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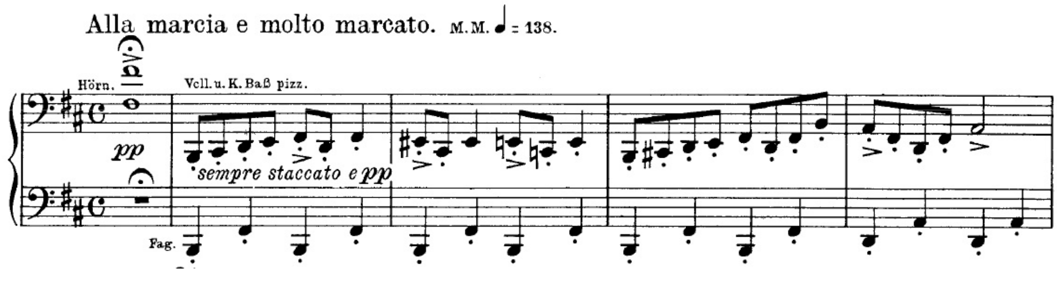
In the Hall of the Flop King

Two Applications of Perverse Coherent Sheaves to Donaldson–Thomas Invariants

A thesis submitted for the degree of
Doctor of Philosophy

Michaelmas Term 2012

John Calabrese: *In the Hall of the Flop King*, DPhil Dissertation, Michaelmas Term 2012.



In the Hall of the Mountain King — Edvard Grieg

It occurred to me that this otherwise empty half of a page could be put to good use by dedicating this document to someone. As far as mathematicians go, the first two who come to mind are certainly Tom and Richard. I cannot thank them enough for all the support they have shown me in the past few years.

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ABSTRACT

Title: In the Hall of the Flop King

Candidate: John Calabrese

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This thesis contains two main results. The first [Cal12a] is a comparison formula for the Donaldson-Thomas invariants of two (complex, smooth and projective) Calabi-Yau threefolds related by a flop; the second [Cal12b] is a proof of the projective case of the Crepant Resolution Conjecture for Donaldson-Thomas invariants, as stated by Bryan, Cadman and Young.

Both results rely on Bridgeland's category of perverse coherent sheaves, which is the heart of a t-structure in the derived category of the given Calabi-Yau variety. The first formula is a consequence of various identities in an appropriate motivic Hall algebra followed by an implementation of the integration morphism (using the technology of Joyce and Song). Our proof of the crepant resolution conjecture is a quick and elegant application of the first formula in the context of the derived McKay correspondence of Bridgeland, King and Reid.

The first chapter is introductory and is followed by two chapters of background material. The last two chapters are devoted to the proofs of the main results.

Possibly an intellectual quote,
perhaps even divided among different lines,
maybe some verses of a poem.

— Mankind at its best

ACKNOWLEDGEMENTS

Despite a PhD thesis being an optimal place where indulging in a corny and endless list of acknowledgements is more or less deemed acceptable, I chose to refrain from doing so. I will solely abide by the routine convention of thanking my supervisor Tom Bridgeland: he is undoubtedly and by far the best rock-climber I know (and a pretty decent mathematician during his spare time).

However, if I were to spend a few more words acknowledging here and there, it would probably go a little something like this...

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I am eternally grateful to Tom Bridgeland for being the best supervisor one could ever hope to have. His patience was endless, his knowledge immense and his guidance invaluable. A good chunk of the algebraic geometry I know I learned from him and inexorably many ideas present in this thesis came from him. Being his student was truly a privilege.

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Oxford, Michaelmas Term 2012 JRC

1

OVERVIEW

The starting point of this thesis is the following question:

> Is there a relationship between the Donaldson-Thomas invariants of two birational Calabi-Yau threefolds?

Our aim is to answer this question and to use the methods developed to prove the Crepant Resolution Conjecture for Donaldson-Thomas invariants of Bryan-Cadman-Young [BCY12], which can be interpreted as an extension of this question where we allow orbifolds.

1.1 INTRODUCTION

In the past years much attention has been devoted to the study of curve-counting invariants, such as Gromov-Witten and Donaldson-Thomas (DT) invariants. Our focus is on Calabi-Yau threefolds which, from both a mathematical and physical perspective, is a case of particular interest. One of the nice features of studying the DT invariants of such varieties is that one can rely on Behrend's microlocal function [Beh09], which in turn makes the use of Hall algebras possible.

The philosophy behind this approach is that (a non-trivial variant of) the Euler characteristic is the correct substitute for counting, even when a space is no longer zero-dimensional. Curve-counting theories are usually defined by finding a suitable (compact) moduli space, equipping it with an obstruction theory, which in turn produces a *virtual fundamental class* and integrating this class to obtain a number, the "virtual count." If the moduli space in question consisted of a finite number of (reduced) points, then this count should coincide with the cardinality of the space, or in other words with its Euler characteristic. We should point out that we shall only concern ourselves with Donaldson-Thomas theory as a *curve-counting* theory, in the spirit of [Mau+06].

In [Beh09], Behrend found a way to attach to any finite type scheme M a constructible function ν_M. The topological Euler characteristic of M can then be weighted by ν_M as follows

$$\chi_B(M) := \sum_k k \cdot \chi_{top}\left(\nu_M^{-1}(k)\right).$$

Behrend also proved that when M comes equipped with a *symmetric obstruction theory* (as is the case in DT theory) the virtual count coincides with the weighted Euler characteristic of M.

The definition of the DT invariants becomes now quite appealing. Let Y be a Calabi-Yau[1] threefold and fix a numerical type α (for example a Chern character). If $\mathrm{Hilb}_Y(\alpha)$ denotes the Hilbert scheme of Y parameterising subschemes whose structure sheaf is of class α, then the *DT number of class α of* Y is defined as

$$\mathrm{DT}_Y(\alpha) := \chi_B\left(\mathrm{Hilb}_Y(\alpha)\right).$$

We will always view Hilb as a Quot functor, parameterising quotients $\mathcal{O}_Y \twoheadrightarrow Q$ of the structure sheaf $\mathcal{O}_Y$. The parameter α then will indicate either the class of Q in the numerical Grothendieck group of Y or the Chern character of Q. Let us expand briefly on this point.

The Grothendieck group $K_0(Y)$ of Y is defined as the free abelian group over isomorphism classes of coherent sheaves on Y modulo the relations given by short

1 For us, this means trivial canonical bundle and torsion fundamental group.

exact sequences. The *numerical* Grothendieck group $N(Y)$ of Y is then defined as the quotient of $K_0(Y)$ modulo the kernel of the Chern character map. Inside these Grothendieck groups, there are filtrations indexed by the dimension of the support. Concretely, we define $F_iN(Y) < N(Y)$ to be the subgroup spanned by those sheaves whose support is of dimension at most i. As we wish to count curves, we are particularly interested in $F_1N(Y)$, although the group $F_0N(Y)$ will also play a small role.

On the other hand, we also have the *numerical Chow group* $N_1(Y)$, which is defined to be the quotient of the Chow group $A_1(Y)$ where we identify two curve classes if they produce the same numbers when intersected with any divisor class. Finally, we can identify $N_1(Y) \oplus \mathbb{Z}$ with $F_1N(Y)$ using the Chern character (see for example [Bri11, Lemma 2.2]).

It is customary to package all the DT numbers into a *generating series*

$$\mathrm{DT}(Y) := \sum_{\alpha \in F_1N(Y)} \mathrm{DT}_Y(\alpha) q^\alpha$$

where q is a formal variable. Using the identification of $F_1N(Y)$ with $N_1 \oplus \mathbb{Z}$, we can alternatively write

$$\mathrm{DT}(Y) = \sum_{\substack{\beta \in N_1(Y) \\ n \in \mathbb{Z}}} \mathrm{DT}_X(\beta, n) q^{(\beta, n)}.$$

1.2 FLOP FORMULA

An interesting question in Donaldson-Thomas theory is whether there exists a relationship between the DT numbers of two birational (smooth and projective) Calabi-Yau threefolds. As Calabi-Yau varieties are minimal models, any birational map between them can be broken down into a sequence of flops [Kaw08]. Hence, in principle, it suffices to understand what happens in this latter case.

In its simplest form, a flop is a birational morphism fitting in a diagram

$$\begin{tikzcd} Y \arrow[rr, dashed] \arrow[dr, "f"'] & & Y^+ \arrow[dl, "f^+"] \\ & X & \end{tikzcd}$$

where f (respectively f^+) is birational and contracts trees of rational curves to points. In this setting it is possible to write down an explicit formula relating the generating series for the DT invariants of both Y and Y^+, as we now explain.

With the flopping contraction $f : Y \to X$ one can also associate the DT series of curves *contracted* by f, that is

$$\mathrm{DT}_{\mathrm{exc}}(Y) := \sum_{\substack{\beta,n \\ f_*\beta=0}} \mathrm{DT}_Y(\beta,n) q^{(\beta,n)}$$

where the subscript $_{\mathrm{exc}}$ stands for *exceptional*. If we wish to index $\mathrm{DT}_{\mathrm{exc}}(Y)$ using the Grothendieck group, we need to introduce the subgroup $F_{\mathrm{exc}}N(Y) < F_1N(Y)$, spanned by sheaves whose support is contracted to a point by f.

For a pair of Calabi-Yau threefolds Y, Y^+ related by a flop we prove the following result.

4.9.4 THEOREM (continuing from p. 93) – If we define the series

$$\mathrm{DT}^{\vee}_{\mathrm{exc}}(Y) := \sum_{\substack{\beta,n \\ f_*\beta=0}} \mathrm{DT}_Y(-\beta,n) q^{(\beta,n)}$$

then the following formula[2] holds:

$$\mathrm{DT}^{\vee}_{\mathrm{exc}}(Y)\cdot\mathrm{DT}(Y)=\mathrm{DT}^{\vee}_{\mathrm{exc}}(Y^+)\cdot\mathrm{DT}(Y^+) \tag{$\bigstar$}$$

where we identify the formal variables $q^{(\beta,n)}$ via the flop.

The key ingredient here is Bridgeland's derived equivalence between Y and Y^+ [Bri02], which we denote by Φ. Inside the derived category $D(Y)$ of Y there is a t-structure whose heart $\mathrm{Per}(Y/X)$ is called the category of *perverse coherent sheaves*.[3] This category is intimately related to the geometry of the flop. In fact, one can construct Y^+ as a moduli space of *point-like objects* in $\mathrm{Per}(Y/X)$. If one defines $\mathrm{Per}(Y^+/X)$ to be the category of perverse coherent sheaves for Y^+, then Φ restricts to an equivalence of abelian categories $\mathrm{Per}(Y^+/X)\cong\mathrm{Per}(Y/X)$. This fact can be exploited to compare DT invariants on both sides of the flop and we explain now how.

It turns out that the structure sheaf of Y is a perverse coherent sheaf, $\mathcal{O}_Y\in\mathrm{Per}(Y/X)$. One can then construct a moduli space $\mathrm{P\text{-}Hilb}(Y/X)$ of quotients (in the abelian category $\mathrm{Per}(Y/X)$) of $\mathcal{O}_Y$. If again we fix a numerical class α in $F_1N(Y)$, it is legitimate to define a *perverse* DT number $\mathrm{DT}_{Y/X}(\alpha)$ as the weighted topological Euler characteristic of the moduli space $\mathrm{P\text{-}Hilb}_{Y/X}(\alpha)$ parameterising perverse quotients of $\mathcal{O}_Y$ of class α.[4]

$$\mathrm{DT}_{Y/X}(\alpha):=\chi_B\left(\mathrm{P\text{-}Hilb}_{Y/X}(\alpha)\right)$$

2 The rigorous meaning of which is explained in Remark 4.8.2.
3 As a matter of fact there are different versions of this category, but we shall momentarily ignore this. It shall be made clear in the following chapters.
4 This definition is slightly imprecise, as we are sweeping under the carpet a few sign issues. This will be fully explained in Section 4.8.

We formally write down a generating series for these perverse DT numbers.

$$\mathrm{DT}(Y/X) := \sum_{\beta,n} \mathrm{DT}_{Y/X}(\beta,n) q^{(\beta,n)}$$

From the discussion so far it's not clear how $\mathrm{DT}(Y/X)$ is related to ordinary DT numbers. However, if we define analogously $\mathrm{DT}(Y^+/X)$ on Y^+ (and once we know that $\Phi(\mathcal{O}_{Y^+}) = \mathcal{O}_X$) it follows immediately that $\mathrm{DT}(Y^+/X)$ matches up with $\mathrm{DT}(Y/X)$ under the equivalence Φ.

To complete the picture, we will prove that $\mathrm{DT}(Y/X)$ is (almost) equal to the left hand side of equation ($\bigstar$). To do this, we will use the incarnation of motivic Hall algebras found in [Bri12], but adapted to the category $\mathrm{Per}(Y/X)$ (see also [KS10; Joy11]). Perverse coherent sheaves are complexes $E \in \mathrm{Per}(Y/X)$ concentrated in degrees $[-1,0]$. Moreover, $H^{-1}(E)[1]$ and $H^0(E)$ are also perverse coherent, so any E sits in an exact sequence

$$H^{-1}(E)[1] \hookrightarrow E \twoheadrightarrow H^0(E)$$

of perverse coherent sheaves. The Hall algebra is designed to encode precisely this kind of information. Given an epimorphism of perverse coherent sheaves $\mathcal{O}_Y \twoheadrightarrow E$ we obtain a surjection of sheaves $\mathcal{O}_Y \to E \to H^0(E)$. In a nutshell, these latter surjections know about the ordinary DT invariants of Y, while $H^{-1}(E)$ is relevant for $\mathrm{DT}^{\vee}_{\mathrm{exc}}(Y)$.

We should mention that Yukinobu Toda has given a different approach to the same problem [Tod09], using Van den Bergh's non-commutative resolution of X [Ber04] and wall-crossing techniques. Our identity ($\bigstar$) is related to [Tod09, Theorem 5.8] via [Tod09, Theorem 5.6] (with slightly different notation). Strictly speaking, Toda's result requires extra assumptions and applies only to the naive counting invariants (defined using the ordinary, unweighted, Euler characteristic) and not to DT invariants, as the proof relies on a yet unproved (but expected) result regarding the local structure of the moduli stack of the objects of the derived category [Tod09, Remark 2.32]. On the other hand Toda is able to prove an additional identity for flops [Tod09, Theorem 1.3 (13)], mainly that $\mathrm{DT}_{\mathrm{exc}}(Y) = \mathrm{DT}^{\vee}_{\mathrm{exc}}(Y^{+})$, which we were not able to duplicate using our techniques.

1.3 CREPANT RESOLUTION CONJECTURE

The second problem we tackle in this thesis is the Crepant Resolution Conjecture for Donaldson-Thomas invariants, as stated in [BCY12, Conjectures 1 and 2]. This conjecture is concerned with understanding the relationship between the DT invariants of a three-dimensional Calabi-Yau orbifold $\mathcal{X}$, satisfying the hard Lefschetz condition, and the DT invariants of a crepant resolution $Y \to X$ of its coarse moduli space X. Concretely, we prove a comparison formula (★★) for the DT generating series of $\mathcal{X}$ and Y. The proof employs a derived equivalence between the two (worked out in [CT08]), which is a "global" version of the McKay correspondence of Bridgeland-King-Reid [BKR01]. We prove that the image of the heart $\mathrm{Coh}(\mathcal{X})$ under this equivalence is Bridgeland's category of perverse coherent sheaves $\mathrm{Per}(Y/X)$.

Let $\mathcal{X}$ be a smooth and projective Calabi-Yau orbifold of dimension three and let X be its coarse moduli space. By [BKR01; CT08] there is a crepant resolution $Y \to X$ of X given by an appropriate Hilbert scheme of points of $\mathcal{X}$.

$$\begin{tikzcd} Y \arrow[dr, "f"'] & & \mathcal{X} \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

The global McKay correspondence tells us that, moreover, Y and $\mathcal{X}$ are derived equivalent via Fourier-Mukai transforms

$$\Phi : D(Y) \rightleftarrows D(\mathcal{X}) : \Psi$$

which automatically induce an isomorphism between the numerical Grothendieck groups.

We also assume that the fibres of f are at most one-dimensional.[5] We denote by $F_{mr}N(\mathcal{X})$ the image of $F_1N(Y)$ via Φ and define[6]

$$DT_{mr}(\mathcal{X}) := \sum_{\alpha \in F_{mr}N(\mathcal{X})} DT_{\mathcal{X}}(\alpha) q^{\alpha}$$

where the DT number of class α for $\mathcal{X}$ is defined in the same way, by taking the weighted Euler characteristic of the Hilbert scheme parameterising quotients of $\mathcal{O}_{\mathcal{X}}$ of class α. The main formula we prove (Corollary 5.2.5) is

$$DT_{mr}(\mathcal{X}) = \frac{DT^{\vee}_{exc}(Y) DT(Y)}{DT_0(Y)} \qquad (\bigstar\bigstar)$$

where $DT^{\vee}_{exc}$ is defined as in the previous section and

$$DT_0(Y) := \sum_{n \in F_0 N(Y)} DT_Y(n) q^n$$

5 By [BG09, Lemma 24], a case-by-case analysis shows that this condition is equivalent to requesting that $\mathcal{X}$ satisfy the hard Lefschetz condition.
6 The subscript $_{mr}$ stands for *multi-regular*, see [BCY12].

is the contribution to the DT series given by points. Notice that in (★★) we have implicitly identified $F_{mr}N(\mathcal{X})$ with $F_1N(Y)$ via Ψ.

To relate the above formula with Conjectures 1 and 2 in [BCY12] we need two more objects: the subgroup $F_0N(\mathcal{X})$ of the numerical K-group of $\mathcal{X}$ spanned by sheaves supported in dimension zero and the subgroup $F_{exc}N(Y) \subset F_1N(Y)$ spanned by sheaves whose support is contracted to a point by f. Notice that $F_{exc}N(Y)$ is the image of $F_0N(\mathcal{X})$ via Ψ and $F_{exc}N(Y)$ is also the image (via the Chern character) of classes (β, n) such that $f_*\beta = 0$. Using this observation we obtain (Corollary 5.2.7) that

$$\mathrm{DT}_0(\mathcal{X}) = \frac{\mathrm{DT}_{exc}(Y)\mathrm{DT}^{\vee}_{exc}(Y)}{\mathrm{DT}_0(Y)} \qquad (★★★)$$

where

$$\mathrm{DT}_{exc}(Y) = \sum_{\alpha \in F_{exc}N(Y)} \mathrm{DT}_Y(\alpha) q^{\alpha} = \sum_{\substack{(\beta,n)\in N_1(Y)\oplus\mathbb{Z} \\ f_*\beta=0}} \mathrm{DT}_Y(\beta,n) q^{(\beta,n)}.$$

The formula (★★★) relates to [BCY12, Conjecture 2] using [Bri11, Theorem 1.1 (b)]. Plugging (★★★) in (★★) we obtain

$$\frac{\mathrm{DT}_{mr}(\mathcal{X})}{\mathrm{DT}_0(\mathcal{X})} = \frac{\mathrm{DT}(Y)}{\mathrm{DT}_{exc}(Y)}$$

which is [BCY12, Conjecture 1].

The key result is identifying the image (via Ψ) of $\mathrm{Coh}(\mathcal{X})$ inside $D(Y)$. It turns out that $\Psi(\mathrm{Coh}(\mathcal{X}))$ is none other than Bridgeland's heart of perverse coherent sheaves $\mathrm{Per}(Y/X)$. We can improve on our previous discussion about the perverse DT series $\mathrm{DT}(Y/X)$ by saying explicitly what $\mathrm{DT}(Y/X)$ looks like, via the following formula.

$$\mathrm{DT}(Y/X) = \frac{\mathrm{DT}^{\vee}_{\mathrm{exc}}(Y)\,\mathrm{DT}(Y)}{\mathrm{DT}_0(Y)}. \tag{1.3.1}$$

The Fourier-Mukai transform Ψ not only identifies $\mathrm{Coh}(\mathcal{X})$ with $\mathrm{Per}(Y/X)$ but also the corresponding Hilbert schemes, so that we obtain

$$\mathrm{DT}_{\mathrm{mr}}(\mathcal{X}) = \mathrm{DT}(Y/X)$$

which implies ($\bigstar\bigstar$).

After the appearance of the first version of [Cal12b], David Steinberg has made available another possible strategy for proving the crepant resolution conjecture [Ste12]. His approach is quite interesting and proceeds via *relative stable pairs*. The idea is that given a morphism (such as the contraction $f: Y \to X$), there is a relative version of the stable pairs of Pandharipande and Thomas and a corresponding DT/PT correspondence holds:

$$f\text{-PT} = \frac{\mathrm{DT}}{\mathrm{DT}_{\mathrm{exc}}}$$

where again exc stands for the numerical classes of curves contracted by f. When f is the identity, one reduces to ordinary stable pairs and the ordinary DT/PT correspondence (in this context $\mathrm{DT}_{\mathrm{exc}} = \mathrm{DT}_0$). In fact, the proof is similar in spirit to the one given by Bridgeland [Bri11]. The hope would then be that the BKR equivalence sends f-stable pairs to stable pairs on the orbifold. Using an orbifold DT/PT correspondence (which has been announced by Arend Bayer), one would then conclude. Unfortunately it seems this is not case, and it looks as though the problem may be traced back to t-structures. Let us elaborate somewhat.[7]

Generalising stable pairs, Steinberg defines a "torsion pair" $(\mathcal{T}_\bullet, \mathcal{Q})$ on $\mathrm{Coh}(Y)$, whose "tilt" $\mathcal{D}$ is home to the stable pairs. The category $\mathcal{T}_\bullet$ is defined as those sheaves T whose pushdown Rf_*T is a skyscraper sheaf. The structure sheaf $\mathcal{O}_Y$ belongs to $\mathcal{D}$ and epimorphisms with source $\mathcal{O}_Y$ are by definition the relative stable pairs. The category $\mathcal{T}_\bullet$ is contained in the larger $\mathcal{T}$, consisting of sheaves T whose pushdown Rf_*T is a sheaf (cfr. page 24). This latter category appears in our proof of the crepant resolution conjecture, as it fits in a torsion pair $(\mathcal{T}, \mathcal{F})$, whose tilt is $\mathrm{Per}(Y/X)$, the category of perverse coherent sheaves (with zero perversity).

Hence we obtain a sort of trisection of $\mathrm{Coh}(Y)$, given by $(\mathcal{T}_\bullet, \mathcal{U}, \mathcal{F})$, where $\mathcal{U}$ is both right orthogonal to $\mathcal{T}_\bullet$ and left orthogonal to $\mathcal{F}$. In turn, general considerations imply the existence of a "torsion pair" on $\mathrm{Per}(Y/X)$ given by $(\mathrm{Per}_\bullet(Y/X), \mathcal{U})$, where $\mathrm{Per}_\bullet(Y/X)$ consists of perverse coherent sheaves whose pushdown to Y is a skyscraper sheaf. It turns out that the "tilt" of $\mathrm{Per}(Y/X)$ along this torsion pair is none other than Steinberg's category shifted by one $\mathcal{D}[1]$.

This seems promising, as under the BKR equivalence Theorem 5.1.4 implies that $\mathrm{Per}_\bullet(Y/X)$ is sent to $\mathrm{Coh}_\bullet(\mathcal{X})$, consisting of coherent sheaves whose pushdown to X is a skyscraper. The first problem arises as $\mathrm{Coh}_\bullet(\mathcal{X})$ needn't be equal to $\mathrm{Coh}_0(\mathcal{X})$ (sheaves with zero-dimensional support), and so a direct application of the DT/PT correspondence on $\mathcal{X}$ is not possible. The second issue is that, while $\mathcal{F}_{\leq 1}[1]$ lies in $\mathrm{Per}_\bullet(Y/X)$, the full $\mathcal{F}[1]$ need not. Finally (and equally problematically), the category $\mathcal{T}_\bullet$ is not closed under quotients in $\mathrm{Coh}(Y)$, and thus cannot be the torsion part of a torsion pair (hence the quotes appearing above).[8] A solution

7 Although not strictly necessary, the following discussion more or less makes the unfair assumption that the reader is already familiar with our proof of the formula for flops (which is summarised in Section 4.1), Theorem 5.1.4 and the statements of the results contained in Section 5.2.

8 David Steinberg points this issue out in [Ste12], in fact his proof avoids the language of t-structures entirely.

suggested by Bryan and Steinberg is to work in a subcategory of $\mathrm{Coh}(Y)$ containing $\mathrm{Coh}_{\leq 1}(Y)$ (sheaves supported on at most curves) and $\mathcal{O}_Y$, where $\mathcal{T}_\bullet$ is closed under quotients. Notwithstanding, passing to a subcategory might destroy the compatibility between $\mathrm{Per}(Y/X)$ and the corresponding category $\mathcal{D}$. At present, the details have yet to be fleshed out.

1.4 CONVENTIONS

In the sequel we shall always work over the field of complex numbers $\mathbb{C}$. All schemes (and stacks) we consider will be locally of finite type over $\mathbb{C}$. For a variety X, we denote by $D(X)$ the *bounded* derived category of *coherent* sheaves (or equivalently of complexes with coherent cohomology). When needed, for a scheme X, we will denote by $D(\mathcal{O}_X)$ the *unbounded* derived category of (Zariski) $\mathcal{O}_X$-modules. Given a complex $E \in D(\mathcal{O}_X)$ we will denote by $H^i(E) \in \mathcal{O}_X\text{–Mod}$ the i-th cohomology sheaf, while $H^i(X, E)$ will denote the i-th (hyper-)cohomology group (which by our assumptions is always automatically endowed with the structure of a complex vector space). Given a fibre product $X_T = T \times_S X$ and a complex $E \in D(\mathcal{O}_X)$, we denote by $E|^L_{X_T}$ the derived pullback $Lu^*E \in D(\mathcal{O}_{X_T})$, where $u : X_T \to X$. Plainly, all we've just written is valid unless we state otherwise.

2

BACKGROUND

In this chapter we collect some background material to be used in the remainder of the thesis. The first section is a very sketchy introduction to Donaldson-Thomas theory, the second is a brief overview of the classical minimal model programme and the third is a reminder on perverse coherent sheaves.

2.1 DONALDSON-THOMAS INVARIANTS

In this section we attempt to give a concise (and ahistorical) account of the ideas surrounding (a small part of) Donaldson-Thomas theory. An ordinary day in the life of a (virtual) enumerative geometer can be summarised as follows:

- he wakes up wishing to count certain geometric objects,
- for this purpose he cooks up a moduli space parameterising those objects (or a compactification thereof),
- the study of the deformation theory of that moduli space produces a so-called perfect obstruction theory,
- using the technology of Behrend and Fantechi [BF97], one obtains a virtual class, which can be integrated to obtain numbers (the virtual count).

Let us somewhat elaborate.[1]

1 The following discussion is largely taken from [PT11], which is recommended reading.

Assume we've managed to construct the relevant moduli space $\mathscr{M}$. Assume moreover that $\mathscr{M}$ sits inside a smooth ambient space $\mathscr{A}$, and that on this ambient space we have a vector bundle E and a section $s \in \Gamma(\mathscr{A}, \mathrm{E})$, such that $\mathscr{M}$ is the zero-locus of that section $\mathrm{Z}(s)$. As locally $\mathscr{M}$ is cut out by $\mathrm{rk}\,\mathrm{E}$ equations, its dimension will be at least $\dim \mathscr{A} - \mathrm{rk}\,\mathrm{E}$.

If the section s is transverse, then $\dim \mathscr{M} = \dim \mathscr{A} - \mathrm{rk}\,\mathrm{E}$, $\mathscr{M}$ is smooth and its fundamental class coincides with the Euler class of the bundle E. When s fails to be transverse, the virtual machinery comes to the rescue, allowing us to pretend transversality to hold. For example, even in the non-transverse case we can speak of the expected (or virtual) dimension of $\mathscr{M}$, which we define to be $\mathrm{vdim}\, \mathscr{M} := \dim \mathscr{A} - \mathrm{rk}\,\mathrm{E}$.

A second scenario we want to consider is when s is not transverse but factors as a transverse section s' of a sub-bundle $\mathrm{E}' \subset \mathrm{E}$. Our moduli space is then the zero locus of s', and is thus of dimension $\dim \mathscr{A} - \mathrm{rk}\,\mathrm{E}' = \mathrm{vdim}\, \mathscr{M} + \mathrm{rk}\,\mathrm{E}/\mathrm{E}'$. If we assume the bundle to split $\mathrm{E} = \mathrm{E}' \oplus \mathrm{E}/\mathrm{E}'$, we can write the section s as $(s', 0)$. If we restrict E/E' to $\mathscr{M}$ and if we can perturb the zero-section of E/E' to a transverse section ϵ, then we can find a space $\mathrm{Z}(\epsilon) \subset \mathrm{Z}(\mathscr{M})$, of dimension $\dim \mathscr{M} - \mathrm{rk}\,\mathrm{E}/\mathrm{E}' = \mathrm{vdim}\, \mathscr{M}$. Although in this setting we had the bundle E' to work with, the idea of virtual classes is that we should always try and find a class of the *expected* dimension, which was defined in terms of the original bundle E. It follows therefore that the natural candidate for the *virtual class* of $\mathscr{M}$ is the fundamental class of $\mathrm{Z}(\epsilon)$, or in other words the Euler class of E/E'. When the bundle E does not split one can use the excess intersection theory of Fulton-MacPherson.

The point of this warm-up discussion is that an obstruction theory provides this description of our moduli space $\mathcal{M}$ at an infinitesimal level. In the setting above, at a point $x \in \mathcal{M}$, we have an exact sequence

$$0 \to T_x\mathcal{M} \to T_x\mathcal{A} \to E_x \to Ob_x \to 0$$

where Ob stands for *obstruction* and Ob_x is the fibre of E/E'. Notice that the virtual dimension of $\mathcal{M}$ is given by the difference of the dimensions of $T_x\mathcal{M}$ and Ob_x. A perfect obstruction theory is a global (taken now in the sheafy sense) version of the above exact sequence. As in algebraic geometry differentials are more natural than tangents (especially in non-smooth cases) one should really take duals.

Governing the deformation theory of the moduli space $\mathcal{M}$ is the cotangent complex $L_{\mathcal{M}}$ of $\mathcal{M}$. This complex is (in an appropriate sense) the result of deriving Kähler differentials, and one always has $H^0(L_{\mathcal{M}}) = \Omega_{\mathcal{M}}$. A *perfect obstruction theory* on $\mathcal{M}$ is the datum of a perfect complex E, concentrated in degrees minus one and zero, together with a morphism $E \to L_{\mathcal{M}}$, which is an isomorphism on H^0 and a surjection on H^{-1}.

