by its image of:

- A generator of $\mathfrak{K}_{(0,0)} = \mathbb{Z}$
- A set of connecting morphisms $p_{kl} : (0,0) \to (k,l)$, one for each object (k,l) in $\mathcal{K}$.

We take p_{kl} to be the union of $k+l$ discs; k discs each as a cobordism from $\emptyset$ to S^1 and l discs each as a cobordism from $\emptyset$ to I. We define p_{00} to be the identity $1_{(0,0)} = \emptyset$. As a generator of $\mathfrak{K}_{(0,0)}$ we choose the free disc. Note that this is indeed a generator, since it maps to -1 under the functor Θ.

Proposition 5.8. *Any functor $F : \mathcal{K} \to \mathbb{Z}$ is determined by where it sends the free disc D^2 and the connecting morphisms p_{kl}. Conversely, for any set of integers $\{b_{ij}\}_{i,j\in\mathbb{N}}$ there exists a functor sending the free disc to b_{00}, and sending p_{kl} to b_{kl} for all k, l not both equal to zero.*

Proof. The first statement follows from the above discussion. For the second statement, let $\Sigma : (m,n) \to (p,q)$ be a morphism in $\mathcal{K}$. Any such functor must satisfy $F(\Sigma) = F(p_{pq}) + \bar{F}(\amalg_\alpha D^2) + F(p_{mn})^{-1}$ where $\alpha = -\bar{\Theta}(p_{pq}^{-1} \circ \Sigma \circ p_{mn}) = -\Theta(\Sigma) \in \mathbb{Z}$. Hence we define

$$F(\Sigma) := \begin{cases} b_{pq} - b_{mn} - b_{00}\Theta(\Sigma) & \text{if } m+n \geq 1,\ p+q \geq 1 \\ b_{pq} - b_{00}\Theta(\Sigma) & \text{if } m+n = 0,\ p+q \geq 1 \\ -b_{mn} - b_{00}\Theta(\Sigma) & \text{if } m+n \geq 1,\ p+q = 0 \\ -b_{00}\Theta(\Sigma) & \text{if } m=n=p=q=0. \end{cases}$$

One checks that F is functorial, and that $F(D^2) = b_{00}$ and $F(p_{kl}) = b_{kl}$. $\square$

For a sequence of rational numbers $(a_{ij})_{i,j\in\mathbb{N}}$ such that $a_{00} \in \mathbb{Z}$ and $(i+j)a_{ij} \in \mathbb{Z}$ for all i and j, we define a functor $\Phi_{(a_{ij})} : \mathcal{K} \to \mathbb{Z}$ which is constant on objects, and sends a morphism $\Sigma : (m,n) \to (p,q)$ to

$$\Phi_{(a_{ij})}(\Sigma) := (p+q)a_{pq} - (m+n)a_{mn} + a_{00}\Theta(\Sigma).$$

One checks that $\Phi_{(a_{ij})}$ is functorial. Since $\Phi_{(a_{ij})}$ sends the free disc to $-a_{00}$, and sends p_{kl} to $(k+l)a_{kl}$, we see that the functors are all distinct. For example, the functor Θ can be written as $\Phi_{(1,0,0,\ldots)}$. Now suppose we are given a set of integers $\{b_{ij}\}_{i,j\in\mathbb{N}}$. If we set $a_{00} = -b_{00}$ and $a_{ij} = \frac{b_{ij}}{i+j}$ for i, j not both equal to 0, then $\Phi_{(a_{ij})}$ is the functor which sends the free disc to b_{00} and sends p_{kl} to b_{kl}.

Theorem 5.9. *Every functor $\mathcal{K} \to \mathbb{Z}$ is of the form $\Phi_{(a_{ij})}$ for some unique sequence $(a_{ij})_{i,j\in\mathbb{N}}$, where $a_{00} \in \mathbb{Z}$ and $(i+j)a_{ij} \in \mathbb{Z}$ for all i,j. Its isomorphism class and homotopy class are both determined by a_{00}, and it is strict monoidal if and only if $a_{ij} = \left(\frac{i}{i+j}\right) a_{10} + \left(\frac{j}{i+j}\right) a_{01}$ whenever i and j are not both equal to zero.*

Proof. The first statement follows from the above discussion. For the second statement, note that the functor $\Phi_{(a_{ij})}$ sends the twice punctured projective plane to a_{00}. Suppose that we have a natural transformation $\tau : \Phi_{(a_{ij})} \to \Phi_{(a'_{ij})}$. Then $\mathrm{B}\Phi_{(a_{ij})}$ and $\mathrm{B}\Phi_{(a'_{ij})}$ are homotopic and hence $a_{00} = a'_{00}$. Conversely we define a natural transformation $\tau : \Phi_{(a_{00},a_{ij})} \to \Phi_{(a_{00},a'_{ij})}$ (where i and j are not both equal to zero) by $\tau_{kl} := (k+l)(a'_{kl} - a_{kl}) \in \mathbb{Z}$ for each object $(k,l) \in \mathcal{K}$. One checks that the appropriate square commutes as required. For the final statement we argue as follows. The functor $\Phi_{(a_{ij})}$ is monoidal if and only if it is monoidal on the connecting morphisms p_{kl}, in other words if and only if $\Phi_{(a_{ij})}(p_{(k+k')(l+l')}) = \Phi_{(a_{ij})}(p_{kl}) + \Phi_{(a_{ij})}(p_{k'l'})$ for all k,l,k',l'. Here the brackets notation $p_{()()}$ on the left hand side is used to distinguish the two indices; it is not intended to denote any sort of multiplication. One checks that this is the case if and only if $\Phi_{(a_{ij})}(p_{kl}) = k\Phi_{(a_{ij})}(p_{10}) + l\Phi_{(a_{ij})}(p_{01})$, in other words if and only if $(k+l)a_{kl} = ka_{10} + la_{01}$ for all k,l. $\square$

5.4.2 Classification of invertible open-closed Klein TQFTs

Definition 5.10. We define a 2-dimensional open-closed KTQFT to be a based symmetric monoidal functor $(F, F_2) : \mathcal{K} \to \mathrm{Vect}_{\mathbb{C}}$. We say that (F, F_2) is *invertible* if

F is a morphism inverting functor.

