

ON A CONJECTURE OF MARTON

W. T. GOWERS, BEN GREEN, FREDDIE MANNERS, AND TERENCE TAO

ABSTRACT. We prove a conjecture of K. Marton, widely known as the polynomial Freiman–Ruzsa conjecture, in characteristic 2. The argument extends to odd characteristic, with details to follow in a subsequent paper.

1. INTRODUCTION

In this paper we prove a conjecture of Katalin Marton (see [28]), widely known in the literature as the *polynomial Freiman–Ruzsa conjecture* in characteristic 2. The conjecture has many equivalent forms, one of which is the following statement.

Conjecture 1.1. *Suppose that $A \subset \mathbf{F}_2^n$ is a set with $|A + A| \leq K|A|$. Then A is covered by at most $2K^C$ cosets¹ of some subgroup $H \leq \mathbf{F}_2^n$ of size at most $|A|$.*

Note that if A is covered by at most r cosets of a subgroup H of size at most $|A|$, then $|A + A| \leq \left(\binom{r}{2} + 1\right)|A|$, so Conjecture 1.1 allows one to pass from the property of having a small doubling constant to the property of being contained in a small union of cosets of a subgroup of size at most $|A|$ and back again with at most a polynomial loss in the parameters.

The first result in this direction was due to Imre Ruzsa [28], who obtained an upper bound of $2K^2 2^{K^4}$ in place of the desired $2K^C$. It

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¹The factor of 2 is needed to deal with the case where K is close to 1. If H_0 is a subgroup of $\mathbf{F}_2^n$ and $A \subseteq H_0$ is a subset of size $(1 - \varepsilon)|H_0|$ for some small $\varepsilon > 0$, then in order to obtain the bound $|H| \leq |A|$, we need to pick a subgroup H of H_0 of index 2 and cover A by the two cosets of that subgroup.

was Ruzsa [28], [30, Conjecture 2.2.2] who attributed Conjecture 1.1 to Marton. The upper bound was improved to $O(K^{O(\sqrt{K})})$ in [11], then to $O(\exp(O(\sqrt{K})))$ in [34]. Sanders [32] made a dramatic breakthrough by proving the first bounds of shape $O(\exp(\log^{C_1} K))$, showing that $C_1 = 4 + \varepsilon$ is permissible here. A variant of Sanders' argument due to Konyagin (not published by Konyagin, but described by Sanders in [33], and see also the comments on page 8 of [9]) showed that $C_1 = 3 + \varepsilon$ is permissible. We also remark that Conjecture 1.1 was previously established in the special case when A was a downset in [10]. We refer to [8, 7, 20] for surveys on this conjecture. In the converse direction, by analysing the case when A is a Hamming ball, it was shown in [10, Theorem 1.8] that the constant C in Conjecture 1.1 is necessarily at least 1.46601.

Our main result is the following.

Theorem 1.2. *Conjecture 1.1 is true with $C = 12$.*

An elaboration of our arguments gives corresponding results in in abelian groups of bounded torsion and in particular in vector spaces over $\mathbf{F}_p$ for arbitrary primes p . This is notationally somewhat heavier and makes the underlying ideas slightly harder to appreciate so we give this argument in a separate paper [6].

Since the appearance of the first preprint of this article, Jyun-Jie Liao has given a refinement of the argument that improves the constant $C = 12$ in Theorem 1.2 to $C = 11$ (with corresponding numerical improvements to our other main results and applications), and a further refinement improves this to $C = 9$; see [18].

Applications. It was shown in [9, Theorem 1.11] that Theorem 1.8 (or Theorem 1.2) in characteristic 2 implies the so-called ‘weak’ polynomial Freiman–Ruzsa conjecture over $\mathbf{Z}$, so that is now a theorem as well.

Theorem 1.3. *Let A be a finite subset of $\mathbf{Z}^D$ for some D , and suppose that $|A + A| \leq K|A|$. Then there is some set $A' \subseteq A$, $|A'| \geq K^{-C_1/2}|A|$, with $\dim A' \leq C_2 \log K$, for some absolute² constants $C_1, C_2 > 0$.*

²If one traces through the arguments in [9] (dropping all terms involving the exponent $1 - 1/C_{\text{PFR}}$, and using the constant $C = 11$ from Theorem 1.8 below), one finds that one can take $C_1 = 20C = 220$ and $C_2 = \frac{40}{\log 2}$. The reason we have stated Theorem 1.3 with a $C_1/2$ in the exponent is so that the constants C_1, C_2 here are the same as the ones in [9]. The Lean4 formalization project mentioned below was able to further improve C_1 to 34.

Theorem 1.3 solves a conjecture stated in [3, 9, 24, 26]. We remark that there is also a ‘strong’ PFR conjecture over $\mathbf{Z}$ (though its formulation requires care; see [21] and [23] for more discussion). This remains a challenging open problem.

As mentioned above, the polynomial Freiman–Ruzsa conjecture is equivalent to many other statements in additive combinatorics. We mention a few of these here.

Corollary 1.4. *There is a constant C_3 such that if $f : \mathbf{F}_2^m \rightarrow \mathbf{F}_2^n$ is a function such that the set $\{f(x) + f(y) - f(x + y) : x, y \in \mathbf{F}_2^m\}$ has cardinality at most K , then f may be written as $g + h$, where g is linear and the range of h is a set of cardinality $O(K^{C_3})$.*

The equivalence of this and Theorem 1.2 is due to Ruzsa [30, Conjecture 2.2.3]. Ruzsa did not publish the proof, but he communicated it to the second author and the details may be found in [7]. Here, as usual, $O(X)$ denotes a quantity bounded by CX for some absolute constant C .

Corollary 1.5. *There is a constant C_4 with the following property. Suppose that $f : \mathbf{F}_2^m \rightarrow \mathbf{F}_2^n$ is a function such that, if x, y are selected uniformly at random from $\mathbf{F}_2^m$, $\mathbf{P}(f(x+y) = f(x) + f(y)) \geq 1/K$. Then there is a homomorphism $g : \mathbf{F}_2^m \rightarrow \mathbf{F}_2^n$ such that $\mathbf{P}(f(x) = g(x)) \gg K^{-C_4}$.*

See [31, Theorem 4.1]. The notation $X \gg Y$ means that there is an absolute constant $c > 0$ such that $X \geq cY$.

The next result is a polynomially effective inverse theorem for the Gowers uniformity norm $U^3(\mathbf{F}_2^n)$; see [12, 19, 31].

Corollary 1.6. *There is a constant C_5 with the following property. Let $f : \mathbf{F}_2^n \rightarrow \mathbf{C}$ be a 1-bounded function such that $\|f\|_{U^3(\mathbf{F}_2^n)} \geq 1/K$. Then there exists a quadratic polynomial $P : \mathbf{F}_2^n \rightarrow \mathbf{F}_2$ such that*

$$\left| \frac{1}{2^n} \sum_{x \in \mathbf{F}_2^n} f(x) (-1)^{P(x)} \right| \gg K^{-C_5}.$$

In principle, explicit values for the exponents C_3, C_4, C_5 (and for the unspecified implied constants in the big-O and $\gg$ notations) could be worked out by tracing through the arguments in the references provided, but we will not do so here³.

³The `Lean4` formalization project mentioned below was able bound the range of h in Corollary 1.4 by K^{12} and the lower bound for $\mathbf{P}(f(x) = g(x))$ in Corollary 1.5 by $2^{-172} K^{-146}$, so one can take $C_3 = 12$ and $C_4 = 146$. Corollary 1.6 was not formalized.

For our arguments a particularly relevant equivalent formulation of the PFR conjecture is the entropic version of the conjecture from [9], which we present with explicit constants (Theorem 1.8 below).

Turning to applications of a more miscellaneous nature, a particularly attractive application of Theorem 1.3 is the following result of Mudgal, which is a special case of [25, Theorem 6.2].

Corollary 1.7 (Mudgal). *Let $A \subset \mathbf{R}$ be a finite set with at least two elements. Then either the m -fold sumset mA or the m -fold product set $A^{(m)}$ has cardinality at least $|A|^{f(m)}$, where $f(m) \rightarrow \infty$ as $m \rightarrow \infty$.*

This generalizes a well-known result of Bourgain and Chang [2], who established this when A is a set of integers. More general results of a similar flavour may be found in [25].

The PFR conjecture has several applications to topics in theoretical computer science such as linearity testing (see Corollary 1.5), construction of extractors, communication complexity, locally decodable codes, and non-malleable codes; see [20] for a survey.

Entropy methods. Ruzsa [29] and the fourth author [37] studied entropy analogues of the notion of sumset. In a recent preprint [9], the second, third and fourth authors further developed the use of entropy techniques in additive combinatorics. Entropy methods, as well as several of the results in both [9] and [37], will be used extensively here. For now, we give only a very brief introduction to these concepts, referring the reader to [9, Section 1] or Appendix A for a full discussion.

A crucial notion is the *entropic Ruzsa distance* between random variables. If X, Y are two finitely supported random variables taking values in an additive group G , then we define

$$d[X; Y] := \mathbf{H}[X' - Y'] - \frac{1}{2}\mathbf{H}[X'] - \frac{1}{2}\mathbf{H}[Y'], \quad (1.1)$$

where X', Y' are independent copies of X, Y . In particular we stress that $d[X; Y]$ depends only on the *individual distributions* of X and Y : it does not require them to be independent, or even that they are defined on a common sample space.

The *Ruzsa triangle inequality* states that $d[X; Y] \leq d[X; Z] + d[Z; Y]$, and one can show that $d[X; Y] \geq 0$ and $d[X; Y] = d[Y; X]$ always hold. However, despite $d[-; -]$ being called a distance, we have $d[X; X] = 0$ only when X is uniformly distributed on a coset of a subgroup (see Lemma 2.2 below). Moreover, $d[X; Y] = 0$ can hold even when X and Y are not identically distributed (specifically, when they are uniform on different cosets of the same subgroup).

In this paper (since we will use no other notion of distance) we write $d[X; Y]$ instead of the more cumbersome $d_{\text{ent}}(X, Y)$ from [9]. We also switch to using square brackets, as many of our random variables X, Y are themselves expressions involving parentheses.

One consequence of the investigation in [9] was another equivalent formulation of the PFR conjecture. This is the version we will focus on in the present paper.

Theorem 1.8. *Let $G = \mathbf{F}_2^n$, and suppose that X_1^0, X_2^0 are G -valued random variables. Then there is some subgroup $H \leq G$ such that*

$$d[X_1^0; U_H] + d[X_2^0; U_H] \leq 11d[X_1^0; X_2^0],$$

where U_H denotes the uniform distribution on H . Furthermore, both $d[X_1^0; U_H]$ and $d[X_2^0; U_H]$ are at most $6d[X_1^0; X_2^0]$.

The fact that Theorem 1.8 is equivalent, up to constants, to the characteristic 2 case of Marton’s original conjecture, as well as to the other combinatorial formulations of the polynomial Freiman–Ruzsa conjecture, was established in [9, Section 8]. For the convenience of the reader, we give the proof that Theorem 1.8 implies Theorem 1.2 in Appendix B.

Variants and further remarks. The polynomial Freiman–Ruzsa conjecture has inspired various analogous investigations in other contexts; for a recent example in the symmetric group, see [16].

One could consider strengthening Corollary 1.4 by requiring the function h to take values in a C_6 -fold sumset of the set $\{f(x) + f(y) - f(x + y) : x, y \in \mathbf{F}_2^m\}$, for some absolute constant C_6 . However, this “strong PFR conjecture” is known to be false: see [1] and [36, Section 1.17].

The *polynomial Bogolyubov conjecture* (see e.g., [20, Conjecture 1.5]) asserts that if A is a non-empty subset of $\mathbf{F}_2^n$ of doubling at most K , then $2A - 2A = 4A$ contains a subspace of cardinality $\gg K^{-O(1)}|A|$. In [32] it was shown that $2A - 2A$ contains a subspace of size at least $\gg K^{-O(\log^3 K)}|A|$; see also a recent bilinear extension of this result in [13]. We were unable to use our methods to improve upon these results. However, we can obtain a weak Bogolyubov type result, in which the number of iterates in the sumset is allowed to grow logarithmically in K ; see Remark 2.4.