In this thesis we are concerned with Donaldson-Thomas (DT) theory as a *curve-*counting theory, very much in the spirit of [Mau+06]. The relevant moduli space is then the Hilbert scheme of points and curves (on a Calabi-Yau threefold Y), or in other words the space parameterising subschemes of dimension at most one. The datum of an obstruction theory, however, is not intrinsic to the moduli space. In fact, when the study of the deformation theory of $\mathrm{Hilb}(Y)$, leads to the deformation space $\mathrm{Hom}(I_Z, \mathcal{O}_Z)$ (for a closed point corresponding to a subscheme $Z \subset Y$) and obstruction space $\mathrm{Ext}^1(I_Z, \mathcal{O}_Z)$. Due to the presence of higher Ext groups this approach does not lead to an obstruction theory. Thomas's idea was to view the

Hilbert scheme as a moduli space of torsion-free sheaves with fixed determinant [Tho00].

If $\mathscr{M}$ is now the Hilbert scheme, viewed now as a moduli space of torsion-free sheaves with fixed determinant, Thomas showed the existence of a perfect obstruction theory. The deformation and obstruction spaces turn out to be $\mathrm{Ext}^1(\mathrm{I_Z},\mathrm{I_Z})_0, \mathrm{Ext}^2(\mathrm{I_Z},\mathrm{I_Z})_0$, where the zero subscript stands the kernel of the trace map $\mathrm{RHom}(\mathrm{I_Z},\mathrm{I_Z}) \to \mathrm{R}\Gamma(\mathrm{Y},\mathcal{O}_\mathrm{Y})$. By Serre duality there aren't any other obstruction spaces and so we do get an obstruction theory. Notice that the virtual dimension is given by the difference of the dimensions of the traceless Ext groups. By using Riemann-Roch and the Calabi-Yau condition on Y we see that the virtual dimension is zero. This implies that the we can take the degree of the virtual fundamental class to obtain a number, the DT number corresponding to the class of $\mathrm{I_Z}$.

The DT obstruction theory has an extra property, that of being *symmetric*, that is there exists an isomorphism of E with $\mathrm{E}^\vee[1]$ [BF08, Lemma 1.22]. This extra property was exploited by Behrend [Beh09], who proved that any space equipped with a symmetric perfect obstruction theory can locally be expressed as the critical locus of a regular function on a smooth ambient space. Using topological techniques, Behrend was able to show that the DT virtual count can be expressed as a weighted topological Euler characteristic of the Hilbert scheme.

To be a little more concrete, Behrend associated to any finite type scheme M a constructible function ν_M, which is commonly called the *Behrend function* of M. The weighted Euler characteristic of M is then defined to be

$$\sum_k k\,\chi_{\mathrm{top}}\left(\nu^{-1}(k)\right).$$

When M is smooth of dimension d, ν_M is the constant function $(-1)^d$. If $f:\mathrm{M}\to\mathrm{N}$ is any morphism, we can precompose ν_N with f to obtain a new constructible function $f^*\nu_N$ on M. When f is smooth of relative dimension d then $f^*\nu_N = (-1)^d\nu_M$. When M is the critical locus of a regular function on a smooth ambient space $\mathscr{A}$, then $\nu_M(x) = (-1)^{\dim \mathscr{A}}(1-\chi_{top}(F_x))$, where F_x is the *Milnor fibre*, that is the intersection of a nearby fibre of f with a small ball in $\mathscr{A}$ centred at x.

The upshot of Behrend's result is that DT numbers depend only on the moduli space, and not on the extra datum of the obstruction theory. Our approach to comparing DT series uses this fact in a crucial and essential fashion. On one hand, the idea is that if two moduli spaces are isomorphic then their weighted Euler characteristics coincide (as happens, for example, for our perverse Hilbert schemes). On the other, the work of this thesis is based on the motivic Hall algebras of Joyce and Song, especially the integration morphism, which is a souped-up version of taking weighted Euler characteristics.

2.2 BIRATIONAL GEOMETRY

The problem of classifying smooth and projective algebraic varieties up to birational equivalence is of unmistakable importance in algebraic geometry. Historically, it has played a prominent role since the very beginning of the italian school. In Miles Reid's words [Rei88]

> "The 'meat' of the varieties [...] is the same, although they may differ a bit around the edges."

The minimal model programme (MMP) aims to isolate special (*minimal*) representatives in each birational class. The proofs for the statements claimed in this section can be found, for example, in [KM98].

Surfaces

One way to motivate the MMP is to start from Castelnuovo's theorem for surfaces.[2] Let S be a smooth surface and C a curve inside it. We call C a (-1)*-curve* if it is rational and has self-intersection $C \cdot C = -1$. Castelnuovo's theorem asserts that given a (-1)-curve C in S one can find another smooth surface S_1 and a birational morphism $S \to S_1$ contracting C to a point, whose exceptional locus is given by C.

One can then ask whether S_1 has any (-1)-curves which can then be contracted. Contracting (-1)-curves is controlled by a discrete parameter, the Picard number of S. Every time we contract a curve, the Picard number decreases by one, so iterated applications of Castelnuovo's theorem must come to a halt. Hence, if we start with a surface S we can keep applying Castelnuovo's theorem, ending with another surface S', with no (-1)-curves. We think of S' as being a *minimal model* for the birational class of S.

This discussion was very special to dimension two, as curves were also divisors. To generalise this picture to higher dimensions we start by noticing that the presence of a (-1)-curve C in S implies that the canonical class K_S is *not* nef,[3] as $K_S \cdot C = -1$ (this is a consequence of the adjunction formula). In fact, one defines a smooth variety to be *minimal* if the canonical class is nef. Notice that Calabi-Yau varieties are automatically minimal.

Going back to surfaces, one might ask if the absence of (-1)-curves implies minimality. This is not the case, unless one excludes rational and uniruled surfaces. One has the following theorem.

2 In dimension one, two smooth curves are birational if and only if they are isomorphic, so the MMP does not have much to say in that case.

3 The acronym *nef* stands for *numerically effective*. A divisor class D is nef if its intersection with any effective curve class C is greater or equal to zero: $D \cdot C \geq 0$.

2.2.1 THEOREM – Let S be a smooth surface. There exists a birational morphism $S \to T$, given by iterated contractions of (-1)-curves, such that:

- K_T is nef (i.e. T is minimal) or
- T is isomorphic to $\mathbb{P}^2$ or
- there exists a morphism $T \to U$, such that U is a smooth curve and T is a minimal ruled surface over U.

The idea is that, given a smooth surface, we first apply Castelnuovo's theorem as many times as we can and then end up with three possibilities: S' is minimal, S' is $\mathbb{P}^2$ or S' is ruled over a curve, which reduces the problem of understanding the geometry of our variety to one dimension less.

It is important to mention that, in the case where this process terminates with a minimal model, the surface T is *unique*. In the other two cases one must simply live with the fact that the original surface did not admit a minimal model.

General Varieties

We would like to have a similar procedure for higher dimensional varieties. The goal would be to start with a variety and perform a series of contractions terminating with a minimal variety (which we recall means that the canonical divisor is nef). To this end, we recall the definition of the *effective cone*.

On a variety X one has a well-defined notion of intersection between a curve and a Cartier divisor, hence one can form the group $N_1(X)$ of 1-cycles modulo numerical equivalence, viz. two curves lie in the same class if and only if their intersection numbers with all Cartier divisors coincide. We denote $N_1(X)_{\mathbb{R}} = N_1(X) \otimes \mathbb{R}$, and inside it we single out the *cone of effective curves* $NE(X)$ given by sums $\sum_i a_i[C_i]$, where C_i is a curve in X and $a_i > 0$. We are interested in the closure (for the Euclidean topology) of NE(X) in $N_1(X)_{\mathbb{R}}$, which is unsurprisingly denoted by $\overline{\mathrm{NE}}(X)$.

Two crucial theorems in the minimal model programme are the *cone theorem* and the *contraction theorem*. Inside $\overline{\mathrm{NE}}(X)$ there is a subspace given by those classes α such that $K_X \cdot \alpha \geq 0$. The cone theorem asserts that there exists an at most countable collection of rational curves L_i, with a bound on their intersection numbers

$$0 < -K_X \cdot L_i \leq 2 \dim X$$

and such that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq 0}[L_i].$$

Furthermore, if we call a ray spanned by $[L_i]$ *extremal*, the collection of all rays $\mathbb{R}_{\geq 0}[L_i]$ is locally discrete. Notice that for a Calabi-Yau, the set of the L_i is empty.

The contraction theorem (often stated as part of the cone theorem) says that, if we pick an extremal ray [L] in $\overline{\mathrm{NE}}(X)$, there exists a unique morphism $\phi\colon X \to Y$, such that an irreducible curve C is contracted to a point if and only if $[C] \in \mathbb{R}_{\geq 0}[L]$. The hope would now be to apply over and over the contraction theorem to end up with a minimal model.

One scenario which might occur is that $\dim Y < \dim X$. In this case X is a *Fano fibration* and one reduces to the study of lower dimensional varieties. Such an X does not admit a minimal model. Notice that for surfaces, this is the case for both when X is uniruled (and Y is a curve) and when X is $\mathbb{P}^2$ (where Y is a point).

Another thing which might happen is that Y is not smooth. However, this can be circumvented by allowing so-called *terminal* $\mathbb{Q}$*-factorial singularities*. Yet another problem arises, we still haven't found a reason why this process should terminate. As the morphism ϕ is designed to contract all curves with corresponding class lying in a given ray of $N_1(X)_{\mathbb{R}}$ it might very well happen that it contracts divisors. In this case, once again the Picard number drops, so we find a discrete invariant ensuring that the process ends.

Unfortunately though, the contraction ϕ might be *small*, i.e. with exceptional locus of codimension at least two. For the resulting variety Y there isn't a naive way to make sense of intersection numbers with K_Y as, if C is a curve contracted by ϕ,

$$0 > K_X \cdot [C] = \phi^* K_X \cdot [C] = K_Y \cdot \phi_* [C] = 0$$

which is absurd. To solve this problem, one must do a *flip*, which we define below.

Given a small contraction

$$\phi \colon X \to Y$$

with $\mathbb{Q}$-Cartier and ϕ-ample anti-canonical divisor K_X, another variety W together with a small contraction $\psi \colon W \to Y$ is called a *flip* of ϕ if K_W is $\mathbb{Q}$-Cartier and ψ-ample.

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr, "\phi"'] & & W \arrow[dl, "\psi"] \\
& Y &
\end{tikzcd}$$

The existence of flips has recently been established in all dimensions [Bir+10], and for a fixed contraction its flip is unique as it is given by $\operatorname{Proj} \bigoplus_{n \geq 0} \phi_* \mathcal{O}_X(n m_0 K_X)$, where m_0 is a positive integer such that $m_0 K_X$ is Cartier.

Hence, we can summarise the MMP as follows:

- start with a variety X,

- if X is minimal we are done, if not
- perform a contraction of an extremal ray,
- if ϕ is a divisorial contraction return to the beginning,
- if ϕ is a Fano fibration stop,
- if ϕ is a small contraction, perform a flip and start over.

The final fact needed to ensure that this algorithm actually terminates, is that we only need perform a finite number of flips. This question is only meaningful in dimension ≥ 3 and was settled for threefolds in the nineties [Kaw92].

The minimal models we obtain through running the MMP might however be quite different from one another. Fortunately, there is a close relationship between them, in that minimal models are related by finite chains of *flops* [Kaw08]. A flop is just like a flip, but where we require the canonical classes K_X (respectively K_W) to be numerically ϕ-trivial (respectively ψ-trivial).

The case of most interest for the purpose of this thesis is the case of Calabi-Yau threefolds. As mentioned, a Calabi-Yau is trivially minimal. Thus, birational equivalence between two Calabi-Yau's implies the existence of a chain of flops connecting them.

2.3 FLOPS

In this section we recall a few facts about the categories of perverse coherent sheaves and construct the corresponding moduli spaces.

2.3.1 *Perverse Coherent Sheaves*

Henceforth we assume to be working within the following setup.

Situation 2.3.1

Fix a smooth and projective variety Y *of dimension three, over* $\mathbb{C}$*, with trivial canonical bundle* $\omega_Y \cong \mathcal{O}_Y$ *and satisfying* $H^1(Y, \mathcal{O}_Y) = 0$*. Fix a map* $f : Y \to X$ *satisfying the following properties:*

- f *is birational and its fibres are at most one-dimensional;*
- X *is projective and Gorenstein;*
- $Rf_*\mathcal{O}_Y = \mathcal{O}_X$.

Notice that from the assumptions it follows that X has rational singularities [Kov00], that its canonical bundle is trivial, $\omega_X \cong \mathcal{O}_X$ and that f is crepant. Also, for any sheaf G on Y, $R^i f_* G = 0$ for $i \geq 2$.

The main protagonist of this thesis is Bridgeland's category of perverse coherent sheaves ${}^p\mathrm{Per}(Y/X)$ of Y over X. As mentioned in the overview chapter there are different versions of it, indexed by an integer p called the *perversity*. We shall only need two of them, corresponding to the -1 and 0 perversity. One way to define these categories is by using a torsion pair [BR07], which we now recall (see also [Ber04, Section 3]).

Notation. For compactness we will often denote $\mathrm{Coh}(Y)$ by $\mathcal{A}$ and ${}^p\mathrm{Per}(Y/X)$ by ${}^p\mathcal{A}$.

Let

$$\mathcal{C} = \{E \in \mathrm{Coh}(Y) \mid Rf_*E = 0\}$$

and consider the following subcategories of $\mathcal{A}$:

$$
\begin{aligned}
{}^{0}\mathcal{T} &= \left\{ T \in \mathcal{A} \,\middle|\, R^1 f_* T = 0 \right\} \\
{}^{-1}\mathcal{T} &= \left\{ T \in \mathcal{A} \,\middle|\, R^1 f_* T = 0, \mathrm{Hom}(T, \mathcal{C}) = 0 \right\} \\
{}^{-1}\mathcal{F} &= \left\{ F \in \mathcal{A} \,|\, f_* F = 0 \right\} \\
{}^{0}\mathcal{F} &= \left\{ F \in \mathcal{A} \,|\, f_* F = 0, \mathrm{Hom}(\mathcal{C}, F) = 0 \right\}.
\end{aligned}
$$

The pair $({}^p\mathcal{T}, {}^p\mathcal{F})$ is a torsion pair on $\mathcal{A}$, for $p = -1, 0$, and the tilt of $\mathcal{A}$ with respect to it is the category of perverse coherent sheaves ${}^p\mathcal{A}$. Notice that we picked the convention where

$$
{}^p\mathcal{F}[1] \subset {}^p\mathcal{A} \subset D^{[-1,0]}(Y).
$$

We mention in passing that the structure sheaf is perverse coherent, $\mathcal{O}_Y \in {}^p\mathcal{T} \subset {}^p\mathcal{A}$.

Notation. For convenience (and unless otherwise stated) we shall adopt the convention where p stands for either -1 or 0 and $q = -(p+1)$. In other words, if p stands for one perversity, q will stand for the other.

Before moving on we state an easy lemma.

2.3.2 LEMMA – For all $T \in {}^p\mathcal{T}$ we have $H^i(Y, T) = H^i(X, f_* T)$, for all i. For all $F \in {}^p\mathcal{F}$ we have $H^i(Y, F) = H^{i-1}(X, R^1 f_* F)$.

For a proof one may use Leray's spectral sequence.

2.3.2 *Moduli*

To define the motivic Hall algebra of ${}^p\mathcal{A}$ in the next section we need, first of all, an algebraic stack ${}^p\mathfrak{A}$ parameterising objects of ${}^p\mathcal{A}$. We build it as a substack of the stack $\mathfrak{Mum}_Y$, which was constructed by Lieblich [Lie06] and christened *the mother of all moduli of sheaves*. For its definition and some further properties we refer the reader to the next section. We only recall that $\mathfrak{Mum}_Y$ parameterises objects in the derived category of Y with no negative self-extensions. This last condition is key to avoid having to enter the realm of higher stacks. We point out that as ${}^p\mathcal{A}$ is the heart of a t-structure its objects satisfy this condition.

Notation. We denote by $\mathfrak{A}$ the stack of coherent sheaves, in other words the stack parameterising objects of $\mathcal{A}$.

Notice that the definition of ${}^p\mathcal{A}$ is independent of the ground field and is stable under field extension. Concretely, take $E \in \mathfrak{Mum}_Y(T)$ a family of complexes over Y parameterised by a scheme T and $t : \operatorname{Spec} k \to T$ a geometric point. We can consider $E|^{L}_{Y_t}$, the derived restriction of E to the fibre Y_t of Y_T over t, and it makes sense to write $E|^{L}_{Y_t} \in {}^p\mathcal{A}$ (where the latter category is interpreted relatively to k).

2.3.3 PROPOSITION – Define a prestack[4] by the rule

$${}^p\mathfrak{A}(T) = \left\{ E \in \mathfrak{Mum}_Y(T) \,\middle|\, \forall t \in T, E|^{L}_{Y_t} \in {}^p\mathcal{A} \right\}$$

with restriction maps induced by $\mathfrak{Mum}_Y$ and where by $t \in T$ we mean that $t : \operatorname{Spec} k \to T$ is a geometric point of T. The prestack ${}^p\mathfrak{A}$ is an open substack of $\mathfrak{Mum}_Y$.

4 We use the term *prestack* in analogy with *presheaf*.

Proof: As mentioned earlier, objects of ${}^p\mathrm{Per}(Y/X)$ have vanishing self-extensions and therefore can be glued. In other words, ${}^p\mathfrak{A}$ satisfies descent. To prove that the inclusion ${}^p\mathfrak{A} \to \mathfrak{Mum}_Y$ is open, we employ van den Bergh's projective generators. For this, we introduce some auxiliary spaces.

If $U \subset X$ is open, we can consider the restriction $g : V = f^{-1}(U) \to U$ of the morphism f. The category of perverse coherent sheaves ${}^p\mathrm{Per}(V/U) =: {}^p\mathcal{A}_U$ still makes sense and the corresponding stack ${}^p\mathfrak{A}_U$ satisfies descent. Notice that ${}^p\mathfrak{A}_X = {}^p\mathfrak{A}$.

When U is affine, there exists a vector bundle P (a projective generator [Ber04]) on V such that an object of the derived category E is perverse coherent (relatively to $g : V \to U$) if and only if $\mathrm{Hom}_V(P, E[i]) = 0$ for $i \neq 0$. In other words, E is perverse coherent if and only if the complex $Rg_* R\underline{\mathrm{Hom}}(P, E)$ is concentrated in degree zero. From this we automatically deduce that the morphism ${}^p\mathfrak{A}_U \to \mathfrak{Mum}_V$ is open as this last condition is open.

To pass from local to global, we recall that in [Ber04] it was also proved that one check whether a complex is a perverse coherent sheaf on an open cover of X. In other words, if $E \in D(Y)$ and if $U \to X$ is an open affine cover and $V = f^{-1}(U)$, then $E \in {}^p\mathrm{Per}(Y/X)$ if and only if $E|V \in {}^p\mathrm{Per}(V/U)$.

We have restriction morphisms $\mathfrak{Mum}_Y \to \mathfrak{Mum}_V$ and ${}^p\mathfrak{A} \to \mathfrak{A}_U$. When $U \to X$ is an open affine cover, we can realise ${}^p\mathfrak{A}$ as the fibre product of ${}^p\mathfrak{A}_U \to \mathfrak{Mum}_U \leftarrow \mathfrak{Mum}_Y$. This is enough to conclude that the inclusion ${}^p\mathfrak{A} \to \mathfrak{Mum}_Y$ is open. ■

It will be important for us to also have moduli spaces for the torsion and torsion-free subcategories ${}^p\mathcal{T}$, ${}^p\mathcal{F}$. We define them similarly as above.

$$
\begin{aligned}
{}^p\mathfrak{F}(T) &= \left\{ E \in \mathfrak{A}(T) \,\middle|\, \forall t \in T, E|^{L}_{Y_t} \in {}^p\mathcal{F} \right\} \\
{}^p\mathfrak{T}(T) &= \left\{ E \in \mathfrak{A}(T) \,\middle|\, \forall t \in T, E|^{L}_{Y_t} \in {}^p\mathcal{T} \right\}
\end{aligned}
$$

Notice that ${}^p\mathfrak{T} = {}^p\mathfrak{A} \cap \mathfrak{A}$ and ${}^p\mathfrak{F}[1] = {}^p\mathfrak{A} \cap \mathfrak{A}[1]$. One has the expected open inclusions of algebraic stacks

$$
\begin{gathered}
{}^p\mathfrak{T}, {}^p\mathfrak{F} \subset \mathfrak{A} \subset \mathfrak{Mum}_Y^{[-1,0]} \\
{}^p\mathfrak{T}, {}^p\mathfrak{F}[1] \subset {}^p\mathfrak{A} \subset \mathfrak{Mum}_Y^{[-1,0]}
\end{gathered}
$$

where $\mathfrak{Mum}_Y^{[-1,0]}$ is the substack of $\mathfrak{Mum}_Y$ parameterising complexes concentrated in degrees -1 and 0.

We conclude this section with a technical result regarding the structure of ${}^p\mathfrak{A}$. This will essentially allow us to carry all the proofs to set up the motivic Hall algebra of ${}^p\mathcal{A}$ from the case of coherent sheaves.

2.3.4 PROPOSITION – Let $p = -1$. There is a collection of open substacks ${}^p\mathfrak{A}_n \subset {}^p\mathfrak{A}$ which jointly cover ${}^p\mathfrak{A}$. Each ${}^p\mathfrak{A}_n$ is isomorphic to an open substack of $\mathfrak{A}$.

To prove this result we start by remarking that, as a consequence of our assumptions on Y, the structure sheaf $\mathcal{O}_Y$ is a spherical object [Huy06, Definition 8.1] in $D^b(Y)$. Thus the Seidel-Thomas spherical twist around it is an autoequivalence of $D^b(Y)$. This functor can be explicitly described as the Fourier-Mukai transform with kernel the ideal sheaf of the diagonal of Y shifted by one. We thus get an exact auto-equivalence τ of $D^b(Y)$ and we notice that the subcategory of complexes with no negative self-extensions is invariant under τ. As Fourier-Mukai transforms behave well in families [BBHR09, Proposition 6.1] we also obtain an automorphism (which by abuse of notation we still denote by τ) of the *stack* $\mathfrak{Mum}_Y$.

Let us now fix an ample line bundle L downstairs on X. Tensoring with f^*L^n also induces an automorphism of $\mathfrak{Mum}_Y$. The automorphism $\tau_n \in \mathrm{Aut}(\mathfrak{Mum}_Y)$ is then defined by $\tau_n(E) = \tau(E \otimes f^*L^n)$. The following lemma tells us how to use the automorphisms τ_n to deduce the proposition above.

2.3.5 LEMMA – Let $p=-1$ and let $E \in {}^p\mathcal{A}$ be a perverse coherent sheaf. Then there exists an n_0 such that for all $n \geq n_0$

$$\widetilde{\tau_n}(E) = \tau_n(E)[-1] \in \mathcal{A}.$$

Proof: The two key properties we use of τ_n are that it is an exact functor and that for a complex G we have an exact triangle

$$H^\bullet(Y, G(n)) \otimes_{\mathbb{C}} \mathcal{O}_Y \xrightarrow{\text{ev}} G(n) \to \tau_n(G) \nrightarrow$$

where $G(n) = G \otimes_{\mathcal{O}_Y} f^*L^n$.

Let now $E \in {}^p\mathcal{A}$ be a perverse coherent sheaf together with its torsion pair exact sequence (in ${}^p\mathcal{A}$)

$$F[1] \hookrightarrow E \twoheadrightarrow T$$

where $F \in {}^p\mathcal{F}$, $T \in {}^p\mathcal{T}$. Using Leray's spectral sequence, the projection formula, Lemma 2.3.2 and Serre vanishing on X we can pick n big enough so that all hypercohomologies involved, $H^\bullet(Y, F[1](n))$, $H^\bullet(Y, T(n))$, $H^\bullet(Y, E(n))$, are concentrated in degree zero.

From the triangle

$$H^\bullet(Y, E(n)) \otimes_{\mathbb{C}} \mathcal{O}_Y \xrightarrow{\text{ev}} E(n) \to \tau_n(E) \nrightarrow$$

we have that $\tau_n(E) \in D^{[-1,0]}(Y)$, similarly for $\tau_n(F[1])$ and $\tau_n(T)$. From the triangle

$$H^\bullet(Y, F[1](n)) \otimes_{\mathbb{C}} \mathcal{O}_Y \xrightarrow{\text{ev}} F[1](n) \to \tau_n(F[1]) \nrightarrow$$

we obtain that $H^0(\tau_n(F[1])) = 0$.

From the triangle

$$\tau_n(F[1]) \to \tau_n(E) \to \tau_n(T) \nrightarrow$$

arising from exactness of τ_n we have that $H^0(\tau_n(T)) \simeq H^0(\tau_n(E))$. Thus to prove the lemma it suffices to show that $H^0(\tau_n(T)) = 0$.

Finally, from the triangle

$$H^\bullet(Y, T(n)) \otimes_{\mathbb{C}} \mathcal{O}_Y \to T(n) \to \tau_n(T) \nrightarrow$$

one obtains the following exact sequence.

$$0 \to H^{-1}(\tau_n(T)) \to H^0(Y, T(n)) \otimes_{\mathbb{C}} \mathcal{O}_Y \xrightarrow{\alpha} T(n) \xrightarrow{\beta} H^0(\tau_n(T)) \to 0$$

Thus we have

$$\tau_n(E)[-1] \in \mathcal{A} \iff H^0(\tau_n(E)) \simeq H^0(\tau_n(T)) = 0 \iff \beta = 0.$$

Let $K = \ker \beta$. We then have two short exact sequences

$$\begin{array}{c} H^{-1}(\tau_n(T)) \hookrightarrow H^0(Y, T(n)) \otimes_{\mathbb{C}} \mathcal{O}_Y \xrightarrow{\gamma} K \\ K \xhookrightarrow{\delta} T(n) \xrightarrow{\beta} H^0(\tau_n(T)) \end{array}$$

and notice that $\delta\gamma = \alpha$. By pushing forward the first sequence via f_* we have that $R^1 f_* K = 0$, as $R^1 f_* \mathcal{O}_Y = 0$. Pushing forward the second sequence yields the exact sequence

$$f_* K \hookrightarrow f_* T(n) \twoheadrightarrow f_* H^0(\tau_n(T))$$

and $R^1 f_* H^0(\tau_n(T)) = 0$, as $R^1 f_* T(n) = 0$ (this last is a consequence of Lemma 2.3.2 and the projection formula).

By taking n even bigger we can assume $f_*\mathrm{T}(n)$ to be generated by global sections and thus we can assume $f_*\alpha$ to be surjective. As $\alpha = \delta\gamma$ we obtain that $f_*\delta$ is surjective and thus $f_*\mathrm{H}^0(\tau_n(\mathrm{T})) = 0$. As a consequence we have that $\mathrm{H}^0(\tau_n(\mathrm{T})) \in \mathcal{C}$.

The sheaf $\mathrm{T}(n)$ is in ${}^p\mathcal{T}$ (this is a simple computation, the key fact to notice is that $\mathcal{C}(n) = \mathcal{C}$). Finally, as $\mathrm{T}(n) \in {}^p\mathcal{T}$ and $\mathrm{H}^0(\tau_n(\mathrm{T})) \in \mathcal{C}$, $\beta = 0$. ■

To prove Proposition 2.3.4 we define ${}^p\mathcal{A}_n$ to be the subcategory of ${}^p\mathcal{A}$ consisting of elements E such that $\tilde{\tau}_n(\mathrm{E}) \in \mathcal{A}$. We can produce a moduli stack for ${}^p\mathfrak{A}_n$ via the following composition of cartesian diagrams.

$$\begin{tikzcd}
{}^p\mathfrak{A}_n \arrow[r, hook] \arrow[d, hook] & \tilde{\tau}_n^{-1}(\mathfrak{A}) \arrow[r] \arrow[d] & \mathfrak{A} \arrow[d, hook] \\
{}^p\mathfrak{A} \arrow[r, hook] & \mathfrak{Mum}_{\mathrm{X}} \arrow[r, "\tilde{\tau}_n"] & \mathfrak{Mum}_{\mathrm{X}}
\end{tikzcd}$$

We obtain that ${}^p\mathfrak{A}_n$ is an open substack of ${}^p\mathfrak{A}$ and is isomorphic to an open substack of $\mathfrak{A}$ via $\tilde{\tau}_n$. From the previous lemma we have that the sum of the inclusions $\coprod_n {}^p\mathfrak{A}_n \to {}^p\mathfrak{A}$ is surjective.

Remark 2.3.6. The proof we just presented here of Proposition 2.3.4 worked for $p = -1$, and we do not know a direct way to extend this result to the zero perversity. However, we can work around this issue by making the following additional assumption (which will hold in the cases which are of interest to us, i.e. for flops and the McKay correspondence): we assume the existence of a Fourier-Mukai equivalence taking ${}^0\mathrm{Per}(\mathrm{Y}/\mathrm{X})$ to ${}^{-1}\mathrm{Per}(\mathrm{W}/\mathrm{X})$, with W a variety over X satisfying the same assumptions as Y. Using this, we obtain a variant of Proposition 2.3.4, namely for $q = 0$ there exists a collection $\{{}^q\mathfrak{A}_n\}_n$ of open substacks of ${}^q\mathfrak{A}$ such that, for every n, ${}^q\mathfrak{A}_n$ is isomorphic to an open substack of the stack of coherent sheaves on W.

Henceforth, we will tacitly assume this extra hypothesis so that this strategy of passing to W can be applied.

2.4 SUBSTACKS

At the core of the construction of the Hall algebra of an abelian category lies the existence of a moduli stack[5] parameterising its objects (and a moduli of short exact sequences). In our case this amounts, first of all, to proving the existence of the moduli stack ${}^p\mathfrak{A}$, parameterising perverse coherent sheaves. We have mentioned in the first section that as the category ${}^p\mathcal{A}$ is the heart of a t-structure, its objects have no negative self-extensions. This simple remark is actually key, as we construct ${}^p\mathfrak{A}$ as an open substack Lieblich's mother of moduli of sheaves $\mathfrak{Mum}$ [Lie06]. Let us recall its definition.