Definition 5.11. We define the category $\mathrm{TFT}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_*$ of 2-dimensional open-closed KTQFTs by

$$\mathrm{TFT}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_* := \mathrm{SymmMon}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_*.$$

Definition 5.12. We define the category $\mathrm{InvTFT}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_*$ of invertible 2-dimensional open-closed KTQFTs by

$$\mathrm{InvTFT}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_* := \mathrm{InvSymmMon}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_*.$$

By Corollary A.13 and Lemma 3.32 we have an equivalence of categories

$$\mathrm{InvSymmMon}[\mathcal{K}, \mathbb{C}]_* \to \mathrm{InvTFT}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_*.$$

As in the previous cases we view an object (F, F_2) in $\mathrm{InvSymmMon}[\mathcal{K}, \mathbb{C}]_*$ as a symmetric monoidal functor $\mathcal{K} \to \mathbb{C}^*$. Via the same methods as were used in Theorem 5.9 we obtain the following.

Theorem 5.13. *Every functor $\mathcal{K} \to \mathbb{C}^*$ is of the form $F^{\boldsymbol{\mu}}$ for some unique sequence $\boldsymbol{\mu} = (\mu_{ij})_{i,j\in\mathbb{N}}$ of non-zero complex numbers. $F^{\boldsymbol{\mu}}$ sends all objects of $\mathcal{K}$ to the only object of $\mathbb{C}^*$, and sends a morphism $\Sigma : (m, n) \to (p, q)$ in $\mathcal{K}$ to*

$$F^{\boldsymbol{\mu}}(\Sigma) := \mu_{pq}^{p+q} \mu_{mn}^{-(m+n)} \mu_{00}^{-\Theta(\Sigma)}. \qquad \square$$

Note that $\mu_{00} = F^{\boldsymbol{\mu}}(D^2)$ and $\mu_{kl}^{(k+l)} = F^{\boldsymbol{\mu}}(p_{kl})$ for k, l not both equal to zero. We now consider how $F^{\boldsymbol{\mu}}$ might be given the structure of a symmetric monoidal functor $(F^{\boldsymbol{\mu}}, F_2^{\boldsymbol{\mu}})$.

By considering the lower two diagrams of Definition A.2, and recalling that $\mathcal{K}$ and $\mathbb{C}^*$ are both strict monoidal categories, we see that the isomorphisms $F_2^{\boldsymbol{\mu}} \in \mathbb{C}^*$ must satisfy

$$F_2^{\boldsymbol{\mu}}((m, n), (0, 0)) = F_2^{\boldsymbol{\mu}}((0, 0), (m, n)) = 1 \text{ for all } m, n \in \mathbb{N}. \qquad (\star)$$

The $F_2^{\boldsymbol{\mu}}$ must also be natural, and so the following diagram must commute for all objects $(m,n),(m',n'),(p,q),(p',q')$ and all morphisms $\Sigma:(m,n)\to(p,q)$ and $\Sigma':(m',n')\to(p',q')$ in $\mathcal{K}$.

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "F_2^{\boldsymbol{\mu}}"] \arrow[d, "\mu_{pq}^{p+q}\mu_{mn}^{-(m+n)}\mu_{00}^{-\Theta(\Sigma)}\mu_{p'q'}^{p'+q'}\mu_{m'n'}^{-(m'+n')}\mu_{00}^{-\Theta(\Sigma')}"'] & \mathbb{C}^* \arrow[d, "\mu_{(p+p')(q+q')}^{p+p'+q+q'}\mu_{(m+m')(n+n')}^{-(m+m'+n+n')}\mu_{00}^{-\Theta(\Sigma\amalg\Sigma')}"] \\ \mathbb{C}^* \arrow[r, "F_2^{\boldsymbol{\mu}}"'] & \mathbb{C}^* \end{tikzcd}$$

By choosing $p=p'=q=q'=0$ and using Equation $(\star)$ we see that

$$F_2^{\boldsymbol{\mu}}((m,n),(m',n')) = \mu_{(m+m')(n+n')}^{m+m'+n+n'}\mu_{mn}^{-(m+n)}\mu_{m'n'}^{-(m'+n')}$$

and one checks that this also satisfies the symmetry axiom (the upper right hand diagram of Definition A.2). Therefore every functor $\mathcal{K}\to\mathbb{C}^*$ can be uniquely given the structure of a symmetric monoidal functor, and we have completely identified the objects of $\mathrm{InvSymmMon}[\mathcal{K},\mathbb{C}]_*$.

Remark. The functor $(F^{\boldsymbol{\mu}},F_2^{\boldsymbol{\mu}})$ is symmetric strict monoidal if and only if $\mu_{ij}^{i+j}=\mu_{10}^{i}\mu_{01}^{j}$ for all $i,j\in\mathbb{N}$.