Let $f(2, K)$ denote the best constant such that every non-empty subset A of $\mathbf{F}_2^n$ of doubling at most K is contained in an affine subspace of cardinality at most $f(2, K)|A|$. It is known that $f(2, K)$ is a piecewise linear function that can be explicitly computed for small K , and has

the asymptotic bounds

$$\frac{2^{2K}}{4K}(1 - o(1)) \leq f(2, K) \leq \frac{2^{2K}}{2K}(1 + o(1))$$

as $K \rightarrow \infty$; see [5] (which builds upon prior work in [4, 10, 17]). The results here do not appear to significantly strengthen these bounds.

A formalization of Theorem 1.2 (as well as the $C = 11$ refinement of Liao), together with Theorem 1.3 and Corollaries 1.4 and 1.5, in the proof assistant language `Lean4` may be found at <https://teorth.github.io/pfr/>.

Notation. All logarithms in this paper will be natural logarithms. We will very extensively use the notation of entropy $\mathbf{H}[-]$ and mutual information $\mathbf{I}[-]$ as well as conditional variants of these concepts. Basic definitions and facts are recalled in Appendix A.

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2. INDUCTION ON ENTROPY DISTANCE

Fix, for the rest of the paper, the group $G = \mathbf{F}_2^n$ and the variables X_1^0, X_2^0 appearing in Theorem 1.8. We shall think of X_1^0, X_2^0 as ‘reference variables’, and not modify them for the rest of the proof; while they do appear in many of the expressions we will need to control, they will play a relatively minor role in the estimates. Our overall strategy can be thought of as a kind of induction on (a slight modification of) the distance $d[X_1; X_2]$, though we will ultimately phrase it as a compactness argument.

For the rest of the paper, η will denote the constant $\frac{1}{9}$. For any two G -valued random variables X_1, X_2 , we introduce the functional

$$\tau[X_1; X_2] := d[X_1; X_2] + \eta d[X_1^0; X_1] + \eta d[X_2^0; X_2]. \quad (2.1)$$

Note that this functional depends only on the distributions p_{X_1}, p_{X_2} of X_1, X_2 , and not on whether X_1, X_2 (or X_1^0, X_2^0) are independent or

dependent (or even on whether they are defined on the same sample space).

Here is the main result from which we will deduce Theorem 1.8.

Proposition 2.1. *Let X_1, X_2 be two G -valued random variables with $d[X_1; X_2] > 0$. Then there are G -valued random variables X'_1, X'_2 such that*

$$\tau[X'_1; X'_2] < \tau[X_1; X_2]. \quad (2.2)$$

If we ignore the two penalty terms involving η on the right-hand side of (2.1), then Proposition 2.1 asserts that if two random variables have a positive Ruzsa distance, then we can ‘improve’ them, finding two new variables X'_1, X'_2 that are closer to each other than the original variables X_1, X_2 . To take care of the penalty terms, we will also need the new variables X'_1, X'_2 to be ‘related’ to the old variables X_1, X_2 in some sense, so that both sets of variables have a comparable Ruzsa distance to the reference variables X_1^0, X_2^0 .

As a starting point for this ‘improvement’ process, we remark that the explicit choice⁴ of random variables $X_1 = X_2^0, X_2 = X_1^0$ gives

$$\begin{aligned} \tau[X_2^0; X_1^0] &= d[X_2^0; X_1^0] + \eta d[X_1^0; X_2^0] + \eta d[X_2^0; X_1^0] \\ &= (1 + 2\eta) d[X_1^0; X_2^0]. \end{aligned} \quad (2.3)$$

In order to deduce Theorem 1.8 from Proposition 2.1 we will use the ‘100 percent’ case of our main theorem, which we note now.

Lemma 2.2. *Suppose that X_1, X_2 are G -valued random variables such that $d[X_1; X_2] = 0$. Then there exists a subgroup $H \leq G$ such that each of p_{X_1}, p_{X_2} is a translate of U_H . In particular, for this H , we have $d[X_1; U_H] = d[X_2; U_H] = 0$.*

This follows from [37, Theorem 1.11(i)] and the triangle inequality; for completeness we record a full proof below.

Proof. Write S_1 for the support of X_1 and S_2 for the support of X_2 . Let X'_1, X'_2 be independent copies of X_1 and X_2 . By (A.4), (A.5), we have

$$\mathbf{H}[X'_1 - X'_2] \geq \mathbf{H}[X'_1 - X'_2 | X'_2] = \mathbf{H}[X'_1 | X'_2] = \mathbf{H}[X'_1]$$

and similarly

$$\mathbf{H}[X'_1 - X'_2] \geq \mathbf{H}[X'_1 - X'_2 | X'_1] = \mathbf{H}[X'_2 | X'_1] = \mathbf{H}[X'_2].$$

⁴It would perhaps be more straightforward to use (X_1^0, X_2^0) but this gives a worse constant.

Since averaging these two inequalities gives $d[X_1; X_2] \geq 0$, equality must hold in each: that is,

$$\mathbf{H}[X'_1 - X'_2 | X'_2] = \mathbf{H}[X'_1 - X'_2] = \mathbf{H}[X'_1 - X'_2 | X'_1].$$

For random variables (A, B) , $\mathbf{H}[A|B] = \mathbf{H}[A]$ if and only if A, B are independent (see (A.4)); i.e., if and only if the conditional random variables $(A|B = b)$ all have the same distribution. Hence, for all $s_1 \in S_1$ and for all $s_2 \in S_2$,

$$p_{X_1 - s_2} = p_{X_1 - X_2} = p_{s_1 - X_2}. \quad (2.4)$$

For a random variable Y supported on a finite set $T \subseteq G$, write $\text{Sym}(Y) := \{t \in G : p_Y = p_{Y+t}\}$. Certainly $\text{Sym}(Y)$ is a subgroup of G , and $\text{Sym}(Y) \subseteq T - T$ (as if $t \notin T - T$ then p_Y, p_{Y+t} have disjoint support) and in particular $\text{Sym}(Y)$ is finite. By (2.4), we have

$$\text{Sym}(X_1) = \text{Sym}(X_2) = \text{Sym}(X_1 - X_2)$$

since $p_{X_1}, p_{-X_2}, p_{X_1 - X_2}$ are all translates of each other. Write H for this common subgroup of G . By (2.4) again, for all $s_2, s'_2 \in S_2$ we have $p_{X_1 - s_2} = p_{X_1 - s'_2}$, so $p_{X_1} = p_{X_1 + s_2 - s'_2}$; hence $S_2 - S_2 \subseteq H$, and therefore $S_2 - S_2 = H$. Similarly, $S_1 - S_1 = H$.

However, $p_{X_1 + U_H} = p_{X_1}$ by definition of H , and so $\mathbf{H}[X_1 + U_H] = \mathbf{H}[X_1]$. Hence, by (A.11),

$$\mathbf{H}[X_1] = \mathbf{H}[X_1 + U_H] \geq \mathbf{H}[U_H] = \log |H|. \quad (2.5)$$

Finally, $\mathbf{H}[X_1] \leq \log |S_1|$ by (A.1), with equality if and only if X_1 is uniform on S_1 . But then, fixing any $a_1 \in S_1$, $S_1 - S_1 \supseteq S_1 - a_1$ and so

$$\log |H| = \log |S_1 - S_1| \geq \log |S_1 - a_1| = \log |S_1| \geq \mathbf{H}[X_1]. \quad (2.6)$$

Comparing (2.5) and (2.6), equality must hold at every step of each. It follows that X_1 is uniform on $S_1 = H + a_1$. By an identical argument, X_2 is uniform on $H + a_2$ for any $a_2 \in S_2$.

This gives the first part of the statement. The second part, namely that $d[X_1; U_H] = 0 = d[X_2; U_H]$, follows immediately from this and the definition of entropic distance. $\square$

Proof of Theorem 1.8, assuming Proposition 2.1. We choose a pair of variables X_1, X_2 (or more precisely, their distributions on G) to be any minimizer⁵ of the functional τ . This optimization problem ranges over the square of the space of probability distributions on the finite set

⁵We remark that an entropy functional minimization was previously employed by the fourth author in [35] to give an alternative proof of the Szemerédi regularity lemma.

G . The usual topology on this space is compact, and it is straightforward that $d[-; -]$ is continuous with respect to this topology, so the minimum exists. By (2.3), we certainly have

$$\tau[X_1; X_2] \leq (1 + 2\eta)d[X_1^0; X_2^0].$$

As (X_1, X_2) minimizes τ , by the contrapositive of Proposition 2.1 we must have $d[X_1; X_2] = 0$, and hence by Lemma 2.2 there exists a subgroup $H \leq G$ such that $d[X_1; U_H] = d[X_2; U_H] = 0$. By the triangle inequality, we then have

$$\begin{aligned} \eta(d[X_1^0; U_H] + d[X_2^0; U_H]) &= \eta(d[X_1^0; X_1] + d[X_2^0; X_2]) \\ &= \tau[X_1; X_2] \leq (1 + 2\eta)d[X_1^0; X_2^0]. \end{aligned}$$

Using the value $\eta = \frac{1}{9}$ we obtain the first claim of Theorem 1.8. Since $|d[X_1^0; U_H] - d[X_2^0; U_H]| \leq d[X_1^0; X_2^0]$ by the triangle inequality, the second statement then follows. $\square$

Remark 2.3. The proof of Proposition 2.1 is completely algorithmic, but our use of compactness here means that the proof of Theorem 1.8 is not. However, our argument can be modified to give a completely algorithmic proof, at the expense of slightly worse constants.

One approach is to prove Proposition 2.1 with (2.2) replaced by the stronger bound

$$d[X'_1; X'_2] \leq (1 - \eta')d[X_1; X_2] - \eta'd[X'_1; X_1] - \eta'd[X'_2; X_2], \quad (2.7)$$

for some value η' rather smaller than $\eta = \frac{1}{9}$. The proof is very similar and follows the same strategy as that outlined in Section 3. This variant is perhaps even slightly easier to follow than the current proof, since the ‘fixed’ variables X_1^0, X_2^0 no longer play a role.

Once (2.7) is established, it can be applied iteratively to give a sequence (X_1^t, X_2^t) , $t = 0, 1, \dots$ for which $d[X_1^t; X_2^t]$ decays exponentially. In particular, for some $T = O(\log d[X_1^0; X_2^0])$ we have $d[X_1^T; X_2^T] \leq \varepsilon_0$, where ε_0 is the absolute constant appearing in [9, Proposition 1.3]. By that result, X_1^T and X_2^T are both close to some variable U_H . One can then control the distance of U_H from the original variables X_1^0, X_2^0 by summing (2.7) over $0 \leq t < T$ and applying the Ruzsa triangle inequality repeatedly.

A version of this argument will be given in detail in our subsequent paper [6].

Remark 2.4. We sketch one further consequence of the variant argument discussed in Remark 2.3 above, namely a weak Bogolyubov-type

result: if $|A + A| \leq K|A|$, then there is a subspace H of G of cardinality $\gg K^{-O(1)}|A|$ that is contained in an iterated sumset mA for some $m = O(\log^{O(1)}(2 + K))$.

Indeed, if takes $X_1^0 = X_2^0 = U_A$ as in Appendix B and runs the above iteration, we see inductively from an inspection of the construction that each pair X_1^t, X_2^t of random variables produced by the iterative process is contained in $m_t A$ for some $m_t = O(1)^t$. Since it only takes $O(\log \log K)$ steps for the doubling constant to get within range of ε_0 , one can check (by inspection of the argument in [9, Proposition 1.3]) that the group H produced is then contained in mA for some $m = O(\log^{O(1)}(2 + K))$, and the claim then follows after some routine verifications.

3. PLAN OF THE REMAINING ARGUMENT

Let k denote the quantity $k := d[X_1; X_2]$. For the following preliminary discussion we write various estimates as $O(\eta k)$ for simplicity. To make the argument actually work with the parameter $\eta = \frac{1}{9}$ one of course needs to be more precise, as indeed we will be from (3.13) onwards.

To prove Proposition 2.1 we will try various choices of X'_1, X'_2 constructed using X_1, X_2 and show that at least one of them works.