First, fix a flat and proper morphism of schemes $\pi : X \to S$.

2.4.1 DEFINITION – An object $E \in D(\mathcal{O}_X)$ is (relatively over S) *perfect and universally gluable* if the following conditions hold.

- There exists an open cover $\{U_i\}$ of X such that $E|_{U_i}$ is quasi-isomorphic to a bounded complex of quasi-coherent sheaves flat over S.
- For any S-scheme $u : T \to S$ we have

  $$R\pi_{T,*}R\underline{\mathrm{Hom}}_{X_T}(Lu_X^*E, Lu_X^*E) \in D^{\geq 0}(\mathcal{O}_T)$$

  where π_T and u_X denote the maps induced by π and u respectively on the base-change X_T.

We denote the category of perfect and universally gluable sheaves on X (over S) as $D_{pug}(\mathcal{O}_X)$.

5 The author would like to thank Fabio Tonini for patiently explaining to him many things about stacks.

If in the definition we take S to be affine and assume $T = S$, then it's clear that gluability has to do with the vanishing of negative self-exts of E. This condition is necessary to avoid having to enter the realm of higher stacks.

A prestack $\mathfrak{Mum}_X$ is defined by associating with an S-scheme $T \to S$ (the associated groupoid of) the category $D_{pug}(\mathcal{O}_{X_T})$ of perfect and universally gluable complexes (relatively over T). The restriction functors are defined by derived pullback.

2.4.2 THEOREM (Lieblich) – The prestack $\mathfrak{Mum}_X$ is an Artin stack, locally of finite presentation over S.

From now on we fix $\pi : X \to S$ flat and projective with S a noetherian scheme. We assume all rings and schemes to be locally of finite type over S.[6]

We want to construct various open substacks of $\mathfrak{Mum}_X$, namely stacks of complexes satisfying additional properties. For example we would like to construct the stack of complexes with cohomology concentrated in degrees less or equal than a fixed integer n. The correct way to proceed is by imposing conditions fibrewise on restrictions to geometric points. Let us illustrate a general recipe first. The following diagram comes in handy.

$$\begin{tikzcd} X_t \arrow[r, "t_X"] \arrow[d, "\pi_t"] & X_T \arrow[r, "u_X"] \arrow[d, "\pi_T"] & X \arrow[d, "\pi"] \\ \operatorname{Spec} k \arrow[r, "t"] & T \arrow[r, "u"] & S \end{tikzcd}$$

6 For what follows, this assumption isn't substantial (as $\mathfrak{Mum}_X$ is locally of finite type over S) but it enables us to use the local criterion of flatness directly. This is essentially a consequence of [LMB00, Corollaire (10.11) (ii)].

Here T is the base space for our family of complexes, together with its structure map to S, and $t \in \mathrm{T}$ is a geometric point. Given a property P, we might define the stack of complexes satisfying P as follows.

$$\mathfrak{Mum}_{\mathrm{X}}^{\mathrm{P}}(\mathrm{T}) = \left\{ \mathrm{E} \in \mathfrak{Mum}_{\mathrm{X}}(\mathrm{T}) \,\middle|\, \forall \text{ geometric } t \in \mathrm{T}, \mathrm{E}|^{\mathrm{L}}_{\mathrm{X}_t} \text{ satisfies P} \right\}$$

We recall that by $\mathrm{E}|^{\mathrm{L}}_{\mathrm{X}_t}$ we mean $\mathrm{L}t_{\mathrm{X}}^{*}\mathrm{E}$.

To construct the substacks of $\mathfrak{Mum}_{\mathrm{X}}$ we are interested in we make use of the following lemma.

2.4.3 LEMMA – Let $\mathrm{T} \to \mathrm{S}$ be an S-scheme, let $t : \operatorname{Spec} k \to \mathrm{T}$ be a point of T and let $\mathrm{E} \in \mathrm{D}^{\mathrm{b}}(\mathcal{O}_{\mathrm{X}_{\mathrm{T}}})$ be a bounded complex of $\mathcal{O}_{\mathrm{X}_{\mathrm{T}}}$-modules flat over T. Let $n \in \mathbb{Z}$ be an integer. The following statements hold.

1. $\mathrm{E}|^{\mathrm{L}}_{\mathrm{X}_t} \in \mathrm{D}^{\leq n}(\mathcal{O}_{\mathrm{X}_t}) \Longleftrightarrow \mathrm{X}_t \subset \mathrm{U}_{>}$, where

$$\mathrm{U}_{>} = \bigcap_{q>n} \mathrm{X}_{\mathrm{T}} \setminus \operatorname{supp} \mathrm{H}^{q}(\mathrm{E}).$$

2. $\mathrm{E}|^{\mathrm{L}}_{\mathrm{X}_t} \in \mathrm{D}^{[n]}(\mathcal{O}_{\mathrm{X}_t}) \Longleftrightarrow \mathrm{X}_t \subset \mathrm{U}$, where

$$\begin{aligned}
\mathrm{U} &= \mathrm{U}_{>} \cap \mathrm{U}_{\mathrm{f}} \cap \mathrm{U}_{<} \\
\mathrm{U}_{>} &= \bigcap_{q>n} \mathrm{X}_{\mathrm{T}} \setminus \operatorname{supp} \mathrm{H}^{q}(\mathrm{E}) \\
\mathrm{U}_{\mathrm{f}} &= \left\{ x \in \mathrm{X}_{\mathrm{T}} \,\middle|\, \mathrm{H}^{n}(\mathrm{E})_{x} \text{ is a flat } \mathcal{O}_{\mathrm{T},\pi_{\mathrm{T}}(x)}\text{-module} \right\} \\
\mathrm{U}_{<} &= \bigcap_{q<n} \mathrm{X}_{\mathrm{T}} \setminus \operatorname{supp} \mathrm{H}^{q}(\mathrm{E}).
\end{aligned}$$

3. $E|^{L}_{X_t} \in D^{\geq n}(\mathcal{O}_{X_t}) \Longleftrightarrow F \in D^{[n]}(\mathcal{O}_{X_t})$, where $F = \sigma_{\leq n}E$ is the stupid truncation of E in degrees less or equal than n.

$$F^p = \begin{cases} E^p, \text{ if } p \leq n \\ 0, \text{ if } p > n \end{cases}$$

Proof: PROOF OF 1. Let t_X be the inclusion of the fibre $X_t \to X_T$. As t_X is an affine map we do not lose information on the cohomologies of $E|^{L}_{X_t}$ after pushing forward back into X_T. We also have isomorphisms

$$t_{X,*}E|^{L}_{X_t} \simeq E \overset{L}{\otimes}_{\mathcal{O}_{X_T}} t_{X,*}\mathcal{O}_{X,t} \simeq E \overset{L}{\otimes}_{\mathcal{O}_{X_T}} \pi_T^* t_* k$$

where the first follows from the projection formula and the second from base change compatibility. As we are interested in the vanishing of $H^q(E|^{L}_{X_t})$ we may restrict to the stalk at a point $x \in X_t$. Taking stalks at x gives us isomorphisms

$$H^q\left(E|^{L}_{X_t}\right)_x \simeq H^q\left(E_x \overset{L}{\otimes}_{\mathcal{O}_{T,t}} k\right). \tag{2.4.4}$$

We have the page two spectral sequence of the pullback

$$L^p t_X^* H^q(E) \Longrightarrow H^{p+q}(E|^{L}_{X_t}). \tag{2.4.5}$$

which, at a point $x \in X_t$ and using the isomorphism (2.4.4), boils down to

$$\mathrm{Tor}_{-p}^{\mathcal{O}_{X_t}}\left(H^q(E)_x, k\right) \Longrightarrow H^{p+q}\left(E|^{L}_{X_t}\right)_x. \tag{2.4.6}$$

Let now q be the largest integer such that $H^q(E) \neq 0$. From the spectral sequence (2.4.6) we have

$$H^q\left(E|^{L}_{X_t}\right)_x \simeq H^q(E)_x \otimes_{\mathcal{O}_{T,t}} k.$$

Hence, by Nakayama, $H^q(E|^{\mathrm{L}}_{X_t})_x = 0$ if and only if $x \in X_T \setminus \operatorname{supp} H^q(E)$ and finally

$$H^q(E|^{\mathrm{L}}_{X_t}) = 0 \Longleftrightarrow X_t \subset X_T \setminus \operatorname{supp} H^q(E).$$

PROOF OF 2. Using 1. we can assume that $E|^{\mathrm{L}}_{X_t} \in D^{\leq n}(\mathcal{O}_{X_t})$. By the spectral sequence (2.4.5) we have that $H^{n-1}(E|^{\mathrm{L}}_{X_t}) \simeq L_1 t_X^* H^n(E)$. Again, we may pass on to the stalk at a point $x \in X_t$ and (2.4.6) yields

$$H^{n-1}\left(E|^{\mathrm{L}}_{X_t}\right)_x \simeq \operatorname{Tor}_1^{\mathcal{O}_{X_t}}(E_x, k)$$

the vanishing of which is equivalent, by the local criterion for flatness, to $H^q(E)_x$ being a flat $\mathcal{O}_{X,t}$-module.

We can thus assume that $X_t \subset U_> \cap U_f$. Once more, from the spectral sequence (2.4.6) we have that

$$H^{n-1}(E|^{\mathrm{L}}_{X_t}) \simeq t_X^* H^{n-1}(E)$$

and we proceed as in the proof of 1.

PROOF OF 3. Consider the page one spectral sequence

$$L^q t_X^* E^p \Longrightarrow H^{p+q}\left(E|^{\mathrm{L}}_{X_t}\right)$$

from which we get isomorphisms

$$H^p\left(E|^{\mathrm{L}}_{X_t}\right) \simeq H^p\left(t_X^* E\right)$$

as a consequence of flatness of the E^q's. Thus, for $p < n$,

$$H^p\left(E|^{\mathrm{L}}_{X_t}\right) = 0 \Longleftrightarrow H^p\left(t_X^* E\right) = 0 \Longleftrightarrow H^p\left(t_X^* F\right) = 0.$$

■

2.4.7 PROPOSITION – Define the prestack $\mathfrak{Mum}_X^{\leq n} = \mathfrak{Mum}_X^{[-\infty,n]}$ by assigning to each S-scheme T the groupoid

$$\mathfrak{Mum}_X^{\leq n}(T) = \left\{ E \in \mathfrak{Mum}_X(T) \,\middle|\, \forall \text{ geometric } t \in T, E|^{L}_{X_t} \in D^{\leq n}(\mathcal{O}_{X_t}) \right\}$$

with restriction functors induced by $\mathfrak{Mum}_X$. The prestack $\mathfrak{Mum}_X^{\leq n}$ is an open substack of $\mathfrak{Mum}_X$.

Proof: That $\mathfrak{Mum}_X^{\leq n}$ satisfies descent is a direct consequence of descent for $\mathfrak{Mum}_X$. To prove that it is indeed an open substack it is sufficient to prove that for any affine S-scheme T, together with a morphism $T \to \mathfrak{Mum}_X$ corresponding to a complex $E \in \mathfrak{Mum}_X(T)$, the set

$$V = \left\{ t \in T \,\middle|\, E|^{L}_{X_t} \in D^{\leq n}(X_t) \right\}$$

is an open subset of T.

By Lemma 2.4.3 1. we know that $t \in V$ if and only if $X_t \subset U_>$ (notice that by our assumptions the complex E is bounded). Thus $\pi_T(X_T \setminus U_>) = \pi_T(X_T) \setminus V$. The set $U_>$ is open as the sheaves $H^q(E)$ are quasi-coherent and of finite type. Finally, the sets $\pi_T(X_T)$ and $\pi_T(X_T \setminus U_>)$ are closed, being the image of closed subsets under a proper map. Thus, V is open. ■

Notice that the condition of being concentrated in degrees less or equal than n is in fact a global condition, i.e. we could have requested $E \in D^{\leq n}(\mathcal{O}_{X_T})$ directly.

We now impose on our complexes the further condition of being concentrated in a fixed degree $n \in \mathbb{Z}$. This stack will be isomorphic to the stack of coherent sheaves shifted by $-n$.

2.4.8 PROPOSITION – Define the prestack $\mathfrak{Num}_X^{[n]}$ by assigning to each S-scheme T the groupoid

$$\mathfrak{Num}_X^{[n]}(T) = \left\{ E \in \mathfrak{Num}_X^{\leq n}(T) \,\middle|\, \forall t \in T, E|^{L}_{X_t} \in D^{[n]}(\mathcal{O}_{X_t}) \right\}$$

with restriction functors induced by $\mathfrak{Num}_X$.The prestack $\mathfrak{Num}_X^{[n]}$ is an open substack of $\mathfrak{Num}_X^{\leq n}$.

Proof: The proof follows along the lines as the previous one. It suffices to show that for any affine scheme T, together with a map $T \to \mathfrak{Num}_X^{\leq n}$ corresponding to a complex $E \in \mathfrak{Num}_X^{\leq n}(T)$, the set

$$V = \left\{ t \in T \,\middle|\, E|^{L}_{X_t} \in D^{[n]}(\mathcal{O}_{X_t}) \right\}$$

is an open subset of T. By Lemma 2.4.3 2. we know that $t \in V$ if and only if $X_t \subset U$. The sets $U_<, U_>$ are open as the sheaves $H^q(E)$ are quasi-coherent and of finite type. The set U_f is open by the open nature of flatness [EGAIV-3, Théorème 11.3.1]. Thus U is open and we conclude as in the previous proof. ∎

When $n = 0$ we get back the ordinary stack of coherent sheaves on X.

We now turn to the opposite condition: being concentrated in degrees greater or equal than a fixed $n \in \mathbb{Z}$.

2.4.9 PROPOSITION – Define the prestack $\mathfrak{Num}_X^{\geq n} = \mathfrak{Num}_X^{[n,\infty]}$ by assigning to each S-scheme T the groupoid

$$\mathfrak{Num}_X^{\geq n}(T) = \left\{ E \in \mathfrak{Num}_X(T) \,\middle|\, \forall t \in T, E|^{L}_{X_t} \in D^{\geq n}(\mathcal{O}_{X_t}) \right\}$$

with restriction functors induced by $\mathfrak{Num}_X$. The prestack $\mathfrak{Num}_X^{\geq n}$ is an open substack of $\mathfrak{Num}_X$.

Proof: As in the previous proofs we consider a complex $E \in \mathfrak{Mum}_X(T)$ corresponding to a morphism $T \to \mathfrak{Mum}_X$ and prove that the set

$$V = \left\{ t \in T \,\middle|\, E|^{L}_{X_t} \in D^{\geq n}(X_t) \right\}$$

is an open subset of T. By Lemma 2.4.3 3. this set is equal to

$$V = \left\{ t \in T \,\middle|\, F|^{L}_{X_t} \in D^{[n]}(X_t) \right\}$$

which is open by the previous proof. ∎

3

HALL ALGEBRAS

Before recalling the definition of the motivic Hall algebras we need, it is instructive (at least to develop some intuition) to start by sloppily defining the sort of algebra we would like to work with. As a reference, we should mention [Rin90; Sch09] and the seminal work [Rei10].

Let $\mathcal{A}$ be an abelian category. We define $\mathrm{H}(\mathcal{A})$ to be the vector space of functions $f : \mathcal{A} \to \mathbb{Q}$, which assign to each object of $\mathcal{A}$ a rational number. This vector space can be equipped with a convolution product $*$ defined as follows:

$$f * g(\mathrm{E}) := \sum_{\mathrm{A} \subset \mathrm{E}} f(\mathrm{A}) g(\mathrm{E}/\mathrm{A}).$$

In other words, $f * g$ applied to an object E is the sum, over all possible isomorphism classes (with E fixed) of short exact sequences

$$0 \to \mathrm{A} \to \mathrm{E} \to \mathrm{E}/\mathrm{A} \to 0,$$

of f applied to the kernel and g applied to the cokernel. With $*$, we call $\mathrm{H}(\mathcal{A})$ the *Hall algebra* of $\mathcal{A}$. What we are blatantly ignoring here is the fact that, for the expressions we wrote above to even make sense, various finiteness conditions should be imposed.

Given $A \in \mathcal{A}$, there is a delta function element $1_A \in H(\mathcal{A})$, defined as

$$1_A(E) := \begin{cases} 1 & \text{if } E \cong A, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the delta of the zero object 1_0 is the neutral element for $*$.

More generally, given a subcategory $\mathcal{B} \subset \mathcal{A}$, we can define its characteristic function

$$1_{\mathcal{B}}(E) := \begin{cases} 1 & \text{if } E \in \mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}$$

Given two subcategories $\mathcal{B}_i$, $i = 1, 2$, the product of the characteristic functions can be written as

$$1_{\mathcal{B}_1} * 1_{\mathcal{B}_2}(E) = \sum_{\substack{B_1 \hookrightarrow E \twoheadrightarrow B_2 \\ B_i \in \mathcal{B}_i}} 1$$

i.e. one counts 1 for every short exact sequence with middle term E, kernel belonging to $\mathcal{B}_1$ and cokernel lying in $\mathcal{B}_2$.

For example, if $\mathcal{A}$ comes equipped with a torsion pair $(\mathcal{T}, \mathcal{F})$, there is an identity

$$1_{\mathcal{A}} = 1_{\mathcal{T}} * 1_{\mathcal{F}}$$

where $1_{\mathcal{A}}$ is the characteristic function of the whole category $\mathcal{A}$. This identity is a consequence of the crucial property of torsion pairs, which can phrased as saying that any object of $\mathcal{A}$ sits in a unique exact sequence with torsion kernel and torsion-free cokernel.

If we fix an object $\mathrm{P} \in \mathcal{A}$, we can define an element

$$1^{\mathrm{P}}(\mathrm{E}) := \#\operatorname{Hom}_{\mathcal{A}}(\mathrm{P}, \mathrm{E})$$

counting the number of maps with domain P. We can also define a sort of *Quot* element as

$$\mathscr{Q}^{\mathrm{P}}(\mathrm{E}) := \#\{\mathrm{P} \twoheadrightarrow \mathrm{E}\}$$

this time counting *epimorphisms* with domain P. The first isomorphism theorem for the abelian category $\mathcal{A}$ translates then into the identity

$$1^{\mathrm{P}}_{\mathcal{A}} = \mathscr{Q}^{\mathrm{P}} * 1_{\mathcal{A}}$$

as any morphism $\mathrm{P} \to \mathrm{E}$ factors through its image $\mathrm{P} \twoheadrightarrow \mathrm{I}$ (and we can then take the cokernel to obtain an exact sequence)

$$0 \to \mathrm{I} \to \mathrm{E} \to \mathrm{E}/\mathrm{P} \to 0.$$

In our context, which broadly speaking is that of a $\mathbb{C}$-linear abelian category, there is no hope that any of the finiteness conditions required to make the previous definitions work are satisfied. The first remedy one takes is to replace the coarse datum of an abelian category $\mathcal{A}$ with a moduli stack $\mathscr{A}$, parameterising its objects. Objects $\mathrm{E} \in \mathcal{A}$ of the original abelian category are now $\mathbb{C}$-points of the stack:

$$\operatorname{Spec}\mathbb{C} \xrightarrow{\mathrm{E}} \mathscr{A}.$$

The elements of the Hall algebra $H(\mathcal{A})$ are now given by $\mathbb{Q}$-linear combinations of (suitable motivic equivalence classes of) morphisms of stacks

$$\left[V \xrightarrow{f} \mathscr{A}\right].$$

One might still think of such an f as a function on the category $\mathcal{A}$, where the evaluation of f at an object E (which we now think as a $\mathbb{C}$-point of $\mathscr{A}$) would be the "size" of the fibre product $V \otimes_{\mathscr{A}} \mathbb{C}$.

We will not define the product structure on the motivic Hall algebra just yet, as it will be explained in detail below. We will, however, describe a special case. If $\mathscr{B} \subset \mathscr{A}$ is an open substack parameterising objects of a corresponding subcategory $\mathcal{B} \subset \mathcal{A}$ then we have an element $1_{\mathcal{B}}$ corresponding to the inclusion of $\mathscr{B}$ in $\mathscr{A}$. Given two such elements $1_{\mathcal{B}_i}$, $i = 1, 2$, their product $1_{\mathcal{B}_1} * 1_{\mathcal{B}_2}$ turns out to be represented by $Z \to \mathscr{A}$, where Z is the space parameterising short exact sequences

$$0 \to B_1 \to E \to B_2 \to 0$$

where $B_i \in \mathcal{B}_i$, and the morphism sends such a sequence to the middle term E. The identity $1_{\mathcal{A}} = 1_{\mathcal{T}} * 1_{\mathcal{F}}$ seen earlier (in the presence of a couple of open substacks corresponding to a torsion pair) immediately follows.

If P is an object of $\mathcal{A}$ (for example the structure sheaf of our variety) and there exists a stack $\mathscr{A}^P$ parameterising objects of $\mathcal{A}$ together with a morphism from P, we can define an element $1_{\mathcal{A}}^P$ as being the forgetful morphism $\mathscr{A}^P \to \mathscr{A}$. Finally, if as well $\mathcal{A}$ admits a *Quot-scheme* for the object P, the first isomorphism theorem again implies the identity $1_{\mathcal{A}}^P = \mathscr{Q}^P * 1_{\mathcal{A}}$.

Now that we have built up some intuition for Hall algebras, we move on to a rigorous treatment of them.

3.1 GROTHENDIECK RINGS

Our goal is to construct the Hall algebra of perverse coherent sheaves, which is a module over $K(\mathrm{St}/\mathbb{C})$, the Grothendieck ring of stacks. We start by recalling the definition of the latter and then proceed to check that we can port the construction of the Hall algebra of coherent sheaves to the perverse case.

Remark. As this porting process relies on Proposition 2.3.4, we remind the reader of Remark 2.3.6.

All the omitted proofs can be found, for example, in [JS08; Bri12].

3.1.1 DEFINITION – The *Grothendieck ring of schemes* $K(\mathrm{Sch}/\mathbb{C})$ is defined to be the $\mathbb{Q}$-vector space spanned by isomorphism classes of schemes of finite type over $\mathbb{C}$ modulo the *cut & paste* relations:

$$[X] = [Y] + [X \setminus Y]$$

for all Y closed in X. The ring structure is induced by $[X \times Y] = [X] \cdot [Y]$.

Notice that the zero element is given by the empty scheme and the unit for the multiplication is given by $[\mathrm{Spec}\,\mathbb{C}]$. Also, the Grothendieck ring disregards any non-reduced structure, as $[X_{\mathrm{red}}] = [X] - 0$. This ring can equivalently be described in terms of geometric bijections and Zariski fibrations.

3.1.2 DEFINITION – A morphism $f : X \to Y$ of finite type schemes is a *geometric bijection* if it induces a bijection on $\mathbb{C}$-points $f(\mathbb{C}) : X(\mathbb{C}) \to Y(\mathbb{C})$.

A morphism $p : X \to Y$ is a *Zariski fibration* if there exists a *trivialising* Zariski open cover of Y. That is, there exists a Zariski open cover $\{Y_i\}_i$ of Y together with schemes F_i such that $p^{-1}(Y_i) \cong Y_i \times F_i$, as Y_i-schemes.

Two Zariski fibrations $p : X \to Y$, $p' : X' \to Y$ *have the same fibres* if there exists a trivialising open cover for both fibrations such that the fibres are isomorphic $F_i \cong F'_i$.

3.1.3 LEMMA – We can describe the ring $K(\mathrm{Sch}/\mathbb{C})$ as the $\mathbb{Q}$-vector space spanned by isomorphism classes of schemes of finite type over $\mathbb{C}$ modulo the following relations.[1]

1. $[X_1 \amalg X_2] = [X_1] + [X_2]$, for every pair of schemes X_1, X_2.
2. $[X_1] = [X_2]$ for every geometric bijection $f : X_1 \to X_2$.
3. $[X_1] = [X_2]$ for every pair of Zariski fibrations $p_i : X_i \to Y$ with same fibres.

We now consider the Grothendieck ring of stacks.

3.1.4 DEFINITION – A morphism of finite type algebraic stacks $f : X_1 \to X_2$ is a *geometric bijection* if it induces an equivalence of groupoids on $\mathbb{C}$-points $f(\mathbb{C}) : X_1(\mathbb{C}) \to X_2(\mathbb{C})$.[2]

A morphism of algebraic stacks $p : X \to Y$ is a *Zariski fibration* if given any morphism from a scheme $T \to Y$ the induced map $X \times_Y T \to T$ is a Zariski fibration of schemes. In particular a Zariski fibration is a schematic morphism.

Two Zariski fibrations between algebraic stacks $p_i : X_i \to Y$ *have the same fibres* if the two maps $X_i \times_Y T \to T$ induced by a morphism from a scheme $T \to Y$ are two Zariski fibrations with the same fibres.

3.1.5 DEFINITION – The *Grothendieck ring of stacks* $K(\mathrm{St}/\mathbb{C})$ is defined to be the $\mathbb{Q}$-vector space spanned by isomorphism classes of Artin stacks of finite type over $\mathbb{C}$ with affine geometric stabilisers, modulo the following relations.

1 The three relations we present here are actually redundant, cf. [Bri12, Lemma 2.9], although the same is not true for stacks.
2 We point out that geometric bijections are relative algebraic spaces [AH11, Lemma 2.3.9].

1. $[X_1 \amalg X_2] = [X_1] + [X_2]$ for every pair of stacks X_1, X_2.

2. $[X_1] = [X_2]$ for every geometric bijection $f : X_1 \to X_2$.

3. $[X_1] = [X_2]$ for every pair of Zariski fibrations $p_i : X_i \to Y$ with the same fibres.

Let us call $\mathbb{L} = [\mathbb{A}^1]$ the element represented by the affine line. The obvious ring homomorphism $K(\mathrm{Sch}/\mathbb{C}) \to K(\mathrm{St}/\mathbb{C})$ becomes an isomorphism after inverting the elements $\mathbb{L}$ and $(\mathbb{L}^k - 1)$, for $k \geq 1$ [Bri12, Lemma 3.9]. Thus the ring homomorphism factors as follows.

$$K(\mathrm{Sch}/\mathbb{C}) \to K(\mathrm{Sch}/\mathbb{C})[\mathbb{L}^{-1}] \to K(\mathrm{St}/\mathbb{C})$$

We also mention that through the lens of the Grothendieck ring one cannot tell apart varieties from schemes or even algebraic spaces [Bri12, Lemma 2.12].

It also makes sense to speak of a *relative* Grothendieck group $K(\mathrm{St}/S)$, where S is a fixed base stack which we assume to be Artin, *locally* of finite type over $\mathbb{C}$ and with affine geometric stabilisers. We define $K(\mathrm{St}/S)$ to be spanned by isomorphism classes of morphisms $[W \to S]$ where W is an Artin stack of finite type over $\mathbb{C}$ with affine geometric stabilisers, modulo the following relations.

1. $[f_1 \amalg f_2 : X_1 \amalg X_2 \longrightarrow S] = [X_1 \xrightarrow{f_1} S] + [X_2 \xrightarrow{f_2} S]$, for every pair of stacks X_i.

2. For a morphism $f : X_1 \to X_2$ over S, with f a geometric bijection, $[X_1 \to S] = [X_2 \to S]$.

3. For every pair of Zariski fibrations with the same fibres $X_1 \to Y \leftarrow X_2$ and every morphism $Y \to S$

$$[X_1 \to Y \to S] = [X_2 \to Y \to S].$$

Given a morphism $a : \mathrm{S} \to \mathrm{T}$ we have a pushforward map

$$\begin{aligned} a_* : \mathrm{K}(\mathrm{St}/\mathrm{S}) &\longrightarrow \mathrm{K}(\mathrm{St}/\mathrm{T}) \\ [\mathrm{X} \to \mathrm{S}] &\longmapsto [\mathrm{X} \to \mathrm{S} \xrightarrow{a} \mathrm{T}] \end{aligned}$$

and given a morphism of finite type $b : \mathrm{S} \to \mathrm{T}$ we have a pullback map

$$\begin{aligned} b^* : \mathrm{K}(\mathrm{St}/\mathrm{T}) &\longrightarrow \mathrm{K}(\mathrm{St}/\mathrm{S}) \\ [\mathrm{X} \to \mathrm{T}] &\longmapsto [\mathrm{X} \times_\mathrm{T} \mathrm{S} \to \mathrm{S}]. \end{aligned}$$

The pushforward and pullback are functorial and satisfy base-change. Furthermore, given a pair of stacks $\mathrm{S}_1, \mathrm{S}_2$ there is a Künneth map

$$\begin{aligned} \varkappa : \mathrm{K}(\mathrm{St}/\mathrm{S}_1) \otimes \mathrm{K}(\mathrm{St}/\mathrm{S}_2) &\longrightarrow \mathrm{K}(\mathrm{St}/\mathrm{S}_1 \times \mathrm{S}_2) \\ [\mathrm{X}_1 \to \mathrm{S}_1] \otimes [\mathrm{X}_2 \to \mathrm{S}_2] &\longmapsto [\mathrm{X}_1 \times \mathrm{X}_2 \to \mathrm{S}_1 \times \mathrm{S}_2]. \end{aligned}$$

Take now $\mathfrak{A}$ to be the stack of coherent sheaves on X, where X is smooth and projective over $\mathbb{C}$, and denote by $\mathrm{H}(\mathcal{A})$ the Grothendieck ring $\mathrm{K}(\mathrm{St}/\mathfrak{A})$ (where $\mathcal{A}$ stands for Coh X). We can endow $\mathrm{H}(\mathcal{A})$ with a *convolution product*, coming from the abelian structure of $\mathcal{A}$. The product is defined as follows. Let $\mathfrak{A}^{(2)}$ be the stack of exact sequences in $\mathcal{A}$. There are three natural morphisms $a_1, b, a_2 : \mathfrak{A}^{(2)} \to \mathfrak{A}$ which take an exact sequence

$$\mathrm{A}_1 \hookrightarrow \mathrm{B} \twoheadrightarrow \mathrm{A}_2$$

to $\mathrm{A}_1, \mathrm{B}, \mathrm{A}_2$ respectively. Consider the following diagram.

$$\begin{tikzcd}
\mathfrak{A}^{(2)} \arrow[r, "b"] \arrow[d, "(a_1,a_2)"'] & \mathfrak{A} \\
\mathfrak{A} \times \mathfrak{A} &
\end{tikzcd}$$

We remark that (a_1, a_2) is of finite type [Bri12, Lemma 4.2]. A *convolution product* can be then defined as follows:

$$m : \mathrm{H}(\mathcal{A}) \otimes \mathrm{H}(\mathcal{A}) \longrightarrow \mathrm{H}(\mathcal{A})$$
$$m = b_*(a_1, a_2)^* \varkappa.$$

Explicitly, given two elements $[X_1 \xrightarrow{f_1} \mathfrak{A}]$, $[X_2 \xrightarrow{f_2} \mathfrak{A}]$ we write $f_1 * f_2 = m(f_1 \otimes f_2)$ for their product which is given by the top row of the following diagram.