We now describe the morphisms in $\mathrm{InvSymmMon}[\mathcal{K},\mathbb{C}]_*$. Suppose we have a based natural transformation $\tau: F^{\boldsymbol{\mu}}\to F^{\boldsymbol{\mu}'}$. Then, by considering the free disc D^2 as a morphism $(0,0)\to(0,0)$, naturality of τ and the fact that $\tau_{00}=1$ give $\mu_{00}=\mu'_{00}$. So suppose this is the case. For a general morphism $\Sigma:(m,n)\to(p,q)$ in $\mathcal{K}$, the following diagram must commute.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tau_{mn}"] \arrow[d, "\mu_{pq}^{p+q}\mu_{mn}^{-(m+n)}\mu_{00}^{-\Theta(\Sigma)}"'] & \mathbb{C} \arrow[d, "(\mu'_{pq})^{p+q}(\mu'_{mn})^{-(m+n)}\mu_{00}^{-\Theta(\Sigma)}"] \\ \mathbb{C} \arrow[r, "\tau_{pq}"'] & \mathbb{C} \end{tikzcd}$$

Choosing $p=q=0$ we deduce

$$\tau_{mn} = (\mu'_{mn}\mu_{mn}^{-1})^{m+n}.$$

Finally, one checks that τ defines a based monoidal natural transformation $(F^{\boldsymbol{\mu}}, F_2^{\boldsymbol{\mu}}) \to (F^{\boldsymbol{\mu}'}, F_2^{\boldsymbol{\mu}'})$; in other words the diagram of Definition A.4 automatically commutes. Hence there is precisely one morphism between any two objects in $\mathrm{InvSymmMon}[\mathcal{K}, \mathbb{C}]_*$ which are indexed by the same μ_{00}, and this morphism is an isomorphism. In particular there are no non-trivial automorphisms. Furthermore, there are no morphisms connecting isomorphism classes of objects. Hence, taking a skeleton, we obtain a discrete category with one object for each $\mu_{00} \in \mathbb{C}^*$. We have proved the following theorem.

Theorem 5.14. *The category* $\mathrm{InvTFT}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_*$ *of invertible open-closed 2-dimensional KTQFTs and based monoidal natural transformations is equivalent to the discrete category* $\mathbb{C}^*$ *of non-zero complex numbers.* □

Remark. We have also shown that $\mathrm{InvTFT}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_*$ is equivalent to the category $\mathrm{Inv}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_*$ of based morphism inverting functors $\mathcal{K} \to \mathrm{Vect}_{\mathbb{C}}$ and based natural transformations.

Remark. We have also shown that $\mathrm{InvTFT}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}]_* = \mathrm{SymmMon}[\mathcal{K}, \mathrm{Vect}_{\mathbb{C}}^*]_*$.

Remark. An open-closed Klein topological quantum field theory (F, F_2) is equivalent to a *structure algebra* [AN06]. A structure algebra $\mathcal{H} = \{H = A \bigoplus B, \langle \cdot, \cdot \rangle, \mathbf{x} \mapsto \mathbf{x}^*, U\}$ is a finite-dimensional $\mathbb{C}$-algebra H with the following additional structure.

- A decomposition $H = A \bigoplus B$ of H into the direct sum of two vector spaces A and B where:
  - (i) A is a subalgebra of the centre of H; the algebra A has unit $1_A \in A$ and 1_A is also the unit of H;
  - (ii) B is a two-sided ideal of H (typically non-commutative); the algebra B has unit $1_B \in B$.

- A symmetric invariant inner product $\langle \cdot, \cdot \rangle : H \bigotimes H \to \mathbb{C}$, that is $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ and $\langle \mathbf{xy}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{yz} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in H$, satisfying the following properties.
  - (i) The restrictions $\langle \cdot, \cdot \rangle|_A$ and $\langle \cdot, \cdot \rangle|_B$ are non-degenerate inner products on A and B respectively.
  - (ii) If $\{a_i\}$ is a basis for A and $\{b_j\}$ is a basis for B then $\langle \sum_{ij} \beta_{ij} b_i \mathbf{b}_1 b_j, \mathbf{b}_2 \rangle = \langle \sum_{ij} \alpha_{ij} a_i \otimes a_j, \mathbf{b}_1 \otimes \mathbf{b}_2 \rangle$ for all $\mathbf{b}_1$, $\mathbf{b}_2 \in B$, where the copairings $\mathbb{C} \to A \bigotimes A$ and $\mathbb{C} \to B \bigotimes B$ are given by $1 \mapsto \sum_{ij} \alpha_{ij} a_i \otimes a_j$ and $1 \mapsto \sum_{ij} \beta_{ij} b_i \otimes b_j$ respectively.
- An involutive anti-automorphism $H \to H$ denoted by $\mathbf{x} \mapsto \mathbf{x}^*$ which preserves the decomposition $H = A \bigoplus B$ and the pairing on H, that is $A^* = A$, $B^* = B$ and $\langle \mathbf{x}^*, \mathbf{y}^* \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}$, $\mathbf{y} \in H$.
- An element $U \in A$ satisfying the following properties.
  - (i) $(\mathbf{a}U)^* = \mathbf{a}U$ for all $\mathbf{a} \in A$.
  - (ii) $U^2 = \sum_{ij} \alpha_{ij} a_i a_j^* \in A$.
  - (iii) $\langle U, \mathbf{b} \rangle = \langle \sum_{ij} \beta_{ij} b_i b_j^*, \mathbf{b} \rangle$ for all $\mathbf{b} \in B$.