To describe the choices we will try, consider a four-tuple of independent random variables $X_1, X_2, \tilde{X}_1, \tilde{X}_2$ where $X_1, \tilde{X}_1$ are copies of X_1 and $X_2, \tilde{X}_2$ are copies of X_2 ; we can in fact assume that all six variables $X_1, X_2, \tilde{X}_1, \tilde{X}_2, X_1^0, X_2^0$ are independent. Our primary choices for (X'_1, X'_2) will be sums

$$X'_1 = X_1 + \tilde{X}_2, \quad X'_2 = X_2 + \tilde{X}_1 \quad (3.1)$$

or

$$X'_1 = X_1 + \tilde{X}_1, \quad X'_2 = X_2 + \tilde{X}_2 \quad (3.2)$$

or alternatively ‘fibres’

$$X'_1 = (X_1 | X_1 + \tilde{X}_2 = g), \quad X'_2 = (X_2 | X_2 + \tilde{X}_1 = g') \quad (3.3)$$

or

$$X'_1 = (X_1 | X_1 + \tilde{X}_1 = g), \quad X'_2 = (X_2 | X_2 + \tilde{X}_2 = g') \quad (3.4)$$

for $g, g' \in G$. A result that we call the *fibring lemma* (which is [9, Proposition 1.4]), concerning the behaviour of entropy doubling under homomorphisms, may be used to relate the distances that come up when we choose the X'_i to be sums or fibres. A self-contained account of this is provided in Section 4.

The conclusion is that either one of these choices succeeds, or they all *narrowly* (by $O(\eta k)$) fail to work, and the latter can happen only if all the inequalities we used were close to equalities. Motivated by this, one can go back and analyse what happens if the inequality in the fibring lemma was almost tight. What we find in this case is that if neither choice (3.1), (3.3) works to prove (2.2) then we have an upper bound

$$I_1 := \mathbf{I}[X_1 + X_2 : \tilde{X}_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] = O(\eta k). \quad (3.5)$$

Informally, this says that $X_1 + X_2$ and $\tilde{X}_1 + X_2$ are almost independent conditioned on $X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2$. Similarly, if neither choice (3.2), (3.4) works to prove (2.2), we obtain a bound

$$I_2 := \mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] = O(\eta k). \quad (3.6)$$

Finally,

$$I_3 := \mathbf{I}[\tilde{X}_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] = O(\eta k), \quad (3.7)$$

since in fact we have $I_2 = I_3$, as may be seen by interchanging the names of $\tilde{X}_1$ and X_1 .

If none of the choices (3.1), (3.2), (3.3), (3.4) works to prove (2.2), all three estimates (3.5), (3.6) and (3.7) hold. In this case, we proceed to a part of the argument we refer to as the ‘endgame’.

Suppose first for simplicity that the mutual informations in (3.5), (3.6) and (3.7) were zero rather than merely small. It follows that for any s in the support of $X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2$, the three random variables

$$\begin{aligned} T_1 &= (X_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2 = s), \\ T_2 &= (X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2 = s), \\ T_3 &= (\tilde{X}_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2 = s) \end{aligned} \quad (3.8)$$

are pairwise independent. We also note that $T_1 + T_2 + T_3$ is constant (in fact it is identically zero). However, for any trio (T_1, T_2, T_3) of pairwise independent random variables such that $T_1 + T_2 + T_3$ is constant, a short calculation shows that

$$d[T_1; T_2] = \mathbf{H}[T_3] - \frac{1}{2}\mathbf{H}[T_1] - \frac{1}{2}\mathbf{H}[T_2]$$

and similarly for cyclic permutations, hence

$$d[T_1; T_2] + d[T_1; T_3] + d[T_2; T_3] = 0 \quad (3.9)$$

and so, taking X'_1 and X'_2 to be some pair of T_1, T_2, T_3 , we have $d[X'_1; X'_2] = 0$. This is a very strong conclusion, and with some further estimation, it allows us to establish (2.2). (The choice of X'_1, X'_2 here

is inspired, albeit rather indirectly, by a step in an argument of Katz and Koester [15].)

If the three variables in (3.8) are merely ‘almost pairwise independent’, as in (3.5) and (3.6), then things are less straightforward. Very roughly, sums of non-independent random variables are to sums of independent random variables as *partial* sumsets $A +_E B$ along a bipartite graph E are to full sumsets $A + B$. Hence, it is natural to apply a variant of the entropic Balog–Szemerédi–Gowers lemma due to the fourth author [37, Lemma 3.3] (reproduced as Lemma A.2 here) to pass from the ‘almost independent’ variables (T_1, T_2) as above, for which $\mathbf{H}[T_1 + T_2] - \frac{1}{2}\mathbf{H}[T_1] - \frac{1}{2}\mathbf{H}[T_2]$ is small (or even negative), to variables X'_1, X'_2 with $d[X'_1; X'_2] = O(\eta k)$. These variables X'_1, X'_2 , which are given to us by Lemma A.2, now establish (2.2).

Examining the proof of Lemma A.2 carefully, the final choice of X'_1 that comes out of this argument is of the form

$$(X_1 + X_2 \mid \tilde{X}_1 + X_2 = t, X_1 + \tilde{X}_2 = s + t) \quad (3.10)$$

or a related quantity obtained by permuting the variables, and similarly for X'_2 (using the same s, t for both X'_1 and X'_2).

Motivating examples. To understand the strategy, it is helpful to consider some cases of the form $X_1 = U_{A_1}$, $X_2 = U_{A_2}$ for various sets $A_1, A_2 \subseteq \mathbf{F}_2^n$, and to discuss which choices of X'_1, X'_2 give the desired estimate (2.2). In this discussion it is convenient to write $K := e^k$.

Consider first the case in which $A_1 = A_2 = A$ and A is a random subset of some subgroup $H \leq \mathbf{F}_2^n$ with density K^{-1} in H . Then (almost surely) $d[X_1; X_2] \approx k$. In this case the choice (3.2), that is to say $X'_1 = X_1 + \tilde{X}_1$ and $X'_2 = X_2 + \tilde{X}_2$, immediately establishes (2.2). Indeed both X'_1 and X'_2 are close to the uniform distribution U_H on H , so $d[X'_1; X'_2] \approx 0$ and $d[X'_i; X_i] \approx k/2$ for $i = 1, 2$, and (2.2) follows, with room to spare, by the triangle inequality.

Consider next the case in which $A_1 = \bigcup_{i=1}^m (x_i + H)$ and $A_2 = \bigcup_{i=1}^m (y_i + H)$, where $H < \mathbf{F}_2^n$ is a subgroup and the $x_i + H$ and $y_i + H$ are linearly independent in G/H . Setting $m := K$ gives $d[X_1; X_2] \approx k$. In this case one can obtain (2.2) with the choice (3.3), that is to say $X'_1 = (X_1 \mid X_1 + \tilde{X}_2 = g)$ and $X'_2 = (X_2 \mid X_2 + \tilde{X}_1 = g')$ for any $g, g' \in A_1 + A_2$. Indeed, X'_1 is the uniform distribution on $A_1 \cap (A_2 + g)$, which if $g \in x_i + y_j + H$ is exactly the coset $x_i + H$ of H (by linear independence). Similarly, X'_2 is uniform on a coset $y'_j + H$, so $d[X'_1; X'_2] \approx 0$

and again (2.2) follows. (Using the fibres in (3.4) also gives a contradiction in this case, but the analysis is slightly more involved, since $A_1 \cap (A_1 + g)$ is typically a union of two cosets of H .)

In both of the above examples, one of the choices (3.1), (3.2), (3.3) or (3.4) already gives the estimate (2.2) and so it is not necessary to proceed to the ‘endgame’.

Here is a third example, a sort of combination of the previous two, for which the choices (3.1), (3.2), (3.3) and (3.4) all fail to work. Let A'_1, A'_2 be (independent) random subsets of density $1/m$ of the sets A_1, A_2 from the previous example, where now $m = \sqrt{K}$. One may then check that the choices $X'_1 = X_1 + \tilde{X}_1$, $X'_2 = X_2 + \tilde{X}_2$ resemble the uniform distribution on the union of $\approx K/2$ cosets of H , so $d[X'_1; X'_2] \approx k - O(1)$ and we do not obtain (2.2) (if k is large) with the choice (3.2). The case of (3.1) is similar.

On the other hand, the variables $X'_1 = (X_1 | X_1 + \tilde{X}_2 = g)$, $X'_2 = (X_2 | X_2 + \tilde{X}_1 = g')$ will be uniform on $A_1 \cap (A_2 + g)$ and $A_2 \cap (A_1 + g')$ respectively, and such sets will typically (when non-empty) resemble random subsets of a coset of H , of density $1/m^2 \approx 1/K$. For these variables we also have $d[X'_1; X'_2] \approx k$, and so we do not obtain (2.2) with the choice (3.3). The case of (3.4) is broadly similar.

We therefore proceed to the endgame and consider the three variables in (3.8). Since $X_1 + X_2$ is roughly uniform on $B := \bigcup_{i,j} (x_i + y_j + H)$, a similar analysis to the second example shows that $(X_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2 = s)$ is roughly uniform on $B \cap (B + s)$, which is typically a union of four cosets of H , and similarly for $(\tilde{X}_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2 = s)$. Moreover, one can check that these two variables are ‘50% independent’ (in that knowing one of them narrows the choice of the other to two cosets of H).

If we then apply Lemma A.2, we find variables X'_1, X'_2 of the form given in (3.10). In other words, X'_1 is a sum of two variables of the form $(X_1 | X_1 + \tilde{X}_2 = a)$ and $(X_2 | X_2 + \tilde{X}_1 = b)$. Each will be uniform on a random subset of a coset of H , and adding them together will give something close to uniform on a coset of H . The case of X'_2 is similar, so we get $d[X'_1; X'_2] \approx 0$ and (2.2) follows. (Again, permuting the variables may give slightly different conclusions.)

Finally, we consider a further example over $G = \mathbf{Z}$ instead of $\mathbf{F}_2^n$. Take X_1, X_2 to have a ‘discrete Gaussian’ distribution $p(x) = C_0 e^{-x^2/2r^2}$ with some large ‘width’ r and a normalizing constant C_0 . By [37, Theorem 1.13], the distance $d[X_1; X_2] \approx \frac{1}{2} \log 2$ is roughly minimal among

random variables on $\mathbf{Z}$ with large entropy, and hence Proposition 2.1 cannot hold over $\mathbf{Z}$ in the form stated.

Concretely, one can show that sums $X_i + \tilde{X}_j$ as in (3.1), (3.2) are (roughly) discrete Gaussians of width $\sqrt{2}r$, and fibres $(X_i | X_i + \tilde{X}_j)$ as in (3.3), (3.4) are (shifted) discrete Gaussians of width about $r/\sqrt{2}$. Hence, all these have essentially the same value of $d[-; -]$, and there is no way of ‘improving’ these variables with any of the choices considered in Sections 5 and 6. One could also see directly that the conditional near-independence statements such as

$$\mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \approx 0$$

do indeed hold for discrete Gaussians.

In the above one-dimensional example, the entropic doubling constant is bounded, and so this example does not, by itself, present a strong obstruction to the entire approach. However, by considering discrete Gaussian examples in a high-dimensional lattice $\mathbf{Z}^d$, the entropic doubling constant is now close to $\frac{d}{2} \log 2$, which is unbounded, and all the moves considered previously do not significantly reduce this constant.

Even though discrete Gaussians over $\mathbf{F}_2$ do not exist, this example places significant constraints on what we can hope to prove in general. In particular, it shows that the endgame argument must ‘see’ the finite characteristic in an essential way.

The contrapositive formulation. The above discussion and the motivating examples which followed it were framed in terms of proving Proposition 2.1 directly, by making various choices of X'_1 and X'_2 , and the authors found this most natural when thinking about the problem.

However, when it comes to recording the proofs, it turns out to be notationally simpler to argue in the contrapositive. Thus, suppose henceforth that we have a pair (X_1, X_2) of G -valued random variables with $d[X_1; X_2] = k$ for some k , and suppose that

$$\tau[X'_1; X'_2] \geq \tau[X_1; X_2] \tag{3.11}$$

for every pair of G -valued random variables (X'_1, X'_2) . Using the definition of τ (see (2.1)) we may rewrite (3.11) as

$$d[X'_1; X'_2] \geq k - \eta(d[X_1^0; X'_1] - d[X_1^0; X_1]) - \eta(d[X_2^0; X'_2] - d[X_2^0; X_2]). \tag{3.12}$$

As in the discussion above, the idea now is to test (3.12) with various choices of (X'_1, X'_2) generated from (X_1, X_2) , the aim being to deduce from this that $k = 0$.

Testing (3.12) with the choices (3.1), (3.3) will lead us to the conclusion that

$$I_1 := \mathbf{I}[X_1 + X_2 : \tilde{X}_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq 2\eta k \quad (3.13)$$

(that is, a more precise version of (3.5)). The details of this deduction are provided in Section 5. Testing (3.12) with the choices (3.2), (3.4) leads to

$$I_2 := \mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq 2\eta k + \frac{2\eta(2\eta k - I_1)}{1 - \eta} \quad (3.14)$$

(a more precise version of (3.6)), and of course the same estimate holds for I_3 , since in fact $I_2 = I_3$. The details of this deduction are provided in Section 6.

