GLACIER DYNAMICS

by

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ABSTRACT

A model is proposed for the study of general two-dimensional 'polythermal' glacial ice flows. It consists of equations of conservation of mass, momentum and energy together with a constitutive relation between stress and strain rate and a full set of boundary conditions. A novel aspect is the consideration of the moisture content of temperate ice as an enthalpy variable: this represents a first attempt at incorporating the glacial hydrology in a dynamical model.

One of the bedrock boundary conditions requires a knowledge of the so-called 'sliding law' relating the basal shear stress to the basal velocity (when the ice is temperate). A detailed model is proposed to determine this law, and upper and lower bounds for the basal velocity in terms of the stress are given by using a variational principle. The effect of cavitation on the sliding law is considered in the case of Newtonian flow, and it is shown to have a dramatic effect on the basal sliding.

We then turn to our analysis of the glacier flow model. Firstly we consider kinematic waves: an analogue of Nye's (1960) equation is derived and analysed from a nonlinear viewpoint. The formation and evolution of surface shocks is studied, and explicit results given for small perturbations to the surface profile. Secondly, we show how the model may be used to predict a finite slope at the glacier snout by use of the method of strained coordinates.

Finally, steady state solutions are examined in the cases of large and small conduction. We demonstrate that there generally exists a large

patch of basal ice which is almost temperate, but which slides at a velocity lower than that predicted by the sliding law: this is probably the most important result of the present work.

CONTENTS

Foreword

The aim of this thesis is rather broad,and as a result a large body of notation is called upon. Inevitably this has led to duplication of certain symbols used in the text,but I have tried to keep such ambiguous usages separated from each other as much as possible. For example,in Chapter II η is used to mean the top surface of the glacier,whereas in Chapter III it refers to the viscosity of ice considered as a Newtonian fluid:both these uses are standard. For ease of reference,a nomenclature indicating the various meanings of the different symbols is included as Appendix 3.

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CHAPTER I

Introduction

On a time-scale of years,ice in a glacier flows like a non-Newtonian fluid (a typical velocity being 100 metres per year). The flow is driven by gravity and is supplied by an accumulation rate of ice formed on the surface from packed fallen snow. At lower altitudes,the ice ablates from the surface,and may be described by a negative accumulation rate.

In general a glacier will consist of two different types of ice, called 'cold' and 'temperate',in which the temperature is respectively below and at the melting point. Following a suggestion by Miller (1976), we shall refer to such a glacier as polythermal. Other types of glacier often referred to in the literature are the so-called 'cold' or 'polar' ones,in which the average temperature of the whole ice-mass is below freezing point,and 'temperate' glaciers,which exist in warmer climates (e.g. the Alps),and in which the whole ice-mass is on the average at the melting point.

It is not immediately obvious how cold and temperate glaciers can exist,since a necessary condition for snowfall (ablation) is that the surface temperature be below (at) the melting point,and so one might expect all glaciers to be polythermal. We can explain this on the basis of the seasonal variation in climate.

A polar glacier has little frictional heating and maintains a mean annual temperature less than freezing point,even on the ablation surface, because this surface is only warmed to the melting point for two or three months of the year during which ablation takes place. This warming penetrates no deeper than a surface layer of ~ 10 metres (Robin 1955),and quickly reverts to sub-freezing temperatures during the remainder of the year. The average surface temperature is thus 'cold'.

Conversely,a temperate glacier is at the melting point for most of the year,except for a winter cold spell when snow falls and ice accumulates. A slightly cold layer thus forms on the surface,but it rapidly becomes temperate for the rest of the year because melt-water produced at the surface during the summer percolates through the layer where it refreezes, thus releasing its latent heat to the surrounding slightly cooler ice. This mechanism provides an effectively instantaneous transport of heat which is able to maintain the mean temperature of the bulk of the ice at the melting point throughout the year.

On the glacier base,cold ice adheres to the bedrock and the usual no-slip condition holds. Temperate ice,however,is able to slide over the bedrock by means of a lubricating water film at the ice-rock interface which is maintained there by pressure melting on the upstream face of protruding obstacles. This phenomenon is known as regelation. The flow is not 'inviscid' however,as the ice experiences a drag due to pressure variations in the flow over small scale undulations in the bedrock profile. An appropriate boundary condition is that the velocity and shear stress are functionally related. This question is considered in some detail in Chapters III and IV.

Many interesting large-scale phenomena have been observed in glacier flow. Of those that occur on a global scale,one should mention slow 'kinematic' waves,fast 'seasonal' waves,and 'surges'. The kinematic waves take the form of undulations of the glacier surface which travel down it at three to four times the surface speed (see Nye (1960) and the references given there). Seasonal waves are waves of velocity which travel at about 20 to 150 times the surface speed of the ice (Hodge 1974, Deeley and Parr 1914). Surging glaciers exhibit a periodic motion like a relaxation oscillation which consists of two phases:one active (surging), when the velocity increases by an order of magnitude,and a large part of the glacier travels far down the valley;and the other quiescent,when the glacier slowly withdraws up its valley to its pre-surge state. These phases last typically 1-2 years and 30-40 years respectively (Meier and

Post 1969).

Of these three phenomena, the first has been considered by Nye (1960, 1963) using Lighthill and Whitham's (1955) theory of kinematic waves. Nye obtains a similar wave speed to those actually observed. For seasonal waves, there is no quantitative theory which can predict the high propagation rates of velocity disturbances. A similar difficulty besets surge models, although for this case the physics is much better understood (Weertman 1969), and various numerical models based on such physical descriptions (e.g. Budd 1975) have exhibited surges of the right period and amplitude. It remains to be shown, however, that any of the proposed physical mechanisms can be explained on the basis of a mathematical theory.

In this thesis, the attempt is made to develop some aspects of the theory of large scale glacier flow from a mathematically consistent model. This is in contrast to previous models, which have generally been proposed to explain only specific features of the flow (e.g. Nye 1960, Budd 1975). The most detailed analytical model is that of Grigoryan, Krass and Shumskiy (1976), but, as discussed by Fowler and Larson (1977), their model omits certain important features.

In Chapter II we propose a system of equations and boundary conditions which describes a two-dimensional incompressible glacial ice flow. The nondimensionalisation of this model is then discussed in some detail, and the important dimensionless parameters identified. It is shown how this complicated model problem may be vastly simplified by making the usual approximations of slow, shallow flow.

In Chapter III we consider the theory of sliding. Firstly the problem is properly formulated, and a consistent model proposed. This is rationally nondimensionalised, and it is shown that for all but the smallest scale obstacles, the drag is determined purely by the ice flow. Nye's (1969) theory is reviewed in the light of this formulation, and then a variational principle is described in order to estimate the drag as a function of the velocity in the non-Newtonian case. Simple bounds are obtained, and the parameter governing the magnitude of the basal sliding velocity is identified.

In Chapter IV, we revert to the Newtonian case to study the sliding law when cavitation occurs and the bedrock slope is small. An exact solution is presented for the first order term in an asymptotic expansion; this solution indicates that we need to solve for the second term in order to obtain the precise information we require. This is not carried out here, but no difficulty is envisaged in utilising the same method of solution again.

In Chapter V, Nye's (1960) kinematic theory is presented from a non-linear viewpoint, and certain errors in his analysis thus corrected. In particular, it is shown that for a small initial perturbation from the steady state depth profile, shocks can form near the glacier snout, although in practice these will be smoothed out by a diffusion-type term.

In Chapter VI, it is shown how a finite slope and finite velocity at the snout of a temperate glacier is predicted by the model.

In Chapter VII, we consider the steady state flow and temperature profiles in the asymptotic limit of large conduction. In this case the temperature equation is uncoupled and can be solved explicitly: we find that the temperature increases monotonely with depth. For certain parameter ranges a temperate region may exist next to the glacier bed. It is also shown how it is essential to take a sliding law that is continuously dependent on temperature, since the limiting solution as the sliding law becomes discontinuous at the melting temperature is <u>not the same</u> as the solution with a discontinuous sliding law. This is directly analogous to the existence of 'mushy' regions in Stefan problems (Atthey 1972). We also demonstrate explicitly that the (unique) steady state solution in this case is linearly unstable if the viscous dissipation parameter is sufficiently large. No such explicit stability result appears to have been given previously in the literature.

Finally. in Chapter VIII, we consider the (more realistic) limit of small heat conduction which is expected to be a good approximation for large glaciers. In this case a thermal boundary layer exists next to the bedrock, which is of fundamental importance in determining the depth and basal velocity. Appropriate outer and inner problems are described, and some initial progress is made in solving them.

CHAPTER II

Glacier Flow Model

§1 Equations and Boundary Conditions.

The general physics of glacier flow has been described in Chapter I. Further details may be found in Paterson (1969), and will be presented in this chapter as the need arises. We restrict our attention to those glaciers whose width is much greater than their depth, so that the flow may be considered to be two-dimensional. This is not a serious restriction. A typical geometry is shown in Figure 2.1.

We take axes (x,y) along and perpendicular to the mean bedrock slope. The position of the origin is arbitrary, and will be chosen for convenience when necessary in later analysis. We shall denote by $y = \eta(x,t)$ the top surface, by $y = h(x)$ the (given) bedrock profile, and by $y = y_M(x,t)$ the surface dividing the cold and temperate zones, which will be called the melting surface. (This curve will generally be considered to be unique.) Then, as shown in the figure, x_M and x_T are the intersections of y_M with h and η respectively. These points will be referred to as the bottom and top melting points. We denote the ends of the glacier by x_0 and x_S, x_0 being the 'head', and x_S being the 'snout'. In general, these points are all functions of time.

We assume the ice is incompressible and isotropic. Let the mean bedrock slope be ε (so the angle of inclination of the x axis to the

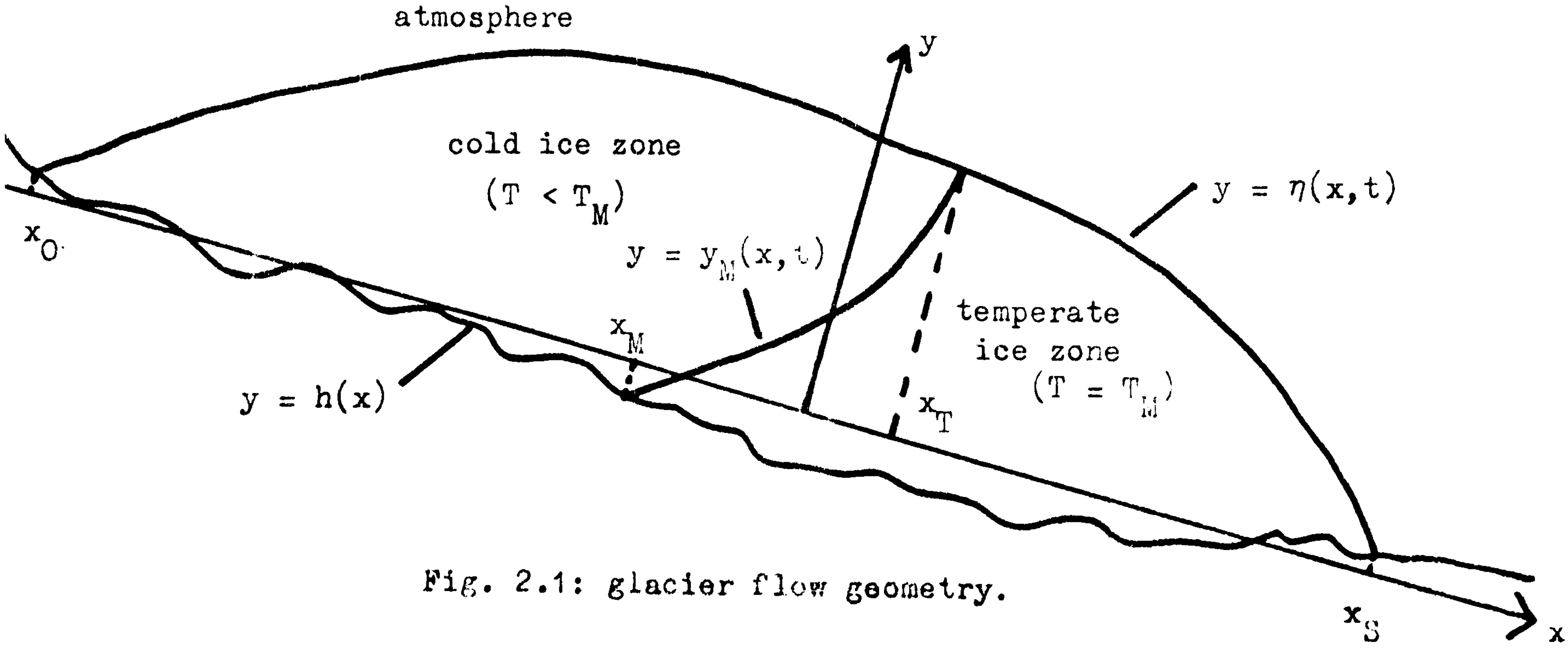


Fig. 2.1: glacier flow geometry.

to the horizontal is $\tan^{-1}\varepsilon$). Then the equations of conservation of mass and momentum for the velocity $\underline{q} \equiv (q_1, q_2) \equiv (u, v)$ and pressure p are

$$u_x + v_y = 0, \tag{2.1}$$

$$\rho(u_t + uu_x + vu_y) = -p_x + \rho g'\varepsilon + \tau_{1x} + \tau_{2y}, \tag{2.2}$$

$$\rho(v_t + uv_x + vv_y) = -p_y - \rho g' + \tau_{2x} - \tau_{1y}, \tag{2.3}$$

where $g' = g(1 + \varepsilon^2)^{-\frac{1}{2}}$, g is the acceleration due to gravity and ρ is the density. Lettered subscripts denote partial differentiation, and τ_1 and τ_2 are the longitudinal and tangential stress deviators, defined for an incompressible isotropic medium by

$$\begin{aligned} \sigma_{ij} &= -p\delta_{ij} + \tau_{ij}, \\ \tau_1 &= \tau_{11} = -\tau_{22}, \\ \tau_2 &= \tau_{12} = \tau_{21}, \end{aligned} \tag{2.4}$$

where σ_{ij} is the stress tensor; suffices 1 and 2 denote x and y components respectively. In (2.4) and henceforth the summation convention is employed. We define the strain rate tensor e_{ij} by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right). \tag{2.5}$$

Then in the cold zone, the energy equation is

$$\rho c(T_t + uT_x + vT_y) = \tau_{ij}e_{ij} + k\nabla^2 T, \tag{2.6}$$

where c is the specific heat of ice and k is its thermal conductivity.

This is not suitable for the temperate zone, where the temperature is approximately constant. Physically, the viscous dissipation $\tau_{ij}e_{ij}$ melts the ice at a rate of $\sim 1\%$ by volume a year, and this moisture may be trapped within the ice, in which case it strongly affects the stress/strain rate law (Lliboutry 1976). In this case the appropriate variable to work with is the moisture content w, expressed as the volume fraction of water in the ice,and the energy equation for temperate ice,following Lliboutry, may be written

$$\rho L\frac{dw}{dt} = \tau_{ij}e_{ij}, \tag{2.7}$$

where $\frac{d}{dt}$ is the material derivative, L is the latent heat,and we have ignored negligible terms involving temperature gradients which are discussed by Lliboutry. (2.7) and (2.10) below constitute a first attempt to describe the dynamic effect of the moisture content on the flow of temperate ice. In postulating (2.7),we have tacitly made the major assumption that water transport within such ice is negligible. No justification for this neglect is offered here,and we shall not discuss the point further.

We require a constitutive relation between e_{ij} and τ_{ij}. For cold ice,we use the generally accepted power law (Glen 1953,1955)

$$\begin{aligned} e_{ij} &= A\tau^{n-1}\tau_{ij}\ , \quad e = A\tau^{n}\ , \\ 2e^2 &= e_{ij}e_{ij}\ , \quad 2\tau^2 = \tau_{ij}\tau_{ij}\ , \end{aligned} \tag{2.8}$$

where A is a function of T,most commonly considered to be an Arrhenius-type term of the form

$$A = B\exp(-Q/RT)\ , \tag{2.9}$$

Q being an activation energy and R the gas constant. Measured values for ice give $n \simeq 3$ for stresses of ~ 1 bar,and $Q \simeq 12$ kcal/mole (Baraty and Tabor 1958—but see also Barnes,Tabor and Walker 1971 and Glen 1955). Apparently n is higher for larger stresses,and decreases to 1 for low stresses (Budd and Radok 1971),as it must do if ice is to have a finite viscosity at zero stress (Bird 1976).

Much less is known of the appropriate relation between e_{ij} and τ_{ij} for temperate ice. Lliboutry (1976) reports a law like (2.8) with $n = 3$ from the experimental work of Duval. He also states that if,for example, the moisture content increases from 0.1% to 1%, then the strain rate e at a given stress increases by a full order of magnitude.

For want of any better information, we therefore assume for temperate ice that (2.8) holds, where now A is a function of the moisture content w, so that

$$A = r(w) \tag{2.10}$$

and we take n to be the same exponent as in Glen's law. The rheological function r(w) is to be determined experimentally.

The boundary conditions to be satisfied are the following ones.

On the surface $y = \eta(x,t)$ (which is to be determined), we specify

(i) continuity of the stress tensor:

$$\sigma_{ij}n_j = -p_A n_i \ , \tag{2.11}$$

Here $\underset{\sim}{n}$ is the unit normal to $y = \eta$, and p_A is the atmospheric pressure, considered given. We also have

(ii) the kinematic condition

$$\frac{d}{dt}(y-\eta) = -a(x,t) \ , \tag{2.12}$$

where a(x,t) is the accumulation rate, which acts as a source (or sink) of ice at the top surface; it will be considered to be given (it is in fact a complicated function of the climatic conditions). A typical graph of a against x (averaged over a year) is shown in Figure 2.2, following

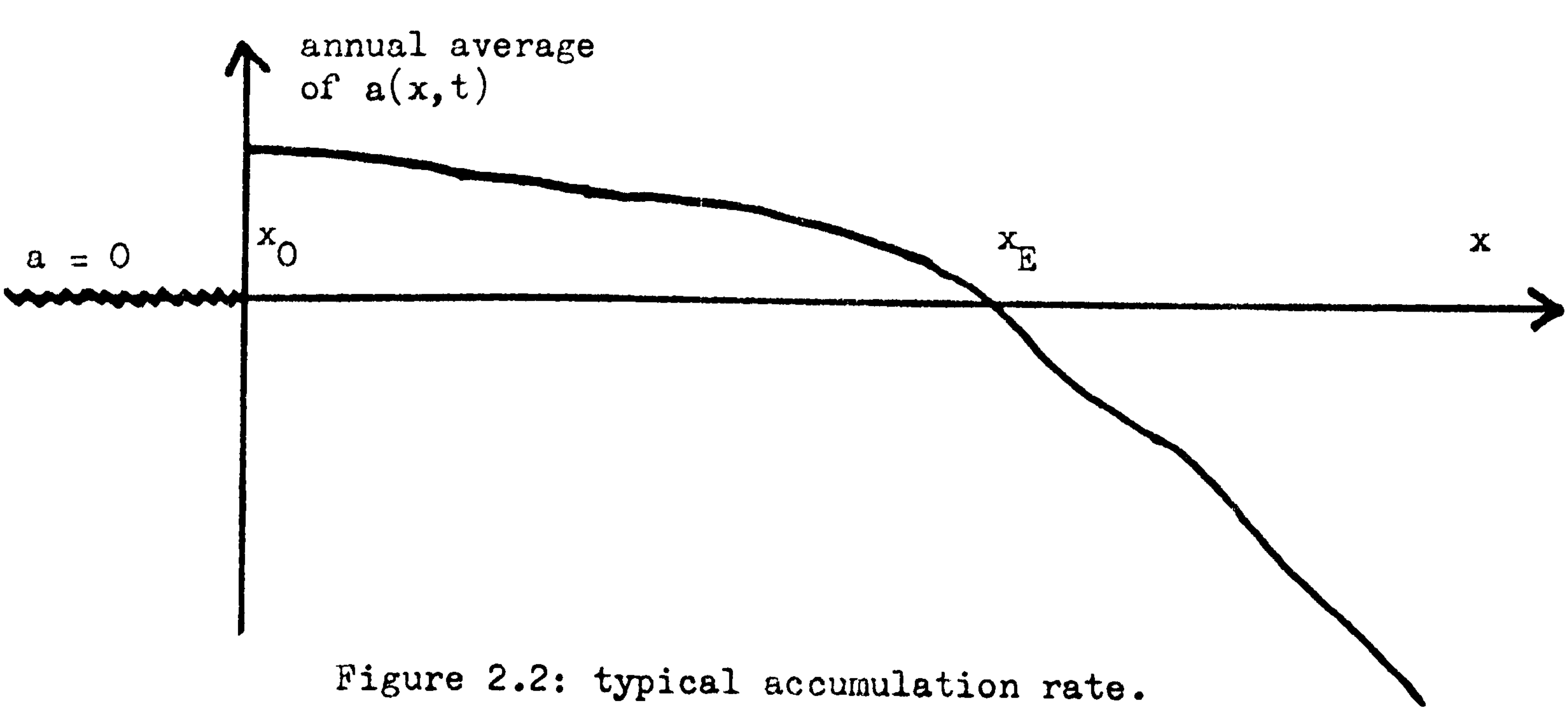


Figure 2.2: typical accumulation rate.

data given for the South Cascade glacier by Nye (1963). In the accumulation area $a > 0$ and in the ablation area $a < 0$. The (assumed unique) point on the glacier surface where $a = 0$ at any time t will be called the equilibrium point, and will be denoted by $x_E(t)$. Note that $a = 0$ in $x < x_0$.

Finally we have in the cold zone

(iii) continuity of temperature:

$$T = T_A \quad \text{while} \quad T < T_M \text{ , i.e. } x < x_T \text{ ,} \tag{2.13}$$

where T_A is the atmospheric temperature, considered given. T_M is the melting temperature, defined by the Clausius-Clapeyron relation

$$T_M = T_F - \theta(p-p_A) \text{ ,} \tag{2.14}$$

where T_F is the melting point at atmospheric pressure and θ is a known constant. T_A and p_A may both be functions of x and t, but we shall consider p_A to be constant. It is reasonable to expect that T_A will be a monotone increasing function of x.

The boundary conditions on the bedrock require a little more consideration. As already described in Chapter I, temperate ice slides on its bedrock, but the flow is not 'inviscid' because the ice experiences a resistance due to the small-scale roughness of the bedrock. It is convenient to consider the ice flow as consisting of an inner and outer part, as in normal boundary layer theory (Batchelor 1967). The inner part contains

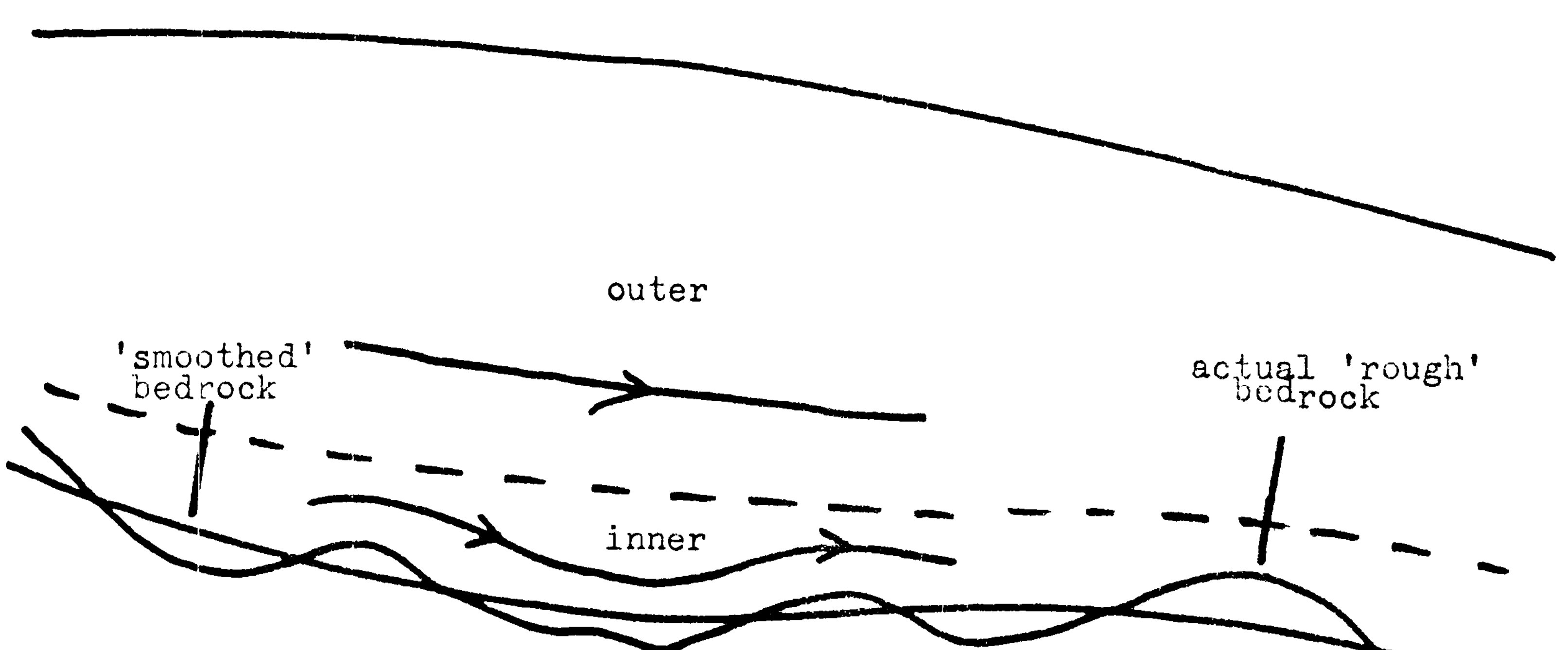


Figure 2.3

the detail of the flow over the roughness of the bedrock, whereas the outer part describes the flow of the bulk of the ice over a smoothed bed as shown in Figure 2.3. The appropriate matching condition (to first order) is that the limiting value of q_{outer} on the smoothed bed should equal the limiting value of q_{inner} as $y_i \to \infty$, where y_i is a scaled coordinate describing the inner flow. (This process is described more fully in Chapter III.) Now in principle one can determine the value of q_{inner} as $y_i \to \infty$ in terms of the drag per unit area exerted by the ice on the bed (Nye 1969). Since this latter term must be equal to the applied stress at $y_i = \infty$, which is itself (by matching) equal to the basal shear stress for the outer flow, it follows that appropriate boundary conditions on the smoothed bed for the outer flow are that

$$q_N = 0 ,$$
$$q_T = f(\tau_{NT}) \quad \text{on } y = h(x), \; x > x_M , \tag{2.15}$$

where h is the smoothed bed, N and T are normal and tangential components respectively, and the function f is to be determined from the solution of the inner flow problem. We might replace the first of (2.15) by $q_N = V(x)$ if there were a run-off of melt-water at the bedrock. However, even if this is the case, typical values of V(x) are completely negligible compared to typical ice flow rates (Lliboutry 1968), and so we shall only consider $q_N = 0$.

(2.15) holds when the ice is temperate. When the ice is cold, the normal no-slip and continuity of heat flux conditions imply

$$u = v = 0 , \quad k(h_x T_x - T_y) = G(1+h_x{}^2)^{\frac{1}{2}} \quad \text{on } y = h, \; T < T_M . \tag{2.16}$$

G is the geothermal heat flux, usually taken as constant. (2.16) states that the ice is frozen to the bedrock.

Now realistic sliding laws have $f(0) = 0$; thus (2.15) and (2.16) predict a discontinuity in the stress (if we have continuous velocity) on the bedrock. In practice, such a discontinuity will not occur, because the no-slip condition in (2.16) becomes invalid when the temperature is near the pressure melting point. This is because the ice will start to slide when the pressure variation on the bedrock is sufficient to cause a small part of the basal ice to reach the pressure melting point, so that the ice becomes lubricated in patches (cf. Robin 1976). Thus the appropriate boundary condition for such ice is a sliding law similar to (2.15) which is dependent on temperature close to the melting point. We can then conveniently prescribe the following bedrock boundary condition which is valid in both cold and temperate regions:

$$q_T = F(\tau_{NT}, T) , \qquad (2.17)$$

where, if T_Q is the temperature at which sliding starts to occur, a typical form of F would be

$$\begin{aligned} &F(\tau_{NT}, T) = 0 , \quad T < T_Q , \\ &F(\tau_{NT}, T) \text{ monotone increasing}, \quad T_Q < T < T_M , \qquad (2.18) \\ &F(\tau_{NT}, T_M) = f(\tau_{NT}) . \end{aligned}$$

As $T_Q \to T_M$, the sliding law (2.17) tends to the discontinuous limit represented by (2.15) and (2.16). However, we shall see in Chapter VII that the limiting behaviour of solutions satisfying (2.18) is not the same as that of solutions satisfying (2.15) and (2.16); since (2.18) is the physically realistic case, we therefore retain the temperature dependence in (2.18).

In a similar manner, we shall assume that the cold ice is supplied with a constant geothermal heat flux G until $T = T_Q$. For $T_Q < T < T_M$, the geothermal heat flux 'felt' by the outer flow becomes progressively less as more and more of it is used up in the lubrication of the bedrock, until $T = T_M$ when it is zero. Thus the heat flux condition may be written

$$-\sigma_{NT}q_T + k\frac{\partial T}{\partial n} = G\Lambda(T) \quad , \; T < T_M \quad , \tag{2.19}$$

where $\underset{\sim}{n}$ is the unit outward normal to $y = h(x)$, and

$$\Lambda(T) = 1 \, , \; T < T_Q \; ; \Lambda \text{ is monotone decreasing, } T_Q < T < T_M \; ; \Lambda(T_M) = 0 \; . \tag{2.20}$$

If we denote by x_Q and x_Z the points where $T = T_Q$ and T first equals T_M, then an appropriate boundary condition in $x > x_Z$ is that $T = T_M$ until $\frac{\partial T}{\partial n} = 0$, which defines x_M and is where the melting surface breaks away from the bedrock. The thermal boundary condition is discussed in more detail in Appendix 2. The final set of bedrock boundary conditions is therefore

$$\begin{gathered} q_N = 0 \; , \; q_T = F(\tau_{NT}, T) \; , \\ -\sigma_{NT}q_T + k\frac{\partial T}{\partial n} = G\Lambda(T) \; (x < x_Z) \, , \; T = T_M \; (x_Z < x < x_M) \; . \end{gathered} \tag{2.21}$$

Lastly, on the melting surface $y = y_M(x,t)$, we specify that

$$\begin{gathered} \text{all variables and the heat flux } k\frac{\partial T}{\partial n} \text{ are continuous,} \\ T = T_M = T_F - \theta(p - p_A) \; , \; w = 0 \; . \end{gathered} \tag{2.22}$$

This completes the model. Formally, we also require a set of initial conditions, but these can conveniently be left arbitrary.

§2 Nondimensionalisation.

We now seek to nondimensionalise the equations and boundary conditions of the previous section by using the naturally occurring dimensional parameters of the problem. The accumulation rate contains two such parameters, namely a typical velocity and a typical length scale: we make these explicit as follows. Let

$$s(x,t) = \int_{x_0(t)}^{x} a(\sigma,t)\, d\sigma \quad ; \tag{2.23}$$

$s(x,t)$ will be called the 'flux' function, or simply the flux, since it is in fact equal to the flux of ice through the line x = constant in the steady state (this is obvious from mass conservation and the no flow through condition on the bedrock). We choose a_0 and ℓ so that ℓa_0 is a typical magnitude of s, and a_0 is a typical magnitude of $\frac{\partial s}{\partial x} = a$: a_0 is thus a mean accumulation/ablation rate, expressed in metres per year, and is the natural scale for the vertical velocity v ; ℓ is the natural length of the problem. A typical profile of s is shown in Figure 2.4.

Let $[\tau]$ be a typical stress scale, d be a typical height scale, and $[w]$ be a typical moisture content. At the moment these are unspecified, and are to be determined. The usual balance of terms in the equation of continuity implies that the natural horizontal velocity scale is $a_0\ell/d$.

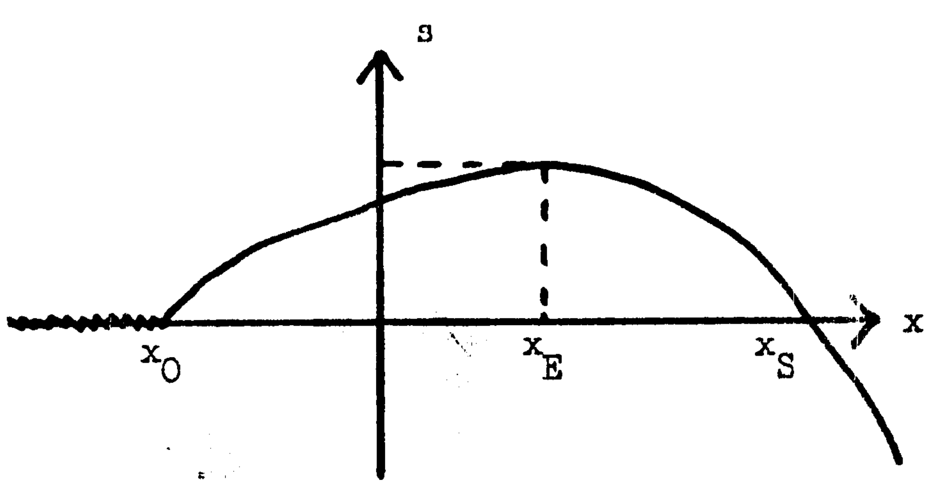


Figure 2.4

We write

$$\begin{aligned}
&x = \ell x^* \ , \ y = dy^* \ , \ v = a_0 v^* \ , \ u = (a_0\ell/d)u^* \ , \ t = (d/a_0)t^* \ , \\
&s(x,t) = \ell a_0 s^*(x^*,t^*) \ , \ y_M(x,t) = dy_M^*(x^*,t^*) \ , \\
&h(x) = dh^*(x^*) \ , \ \eta(x,t) = d\eta^*(x^*,t^*) \ , \ e = (a_0\ell/2d^2)e^* \ , \\
&T = T_F + T_0 T^* \ , \ p = p_A + \rho g' d[\eta^* - y^*] + \delta[\tau]p^* \ , \ w = [w]w^* \ .
\end{aligned} \tag{2.24}$$

T_F is the freezing temperature ≃ 273 K at atmospheric pressure p_A. T_0 is the temperature range of the climate. Since generally we expect the surface temperature to be coldest at the glacier head, T_0 may be taken to be the temperature there, say $-T_0$ °C. In (2.24), δ is defined by

$$\delta = d/\ell \ . \tag{2.25}$$

Since we anticipate negligible inertia terms, the natural scale for the pressure (minus its hydrostatic component) is $[\tau]$. The factor δ in (2.24) is inserted by foresight of the result. It will be justified a posteriori by showing that it leads to a sensibly scaled problem for p^*. We will assume $h^{*\prime}$ and therefore h^* are $O(1)$: for reasonably smooth landforms this is not a drastic assumption.

Let us suppose that the (assumed known) rheological function A in (2.8) is scaled by A_0 in both cold and temperate zones so that

$$A = A_0 A^* \,, \tag{2.26}$$

where A^* is a dimensionless $O(1)$ function of an $O(1)$ argument, this being the dimensionless temperature or moisture as appropriate.

From (2.8), we have

$$e_{ij} = A\tau^{n-1}\tau_{ij} \,, \tag{2.27}$$

whence

$$\tau_{ij} = \frac{1}{A} e_{ij}\left(\frac{A}{e}\right)^{\frac{n-1}{n}} = e_{ij}A^{\frac{-1}{n}} e^{-\left(\frac{n-1}{n}\right)} \tag{2.28}$$

and

$$\tau_{ij}e_{ij} = 2A^{\frac{-1}{n}} e^{\frac{n+1}{n}} \,. \tag{2.29}$$

Now from (2.5)

$$e_{11} = u_x \,, \; e_{12} = e_{21} = \tfrac{1}{2}(u_y+v_x) \,, \; e_{22} = v_y \,, \tag{2.30}$$

so that

$$e = \tfrac{1}{2}\left[4u_x^{\,2} + (u_y+v_x)^2\right]^{\frac{1}{2}} . \tag{2.31}$$

Thus (2.5) gives

$$\begin{aligned} \tau_1 = \tau_{11} &= A^{\frac{-1}{n}} u_x \, e^{-\left(\frac{n-1}{n}\right)} \,, \\ \tau_2 = \tau_{12} &= A^{\frac{-1}{n}} \tfrac{1}{2}(u_y+v_x)\, e^{-\left(\frac{n-1}{n}\right)} \,. \end{aligned} \tag{2.32}$$

From (2.24) and (2.31),

$$e^* = \left[(u^*_{y^*}+\delta^2 v^*_{x^*})^2 + 4\delta^2 u^{*2}_{x^*} \right]^{1/2} . \qquad (2.33)$$

From (2.24),(2.26) and (2.32),

$$\tau_1 = \left(\frac{a_0 \ell}{2A_0 d^2} \right)^{\frac{1}{n}} 2\delta A^{*-\frac{1}{n}} u^*_{x^*} e^{*-(\frac{n-1}{n})} , \qquad (2.34)$$

$$\tau_2 = \left(\frac{a_0 \ell}{2A_0 d^2} \right)^{\frac{1}{n}} (u^*_{y^*}+\delta^2 v^*_{x^*}) e^{*-(\frac{n-1}{n})} A^{*-\frac{1}{n}} . \qquad (2.35)$$

Thus, assuming $\delta \lesssim 1$, it is natural to define the stress scale as

$$[\tau] = \left(\frac{a_0 \ell}{2A_0 d^2} \right)^{\frac{1}{n}} , \qquad (2.36)$$

and for abbreviation we write

$$\tau_1 = \delta[\tau]\tau^*_1 \quad , \quad \tau_2 = [\tau]\tau^*_2 \quad . \qquad (2.37)$$

Of all the parameters in (2.24), only d and [w] remain to be defined. We choose d by considering the appropriate balance of terms in the x-momentum equation. We have already stated that the flow is driven by gravity, hence (neglecting the inertia terms) the x-component of gravity must be balanced by the largest of the stress terms which, as we shall see, is the vertical gradient of the shear stress: that is, we must balance the terms τ_{2y} and $\rho g'\varepsilon$ in (2.2). Thus from (2.24) and (2.37) we set

$$d = \left(\frac{[\tau]}{\rho g'\varepsilon} \right) = \left[\frac{a_0 \ell}{2A_0 (\rho g'\varepsilon)^n} \right]^{\frac{1}{n+2}} , \qquad (2.38)$$

using (2.36), and we show below that this does indeed give a consistent scaling.

We should note that there is a certain ambiguity in the definition of A_0, since we could choose it to be either A_T, the natural scale for the (cold) temperature-dependent flow law (2.9), or A_w, the scale for the (temperate) moisture-dependent flow law (2.10). This is really a matter of convenience, and we shall take

$$A_0 = A_T \tag{2.39}$$

in (2.26) for a cold or polythermal glacier. We return to this point later.

From (2.9) and (2.24), we have in the cold zone

$$A = B \exp\left[\frac{-Q}{RT}\right] = B \exp\left[\frac{-Q}{RT_F}\left(1+\frac{T_0}{T_F}T^*\right)^{-1}\right]$$

$$= B \exp\left[-\frac{Q}{RT_F}\left\{1 - \frac{T_0}{T_F}T^* + O\left(\frac{T_0^2}{T_F^2}\right)\right\}\right] \quad . \tag{2.40}$$

Now typically $T_0 \lesssim 20$ K, and $T_F \simeq 273$ K, hence $T_0 \ll T_F$ and we may neglect $O(T_0^2/T_F^2)$ in (2.40). Hence it is justifiable to approximate A by

$$A = A_T \exp(\kappa T^*) \ , \tag{2.41}$$

$$\kappa = QT_0/RT_F^2 \ , \ A_T = B \exp(-Q/RT_F) \ . \tag{2.42}$$

Some authors assume a law of the form (2.41) ab initio.

In order to consider appropriate values for $[w]$ and A_w, let us consider a completely temperate glacier. The appropriate choice for A_0 is then

$$A_0 = A_w \tag{2.43}$$

and we choose $[w]$ by balancing the terms in the energy equation. From (2.7), (2.29) and (2.24),

$$[w] = \frac{2A_w[\tau]^{n+1}d}{\rho La_0} = \frac{2A_w}{\rho La_0}\left(\frac{a_0\ell}{2A_w}\right)^{\frac{n+1}{n}}\left(\frac{2A_w}{a_0\ell}\right)^{\frac{1}{n}}\rho g'\varepsilon$$

$$= \frac{\ell g'\varepsilon}{L} \tag{2.44}$$

and is thus independent of A_w. Given this typical moisture content, A_w is then chosen so that $r^*(w^*)$ defined by

$$A = A_w r^*(w^*) \tag{2.45}$$

is O(1). Since (2.44) is independent of A_w, we define [w] by (2.44) for a polythermal glacier also.

Under the nondimensionalisation (2.24), and using the results given above, we may (after some algebra) rewrite the nonlinear free boundary problem posed in §1 as follows:

$$u^*_{x^*} + v^*_{y^*} = 0 , \tag{2.46}$$

$$\tau^*_{2y^*} = -1 + \mu\eta^*_{x^*} + \delta^2 [p^*_{x^*} - \tau^*_{1x^*}] + \mathrm{Re}\,\frac{du^*}{dt^*} , \tag{2.47}$$

$$p^*_{y^*} = \tau^*_{2x^*} - \tau^*_{1y^*} - \mathrm{Re}\,\frac{dv^*}{dt^*} , \tag{2.48}$$

$$\frac{dT^*}{dt^*} = \beta_1 A^{*\frac{-1}{n}} e^{*\frac{n+1}{n}} + \beta_2 [T^*_{y^*y^*} + \delta^2 T^*_{x^*x^*}] , \quad T^* < T^*_M , \tag{2.49}$$

$$\frac{dw^*}{dt^*} = A^{*\frac{-1}{n}} e^{*\frac{n+1}{n}} , \quad T^* = T^*_M , \tag{2.50}$$

where $\frac{d}{dt^*}$ is the material derivative, and T^*_M is defined in (2.56).

$$\tau^*_1 = 2u^*_{x^*} A^{*\frac{-1}{n}} e^{*-(\frac{n-1}{n})} , \tag{2.51}$$

$$\tau^*_2 = (u^*_{y^*} + \delta^2 v^*_{x^*}) A^{*\frac{-1}{n}} e^{*-(\frac{n-1}{n})} , \tag{2.52}$$

$$e^* = \left[(u^*_{y^*} + \delta^2 v^*_{x^*})^2 + 4\delta^2 u^{*2}_{x^*} \right]^{\frac{1}{2}} , \tag{2.53}$$

where from (2.39) and (2.41)

$$A^* = \exp(\kappa T^*) , \quad T^* < T^*_M , \tag{2.54}$$

and from (2.26) and (2.45)

$$A^* = \gamma r^*(w^*) , \quad T^* = T^*_M . \tag{2.55}$$

T^*_M is defined from (2.14) by

$$T^*_M = -\theta^* [\eta^* - y^* + \delta\varepsilon p^*] . \tag{2.56}$$

The parameters occurring in (2.46)-(2.56) are defined by

$$\delta = \frac{d}{\ell} = \frac{1}{\ell}\left[\frac{a_0 \ell}{2A_0(\rho g'\varepsilon)^n}\right]^{\frac{1}{n+2}} ; \tag{2.57}$$

$$\mu = \frac{\delta}{\varepsilon} ; \tag{2.58}$$

the Reynolds number

$$Re = \frac{\rho a_0^2 \ell}{[\tau] d} = \frac{a_0^2 \ell}{\varepsilon g'}\left[\frac{a_0 \ell}{2A_0(\rho g'\varepsilon)^n}\right]^{\frac{-2}{n+2}} ; \tag{2.59}$$

$$\beta_1 = \frac{[\tau]}{\rho c T_0 \delta} = \frac{\varepsilon g' \ell}{c T_0} ; \tag{2.60}$$

$$\beta_2 = \frac{k}{\rho c a_0 d} = \frac{k}{\rho c a_0}\left[\frac{a_0 \ell}{2A_0(\rho g'\varepsilon)^n}\right]^{\frac{-1}{n+2}} ; \tag{2.61}$$

$$\kappa = \frac{Q T_0}{R T_F^2} ; \tag{2.62}$$

$$\gamma = \frac{A_w}{A_0} ; \tag{2.63}$$

$$A_0 = \begin{cases} A_T \text{ (cold or polythermal glacier) :} \\ A_w \text{ (temperate glacier);} \end{cases} \tag{2.64}$$

$$\theta^* = \frac{\theta \rho g' d}{T_0} . \tag{2.65}$$

The boundary conditions for (2.46)-(2.56) are these:

on the surface $y^* = \eta^*(x^*,t^*)$:

$$\tau_2^* + \delta^2(p^* - \tau_1^*)\eta_{x^*}^* = 0 , \tag{2.66}$$

$$\tau_1^* + p^* + \tau_2^* \eta_{x^*}^* = 0 , \tag{2.67}$$

$$\eta_{t^*}^* + u^* \eta_{x^*}^* - v^* = s_{x^*}^*(x^*,t^*) , \tag{2.68}$$

$$T^* = T_A^*(x^*,t^*) , \quad x^* < x_T^* , \tag{2.69}$$

where

$$x_T^* = x_T/\ell , \quad T_A^* = \frac{T_A - T_F}{T_0} ; \tag{2.70}$$

on the bedrock $y^* = h^*(x^*)$,

$$u^*h^{*\prime} - v^* = 0 \ , \quad (2.71)$$

$$\tau^*_{NT} = F^*(q^*_T, T^*) \ , \quad (2.72)$$

where F^* is a dimensionless form of F in (2.16), and

$$\beta_1 \tau^*_{NT} q^*_T + \beta_2 \frac{(T^*_{y^*} - \delta^2 h^{*\prime} T^*_{x^*})}{(1 + \delta^2 h^{*\prime 2})^{\frac{1}{2}}} = - \lambda\beta_2 \Lambda^*(T^*) \ , \quad x^* < x^*_Z \ ,$$
$$T^* = T^*_M \ , \quad x^*_Z < x^* < x^*_M \ , \quad (2.73)$$

where

$$\tau^*_{NT} = \frac{(1 - \delta^2 h^{*\prime 2})\tau^*_2 - 2\delta^2 h^{*\prime}\tau^*_1}{1 + \delta^2 h^{*\prime 2}} \ , \quad q^*_T = \frac{u^* + \delta^2 v^* h^{*\prime}}{(1 + \delta^2 h^{*\prime 2})^{\frac{1}{2}}} \ ,$$
$$x^*_M = x_M/\ell \ , \quad \lambda = Gd/kT_0 \ , \quad \Lambda^*(T^*) \equiv \Lambda(T) \ ; \quad (2.74)$$

on the melting surface $y^* = y^*_M(x^*, t^*)$,

$$\text{all variables and } (T^*_{y^*} - \delta^2 T^*_{x^*} y^*_{Mx^*}) \text{ are continuous,}$$
$$T^* = T^*_M \ , \quad w^* = 0 \ . \quad (2.75)$$

The physical constants appearing in the model are given in Table 1.

Constant	Value	Units	Source
T_F	273	K	Paterson (1969)
ρ	900	kg m^{-3}	"
g	9.8	m s^{-2}	"
c	2×10^3	J kg^{-1} K^{-1}	"
k	7×10^7	J K^{-1} m^{-1} y^{-1}	"
L	3.3×10^5	J kg^{-1}	"
R	8.3	J $mole^{-1}$ K^{-1}	"
θ	0.74×10^{-2}	K bar^{-1}	"
n	3.2 (cold)	-	Glen (1955)
	3 (temperate)		Lliboutry (1976)
A_T	0.17	bar^{-n} y^{-1}	Glen (1955)
A_W	$\gtrsim 10A_T$ (?)	"	Lliboutry (1976)
Q	6×10^4	J $mole^{-1}$	Raraty and Tabor (1958)
1 y = 1 year = 3×10^7 s ; 1 bar = 10^5 N m^{-2}			

Table 1.

The values of the last four constants in Table 1 are not known as accurately as the others, but it is considered that they are at least of the right orders of magnitude.

There are five dimensional inputs to the problem: $a_0, \ell, G, \varepsilon, T_0$. Typical values of these are shown in Table 2 (Paterson 1969). With these values,

a_0	1 m y^{-1}
ℓ	10 km
G	$1.6 \times 10^6 \text{ J m}^{-2} \text{ y}^{-1}$
ε	10^{-1}
T_0	20 K

Table 2. Physical inputs.

we calculate that a typical depth of the glacier is $d \sim 100$ m, a typical velocity $a_0 \ell/d \sim 100 \text{ m y}^{-1}$, a typical moisture content 3%, and a typical stress $[\tau] \sim 1$ bar. Such values are commonly observed, and are a useful check of the validity of our nondimensionalisation procedure.

Using the values given in Tables 1 and 2, we find that typical values of the dimensionless parameters which occur are

$$
\begin{aligned}
&\delta \sim 10^{-2} , \\
&\mu \sim 10^{-1} , \\
&Re \sim 10^{-13} , \\
&\beta_1 \sim 0.25 , \\
&\beta_2 \sim 0.3 , \\
&\kappa \sim 1 , \\
&\gamma \sim 10 \ (?) , \\
&\theta^* \sim 10^{-2} , \\
&\lambda \sim 10^{-1} .
\end{aligned}
\tag{2.76}
$$

In what follows, we shall use (2.76) to neglect certain terms in the complete model. However we shall stipulate, following the discussion leading up to (2.39), that γ is to be considered $O(1)$ — its only effect is to alter slightly (by $\gamma^{\frac{1}{n+2}}$) certain of the scales we introduced, as can be inferred from, for example, (2.38). Since in any real situation $\gamma^{1/(n+2)}$ is not expected to be large, this is a reasonable procedure.

§3 The Reduced Model.

Motivated by (2.76), we make the slow flow approximation

$$Re \to 0 \tag{2.77}$$

and the shallow ice approximation

$$\delta \to 0 \ . \tag{2.78}$$

We note that (2.78) leads to the loss of highest derivatives in the equations, and we may expect the approximation to break down in certain circumstances, when the problem is of singular type. As usual in singular perturbation theory (Cole 1968), this problem will be dealt with if and when it arises.

We finally make the approximation

$$\theta^* \to 0 \ ; \tag{2.79}$$

then the equations (2.46)-(2.55) become, using (2.56) and dropping the asterisks on the variables for convenience,

$$u_x + v_y = 0 \ , \tag{2.80}$$

$$\tau_{2y} = -1 + \mu\eta_x \ , \tag{2.81}$$

$$p_y = \tau_{2x} - \tau_{1y} \ , \tag{2.82}$$

$$\tau_1 = 2u_x A^{\frac{-1}{n}} e^{-(\frac{n-1}{n})} \ , \tag{2.83}$$

$$\tau_2 = u_y A^{\frac{-1}{n}} e^{-(\frac{n-1}{n})} \ , \tag{2.84}$$

$$e = |u_y| \ , \tag{2.85}$$

(note e here is <u>not</u> the exponential function);

in the cold zone $T < 0$ and

$$T_t + uT_x + vT_y = \beta_1 A^{\frac{-1}{n}} e^{(\frac{n+1}{n})} + \beta_2 T_{yy} , \tag{2.86}$$

$$A = \exp(\kappa T) ; \tag{2.87}$$

in the temperate zone $T \equiv 0$ and

$$w_t + uw_x + vw_y = A^{\frac{-1}{n}} e^{(\frac{n+1}{n})} , \tag{2.88}$$

$$A = \gamma r(w) \equiv r_1(w) , \tag{2.89}$$

say. The boundary conditions are the following:

on $y = \eta(x,t)$,

$$\tau_2 = 0 , \tag{2.90}$$

$$\tau_1 + p + \tau_2\eta_x = 0 , \tag{2.91}$$

$$\eta_t + u\eta_x - v = s_x(x,t) , \tag{2.92}$$

$$T = T_A(x,t) ; \tag{2.93}$$

on the bedrock $y = h(x)$,

$$v = uh' , \tag{2.94}$$

$$u = F(\tau_2, T) , \tag{2.95}$$

$$\beta_1 u\tau_2 + \beta_2 T_y = \lambda\beta_2 A(T) , \; x < x_Z : \qquad T = 0 , \; x_Z < x < x_M ; \tag{2.96}$$

on the melting surface $y = y_M(x,t)$,

all variables are continuous, (2.97)

$$w = 0 , \tag{2.98}$$

$$T = T_y = 0 . \tag{2.99}$$

We can reduce the system somewhat further as follows. First notice that the equations and boundary conditions for p and τ_1 uncouple from the rest: these variables will therefore not be considered further here. We integrate (2.81) using (2.90) to obtain

$$\tau_2 = (\eta - y)(1-\mu\eta_x) \ . \qquad (2.100)$$

Thus $\tau_2 > 0$ (assuming $\eta_x < 1/\mu$), and so from (2.85), $e = u_y$ and (2.84) gives

$$u_y = A\tau_2^{\ n} = A(\eta-y)^n(1-\mu\eta_x)^n \ , \qquad (2.101)$$

using (2.100). The dissipation term in (2.86) and (2.88) is then

$$A^{-\frac{1}{n}} e^{(\frac{n+1}{n})} = A\tau_2^{\ n+1} = A(\eta-y)^{n+1}(1-\mu\eta_x)^{n+1} \ . \qquad (2.102)$$

Now we introduce the stream function ψ by writing

$$u = \psi_y \ , \ v = -\psi_x \ , \qquad (2.103)$$

and make the following change of variables for convenience:

$$\xi = \eta(x,t) - y \ , \ H(x,t) = \eta(x,t) - h(x) \ ,$$
$$\Psi = \psi + \frac{\partial}{\partial t}\int_{x_0(t)}^{x} H(x',t)\ dx' \ . \qquad (2.104)$$

The equations (2.101), (2.86) and (2.88) become, using (2.102), (2.103), (2.100) and (2.104),

$$\Psi_{\xi\xi} = \xi^n \left[1 - \mu(H_x+h_x)\right]^n A \ , \qquad (2.105)$$

$$A = \begin{cases} \exp(\kappa T) \ , & T < 0 \ , \\ r_1(w) \ , & T = 0 \ , \end{cases} \qquad (2.106)$$

$$T_t + \Psi_x T_\xi - \Psi_\xi T_x = \beta_1 \xi^{n+1}\left[1 - \mu(H_x+h_x)\right]^{n+1} \exp(\kappa T) + \beta_2 T_{\xi\xi} \ , \ T < 0 \ , \qquad (2.107)$$

$$w_t + \Psi_x w_\xi - \Psi_\xi w_x = \xi^{n+1}\left[1 - \mu(H_x+h_x)\right]^{n+1} r_1(w) \ , \ T = 0 \ , \qquad (2.108)$$

subject to the boundary conditions:

on $\xi = 0$,

$$\Psi = s(x,t) \tag{2.109}$$

(by choosing $\Psi = 0$ at $x = x_0(t)$ for convenience);

$$T = T_A(x,t) \ , \ x < x_T(t) \ ; \tag{2.110}$$

on $\xi = H(x,t)$,

$$\Psi = \frac{\partial}{\partial t}\int_{x_0(t)}^{x} H(x',t)\, dx' \ , \tag{2.111}$$

$$\Psi_\xi = -F\left[H(1-\mu(H_x+h_x)),T\right] \ , \tag{2.112}$$

$$\begin{aligned} &\beta_2 T_\xi = -\beta_1 \Psi_\xi H[1-\mu(H_x+h_x)] + \lambda\beta_2\Lambda(T) \ , \ x < x_Z : \\ &T = 0 \ , \ x_Z < x < x_M \ ; \end{aligned} \tag{2.113}$$

on $\xi = \eta(x,t) - y_M(x,t) \equiv \xi_M(x,t)$,

$$\Psi, \underset{\sim}{\nabla}\Psi \text{ and } H \text{ are continuous} \tag{2.114}$$

and

$$w = T = T_\xi = 0 \ . \tag{2.115}$$

The set of equations and boundary conditions (2.105)-(2.115) will henceforth be referred to as the <u>reduced model</u>. The unknowns to be found in a solution of this system are Ψ, T, w, H and ξ_M.

CHAPTER III

The Theory of Sliding: (i) Without Cavitation

As discussed in the previous chapters, temperate glaciers can slide on their beds, but the shear stress there is non-zero due to the drag offered to the motion by the roughness of the bedrock. The appropriate boundary condition is then that the tangential velocity is a function of the shear stress at the bedrock: this function must be determined by examination of the flow conditions at the bedrock, and is the subject of the present chapter.