For an open-closed KTQFT (F, F_2), the algebras A and B are given by $F(S^1)$ and $F(I)$ respectively. The algebraic structure corresponds to the following cobordisms. The multiplications $A \bigotimes A \to A$, $B \bigotimes B \to B$ and $A \bigotimes B \to B$ correspond respectively to the pair of pants surface as a morphism $S^1 \amalg S^1 \to S^1$, the disc as a morphism $I \amalg I \to I$, and the composition of the cylinder : $S^1 \to I$ with the disc : $I \amalg I \to I$. The unit 1_A is given by the image of 1 under the linear map $F(p_{10}) : \mathbb{C} \to A$, and the unit 1_B is given by the image of 1 under $F(p_{01})$. The inner product corresponds to the composition of the multiplication with the appropriate disc as a cobordism to $\emptyset$. The involutive anti-automorphism corresponds to $C_r^{S^1}$ on A and to C_r^I on B. Finally, the element U is given by the image of 1 under the map $\mathbb{C} \to A$ corresponding to the Möbius band as a morphism $\emptyset \to S^1$. For the invertible field theory defined by $F^{\mu_{00}} := F^{\boldsymbol{\mu}}$ with $\mu_{ij} = 1$ for all i, j not both equal to zero we make the following observation.

Corollary 5.15. *The structure algebra corresponding to the invertible open-closed KTQFT defined by $F^{\mu_{00}}$ is commutative. The algebras $A = F^{\mu_{00}}(S^1)$ and $B = F^{\mu_{00}}(I)$ are both $\mathbb{C}$ with its usual algebra structure, and the multiplication $A \bigotimes B \to B$ is again just multiplication in $\mathbb{C}$. The pairing is defined by $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = \mu_{00}^2 \mathbf{a}_1 \mathbf{a}_2$, $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle = \mu_{00} \mathbf{b}_1 \mathbf{b}_2$ and $\langle \mathbf{a}_1, \mathbf{b}_1 \rangle = \mu_{00} \mathbf{a}_1 \mathbf{b}_1$ where $\mathbf{a}_i \in A$ and $\mathbf{b}_i \in B$ for $i \in \{1, 2\}$. The involutive anti-automorphism is the identity, and the element $U \in A$ is equal to μ_{00}^{-1}.* □

Appendix A Symmetric monoidal categories

A.1 Symmetric monoidal equivalences

Notation. $\langle \mathscr{C}, \otimes, e, \alpha, \lambda, \rho, \gamma \rangle$ will denote the symmetric monoidal category $\mathscr{C}$ with product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, unit $e \in \mathscr{C}$, associator α, left and right unitors λ and ρ, and braiding γ. Recall that for all objects $a, b, c \in \mathscr{C}$ the components

$$\alpha_{a,b} : a \otimes (b \otimes c) \to (a \otimes b) \otimes c$$

$$\lambda_a : e \otimes a \to a$$

$$\rho_a : a \otimes e \to a$$

$$\gamma_{a,b} : a \otimes b \to b \otimes a$$

are isomorphisms in $\mathscr{C}$, natural in a, b and c, and must satisfy certain commutativity requirements as described by Mac Lane [ML98]. Recall also that

$$\gamma_{b,a} \circ \gamma_{a,b} = 1_{a \otimes b}.$$

Finally, recall that $\langle \mathscr{C}, \otimes, e, \alpha, \lambda, \rho, \gamma \rangle$ is a symmetric *strict* monoidal category if the morphisms α, ρ and λ are all identities.

We shall often denote the symmetric monoidal category $\langle \mathscr{C}, \otimes, e, \alpha, \lambda, \rho, \gamma \rangle$ simply by $\mathscr{C}$.

Definition A.1. Let $\mathscr{C}$ and $\mathscr{C}'$ be symmetric monoidal categories with respective units e and e'. Then we say that

(i) a functor $F : \mathscr{C} \to \mathscr{C}'$ is *based* if $F(e) = e'$;

(ii) a natural transformation $\tau : F \to G$ between two based functors $F, G : \mathscr{C} \to \mathscr{C}'$ is *based* if $\tau_e = 1_{e'}$.

Definition A.2. Let $\mathscr{C}$ and $\mathscr{C}'$ be symmetric monoidal categories. We define a *based symmetric monoidal functor*[4] $(F, F_2) : \mathscr{C} \to \mathscr{C}'$ to be a pair consisting of

(i) an (ordinary) based functor $F : \mathscr{C} \to \mathscr{C}'$;

(ii) for each pair of objects $a, b \in \mathscr{C}$ an isomorphism

$$F_2(a, b) : F(a) \otimes' F(b) \to F(a \otimes b)$$

in $\mathscr{C}'$ which is natural in a and b.

Together these must make the following four diagrams commute in $\mathscr{C}'$.

$$\begin{tikzcd}
F(a) \otimes' (F(b) \otimes' F(c)) \arrow[r, "\alpha'"] \arrow[d, "1 \otimes' F_2"'] & (F(a) \otimes' F(b)) \otimes' F(c) \arrow[d, "F_2 \otimes' 1"] \\
F(a) \otimes' F(b \otimes c) \arrow[d, "F_2"'] & F(a \otimes b) \otimes' F(c) \arrow[d, "F_2"] \\
F(a \otimes (b \otimes c)) \arrow[r, "F(\alpha)"'] & F((a \otimes b) \otimes c)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(a) \otimes' F(b) \arrow[r, "\gamma'"] \arrow[d, "F_2"'] & F(b) \otimes' F(a) \arrow[d, "F_2"] \\
F(a \otimes b) \arrow[r, "F(\gamma)"'] & F(b \otimes a)
\end{tikzcd}$$

$$\begin{tikzcd}
F(a) \otimes' e' \arrow[r, "\rho'"] \arrow[d, equal] & F(a) \\
F(a) \otimes' F(e) \arrow[r, "F_2"'] & F(a \otimes e) \arrow[u, "F(\rho)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
e' \otimes' F(a) \arrow[r, "\lambda'"] \arrow[d, equal] & F(a) \\
F(e) \otimes' F(a) \arrow[r, "F_2"'] & F(e \otimes a) \arrow[u, "F(\lambda)"']
\end{tikzcd}$$

In the case of a strict source and target, we say that (F, F_2) is a based symmetric *strict* monoidal functor if all F_2 are identities.