Finally, armed with the knowledge that (3.13), (3.14) hold, we proceed to the endgame, using choices of X'_1, X'_2 arising from (3.8) and entropic Balog–Szemerédi–Gowers (i.e., of forms such as (3.10)) to finally show that $k = 0$. The details are given in Section 7.

The main reason for arguing by contradiction is that it greatly simplifies our discussions of conditioned random variables. Note that (3.12) implies a conditioned version of itself: for any random variables (X'_1, Y_1) and (X'_2, Y_2) we may apply (3.12) with X'_1, X'_2 replaced by each of the conditioned random variables $(X'_1 | Y_1 = y_1), (X'_2 | Y_2 = y_2)$, obtaining

$$\begin{aligned} & d[(X'_1 | Y_1 = y_1); (X'_2 | Y_2 = y_2)] \\ & \geq k - \eta(d[X_1^0; (X'_1 | Y_1 = y_1)] - d[X_1^0; X_1]) \\ & \quad - \eta(d[X_2^0; (X'_2 | Y_2 = y_2)] - d[X_2^0; X_2]). \end{aligned}$$

Multiplying by $p_{Y_1}(y_1)p_{Y_2}(y_2)$ and summing, we have

$$\begin{aligned} d[X'_1 | Y_1; X'_2 | Y_2] & \geq k - \eta(d[X_1^0; X'_1 | Y_1] - d[X_1^0; X_1]) \\ & \quad - \eta(d[X_2^0; X'_2 | Y_2] - d[X_2^0; X_2]). \end{aligned} \quad (3.15)$$

More discussion of the notion of conditional distance may be found at (A.14).

To make the same argument in the direct (non-contrapositive) direction we would pick some particular (y_1, y_2) such that

$$\begin{aligned} & d[(X'_1 | Y_1 = y_1); (X'_2 | Y_2 = y_2)] + \eta(d[X_1^0; (X'_1 | Y_1 = y_1)] - d[X_1^0; X_1]) \\ & \quad + \eta(d[X_2^0; (X'_2 | Y_2 = y_2)] - d[X_2^0; X_2]) \end{aligned}$$

is at most the (weighted) average value of the same quantity over all (y_1, y_2) . If we have to perform many such steps in sequence, this becomes notationally very taxing. However, this is purely a matter of

preference: for instance, in Lemma 7.2 and its proof we instead choose to argue directly.

Remarks on odd p . We will handle the case p odd in our forthcoming paper [6]. All of the above steps go through, except for the final contradiction in the ‘endgame’ where we used an argument specific to characteristic 2. To get a contradiction in characteristic p , it is necessary to run a variant of the argument with a p -partite ‘distance’ function $D[X_1; \dots; X_p]$ replacing $d[X_1; X_2]$ as the main term in the functional to be minimized. This introduces additional notational complexity: for instance we now have an array $(X_{ij})_{i,j \in \mathbb{F}_p}$ of random variables, and the fibring lemma must be generalized to this p -partite distance and applied $p - 1$ times.

4. FIBRING LEMMA

Here we record the fibring lemma, that is to say [9, Proposition 1.4], with an explicit error term. (The latter was mentioned in [9], but only as a casual remark.)

Proposition 4.1. *Let $\pi: H \rightarrow H'$ be a homomorphism between abelian groups and let Z_1, Z_2 be H -valued random variables. Then we have*

$$d[Z_1; Z_2] \geq d[\pi(Z_1); \pi(Z_2)] + d[Z_1 | \pi(Z_1); Z_2 | \pi(Z_2)]$$

where the notation is as in (A.14). Moreover, if Z_1, Z_2 are taken to be independent, then the difference between the two sides is

$$\mathbf{I}[Z_1 - Z_2 : (\pi(Z_1), \pi(Z_2)) \mid \pi(Z_1 - Z_2)]. \quad (4.1)$$

Proof. Let Z_1, Z_2 be independent throughout. We have

$$\begin{aligned} & d[Z_1 | \pi(Z_1); Z_2 | \pi(Z_2)] \\ &= \mathbf{H}[Z_1 - Z_2 | \pi(Z_1), \pi(Z_2)] - \frac{1}{2} \mathbf{H}[Z_1 | \pi(Z_1)] - \frac{1}{2} \mathbf{H}[Z_2 | \pi(Z_2)] \\ &\leq \mathbf{H}[Z_1 - Z_2 | \pi(Z_1 - Z_2)] - \frac{1}{2} \mathbf{H}[Z_1 | \pi(Z_1)] - \frac{1}{2} \mathbf{H}[Z_2 | \pi(Z_2)] \\ &= d[Z_1; Z_2] - d[\pi(Z_1); \pi(Z_2)]. \end{aligned}$$

In the middle step, we used submodularity of entropy, and in the last step we used the fact that

$$\mathbf{H}[Z_1 - Z_2 | \pi(Z_1 - Z_2)] = \mathbf{H}[Z_1 - Z_2] - \mathbf{H}[\pi(Z_1 - Z_2)]$$

(since $Z_1 - Z_2$ determines $\pi(Z_1 - Z_2)$) and that

$$\mathbf{H}[Z_i | \pi(Z_i)] = \mathbf{H}[Z_i] - \mathbf{H}[\pi(Z_i)]$$

(since Z_i determines $\pi(Z_i)$). This gives the claimed inequality. The difference between the two sides is precisely

$$\mathbf{H}[Z_1 - Z_2 | \pi(Z_1 - Z_2)] - \mathbf{H}[Z_1 - Z_2 | \pi(Z_1), \pi(Z_2)].$$

To rewrite this in terms of (conditional) mutual information, we use the identity

$$\mathbf{H}[A|B] - \mathbf{H}[A|B, C] = \mathbf{I}[A : C|B],$$

taking $A := Z_1 - Z_2$, $B := \pi(Z_1 - Z_2)$ and $C := (\pi(Z_1), \pi(Z_2))$, and noting that in this case $\mathbf{H}[A|B, C] = \mathbf{H}[A|C]$ since C uniquely determines B . This completes the proof. $\square$

We extract the specific special case of this result that we will need in our arguments.

Corollary 4.2. *Let Y_1, Y_2, Y_3 and Y_4 be independent random variables taking values in some abelian group G . Then*

$$\begin{aligned} d[Y_1 - Y_3; Y_2 - Y_4] + d[Y_1 | Y_1 - Y_3; Y_2 | Y_2 - Y_4] \\ + \mathbf{I}[Y_1 - Y_2 : Y_2 - Y_4 | Y_1 - Y_2 - Y_3 + Y_4] = d[Y_1; Y_2] + d[Y_3; Y_4]. \end{aligned}$$

Proof. We apply Proposition 4.1 with $H := G \times G$, $H' := G$, π the subtraction homomorphism $\pi(x, y) := x - y$, and with the random variables $Z_1 := (Y_1, Y_3)$ and $Z_2 := (Y_2, Y_4)$. Then by independence we easily calculate

$$d[Z_1; Z_2] = d[Y_1; Y_2] + d[Y_3; Y_4]$$

while by definition

$$d[\pi(Z_1); \pi(Z_2)] = d[Y_1 - Y_3; Y_2 - Y_4].$$

Furthermore,

$$d[Z_1 | \pi(Z_1); Z_2 | \pi(Z_2)] = d[Y_1 | Y_1 - Y_3; Y_2 | Y_2 - Y_4],$$

since $Z_1 = (Y_1, Y_3)$ and Y_1 are linked by an invertible affine transformation once $\pi(Z_1) = Y_1 - Y_3$ is fixed, and similarly for Z_2 and Y_2 . Finally, we have

$$\begin{aligned} & \mathbf{I}[Z_1 - Z_2 : (\pi(Z_1), \pi(Z_2)) | \pi(Z_1) + \pi(Z_2)] \\ &= \mathbf{I}[(Y_1 - Y_2, Y_3 - Y_4) : (Y_1 - Y_3, Y_2 - Y_4) | Y_1 - Y_2 - Y_3 + Y_4] \\ &= \mathbf{I}[Y_1 - Y_2 : Y_2 - Y_4 | Y_1 - Y_2 - Y_3 + Y_4] \end{aligned}$$

where in the last line we used the fact that $(Y_1 - Y_2, Y_1 - Y_2 - Y_3 + Y_4)$ uniquely determine $Y_3 - Y_4$ and similarly $(Y_2 - Y_4, Y_1 - Y_2 - Y_3 + Y_4)$ uniquely determine $Y_1 - Y_3$. $\square$

Remark 4.3. Of course, in our main characteristic 2 setting, the minus signs in Proposition 4.1 and Corollary 4.2 may be replaced with plus signs.

5. FIRST ESTIMATE

Recall that in this and subsequent sections we are working on the assumption that (3.12) and its conditioned variant (3.15) hold, aiming to prove that $k = 0$ and hence conclude (the contrapositive of) Proposition 2.1.

As above, $X_1, X_2, \tilde{X}_1, \tilde{X}_2$ are independent random variables, with $X_1, \tilde{X}_1$ copies of X_1 and $X_2, \tilde{X}_2$ copies of X_2 . In this section we establish the upper bound (3.13), which was that

$$I_1 := \mathbf{I}[X_1 + X_2 : \tilde{X}_1 + X_2 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq 2\eta k.$$

If we apply Corollary 4.2 (and Remark 4.3) with the choice

$$(Y_1, Y_2, Y_3, Y_4) := (X_1, X_2, \tilde{X}_2, \tilde{X}_1)$$

we obtain

$$d[X_1 + \tilde{X}_2; X_2 + \tilde{X}_1] + d[X_1 | X_1 + \tilde{X}_2; X_2 | X_2 + \tilde{X}_1] + I_1 = 2k, \quad (5.1)$$

for I_1 as above, since $d[X_1; X_2] = k$. This forces one of two possibilities: either the term I_1 is small, or if not, one of

$$d[X_1 + \tilde{X}_2; X_2 + \tilde{X}_1], \quad d[X_1 | X_1 + \tilde{X}_2; X_2 | X_2 + \tilde{X}_1]$$

must be significantly smaller than k . However, we will argue that the latter possibility is incompatible with our standing assumptions (3.12) and (3.15), which informally assert that no variables X'_1, X'_2 which are “reasonably close” to X_1, X_2 can have $d[X'_1; X'_2]$ significantly smaller than k .

Specifically, applying (3.12), (3.15), we have

$$\begin{aligned} d[X_1 + \tilde{X}_2; X_2 + \tilde{X}_1] &\geq k - \eta(d[X_1^0; X_1 + \tilde{X}_2] - d[X_1^0; X_1]) \\ &\quad - \eta(d[X_2^0; X_2 + \tilde{X}_1] - d[X_2^0; X_2]) \end{aligned}$$

and

$$\begin{aligned} d[X_1 | X_1 + \tilde{X}_2; X_2 | X_2 + \tilde{X}_1] &\geq k - \eta(d[X_1^0; X_1 | X_1 + \tilde{X}_2] - d[X_1^0; X_1]) \\ &\quad - \eta(d[X_2^0; X_2 | X_2 + \tilde{X}_1] - d[X_2^0; X_2]). \end{aligned} \quad (5.2)$$

Combining these with (5.1), to deduce (3.13) it would therefore suffice to prove the upper bounds

$$(d[X_1^0; X_1 + \tilde{X}_2] - d[X_1^0; X_1]) + (d[X_2^0; X_2 + \tilde{X}_1] - d[X_2^0; X_2]) \leq k \quad (5.3)$$

and

$$\begin{aligned} & (d[X_1^0; X_1 | X_1 + \tilde{X}_2] - d[X_1^0; X_1]) \\ & + (d[X_2^0; X_2 | X_2 + \tilde{X}_1] - d[X_2^0; X_2]) \leq k. \end{aligned} \quad (5.4)$$

Remark 5.1. To prove Theorem 1.8 for *some* constant C replacing 11 or 6, it would be enough to show $I_1 = O(\eta k)$ here (as in (3.5)): we only need the sharper $I_1 \leq 2\eta k$ to obtain those exact final constants. For this cruder bound, we could also replace k with $O(k)$ on the right-hand sides of (5.3) and (5.4).

There are many ways to combine the “entropic Ruzsa calculus” inequalities from Appendix A to prove *some* upper bound of this weaker type: the more delicate part is avoiding unnecessary constant factor losses. The reader who is not concerned with constants could therefore spare themselves some of the technical details that follow.⁶

We pause to state some lemmas which we will use for bounding a number of expressions similar to (5.3), (5.4), both here and in the next two sections.