Knowledge of the sliding law is clearly necessary in order to complete the model put forward in Chapter II, and to examine some of the global flow properties observed in glaciers. The sliding law plays an important part in the steady solutions presented in Chapters VII and VIII, and from Weertman's (1969) viewpoint, it provides a mechanism for the occurrence of surges due to a periodic fluctuation between states of high and low sliding velocities. The sliding law thus plays an essential rôle in the behaviour of a glacier, and its elucidation is of great importance.

Experimental work has been limited by the difficulty involved in obtaining results. The only way to observe the sliding velocity directly is to tunnel to the bedrock (which has been done), but clearly this can only give a local idea of the relation of shear stress to sliding velocity. Observations relying on semi-theoretic methods (e.g. Hodge 1974) are limited in accuracy by the assumptions of the theory. No detailed observations of the velocity in surges (when the sliding velocity is high) have been made, and it seems unlikely that such measurements can be made in the near future. Thus a detailed theoretical analysis is important to improve our understanding of the sliding law, and hence some of the more interesting properties of glaciers. In this chapter we give a detailed description of the physics involved, and then review the previous work of Weertman (1957, 1964), Lliboutry (1968, 1975), Nye (1969, 1970) and Kamb (1970).

A more general model of the flow near the bedrock is then presented. This incorporates a description of the heat flow problem in the bedrock and the water flow in the lubricating film. It is shown by dimensional arguments that these problems may be uncoupled from the ice flow, and the remainder of the chapter is concerned with this topic. Nye's work on a Newtonian ice flow is reviewed in some detail, since we shall have recourse to it in the next chapter. A variational principle for the nonlinear flow problem is then given which allows us to estimate the drag on the bedrock in terms of the sliding velocity. It is shown how to find suitable trial functions, and in Appendix 1 bounds are obtained for the drag when no assumptions at all are made about the bedrock topography, except that the slope is everywhere small. The usefulness of this approximation is made clear by the fact that it is a necessary assumption to make if the sliding velocity is to be comparable with the velocity due to shearing within the ice mass.

§1 The Physics of Glacier Sliding.

If temperate ice moves over a flat surface, then the frictional heat produced and geothermal heat input will maintain a water film there which lubricates the ice-rock interface. If we introduce protuberances in the bed, the moving ice will flow round them by a combination of two processes: ordinary slow viscous flow round an obstacle (called 'enhanced plastic flow' in the literature (Lliboutry 1968, Weertman 1957), and the process of 'regelation', whereby the ice experiences an increase in pressure as it approaches the obstacle which lowers the melting point, so that the ice is melted and squirts round the obstacle until refrozen by the eventual decrease of pressure. This process induces a transport of heat in the bedrock. Note that if the pressure decreases further, the ice becomes 'cold' (though only by 10^{-2} degrees if the pressure decrease is 1 bar), so that the ice becomes frozen to the obstacle. However, this will be counteracted by the frictional heating. In this case the ice must slide frictionally like a normal solid and by the same mechanism. The existence of such

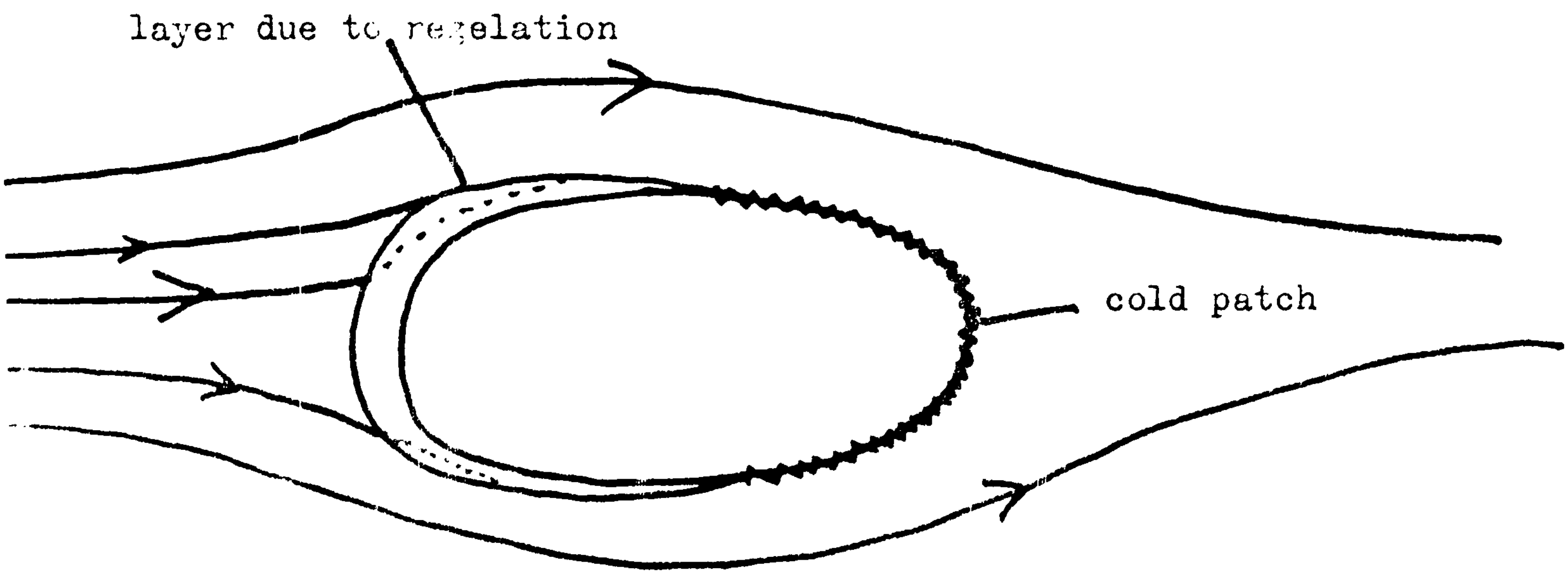


Figure 3.1: Plan (view from top) of ice flow round cylindrical obstacle.

'cold patches' has been considered by Robin (1976). The streamlines of a possible flow are shown in Figure 3.1. The relative contributions of regelation and Stokes flow are determined by the dimensions of the obstacle and the velocity. For a given velocity, regelation dominates for smaller obstacles, and vice versa.

In what follows, we shall be concerned with evaluating the drag which the above flow generates on a two-dimensional bed; the flow over such a bed could be as in Figure 3.2.

One might think that the main difference between a two- and three-dimensional bedrock would be that in the two-dimensional case, the overburden (hydrostatic) pressure of the ice (typically 10 bars) would be sufficient to constrain the ice to flow only by regelation, so that the

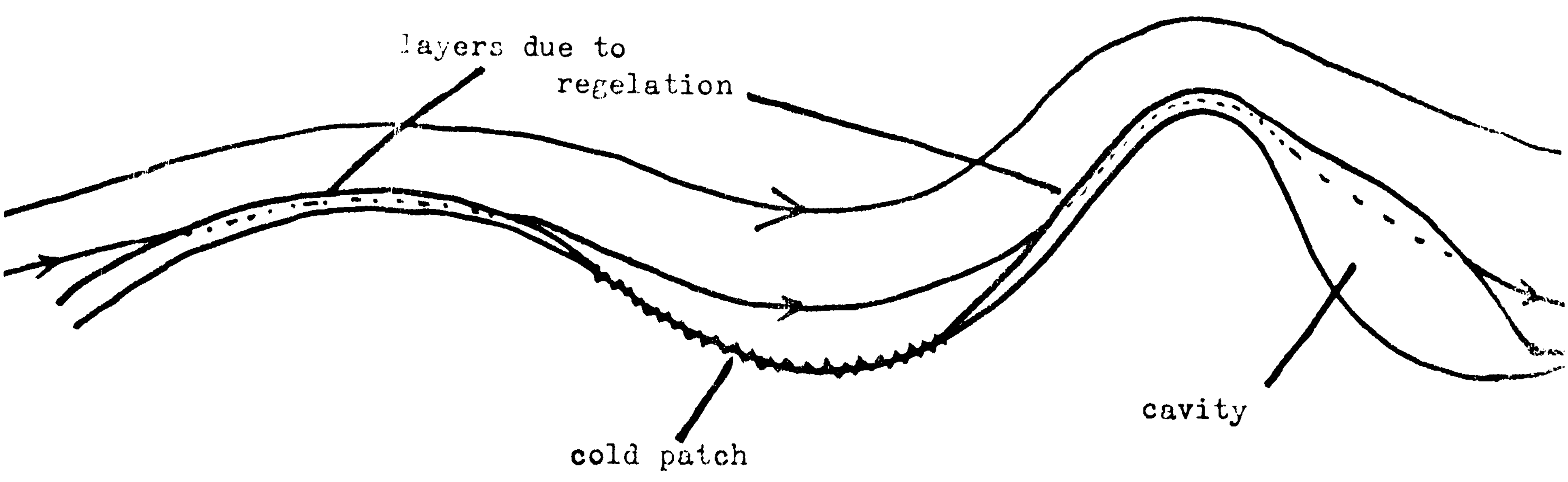


Figure 3.2: Flow over a two-dimensional bedrock.

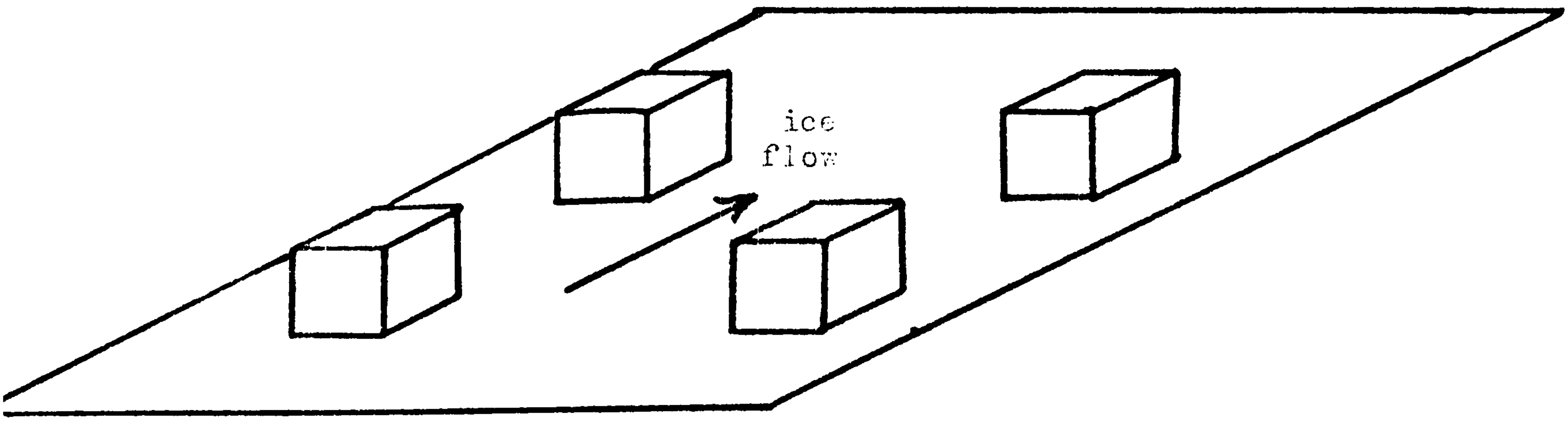


Figure 3.3: Weertman bedrock.

two- and three-dimensional flow mechanisms would be essentially different. In fact this is not so, as will be seen below. The reason for this is that the latent heat of ice is so 'large' that only a small amount of ice can be melted, and so even for sizeable obstacles very little of the flow is due to regelation. Hence we suspect that there may be little qualitative difference between the cases of two and three dimensions.

If the pressure in the water film becomes negative, then the ice cannot maintain contact with the bedrock, and a cavity forms. This is illustrated in Figure 3.2: the question of cavity formation and its effect on the drag is discussed in detail in Chapter IV.

§2 Previous Work.

Weertman (1957) was the first to give a quantitative theory of glacier sliding. He considered the flow of ice over an idealised bed consisting of a regular array of cubical obstacles on a flat plane (see Figure 3.3). For a given shear stress, he estimated the velocities due to pure regelation and pure 'enhanced plastic flow' dimensionally. This led him to

$$u_{reg} \propto \tau_b \quad , \tag{3.1}$$

$$u_{plas} \propto \tau_b^{\,n} \quad , \tag{3.2}$$

where τ_b is the basal shear stress, n is the exponent in Glen's law, and u_{reg}, u_{plas} are velocities due to regelation only and plastic flow only. Since these have different magnitudes for different-sized obstacles, he was led to the concept of a controlling obstacle size, and thence to an approximate intermediate law

$$u_b = C\tau_b^{\frac{n+1}{2}} . \qquad (3.3)$$

Weertman later refined his ideas (e.g. 1964) by considering a more realistic bedrock with a varying size of obstacles, and introduced the idea of cavitation behind obstacles: however, his basic approach remains non-mathematical and numerical values of C in (3.3) should be treated with some caution. For certain problems a simple approach may be of more use than other, more complicated theories: for this reason Weertman has since defended his ideas (1971).

Lliboutry is the other major exponent of sliding theory. In a long paper (1968) he reviews previous work and proposes his own theory. In this he envisages sliding over a two-dimensional bedrock (of small slope) and introduces the effect of cavitation. His method is, like Weertman's, semi-theoretical (there is much use of physically motivated approximation), but nevertheless it represents a useful first attempt. Inclusion of cavitation gives him a two-valued function for the velocity in terms of the stress, an important result that will be considered in Chapter IV.

Nye (1969, 1970) and Kamb (1970) independently took a more mathematical viewpoint. They considered the slow flow of a Newtonian fluid over a slowly varying bedrock, with a suction velocity at the bed due to melting and refreezing, which may be found by solving the rock temperature problem, since (as we show below) the Stefan condition on the unknown ice-water boundary gives an extra condition on the ice-rock interface.

Nye's and Kamb's theories are subject to certain criticisms: their models are not well formulated as they do not consider the induced drag on the bedrock to be balanced by an imposed stress at infinity, and thus conservation of momentum is not satisfied. In fact the results are correct to first order, but only because the bedrock slope is assumed small. The removal of this restriction immediately invalidates their models.

Both Nye and Kamb consider a bedrock of white roughness,that is to say one in which the 'roughness' of the undulations is independent of their scale:the bed has the same aspect when viewed at different length scales (except very long ones). Short of direct observation,such a bedrock may be the most realistic one to consider. Nye in particular treats a wide variety of bedrock topographies (1970).

In his paper,Kamb extends his solution for a Newtonian fluid to a power-law fluid by assuming that the viscosity is a function of the vertical coordinate only. His predicted results are generally,but not always,in agreement with observation. He finally considers cavitation, and discusses the effect of this on sliding,though without giving any quantitative analysis.

All these models consider a small bedrock slope,but do not give any mathematical justification for such a choice.

More recently,Lliboutry (1975,1976) has summarised and extended these theories. For sliding without cavitation,he obtains

$$u_b = C\tau_b^{\frac{n+1}{2}} \qquad (3.4)$$

as in Weertman's theory,but C is such that only small velocities are predicted. With dominant cavitation at high velocities,the flow becomes similar to frictional sliding of a solid,and then

$$\tau_b \propto N , \qquad (3.5)$$

where N is the difference between the mean ice pressure and the mean cavity water pressure. A complete sliding law would then look like Figure 3.4.

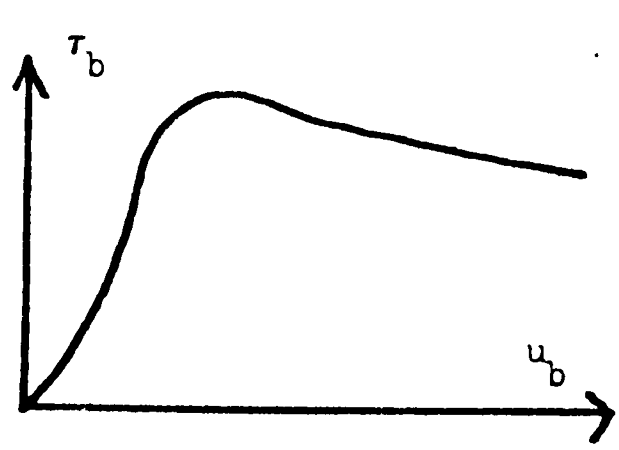


Figure 3.4: Typical sliding law.

Morland (1976) has used the methods of complex variable theory to estimate the drag exerted on a Newtonian fluid when there is no cavitation. He includes the effect of gravity and the existence of the top surface, which gives him a consistent model problem. His work is essentially a correction and extension of Nye's (1969) theory.

§3 Problem Specification.

We wish to find a relation between u_b and τ_b. How do we distinguish bedrock roughness from overall variations in the topography? Further, how do we define 'the' basal velocity u_b, since it must vary as the ice flows over the protuberances in the bedrock?

To answer these questions, we consider the bed $h(x)$ to be composed of a smooth component $h_S(x)$ and a rough component $h_R(x)$ (cf. Nye 1970). These are such that the flow of the bulk of the glacier follows the mean profile h_S (changes in h_S affect the whole depth of the glacier), whereas the roughness h_R only affects the flow in a small 'inner' layer close to the bedrock. h_R is considered to be periodic, in order to have a constant imposed stress at infinity. On the scale of the roughness (say 5 metres), the flow can be thought of as being similar to a composition of Stokes and Oseen flows. The 'Oseen' outer flow satisfies the bulk equations given in the previous chapter, whereas the 'Stokes' inner flow satisfies differently scaled equations, and must satisfy appropriate boundary conditions on the rough bedrock. Viewed like this, the sliding condition is a boundary condition for the outer flow, and the appropriate problem to be solved is that for the inner flow, with matching conditions into the outer flow in some matching region.

§4 Formulation.

We define the locally rough bedrock by the equation

$$h_D = dh_S\left(\frac{x}{\ell}\right) + [y]h_R\left(\frac{x}{[x]}\right) , \qquad (3.6)$$

where $[\cdot]$ denotes a typical scale of the bracketed variable, d and ℓ are the height and length scales defined in Chapter II, h_D is the actual dimensional bedrock, h_S is the dimensionless smooth bedrock (corresponding to h of Chapter II), and h_R is the local roughness.

Let

$$\nu = \frac{[y]}{[x]} \tag{3.7}$$

be the mean roughness slope, and

$$\sigma = \frac{[x]}{d} \quad . \tag{3.8}$$

We assume $\sigma \ll 1$.
In the notation of Chapter II, the dimensionless outer stress is

$$\tau_2 = \eta - y \,, \tag{3.9}$$

$$u_y = (\eta - y)^n \, r_1(W) \,, \tag{3.10}$$

where W denotes the moisture at the bedrock; hence near $y = h_S$

$$u = u(h_S) + (y-h_S)u_y(h_S) + \ldots$$

$$= u_b + H^n r_1(W)(y-h_S) + O(y-h_S)^2 \,, \tag{3.11}$$

or dimensionally

$$u_D = U\,[u_b + H^n r_1(W)(y_D - h_{SD})/d + \ldots] \,, \tag{3.12}$$

where U is the longitudinal velocity scale $a_0 l/d$ introduced in (2.24), and D denotes a dimensional variable. If we write

$$y_D = h_{SD} + [x]y_i \ , \ x_D = [x]x_i \tag{3.13}$$

(i for inner), then

$$u_D = U\,[u_b + H^n r(W)\sigma y_i + O(\sigma^2)] \,, \tag{3.14}$$

where in (3.14) and for the remainder of this chapter we omit the suffix one from $r_1(W)$. Hence the appropriate boundary condition on the inner solution is that

$$u_i = u_D/U \sim u_b + \sigma S y_i + O(\sigma^2) \tag{3.15}$$

as $y_i \to \infty$, where

$$S = H^n r(W) \tag{3.16}$$

is the dimensionless (outer) shearing at the (smoothed) bedrock.

§5 Sliding Model.

We consider the situation shown in Figure 3.5. We assume that h_R is periodic. The equations of motion for the ice are

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{q} = 0 , \tag{3.17}$$

$$p_x = \rho g'\varepsilon + \tau_{1x} + \tau_{2y} , \tag{3.18}$$

$$p_y = -\rho g' + \tau_{2x} - \tau_{1y} , \tag{3.19}$$

where ε and g' are as defined in Chapter II. We neglect the energy equation, since on a local scale the moisture is effectively constant*, and it has been shown elsewhere (Lliboutry 1968) that advection of heat by the ice is

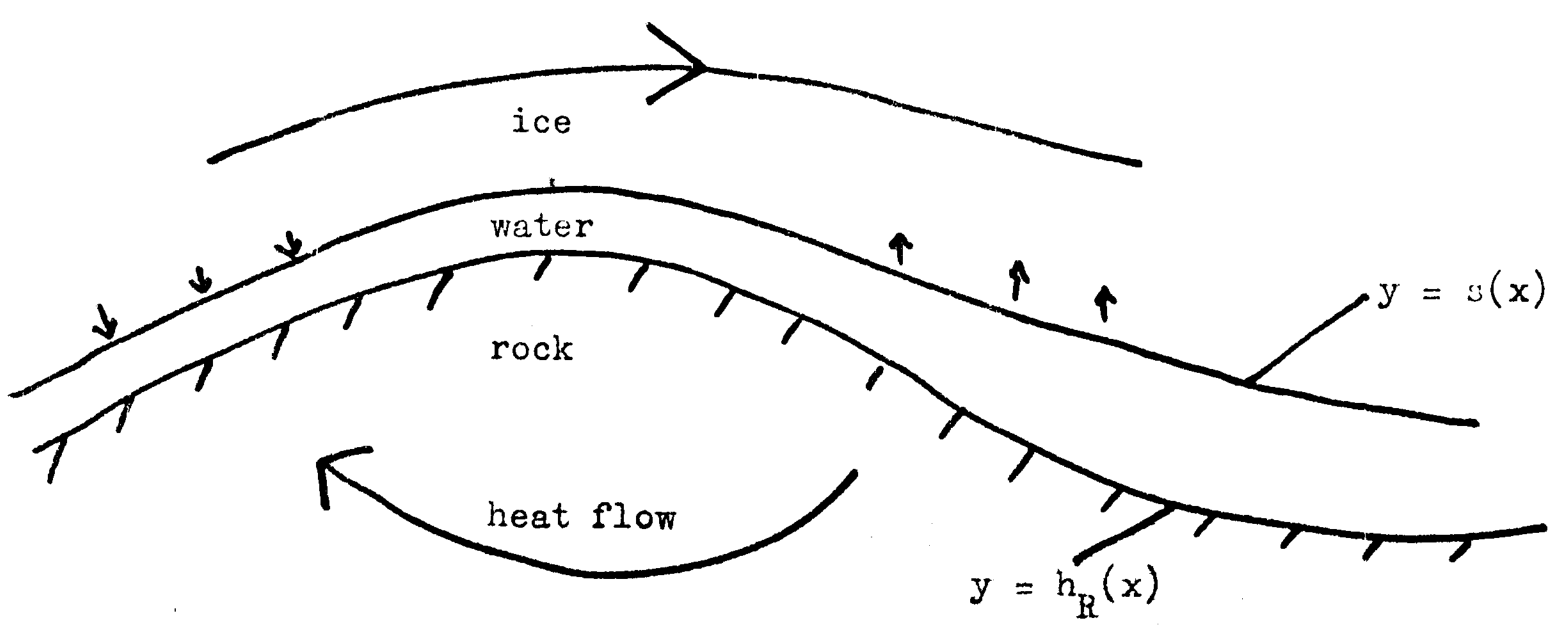


Figure 3.5: bedrock geometry.

negligible. (We are neglecting any hydrological considerations in the present treatment.)

The equations for the water film are

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{q} = 0 , \tag{3.20}$$

$$\rho(\underset{\sim}{q}\cdot\underset{\sim}{\nabla})\underset{\sim}{q} + \underset{\sim}{\nabla}(p + \rho g'y - \varepsilon\rho g'x) = \mu\nabla^2\underset{\sim}{q} , \tag{3.21}$$

$$\underset{\sim}{q}\cdot\underset{\sim}{\nabla}T = \kappa_W\nabla^2 T , \tag{3.22}$$

where μ is the viscosity of water, and κ_W its thermal conductivity.

* But see Appendix 2.

Finally the temperature in the rock satisfies

$$\nabla^2 T = 0 \ . \qquad (3.23)$$

The boundary conditions are:

in the ice, the velocity satisfies the matching condition (3.15) as $y \to \infty$;

on the ice-water interface $y = s(x)$ (not to be confused with the flux of Chapter II), we have

(i) continuity of stress:

$$s_x(p-\tau_1) + \tau_2 = p_W s_x \ , \qquad (3.24)$$

$$s_x \tau_2 + p + \tau_1 = p_W \ ; \qquad (3.25)$$

(ii) conservation of mass:

$$\rho \underset{\sim}{q} \text{ is continuous;} \qquad (3.26)$$

(iii) the Stefan condition:

$$-\rho_I L(v_I - u_I s_x) = [k(s_x T_x - T_y)]_{ice}^{water} \ , \qquad (3.27)$$

(iv) the Clausius-Clapeyron condition:

$$T = T_M - \theta(p-p_A) \ ; \qquad (3.28)$$

on the water-rock interface $y = h(x)$,

$$[k\frac{\partial T}{\partial n}] = [T] = \underset{\sim}{q} = 0 \ ; \qquad (3.29)$$

in the rock,

$$\frac{\partial T_R}{\partial y} \to -G/k_R \quad \text{as } y \to -\infty \ . \qquad (3.30)$$

Let us first consider the water problem. (Note that the lubrication film is not the same as Kamb's regelation layer, which is the layer of ice adjoining the water film which has been melted and refrozen (Figure 3.6).

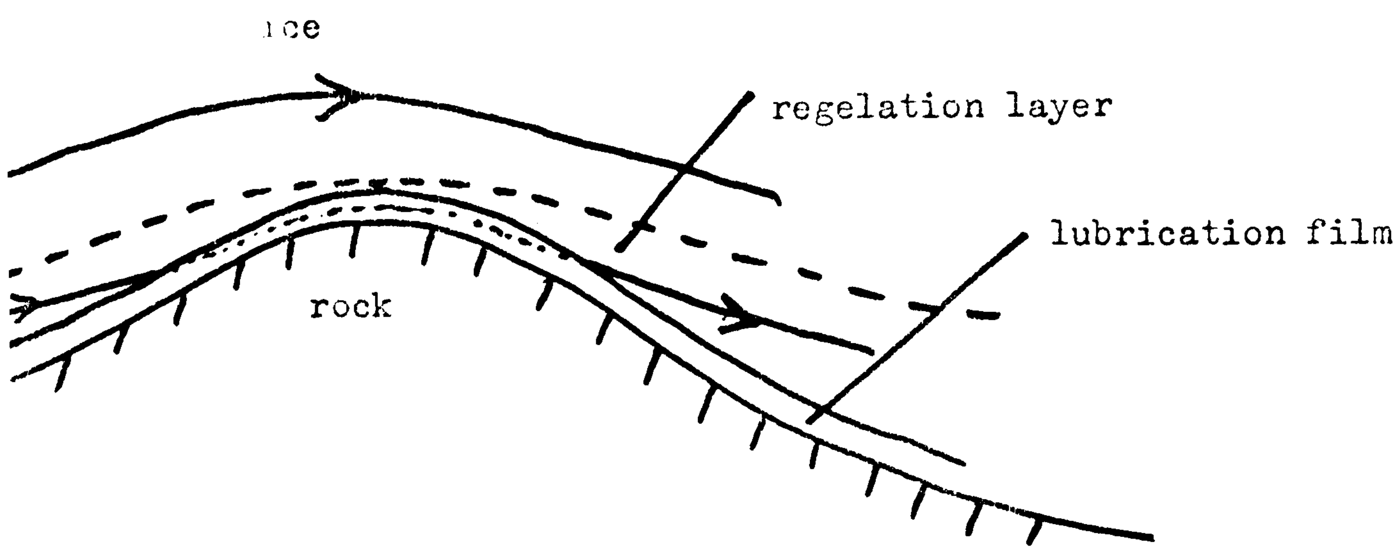


Figure 3.6: regelation layer.

Its upper boundary is the lowest streamline which does not intersect the film.)

First we nondimensionalise the geometry by writing

$$y = [y]\{h_R(X)+\delta Y\} \ , \tag{3.31}$$

$$x = [x]X \ , \tag{3.32}$$

so that δ is the dimensionless film thickness, undetermined at present. Also let

$$s(x) = [y]\{h_R(X)+\delta\Sigma(X)\} \ . \tag{3.33}$$

For the remainder of this chapter we omit the R on h_R. From (3.33)

$$s'(x) = \nu(h'+\delta\Sigma') \ . \tag{3.34}$$

From (3.31) and (3.32),

$$\frac{\partial}{\partial x} = \frac{1}{[x]}\frac{\partial}{\partial X} - \frac{h'}{\delta[x]}\frac{\partial}{\partial Y} \ , \quad \frac{\partial}{\partial y} = \frac{1}{\delta[y]}\frac{\partial}{\partial Y} \ . \tag{3.35}$$

The equation of continuity becomes

$$u_X + \frac{1}{\delta}\left(\frac{v}{\nu} - uh'\right)_Y = 0 \ . \tag{3.36}$$

Now suppose the melting velocity

$$v_I - u_I s' = -\alpha u_b V_M(x) \ , \tag{3.37}$$

where $V_M \sim 1$, and $\alpha \lesssim 1$ is a dimensionless number to be determined. Using

(3.34),(3.37) is

$$v - \nu u(h'+\delta\Sigma') = -\alpha u_b V_M(x) \ . \tag{3.38}$$

We scale $\underset{\sim}{q}$ with $[u]$ (to be determined) by writing

$$u = [u]U \ , \ v = \nu[u]Uh' + \nu\delta[u]V \ . \tag{3.39}$$

Then (3.36) is

$$U_X + V_Y = 0 \ , \tag{3.40}$$

and the boundary condition (3.38) becomes

$$V - U\Sigma' = -V_M(x) \text{ on } Y = \Sigma \ , \tag{3.41}$$

provided we choose

$$[u] = \frac{\alpha u_b}{\nu\delta} \ . \tag{3.42}$$

The other condition on $Y = \Sigma$ is that

$$[u]U = \frac{\rho_I}{\rho_W}u_I \ , \text{ or } U = \frac{\rho_I}{\rho_W}\frac{\nu\delta}{\alpha}\frac{u_I}{u_b} \ . \tag{3.43}$$

Now $\dfrac{\rho_I u_I}{\rho_W u_b} \lesssim 1$, so if

$$\frac{\nu\delta}{\alpha} << 1 \ , \tag{3.44}$$

then approximately

$$U = 0 \text{ on } Y = \Sigma \ . \tag{3.45}$$

In this case, (3.41) becomes

$$V = -V_M(x) \text{ on } Y = \Sigma \ . \tag{3.46}$$

The boundary condition on $Y = 0$ is that

$$U = V = 0 \ , \ Y = 0 \ . \tag{3.47}$$

Now in (3.21)

$$\frac{\rho(\underset{\sim}{q}\cdot\underset{\sim}{\nabla})\underset{\sim}{q}}{\mu\nabla^2\underset{\sim}{q}} \sim \frac{\rho[u][y]^2\delta^2}{[x]\mu} . \tag{3.48}$$

We anticipate that $\delta << 1$ and therefore we assume for the moment that the expression in (3.48) is << 1;the consistency of this assumption is confirmed later. In this case the equations become those of lubrication theory,and we must as usual (e.g. Batchelor 1967) balance

$$p_x \sim \mu u_{yy} \tag{3.49}$$

in the first momentum equation. This requires the pressure to vary as

$$[p] \sim \frac{\mu\alpha u_b[x]}{\nu\delta^3[y]^2} , \tag{3.50}$$

using (3.42). The Clausius Clapeyron relation implies that the temperature change along the film is

$$[T] \sim \theta[p] , \tag{3.51}$$

where $\theta \simeq .0074$ K bar^{-1}. We assume that this is also effectively the temperature variation on the rock-water interface (i.e. the temperature gradient in the water is 'small'). Then

$$[T_R] \sim \theta[p] . \tag{3.52}$$

Dimensional balance of the heat flux on h(X) and the Stefan condition on $\Sigma(X)$ imply

$$\frac{k_R T_R}{[x]} \sim \rho L\alpha u_b . \tag{3.53}$$

Combining (3.50),(3.52) and (3.53),we obtain

$$\delta^3 \sim \frac{\mu\theta k_R}{\rho L\nu[y]^2} . \tag{3.54}$$

A similar result has been obtained by other authors (e.g. Lliboutry 1968, Nye 1967). Note that δ is independent of α. With the values given in

Table 1 (page 20), $\mu \simeq 10^{-2}$ c.g.s. units, and assuming $k_R \sim k_I$, we find

$$\delta \sim 10^{-6}[x]^{1/3}[y]^{-1} , \tag{3.55}$$

where $[x],[y]$ are expressed in metres. Thus a typical film thickness $\delta[y]$ is the order of a micron.

In order to determine α, we must consider the ice flow. The drag per period on the bedrock is

$$D \simeq \int p_W \, dy \sim \nu[p][x] , \tag{3.56}$$

where p_W is the water pressure. But also

$$D \sim [\tau]_o[x] , \tag{3.57}$$

(o for outer), since the stress at infinity balances the drag on the bed. Hence

$$[p] \sim \frac{[\tau]_o}{\nu} . \tag{3.58}$$

Thus from (3.52), (3.53) and (3.58) we obtain

$$\alpha \sim \frac{k_R \theta [\tau]_o}{[y]\rho L u_b} . \tag{3.59}$$

Taking $[\tau]_o \sim 1$ bar, we find

$$\alpha \sim \frac{2}{[y]u_b} \times 10^{-3} , \tag{3.60}$$

where $[y]$ is in metres and u_b is in metres per year. Thus $\alpha \ll 1$ except for the smallest obstacles and slowest flows. The ice flows over the protrusions and it is justifiable to estimate the drag by solving the ice flow problem with no tangential traction on $y = h$ (since the film is negligibly thin).

From the above, we find

$$[p] \sim \frac{1}{\nu} \text{ bars} \tag{3.61}$$

which is large if $\nu \ll 1$, i.e. the bedrock slope is small; thus for a given applied stress at infinity, cavities will form more easily as ν is reduced. It is to this end that we retain an interest in the lubrication problem, since cavity formation may be determined by the pressure variation in the water (see Chapter IV).

The temperature variation is

$$[T_R] \sim \frac{10^{-2}}{\nu} \text{ degrees,} \tag{3.62}$$

whereas the change in temperature across the film is

$$\frac{k_R[T_R]}{k_W[x]}\delta[y] \sim 4\delta\nu[T_R] \ll [T_R] , \tag{3.63}$$

using (3.29). This justifies the assumption stated between equations (3.51) and (3.52). We find

$$\frac{\nu\delta}{\alpha} \sim \frac{2 \times 10^{-9}}{[x]^{2/3}[y]u_b} \ll 1 \tag{3.64}$$

for all physically sensible velocities and obstacles. This justifies (3.45) and (3.46).

We now wish to use the lubrication equations for the water film. From (3.48),

$$\frac{\rho(\underset{\sim}{q}.\underset{\sim}{\nabla})\underset{\sim}{q}}{\mu\nabla^2\underset{\sim}{q}} \sim 10^{-10}\frac{[x]^{1/3}}{[y]} \ll 1 . \tag{3.65}$$

We write

$$p = p_A + \rho g' dH + \frac{[\tau]_o}{\nu} p^* ; \tag{3.66}$$

then the momentum equation (3.21) is, using (3.35) and (3.39) and neglecting inertia,

$$-\varepsilon\rho g' + \frac{[\tau]_o}{\nu[x]}(p^*_X - \frac{h'}{\delta}p^*_Y) = \frac{\mu[u]}{\delta^2[y]^2}\left[(1+\nu^2h'^2)u_{YY} + O(\delta)\right] . \tag{3.67}$$

Since we have already chosen δ such that $p_x \sim \mu u_{yy}$, (3.67) becomes

$$-\sigma\nu + (p^*_X - \frac{h'}{\delta}p^*_Y) = (1+\nu^2h'^2)u_{YY} + \dots , \tag{3.68}$$

where we have used (2.38) ($[\tau]_o = \rho g'\varepsilon d$) and (3.8). The second component is

$$\rho g' + \frac{[\tau]_0}{\nu}\frac{1}{\delta[y]}\, p^*_Y = \frac{\mu\nu[u]}{\delta^2[y]^2}\left[(1+\nu^2 h'^2)h'U_{YY} + O(\delta)\right] . \qquad (3.69)$$

But from (3.42),(3.50) and (3.58),

$$\frac{[\tau]_0}{\nu[x]} = \frac{\mu[u]}{\delta^2[y]^2} , \qquad (3.70)$$

whence

$$\sigma\nu^2/\varepsilon + \frac{1}{\delta}\, p^*_Y = \nu^2(1+\nu^2 h'^2)h'U_{YY} + \dots \qquad (3.71)$$

Substituting (3.71) into (3.68),

$$-\sigma\nu + \frac{\nu^2\sigma}{\varepsilon}h' + p^*_X = (1+\nu^2 h'^2)^2 U_{YY} + \dots \qquad (3.72)$$

As usual,(3.71) states that

$$p^* \simeq p^*(X) , \qquad (3.73)$$

and if we write

$$\Pi(X) = p^* + \frac{\sigma\nu^2}{\varepsilon}h - \sigma\nu X , \qquad (3.74)$$

then

$$\Pi'(X) = (1+\nu^2 h'^2)U_{YY} . \qquad (3.75)$$

Integrating (3.40) between 0 and Σ,and using (3.45),(3.46) and (3.47), we obtain

$$\frac{\partial}{\partial X}\int_0^\Sigma U\, dY = \int_0^\Sigma U_X\, dY = [-V]_0^\Sigma = V_M(X) ,$$

so that

$$\int_0^\Sigma U\, dY = \int^X V_M(X')\, dX' . \qquad (3.76)$$

From (3.75),

$$(1+\nu^2 h'^2)^2 U = \tfrac{1}{2}\Pi'(X)Y(Y-\Sigma) . \qquad (3.77)$$

Combining this with (3.76),

$$(1+\nu^2 h'^2)^2 \int^X V_M(X')\, dX' = \tfrac{1}{2}\Pi'(X)[\tfrac{1}{3}Y^3 - \tfrac{1}{2}\Sigma Y^2]_0^\Sigma$$

$$= -\frac{1}{12}\Sigma^3\Pi'(X) \; ; \tag{3.78}$$

for small slope bedrock ($\nu << 1$),(3.78) reduces to

$$\int^X V_M(X')\, dX' = -\frac{1}{12}\Sigma^3\Pi'(X) \; . \tag{3.79}$$

The dimensionless bedrock temperature problem is obtained by writing

$$T = T_F - \theta\rho g' dH + \frac{\theta[\tau]_0}{\nu} T^* , \tag{3.80}$$

where T_F is the melting point at atmospheric pressure. Then

$$\Delta T^* = 0 \; . \tag{3.81}$$

The boundary conditions are:

on $y = \nu h$,

$$T^* = -p^*(x) \; , \; T^*_y - \nu h' T^*_x = -V_M(X) \tag{3.82}$$

(using (3.53) and (3.27)),and

as $y \to -\infty$,

$$\frac{\partial T^*}{\partial y} \to \frac{-G}{k_R} \cdot \theta\frac{[y]}{[\tau]_0} = -\Lambda^* . \tag{3.83}$$

With typical values,

$$\Lambda^* \sim 2[y] \tag{3.84}$$

if $[y]$ is in metres, $[\tau]_0 = 1$ bar. Thus Λ^* is not normally negligible: this contradicts previous authors (e.g. Morland 1976). However,we need only consider the bedrock temperature field if $\alpha \sim 1$ (regelation is important). From (3.60),this implies $[y] \sim \frac{2}{u_b} \times 10^{-3}$,whence $\Lambda \sim \frac{4}{u_b} \times 10^{-3}$,and so in this case Λ^* is indeed negligible.

There are thus three unknowns that arise in the boundary conditions: Σ, p^* and V_M. Solution of the rock temperature profile gives a consistency condition between p^* and V_M. Solution of the ice problem with boundary conditions of no flow through, and no traction on, the boundary $h(x)$ gives the pressure p^* via (3.25): lastly solution of the lubrication equation (2.78) gives the ice-water interface $\Sigma(X)$. Without cavitation, it is sufficient to solve the ice flow problem. This is the most difficult problem to solve: we now turn our attention to it.

§6 Ice Flow Problem: Newtonian Flow.

Let us first of all review Nye's (1969, 1970) work from the standpoint of the present formulation. We consider the flow to be that of a Newtonian fluid, with viscosity η defined by

$$\tau_{ij} = 2\eta e_{ij} \quad . \tag{3.85}$$

In the notation of Chapter II, $A = 1/2\eta$, and so

$$[\tau]_o = \rho g'\varepsilon d = \frac{\eta U}{d} \quad . \tag{3.86}$$

The dimensional stresses are given by

$$\tau_1 = 2\eta\psi_{xy} \ , \tag{3.87}$$

$$\tau_2 = \eta(\psi_{yy}-\psi_{xx}) \ , \tag{3.88}$$

where ψ is the stream function, and we have to solve

$$\Delta\Delta\psi = 0 \ , \tag{3.89}$$

$$p_x = \rho g'\varepsilon + \eta\Delta\psi_y \ , \tag{3.90}$$

$$p_y = -\rho g' - \eta\Delta\psi_x \ , \tag{3.91}$$

subject to

$$\begin{aligned} p &\sim p_A + \rho g'(\eta - y) \ , \\ \psi_y &\sim U[u_b + Sy/d \ \dots] \ , \\ \psi_x &\to 0 \ , \end{aligned} \tag{3.92}$$

as $y \to \infty$, and

$$\psi = \sigma_{NT} = 0 \tag{3.93}$$

on $y = h_{RD}$. The latter condition in (3.93) is that of zero traction at the bedrock. Since the normal and tangential dimensionless vectors are respectively

$$\underset{\sim}{N} = (N_1,N_2) \propto (-\nu h',1) \;,\; \underset{\sim}{T} = (T_1,T_2) \propto (1,\nu h') \;, \tag{3.94}$$

we obtain, since $\sigma_{NT} = \sigma_{ij}N_iT_j$,

$$(1-\nu^2h'^2)\tau_2 - 2\nu h'\tau_1 = 0 \;, \tag{3.95}$$

whence, using (3.87) and (3.88),

$$(1-\nu^2h'^2)(\psi_{yy}-\psi_{xx}) - 4\nu h'\psi_{xy} = 0 \;,\; y = \nu h \;. \tag{3.96}$$

Using the considerations of the previous section, we nondimensionalise the variables by using (3.13) and writing

$$\begin{aligned} p &= p_A + \rho g' dH - \rho g'[x]y_i + \frac{[\tau]_0}{\nu}\tilde{p} \;, \\ \psi &= U[x]\tilde{\psi} \;. \end{aligned} \tag{3.97}$$

Using (3.86), the equations (3.89)-(3.91) become

$$\begin{aligned} \Delta\Delta\tilde{\psi} &= 0 \;, \\ \tilde{p}_{x_i} &= \sigma\nu + \frac{\nu}{\sigma}\Delta\tilde{\psi}_{y_i} \;, \\ \tilde{p}_{y_i} &= \frac{-\nu}{\sigma}\Delta\tilde{\psi}_{x_i} \;, \end{aligned} \tag{3.98}$$

to be solved subject to

$$\tilde{p} \to 0 \;,\; \tilde{\psi} \sim u_b y_i + \tfrac{1}{2}\sigma S y_i^2 \ldots \tag{3.99}$$

as $y_i \to \infty$, and

$$\tilde{\psi} = (1-\nu^2h'^2)(\tilde{\psi}_{y_iy_i}-\tilde{\psi}_{x_ix_i}) - 4\nu h'\tilde{\psi}_{x_iy_i} = 0 \tag{3.100}$$

on $y_i = \nu h$. We henceforth drop suffices and overtildes for convenience.

Nye considers the case $\nu \ll 1$, and seeks a series expansion in ν. The parameter σ occurring here is not considered by Nye: we shall assume that $\sigma \ll 1$; in fact, for reasons that will become more clear in §7, we assume

$$\sigma \sim \nu^2 \ll 1 \ , \tag{3.101}$$

and define

$$u_b = \frac{\sigma}{\nu^2} u_b^* \tag{3.102}$$

(compare (3.190)).

We seek an asymptotic solution to (3.98) and (3.99) in the form

$$\begin{aligned} \psi &= \psi_0 + \nu\psi_1 + \dots \\ p &= p_0 + \nu p_1 + \dots \end{aligned} \tag{3.103}$$

Expanding the conditions (3.100) about $y = 0$, we obtain successively on $y = 0$,

$$O(1) : \quad \psi_0 = \psi_{0yy} - \psi_{0xx} = 0 \ , \tag{3.104}$$

$$\begin{aligned} O(\nu) : \quad & h\psi_{0y} + \psi_1 = 0 \ , \\ & \psi_{1yy} - \psi_{1xx} + h'[\psi_{0yyy} - \psi_{0xxy}] - 4h'\psi_{0xy} = 0 \ , \end{aligned} \tag{3.105}$$

$$\begin{aligned} O(\nu^2) : \quad & \tfrac{1}{2}h^2\psi_{0yy} + h\psi_{1y} + \psi_2 = 0 \ , \\ & \psi_{2yy} - \psi_{2xx} + h[\psi_{1yyy} - \psi_{1xxy}] + \tfrac{1}{2}h^2[\psi_{0yyy} - \psi_{0xxy}] \\ & - h'^2[\psi_{0yy} - \psi_{0xx}] - 4h'\psi_{1xy} - 4hh'\psi_{1yy} = 0 \ , \end{aligned} \tag{3.106}$$

etc. (3.99) gives, as $y \to \infty$,

$$\psi_0 \sim u_b y \ , \ \psi_1 \to 0 \ , \ \psi_2 \sim \tfrac{1}{2}u_b^* S y^2 \ , \tag{3.107}$$

and so on. From (3.104) and (3.107), the $O(1)$ solution is

$$\psi_0 = u_b y. \tag{3.108}$$

Thus ψ_1 is the solution of

$$\begin{aligned} &\Delta\Delta\psi_1 = 0 , \\ &\psi_1 \to 0 \text{ as } y \to \infty , \\ &\psi_1 = -u_b h(x) , \ \psi_{1yy} - \psi_{1xx} = 0 , \text{ on } y = 0. \end{aligned} \tag{3.109}$$

Since $\Delta\psi_0 = 0$, choosing $\sigma \sim \nu^2$ provides a balance in the momentum equations in (3.98) and justifies our choice of nondimensionalisation.

Nye's solution of (3.109) is given in terms of the Fourier components of the periodic bedrock h. We prefer to present the results in terms of the theory of complex variables since this is more elegant, and is of the form required for the consideration of cavitation presented in the next chapter.

We introduce the complex variables

$$z = x + iy , \ z^* = x - iy , \tag{3.110}$$

so that

$$\begin{aligned} &\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*} , \ \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right) , \\ &\Delta \equiv 4\frac{\partial^2}{\partial z \partial z^*} , \ \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} = -2\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z^{*2}}\right) . \end{aligned} \tag{3.111}$$

It is well known that the solution of the biharmonic equation in (3.109) can be written

$$\psi_1 = (z^*-z)f(z) + B(z) + (z-z^*)f^*(z^*) + B^*(z^*), \tag{3.112}$$

where f and B are analytic in Im $z > 0$. Satisfaction of the zero stress condition requires

$$(z^*-z)f'' - B'' - 2f' + (z-z^*)f^{*''} - B^{*''} - 2f^{*'} = 0 \tag{3.113}$$

on $y = 0$, i.e. $z = z^*$. We can satisfy this by choosing

$$f(z) = -\tfrac{1}{2}B'(z) . \tag{3.114}$$

The other condition on $y = 0$ requires

$$B(x) + B^*(x) = u_b h(x) , \tag{3.115}$$

and we require B satisfying (3.115) to be such that

$$B \to 0 , \ \text{Im } z \to \infty . \tag{3.116}$$

B may be uniquely determined by reformulating (3.115) and (3.116) as a Hilbert problem: this approach is adopted in Chapter IV. Here we note that if h is of period 2π and possesses a convergent Fourier series of zero mean,

$$h \sim \sum_{-\infty}^{+\infty} a_k e^{ikx} , \ a_0 = 0 , \tag{3.117}$$

say, then by inspection

$$B(z) = u_b \sum_1^{\infty} a_k e^{ikz} \tag{3.118}$$

satisfies (3.115) and (3.116). Obviously, we can always choose $y = 0$ such that h has zero mean, and in fact this prescribes how we should choose h_s, that is, as a running mean of the bedrock over a distance x_{av} such that $[x] \ll x_{av} \ll \ell$ (cf. Nye 1970). We neglect $\sigma\nu$ in (3.98), and using the relations (3.111), we find

$$\begin{aligned} p_z + p_{z*} &= 4i\frac{\nu^2}{\sigma}(\psi_{1zzz*} - \psi_{1zz*z*}) , \\ i(p_z - p_{z*}) &= -4\frac{\nu^2}{\sigma}(\psi_{1zzz*} + \psi_{1zz*z*}) , \end{aligned} \tag{3.119}$$

whence

$$p_z = 4i\frac{\nu^2}{\sigma}\psi_{1zzz*} , \ p_{z*} = -4i\frac{\nu^2}{\sigma}\psi_{1zz*z*} , \tag{3.120}$$

so that

$$p = 4i\frac{\nu^2}{\sigma}\psi_{1zz*} + p_1(z^*) = -4i\frac{\nu^2}{\sigma}\psi_{1zz*} + p_2(z) . \tag{3.121}$$

Using the solution (3.112), we obtain

$$\begin{aligned} p &= 4i\frac{\nu^2}{\sigma}[f'(z)+f^{*\prime}(z^*)] + p_1(z^*) \\ &= -4i\frac{\nu^2}{\sigma}[f'+f^{*\prime}] + p_2(z) \end{aligned} \tag{3.122}$$

which implies

$$p_1(z^*) = -8i\frac{\nu^2}{\sigma}f^{*\prime} \tag{3.123}$$

up to addition of a constant, and so

$$\begin{aligned} p &= 4i\frac{\nu^2}{\sigma}[f'-f^{*\prime}] \\ &= \frac{2\nu^2}{i\sigma}[B''(z)-B^{*\prime\prime}(z^*)] \end{aligned} \tag{3.124}$$

from (3.114).

For convenience we now consider h to have period 2π. Equating the dimensional stress at infinity with the drag exerted on the bedrock by the water film, we obtain

$$\frac{\eta U}{d}H = \frac{1}{2\pi[x]}\int_0^{2\pi} p_W\ \nu h'\ [x]\ dx = \frac{1}{2\pi}\int_0^{2\pi} p_W \nu h'\ dx\ , \tag{3.125}$$

where the dimensional water pressure p_W is given from (3.24), (3.87), (3.88) and (3.97) by

$$\nu p_r h' = \frac{\eta U}{[x]}(\psi_{yy}-\psi_{xx}) + \nu h'\left[\frac{-2\eta U}{[x]}\psi_{xy}+p_A+\rho g' dH - \frac{\sigma}{\varepsilon}\frac{\eta U}{d}\nu h + \frac{\eta U}{\nu d}p\right], \tag{3.126}$$

where the right hand side of (3.126) is evaluated on $y = \nu h$. Thus

$$\begin{aligned} H &= \frac{1}{2\pi}\int_0^{2\pi} [ph' - \frac{\sigma\nu^2}{\varepsilon}hh' + \frac{d}{\eta U}\nu h'(p_A+\rho g' dH) + \frac{1}{\sigma}\{\psi_{yy} - \psi_{xx} - 2\nu h'\psi_{xy}\}]\ dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} p_0\Big|_{y=0} h'\ dx + O(\nu)\ , \end{aligned} \tag{3.127}$$

using (3.96) and that from (3.103) and (3.108), $\psi_{xy} = \nu\psi_{1xy} + \dots$ on $y = \nu h$, and from (3.111), (3.112), (3.114) and (3.115) that

$$\psi_{1xy}\Big|_{y=0} \propto h''\ . \tag{3.128}$$

Using the Fourier series (3.117) and (3.118) with $a_0 = 0$, we have on $y = 0$

$$p = \frac{2\nu^2}{i\sigma}\left[B''(x) - B^{*''}(x)\right]$$

$$= \frac{2\nu^2}{i\sigma} u_b \left[\sum_1^\infty (-k^2 a_k e^{ikx}) - \sum_1^\infty (-k^2 a_{-k} e^{-ikx})\right] \qquad (3.129)$$

and

$$h' = i\left[\sum_1^\infty k a_k e^{ikx} + \sum_1^\infty (-k a_{-k} e^{-ikx})\right] . \qquad (3.130)$$

The only contributions to (3.127) come from the non-oscillatory terms in the product. Therefore

$$H = \frac{1}{2\pi} \cdot 2\pi \cdot \frac{2\nu^2}{\sigma} u_b \sum_1^\infty (k^3 a_k a_{-k} + k^3 a_k a_{-k})$$

$$= 4u_b^* \sum_1^\infty k^3 |a_k|^2 \qquad (3.131)$$

using (3.101), and that $a_k^* = a_{-k}$ (since h is real). (3.131) is the counterpart of Nye's formula (33) (1969). For a sinusoidal bed $h = \cos x$, (3.118), (3.124) and (3.131) give

$$p = -2u_b^* \sin x ,$$
$$H = u_b^* . \qquad (3.132)$$

Nye and Kamb both devote a great deal of attention to the appropriate form to take for the bedrock profile. An inspection of (3.131) reveals why this is necessary. (They both use Fourier integral decomposition, but this distinction is irrelevant to the present discussion.) Unless the Fourier series for h converges very rapidly, (3.131) will not converge at all. Thus roughness on the finest scale could render u_b^* negligible. There are two mechanisms that may counteract this dependence on large wavenumber. One is regelation, which has the effect of making the large k terms in (3.131) $\sim k|a_k|^2$ (Nye 1969): (3.131) will then converge more rapidly;but for a white roughness,defined so that $a_k \sim 1/k$ as $k \to \infty$ (the roughness is independent of scale),the convergence is very slow and u_b^* may again be negligible. The second mechanism is the onset of cavitation when the pressure from (3.129) becomes sufficiently negative. In this case the smaller bumps may be swamped by the cavities and thus there would be an effective 'cut-off wavenumber' k_c above which $a_k \simeq 0$. However,both these explanations are circumferential. An alternative explanation is that the infinite series (3.131) may not be a very useful form in which to present the solution,since we require a detailed knowledge of the bedrock variation on the finest scale in order to predict the drag,which is essentially a global phenomenon.

Two questions thus present themselves: (1) is it possible to show that the proposed model with neither regelation nor cavitation occurring gives sliding velocities which do not require detailed knowledge of a_k for large k; (2) what is the effect of cavitation on the sliding law? The first question forms the subject of the remainder of this chapter:the second will be considered in Chapter IV.

§7 Ice Flow Problem:Non-Newtonian Flow.

In his paper,Kamb (1970) considers the effect of the nonlinear flow law of ice by assuming that the mean stress of a single Fourier component is dependent only on the height y. This assumption allows the Fourier-analytic technique to be used since,as in the linear case,the different components do not couple when the boundary conditions are applied. The merits of this assumption are questionable. Kamb refers to another paper for a detailed explanation of his assumption,but no location for this paper is given.

The present approach,and the central part of this chapter,is to consider a variational principle for the flow of the ice over the bedrock. Such principles for slow non-Newtonian flow were first comprehensively put forward by Johnson (1960,1961). For certain stress-strain relations he stated a general variational principle which has as its Euler equations and natural boundary conditions the equations and boundary conditions of steady slow motion of a non-Newtonian fluid bounded by a surface on which appropriate velocity and stress conditions are given. Specifically, he considered the bounding surface S to be composed of non-overlapping components S_t and S_v, on which respectively the stress and velocity were specified. In this case, by adding natural admissibility conditions, velocity and stress principles may be deduced. For certain flow laws (of which the power law model is one) these give a global maximum and minimum for the variational functional. For the power law model, this functional is just a multiple of the drag : hence by finding appropriate trial functions, we can estimate the drag on the bedrock.