Lemma A.3. *Let $(F, F_2) : \mathscr{C} \to \mathscr{C}'$ and $(G, G_2) : \mathscr{C}' \to \mathscr{C}''$ be based symmetric monoidal functors. Then the composite $(G \circ F, (G \circ F)_2) : \mathscr{C} \to \mathscr{C}''$ is a based symmetric monoidal functor, where we define*

$$(G \circ F)_2(a, b) := G(F_2(a, b)) \circ G_2(F(a), F(b))$$

for all objects $a, b \in \mathscr{C}$.

[4] In the terminology of Mac Lane [ML98], (F, F_2) is a symmetric *strong* monoidal functor which is strict with respect to units.

Proof. It is clear that the composite $(G \circ F)$ is based and that all $(G \circ F)_2$ are invertible. Checking that the remaining properties hold goes through in the usual way. The appropriate diagrams commute for $(G \circ F, (G \circ F)_2)$ as a consequence of the fact that the analogous diagrams commute for (G, G_2) and (F, F_2), along with the fact that G_2 and F_2 are natural. □

Definition A.4. Let $\mathscr{C}, \mathscr{C}'$ be symmetric monoidal categories, and let $(F, F_2), (G, G_2) : \mathscr{C} \to \mathscr{C}'$ be based symmetric monoidal functors. A *based monoidal natural transformation* $\tau : (F, F_2) \to (G, G_2)$ between (F, F_2) and (G, G_2) is a based natural transformation $\tau : F \to G$ between the underlying ordinary functors, such that the following diagram commutes in $\mathscr{C}'$ for all objects $a, b \in \mathscr{C}$.

$$\begin{tikzcd}
F(a) \otimes' F(b) \arrow[r, "F_2"] \arrow[d, "\tau_a \otimes' \tau_b"'] & F(a \otimes b) \arrow[d, "\tau_{a\otimes b}"] \\
G(a) \otimes' G(b) \arrow[r, "G_2"'] & G(a \otimes b)
\end{tikzcd}$$

Remark. One could ask that τ be in some sense symmetric, in that it should satisfy $(\tau_b \otimes' \tau_a) \circ \gamma'_{F(a),F(b)} = \gamma'_{G(a),G(b)} \circ (\tau_a \otimes' \tau_b)$ and $\tau_{b\otimes a} \circ F(\gamma_{a,b}) = G(\gamma_{a,b}) \circ \tau_{a\otimes b}$. But this is redundant; both properties hold automatically by naturality of γ' and τ.

Lemma A.5. *Let $(F, F_2), (G, G_2), (H, H_2) : \mathscr{C} \to \mathscr{C}'$ be based symmetric monoidal functors between symmetric monoidal categories. Let $\tau : (F, F_2) \to (G, G_2)$ and $\sigma : (G, G_2) \to (H, H_2)$ be based monoidal natural transformations. Then the composite $\sigma \circ \tau : (F, F_2) \to (H, H_2)$ is a based monoidal natural transformation, where $(\sigma \circ \tau)_a := \sigma_a \circ \tau_a$ for each object $a \in \mathscr{C}$.*

Proof. Again the proof is routine. The composite $\sigma \circ \tau$ is clearly based, and is natural as a composite of natural transformations. Monoidality follows from monoidality of σ and τ. □

Definition A.6. For symmetric monoidal categories $\mathscr{C}$ and $\mathscr{D}$, denote by $\mathrm{SymmMon}[\mathscr{C},\mathscr{D}]_*$ the category whose objects are based symmetric monoidal functors $\mathscr{C} \to \mathscr{D}$, and whose morphisms are based monoidal natural transformations.

Definition A.7. Let $\mathscr{D}$ and $\mathscr{D}'$ be symmetric monoidal categories. A *based symmetric monoidal equivalence* between $\mathscr{D}$ and $\mathscr{D}'$ is a quadruple $((\phi,\phi_2),(\psi,\psi_2),\eta,\varepsilon)$ where

(i) $(\phi,\phi_2) : \mathscr{D} \to \mathscr{D}'$ and $(\psi,\psi_2) : \mathscr{D}' \to \mathscr{D}$ are based symmetric monoidal functors;

(ii) $\phi : \mathscr{D} \to \mathscr{D}'$ and $\psi : \mathscr{D}' \to \mathscr{D}$ form an equivalence of categories in the usual way;

(iii) the unit $\eta : 1_{\mathscr{D}} \to \psi \circ \phi$ and counit $\varepsilon : \phi \circ \psi \to 1_{\mathscr{D}'}$ of this equivalence are based monoidal natural isomorphisms.

We often say that a single functor (ϕ,ϕ_2) is a based symmetric monoidal equivalence if the corresponding $(\psi,\psi_2),\eta$ and ε exist.