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] \arrow[rr, bend left, dashed, "f_1 * f_2"] \arrow[dr, phantom, "\square"] & \mathfrak{A}^{(2)} \arrow[r, "b"] \arrow[d, "(a_1,a_2)"] & \mathfrak{A} \\
X_1 \times X_2 \arrow[r, "f_1 \times f_2"] & \mathfrak{A} \times \mathfrak{A} &
\end{tikzcd}$$

The convolution product endows $\mathrm{H}(\mathcal{A})$ with an associative $\mathrm{K}(\mathrm{St}/\mathbb{C})$-algebra structure with unit element given by $[\operatorname{Spec} \mathbb{C} = \mathfrak{A}_0 \subset \mathfrak{A}]$, the inclusion of the zero object.

3.2 PERVERSE COHERENT SHEAVES

We now assume to be working in Situation 2.3.1. We want to replace $\mathcal{A}$ by ${}^p\mathcal{A}$ and construct the analogous algebra $\mathrm{H}({}^p\mathcal{A})$. We first need the moduli stack ${}^p\mathfrak{A}^{(2)}$

parameterising short exact sequences in ${}^p\mathcal{A}$. Define a prestack ${}^p\mathfrak{A}^{(2)}$ by assigning to each scheme T the groupoid ${}^p\mathfrak{A}^{(2)}(\mathrm{T})$ whose objects are exact triangles

$$\mathrm{E}_1 \to \mathrm{E} \to \mathrm{E}_2 \nrightarrow$$

with vertices belonging to ${}^p\mathfrak{A}(\mathrm{T})$ and whose morphisms are isomorphisms of triangles. The restriction functors are given by derived pullback, which is an exact functor so takes exact triangles to exact triangles.

3.2.1 PROPOSITION – The prestack ${}^p\mathfrak{A}^{(2)}$ is an Artin stack locally of finite type over $\mathbb{C}$ with affine stabilisers.

Proof: This prestack is well-defined and satisfies descent. In fact, given the existence of the stack of objects of ${}^p\mathcal{A}$, the only issue arises in gluing automorphisms. This is taken care of by noticing that $\mathrm{Ext}^{<0}_{\mathcal{A}}(\mathrm{A},\mathrm{B})$ vanishes for any two objects $\mathrm{A},\mathrm{B} \in {}^p\mathcal{A}$ [AP06][Lemma 2.1.10]. Take now $p = -1$. We want to use the functors $\widetilde{\tau}_n$ of Lemma 2.3.5. Notice that the subcategory ${}^p\mathcal{A}_n \subset {}^p\mathcal{A}$, of objects which become coherent after a twist by $\widetilde{\tau}_n$, is extension-closed and hence we have a well-defined stack of exact sequences ${}^p\mathfrak{A}^{(2)}_n$, which is an open substack of ${}^p\mathfrak{A}^{(2)}$. Using Proposition 2.3.4 and the fact that $\widetilde{\tau}_n$ is an exact functor we can embed ${}^p\mathfrak{A}^{(2)}_n$ inside $\mathfrak{A}^{(2)}$, thus proving that ${}^p\mathfrak{A}^{(2)}_n$ is algebraic.

The sum $\coprod_n {}^p\mathfrak{A}^{(2)}_n \to {}^p\mathfrak{A}^{(2)}$ is surjective and thus the stack ${}^p\mathfrak{A}^{(2)}$ is algebraic. All other properties are deduced by the fact that ${}^p\mathfrak{A}^{(2)}_n$ is an open substack of $\mathfrak{A}^{(2)}$. To deal with the $p = 0$ case one appeals to Remark 2.3.6. ∎

The proof actually produces more: it gives an analogue of Proposition 2.3.4.

As for coherent sheaves, the stack ${}^p\mathfrak{A}^{(2)}$ comes equipped with three morphisms a_1, b, a_2, sending a triangle of perverse coherent sheaves

$$\mathrm{E}_1 \to \mathrm{E} \to \mathrm{E}_2 \nrightarrow$$

to $\mathrm{E}_1, \mathrm{E}, \mathrm{E}_2$ respectively. The exact functor $\widetilde{\tau}_n$ yields a commutative diagram

$$\begin{tikzcd} {}^p\mathfrak{A}_n^{(2)} \arrow[r, hook] \arrow[d, "(a_1,a_2)"'] & \mathfrak{A}^{(2)} \arrow[d] \\ {}^p\mathfrak{A}_n \times {}^p\mathfrak{A}_n \arrow[r, hook] & \mathfrak{A} \times \mathfrak{A} \end{tikzcd}$$

where the vertical arrow on the right is the corresponding morphism for coherent sheaves, which is of finite type. From this last observation and the fact that being of finite type is local on the target, we automatically have that the (global) morphism $(a_1, a_2) : {}^p\mathfrak{A}^{(2)} \to {}^p\mathfrak{A}^2$ is of finite type. To define the convolution product on $\mathrm{K}(\mathrm{St}/{}^p\mathfrak{A})$ (or equivalently the algebra structure of $\mathrm{H}({}^p\mathcal{A})$) we may proceed analogously as for coherent sheaves. As usual, this discussion is valid for $p = -1$, but an entirely parallel one can be carried out for $p = 0$ using Remark 2.3.6.

3.3 MORE STRUCTURE

There is a natural way to bestow a grading upon our Hall algebras. Recall that for a triangulated category $\mathcal{T}$ and the heart $\mathcal{H}$ of a bounded t-structure on $\mathcal{T}$, the Grothendieck groups $\mathrm{K}(\mathcal{T})$ and $\mathrm{K}(\mathcal{H})$ coincide (by taking alternating sums of cohomology objects). In particular, $\mathrm{K}(\mathrm{D^b}(\mathrm{Y}))$ can be viewed as both $\mathrm{K}(\mathcal{A})$ or $\mathrm{K}({}^p\mathcal{A})$. The *Euler form* χ is defined as

$$\chi(\mathrm{E}, \mathrm{F}) = \sum_j (-1)^j \dim_{\mathbb{C}} \mathrm{Ext}^j_{\mathrm{Y}}(\mathrm{E}, \mathrm{F})$$

on coherent sheaves E, F and then extended to the whole of $\mathrm{K}(\mathrm{Y})$. By Serre duality the left and right radicals of χ are equal and we define the *numerical* Grothendieck group of Y as $N(Y) = K(Y)/K(Y)^{\perp}$. As the numerical class of a complex stays constant in families we have a decomposition

$$\mathfrak{Mum}_Y = \coprod_{\alpha \in N(Y)} \mathfrak{Mum}_{Y,\alpha}$$

where $\mathfrak{Mum}_{Y,\alpha}$ parameterises complexes of class α. Let Γ denote the *positive cone* of coherent sheaves, i.e. the image of objects of $\mathcal{A}$ inside $N(Y)$. It is a submonoid of $N(Y)$ and for $\mathfrak{A}$ the previous decomposition can be refined to

$$\mathfrak{A} = \coprod_{\alpha \in \Gamma} \mathfrak{A}_\alpha.$$

We can also define sub-modules $H(\mathcal{A})_\alpha \subset H(\mathcal{A})$, where $H(\mathcal{A})_\alpha$ denotes $K(\mathrm{St}/\mathfrak{A}_\alpha)$ (which can be thought as spanned by classes of morphisms $[W \to \mathfrak{A}]$ factoring through $\mathfrak{A}_\alpha$). We then get a Γ-grading

$$H(\mathcal{A}) = \bigoplus_{\alpha \in \Gamma} H(\mathcal{A})_\alpha.$$

Analogously, we have a positive cone ${}^p\Gamma \subset N(X)$ of perverse coherent sheaves. The Hall algebra thus decomposes as

$$H({}^p\mathcal{A}) = \bigoplus_{\alpha \in {}^p\Gamma} H({}^p\mathcal{A})_\alpha.$$

We mentioned earlier that the morphism from the Grothendieck ring of varieties to the Grothendieck ring of stacks factors as follows

$$K(\mathrm{Sch}/\mathbb{C}) \to K(\mathrm{Sch}/\mathbb{C})[\mathbb{L}^{-1}] \to K(\mathrm{St}/\mathbb{C}).$$

Let $R = K(\mathrm{Sch}/\mathbb{C})[\mathbb{L}^{-1}]$. One can define a subalgebra [Bri12, Theorem 5.1] $H_{\mathrm{reg}}(\mathcal{A})$ of *regular elements* as the R-module spanned by classes $[W \to \mathfrak{A}]$ with W a scheme. We have an analogous setup for perverse coherent sheaves.

3.3.1 PROPOSITION – Let $H_{\mathrm{reg}}({}^p\mathcal{A})$ to be the sub-R-module spanned by classes $[W \to {}^p\mathfrak{A}]$ with W a scheme. Then $H_{\mathrm{reg}}({}^p\mathcal{A})$ is closed under the convolution product and the quotient

$$H_{sc}({}^p\mathcal{A}) = H_{\mathrm{reg}}({}^p\mathcal{A})/(\mathbb{L}-1)H_{\mathrm{reg}}({}^p\mathcal{A})$$

is a commutative $K(\mathrm{Sch}/\mathbb{C})$-algebra.

Proof: Once again, we may appeal to the case of coherent sheaves by using the functors $\widetilde{\tau}_n$. Let $p = -1$. Let $[f_1 : S_1 \to {}^p\mathfrak{A}]$, $[f_2 : S_2 \to {}^p\mathfrak{A}]$ be two elements of $H({}^p\mathcal{A})$ such that the S_i are schemes. Consider the two morphisms

$$f_1 \times f_2 : S_1 \times S_2 \to {}^p\mathfrak{A} \times {}^p\mathfrak{A}$$
$$(a_1, a_2) : {}^p\mathfrak{A}^{(2)} \to {}^p\mathfrak{A} \times {}^p\mathfrak{A}$$

used to define the product $f_1 * f_2$ in $H({}^p\mathcal{A})$. It suffices to show that the fibre product

$$T = (S_1 \times S_2) \times_{{}^p\mathfrak{A} \times {}^p\mathfrak{A}} {}^p\mathfrak{A}^{(2)}$$

is a regular element. Consider the open cover $\{{}^p\mathfrak{A}_n\}_n$ of ${}^p\mathfrak{A}$ given in Proposition 2.3.4. The first thing we notice is that the collection $\{{}^p\mathfrak{A}_n \times {}^p\mathfrak{A}_n\}_n$ is an open cover of ${}^p\mathfrak{A} \times {}^p\mathfrak{A}$ (it covers the whole product via Lemma 2.3.5). Pulling it back via $f_1 \times f_2$ yields open covers $\{S_{i,n}\}_n$ for each of the S_i and an open cover $\{S_{1,n} \times S_{2,n}\}_n$ of $S_1 \times S_2$.

On the other hand, by the proof of Proposition 3.2.1 we have an open cover $\{{}^p\mathfrak{A}^{(2)}_n\}_n$ of ${}^p\mathfrak{A}^{(2)}$. By pulling back we obtain an open cover $\{\mathrm{T}_n\}_n$ of T. By chasing around base-changes one can see that

$$\mathrm{T}_n = (\mathrm{S}_{1,n} \times \mathrm{S}_{2,n}) \times_{{}^p\mathfrak{A}_n \times {}^p\mathfrak{A}_n} {}^p\mathfrak{A}^{(2)}_n.$$

The functor $\widetilde{\tau}_n$ induces morphisms ${}^p\mathfrak{A}_n \times {}^p\mathfrak{A}_n \to \mathfrak{A} \times \mathfrak{A}$, ${}^p\mathfrak{A}^{(2)}_n \to \mathfrak{A}^{(2)}$ and it is easy to check that

$${}^p\mathfrak{A}^{(2)}_n = ({}^p\mathfrak{A}_n \times {}^p\mathfrak{A}_n) \times_{\mathfrak{A}\times\mathfrak{A}} \mathfrak{A}^{(2)}$$

thus $\mathrm{T}_n = (\mathrm{S}_{1,n} \times \mathrm{S}_{2,n}) \times_{\mathfrak{A}\times\mathfrak{A}} \mathfrak{A}^{(2)}$ and by [Bri12, Theorem 5.1] it is a regular element. We conclude that T is also a regular element.

When $p = 0$ one may use Remark 2.3.6. ■

We now briefly turn back to the case of coherent sheaves. The *semi-classical* Hall algebra of coherent sheaves $\mathrm{H}_{\mathrm{sc}}(\mathcal{A})$, defined as $\mathrm{H}_{\mathrm{reg}}(\mathcal{A})/(\mathbb{L}-1)\mathrm{H}_{\mathrm{reg}}(\mathcal{A})$, can be equipped with a Poisson bracket given by

$$\{f, g\} = \frac{f * g - g * f}{\mathbb{L} - 1}.$$

There is another Poisson algebra $\mathbb{Q}_\sigma[\Gamma]$, which depends on a choice $\sigma \in \{-1, 1\}$, defined as the $\mathbb{Q}$-vector space spanned by symbols q^α, with $\alpha \in {}^p\Gamma$, together with a product

$$q^{\alpha_1} * q^{\alpha_2} = \sigma^{\chi(\alpha_1,\alpha_2)} q^{\alpha_1+\alpha_2}.$$

and a Poisson bracket

$$\{q^{\alpha_1}, q^{\alpha_2}\} = \sigma^{\chi(\alpha_1,\alpha_2)} \chi(\alpha_1, \alpha_2) q^{\alpha_1+\alpha_2} = \chi(\alpha_1, \alpha_2)(q^{\alpha_1} * q^{\alpha_2}).$$

Given a locally constructible function [JS08, Chapter 2] $\lambda : \mathfrak{A}(\mathbb{C}) \to \mathbb{Z}$, there exists a so-called *integration morphism*

$$I : H_{sc}(\mathcal{A}) \to \mathbb{Q}_\sigma[\Gamma].$$

For convenience of the reader we compactly recall its properties [Bri12, Theorem 5.2]. The map I is the unique homomorphism of rational vector spaces such that if V is a variety and $f : V \to \mathfrak{A}$ factors through $\mathfrak{A}_\alpha$, for $\alpha \in \Gamma$, then

$$I([f]) = \chi(V, f^*\lambda)\, q^\alpha$$

where

$$\chi_{\mathrm{top}}(V, f^*\lambda) = \sum_{n \in \mathbb{Z}} n\, \chi_{\mathrm{top}}((\lambda \circ f)^{-1}(n))$$

and where, for a variety V, $\chi_{\mathrm{top}}(V)$ denotes the topological Euler characteristic. Moreover, I is a homomorphism of commutative algebras if, for all $F, G \in (A)$,

$$\lambda(F \oplus G) = \sigma^{\chi(F,G)} \lambda(F)\lambda(G)$$

and is a homomorphism of Poisson algebras if the expression

$$\chi\left(\mathbb{P}\,\mathrm{Ext}^1_{\mathcal{A}}(F,G), \lambda(E_\theta) - \lambda(E_0)\right)$$

is symmetric in F and G. The notation E_θ stands for the extension

$$0 \to G \to E_\theta \to F \to 0$$

corresponding to a class $\theta \in \mathrm{Ext}^1_{\mathcal{A}}(F,G)$.

For $\sigma = 1$ one can choose λ to be identically equal to 1. This gives a well-defined integration morphism which in turn leads to *naive* curve counting invariants. We are more interested in the case $\sigma = -1$ (although what follows certainly holds for the naive invariants as well) where one takes Behrend's microlocal function ν. For $\mathrm{H}_{\mathrm{sc}}(\mathcal{A})$ we know [JS08, Theorem 5.5] that the Behrend function satisfies the necessary hypotheses and thus yields an integration morphism.

To define an integration morphism in the context of perverse coherent sheaves we first define $\mathbb{Q}_\sigma[{}^p\Gamma]$ analogously as $\mathbb{Q}_\sigma[\Gamma]$, but using the cone of perverse coherent sheaves. In this context, we may still use Behrend's function. More precisely, every Artin stack $\mathfrak{M}$ locally of finite type over $\mathbb{C}$ comes equipped with a Behrend function $\nu_{\mathfrak{M}}$ and given any smooth morphism $f : \mathfrak{M}' \to \mathfrak{M}$ of relative dimension d we have $f^*\nu_{\mathfrak{M}} = (-1)^d \nu_{\mathfrak{M}'}$. To obtain an integration morphism on $\mathrm{H}({}^p\mathcal{A})$ the Behrend function must satisfy the assumptions of [Bri12, Theorem 5.2]. But these concern only the *points* of ${}^p\mathfrak{A}$ and we know that ${}^p\mathfrak{A}$ is locally isomorphic to $\mathfrak{A}$, so the assumptions are satisfied and we have a well-defined integration morphism

$$\mathrm{I} : \mathrm{H}({}^p\mathcal{A}) \to \mathbb{Q}_\sigma[{}^p\Gamma].$$

4

FLOPS

As hinted at in the overview chapter, the proof of our main result can be roughly divided into two blocks: the first is concerned with proving a formula relating 'perverse' DT invariants with ordinary ones; the second uses this formula to compare the DT invariants over a flop. We will start by focusing on the former.

Recall that we denote by $\mathcal{A}$ the category of coherent sheaves of Y. In the previous sections we reminded ourselves of the category of perverse coherent sheaves ${}^p\mathcal{A}$ and of the subcategories ${}^p\mathcal{T}, {}^p\mathcal{F}$. We also reminded ourselves of the motivic Hall algebra of coherent sheaves $\mathrm{H}(\mathcal{A})$, defined as the Grothendieck ring $\mathrm{K}(\mathrm{St}\,/\mathfrak{A})$ of stacks over the stack of coherent sheaves $\mathfrak{A}$ equipped with the convolution product. We also constructed a moduli stack ${}^p\mathfrak{A}$ parameterising objects in ${}^p\mathcal{A}$ and the Hall algebra $\mathrm{H}({}^p\mathcal{A})$ of perverse coherent sheaves, together with the subalgebra of regular elements $\mathrm{H}_{\mathrm{reg}}({}^p\mathcal{A})$, its semi-classical limit $\mathrm{H}_{\mathrm{sc}}({}^p\mathcal{A})$ and the integration morphism $\mathrm{I} : \mathrm{H}_{\mathrm{sc}}({}^p\mathcal{A}) \to \mathbb{Q}_\sigma[{}^p\Gamma]$. Recall that ${}^p\Gamma$ is the cone of perverse coherent sheaves sitting inside the numerical Grothendieck group $\mathrm{N}(\mathrm{Y})$ and we take $\sigma = -1, 1$ depending on the choice of a locally constructible function on ${}^p\mathfrak{A}$ (either the function identically equal to one or the Behrend function).

4.1 A ROUTE

Before we start off, we would like to give a moral proof our main result, which will later guide us through the maze of technical details. As we are interested in counting *curves*, we will restrict to sheaves (and complexes) supported in dimension at most one. All the constructions and definitions so far restrict to this setting, and we will append a ≤ 1 subscript to notify this change (e.g. we deal with the Hall algebra $\mathrm{H}({}^p\mathcal{A}_{\leq 1})$ of perverse coherent sheaves supported in dimension at most one). The two key results are the identities (4.1.1), (4.1.2). Continuing from the overview chapter, our goal is to understand the relationship between perverse DT numbers ${}^p\mathrm{DT}(\mathrm{Y}/\mathrm{X})$ and ordinary DT numbers $\mathrm{DT}(\mathrm{Y})$.

The Hilbert scheme of curves and points $\mathrm{Hilb}_{\leq 1}(\mathrm{Y})$ maps to $\mathfrak{A}_{\leq 1}$ by taking a quotient $\mathcal{O}_\mathrm{Y} \twoheadrightarrow \mathrm{E}$ to E, thus defining an element $\mathscr{H}_{\leq 1} \in \mathrm{H}(\mathcal{A}_{\leq 1})$.[1] From the previous section we know that the integration morphism is related to taking weighted Euler characteristics and in fact integrating $\mathscr{H}_{\leq 1}$ gives the generating series for the DT invariants[2]

$$\mathrm{I}(\mathscr{H}_{\leq 1})\text{“}=\text{”}\mathrm{DT}(\mathrm{Y}) := \sum_{\beta,n} \mathrm{DT}_\mathrm{Y}(\beta,n) q^{(\beta,n)}$$

where $\beta \in \mathrm{N}_1(\mathrm{Y})$ ranges among curve-classes in Y and $n \in \mathbb{Z}$ is a zero-cycle. The perverse Hilbert scheme ${}^p\mathrm{Hilb}_{\leq 1}(\mathrm{Y}/\mathrm{X})$ produces a corresponding element ${}^p\mathscr{H}_{\leq 1}$ of $\mathrm{H}({}^p\mathcal{A}_{\leq 1})$, which upon being integrated produces ${}^p\mathrm{DT}(\mathrm{Y}/\mathrm{X})$.

The first thing we remark is that, as quotients (in $\mathcal{A}$) of $\mathcal{O}_\mathrm{Y}$ lie in ${}^p\mathcal{T}$ and ${}^p\mathcal{T} \subset {}^p\mathcal{A}$, we can interpret $\mathscr{H}_{\leq 1}$ as an element of $\mathrm{H}({}^p\mathcal{A}_{\leq 1})$. There is an element $1_{{}^p\mathcal{F}_{\leq 1}[1]}$ in $\mathrm{H}({}^p\mathcal{A}_{\leq 1})$ represented by the inclusion ${}^p\mathfrak{F}_{\leq 1}[1] \subset {}^p\mathfrak{A}_{\leq 1}$. There is also a stack parameterising objects of ${}^p\mathcal{F}_{\leq 1}[1]$ together with a morphism from $\mathcal{O}_\mathrm{Y}$. This stack

1 Strictly speaking this is false as $\mathrm{Hilb}(\mathrm{Y})$ is not of finite type. We shall later enlarge our Hall algebra precisely to deal with this issue.
2 Again, this is slightly imprecise, there is a sign issue to be explained at the beginning of Section (4.8).

maps down to ${}^p\mathfrak{A}_{\leq 1}$ by forgetting the morphism, yielding an element $1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]}$. We will prove that there is an identity

$$ {}^p\mathscr{H}_{\leq 1} * 1_{{}^p\mathcal{F}_{\leq 1}[1]} = 1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathscr{H}_{\leq 1} \tag{4.1.1} $$

in the Hall algebra of perverse coherent sheaves. Let us see how one might deduce this.

We extend the notation $1_{{}^p\mathcal{F}[1]}, 1^{\mathcal{O}}_{{}^p\mathcal{F}[1]}$ to general subcategories $\mathcal{B} \subset {}^p\mathcal{A}$ (whenever we have an open inclusion of stacks $\mathfrak{B} \subset {}^p\mathfrak{A}$) producing elements $1_{\mathcal{B}}, 1^{\mathcal{O}}_{\mathcal{B}}$ in $\mathrm{H}({}^p\mathcal{A})$, and similarly for $\mathrm{H}(\mathcal{A})$. As $({}^p\mathcal{T}, {}^p\mathcal{F})$ is a torsion pair in $\mathcal{A}$, we have an identity $1_{\mathcal{A}} = 1_{{}^p\mathcal{T}} * 1_{{}^p\mathcal{F}}$. This follows from the fact that for any coherent sheaf E there is a unique exact sequence $\mathrm{T} \hookrightarrow \mathrm{E} \twoheadrightarrow \mathrm{F}$ with $\mathrm{T} \in {}^p\mathcal{T}$, $\mathrm{F} \in {}^p\mathcal{F}$. Notice that the product $1_{{}^p\mathcal{T}} * 1_{{}^p\mathcal{F}}$ is given by $[\mathrm{Z} \to \mathfrak{A}]$ where Z parameterises exact sequences $\mathrm{T} \hookrightarrow \mathrm{E} \twoheadrightarrow \mathrm{F}$ and the morphism $\mathrm{Z} \to \mathfrak{A}$ sends such an exact sequence to E.

We also have an identity $1^{\mathcal{O}}_{\mathcal{A}} = 1^{\mathcal{O}}_{{}^p\mathcal{T}} * 1^{\mathcal{O}}_{{}^p\mathcal{F}}$. This is a consequence of the previous identity plus the fact that $\mathrm{Hom}(\mathcal{O}_{\mathrm{Y}}, {}^p\mathcal{F}) = 0$ (Lemma 2.3.2). This last fact also tells us that $1^{\mathcal{O}}_{{}^p\mathcal{F}} = 1_{{}^p\mathcal{F}}$. Moreover, the first isomorphism theorem for the abelian category $\mathcal{A}$ is reflected in the identity $1^{\mathcal{O}}_{\mathcal{A}} = \mathscr{H} * 1_{\mathcal{A}}$ (any morphism $\mathcal{O}_{\mathrm{Y}} \to \mathrm{E}$ factors through its image). Combining everything together (and restricting to sheaves supported in dimension at most one) we see that $\mathscr{H}_{\leq 1} = 1^{\mathcal{O}}_{\mathcal{A}_{\leq 1}} * 1^{-1}_{\mathcal{A}_{\leq 1}} = 1^{\mathcal{O}}_{{}^p\mathcal{T}_{\leq 1}} * 1^{-1}_{{}^p\mathcal{T}_{\leq 1}}$.

A parallel argument can be carried out for ${}^p\mathcal{A}$ yielding

$$ {}^p\mathscr{H}_{\leq 1} = 1^{\mathcal{O}}_{{}^p\mathcal{A}_{\leq 1}} * 1^{-1}_{{}^p\mathcal{A}_{\leq 1}} = 1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]} * (1^{\mathcal{O}}_{{}^p\mathcal{T}_{\leq 1}} * 1^{-1}_{{}^p\mathcal{T}_{\leq 1}}) * 1^{-1}_{{}^p\mathcal{F}_{\leq 1}[1]} = 1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathscr{H}_{\leq 1} * 1^{-1}_{{}^p\mathcal{F}_{\leq 1}[1]} $$

from which we extract (4.1.1). Notice that for the identity $1^{\mathcal{O}}_{{}^p\mathcal{A}_{\leq 1}} = 1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]} * 1^{\mathcal{O}}_{{}^p\mathcal{T}_{\leq 1}}$ one uses

$$\mathrm{Hom}_Y(\mathcal{O}_Y, {}^p\mathcal{F}_{\leq 1}[2]) = 0.$$

We now want to understand how to rewrite $1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]}$ in a more familiar form. It turns out that duality almost interchanges ${}^q\mathcal{T}$ and ${}^p\mathcal{F}$, where $q = -(p+1)$. Precisely, let $\mathcal{Q}$ be the subcategory of $\mathcal{A}$ consisting of sheaves with no subsheaves supported in dimension zero. Let $\mathcal{Q}_{\mathrm{exc}}$ denote the subcategory of $\mathcal{Q}$ made up of sheaves Q such that Rf_*Q is supported in dimension zero and let ${}^q\mathcal{T}_\bullet = \mathcal{Q}_{\mathrm{exc}} \cap {}^q\mathcal{T}$. It is a simple computation (Lemma 4.5.1) to check that the duality functor $\mathbb{D} = R\underline{\mathrm{Hom}}_Y(-, \mathcal{O}_Y)[2]$ takes ${}^q\mathcal{T}_\bullet$ to ${}^p\mathcal{F}_{\leq 1}$. The category $\mathcal{Q}$ is related to DT invariants in the following way.

There is an identity $1^{\mathcal{O}}_{\mathcal{Q}} = \mathcal{H}^\# * 1_{\mathcal{Q}}$ in $H(\mathcal{A})$, where $\mathcal{H}^\#$ corresponds to (yet another) Hilbert scheme of a tilt $\mathcal{A}^\#$ of $\mathcal{A}$, where $\mathcal{A}^\#$ is the category in which quotients of $\mathcal{O}_Y$ are the so-called *stable pairs* of Pandharipande and Thomas [PT09] (see also [Bri11]). We can restrict to sheaves with zero-dimensional pushdown, which yields an identity $1^{\mathcal{O}}_{\mathcal{Q}_{\mathrm{exc}}} = \mathcal{H}^\#_{\mathrm{exc}} * 1_{\mathcal{Q}_{\mathrm{exc}}}$, which can be refined to $1^{\mathcal{O}}_{{}^q\mathcal{T}_\bullet} = \mathcal{H}^\#_{\mathrm{exc}} * 1_{{}^q\mathcal{T}_\bullet}$. Integrating $\mathcal{H}^\#_{\mathrm{exc}}$ gives the generating series for the Pandharipande-Thomas (PT) invariants of Y [Bri11, Lemma 5.5]

$$I(\mathcal{H}^\#_{\mathrm{exc}}) \text{``="} \mathrm{PT}_{\mathrm{exc}}(Y) := \sum_{\substack{\beta, n \\ f_*\beta = 0}} \mathrm{PT}_Y(\beta, n) q^{(\beta, n)}$$

where β ranges over the curve-classes contracted by f. If we let

$$\mathrm{DT}_0(Y) := \sum_n \mathrm{DT}_Y(0, n) q^n$$

we know [Bri12, Theorem 1.1] that the *reduced* DT invariants $\mathrm{DT}'(Y) := \mathrm{DT}(Y)/\mathrm{DT}_0(Y)$ coincide with the PT invariants $\mathrm{PT}(Y)$.