Johnson's theory has been widely applied. Wasserman and Slattery (1964) used it to estimate the drag on a sphere flowing slowly in an unbounded fluid. Astarita and Apuzzo (1965) considered the same geometry, but with a gaseous sphere; Nakano and Tien (1968) and later Mohan (1974) considered a Newtonian fluid sphere. Hopke and Slattery (1970) obtained bounds on the drag on a sphere moving slowly in an Ellis model fluid, which is a simple generalisation of a power law model fluid, but with finite viscosity at low stresses. This suggests itself as a useful model for ice (see Budd and Radok 1971) but is not a necessary refinement in the present problem since the shear stress approaches a finite limit as $y \to \infty$.

None of the above authors has considered a variational principle for a problem such as flow past a bubble which requires mixed boundary conditions (no normal velocity, no tangential stress): a slight modification of Johnson's principle is necessary, using a method well known in linear elasticity.

That all the work has been done on spherical geometries is not surprising, since the Stokes paradox does not appear explicitly for this case. The only work that incorporates the Oseen flow past a sphere is that by Caswell and Schwarz (1962) for a Rivlin-Erickson fluid. In order to state a rigorous variational principle, it is necessary to solve the outer (Oseen) flow problem, find the matching conditions on the inner (Stokes) flow, and prescribe these conditions on a 'boundary' in the matching region. This 'asymptotic variational principle' will give bounds which are correct to the order that the matching boundary conditions are correct.

This is essentially how the bounding 'surface' is chosen in the present problem. In this case, however, the whole glacier is well within the Stokes region (since the Reynolds number $Re \sim 10^{-13}$), and the inner and outer regions correspond to expansions of the flow solution in terms of $\sigma = [x]/d << 1$. We thus choose the upper surface of the bounding region as

$$y = [x]y^* \ , \ 1 << y^* << 1/\sigma \ . \qquad (3.133)$$

We consider a bedrock periodic over a length L; the geometry is shown in Figure 3.7. The bedrock is denoted by S_b, and the upper surface by S_∞.

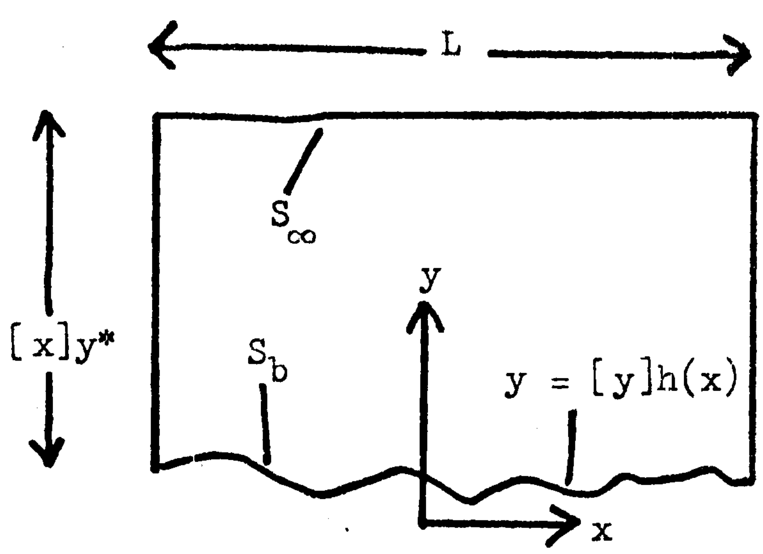


Figure 3.7.

The boundary conditions we shall consider are

$$u = U[u_b + \frac{\sigma S}{[x]}y + \ldots] \ , \ v = 0 \text{ on } y = [x]y^* \ , \qquad (3.134)$$

where S is given by (3.16). On $y = [y]h$,

$$q_n \text{ and } \sigma_{nt} \text{ are given,} \qquad (3.135)$$

where $\underset{\sim}{n}$ and $\underset{\sim}{t}$ are the normal and tangential unit vectors, respectively. Following Johnson (1960), we use suffices i to identify components in cartesian geometry. Commas denote partial differentiation with respect to the indicated coordinate, and we use the summation convention. We thus write (3.135) in the form

$$v_i n_i = V(x) \ , \qquad \text{(a)}$$
$$\sigma_{ij} n_i t_j = T(x) \ , \qquad \text{(b)} \qquad (3.136)$$

on $y = [y]h$. For reasonably large obstacles, $\alpha \ll 1$ (from §5) and $V \approx 0$: also if there are no basal cold patches, the existence of the lubrication film implies $T = 0$. We leave these functions in for the moment, however.

We also stipulate that the flow and stress fields are periodic in L.

The equations of motion (3.17), (3.18) and (3.19) may be written

$$v_{i,i} = 0 \ , \qquad (3.137)$$

$$\sigma_{ij,j} + \rho f_i = 0 \qquad (3.138)$$

where

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \qquad (3.139)$$

and

$$\underset{\sim}{f} = (\varepsilon g', -g') \ . \qquad (3.140)$$

The rate of deformation tensor is

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \qquad (3.141)$$

(we use d_{ij} rather than e_{ij} for ease of comparison to Johnson's work). We assume there is a function $\Gamma(d_{rs})$ such that the flow law is

$$\tau_{ij} = \frac{\partial \Gamma}{\partial d_{ij}} = \tau_{ji} \ . \qquad (3.142)$$

Equations (3.137), (3.138), (3.139), (3.141) and (3.146) are to be solved subject to periodicity, (3.134) and (3.136).

§8 Variational Principle.

Consider the functional

$$J = \int_V [\{\tfrac{1}{2}(v_{i,j} + v_{j,i}) - d_{ij}\}\tau_{ij} + \Gamma - \rho f_i v_i - p v_{i,i}]\, dV$$
$$- \int_{S_b} [\sigma_{kj} n_k n_j (v_i n_i - V) + T v_i t_i]\, dS$$
$$- \int_{S_\infty} \sigma_{ij} n_j (v_i - U_i)\, dS \quad , \tag{3.143}$$

where

$$U_1 = U\,[u_b + \sigma S y^* + \ldots]\ , \ U_2 = 0\ . \tag{3.144}$$

This corresponds to (3.1) of Johnson (1961) except that only the components corresponding to the terms that are known in the surface integral are taken. The second integrand may be written more briefly as

$$\sigma_{nn}(v_n - V) + v_t T\ . \tag{3.145}$$

Now let v_i, τ_{ij}, etc. be a solution of the boundary value problem, and let $\delta v_i, \delta\tau_{ij}$, etc. be arbitrary variations. As admissibility conditions we stipulate only that the variations be periodic in L, and sufficiently differentiable for what follows to be valid. The periodicity enables us to cancel out integrals on the ends $y = \pm L/2$ of our surface. The variation

$$\delta J = \int_V [\{\tfrac{1}{2}(v_{i,j} + v_{j,i}) - d_{ij}\}\delta\tau_{ij} + (\frac{\partial\Gamma}{\partial d_{ij}} - \tau_{ij})\delta d_{ij}$$
$$+ \tau_{ij}\delta v_{i,j} - p\delta v_{i,i} - \rho f_i \delta v_i - \delta p v_{i,i}]\, dV$$
$$- \int_{S_\infty} (v_i - U_i)\delta\sigma_{ij} n_j\, dS - \int_{S_\infty} \sigma_{ij}\delta v_i n_j\, dS$$
$$- \int_{S_b} [(v_i n_i - V)\delta\sigma_{kj} n_j n_k + \delta v_i (T t_i + \sigma_{kj} n_k n_j n_i)]\, dS$$
$$= \int_V \Big[(\tau_{ij}\delta v_i)_{,j} - (p\delta v_i)_{,i} - \{\tau_{ij,j} - p_{,i} - \rho f_i\}\delta v_i\Big]\, dV$$
$$- \int_{S_\infty} \sigma_{ij} n_j \delta v_i\, dS - \int_{S_b} \delta v_i (T t_i + \sigma_{kj} n_k n_j n_i)\, dS\ ,$$

using the equations and boundary conditions. Thus, using Green's theorem and periodicity of τ_{ij}, we find

$$\delta J = \int_{S_b} \delta v_i [\sigma_{ij} n_j - T t_i - \sigma_{kj} n_k n_j n_i] \, dS \, . \tag{3.146}$$

Now

$$\sigma_{nt} = \sigma_{ij} n_i t_j = T \, , \tag{3.147}$$

therefore

$$\begin{aligned} & T t_i + \sigma_{kj} n_k n_j n_i \\ &= \sigma_{kj} n_k t_j t_i + \sigma_{kj} n_k n_j n_i \\ &= \sigma_{nt} t_i + \sigma_{nn} n_i \\ &= \sigma_{ni} = \sigma_{in} = \sigma_{ij} n_j \, . \end{aligned} \tag{3.148}$$

Hence

$$\delta J = 0 \; : \tag{3.149}$$

thus J is stationary at a solution of the problem.

If, instead of (3.142), we can write the flow law in the form

$$d_{ij} = d_{ji} = \frac{\partial \hat{\Gamma}(\tau_{rs})}{\partial \tau_{ij}} \tag{3.150}$$

then we can write

$$\Gamma - d_{ij} \tau_{ij} = -\hat{\Gamma} \quad . \tag{3.151}$$

If we define $\mathscr{H}$ by the functional (3.143) with Γ replaced by $d_{ij}\tau_{ij} - \hat{\Gamma}$, then similarly

$$\delta \mathscr{H} = 0 \tag{3.152}$$

at a solution of the boundary value problem.

§9 Velocity and Stress Principles.

We now restrict ourselves to consideration of the functional J when equations (3.137),(3.141) and (3.142) are satisfied together with the velocity boundary conditions (3.134) and (3.136a). In this case the functional is

$$J_V = \int_V [\Gamma - \rho f_i v_i]\, dV - \int_{S_b} T v_i t_i \, dS \,, \tag{3.153}$$

and writing

$$J_V = J_0 + \delta J + J_2 \,, \tag{3.154}$$

$\delta J = 0$ and

$$J_2 = \int_V [\Gamma(d_{ij} + \delta d_{ij}) - \Gamma(d_{ij}) - \frac{\partial \Gamma}{\partial d_{ij}} \delta d_{ij}]\, dV \,. \tag{3.155}$$

Johnson (1961) shows that $J_2 \geq 0$ for a pseudoplastic fluid such as ice for which Γ is convex, and so J is globally minimised by the solution for all admissible trial velocity fields,

$$J_0 \leq J_V \,. \tag{3.156}$$

It is similarly straightforward to show that if we consider the functional (3.143) with Γ defined by (3.151), and admit only trial fields satisfying (3.150),(3.138) and (3.136b), then

$$\mathcal{H} = \mathcal{H}_T = \int_V -\hat{\Gamma}\, dV + \int_{S_\infty} \sigma_{ij} U_i n_j \, dS + \int_{S_b} \sigma_{kj} n_k n_j V \, dS \quad , \tag{3.157}$$

and if

$$\mathcal{H}_T = \mathcal{H}_0 + \delta\mathcal{H} + \mathcal{H}_2 \,, \tag{3.158}$$

then $\mathcal{H}_0 = J_0$, $\delta\mathcal{H} = 0$ and

$$\mathcal{H}_2 = -\int_V [\hat{\Gamma}(\tau_{ij} + \delta\tau_{ij}) - \hat{\Gamma}(\tau_{ij}) - \delta\tau_{ij}\frac{\partial\hat{\Gamma}}{\partial\tau_{ij}})]\, dV \,. \tag{3.159}$$

Just as $J_2 \geq 0$, so $\mathcal{H}_2 \leq 0$ and hence for all admissible trial functions,

$$\mathcal{H}_T \leq \mathcal{H}_0 \,. \tag{3.160}$$

Thus we have global bounds on J_0,

$$\mathcal{H}_\tau \leqslant J_0 \leqslant J_V \ . \tag{3.161}$$

§10 Bounds for the Drag.

We proceed to describe $\Gamma, \hat{\Gamma}$ and J_0 for a power law model fluid. We use Johnson's notation

$$II = d_{ij}d_{ij} = 2e^2 \ , \ II_\tau = \tau_{ij}\tau_{ij} = 2\tau^2 \ ; \tag{3.162}$$

thus the viscosity is given by

$$\tau_{ij} = 2\eta d_{ij} \ , \tag{3.163}$$

whence

$$\tau = 2\eta e \ , \ \eta = \tau/2e = 2^{-1}e^{-1}e^{\frac{1}{n}}A^{\frac{-1}{n}} \tag{3.164}$$

since $e = A\tau^n$. Hence from (3.162)

$$\begin{aligned} \eta &= 2^{-1} II^{-\frac{1}{2}} 2^{\frac{1}{2}} II^{\frac{1}{2n}} 2^{\frac{-1}{2n}} A^{\frac{-1}{n}} \\ &= 2^{-\frac{1}{2}(1+\frac{1}{n})} A^{\frac{-1}{n}} II^{\frac{1}{2}(\frac{1}{n}-1)} \ . \end{aligned} \tag{3.165}$$

Using (4.8) and (4.9) of Johnson (1961),

$$\Gamma = \frac{n}{n+1} 2^{\frac{1}{2}(1-\frac{1}{n})} A^{\frac{-1}{n}} II^{\frac{1}{2}(\frac{1}{n}+1)} \ , \tag{3.166}$$

$$\hat{\Gamma} = \frac{1}{n+1} 2^{-\frac{1}{2}(n-1)} A \ II_\tau^{\frac{1}{2}(n+1)} \ . \tag{3.167}$$

Now $2^{\frac{1}{2}(1-\frac{1}{n})} A^{\frac{-1}{n}} II^{\frac{1}{2}(\frac{1}{n}+1)} = 2A^{\frac{-1}{n}} e^{\frac{1}{n}+1} = 2e\tau$, and $\tau_{ij}e_{ij} = \tau_{ij}A\tau^{n-1}\tau_{ij} = 2A\tau^{n+1} = 2\tau e$, hence

$$\begin{aligned} \int_V \Gamma \, dV &= \frac{n}{n+1} \int_V \sigma_{ij}v_{i,j} \, dV \\ &= \frac{n}{n+1} \left[\int_{S_b, S_\infty} \sigma_{ij}v_i n_j \, dS + \int_V \rho f_i v_i \, dV \right] \end{aligned} \tag{3.168}$$

and so

$$J_0 = \frac{n}{n+1}\int_{S_\infty} \sigma_{ij} n_j U_i \, dS - \frac{1}{n+1}\int_V \rho f_i v_i \, dV + \int_{S_b} [\frac{n}{n+1}\sigma_{ij} v_i n_j - T v_i t_i] \, dS. \tag{3.169}$$

Also

$$\sigma_{ij} v_i n_j = v_i \sigma_{in} = v_i(\sigma_{nt} t_i + \sigma_{nn} n_i)$$

$$= \sigma_{nt} t_i v_i = T t_i v_i \; ; \tag{3.170}$$

since $v_i n_i = 0$ for the velocity principle,it follows that

$$J_0 = \frac{n}{n+1}[\tau]_0 HLU[u_b + \sigma S y^* \ldots]$$

$$- \frac{1}{n+1}\int_V \rho f_i v_i \, dV - \frac{1}{n+1}\int_{S_b} T v_t \, dS \; . \tag{3.171}$$

We now put $T = 0$ (zero traction on the bedrock);we may further simplify (3.171) since if,for example, $\nu \sim \varepsilon$,then $\rho f_i v_i \sim \rho g' U u_b \varepsilon \sim \frac{[\tau]_0 U u_b}{d}$,and so (3.171) may be written

$$J_0 = \frac{n}{n+1}[\tau]_0 ULH \, [u_b + O(\sigma y^*)] \; . \tag{3.172}$$

Using (3.153),(3.156) and (3.166),

$$[\tau]_0 ULHu_b + \ldots \leqslant 2^{\frac{1}{2}(1 - \frac{1}{n})} A^{\frac{-1}{n}} \int_V II^{\frac{1}{2}(\frac{1}{n} + 1)} \, dV \; . \tag{3.173}$$

We automatically satisfy (3.136a) and (3.137) by writing

$$u = \psi_y \; , \; v = -\psi_x \; , \tag{3.174}$$

and choosing

$$\psi = 0 \text{ on } y = [y]h \; . \tag{3.175}$$

We should also satisfy (3.134). However (compare the linear theory) the second order shearing term will not affect the right hand side of (3.173) to leading order,and so it is sufficient to have

$$\psi \sim U u_b y \; , \; y \to \infty \; . \tag{3.176}$$

We have

$$II = 2e^2 = 2u_x{}^2 + \tfrac{1}{2}(u_y+v_x)^2 \quad \text{from } (2.31)$$

$$= 2^{-1}[4\psi_{xy}^2 + (\psi_{yy}-\psi_{xx})^2] \,, \tag{3.177}$$

so

$$[\tau]_o ULHu_b + \ldots \leq (2A)^{\frac{-1}{n}} \int_{-L/2}^{L/2} \int_{[y]h}^{[x]y^*} [(\psi_{yy} - \psi_{xx})^2 + 4\psi_{xy}^2]^{\frac{n+1}{2n}} \, dx \, dy \,. \tag{3.178}$$

We now turn to the stress principle. Let us assume V = 0. We must satisfy the equations

$$\hat{\sigma}_{11x} + \hat{\sigma}_{12y} = 0 \,,$$
$$\hat{\sigma}_{2\ x} + \hat{\sigma}_{22y} = 0 \,, \tag{3.179}$$

where we have incorporated ρf_i into the pressure by writing

$$\hat{p} = p - \rho g' \varepsilon x + \rho g' y \tag{3.180}$$

and

$$\hat{\sigma}_{ij} = -\hat{p}\delta_{ij} + \tau_{ij} \,. \tag{3.181}$$

In this case there exists the Airy stress function ϕ satisfying

$$\hat{\sigma}_{11} = -\hat{p} + \tau_1 = \phi_{yy} \,,$$
$$\hat{\sigma}_{12} = \hat{\sigma}_{21} = \tau_2 = -\phi_{xy} \,, \tag{3.182}$$
$$\hat{\sigma}_{22} = -\hat{p} - \tau_1 = \phi_{xx} \,,$$

and we wish to pick any ϕ such that

$$\sigma_{nt} = 0 \text{ on } y = [y]h \,. \tag{3.183}$$

Let us suppose the stress at infinity of our trial function is $\tau_2 = [\tau]_o H$, i.e.

$$\phi \sim -[\tau]_o Hxy \ , \ y \to \infty \ . \tag{3.184}$$

Then we have, from (3.157), (3.161), (3.167) and (3.172),

$$J_O = \frac{n}{n+1}[\tau]_o ULHu_b + \ldots \geq -\frac{1}{n+1} 2^{-\frac{1}{2}(n-1)} A\int_V II_\tau^{\frac{n+1}{2}} + [\tau]_o HUL(u_b + ..) \ ,$$

whence

$$2^{-\frac{1}{2}(n-1)} A\int_V II_\tau^{\frac{n+1}{2}} \leq [\tau]_o ULHu_b + \ldots \tag{3.185}$$

The inclusion of a stress $> \frac{n}{n+1}[\tau]_o H$ at infinity is necessary to obtain a nontrivial bound: this is because the driving stress enters the stress solution at an earlier stage than it enters the velocity solution. Now

$$II_\tau = \tau_{ij}\tau_{ij} = 2(\tau_1^2 + \tau_2^2)$$

$$= 2[\tfrac{1}{4}(\phi_{yy} - \phi_{xx})^2 + \phi_{xy}^2] \ , \tag{3.186}$$

whence

$$\int_V [\phi_{xy}^2 + \tfrac{1}{4}(\phi_{yy} - \phi_{xx})^2]^{\frac{n+1}{2}} dV \geq [\tau]_o^{n+1} dLHu_b \ .. \tag{3.187}$$

since $[\tau]_o = (U/2dA)^{1/n}$.

§11 Dimensional Estimates.

So far we have not introduced the bedrock roughness ν: (3.187) and (3.178) are inequalities for the leading terms in expansions in σ. It is not therefore obvious what size u_b will be. For example, the Newtonian flow gave $u_b \sim 1$ if $\sigma \sim \nu^2$. We can estimate u_b by use of the fact that (3.187) and (3.178) are equalities when the trial functions are solutions. Now since the drag at infinity equals the drag on the bedrock, we know (with an obvious notation) that

$$[\tau]_i \sim [\tau]_o/\nu \tag{3.188}$$

and that

$$e_i \sim A[\tau]_i^n \ . \tag{3.189}$$

If we formally write

$$
\begin{aligned}
u_b &= \frac{\sigma}{\nu^{n+1}}\, u_b^* , \\
\psi &= \nu[x]Uu_b\psi^*(x_i,y_i) , \\
\phi &= \frac{[x]^2[\tau]_o}{\nu}\, H\phi^*(x_i,y_i) , \\
y &= [x]y_i , \\
x &= [x]x_i , \\
h &= [y]h(x_i) , \\
M &= \frac{L}{[x]} ,
\end{aligned}
\tag{3.190}
$$

then (3.187) and (3.178) become, on dropping the suffix i,

$$H \leqslant u_b^{*1/n}\frac{1}{M}\int_{-M/2}^{M/2} dx \int_{\nu h}^{y^*} [(\psi^*_{yy} - \psi^*_{xx})^2 + 4\psi^{*2}_{xy}]^{\frac{n+1}{2n}}\, dy , \tag{3.191}$$

$$u_b^* \leqslant H^n \frac{1}{M}\int_{-M/2}^{M/2} dx \int_{\nu h}^{y^*} [\phi^{*2}_{xy} + \tfrac{1}{4}(\phi^*_{yy} - \phi^*_{xx})^2]^{\frac{n+1}{2}}\, dy . \tag{3.192}$$

Hence $u_b^* \sim 1$, and so

$$u_b \sim \frac{\sigma}{\nu^{n+1}} , \tag{3.193}$$

and (3.190) satisfies (3.188) and (3.189). This confirms that $u_b \sim 1$ when $\sigma \sim \nu^2$ $(n = 1)$ for Newtonian flow. It may be seen from (3.193) that we are justified in formally considering $\nu \ll 1$ to obtain $O(1)$ values of u_b. However in practice if for example $\sigma \sim 10^{-1}$-10^{-2}, $\nu \sim \sigma^{1/(n+1)} \sim 1/3$ for $O(1)$ u_b; it is evident that u_b is highly sensitive to the precise magnitude of ν.

If $\sigma/\nu^{n+1} \gg 1$, (3.193) seems to imply a large ($\gg O(1)$) basal velocity. Since the velocity was specifically scaled in Chapter II from the accumulation rate to be $O(1)$, this appears to be contradictory. In fact (3.193) implies that H is small: if we then denote by U_{shear} the change in velocity in the outer flow due to shearing, then $U_{shear} \sim Hu_y \sim H^{n+1}$, provided (2.101) still holds. Then since $u_b \sim \sigma H^n/\nu^{n+1}$ from (3.191)-(3.193), $U_{shear} << u_b$, and so $H \sim (\nu^{n+1}/\sigma)^{1/n}$. The outer flow velocity may thus be properly written

$$u = u_b(x) + (\frac{\nu^{n+1}}{\sigma})^{(\frac{n+1}{n})} u_1(x,y) + \ldots \quad (3.194)$$

The same reduction for the outer flow as in Chapter II can be carried out to find u_1 provided

$$\nu^{n+1}/\sigma >> \delta^{\frac{2n}{n+1}} , \quad (3.195)$$

and in this case (2.101) is indeed valid and thus also (3.194). However, if (3.194) holds, we shall not in general be interested in u_1, and will then neglect the momentum equation (2.105): we still require (3.195) to be valid if we are to obtain the same energy equation (2.107) as in the reduced model.

§12 Modifications.

(3.191) and (3.192) predict $u_b^* \sim H^n$ as would be expected from dimensional considerations when we neglect regelation. Weertman's heuristic $u_b \sim H^{(n+1)/2}$ could arise if $[x] \sim \lambda_*$ (Morland 1976), the controlling obstacle size where regelation and plastic flow are equally important (since one can easily show $u_b \sim H$ for pure regelation); in fact $[x] >> \lambda_*$, and we expect regelation to play a small part in the sliding process. To include its effect, we could modify our principles (3.191) and (3.192) in two ways: firstly, we could include the suction velocity in (3.143). However, since this depends on the pressure which is a result of solving the ice flow, such a method is not entirely satisfactory. It is probably better to utilise the fact that regelation only occurs significantly past obstacles asymptotically smaller than the scale $[x]$. Then, if we can solve the small scale regelation flow for a given tangential velocity, this leads asymptotically in the large scale $[x]$ problem, to the imposition of an effective regelation drag

$$\sigma_{nt} = T_{reg}(v_t) \qquad (3.196)$$

over the surface. Thus regelation can be incorporated into the general principle (3.143) via the traction T. Similarly, if cold patches (giving rise to short time-scale stick-slip ("stictional") motion exist due to pressure reduction in the lee of obstacles, their effect can be included by imposing an additional tangential stress due to Coulomb friction. These effects will not be considered further here; we restrict ourselves to the consideration of appropriate trial functions for ψ^* and ϕ^*.

§13 Trial Stream and Stress Functions.

For the Newtonian flow over a perfect sinusoid $y = \nu ae^{ikx}$, the solution to $O(\nu)$ is

$$\psi^* = \frac{y}{\nu} - (1+ky)e^{-ky}ae^{ikx} \;.. \qquad (3.197)$$

as may be verified by direct substitution into (3.98)-(3.100). The most obvious approximation to (3.197) which satisfies the trial function condition

$$\psi^* = 0 \text{ on } y = \nu h \qquad (3.198)$$

is

$$\psi^* = \frac{y}{\nu} - h[1 + k_1(y-\nu h)]e^{-k_2(y-\nu h)} \;, \qquad (3.199)$$

and others may be written down simply. A calculation using (3.199) is carried out in Appendix 1.

The dimensionless trial stress function ϕ^* has to satisfy the boundary condition $\sigma_{nt} = 0$ on $y = \nu h$. Since $\underset{\sim}{t} \propto (1, \nu h')$, $\underset{\sim}{n} \propto (-\nu h', 1)$, this condition becomes, using (3.182),

$$\nu h'(\phi^*_{yy} - \phi^*_{xx}) + (1 - \nu^2 h'^2)\phi^*_{xy} = 0 \text{ on } y = \nu h. \quad (3.200)$$

We will also try to satisfy

$$\phi^* \sim -\nu xy \;, \; y \to \infty \;, \qquad (3.201)$$

in order to obtain accuracy. Unlike the case of the stream function, it is a non-trivial matter even to satisfy the boundary condition (3.200) for ϕ^*. To see how to do so, we write

$$Y = y - \nu h \ , \ \phi^* = \theta(x,Y) \ ; \tag{3.202}$$

then

$$\begin{aligned}
\phi^*_x &= \theta_x - \nu h'\theta_Y \ , \\
\phi^*_{xx} &= \theta_{xx} - 2\nu h'\theta_{xY} - \nu h''\theta_Y + \nu^2 h'^2\theta_{YY} \ , \\
\phi^*_{xy} &= \theta_{xY} - \nu h'\theta_{YY} \ , \\
\phi^*_{yy} &= \theta_{YY} \ ;
\end{aligned} \tag{3.203}$$

the transformation (3.202) moves the boundary to Y = 0. Thus (3.200) becomes

$$\nu h'[\theta_{YY} - \theta_{xx} + 2\nu h'\theta_{xY} + \nu h''\theta_Y - \nu^2 h'^2\theta_{YY}] + (1 - \nu^2 h'^2)(\theta_{xY} - \nu h'\theta_{YY}) = 0 \ ,$$

i.e.

$$\theta_{xY} = \nu h'[\theta_x - \nu h'\theta_Y]_x \text{ on } Y = 0 \ . \tag{3.204}$$

Now notice that for the accuracy condition (3.201) we really only require

$$\int_{-L/2}^{L/2} \phi^*_{xy} \, dx = -\nu L \ , \ y \to \infty \ , \tag{3.205}$$

and we still obtain the inequality (3.192); (3.205) may be written

$$\theta_Y \Big|_{-L/2}^{L/2} = -\nu L \ , \ y \to \infty \ . \tag{3.206}$$

We can satisfy (3.206) by stipulating that θ_Y is periodic, but with a non-zero mean. This rules out various simple forms for θ we might try.

We now write

$$\theta_{xY} = \nu f'(x) \quad \text{on } Y = 0 , \tag{3.207}$$

so

$$\theta_Y = \nu f(x) \quad \text{on } Y = 0 . \tag{3.208}$$

Substituting into (3.204),

$$f' = h'[\theta_x - \nu^2 h'f]_x \quad \text{on } Y = 0 . \tag{3.209}$$

Let

$$f' = h'g' , \tag{3.210}$$

($g'(x)$ is unrelated to g' of Chapter II), then

$$\theta_x = \nu^2 h'f + g \quad \text{on } Y = 0 . \tag{3.211}$$

Hence we may write

$$\theta = k(x) , \; \theta_Y = \nu f(x) \quad \text{on } Y = 0 , \tag{3.212}$$

where

$$k'(x) = g(x) + \nu^2 h'f , \quad f' = h'g' . \tag{3.213}$$

(3.212) and (3.213) define boundary conditions on Y = 0 for an appropriate trial function. We also try to satisfy the accuracy condition

$$\theta_{xY} - \nu h'\theta_{YY} \to -\nu\alpha(x) , \; Y \to \infty , \tag{3.214}$$

or simply, from (3.206),

$$\theta \sim -\nu Y \int^x \alpha(x)\, dx , \; Y \to \infty , \tag{3.215}$$

where α is periodic with mean 1.

From (3.203), we find

$$4\phi^{*2}_{xy} + (\phi^*_{yy} - \phi^*_{xx})^2 =$$

$$(1+\nu^2h'^2)^2\theta_{YY}^2 - 4\nu h'(1+\nu^2h'^2)\theta_{YY}\theta_{xY} + 4(1+\nu^2h'^2)\theta_{xY}^2$$

$$+ \theta_{xx}^2 - 4\nu h'\theta_{xY}\theta_{xx} + 2(1-\nu^2h'^2)\theta_{YY}(\nu h''\theta_Y - \theta_{xx})$$

$$+ \nu^2h''^2\theta_Y^2 + 2\nu h''\theta_Y(2\nu h'\theta_{xY} - \theta_{xx}) = \Box \quad , \qquad (3.216)$$

say.

In order to select a reasonable trial function, we consider the Newtonian stream function (3.197). With p given by (3.129), $\tau_1 = 2u_x$, $\tau_2 = u_y - v_x$ and using (3.182), we find that the stress function in this case is

$$\phi^* = 2i(1+ky)e^{-ky}e^{ikx} \ . \qquad (3.217)$$

An obvious choice for θ is thus

$$\theta = K(x)(1+ky)e^{-ky} + \nu Yf(x) \ , \qquad (3.218)$$

where $f(x) = -\int^x \alpha(x)\,dx$. The form of (3.217) suggests we try $\int^x g\,dx = h, h', \int^x h\,dx$, etc., where g is defined in (3.210) or (3.213). In fact from (3.213) the simplest choice that gives appropriate behaviour at $Y \to \infty$ is

$$g' = ah' \Rightarrow g = ah, \qquad (3.219)$$

where a is constant. With this choice,

$$f = a\int^x h'^2\,dx \ , \qquad (3.220)$$

$$K(x) = a\int^x h\,dx + a\nu^2h'\int^x h'^2\,dx \ , \qquad (3.221)$$

and

$$\theta = -\frac{1}{[\frac{1}{M}\int_{-M/2}^{M/2} h'^2\,dx]}\Big[\{\int^x h\,dx + \nu^2h'\int^x h'^2\,dx\}(1+kY)e^{-kY}$$
$$+ \nu Y\int^x h'^2\,dx\Big], \qquad (3.222)$$

where a has been chosen appropriately. Other more complicated examples could be found.

In Appendix 1 we use the stress function estimate (3.222) in order to obtain the crudest possible bounds on u_b^*. We find that, if no detail of the topography is assumed at all, then we have numerically with n = 3,

$$0.0688\, H^n \leq u_b^* \leq \frac{16.414}{(2\overline{h'^2})^{\frac{1}{2}(n+1)}}\, H^n\,, \qquad (3.223)$$

where $\overline{h'^2}$ is the average value of h'^2. If some topographic detail is assumed, these bounds could obviously be very much refined. For n = 1, the corresponding result is

$$0.447\, H \leq u_b^* \leq \frac{3.51}{2\overline{h'^2}}\, H\,. \qquad (3.224)$$

This should be compared with the rather uninformative (3.131).

§14 Conclusions.

We have considered a complete model for glacier sliding without cavitation. This shows that for 'large scale' irregularities (e.g. ~ 1 m) regelation is unimportant and can be neglected. For a Newtonian fluid, a straightforward asymptotic expansion (following Nye) of the stream function gives a sliding law which is sensitively dependent on the higher frequency Fourier coefficients of the bedrock. In order to obtain a more specific estimate of the sliding law, and also to consider rigorously the nonlinear problem, a variational principle was introduced which gives upper and lower bounds for u_b as $\nu, \sigma \to 0$. Even for very general bedrocks this gives useful estimates, and incidentally shows that the sliding velocity

$$u_b \sim \sigma/\nu^{n+1}\,. \qquad (3.225)$$

(3.225) is to be interpreted as follows: if $\sigma << \nu^{n+1}$, the sliding velocity is negligible; if $\sigma \sim \nu^{n+1}$, it is comparable to the velocity due to shear; finally if $\sigma >> \nu^{n+1}$, the depth is reduced by a factor $(\nu^{n+1}/\sigma)^{1/n}$, basal sliding is dominant and the velocity can be written as in (3.194), provided (3.195) is also valid;in this case the approximations used in deriving (2.84) and (2.85) are still valid,the dissipation term is still given by (2.102),and thus the energy equation is given by (2.108).

CHAPTER IV

The Theory of Sliding:

(ii) Cavitation in Newtonian Flow

§1 Cavitation Mechanisms.

We now turn our attention to the effect of cavity formation on the sliding law of a Newtonian fluid. First of all we need to know why cavities should form at all. There are several possible reasons.

If the stress is sufficiently high, the ice may rupture. The flow will maintain a cavity, on the unknown roof of which an extra stress condition will hold (for example the Coulomb failure criterion). However, typical stresses occurring in glacial ice are too low for failure to take place, even near the bedrock (cf. (3.128)), and this mechanism is therefore not considered to be physically relevant.

If the ice-water interface is in thermodynamic equilibrium, then the pressure p_w in the water film is given by the Clausius-Clapeyron relation on the equilibrium phase diagram (see Figure 4.1). If p_w (determined by the ice flow) decreases to the ice-water-vapour triple point, a cavity forms since the water vapourises for further reduction in the pressure. The boundary condition in such a cavity is

$$p_w = \text{constant } (\approx 0 \text{ bars}) . \qquad (4.1)$$

This kind of cavity is directly equivalent to the cavities normally observed in fluid flow past bluff bodies (Batchelor 1967); (4.1) is the cavity boundary condition used by Kamb (1970 p 720) and Nye (1969). Lliboutry has this cavitation pressure unknown because he post-

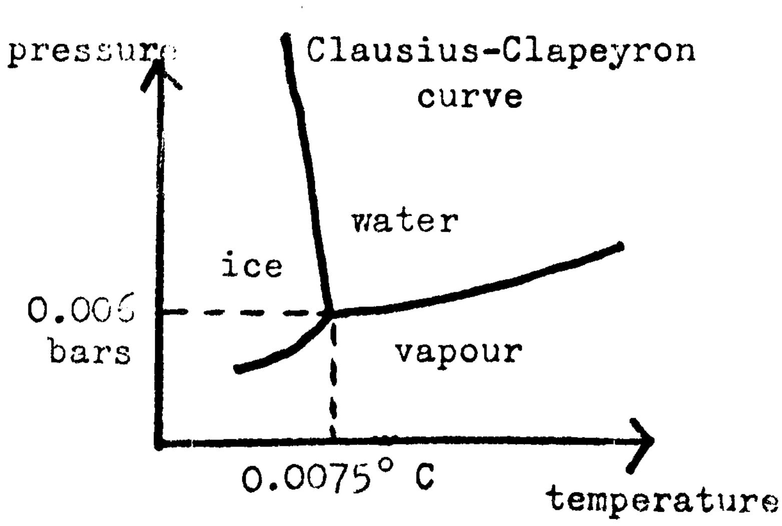


Figure 4.1

ulates, rather than predicts, cavities.

A cavity will also form if the water film thickness $\Sigma \to \infty$. From (3.78), we see that this is the case if $\Pi' \to 0$ while $\int^x V_M\, dx$ is finite. Although Π and V_M are related by (3.82) and (3.74), the zeros of Π' and $\int^x V_M\, dx$ will not in general be the same, and so cavities and cold patches (when $\int^x V_M\, dx \to 0$ with $\Pi' \neq 0$) will generally exist on the bedrock. There are bedrocks for which this is not true: for example if $h = \cos x$, then from (3.132) $p^* \simeq \Pi \propto -\sin x$, whence $T^*|_{y=0} \propto \sin x$, $T^* \propto e^y \sin x$, $V_M \propto -T^*_y|_{y=0} \propto -\sin x$, and so from (3.79)

$$\Sigma^3 \propto \frac{\int^x -\sin x\, dx}{\cos x}$$

$$\Rightarrow \Sigma = \text{constant} . \qquad (4.2)$$

(The constant of integration must be zero in order that a physically meaningful solution be given.)

We shall ignore this cavitation mechanism for the rest of this chapter, and therefore we formally assume that Σ, as calculated from (3.79) is always finite.

Lliboutry (1968) is the only author who has considered the effect of cavitation on the sliding law in any detail. It is reasonable to suppose that the effect of cavitation is to reduce the friction for increasing velocity. The sliding law might then resemble one of the forms shown in Figure 4.2. It is important to gain some idea which of the profiles (a), (b) or (c) is most likely to occur.

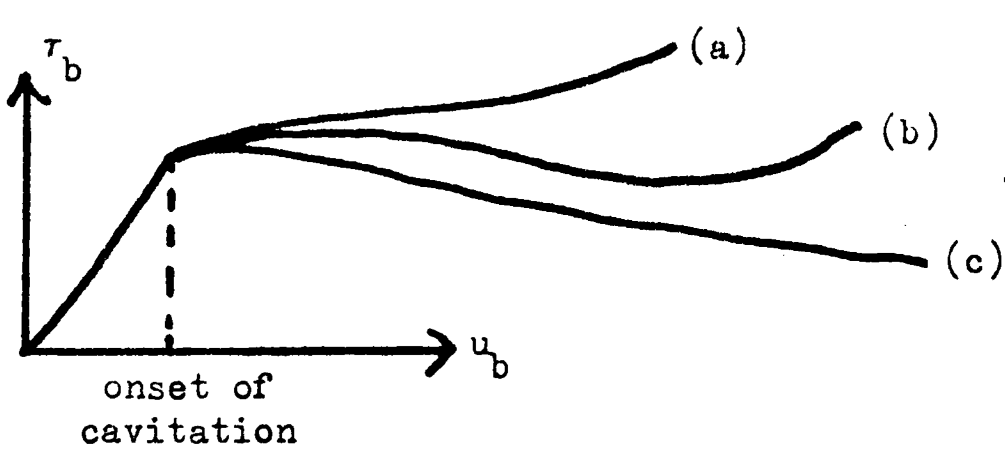


Figure 4.2: Schematic Sliding Laws.

Lliboutry (1968) derives a law like Figure 4.3 for a sinusoidal bed. His theory is based on rather arbitrary assumptions, for example that a cavity roof is of constant slope, and thus cannot be considered to give reliable results. It is, however, physically well motivated, and a useful first theoretical attempt.

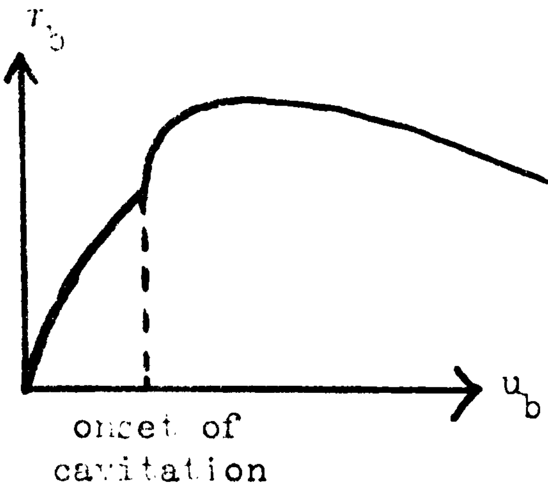


Figure 4.3: Lliboutry's law.

§2 Mathematical Model.

Let us once again consider ice as a Newtonian fluid flowing over a 2π-periodic bedrock of small slope with cavities formed purely due to the water pressure p_w reaching the triple point pressure (Figure 4.4). We take the bedrock profile along the roof of the cavities. In this case there is a section of the boundary which is unknown, but on which the pressure is a given constant. In fact, denoting the triple point pressure by p_T, and using (3.97),

$$p_A + \rho g' dH - \rho g'[x]\nu h + \frac{[\tau]}{\nu} o\; p = p_T \,, \tag{4.3}$$

or

$$p = -\,p_c \,, \tag{4.4}$$

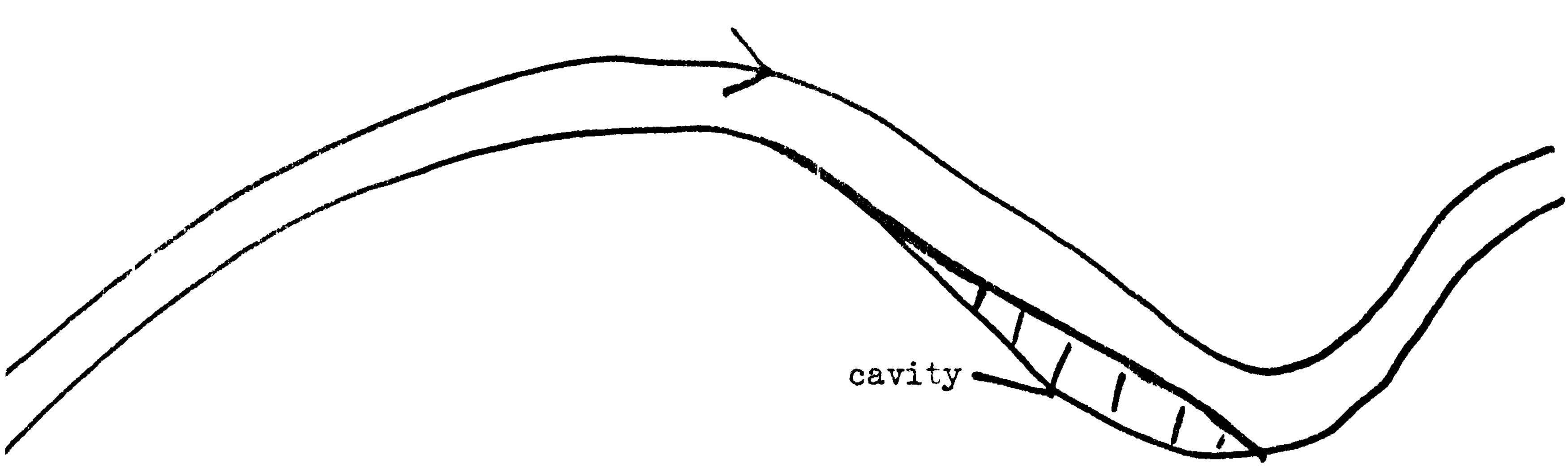


Figure 4.4: Geometry.

where

$$p_c = \frac{\nu}{\varepsilon}H + \frac{\nu}{[\tau]_0}(p_A - p_T) \ , \qquad (4.5)$$

and we have neglected $O(\sigma\nu^2/\varepsilon)$ in (4.5). The dominant term on the right hand side of (4.5) is the first one, and hence the critical roughness for onset of cavitation is when $\nu \sim \varepsilon$. For a sinusoidal bedrock, $p = -2H \sin x$ from (3.132) and thus cavitation occurs if

$$\nu \lesssim 2\varepsilon \ . \qquad (4.6)$$

A similar result was obtained by Morland (1976 equation 102). Note that the appearance of cavitation depends only on the relative size of the mean and local bedrock slopes.

Now the analysis of §7 of Chapter III is still valid, and thus to leading order, h and p satisfy

$$B(x) + B^*(x) = u_b h(x) \ , \qquad (4.7)$$

$$B''(x) - B^{*''}(x) = \frac{i\sigma}{2\nu^2} p(x) \ , \qquad (4.8)$$

from (3.115) and (3.124), where $h(x)$ is taken along the roof of the cavity. We denote by C and C' respectively the cavitated and uncavitated bedrock in a single period. Suppose for simplicity that

$$C = (a,b) \ , \ C' = (b,A) \ , \ A = a + 2\pi \ . \qquad (4.9)$$

Let h_R be the actual bedrock, which is a given function: differentiating (4.7) twice, we have to solve

$$\begin{aligned} B'' + B^{*''} &= u_b^* \frac{\sigma}{\nu^2} h_R'' \ , \quad x \in C \ , \\ B'' - B^{*''} &= -\frac{i}{2} p_c \frac{\sigma}{\nu^2} \ , \quad x \in C' \ , \end{aligned} \qquad (4.10)$$

for a periodic function $B''(z)$ analytic in $\text{Im } z > 0$, and such that $B'' \to 0$, $\text{Im } z \to \infty$: a and b are to be found as part of the solution, and we shall try to satisfy p continuous at a and b and h' continuous at a. These conditions are the physically appropriate ones to take.

We transform (4.10) to the form of a Hilbert problem as follows: let

$$
\begin{aligned}
H(z) &= \frac{\nu^2}{\sigma} B''(z) \ , \ \mathrm{Im}\, z > 0 \ , \\
H(z) &= \frac{\nu^2}{\sigma} B^{*''}(z) \ , \ \mathrm{Im}\, z < 0 \ .
\end{aligned} \tag{4.11}
$$

Then (4.10) is

$$H^+ + H^- = u_b^* h_R'' \ , \quad x \text{ in } C \ , \tag{4.12}$$

$$H^+ - H^- = \frac{-i}{2} p_c \ , \ x \text{ in } C' \ , \tag{4.13}$$

where H^+, H^- are the limiting values of H as Im $z \to 0 \pm$. We have to solve (4.12) and (4.13) subject to

$$
\begin{aligned}
&H(z) \to 0 \ , \ \mathrm{Im}\, z \to \infty \ , \\
&H^*(z^*) = H(z^*) \ , \\
&H(z) \text{ periodic} \ , \\
&\lim_{\substack{x \to a,b \\ x \text{ in } C'}} (H^+ - H^-) = \frac{-i}{2} p_c \ .
\end{aligned} \tag{4.14}
$$

If we can satisfy (4.12),(4.13),(4.14) with one undetermined constant in the solution, then since

$$
\begin{aligned}
u_b^* h'' &= H^+ + H^- \ , \ x \text{ in } C' \ , \\
\frac{i}{2} p &= H^+ - H^- \ , \ x \text{ in } C \ ,
\end{aligned} \tag{4.15}
$$

we can integrate h'' twice using continuity of h and h' at a and continuity of h at b. The extra condition for h then determines the unknown constant in the solution; h' will not in general be continuous at b.

In order to deal with the periodicity of H, we introduce

$$
\begin{aligned}
&\zeta = e^{iz} \ , \ \xi = e^{ix} \ , \ \xi_a = e^{ia} \ , \ \xi_b = e^{ib} \ , \\
&F(\zeta) \equiv H(z) \ , \ C \equiv \{e^{i\theta} : a < \theta < b\} \ , \ C' \equiv \{e^{i\theta} : b < \theta < A\} \ .
\end{aligned} \tag{4.16}
$$

By p and h'' will be understood their functional form in the new variables (e.g. $h''(x) = -\cos x \Longleftrightarrow h''(\xi) = -\frac{1}{2}(\xi + \frac{1}{\xi})$). Then (4.12), (4.13), (4.14) are

$$F^+ + F^- = u_b^* h_R'' \ , \quad \xi \text{ in } C \ , \tag{4.17}$$

$$F^+ - F^- = \frac{-i}{2} p_c \ , \quad \xi \text{ in } C' \ , \tag{4.18}$$

where + and - refer to the interior and exterior of the unit circle respectively (Figure 4.5). We require F to satisfy (4.17) and (4.18) together with the conditions that, from (4.14),

$$F \to 0 \ , \quad \zeta \to \infty \ ,$$

$$\lim_{\substack{\xi \to \xi_a, \xi_b \\ \xi \text{ in } C'}} (F^+ - F^-) = \frac{-i}{2} p_c \ , \tag{4.19}$$

$$F^*(\zeta^*) = F(1/\zeta^*) \ .$$

§3 The Exact Solution.

Since ξ_a and ξ_b are to be determined in the solution, the problem is nonlinear: it is not clear whether a solution is unique. We shall solve (4.17)-(4.19) explicitly, and then the uniqueness of the solution becomes obvious.

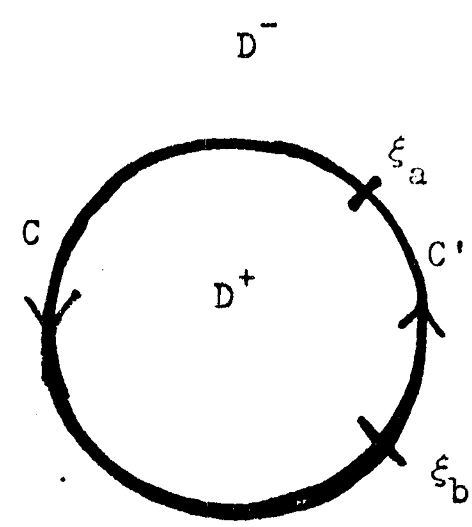


Figure 4.5: ζ-plane.

We define

$$\begin{aligned} G &= F + \frac{ip_c}{4} + \frac{\alpha}{2} \ , \quad z \text{ in } D^+ \ , \\ G &= F - \frac{ip_c}{4} + \frac{\alpha}{2} \ , \quad z \text{ in } D^- \ , \end{aligned} \tag{4.20}$$

where α is a constant to be specified later. Then (4.17), (4.18) and (4.19) are

$$G^+ + G^- = \alpha + u_b^* h_R'' \ , \ \xi \text{ in } C \ , \tag{4.21}$$

$$G^+ - G^- = 0 \ , \ \xi \text{ in } C' \ , \tag{4.22}$$

$$G \to \frac{\alpha}{2} - \frac{ip_c}{4} \ , \ \zeta \to \infty \ , \tag{4.23}$$

$$G^*(\zeta^*) = G(1/\zeta^*) + \tfrac{1}{2}(\alpha^* - \alpha) \ , \tag{4.24}$$

$$\lim_{\substack{\xi \to \xi_a, \xi_b \\ \xi \text{ in } C'}} (G^+ - G^-) = 0 \ . \tag{4.25}$$

The unique solution of (4.21) and (4.22) satisfying (4.25) is

$$G(\zeta) = \frac{\chi(\zeta)}{2\pi i}\int_{C'} \frac{[\alpha + u_b^* h_R''] \, d\tau}{\chi^+(\tau)(\tau - \zeta)} \tag{4.26}$$

where

$$\chi(\zeta) = [(\zeta - \xi_a)^{\frac{1}{2}}(\zeta - \xi_b)^{\frac{1}{2}}] \ . \tag{4.27}$$

(An additive constant in (4.26) can be absorbed into α.) In the definition of χ, we take the branch cut along C', and choose that branch that has

$$\chi \sim \zeta \ , \ \zeta \to \infty \ . \tag{4.28}$$

Satisfaction of (4.23) requires

$$-\frac{1}{2\pi i}\int_{C'} [\alpha + u_b^* h_R''(\tau)] \frac{d\tau}{\chi^+(\tau)} = \frac{\alpha}{2} - \frac{ip_c}{4} \ ; \tag{4.29}$$

we return to this later.

Let us consider the transformation

$$w = 1/\zeta^* \ . \tag{4.30}$$

We have

$$\chi^*(\zeta^*) = [\zeta^{*2} e^{-ia} e^{-ib} (\frac{1}{\zeta^*} - e^{ia})(\frac{1}{\zeta^*} - e^{ib})]^{\frac{1}{2}} \ . \tag{4.31}$$

We refer to the geometry in Figure 4.6. As $\zeta \to \infty$ along $\arg \zeta = 0$,

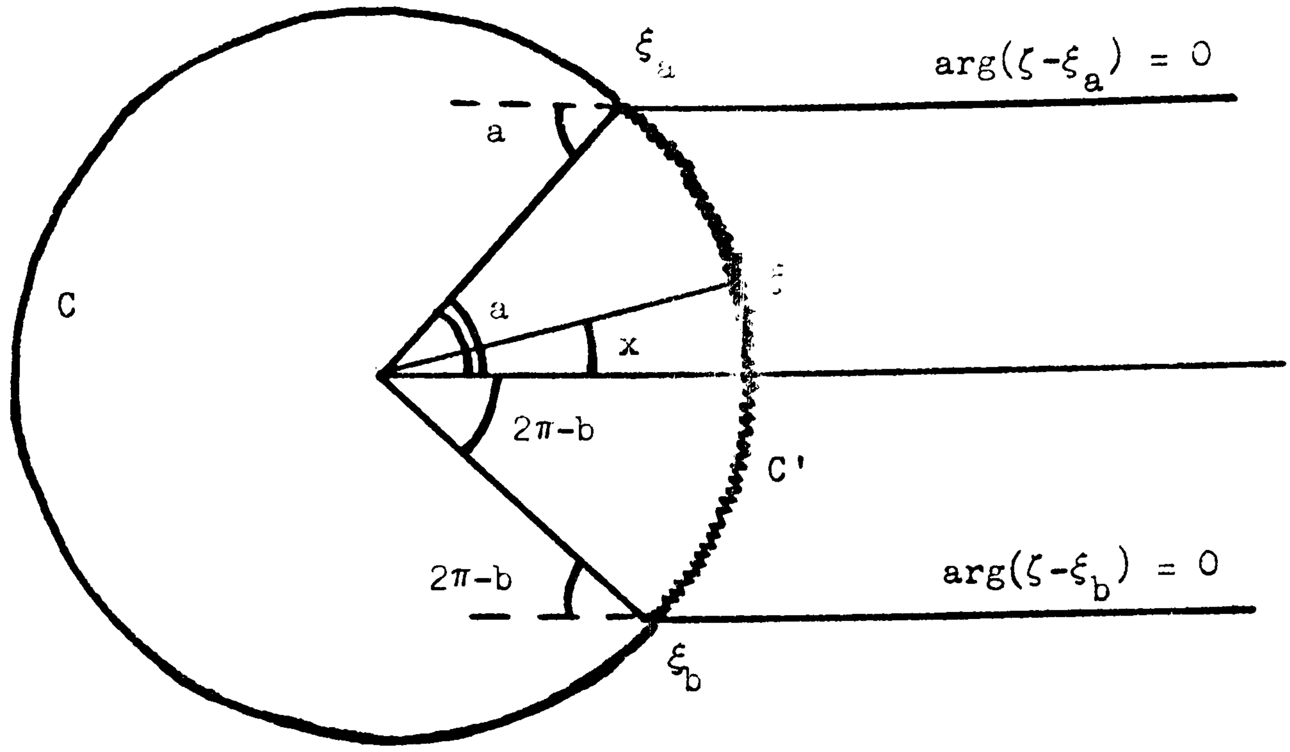


Figure 4.6.