The first is a bound relating conditioned and unconditioned variants of the Ruzsa distance.

Lemma 5.2. *Suppose that (X, Z) and (Y, W) are random variables, where X, Y take values in some abelian group. Then*

$$d[X|Z; Y|W] \leq d[X; Y] + \frac{1}{2}\mathbf{I}[X : Z] + \frac{1}{2}\mathbf{I}[Y : W].$$

Proof. The definition of conditional distance is given in (A.14).

Using the alternative expression (A.15), if $(X', Z'), (Y', W')$ are independent copies of the variables $(X, Z), (Y, W)$, we have

$$\begin{aligned} d[X|Z; Y|W] &= \mathbf{H}[X' - Y' | Z', W'] - \frac{1}{2}\mathbf{H}[X' | Z'] - \frac{1}{2}\mathbf{H}[Y' | W'] \\ &\leq \mathbf{H}[X' - Y'] - \frac{1}{2}\mathbf{H}[X' | Z'] - \frac{1}{2}\mathbf{H}[Y' | W'] \\ &= d[X'; Y'] + \frac{1}{2}\mathbf{I}[X' : Z'] + \frac{1}{2}\mathbf{I}[Y' : W']. \end{aligned}$$

Here, in the middle step we used (A.5), and in the last step we used the definitions of $d[-; -]$ and $\mathbf{I}[-]$. $\square$

⁶An expository account of the proof avoiding such details can be found at <https://terrytao.wordpress.com/2024/06/22/an-abridged-proof-of-martons-conjecture/>.

In the proof of the next lemma we will use the fact that, for any independent random variables X, Y, Z taking values in an abelian group, we have the inequality

$$\mathbf{H}[X + Y + Z] - \mathbf{H}[X + Y] \leq \mathbf{H}[Y + Z] - \mathbf{H}[Y]. \quad (5.5)$$

This is a result of Madiman [22, Theorem I], and is a more general form of an inequality of Kaimanovich and Vershik [14, Proposition 1.3]. It can be viewed as an entropy analogue of an inequality of Plünnecke [27]. For the convenience of the reader, we give the proof in Appendix A.

Lemma 5.3. *Let X, Y, Z be random variables taking values in some abelian group, and with Y, Z independent. Then we have⁷*

$$\begin{aligned} d[X; Y - Z] - d[X; Y] &\leq \tfrac{1}{2}(\mathbf{H}[Y - Z] - \mathbf{H}[Y]) \\ &= \tfrac{1}{2}d[Y; Z] + \tfrac{1}{4}\mathbf{H}[Z] - \tfrac{1}{4}\mathbf{H}[Y] \end{aligned} \quad (5.6)$$

and

$$\begin{aligned} d[X; Y|Y - Z] - d[X; Y] &\leq \tfrac{1}{2}(\mathbf{H}[Y - Z] - \mathbf{H}[Z]) \\ &= \tfrac{1}{2}d[Y; Z] + \tfrac{1}{4}\mathbf{H}[Y] - \tfrac{1}{4}\mathbf{H}[Z]. \end{aligned} \quad (5.7)$$

Proof. We first prove (5.6). We may assume (taking an independent copy) that X is independent of Y, Z . Then we have

$$\begin{aligned} d[X; Y - Z] - d[X; Y] &= \mathbf{H}[X - Y + Z] - \mathbf{H}[X - Y] - \tfrac{1}{2}\mathbf{H}[Y - Z] + \tfrac{1}{2}\mathbf{H}[Y]. \end{aligned}$$

Combining this with (5.5) (applied with Y replaced by $-Y$) gives the required bound. The second form of the result is immediate from the definition of $d[Y; Z]$, since Y, Z are independent.

Turning to (5.7), we have

$$\begin{aligned} \mathbf{I}[Y : Y - Z] &= \mathbf{H}[Y] + \mathbf{H}[Y - Z] - \mathbf{H}[Y, Y - Z] \\ &= \mathbf{H}[Y] + \mathbf{H}[Y - Z] - \mathbf{H}[Y, Z] = \mathbf{H}[Y - Z] - \mathbf{H}[Z], \end{aligned}$$

and so (5.7) is a consequence of Lemma 5.2, applied to the random variables $(X, *)$, $(Y, Y - Z)$ in place of (X, Z) , (Y, W) , where $*$ denotes a trivial (i.e., constant or deterministic) random variable. Once again the second form of the result is immediate from the definition of $d[Y; Z]$. $\square$

⁷We thank Floris van Doorn for noting a sign discrepancy in a previous version of this statement, which was uncovered as part of the effort to formalize the results of this paper in `Lean 4`.

We return to our main task of establishing (5.3), (5.4), which will imply (3.13).

As usual, in the main setting $G = \mathbf{F}_2^n$ we may replace all minus signs with plus signs in the statement of Lemma 5.3. Hence by Lemma 5.3 (and recalling that k is defined to be $d[X_1; X_2]$) we have

$$d[X_1^0; X_1 + \tilde{X}_2] - d[X_1^0; X_1] \leq \frac{1}{2}k + \frac{1}{4}\mathbf{H}[X_2] - \frac{1}{4}\mathbf{H}[X_1],$$

$$d[X_2^0; X_2 + \tilde{X}_1] - d[X_2^0; X_2] \leq \frac{1}{2}k + \frac{1}{4}\mathbf{H}[X_1] - \frac{1}{4}\mathbf{H}[X_2],$$

and

$$d[X_1^0; X_1 | X_1 + \tilde{X}_2] - d[X_1^0; X_1] \leq \frac{1}{2}k + \frac{1}{4}\mathbf{H}[X_1] - \frac{1}{4}\mathbf{H}[X_2]$$

$$d[X_2^0; X_2 | X_2 + \tilde{X}_1] - d[X_2^0; X_2] \leq \frac{1}{2}k + \frac{1}{4}\mathbf{H}[X_2] - \frac{1}{4}\mathbf{H}[X_1].$$

Adding these inequalities in pairs gives (5.3), (5.4) respectively.

For use in the next two sections, we note that subtracting (5.2) from (5.1), and combining the resulting inequality with (5.4), gives the bound

$$d[X_1 + \tilde{X}_2; X_2 + \tilde{X}_1] \leq (1 + \eta)k - I_1,$$

which is equivalent to

$$\mathbf{H}[X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq \frac{1}{2}\mathbf{H}[X_1] + \frac{1}{2}\mathbf{H}[X_2] + (2 + \eta)k - I_1. \quad (5.8)$$

One could also bound the left-hand side using (5.5) (twice), but by making use of our hypothesis (3.15) as above we obtain a slightly better constant.

6. SECOND ESTIMATE

In this section we establish the upper bound (3.14), which was

$$I_2 := \mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq 2\eta k + \frac{2\eta(2\eta k - I_1)}{1 - \eta}.$$

The overall approach is very similar to that in Section 5. We apply Corollary 4.2 (and Remark 4.3), but now with the choice

$$(Y_1, Y_2, Y_3, Y_4) := (X_2, X_1, \tilde{X}_2, \tilde{X}_1).$$

Now Corollary 4.2 can be rewritten as

$$\begin{aligned} & d[X_1 + \tilde{X}_1; X_2 + \tilde{X}_2] + d[X_1 | X_1 + \tilde{X}_1; X_2 | X_2 + \tilde{X}_2] \\ & + \mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] = 2k, \end{aligned}$$

recalling once again that $k := d[X_1; X_2]$. From (3.12) and (3.15) as before, one has

$$\begin{aligned} d[X_1 + \tilde{X}_1; X_2 + \tilde{X}_2] &\geq k - \eta(d[X_1^0; X_1 + \tilde{X}_1] - d[X_1^0; X_1]) \\ &\quad - \eta(d[X_2^0; X_2 + \tilde{X}_2] - d[X_2^0; X_2]) \end{aligned} \quad (6.1)$$

and

$$\begin{aligned} d[X_1|X_1 + \tilde{X}_1; X_2|X_2 + \tilde{X}_2] &\geq k - \eta(d[X_1^0; X_1|X_1 + \tilde{X}_1] - d[X_1^0; X_1]) \\ &\quad - \eta(d[X_2^0; X_2|X_2 + \tilde{X}_2] - d[X_2^0; X_2]). \end{aligned}$$

The remaining technical task is to find upper bounds of the shape $O(k)$ for the various quantities such as $d[X_1^0; X_1 + \tilde{X}_1] - d[X_1^0; X_1]$ on the right-hand side; as in Remark 5.1, the task would be simpler if we paid no attention to constant factors. This time, Lemma 5.3 gives

$$d[X_1^0; X_1 + \tilde{X}_1] - d[X_1^0; X_1] \leq \frac{1}{2}d[X_1; X_1], \quad (6.2)$$

$$d[X_2^0; X_2 + \tilde{X}_2] - d[X_2^0; X_2] \leq \frac{1}{2}d[X_2; X_2], \quad (6.3)$$

$$d[X_1^0; X_1|X_1 + \tilde{X}_1] - d[X_1^0; X_1] \leq \frac{1}{2}d[X_1; X_1],$$

and

$$d[X_2^0; X_2|X_2 + \tilde{X}_2] - d[X_2^0; X_2] \leq \frac{1}{2}d[X_2; X_2].$$

Combining all these inequalities and cancelling terms, we obtain

$$\mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1|X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq \eta(d[X_1; X_1] + d[X_2; X_2]). \quad (6.4)$$

One could bound the right-hand side by $4\eta k$ using the Ruzsa triangle inequality, but a more efficient approach is as follows. First, by combining (6.1), (6.2) and (6.3), we obtain

$$d[X_1 + \tilde{X}_1; X_2 + \tilde{X}_2] \geq k - \frac{\eta}{2}(d[X_1; X_1] + d[X_2; X_2]). \quad (6.5)$$

We may also expand

$$\begin{aligned} &d[X_1 + \tilde{X}_1; X_2 + \tilde{X}_2] \\ &= \mathbf{H}[X_1 + \tilde{X}_1 + X_2 + \tilde{X}_2] - \frac{1}{2}\mathbf{H}[X_1 + \tilde{X}_1] - \frac{1}{2}\mathbf{H}[X_2 + \tilde{X}_2] \\ &= \mathbf{H}[X_1 + \tilde{X}_1 + X_2 + \tilde{X}_2] - \frac{1}{2}\mathbf{H}[X_1] - \frac{1}{2}\mathbf{H}[X_2] \\ &\quad - \frac{1}{2}(d[X_1; X_1] + d[X_2; X_2]), \end{aligned}$$

and hence by (5.8)

$$d[X_1 + \tilde{X}_1; X_2 + \tilde{X}_2] \leq (2 + \eta)k - \frac{1}{2}(d[X_1; X_1] + d[X_2; X_2]) - I_1.$$

Combining this bound with (6.5) we obtain

$$d[X_1; X_1] + d[X_2; X_2] \leq 2k + \frac{2(2\eta k - I_1)}{1 - \eta}. \quad (6.6)$$

Therefore by (6.4) we have the desired bound

$$\mathbf{I}[X_1 + X_2 : X_1 + \tilde{X}_1 | X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2] \leq 2\eta k + \frac{2\eta(2\eta k - I_1)}{1 - \eta}.$$

7. ENDGAME

In this section we conclude the proof of Proposition 2.1. Let us begin by recording an inequality which will be used several times in the calculations below.