Now, the (shifted) duality functor $\mathbb{D}' = \mathbb{D}[1]$ induces a anti-homomorphism between Hall algebras[3] and takes ${}^q\mathcal{T}_\bullet$ to ${}^p\mathcal{F}_{\leq 1}[1]$ and so we have $\mathbb{D}'(1_{{}^q\mathcal{T}_\bullet}) = 1_{{}^p\mathcal{F}_{\leq 1}[1]}$. Furthermore, as a consequence of Serre duality, $\mathbb{D}'(1^{\mathcal{O}}_{{}^q\mathcal{T}_\bullet}) = 1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]}$. As a result we have

$$1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]} = 1_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathbb{D}'\left(\mathcal{H}^{\#}_{\mathrm{exc}}\right) \tag{4.1.2}$$

as $1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]} = \mathbb{D}'\left(1^{\mathcal{O}}_{{}^q\mathcal{T}_\bullet}\right) = \mathbb{D}'\left(\mathcal{H}^{\#}_{\mathrm{exc}} * 1_{{}^q\mathcal{T}_\bullet}\right) = \mathbb{D}'\left(1_{{}^q\mathcal{T}_\bullet}\right) * \mathbb{D}'\left(\mathcal{H}^{\#}_{\mathrm{exc}}\right) = 1_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathbb{D}'\left(\mathcal{H}^{\#}_{\mathrm{exc}}\right)$ (notice that duality is an *anti*-equivalence and thus swaps extensions). We can rewrite (4.1.1) as follows.

$${}^p\mathcal{H}_{\leq 1} * 1_{{}^p\mathcal{F}_{\leq 1}[1]} = 1_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathbb{D}'\left(\mathcal{H}^{\#}_{\mathrm{exc}}\right) * \mathcal{H}_{\leq 1} \tag{4.1.3}$$

Duality and integration can be interchanged up to a flip in signs. Precisely

$$I\left(\mathbb{D}'\left(\mathcal{H}^{\#}_{\mathrm{exc}}\right)\right) \text{“="} \mathrm{PT}^{\vee}_{\mathrm{exc}}(Y) := \sum_{\substack{\beta, n \\ f_*\beta = 0}} \mathrm{PT}_Y(-\beta, n) q^{(\beta, n)}.$$

Upon integrating the two sides of (4.1.3) the two $1_{{}^p\mathcal{F}_{\leq 1}[1]}$ cancel out[4] and we are left with the identity

$${}^p\mathrm{DT}(Y/X) = \mathrm{PT}^{\vee}_{\mathrm{exc}}(Y) \cdot \mathrm{DT}(Y).$$

3 More precisely it induces a morphism between certain subalgebras to be defined below.
4 This is the content of Proposition 4.7.4, a consequence of an important result of Joyce.

4.2 THE PERVERSE HILBERT SCHEME

We now proceed along the route traced in the previous section, but taking care of technical details. Let us start by working in *infinite-type* versions $H_\infty(\mathcal{A}_{\leq 1})$, $H_\infty({}^p\mathcal{A}_{\leq 1})$ of our Hall algebras. The advantage of H_∞ is that we include stacks *locally* of finite type over $\mathbb{C}$ (e.g. ${}^p\mathfrak{A}_{\leq 1}$), the disadvantage is that we do not have an integration morphism at our disposal. To define this algebra we proceed exactly as in the previous section: the only differences being that we allow our stacks to be *locally* of finite type over $\mathbb{C}$, we insist that geometric bijections be finite type morphisms and we disregard the disjoint union relation.[5]

The first element we consider is $\mathscr{H}_{\leq 1} \in H_\infty(\mathcal{A}_{\leq 1})$ corresponding to the Hilbert scheme of Y, which parameterises quotients of $\mathcal{O}_Y$ in $\mathcal{A}_{\leq 1}$. To be precise, $\mathscr{H}_{\leq 1}$ is represented by the forgetful morphism $\mathrm{Hilb}_{\leq 1}(Y) \to \mathfrak{A}_{\leq 1}$, which takes a quotient $\mathcal{O}_Y \twoheadrightarrow E$ to E. For us, the important thing to notice is that if $\mathcal{O}_Y \twoheadrightarrow E$ is a quotient in $\mathcal{A}_{\leq 1}$, then $E \in {}^p\mathcal{T}$. This is a consequence of $\mathcal{O}_Y \in {}^p\mathcal{T}$ and of the fact that the torsion part of a torsion pair is closed under quotients. Thus the morphism $\mathrm{Hilb}_{\leq 1}(Y) \to \mathfrak{A}_{\leq 1}$ factors through ${}^p\mathfrak{T}_{\leq 1}$. As ${}^p\mathfrak{T}_{\leq 1} \subset {}^p\mathfrak{A}_{\leq 1}$, $\mathscr{H}_{\leq 1}$ can be interpreted as an element of $H_\infty({}^p\mathcal{A}_{\leq 1})$.

Once and for all we establish some general notation. For $\mathcal{B} \subset \mathcal{A}$ a subcategory we denote $1_\mathcal{B}$ the element of $H_\infty(\mathcal{A})$ represented by the inclusion of stacks $\mathfrak{B} \subset \mathfrak{A}$, when this is an open immersion (analogous notation for $\mathcal{A}_{\leq 1}$ and ${}^p\mathcal{A}_{\leq 1}$). Another important stack is $\mathfrak{A}^{\mathcal{O}}_{\leq 1}$, the stack of *framed* coherent sheaves [Bri11, Section 2.3], which parameterises sheaves with a fixed global section $\mathcal{O}_Y \to E$. By considering surjective sections we can realise $\mathrm{Hilb}_{\leq 1}(Y)$ as an open subscheme of $\mathfrak{A}^{\mathcal{O}}_{\leq 1}$. We

5 If we allowed both the disjoint union relation and spaces of infinite type then we would be left with the zero ring. Indeed, if Z is an infinite disjoint union of points, then $Z \setminus \{pt\} \cong Z$ and thus $[Z] = [Z] - 1$ so $1 = 0$. The finite type assumption for geometric bijections is there to avoid pathologies such as an infinite disjoint union of points representing the same class as a line.

have a forgetful map $\mathfrak{A}^{\mathcal{O}}_{\leq 1} \to \mathfrak{A}_{\leq 1}$, which takes a morphism $\mathcal{O}_Y \to E$ to E. Given an open substack $\mathfrak{B} \subset \mathfrak{A}_{\leq 1}$, we can consider the fibre product $\mathfrak{B}^{\mathcal{O}} = \mathfrak{B} \times_{\mathfrak{A}_{\leq 1}} \mathfrak{A}^{\mathcal{O}}_{\leq 1}$, which gives an element $1^{\mathcal{O}}_{\mathcal{B}} \in H_\infty(\mathcal{A}_{\leq 1})$.

We want to emulate this last construction for $H_\infty({}^p\mathcal{A}_{\leq 1})$. We define the stack ${}^p\mathfrak{A}^{\mathcal{O}}$ of *framed perverse coherent sheaves* as the prestack taking a base S to a family of perverse coherent sheaves P together with a morphism $\mathcal{O}_{S\times Y} \to P$. It is useful for us to realise ${}^p\mathfrak{A}^{\mathcal{O}}$ as a fibre product as follows.

Note first that we also have a stack $\mathfrak{C}$ parameterising coherent sheaves on X. Pushforward of complexes induces a morphism of stacks ${}^p\mathfrak{A} \to \mathfrak{C}$. In fact, for this to be well-defined, we simply need to check that given a family of perverse coherent sheaves P over a base S, the pushforward $Rf_{S,*}P$ is a coherent sheaf. This can be verified on fibres. If $s \in S$ is a point, then $Ls^*Rf_{S,*}P = Rf_{s,*}P|^{L}_{Y_s}$ [6], which is a coherent sheaf as $P|^{L}_{Y_s}$ is a perverse coherent sheaf.

Moreover, there is a corresponding stack of framed sheaves $\mathfrak{C}^{\mathcal{O}}$ [Bri11, p. 2.3]. For $P \in {}^p\mathcal{A}$, morphisms $\mathcal{O}_Y \to P$ correspond (by adjunction) to morphisms $\mathcal{O}_X \to Rf_*P$. We know that Rf_*P is a sheaf, so morphisms $\mathcal{O}_Y \to P$ correspond to points of $\mathfrak{C}^{\mathcal{O}}$.

To make the argument work in families, we notice that over a base S we still have $Rf_{S,*}\mathcal{O}_{S\times Y} = \mathcal{O}_{S\times X}$ (this follows from flatness of $S \to \operatorname{Spec}\mathbb{C}$ and base change). Hence, the considerations made above still apply and ${}^p\mathfrak{A}^{\mathcal{O}}$ sits in the cartesian diagram below.

$$\begin{tikzcd} {}^p\mathfrak{A}^{\mathcal{O}} \arrow[r] \arrow[d] & \mathfrak{C}^{\mathcal{O}} \arrow[d] \\ {}^p\mathfrak{A} \arrow[r] & \mathfrak{C} \end{tikzcd}$$

6 For a proof of this non-flat base-change refer the reader to [Har12, Proposition 6.3].

Once again, we have an obvious substack ${}^p\mathfrak{A}^{\mathcal{O}}_{\leq 1}$, which can also be described as the preimage of ${}^p\mathfrak{A}_{\leq 1}$.

We have elements $1_{{}^p\mathcal{F}_{\leq 1}[1]}, 1_{{}^p\mathcal{T}_{\leq 1}} \in \mathrm{H}_\infty({}^p\mathcal{A}_{\leq 1})$ corresponding to the subcategories ${}^p\mathcal{F}_{\leq 1}[1], {}^p\mathcal{T}_{\leq 1}$ of ${}^p\mathcal{A}_{\leq 1}$. By taking fibre products with ${}^p\mathcal{A}_{\leq 1}{}^{\mathcal{O}} \to {}^p\mathcal{A}_{\leq 1}$ we produce elements $1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]}, 1^{\mathcal{O}}_{{}^p\mathcal{T}_{\leq 1}} \in \mathrm{H}_\infty({}^p\mathcal{A}_{\leq 1})$.

We also want a *perverse Hilbert scheme* ${}^p\mathrm{Hilb}_{\leq 1}(\mathrm{Y}/\mathrm{X})$ of Y over X parameterising quotients of $\mathcal{O}_\mathrm{Y}$ in ${}^p\mathcal{A}_{\leq 1}$. One can realise it as an open substack of ${}^p\mathfrak{A}^{\mathcal{O}}_{\leq 1}$. Indeed, for $\alpha : \mathcal{O}_\mathrm{Y} \to \mathrm{P}$ with $\mathrm{P} \in {}^p\mathcal{A}_{\leq 1}$, being surjective is equivalent to the cone of α lying in ${}^p\mathcal{A}_{\leq 1}[1]$, which we know to be an open condition on ${}^p\mathfrak{A}^{\mathcal{O}}_{\leq 1}$. Thus we have an element ${}^p\mathscr{H}_{\leq 1} \in \mathrm{H}_\infty({}^p\mathcal{A}_{\leq 1})$.

4.3 A FIRST IDENTITY

We want to prove the identity

$$ {}^p\mathscr{H}_{\leq 1} * 1_{{}^p\mathcal{F}_{\leq 1}[1]} = 1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathscr{H}_{\leq 1} \tag{4.3.1} $$

which we motivated in the beginning of this section. The left hand side is represented by a stack $\mathfrak{M}_\mathrm{L}$, parameterising diagrams

$$\begin{tikzcd} \mathcal{O}_\mathrm{Y} \arrow[d, two heads] & & \\ \mathrm{P}_1 \arrow[r, hook] & \mathrm{E} \arrow[r, two heads] & \mathrm{P}_2 \end{tikzcd}$$

where all objects are in ${}^p\mathcal{A}_{\leq 1}$, the sequence $\mathrm{P}_1 \hookrightarrow \mathrm{E} \twoheadrightarrow \mathrm{P}_2$ is exact in ${}^p\mathcal{A}_{\leq 1}$, $\mathcal{O}_\mathrm{Y} \twoheadrightarrow \mathrm{P}_1$ is surjective in ${}^p\mathcal{A}_{\leq 1}$ and $\mathrm{P}_2 \in {}^p\mathcal{F}_{\leq 1}[1]$.[7]

7 To be precise, over a base U, the groupoid $\mathfrak{M}_\mathrm{L}(\mathrm{U})$ consists of diagrams as above which, upon restricting to fibres of points of U, satisfy the required properties. Similar remarks will be implicit for the other stacks we define below.

The right hand side is represented by a stack $\mathfrak{M}_R$ parameterising diagrams

$$\begin{tikzcd} \mathcal{O}_Y \arrow[d] & & \mathcal{O}_Y \arrow[d, "\mathrm{sur}"] \\ F[1] \arrow[r, hook] & E \arrow[r, two heads] & T \end{tikzcd}$$

where the horizontal maps form a short exact sequence in ${}^p\mathcal{A}_{\leq 1}$, $F \in {}^p\mathcal{F}_{\leq 1}, T \in {}^p\mathcal{T}_{\leq 1}$ and the map $\mathcal{O}_Y \to T$ is surjective as a morphism in $\mathcal{A}_{\leq 1}$. We remind ourselves that $({}^p\mathcal{F}_{\leq 1}[1], {}^p\mathcal{T}_{\leq 1})$ is a torsion pair in ${}^p\mathcal{A}_{\leq 1}$ so that given a perverse coherent sheaf E, there is a unique exact sequence $F[1] \hookrightarrow E \twoheadrightarrow T$, with $F \in {}^p\mathcal{F}_{\leq 1}, T \in {}^p\mathcal{T}_{\leq 1}$.

As the proof of the required identity goes through a chain of geometric bijections and Zariski fibrations, we draw a diagram for future reference.

$$\begin{tikzcd} \mathfrak{M}_L \arrow[dr] & & \mathfrak{M} \arrow[dl] \arrow[dr] & & \mathfrak{M}_R \arrow[dl] \\ & \mathfrak{M}' & & \mathfrak{N} & \end{tikzcd}$$

In what follows, we shall make use of the next lemma. By *perverse* kernel, cokernel, surjection etc. we mean kernel, cokernel, surjection etc. in the abelian category ${}^p\mathcal{A}$.

4.3.2 LEMMA – Let $\varphi : \mathcal{O}_Y \to E$ be a morphism from the structure sheaf to a perverse coherent sheaf. Then the following are equivalent: the perverse cokernel of φ lies in ${}^p\mathcal{F}[1]$; the cone of φ belongs to $D^{\leq -1}(Y)$; the morphism $H^0(\varphi)$ is surjective.

Proof: Let $\sigma : E \to T$ be the surjection from E to its torsion-free part. We first show that the statement ${}^p\mathrm{coker}\,\varphi \in {}^p\mathcal{F}[1]$ is equivalent to $\sigma\varphi$ being surjective as a morphism of coherent sheaves. First of all notice that as $H^0(\sigma)$ is an isomorphism then $H^0(\sigma\varphi)$ is surjective if and only if $H^0(\varphi)$ is surjective. Consider the diagram obtained by factoring φ through the perverse image and by taking the perverse cokernel.

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr, "\varphi"] \arrow[dr, two heads, "\alpha"'] & & E \arrow[dr, two heads] & \\ & I \arrow[ur, hook, "\beta"'] & & K \end{tikzcd}$$

Glancing at the cohomology sheaves long exact sequence we see that $H^0(\alpha)$ is surjective. Thus $H^0(\varphi)$ is surjective if and only if $H^0(\beta)$ is surjective if and only if $H^0(K) = 0$ if and only if $K \in {}^p\mathcal{F}[1]$.

Let now C be the cone of φ. By taking the cohomology sheaves long exact sequence we immediately see that $H^0(\varphi)$ is surjective if and only if $C \in D^{\leq -1}(Y)$. ■

We now define a stack $\mathfrak{M}'$ parameterising diagrams of the form

$$\begin{tikzcd} \mathcal{O}_Y \arrow[d, "\varphi"] \\ E \end{tikzcd}$$

where ${}^p\mathrm{coker}\,\varphi \in {}^p\mathcal{F}_{\leq 1}[1]$. By the previous lemma this last condition is equivalent to $\mathrm{cone}(\varphi) \in D^{\leq -1}(Y)$, which is open. Thus $\mathfrak{M}'$ is an open substack of the stack of framed perverse sheaves ${}^p\mathfrak{A}^{\mathcal{O}}_{\leq 1}$.

4.3.3 Proposition – There is a map $\mathfrak{M}_L \to \mathfrak{M}'$ induced by the composition $\mathcal{O}_Y \twoheadrightarrow P_1 \hookrightarrow E$. This map is a geometric bijection.

Proof: This map induces an equivalence on $\mathbb{C}$-points. To prove finite typeness of the morphism we use a fact that shall be proved later: namely, the stack $\mathfrak{M}_L$ represents a Laurent element of our Hall algebra. This follows as $\mathfrak{M}_L$ is the product ${}^p\mathcal{H}_{\leq 1} * 1_{{}^p\mathcal{F}_{\leq 1}[1]}$, and both these elements are Laurent. Thus, for any numerical class α, we have a morphism $\mathfrak{M}_{L,\alpha} \to \mathfrak{M}'_\alpha$. As $\mathfrak{M}_{L,\alpha}$ is of finite type, we are done. ■

We define another stack $\mathfrak{M}$ parameterising diagrams of the form

$$\begin{tikzcd} & \mathcal{O}_Y \arrow[d, "\varphi"] & \\ F[1] \arrow[r, hook] & E \arrow[r, two heads] & T \end{tikzcd}$$

where the horizontal maps form a short exact sequence of perverse sheaves, $F \in {}^p\mathcal{F}_{\leq 1}$, $T \in {}^p\mathcal{T}_{\leq 1}$ and ${}^p\mathrm{coker}\,\varphi \in {}^p\mathcal{F}_{\leq 1}[1]$. This stack can be obtained as a fibre product as follows. The element $1_{{}^p\mathcal{F}_{\leq 1}[1]} * 1_{{}^p\mathcal{T}_{\leq 1}}$ is represented by a morphism $Z \to {}^p\mathfrak{A}_{\leq 1}$ and $\mathfrak{M}$ is the top left corner of the following cartesian diagram.

$$\begin{tikzcd} \mathfrak{M} \arrow[r] \arrow[d] & \mathfrak{M}' \arrow[d] \\ Z \arrow[r] & {}^p\mathfrak{A}_{\leq 1} \end{tikzcd}$$

4.3.4 PROPOSITION – The morphism $\mathfrak{M} \to \mathfrak{M}'$ defined by forgetting the exact sequence is a geometric bijection.

Proof: The morphism in question is precisely the top row of the previous diagram. The bottom row is obtained by composing the top arrows of the following diagram.

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & {}^p\mathfrak{A}^{(2)}_{\leq 1} \arrow[r, "b"] \arrow[d] & {}^p\mathfrak{A}_{\leq 1} \\ {}^p\mathfrak{F}_{\leq 1}[1] \times {}^p\mathfrak{T}_{\leq 1} \arrow[r] & {}^p\mathfrak{A}_{\leq 1} \times {}^p\mathfrak{A}_{\leq 1} & \end{tikzcd}$$

where the bottom row is an open immersion (and thus of finite type) and the morphism b is of finite type (this follows from the fact that b locally is isomorphic to the analogous morphism for coherent sheaves). The morphism $Z \to {}^p\mathfrak{A}_{\leq 1}$ induces an equivalence on $\mathbb{C}$-points because $({}^p\mathcal{F}_{\leq 1}[1], {}^p\mathcal{T}_{\leq 1})$ is a torsion pair in ${}^p\mathcal{A}_{\leq 1}$ (and thus any perverse coherent sheaf has a unique short exact sequence with torsion kernel and torsion-free cokernel) and because an automorphism of a short exact sequence which is the identity on the middle term is trivial. As $\mathfrak{M} \to \mathfrak{M}'$ is a base change of $Z \to {}^p\mathfrak{A}_{\leq 1}$ we are done. ■

Thus the identity (4.3.1) boils down to proving that $\mathfrak{M}$ and $\mathfrak{M}_R$ represent the same element in the Hall algebra $H_\infty({}^p\mathcal{A})$. To do this we use one last stack $\mathfrak{N}$ and build a pair of Zariski fibrations with same fibres. We define the stack $\mathfrak{N}$ to be the moduli of the following diagrams

$$\begin{tikzcd} & & \mathcal{O}_Y \arrow[d, two heads, "\text{sur}"] \\ F[1] \arrow[r, hook] & E \arrow[r, two heads] & T \end{tikzcd} \tag{4.3.5}$$

where the horizontal maps form a short exact sequence of perverse sheaves, $F \in {}^p\mathcal{F}_{\leq 1}$, $T \in {}^p\mathcal{T}_{\leq 1}$ and the map $\mathcal{O}_Y \to T$ is surjective on H^0. This stack is also a fibre product of known stacks (compare with the element $1_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathscr{H}_{\leq 1}$). Notice that there are two maps $\mathfrak{M} \to \mathfrak{N} \leftarrow \mathfrak{M}_R$. The map $\mathfrak{M}_R \to \mathfrak{N}$ is given by forgetting the morphism $\mathcal{O}_Y \to F[1]$. The map $\mathfrak{M} \to \mathfrak{N}$ is given by composition $\mathcal{O}_Y \to E \to T$ (which is a surjective morphism thanks to the previous lemma).

4.3.6 PROPOSITION – The maps $\mathfrak{M} \to \mathfrak{N} \leftarrow \mathfrak{M}_R$ are two Zariski fibrations with the same fibres.

Proof: Keeping in mind diagram (4.3.5), the idea is that over a perverse coherent sheaf E the morphism $\mathfrak{M}_R \to \mathfrak{N}$ has fibres $\mathrm{Hom}_Y(\mathcal{O}_Y, F[1])$ while $\mathfrak{M} \to \mathfrak{N}$ has fibres lifts $\mathcal{O}_Y \to E$. The long exact sequence

$$0 \to \mathrm{Hom}_Y(\mathcal{O}_Y, F[1]) \to \mathrm{Hom}_Y(\mathcal{O}_Y, E) \to \mathrm{Hom}_Y(\mathcal{O}_Y, T) \to 0$$

tells us that given a choice of a lift of $\mathcal{O}_Y \to T$ all lifts are in bijection with $\mathrm{Hom}_Y(\mathcal{O}_Y, F[1])$.

Let's see how to make this argument work in families. Let S be an affine and connected scheme and let $S \to \mathfrak{N}$ correspond to a diagram

$$\begin{tikzcd} & & \mathcal{O}_Y \arrow[d, "\text{sur}"] \\ F[1] \arrow[r, hook] & E \arrow[r, two heads] & T \end{tikzcd}$$

on Y_S. First of all notice that base change and Lemma 2.3.2 (and the proof of Proposition 2.4.7) tell us that $Rp_{S,*}F$ is just $H^1(Y_S, F)$ shifted by one, where $p_S : Y_S \to S$ is the projection. In addition, $H^1(Y_S, F)$ is flat over S, or in other words $\mathcal{O}_{Y_S}$ and F have constant Ext groups in the sense of [Bri12, Section 6.1] (all the others vanish).

Let W be the fibre product $\mathfrak{M}_R \times_{\mathfrak{N}} S$. This is actually a functor which associates to an affine S-scheme $q : T \to S$ the group $H^1(Y_T, q_Y^* F)$ and we know by loc. cit. that it is represented by a vector bundle over S of rank the rank of $H^1(Y_S, F)$.

Similarly, the fibre product $\mathfrak{M} \times_{\mathfrak{N}} S$ is represented by an affine bundle of rank the rank of $H^1(Y_S, F)$ (notice that because of the previous arguments the exact sequence at the beginning of the proof still holds over S). This concludes the proof. ■

4.4 PT INVARIANTS

We are still left with the task of understanding what we obtain by integrating ${}^p\mathcal{H}_{\leq 1}$. To achieve this goal we first substitute $1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]}$ with something more recognisable (from the point of view of the integration morphism I). Recall [Bri11, Section 2.2] that on $\mathcal{A}$ there is a torsion pair $(\mathcal{P}, \mathcal{Q})$, where $\mathcal{P}$ consists of sheaves supported in dimension zero and $\mathcal{Q}$ is the right orthogonal of $\mathcal{P}$. In particular, an element $Q \in \mathcal{Q}$ which is supported in dimension one is pure. Notice also that $\mathcal{O}_Y \in \mathcal{Q}$. We denote by $\mathcal{A}^\#$ the tilt with respect to $(\mathcal{P}, \mathcal{Q})$, but with the convention

$$\mathcal{P}[-1] \subset \mathcal{A}^\# \subset D^{[0,1]}(Y).$$

There exists a scheme $\mathrm{Hilb}^{\#}_{\leq 1}(Y)$ parameterising quotients of $\mathcal{O}_Y$ in $\mathcal{A}^{\#}$ supported in dimension at most one. Using [Bri11, Lemma 2.3] one constructs an element $\mathcal{H}^{\#}_{\leq 1} \in H_{\infty}(\mathcal{A}_{\leq 1})$ which eventually leads to the PT invariants of Y. We recall that quotients of $\mathcal{O}_Y$ in $\mathcal{A}^{\#}$ are exactly morphisms $\mathcal{O}_Y \to Q$, with cokernel in $\mathcal{P}$ and $Q \in \mathcal{Q}$.

In $H_{\infty}(\mathcal{A}_{\leq 1})$ we have an element $1_{\mathcal{Q}_{\leq 1}}$ given by the inclusion of the stack parameterising objects in $\mathcal{Q}_{\leq 1}$ inside $\mathfrak{A}_{\leq 1}$ and its framed version $1^{\mathcal{O}}_{\mathcal{Q}_{\leq 1}}$. There is also an identity [Bri11, Section 4.5]

$$1^{\mathcal{O}}_{\mathcal{Q}_{\leq 1}} = \mathcal{H}^{\#}_{\leq 1} * 1_{\mathcal{Q}_{\leq 1}}.$$

We want to restrict the element $\mathcal{H}^{\#}_{\leq 1}$ further by considering only quotients whose derived pushforward Rf_* is supported in dimension zero. We thus define the following subcategories.

$$\begin{aligned}
\mathcal{Q}_{\mathrm{exc}} &= \{Q \in \mathcal{Q} \mid \dim \operatorname{supp} Rf_* Q = 0\} \\
{}^p\mathcal{A}_{\mathrm{exc}} &= \{E \in {}^p\mathcal{A} \mid \dim \operatorname{supp} Rf_* E = 0\} \\
{}^p\mathcal{T}_{\mathrm{exc}} &= {}^p\mathcal{T} \cap {}^p\mathcal{A}_{\mathrm{exc}} \\
{}^p\mathcal{T}_{\bullet} &= {}^p\mathcal{T}_{\mathrm{exc}} \cap \mathcal{Q}_{\mathrm{exc}}
\end{aligned}$$

We can also consider the scheme $\mathrm{Hilb}^{\#}_{\mathrm{exc}}(Y)$ parameterising quotients of $\mathcal{O}_Y$ in $\mathcal{A}^{\#}_{\leq 1}$ with target having zero-dimensional pushdown (it is indeed an open subscheme of $\mathrm{Hilb}^{\#}_{\leq 1}(Y)$ as we are imposing a restriction on the numerical class of the quotients). From it we obtain an element $\mathcal{H}^{\#}_{\mathrm{exc}} \in H_{\infty}(\mathcal{A})$. Before we move on to the following result, we point out that ${}^p\mathcal{T}_{\bullet} \subset \mathcal{A}^{\#}_{\leq 1}$

4.4.1 PROPOSITION – The following identity in $H_\infty(\mathcal{A})$ is true.

$$1^{\mathcal{O}}_{{}^p\mathcal{T}_\bullet} = \mathcal{H}^{\#}_{\mathrm{exc}} * 1_{{}^p\mathcal{T}_\bullet} \tag{4.4.2}$$

Proof: We start with a remark. If we have a morphism $\mathcal{O}_Y \to T$ in $\mathcal{A}^{\#}$, with $T \in {}^p\mathcal{T}_\bullet$, we can factor it through its image (in $\mathcal{A}^{\#}$) $\mathcal{O}_Y \to I \to T$ and we denote by Q the quotient, again in $\mathcal{A}^{\#}$. We already know [Bri11, Lemma 2.3] that I is a sheaf and that the morphism $\mathcal{O}_Y \to I$, as a morphism in $\mathcal{A}$, has cokernel P supported in dimension zero.

Glancing at the cohomology sheaves long exact sequence reveals that Q is also a sheaf, thus the sequence $I \hookrightarrow T \twoheadrightarrow Q$ is actually a short exact sequence of sheaves. The sheaf Q is in ${}^p\mathcal{T}$, as it is a quotient of T, and it lies in $\mathcal{Q}$ as it is an object of $\mathcal{A}^{\#}$. Also, Rf_*Q is supported on points as Rf_*T is, thus $Q \in {}^p\mathcal{T}_\bullet$.

On the other hand, given a morphism of sheaves $\mathcal{O}_Y \to I$, which is an epimorphism in $\mathcal{A}^{\#}$, and given a short exact sequence of coherent sheaves $I \hookrightarrow T \twoheadrightarrow Q$, with $I \in \mathcal{Q}_{\mathrm{exc}}$ and $Q \in {}^p\mathcal{T}_\bullet$, we claim that $T \in {}^p\mathcal{T}_\bullet$. The fact that $T \in \mathcal{Q}_{\mathrm{exc}}$ is clear, if we prove that $I \in {}^p\mathcal{T}$ then we are done.

We know there is an exact sequence $\mathcal{O}_Y \to I \twoheadrightarrow P$, with P supported in dimension zero, viz. a skyscraper sheaf. Let $I \twoheadrightarrow F$ be the projection to the torsion-free part of I (for the $({}^p\mathcal{T}, {}^p\mathcal{F})$ torsion pair). The morphism $\mathcal{O}_Y \to I \twoheadrightarrow F$ is zero, as objects of ${}^p\mathcal{F}$ have no sections. Thus there is a morphism $P \to F$ such that $I \twoheadrightarrow P \to F$ is equal to $I \twoheadrightarrow F$. As P is a skyscraper sheaf, the morphisms from it are determined on global sections, thus $P \to F$ is zero, which in turn implies that $I \twoheadrightarrow F$ is zero. Thus $F = 0$ and $I \in {}^p\mathcal{T}$.

Using the remark above we can see that there exists a morphism from the stack parameterising diagrams

$$\begin{tikzcd} \mathcal{O}_Y \arrow[d] & & \\ I \arrow[r, hook] & T \arrow[r, two heads] & Q \end{tikzcd}$$

with $\mathcal{O}_Y \to I$ an epimorphism in $\mathcal{A}^{\#}$, $I \in \mathcal{Q}_{exc}$, $Q \in {}^p\mathcal{T}_\bullet$, to the stack parameterising morphisms $\mathcal{O}_Y \to T$, with $T \in {}^p\mathcal{T}_\bullet$. This morphism induces an equivalence on $\mathbb{C}$-points and the fact that it is of finite type will follow from Proposition 4.6.5 and Proposition 4.5.2. ∎

4.5 DUALITY

We will see now how to link everything together via the duality functor.