$1/\zeta^* \to 0$ along $\arg(1/\zeta^*) = 0$. By the definition of χ, $\arg \chi \to 0$ and also

$$\arg\left[\zeta^{*2}e^{-ia}e^{-ib}\left(\frac{1}{\zeta^*} - e^{ia}\right)\left(\frac{1}{\zeta^*} - e^{ib}\right)\right]^{\frac{1}{2}}$$

$$= \tfrac{1}{2}[-a - b + (\pi+a) + (\pi-2\pi+b)] = 0 , \qquad (4.32)$$

if we define the branch in w-space so that $[(w-e^{ia})(w-e^{ib})]^{\frac{1}{2}} \sim w,\ w \to \infty$.
It follows that under the transformation (4.30),

$$\chi^*(\zeta^*) = e^{-ia/2}e^{-ib/2}\,\frac{1}{w}\,\chi(w) \equiv \xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}\,\frac{1}{w}\,\chi(w) . \qquad (4.33)$$

By similar reasoning, or directly from (4.33), we find

$$\chi^{+*}(\zeta^*) = -\zeta^* e^{-ia/2}e^{-ib/2}\chi^{+}(1/\zeta^*) . \qquad (4.34)$$

From (4.26),

$$[G(1/\zeta^*)]^* = -\frac{\chi^*(1/\zeta^*)}{2\pi i}\int_{C'^*}\frac{[\alpha^* + u_b^* h_R''^*(\tau)]}{\chi^{+*}(\tau)(\tau - \frac{1}{\zeta})}\,d\tau \qquad (4.35)$$

since τ is a dummy variable. We make the transformation $w = 1/\tau$, so that

$$[G(1/\zeta^*)]^* = -\frac{\chi^*(1/\zeta^*)}{2\pi i}\int_{C'}\frac{[\alpha^* + u_b^* h_R''^*(\frac{1}{w})]\,(-1/w^2)\,dw}{\chi^{+*}(\frac{1}{w})\,(\frac{1}{w} - \frac{1}{\zeta})} . \qquad (4.36)$$

Using (4.33) and (4.34), we have, since $\chi^{+*}(1/w) = \chi^{+}(1/w^*)^*$,

$$[G(1/\zeta^*)]^* = -\frac{\xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}\chi(\zeta)}{2\pi i\zeta}\int_{C'}\frac{[\alpha^* + u_b^* h_R''^*(\frac{1}{w})].(-\,dw)}{w^2.(-1/w)\xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}\chi^+(w).(-\frac{1}{w\zeta})(w-\zeta)}$$

$$= \frac{\chi(\zeta)}{2\pi i}\int_{C'}\frac{[\alpha^* + u_b^* h_R''(w)]\,dw}{\chi^+(w)\;(w-\zeta)}\quad, \tag{4.37}$$

since on C' $ww^* = 1$, whence $h_R''^*(1/w) = h_R''^*(w^*) = [h_R''(w)]^* = h_R''(w)$ as this last function is real. From (4.24) we require, using (4.37) and (4.26),

$$\tfrac{1}{2}(\alpha-\alpha^*) + \frac{\chi(\zeta)}{2\pi i}\int_{C'}\frac{[\alpha^* + u_b^* h_R'']\,dw}{\chi^+(w)(w-\zeta)} = \frac{\chi(\zeta)}{2\pi i}\int_{C'}\frac{[\alpha + u_b^* h_R'']\,d\tau}{\chi^+(\tau)(\tau-\zeta)}\;, \tag{4.38}$$

whence

$$(\alpha-\alpha^*)\left[\frac{1}{2} - \frac{\chi(\zeta)}{2\pi i}\int_{C'}\frac{dw}{\chi^+(w)(w-\zeta)}\right] = 0\;. \tag{4.39}$$

(4.39) is satisfied for any α since the term in square brackets is easily shown to be identically zero. Thus G defined in (4.26) provides a solution to the problem as long as α, ξ_a, ξ_b satisfy (4.29).

Given this solution, we determine p in C' and h in C by means of the formulae (from (4.15) and (4.16))

$$\tfrac{1}{2}ip = F^+ - F^- = -\tfrac{1}{2}ip_c + G^+ - G^-\;,\; \xi \text{ in } C', \tag{4.40}$$

$$u_b^* h'' = F^+ + F^- = -\,\alpha + G^+ + G^-\;, \xi \text{ in } C\;. \tag{4.41}$$

Writing

$$G(z) = \chi(z)K(z)\;, \tag{4.42}$$

we have

$$\begin{aligned} G^+ + G^- &= \chi^+K^+ + \chi^-K^- = \chi^+(K^+ - K^-)\;,\\ G^+ - G^- &= \chi^+K^+ - \chi^-K^- = \chi^+(K^+ + K^-)\;, \end{aligned} \tag{4.43}$$

and from (4.40) and (4.41), using the Plemelj formulae for K (Carrier, Krook and Pearson 1966),

$$p = -p_c - \frac{2\chi^+(\xi)}{\pi} ⨍_{C'} \frac{[\alpha + u_b^* h_R''] \, d\tau}{\chi^+(\tau)(\tau - \xi)} \quad , \; \xi \text{ in } C' \; , \tag{4.44}$$

$$u_b^* h'' = -\alpha \; , \; \xi \text{ in } C \; , \tag{4.45}$$

since K is analytic across C: ⨍ represents the principal value of the integral. Thus the cavity roof is a quadratic since α is constant in (4.45), and we require α to be real (and so (4.39) is valid irrespective of the value of the integral).

§4 Reduction for a Sinusoidal Bedrock.

To be specific, let us now consider $h_R(x) = \cos x$. Then

$$h_R'' = -\cos x \equiv -\tfrac{1}{2}(\xi + \tfrac{1}{\xi}) \; , \; \xi \text{ in } C' \quad . \tag{4.46}$$

We can simplify (4.44) and (4.29) as follows. We define

$$I_n = \int_{C'} \frac{\tau^n \, d\tau}{\chi^+(\tau)} \quad , \tag{4.47}$$

and so

$$I_n^* = \int_{C'^*} \frac{\tau^n \, d\tau}{\chi^{+*}(\tau)} \quad . \tag{4.48}$$

Putting $w = 1/\tau$, and using (4.34), we obtain

$$I_n^* = \int_{C'} \frac{1}{w^n} \cdot \frac{1}{(-1/w)\xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}\chi^+(w)} \cdot -\frac{dw}{w^2}$$

$$= \xi_a^{\frac{1}{2}}\xi_b^{\frac{1}{2}} I_{-(n+1)} \; . \tag{4.49}$$

Now consider

$$g(z) = \frac{z^n}{\chi(z)} \; , \; n \quad 0 \; . \tag{4.50}$$

As $z \to \infty$,

$$g \sim z^{n-1}[(1 - \tfrac{\xi_a}{z})(1 - \tfrac{\xi_b}{z})]^{-\frac{1}{2}} \sim P_{n-1}(z) \; , \tag{4.51}$$

say, where P_{n-1} is a polynomial of order n-1. On C',

$$g^+ = \frac{\tau^n}{\chi^+} \ , \ g^- = -\frac{\tau^n}{\chi^+}$$

$$\Rightarrow g^+ - g^- = \frac{2\tau^n}{\chi^+} \ . \qquad (4.52)$$

By the discontinuity theorem, it follows that

$$g = \frac{1}{\pi i}\int_{C'} \frac{\tau^n \, d\tau}{\chi^+(\tau)(\tau - z)} + P_{n-1}(z) \ . \qquad (4.53)$$

Putting z = 0, we have from (4.53) and (4.50)

$$I_{-1} = \int_{C'} \frac{d\tau}{\tau\chi^+(\tau)} = i\pi\xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}} \ ,$$

$$I_m = \int_{C'} \frac{\tau^m \, d\tau}{\chi^+(\tau)} = -i\pi a_m \ , \ a_m \geqslant 0 \ , \qquad (4.54)$$

where a_m is the coefficient of z^{-m} in $(1 - \xi_a/z)^{-\frac{1}{2}}(1 - \xi_b/z)^{-\frac{1}{2}}$. Alternatively, using the Plemelj formulae, we have from (4.52) and (4.53)

$$g^+ + g^- = 0 = \frac{2}{\pi i} \fint_{C'} \frac{\tau^n \, d\tau}{\chi^+(\tau)(\tau-\xi)} + 2P_{n-1}(\xi)$$

$$\Rightarrow \fint_{C'} \frac{\tau^n \, d\tau}{\chi^+(\tau)(\tau-\xi)} = -i\pi P_{n-1}(\xi) \ , \ n \geqslant 0 \ . \qquad (4.55)$$

Using these results, we obtain

$$\int_{C'} \frac{d\tau}{\chi^+(\tau)} = I_0 = -i\pi \ , \qquad (4.56)$$

$$\int_{C'} \frac{d\tau}{\tau\chi^+(\tau)} = I_{-1} = i\pi\xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}} \ , \qquad (4.57)$$

$$\int_{C'} \frac{\tau \, d\tau}{\chi^+(\tau)} = I_1 = -\tfrac{1}{2}i\pi(\xi_a + \xi_b) \ , \qquad (4.58)$$

$$\fint_{C'} \frac{d\tau}{\chi^+(\tau)(\tau - \xi)} = 0 \ , \qquad (4.59)$$

$$\int_{C'} \frac{\tau\, d\tau}{\chi^{+}(\tau)(\tau - \xi)} = -i\pi , \tag{4.60}$$

$$\int_{C'} \frac{d\tau}{\tau\chi^{+}(\tau)(\tau - \xi)} = \int_{C'} \frac{1}{\xi\chi^{+}(\tau)} \left[\frac{1}{(\tau - \xi)} - \frac{1}{\tau}\right] d\tau$$

$$= -\frac{1}{\xi}I_{-1} = -\frac{i\pi}{\xi}\xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}} . \tag{4.61}$$

It is obvious how to extend the above to integrals like $\int_{C'} \frac{d\tau}{\tau^n\chi^{+}(\tau)(\tau-\xi)}$.

For the sinusoidal bedrock (4.46), the pressure in C' is determined by (4.44) and (4.56)-(4.61) as

$$p = - p_c + \frac{1}{\pi}u_b^*\chi^{+}(\xi) \;.\; -i\pi \left[1 + \xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}/\xi\right]$$

$$= - p_c - iu_b^*\chi^{+}(\xi) \left[1 + \xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}/\xi\right] . \tag{4.62}$$

By inspection of Figure 4.6, we find that

$$\arg \chi^{+} = 3\pi/2 + x/2 + (a+b)/4 \;,\; x \text{ in } C' , \tag{4.63}$$

if $\arg \xi$ is chosen so that $b < \arg \xi < a + 2\pi$, i.e. $b < x < a + 2\pi$. Now

$$\chi^{+} = [(\xi - \xi_a)(\xi - \xi_b)]^{\frac{1}{2}}$$

$$= -\left[4e^{i(x+a)/2} \left\{\frac{e^{i(A-x)/2} - e^{-i(A-x)/2}}{2i}\right\} e^{i(x+b)/2} \times \left\{\frac{e^{i(x-b)/2} - e^{-i(x-b)/2}}{2i}\right\}\right]^{\frac{1}{2}}$$

$$= - 2ie^{i(\frac{x}{2} + \frac{a}{4} + \frac{b}{4})}\left[\sin(\tfrac{A-x}{2})\sin(\tfrac{x-b}{2})\right]^{\frac{1}{2}} \;,\; b < x < A \;, \tag{4.64}$$

using (4.63), where the positive square root is to be taken. From (4.62),

$$p = - p_c - 2u_b^* \left[\sin(\tfrac{A-x}{2})\sin(\tfrac{x-b}{2})\right]^{\frac{1}{2}} e^{i(\frac{x}{2} + \frac{a}{4} + \frac{b}{4})}\left[1 + e^{-i(x + \frac{a}{2} + \frac{b}{2})}\right]$$

$$= - p_c - 4u_b^* \left[\sin(\tfrac{A-x}{2})\sin(\tfrac{x-b}{2})\right]^{\frac{1}{2}} \cos(\tfrac{x}{2} + \tfrac{a}{4} + \tfrac{b}{4}) \;,\; b < x < A . \tag{4.65}$$

From (4.56)-(4.58), the condition (4.29) becomes

$$-\frac{1}{2\pi i}[-\alpha i\pi - \tfrac{1}{2}u_b^*\{-\tfrac{1}{2}\pi i(\xi_a+\xi_b) + i\pi\xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}\}]$$

$$= \frac{\alpha}{2} - \frac{ip_c}{4}\ , \tag{4.66}$$

which gives

$$p_c = iu_b^*[-\tfrac{1}{2}(\xi_a + \xi_b) + \xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}]$$

$$= iu_b^*[-\tfrac{1}{2}e^{i(b+a)/2}\{e^{i(b-a)/2} + e^{-i(b-a)/2}\} + e^{-i(a+b)/2}]$$

$$= iu_b^*[-\{\cos(\tfrac{a+b}{2}) + i\sin(\tfrac{a+b}{2})\}\cos(\tfrac{b-a}{2}) + \cos(\tfrac{a+b}{2}) - i\sin(\tfrac{a+b}{2})]\ . \tag{4.67}$$

Equating real and imaginary parts,

$$p_c = u_b^*\sin(\tfrac{a+b}{2})[1 + \cos(\tfrac{b-a}{2})]\ , \tag{4.68}$$

$$\cos(\tfrac{a+b}{2})[1 - \cos(\tfrac{b-a}{2})] = 0\ . \tag{4.69}$$

(4.69) has solutions

$$\begin{aligned} a + b &= \pi\ ,\ 3\pi\ ,\ 5\pi\ ,\ \ldots \\ b - a &= 0\ ,\ 4\pi\ ,\ \ldots \end{aligned} \tag{4.70}$$

Physically $p_c > 0$, also $0 \leq a + b \leq 4\pi$, and so from (4.68) we must have

$$a + b = \pi\ , \tag{4.71}$$

or $b - a = 0$, as the only realistic solutions. $b - a = 0$ is a degenerate case at inception of cavitation: in fact (4.68) and (3.126) then imply that $a + b = \pi$ in any case. Thus the cavity length is

$$b - a = \pi - 2a\ . \tag{4.72}$$

Cavitation sets in at $a = \pi/2$; the maximum cavity length is π when $p_c = 0$, although in practice this limiting case will not be reached.

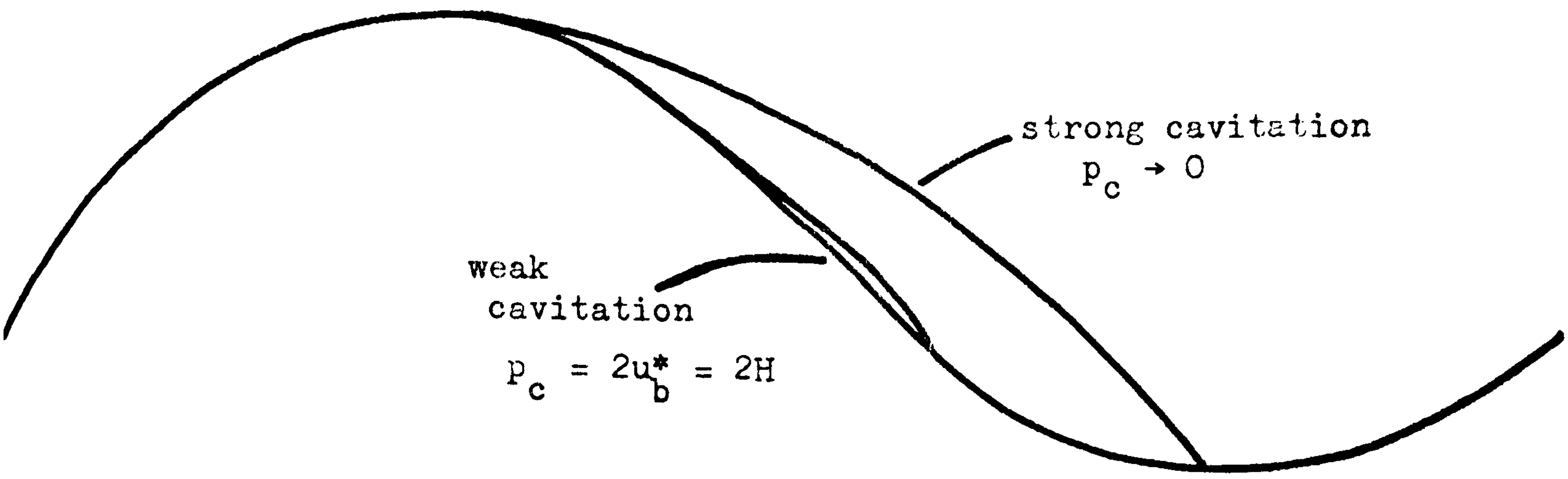


Figure 4.7: strong and weak cavitation.

These situations are depicted in Figure 4.7. We are led to the rather surprising conclusion that no more than half the bedrock can become cavitated, however fast the flow.

From (4.68) and (4.72)

$$p_c = u_b^*(1 + \sin a) , \qquad (4.73)$$

or

$$a = \sin^{-1}\left\{\frac{p_c}{u_b^*} - 1\right\} , \quad b = \pi - a . \qquad (4.74)$$

The solution is now completed by choosing the unique α (> 0) such that h' can be continuous at a, and it is clear that the solution so presented is unique. We do not need to know α to find the drag, however. From (3.127) the dimensionless drag is

$$H = \frac{1}{2\pi}\int_0^{2\pi} ph'\,dx = \frac{1}{2\pi}\int_0^{2\pi}(p + p_c)h'\,dx$$

$$= \frac{2u_b^*}{\pi}\int_b^A \sin x \cos\left\{\frac{x}{2} + \frac{\pi}{4}\right\}\left[\sin\left(\frac{A-x}{2}\right)\sin\left(\frac{x-b}{2}\right)\right]^{\frac{1}{2}} dx , \qquad (4.75)$$

using (4.65), where $A = a + 2\pi$, and a, b are defined by (4.74). We can evaluate (4.75) by using (4.62) and that $h' = -\frac{1}{2i}(\xi - \frac{1}{\xi})$, ξ in C',

$\xi = e^{ix}$, $d\xi = i\xi\, dx$;thus

$$H = \frac{1}{2\pi}\int_{C'} - iu_b^* \chi^+(\tau)[1 + \xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}/\tau]. - \frac{1}{2i}(\tau - \frac{1}{\tau})\frac{d\tau}{i\tau}$$

$$= \frac{u_b^*}{2\pi i}\int_{C'} \tfrac{1}{2}\chi^+(\tau)[1 + \xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}/\tau - 1/\tau^2 - \xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}/\tau^3]\, d\tau \ . \quad (4.76)$$

We can integrate (4.76) using the same methods as before. Let

$$J_m = \int_{C'} \tau^m \chi^+(\tau)\, d\tau, \quad (4.77)$$

then

$$J_m^* = \int_{C'^*} \tau^m \chi^{+*}(\tau)\, d\tau \ ; \quad (4.78)$$

under the transformation $w = 1/\tau$, (4.78) becomes,using (4.34),

$$J_m^* = \int_{C'} \xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}} w^{-(m+3)}\chi^+(w)\, dw$$

$$= \xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}} J_{-(3+m)} \ . \quad (4.79)$$

To evaluate J_m,consider

$$g(z) = \tau^{m+1}\chi(\tau) \ , \ m \geqslant -1 \ . \quad (4.80)$$

As before,on C'

$$g^+ - g^- = 2\tau^{m+1}\chi^+(\tau) \ , \quad (4.81)$$

whence the discontinuity theorem implies

$$g(z) = \frac{1}{\pi i}\int_{C'} \frac{\tau^{m+1}\chi^+(\tau)}{\tau - z}\, d\tau + P_{m+2}(z) \ , \quad (4.82)$$

where P_{m+2} is the (m+2)-order polynomial such that

$$z^{m+1}\chi^+(z) \sim P_{m+2}(z) \text{ as } z \to \infty \ . \quad (4.83)$$

Thus (4.80) and (4.82) prove

$$\frac{1}{\pi i}\int_{C'} \tau^m \chi^+(\tau)\, d\tau + P_{m+2}(0) = g(0) = \begin{cases} 0 , & m \geq 0 \\ \xi_a^{\frac{1}{2}}\xi_b^{\frac{1}{2}} , & m = -1, \end{cases} \tag{4.84}$$

and so

$$\begin{aligned} J_{-1} &= i\pi[\xi_a^{\frac{1}{2}}\xi_b^{\frac{1}{2}} - b_1] , \\ J_m &= - i\pi b_{m+2} , \quad m \geq 0 , \end{aligned} \tag{4.85}$$

where b_r is the coefficient of z^{-r} in the expansion of $(1 - \xi_a/z)^{\frac{1}{2}}(1 - \xi_b/z)^{\frac{1}{2}}$. Using (4.76), (4.79) and (4.85), we have

$$H = \frac{u_b^*}{4\pi i}[\xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}J_{-1} - \xi_a^{\frac{1}{2}}\xi_b^{\frac{1}{2}}J_{-1}^*] . \tag{4.86}$$

From (4.85),

$$\begin{aligned} H &= \frac{u_b^*}{4\pi i}[i\pi(1 - b_1\xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}}) + i\pi(1 - b_1^*\xi_a^{\frac{1}{2}}\xi_b^{\frac{1}{2}})] \\ &= \tfrac{1}{2}u_b^*[1 - \mathrm{Re}(b_1\xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}})] . \end{aligned} \tag{4.87}$$

Now

$$b_1 = - \tfrac{1}{2}(\xi_a + \xi_b) , \tag{4.88}$$

so

$$\begin{aligned} - b_1\xi_a^{-\frac{1}{2}}\xi_b^{-\frac{1}{2}} &= \tfrac{1}{2}[\xi_a^{\frac{1}{2}}/\xi_b^{\frac{1}{2}} + \xi_b^{\frac{1}{2}}/\xi_a^{\frac{1}{2}}] \\ &= \mathrm{Re}[(\xi_a/\xi_b)^{\frac{1}{2}}] = \cos(\tfrac{b-a}{2}) . \end{aligned} \tag{4.89}$$

Thus the drag law becomes the simple expression

$$H = \tfrac{1}{2}u_b^*[1 + \cos(\tfrac{b-a}{2})] , \tag{4.90}$$

or using (4.72),

$$H = \tfrac{1}{2}u_b^*(1 + \sin a) , \tag{4.91}$$

with a given by (4.74) and p_c by (4.5).

From (4.73), we obtain

$$H = \tfrac{1}{2}p_c \ . \tag{4.92}$$

Writing (4.5) in the form

$$p_c = \frac{\nu}{\varepsilon}\left[H + \frac{p_A - p_T}{\rho g' d}\right] , \tag{4.93}$$

(using the definition of $[\tau]_0$), we obtain the complete sliding law

$$\begin{aligned} u_b^* &= H \ , \quad u_b^* \leq \gamma(H+\beta) \ , \\ H &= \frac{\beta\gamma}{1-\gamma} \ , \quad u_b^* \geq \gamma(H+\beta) \ , \end{aligned} \tag{4.94}$$

where the cavitation parameters β and γ are defined by

$$\gamma = \frac{\nu}{2\varepsilon} \quad , \ \beta = \frac{p_A - p_T}{\rho g' d} \approx \frac{p_A}{\rho g' d} \ . \tag{4.95}$$

For cavitation to occur at all, we require $\gamma \sim 1$ (more specifically $0 < \frac{\beta\gamma}{1-\gamma} \sim 1$, from (4.94)). From (4.95), $\beta \sim 10/d$, where d is measured in metres. Thus typically $\beta \sim 0.1$. The form of this sliding law is shown in Figure 4.8.

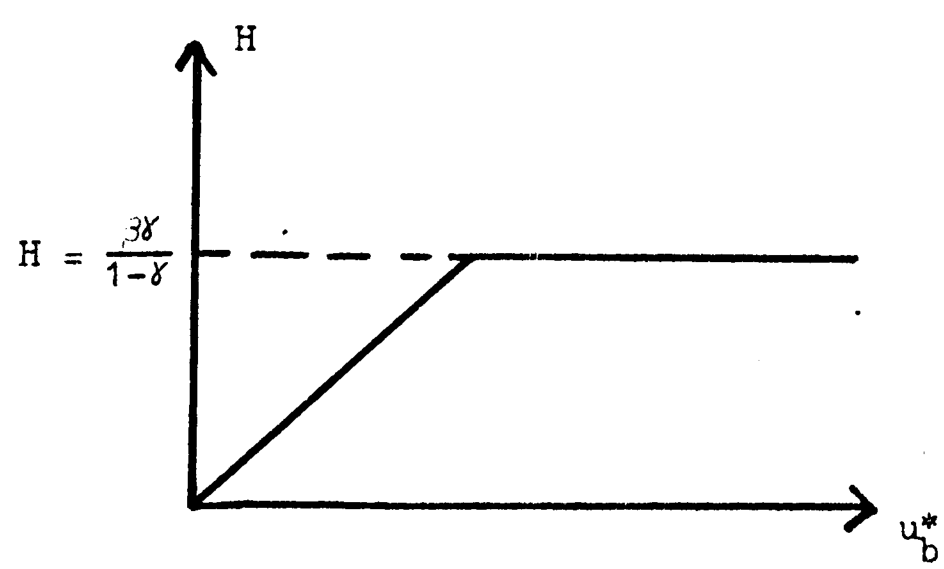


Figure 4.8: sliding law.

§5 Conclusions.

The form of Figure 4.8 is not completely satisfactory: there are two reasons for this. One is that u_b^* cancels out of the equation (4.92) fortuitously due to its linearity: thus a non-Newtonian fluid would presumably give a more sensible law. The other reason is that this is only a leading order approximation in ν. Successive approximations in the asymptotic expansion for ψ would hopefully lead to a correction term for the drag that would indicate which of the graphs (a),(b) or (c) of Figure 4.2 is the appropriate law. It is at least clear from Figure 4.8 that one of them must be, and that cavitation plays a crucial part in deciding which it is. One should without difficulty be able to extend the method of solution presented in this chapter to the consideration of higher order terms in the asymptotic expansion. This is not done in the present work.

CHAPTER V

Kinematic Surface Waves

The aim of the present chapter is to consider the phenomenon of surface waves, which are observed to travel down glaciers at a speed approximately four times the surface speed of the ice.

A theory of such waves based on the kinematic wave equation

$$\frac{\partial Q}{\partial x} + \frac{\partial H}{\partial t} = a \qquad (5.1)$$

where Q, H, a are the flux, depth and accumulation rate respectively, was developed by Nye (1960, 1963, 1963a). His procedure, following Lighthill and Whitham (1955), is to assume Q is a function of H, x and α, where α is the surface slope, and then to linearise the equation about its steady state solution. The resultant equation has wave-like solutions with a wave speed of about four times the surface speed; however these solutions are not uniformly small near the snout, as may be seen from Figure 5.1 (after Nye 1960), which shows the evolution in time of a small, initially

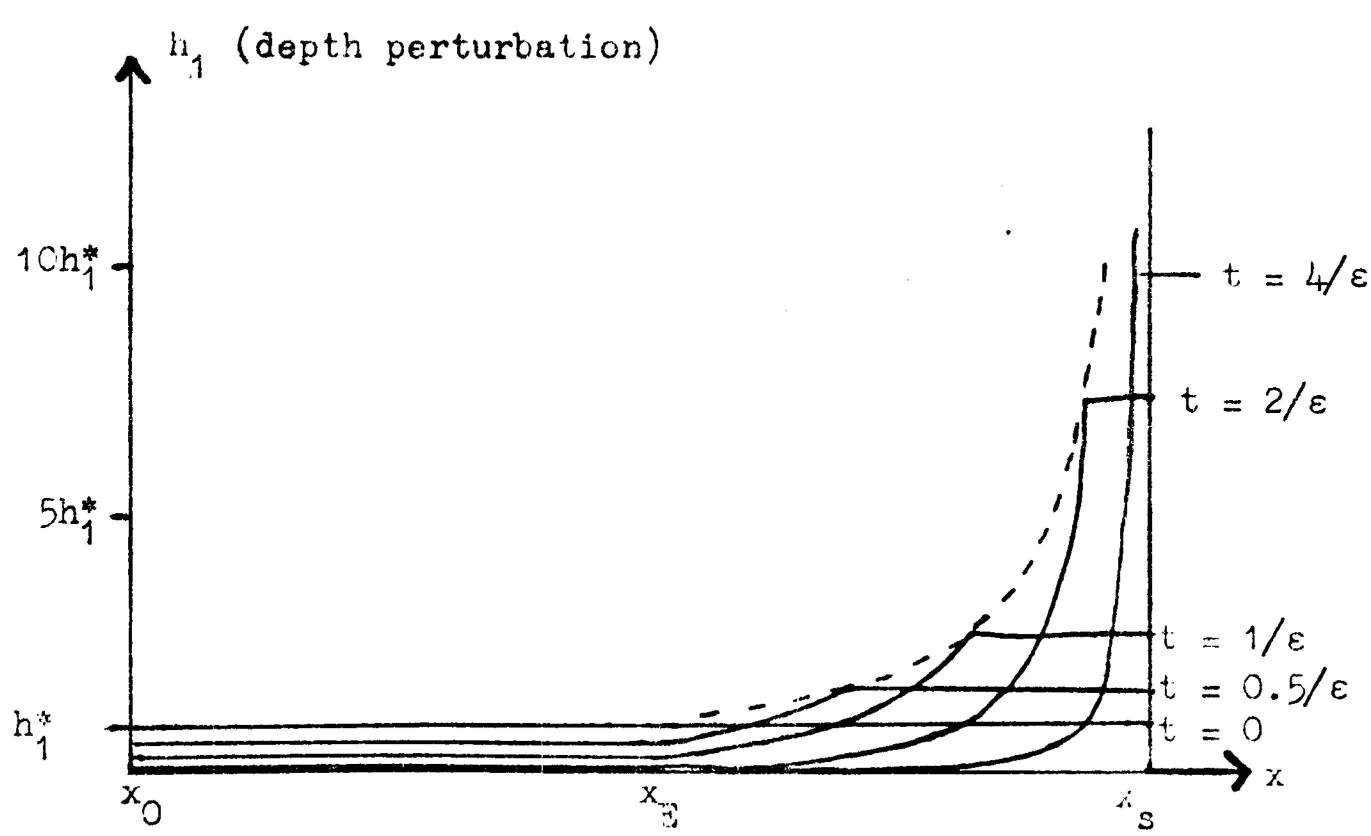


Figure 5.1. (Nye's notation).

uniform disturbance from the steady state depth profile. He observes that "the lower parts thicken unstably until a kinematic wave arrives to restore stability." The evident nonuniformity results from the linearisation of an equation which is essentially nonlinear. Nye accounts for this difficulty by observing that the ice velocity (and hence the wave speed) at the snout of a temperate glacier is not zero (see Chapter VI); nevertheless it is still small, and the explanation is an unsatisfactory one. For cold glaciers, which are frozen to the bedrock, the snout velocity must be considered to be zero and this explanation is then inadequate.

In the following sections, we derive from the reduced model of Chapter II a nonlinear equation for the glacier depth which is valid for a cold glacier in the limit $\kappa \to 0$, and is expected to have qualitative validity when $\kappa \neq 0$. This equation is analogous to that considered by Nye. The solution can be written in terms of a characteristic parameter σ measuring distance on $t = 0$, and it is then clear that no such growth near the snout as that predicted by Nye will in fact occur.

The possibility of the formation and subsequent evolution of shocks (i.e. discontinuities in the surface profile) is considered, and it is shown that shocks will form from an initial perturbation of the depth if and only if this perturbation has negative gradient at some point. Two particular cases are then discussed: firstly a shock in the initial data, such as might model a sudden influx of ice on to the top surface due to a tributary glacier surge, and secondly an initial small perturbation to the surface. In the latter case a shock forms at the snout, which then advances forward before retreating once more to its steady state position.

The effect of diffusion is briefly discussed, and then a uniformly valid approximate result for small perturbations obtained by linearising the characteristics. Finally we show that seasonal changes in accumulation and ablation have a negligible effect on the behaviour of the glacier depth over the natural glacial time scale.

§1 Derivation of the Kinematic Wave Equation.

For the purposes of the present chapter, we assume that the glacier is completely cold. In fact, we shall assume that

$$T < T_Q \ , \tag{5.2}$$

where T_Q is the temperature at which the ice starts to slide. In this case the no-slip condition is valid on the whole bedrock. We further assume

$$\kappa = 0 \tag{5.3}$$

in (2.106). Then the equation and boundary conditions for the stream function uncouple from those for the temperature, and Ψ is to be found as the solution of

$$\begin{aligned} &\Psi_{\xi\xi} = \xi^n \, [1 - \mu(H_x + h_x)]^n \ , \\ &\Psi = s(x,t) \text{ on } \xi = 0 \ , \\ &\Psi_\xi = 0 \ , \ \Psi = \frac{\partial}{\partial t}\int_{x_0(t)}^{x} H(\sigma,t)\, d\sigma \text{ on } \xi = H(x,t) \ . \end{aligned} \tag{5.4}$$

Two immediate integrations of (5.4) yield

$$\Psi = s(x,t) - [1 - \mu(H_x + h_x)]^n \left[\frac{H^{n+1}\xi}{n+1} - \frac{\xi^{n+2}}{(n+1)(n+2)}\right] . \tag{5.5}$$

Application of the remaining boundary condition for Ψ on $\xi = H$ gives, from (5.4),

$$\frac{\partial}{\partial t}\int_{x_0(t)}^{x} H(\sigma,t)\, d\sigma = s(x,t) - [1 - \mu(H_x + h_x)]^n \, \frac{H^{n+2}}{n+2} \ . \tag{5.6}$$

Differentiating with respect to x, we obtain

$$H_t + \left[\theta\{1 - \mu(H_x + h_x)\}^n \frac{H^{n+2}}{n+2}\right]_x = s_x(x,t) \ , \tag{5.7}$$

where $\theta \equiv 1$.

Now the assumption that $\kappa = 0$ is not a realistic one. However, we expect (5.7) to be qualitatively valid even in the case $\kappa = O(1)$ for the following heuristic reason. Let us assume that a steady state solution to the reduced problem exists and is stable to small disturbances. Then for such small disturbances to the steady state temperature and flow fields, we may write

$$\theta_0 < \exp(\kappa T) < 1 \tag{5.8}$$

everywhere; here $\theta_0 \approx \exp(-\kappa)$ since min $T = -1$ (at $x = x_0$) in the steady state. Assuming T to be approximately given by its steady state solution, we may integrate (5.4) twice as before to obtain (5.7), where $\theta[x,H(x,t)]$ is an appropriately defined average of T, and

$$\theta_0 < \theta[x,H(x,t)] < 1 \ . \tag{5.9}$$

The general procedure of the kinematic wave theory is still valid in this case, since Nye's assumption that the flux term Q given by the term in square brackets in (5.7) is a function of H, x and H_x (our notation) still holds. We therefore expect qualitative conclusions of the ensuing analysis to hold for $\kappa \neq 0$, with the proviso that the steady state temperature and flow fields are stable. We henceforth consider (5.7) with $\theta = 1$.

Motivated by (2.76), we further assume that

$$\mu \ll 1 \ , \tag{5.10}$$

and so on neglecting μ in (5.7), we have to solve the much simpler equation

$$H_t + H^{n+1} H_x = s_x(x,t) \ . \tag{5.11}$$

The effect of the approximation made in (5.10) will be discussed briefly in §8. Putting $x = x_0(t)$ in (5.6), we find the boundary condition for (5.11) is

$$H^{n+1}(x_0(t),t) = (n+2)x_0'(t) \text{ for } x_0' > 0 \\ = 0 \text{ for } x_0' \leq 0 \Bigg\} . \tag{5.12}$$

We also prescribe the initial condition

$$H(x,0) = A(x) . \tag{5.13}$$

§2 Steady State Solution.

First we describe the steady solution of (5.11), since we shall be concerned with perturbations from this 'datum' state. In this case $s(x,t) = s(x), x_0(t) = x_0$, and (5.12) is

$$H(x_0,t) = 0 . \tag{5.14}$$

The initial condition (5.13) is irrelevant. The solution of (5.11) and (5.14) is

$$\frac{H_0^{\,n+2}}{n+2} = s(x) , \tag{5.15}$$

where H_0 is the datum profile: this incidentally confirms that our choice of ℓ as a length scale for the problem is an appropriate one. Such a simple profile as (5.15) appears not to have been given previously. We observe that if (as is physically reasonable) s' is finite at x_0 and x_s, then the depth H has infinite slope there. Although (5.15) satisfies all the boundary conditions of the reduced model, consideration of (2.66) shows that neglect of terms in δ^2 is not uniformly valid at the snout. In fact, this is a second-order effect: inclusion of these second order terms 'corrects' the solution and provides a finite slope at x_0 and x_s: the details of the procedure are given for the case of a temperate glacier in Chapter VI.

§3 Characteristic Solution.

In §3-§9, we shall consider the case of a constant flux function $s(x,t) \equiv s(x)$: that is, we consider the accumulation rate and x_0 to be independent of time. There are then two interpretations of (5.13). We may either consider a datum glacier suddenly subject to a perturbation in its thickness, so that the new profile is $A(x)$: or we may consider a datum glacier with a flux function $s(x) - s_1(x)$ which is suddenly subject to a climatic change so that its new flux is $s(x)$. These two interpretations may be related by using (5.15), whence we find

$$\frac{A^{n+2}(\sigma)}{n+2} = s(\sigma) - s_1(\sigma) \ , \ x_0 < \sigma < x_s \ , \tag{5.16}$$

where we require $s_1(x_0) = 0$ by virtue of the assumption that x_0 is constant.

The equation (5.11), with boundary and initial conditions (5.13) and (5.14), may be written in characteristic form as

$$\frac{dx}{dt} = H^{n+1} \ , \tag{5.17}$$

$$\frac{dH}{dt} = s'(x) \ , \tag{5.18}$$

$$H(x_0,t) = 0 \ , \ t \geqslant 0 \ , \tag{5.19}$$

$$H = A(\sigma) \ , \ x = \sigma \text{ on } t = 0 \ , \tag{5.20}$$

where $\sigma \in [x_0, x_s(0)]$ is a characteristic parameter.

Differentiating (5.5) with respect to ξ, neglecting μ and evaluating on $\xi = 0$, we find that the surface velocity u_s is given by

$$u_s = -\Psi_\xi\big|_{\xi=0} = \frac{H^{n+1}}{n+1} \ . \tag{5.21}$$

Comparing (5.21) with (5.17), we immediately see that the wave speed of solutions of (5.17)-(5.20) is equal to (n+1) times the surface speed, i.e. about $4u_s$.

The first integral of (5.17) and (5.18) satisfying (5.19) and (5.20) is, using (5.16),

$$\frac{H^{n+2}}{n+2} = s(x) - s_1(\sigma) \ . \tag{5.22}$$

Substituting (5.22) into (5.17) and integrating using (5.20), we obtain the equation of the characteristics as

$$t = \int_{\sigma}^{x} \frac{dx}{[(n+2)\{s(x)-s_1(\sigma)\}]^{(n+1)/(n+2)}} \quad . \qquad (5.23)$$

(5.22) and (5.23) define the solution in terms of the characteristic parameter σ.

To give some idea of the shape of the characteristics, a typical characteristic diagram in the steady state is shown in Figure 5.2. Here $s_1(\sigma) = 0$, and the curves are given by

$$t = \int_{\sigma}^{x} \frac{dx}{[(n+2)s(x)]^{(n+1)/(n+2)}} \quad , \; x_0 < \sigma < x_s \; . \qquad (5.24)$$

Figure 5.2 is similar to a diagram given in Nye (1960).

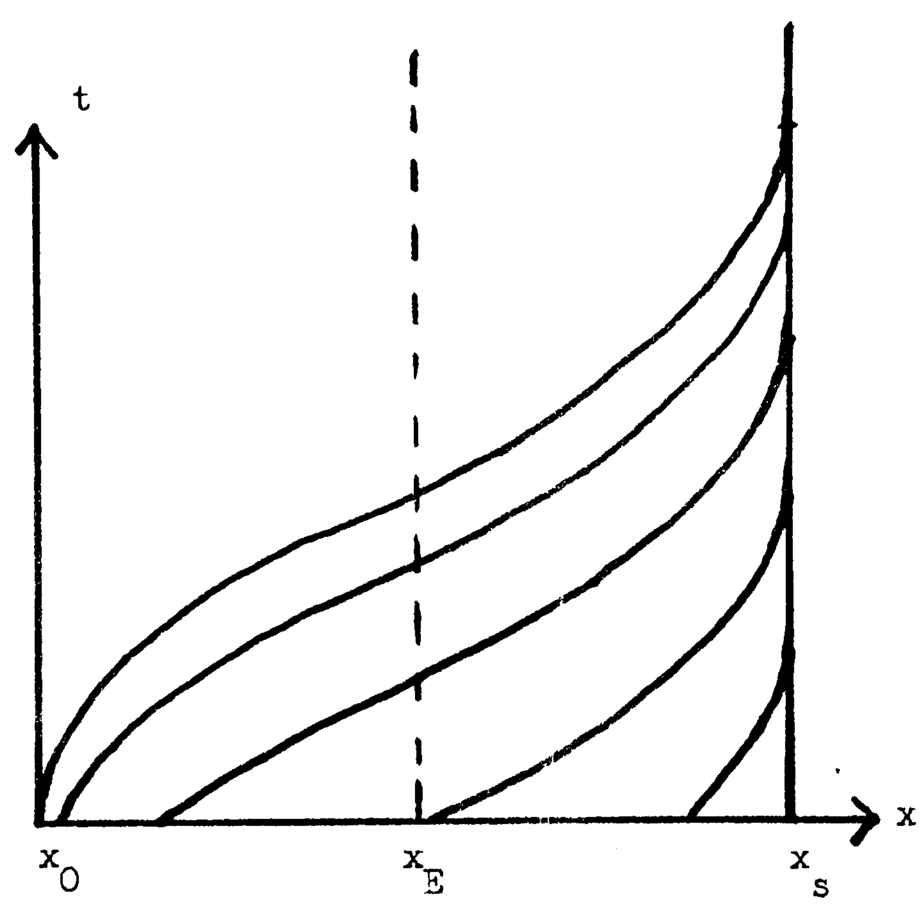


Figure 5.2

There are two main questions concerning the solution (5.22) and (5.23) that we wish to answer: firstly, under what conditions will shocks form and develop, and secondly, can we write a uniformly valid approximate solution for the case when the initial perturbation to the datum state is small? The answer to the second question is given in §9, but before examining this, we must consider the essentially nonlinear phenomenon of shock formation. As a preliminary, we consider some basic properties of the characteristic diagram given by (5.23).

Along any characteristic (σ constant), (5.23) implies that as we increase t, x increases until $s(x) = s_1(\sigma)$, at which point $H = 0$ and the characteristic slope $\frac{dt}{dx}$ is infinite. The curve $s(x) = s_1(\sigma)$, with σ given as $\sigma(x,t)$ from (5.23), thus defines the snout of the glacier: the characteristics may be considered to terminate there. The parametric equation for the snout position $x_s(t)$ is thus

$$\left\{\begin{aligned} &s[x_s(t)] = s_1(\sigma) \\ &t = \int_\sigma^{x_s(t)} \frac{dx}{[(n+2)\{s(x)-s_1(\sigma)\}]^{(n+1)/(n+2)}} \,. \end{aligned}\right. \qquad (5.25)$$

The initial disturbance at $x = \sigma$ propagates along the characteristic through σ until it reaches $x = x_s$. The time taken for the disturbance to reach the snout is then given implicitly by (5.25). This time is finite or infinite according to whether $[s(x)-s_1(\sigma)]^{-(n+1)/(n+2)}$ is integrable or not at $s(x) = s_1(\sigma)$. The former case is generally true since the ablation rate at the snout is finite, and so an initial disturbance to the datum state decays in a finite time. We shall assume this to be the case. This time may be given explicitly, since the bounding characteristic which reaches x_s 'last' is that through x_0. Hence the time for final decay of an initial perturbation is given by

$$t = \int_{x_0}^{x_s} \frac{dx}{[(n+2)s(x)]^{(n+1)/(n+2)}} \,, \qquad (5.26)$$

where x_s has its steady state value given by $s(x_s) = 0$. A typical characteristic diagram illustrating the above remarks is shown in Figure 5.3.

We have assumed so far that (5.22) and (5.23) define a single-valued function $H(x,t)$, that is, none of the characteristics intersect. With a

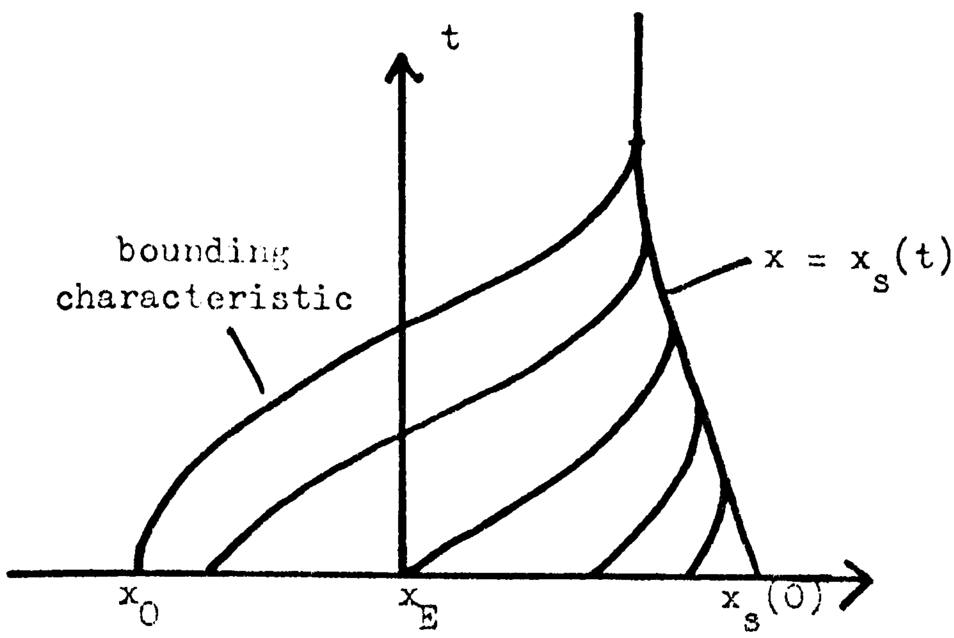


Figure 5.3.

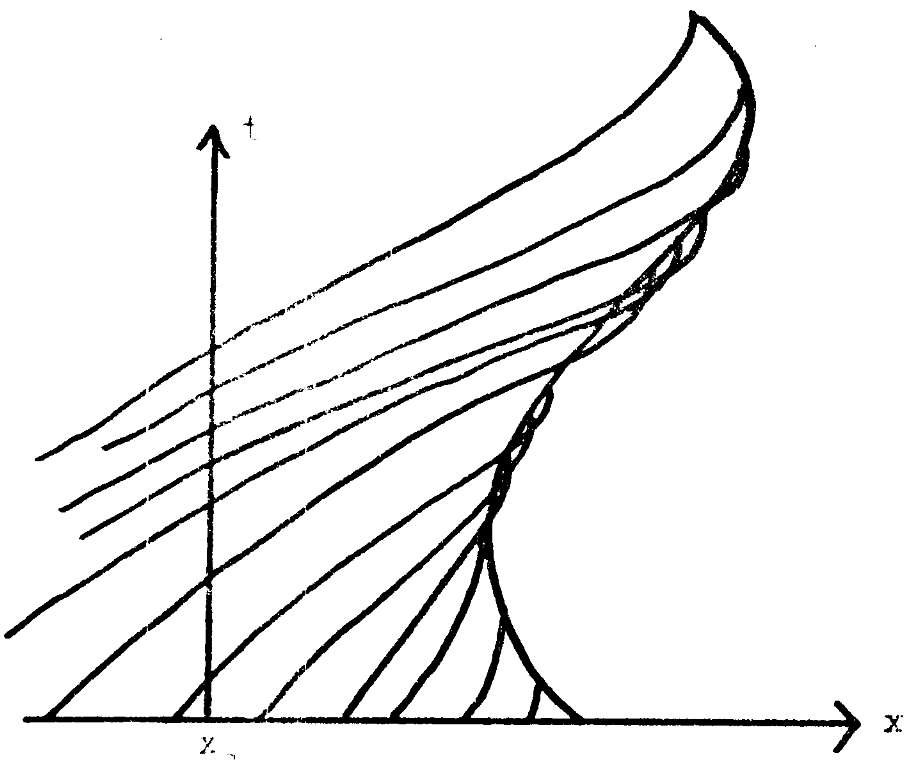


Figure 5.4.

geometry as shown in Figure 5.3 where $x_s'(t) < 0$, this is obviously possible: however, if we try to construct a similar diagram in which $x_s'(t) > 0$, we cannot do so without having characteristics intersect (Figure 5.4), essentially because characteristics which terminate on x_s must have infinite slope there. We observe from Figure 5.3 that on $x = x_s$, $\frac{d\sigma}{dt} < 0$, and since $s'(x) < 0$ in $x > x_E$, we have from (5.25)

$$s'(x_s)x_s'(t) - s_1'(\sigma)\frac{d\sigma}{dt} = 0,$$

whence $s_1'(\sigma) < 0$ in Figure 5.3. Conversely, the shock formation in Figure 5.4 is associated with characteristics on which $s_1'(\sigma) > 0$. This simple heuristic association of shocks with regions where $s_1'(\sigma) > 0$ is made rigorous in the following section.

§4 Shock Formation.

A shock forms when the characteristic solution (5.22) and (5.23) becomes multivalued for $H(x,t)$. This occurs when two neighbouring characteristics intersect; in this case the usual condition is satisfied that the characteristic should satisfy its envelope equation, that is (5.23) differentiated partially with respect to σ:

$$\frac{\partial}{\partial\sigma}\left[\int_\sigma^x \frac{dx}{[(n+2)\{s(x)-s_1(\sigma)\}]^{(n+1)/(n+2)}}\right] = 0. \quad (5.27)$$

Performing the differentiation, and using (5.16), (5.27) may be written

$$\frac{1}{A^{n+1}(\sigma)} = (n+1)s_1'(\sigma)\int_\sigma^x \frac{dx}{[(n+2)\{s(x)-s_1(\sigma)\}]^{(1+\frac{n+1}{n+2})}} \quad . \tag{5.28}$$

Thus the characteristic through σ intersects its neighbour if and only if there is a solution of (5.28) for $x < x_s$, x_s being given by the first of (5.25).

We see that the remarks of §3 are immediately justified. The integral in (5.28) is positive and A is positive, hence no solution exists if $s_1' < 0$: in this case no shocks form. Conversely if $s_1' > 0$, then $s(x)-s_1(\sigma) = s(x)-s(x_s) \sim x_s - x$ as $x \to x_s$, and so the integral in (5.28) $\sim (x_s - x)^{-(\frac{n+1}{n+2})} \to \infty$ as $x \to x_s$. Thus the right hand side increases continuously from 0 (at σ) to ∞ (at x_s) and so there is a solution for x of (5.28).

This shows that if $s_1'(\sigma) > 0$ for a range of σ in $(x_0, x_s(0))$, then at least one shock must form: it forms at time t_c, where t_c is the minimum value of t given by (5.23) for $\sigma \in (x_0, x_s(0))$, and $x(\sigma)$ is defined as the solution (when it exists) of (5.28). Subsequent shocks form at the earliest time when two neighbouring characteristics intersect which have not previously intersected a shock.

§5 Shock Evolution.

We now describe the equations for the velocity of, and the jump discontinuity in H across such a shock. This has been done for a similar class of equations by Murray (1970). We follow his notation and denote the position of a shock at time t as

$$x = x_d(t) \quad . \tag{5.29}$$

To find x_d, we specify that mass is conserved across the shock. In the usual way, the condition for this to be true is found from an integral formulation of the equation (5.11). We obtain

$$\frac{dx_d}{dt} = \frac{H_1^{n+2} - H_2^{n+2}}{(n+2)(H_1 - H_2)} \quad , \tag{5.30}$$

where

$$H_1(x_d,t) = H(x_d-,t) \ , \ H_2(x_d,t) = H(x_d+,t) \ . \tag{5.31}$$

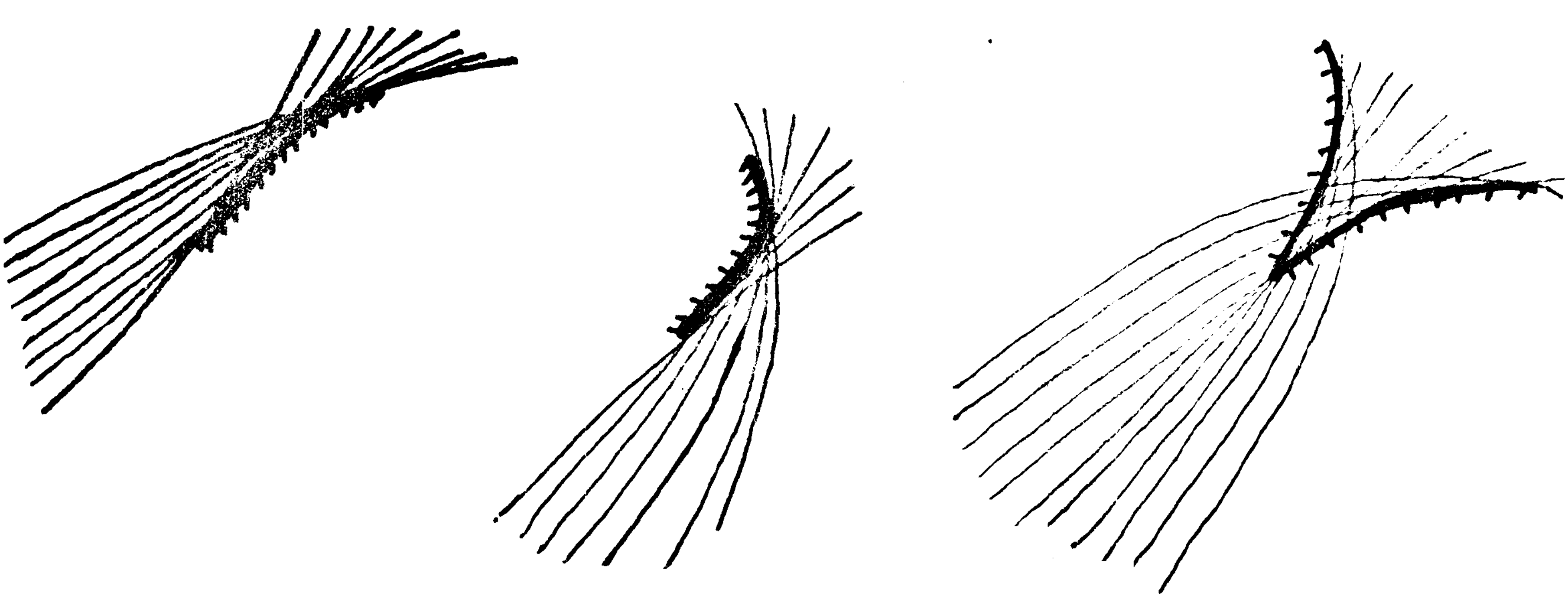
We further specify that characteristics terminate on the shock, in other words that H_1 and H_2 are themselves characteristic solutions of the form (5.22) and (5.23). In this case (5.30) is a first order differential equation for $x_d(t)$. The initial condition to be satisfied is that at time $t = t_c$, defined at the end of §4, $x_d(t_c)$ should be the solution of (5.28) with σ being determined by the condition that t_c be the minimum time at which intersection of characteristics occurs.

We now wish to show that $x_d(t)$, as defined by (5.30) and the above condition, is a valid expression for the shock formed at $t = t_c$; to do this we must prove that characteristics which terminate on x_d have not already intersected: or, equivalently, that the envelope of characteristics terminating on $x_d \pm$ lies in $x \lessgtr x_d$.

Let us denote the envelope of the characteristics defined by (5.23) and (5.28) by

$$x = x_e(t) \ ; \tag{5.32}$$

x_e may have one or two branches as shown in Figure 5.5. We wish to show



(a) (b) (c)

Figure 5.5.

that the shock in these cases lies respectively above, below and between the envelopes in (a), (b) and (c) of Figure 5.5.

We can evaluate $\frac{dx_e}{dt}$ by differentiating (5.23) with respect to t. Using the envelope condition (5.27), we find

$$1 = \left[\frac{dx_e}{dt}\frac{\partial}{\partial x_e} + \frac{d\sigma}{dt}\frac{\partial}{\partial \sigma}\right]\int_{\sigma}^{x_e} \frac{dx}{[(n+2)\{s(x)-s_1(\sigma)\}]^{\left(\frac{n+1}{n+2}\right)}}$$

whence

$$\frac{dx_e}{dt} = [(n+2)\{s(x_e)-s_1(\sigma)\}]^{\left(\frac{n+1}{n+2}\right)} . \tag{5.33}$$

Let us consider the characteristics which terminate on x_d+: assume these form an envelope x_e (if not there is nothing to prove). We have to show $x_e \leq x_d$ in $t \geq t_c$. Certainly $x_e = x_d$ at $t = t_c$. If $x_e = x_d$ for any $t > t_c$, then from (5.33), (5.22) and (5.31)

$$\frac{dx_e}{dt} = [(n+2)\{s(x_d)-s_1(\sigma)\}]^{(n+1)/(n+2)} = H_2^{n+2} \tag{5.34}$$

since x_e is the envelope of characteristics from $x > x_d$. H_2^{n+1} is also the slope of the characteristics on x_d+, and since these must terminate on x_d, we require that this slope be <u>less</u> than $\frac{dx_d}{dt}$: hence

$$\frac{dx_d}{dt} > \frac{dx_e}{dt}, \quad t > t_c , \tag{5.35}$$

and so $x_d \geq x_e$ in $t \geq t_c$, as required. The geometry is shown in Figure 5.6. A similar argument can be applied to characteristics terminating on x_d-; thus we have shown that $x_d(t)$ is a valid description of the shock formed at $t = t_c$.