Proposition A.8. *Let $\mathscr{C},\mathscr{D}$ and $\mathscr{D}'$ be symmetric monoidal categories. Suppose we have a based symmetric monoidal equivalence $((\phi,\phi_2),(\psi,\psi_2),\eta,\varepsilon)$ between $\mathscr{D}$ and $\mathscr{D}'$ as in Definition A.7. Then the categories $\mathrm{SymmMon}[\mathscr{C},\mathscr{D}]_*$ and $\mathrm{SymmMon}[\mathscr{C},\mathscr{D}']_*$ are equivalent.*

Proof. For objects $(F,F_2),(G,G_2)$ and morphisms $\tau : (F,F_2) \to (G,G_2)$ in $\mathrm{SymmMon}[\mathscr{C},\mathscr{D}]_*$ we define a functor $\Phi : \mathrm{SymmMon}[\mathscr{C},\mathscr{D}]_* \to \mathrm{SymmMon}[\mathscr{C},\mathscr{D}']_*$ as follows.

$$\begin{aligned}
\Phi : \mathrm{SymmMon}[\mathscr{C},\mathscr{D}]_* &\to \mathrm{SymmMon}[\mathscr{C},\mathscr{D}']_* \\
(F,F_2) &\mapsto (\phi \circ F, (\phi \circ F)_2) \\
\tau &\mapsto \Phi(\tau) \text{ where } \Phi(\tau)_a := \phi(\tau_a) \text{ for all objects } a \in \mathscr{C}.
\end{aligned}$$

It is routine to check that Φ is functorial, and by Lemma A.3 $\Phi((F,F_2))$ is indeed an object in $\mathrm{SymmMon}[\mathscr{C},\mathscr{D}']_*$. Naturality of $\Phi(\tau)$ follows from naturality of τ, and $\Phi(\tau)$ is based since ϕ and τ are based. To check that $\Phi(\tau)$ is monoidal we check the commutativity of the following diagram. For brevity we denote the tensor products in $\mathscr{C}, \mathscr{D}$ and $\mathscr{D}'$ all by $\otimes$.

$$\begin{tikzcd}
(\phi \circ F)(a) \otimes (\phi \circ F)(b) \arrow[r, "(\phi\circ F)_2"] \arrow[d, "\phi(\tau_a)\otimes\phi(\tau_b)"'] & (\phi \circ F)(a \otimes b) \arrow[d, "\phi(\tau_{a\otimes b})"] \\
(\phi \circ G)(a) \otimes (\phi \circ G)(b) \arrow[r, "(\phi\circ G)_2"'] & (\phi \circ G)(a \otimes b)
\end{tikzcd}$$

Moving along the top row and then down we have

$$\begin{aligned}
\phi(\tau_{a\otimes b}) \circ (\phi \circ F)_2(a,b) &= \phi(\tau_{a\otimes b}) \circ \phi(F_2(a,b)) \circ \phi_2(F(a), F(b)) \\
&= \phi(\tau_{a\otimes b} \circ F_2(a,b)) \circ \phi_2(F(a), F(b)) \\
&= \phi(G_2(a,b) \circ (\tau_a \otimes \tau_b)) \circ \phi_2(F(a), F(b)) && \text{since } \tau \text{ is monoidal} \\
&= \phi(G_2(a,b)) \circ \phi(\tau_a \otimes \tau_b) \circ \phi_2(F(a), F(b)) \\
&= \phi(G_2(a,b)) \circ \phi_2(G(a), G(b)) \circ (\phi(\tau_a) \otimes \phi(\tau_b)) && \text{by naturality of } \phi_2 \\
&= (\phi \circ G)_2(a,b) \circ (\phi(\tau_a) \otimes \phi(\tau_b))
\end{aligned}$$

as required. In the other direction we define a functor

$$\begin{aligned}
\Psi : \mathrm{SymmMon}[\mathscr{C}, \mathscr{D}']_* \to &\mathrm{SymmMon}[\mathscr{C}, \mathscr{D}]_* \\
(F, F_2) \mapsto &(\psi \circ F, (\psi \circ F)_2) \\
\tau \mapsto &\Psi(\tau) \text{ where } \Psi(\tau)_a := \psi(\tau_a) \text{ for all objects } a \in \mathscr{C}
\end{aligned}$$

and by similar arguments this satisfies all the required properties. It remains to show that Φ and Ψ define an equivalence of categories. Define a natural isomorphism

$$\bar{\eta} : 1_{\mathrm{SymmMon}[\mathscr{C},\mathscr{D}]_*} \to \Psi \circ \Phi$$

by

$$(\bar{\eta}_{(F,F_2)})_a := \eta_{F(a)}$$

for objects $a \in \mathscr{C}$. We have two things to check: that the components $\bar{\eta}_{(F,F_2)}$ are indeed isomorphisms in $\mathrm{SymmMon}[\mathscr{C}, \mathscr{D}]_*$, and that $\bar{\eta}$ is natural.

To show that $\bar{\eta}_{(F,F_2)} : (F, F_2) \to (\psi \circ \phi \circ F, (\psi \circ \phi \circ F)_2)$ is a based monoidal natural isomorphism we argue as follows. Clearly $(\bar{\eta}_{(F,F_2)})_a$ is invertible, since η is a natural isomorphism. We also see that $\bar{\eta}_{(F,F_2)}$ is based since F and η are based, and naturality follows from naturality of η. It remains to check that $\bar{\eta}_{(F,F_2)}$ is monoidal, for which we must verify the commutativity of the following diagram.

$$\begin{tikzcd}
F(a) \otimes F(b) \arrow[r, "F_2"] \arrow[d, "\eta_{F(a)} \otimes \eta_{F(b)}"'] & F(a \otimes b) \arrow[d, "\eta_{F(a \otimes b)}"] \\
(\psi \circ \phi \circ F)(a) \otimes (\psi \circ \phi \circ F)(b) \arrow[r, "(\psi \circ \phi \circ F)_2"'] & (\psi \circ \phi \circ F)(a \otimes b)
\end{tikzcd}$$