4.5.1 LEMMA – Let $\mathbb{D} : D(Y) \to D(Y)$ be the anti-equivalence defined by

$$E \longmapsto \mathbb{D}(E) = R\underline{Hom}_Y(E, \mathcal{O}_Y)[2].$$

Then

$$\mathbb{D}({}^q\mathcal{T}_\bullet) = {}^p\mathcal{F}_{\leq 1}$$

for $q = -(p+1)$.

The shift [2] in the definition of $\mathbb{D}$ is due to the fact we are dealing with pure sheaves supported in codimension two. Indeed, if $\mathcal{Q}_1$ is the category of pure sheaves supported in dimension one, then $\mathbb{D}(\mathcal{Q}_1) = \mathcal{Q}_1$ [Bri11, Lemma 5.6]. Notice that any sheaf $F \in {}^p\mathcal{F}_{\leq 1}$ is automatically pure, as the existence of a zero-dimensional subsheaf would contradict the condition $f_*F = 0$.

Proof: We will prove the two inclusions $\mathbb{D}({}^q\mathcal{T}_\bullet) \subset {}^p\mathcal{F}_{\leq 1}$, $\mathbb{D}({}^p\mathcal{F}_{\leq 1}) \subset {}^q\mathcal{T}_\bullet$, but first let us make a consideration about the category $\mathcal{C}_{\leq 1}$ of coherent sheaves supported in dimension at most one with vanishing derived pushforward. We have $\mathbb{D}(\mathcal{C}_{\leq 1}) = \mathcal{C}_{\leq 1}$. In fact, as $\mathcal{C}_{\leq 1} \subset \mathcal{Q}_1$, one has $\mathbb{D}(\mathcal{C}_{\leq 1}) \subset \mathcal{Q}_1$, thus one needs only check $Rf_*\mathbb{D}(C) = 0$, for all $C \in \mathcal{C}_{\leq 1}$.

$$\begin{aligned} Rf_*\mathbb{D}(C) &= Rf_*R\underline{\mathrm{Hom}}_Y(C, \mathcal{O}_Y)[2] \\ &= Rf_*R\underline{\mathrm{Hom}}_Y(C, f^!\mathcal{O}_X)[2] \\ &= R\underline{\mathrm{Hom}}_X(Rf_*C, \mathcal{O}_X)[2] = 0 \end{aligned}$$

Let $F \in {}^p\mathcal{F}_{\leq 1}$. We first check that $R^1f_*\mathbb{D}(F) = 0$.

$$\begin{aligned} R^1f_*\mathbb{D}(F) &= H^1\left(Rf_*R\underline{\mathrm{Hom}}_Y(F, \mathcal{O}_Y))[2]\right) \\ &= H^3\left(R\underline{\mathrm{Hom}}_X(Rf_*F, \mathcal{O}_X)\right) \\ &= H^3\left(R\underline{\mathrm{Hom}}_Y(R^1f_*F[-1], \mathcal{O}_X)\right) \\ &= \underline{\mathrm{Ext}}^4_X\left(R^1f_*F, \mathcal{O}_X\right) \\ &= \mathrm{Ext}^4_X\left(R^1f_*F, \mathcal{O}_X\right) = 0 \end{aligned}$$

where the last equality follows from Serre duality and the second to last is a consequence of the local-to-global spectral sequence and the fact that R^1f_*F (and thus $\underline{\mathrm{Ext}}^4_Y(R^1f_*F, \mathcal{O}_Y)$) is supported in dimension zero. When $p = -1$ this is enough to show that $\mathbb{D}({}^p\mathcal{F}_{\leq 1}) \subset {}^q\mathcal{T}_\bullet$. When $p = 0$ we are still left to check that $\mathrm{Hom}_Y(\mathbb{D}({}^p\mathcal{F}_{\leq 1}), \mathcal{C}) = 0$. If $F \in {}^p\mathcal{F}_{\leq 1}$, then (using the fact that $\mathbb{D}$ is an antiequivalence of $D(Y)$)

$$\mathrm{Hom}_Y(\mathbb{D}(F), \mathcal{C}_{\leq 1}) = \mathrm{Hom}_Y(\mathcal{C}_{\leq 1}, F) \subset \mathrm{Hom}_Y(\mathcal{C}, F) = 0$$

where the last equality is by definition of ${}^0\mathcal{F}$. To complete the proof, we show that if $T \in \mathcal{A}_{\leq 1}$ is such that $R^1f_*T = 0$ and $\mathrm{Hom}_Y(T, \mathcal{C}_{\leq 1}) = 0$, then $\mathrm{Hom}_Y(T, \mathcal{C}) = 0$. In fact, let $T \to C$ be a morphism with $C \in \mathcal{C}$. The image I satisfies $R^1f_*I = 0$ as it is a quotient of T and $f_*I = 0$ as it is a subobject of C. Observing that $T \twoheadrightarrow I$ is surjective implies that $I \in \mathcal{C}_{\leq 1}$ and that $T \to C$ is the zero morphism.

Let now $T \in {}^q\mathcal{T}_\bullet$, we check that $f_*\mathbb{D}(T) = 0$.

$$\begin{aligned} f_*\mathbb{D}(T) &= H^0\left(Rf_*R\underline{\mathrm{Hom}}_Y\left(T, \mathcal{O}_Y\right)[2]\right) \\ &= H^2\left(R\underline{\mathrm{Hom}}_X\left(f_*T, \mathcal{O}_X\right)\right) \\ &= \underline{\mathrm{Ext}}^2_X\left(f_*T, \mathcal{O}_X\right) \\ &= \mathrm{Ext}^2_X\left(f_*T, \mathcal{O}_X\right) = 0 \end{aligned}$$

where the last two equalities again follow from Serre duality and the dimension of the support of f_*T. Analogously as above, this is enough for $p = -1$, and for $p = 0$ we see that $\mathrm{Hom}_Y(\mathcal{C}_{\leq 1}, \mathbb{D}({}^q\mathcal{T}_\bullet)) = 0$.

Let now $F \in \mathcal{A}_{\leq 1}$ be such that $f_*F = 0$ and $\mathrm{Hom}_Y(\mathcal{C}_{\leq 1}, F) = 0$. It follows that $\mathrm{Hom}_Y(\mathcal{C}, F) = 0$. In fact, if $C \to F$ is a morphism with $C \in \mathcal{C}$, then the image I satisfies $f_*I = 0$ as it is a subobject of F and satisfies $R^1f_*I = 0$ as it is a quotient of C. As $I \hookrightarrow F$ is injective, $I \in \mathcal{C}_{\leq 1}$ which implies that $C \to F$ is the zero morphism. ■

We now want to apply the duality functor, or better $\mathbb{D}' = \mathbb{D}[1]$, to our Hall algebras. As the category ${}^p\mathcal{F}_{\leq 1}[1]$ (respectively ${}^q\mathcal{T}_\bullet$) is closed by extensions we have an algebra $H_\infty({}^p\mathcal{F}_{\leq 1}[1])$ (respectively $H_\infty({}^q\mathcal{T}_\bullet)$) spanned by morphisms $[W \to {}^p\mathfrak{F}_{\leq 1}[1]]$ (respectively $[W \to {}^q\mathfrak{T}_\bullet]$). Notice that while the first is a subalgebra of $H_\infty({}^p\mathcal{A}_{\leq 1})$, the second can be viewed as a subalgebra of both $H_\infty({}^q\mathcal{A}_{\leq 1})$ and $H_\infty(\mathcal{A}_{\leq 1})$, as a distinguished triangle with vertices lying in ${}^q\mathcal{T}_\bullet$ is an exact sequence in both ${}^q\mathcal{A}$ and $\mathcal{A}$.

4.5.2 PROPOSITION – The functor $\mathbb{D}'$ induces an anti-isomorphism between the algebras $H_\infty({}^q\mathcal{T}_\bullet)$ and $H_\infty({}^p\mathcal{F}_{\leq 1}[1])$. Furthermore the following identities hold.

$$\begin{aligned} \mathbb{D}'\left(1_{{}^q\mathcal{T}_\bullet}\right) &= 1_{{}^p\mathcal{F}_{\leq 1}[1]} \\ \mathbb{D}'\left(1^{\mathcal{O}}_{{}^q\mathcal{T}_\bullet}\right) &= 1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]} \end{aligned}$$

Proof: Duality $\mathbb{D}'$ induces an isomorphism between stacks ${}^q\mathfrak{T}_\bullet$ and ${}^p\mathfrak{F}_{\leq 1}[1]$. The anti-isomorphism between the Hall algebras is then defined by taking a class $[W \to {}^q\mathfrak{T}_\bullet]$ to $[W \to {}^q\mathfrak{T}_\bullet \to {}^p\mathfrak{F}_{\leq 1}[1]]$ and noticing that duality flips extensions [Bri11, Section 5.4]. Clearly this takes the element $1_{{}^q\mathcal{T}_\bullet}$ to $1_{{}^p\mathcal{F}_{\leq 1}[1]}$, while the second identity requires a bit of work.

Two remarks are in order. The first is that given any $T \in {}^q\mathcal{T}_\bullet$,

$$\mathrm{Hom}_Y(\mathcal{O}_Y, T) = \mathrm{Hom}_Y(\mathbb{D}'(T), \mathcal{O}_Y[3]) = \mathrm{Hom}_Y(\mathcal{O}_Y, \mathbb{D}'(T))^\vee.$$

The second is that, if $T \in {}^q\mathcal{T}_\bullet$ and $F \in {}^p\mathcal{F}_{\leq 1}$, then $\dim_{\mathbb{C}} H^0(Y,T) = \chi(T)$ and similarly $\dim_{\mathbb{C}} H^1(Y,F) = -\chi(F)$. This is useful since, for a family of coherent sheaves, the Euler characteristic is locally constant on the base. Thus we can decompose the stack ${}^q\mathfrak{T}_\bullet$ as a disjoint union according to the value of the Euler characteristic. We have a corresponding decomposition of ${}^q\mathfrak{T}_\bullet{}^{\mathcal{O}}$ and we write ${}^q\mathfrak{T}^{\mathcal{O}}_{\bullet,n}$ for the nth component of this disjoint union. This space maps down to ${}^q\mathfrak{T}_{\bullet,n}$ by forgetting the section. Similarly, the space $\mathbb{A}^n \times {}^q\mathfrak{T}_{\bullet,n}$ projects onto ${}^q\mathfrak{T}_{\bullet,n}$. As these two maps are Zariski fibrations with same fibres the stacks ${}^q\mathfrak{T}^{\mathcal{O}}_{\bullet,n}$ and $\mathbb{A}^n \times {}^q\mathfrak{T}_{\bullet,n}$ represent the same element in the Grothendieck ring. This argument is then extended to the whole ${}^q\mathfrak{T}^{\mathcal{O}}_{\bullet,n}$ proving that

$$\left[{}^q\mathfrak{T}^{\mathcal{O}}_\bullet\right] = \left[\coprod_n \mathbb{A}^n \times {}^q\mathfrak{T}_{\bullet,n}\right].$$

We can proceed analogously for ${}^p\mathfrak{F}_{\leq 1}[1]$. The component ${}^p\mathfrak{F}_{\leq 1}[1]^{\mathcal{O}}_n$ represents the same element as $\mathbb{A}^n \times {}^p\mathfrak{F}_{\leq 1}[1]_n$. The first remark above implies that duality $\mathbb{D}'$ takes ${}^q\mathfrak{T}_{\bullet,n}$ to ${}^p\mathfrak{F}_{\leq 1}[1]_n$, which lets us conclude. ■

Thus in our infinite-type Hall algebra we deduce that $1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]} = \mathbb{D}'(1^{\mathcal{O}}_{{}^q\mathcal{T}_\bullet}) = \mathbb{D}'(\mathscr{H}^{\#}_{\mathrm{exc}} * 1_{{}^q\mathcal{T}_\bullet}) = 1_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathbb{D}'(\mathscr{H}^{\#}_{\mathrm{exc}})$. Accordingly, we have the following identities.

$$1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]} = 1_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathbb{D}'(\mathscr{H}^{\#}_{\mathrm{exc}})$$

and

$${}^p\mathscr{H}_{\leq 1} * 1_{{}^p\mathcal{F}_{\leq 1}[1]} = 1_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathbb{D}'(\mathscr{H}^{\#}_{\mathrm{exc}}) * \mathscr{H}_{\leq 1}.$$

4.6 LAURENT ELEMENTS

Our objective is to get rid of the spurious $1_{{}^p\mathcal{F}_{\leq 1}[1]}$'s in the identity above. This is achieved by constructing a (weak) stability condition (in the sense of [JS08, Definition 3.5]) with values in the ordered set $\{1,2\}$, such that ${}^p\mathcal{F}_{\leq 1}[1]$ manifests as the class of semi-stable objects of $\mu = 2$. Before we do that, however, we want to define a sort of completed Hall algebra $\mathrm{H}({}^p\mathcal{A})_\Lambda$ (parallel to the one in [Bri11, Section 5.2]) which morally sits in between $\mathrm{H}({}^p\mathcal{A}_{\leq 1})$ and $\mathrm{H}_\infty({}^p\mathcal{A}_{\leq 1})$. The reason we need to do so is simple. On one hand the Hall algebra constructed in the previous section only includes spaces that are of finite type, on the other the infinite type Hall algebra is much too big to support an integration morphism. To deal with objects such as the Hilbert scheme of curves and points of Y we allow our spaces to be *locally* of finite type while imposing a *Laurent* condition.

We previously mentioned that $\mathrm{H}({}^p\mathcal{A})$ is graded by the numerical Grothendieck group $\mathrm{N}(\mathrm{Y})$. There is a subgroup $\mathrm{N}_{\leq 1}(\mathrm{Y})$ generated by sheaves supported in dimension at most one and $H({}^p\mathcal{A}_{\leq 1})$ is graded by it. We also notice [Bri11, Lemma 2.2] that the Chern character induces an isomorphism

$$N_{\leq 1}(Y) \ni [E] \longmapsto (\mathrm{ch}_2 E, \mathrm{ch}_3 E) \in N_1(Y) \oplus N_0(Y)$$

where by $N_1(Y)$ we mean the group of curve-classes modulo numerical equivalence, and $N_0(Y) \simeq \mathbb{Z}$. Henceforth we tacitly identify $N_{\leq 1}(Y)$ with $N_1(Y) \oplus \mathbb{Z}$.

We have a pushforward morphism $f_*\colon N_1(Y) \to N_1(X)$. This morphism is surjective and we denote its kernel by $N_1(Y/X)$. The short exact sequence

$$N_1(Y/X) \hookrightarrow N_1(Y) \overset{f_*}{\twoheadrightarrow} N_1(X)$$

is of free abelian groups (of finite rank) therefore it splits (non-canonically)

$$N_1(Y) \cong N_1(X) \oplus N_1(Y/X).$$

Elements of $N_{\leq 1}(Y)$ can then be described by triples $(\gamma, \delta, n) \in N_1(X) \oplus N_1(Y/X) \oplus \mathbb{Z}$. We denote the image of ${}^p\mathcal{A}_{\leq 1}$ (via the Chern character) in $N_{\leq 1}(Y)$ by ${}^p\Delta$ (this is the cone of perverse coherent sheaves supported in dimensions ≤ 1). The algebra $H({}^p\mathcal{A}_{\leq 1})$ is graded by ${}^p\Delta$. Finally, by $\mathscr{E} \subset N_1(Y/X)$ we denote the *effective* curve classes in Y which are contracted by f.

4.6.1 DEFINITION – We define a subset $L \subset {}^p\Delta$ to be *Laurent* if the following conditions hold:

- for all γ there exists an $n(\gamma, L)$ such that for all δ, n with $(\gamma, \delta, n) \in L$ one has that $n \geq n(\gamma, L)$;

- for all γ, n there exists a $\delta(\gamma, n, \mathrm{L}) \in \mathscr{E}$ such that for all δ with $(\gamma, \delta, n) \in \mathrm{L}$ one has that $\delta \leq \delta(\gamma, n, \mathrm{L})$.[8]

We denote by Λ the set of all Laurent subsets of ${}^{p}\Delta$.

Notice that Λ does not depend on the choice of the above splitting. We have the following lemma.

4.6.2 LEMMA – The set Λ of Laurent subsets of ${}^{p}\Delta$ satisfies the two following properties.

1. If $\mathrm{L}_1, \mathrm{L}_2 \in \Lambda$ then $\mathrm{L}_1 + \mathrm{L}_2 \in \Lambda$.

2. If $\alpha \in {}^{p}\Delta$ and $\mathrm{L}_1, \mathrm{L}_2 \in \Lambda$ then there exist only finitely many decompositions $\alpha = \alpha_1 + \alpha_2$ with $\alpha_j \in \mathrm{L}_j$.

Proof: We start by proving (1). Fix a γ and let $(\gamma, \delta, n) \in \mathrm{L}_1 + \mathrm{L}_2$. By [KM98, Corollary 1.19] there are only finitely many decompositions $\gamma = \gamma_1 + \gamma_2$ with $\gamma_i \geq 0$ (i.e. with γ_i effective). Given a decomposition $(\gamma, \delta, n) = (\gamma_1 + \gamma_2, \delta_1 + \delta_2, n_1 + n_2)$, with $(\gamma_i, \delta_i, n_i) \in \mathrm{L}_i$, we know that $n_i \geq n(\gamma_i, \mathrm{L}_i)$ so $n = n_1 + n_2 \geq n(\gamma_1, \mathrm{L}_1) + n(\gamma_2, \mathrm{L}_2)$. By letting the γ_i's vary we obtain the desired lower bound for n.

Fix now γ, n. We want to find an upper bound for the possible δ's such that $(\gamma, \delta, n) \in \mathrm{L}_1 + \mathrm{L}_2$. By the argument above we know that for decompositions $(\gamma, \delta, n) = (\gamma_1 + \gamma_2, \delta_1 + \delta_2, n_1 + n_2)$ with $(\gamma_i, \delta_i, n_i) \in \mathrm{L}_i$ the possible combinations of γ_i and n_i are finite. Fix such a decomposition $(\gamma_1 + \gamma_2, \delta_1 + \delta_2, n_1 + n_2)$. We know that $\delta_i \leq \delta(\gamma_i, n_i, \mathrm{L}_i)$. Thus $\delta = \delta_1 + \delta_2 \leq \delta(\gamma_1, n_1, \mathrm{L}_1) + \delta(\gamma_2, n_2, \mathrm{L}_2)$. Take now another decomposition $(\gamma_1' + \gamma_2', \delta_1' + \delta_2', n_1' + n_2')$. Running the same argument we have that $\delta \leq \delta(\gamma_1', n_1', \mathrm{L}_1) + \delta(\gamma_2', n_2', \mathrm{L}_2)$. Finally, as $\delta(\gamma_i, n_i, \mathrm{L}_i), \delta(\gamma_i', n_i', \mathrm{L}_i) \geq 0$, we conclude $\delta \leq \sum_i \delta(\gamma_i, n_i, \mathrm{L}_i) + \delta(\gamma_i', n_i', \mathrm{L}_i)$. By taking the sum for all possible decompositions we have our upper bound for δ.

8 For $\delta, \delta' \in \mathrm{N}_1(\mathrm{Y}/\mathrm{X})$, by the notation $\delta \leq \delta'$ we mean $\delta' - \delta \in \mathscr{E}$ or equivalently $\delta - \delta' \in -\mathscr{E}$. In general we will write $\alpha \geq 0$ to denote that a certain class is effective.

Let us now prove (2). Fix a class $\alpha = (\gamma, \delta, n) \in {}^p\Delta$ and two Laurent subsets L_1, L_2. Again by [KM98, Corollary 1.19] we know that there are only finitely many possible decompositions $\gamma = \gamma_1 + \gamma_2$. Thus we may fix γ_1 and γ_2. Given a decomposition $(\gamma, \delta, n) = (\gamma_1 + \gamma_2, \delta_1 + \delta_2, n_1 + n_2)$, there are again finitely many possible values occurring for n_1, n_2, as $n_i \geq n(\gamma_i, L_i)$. Thus we may take n_1, n_2 also to be fixed. Finally, the combinations $(\gamma, \delta, n) = (\gamma_1 + \gamma_2, \delta_1 + \delta_2, n_1 + n_2)$ are again a finite number, as $\delta = \delta_1 + \delta_2$ lives in $\delta(\gamma_1, n_1, L_1) + \delta(\gamma_2, n_2, L_2) - \mathscr{E}$ (thus we can apply [KM98, Corollary 1.19] again). ■

We now have all the ingredients to define a Λ-completion $H({}^p\mathcal{A}_{\leq 1})_\Lambda$ of $H({}^p\mathcal{A}_{\leq 1})$. Let us give a general definition.

4.6.3 DEFINITION – Let R be a ${}^p\Delta$-graded associative $\mathbb{Q}$-algebra. We define R_Λ to be the vector space of formal series

$$\sum_{(\gamma,\delta,n)} x_{(\gamma,\delta,n)}$$

with $x_{(\gamma,\delta,n)} \in R_{(\gamma,\delta,n)}$ and $x_{(\gamma,\delta,n)} = 0$ outside a Laurent subset. We equip this vector space with a product

$$x \cdot y = \sum_{\alpha \in {}^p\Delta} \sum_{\alpha_1 + \alpha_2 = \alpha} x_{\alpha_1} \cdot y_{\alpha_2}.$$

The algebra R is included in R_Λ as any finite set is Laurent. To a morphism $R \to S$ of ${}^p\Delta$-graded algebras corresponds an obvious morphism $R_\Lambda \to S_\Lambda$.

There is a subalgebra

$$\mathbb{Q}_\sigma[{}^p\Delta] \subset \mathbb{Q}_\sigma[{}^p\Gamma]$$

spanned by symbols q^α with $\alpha \in {}^p\Delta$. Notice that the Poisson structure on $\mathbb{Q}_\sigma[{}^p\Delta]$ is trivial as the Euler form on $\mathrm{N}_{\leq 1}(\mathrm{Y})$ is identically zero. The integration morphism restricts to $\mathrm{I} : \mathrm{H}_{\mathrm{sc}}({}^p\mathcal{A}_{\leq 1}) \to \mathbb{Q}_\sigma[{}^p\Delta]$ and so, by taking Λ-completions, we have a morphism

$$\mathrm{I}_\Lambda : \mathrm{H}_{\mathrm{sc}}({}^p\mathcal{A}_{\leq 1})_\Lambda \longrightarrow \mathbb{Q}_\sigma[{}^p\Delta]_\Lambda.$$

Remark 4.6.4. Notice that given an algebra R as above and an element $r \in \mathrm{R}$ with $r_{(0,0,0)} = 0$, the element $1 - r$ is invertible in R_Λ. This is due to the fact that the series

$$\sum_{k \geq 0} r^k$$

makes sense in R_Λ.

Now it's time to have a look at what the elements of $\mathrm{H}({}^p\mathcal{A}_{\leq 1})_\Lambda$ look like. Let $\mathfrak{M}$ be an algebraic stack locally of finite type over $\mathbb{C}$ mapping down to ${}^p\mathfrak{A}_{\leq 1}$ and denote by $\mathfrak{M}_\alpha$ the preimage under ${}^p\mathfrak{A}_\alpha$, for $\alpha \in {}^p\Delta$. We say that

$$\left[\mathfrak{M} \to {}^p\mathfrak{A}_{\leq 1}\right] \in \mathrm{H}_\infty({}^p\mathcal{A}_{\leq 1})$$

is *Laurent* if $\mathfrak{M}_\alpha$ is a stack of finite type for all $\alpha \in {}^p\Delta$ and if $\mathfrak{M}_\alpha$ is empty for α outside a Laurent subset. Such a Laurent element gives an element of $\mathrm{H}({}^p\mathcal{A}_{\leq 1})_\Lambda$ by considering $\sum_\alpha \mathfrak{M}_\alpha$.

4.6.5 PROPOSITION – The elements $1_{{}^p\mathcal{F}_{\leq 1}[1]}$, $1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]}$ are Laurent.

Proof: Let $F \in {}^p\mathcal{F}_{\leq 1}$ and let (γ, δ, n) be the class in $N_{\leq 1}(Y)$ corresponding to $[F[1]] = -[F]$. By [SGA6, Proposition X-1.1.2] we know that in the Grothendieck group F decomposes as

$$F = \sum_i l_i [\mathcal{O}_{C_i}] + \tau$$

where the C_i are the curves comprising the irreducible components of the support of F (which is contained in the exceptional locus of f), where $l_i \geq 0$ and where τ is is supported in dimension zero. From this decomposition we infer that $\gamma = 0$ and $\delta \leq 0$. Finally, Riemann-Roch tells us that n is minus the Euler characteristic of F and Lemma 2.3.2 gives us that $n \geq 0$. To conclude, the finite type axiom is deduced using Lemma 4.7.3, combined with Lemma 4.5.1.

For $1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]}$, it is enough to notice that for $F \in {}^p\mathcal{F}$, $H^1(Y, F)$ is finite-dimensional. ■

Notice also that by the remark above both $1_{{}^p\mathcal{F}_{\leq 1}[1]}$ and $1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]}$ are invertible in $H({}^p\mathcal{A}_{\leq 1})_\Lambda$.

4.6.6 PROPOSITION – The element ${}^p\mathcal{H}_{\leq 1}$ is Laurent.

Proof: By [Bri02, Theorem 5.5] if we fix a numerical class $\alpha \in N_{\leq 1}(Y)$ then the space ${}^p\mathrm{Hilb}_{Y/X}(\alpha)$ is of finite type (it is in fact a projective scheme). Thus we are left with checking the second half of the Laurent property. Fix then a class $\gamma \in N_1(X)$ and consider a possible quotient $\mathcal{O}_Y \twoheadrightarrow P$ in ${}^p\mathcal{A}$, with $\dim \operatorname{supp} P \leq 1$ and with P of class (γ, δ, n). We need to show that there exists a lower bound on the possible values of n. By pushing down to X we obtain a quotient (in Coh(X)) $\mathcal{O}_X \twoheadrightarrow Rf_*P$, and we note that the sheaf Rf_*P is of class (γ, n). If a class γ is fixed, it is known that the possible values of the Euler characteristic of a quotient $\mathcal{O}_X \twoheadrightarrow Q$ are bounded below (this follows from boundedness of the Hilbert scheme), hence we have the required bound.

To proceed, we let γ and n both be fixed and notice that we only really need to focus on exact sequences of both coherent and perverse sheaves, that is on points of $\mathrm{Hilb}_Y \cap {}^p\mathrm{Hilb}_{\leq 1}(Y/X)$ (which we temporarily denote by $\mathrm{Pilb}(Y)$). This is a consequence of the fact that given a quotient $\mathcal{O}_Y \twoheadrightarrow P$ in ${}^p\mathcal{A}$, with P of class (γ, δ, n), we can consider the torsion torsion-free exact sequence

$$F[1] \hookrightarrow P \twoheadrightarrow T.$$

In fact, $F[1]$ does not contribute towards γ, contributes negatively towards δ and positively towards n, as seen in the previous proposition. Thus we just need to study the possible classes of T. Finally, $\mathcal{O}_Y \twoheadrightarrow P \twoheadrightarrow T$ is a quotient in ${}^p\mathcal{A}$ but glancing at the cohomology sheaves long exact sequence tells us that it is indeed a quotient in $\mathcal{A}$ as well. Thus we only need to check that, having chosen a γ and an n, there exists an upper bound δ_0 such that $\mathrm{Pilb}_Y(\gamma, \delta, n)$ is empty for $\delta \geq \delta_0$.

Notice that the pushforward induces a morphism from ${}^p\mathrm{Hilb}(Y/X)$ to $\mathrm{Hilb}(X)$. We consider its restriction to $\mathrm{Pilb}(Y)$. We would like for the pullback functor to induce a morphism going in the opposite direction. A flat family of sheaves on X might, however, cease to be flat once pulled back on Y. To remedy we impose this condition by hand. We define a subfunctor Filb_X of Hilb_X by the rule

$$\mathrm{Filb}_X(S) = \left\{ \mathcal{O}_{X_S} \twoheadrightarrow G \,\middle|\, G, f_S^* G \text{ flat over } S \right\}.$$

If U is the structure sheaf of the universal subscheme for $\mathrm{Hilb}(X)$ on $X \times \mathrm{Hilb}(X)$ then one can see that $\mathrm{Filb}(X)$ is represented by the flattening stratification of $\mathrm{Hilb}(X)$ with respect to $f^*_{\mathrm{Hilb}(X)} U$. From this we deduce that if we fix a numerical class (γ, n) on X then $\mathrm{Filb}_X(\gamma, n)$ is of finite type.

We claim that the composition of pushing forward and pulling up as just described, $\mathrm{Pilb}(Y) \to \mathrm{Filb}(X) \to \mathrm{Pilb}(Y)$, is the identity. Let us see first why this is true on geometric points. Take an exact sequence of both coherent and perverse sheaves

$$I \hookrightarrow \mathcal{O}_Y \twoheadrightarrow E.$$

Applying the counit of the adjunction $f^* \dashv f_*$ (and using the fact that the objects above are both sheaves and perverse sheaves) we obtain a commutative diagram

$$\begin{tikzcd}
 & f^*f_*\mathrm{I} \arrow[r] \arrow[d] & \mathcal{O}_\mathrm{Y} \arrow[r] \arrow[d, "\mathrm{id}"] & f^*f_*\mathrm{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{I} \arrow[r] & \mathcal{O}_\mathrm{Y} \arrow[r] & \mathrm{E} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By [Bri02, Proposition 5.1] we have that $f^*f_*\mathrm{I} \to \mathrm{I}$ is surjective and so, by a simple diagram chase, $f^*f_*\mathrm{E} \to \mathrm{E}$ is an isomorphism. This argument indeed works in families, as surjectivity can be checked fibrewise.