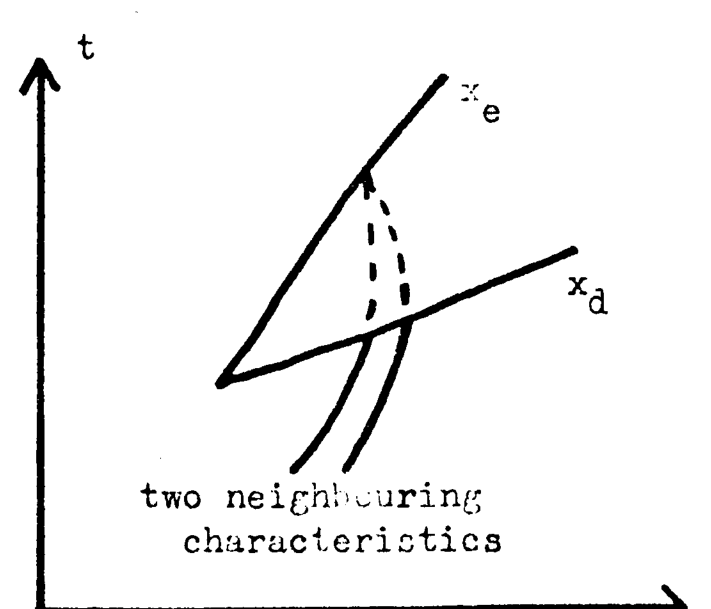


Figure 5.6.

§6 Initial Shock.

In order to gain some understanding of the characteristic diagram in the presence of shocks, we consider in this section and the next two particular forms of initial profile.

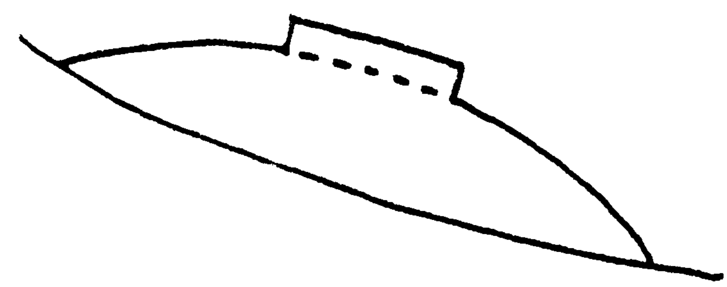


Figure 5.7: Initial Shock.

Let us consider an initial shock as shown in Figure 5.7, defined by

$$s_1(\sigma) = -c \ , \ \sigma_1 < \sigma < \sigma_2 \ , \ c > 0 \ , \qquad (5.36)$$
$$= 0 \text{ otherwise} \ .$$

This initial condition models a physical situation such as a surge of a tributary glacier which deposits a large quantity of material on the

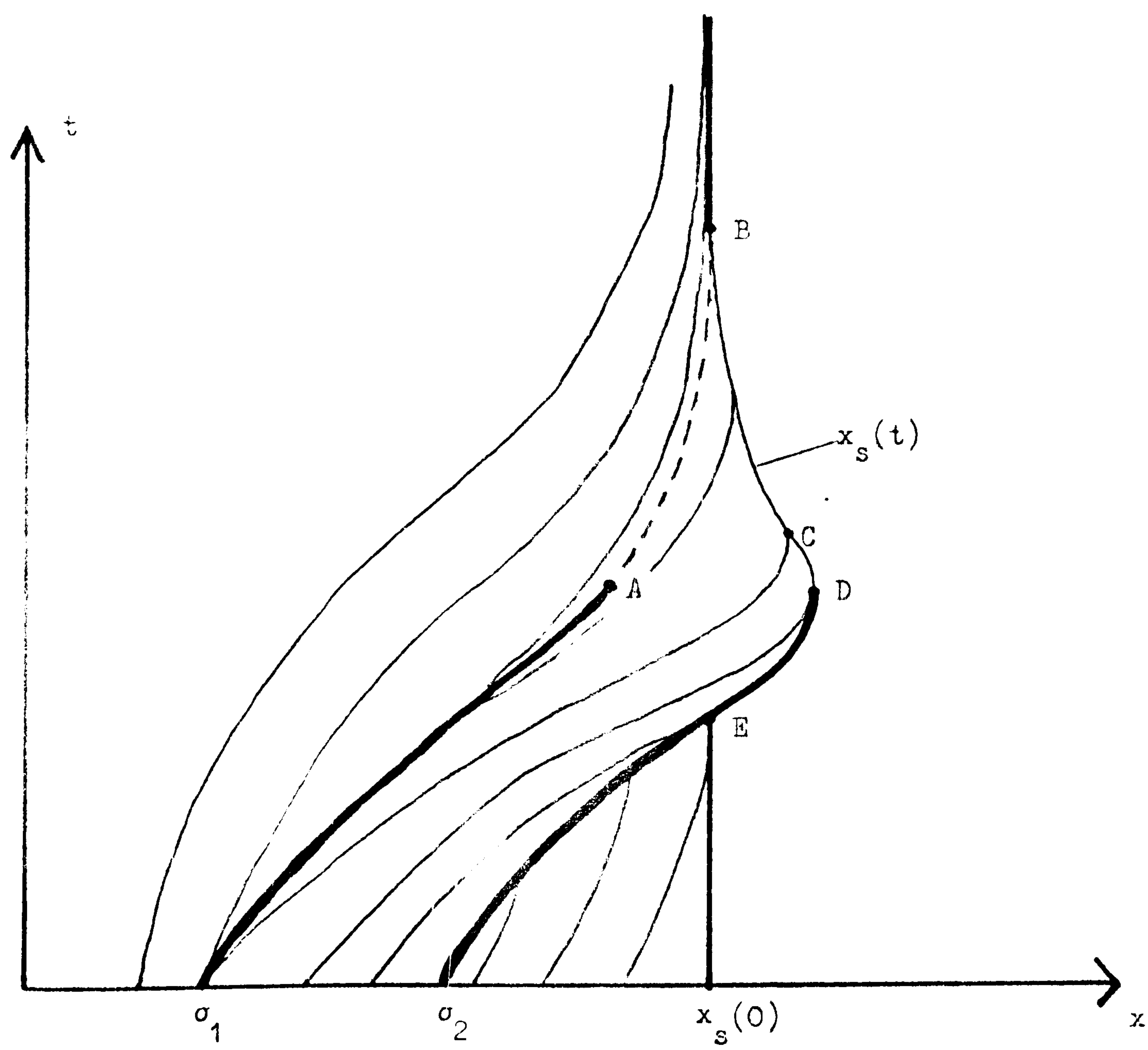


Figure 5.8.

glacier surface.

A typical characteristic diagram is displayed in Figure 5.8, where we have argued as follows. There are two shocks which emanate from σ_1 and σ_2. We denote these by $\sigma_1 A$ and $\sigma_2 D$, A and D being the points where the shocks decay. Let us assume for the moment that $\sigma_1 A$ never intersects $\sigma_2 D$. 'Steady state' characteristics of the form of (5.24) originate in $\sigma_1 A-$ and terminate on $\sigma_2 E+$ (with an obvious notation). At A, $H^{n+2}/(n+2) = s(x)$ and so $\sigma_1 A$ is tangential there to a steady state characteristic AB.

If we suppose that D occurs in $x < x_s(0)$, that is the shock from σ_2 decays before it reaches the snout, then similarly $\sigma_2 D$ is tangential at D to a steady state characteristic coming from $x > \sigma_2$. However it is also tangential to a characteristic from $\sigma_2 D-$. This charscteristic cannot come from $\sigma_1 \sigma_2$ or $\sigma_1 A$, since $H^{n+2}/(n+2) = s(x)$ at D, but $H^{n+2}/(n+2) > s(x)$ in $\sigma_1 A$, and so it must come from A. This is illustrated in Figure 5.9,

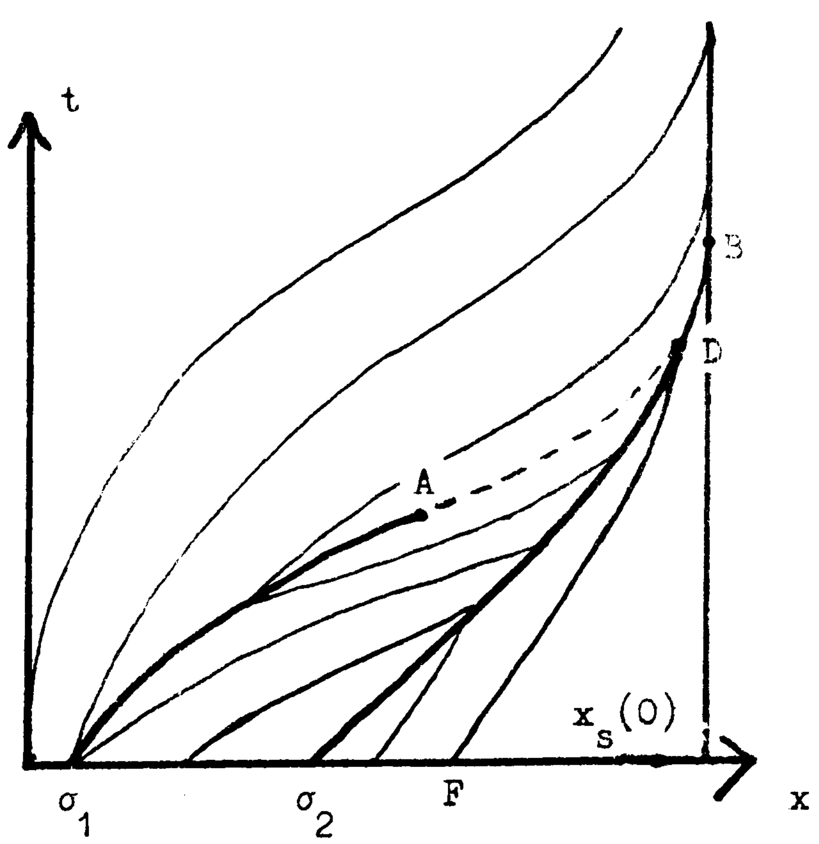


Figure 5.9.

and represents a contradiction, since ADB and FDB are <u>both</u> steady-state characteristics. Our supposition that D occurs in $x < x_s(0)$ is therefore wrong, and we must have the diagram as given in Figure 5.8. Thus an initial shock inevitably leads to a temporary advance of the snout. (This applies whether the second shock $\sigma_1 A$ is present or not.) It is then of interest to find out how far the snout advances, and how this is related to σ_1, σ_2 and c (defined in (5.35)).

We shall see in §7 that D is where $x_s'(t) = 0$, since this is where

$H = 0$. The shock has decayed at D, and the line DB is not a characteristic, but represents only the position of the snout. x_s then returns to its initial (steady state) value precisely at B, since AB is the first steady state characteristic to reach $x_s(t)$ after the advance.

Let $\sigma_1 C$ be the characteristic from σ_1 as shown in Figure 5.8. We have drawn C on DB, but it could also possibly lie on $\sigma_2 D$. Let us suppose that it is in $\sigma_2 D$ at $x = X$, say. Then on $\sigma_2 C-$, all the characteristics come from $\sigma_1 \sigma_2$, and so

$$\frac{H^{n+2}}{n+2} = s(x) + c \text{ on } \sigma_2 C- . \tag{5.37}$$

It follows from (5.30) that $\sigma_2 C$ is given by $x = x_d(t)$, where

$$\frac{dx_d}{dt} = \frac{c + s(x_d) - s^*(x_d)}{(n+2)^{1/(n+2)}[\{c+s(x_d)\}^{1/(n+2)} - \{s^*(x_d)\}^{1/(n+2)}]}, \tag{5.38}$$

and

$$s^*(x) = \begin{cases} s(x), & x < x_s(0) \\ 0, & x > x_s(0) \end{cases} . \tag{5.39}$$

The initial condition for (5.38) is

$$x_d(0) = \sigma_2 . \tag{5.40}$$

The equation of the characteristic through σ_1 is, from (5.23),

$$t = \int_{\sigma_1}^{x} \frac{dx}{[(n+2)\{c+s(x)\}]^{(n+1)/(n+2)}} . \tag{5.41}$$

The solution of (5.38) and (5.40) is

$$t = \int_{\sigma_2}^{x_d} \frac{(n+2)^{1/(n+2)}[\{c+s(x)\}^{1/(n+2)} - \{s^*(x)\}^{1/(n+2)}]}{c + s(x) - s^*(x)}\, dx. \tag{5.42}$$

These meet at C where $x = X$ and t has the same value. Thus putting $x = X = x_d$ and equating (5.41) and (5.42), we find

$$\int_{\sigma_1}^{X} \frac{dx}{[(n+2)\{c+s(x)\}]^{\frac{n+1}{n+2}}} = \int_{\sigma_2}^{X} \frac{(n+2)^{\frac{1}{n+2}}[\{c+s(x)\}^{\frac{1}{n+2}} - \{s^*(x)\}^{\frac{1}{n+2}}]}{c + s(x) - s^*(x)}\, dx . \tag{5.43}$$

Let us denote by x_c the point where $s(x) = -c$. In this case (5.43) has the following meaning: if there is a solution $\sigma_2 < X < x_c$ of (5.43), then the characteristic $\sigma_1 C$ in Figure 5.8 intersects the shock $\sigma_2 D$ at C where $x = X$ and t is given by (5.41) or (5.42); in this case the characteristic meeting the shock at D comes from $\sigma_1 A+$. If there is no such solution, then C lies on DB as shown in Figure 5.8 and the characteristic meeting the shock at D comes from $\sigma_1 \sigma_2$.

Defining $\phi(\theta) = \theta^{1/(n+2)}$, $\theta > 0$, we have since ϕ is concave

$$\phi'(\theta_2) \leq \frac{\phi(\theta_2) - \phi(\theta_1)}{\theta_2 - \theta_1} \leq \phi'(\theta_1) \; , \; \theta_1 < \theta_2 \; . \tag{5.44}$$

Since by definition $c + s(x) > s^*(x)$ for $\sigma_2 < x < x_c$, it immediately follows upon application of (5.44) to (5.43) that the integrand of the right hand side is greater than that of the left hand side; since the left hand integral is positive at $X = \sigma_2$ whereas the right hand one is zero, it is evident that the right hand side 'catches up' with the left hand side as X increases, and therefore no solution to (5.43) exists only if

$$\frac{1}{n+2} \int_{\sigma_1}^{x_c} \frac{dx}{[c+s(x)]^{\frac{n+1}{n+2}}} \geq \int_{\sigma_2}^{x_c} \frac{[\{c+s(x)\}^{\frac{1}{n+2}} - s^*(x)^{\frac{1}{n+2}}]}{c + s(x) - s^*(x)} \, dx$$

$$= \int_{\sigma_2}^{x_s} \frac{[\{c+s(x)\}^{\frac{1}{n+2}} - s(x)^{\frac{1}{n+2}}]}{c} \, dx + \int_{x_s}^{x_c} \frac{dx}{[c+s(x)]^{\frac{n+1}{n+2}}} \; . \tag{5.45}$$

Given $s(x)$, c can then be computed as a function of σ_1 and σ_2; for the present, we shall obtain explicit analytic estimates for c in terms of σ_1 and σ_2.

Let us suppose that s is concave, i.e. s' is monotone decreasing, and also that $s_1'(\sigma_1) < 0$, so that we consider only shocks initially present in the ablation zone. For $\sigma_1 \leq x_1 < x < x_2$, we then have

$$-\frac{1}{s'(x_1)} > -\frac{1}{s'(x)} > -\frac{1}{s'(x_2)} \; , \tag{5.46}$$

and on rearranging (5.45) and applying (5.44) and (5.46) as appropriate, we find that certainly no solution of (5.43) exists provided

$$\frac{1}{|s'(\sigma_2)|}\int_{s(\sigma_2)}^{s(\sigma_1)} [c+s]^{-(\frac{n+1}{n+2})} ds \geq \frac{1}{|s'(\sigma_2)|}\int_0^{s(\sigma_2)} [s^{-(\frac{n+1}{n+2})} - (c+s)^{-(\frac{n+1}{n+2})}] ds$$

$$+ \frac{(n+1)}{a_s}\int_{-c}^{0} (c+s)^{-(\frac{n+1}{n+2})} ds ,$$

i.e.

$$[c+s(\sigma_1)]^{\frac{1}{n+2}} \geq s(\sigma_2)^{\frac{1}{n+2}} + \left[1 + \frac{(n+1)|s'(\sigma_2)|}{a_s}\right] c^{\frac{1}{n+2}} , \tag{5.47}$$

where a_s is the snout ablation rate. A physically interesting case is

$$\sigma_2 - \sigma_1 << 1 , \ c << 1 , \tag{5.48}$$

e.g. an influx of ten metres of ice on a hundred metre deep glacier over a length of a few hundred metres. Provided $x_s - \sigma_2 \sim 1$, we may expand (5.47) by writing

$$\begin{aligned} \sigma_1 &= \sigma , \\ \sigma_2 &= \sigma + \Omega , \ \Omega << 1 , \\ c &<< 1 , \end{aligned} \tag{5.49}$$

and then (5.47) becomes

$$s(\sigma)^{\frac{1}{n+2}}\left[1 + \frac{c}{(n+2)s(\sigma)} + O(c^2)\right] \geq \left[s(\sigma) + \Omega s'(\sigma) + O(\Omega^2)\right]^{\frac{1}{n+2}}$$

$$+ \left[1 + \frac{(n+1)|s'(\sigma)|}{a_s}\right] c^{\frac{1}{n+2}} ,$$

whence

$$c + \Omega|s'(\sigma)| \geq \left[1 + \frac{(n+1)|s'(\sigma)|}{a_s}\right] c^{\frac{1}{n+2}} . \tag{5.50}$$

As $c \to 0, c^{1/(n+2)} >> c$, and so an approximate criterion for there to be no solution to (5.43) is

$$\Omega \geqslant \left[\frac{1}{|s_1'(\sigma)|} + \frac{n+1}{a_s} \right] c^{\frac{1}{n+2}} \quad . \tag{5.51}$$

Therefore, even if c is fairly small, Ω must be comparatively large for (5.51) to hold: for example if $|s_1'(\sigma)| = 1$, $a_s = 2$, $c = 1/32$, (5.51) implies $\Omega \geqslant 3/2$, in other words the disturbance must be longer than the glacier!

Let us suppose that (5.45) holds, so that no solution of (5.43) exists. In this case the characteristic through D comes from $\sigma_1\sigma_2$, and hence on this characteristic

$$\frac{H^{n+2}}{n+2} = s(x) + c \ , \tag{5.52}$$

and the shock terminates at $x = x_c$. If c is small, so that $s(x) \approx - a_s[x-x_s]$, then

$$x_c = x_s + \frac{c}{a_s} \quad . \tag{5.53}$$

Denoting dimensional variables by a suffix d, the advance Δx_{sd} of the snout is given by

$$\begin{aligned} \Delta x_{sd} &= \frac{c_d}{a_{sd}} \frac{\ell a_0}{d} \\ &\approx \frac{Uc_d}{a_{sd}} \ , \end{aligned} \tag{5.54}$$

where U is a typical longitudinal velocity. For example, if $U = 100 \text{ m y}^{-1}$, $a_{sd} \approx 5 \text{ m y}^{-1}$, and $c_d = 10$ m of ice is suddenly supplied to a region of the glacier surface, the snout will eventually advance a distance $\Delta x_{sd} \approx 200$ m before retreating.

If (5.45) does not hold, then C lies in σ_2D and the characteristic meeting the shock at D emanates from $\sigma_1 A+$ along which $H^{n+2}/(n+2) - s(x) < c$. It follows that $x < x_c$ at D, and thus in either case (5.54) gives the farthest snout advance possible. No simple quantitative analysis seems possible if C lies in σ_2D.

In drawing Figure 5.8, we have tacitly assumed A occurs in $x < x_s(0)$, i.e. that the σ_1 shock decays before the slopes $\frac{dt}{dx}$ of the characteristics originating on σ_1A- become infinite. This is not a necessary restriction. If A is in $x > x_s(0)$, the same reasoning as above is valid and we find that the appropriate characteristic diagram is as in Figure 5.10: the only real difference is that A and B are now identical. This figure represents the splitting off (at F) of a section of the glacier at the snout.

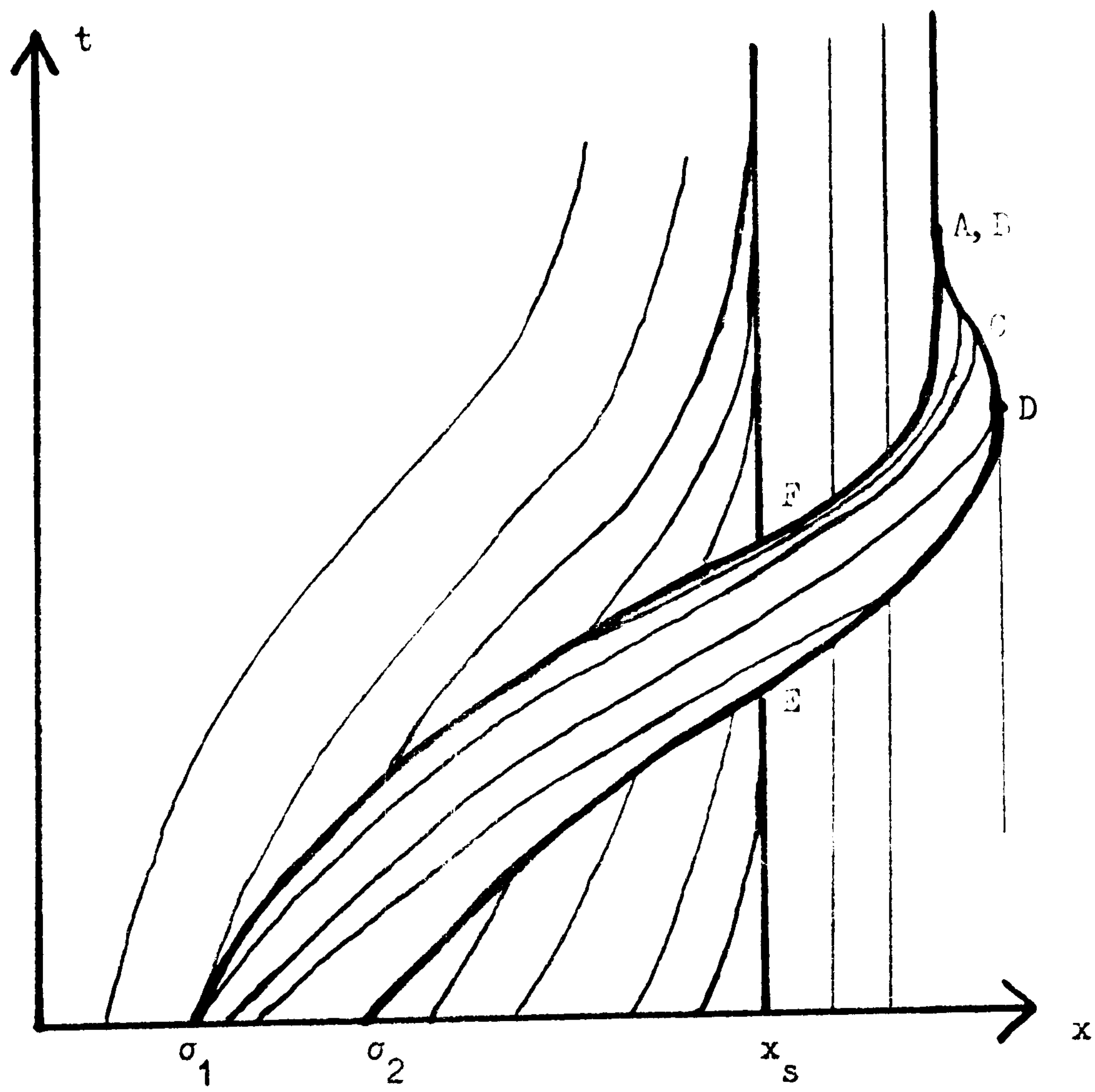


Figure 5.10.

§7 Small Initial Perturbations.

We now consider an initial condition representing a slight change in the climate. Specifically we shall assume

$$s_1'(\sigma) << 1 \ , \ x_0 < \sigma < x_s(0) \ , \tag{5.55}$$

whence also $s_1(\sigma) << 1$. The characteristics are, from (5.23),

$$t(x,\sigma) = \int_\sigma^x \frac{dx}{[(n+2)\{s(x)-s_1(\sigma)\}]^{(n+1)/(n+2)}} \ , \tag{5.56}$$

and these intersect when, from (5.28), $\partial t/\partial\sigma = 0$, i.e.

$$\frac{1}{A^{n+1}(\sigma)} = (n+1)s_1'(\sigma)\int_\sigma^x \frac{dx}{[(n+2)\{s(x)-s_1(\sigma)\}]^{1+(\frac{n+1}{n+2})}} \quad . \qquad (5.57)$$

It is evident from (5.57) that, since $1/A^{n+1} \gtrsim O(1)$ and $s_1' \ll 1$, shocks only form when the integral in (5.52) is large. Since $s_1 \ll 1$, this implies that $s(x) \approx 0$, i.e. that x is near x_0 or x_s. We shall restrict our attention to the more physically interesting case when $x \approx x_s$. In this case shocks must form near the snout. In fact, if s_1' is sufficiently small, one may suppose they form <u>at</u> the snout. The reason for this is as follows.

Consider an initial perturbation $s_1(\sigma)$. Suppose that $s_1'(\sigma) > 0$ for $\sigma_1 < \sigma < \sigma_2$, and $s_1' < 0$ just outside these limits. Suppose also that the characteristics from σ_1 and σ_2 do not intersect any shocks that may form from other regions of the initial curve $t = 0$. The time for the characteristic from $\sigma \in (\sigma_1, \sigma_2)$ to intersect its neighbours is given from (5.56) by

$$T(\sigma) = t[X(\sigma),\sigma] = \int_\sigma^{X(\sigma)} \frac{dx}{[(n+2)\{s(x)-s_1(\sigma)\}]^{(n+1)/(n+2)}} \, , \qquad (5.58)$$

where $x = X(\sigma)$ is where the intersection takes place and X is given by (5.57). From (5.57) we see that as $\sigma \to \sigma_2$, $X(\sigma) \to x_s$ defined by $s(x_s) = s_1(\sigma_2)$, and so the σ_2 characteristic intersects its neighbour just when $x = x_s$. Thus if we can show that for $\sigma \in (\sigma_1,\sigma_2)$, $T(\sigma) > T(\sigma_2)$, then the shock must actually form at the snout, and the previously established method for determining its evolution may be called upon. Now certainly $T(\sigma) > T(\sigma_2)$ in (σ_1,σ_2) if $T'(\sigma) < 0$ there. We shall prove (if s_1'' and s_1' are sufficiently small) that this is indeed the case.

Recall the definition of $X(\sigma)$ from (5.57),

$$\frac{1}{A^{n+1}(\sigma)} = (n+1)s_1'(\sigma)\int_\sigma^{X(\sigma)} \frac{dx}{[(n+2)\{s(x)-s_1(\sigma)]^{1+(\frac{n+1}{n+2})}} \quad . \qquad (5.59)$$

Integrating the right hand side by parts, we obtain

$$\frac{1}{A^{n+1}(\sigma)}\left[1 - \frac{s_1'(\sigma)}{s(\sigma)}\right] = \frac{-s_1'(\sigma)}{s'(X)H^{n+1}(X)} + s_1'(\sigma)\int_\sigma^X \left[\frac{1}{s'(x)}\right]' \frac{dx}{H^{n+1}} \, , \qquad (5.60)$$

and rearranging (5.60),

$$\frac{H^{n+2}}{n+2} = s(X) - s_1(\sigma) = \left[\frac{s_1'(\sigma)}{|s'(X)|}\right]^{\frac{n+2}{n+1}} [s(\sigma) - s_1(\sigma)]\left[1 - \frac{s_1'(\sigma)}{s'(\sigma)}\right]^{-(\frac{n+1}{n+2})}$$

$$\times \left[1 + |s'(X)|H^{n+1}\{1 - \frac{s_1'(\sigma)}{s'(X)}\}\int_\sigma^X (\frac{1}{s'})' \frac{dx}{H^{n+1}}\right]^{\frac{n+2}{n+1}}, \qquad (5.61)$$

which simply says

$$s(X) = s_1(\sigma) + \left[\frac{s_1'(\sigma)}{|s'(X)|}\right]^{\frac{n+2}{n+1}}\left[s(\sigma) + 0(s_1)\right]. \qquad (5.62)$$

Now note that from (5.56),(5.57) and (5.58) we have

$$\frac{dT}{d\sigma} = \frac{\partial t}{\partial \sigma} + \frac{\partial t}{\partial x}\frac{dX}{d\sigma} = \frac{1}{H^{n+1}}\frac{dX}{d\sigma} \quad . \qquad (5.63)$$

Differentiating (5.62) and rearranging,we find

$$\left[s'(X) - \{s_1'(\sigma)\}^{\frac{n+2}{n+1}}\{s(\sigma)+0(s_1)\}\left\{\frac{1}{|s'(X)|^{\frac{n+2}{n+1}}}\right\}'\right]\frac{dX}{d\sigma}$$

$$- \left[\frac{s_1'(\sigma)}{|s'(X)|}\right]^{\frac{n+2}{n+1}}[s'(\sigma)+0(s_1)] = s_1'(\sigma) + \frac{[s(\sigma)+0(s_1)]}{|s'(X)|^{\frac{n+2}{n+1}}}\frac{d}{d\sigma}\{s_1'(\sigma)\}^{\frac{n+2}{n+1}}.$$

(5.64)

Now define

$$\max_{\sigma \in (\sigma_1,\sigma_2)} s_1'(\sigma) = \Sigma , \qquad (5.65)$$

and let us prescribe Σ so that

$$\Sigma^{\frac{n+2}{n+1}} s(\sigma)\left[\frac{1}{s'(X)}\right]' < |s'(X)| \qquad (5.66)$$

for $X > x_s, \sigma \in (\sigma_1,\sigma_2)$. Then,neglecting $0(s_1)$ terms in (5.64),we see that the coefficient of $\frac{dX}{d\sigma}$ is negative,and hence a sufficient condition that X' (and hence T') be negative is that

$$s_1'(\sigma) + \frac{s(\sigma)}{|s'(X)|^{\frac{n+2}{n+1}}} \frac{d}{d\sigma}\left[\{s_1'(\sigma)\}^{\frac{n+2}{n+1}}\right] > |s'(\sigma)| \left[\frac{s_1'(\sigma)}{|s'(X)|}\right]^{\frac{n+2}{n+1}} . \quad (5.67)$$

Note that including $O(s_1)$ will only have the effect of requiring a small correction term to be included in the bounds on Σ and s_1'' below.

If $s_1'' > 0$, then (5.67) holds if

$$s_1'(\sigma) < \alpha^{n+1} \frac{|s'(X)|^{n+2}}{|s'(\sigma)|^{n+1}} , \quad \alpha < 1 . \quad (5.68)$$

If $s_1'' < 0$ (as it must be in a neighbourhood of σ_2), then define M by

$$\max_{\substack{X > x_s \\ \sigma \in (\sigma_1, \sigma_2)}} \frac{s(\sigma)}{|s'(X)|^{\frac{n+2}{n+1}}} = M . \quad (5.69)$$

When $s_1'' < 0$, (5.67) is certainly valid if

$$- M \frac{d}{d\sigma}\left[\{s_1'(\sigma)\}^{\frac{n+2}{n+1}}\right] < s_1'(\sigma) - \frac{|s'(\sigma)|}{|s'(X)|^{\frac{n+2}{n+1}}} [s'(\sigma)]^{\frac{n+2}{n+1}} ,$$

and hence using (5.68) when

$$M(\tfrac{n+2}{n+1})|s_1''(\sigma)| < [s_1'(\sigma)]^{\frac{n}{n+1}}(1-\alpha) . \quad (5.70)$$

If we now assume that s_1 satisfies the conditions (5.65), (5.66), (5.68) and (5.70) for some $\alpha > 0$, then $X' < 0$, $T' < 0$ and so the shock does in fact form at $x = x_s$. Note that if (5.70) holds, it <u>implies</u>

$$s_1' < \left[\frac{(1-\alpha)}{M(n+2)}(\sigma_2 - \sigma)\right]^{n+1} , \quad (5.71)$$

which is quite restrictive near σ_2. This kind of restriction on s_1' appears to be a necessary one.

Let us now denote the shock position by $x_s(t)$, since it is still the terminus of the glacier (though not snout-shaped). The equation for x_s is given by (5.30) with $H_2 = 0$. Thus, writing $H_1 = H$,

$$\frac{dx_s}{dt} = \frac{H^{n+1}}{n+2} . \quad (5.72)$$

On $x = x_s$, the characteristic solutions are still valid, and hence

$$t = \int_{\sigma}^{x_s} \frac{dx}{[(n+2)\{s(x)-s_1(\sigma)\}]^{(n+1)/(n+2)}} , \qquad (5.73)$$

$$\frac{H^{n+2}}{n+2} = s(x_s) - s_1(\sigma) . \qquad (5.74)$$

As mentioned in §6, we see from (5.72) that the shock decays when $H = 0$, i.e. $\frac{dx_s}{dt} = 0$. The shock geometry is shown in Figure 5.11.

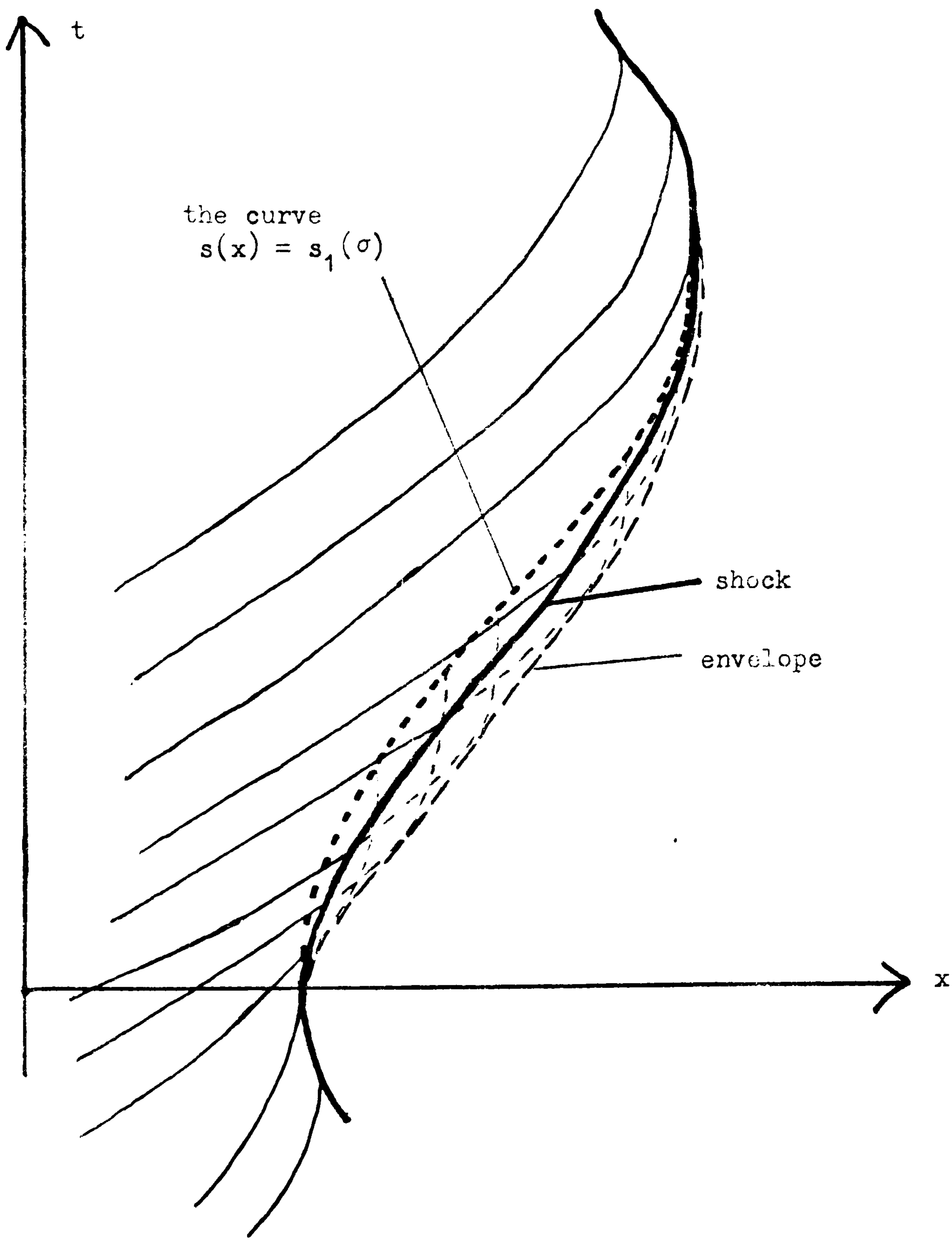


Figure 5.11.

Since from (5.21) the surface speed of a cold glacier is

$$u_s = \frac{H^{n+1}}{n+1} , \tag{5.75}$$

we have

$$\frac{dx_s}{dt} = \frac{n+1}{n+2} u_s : \tag{5.76}$$

hence the rate of advance of a cold snout is about 4/5 the surface speed; this should be capable of verification, although the effect may be blurred by the effects of diffusion and seasonal warming.

§8 Diffusive Effects.

In common with other physical systems in which shocks occur (e.g. gas dynamics) there is a diffusive mechanism (which we have neglected) which acts physically to 'smooth out' the shocks. In fact from (5.7) we can see that μ represents just such a mechanism, since in neglecting it we lose a term $-\frac{\mu n}{n+2} H^{n+2} H_{xx}$ from the left hand side (as well as lower derivative terms). This represents a diffusion term with small diffusion coefficient $\frac{\mu n}{n+2} H^{n+2}$. From our knowledge of the effect of such terms, we expect that when $H \sim 1$, the effect of the diffusion term on shocks will be to smooth them out over a short distance. For shocks at the snout, the effect is not clear since the diffusion coefficient is 'degenerate', i.e. tends to zero there, and the diffusional effect disappears.

We note that with inclusion of the term in μ, we have apparently not enough boundary conditions for the problem: in fact, there is an implicit condition in the reduced model which is that H_x should be finite at $H = 0$, so that the approximations involved in ignoring terms like $\delta^2 \eta_x$ in (2.66) are consistent. However, (at least in the steady state—see Chapter VI), it appears that this implicit condition cannot be satisfied even with the diffusion term included: this seems to be associated with the degeneracy of the coefficient of the highest derivative, which precludes the necessity of applying a boundary condition on $x = x_s$ as well as on $x = x_0$. Appropriate boundary conditions for linear equations of this type have been discussed by Smirnova (1963) and also (heuristically) by Nye (1963a).

In Chapter VI we shall see that a further correction term due to longitudinal stress, which would here be proportional to H_{xxx}, can also be introduced. This term is similarly 'degenerate', but we find that (in the steady state) its effect is to make the snout slope finite, thus satisfying the implicit boundary condition. We might reasonably expect similar phenomena to occur in the present case.

§9 'Linearised' Results.

We consider again the equation

$$H_t + H^{n+1}H_x = s'(x) \tag{5.77}$$

with solution

$$t = \int_\sigma^x \frac{dx}{[(n+2)\{s(x)-s_1(\sigma)\}]^{(n+1)/(n+2)}}\ , \tag{5.78}$$

$$\frac{H^{n+2}}{n+2} = s(x) - s_1(\sigma)\ , \tag{5.79}$$

and where $s_1, s_1' \ll 1$. It is obvious from (5.79) that a uniform approximate solution exists,

$$\frac{H^{n+2}}{n+2} = s(x) + \dots\ , \tag{5.80}$$

thus contradicting the linearised results of Nye, which were invalid at $x = x_s$. We can obtain a uniform approximation as follows. Write

$$\frac{H_0^{\ n+2}}{n+2} = s(x)\ , \tag{5.81}$$

so that H_0 is the steady-state profile. Then (5.78) is

$$t = \int_\sigma^x \frac{dx}{H_0^{\ n+1}} + \text{smaller terms.} \tag{5.82}$$

This simply states that the characteristics are uniformly approximated by the steady state ones. But now (5.79) and (5.82) are the solutions of the system

$$H_t + H_0^{n+1} H_x = s'(x) \; , \qquad H = A(\sigma) \; , \; x = \sigma \text{ on } t = 0 \; . \tag{5.83}$$

They are also clearly the solutions of (5.77) to a first approximation. Writing

$$\xi = \int_0^x \frac{dx}{H_0^{n+1}} \tag{5.84}$$

and

$$H_0(x) \equiv f(\xi) \; , \tag{5.85}$$

whence

$$s'(x) = H_0^{n+1} H_0'(x) = H_0^{n+1} f'(\xi) \frac{1}{H_0^{n+1}} = f'(\xi) \; , \tag{5.86}$$

then (5.83) is

$$H_t + H_\xi = f'(\xi) \; , \tag{5.87}$$

with solution

$$H = f(\xi) + \phi(\xi - t) \tag{5.88}$$

where

$$\phi(\xi) \equiv A(x) - H_0(x) \tag{5.89}$$

is the initial perturbation in H. This shows up the travelling wave form of H, and is a uniformly valid approximation right up to the snout $x = x_s$. The question of shock formation at x_s has been considered in §7.

§10 Seasonal Changes.

As an example of a time-dependent flux we show, using the above linearisation of characteristics, that the fast oscillation due to seasonal flux changes has a negligible effect on the longer term evolution of the profile. We consider by way of illustration

$$H_t + H^{n+1}H_x = s'(x) + s_1'(x)e^{i\omega t} \tag{5.90}$$

where $\omega >> 1$. Anticipating that $H \simeq H_0$, we write

$$H = H_0 + \phi e^{i\omega t} \quad , \quad \phi << 1 \tag{5.91}$$

and, using (5.84) and (5.86), we approximate (5.90) by

$$i\omega\phi + \phi_\xi = s_1'(x) = f_1(\xi) \ , \tag{5.92}$$

say, whence

$$\phi = e^{-i\omega\xi}\int_0^\xi f_1(\xi)e^{i\omega\xi}\,d\xi \ . \tag{5.93}$$

Since $\omega >> 1$, (5.93) implies $\phi << 1$, as we assumed. In fact

$$\int_0^\xi f_1(\xi)e^{i\omega\xi}\,d\xi = \frac{1}{i\omega}[f_1(\xi)e^{i\omega\xi} - f_1(0)] + O(\frac{1}{\omega^2})$$

$$= \frac{1}{i\omega} f_1(\xi)e^{i\omega\xi} + O(\frac{1}{\omega^2}) \tag{5.94}$$

if there is no perturbation in the accumulation rate at the glacier head. Hence from (5.91), (5.93) and (5.94),

$$H(x,t) = H_0(x) + \frac{1}{i\omega}f_1(\xi)e^{i\omega t} + O(\frac{1}{\omega^2}) \ , \tag{5.95}$$

whence to $O(1/\omega)$, the oscillation has no wave effect at all. If $f_1(0) \neq 0$, (5.95) contains a term $-\frac{1}{i\omega}f_1(0)e^{i\omega(t-\xi)}$ as well, which represents a travelling wave due to variations in the accumulation rate at the head.

CHAPTER VI

Analysis of the Snout

In the previous chapter we saw that

$$\frac{H^{n+2}}{n+2} = s(x) \tag{6.1}$$

was the steady state solution of the reduced model with the additional assumptions that $\mu = 0, A = 1$. If the glacier is sliding, a simple extension gives the steady state profile as

$$Hu_b + \frac{H^{n+2}}{n+2} = s(x) \tag{6.2}$$

where

$$u_b = F[H, T(x,H)] , \tag{6.3}$$

and F is defined in (2.18). Since $u_b \leqslant f(H)$ which $\to 0$ as $H \to 0$, it follows immediately from (6.2) that if the ablation rate at the snout is finite, then the glacier has infinite slope there.

Although this statement does not contradict any of the boundary conditions imposed in the reduced model, we see on examining (2.66) (for example) that the validity of the approximate solution with $\delta = 0$ breaks down at the snout, and hence the sliding law in (6.3) becomes invalid there. This failure is due to a nonuniformity in the expansion in powers of δ of terms like e in (2.53) near the snout, rather than because a boundary condition cannot be satisfied by the first order solution. For this reason, the appropriate asymptotic technique to use in order to obtain a uniform first order solution is the method of strained coordinates (Van Dyke 1975) and the δ^2 terms in (2.66) and elsewhere act as a correction to the basic solution of the reduced problem.

There are two facts that we wish to explain. In the first place glaciers are observed to have finite slopes at their snouts. For temperate glaciers, these slopes are generally in the range 5° - 30° :a typical example is the Athabasca glacier in Alberta, Canada, with a snout angle of 15° (Savage and Paterson 1963). Secondly, the ice in a temperate snout is observed to have a small but non-zero velocity right up to the front tip. One can see from (6.2) that this phenomenon is an immediate consequence of the first, and so we only need to show that inclusion of $O(\delta^2)$ terms predicts a finite snout slope.

In so doing, we shall show that inclusion of a non-zero μ is not sufficient to make the slope finite, so that it is indeed necessary to include terms of $O(\delta^2)$. One might also point out that the sliding law becomes invalid at the snout for another reason, namely that the supposed 'inner flow' region merges with the 'outer flow' region, so that the formulation in Chapter III breaks down. However, this will probably only occur in the last few metres of the glacier, and is thus neglected here: as pointed out by Lliboutry (1958), it is somewhat over academic to discuss the details of this region on the basis of a large scale mathematical model.

The only previous work on the snout of a temperate glacier appears to be that of Lliboutry (1956) who obtained a correction to the profile predicted by plasticity theory. His approximations were not rigorously followed through (Nye 1958), but he obtained good agreement with observations on the Glacier de Saint-Sorlin (Lliboutry 1958).

Some work has been done on the surface perturbations in ice sheets due to flow over large irregularities in the bedrock (Robin 1967, Budd and Radok 1971) which appears to have some similarity to the results of the present work in that the largest corrective term is $\delta^2 \tau_{1x}$; in their work this is an assumption, whereas in the present chapter it is deduced from dimensional arguments: these may be of some value in relating the two theories.

§1 Rescaling Procedure.

In the present chapter we shall consider only the steady state snouts of temperate glaciers with an important basal sliding velocity (in the notation of Chapter III, $\sigma \gtrsim \nu^{n+1}$): specifically we shall take the sliding law to be of the following Weertman-type form

$$u_b = C\tau_2^{\,m} , \quad m \leqslant n , \tag{6.4}$$

where $C \gtrsim 1$. Over the whole bedrock C may depend on moisture, bedrock 'roughness', etc., but will be considered locally in the snout as a constant. If $m = n$, (6.4) is of the type considered in §7 of Chapter III, where the sliding has no regelative component. It is simple to show from dimensional arguments that pure regelation leads to a law with $m = 1$ in (6.4), and so it is realistic to suppose that $m \leqslant n$ in (6.4), since regelation has the effect of decreasing the power in (6.4), which approximates the 'real' sliding law. In fact it will emerge that the ensuing analysis is valid if the snout flow is dominated by its basal sliding component: we shall see that (6.4) implies this result.

Let the (steady) snout position be $x = 0$ for convenience. In what follows, u, τ, x, y denote the orders of magnitude of u, τ_2, x, y; generally we shall make use of the assumption that for example $u_x \sim \frac{u}{x}$, $\tau_{2y} \sim \frac{\tau}{y}$, etc., except where this leads to contradiction, as with u_y. Formally this procedure must be checked at the end to verify that the approximations used are indeed valid.

To make sure we are near the snout, we specify

$$x \ll 1 , \quad y \ll 1 . \tag{6.5}$$

The vertical ablation rate is unaffected by the dimensions of the snout, therefore we choose

$$v \sim 1 , \tag{6.6}$$

and then in order to balance the continuity equation,

$$u \sim \frac{x}{y} . \qquad (6.7)$$

To balance the terms in the momentum equation, we require

$$\tau \sim y . \qquad (6.8)$$

Now the sliding law (6.4) says that $u_b \gtrsim \tau^m$, and hence certainly

$$u \gtrsim \tau^m . \qquad (6.9)$$

We contend that in fact $u \sim u_b$: to show this we reason as follows.

Certainly we must have

$$u_y \lesssim \frac{u}{y} , \quad u_y \overset{\gg}{\underset{\ll}{\sim}} \delta^2 v_x , \qquad (6.10)$$

where multiple signs express alternatives. Let us suppose that

$$u_y \sim \frac{u}{y} , \quad u_y \ll \delta^2 v_x ; \qquad (6.11)$$

then

$$\frac{u}{y} \gg u \gtrsim \tau^m \gtrsim \tau^n \sim \delta^2 v_x \sim \frac{\delta^2}{x} , \qquad (6.12)$$

using (6.5),(6.9),(2.52) and (6.6), and hence

$$u \gg \delta\left(\frac{\delta y}{x}\right) . \qquad (6.13)$$

However, from (6.11) and (6.7),

$$u_y \sim x/y^2 \ll \delta^2/x$$

$$\Rightarrow \frac{\delta y}{x} \gg 1 , \qquad (6.14)$$

so that

$$u \sim x/y \ll \delta ; \qquad (6.15)$$

(6.13) and (6.14) imply

$$u \gg \delta , \qquad (6.16)$$

contradicting (6.15): hence (6.11) cannot hold.

If we now suppose that

$$u_y \sim u/y \ , \ u_y \gtrsim \delta^2 v_x \ , \tag{6.17}$$

then from (6.5),(6.9) and (2.52),

$$u/y >> u \gtrsim \tau^m \gtrsim \tau^n \sim u_y \tag{6.18}$$

contradicting (6.17). Since neither (6.11) nor (6.17) can hold, it follows from (6.10) that we must have

$$u_y << u/y \ . \tag{6.19}$$

Because of this, we see that indeed

$$u \simeq u(x) = u_b(x) \sim \tau^m \ , \tag{6.20}$$

so that the snout flow is dominated by sliding. As stated above, this is the only assumption that we need to use below in establishing a corrected equation for the surface profile—the precise asymptotic form of u in (6.20) is not important.

Assuming for the moment (this will be justified in §2) that inclusion of μ does not give finite slope at the snout, it is necessary to consider a scaling at the snout such that

$$\tau_{2y} \sim \delta^2 \tau_{1x} \tag{6.21}$$

in (2.47). Since, from (2.51) and (2.52), $\tau_2/\tau_1 \sim u_y/u_x$, this requirement may be written

$$u_y \sim \delta^2/x \ , \tag{6.22}$$

using (6.7). Hence $u_y \sim (\delta^2 y^2/x^2)(u/y)$, again using (6.7), and so (6.19) implies

$$\frac{\delta y}{x} << 1 \ . \tag{6.23}$$

From (6.22),(6.23),(6.7) and (6.6), we find

$$u_y/\delta u_x \sim \delta y/x << 1 \ , \tag{6.24}$$

$$\delta^2 v_x/\delta u_x \sim \delta y/x << 1 \ , \tag{6.25}$$

so that from (2.53), e* is given approximately by

$$e^* \simeq 2\delta|u_x| \tag{6.26}$$

near the snout: from (6.20) e* is a function of x, and hence to first order

$$\tau_1 = \tau_1(x) = 2u_x A^{-1/n}[2\delta|u_x|]^{\frac{1}{n}-1} \ , \tag{6.27}$$

using (2.51); near the snout A may be considered constant.

Now from (2.51),(2.52) and (6.22),

$$\frac{\tau_{2x}}{\tau_{1y}} \sim \frac{\delta^2 y^2}{x^2} << 1 \ , \tag{6.28}$$

hence to first order equation (2.48) is

$$(p + \tau_1)_y = 0 \tag{6.29}$$

with approximate boundary condition (from (2.67))

$$p + \tau_1 = 0 \text{ on } y = \eta \ . \tag{6.30}$$

(6.29) and (6.30) imply

$$p + \tau_1 = 0 \ . \tag{6.31}$$

Substituting this into (2.47), we have

$$\tau_{2y} \doteq -(1 - \mu\eta_x) - 2\delta^2\tau_{1x} \ , \tag{6.32}$$

whence

$$\tau_2 = (\eta - y)(1 - \mu\eta_x) + 2\delta^2[(\eta - y)\tau_1]_x \ , \tag{6.33}$$

using the boundary condition (2.67). The term in $O(\delta^2)$ is similar to that considered by Robin (1967).

We neglect h_x compared with H_x in the snout; then using (6.27), we find that the tangential stress on the bed is given by

$$\tau_b = H[1 - \mu H_x] + [(2\delta)^{1+1/n}A^{-1/n}\mathrm{sgn}(u_x)H|u_x|^{1/n}]_x . \qquad (6.34)$$

The correction term to the basal stress is the last term in (6.34). The sliding law (6.4) is thus

$$u = C\tau_b{}^m = C\ [H\{1 - \mu H_x\} + 2\delta(\frac{2\delta}{A})^{1/n}\mathrm{sgn}(u_x)\{H|u_x|^{1/n}\}_x]^m . \qquad (6.35)$$

From (6.2), or directly from (6.20), conservation of mass gives approximately

$$uH = s(x) \simeq - a_s x . \qquad (6.36)$$

We have to solve (6.35) and (6.36) for the profile H and velocity u, subject to the condition that as $x \to -\infty$, H tends to the 'outer' solution; in order to ensure that our solution of (6.35) and (6.36) provides a valid approximation at the snout, we also stipulate that H_x is finite when H = 0 (so that (2.67) is approximated by (6.31), for example). As discussed in Chapter V, §8, this type of condition is associated with the fact that (6.35), considered (using (6.36)) as an equation for H, has a highest derivative with a 'degenerate' coefficient. Similar regularity conditions were considered by Nye (1963a) for a linear degenerate diffusion-type equation, although for a slightly different reason.

The problem to be solved is thus the equations (6.35) and (6.36) with the boundary conditions

$$\left.\begin{array}{l} H \sim [\frac{-s(x)}{C}]^{\frac{1}{2}} \quad \text{as } x \to -\infty , \\ H_x \text{ finite at } H = 0 . \end{array}\right\} \qquad (6.37)$$

In the following sections we shall show that an approximate solution of (6.35)-(6.37) may be developed using the method of strained coordinates. In principle this may be applied to any sliding law. For the purposes of initial theoretical consideration, we shall use the Weertman law (6.4) with m = 1.

§2 First Order Solution, $\delta = 0$.

Let us for convenience reverse the direction of x so that the snout lies in x > 0 (Figure 6.1). We replace s(x) by its approximation near the snout

$$s(x) = a_s x \tag{6.38}$$

where a_s is the ablation rate at the snout. With this geometry, the outer solution ($\delta = 0$) has $u_x > 0$ (as we show below): we therefore assume that $u_x > 0$ even with $\delta \neq 0$. Then (6.36) is

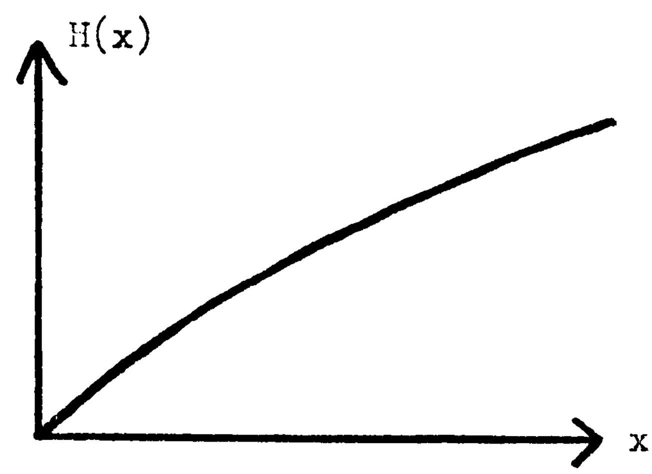


Figure 6.1: Snout Geometry.

$$H = \frac{a_s x}{u} \tag{6.39}$$

and so (6.35) is

$$u = a_s C \frac{x}{u} [1 + \mu a_s (\frac{x}{u})'] + 2\delta C a_s (\frac{2\delta}{A})^{1/n} \{\frac{x u'^{1/n}}{u}\}' \quad , \tag{6.40}$$

where ' denotes differentiation with respect to x. The boundary conditions on (6.40) are

$$u \sim (a_s C x)^{\frac{1}{2}} \text{ as } x \to \infty ,$$
$$u' \text{ bounded on } x/u = 0 , \tag{6.41}$$

using (6.39). We write

$$u = a_s \mu \bar{u} \; , \; x = \frac{\mu^2 a_s}{C} \bar{x} \; ; \tag{6.42}$$

then (6.40) becomes, (dropping dashes for convenience),

$$u = \frac{x}{u}\left[1 + \left(\frac{x}{u}\right)'\right] + \nu\left[\frac{xu'^{1/n}}{u}\right]' , \tag{6.43}$$

where

$$\nu = \frac{\Lambda}{a_s\mu}\left(\frac{2\delta C}{\Lambda\mu}\right)^{\frac{n+1}{n}} . \tag{6.44}$$

We shall consider (6.43) on the basis that δ is sufficiently small that ν as defined by (6.44) is also small: that is, we assume

$$\nu \ll 1 . \tag{6.45}$$

(Note that ν is not the same as the ν used in Chapters III and IV.) The boundary conditions on (6.43) are that

$$u \sim \sqrt{x} , \quad x \to \infty , \qquad u' \text{ bounded on } x/u = 0 . \tag{6.46}$$

We observe that the scaling (6.42) is appropriate for the snout, since if $\mu \sim 10^{-1}$, $a_s = 3$, $C = 1$, $\ell = 10$ km, then the length scale considered by (6.43) is the final 300 m of the glacier.