Reading down the left hand column and then along we have

$$\begin{aligned}
(\psi \circ \phi \circ F)_2(a,b) \circ (\eta_{F(a)} \otimes \eta_{F(b)}) &= (\psi \circ \phi)(F_2(a,b)) \circ (\psi \circ \phi)_2(F(a), F(b)) \circ (\eta_{F(a)} \otimes \eta_{F(b)}) \\
&= (\psi \circ \phi)(F_2(a,b)) \circ \eta_{F(a) \otimes F(b)} \qquad \text{since } \eta \text{ is monoidal} \\
&= \eta_{F(a \otimes b)} \circ F_2(a,b) \qquad \text{by naturality of } \eta
\end{aligned}$$

as required. Finally, one checks that naturality of $\bar{\eta}$ again follows from naturality of η. Conversely we define

$$\bar{\varepsilon} : \Phi \circ \Psi \to 1_{\mathrm{SymmMon}[\mathscr{C}, \mathscr{D}']_*}$$

by

$$(\bar{\varepsilon}_{(F,F_2)})_a := \varepsilon_{F(a)}.$$

Via similar arguments we see that $\bar{\varepsilon}$ is a natural isomorphism, and hence $(\Phi, \Psi, \bar{\eta}, \bar{\varepsilon})$ defines an equivalence of categories. $\square$

A.2 The skeleton of a symmetric monoidal category

Proposition A.9. *Let $\langle \mathscr{C}, \otimes, e, \alpha, \lambda, \rho, \gamma \rangle$ be a symmetric monoidal category, and let $\mathscr{C}^s \subseteq \mathscr{C}$ be a skeleton of $\mathscr{C}$. Denote by $\phi(a)$ the object in $\mathscr{C}^s$ representing the isomorphism class of the object $a \in \mathscr{C}$, and set $\phi(e) \coloneqq e$. Choose one fixed isomorphism $\phi_a : a \to \phi(a)$ for each a, setting $\phi_{\phi(a)} \coloneqq 1_{\phi(a)}$. Then $\mathscr{C}^s$ can be given a monoidal structure $\langle \mathscr{C}^s, \otimes^s, e, \alpha^s, \lambda^s, \rho^s, \gamma^s \rangle$ where the structure maps are defined as follows. For objects u, u', v, v', w and morphisms $f : u \to v$, $f' : u' \to v'$ in $\mathscr{C}^s$ we define*

$$
\begin{array}{rll}
\textbf{Product} & u \otimes^s u' & \coloneqq \phi(u \otimes u') \\
 & f \otimes^s f' & \coloneqq \phi_{v \otimes v'} \circ (f \otimes f') \circ \phi^{-1}_{u \otimes u'} \\
\textbf{Associator} & \alpha^s_{u,v,w} & \coloneqq \phi_{\phi(u \otimes v) \otimes w} \circ (\phi_{u \otimes v} \otimes 1_w) \circ \alpha_{u,v,w} \circ (1_u \otimes \phi^{-1}_{v \otimes w}) \circ \phi^{-1}_{u \otimes \phi(v \otimes w)} \\
\textbf{Unitors} & \lambda^s_u & \coloneqq \lambda_u \circ \phi^{-1}_{e \otimes u} \\
 & \rho^s_u & \coloneqq \rho_u \circ \phi^{-1}_{u \otimes e} \\
\textbf{Braiding} & \gamma_{u,v} & \coloneqq \phi_{v \otimes u} \circ \gamma_{u,v} \circ \phi^{-1}_{u \otimes v}.
\end{array}
$$

Proof. Checking that the appropriate diagrams commute is a long but standard process. The required properties hold for the skeletal structure maps in $\mathscr{C}^s$ as a consequence of the fact that they hold for the analogous maps in $\mathscr{C}$. $\square$

Proposition A.10. *The inclusion $(i, i_2) : \mathscr{C}^s \hookrightarrow \mathscr{C}$ where $i_2(u, v) \coloneqq \phi_{u \otimes v}$ is a based symmetric monoidal equivalence of categories.*

Proof. Clearly i is a based functor and all i_2 are invertible. Naturality of the i_2 is immediate from the definition of $\otimes^s$, and the four diagrams of Definition A.2 all commute by definition of the structure maps $\alpha^s, \lambda^s, \rho^s$ and γ^s. Hence (i, i_2) is a based symmetric monoidal functor.

For objects $a, b \in \mathscr{C}$ and morphisms $f : a \to b$ we define $(\phi, \phi_2) : \mathscr{C} \to \mathscr{C}^s$ by

$$\begin{aligned} \phi : \mathscr{C} &\longrightarrow \mathscr{C}^s \\ a &\longmapsto \phi(a) \\ f &\longmapsto \phi_b \circ f \circ \phi_a^{-1} \end{aligned}$$

and

$$\phi_2(a, b) := \phi_{a \otimes b} \circ (\phi_a^{-1} \otimes \phi_b^{-1}) \circ \phi^{-1}_{\phi(a) \otimes \phi(b)}.$$

Clearly ϕ is a based functor and all ϕ_2 are invertible. Naturality of the ϕ_2 is immediate from the definitions, and the diagrams of Definition A.2 commute by naturality of the structure maps α, λ, ρ and γ. Therefore (ϕ, ϕ_2) is a based symmetric monoidal functor.