Finally, let us fix a γ and an n and let $\mathrm{Pilb}_\mathrm{Y}(\gamma, n)$ be the subspace of $\mathrm{Pilb}(\mathrm{Y})$ where we've fixed γ and n but we let δ vary. By the previous arguments we know that $\mathrm{Pilb}_\mathrm{Y}(\gamma, n) \to \mathrm{Filb}_\mathrm{X}(\gamma, n) \to \mathrm{Pilb}_\mathrm{Y}(\gamma, n)$ composes to the identity. As the retract of a quasi-compact space is quasi-compact[9] we obtain that $\mathrm{Pilb}_\mathrm{Y}(\gamma, n)$ is of finite type, which is enough to conclude. ■

4.6.7 PROPOSITION – The element $\mathscr{H}_{\leq 1}$ is Laurent.

Proof: It is a known fact that for a fixed numerical class $\alpha \in \mathrm{N}_{\leq 1}(\mathrm{Y})$ the scheme $\mathrm{Hilb}_\mathrm{Y}(\alpha)$ is of finite type (it is in fact a projective scheme). To prove the second half of the Laurent property we start by fixing a class $\gamma \in \mathrm{N}_1(\mathrm{X})$. If $\mathcal{O}_\mathrm{Y} \twoheadrightarrow \mathrm{T}$ is a quotient in $\mathcal{A}$ with kernel I, we have an exact sequence

$$0 \to f_*\mathrm{I} \to \mathcal{O}_\mathrm{X} \to f_*\mathrm{T} \to \mathrm{R}^1 f_*\mathrm{I} \to 0.$$

If T is of class (γ, δ, n) then $f_*\mathrm{T}$ is of class (γ, n) and $\mathrm{R}^1 f_*\mathrm{I}$ is supported in dimension zero. The image Q of $\mathcal{O}_\mathrm{X} \to f_*\mathrm{T}$ is of class (γ, m) with $m \leq n$. As γ is fixed we have a lower bound on the possible values of m and a fortiori on the values of n.

9 If $\mathrm{A} \to \mathrm{B} \to \mathrm{A}$ composes to the identity, one can start with an open cover $\{\mathrm{A}_i\}$ and pull it back to a cover $\{\mathrm{B}_i\}$ of B. Pick a finite subcover $\{\mathrm{B}_j\}$ and pull it back to A. This is a finite subcover of $\{\mathrm{A}_i\}$.

Let now γ and n be fixed. We start off with the identity

$$ {}^p\mathcal{H}_{\leq 1} * 1_{{}^p\mathcal{F}_{\leq 1}[1]} = 1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathcal{H}_{\leq 1} $$

in $\mathrm{H}_\infty({}^p\mathcal{A}_{\leq 1})$. By directly applying our definition of $*$ we see that the right hand side is represented by a morphism $[\mathrm{W} \to {}^p\mathfrak{A}_{\leq 1}]$, given by the top row of the following diagram.

$$\begin{tikzcd}
\mathrm{W} \arrow[r] \arrow[d] & {}^p\mathfrak{A}^{(2)}_{\leq 1} \arrow[r, "b"] \arrow[d, "(a_1,a_2)"] & {}^p\mathfrak{A}_{\leq 1} \\
{}^p\mathfrak{F}_{\leq 1}[1]^{\mathcal{O}} \times \mathrm{Hilb}_{\mathrm{Y},\leq 1} \arrow[r] & {}^p\mathfrak{A}_{\leq 1} \times {}^p\mathfrak{A}_{\leq 1} &
\end{tikzcd}$$

Similarly, the left hand side is represented by a morphism $[\mathrm{Z} \to {}^p\mathfrak{A}_{\leq 1}]$. The main tool we use for the proof is the cover $\{{}^p\mathfrak{A}_\alpha\}_\alpha$ of ${}^p\mathfrak{A}_{\leq 1}$, with $\alpha \in {}^p\Delta$ ranging inside the cone of perverse coherent sheaves.

By taking preimages through b we obtain an open cover $\{\mathrm{U}_\alpha\}_\alpha$ of ${}^p\mathfrak{A}^{(2)}_{\leq 1}$. Concretely, U_α parameterises exact sequences $\mathrm{P}_1 \hookrightarrow \mathrm{P} \twoheadrightarrow \mathrm{P}_2$ in ${}^p\mathcal{A}_{\leq 1}$ with P of class α.

On the other hand, we can cover ${}^p\mathfrak{A}_{\leq 1} \times {}^p\mathfrak{A}_{\leq 1}$ by taking products ${}^p\mathfrak{A}_{\alpha_1} \times {}^p\mathfrak{A}_{\alpha_2}$. By pulling back via (a_1, a_2) we produce an open cover $\{\mathrm{U}_{\alpha_1,\alpha_2}\}_{\alpha_1,\alpha_2}$ of ${}^p\mathfrak{A}^{(2)}_{\leq 1}$. The space $\mathrm{U}_{\alpha_1,\alpha_2}$ parameterises exact sequences $\mathrm{P}_1 \hookrightarrow \mathrm{P} \twoheadrightarrow \mathrm{P}_2$ in ${}^p\mathcal{A}_{\leq 1}$ with P_1 of class α_1 and P_2 of class α_2. Notice that the collection $\{\mathrm{U}_{\alpha_1,\alpha_2}\}_{\alpha_1+\alpha_2=\alpha}$ is an open cover of U_α.

By pulling back these covers of ${}^p\mathfrak{A}^{(2)}_{\leq 1}$ we obtain open covers $\{\mathrm{W}_\alpha\}_\alpha$ and $\{\mathrm{W}_{\alpha_1,\alpha_2}\}_{\alpha_1,\alpha_2}$ of W. The same can be done for Z.

We remind ourselves that we think of a class α as a triple (γ, δ, n). If we fix a γ and an n, it is a consequence of ${}^p\mathcal{H}_{\leq 1} * 1_{{}^p\mathcal{F}_{\leq 1}[1]}$ being Laurent that there exists a δ' such that $\mathrm{Z}_{(\gamma,\delta,n)} = \emptyset$ for $\delta \geq \delta'$. Because of the identity above, the same holds for $\mathrm{W}_{(\gamma,\delta,n)}$.

What we need to prove is that, once we fix γ and n_2, the space $\mathrm{Hilb}_\mathrm{Y}(\gamma, \delta_2, n_2)$ is empty for large δ_2. Fix δ_1, n_1 such that ${}^p\mathfrak{F}_{\leq 1}[1]^{\mathcal{O}}_{(0,\delta_1,n_1)} \neq \emptyset$. The space representing the product

$$ 1^{\mathcal{O}}_{{}^p\mathfrak{F}_{\leq 1}[1]_{(0,\delta_1,n_1)}} * \mathrm{Hilb}_\mathrm{Y}(\gamma, \delta_2, n_2) $$

is $W_{(0,\delta_1,n_1),(\gamma,\delta_2,n_2)} \subset W_{(\gamma,\delta_1+\delta_2,n_1+n_2)}$. We have already remarked that for fixed γ, n_1, n_2 we have an upper bound δ' such that $W_{(\gamma,\delta_1+\delta_2,n_1+n_2)} = \emptyset$ for $\delta_1 + \delta_2 \geq \delta'$. As ${}^p\mathfrak{F}_{\leq 1}[1]^{\mathcal{O}}_{(0,\delta_1,n_1)} \neq \emptyset$, we conclude that $\mathrm{Hilb}_Y(\gamma, \delta_2, n_2) = \emptyset$ for $\delta_2 \geq \delta' - \delta_1$, in particular the same is true for $\delta_2 \geq 0$. ■

Remark 4.6.8. We need to interpret Proposition 4.5.2 in the Laurent setting. Duality $\mathbb{D}'$ acts on $N_{\leq 1}(Y)$ by taking a class (γ, δ, n) to $(-\gamma, -\delta, n)$. Even more concretely, an element $T \in {}^q\mathcal{T}_\bullet$ of class $(0, \delta, n)$ is sent to an element $\mathbb{D}'(T) \in {}^p\mathcal{F}_{\leq 1}$ of class $(0, -\delta, n)$. This suggests that we should complete the algebra $H({}^q\mathcal{T}_\bullet)$ with respect to a sort of dual Laurent subsets.

Let ${}^q\Delta_{\mathrm{exc}}$ be the subcone of ${}^q\Delta$ consisting of elements of the form $(0, \delta, n)$, with $n \geq 0$. We define Λ' as the collection of subsets $L \subset {}^q\Delta_{\mathrm{exc}}$ such that:

- for all $n \in \mathbb{Z}$, there exists $\delta(n) \in \mathscr{E}$ such that for all δ, with $(0, \delta, n) \in L$, $\delta \geq \delta(n)$.

We can complete the algebra $H({}^q\mathcal{T}_\bullet)$ with respect to Λ', just as we complete the Hall algebra of perverse coherent sheaves with respect to Λ. We denote this completion by $H({}^q\mathcal{T}_\bullet)_{\Lambda'}$.

The elements $1_{{}^q\mathcal{T}_\bullet}$ and $1^{\mathcal{O}}_{{}^q\mathcal{T}_\bullet}$ belong $H({}^q\mathcal{T}_\bullet)_{\Lambda'}$ by Proposition 4.6.5 and duality. The element $\mathscr{H}^{\#}_{\mathrm{exc}}$ also belongs to $H({}^q\mathcal{T}_\bullet)_{\Lambda'}$ by running a similar proof to the one above, using (4.4.2). Proposition 4.5.2 now implies $\mathbb{D}'$ defines an isomorphism between $H({}^q\mathcal{T}_\bullet)_{\Lambda'}$ and $H({}^p\mathcal{F}_{\leq 1}[1])_\Lambda$, taking $1_{{}^q\mathcal{T}_\bullet}$ to $1_{{}^p\mathcal{F}_{\leq 1}[1]}$ and $1^{\mathcal{O}}_{{}^q\mathcal{T}_\bullet}$ to $1^{\mathcal{O}}_{{}^p\mathcal{F}_{\leq 1}[1]}$.

Going back to $H({}^p\mathcal{A}_{\leq 1})_\Lambda$, the remark above implies the identity

$$ {}^p\mathscr{H}_{\leq 1} = 1_{{}^p\mathcal{F}_{\leq 1}[1]} * \mathbb{D}'(\mathscr{H}^{\#}_{\mathrm{exc}}) * \mathscr{H}_{\leq 1} * 1^{-1}_{{}^p\mathcal{F}[1]_{\leq 1}}. \tag{4.6.9} $$

What keeps us from simply applying the integration morphism I_Λ is that, although $\mathbb{D}'(\mathscr{H}^{\#}_{\mathrm{exc}})$ and $\mathscr{H}_{\leq 1}$ are regular, $1_{{}^p\mathcal{F}_{\leq 1}[1]}$ is not.

4.7 A STABILITY CONDITION

We want to proceed analogously as in [Bri11, Section 6.3], proving that

$$\mathrm{I}_\Lambda({}^p\mathscr{H}_{\leq 1}) = \mathrm{I}_\Lambda(\mathbb{D}'(\mathscr{H}^{\#}_{\mathrm{exc}})) \cdot \mathrm{I}_\Lambda(\mathscr{H}_{\leq 1})$$

holds nevertheless. The key is to show that $(\mathbb{L}-1)\cdot \log(1_{{}^p\mathcal{F}_{\leq 1}[1]}) \in \mathrm{H}_{\mathrm{reg}}({}^p\mathcal{A})$. This can be achieved by constructing an appropriate stability condition such that ${}^p\mathcal{F}_{\leq 1}[1]$ manifests as the set of objects of some fixed slope. For convenience we work within the category ${}^p\mathcal{A}_{\mathrm{exc}}$, whose objects are those perverse coherent sheaves $\mathrm{P} \in {}^p\mathcal{A}_{\leq 1}$ whose pushforward to X is supported on points (in other words such a P is of class $(0, \delta, n)$, for some $\delta \in \mathrm{N}_1(\mathrm{Y}/\mathrm{X})$ and $n \in \mathbb{Z}$). We define a stability condition μ, taking values in the ordered set $\{1, 2\}$ as follows.

$$(0, \delta, n) \longmapsto \begin{cases} 1 \text{ if } \delta \geq 0 \\ 2 \text{ if } \delta < 0. \end{cases}$$

It is immediate that μ is indeed a weak stability condition (in the sense of [JS08, Definition 3.5]), as the only axiom one needs to check is the (weak) see-saw property.

4.7.1 LEMMA – The set of μ-semistable objects of slope $\mu = 2$ is ${}^p\mathcal{F}_{\leq 1}[1]$. The set of μ-semistable objects with $\mu = 1$ is ${}^p\mathcal{T}_{\mathrm{exc}}$.

Recall that an object P is said to be *semistable* if for all proper subobjects $\mathrm{P}' \subset \mathrm{P}$ we have $\mu(\mathrm{P}') \leq \mu(\mathrm{P}/\mathrm{P}')$.

Proof: Let P be any semistable perverse coherent sheaf. Consider the torsion torsion-free exact sequence

$$F[1] \hookrightarrow P \twoheadrightarrow T.$$

If $F[1] \neq 0$ and $T \neq 0$ then, by semistability, $2 = \mu(F[1]) \leq \mu(T) = 1$ which is impossible. Thus a semistable object must be either torsion or torsion-free.

On the other hand, as ${}^p\mathcal{F}_{\leq 1}[1]$ is stable under quotients and ${}^p\mathcal{T}_{\text{exc}}$ is stable under subobjects we conclude. ■

The last property we need is permissibility, in the sense of [Joy07, Definition 4.7].

4.7.2 PROPOSITION – The stability condition μ is permissible.

Proof: The first fact we check is that the category ${}^p\mathcal{A}_{\text{exc}}$ is noetherian. More generally, this follows from Noetherianness of ${}^p\mathcal{A}$. The latter can be seen as a consequence of [Ber04], as ${}^p\mathcal{A}$ is equivalent to the category of finitely generated modules over a noetherian coherent $\mathcal{O}_X$-algebra.

Now we want to check that if $P \in {}^p\mathcal{A}_{\text{exc}}$ and $[P] = 0$ in $N_{\leq 1}(Y)$ then $P = 0$. By pushing forward via f we have that $[Rf_*P] = 0$ and as $Rf_*P \in \text{Coh}(X)$ it follows that $Rf_*P = 0$. Now, from Leray's spectral sequence we obtain that $f_*H^{-1}(P) = f_*H^0(P) = 0$. Thus, if $p = -1$, $P = H^{-1}(P)[1]$ and, if $p = 0$, $P = H^0(P)$. In either case we reduce to dealing with a coherent sheaf and so $P = 0$.

Let now ${}^p\mathcal{A}_\alpha(i)$ be the subset of ${}^p\mathfrak{A}_{\text{exc}}(\mathbb{C})$ consisting of perverse coherent sheaves which are of numerical class α and semistable with $\mu = i$. We now check that these subsets are constructible.

In light of Lemma 4.5.1 and Lemma 4.7.1, what remains to be proved is that, given a δ and an n, the stack ${}^p\mathfrak{T}_{(0,\delta,n)}$ is of finite type. This is the content of the following lemma.

To finish, we show that μ is artinian. Consider a chain of subobjects

$$\cdots \hookrightarrow P_2 \hookrightarrow P_1$$

with $\mu(P_{n+1}) \geq \mu(P_n/P_{n+1})$. Let $P' \hookrightarrow P$ be any two consecutive elements in the chain above and let Q be the quotient P'/P so that we have an exact sequence

$$P' \hookrightarrow P \twoheadrightarrow Q$$

with $\mu(P') \geq \mu(Q)$, which corresponds to the relation $(\delta', n') + (\delta_q, n_q) = (\delta, n)$ in ${}^p\Delta$. As the sheaves we are considering have support contracted by f we see that $n, n', n_q \geq 0$ hence we can assume (by going further down the chain if necessary) that $n = n'$, which in turn implies $n_q = 0$.

When $p = -1$, this implies that $Q \in {}^p\mathcal{F}_{\leq 1}[1]$ and $\mu(Q) = 2$. As a consequence, we have $\mu(P') = \mu(P) = 2$ and so $\delta, \delta' < 0$. Finally, as $\delta' \geq \delta$, we can assume $\delta' = \delta$ and so $\delta_q = 0$, from which we gather that $Q = 0$, which concludes the proof.

When $p = 0$, the condition $n_q = 0$ implies $Q \in {}^p\mathcal{T}_{\mathrm{exc}}$. Glancing at the cohomology sheaves long exact sequence we see that $P' \to P$ is an isomorphism on H^{-1} and an injection on H^0. If we denote $\delta_0 = \mathrm{ch}_2(H^{-1}(P))$, we see that $\delta_0 \leq \delta' \leq \delta$. Thus, again by descending further down the chain if necessary, we can assume $\delta = \delta'$ and we are done.

■

4.7.3 LEMMA – Let $\delta \in N_1(Y/X)$ and let $n \in \mathbb{Z}$. Then, the stack ${}^p\mathfrak{T}_{(0,\delta,n)}$ is of finite type.

Proof: We can use the criterion found for example in [HL10, Lemma 1.7.6]. Let S be a finite type scheme and let $T \in \mathrm{Coh}(S \times Y)$ be a flat family of sheaves such that for any closed point $s \in S$ the restriction to the fibre T_s lies in ${}^p\mathcal{T}_{\mathrm{exc}}$ and is of class $(0, \delta, n)$. We show that there exists a sheaf surjecting onto all the T_s. It is sufficient to prove that T_s is generated by global sections, as then the sheaf $H^0(Y, T_s) \otimes_{\mathbb{C}} \mathcal{O}_Y$ will surject onto T_s and $\dim H^0(Y, T_s) = n$ is independent of s.

Let I and C be respectively the image and the cokernel of the evaluation morphism $H^0(Y, T_s) \otimes_{\mathbb{C}} \mathcal{O}_Y \to T_s$. The sheaf I belongs to ${}^p\mathcal{T}$ and together with the exact sequence $I \hookrightarrow T_s \twoheadrightarrow C$ we have

$$f_*I \hookrightarrow f_*T_s \twoheadrightarrow f_*C.$$

As f_*T_s is supported on points, the morphism $H^0(X, f_*T_s) \otimes_{\mathbb{C}} \mathcal{O}_X \to f_*T_s$ is surjective, which (by adjunction) implies that $f_*I \to f_*T_s$ is surjective, which by the above exact sequence implies that $f_*C = 0$. As $R^1 f_*C = 0$ as well, by the properties of perverse coherent sheaves it follows that $T_s \to C$ is the zero morphism, which in turn implies $C = 0$. Hence the claim. ■

4.7.4 PROPOSITION – In $H({}^p\mathcal{A}_{\leq 1})_\Lambda$ we have $1_{{}^p\mathcal{F}_{\leq 1}[1]} = \exp(\epsilon)$, with $\eta = (\mathbb{L} - 1) \cdot \epsilon \in H_{\mathrm{reg}}({}^p\mathcal{A}_{\leq 1})_\Lambda$ a regular element. Furthermore the automorphism

$$\mathrm{Ad}_{1_{{}^p\mathcal{F}_{\leq 1}[1]}} : H({}^p\mathcal{A}_{\leq 1})_\Lambda \longrightarrow H({}^p\mathcal{A}_{\leq 1})_\Lambda$$

preserves regular elements. The induced Poisson automorphism of $H_{\mathrm{sc}}({}^p\mathcal{A}_{\leq 1})_\Lambda$ is given by

$$\mathrm{Ad}_{1_{{}^p\mathcal{F}_{\leq 1}[1]}} = \exp\{\eta, -\}.$$

Proof: We can draw an argument entirely parallel to the one in Theorem 6.3 and Corollary 6.4 of [Bri11]. The only thing to check here is that the class $[\mathbb{C}^*] \cdot \log(1_{{}^p\mathcal{F}_{\leq 1}[1]})$ is a regular element, which can be done in the subalgebra $H({}^p\mathcal{A}_{\mathrm{exc}})_\Lambda$, exploiting the stability condition we just described. ■

4.8 MAIN IDENTITY

At last, we have all the ingredients to prove our main result. Before we proceed, we must deal with an issue of signs.

The Hilbert scheme $\mathrm{Hilb}_Y(\beta, n)$ comes with two constructible functions which are of interest to us. The first (ν) is Behrend's microlocal function. The second (μ) is the pullback along $\mathrm{Hilb}_Y(\beta, n) \to \mathfrak{A}$ of the Behrend function of the stack $\mathfrak{A}$. Given a homology class β and an integer n, the DT number of class (β, n) is defined to be

$$\mathrm{DT}_Y(\beta, n) := \chi_\nu(\mathrm{Hilb}_Y(\beta, n)) := \sum_{k \in \mathbb{Z}} k \chi_{\mathrm{top}}(\nu^{-1}(k))$$

where χ_{top} is the topological Euler characteristic. We package all these numbers into a generating series

$$\mathrm{DT}(Y) = \sum_{\beta, n} \mathrm{DT}_Y(\beta, n) q^{(\beta, n)}$$

which can be interpreted as a Laurent series according to our definition. As we work with the integration morphism, it is convenient for us to define a variant of the DT series:

$$\underline{\mathrm{DT}}(Y) := I_\Lambda(\mathscr{H}_{\leq 1}).$$

Unpacking the definitions, we see that, if we write

$$\underline{\mathrm{DT}}_Y(\beta, n) := \chi_\mu(\mathrm{Hilb}_Y(\beta, n)) := \sum_{k \in \mathbb{Z}} k \chi_{\mathrm{top}}(\mu^{-1}(k))$$

then

$$\mathrm{I}_\Lambda\left(\mathscr{H}_{\leq 1}\right) = \underline{\mathrm{DT}}(\mathrm{Y}) = \sum_{\beta,n} \underline{\mathrm{DT}}_\mathrm{Y}(\beta,n) q^{(\beta,n)}.$$

Remark 4.8.1. It is shown in [Bri11, Theorem 3.1] that there is a simple relationship between DT and $\underline{\mathrm{DT}}$, given as follows.

$$\underline{\mathrm{DT}}_\mathrm{Y}(\beta,n) = (-1)^n \, \mathrm{DT}_\mathrm{Y}(\beta,n)$$

We pause a moment to notice that on Hilb(Y), not only do we have the pullback of the Behrend function of $\mathfrak{A}$, but also the pullback of the Behrend function of ${}^p\mathfrak{A}$. However, no ambiguity arises, as the two morphisms factor through ${}^p\mathfrak{T}$, which is open in both $\mathfrak{A}$ and ${}^p\mathfrak{A}$.

Proceeding analogously as above, we define the perverse DT series as

$${}^p\underline{\mathrm{DT}}(\mathrm{Y}/\mathrm{X}) := \mathrm{I}_\Lambda({}^p\mathscr{H}_{\leq 1})$$

which can be seen as a sum of perverse DT numbers

$${}^p\underline{\mathrm{DT}}(\mathrm{Y}/\mathrm{X}) = \sum_{\beta,n} {}^p\underline{\mathrm{DT}}_{\mathrm{Y}/\mathrm{X}}(\beta,n) q^{(\beta,n)}$$

where

$${}^p\underline{\mathrm{DT}}_{\mathrm{Y}/\mathrm{X}}(\beta,n) := \chi_\mu\left({}^p\mathrm{Hilb}_{\mathrm{Y}/\mathrm{X}}(\beta,n)\right).$$

For the purpose of this thesis, however, we needn't be concerned with comparing $\chi_\mu({}^p\mathrm{Hilb}_{\mathrm{Y}/\mathrm{X}}(\beta,n))$ with $\chi_\nu({}^p\mathrm{Hilb}_{\mathrm{Y}/\mathrm{X}}(\beta,n))$.

Remark 4.8.2. We point out that the identities we write down below should be interpreted as taking place in the algebra $\mathbb{Q}_\sigma[{}^p\Delta]_\Lambda$, defined in Section 4.6.

We introduce the following sums,

$$\mathrm{DT}_0(\mathrm{Y}) := \sum_{n\in\mathbb{Z}} \mathrm{DT}_\mathrm{Y}(0,n)q^{(0,n)}$$
$$\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y}) := \sum_{\substack{\beta,n\\ f_*\beta=0}} \mathrm{DT}_\mathrm{Y}(\beta,n)q^{(\beta,n)}$$
$$\mathrm{DT}^\vee_{\mathrm{exc}}(\mathrm{Y}) := \sum_{\substack{\beta,n\\ f_*\beta=0}} \mathrm{DT}_\mathrm{Y}(-\beta,n)q^{(\beta,n)}$$

and their $\underline{\mathrm{DT}}$ analogues.

4.8.3 THEOREM – Assume to be working in Situation 2.3.1. The following identity holds.

$$ {}^p\underline{\mathrm{DT}}(\mathrm{Y}/\mathrm{X}) = \frac{\underline{\mathrm{DT}}^\vee_{\mathrm{exc}}(\mathrm{Y})\cdot\underline{\mathrm{DT}}(\mathrm{Y})}{\mathrm{DT}_0(\mathrm{Y})} \tag{4.8.4}$$

Proof: The Poisson bracket on $\mathbb{Q}_\sigma[{}^p\Delta]$ is trivial, so Proposition 4.7.4, together with (4.6.9), yields the identity

$$\mathrm{I}_\Lambda({}^p\mathscr{H}_{\leq 1}) = \mathrm{I}_\Lambda(\mathbb{D}'(\mathscr{H}^\#_{\mathrm{exc}}))\cdot\mathrm{I}_\Lambda(\mathscr{H}_{\leq 1}).$$

The left hand side is equal to ${}^p\underline{\mathrm{DT}}(\mathrm{Y}/\mathrm{X})$ and $\mathrm{I}_\Lambda(\mathscr{H}_{\leq 1}) = \underline{\mathrm{DT}}(\mathrm{Y})$. As remarked in Section 4.1, [Bri11, Lemma 5.5 and Theorem 1.1] tell us how $\mathscr{H}^\#$ is related to DT invariants. In fact, combining these with Remark 4.6.8 we see that

$$\mathrm{I}_\Lambda(\mathbb{D}'(\mathscr{H}^\#_{\mathrm{exc}})) = \frac{\underline{\mathrm{DT}}^\vee_{\mathrm{exc}}(\mathrm{Y})}{\mathrm{DT}_0(\mathrm{Y})}$$

and hence the claim. ∎

Notice that, as we are working with the assumption of Remark 2.3.6, the Theorem above holds for both perversities, hence the series ${}^p\underline{\mathrm{DT}}(Y/X)$ is independent of the perversity p. We will therefore drop the superscript p.

4.9 CONCLUSION

Now that we understand how the category of perverse coherent sheaves relates to DT invariants we can prove our promised formula for flops.

Situation 4.9.1

Recall Situation 2.3.1 and assume moreover $f: Y \to X$ to be an isomorphism in codimension one. Let $f^+: Y^+ \to X$ be the flop of f.

$$\begin{tikzcd} Y \arrow[dr, "f"'] & & Y^+ \arrow[dl, "f^+"] \\ & X & \end{tikzcd}$$

Notice that with these additional assumptions it follows automatically that ${}^p\mathcal{F} = {}^p\mathcal{F}_{\leq 1}$ (for $p = -1, 0$).

Following [Bri02], we know that the variety Y^+ can be constructed as the moduli space of point-like objects of ${}^{-1}\mathrm{Per}(Y/X) = {}^{-1}\mathcal{A}$, the category of perverse coherent sheaves with minus one perversity. The pair (Y^+, f^+) satisfies the same assumptions as (Y, f), so the categories of perverse coherent sheaves ${}^q\mathrm{Per}(Y^+/X) = {}^q\mathcal{A}^+$ (for $q = -1, 0$) make sense as well. Moreover, Bridgeland proved that there is a derived equivalence Φ (with inverse Ψ) between Y and Y^+ restricting to an equivalence

$$\Phi\colon {}^{-1}\mathcal{A}^+ \leftrightarrows {}^0\mathcal{A} : \Psi$$

which is the key to transport DT invariants from one side of the flop to the other.

The following lemma will be useful.

4.9.2 LEMMA – Assume to be working in Situation 4.9.1, then $\Phi(\mathcal{O}_{Y^+}) = \mathcal{O}_Y$.

Proof: First of all, it is shown in [Bri02, (4.4)] that the equivalence Φ commutes with pushing down to X. The object $\Phi(\mathcal{O}_{Y^+}) =: L$ is a line bundle as, for any closed point $y \in Y$, the complex

$$\mathrm{RHom}_Y(\Phi(\mathcal{O}_{Y^+}), \mathcal{O}_y) = \mathrm{RHom}_{Y^+}(\mathcal{O}_{Y^+}, \Psi(\mathcal{O}_y)) = \mathrm{RHom}_X(\mathcal{O}_X, Rf^+_*\Psi(\mathcal{O}_y)) = \mathrm{RHom}_X(\mathcal{O}_X, Rf_*\mathcal{O}_y)$$

is concentrated in degree zero and has dimension one. The bundle L pushes down to the structure sheaf, $Rf_*L = \mathcal{O}_X$. By adjunction, morphisms $\mathcal{O}_X \to Rf_*L$ correspond to morphisms $\mathcal{O}_Y \to L$, so that we deduce the existence of a non-zero section of L. Using Grothendieck duality for f, we see that $Rf_*L^\vee = Rf_*\underline{\mathrm{RHom}}(L, f^!\mathcal{O}_X) = (Rf_*L)^\vee = \mathcal{O}_X$, hence $L^\vee$ has a non-zero section as well. As Y is proper and integral, it follows that $L = \Phi(\mathcal{O}_{Y^+})$ must be the structure sheaf $\mathcal{O}_Y$. ■

Gathering all the results so far, the only task left to accomplish is to compare the generating series for the perverse DT invariants on both sides of the flop: $\underline{\mathrm{DT}}(Y/X), \underline{\mathrm{DT}}(Y^+/X)$.

The functor Φ induces an isomorphism between the numerical K-groups of Y and Y^+, which restricts to an isomorphism

$$\phi\colon N_{\leq 1}(Y^+) \leftrightarrows N_{\leq 1}(Y) : \psi.$$

We can sharpen this result, by noticing that a class $(\beta, n) \in N_{\leq 1}(Y^+)$ is sent to $(\varphi(\beta), n)$, where φ can be described as follows. The smooth locus U of X is a common open subset of both Y and Y^+. By the Gysin exact sequence, we have an identification between the numerical groups of divisors of Y and Y^+, via pulling back to U. The inverse of the transpose of this identification is precisely φ, as the equivalence Φ restricts to the identity on U.

As the Fourier-Mukai equivalence Φ is an exact functor, and in light of Lemma 4.9.2, we deduce an isomorphism of perverse Hilbert schemes ${}^{-1}\mathrm{Hilb}(Y^+/X) \simeq {}^{0}\mathrm{Hilb}(Y/X)$. We can sharpen this result by noticing that, for a class $(\beta, n) \in N_1(Y^+) \oplus \mathbb{Z}$, we have

$$ {}^{-1}\mathrm{Hilb}_{Y^+/X}(\beta, n) \simeq {}^{0}\mathrm{Hilb}_{Y/X}(\varphi(\beta), n). $$

Taking weighted Euler characteristics and summing over all β's and n's we obtain

$$ \sum_{\beta,n} \underline{\mathrm{DT}}_{Y^+/X}(\beta, n) q^{(\beta,n)} = \sum_{\beta,n} \underline{\mathrm{DT}}_{Y/X}(\varphi(\beta), n) q^{(\beta,n)} $$

which can be rephrased as a theorem.