Let us first consider in detail the 'outer' solution of (6.43) and (6.41) with $\nu = 0$. Defining

$$\phi = \frac{x}{u} \tag{6.47}$$

so that

$$H = \frac{a_s\mu}{C}\phi , \tag{6.48}$$

(6.43) becomes

$$x = \phi^2(1 + \phi') \tag{6.49}$$

subject to

$$u \sim \sqrt{x} , \quad x \to \infty : \tag{6.50}$$

the second of (6.46) cannot be satisfied, as shown below.

By inspection of (6.49) either $\phi = 0$ or $\phi' = -1$ at $x = 0$. There appears to be nothing which is obviously physically invalid about the second case. However, if the glacier terminates in a shock, this case is unsteady as shown in Chapter V. If the glacier does not terminate, then since $\phi' = x/\phi^2 - 1 < 0$ in $x < 0$, the profile increases continually: we may obviously rule out such a solution, by requiring for example that there is no source of ice at infinity in the positive direction. Therefore $\phi = 0$ at $x = 0$. We seek an asymptotic solution for ϕ in the form

$$\phi \sim \phi_0 + \phi_1 + \phi_2 \ldots , \quad x \to 0 ; \tag{6.51}$$

this will show that we cannot satisfy the second boundary condition in (6.46).

Substituting (6.51) into (6.49), we obtain

$$x = [\phi_0^2 + 2\phi_0\phi_1 + \{\phi_1^2 + 2\phi_0\phi_2\} \ldots][\phi_0' + (1+\phi_1') + \phi_2' \ldots] . \tag{6.52}$$

The appropriate leading term is

$$x = \phi_0^2\phi_0'$$

$$\Rightarrow \phi_0 = (3/2)^{1/3}\, x^{2/3} \tag{6.53}$$

since $\phi = 0$ on $x = 0$. The next order term gives

$$0 = 2\phi_0\phi_0'\phi_1 + (1+\phi_1')\phi_0^2 , \tag{6.54}$$

whence we find

$$\phi_1 = -3x/7 . \tag{6.55}$$

The next term is

$$0 = \{\phi_1^2 + 2\phi_0\phi_2\}\phi_0' + (1+\phi_1').2\phi_0\phi_1 + \phi_0^2\phi_2' \tag{6.56}$$

which gives

$$\phi_2 = \frac{27}{196}(2/3)^{1/3}x^{4/3} \quad . \tag{6.57}$$

Thus

$$\phi \sim (3/2)^{1/3}x^{2/3} - \frac{3}{7}x + (2/3)^{1/3}\frac{27}{196}x^{4/3} \ldots \tag{6.58}$$

as $x \to 0$, and further terms may be obtained systematically. We have from (6.58)

$$u = \frac{x}{\phi} \sim (2x/3)^{1/3} + \frac{3}{7}(2x/3)^{2/3} + \frac{9}{196}(2x/3) \ldots \tag{6.59}$$

as $x \to 0$. It is now clear from (6.59) that the 'outer' solution has unbounded u' on $x = 0$. The effect of μ is only to weaken the singularity in ϕ' from $x^{-\frac{1}{2}}$ to $x^{-1/3}$, hence we must consider the $O(\nu)$ term. In fact from (6.58) and (6.59) we have that

$$\begin{aligned}[\frac{x}{u}u'^{1/n}] = \phi u'^{1/n} &\sim [(3/2)^{1/3}x^{2/3} - \frac{3}{7}x + \ldots] \\ &\quad\times [\frac{2}{9}(2x/3)^{-2/3} + \frac{4}{21}(2x/3)^{-1/3} \ldots]^{1/n} \\ &= \frac{3}{2}(2/9)^{1/n}(2x/3)^{\frac{2}{3}(1-\frac{1}{n})} + (2/9)^{1/n}\frac{9}{14}(\frac{2}{n} - 1)(2x/3)^{1-\frac{2}{3n}} + \ldots\end{aligned}$$

and so

$$\begin{aligned}[\frac{x}{u}u'^{1/n}]' &\sim (1-\frac{1}{n})\,\frac{2}{3}(2/9)^{1/n}(2x/3)^{-\frac{1}{3}-\frac{2}{3n}} \\ &\quad + \frac{3}{7}(2/9)^{1/n}(\frac{2}{n}-1)(1-\frac{2}{3n})(2x/3)^{-\frac{2}{3n}} \ldots\end{aligned} \tag{6.60}$$

i.e. the $O(\nu)$ term in (6.43) is unbounded. (If $n = 1$, the leading order term in (6.60) is zero, but then the next order term is unbounded.)

§3 Corrected First Order Solution, $\delta \neq 0$.

The singularity at the end point suggests that the method of strained coordinates (Van Dyke 1975) may be a suitable tool to employ. We therefore look for a uniformly valid expansion for u as

$$u = u_0(s) + \nu u_1(s) \ldots \tag{6.61}$$

where

$$x = s + \nu x_1(s) + \dots \tag{6.62}$$

and the straining x_1 is to be chosen by the condition that u_1/u_0 should be bounded as $s \to 0$, so that the expansion (6.61) is uniform. Then

$$\frac{d}{dx} = (1 + \nu x_1' + \dots)\frac{d}{ds} \tag{6.63}$$

and (6.43) gives

$$u_0 + \nu u_1 \dots = \frac{(s + \nu x_1 + \dots)}{(u_0 + \nu u_1 + \dots)}\left[1 + (1 + \nu x_1' \dots)\left\{\frac{s + \nu x_1 \dots}{u_0 + \nu u_1 \dots}\right\}'\right]$$

$$+ \nu(1 + \nu x_1' \dots)\left[\frac{(s + \nu x_1 \dots)(1 + \nu x_1' \dots)^{1/n}(u_0' + \nu u_1' \dots)^{1/n}}{(u_0 + \nu u_1 \dots)}\right]' . \tag{6.64}$$

The $O(1)$ terms give

$$u_0 = \frac{s}{u_0}\left[1 + \left(\frac{s}{u_0}\right)'\right], \tag{6.65}$$

of which the asymptotic solution as $s \to 0$ is described above in (6.59). The terms of $O(\nu)$ give

$$u_1 = \frac{s}{u_0}\left[x_1'\left(\frac{s}{u_0}\right)' + \left\{\frac{s}{u_0}\left(\frac{x_1}{s} - \frac{u_1}{u_0}\right)\right\}'\right] + \frac{s}{u_0}\left(\frac{x_1}{s} - \frac{u_1}{u_0}\right)\left[1 + \left(\frac{s}{u_0}\right)'\right]$$

$$+ \left(\frac{s u_0'^{1/n}}{u_0}\right)' . \tag{6.66}$$

Thus from (6.58), (6.59) and (6.60), and using (6.65),

$$u_1 = x_1'[(3/2)^{1/3}s^{2/3}\dots][(3/2)^{1/3}(2/3)s^{-1/3}\dots]$$

$$+ [(3/2)^{1/3}s^{2/3}\dots][(3/2)^{1/3}x_1/s^{1/3}\dots]'$$

$$- [(3/2)^{1/3}s^{2/3}\dots][(3/2)^{2/3}s^{1/3}u_1]' + x_1(2/3)^{1/3}s^{-2/3} - u_1$$

$$+ \left(1 - \frac{1}{n}\right)\frac{2}{3}(2/9)^{1/n}(2s/3)^{-\frac{1}{3} - \frac{2}{3n}},$$

whence

$$\frac{3}{2}[\, s^{5/3}u_1\,]' = (2/3)^{1/3}[\,\tfrac{5}{2}sx_1' + \tfrac{1}{2}x_1 + (1-\tfrac{1}{n})(2/9)^{1/n}(2s/3)^{\frac{1}{3}-\frac{2}{3n}} \ldots] + 0(s^{\frac{2}{3}-\frac{2}{3n}}) \,. \qquad (6.67)$$

Integrating,we obtain

$$\frac{3}{2}\, s^{5/3}u_1 = (2/3)^{1/3}[\tfrac{5}{2}sx_1 - 2\int_0^s x_1\, ds + \frac{9(n-1)}{4(2n-1)}(2/9)^{1/n}(2s/3)^{\frac{4}{3}-\frac{2}{3n}}\,] + 0(s^{\frac{5}{3}-\frac{2}{3n}}) \,. \qquad (6.68)$$

We wish to choose x_1 so that u_1/u_0 is bounded,i.e. such that

$$u_1 \lesssim 0(s^{1/3}) \qquad (6.69)$$

from (6.59). An appropriate choice of x_1 is clearly

$$x_1 \sim -Ks^{\frac{1}{3}-\frac{2}{3n}} + 0(s^{\frac{2}{3}-\frac{2}{3n}}) \qquad (6.70)$$

as $s \to 0$,where K is defined so that the coefficient of $s^{\frac{4}{3}-\frac{2}{3n}}$ in the square-bracketed term in (6.68) vanishes;the coefficients of $s^{\frac{2}{3}-\frac{2}{3n}},\ldots s^{2-\frac{2}{3n}}$ are chosen similarly:thus

$$-K\,[\tfrac{5}{2} - \frac{2}{\frac{4}{3}-\frac{2}{3n}}\,] + \frac{9(n-1)}{4(2n-1)}(2/9)^{1/n}(2/3)^{\frac{4}{3}-\frac{2}{3n}} = 0 \,,$$

and so

$$K = (\frac{n-1}{4n-5})(9/2)^{\frac{n-1}{n}}(2/3)^{\frac{2(2n-1)}{3n}} \,, \qquad (6.71)$$

and the first order solution may be written

$$u = u_0(s) \sim s^{1/3},$$
$$x = s + \nu x_1(s) \ldots \sim s - \nu Ks^{\frac{1}{3}-\frac{2}{3n}} \,, \qquad (6.72)$$

as $s \to 0$.

The validity of the method of strained coordinates has been discussed by Van Dyke (1975). In general it is less reliable than the method of matched asymptotic expansions, and in some cases leads to erroneous results (Levey 1959). The essential criterion for its validity appears to be that the singularity occurring in the 'linearised' or 'unperturbed' equation should correspond to one of the same type in the complete equation. Levey gives an example of a first order differential equation whose solution is everywhere analytic, but whose 'outer' solution (with the derivative term not present) is singular at zero: he shows that straining the coordinates does not work, whereas matched asymptotic expansions do. On this basis Van Dyke recommends that the method of strained coordinates never be used for problems of singular perturbation type, that is, where the small parameter multiplies the highest derivative; however, this is precisely what we have attempted to do: we therefore examine the validity of the solution (6.72).

It may be seen from (6.71), (6.72) and Figure 6.2 that since $x \sim s$ as $x \to \infty$, s tends to a finite limit as $x \to 0$ for all $n > 5/4$: we denote this limit by s_0. We ignore the special case $n = 1$ (Newtonian fluid) since (6.60) implies that the leading term in an asymptotic expansion for $\left(\frac{xu'}{u}\right)^{1/n}{}'$ near $x = 0$ is zero: this in turn means that our assumptions in §1 about the magnitude of τ_{1x} are invalid, and hence (6.43) is not necessarily the appropriate 'corrected' equation to consider.

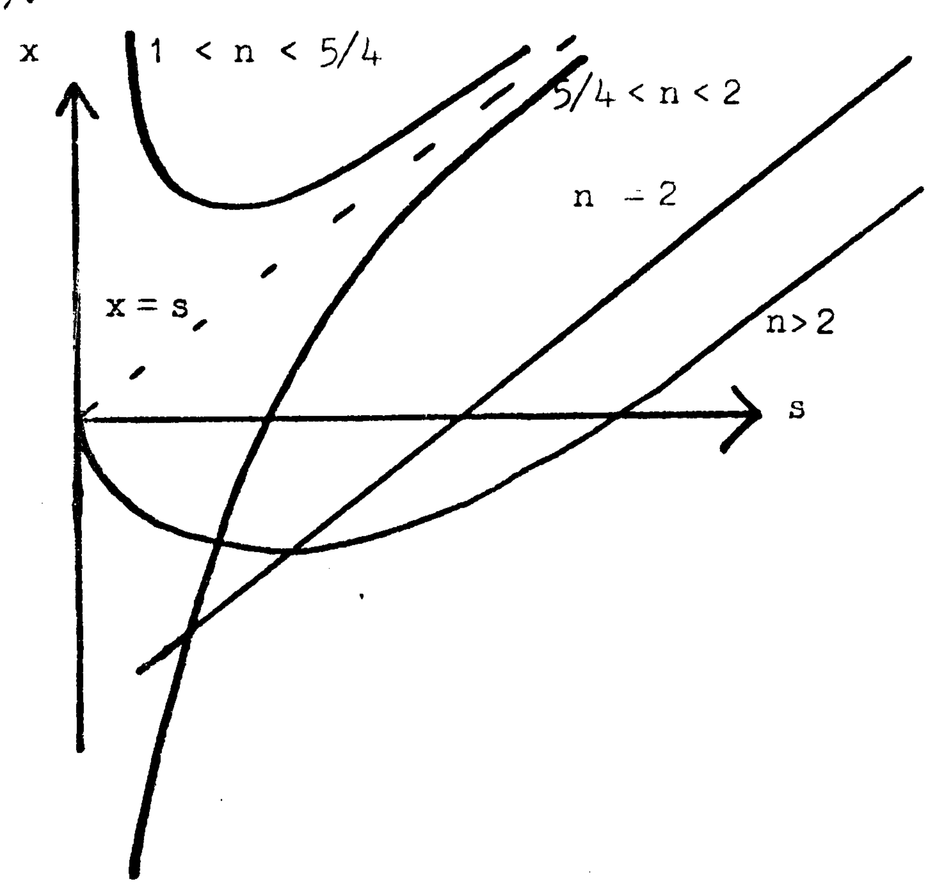


Figure 6.2.

In order to show that the strained solution is a uniformly approximate solution, we wish to show that the full equation (6.43) has the same type of singularity as the reduced equation. We know the reduced equation has the solution $u \sim x^{1/3}$ as $x \to 0$. Since the strained solution 'shifts' this singularity to $x = s_0$, it is reasonable to seek a solution of the full equation which also has this property: that is, we define a new space variable s_1 by $x = s_0 + s_1$, and seek a solution of the full equation (6.43) such that $u \sim s_1^{1/3}$ as $s_1 \to 0$.

Holding s_0 fixed, and letting $s_1 \to 0$, we find that if $u \sim s_1^{1/3}$, then also

$$x/u \sim s_1^{-1/3} \; , \; (x/u)' \sim s_1^{-4/3} \; , \; \left(\frac{xu'^{1/n}}{u}\right)' \sim s_1^{-\frac{4}{3}-\frac{2}{3n}} \; .$$

Balancing the terms of largest magnitude then requires

$$\left(\frac{x}{u}\right)\left(\frac{x}{u}\right)' \sim \left(\frac{xu'^{1/n}}{u}\right)' \quad ,$$

$$\Rightarrow \; s_1^{-5/3} \sim s_1^{-\frac{4}{3}-\frac{2}{3n}} \, ,$$

$$\text{i.e.} \quad n = 2 \; . \tag{6.73}$$

(6.73) is a necessary condition that (6.43) and its approximate form should have a strained solution of the form (6.72). Substituting (6.73) into (6.71), we find

$$K = \tfrac{1}{3}\sqrt{2} \quad , \tag{6.74}$$

and so in this case

$$\begin{aligned} x &= s - \tfrac{\nu}{3}\sqrt{2} \; \ldots \; , \\ u &\simeq u_0(s) \simeq u_0\left(x + \tfrac{\nu}{3}\sqrt{2}\right) \; , \end{aligned} \tag{6.75}$$

is a uniformly valid approximate solution.

If $n \neq 2$, the singularities of the full and reduced equations are of different type, and we should not expect a strained solution to be valid. However, we can surmount this difficulty as follows. We put

$$x = X^{\alpha} \; , \; \alpha > 0 \tag{6.76}$$

so that

$$\frac{d}{dx} = \frac{1}{\alpha X^{\alpha-1}} \frac{d}{dX} \quad , \tag{6.77}$$

and (6.43) becomes

$$u = \frac{X^\alpha}{u}\left[1 + \frac{1}{\alpha X^{\alpha-1}}\left(\frac{X^\alpha}{u}\right)' + \frac{\nu}{\alpha X^{\alpha-1}}\left[\frac{X^\alpha(\alpha X^{\alpha-1})^{-1/n} u'^{1/n}}{u}\right]' \right. \quad , \tag{6.78}$$

where ' in (6.78) refers to differentiation with respect to X. The 'outer' solution has $u \sim x^{1/3} \sim X^{\alpha/3}$ as $X \to 0$: thus if we require the complete equation to have a similar singularity at X_0, say, we put $X = X_0 + X_1$ and then as before we find that the largest terms as $X_1 \to 0$ come from $(x/u)(x/u)'$ and $(xu'^{1/n}/u)'$. A balance of these requires

$$X_1^{-\frac{2}{3}\alpha - 1} \sim X_1^{-1 - \frac{1}{3}\alpha + \frac{1}{n}(\frac{\alpha}{3} - 1)}$$

$$\Rightarrow \quad \alpha = \frac{3}{n+1} \ . \tag{6.79}$$

When n = 2, $\alpha = 1$ corresponding to our previous derivation. Thus in principle using (6.79) we can strain the coordinates in (6.78) and obtain an approximate solution for any n. The details are not considered here. Instead, we calculate the slope for the case n = 2, as this is probably quite an accurate value for the low stresses (say ~ 0.1 bar) which can occur at the snout.

§4 Velocity and Slope at the Snout.

For n = 2, we have from (6.75) and (6.59)

$$u \simeq u_0[x + \frac{\nu}{3}\sqrt{2}] \simeq [\frac{2}{3}\{x + \frac{\nu}{3}\sqrt{2}\}]^{1/3} + \ldots \tag{6.80}$$

Thus (in these variables) the snout velocity is

$$u(0) \simeq (\frac{2\sqrt{2}}{9}\nu)^{1/3} \tag{6.81}$$

and the snout velocity gradient is

$$u'(0) \simeq (3\nu)^{-2/3} \ . \tag{6.82}$$

Returning to mainstream variables (via (6.42)), (6.81) and (6.82) become

$$u(0) = a_s \mu \left(\frac{2\sqrt{2}\nu}{9}\right)^{1/3} , \tag{6.83}$$

$$u_x(0) = \frac{C}{\mu}(3\nu)^{-2/3} . \tag{6.84}$$

From (6.27),

$$\delta\tau_1(0) = (2\delta u_x/A)^{1/n}$$
$$= \left[\frac{2\delta C}{A\mu}(3\nu)^{-2/3}\right]^{\frac{1}{2}} \tag{6.85}$$

with n = 2. From (6.44),

$$\frac{2\delta C}{A\mu} = \left(\frac{a_s \mu\nu}{A}\right)^{2/3} ,$$

whence

$$\delta\tau_1 = (a_s \mu/3A)^{1/3} , \tag{6.86}$$

which is the dimensional longitudinal stress divided by $[\tau]_0$. From (6.35)

$$\tau_2 = \frac{u}{C} \approx \frac{a_s \mu}{C}\left(\frac{2\sqrt{2}}{9}\nu\right)^{1/3} \tag{6.87}$$

using (6.83). From (6.44) with n = 2,

$$\nu^{1/3} = (A/a_s\mu)^{1/3}(2\delta C/A\mu)^{1/2}$$

$$\Rightarrow \quad \tau_2 \approx (a_s\mu/3A)^{1/3}(\delta/\mu)^{1/2}(2\sqrt{2}/3)^{1/3}A^{-1/6}(2C)^{1/2}(a_s\mu)^{1/3} . \tag{6.88}$$

Comparing this with (6.87),we see that $\tau_2 \lesssim \delta\tau_1$ since δ/μ and μ are both small,and so a typical dimensionless stress at the snout is $\delta\tau_1$. With (as before) $a_s = 3, \mu = 10^{-1}$ and $A_w = 10$ for a temperate snout, $\delta\tau_1 \sim 10^{-2/3} \sim 0.215$. One may thus reasonably obtain a typical stress of 1/5 bar at the snout,and n = 2 is probably a good approximation.

The 'dimensional' snout slope α_s is given by

$$\alpha_s = \delta H'(0) = \delta a_s/u(0)$$

from (6.39), whence

$$\alpha_s \simeq \frac{\delta}{\mu}\left(\frac{9}{2\sqrt{2}\nu}\right)^{1/3} . \tag{6.89}$$

With $A = 10, a_s = 3, \mu = 10^{-1}, \delta = 10^{-2}, C = 1, n = 2$, (6.44) gives

$$\nu = \frac{2}{3\sqrt{5}} , \tag{6.90}$$

and so from (6.89)

$$\alpha_s \simeq 0.246 , \tag{6.91}$$

and the angle of inclination of the snout is

$$\tan^{-1}\alpha_s \simeq 14^\circ . \tag{6.92}$$

The values in (6.90),(6.91) and (6.92) are purely illustrative, but do show that the observed orders of magnitude can easily be predicted. We have only taken leading order terms, since detailed numerical predictions are of little help unless values of C and m in (6.35) are known with reasonable certainty. However, it is clear that further terms in the expansion for u could easily be found.

Lastly, with the above-mentioned values for a_s, etc., we find that the snout velocity (6.83) is

$$u_s \simeq 0.12 , \tag{6.93}$$

and so the snout velocity in this case is roughly a tenth of the typical glacier velocity. One can see from (6.83) that we generally expect u_s to be small.

CHAPTER VII

Large Conduction Limit, $\beta_2 >> 1$

In this chapter, we consider the asymptotic limit of large conduction. In this case, we neglect the convection terms in the energy equation, and the temperature (and hence the flow field) can then be found explicitly in the steady state. From equation (2.61) and Table 1, we have that

$$\beta_2 \simeq \frac{38}{a_0 d} \ , \tag{7.1}$$

where the depth scale d is expressed in metres, and the accumulation rate a_0 is in metres per year; thus the limit $\beta_2 >> 1$ can only be realistically obtained for shallow glaciers of low accumulation rates: for example if $d \sim 10$ m and $a_0 \sim 0.3$ m y^{-1}, then $\beta_2 \sim 12$. Although this limit is of less interest as regards the behaviour of large glacial ice-masses, we consider it here as a first attempt at an approximate solution because it is the simplest approximation (apart from $\kappa = 0$) that we can make to the reduced model, and also because we gain much valuable qualitative information therefrom.

§1 Model Equations in the Large Conduction Limit.

We shall use the notation of Chapter II. The equations of the reduced model (2.105)-(2.115) for the stream function Ψ and temperature T of the cold zone are, with the assumption that

$$\mu = 0 \ , \ \kappa \sim 1 \ , \ \beta_2 >> 1 \ , \tag{7.2}$$

$$\Psi_{\xi\xi} = \xi^n \exp[\kappa T] \ , \tag{7.3}$$

$$T_t + \Psi_x T_\xi - \Psi_\xi T_x = \beta_1 \xi^{n+1} \exp[\kappa T] + \beta_2 T_{\xi\xi} \ , \tag{7.4}$$

where on $\xi = 0$,

$$\begin{aligned} \Psi &= s(x,t) \ , \\ T &= T_A(x,t) \ , \ x < x_m \ ; \end{aligned} \tag{7.5}$$

on $\xi = H$,

$$\begin{aligned} &\Psi = \frac{\partial}{\partial t}\int_{x_0(t)}^{x} W(x',t)\,dx' , \\ &\Psi_\xi = -F[H,T] = -u_b , \\ &\beta_2 T_\xi = \lambda\beta_2\Lambda(T) + \beta_1 H u_b ,\ T < 0\ (x < x_Z) , \\ &T = 0\ ,\ T_\xi > 0\ (x_Z < x < x_M) ; \end{aligned} \tag{7.6}$$

on $\xi = \xi_M$,

$$T = T_\xi = 0 . \tag{7.7}$$

The equations (7.3)-(7.7) will be found to completely determine the temperature field and position of the melting surface when $\beta_2 \gg 1$,although to find the depth H in $x > x_M$ we require also the solution for Ψ in the temperate region.

Since $\kappa \sim 1$ and $-1 < T < 0$,it is reasonable (especially since the term is an empiricism anyway) to approximate the exponential in (7.3) and (7.4) by

$$e^{\kappa T} \simeq 1 + \kappa_1 T \equiv \chi , \tag{7.8}$$

where $\kappa_1 < \kappa$. We introduce (7.8) in order to simplify the ensuing analysis:it is not an essential approximation to make,but is considered to be useful if $\kappa \sim 1$ (not if $\kappa \gg 1$). Using (7.8) does not markedly affect the structure or stability of solutions:this is firstly because T is bounded,and secondly the instability to be described in §6 is basically due to the convexity in ξ of the function $\xi^n \exp[\kappa T]$ — which is <u>not</u> crucially due to the convexity of the exponential.

We change to a conductive time scale

$$\tau = \beta_2 t , \tag{7.9}$$

and define

$$\alpha = \frac{\kappa_1 \beta_1}{\beta_2} \tag{7.10}$$

where κ_1 is defined in (7.8). If we also take the inputs s and T_A to be time-independent, then the equations and boundary conditions (7.3)-(7.7) may be written, using (7.8)-(7.10),

$$\Psi_{\xi\xi} = \xi^n \chi \quad , \tag{7.11}$$

$$\chi_\tau + \frac{1}{\beta_2}[\Psi_x \chi_\xi - \Psi_\xi \chi_x] = \alpha \xi^{n+1} \chi + \chi_{\xi\xi} \; , \tag{7.12}$$

where

$$\text{on } \xi = 0, \qquad \Psi = s(x) \; , \quad \chi = \chi_A(x) \; , \; x < x_T \; ; \tag{7.13}$$

$$\begin{aligned} \text{on } \xi = H, \qquad & H_\tau = \frac{1}{\beta_2} \Psi_x \; , \\ & \Psi_\xi = - u_b = - F[H,\chi] \; , \\ & \chi_\xi - \alpha H u_b = \lambda \kappa_1 \Lambda(\chi) = \frac{\kappa_1 G d}{k T_0} \Lambda(\chi) \; , \chi < 1 \; , \; x < x_Z \; , \\ & \chi = 1 \; , \; \chi_\xi > 0 \; , \; x_Z < x < x_M \; ; \end{aligned} \tag{7.14}$$

$$\text{on } \xi = \xi_M \; , \qquad \chi = 1 \; , \; \chi_\xi = 0 \; . \tag{7.15}$$

In (7.14), the geothermal heat flux parameter $\kappa_1 Gd/kT_0 \lesssim 10^{-1}$ (since $\kappa_1 \lesssim O(1)$) and we therefore neglect it altogether. We do not wish to specify the size of the viscous dissipation, and therefore we formally consider

$$\frac{1}{\beta_2} \to 0 \; , \; \alpha \text{ fixed} \tag{7.16}$$

in the equations and boundary conditions. Neglecting $O(1/\beta_2)$ and the geothermal heat flux, (7.12) becomes

$$\chi_\tau = \chi_{\xi\xi} + \alpha \xi^{n+1} \chi \; , \tag{7.17}$$

with the boundary conditions that

$$\text{on } \xi = 0 , \quad \chi = \chi_A(x) , \quad x < x_M ;$$

$$\text{on } \xi = H , \quad \chi_\xi = \alpha H u_b = \alpha H F[H,\chi] , \quad x < x_Z , \qquad (7.18)$$

$$\chi = 1 , \quad x_Z < x < x_M .$$

The solution of (7.17) and (7.18) includes the unknown H, which must be found from solving the flow problem: in the steady state this is

$$\Psi_{\xi\xi} = \xi^n \chi , \qquad (7.19)$$

with

$$\Psi = s(x) \text{ on } \xi = 0 ,$$

$$\Psi = 0 , \quad \Psi_\xi = - u_b = - F[H,\chi] \text{ on } \xi = H , \qquad (7.20)$$

where χ is given from the solution of (7.17) and (7.18). In the unsteady state, the first of (7.20) is replaced by

$$H_\tau = \frac{1}{\beta_2} \Psi_x \approx 0 . \qquad (7.21)$$

Lastly, if $x > x_M$, then (7.17) is to be solved subject to the first of (7.18), and

$$\chi = 1 , \quad \chi_\xi = 0 \text{ on } \xi = \xi_M , . \qquad (7.22)$$

which therefore determines χ and ξ_M on $x > x_M$.

§2 Explicit Steady State Solutions.

As explained in Chapter II, the sliding law is generally taken to be discontinuous at T = 0 (e.g. Grigoryan et al. 1976), and this implies that there exists a discontinuity in either the stress or the velocity (or both) at the bedrock. We here show by explicit solution that a continuous sliding law like (2.17) leads in the limit as $T_Q \to 0$ to a solution which is continuous and has a large basal zone of almost temperate ice over which the basal velocity increases to the value predicted by the (temperate) sliding law: this 'sub-temperate' zone is directly comparable to the mushy regions encountered in heat transfer problems involving phase transition (Atthey 1972).

We consider first the steady state version of (7.17),

$$\chi_{\xi\xi} + \alpha\xi^{n+1}\chi = 0 \qquad (7.23)$$

with

$$\chi = \chi_A \text{ on } \xi = 0 ,$$

$$\left.\begin{array}{l} \chi_\xi = \alpha H u_b , \quad x < x_Z \\ \chi = 1 , \quad x_Z < x < x_M \end{array}\right\} \text{ on } \xi = H . \qquad (7.24)$$

The two independent solutions of the ordinary differential equation (7.23) are

$$\chi_1(\xi) = \xi^{\frac{1}{2}} J_{-1/(n+3)}[\frac{2\sqrt{\alpha}}{n+3} \xi^{(n+3)/2}]$$

$$= 1 - \frac{\alpha\xi^{n+3}}{(n+2)(n+3)} + \frac{\alpha^2\xi^{2(n+3)}}{2(2n+5)(n+2)(n+3)^2} + \cdots , \qquad (7.25)$$

$$\chi_2(\xi) = \xi^{\frac{1}{2}} J_{1/(n+3)}[\frac{2\sqrt{\alpha}}{n+3} \xi^{(n+3)/2}]$$

$$= \xi - \frac{\alpha\xi^{n+4}}{(n+3)(n+4)} + \frac{\alpha^2\xi^{2n+7}}{2(2n+7)(n+4)(n+3)^2} + \cdots \qquad (7.26)$$

The behaviour of χ_1 and χ_2 is shown in Figure 7.1. $J_{\pm 1/(n+3)}$ are the Bessel functions of the first kind of order $\pm 1/(n+3)$. With $n = 3$, the series (7.25) and (7.26) are very rapidly convergent, and their leading terms will be used subsequently in §5 to give useful represent-

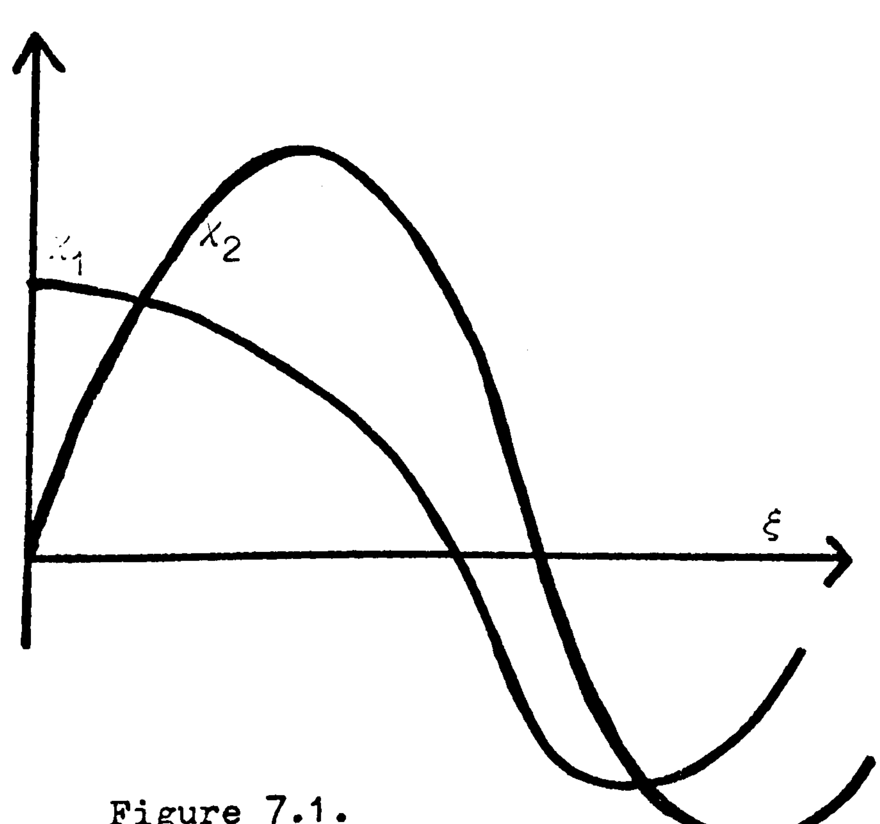


Figure 7.1.

ations of the temperature and flow fields. (This incidentally suggests that solution of the full reduced model as a power series in ξ may be a fruitful approach to adopt.) In what follows, we make use of the properties

$$\chi_1(0) = \chi_2'(0) = 1 \ , \ \chi_1'(0) = \chi_2(0) = 0 \ ,$$
$$\chi_1(\xi)\chi_2'(\xi) - \chi_1'(\xi)\chi_2(\xi) \equiv 1 \ . \tag{7.27}$$

The solution for χ satisfying the first boundary condition in (7.24) is

$$\chi = \chi_A(x)\chi_1(\xi) + A(x)\chi_2(\xi) \ . \tag{7.28}$$

In $x < x_Z$, A must be such that on $\xi = H$,

$$\chi_\xi = \chi_A(x)\chi_1'(H) + A(x)\chi_2'(H) = \alpha u_b H \ ,$$

i.e.

$$A(x) = \frac{[\alpha u_b H - \chi_A(x)\chi_1'(H)]}{\chi_2'(H)} \ , \tag{7.29}$$

and therefore the solution for χ is

$$\chi = \frac{\chi_A(x)}{\chi_2'(H)}[\chi_1(\xi)\chi_2'(H) - \chi_1'(H)\chi_2(\xi)] + \frac{\alpha u_b H \chi_2(\xi)}{\chi_2'(H)} \ ,$$
$$x < x_Z \ , \tag{7.30}$$

where H is yet to be found. In $x_Z < x < x_M$, we have to satisfy $\chi = 1$ on $\xi = H$, and so A must satisfy

$$A = \frac{1 - \chi_A\chi_1(H)}{\chi_2(H)} \ , \tag{7.31}$$

whence the temperature is given by

$$\chi = \frac{\chi_2(\xi)}{\chi_2(H)} + \frac{\chi_A}{\chi_2(H)}[\chi_1(\xi)\chi_2(H) - \chi_1(H)\chi_2(\xi)] \ . \tag{7.32}$$

We may find x_c, x_Z and x_M in terms of H as follows. When $\xi = H$ and

$x = x_Q, u_b = 0$, $\chi = \chi_Q$ ($= \exp[\kappa T_Q]$), and thus, using (7.27),

$$\chi_Q = \frac{\chi_A(x)}{\chi_2'(H)} . \tag{7.33}$$

Similarly on $\xi = H, x = x_Z$, $\chi = 1$ and $u_b = u_b(H)$, and so

$$1 = \frac{\chi_A + \alpha H u_b(H)\chi_2(H)}{\chi_2'(H)} . \tag{7.34}$$

On $x = x_M$, we have $\chi_\xi = 0$, and so from (7.32) and (7.27)

$$\chi_2'(H) = \chi_A(x_M) . \tag{7.35}$$

These equations define x_Q, x_Z and x_M in terms of $H(x_Q), H(x_Z)$ and $H(x_M)$ (or vice versa). To gain further information we must examine the flow field.

The equation (7.19) for Ψ may be written

$$\xi\Psi_{\xi\xi} = \xi^{n+1}\chi = -\frac{1}{\alpha}\chi_{\xi\xi} . \tag{7.36}$$

The first integral of (7.36) is, applying the boundary conditions (7.18) and (7.20) on $\xi = H$,

$$\xi\Psi_\xi - \Psi + \frac{1}{\alpha}\chi_\xi = -Hu_b + Hu_b = 0, \; x < x_Z . \tag{7.37}$$

Putting $\xi = 0$ in (7.37), and using (7.30), (7.27) and (7.20), we obtain

$$s(x) = \frac{1}{\alpha}\chi_\xi\Big|_{\xi=0} = \frac{1}{\chi_2'(H)}[Hu_b - \frac{1}{\alpha}\chi_A\chi_1'(H)] , \; x < x_Z . \tag{7.38}$$

In $x_Z < x < x_M$, we have from (7.32) and (7.27) that

$$\chi'(H) = \frac{\chi_2'(H) - \chi_A}{\chi_2(H)} , \tag{7.39}$$

and thus the first integral of (7.36) is

$$\xi\Psi_\xi - \Psi + \frac{1}{\alpha}\chi_\xi = -Hu_b + \frac{[\chi_2'(H) - \chi_A]}{\chi_2(H)} , \tag{7.40}$$

and putting $\xi = 0$ in (7.40), we have, using (7.40), (7.32) and (7.27),

$$s(x) = Hu_b - \frac{[\chi_2'(H) - \chi_A]}{\alpha\chi_2(H)} + \frac{1}{\alpha}\left[\frac{1}{\chi_2(H)} - \frac{\chi_A\chi_1(H)}{\chi_2(H)}\right]$$

$$= Hu_b + \frac{[1 - \chi_2'(H)] + \chi_A[1 - \chi_1(H)]}{\alpha\chi_2(H)}, \quad x_Z < x < x_M . \qquad (7.41)$$

(7.38) and (7.41) define H(x), and may be used to find x_Q and x_M from (7.33), (7.34) and (7.35).

In particular (7.33) and (7.38) imply that at $x = x_Q$

$$\chi_Q = - \frac{\alpha s(x_Q)}{\chi_1'(H)} , \qquad (7.42)$$

and on x_Z, from (7.34) and (7.38),

$$1 = - \frac{\alpha s(x_Z)}{\chi_1'(H)} + \frac{\alpha Hu_b(H)\chi_2(H)}{\chi_2'(H)} . \qquad (7.43)$$

Now let $\chi_Q \to 1$. We claim $x_Q - x_Z$ does not tend to zero: for if it did, then from (7.38) (since s, χ_1', χ_2' are continuous) $H(x_Q) - H(x_Z) \to 0$ (i.e. H would be continuous). Then as $x_Q \to x_Z$ and $\chi_Q \to 1$, we have from (7.42) and (7.4?) that

$$0 = - \frac{\alpha s(x_Q)}{\chi_1'[H(x_Q)]} - \chi_Q \to - \frac{\alpha s(x_Z)}{\chi_1'\ H(x_Z)} - 1$$

$$= - \frac{\alpha Hu_b(H)\chi_2(H)}{\chi_2'(H)} , \qquad (7.44)$$

which is patently absurd if H > 0. Thus $x_Q \not\to x_Z$ as $\chi_Q \to 1$, and in the limit there exists a finite region of the bedrock on which the ice is (almost) temperate, but the basal velocity is <u>not</u> that predicted by the temperate sliding law. This fundamental physical fact, here predicted analytically, was first pointed out by D.A.Larson in private communication. The existence of such a region of 'sub-temperate' sliding and its effect on the velocity has not previously been considered.

§3 <u>Effective Boundary Condition in $x_Q < x < x_Z$.</u>

The 'real' boundary condition we should impose on $\xi = H, x_Q < x < x_Z$

for the thermal flux is given by the second of (7.24), that is

$$\chi_{\xi} = \alpha H F[H, \chi] \quad . \qquad (7.45)$$

In the limit, this condition is difficult to impose (especially numerically) since it is very delicately dependent on the precise form of F, and hence on a detailed analysis of the sub-temperate sliding law F.

However, since the sub-temperate region does not vanish as $\chi_Q \to 1$, it is relatively insensitive to the precise temperature on the bedrock, and we are therefore led in the limit as $\chi_Q \to 1$ to replace (7.45) by the effective boundary condition

$$\chi = 1 \ , \ x_Q < x < x_Z \ . \qquad (7.46)$$

H is defined by (7.38), and putting $\xi = H$ in (7.30) and letting $\chi_Q \to 1$, we have that u_b is given by

$$u_b = \frac{[\chi_2'(H) - \chi_A(x)]}{\alpha H \chi_2(H)} \ , \ x_Q < x < x_Z \ , \qquad (7.47)$$

and the rôle of (7.45) is now (assuming F is known) to determine the precise magnitude of χ_Q—which is however of no great interest.

Although we have in fact solved the equations here, and do not need to impose (7.46), we shall by analogy define the effective boundary condition

$$\chi = 1 \ , \ 0 < u_b < u_b(H) \text{ in } x_Q < x < x_Z \qquad (7.48)$$

to replace (7.45) even when $\beta_2 \lesssim O(1)$.

§4 Melting Surface.

ξ_M may be found from the temperature solution (7.28). We impose the boundary conditions (7.22) on $\xi = \xi_M$. Then

$$\begin{aligned} \chi_A \chi_1(\xi_M) + a\chi_2(\xi_M) &= 1 \ , \\ \chi_A \chi_1'(\xi_M) + a\chi_2'(\xi_M) &= 0 \ ; \end{aligned} \qquad (7.49)$$

eliminating a and using (7.27), we obtain the implicit definition of ξ_M by the single formula

$$\chi_2'(\xi_M) = \chi_A(x) . \qquad (7.50)$$

Now the Bessel-type functions χ_1 and χ_2 have interlacing zeros as shown in Figure 7.1. We wish to prove that the domain of present interest is such that ξ is less than the first zero of χ_2', say ξ_0 (Figure 7.2).

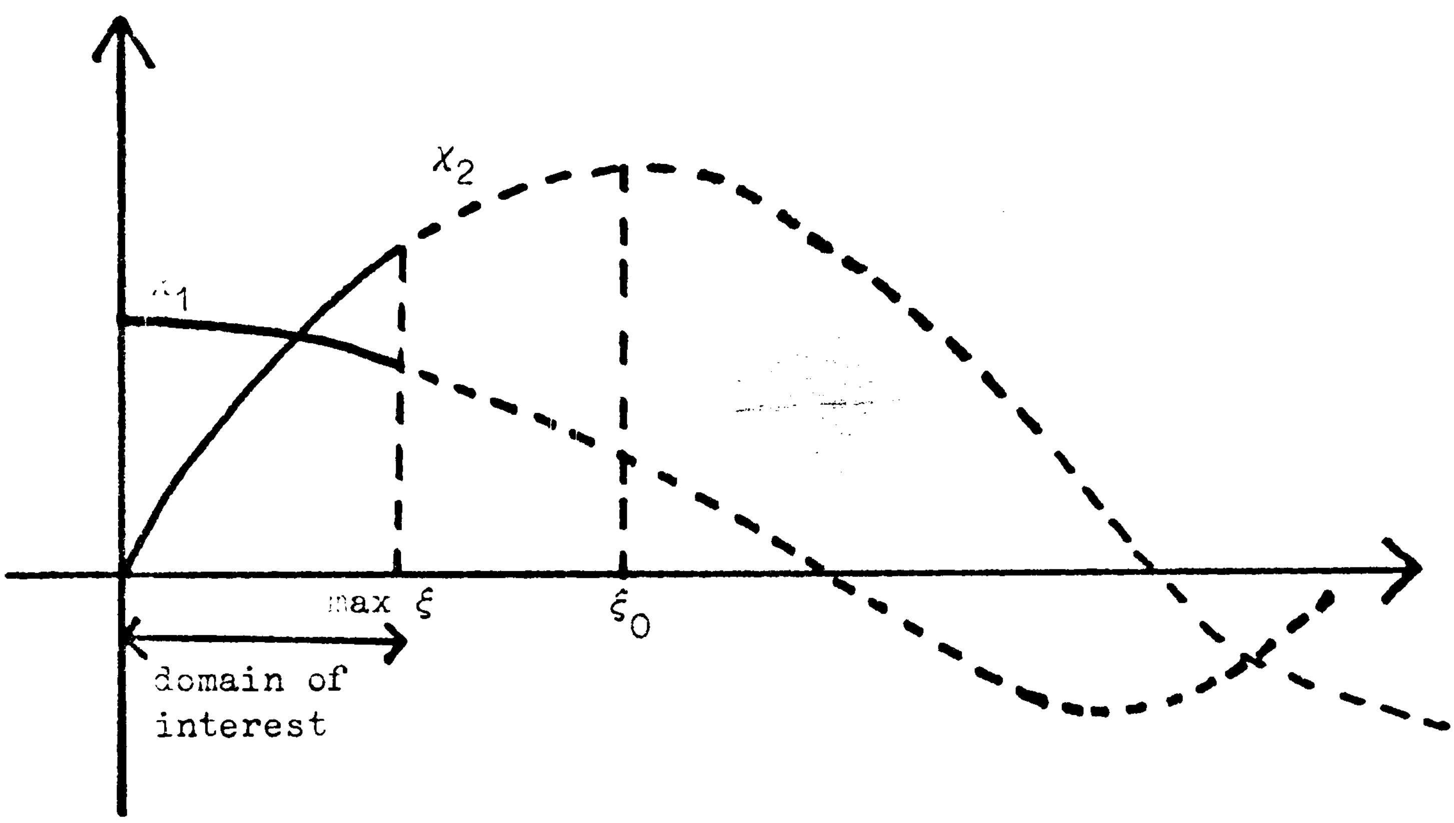


Figure 7.2.

For $x < x_Z$, (7.30) gives, using (7.27),

$$\chi = \frac{1}{\chi_2'(H)}[\chi_A + \alpha u_b H \chi_2(H)] \qquad (7.51)$$

on the bedrock. At $H = 0$, χ is positive and increases to infinity as H tends to ξ_0. Since $\chi \leq 1$, it follows that max $H < \xi_0$, $x < x_M$. In $x_Z < x < x_M$, (7.39) implies that $\chi_2'(H) > \chi_A > 0$, since $\chi_\xi > 0$ on the bedrock. Hence $H < \xi_0$ for $x_Z < x < x_M$. Finally in $x > x_M$, (7.50) implies that $\chi_2'(\xi_M) > 0$, and therefore $\xi_M < \xi_0$. Since $\xi < \xi_M, H$, it follows that $0 < \xi < \xi_0$, and so the domain of interest is such that $\chi_2' > 0$ and $\chi_2 > 0$.

Since χ_2 is monotone in ξ throughout the cold zone, (7.50) defines a unique curve $\xi_M(x)$. If we take the climatic 'temperature' χ_A to be a monotone increasing function of x, it follows from (7.44) that $\xi_M(x)$ is monotone decreasing.

We can qualitatively describe the melting surface profile in the following way. If the top surface is polythermal, i.e. $\chi_A = 1$ for some $x < x_S$, then ξ_M increases until it reaches the bedrock: this is illustrated in (i) of Figure 7.3. If we consider successively cooler surface temperatures for which x_T gradually increases, then we obtain the sequence

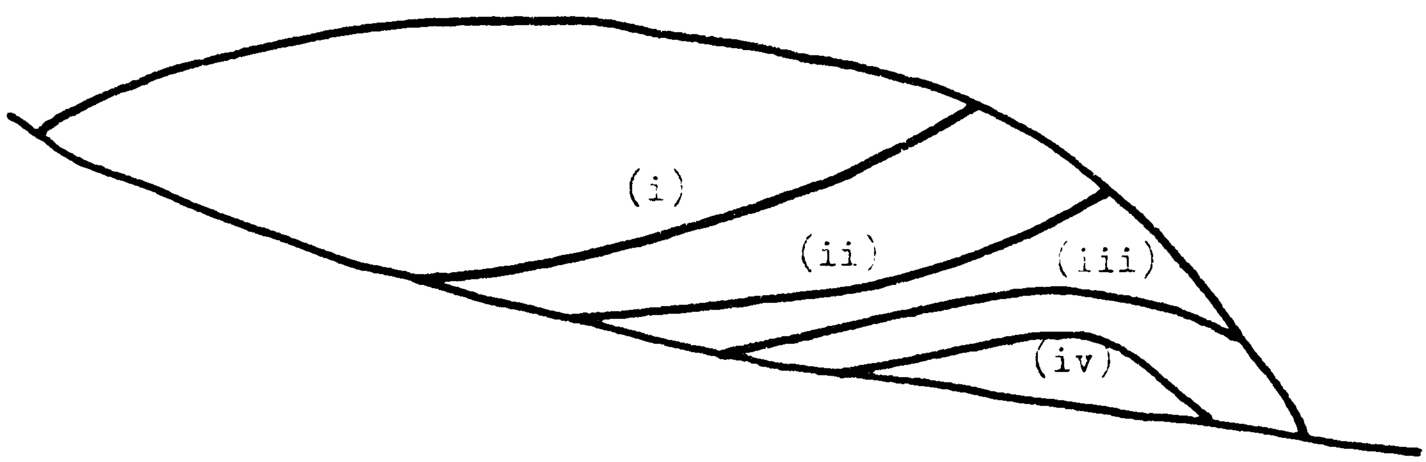


Figure 7.3: Melting Surfaces for polythermal surface ((i)-(iii)) and cold surface (iv).

of profiles (i) to (iv). In (iii) and (iv) the melting surface $\xi_M(x)$ is still monotone decreasing, although the function $y_M = \eta - \xi_M$ is not. In (iv) the surface is completely cold, but a basal zone of temperate ice exists. We can describe this dependence of the melting surface geometry on χ_A and α as follows.

The solutions for the depth H and melting surface are, from (7.38), (7.47), (7.27), (7.32), (7.41) and (7.39)

$$s(x) = -\frac{\chi_A \chi_1'(H_C)}{\alpha \chi_2'(H_C)}, \quad x < x_Q, \qquad (7.52)$$

$$s(x) = \frac{1}{\chi_2'(H_T)} \left[\frac{\chi_2'(H_T) - \chi_A(x)}{\alpha \chi_2(H_T)} - \frac{\chi_A(x)\chi_1'(H_T)}{\alpha} \right]$$

$$= \frac{1 - \chi_A(x)\chi_1(H_T)}{\alpha \chi_2(H_T)}, \quad x_Q < x < x_Z, \qquad (7.53)$$

$$s(x) = H_T u_b(H_T) + \frac{[1 - \chi_2'(H_T)] + \chi_A(x)[1 - \chi_1(H_T)]}{\alpha \chi_2(H_T)} ,$$

$$x_Z < x < x_M . \qquad (7.54)$$

The suffices C and T are appended to denote cold ~~(or sub-temperate)~~ and temperate (or sub-temperate) ice, respectively. We also define H_M by

$$\chi_2'(H_M) = \chi_A(x) . \qquad (7.55)$$

Thus $H = H_M$ at $x = x_Q$ (from (7.33)) and also the melting surface is given by $\xi_M = H_M$, $x > x_M$ using (7.50). If we assume s is concave ($s'' < 0$) and χ_A is monotone increasing, then H_M is monotone decreasing and one may realistically suppose that H_C as given by (7.52) is convex provided χ_A' is not too large: we take this to be the case. Then typical forms of H_C and H_M are given in Figure 7.4. The curves are to be interpreted

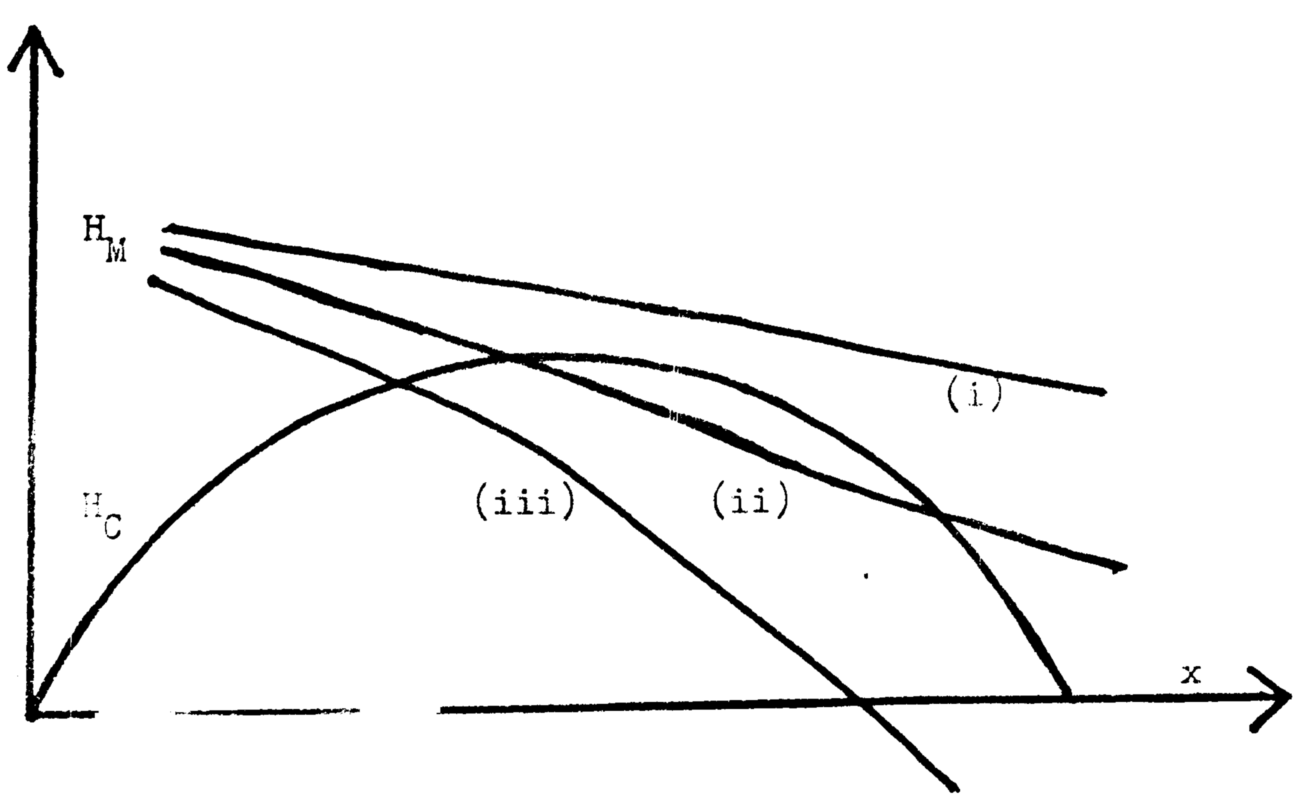


Figure 7.4.

as follows. The intersections of H_C and H_M are the possible points x_Q where sliding is just beginning (or just ending). Thus in case (i) there is no intersection, and so the bedrock is uniformly cold ($x < x_Q$) and no sliding occurs. If there is an intersection, then either case (ii) holds, i.e. H_M remains positive which implies that $\chi_A < 1$ (the top surface is cold) or (iii) holds, in which case the top surface is polythermal, and $\chi = 1$ for $x > x_T$.