Lemma 7.1. *Let X, Y, Z, Z' be random variables taking values in some abelian group, and with Y, Z, Z' independent. Then we have*

$$\begin{aligned} & d[X; Y - Z | Y - Z - Z'] - d[X; Y] \\ & \leq \frac{1}{2}(\mathbf{H}[Y - Z - Z'] + \mathbf{H}[Y - Z] - \mathbf{H}[Y] - \mathbf{H}[Z']). \end{aligned} \quad (7.1)$$

Proof. By (5.7) (with a change of variables) we have

$$d[X; Y - Z | Y - Z - Z'] - d[X; Y - Z] \leq \frac{1}{2}(\mathbf{H}[Y - Z - Z'] - \mathbf{H}[Z']).$$

Adding this to (5.6) gives the result. $\square$

Turning now to the main argument, let $X_1, X_2, \tilde{X}_1, \tilde{X}_2$ be as before, and introduce the random variables

$$U := X_1 + X_2, \quad V := \tilde{X}_1 + X_2, \quad W := X_1 + \tilde{X}_1$$

and

$$S := X_1 + X_2 + \tilde{X}_1 + \tilde{X}_2.$$

From the definitions (3.5), (3.6), (3.7) of I_1, I_2, I_3 and the above notation, we see that

$$I_1 = \mathbf{I}[U : V | S], \quad I_2 = \mathbf{I}[W : U | S], \quad I_3 = \mathbf{I}[V : W | S].$$

Recall from (3.13) that $I_1 \leq 2\eta k$. From (3.14) (and since $I_2 = I_3$) we have the inequalities

$$\mathbf{I}[V : W | S], \mathbf{I}[W : U | S] \leq 2\eta k + \frac{2\eta(2\eta k - I_1)}{1 - \eta}.$$

Summing these two inequalities and the equality $\mathbf{I}[U : V | S] = I_1$ gives

$$\begin{aligned} & \mathbf{I}[U : V | S] + \mathbf{I}[V : W | S] + \mathbf{I}[W : U | S] \\ & \leq I_1 + 4\eta k + \frac{4\eta(2\eta k - I_1)}{1 - \eta} \\ & = 6\eta k - \frac{1 - 5\eta}{1 - \eta}(2\eta k - I_1). \end{aligned} \quad (7.2)$$

We encourage the reader to read the argument assuming first that $I_1 = 2\eta k$, in which case the calculations are cleaner.

We assemble some preliminary estimates on distances. By Lemma 7.1 (again replacing all minus signs with plus signs, and taking $X = X_1^0$, $Y = X_1$, $Z = X_2$ and $Z' = \tilde{X}_1 + \tilde{X}_2$, so that $Y + Z = U$ and $Y + Z + Z' = S$) we have, noting that $\mathbf{H}[Y + Z] = \mathbf{H}[Z']$,

$$d[X_1^0; U|S] - d[X_1^0; X_1] \leq \frac{1}{2}(\mathbf{H}[S] - \mathbf{H}[X_1]).$$

Further applications of Lemma 7.1 give

$$\begin{aligned} d[X_2^0; U|S] - d[X_2^0; X_2] &\leq \frac{1}{2}(\mathbf{H}[S] - \mathbf{H}[X_2]) \\ d[X_1^0; V|S] - d[X_1^0; X_1] &\leq \frac{1}{2}(\mathbf{H}[S] - \mathbf{H}[X_1]) \\ d[X_2^0; V|S] - d[X_2^0; X_2] &\leq \frac{1}{2}(\mathbf{H}[S] - \mathbf{H}[X_2]) \end{aligned}$$

and

$$d[X_1^0; W|S] - d[X_1^0; X_1] \leq \frac{1}{2}(\mathbf{H}[S] + \mathbf{H}[W] - \mathbf{H}[X_1] - \mathbf{H}[W']),$$

where $W' := X_2 + \tilde{X}_2$. To treat $d[X_2^0; W|S]$, first note that this equals $d[X_2^0; W'|S]$, since $W' = W + S$. Now we may apply Lemma 7.1 to obtain

$$d[X_2^0; W'|S] - d[X_2^0; X_2] \leq \frac{1}{2}(\mathbf{H}[S] + \mathbf{H}[W'] - \mathbf{H}[X_2] - \mathbf{H}[W]).$$

Summing these six estimates and using (5.8), we conclude that

$$\begin{aligned} \sum_{i=1}^2 \sum_{A \in \{U, V, W\}} (d[X_i^0; A|S] - d[X_i^0; X_i]) \\ \leq 3\mathbf{H}[S] - \frac{3}{2}\mathbf{H}[X_1] - \frac{3}{2}\mathbf{H}[X_2] \\ \leq (6 - 3\eta)k + 3(2\eta k - I_1). \end{aligned} \quad (7.3)$$

Now we come to the key observation we will exploit, which is that

$$U + V + W = 0. \quad (7.4)$$

Here, of course, we are using the fact that we are in characteristic 2. This is the only critical use of this fact in the argument, in the sense that it cannot be avoided by judicious insertion of negative signs or by accepting slightly worse constants by invoking the inequality $d[X; -Y] \leq 3d[X; Y]$ (see [37, Equation 17]) and related estimates.

To see the force of (7.2) and (7.4), we state the following general claim.

Lemma 7.2. *Let $G = \mathbf{F}_2^n$ and let (T_1, T_2, T_3) be a G^3 -valued random variable such that $T_1 + T_2 + T_3 = 0$ holds identically. Set*

$$\delta := \sum_{1 \leq i < j \leq 3} \mathbf{I}[T_i; T_j]. \quad (7.5)$$

Then there exist random variables T'_1, T'_2 such that

$$\begin{aligned} & d[T'_1; T'_2] + \eta(d[X_1^0; T'_1] - d[X_1^0; X_1]) + \eta(d[X_2^0; T'_2] - d[X_2^0; X_2]) \\ & \leq \delta + \frac{\eta}{3} \left(\delta + \sum_{i=1}^2 \sum_{j=1}^3 (d[X_i^0; T_j] - d[X_i^0; X_i]) \right). \end{aligned}$$

We recommend that the reader first work through this proof in the case $\delta = 0$, when the variables T_1, T_2, T_3 are independent. Recalling (3.9), the proof below collapses to taking T'_1, T'_2 to be some two of $\{T_1, T_2, T_3\}$, without the need to apply Lemma A.2.

Proof. We apply the variant of the entropic Balog–Szemerédi–Gowers theorem stated in Lemma A.2, taking $(A, B) = (T_1, T_2)$ there. Since $T_1 + T_2 = T_3$, the conclusion is that

$$\begin{aligned} & \sum_{t_3} p_{T_3}(t_3) d[(T_1|T_3 = t_3); (T_2|T_3 = t_3)] \\ & \leq 3\mathbf{I}[T_1 : T_2] + 2\mathbf{H}[T_3] - \mathbf{H}[T_1] - \mathbf{H}[T_2]. \end{aligned} \quad (7.6)$$

The right-hand side in (7.6) can be rearranged as

$$\begin{aligned} & 2(\mathbf{H}[T_1] + \mathbf{H}[T_2] + \mathbf{H}[T_3]) - 3\mathbf{H}[T_1, T_2] \\ & = 2(\mathbf{H}[T_1] + \mathbf{H}[T_2] + \mathbf{H}[T_3]) - \mathbf{H}[T_1, T_2] - \mathbf{H}[T_2, T_3] - \mathbf{H}[T_1, T_3] = \delta, \end{aligned}$$

using the fact that all three terms $\mathbf{H}[T_i, T_j]$ are equal to $\mathbf{H}[T_1, T_2, T_3]$ and hence to each other. We also have

$$\begin{aligned} & \sum_{t_3} p_{T_3}(t_3) (d[X_1^0; (T_1|T_3 = t_3)] - d[X_1^0; X_1]) \\ & = d[X_1^0; T_1|T_3] - d[X_1^0; X_1] \leq d[X_1^0; T_1] - d[X_1^0; X_1] + \frac{1}{2}\mathbf{I}[T_1 : T_3] \end{aligned}$$

by Lemma 5.2, and similarly

$$\begin{aligned} & \sum_{t_3} p_{T_3}(t_3) (d[X_2^0; (T_2|T_3 = t_3)] - d[X_2^0; X_2]) \\ & \leq d[X_2^0; T_2] - d[X_2^0; X_2] + \frac{1}{2}\mathbf{I}[T_2 : T_3]. \end{aligned}$$

Temporarily define

$$\psi[Y_1; Y_2] := d[Y_1; Y_2] + \eta(d[X_1^0; Y_1] - d[X_1^0; X_1]) + \eta(d[X_2^0; Y_2] - d[X_2^0; X_2]).$$

Putting the above observations together, we have

$$\begin{aligned} \sum_{t_3} p_{T_3}(t_3) \psi[(T_1|T_3 = t_3); (T_2|T_3 = t_3)] &\leq \delta + \eta(\mathrm{d}[X_1^0; T_1] - \mathrm{d}[X_1^0; X_1]) \\ &\quad + \eta(\mathrm{d}[X_2^0; T_2] - \mathrm{d}[X_2^0; X_2]) + \tfrac{1}{2}\eta\mathbf{I}[T_1 : T_3] + \tfrac{1}{2}\eta\mathbf{I}[T_2 : T_3]. \end{aligned}$$

Choosing some t_3 in the support of T_3 that minimizes the $\psi[-; -]$ value, and setting $T'_{1,3} := (T_1|T_3 = t_3)$, $T'_{2,3} := (T_2|T_3 = t_3)$, we have

$$\begin{aligned} \psi[T'_{1,3}; T'_{2,3}] &\leq \delta + \eta(\mathrm{d}[X_1^0; T_1] - \mathrm{d}[X_1^0; X_1]) + \eta(\mathrm{d}[X_2^0; T_2] - \mathrm{d}[X_2^0; X_2]) \\ &\quad + \tfrac{1}{2}\eta\mathbf{I}[T_1 : T_3] + \tfrac{1}{2}\eta\mathbf{I}[T_2 : T_3]. \end{aligned} \quad (7.7)$$

We now repeat this analysis for all permutations of $\{T_1, T_2, T_3\}$ to get variables $T'_{\alpha,\gamma}, T'_{\beta,\gamma}$ for $\{\alpha, \beta, \gamma\}$ ranging over all six permutations of $\{1, 2, 3\}$. Averaging the resulting inequalities (7.7), and recalling the definition (7.5) of δ , we get

$$\tfrac{1}{6} \sum_{\alpha,\beta,\gamma} \psi[T'_{\alpha,\gamma}; T'_{\beta,\gamma}] \leq \delta + \frac{\eta}{3} \left(\delta + \sum_{i=1}^2 \sum_{j=1}^3 (\mathrm{d}[X_i^0; T_j] - \mathrm{d}[X_i^0; X_i]) \right),$$

from which the result follows (taking T'_1, T'_2 to be $T'_{\alpha,\gamma}, T'_{\beta,\gamma}$ for some (α, β, γ) that gives at most the average value). $\square$

Applying Lemma 7.2 with any random variables (T_1, T_2, T_3) such that $T_1 + T_2 + T_3 = 0$ holds identically, and applying (3.12) with $X'_1 = T'_1$, $X'_2 = T'_2$, we deduce that

$$k \leq \delta + \frac{\eta}{3} \left(\delta + \sum_{i=1}^2 \sum_{j=1}^3 (\mathrm{d}[X_i^0; T_j] - \mathrm{d}[X_i^0; X_i]) \right).$$

Note that δ is still defined by (7.5) and thus depends on T_1, T_2, T_3 . In particular we may apply this for

$$T_1 = (U|S = s), \quad T_2 = (V|S = s), \quad T_3 = (W|S = s)$$

for s in the range of S (which is a valid choice by (7.4)) and then average over s with weights $p_S(s)$, to obtain

$$k \leq \tilde{\delta} + \frac{\eta}{3} \left(\tilde{\delta} + \sum_{i=1}^2 \sum_{A \in \{U, V, W\}} (\mathrm{d}[X_i^0; A|S] - \mathrm{d}[X_i^0; X_i]) \right), \quad (7.8)$$

where

$$\tilde{\delta} := \mathbf{I}[U : V|S] + \mathbf{I}[V : W|S] + \mathbf{I}[W : U|S].$$

Putting this together with (7.2) and (7.3), we conclude that

$$\begin{aligned} k &\leq \left(1 + \frac{\eta}{3}\right) \left(6\eta k - \frac{1-5\eta}{1-\eta}(2\eta k - I_1)\right) + \frac{\eta}{3} \left((6-3\eta)k + 3(2\eta k - I_1)\right) \\ &= (8\eta + \eta^2)k - \left(\frac{1-5\eta}{1-\eta} \left(1 + \frac{\eta}{3}\right) - \eta\right) (2\eta k - I_1) \\ &\leq (8\eta + \eta^2)k \end{aligned}$$

since the quantity $2\eta k - I_1$ is non-negative (by (3.13)), and its coefficient in the above expression is non-positive provided that $\eta(2\eta + 17) \leq 3$, which is certainly the case for our choice $\eta = \frac{1}{9}$ (and in fact for any $\eta \leq 0.1729$). Moreover, for⁸ $\eta = \frac{1}{9}$ we have $8\eta + \eta^2 < 1$. It follows that $k = 0$, as desired. The proof of Proposition 2.1 (and hence of all our results) is complete.

APPENDIX A. ENTROPY AND ADDITIVE COMBINATORICS

In this appendix we record some standard inequalities regarding Shannon entropy, as well as the standard ‘entropic Ruzsa calculus’ concerning entropies of sums of random variables.