Note that $(\phi \circ i, (\phi \circ i)_2) = (1_{\mathscr{C}^s}, 1)$ and so it remains to define a based monoidal counit $\varepsilon : i \circ \phi \to 1_{\mathscr{C}}$. We do this by setting $\varepsilon_a := \phi_a^{-1}$ for $a \in \mathscr{C}$, and this satisfies all the required properties by construction. $\square$

Putting together Proposition A.8 and Proposition A.10 we obtain the following.

Corollary A.11. *Let $\mathscr{C}$ and $\mathscr{D}$ be symmetric monoidal categories, and let $\mathscr{D}^s$ be a skeleton of $\mathscr{D}$ with symmetric monoidal structure as in Proposition A.9. Then the inclusion $\mathscr{D}^s \hookrightarrow \mathscr{D}$ induces an equivalence of categories* $\mathrm{SymmMon}[\mathscr{C}, \mathscr{D}^s]_* \to \mathrm{SymmMon}[\mathscr{C}, \mathscr{D}]_*$. $\square$

We derive a version of Corollary A.11 for morphism inverting based symmetric monoidal functors. For any symmetric monoidal category $\mathscr{C}$, we can extend the product on $\mathscr{C}$ to a product on its localisation $\mathscr{C}[\mathscr{C}^{-1}]$ by defining $f^{-1} \otimes g^{-1} := (f \otimes g)^{-1}$ for inverses and extending bifunctorially. The localised category then has symmetric monoidal structure inherited from $\mathscr{C}$ by transferring the unit and structure maps along the canonical projection $\mathscr{C} \to \mathscr{C}[\mathscr{C}^{-1}]$. Furthermore, for a second symmetric monoidal category $\mathscr{D}$, a morphism inverting functor $F : \mathscr{C} \to \mathscr{D}$ is based symmetric monoidal if and only if the corresponding functor $\bar{F} : \mathscr{C}[\mathscr{C}^{-1}] \to \mathscr{D}$ is based symmetric monoidal.

Definition A.12. Let $\mathscr{C}$ and $\mathscr{D}$ be symmetric monoidal categories. Define $\mathrm{InvSymmMon}[\mathscr{C}, \mathscr{D}]_*$ to be the full subcategory of $\mathrm{SymmMon}[\mathscr{C}, \mathscr{D}]_*$ whose objects are morphism inverting functors.

Note that the categories $\mathrm{InvSymmMon}[\mathscr{C}, \mathscr{D}]_*$ and $\mathrm{SymmMon}[\mathscr{C}[\mathscr{C}^{-1}], \mathscr{D}]_*$ are canonically isomorphic. Hence by Corollary A.11 we obtain the following.

Corollary A.13. *Let $\mathscr{C}$ and $\mathscr{D}$ be symmetric monoidal categories, and let $\mathscr{D}^s$ be a skeleton of $\mathscr{D}$ with symmetric monoidal structure as in Proposition A.9. Then the inclusion $\mathscr{D}^s \hookrightarrow \mathscr{D}$ induces an equivalence of categories* $\mathrm{InvSymmMon}[\mathscr{C}, \mathscr{D}^s]_* \to \mathrm{InvSymmMon}[\mathscr{C}, \mathscr{D}]_*$. □

A.2.1 A skeleton for $\mathrm{Vect}_{\mathbb{C}}$

Define $\mathrm{Vect}_{\mathbb{C}}^s \subset \mathrm{Vect}_{\mathbb{C}}$ to be the skeleton of $\mathrm{Vect}_{\mathbb{C}}$ whose objects are $\{\mathbb{C}^n | n \in \mathbb{N}_{>0}\}$ (so for an n-dimensional vector space V^n we have $\phi(V^n) := \mathbb{C}^n$). Choose the isomorphism $\phi_{\mathbb{C}\otimes\mathbb{C}} : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ in the canonical way, that is to say $z \otimes z' \mapsto zz'$. We give $\mathrm{Vect}_{\mathbb{C}}^s$ a symmetric monoidal structure as in Proposition A.9. Explicitly this means that for linear maps $\sigma : \mathbb{C}^n \to \mathbb{C}^m$ and $\sigma' : \mathbb{C}^{n'} \to \mathbb{C}^{m'}$ in $\mathrm{Vect}_{\mathbb{C}}^s$ we have

$$\sigma \otimes^s \sigma' := \phi_{\mathbb{C}^m \otimes \mathbb{C}^{m'}} \circ (\sigma \otimes \sigma') \circ \phi^{-1}_{\mathbb{C}^n \otimes \mathbb{C}^{n'}} : \mathbb{C}^{nn'} \to \mathbb{C}^{mm'}.$$

In particular if we consider two linear maps $\sigma, \sigma' : \mathbb{C} \to \mathbb{C}$, that is to say two elements $\sigma, \sigma' \in \mathbb{C}$, the canonical choice of $\phi_{\mathbb{C}\otimes\mathbb{C}}$ has two consequences. Firstly we have $\sigma \otimes^s \sigma' = \sigma\sigma' \in \mathbb{C}$. Secondly, the structure map components $\alpha^s_{\mathbb{C},\mathbb{C},\mathbb{C}}$, $\lambda^s_{\mathbb{C}}, \rho^s_{\mathbb{C}}$ and $\gamma^s_{\mathbb{C},\mathbb{C}}$ are all identities, and so the monoidal structure on $\mathbb{C} \subset \mathrm{Vect}_{\mathbb{C}}^s$ is strict. Hence, if we exclude the trivial map and consider just the non-zero complex numbers $\mathbb{C}^*$, the monoidal structure coincides with the monoidal structure on $\mathbb{C}^*$ thought of as an abelian group.

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