Now when H_C first intersects H_M, the solution for the depth becomes H_T along which $\chi_2' > \chi_A$ since $\chi' > 0$ on $\xi = H$, and using (7.39). One easily sees that $\chi_2' > \chi_A$ corresponds to $H < H_M$, and thus for $x > x_Q$ the H_C curve must lie initially underneath that for H_M. Furthermore, using (7.27) and by inspection of (7.53), we can see that H_T is ~~zero precisely~~ finite when s(x) is zero. A typical geometry is shown in Figure 7.5. We see that

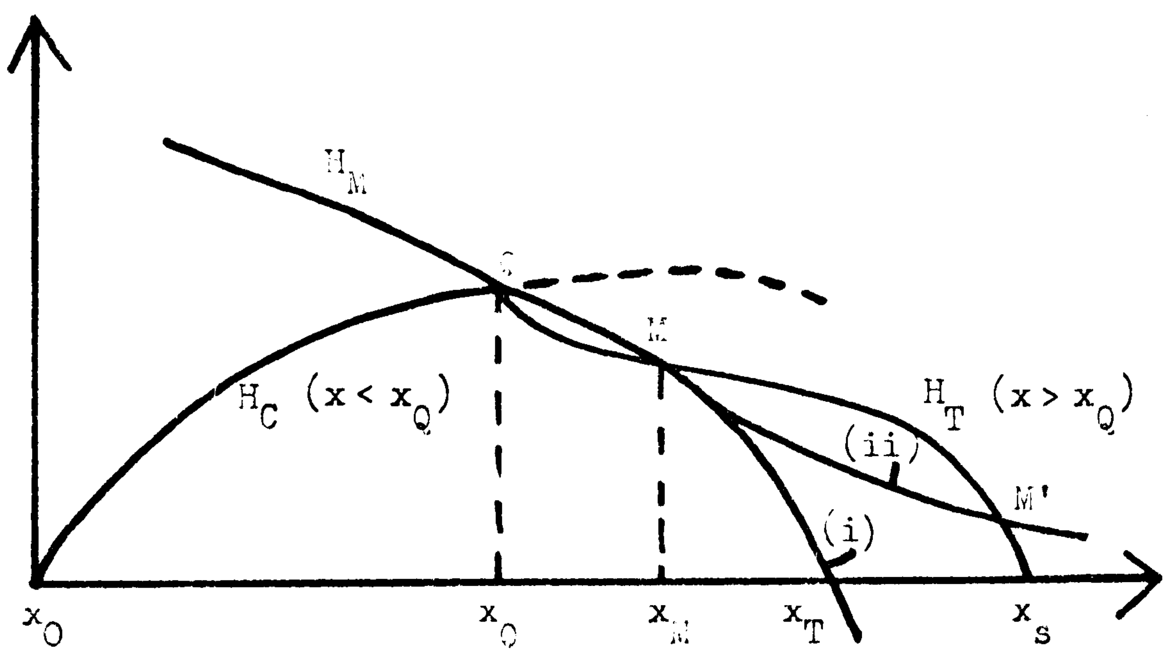


Figure 7.5.

H_C intersects H_M at Q $(x = x_Q)$: the depth then follows the curve H_T, which is initially beneath H_M. If H_T then intersects H_M at a point M where $H_T \neq H_C$, the melting surface breaks off the bedrock at $x = x_M$ and is described by $\xi_M = H_M$ as long as $H_M < H_T$. In (i) this is until $\xi_M = 0$, and hence corresponds to (i)-(iii) of Figure 7.3, whereas the curve marked (ii) reintersects H_T at M', i.e. ξ_M is as shown in Figure 7.3 (iv), and in this case a basal region of temperate ice exists beneath an otherwise cold glacier.

Further, more complicated situations can obviously occur in which a whole series of temperate regions exist on the bedrock: this depends critically on the precise forms of χ_A and s.

We can see in Figure (iv) that the sequence of H_M curves (i)-(iii) corresponds to increasing χ_A or α. It is of interest to find the critical value of α (for given χ_A and s) at which we pass from (i) to (ii), i.e. at which basal sliding begins to occur, and at which a temperate zone may start to appear. This critical value α_0 is given parametrically by the condition that the equation $H_M = H_C$ should have a double root at $x_Q(\alpha_0)$:

$$H_M - H_C = H_M' - H_C' = 0 \quad . \tag{7.56}$$

An approximate result is given in the next section.

§5 Approximate Results.

We here collect the main results we have found so far, and then give approximate formulae for them based on the series expansions (7.25) and (7.26) for χ_1 and χ_2. These formulae provide detailed relations between the surface temperature, flux, velocity and depth of a glacier. A comparison with observations in the field would be of considerable interest.

The temperature field is described by (7.30) and (7.32), that is

$$\chi = \frac{\chi_A[\chi_1(\xi)\chi_2'(H) - \chi_1'(H)\chi_2(\xi)] + \alpha u_b H\chi_2(\xi)}{\chi_2'(H)}, \quad x < x_Z ,$$

$$\chi = \frac{\chi_2(\xi)}{\chi_2(H)} + \frac{\chi_A}{\chi_2(H)}[\chi_1(\xi)\chi_2(H) - \chi_1(H)\chi_2(\xi)] , \quad x_Z < x < x_M . \tag{7.57}$$

The flow field is given by integrating (7.37) whence we obtain

$$\Psi = \frac{\xi^2}{\alpha}\int_\xi^H \left(\frac{\chi_\xi}{\xi^2}\right) d\xi, \quad x < x_Z ,$$

$$\Psi = \frac{\xi^2}{\alpha}\int_\xi^H \left(\frac{\chi_\xi}{\xi^2}\right) d\xi + [Hu_b - \frac{\{\chi_2'(H) - \chi_A\}}{\alpha\chi_2}]\,\xi(H-\xi) , \quad x_Z < x < x_M . \tag{7.58}$$

In (7.57) and (7.58), the depth H is defined by (7.52)-(7.54), that is

$$s(x) = -\frac{\chi_A\chi_1'(H)}{\alpha\chi_2'(H)} , \quad x < x_Q , \tag{7.59}$$

$$s(x) = \frac{1}{\alpha\chi_2}[1 - \chi_A\chi_1] , \; x_Q < x < x_Z , \tag{7.60}$$

$$s(x) = Hu_b(H) + \frac{(1-\chi_2') + \chi_A(1-\chi_1)}{\alpha\chi_2} , \; x_Z < x < x_M . \tag{7.61}$$

It is clear from the behaviour of χ_1 and χ_2 that (7.59) defines H uniquely, and the same is easily shown to be true of (7.60) and (7.61), since the assumption that $\frac{dx}{dH} = 0$ (a necessary precondition that H should be multivalued) leads to a contradiction.

From (7.57) and (7.27), the bedrock temperature in $x < x_Q$ is

$$\chi = \frac{\chi_A(x)}{\chi_2'(H)} , \tag{7.62}$$

and so x_Q is defined to be where

$$\chi_A(x_Q) = \chi_2'(H) \tag{7.63}$$

and also (7.59) holds ($H = H_C$ there).

In $x_Q < x < x_Z$, the sliding velocity is given by

$$u_b = \frac{\chi_2'(H) - \chi_A}{\alpha H} : \tag{7.64}$$

x_Z is then simply defined to be where $u_b = u_b(H)$.

The bottom melting point x_M is given, like x_Q, by

$$\chi_2'(H) = \chi_A(x_M) , \tag{7.65}$$

but we require that (7.61) should also hold, i.e. $H = H_T$ there.

We now consider approximate solutions for (7.59)-(7.65). From (7.25) and (7.26), we have to an adequate approximation

$$\begin{aligned} \chi_1 &\simeq 1 - \frac{\alpha\xi^{n+3}}{(n+2)(n+3)} , \\ \chi_2 &\simeq \xi[1 - \frac{\alpha\xi^{n+3}}{(n+3)(n+4)}] , \\ -\chi_1' &\simeq \alpha\xi^{n+2}/(n+2) , \\ \chi_2' &\simeq 1 - \alpha\xi^{n+3}/(n+3) , \end{aligned} \tag{7.66}$$

and therefore we have from (7.59)-(7.61),

$$s(x)[1 - \frac{\alpha H^{n+3}}{n+3}] = \frac{\chi_A H^{n+2}}{n+2}, \quad x < x_Q, \tag{7.67}$$

$$s(x) = \frac{[(1-\chi_A) + \frac{\alpha\chi_A H^{n+3}}{(n+2)(n+3)}]}{\alpha H[1 - \frac{\alpha H^{n+3}}{(n+3)(n+4)}]}, \quad x_Q < x < x_Z,$$

or more simply

$$H \simeq \frac{(1-\chi_A)}{\alpha s(x)}, \quad x_Q < x < x_Z, \tag{7.68}$$

and

$$s(x) = Hu_b(H) + \frac{\frac{\alpha^2 H^{n+3}}{n+3} + \chi_A \cdot \frac{\alpha^2 H^{n+3}}{(n+2)(n+3)}}{\alpha^2 H[1 - \frac{\alpha^2 H^{n+3}}{(n+3)(n+4)}]}$$

$$\simeq Hu_b(H) + \frac{H^{n+2}}{n+2}, \quad x_Z < x < x_M. \tag{7.69}$$

The bedrock temperature in $x < x_Q$ is given from (7.62) by

$$\chi \simeq \frac{\chi_A}{1 - \frac{\alpha H^{n+3}}{n+3}}, \tag{7.70}$$

and x_Q is given by

$$\chi_A(x_Q) \simeq 1 - \frac{\alpha H^{n+3}}{n+3}. \tag{7.71}$$

In $x_Q < x < x_Z$ the sliding velocity is

$$u_b = \frac{(1-\chi_A) - \frac{\alpha H^{n+3}}{n+3}}{\alpha H}. \tag{7.72}$$

The bottom melting point x_M is the solution of

$$1 - \frac{\alpha H^{n+3}}{n+3} = \chi_A(x_M), \tag{7.73}$$

where $H(x_M)$ is given by (7.69).

Finally, α_0 can be determined by the requirement (7.56), i.e. that

x_Q is a double root of (7.67) and (7.71). x_Q is thus found by solving

$$s(x)\left[1 - \frac{\alpha H^{n+3}}{n+3}\right] = \frac{\chi_A H^{n+2}}{n+2} ,$$
$$1 - \frac{\alpha H^{n+3}}{n+3} = \chi_A , \tag{7.74}$$

whence approximately

$$\frac{H^{n+2}}{n+2} \approx s(x) \tag{7.75}$$

defines $H(x_Q)$, and x_Q is then defined from

$$1 - \chi_A(x_Q) = \frac{\alpha}{n+3}\left[\{n+2\}s(x_Q)\right]^{\frac{n+3}{n+2}} . \tag{7.76}$$

α_0 is such that x_Q is a double root of (7.76). We immediately see that $s'(x_Q) < 0$ since $\chi_A' > 0$, and so x_Q lies in the ablation area. For example, if (taking the origin at x_E)

$$\chi_A \approx a + cx , \quad \frac{[(n+2)s(x)]^{(n+3)/(n+2)}}{(n+3)} \approx 1 - x^2 , \tag{7.77}$$

then α is found by solving

$$(1 - a) - cx = \alpha(1 - x^2) ,$$
$$- c = - 2\alpha x , \tag{7.78}$$

which imply

$$x_Q = \frac{(1-a) - [(1-a)^2 - c^2]^{\frac{1}{2}}}{c} . \tag{7.79}$$

We choose the minus sign in (7.79) because we require $1 - a > c$ for real x_Q (which implies that the top surface is cold) but also $x_Q < 1$ as this is the relevant value.

From (7.76) and (7.79)

$$\alpha = \frac{c^2}{[(1-a) - \{(1-a)^2 - c^2\}^{\frac{1}{2}}]} . \tag{7.80}$$

A typical result would be for a surface temperature for which $a = \frac{1}{2}$, $c = \sqrt{5}/6 \simeq 0.37$: in this case $\alpha = 5/6$ and $x_Q = \sqrt{5}/10 = 0.2236$, and the sliding zone starts to appear about a fifth of the way along the ablation zone from the equilibrium point towards the snout.

§6 Linear Stability.

In this last section we consider the linear stability of the steady state solutions described in the preceding sections. We put

$$\Psi = \Psi_0 + \Psi_1 e^{\sigma t}, \quad \chi = \chi_0 + \chi_1 e^{\sigma t}, \qquad H = H_0 + H_1 e^{\sigma t}, \quad \xi_M = \xi_{M0} + \xi_{M1} e^{\sigma t}, \tag{7.81}$$

where 0 and 1 refer to steady state and perturbed quantities respectively, and σ is a (complex) constant to be determined. The perturbations are assumed to be sufficiently small that a linear analysis is applicable, but in order that we do not consider here the 'perturbed' stability problem involving terms of $O(1/\beta_2)$, we formally specify that

$$1/\beta_2 \ll \Psi_1, \; \chi_1, \; H_1, \; \xi_{M1} \ll 1 . \tag{7.82}$$

As before, we therefore neglect $1/\beta_2$ altogether in (7.11)-(7.15). The first equation in (7.14) then immediately gives

$$H_1 = 0 \tag{7.83}$$

(any given initial perturbation in H is simply absorbed into H_0). Thus the solution for the temperature once more uncouples from that for Ψ, and the equation describing the perturbed 'temperature' may be written, on dropping the suffix 1,

$$\chi_{yy} + \delta^2 \chi_{xx} + [\alpha(\eta - y)^{n+1} - \sigma] = 0 \tag{7.84}$$

with boundary conditions

$$\chi = 0 \text{ on } y = \eta \equiv h + H_0 + H_1 , \tag{7.85}$$

$$\chi_y = \delta^2 h' \chi_x \text{ on } y = 0 \,, \tag{7.86}$$

using (2.73), and

$$\chi = 0 \text{ on } y = y_{MO} \,, \text{ i.e. } \xi = \xi_{MO} \,, \tag{7.87}$$

the last of which comes from expanding the first of (7.15) about ξ_{MO}. The melting surface perturbation ξ_{M1} also uncouples from the problem: it is described by expanding the second condition in (7.15), whence we find

$$\xi_{M1} = - \left. \frac{\chi_\xi}{\chi_{0\xi\xi}} \right|_{\xi = \xi_{MO}} = \frac{\chi_\xi |_{\xi = \xi_{MO}}}{\alpha \xi_{MO}^{n+1}} \,, \tag{7.88}$$

using (7.15) and (7.23). If there exist non-trivial solutions of (7.84)-(7.87) for which Re(σ) > 0, then the steady state solution of (7.11)-(7.15) is said to be linearly unstable: otherwise it is linearly stable.

The reason we have reintroduced the longitudinal diffusion term (and therefore (x,y) coordinates) in (7.84) is as follows. (7.84)-(7.87) constitute an eigenvalue problem for σ, and standard theorems (e.g. Courant and Hilbert 1953) guarantee the existence of a decreasing sequence $\{\sigma_n\}$ of real eigenvalues such that $\sigma_n \to -\infty$ as $n \to \infty$. If, however, we remove the term in δ^2 from (7.84), then it is essentially an ordinary differential equation and so, although the eigenvalues still exist, they are now functions of x, $\{\sigma_n(x)\}$. While such solutions are physically meaningful as long as $\sigma_n(x) < 0$, this is no longer so when the maximum eigenvalue $\sigma_0(x)$ becomes positive for some x, as it implies that disturbances may grow in some parts of the glacier, and decay in others! In such a case, x derivatives become large at points where $\sigma = 0$, and thus it is necessary to retain the term in δ^2 in order to cope with such large derivatives.

If we define the volume of cold ice to be V and its outer surface ($\xi = 0$, $\xi = \xi_M$, $\xi = H$) to be S, then one easily derives from a variational formulation of (7.84) (e.g. Morse and Feshbach 1953) that the highest (most unstable) eigenvalue σ satisfies the bound

$$\sigma \geqslant \frac{\int_V [\alpha\{\eta - y\}^{n+1}\phi^2 - \{\phi_y{}^2 + \delta^2\phi_x{}^2\}]\, dV}{\int_V \phi^2\, dV}, \qquad (7.89)$$

where ϕ is any (non-zero) function which is continuous in V and has piecewise continuous first derivatives in V + S, and

$$\phi = 0 \text{ on } \xi = 0 \text{ and } \xi = \xi_{MO}\,. \qquad (7.90)$$

Equality in (7.89) is obtained when ϕ is the corresponding eigenfunction and satisfies the equations and boundary conditions (7.84)-(7.87).

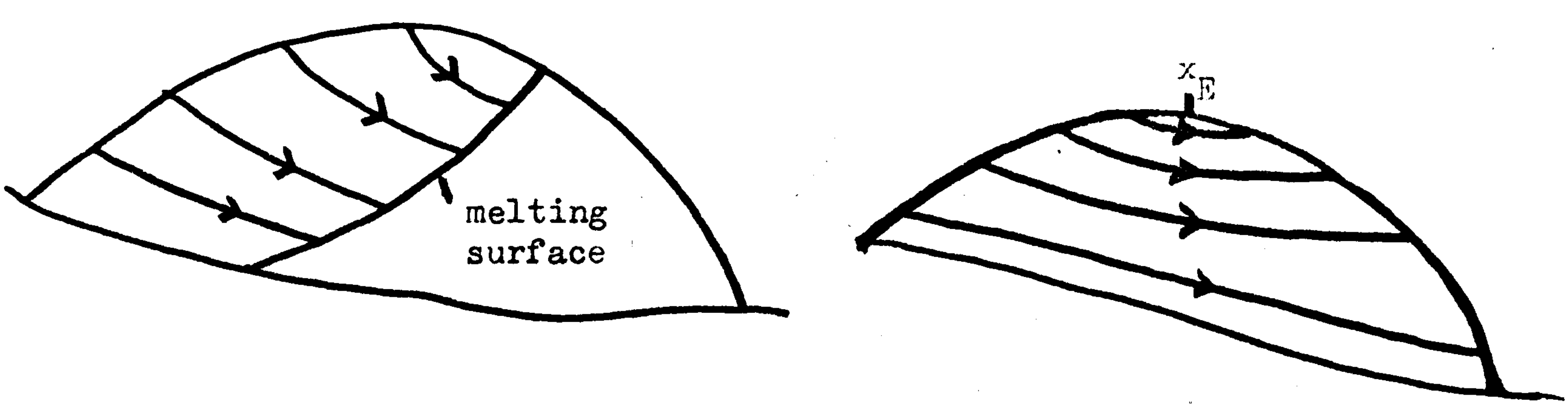
When $\alpha = 0$, i.e. $\kappa = 0$ (the viscosity is independent of temperature), it is obvious from (7.89) that the highest eigenvalue is negative (since (7.89) is an equality when ϕ is equal to the eigenfunction): thus the temperature field is stable at $\alpha = 0$. As α is increased, the highest eigenvalue σ will become zero for some $\alpha = \alpha_c$, say, and be positive in $\alpha > \alpha_c$. At $\alpha = \alpha_c$ the temperature field is said to be marginally stable: it follows from standard theory (Courant and Hilbert, op. cit.) that σ is negative for all $\alpha < \alpha_c$: it similarly follows from a consideration of (7.89) that $\sigma > 0$ for all $\alpha > \alpha_c$, and hence α_c is the <u>critical stability parameter</u>. (7.89) can then be used to obtain an upper bound for α_c: this is done using a simple trial function in Appendix 4. We find that the estimate is too crude to be of any direct use, as it gives an upper bound for α_c in terms of H which cannot be attained by the α corresponding to the steady solution. The form of the bound suggests that increasing H may have a destabilising effect on the steady solution, whereas the 'high' value of n in Glen's power law would seem to be a stabilising feature.

CHAPTER VIII

Small Conduction Limit: $\beta_2 << 1$

In the previous chapter we examined the case of large conduction ($\beta_2 >> 1$) and found that the steady state solution could be uniformly approximated by the leading order term in an asymptotic expansion in $1/\beta_2$. In the present chapter we turn our attention to the steady state in the limit of small conduction ($\beta_2 << 1$), which is expected to be a more physically realistic approximation. Since neglect of β_2 in this case is associated with the removal of the highest derivative, the problem is of singular perturbation type and we therefore expect thermal boundary layers to exist in certain regions of the flow (Cole 1968).

The outer flow problem for the temperature, (2.107) with $\beta_2 = 0$, is hyperbolic: in Cole's terminology the streamlines Ψ = constant are subcharacteristics of the outer problem; they begin on the top surface and terminate either on the melting surface (for a polythermal glacier), or on the top surface in the ablation zone $x > x_E$ (for a cold glacier): see Figure 8.1.



(a): polythermal glacier. (b): cold glacier.

Figure 8.1: subcharacteristics.

As discussed by Cole, we expect boundary layers to occur where the subcharacteristics terminate, and also on the bedrock, which is itself the subcharacteristic $\Psi = 0$: no boundary layer will exist on $\xi = 0$, $x < x_E$, since such a layer would not be able to satisfy the matching condition into the mainstream flow. (We are here neglecting temporal effects such as the winter 'cold wave' due to the seasonal variation in the surface temperature.)

Now let us consider the boundary layer on the melting surface. The outer solution for T cannot satisfy both conditions $T = T_\xi = 0$ on $\xi = \xi_M$, and we shall require a boundary layer to exist there. However, it is easy to see that the condition that the outer solution 'loses' is $T_\xi = 0$, and thus the boundary layer provides an O(1) correction to the temperature gradient, but only a small correction to T. In other words, for small β_2 the melting surface is given to first order by the single condition T = 0 applied to the equations with $\beta_2 = 0$. Since we are not initially interested in higher order approximations, we may use this as an effective boundary condition on the melting surface: the stream function is unaffected (to first order) by the boundary layer.

In a similar fashion, the boundary layer on the top surface $\xi = 0$, $x > x_E$ of a cold glacier gives a local correction to the temperature field, but does not affect the stream function to first order. As we are interested in global properties of the flow and temperature fields, we can ignore this boundary layer by simply dropping the surface temperature condition in $x > x_E$. Then, writing as before

$$\chi = 1 + \kappa_1 T \simeq e^{\kappa T}, \quad \beta = \kappa_1 \beta_1, \quad \mu = 0, \tag{8.1}$$

the first order outer flow problem ($\beta_2 = 0$) in the cold zone may be written, from (2.105)-(2.107),

$$\Psi_{\xi\xi} = \xi^n \chi, \tag{8.2}$$

$$\Psi_x \chi_\xi - \Psi_\xi \chi_x = \beta \xi^{n+1} \chi, \tag{8.3}$$

and the effective boundary conditions we impose are, from the above considerations, on $\xi = 0$:

$$\Psi = s(x) \,, \qquad \chi = \chi_A(x) \,,\ x < x_E \,; \tag{8.4}$$

on $\xi = \xi_M(x)$:

$$\chi = 1 \,; \tag{8.5}$$

on $\xi = H(x)$:

$$\Psi = 0 \,, \qquad \Psi_\xi = -\, u_b \,, \tag{8.6}$$

where u_b is determined from the solution in the boundary layer. The remaining boundary conditions on $\xi = H$ are discussed below.

We can consider the bedrock to be composed of four kinds of region: $(x_0,x_Q),(x_Q,x_Z),(x_Z,x_M),x > x_M$, characterised by the following physical descriptions, based on the discussion in §3 of the previous chapter.

(a) $x < x_Q$. The ice is cold and frozen to the rock, so that the no-slip condition is satisfied.

(b) $x_Q < x < x_Z$. The ice is almost at the melting point and has started to slide, though at less than the speed given by the full temperate sliding law: the heat generated by this sliding is conducted into the cold ice above.

(c) $x_Z < x < x_M$. The basal ice reaches the melting point and the full sliding law is operative. The heat conducted into the ice decreases to zero as the melting surface rises through the bedrock flow layer (cf. Appendix 2).

(d) $x > x_M$. The melting surface is 'far' from the (rough) bedrock flow layer, but 'near' the (smooth) bedrock of the mainstream flow (see §§3,4, Chapter II). Thus the melting surface breaks off the bedrock at x_M, and the viscous heating term determines the basal moisture (which does not 'diffuse' upwards, since we are not considering any 'conduction' of the moisture due to the hydrology) - see Appendix 2.

Now from the considerations of the previous chapter, we see that though for example u_b is formally prescribed as a function of χ and H in the region $x_Q < x < x_Z$, the <u>effective</u> boundary condition as $T_Q \to 0$ is that $\chi = 1$, and u_b is then determined from the solution for the temperature field. Thus we prescribe the following effective boundary conditions on the bedrock $x < x_M$ in place of (2.111)-(2.113), in the limit of negligible geothermal heat flux and negligible T_c:

on $x < x_Q$:

$$\Psi = \Psi_\xi = \chi_\xi = 0 \ , \ \chi < 1 \tag{8.7}$$

until $\chi = 1$ $(x = x_Q)$, then

on $x_Q < x < x_Z$:

$$\Psi = 0 \ , \ \chi = 1 \ , \ \beta H u_b = \beta_2 \chi_\xi \ , \ u_b < u_b(H) \tag{8.8}$$

until $u_b = u_b(H)$, $(x = x_Z)$, then

on $x > x_Z$:

$$\Psi = 0 \ , \ \chi = 1 \ , \ u_b = u_b(H) \ , \ \beta_2 \chi_\xi < \beta H u_b \tag{8.9}$$

until $\chi_\xi = 0$ $(x = x_M)$, at which point the melting surface ξ_M breaks away from the bedrock. The boundary conditions (8.7)-(8.9) are illustrated in Figure 8.2.

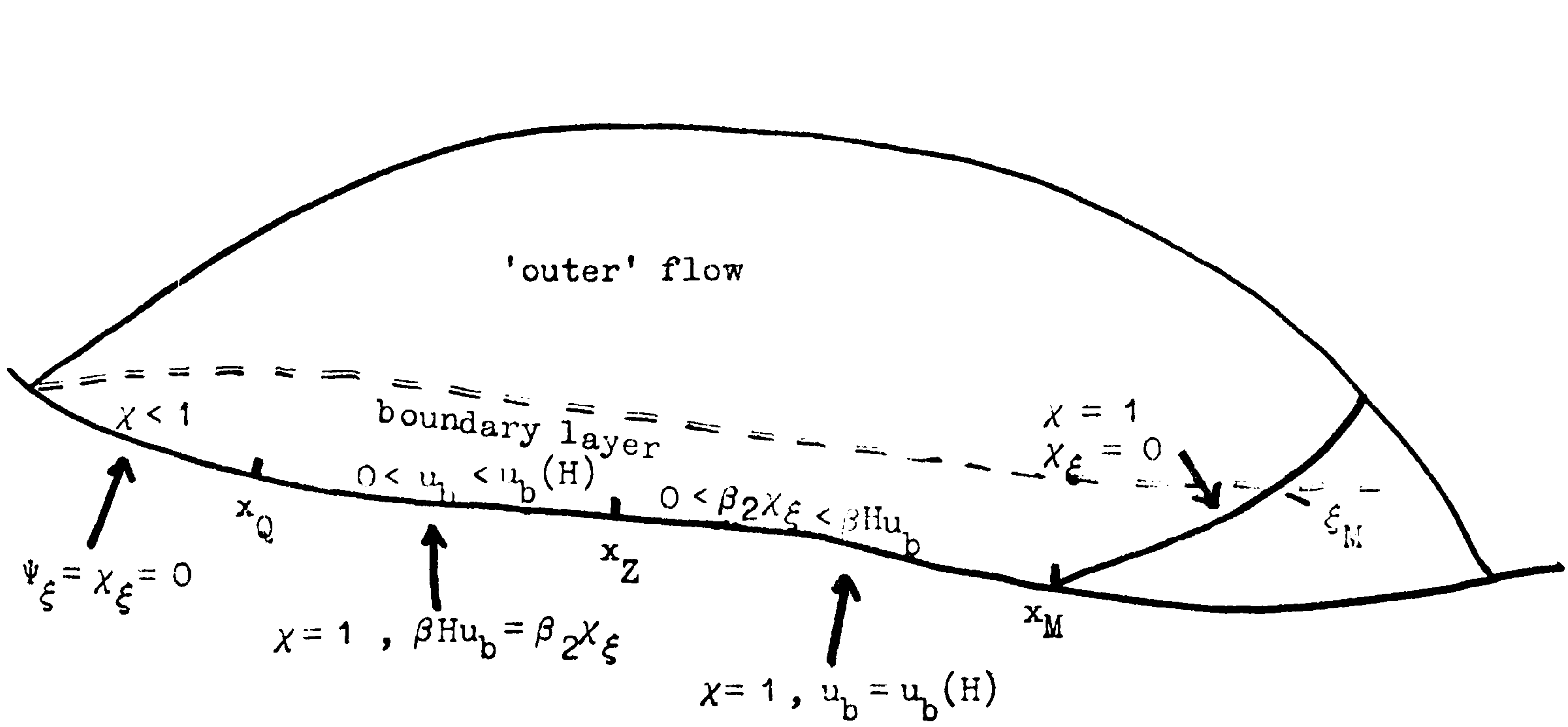


Figure 8.2: Effective Bedrock Boundary Conditions (and $\Psi = 0$).

The outer solution of (8.2)-(8.6) cannot be given until H and u_b are known, and to find these we have to solve the boundary layer equations. In this chapter we show how to bring the conduction term back into (8.3) by appropriately rescaling the equations near the bedrock. In this manner we are in principle able to determine the positions of x_Q and x_M.

It is worth emphasising that an analytical description of these points is vitally necessary to our understanding of the dynamics of glacier flow. For example, if $x_Q < x < x_Z$, the sliding velocity will not exhibit any obvious dependence on the basal stress (i.e. the depth), since the temperature is slightly cold. Such a lack of dependence has been noted by Meier and Post (quoted by Hodge 1974): however, it may not be a straightforward matter to ascertain in the field whether this is due to slightly cold basal ice; if T_Q is sufficiently near T_M, it may be very difficult indeed.

The analytical aid to the numerical solution of the reduced model is clearly very important.

§1 Boundary Layer Equations.

It is apparent that the outer flow solution of (8.2)-(8.6) cannot satisfy any of the boundary conditions (8.7)-(8.9). In fact, when $x < x_Q$ the convective terms Ψ_ξ and Ψ_x tend to zero as $\Psi \to 0$, and so from (8.3) χ must become infinite: this reflects the fact that the dissipative heating does not vanish at the bedrock, whereas the heat transport mechanism (convection) does. Thus a boundary layer analysis is essential, and this will show that the heat generated by viscosity must be transported away by conduction.

In rescaling the variables to consider the boundary layer, we must bring back the conduction term to balance the dissipation. Since the convective term is small, we might suppose that it does not appear to first order in the scaled temperature equation. However in this case it is found that the inner solution cannot match with the outer solution, and hence it is necessary to balance the convection term with the other two in the equation. (This result is also obtained when χ is replaced by the exponential form $e^{\kappa T}$.) In order to achieve such a double balance, we have to scale both ξ <u>and</u> x in the equations. We do this as follows.

Let us define $x_0 = 0$ for convenience, and introduce

$$x = \varepsilon x^* \ , \ \xi = \varepsilon^{1/(n+2)}\xi^* \ , \ \Psi = \varepsilon\Psi^* \ , \tag{8.11}$$

where ε here is nothing to do with the bedrock slope as defined in Chapter II, and is as yet unspecified. Using (8.11) and (8.1), the equations (2.105)-(2.107) become, on dropping the asterisks for convenience,

$$\Psi_{\xi\xi} = \xi^n\chi \ , \tag{8.12}$$

$$\Psi_x\chi_\xi - \Psi_\xi\chi_x = \beta\varepsilon\xi^{n+1}\chi + \beta_2\varepsilon^{-1/(n+2)}\chi_{\xi\xi} + O(\beta_2\delta^2\varepsilon^{-1/(n+2)}\varepsilon^{-2}) \ , \tag{8.13}$$

where the order of magnitude term represents the longitudinal diffusion present in (2.49): it is included in (8.13) so that we may ensure it is indeed negligible under the rescalings we introduce.

We now seek to define a thermal boundary layer by writing

$$Y = \frac{H-\xi}{\omega} \ , \ \omega \ll 1 \ , \tag{8.14}$$

so that

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \frac{1}{\omega}H'\frac{\partial}{\partial Y} \ , \quad \frac{\partial}{\partial \xi} = -\frac{1}{\omega}\frac{\partial}{\partial Y} : \tag{8.15}$$

ω is to be determined by balancing appropriate terms. Since $\Psi = \Psi_Y = 0$ on the bedrock in $x < x_Q$, we scale Ψ by writing

$$\Psi = \omega^2\psi \ ; \tag{8.16}$$

then to leading order the equations (8.12) and (8.13) become

$$\psi_{YY} = H^n\chi \ , \tag{8.17}$$

$$\omega[\psi_Y\chi_x - \psi_x\chi_Y] = \beta\varepsilon H^{n+1}\chi + \beta_2\omega^{-2}\varepsilon^{-1/(n+2)}[\chi_{YY} + O(\delta^2\varepsilon^{-2(\frac{n+1}{n+2})})] \ , \tag{8.18}$$

with an error of $O(\omega)$ in (8.17) and $O(\omega^2)$ in (8.18).

In order to obtain a balance in the energy equation (8.18),we define ε and ω by

$$\omega = \beta\varepsilon = \beta_2\varepsilon^{-1/(n+2)}\omega^{-2} , \qquad (8.19)$$

assuming that $\delta << \varepsilon^{(n+1)/(n+2)}$. Solving for ω and ε ,we find

$$\varepsilon = \left(\frac{\beta_2}{\beta^3}\right)^{\left(\frac{n+2}{3n+7}\right)} \qquad (8.20)$$

and

$$\omega = [\beta\beta_2{}^{n+2}]^{1/(3n+7)} . \qquad (8.21)$$

Our assumption that $\delta << \varepsilon^{(n+1)/(n+2)}$ thus requires

$$\delta << \left(\frac{\beta_2}{\beta^3}\right)^{\left(\frac{n+1}{3n+7}\right)} \sim \left(\frac{\beta_2}{\beta^3}\right)^{\frac{1}{4}} \qquad (8.22)$$

for $n = 3$;(8.22) is a valid approximation in practical cases. The scales ε and ω define the extent of the boundary layer:thus in order that (8.17) and (8.18) should provide an accurate approximation,we require $\omega << 1$:this is formally true as $\beta_2 \to 0$,although even if $\beta_2 \approx 1/30$, $\beta = 1$, we only have that $\omega \approx 1/3$. The present analysis is therefore more qualitative than quantitative,but we shall proceed with the formal assumption that $\omega << 1$. The boundary layer equations are then,from (8.17) and (8.18) and with error $O(\omega)$,

$$\psi_{YY} = H^n\chi , \qquad (8.23)$$

$$\psi_Y\chi_X - \psi_X\chi_Y = H^{n+1}\chi + \chi_{YY} . \qquad (8.24)$$

The basal velocity in ε-scale variables is,from (8.11)

$$-\Psi_\xi = \varepsilon^{-(n+1)/(n+2)}u_b = u_b^{\dagger} , \qquad (8.2\ )$$

where u_b refers to the original variables. Since $-\Psi_\xi = \omega\psi_Y$,we are led to define $\tilde{u}_b$ by

$$u_b^{\dagger} = \omega \tilde{u}_b \ , \tag{8.26}$$

so that the basal velocity in x,Y coordinates is $\tilde{u}_b$: then the boundary conditions for (8.23) and (8.24) may be written:

$$\begin{aligned}
&\text{on } x = 0 \ , \\
&\qquad \psi = 0 \ , \ \chi = \chi_0 \ ; \\
&\text{on } Y = 0 \ , \\
&\qquad \psi = 0 \ , \ \psi_Y = \tilde{u}_b \ , \text{ and} \\
&\qquad \chi_Y = 0 \ , \ \tilde{u}_b = 0 \ , \ x < x_Q \ ; \\
&\qquad \chi = 1 \ , \ \chi_Y = - H\tilde{u}_b \ , \ x_Q < x < x_Z \ ; \\
&\qquad \chi = 1 \ , \ \tilde{u}_b = \tilde{f}(H) \equiv \omega^{-1}\varepsilon^{-(n+1)/(n+2)} f[\varepsilon^{1/(n+2)}H] \ , \\
&\qquad\qquad\qquad x_Z < x < x_M \ ,
\end{aligned} \tag{8.27}$$

where f is the temperate sliding law defined in (2.15). We also require a matching condition for χ as $Y \to \infty$.

Now let us reconsider the outer solution. In the ε-scaled variables, and using (8.19), the equations for the outer stream function and 'temperature' are, from (8.12) and (8.13),

$$\Psi_{\xi\xi} = \xi^n \chi \ , \tag{8.28}$$

$$\Psi_x \chi_\xi - \Psi_\xi \chi_x = \omega \xi^{n+1} \chi + \omega^3 \chi_{\xi\xi} \ . \tag{8.29}$$

To first order we neglect $O(\omega)$, and then the solution of (8.29) is

$$\chi = G(\Psi) \ , \tag{8.30}$$

where G is determined from the boundary condition on $\xi = 0$. From (8.4) we see that $\Psi = s$ is increasing in $0 < x < x_E$ and thus G as given by (8.4) is a well defined function. If we assume χ_A is monotone increasing, then so also is G.

The outer problem for Ψ is therefore, from (8.28) and (8.30),

$$\Psi_{\xi\xi} = \xi^n G(\Psi) , \tag{8.31}$$

with the boundary conditions that

$$\begin{aligned} \Psi &= s \text{ on } \xi = 0 , \\ \Psi &= 0 \text{ on } \xi = H , \end{aligned} \tag{8.32}$$

and a matching condition for Ψ_ξ as $\xi \to H$. The matching condition to first order may now be determined. As $\Psi \to 0$, the outer solution $\chi \to G(0) = \chi_0$, and thus the extra condition we impose on (8.23) and (8.24) is

$$\chi \to \chi_0 \text{ as } Y \to \infty . \tag{8.33}$$

From (8.23) and (8.27),

$$\begin{aligned} \psi_Y &= \tilde{u}_b + \int_0^Y H^n \chi \, dY \\ &\sim \tilde{u}_b + \chi_0 H^n Y + O(1) \end{aligned} \tag{8.34}$$

as $Y \to \infty$, using (8.32). Rewriting (8.34) in outer (ε-scale) variables via (8.26),(8.16) and (8.14),we obtain

$$- \Psi_\xi \sim u_b^\dagger + \chi_0 H^n (H - \xi) + O(\omega) . \tag{8.35}$$

Similarly, expanding (8.31) near $\Psi = 0$ and integrating, we obtain (to first order)

$$- \Psi_\xi = - \Psi_\xi \Big|_{\xi = H} + \chi_0 [H^{n+1} - \xi^{n+1}]/(n+1) . \tag{8.36}$$

From (8.35) and (8.36), it is clear that the usual matching procedure gives the extra condition for (8.31) as

$$\Psi_\xi = - u_b^\dagger \text{ on } \xi = H , \tag{8.37}$$

that is, the (outer) velocity field is unaffected to first order by the boundary layer.

The interpretation of these equations is this: in $x < x_Q$, and also

$x_Z < x < x_M$,(8.31),(8.32) and (8.37) serve to determine H:the boundary layer equations (8.23) and (8.24) then determine x_Q and x_M. In $x_Q < x < x_Z$, the outer flow solution for Ψ gives $\tilde{u}_b$ as a functional of H,and another relation is provided by a solution of the boundary layer equations: solving these simultaneous relations determines $\tilde{u}_b$ and H explicitly.

The outer equation for the stream function (8.31) is essentially an ordinary differential equation in ξ with parametric dependence on x via the boundary conditions (8.32). It is straightforward to show using continuity and monotonicity arguments that (8.31) and (8.32) have a unique solution in $x < x_Q$ and $x_Z < x < x_M$. The main qualitative result on the solution is that H increases with s(x). Details of the proofs are given in Appendix 5.

We now transform the boundary layer equations (8.23) and (8.24) to a slightly different form. Firstly,we integrate (8.23) using the boundary condition on Y = 0 to obtain

$$\psi_Y = \tilde{u}_b + H^n \int_0^Y \chi \, dY \tag{8.38}$$

and

$$\psi = \tilde{u}_b Y + H^n \int_0^Y \int_0^{Y'} \chi \, dY' \, dY \, . \tag{8.39}$$

We then change to Von Mises' variables x and ψ and hence obtain (Levich 1962 p. 79)

$$\frac{\partial \theta}{\partial x} = \frac{H^{n+1}\theta}{U} + \frac{\partial}{\partial \psi}\left[U \frac{\partial \theta}{\partial \psi} \right] , \tag{8.40}$$

where

$$\begin{aligned} U(x,\psi,\theta) &\equiv \psi_Y \, , \\ \theta(x,\psi) &\equiv \chi(x,Y) \, . \end{aligned} \tag{8.41}$$

(8.40) is in the form of a diffusion equation with a variable diffusion coefficient and a source term H^{n+1}/U. In what follows,we identify θ

with χ for convenience. After some algebra, we find that the boundary conditions (8.27) for (8.40) are

$$\text{on } x = 0 \text{ , } \chi = \chi_0 \text{ ;} \tag{8.42}$$

$$\begin{aligned} \text{on } \psi = 0 \text{ , } \chi_\psi &= -H \text{, } 0 < x < x_Z \text{ ,} \\ \chi &= 1 \text{ , } x_Q < x < x_M \text{ ;} \end{aligned} \tag{8.43}$$

$$\text{as } \psi \to \infty \text{ , } \chi \to \chi_0 \text{ .} \tag{8.44}$$

An analytic solution of (8.39) and (8.40) is of course intractable, but there are certain limiting cases which will be of some interest in practical applications. Some of these are described in the remainder of the chapter: in view of the difficulty of the problems considered, we do not attempt to give an exhaustive discussion.

§2 Bounds on the Solution: Estimate for x_Q.

Let us first consider the region $x < x_Q$ in which $\tilde{u}_b = 0$ and $\chi < 1$. One method of obtaining bounds on the solution for χ is to consider maximum principles for the equation (8.24) in the manner of Protter and Weinberger (1967). However, such principles are exceedingly difficult to find for equations of this form: on the other hand a physical interpretation of (8.24) leads us to expect that an 'increase' of the viscous dissipation term and/or a 'decrease' in the convection term will give an upper bound for the 'temperature' χ, and vice versa.

Specifically, if ψ^* is such that $\psi^* = 0$ on $Y = 0$, $\psi^*_Y > 0$, $\psi^*_x > 0$, and we define $\underset{\sim}{q}^*$ by

$$\underset{\sim}{q}^* = (\psi_Y - \psi^*_Y \text{ , } -(\psi_x - \psi^*_x)) \text{ ,} \tag{8.45}$$

where ψ is determined by (8.23), (8.24) and (8.27), then we expect χ^*, the solution of

$$\underset{\sim}{q}^* \cdot \underset{\sim}{\nabla} \chi^* = H^{n+1} + \chi^*_{YY} \tag{8.46}$$

with boundary conditions (8.27), to be an upper bound for χ as long $\chi < 1$, which is the region of interest. This is because the source term in (8.46) is larger than that in (8.24); also (8.45) implies that the extra heat is convected away less quickly than in (8.24) ($\psi^*_Y > 0$), and there is a smaller influx of cold ice from infinity ($\psi^*_x > 0$). We may provide a partial mathematical justification for this as follows.

Since χ approximates an exponential, we may take $\chi > 0$. In fact we have $\chi > \chi_0$, since at any turning point of χ (8.24) implies that $\chi_{YY} < 0$, and hence χ is a maximum there. Thus the minimum value of χ must occur on the boundary Y = 0 or x = 0. It cannot occur on Y = 0 (for the same reason), and so the minimum value of χ is its value on x = 0, that is χ_0. Since we are only interested in the solution while $\chi < 1$, we may therefore take

$$\chi_0 < \chi < 1 \ . \tag{8.47}$$

We can apply similar reasoning to the pair of equations

$$\begin{aligned} \underset{\sim}{q}^* \cdot \underset{\sim}{\nabla} \chi^* &= H^{n+1} + \chi^*_{YY} \ , \\ \underset{\sim}{q} \cdot \underset{\sim}{\nabla} \chi &= H^{n+1} \chi + \chi_{YY} \ , \end{aligned} \tag{8.48}$$

where $\underset{\sim}{q}$, $\underset{\sim}{q}^*$ are defined by (8.23) and (8.45) and the boundary conditions for both equations are given by (8.27). Let

$$\phi = \chi^* - \chi \ ; \tag{8.49}$$

we want to show that $\phi \geq 0$. From (8.27) we have that $\phi = 0$ on x = 0 and $Y = \infty$, and $\phi_Y = 0$ on Y = 0. If $\phi < 0$ in x > 0, Y ≥ 0, then there exists a minimum in x > 0, Y > 0 (since on Y = 0, $\phi_{YY} < 0$ from (8.48)), and at such a point $\underset{\sim}{\nabla}\chi = \underset{\sim}{\nabla}\chi^*$ and $\phi_{YY} \geq 0$, whence it follows from (8.48) that

$$(\underset{\sim}{q} - \underset{\sim}{q}^*) \cdot \underset{\sim}{\nabla} \chi^* \leq 0 \ . \tag{8.50}$$

Thus if

$$(\underset{\sim}{q} - \underset{\sim}{q}^*) \cdot \underset{\sim}{\nabla} \chi^* > 0 \qquad (8.51)$$

everywhere (8.50) is impossible, and hence $\phi \geq 0$. This is the motivation behind the definition of $\underset{\sim}{q}^*$, and we now show that (8.51) is a reasonable assumption to make.

Taking note of (8.47), the obvious choice for ψ^* is by inspection of (8.38) with $\tilde{u}_b = 0$,

$$\psi^* = H^n \int_0^Y \int_0^{Y'} (\chi - \chi_0) \, dY' \, dY \; , \qquad (8.52)$$

that is, our estimate $\underset{\sim}{q}^*$ of the velocity field is obtained from (8.38) by replacing χ with χ_0. If we assume that $\chi^*_x > 0$, $\chi^*_Y < 0$ (this may be checked subsequently), then (8.51) is certainly valid if

$$\psi^*_Y > 0 \; , \; \psi^*_x > 0 \; . \qquad (8.53)$$

From the definition of ψ^* in (8.52), it is clear that $\psi^*_Y > 0$, and since (as shown in Appendix 5) $H_x > 0$, a sufficient condition that $\psi^*_x > 0$ is that $\chi_x > 0$.

Thus with the single (reasonable) assumption that $\chi_x > 0$, it follows that χ^* is an upper bound for χ as defined by (8.23), (8.24) and (8.27). In a similar manner, we may obtain a lower bound for χ by replacing χ in (8.38) by 1, and $H^{n+1}\chi$ in (8.24) by $H^{n+1}\chi_0$; a sufficient condition for this solution to be a lower bound is that $\frac{\partial}{\partial x}[H^n(1-\chi)] > 0$. The conditions for these estimates to be upper and lower bounds may be written jointly as

$$\chi_0 \frac{\partial}{\partial x}[H^n] < \frac{\partial}{\partial x}[H^n \chi] < \frac{\partial}{\partial x}[H^n] \; . \qquad (8.54)$$

We shall assume that (8.54) is true.

In this case an upper bound for χ is obtained by solving

$$\psi_Y \chi_x - \psi_x \chi_Y = H^{n+1} + \chi_{YY} \; , \qquad (8.55)$$

where

$$\psi = \tfrac{1}{2}H^n \chi_0 Y^2 \tag{8.56}$$

and the boundary conditions (8.27) are satisfied. In the notation of (8.41), the horizontal velocity is

$$\begin{aligned} U = \psi_Y &= H^n \chi_0 Y \\ &= [2\chi_0 H^n \psi]^{\frac{1}{2}} , \end{aligned} \tag{8.57}$$

using (8.56). Thus introducing the Von Mises variables, (8.55) becomes

$$[2\chi_0 H^n \psi]^{\frac{1}{2}} \frac{\partial \chi}{\partial x} = H^{n+1} + [2\chi_0 H^n \psi]^{\frac{1}{2}} \frac{\partial}{\partial \psi} [\{2\chi_0 H^n \psi\}^{\frac{1}{2}} \frac{\partial \chi}{\partial \psi}] . \tag{8.58}$$

The boundary conditions are that

$$\begin{aligned} &\text{on } x = 0 , \ \chi = \chi_0 , \\ &\text{as } \psi \to \infty , \ \chi \to \chi_0 , \\ &\text{on } Y = 0 , \ \chi_\psi = \lim_{Y \to 0} \frac{\chi_Y}{\psi_Y} = \left.\frac{\chi_{YY}}{\psi_{YY}}\right|_{Y=0} = -\frac{H}{\chi_0} , \end{aligned} \tag{8.59}$$

from (8.57) and (8.55).

We define

$$Z = \sqrt{\psi} , \Rightarrow \psi^{\frac{1}{2}} \frac{\partial}{\partial \psi} = \frac{1}{2} \frac{\partial}{\partial Z} ; \tag{8.60}$$

then (8.58) is

$$Z \frac{\partial \chi}{\partial x} = \frac{H^{n+1}}{[2\chi_0 H^n]^{\frac{1}{2}}} + \frac{[2\chi_0 H^n]^{\frac{1}{2}}}{4} \frac{\partial^2 \chi}{\partial Z^2} . \tag{8.61}$$

Putting

$$X = \frac{1}{4}\int_0^x [2\chi_0 H^n]^{\frac{1}{2}} dx , \ \beta(X) \equiv 2H/\chi_0 , \tag{8.62}$$

(8.61) becomes

$$Z \frac{\partial \chi}{\partial X} = \beta(X) + \frac{\partial^2 \chi}{\partial Z^2} , \tag{8.63}$$

and the boundary conditions are, from (8.51) and (8.60),

$$\chi = \chi_0 \text{ on } x = 0 \text{ and } Z = \infty , \qquad \chi_Z = 0 \text{ on } Z = 0 . \tag{8.64}$$

Similarly, we obtain a lower bound by solving

$$\psi_Y \chi_x - \psi_x \chi_Y = H^{n+1} \chi_0 + \chi_{YY} , \tag{8.65}$$

$$\psi = \tfrac{1}{2} H^n Y^2 \tag{8.66}$$

and (8.27). With the notation of (8.62), we find that

$$Z \frac{\partial \chi}{\partial X} = \chi_0^{3/2} \beta(X) + \chi_0^{-1/2} \frac{\partial^2 \chi}{\partial Z^2} \tag{8.67}$$

with the boundary conditions (8.64). Denoting the upper bound by χ_U and the lower bound by χ_L, we can put

$$\chi_U = \chi_0 + \theta(X,Z) , \qquad \chi_L = \chi_0 + \chi_0^{5/3} \theta(X,Z_1) , \tag{8.68}$$

where

$$Z_1 = \chi_0^{1/6} Z \tag{8.69}$$

and θ satisfies

$$Z \frac{\partial \theta}{\partial X} = \beta(X) + \frac{\partial^2 \theta}{\partial Z^2} , \tag{8.70}$$

$$\theta = 0 \text{ on } X = 0,\ Z = \infty , \qquad \theta_Z = 0 \text{ on } Z = 0 . \tag{8.71}$$

Taking Laplace transforms of (8.70) using (8.71), we obtain

$$\Theta'' - pZ\Theta = - \beta^*(p) , \tag{8.72}$$

where Θ, β^* are the Laplace transforms of θ and β, and p is the Laplace transform variable. The boundary conditions on (8.72) are that

$$\begin{aligned} &\Theta \to 0 \ , \ Z \to \infty \ , \\ &\Theta' = 0 \ , \ Z = 0 \ . \end{aligned} \tag{8.73}$$

(8.72) is Airy's equation with a constant right hand side. The general solution which tends to zero as $Z \to \infty$ is

$$\Theta = a\,\mathrm{Ai}[p^{1/3}Z] + \pi p^{-2/3}\beta^*(p)\,\mathrm{Gi}[p^{1/3}Z] \ , \tag{8.74}$$

where Ai is the Airy function and Gi is defined by

$$\mathrm{Gi}(z) = \frac{1}{\pi}\int_0^\infty \sin\{zt + \tfrac{1}{3}t^3\}\,dt \tag{8.75}$$

(Abramowitz and Stegun 1968,p.448). We require also $\Theta' = 0$ on $Z = 0$, and therefore

$$a = -\pi p^{-2/3}\beta^* \frac{\mathrm{Gi}'(0)}{\mathrm{Ai}'(0)} = \frac{\pi\beta^*}{p^{2/3}\sqrt{3}} \ . \tag{8.76}$$

The solution for θ can in principle be obtained by inverting (8.74). We here restrict our attention to the bedrock $Z = 0$,which is of most interest.

We have that

$$\begin{aligned} \mathrm{Ai}(0) &= 3^{-2/3}/\Gamma(2/3) \ , \\ \mathrm{Gi}(0) &= \frac{1}{\sqrt{3}}\mathrm{Ai}(0) = 3^{-7/6}/\Gamma(2/3) \ : \end{aligned} \tag{8.77}$$

therefore

$$\Theta(0,p) = \frac{2\pi\beta^*}{p^{2/3}3^{7/6}\Gamma(2/3)} \ . \tag{8.78}$$

The inversion of (8.78) is a convolution integral,thus

$$\theta(X,0) = \frac{2\pi}{3^{7/6}\{\Gamma(2/3)\}^2}\int_0^X \frac{\beta(t)}{(X-t)^{1/3}}\,dt \ , \tag{8.79}$$

and (8.79) defines the bedrock temperature. Upper and lower bounds on x_Q are defined by the solutions of $\chi_L = 1$ and $\chi_U = 1$ respectively. Defining

$$X_Q = \frac{1}{4}\int_0^{x_Q} [2\chi_0 H^n]^{\frac{1}{2}}\, dx , \qquad (8.80)$$

this implies, from (8.68),

$$\chi_0 + \chi_0^{5/3}\theta(X_Q,0) < 1 < \chi_0 + \theta(X_Q,0) , \qquad (8.81)$$

and hence using (8.79)

$$1 - \chi_0 < \frac{2\pi}{3^{7/6}\{\Gamma(2/3)\}^2}\int_0^{X_Q} \frac{\beta(t)\, dt}{(X_Q - t)^{1/3}} < \frac{1-\chi_0}{\chi_0^{5/3}} . \qquad (8.82)$$

As an example, let us suppose ε as defined by (8.20) is small: then the outer solution (8.30) for χ is $\chi = \chi_0$ to first order, and the equation (8.31) for Ψ becomes

$$\Psi_{\xi\xi} = \xi^n \chi_0 , \qquad (8.83)$$

with solution

$$\Psi = s(x) - \chi_0 \frac{H^{n+1}\xi}{n+1} + \chi_0 \frac{\xi^{n+2}}{(n+1)(n+2)} , \qquad (8.84)$$

where H satisfies

$$\frac{\chi_0 H^{n+2}}{n+2} = s(x) \approx a_0 x ; \qquad (8.85)$$

a_0 is the accumulation rate at the head. From (8.85) we obtain

$$X = \tfrac{1}{2}\left(\frac{n+2}{3n+4}\right)\left[2\chi_0\left\{\frac{(n+2)a_0}{\chi_0}\right\}^{\frac{n}{n+2}}\right]^{\frac{1}{2}} x^{\frac{3n+4}{2(n+2)}} , \qquad (8.86)$$

and therefore

$$\beta(X) = \frac{2}{\chi_0}\left[\frac{(n+2)a_0}{\chi_0}\right]^{\frac{1}{n+2}}\left[\tfrac{1}{2}\left(\frac{n+2}{3n+4}\right)\left\{2\chi_0\left\{\frac{(n+2)a_0}{\chi_0}\right\}^{\frac{n}{n+2}}\right\}^{\frac{1}{2}}\right]^{\frac{-2}{(3n+4)}} X^{\frac{2}{(3n+4)}} .$$

(8.87)

Writing for the moment $\beta = bX^{2/(3n+4)}$, it follows that

$$\beta^* = b\Gamma\left(\frac{3n+6}{3n+4}\right)/p^{\left(\frac{3n+6}{3n+4}\right)} ,$$

and so

$$\beta^*/p^{2/3} = \frac{b\Gamma\left(\frac{3n+6}{3n+4}\right)}{p^{\left\{\frac{15n+26}{3(3n+4)}\right\}}} .$$

The inverse of $\beta^*/p^{2/3}$ is therefore

$$\frac{b\Gamma\left(\frac{3n+6}{3n+4}\right)}{\Gamma\left(\frac{15n+26}{9n+12}\right)} X^{\frac{6n+14}{9n+12}} . \tag{8.88}$$

After some algebra, we find that (8.82) becomes

$$\chi_0^{\frac{5(n+2)}{3n+7}} \Xi < x_Q < \Xi , \tag{8.89}$$

where

$$\Xi = \left[\left\{ \frac{3^{7/6}\Gamma(2/3)\Gamma\left(\frac{15n+26}{9n+12}\right)(1-\chi_0)}{2\pi\Gamma\left(\frac{3n+6}{3n+4}\right)\chi_0} \right\}^{3(n+2)} \left\{\frac{3n+4}{2(n+2)}\right\}^{2(n+2)} \left\{\frac{(n+2)a_0}{\chi_0}\right\}^{(n+3)} \right]^{\frac{1}{3n+7}} . \tag{8.90}$$

Since $\chi_0 \sim 1$, (8.89) provides a reasonable estimate of x_Q. Putting n = 3, (8.90) is

$$\Xi \simeq 1.72 \left[\left(\frac{1-\chi_0}{\chi_0}\right)^{3(n+2)} \left(\frac{a_0}{\chi_0}\right)^{n+3} \right]^{\frac{1}{3n+7}} . \tag{8.91}$$

A typical case is $\chi_0 = \frac{1}{2}$, $a_0 = 1$. Then (8.89) and (8.91) imply

$$0.94 < x_Q < 2.23 . \tag{8.92}$$

§3 $x_Q < x < x_Z$: Similarity Solutions?