First we remark that if X takes values in a set S then, by Jensen’s inequality,

$$\mathbf{H}[X] \leq \log |S| \tag{A.1}$$

with equality if and only if X is uniform on S . Also, denoting by p_X the density function of X ,

$$\mathbf{H}[X] = \sum_x p_X(x) \log \frac{1}{p_X(x)} \geq \min_{x: p_X(x) > 0} \log \frac{1}{p_X(x)},$$

and therefore

$$\max_x p_X(x) \geq e^{-\mathbf{H}[X]}. \tag{A.2}$$

Given a pair (X, Y) of random variables, the *conditional entropy* $\mathbf{H}[X|Y]$ is defined by the formula

$$\mathbf{H}[X|Y] := \sum_y p_Y(y) \mathbf{H}[X|Y = y]$$

where y ranges over the support of p_Y , and $X|Y = y$ denotes the random variable X conditioned on the event $Y = y$. We have the

⁸In fact we can take any $\eta < \frac{1}{4+\sqrt{17}} = \frac{1}{8.1231\dots}$, and the other constants in the paper can be improved accordingly. In [18] the arguments were modified to also apply in the regime $\eta < \frac{1}{8}$.

fundamental *chain rule*

$$\mathbf{H}[X, Y] = \mathbf{H}[X|Y] + \mathbf{H}[Y]. \quad (\text{A.3})$$

Here we abbreviate $\mathbf{H}[(X, Y)]$ as $\mathbf{H}[X, Y]$, and will make similar abbreviations regarding other information-theoretic quantities in this paper without further comment; for instance, $\mathbf{H}[(X, Y)|(Z, W)]$ becomes $\mathbf{H}[X, Y|Z, W]$. Note that (A.3) implies a conditional generalization

$$\mathbf{H}[X, Y|Z] = \mathbf{H}[X|Y, Z] + \mathbf{H}[Y|Z].$$

for all random variables X, Y, Z .

The *mutual information* $\mathbf{I}[X : Y]$ is defined by the formula

$$\begin{aligned} \mathbf{I}[X : Y] &= \mathbf{H}[X] + \mathbf{H}[Y] - \mathbf{H}[X, Y] \\ &= \mathbf{H}[X] - \mathbf{H}[X|Y] \\ &= \mathbf{H}[Y] - \mathbf{H}[Y|X], \end{aligned}$$

and is non-negative by a standard application of Jensen's inequality, vanishing precisely when X, Y are independent; in particular

$$\mathbf{H}[X, Y] = \mathbf{H}[X] + \mathbf{H}[Y] \quad (\text{A.4})$$

if and only if X, Y are independent, and

$$\mathbf{H}[X|Y] \leq \mathbf{H}[X], \quad \mathbf{H}[X, Y] \leq \mathbf{H}[X] + \mathbf{H}[Y] \quad (\text{A.5})$$

always.

Suppose now that (X, Y, Z) is a triple of random variables. Applying (A.5) to $(X|Z = z)$ and summing over z (weighted by $p_Z(z)$) gives

$$\mathbf{H}[X|Y, Z] \leq \mathbf{H}[X|Z], \quad (\text{A.6})$$

which is known as *submodularity*. It may equivalently be written as

$$\mathbf{H}[X, Y, Z] + \mathbf{H}[Z] \leq \mathbf{H}[X, Z] + \mathbf{H}[Y, Z]. \quad (\text{A.7})$$

The *conditional mutual information* $\mathbf{I}[X : Y|Z]$ is defined by

$$\mathbf{I}[X : Y|Z] := \sum_z p_Z(z) \mathbf{I}[(X|Z = z) : (Y|Z = z)].$$

Submodularity is equivalent to the statement that

$$\mathbf{I}[X : Y|Z] \geq 0, \quad (\text{A.8})$$

since

$$\mathbf{I}[X : Y|Z] = \mathbf{H}[X, Z] + \mathbf{H}[Y, Z] - \mathbf{H}[X, Y, Z] - \mathbf{H}[Z]. \quad (\text{A.9})$$

Equality occurs in (A.8) (and hence in (A.7)) if and only if X, Y are conditionally independent relative to Z .

G-valued random variables. We turn now to additional properties enjoyed by random variables taking values in an additive group G .

Whenever X, Y are G -valued random variables, we have that

$$\mathbf{H}[X \pm Y] \geq \mathbf{H}[X \pm Y|Y] = \mathbf{H}[X|Y] = \mathbf{H}[X] - \mathbf{I}[X : Y]$$

and similarly with the roles of X, Y reversed, thus

$$\max(\mathbf{H}[X], \mathbf{H}[Y]) - \mathbf{I}[X : Y] \leq \mathbf{H}[X \pm Y]. \quad (\text{A.10})$$

We note also the conditional variant of (A.10), namely

$$\max(\mathbf{H}[X|Z], \mathbf{H}[Y|Z]) - \mathbf{I}[X : Y|Z] \leq \mathbf{H}[X \pm Y|Z],$$

which follows from (A.10) by conditioning on $Z = z$ and summing over z (weighted by $p_Z(z)$).

A particular consequence of (A.10) is that

$$\max(\mathbf{H}[X], \mathbf{H}[Y]) \leq \mathbf{H}[X \pm Y] \quad (\text{A.11})$$

when X, Y are independent.

Continuing to suppose that X, Y are independent, recall the definition (1.1) of the Ruzsa distance $d[X; Y]$ which, since X and Y are independent, is that

$$d[X; Y] = \mathbf{H}[X - Y] - \frac{1}{2}\mathbf{H}[X] - \frac{1}{2}\mathbf{H}[Y].$$

Comparing this with (A.11) we see that

$$|\mathbf{H}[X] - \mathbf{H}[Y]| \leq 2d[X; Y]. \quad (\text{A.12})$$

We may also deduce that

$$\mathbf{H}[X - Y] - \mathbf{H}[X], \mathbf{H}[X - Y] - \mathbf{H}[Y] \leq 2d[X; Y].$$

The most important property of the Ruzsa distance is the (Ruzsa) triangle inequality

$$d[X; Y] \leq d[X; Z] + d[Z; Y].$$

This was shown in [29] and [37, (16)]; we recall a proof for completeness. This is equivalent to establishing

$$\mathbf{H}[X - Y] \leq \mathbf{H}[X - Z] + \mathbf{H}[Z - Y] - \mathbf{H}[Z] \quad (\text{A.13})$$

whenever X, Y, Z are independent. To prove this, the key step is to apply (A.6) in the form

$$\mathbf{H}[Y - Z|X - Y] \geq \mathbf{H}[Y - Z|X - Y, Y] = \mathbf{H}[Z|X, Y] = \mathbf{H}[Z]$$

where in the last step we used independence. Moreover,

$$\begin{aligned}\mathbf{H}[Y - Z|X - Y] &= \mathbf{H}[Y - Z, X - Y] - \mathbf{H}[X - Y] \\ &= \mathbf{H}[X - Z, Y - Z] - \mathbf{H}[X - Y] \\ &\leq \mathbf{H}[X - Z] + \mathbf{H}[Y - Z] - \mathbf{H}[X - Y]\end{aligned}$$

by (A.5). Combining these gives (A.13). (As observed in [9], we do not in fact use the independence of X and Y here.)

We will also need conditional variants of the distance. If (X, Z) and (Y, W) are random variables (where X and Y are G -valued) we define

$$d[X|Z; Y|W] := \sum_{z,w} p_Z(z)p_W(w)d[(X|Z=z); (Y|W=w)]. \quad (\text{A.14})$$

Alternatively, if $(X', Z'), (Y', W')$ are independent copies of the variables $(X, Z), (Y, W)$,

$$d[X|Z; Y|W] = \mathbf{H}[X' - Y'|Z', W'] - \frac{1}{2}\mathbf{H}[X'|Z'] - \frac{1}{2}\mathbf{H}[Y'|W']. \quad (\text{A.15})$$

To conclude this appendix we give the proofs of two results from the literature which were used in the main text. The first is an inequality of Madiman [22, Theorem I], which is a more general form of an inequality of Kaimanovich and Vershik [14, Proposition 1.3].

Lemma A.1. *Suppose that X, Y, Z are independent random variables taking values in an abelian group. Then*

$$\mathbf{H}[X + Y + Z] - \mathbf{H}[X + Y] \leq \mathbf{H}[Y + Z] - \mathbf{H}[Y].$$

Proof. By (A.9) we have

$$\begin{aligned}\mathbf{I}[X : Z|X + Y + Z] &= \mathbf{H}[X, X + Y + Z] + \mathbf{H}[Z, X + Y + Z] \\ &\quad - \mathbf{H}[X, Z, X + Y + Z] - \mathbf{H}[X + Y + Z].\end{aligned}$$

However, using (A.4) three times we have $\mathbf{H}[X, X + Y + Z] = \mathbf{H}[X, Y + Z] = \mathbf{H}[X] + \mathbf{H}[Y + Z]$, $\mathbf{H}[Z, X + Y + Z] = \mathbf{H}[Z, X + Y] = \mathbf{H}[Z] + \mathbf{H}[X + Y]$ and $\mathbf{H}[X, Z, X + Y + Z] = \mathbf{H}[X, Y, Z] = \mathbf{H}[X] + \mathbf{H}[Y] + \mathbf{H}[Z]$.

After a short calculation, we see that the claimed inequality is equivalent to the assertion that $\mathbf{I}[X : Z|X + Y + Z] \geq 0$, which of course is an instance of (A.8). $\square$

The next lemma is not quite in the literature but is very closely related to the entropic version of the Balog–Szemerédi–Gowers lemma due to the fourth author [37, Lemma 3.3]. Here we provide slightly better constants and a slightly simpler proof.

Lemma A.2. *Let (A, B) be a G^2 -valued random variable, and set $Z := A + B$. Then*

$$\sum_z p_Z(z) d[(A|Z=z); (B|Z=z)] \leq 3\mathbf{I}[A : B] + 2\mathbf{H}[Z] - \mathbf{H}[A] - \mathbf{H}[B]. \quad (\text{A.16})$$

We stress that the quantity $2\mathbf{H}[Z] - \mathbf{H}[A] - \mathbf{H}[B]$ is *not* the same as $2d[A; B]$, because (A, B) are given a joint distribution which may not be independent. In particular, $\mathbf{H}[Z] = \mathbf{H}[A + B]$ may not match the entropy of a sum of independent copies of A and B .

Proof. In the proof we will need the notion of *conditionally independent trials* of a pair of random variables (X, Y) (not necessarily independent). We say that X_1, X_2 are *conditionally independent trials of X relative to Y* to mean that (X_1, X_2, Y) is a random variable with distribution

$$p_{X_1, X_2, Y}(x_1, x_2, y) = p_Y(y) p_{X|Y=y}(x_1) p_{X|Y=y}(x_2);$$

in other words, $(X_1|Y=y)$ and $(X_2|Y=y)$ are independent copies of $(X|Y=y)$ for all y in the support of Y . We then have

$$\mathbf{H}[(X_1|Y=y), (X_2|Y=y)] = 2\mathbf{H}[X|Y=y]$$

for all y , which upon summing over y (weighted by $p_Y(y)$) gives

$$\mathbf{H}[X_1, X_2|Y] = 2\mathbf{H}[X|Y]$$

and hence

$$\begin{aligned} \mathbf{H}[X_1, X_2, Y] &= \mathbf{H}[X_1, X_2|Y] + \mathbf{H}[Y] = 2\mathbf{H}[X|Y] + \mathbf{H}[Y] \\ &= 2\mathbf{H}[X, Y] - \mathbf{H}[Y]. \end{aligned} \quad (\text{A.17})$$

Note also that the marginal distributions of (X_1, Y) and (X_2, Y) each match the original distribution (X, Y) .

Turning to the proof of Lemma A.2 itself, let (A_1, B_1) and (A_2, B_2) be conditionally independent trials of (A, B) relative to Z , thus (A_1, B_1) and (A_2, B_2) are coupled through the random variable $A_1 + B_1 = A_2 + B_2$, which by abuse of notation we shall also call Z .

Observe that the left-hand side of (A.16) is

$$\mathbf{H}[A_1 - B_2|Z] - \frac{1}{2}\mathbf{H}[A_1|Z] - \frac{1}{2}\mathbf{H}[B_2|Z]. \quad (\text{A.18})$$

since, crucially, $(A_1|Z=z)$ and $(B_2|Z=z)$ are independent for all z .