4.9.3 THEOREM – Assume to be working in Situation 4.9.1. Then, identifying variables via ϕ, the following identity holds.

$$ \underline{\mathrm{DT}}(Y^+/X) = \underline{\mathrm{DT}}(Y/X) $$

The identity (★) promised in the overview chapter now follows.

4.9.4 COROLLARY – The following identity holds.

$$ \mathrm{DT}^{\vee}_{\mathrm{exc}}(Y^+)\,\mathrm{DT}(Y^+) = \mathrm{DT}^{\vee}_{\mathrm{exc}}(Y)\,\mathrm{DT}(Y) $$

Concretely, for a class $\beta = (\gamma, \delta) \in N_1(X) \oplus N_1(Y/X)$ and an integer n we have

$$\sum_{\substack{\delta_1+\delta_2=\delta \\ n_1+n_2=n}} DT_{Y^+}(0, -\delta_1, n_1) DT_{Y^+}(\gamma, \delta_2, n_2) - DT_Y(0, -\varphi(\delta_1), n_1) DT_Y(\gamma, \varphi(\delta_2), n_2) = 0.$$

Proof: The hard work is done, as we already have Theorem 4.8.3. To prove this last identity we first observe that $DT_0(Y)$ is an expression depending only on the topological Euler characteristic of Y [BF08]. A result of Batyrev [Bat99] tells us that $\chi_{top}(Y) = \chi_{top}(Y^+)$, so that the combination of Theorem 4.9.3, Theorem 4.8.3 and Remark 4.8.1 imply the desired identity. ∎

5

CREPANT RESOLUTION CONJECTURE

In this chapter we present our proof of the projective case of the Crepant Resolution Conjecture for Donaldson-Thomas invariants, as stated in [BCY12, Conjectures 1 & 2].

5.1 THE EQUIVALENCE BETWEEN $\mathrm{Per}(Y/X)$ AND $\mathrm{Coh}(\mathcal{X})$

We work in the following setup.

Situation 5.1.1

Let $\mathcal{X}$ be a smooth, quasi-projective, Deligne-Mumford stack of dimension n. Assume the canonical bundle $\omega_{\mathcal{X}}$ to be Zariski-locally trivial and denote by X *the coarse moduli space of $\mathcal{X}$.*

Remark 5.1.2. The bundle $\omega_{\mathcal{X}}$ on $\mathcal{X}$ is *Zariski*-locally trivial if there exists a Zariski open cover $\mathcal{X}' \to \mathcal{X}$ (where we allow $\mathcal{X}'$ to be a *stack*) such that the restriction $\omega_{\mathcal{X}}|\mathcal{X}'$ is trivial. This is a technical condition which, by working locally on the coarse space X, allows us to reduce to the setting of [BKR01]. In fact, in the case where $\mathcal{X} = [V/G]$, it amounts to requiring that the canonical bundle of V be G-equivariantly locally trivial. This condition seems to be missing in [CT08].

It is beneficial to recall the framework of [CT08]. A candidate for a resolution of X (and a replacement for the equivariant Hilbert scheme found in [BKR01]) is given by the irreducible component Y of the Hilbert scheme $\mathrm{Hilb}(\mathcal{X})$ containing the non-stacky points of $\mathcal{X}$.[1] The morphism $g : \mathcal{X} \to X$ induces a morphism $f : Y \to X$ and we draw a diagram.

$$\begin{tikzcd} & Y \times \mathcal{X} \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_{\mathcal{X}}"] & \\ Y \arrow[dr, "f"'] & & \mathcal{X} \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

Under the additional assumption that $Y \times_X Y$ is at most of dimension $n + 1$ it is proved in [CT08] that Y is smooth and that f is a crepant resolution. Furthermore, the scheme Y represents a moduli functor and its corresponding universal object is a quotient $\mathcal{O}_{Y \times \mathcal{X}} \twoheadrightarrow \mathcal{O}_{\mathcal{Z}}$. Finally, it is shown that one has a Fourier-Mukai equivalence $D(Y) \simeq D(\mathcal{X})$ with kernel given by $\mathcal{O}_{\mathcal{Z}}$.

We recall three key results involved in the proof: the Hilbert scheme $\mathrm{Hilb}(\mathcal{X})$ commutes with étale base-change on X [CT08, Proposition 2.3]; étale-locally on X the space $\mathcal{X}$ is isomorphic to a quotient stack $[V/G]$, with V smooth and affine and G a finite group (whose coarse space is thus the quotient V/G) [AV02, Lemma 2.2.3]; the Hilbert scheme of $[V/G]$ is isomorphic to Nakamura's G-equivariant Hilbert scheme $G\text{-}\mathrm{Hilb}(V)$ [CT08, Lemma 2.2]. Exploiting these facts one may reduce to [BKR01], as checking that the given kernel produces an equivalence may be done locally [CT08, Proposition 3.3].

Remark 5.1.3. As is usual with integral transforms, the kernel $\mathcal{O}_{\mathcal{Z}}$ may be interpreted as giving a functor in two different directions. The standard Mukai-implies-McKay convention is to take $\mathcal{O}_{\mathcal{Z}}$ to define a functor $\Phi \colon D(Y) \to D(\mathcal{X})$ [BKR01; CT08]. To deal with a technical issue (caused by the previous chapter), we will

1 It is probably helpful to remark that for a stack $\mathcal{X}$ there might be some ambiguity in the term *Hilbert scheme* (see [Ryd11]). However, we shall always interpret Hilbert schemes as Quot functors, which for Deligne-Mumford stacks were studied by Olsson and Starr [OS03].

also consider $\hat{\Phi} = \mathbb{D}\Phi\mathbb{D}\colon D(Y) \to D(\mathcal{X})$, where $\mathbb{D} = R\underline{\mathrm{Hom}}(-, \mathcal{O})$ is the duality functor. We denote by Ψ the inverse of Φ and by $\hat{\Psi}$ the inverse of $\hat{\Phi}$. When Y and X are projective, the relationship between Φ and $\hat{\Phi}$ is quite simple, as $\hat{\Psi}$ is given by the Fourier-Mukai transform with kernel $\mathcal{O}_{\mathcal{Z}}$ (this is a standard consequence of [BBHR09, Propositions 1.13 and 1.15]).

It is in some sense a reflection of the ambiguity revolving around the kernel $\mathcal{O}_{\mathcal{Z}}$ that we consider perverse coherent sheaves of both perversities: ${}^{p}\mathrm{Per}(Y/X)$, with $p = -1, 0$.

The rest of this section is devoted to the proof of the following statement.

5.1.4 THEOREM – Assume to be working in Situation 5.1.1 and assume in addition f to have relative dimension at most one. Then the equivalence Φ between $D(Y)$ and $D(\mathcal{X})$ restricts to an equivalence of abelian categories between ${}^{0}\mathrm{Per}(Y/X)$ and $\mathrm{Coh}(\mathcal{X})$, while the equivalence $\hat{\Phi}$ restricts to an equivalence between ${}^{-1}\mathrm{Per}(Y/X)$ and $\mathrm{Coh}(\mathcal{X})$.

Remark 5.1.5. Notice that the condition $\dim Y \times_X Y \leq n+1$ follows automatically from the condition on the fibres of f.

In particular ${}^{0}\mathrm{Per}(Y/X)$ is equivalent to ${}^{-1}\mathrm{Per}(Y/X)$. We also point out that the composition $\hat{\Phi}\Phi^{-1}$ gives a non-trivial autoequivalence of $D(\mathcal{X})$, which seems related to the *window shifts* of Donovan-Segal [DS12]. It might be worthwhile to compute this equivalence in explicit examples.

Let us now begin the proof of the theorem, which will be divided into small steps. We start by considering Φ.

STEP 1 Given an object of the derived category, membership of either of the categories in question can be checked étale-locally on X [Ber04, Proposition 3.1.6]. Thus, by base-changing over étale patches of X, we can reduce to the case where X is affine and furthermore $\mathcal{X} = [V/G]$ with V a smooth affine scheme and G finite. Moreover, the functors Φ and Ψ (being Fourier-Mukai) commute with this base-change [BBHR09, Proposition 6.1].

STEP 2 It suffices to prove $\Psi(\mathrm{Coh}(\mathcal{X})) \subset {}^0\mathrm{Per}(Y/X)$. In fact, an inclusion $\mathcal{A} \subset \mathcal{A}'$ of two hearts of bounded t-structures in a triangulated category is necessarily an equality (it is a consequence of the uniqueness of the cohomological filtrations).

STEP 3 To prove the mentioned inclusion we will exhibit two systems of generators (see definition below), one for ${}^0\mathrm{Per}(Y/X)$ and one for $\mathrm{Coh}(\mathcal{X})$, and show that elements of the first system are sent to the second.

5.1.6 DEFINITION – Let D be a triangulated category and let $\mathcal{A}$ be the heart of a bounded t-structure. A collection $\mathscr{P}$ of objects of $\mathcal{A}$ is a *system of projective generators* if, for all $A \in \mathcal{A} \setminus \{0\}$ and all $P \in \mathscr{P}$, $\mathrm{Ext}^\bullet_{\mathsf{D}}(P, A)$ is concentrated in degree zero and for all $A \in \mathcal{A}$ there exists $P_A \in \mathscr{P}$ such that $\mathrm{Hom}_{\mathsf{D}}(P_A, A) \neq 0$.

By [Ber04, Lemma 3.2.4], when X is affine, we have a system of generators $\mathscr{P}$ for ${}^0\mathrm{Per}(Y/X)$ consisting of vector bundles P such that

- $R^1 f_* P = 0$,
- $P^\vee$ is generated by global sections.

For $\mathrm{Coh}(\mathcal{X})$ we also have a nice system of generators.

5.1.7 LEMMA – The collection $\mathscr{Q}$ of vector bundles on $\mathcal{X}$ is a system of generators for $\mathrm{Coh}(\mathcal{X})$.

Proof: As we are working in the case $\mathcal{X} = [V/G]$, it is easy to reduce the problem to bundles on V. In fact, coherent sheaves on $\mathcal{X}$ are G-equivariant coherent sheaves on V.

Given an equivariant vector bundle P and an equivariant sheaf E on V we have that $G\text{-}\mathrm{Ext}^i_V(P,E) = \mathrm{Ext}^i_V(P,E)^G$, where the latter is the G-invariant part [BKR01, Section 4.1]. As V is affine, these groups vanish for $i > 0$.

Fix now an equivariant sheaf E, we want to find an equivariant vector bundle P such that $\mathrm{Hom}_V(P,E)^G \neq 0$. By [BKR01, Lemma 4.1] $\mathrm{Hom}_V(P,E)$ splits as a direct sum of $\mathrm{Hom}_V(P \otimes \rho, F)^G \otimes \rho$, where ρ ranges among the irreducible representations of G. The claim thus follows as $P \otimes \rho$ is a vector bundle. ∎

STEP 4 We now conclude the proof by showing that elements of $\mathscr{P}$ are sent to elements of $\mathscr{Q}$. First we remark that we can check whether a complex on $\mathcal{X} = [V/G]$ is a vector bundle by pulling back to the étale atlas $V \to [V/G]$. Thus, if $P \in \mathscr{P}$, we are interested in the pullback of $\Phi(P)$ to V. This allows us to reduce to the setup of [BKR01], where one has the following diagram.

$$\begin{tikzcd}
& Z \arrow[d, "i"] \arrow[ddl, bend right, "p"'] \arrow[ddr, bend left, "q"] & \\
& Y \times V \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_V"] & \\
Y \arrow[dr, "f"'] & & V \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

Here Z is the universal G-cluster for the action of G on V, q and f are proper and birational, p and g are finite and p is also flat. Moreover, the quotient $\mathcal{O}_{Y\times V} \twoheadrightarrow \mathcal{O}_Z$ is precisely the pullback, under the morphism $Y \times V \to Y \times [V/G] = Y \times \mathcal{X}$, of the universal quotient $\mathcal{O}_{Y\times\mathcal{X}} \twoheadrightarrow \mathcal{O}_{\mathcal{Z}}$, which we used to define Φ. It follows that applying Φ followed by pulling back to V is the same as applying $Rq_* p^*$.

We have thus reduced our final step to checking that, given an element $P \in \mathscr{P}$, the complex Rq_*p^*P is actually a vector bundle.

5.1.8 LEMMA – Let $P \in \mathrm{Coh}(Y)$ satisfy $R^1 f_* P = 0$. Then $Rq_*p^*P \in \mathrm{Coh}(\mathcal{X})$.

Proof: Notice that $Rq_*p^*P = R\pi_{V,*}i_*p^*P = R\pi_{V,*}(\pi_Y^*P \otimes \mathcal{O}_Z)$, where we made the standard identification $\mathcal{O}_Z = i_*\mathcal{O}_Z$. We point out that, as a consequence of our assumption on f, $\pi_{V,*}$ is of homological dimension at most one (we remind the reader that we work under the reduction done in the Step 1, in particular X is affine).

By tensoring the quotient $\mathcal{O}_{Y\times V} \twoheadrightarrow \mathcal{O}_Z$ with π_Y^*P we produce a surjection $\pi_Y^*P \twoheadrightarrow \pi_Y^*P \otimes \mathcal{O}_Z$. Applying $\pi_{V,*}$ yields a surjection $R^1\pi_{V,*}\pi_Y^*P \twoheadrightarrow R^1\pi_{V,*}(\pi_Y^*P \otimes \mathcal{O}_Z)$. But $R^1\pi_{V,*}\pi_Y^*P = H^1(Y,P) \otimes_{\mathbb{C}} \mathcal{O}_V$ and $H^1(Y,P) = 0$ as $R^1 f_*P = 0$, hence the claim. ■

5.1.9 LEMMA – Let $P \in \mathscr{P}$, then Rq_*p^*P is a vector bundle on V.

Proof: We know that the dual of P is generated by global sections, hence there exists a short exact sequence

$$K \hookrightarrow \mathcal{O}_Y^{\oplus m} \twoheadrightarrow P^\vee.$$

From the fact that P and $\mathcal{O}_Y$ are vector bundles it follows that K is also a vector bundle. We therefore have a dual sequence

$$P \hookrightarrow \mathcal{O}_Y^{\oplus m} \twoheadrightarrow K^\vee.$$

It follows from the previous lemma, plus the fact that $q_*\mathcal{O}_Z = \mathcal{O}_V$, that applying Rq_*p^* yields an exact sequence

$$q_*p^*P \hookrightarrow \mathcal{O}_V^{\oplus m} \twoheadrightarrow q_*p^*K^\vee.$$

To prove our claim it suffices to check that $\mathrm{Ext}^1_V(q_* p^* P, M) = 0$ for all modules M on V. By the above short exact sequence this is the same as showing that $\mathrm{Ext}^2_V(q_* p^* K^\vee, M) = 0$ for all modules M. Using Grothendieck duality [Har66; Lip09] for q we have

$$\begin{aligned}\mathrm{Ext}^2_V(q_* p^* K^\vee, M) &= \mathrm{Ext}^2_Z(p^* K^\vee, q^! M)\\ &= H^2(Z, p^* K \otimes q^! M).\end{aligned}$$

The scheme Z admits a finite and flat map to a smooth variety ($f : Z \to Y$) thus it is Cohen-Macaulay. Moreover, as $\dim Z - \dim V = 0$ and q is of finite tor-dimension, the complex $q^! M$ is concentrated in non-positive degrees. As our assumption on f implies that $H^i(Z, E) = 0$ for all $i > 1$ and all sheaves E, the hypercohomology spectral sequence tells us that $H^2(Z, p^* K \otimes q^! M) = 0$. Hence we are done. ∎

The previous lemma concludes the first half of the proof. As is often the case, the second half is much shorter than the first. In fact, to prove the statement for ${}^{-1}\mathrm{Per}(Y/X)$ and Φ, one need only notice the following:

- $\hat{\Phi} = \mathbb{D}\Phi\mathbb{D}$,
- the dual system $\mathscr{P}^\vee = \mathbb{D}\mathscr{P}$ is a system of generators for ${}^{-1}\mathrm{Per}(Y/X)$ [Ber04, p. 3.2.3],
- the system $\mathscr{Q}$ is self-dual $\mathbb{D}\mathscr{Q} = \mathscr{Q}$.

This concludes the proof and we can now move on to comparing the DT invariants of $\mathcal{X}$ and Y.

Remark 5.1.10. For the next section, it will be important to know that $\Phi(\mathcal{O}_Y) = \mathcal{O}_\mathcal{X}$. We already know that $\Phi(\mathcal{O}_Y)$ is a vector bundle given by $Rq_*\mathcal{O}_\mathcal{Z}$. By restricting to the smooth locus of X (viz. to an open where Φ is the identity) we see that $Rq_*\mathcal{O}_\mathcal{Z}$ is in fact a line bundle. In turn this implies that $\Phi(\mathcal{O}_Y) = \mathcal{O}_\mathcal{X}$ as the unit $\mathcal{O}_{\mathcal{X}} \to Rq_*q^*\mathcal{O}_{\mathcal{X}}$ is an isomorphism. The same statement obviously holds for $\hat{\Phi} = \mathbb{D}\Phi\mathbb{D}$ as well.

Remark 5.1.11. It can be useful to know that when Y and X are projective the equivalences described above commute with pushing down to X. For example, let us check that $g_*\Phi = Rf_*$. We have $g_*\Phi = Rf_* Rp_*p^*$. If we proved that $Rp_*\mathcal{O}_{\mathcal{Z}} = \mathcal{O}_Y$, then by the projection formula we would be done. Thankfully, the previous remark together with Remark 5.1.3 already tell us that $Rp_*\mathcal{O}_{\mathcal{Z}} = \Psi(\mathcal{X}) = \mathcal{O}_Y$.

5.2 THE FORMULA FOR DT INVARIANTS

We now impose further restrictions on our spaces.

Situation 5.2.1

Recall Situation 5.1.1 and assume in addition $\mathcal{X}$ to be projective and of dimension three. Assume moreover $\mathcal{X}$ to be Calabi-Yau, i.e. $\omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$ and $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$. Finally, assume the crepant resolution $f : Y \to X$ of the previous section to have relative dimension at most one.

Remark 5.2.2. We follow the convention where a Deligne-Mumford stack is projective if its coarse moduli space is. From the assumptions above it follows that X is of dimension three, projective, Gorenstein with quotient singularities and with trivial canonical bundle. In turn it follows that Y is Calabi-Yau of dimension three and that X has rational singularities, and so $Rf_*\mathcal{O}_Y = \mathcal{O}_X$ [Kov00].

As the functor Φ is more natural from the perspective of the McKay correspondence we shall focus on the zero perversity.

Notation. We will drop the superscript ^{0} from $^0\mathrm{Per}(Y/X) =: \mathrm{Per}(Y/X)$.

Let us recall some definitions we gave earlier. We denote by $N(Y)$ the *numerical K-group* of coherent sheaves of Y. We remind ourselves that we can define a bilinear form on $K_0(\mathrm{Coh}(Y))$

$$\chi(E,F) := \sum_k (-1)^k \dim_{\mathbb{C}} \mathrm{Ext}^k_Y(E,F)$$

and that $N(Y)$ is obtained by quotienting out its radical. Inside $N(Y)$ we can single out $F_1N(Y)$, which is the subgroup generated by sheaves supported in dimensions at most one. We also define $F_{\mathrm{exc}}N(Y)$ to be the subgroup of $F_1N(Y)$ spanned by sheaves with derived pushforward to X supported in dimension zero.

To Y one can also attach the numerical Chow groups $N_*(Y)$, which are the groups of cycles modulo numerical equivalence. We write $N_{\leq 1}(Y) := N_1(Y) \oplus N_0(Y)$ and recall that $N_0(Y) \cong \mathbb{Z}$. In [Bri11, Lemma 2.2] it is shown that the Chern character induces an isomorphism $F_1N(Y) \cong N_{\leq 1}(Y) \cong N_1(Y) \oplus \mathbb{Z}$, which allows us to pass from one group to the other. Using this identification, $F_{\mathrm{exc}}N(Y)$ can be rewritten as

$$F_{\mathrm{exc}}N(Y) = \{(\beta, n) \in N_1(Y) \oplus \mathbb{Z} \,|\, f_*\beta = 0\}$$

where f_* here stands for the proper pushforward on cycles (the subscript exc is short for *exceptional*).

For the orbifold $\mathcal{X}$ we can also define a numerical K-group $N(\mathcal{X})$. Inside it lies $F_0N(\mathcal{X})$, the subgroup spanned by sheaves supported in dimension zero. The two Fourier-Mukai functors

$$\Phi : D(Y) \rightleftarrows D(\mathcal{X}) : \Psi$$

induce an isomorphism on the level of numerical K-groups.

$$\phi : \mathrm{N}(\mathrm{Y}) \rightleftarrows \mathrm{N}(\mathcal{X}) : \psi$$

The group $\mathrm{F}_0\mathrm{N}(\mathcal{X})$ is sent isomorphically to $\mathrm{F}_{\mathrm{exc}}\mathrm{N}(\mathrm{Y})$ via ψ and we define $\mathrm{F}_{\mathrm{mr}}\mathrm{N}(\mathcal{X})$ to be the image under ϕ of $\mathrm{F}_1\mathrm{N}(\mathrm{Y})$ (the subscript mr stands for *multi-regular* [BCY12]). As it will be useful later, we analogously define $\mathrm{F}_0\mathrm{N}(\mathrm{Y})$ to be the subgroup spanned by sheaves supported in dimension zero.

$$\begin{tikzcd}
\mathrm{F}_0\mathrm{N}(\mathrm{Y}) \subset \mathrm{F}_{\mathrm{exc}}\mathrm{N}(\mathrm{Y}) \arrow[d, "\wr"] & \subset & \mathrm{F}_1\mathrm{N}(\mathrm{Y}) \arrow[d, "\wr"] \\
\mathrm{F}_0\mathrm{N}(\mathcal{X}) & \subset & \mathrm{F}_{\mathrm{mr}}\mathrm{N}(\mathcal{X}) \subset \mathrm{F}_1\mathrm{N}(\mathcal{X})
\end{tikzcd}$$

It is important now to duplicate Remark 4.8.1 and the discussion preceding it for the orbifold $\mathcal{X}$. We recall that DT invariants are defined using Behrend's microlocal function ν. However, from the previous chapter we know the importance of also being able to work with a different constructible function. The Hilbert scheme $\mathrm{Hilb}(\mathcal{X})$ admits a forgetful morphism to the stack of coherent sheaves on $\mathcal{X}$. We denote the pullback along this morphism of the Behrend function by μ. As in Section 4.8, we define the *DT number of class* α, for $\alpha \in \mathrm{F}_1\mathrm{N}(\mathcal{X})$, of $\mathcal{X}$ to be

$$\mathrm{DT}_{\mathcal{X}}(\alpha) = \chi_{\mathrm{top}}\left(\mathrm{Hilb}_{\mathcal{X}}(\alpha), \nu\right)$$

and we also define the underlined analogue

$$\underline{\mathrm{DT}}_{\mathcal{X}}(\alpha) = \chi_{\mathrm{top}}\left(\mathrm{Hilb}_{\mathcal{X}}(\alpha), \mu\right).$$

We package all these numbers into generating series $\mathrm{DT}(\mathcal{X})$, $\underline{\mathrm{DT}}(\mathcal{X})$.

Remark 5.2.3. Remark 4.8.1 remains valid for $\mathcal{X}$, that is the following identity holds

$$\underline{\mathrm{DT}}_{\mathcal{X}}(\alpha) = (-1)^{\chi(\alpha)} \mathrm{DT}_{\mathcal{X}}(\alpha).$$

To prove this, one can choose an appropriate divisor D on the coarse space X, and its pullback to $\mathcal{X}$ plays the role of H in the proof of [Bri11, Theorem 3.1]. The affine U can then be chosen to be an étale open in $\mathcal{X}$, so that [Bri11, Lemma 3.2] can be applied.

Passing over to Y now, recall that the structure sheaf $\mathcal{O}_Y$ belongs to the category $\mathrm{Per}(Y/X)$ of perverse coherent sheaves. Given a class $\alpha \in F_1N(Y)$, we denote $\text{P-Hilb}_{Y/X}(\alpha)$ the corresponding perverse Hilbert scheme parameterising quotients of $\mathcal{O}_Y$ in $\mathrm{Per}(Y/X)$ of numerical class α. We can define a *perverse* DT number of Y over X of class α as the weighted Euler characteristic

$$\underline{\mathrm{DT}}_{Y/X}(\alpha) := \chi_{\mathrm{top}}\left(\text{P-Hilb}_{Y/X}(\alpha), \mu\right)$$

where, again, μ is the pullback of the Behrend function of the stack of perverse coherent sheaves on Y. We also collect these numbers into a partition function

$$\underline{\mathrm{DT}}(Y/X) := \sum_{\alpha \in F_1N(Y)} \underline{\mathrm{DT}}_{Y/X}(\alpha) q^{\alpha}.$$

At this point, we should mention again Remark 4.8.2, which says that all the identities we write down below take place in $\mathbb{Q}_\sigma[{}^p\Delta]_\Lambda$.

5.2.4 THEOREM – Assume to be working in Situation 5.2.1. The following formula holds

$$\underline{\mathrm{DT}}_{\mathrm{mr}}(\mathcal{X}) = \underline{\mathrm{DT}}(\mathrm{Y}/\mathrm{X})$$

after an identification of variables via ϕ. In particular, for each $\alpha \in \mathrm{F}_1\mathrm{N}(\mathrm{Y})$

$$\underline{\mathrm{DT}}_{\mathrm{Y}/\mathrm{X}}(\alpha) = \underline{\mathrm{DT}}_{\mathcal{X}}(\phi(\alpha)).$$

Proof: Form the previous section we know that the Fourier-Mukai equivalences Φ and Ψ restrict to an equivalence of abelian categories between $\mathrm{Per}(\mathrm{Y}/\mathrm{X})$ and $\mathrm{Coh}(\mathcal{X})$. Using Remark 5.1.10, which tells us that $\Phi(\mathcal{O}_{\mathrm{Y}}) = \mathcal{O}_{\mathcal{X}}$, we have an induced isomorphism of Quot functors (or Hilbert schemes), hence

$$\mathrm{Hilb}_{\mathcal{X}}(\alpha) \simeq \text{P-}\mathrm{Hilb}_{\mathrm{Y}/\mathrm{X}}(\psi(\alpha)).$$

By definition $\psi(\mathrm{F}_{\mathrm{mr}}\mathrm{N}(\mathcal{X})) = \mathrm{F}_1\mathrm{N}(\mathrm{Y})$, from which the result for the generating series follows. ∎

5.2.5 COROLLARY – Assume to be working in Situation 5.2.1 and recall the identification of variables from the previous theorem. The following formula is true

$$\mathrm{DT}_{\mathrm{mr}}(\mathcal{X}) = \frac{\mathrm{DT}^{\vee}_{\mathrm{exc}}(\mathrm{Y})\,\mathrm{DT}(\mathrm{Y})}{\mathrm{DT}_0(\mathrm{Y})} \tag{5.2.6}$$

where

$$\mathrm{DT}_0(\mathrm{Y}) := \sum_{n\in \mathrm{F}_0\mathrm{N}(\mathrm{Y})} \mathrm{DT}_{\mathrm{Y}}(n)q^n,$$

$$\mathrm{DT}^{\vee}_{\mathrm{exc}} := \sum_{\substack{(\beta,n)\in \mathrm{N}_1(\mathrm{Y})\oplus\mathbb{Z}\\ f_*\beta=0}} \mathrm{DT}_{\mathrm{Y}}(-\beta,n)q^{(\beta,n)}.$$

Proof: First we notice that we can get rid of the underlines thanks to Remark 5.2.3. From the previous theorem the statement we wish to prove is equivalent to proving that $\underline{\mathrm{DT}}(Y/X)$ is equal to the right hand side of (5.2.6) (modulo the underlines). What prevents us from simply applying the formula (1.3.1) from the overview chapter is that in the previous chapter the result is, strictly speaking, only proved for the minus one perversity.

Nonetheless, we can verify that the context we work in here satisfies the hypothesis of Remark 2.3.6. The only thing to prove is that the stack of perverse coherent sheaves is locally isomorphic to the stack of coherent sheaves (via an *exact* functor). However, using the Fourier-Mukai equivalence $\hat{\Phi}$, we have that the stack parameterising objects in ${}^{0}\mathrm{Per}(Y/X)$ is isomorphic to the stack parameterising objects in ${}^{-1}\mathrm{Per}(Y/X)$. As $\hat{\Phi}$ is also an exact functor, all the constructions of the previous chapters go through and (1.3.1) does indeed hold. ■

Finally, we relate this Corollary with the Crepant Resolution Conjecture of [BCY12].

5.2.7 COROLLARY – Once again, we assume to be working in Situation 5.2.1, while also bearing in mind the identification of variables from the previous theorem. The following formulae hold.

$$\mathrm{DT}_0(\mathcal{X}) = \frac{\mathrm{DT}^{\vee}_{\mathrm{exc}}(Y)\,\mathrm{DT}_{\mathrm{exc}}(Y)}{\mathrm{DT}_0(Y)}$$

$$\frac{\mathrm{DT}_{\mathrm{mr}}(\mathcal{X})}{\mathrm{DT}_0(\mathcal{X})} = \frac{\mathrm{DT}(Y)}{\mathrm{DT}_{\mathrm{exc}}(Y)}$$

where

$$\mathrm{DT}_0(\mathcal{X}) := \sum_{\alpha \in \mathrm{F}_0\mathrm{N}(\mathcal{X})} \mathrm{DT}_{\mathcal{X}}(\alpha) q^{\alpha},$$

$$\mathrm{DT}_{\mathrm{exc}}(\mathrm{Y}) := \sum_{\substack{(\beta,n) \in \mathrm{N}_1(\mathrm{Y}) \oplus \mathbb{Z} \\ f_*\beta = 0}} \mathrm{DT}_{\mathrm{Y}}(\beta, n) q^{(\beta,n)}.$$

Proof: As previously mentioned, ψ identifies $\mathrm{F}_0\mathrm{N}(\mathcal{X})$ with $\mathrm{F}_{\mathrm{exc}}\mathrm{N}(\mathrm{Y})$, from which we deduce the first identity. The second is obtained by combining the first identity with (5.2.6). ■

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