Applying submodularity (A.7) gives

$$\begin{aligned} \mathbf{H}[A_1 - B_2] + \mathbf{H}[A_1 - B_2, A_1, B_1] \\ \leq \mathbf{H}[A_1 - B_2, A_1] + \mathbf{H}[A_1 - B_2, B_1]. \end{aligned} \quad (\text{A.19})$$

We estimate the second, third and fourth terms appearing here. First note that, by (A.17) (noting that the tuple $(A_1 - B_2, A_1, B_1)$ determines the tuple (A_1, A_2, B_1, B_2) since $A_1 + B_1 = A_2 + B_2$)

$$\mathbf{H}[A_1 - B_2, A_1, B_1] = \mathbf{H}[A_1, B_1, A_2, B_2] = 2\mathbf{H}[A, B] - \mathbf{H}[Z]. \quad (\text{A.20})$$

Next observe that

$$\mathbf{H}[A_1 - B_2, A_1] = \mathbf{H}[A_1, B_2] \leq \mathbf{H}[A] + \mathbf{H}[B]. \quad (\text{A.21})$$

Finally, since $A_1 - B_2 = A_2 - B_1$, we have

$$\mathbf{H}[A_1 - B_2, B_1] = \mathbf{H}[A_2 - B_1, B_1] = \mathbf{H}[A_2, B_1] \leq \mathbf{H}[A] + \mathbf{H}[B]. \quad (\text{A.22})$$

Substituting (A.20), (A.21) and (A.22) into (A.19) yields

$$\mathbf{H}[A_1 - B_2] \leq 2\mathbf{I}[A : B] + \mathbf{H}[Z]$$

and so by (A.5)

$$\mathbf{H}[A_1 - B_2 | Z] \leq 2\mathbf{I}[A : B] + \mathbf{H}[Z].$$

Since

$$\begin{aligned} \mathbf{H}[A_1 | Z] &= \mathbf{H}[A_1, A_1 + B_1] - \mathbf{H}[Z] \\ &= \mathbf{H}[A, B] - \mathbf{H}[Z] \\ &= \mathbf{H}[A] + \mathbf{H}[B] - \mathbf{I}[A : B] - \mathbf{H}[Z] \end{aligned}$$

and similarly for $\mathbf{H}[B_2 | Z]$, we see that (A.18) is bounded by $3\mathbf{I}[A : B] + 2\mathbf{H}[Z] - \mathbf{H}[A] - \mathbf{H}[B]$ as claimed. $\square$

APPENDIX B. FROM ENTROPIC PFR TO COMBINATORIAL PFR

In this appendix we repeat the arguments from [9] showing that Theorem 1.8 with some constant C' in place of 11 implies Theorem 1.2 with $C = C' + 1$. In particular, with $C' = 11$ in Theorem 1.8, this gives the claimed constant $C = 12$ for Theorem 1.2.

Let A, K be as in Theorem 1.2. Let U_A be the uniform distribution on A , thus $\mathbf{H}[U_A] = \log |A|$. By (A.1) and the fact that $U_A + U_A$ is supported on $A + A$, $\mathbf{H}[U_A + U_A] \leq \log |A + A|$. The doubling condition $|A + A| \leq K|A|$ therefore gives

$$\mathbf{d}[U_A; U_A] \leq \log K.$$

By Theorem 1.8, we may thus find a subspace H of $\mathbf{F}_2^n$ such that

$$\mathbf{d}[U_A; U_H] \leq \frac{1}{2}C' \log K. \quad (\text{B.1})$$

By (A.12) we conclude that

$$|\log |H| - \log |A|| \leq C' \log K. \quad (\text{B.2})$$

From the definition of Ruzsa distance, (B.1) is equivalent to

$$\mathbf{H}[U_A - U_H] \leq \log(|A|^{1/2}|H|^{1/2}) + \frac{1}{2}C' \log K.$$

By (A.2) we conclude the existence of a point $x_0 \in \mathbf{F}_2^n$ such that

$$p_{U_A - U_H}(x_0) \geq |A|^{-1/2}|H|^{-1/2}K^{-C'/2},$$

or equivalently

$$|A \cap (H + x_0)| \geq K^{-C'/2}|A|^{1/2}|H|^{1/2}.$$

Applying the Ruzsa covering lemma [38, Lemma 2.14], we may thus cover A by at most

$$\frac{|A + (A \cap (H + x_0))|}{|A \cap (H + x_0)|} \leq \frac{K|A|}{K^{-C'/2}|A|^{1/2}|H|^{1/2}} = K^{C'/2+1} \frac{|A|^{1/2}}{|H|^{1/2}}$$

translates of

$$(A \cap (H + x_0)) - (A \cap (H + x_0)) \subseteq H.$$

If $|H| \leq |A|$ then we are already done thanks to (B.2). If $|H| > |A|$ then we can cover H by at most $2|H|/|A|$ translates of a subspace H' of H with $|H'| \leq |A|$. We can thus cover A by at most

$$2K^{C'/2+1} \frac{|H|^{1/2}}{|A|^{1/2}}$$

translates of H' , and the claim again follows from (B.2).

REFERENCES

- [1] James Aaronson, *A counterexample to a strong variant of the Polynomial Freiman–Ruzsa conjecture*, preprint available at <https://arxiv.org/abs/1902.00353>.
- [2] Jean Bourgain and Mei-Chu Chang, *On the size of k -fold sum and product sets of integers*, J. Amer. Math. Soc. **17** (2004), no. 2, 473–497.
- [3] Mei-Chu Chang, *Some consequences of the polynomial Freiman–Ruzsa conjecture*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 11–12, 583–588.
- [4] Hansheng Diao, *Freiman–Ruzsa-type theory for small doubling constant*, Math. Proc. Cambridge Philos. Soc. **146** (2009), no. 2, 269–276.
- [5] Chaim Even-Zohar, *On sums of generating sets in $\mathbf{Z}_2^n$* , Combin. Probab. Comput. **21** (2012), no. 6, 916–941.
- [6] W. T. Gowers, Ben Green, Freddie Manners, and Terence Tao, *Marton’s Conjecture in abelian groups with bounded torsion*, preprint available at <https://arxiv.org/abs/2404.02244>.
- [7] Ben Green, *Notes on the polynomial Freiman–Ruzsa conjecture*, unpublished note available at <https://people.maths.ox.ac.uk/greenbj/papers/PFR.pdf>.

- [8] ———, *Finite field models in additive combinatorics*, Surveys in combinatorics 2005, London Math. Soc. Lecture Note Ser., vol. 327, Cambridge Univ. Press, Cambridge, 2005, pp. 1–27.
- [9] Ben Green, Freddie Manners, and Terence Tao, *Sumsets and entropy revisited*, Random Structures & Algorithms (2024).
- [10] Ben Green and Terence Tao, *Freiman’s theorem in finite fields via extremal set theory*, Combin. Probab. Comput. **18** (2009), no. 3, 335–355.
- [11] ———, *A note on the Freiman and Balog-Szemerédi-Gowers theorems in finite fields*, J. Aust. Math. Soc. **86** (2009), no. 1, 61–74.
- [12] ———, *An equivalence between inverse sumset theorems and inverse conjectures for the U^3 norm*, Math. Proc. Cambridge Philos. Soc. **149** (2010), no. 1, 1–19.
- [13] Kaave Hosseini and Shachar Lovett, *A bilinear Bogolyubov-Ruzsa lemma with polylogarithmic bounds*, Discrete Anal. (2019), Paper No. 10, 14pp.
- [14] V. A. Kaimanovich and A. M. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Probab. **11** (1983), no. 3, 457–490.
- [15] Nets Hawk Katz and Paul Koester, *On additive doubling and energy*, SIAM J. Discrete Math. **24** (2010), no. 4, 1684–1693.
- [16] Peter Keevash and Noam Lifshitz, *Sharp hypercontractivity for symmetric groups and its applications*, preprint available at <https://arxiv.org/abs/2307.15030>.
- [17] S. V. Konyagin, *On Freiman’s theorem in finite fields*, Mat. Zametki **84** (2008), no. 3, 472–474.
- [18] Jyun-Jie Liao, *Improved exponent for Marton’s Conjecture in $\mathbb{F}_2^n$* , preprint available at <https://arxiv.org/abs/2404.09639>.
- [19] Shachar Lovett, *Equivalence of polynomial conjectures in additive combinatorics*, Combinatorica **32** (2012), no. 5, 607–618.
- [20] ———, *An exposition of Sanders’ quasi-polynomial Freiman–Ruzsa theorem*, Graduate Surveys, no. 6, Theory of Computing Library, 2015.
- [21] Shachar Lovett and Oded Regev, *A counterexample to a strong variant of the polynomial Freiman–Ruzsa conjecture in Euclidean space*, Discrete Anal. (2017), Paper No. 8, 6pp.
- [22] Mokshay Madiman, *On the entropy of sums*, 2008 IEEE Information Theory Workshop, 2008, pp. 303–307.
- [23] Freddie Manners, *Finding a low-dimensional piece of a set of integers*, Int. Math. Res. Not. IMRN (2017), no. 15, 4673–4703.
- [24] ———, *A note on the polynomial Freiman–Ruzsa conjecture over $\mathbb{Z}$* , Math. Proc. Cambridge Philos. Soc. **166** (2019), no. 2, 243–245.
- [25] Akshat Mudgal, *An Elekes–Rónyai Theorem for Sets With Few Products*, Int. Math. Res. Not. IMRN **2024** (2024), no. 13, 10410–10424.
- [26] Dömötör Pálvölgyi and Dmitrii Zhelezov, *Query complexity and the polynomial Freiman–Ruzsa conjecture*, Adv. Math. **392** (2021), Paper No. 108043, 18.
- [27] Helmut Plünnecke, *Eine zahlentheoretische Anwendung der Graphentheorie*, J. Reine Angew. Math. **243** (1970), 171–183.
- [28] Imre Z. Ruzsa, *An analog of Freiman’s theorem in groups*, Astérisque (1999), no. 258, 323–326.
- [29] ———, *Sumsets and entropy*, Random Struct. Alg. **34** (2009), no. 1, 1–10.

- [30] ———, *Sumsets and structure*, Combinatorial number theory and additive group theory, Adv. Courses Math. CRM Barcelona, Birkhäuser Verlag, Basel, 2009, pp. 87–210.
- [31] Alex Samorodnitsky, *Low-degree tests at large distances*, STOC’07—Proceedings of the 39th Annual ACM Symposium on Theory of Computing, ACM, New York, 2007, pp. 506–515.
- [32] Tom Sanders, *On the Bogolyubov-Ruzsa lemma*, Anal. PDE **5** (2012), no. 3, 627–655.
- [33] ———, *The structure theory of set addition revisited*, Bull. Amer. Math. Soc. (N.S.) **50** (2013), no. 1, 93–127.
- [34] Tomasz Schoen, *Near optimal bounds in Freiman’s theorem*, Duke Math. J. **158** (2011), no. 1, 1–12.
- [35] Terence Tao, *Szemerédi’s regularity lemma revisited*, Contrib. Discrete Math. **1** (2006), no. 1, 8–28.
- [36] ———, *Poincaré’s legacies, pages from year two of a mathematical blog. Part I*, American Mathematical Society, Providence, RI, 2009.
- [37] ———, *Sumset and inverse sumset theory for Shannon entropy*, Combin. Probab. Comput. **19** (2010), no. 4, 603–639.
- [38] Terence Tao and Van Vu, *Additive combinatorics*, Cambridge Studies in Advanced Mathematics, vol. 105, Cambridge University Press, Cambridge, 2006.

COLLÈGE DE FRANCE, 11, PLACE MARCELIN-BERTHELOT, 75231 PARIS 05, FRANCE, AND CENTRE FOR MATHEMATICAL SCIENCES, WILBERFORCE ROAD, CAMBRIDGE CB3 0WB, UK

Email address: `wtg10@dpms.cam.ac.uk`

MATHEMATICAL INSTITUTE, ANDREW WILES BUILDING, RADCLIFFE OBSERVATORY QUARTER, WOODSTOCK RD, OXFORD OX2 6QW, UK

Email address: `ben.green@maths.ox.ac.uk`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO (UCSD), 9500 GILMAN DRIVE # 0112, LA JOLLA, CA 92093-0112, USA

Email address: `fmanners@ucsd.edu`

MATH SCIENCES BUILDING, 520 PORTOLA PLAZA, BOX 951555, LOS ANGELES, CA 90095, USA

Email address: `tao@math.ucla.edu`