In this section we consider the thermal boundary layer equation in the Von Mises form (8.40). Levich (op.cit.) shows how equations of this type often have similarity solutions, and we here pose the question whether such solutions (if they exist) have any relevance to the present problem. We shall largely be concerned with the sub-temperate region $x_Q < x < x_Z$, though the analysis may be extended to include both $x < x_Q$ and $x > x_Z$.

In order to simplify (8.40), we must express U in terms of x and ψ. This is not possible with ψ given by (8.39) and therefore we consider the approximate form

$$\psi = \tilde{u}_b Y + \tfrac{1}{2}\tilde{\chi} H^n Y^2 \tag{8.93}$$

where

$$\chi_0 \leq \tilde{\chi} \leq 1 , \tag{8.94}$$

and $\tilde{\chi}$ is some mean value of χ. As before, various choices of $\tilde{\chi}$ may be expected to give bounds on the solution χ of (8.40). From (8.93)

$$U = \psi_Y = \tilde{u}_b + \tilde{\chi} H^n Y , \tag{8.95}$$

and also solving (8.93) for Y,

$$Y = \frac{- \tilde{u}_b + [\tilde{u}_b^{\,2} + 2\tilde{\chi} H^n \psi]^{\frac{1}{2}}}{\tilde{\chi} H^n} , \tag{8.96}$$

whence from (8.95) U may be expressed explicitly as

$$U = [\tilde{u}_b^{\,2} + 2\tilde{\chi} H^n \psi]^{\frac{1}{2}} . \tag{8.97}$$

The energy equation (8.40) is therefore

$$[\tilde{u}_b^{\,2} + 2\tilde{\chi} H^n \psi]^{\frac{1}{2}} \frac{\partial \chi}{\partial x} = H^{n+1} \chi + [\tilde{u}_b^{\,2} + 2\tilde{\chi} H^n \psi]^{\frac{1}{2}} \frac{\partial}{\partial \psi} \left[\{\tilde{u}_b^{\,2} + 2\tilde{\chi} H^n \psi\}^{\frac{1}{2}} \frac{\partial \chi}{\partial \psi} \right] \tag{8.98}$$

with the boundary conditions, from (8.42)-(8.44), that

$$\begin{aligned} &\chi = \chi_0 \text{ on } x = x_Q \text{ and } \psi \to \infty , \\ &\chi_\psi = - H , \ \chi = 1 \text{ on } \psi = 0 \ (x_Q < x < x_Z) . \end{aligned} \tag{8.99}$$

As mentioned on page 161, both H and $\tilde{u}_b$ are unknown in (8.98) and (8.99).

The outer solution gives one functional relation between $\tilde{u}_b$ and H, and the solution of (8.98) and (8.99) with the extra condition on $\psi = 0$ gives another. The intersection of these two functionals then determines $\tilde{u}_b$ and H. In general this is a very difficult procedure. However, we see from (8.26) that the 'outer' and 'inner' basal velocities $u_b^\dagger$ and $\tilde{u}_b$ are related by

$$u_b^\dagger = \omega\tilde{u}_b \ , \tag{8.100}$$

and so if the sliding velocity $\tilde{u}_b \lesssim 1$, we may neglect the outer sliding velocity to first order. That is, the outer stream function Ψ is to $O(\omega)$ the solution of (8.31) and (8.32) with the no slip condition

$$\Psi_\xi = 0 \text{ on } \xi = H \ . \tag{8.101}$$

Thus H is determined, just as in $x < x_Q$, from the outer solution, and no surface 'kink' is evident between the regions of cold and sub-temperate basal ice. Given this H, we have to solve (8.98) and (8.99) in order to determine the basal sliding velocity $\tilde{u}_b$.

If on the other hand $u_b^\dagger \gtrsim 1$, then the depth is not determined by the outer flow. However we may then consider $\tilde{u}_b >> 1$, and the equation (8.98) is much simplified: this case is examined in §4.

Let us consider (8.98). In seeking solutions of similarity type, an obvious transformation to make is

$$Z = g(x)U \ , \Rightarrow \psi = \frac{Z^2 - \tilde{u}_b^2 g^2}{2g^2\tilde{\chi}H^n} \ , \tag{8.102}$$

where g is to be chosen for convenience. We obtain

$$\begin{aligned} \frac{\partial}{\partial x} &\to \frac{\partial}{\partial x} + [g'U + \frac{g}{2U}\{(\tilde{u}_b^{\,2})' + 2\tilde{\chi}(H^n)'\psi\}]\ \frac{\partial}{\partial Z} \ , \\ \frac{\partial}{\partial \psi} &\to \frac{gH^n\tilde{\chi}}{U}\ \frac{\partial}{\partial Z} \ , \end{aligned} \tag{8.103}$$

and so (8.98) becomes, using (8.102),

$$Z\frac{\partial\chi}{\partial x} + \left[Z^2\left\{\frac{g'}{g} + \frac{(H^n)'}{2H^n}\right\} + \tilde{u}_b^{\,2} g^2\left\{\frac{\tilde{u}_b'}{u_b} - \frac{(H^n)'}{2H^n}\right\}\right]\frac{\partial\chi}{\partial Z}$$

$$= gH^{n+1}\chi + g^3\tilde{\chi}^2H^{2n}\chi_{ZZ} \quad , \qquad (8.104)$$

and the boundary conditions are that

$$\begin{aligned} &\chi = \chi_0 \text{ on } x = x_Q \text{ and } Z = \infty \, , \\ &\chi_Z = \frac{-\tilde{u}_b}{\tilde{\chi} g H^{n-1}} \; , \; \chi = 1 \text{ on } Z = g\tilde{u}_b \; (x_Q < x < x_Z). \end{aligned} \qquad (8.105)$$

Let us attempt to find a 'similarity' solution of (8.104) and (8.105). We take the origin to be x_Q, and put

$$g = \tilde{u}_b^{\,-1} \; , \; \tilde{u}_b = BH^{(n-1)/2} \; , \qquad (8.106)$$

and seek solutions $\chi = \chi(Z)$ of (8.104), which may be written as

$$Z\frac{\partial\chi}{\partial x} + \tfrac{1}{2}[Z^2 - 1]\frac{H'}{H}\frac{\partial\chi}{\partial Z} = \frac{H^{(n+3)/2}}{B}[\chi + \frac{\tilde{\chi}^2}{B^2}\chi_{ZZ}] \; ,$$

or, if $\chi = \chi(Z)$,

$$\frac{\tilde{\chi}^2}{B^2}\chi'' - (Z^2 - 1)\chi' + \chi = 0 \; , \qquad (8.107)$$

provided

$$H'H^{-(n+5)/2} = 2/B \; . \qquad (8.108)$$

(8.107) is to be solved subject to

$$\begin{aligned} &\chi = \chi_0 \text{ on } Z = \infty \, , \\ &\chi = 1 \; , \; \chi' = -B/\tilde{\chi} \text{ on } Z = 1 \; , \end{aligned} \qquad (8.109)$$

and B is to be determined. Unfortunately the requirement (8.108) implies $H \sim x^{-2/(n+3)}$ as $x \to 0$, and this solution is therefore an unrealistic one. This occurs because if $\tilde{u}_b \propto H^{n-1}$ and $Z = [1 + 2\tilde{\chi}H^n\psi/\tilde{u}_b]^{\frac{1}{2}}$, then Z tends to infinity as $x \to 0$ and $\psi \to \infty$ only if $H \to \infty$. We can see that a similarity solution in which $\tilde{u}_b \propto H^m$ must be unrealistic if $m \leq n$, and also because in reality $\tilde{u}_b = 0$ on $x = x_Q$ where H is finite. We might expect, however, that a similarity solution could provide a valid 'intermediate asymptotic' solution in the sense of Barenblatt and Zel'dovich (1972) if ε defined by (8.20) is $\ll 1$, and (in the ε-scaled variables) $x - x_Q \gg 1$. However in the latter case we may have $\tilde{u}_b \gg 1$, and the discussion in §4 is appropriate.

We note here that we cannot hope to obtain a finite value of H at x_Q with the 'switch on' boundary conditions (8.99): 'switching on' the bedrock temperature $\chi = 1$ at $x = x_Q$ ensures that $-\chi_\psi = H$ is infinite there.

In $x < x_Q$, $\tilde{u}_b = 0$, and so we put

$$g = H^{-n/2} \tag{8.110}$$

in (8.104) and (8.105). Then we have, on writing

$$X = \int_0^x H^{n/2}\,dx \;, \tag{8.111}$$

$$Z\frac{\partial\chi}{\partial X} = H\chi + \tilde{\chi}^2\chi_{ZZ} \;, \tag{8.112}$$

with

$$\left.\begin{aligned} &\chi = \chi_0 \text{ on } X = 0 \text{ and } Z = \infty \;, \\ &\chi_Z = 0 \text{ on } Z = 1 \;, \end{aligned}\right\} \tag{8.113}$$

and estimates for χ may be obtained as in §2.

If $x > x_Z$, suppose the sliding law is that derived in Chapter III,

$$\tilde{u}_b = \tilde{C}H^n \;. \tag{8.114}$$

We again define g by (8.106), and then (8.104) is

$$Z\frac{\partial\chi}{\partial x} + \frac{n}{2}\frac{H'}{H}(1 - Z^2)\frac{\partial\chi}{\partial Z} = \frac{H}{\tilde{C}}\chi + \frac{\tilde{\chi}^2}{\tilde{C}^3H^n}\chi_{ZZ} \;, \tag{8.115}$$

which on multiplying by H^n is seen to be almost identical to the original equation (8.24).

§4 'Large' Basal Sliding: $\tilde{u}_b >> 1$.

If the sliding law in the original coordinates is

$$u_b = CH^n , \tag{8.116}$$

then in the ε-scaled coordinates it may be written ,using (8.11) and (8.26),as

$$u_b^{\dagger} = C\varepsilon^{-1/(n+2)}H^n , \tag{8.117}$$

and so if $C \gtrsim 1$ the full sliding law (8.117) only becomes valid (i.e. x reaches x_Z) if the 'inner' basal sliding velocity $\tilde{u}_b >> 1$. Scaling considerations imply that it is unlikely that $\tilde{u}_b$ can remain O(1) if $x >> \varepsilon$: hence if $\varepsilon << 1$ it is reasonable to consider the boundary layer equation (8.98) with $\tilde{\chi} = 0$ (that is,negligible shearing) as a reasonable approximation over regions where $x_Q << x < x_Z$. In this case,the thermal boundary layer equation may be written (still on the ε-scale)

$$\psi_Y \chi_x - \psi_x \chi_Y = H^{n+1}\chi + \chi_{YY} , \tag{8.118}$$

where

$$\psi = \tilde{u}_b Y . \tag{8.119}$$

In the Von Mises variables,(8.118) is by comparison with (8.98)

$$\tilde{u}_b \frac{\partial \chi}{\partial x} = H^{n+1}\chi + \tilde{u}_b \frac{\partial}{\partial \psi}\left[\tilde{u}_b \frac{\partial \chi}{\partial \psi}\right] . \tag{8.120}$$

The boundary conditions to be applied on (8.120) are that

$$\chi_\psi \to 0 \text{ as } \psi \to \infty , \tag{8.121}$$

$$\chi = 1 , \chi_\psi = - H \text{ on } \psi = 0 , x_Q < x < x_Z , \tag{8.122}$$

and an initial condition on x_Q. Note that a 'switch-on' initial condition (e.g. $\chi = \chi_0$ at $x = x_Q$) is bound to give an infinite depth at x_Q for the same reason given in §3. In fact the boundary layer at x_Q is of thickness (assuming $\beta \sim 1$) $\omega\varepsilon^{1/(n+2)} \sim \beta_2^{(n+3)/(3n+7)} \sim \beta_2^{3/8}$, which is notionally larger than $\beta_2^{\frac{1}{2}}$, and thus such a condition is probably not very realistic. We therefore leave the initial condition unspecified for the moment, and simply require

$$\chi = \chi_Q(\psi) \text{ on } x = x_Q , \tag{8.124}$$

where $\chi_Q \to \chi_0$ as $\psi \to \infty$.

Let us define

$$\iota = \chi \exp[-\int^x \frac{H^{n+1}}{\tilde{u}_b} dx] \tag{8.125}$$

and

$$\eta = \int_{x_Q}^x \tilde{u}_b(x)\, dx ; \tag{8.126}$$

then the equation (8.121) becomes simply the diffusion equation

$$\iota_\eta = \iota_{\psi\psi} \tag{8.127}$$

and the boundary conditions are

$$\begin{aligned} &\iota \to \chi_Q(\psi) \exp[-\int^{x_Q} \frac{H^{n+1}}{\tilde{u}_b} dx] \text{ as } \eta \to 0 , \\ &\iota = \iota_0(\eta) \equiv \exp[-\int^x \frac{H^{n+1}}{\tilde{u}_b} dx] \text{ on } \psi = 0 , \\ &\iota_\psi = - H\iota_0(\eta) \text{ on } \psi = 0 , \\ &\iota_\psi \to 0 \text{ as } \psi \to \infty . \end{aligned} \tag{8.128}$$

Now in reality since $\tilde{u}_b \to 0$ at x_Q the model (8.127) and (8.128) becomes invalid near x_Q, and we can see from (8.125) that if H is finite at x_Q then $\iota \to \infty$ at x_Q: on the other hand we are interested in solving for x much larger than x_Q rather than near x_Q, and therefore we expect that the details of the initial condition for ι are not too important so long as the essential physical features are preserved: this effectively means ensuring that H is small at x_Q.

In this spirit, we replace the boundary conditions (8.128) by the following:

$$\begin{aligned} \iota &= \chi_Q(\psi) \text{ on } \eta = 0 , \\ \iota &= \iota_0(\eta) \text{ on } \psi = 0 , \\ \iota_\psi &= - H\iota_0(\eta) \text{ on } \psi = 0 , \\ \iota_\psi &\to 0 \text{ as } \psi \to \infty , \end{aligned} \tag{8.129}$$

where

$$\iota_0(\eta) = \exp[-\int_0^x \frac{H^{n+1}}{\tilde{u}_b} dx] , \tag{8.130}$$

$$\eta = \int_0^x \tilde{u}_b(x)\, dx , \tag{8.131}$$

and we require $H^{n+1}/\tilde{u}_b$ to be integrable at x = 0. We have removed the singularity in H by letting $x_Q \to 0$: we still require $\chi_Q'(0) = 0$ in order that H = 0 at x = 0, and thus a 'switch-on' initial condition is still excluded. Taking the Laplace transform of (8.127) and using the boundary conditions (8.129), we obtain

$$s\iota^* - \chi_Q = \iota^{*''} , \tag{8.132}$$

where ι^* is the Laplace transform of ι, s is the transform variable and ' denotes differentiation with respect to ψ. We write (8.132) in the form

$$(\iota^* - \chi_0/s)'' - s(\iota^* - \chi_0/s) = - (\chi_Q - \chi_0) , \tag{8.133}$$

and the boundary conditions on (8.133) are then

$$\begin{aligned} \iota^* - \chi_0/s &\to 0 \text{ as } \psi \to \infty , \\ \iota^* - \chi_0/s &= \iota_0^* - \chi_0/s \text{ on } \psi = 0 . \end{aligned} \tag{8.134}$$

The Green's function for the left hand side of (8.133) with the boundary conditions (8.134) is

$$K(\psi,\Omega) = \begin{cases} [\iota_0^* - \chi_0/s]e^{-\sqrt{s}\psi} + \frac{1}{\sqrt{s}} e^{-\sqrt{s}\psi} \sinh \Omega\sqrt{s} \ , \ \psi > \Omega \ , \\ [\iota_0^* - \chi_0/s]e^{-\sqrt{s}\psi} + \frac{1}{\sqrt{s}} e^{-\sqrt{s}\Omega} \sinh \psi\sqrt{s} \ , \ \psi < \Omega \ , \end{cases} \quad (8.135)$$

and the solution of (8.133) and (8.134) is therefore

$$\iota^* = \chi_0/s + \int_0^\infty K(\psi,\Omega)[\chi_Q(\Omega) - \chi_0]\, d\Omega$$

$$= \chi_0/s + [\iota_0^* - \chi_0/s]e^{-\sqrt{s}\psi} \int_0^\infty \iota_Q(\Omega)\, d\Omega$$

$$+ \frac{1}{\sqrt{s}} e^{-\sqrt{s}\psi} \int_0^\psi \iota_Q(\Omega) \sinh \Omega\sqrt{s}\, d\Omega$$

$$+ \frac{1}{\sqrt{s}} \sinh \psi\sqrt{s} \int_\psi^\infty e^{-\sqrt{s}\Omega}\, \iota_Q(\Omega)\, d\Omega \ , \quad (8.136)$$

where we have written

$$\iota_Q(\Omega) = \chi_Q(\Omega) - \chi_0 \ . \quad (8.137)$$

We are interested in determining the value of ι_ψ on $\psi = 0$. From (8.136),

$$\iota^*_\psi\big|_{\psi=0} = - \sqrt{s}[\iota_0^* - \chi_0/s] \int_0^\infty \iota_Q(\Omega)\, d\Omega + \int_0^\infty e^{-\sqrt{s}\Omega}\, \iota_Q(\Omega)\, d\Omega \ . \quad (8.138)$$

Interchanging the order of integration, we can invert (8.138) to obtain, after some manipulation,

$$\iota_\psi\big|_{\psi=0} = - \frac{(1-\chi_0)}{\sqrt{\pi\eta}} \int_0^\infty \iota_Q(\Omega)\, d\Omega + \frac{1}{2(\pi\eta^3)^{\frac{1}{2}}} \int_0^\infty \iota_Q(\Omega) e^{-\frac{\Omega^2}{4\eta}} \frac{d\Omega}{\Omega}$$

$$- \frac{1}{\sqrt{\pi}} \int_0^\eta \frac{\iota_0'(t)\, dt}{\sqrt{\eta - t}} \int_0^\infty \iota_Q(\Omega)\, d\Omega \ . \quad (8.139)$$

Using the third boundary condition in (8.129), (8.139) becomes an integral equation for the unknown ι_0:

$$H\iota_0(\eta) = \frac{\int_0^\infty \iota_Q(\Omega)\, d\Omega}{\sqrt{\pi}} \left[\frac{(1-\chi_0)}{\sqrt{\eta}} + \int_0^\infty \frac{\iota_0'(t)\, dt}{\sqrt{\eta - t}} \right]$$

$$+ \frac{1}{2(\pi\eta^3)^{\frac{1}{2}}} \int_0^\infty \iota_Q(\Omega) \exp(-\Omega^2/4\eta) \frac{d\Omega}{\Omega} \ . \quad (8.140)$$

We can translate (8.140) to a standard form as follows:let

$$\iota_0(\eta) = v(\eta) + \chi_0 \; ; \tag{8.141}$$

then (8.140) may be written

$$H(v + \chi_0) = \frac{\int_0^\infty \iota_Q(\Omega)\, d\Omega}{\sqrt{\pi}} \frac{d}{d\eta}\left[\int_0^\eta \frac{v(t)}{\sqrt{\eta - t}} dt\right]$$

$$+ \frac{1}{2(\pi\eta^3)^{\frac{1}{2}}} \int_0^\infty \iota_Q(\Omega) \exp(-\Omega^2/4\eta) \frac{d\Omega}{\Omega} \tag{8.142}$$

(cf. Carrier, Krook and Pearson, op.cit. p 357). Integrating from 0 to η gives

$$\sqrt{\pi}\int_0^\eta Hv\, d\eta + \sqrt{\pi}\chi_0\int_0^\eta H\, d\eta = \int_0^\infty \iota_Q\, d\Omega\int_0^\eta \frac{v(t)}{\sqrt{\eta - t}}\, dt$$

$$+ \int_0^\eta \frac{d\eta}{2\eta^{3/2}}\int_0^\infty \iota_Q \exp(-\Omega^2/4\eta) \frac{d\Omega}{\Omega}\,, \tag{8.143}$$

and we can invert (8.143) to find that v is the solution of

$$\left[\int_0^\infty \iota_Q\, d\Omega\right] v = \frac{1}{\sqrt{\pi}}\int_0^\eta \frac{H(v + \chi_0)}{\sqrt{\eta - t}}\, dt - \frac{1}{2\pi}\int_0^\eta \frac{dt}{t^{3/2}(\eta-t)^{\frac{1}{2}}}\int_0^\infty \iota_Q \exp(-\Omega^2/4\eta) \frac{d\Omega}{\Omega}\,. \tag{8.144}$$

Defining

$$w(\eta) = H(v + \chi_0) = H\iota_0\,, \tag{8.145}$$

$$a(\eta) = \frac{H}{\sqrt{\pi}\int_0^\infty \iota_Q\, d\Omega}\,, \tag{8.146}$$

$$b(\eta) = H\left[\chi_0 - \frac{1}{2\pi\int_0^\infty \iota_Q\, d\Omega}\int_0^\eta \frac{dt}{t^{3/2}(\eta-t)^{\frac{1}{2}}}\int_0^\infty \iota_Q \exp(-\Omega^2/4\eta) \frac{d\Omega}{\Omega}\right], \tag{8.147}$$

(8.144) may be written as a singular Volterra equation of the second kind with a kernel of Abel type,

$$w(\eta) = a(\eta)\int_0^\eta \frac{w(t)}{\sqrt{\eta - t}}\, dt + b(\eta)\,, \tag{8.148}$$

where a and b are given functions. The solution of (8.148) determines a relationship between u_b and H via (8.130) and (8.131). If $x > x_Z$, $w(\eta)$

determines the heat flux away from the bedrock, and x_M is determined by the solution for η of $w(\eta) = 0$.

We can in the usual way obtain a series solution for w of (8.148), but a solution in closed form is not easily obtainable. Further analysis of the equation is not pursued here.

Afterword

The scope of this thesis is somewhat large. Rather than concentrate on one particular subject, I have tried to give an account of how the proposed glacier flow model may be applied to the study of a wide variety of phenomena. Even so, there are many topics which remain unconsidered. Some of these are discussed below.

The model proposed in Chapter II is specifically concerned with two-dimensional valley glaciers. What of cirque glaciers, ice sheets, etc.? Most of the considerations of the model will carry over: for ice sheets we will require a cylindrical geometry with coordinates z and r; also if the base is flat, the flow is not driven by gravity ($\varepsilon = 0$ in the notation of Chapter II), but by the surface slope: thus the scaling must be modified so that $\tau_{2y} \sim \mu\eta_x$ in the first momentum equation: apart from this, the basic ideas are the same, and analysis should be similar. In particular, we can expect that the thermal boundary layer analysis of Chapter VIII will be quantitatively applicable, since the depth of ice sheets is much greater than that of glaciers and hence $\beta_2 << 1$ should be an accurate assumption.

In Chapter II, emphasis was laid on the necessity of considering the moisture content of ice as an independent variable. However, the effect of the moisture content was not directly considered in the ensuing chapters. Furthermore the phenomena of seasonal waves and surges described in Chapter I, possibly the most interesting phenomena requiring explanation, have not been considered at all. These omissions concern work which has not been done at present. We give a brief outline of how these different topics are felt to be related.

Let us suppose that the sliding law with cavitation is of the form shown in Figure 4.2 (b). Consider the kinematic wave equation (5.1) when the flux Q is of the form $u_b H$, corresponding to a large basal sliding component of the velocity, and let H_c be the depth at the first turning point of $H(u_b)$. Then if H is perturbed so that $H > H_c$ at some point, the sliding law implies that the velocity there is much higher than when $H < H_c$:it is reasonable to expect that such regions of perturbed velocity may propagate downstream without significant alteration of the depth. Further,if the discrepancy between the two values of u_b at $H = H_c$ is large, the propagation speed of the velocity wave may be correspondingly large: these are precisely the characteristics observed in fast seasonal waves (Hodge 1974,Deeley and Parr 1914).

This explains how the sliding law may give rise to the fast propagation speeds observed in both seasonal waves and surges. It is unlikely that a single equation such as (5.1) can explain the cyclic phenomenon of surges,which apparently require some mechanism of wave propagation up the glacier (Meier and Post 1969). However,if we consider the energy equation for temperate ice as well as the continuity equation (5.1), with u_b given as a function of H and W (the basal moisture),then the two equations are closely akin to the shallow water equations of fluid mechanics;there are now two wave speeds,one of which may be negative. We envisage the surge as being described by a fast mode and a slow mode, when u_b is respectively at the larger and smaller value corresponding to H_c:these two modes could be bridged by the passage of a wave similar to that of seasonal type. The physical interpretation of the above description is roughly the same as that of Weertman (1969),except that his reference to large amounts of basal water is irrelevant:the 'triggering' of the surge would occur due to an increase in the depth and/or the basal moisture content.

Finally,we should emphasise that the inclusion of the moisture content in the flow model of Chapter II is intended only as a first attempt at representing what may well be very important dynamic effects due to the hydrology. A fuller discussion of this subject is beyond the scope of this work.

APPENDIX 1

Bounds on the Sliding Law

We here derive upper and lower bounds for the sliding law using the variational principles (3.191) and (3.192). As a trial stream function we take (3.199),

$$\psi^* = \frac{y}{\nu} - h[1 + k_1(y - \nu h)]e^{-k_2(y - \nu h)} , \qquad (A1.1)$$

where ν is defined by (3.7), and without loss of generality can be chosen so that the (rough) bedrock h satisfies

$$|h| \; , \; |h'| \leq 1 \; ; \qquad (A1.2)$$

we also assume that

$$|h''| \leq 1 \; : \qquad (A1.3)$$

(A1.3) may be thought of as a kind of white roughness specification.

From (A1.1), we find

$$\psi^*_y = \frac{1}{\nu} - h\, e^{-k_2(y - \nu h)}[k_1 - k_2\{1 + k_1(y - \nu h)\}] \; ; \qquad (A1.4)$$

$$\psi^*_{yy} = k_2 h\, e^{-k_2(y - \nu h)}[2k_1 - k_2 - k_1 k_2(y - \nu h)] \; ; \qquad (A1.5)$$

$$\psi^*_x = - e^{-k_2(y - \nu h)}[h'\{1 + k_1(y - \nu h)\} + \nu h h'\{k_2 - k_1 + k_1 k_2(y - \nu h)\}] \; ; \qquad (A1.6)$$

$$\psi^*_{xy} = h' e^{-k_2(y - \nu h)}[k_2\{1 + k_1(y - \nu h) + \nu h[k_2 - 2k_1 + k_1 k_2(y - \nu h)]\} - k_1] \; ; \qquad (A1.7)$$

$$\psi^*_{xx} = - e^{-k_2(y - \nu h)}[h''\{1 + k_1(y - \nu h)\} + \nu k_1 k_2(y - \nu h)\{2h'^2 + hh'\} + \nu(k_2 - k_1)\{2h'^2 + hh''\} + \nu^2 k_2 h h'^2\{k_2 - 2k_1 + k_1 k_2(y - \nu h)\}] \, . \qquad (A1.8)$$

Therefore

$$\psi^*_{yy} - \psi^*_{xx} = e^{-k_2(y-\nu h)}[k_2 h\{2k_1 - k_2 - k_1k_2(y-\nu h)\} + h''\{1 + k_1(y-\nu h)\}$$

$$+ \nu(2h'^2 + hh'')\{k_2 - k_1 + k_1k_2(y-\nu h)\}$$

$$+ \nu^2 k_2 hh'^2\{k_2 - 2k_1 + k_1k_2(y-\nu h)\}] \,, \qquad (A1.9)$$

and

$$\psi^*_{xy} = e^{-k_2(y-\nu h)}[h'\{k_2 - k_1 + k_1k_2(y-\nu h)\}$$

$$+ \nu k_2 hh'\{k_2 - 2k_1 + k_1k_2(y-\nu h)\}] \,. \qquad (A1.10)$$

By inspection of (A1.9) and (A1.10) we see that

$$II^{(n+1)/2n} = e^{-\alpha Y}|A(x) + B(x)Y + C(x)Y^2|^q \,, \qquad (A1.11)$$

where

$$\alpha = (\frac{n+1}{n})k_2 \,,\; Y = y - \nu h \,,\; q = \frac{n+1}{2n} \,, \qquad (A1.12)$$

and A,B,C are functions of k_1,k_2 and $h(x)$. We are interested in the integral of (A1.11) from $Y = 0$ to $Y = \infty$: in general this is not a trivial computation. Various cases can be treated analytically,e.g. $A = B = 0, B^2 = 4AC$, $B = 0$ if A and C have the same sign,by means of the formula

$$\int_0^\infty (x^2 + u^2)^{\lambda-1} e^{-\mu x}\, dx = \frac{\sqrt{\pi}}{2}(\frac{2\mu}{u})^{\lambda-\frac{1}{2}}\Gamma(\lambda)[\mathbf{H}_{\lambda-\frac{1}{2}}(u\mu) - Y_{\lambda-\frac{1}{2}}(u\mu)] \qquad (A1.13)$$

(Gradshteyn and Ryzhik 1965 p 322),where $\mathbf{H}$ is Struve's function and Y is Neumann's function. Such formulae are of little advantage,and direct numerical integration of (A1.11) would be better.

A simple choice that is of some use is $k_1 = 0$. Then $B = C = 0$ and

$$II^{(n+1)/2n} = e^{-\alpha Y}|A|^q \,, \qquad (A1.14)$$

where

$$A = [\{h'' - k_2{}^2 h + \nu k_2(2h'^2 + hh'') + \nu^2 k_2{}^2 hh'\}^2 + 4k_2{}^2 h'^2(1 + \nu k_2 h)^2] \,, \qquad (A1.15)$$

and (3.191) is

$$H \leq u_b^{*\,1/n} \frac{n}{(n+1)k_2} \frac{1}{M} \int_{-M/2}^{M/2} |A|^q \, dx \,. \tag{A1.16}$$

From (A1.2) and (A1.3),

$$|h|, |h'| \,, |h''| \leq 1 \,, \tag{A1.17}$$

and so, neglecting $O(\nu)$ in (A1.15),

$$H \leq u_b^{*\,1/n} \frac{n}{(n+1)k_2} (1 + 5k_2{}^2)^{(n+1)/2n} \,. \tag{A1.18}$$

$(1 + 5k^2)^{n+1}/k^{2n}$ is a minimum when $k^2 = n/5$, whence we obtain

$$u_b^* \geq \left(\frac{n+1}{5n}\right)^{n/2} (n+1)^{-1/2} H^n \,: \tag{A1.19}$$

numerically, with $n = 3$,

$$u_b^* \geq 0.0688 \, H^n \,. \tag{A1.20}$$

This is about the crudest estimate we could make, as is evident from the smallness of the numerical coefficient in (A1.20). It could be much refined by better numerical approximations. For the Newtonian case $n = 1$, we obtain

$$u_b^* \geq \frac{1}{\sqrt{5}} H = 0.45 \, H \,. \tag{A1.21}$$

We now turn our attention to the trial stress function θ: we choose the form given by (3.218), and obtain

$$\theta_Y = - K(x)k^2 Y e^{-kY} + \nu f(x) \,, \tag{A1.22}$$

$$\theta_{YY} = - K(x)k^2(1 - kY)e^{-kY}, \tag{A1.23}$$

$$\theta_{xY} = - K'(x)k^2 Y e^{-kY} + \nu f'(x) \,, \tag{A1.24}$$

$$\theta_{xx} = K''(x)(1 + kY)e^{-kY} + \nu Y f''(x) \,. \tag{A1.25}$$

For the moment, we shall derive a very crude estimate. Note that we require $\nu y^* << 1$ so that the integral in (3.192) is bounded. Then if we neglect

$O(\nu)$ in (3.216), we have from (3.192)

$$u_b^* \leqslant H^n \frac{1}{M} \int_{-M/2}^{M/2} dx \int_0^{\infty} 2^{-(n+1)/2} \Box^{(n+1)/2} dy \qquad (A1.26)$$

approximately, where

$$\Box \simeq 4\theta_{xY}^{\ 2} + (\theta_{YY} - \theta_{xx})^2$$
$$\simeq 4K'^2k^4Y^2e^{-2kY} + \{K''(1+kY) + k^2K(1-kY)\}^2e^{-2kY} + O(\nu) \ . \qquad (A1.27)$$

Thus, a leading order approximation is

$$u_b^* \leqslant 2^{-(n+1)/2}H^n \frac{1}{M} \int_{-M/2}^{M/2} dx \int_0^{\infty} e^{-(n+1)kY}\{k^2Y^2(4K'^2k^2 + K''^2 + k^4K^2 - 2k^2KK'') + 2kY(K''^2 - k^4K^2) + (K''^2 + k^4K^2 + 2k^2KK'')\}^{(n+1)/2} dy \ . \qquad (A1.28)$$

As already mentioned, this integral is non-trivial to evaluate. Let us take K as defined by (3.221), and write

$$\overline{h'^2} = \frac{1}{M} \int_{-M/2}^{M/2} h'^2 \, dx \ , \qquad (A1.29)$$

so that

$$|K| \ , \ |K'| \ , \ |K''| \leqslant (\overline{h'^2})^{-1} \qquad (A1.30)$$

to $O(\nu^2)$. A simple bound for $\Box$ is obtained by using Minkowski's inequality, viz.

$$\Box^{\frac{1}{2}} \leqslant 2|\theta_{xy}| + |\theta_{yy}| + |\theta_{xx}|$$
$$\leqslant \frac{1}{(\overline{h'^2})^{\frac{1}{2}}} e^{-kY}[2k^2Y + (1+k^2)(1+kY)] \ . \qquad (A1.31)$$

Thus

$$u_b^* \leqslant (2\overline{h'^2})^{-(n+1)/2}H^n \int_0^{\infty} \frac{1}{k} e^{-(n+1)\theta}[(1+k^2) + (1+k)^2\theta]^{n+1} d\theta \ , \qquad (A1.32)$$

or

$$u_b^* \leq (2\overline{h'^2})^{-(n+1)/2} H^n \frac{1}{k}\left[\frac{(1+k)^2}{n+1}\right]^{n+1} \frac{e^{\frac{(n+1)(1+k^2)}{(1+k)^2}}}{(1+k)^2}\,\Gamma\left(n+2, \frac{(n+1)(1+k^2)}{(1+k)^2}\right) , \tag{A1.33}$$

where

$$\Gamma(a,x) = \Gamma(a) - \gamma(a,x) , \tag{A1.34}$$

Γ and γ being the ordinary and incomplete gamma functions respectively. For integral values of n,(A1.33) may be written (by direct integration of (A1.32))

$$u_b^* \leq H^n(2\overline{h'^2})^{-(n+1)/2} \frac{1}{k}(1+k)^{2(n+1)} \sum_{r=0}^{n+1} \frac{n!}{r!(n+1)^{n+1-r}} \left\{\frac{1+k^2}{(1+k)^2}\right\}^r \tag{A1.35}$$

Now $1 + k^2 \leq (1+k)^2$, and $\frac{1}{k}(1+k)^{2(n+1)}$ is a minimum when $k = 1/(2n+1)$, so

$$u_b^* \leq H^n(2\overline{h'^2})^{-(n+1)/2}(2n+1)\left(\frac{2n+2}{2n+1}\right)^{2(n+1)} \frac{n!}{(n+1)^{n+1}} \sum_{r=0}^{n+1} \frac{(n+1)^r}{r!} . \tag{A1.36}$$

Numerically, with n = 3, this gives

$$u_b^* \leq \frac{16.414\, H^n}{(2\overline{h'^2})^2} . \tag{A1.37}$$

Combining (A1.20) and (A1.37) as an inequality for H,

$$(2\overline{h'^2})^{2/n}\, 0.4\, u_b^{*\,1/n} \leq H \leq 2.4\, u_b^{*\,1/n} . \tag{A1.38}$$

When n = 1, the corresponding result is

$$0.447\, H \leq u_b^* \leq \frac{3.51}{2\overline{h'^2}} H . \tag{A1.39}$$

APPENDIX 2

The Thermal Boundary Condition

In this appendix we give a detailed discussion of the thermal boundary condition on the bedrock (2.19). We consider the following three cases: firstly $x_Q < x < x_Z$, where the ice is sliding and $T < T_M$; secondly $x_Z < x < x_M$, where $T = T_M$ but the ice immediately above the bedrock is cold, and lastly $x > x_M$, where the melting surface breaks away from the bedrock, and the basal layer is temperate. In the last case, it may not perhaps be immediately obvious what the bedrock viscous heating source is doing. In discussing this we are led to a slight refinement of the ideas on sliding presented in Chapter III.

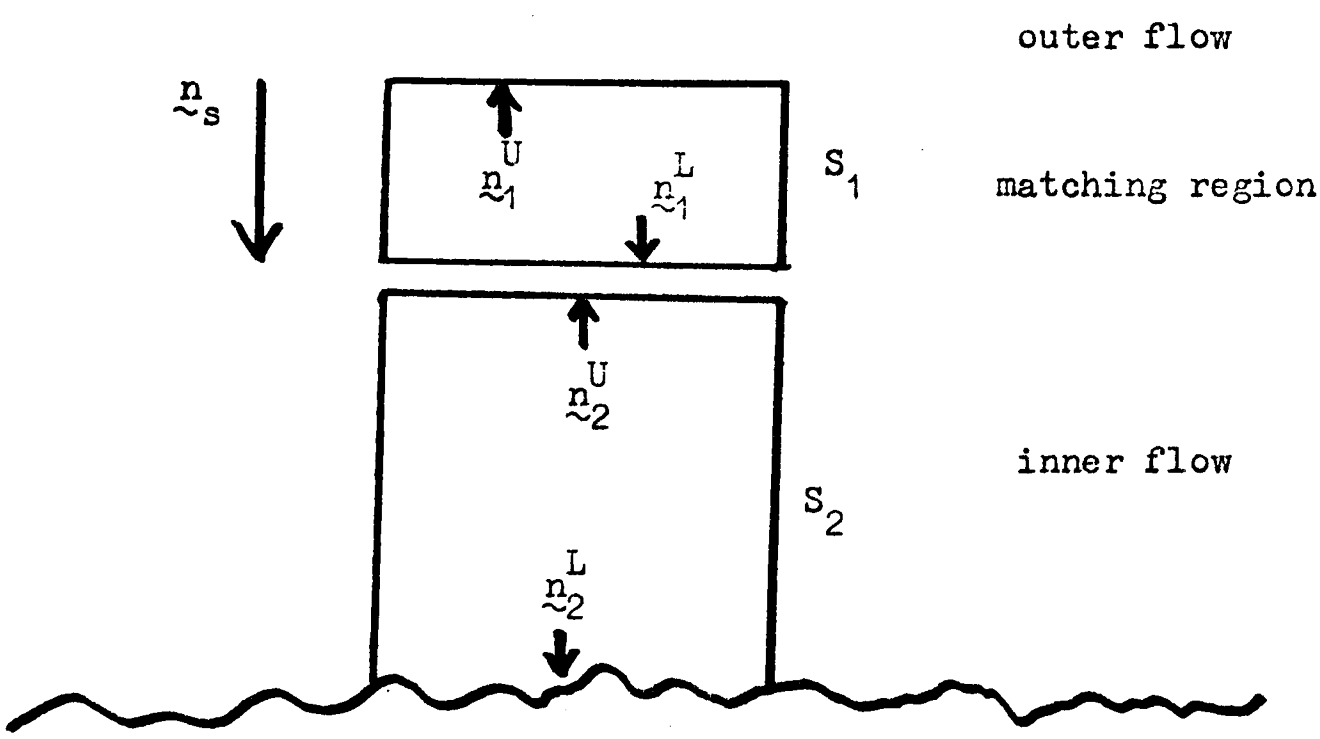


Figure A2.1: basal flow geometry.

Let us consider the geometry shown in Figure A2.1. The lower surface S_2 extends from the bedrock to the matching region between the inner and outer flows of Chapter III, and is of length $\sim [x]$ in the x direction. S_2 lies in the matching region, and is the equivalent of the usual limiting cylinder one considers in obtaining the continuity of heat flux condition (Batchelor 1967 p 69). Here we consider S_1 to be much shorter in the y direction than it is in the x direction. Integration of the energy equation and subsequent shrinkage of S_1 to a point gives the condition

$$k\frac{\partial T_o}{\partial n_1^U} = k\frac{\partial T_I}{\partial n_1^L} \;, \tag{A2.1}$$

where o and I denote outer and inner variables, and $\underset{\sim}{n}_i^{U,L}$ are the normals as depicted in Figure A2.1.

We also let $\underset{\sim}{n}_s$ be the outward normal at the smooth bedrock to the whole outer flow. Thus from the figure,

$$\underset{\sim}{n}_s = -\underset{\sim}{n}_1^U = \underset{\sim}{n}_1^L = -\underset{\sim}{n}_2^U \tag{A2.2}$$

and so

$$\begin{aligned} k\frac{\partial T_o}{\partial n_s} &= k\frac{\partial T_I}{\partial n_1^L} \\ &= -k\frac{\partial T_I}{\partial n_2^U} \;, \end{aligned} \tag{A2.3}$$

by continuity of $\frac{\partial T}{\partial n}$. We now integrate the steady state temperature equation over the interior of S_2. Using Green's theorem, we obtain

$$\int_{S_2} \left[k\frac{\partial T}{\partial n} + \sigma_{in} v_i - \rho c T q_n \right] dS = 0 \;. \tag{A2.4}$$

Although we cannot let $[x] \to 0$, it is nevertheless true that the change in the integrand over a length $[x]$ is negligibly small, since the variables change by $O(1)$ over a much longer length scale. Equating the top and bottom contributions to (A2.4) and using (A2.3), we have that the boundary condition for the outer flow is (dropping suffices)

$$-k\frac{\partial T}{\partial n} + \sigma_{nt} v_t \simeq \left[\int_{S_b} -(k\partial T/\partial n_2^L)\, dx \right] / \delta x \;, \tag{A2.5}$$

where S_b is that part of the surface S_2 which lies on the bedrock. We have used the fact that $v_n = 0$ on the top surface, and $\sigma_{in} v_i = q_n = 0$ on the bedrock. The last function in (A2.5) is a mean thermal flux escaping from the bedrock, and is precisely the definition of $A(T)$ in (2.19) (apart from a scale factor of G).

In $x_Z < x < x_M$, the melting surface leaves the bedrock, but remains in the inner flow layer until x_M; in this case the appropriate form of S_2 to take is as shown in Figure A2.2. The thermal boundary condition is

$$- k\frac{\partial T}{\partial n} + \sigma_{nt} v_t \simeq [\int_{S_M} \xi_M' \sigma_{in} v_i \, dx]/\delta x \, , \qquad (A2.6)$$

where S_M is the part of the surface S_2 coincident with ξ_M. The right

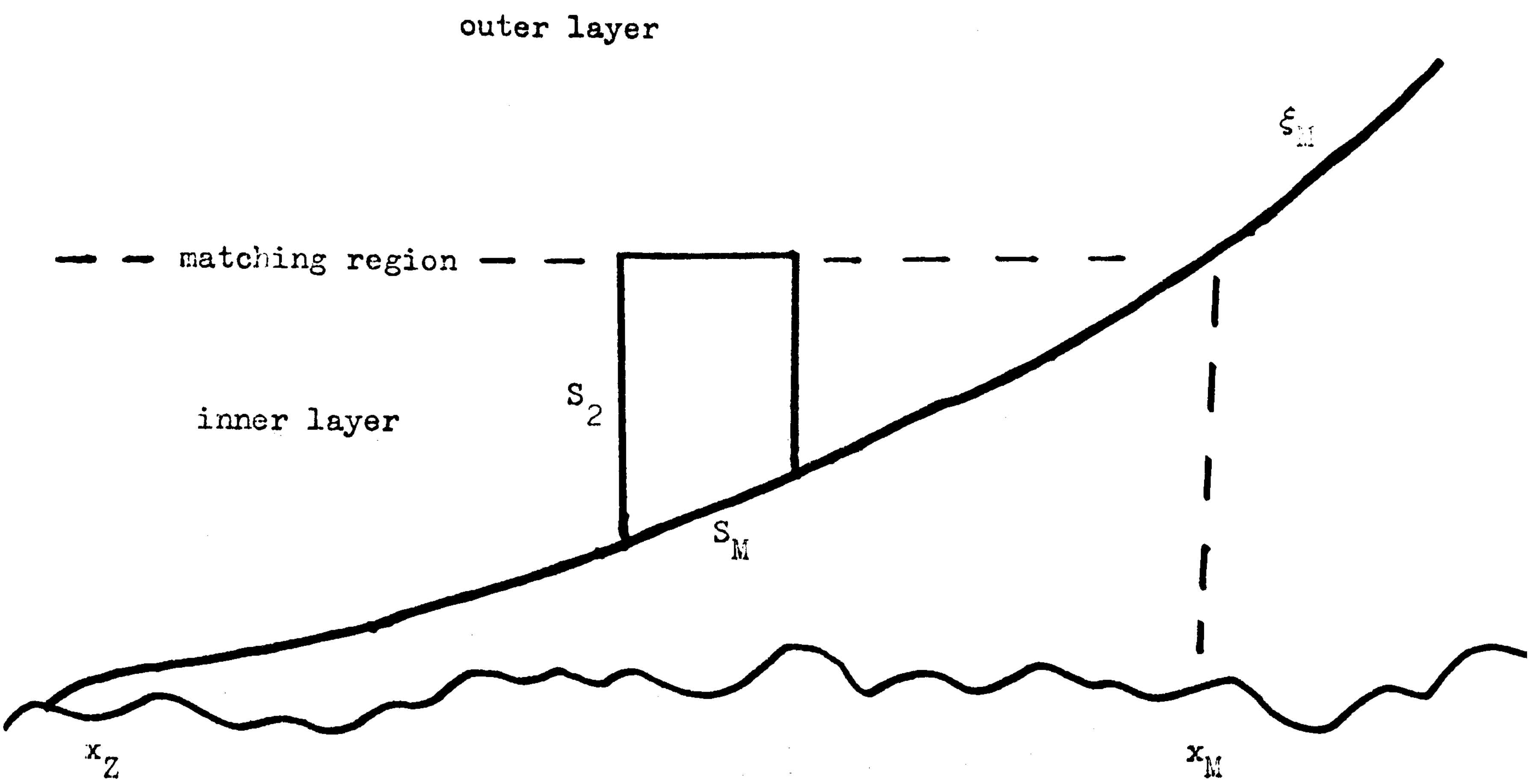


Figure A2.2: $x_Z < x < x_M$.

hand term must formally be determined from the inner flow problem: we see that $\frac{\partial T}{\partial n}$ decreases as $x \to x_M$ and equals zero at x_M, when the melting surface 'detaches' from the smooth bedrock (as seen by the outer flow).

In $x > x_M$, what is the viscous heating term $\sigma_{nt} q_t$ balanced with? Since the temperature gradient is (effectively) zero and there is no moisture diffusion term, it is idle to integrate the energy equation (2.7) over S_1, since (2.7) does not require a boundary condition to be given on the characteristic $\psi = 0$. We see from (2.7) that (since $[\tau]_i \sim [\tau]_o/\nu$ from Chapter III) the rate of moisture generation in the inner layer is much larger than in the outer layer. In fact, using (3.193), (2.44) and

(2.24), we find

$$(\rho L\frac{dw}{dt})/(\tau_{ij}e_{ij}) \sim \sigma \, , \tag{A2.7}$$

and so moisture production in the inner layer is $O(1/\sigma)$ times as great as in the mainstream. This may be the reason why basal ice has been found to be much more permeated with water veins than that away from the bedrock (Carol 1947).

The moisture may vary considerably in the inner layer, and thus the viscosity cannot practically be considered as a constant, as was assumed in Chapter III. However,

$$\frac{\ell}{\sigma} = \delta^{-1}\sigma^{-2}[x] >> [x] \, , \tag{A2.8}$$

and so on the $[x]$ scale the moisture is still effectively constant along a stream line: thus in the sliding theory we can write

$$w \simeq w(\psi) \, . \tag{A2.9}$$

The form of this function depends on the shape of the melting surface within the inner layer, and is thus non-trivial to find. We might for example try to solve the Newtonian flow problem with the viscosity η as

$$\eta = \eta(\psi) \, ; \tag{A2.10}$$

this is markedly similar to Kamb's use of a viscosity depending only on y, although introduced here for a totally different reason. It is unclear whether a stream function dependent viscosity like (A2.10) can be accommodated in the variational principle of Chapter III.

APPENDIX 3

Nomenclature

This is not the most complete list of symbols possible, but it contains all the more important ones, and indicates when certain symbols have two or more meanings in different chapters.

$a(x,t)$: accumulation/ablation rate.

a_0 : typical accumulation rate.

: accumulation rate at the head of the glacier (VIII).

a_s : value of a at snout of glacier.

A : rheological function in Glen's power law (2.8).

A_0 : typical value of A.

A_T, A_W : typical values of A for cold and temperate ice.

$A(\sigma)$: initial glacier profile (V).

c : specific heat of ice (II).

: height of initial shock (V).

C : constant in sliding law (III, VI).

C, C' : contours in complex plane (IV).

d : typical depth, (2.38).

$d_{ij} = e_{ij}$: used in IV.

e, e_{ij} : strain rate invariant and tensor.

f : temperate sliding law.

: used in V for the datum state as a function of the 'characteristic' coordinate.

f_i : gravity terms in variational principle (III).

F : temperature dependent sliding law .

: analytic function in Hilbert problem (IV).

g : gravity.

g' : gravity component perpendicular to mean bedrock slope.

G : geothermal heat flux.

: analytic function (IV).

$G(\Psi)$: outer solution for 'temperature' χ in VIII.

$h(x)$: bedrock.

h_R, h_S, h_D : rough, smooth, dimensional bedrock.

$\mathcal{H}$: functional for stress principle (III).

$H(x,t)$: dimensionless glacier depth.

H_C, H_T, H_M : varieties of H introduced in VII.

H_O : datum profile (VIII).

II, II_τ : tensor invariants in III.

I_n : integrals considered in IV.

J : functional for velocity principle (III).

k, k_R : thermal conductivities of ice and rock.

L : latent heat of ice.

: dimensional period of rough bedrock (III).

ℓ : typical length scale.

M : dimensionless period of rough bedrock (III).

n : exponent in Glen's law.

$\underset{\sim}{n}$: unit outward normal at a boundary.

p : pressure (II).

[p] : pressure scale near the bedrock (III).

p_A, p_W, p_T : atmospheric, water (in the bedrock lubrication layer) and triple point pressures.

p_c : cavitation pressure (4.5).

$\underset{\sim}{q}, q_i$: velocity and components.

Q : activation energy.

R : gas constant.

Re : Reynolds number.

r, r_1 : rheological functions for temperate ice.

$s(x,t)$: flux function (2.23).

$s(x)$: unknown ice-water boundary (III).

s : strained coordinate (VI).

: Laplace transform variable in VIII.

s_1 : perturbation to flux function (V).

t : time.

T : temperature.

$\underset{\sim}{T}$: tangential vector (3.94).

T(x) : applied stress at bedrock (III).

T(σ) : time for neighbouring characteristics to intersect (5.58).

T_M : melting temperature (2.14).

T_O : temperature scale.

T_A : atmospheric temperature.

T_F : atmospheric melting temperature of ice.

T_Q : temperature at which basal sliding starts to occur.

u : velocity in x direction.

u_b : basal velocity.

u_b^* : scaled basal velocity (3.190).

$\tilde{u}_b, u_b^+$: basal velocity in inner and outer regions on ε-scale (VIII).

v : velocity in y direction.

V_M : suction velocity on bedrock due to pressure melting.

w : moisture content.

[w] : scale for w.

W : basal moisture.

x : length coordinate.

[x] : longitudinal dimension of bedrock roughness.

x_T : top melting point.

x_M : bottom melting point.

x_O : glacier head.

x_s : glacier snout.

x_E : equilibrium point.

x_Q : point where sliding begins.

x_Z : point where full sliding law becomes operative.

x_d : shock position (V).

x_e : envelope position (V).

y : height coordinate.

y_M : melting surface.

[y] : transverse dimension of bedrock roughness.

z : complex variable, = x + iy (III,IV).

α : regelation parameter (III);stability parameter (VII);and sundry other meanings.

β_1 : viscous dissipation parameter.

β_2 : thermal conduction parameter.

β : cavitation parameter (IV).

: = $\kappa_1\beta_1$;modified dissipation parameter (VII & VIII).

$\beta(x)$: dissipation function in VIII.

γ : ratio of A_w and A_0,(II).

: cavitation parameter (IV).

δ : shallow ice parameter (2.25).

:film thickness parameter (III).

ε : bedrock slope (II,III,IV).

: x-scale in thermal boundary layer (VIII).

ζ : = e^{iz} (4.16).

η : top surface of glacier (II).

: viscosity of ice considered as a Newtonian fluid.(III).

: distance coordinate (VIII).

θ : Clausius-Clapeyron constant (2.14).

: mean temperature (5.9).

θ^* : dimensionless form of θ (2.65).

ι : 'temperature' function (VIII).

κ : thermal parameter defined by (2.62).

$\Lambda(T)$: scaled temperature-dependent geothermal heat flux function.

λ : geothermal heat flux parameter.

μ : surface slope parameter (2.58).

: viscosity of water (III).

ν : bedrock roughness parameter (III & IV).

: small coefficient in snout equation (VI).

$\Pi(x)$: pressure in lubrication layer (III).

ξ : depth coordinate .

: $= e^{ix}$, complex variable (IV).

: linearised characteristic coordinate (V).

ξ_M : melting surface.

ρ : density of ice.

σ_{ij} : stress tensor.

σ : small parameter in sliding theory (III).

: characteristic parameter (V).

: growth constant in stability analysis (VII).

Σ : dimensionless ice-water interface (III).

τ : second invariant of stress tensor.

: conductive time scale (VII).

$[\tau]$: typical stress.

τ_b : basal stress.

ϕ : Airy stress function (III).

χ : temperature function (VII & VIII).

ψ, Ψ : stream functions.

ω : seasonal frequency (V).

: thermal boundary layer scale (VIII).

APPENDIX 4

An Initial Estimate of the Stability Criterion for Large Conduction

In this appendix, we consider an estimate of the stability criterion for the steady state temperature field in the case of large conduction. The notation used is that of Chapter VII, §6. Generally speaking, estimates of σ can be obtained from (7.89) by computation using the Rayleigh-Ritz criterion (Hildebrand 1965): for the present we shall consider only a very simple bound on the stability parameter α. From (7.89) the stability criterion may be written

$$\sigma \geq 0 \text{ if } \int_V [\alpha(\eta - y)^{n+1}\phi^2 - \phi_y^{\,2}] \geq \delta^2 \int_V \phi_x^{\,2}\, dV\,. \tag{A.4.1}$$

Let us define

$$\begin{aligned} H^* &= H\,,\ x < x_M\,, \\ &= \xi_M\,,\ x > x_M\,, \end{aligned} \tag{A4.2}$$

so that H^* is the depth of the cold ice region. It is equal to the depth H except where there is a temperate layer adjoining the bedrock.

The simplest trial function which satisfies the conditions that $\phi = 0$ on $\xi = 0$ and $\xi = \xi_M$ is then

$$\phi = (\eta - y)(y - h^*) \equiv \xi(H^* - \xi)\,, \tag{A4.3}$$

where $h^* = \eta - H^*$. Now H^* is continuous and piecewise continuously differentiable, and so it follows that ϕ is continuous on V + S, and has piecewise continuous first derivatives in V in the sense of Courant and Hilbert (op.cit.), where V is the interior and S the boundary of the domain of interest.

In order that the integrals in A4.1 be finite, we require $H^{*\prime}$ to be finite at $H^* = 0$: although this is not predicted by the steady state solution of Chapter VII (e.g. $H^* \sim (x - x_0)^{1/(n+2)}$ as $x \to x_0$), it is a realistic assumption, as shown in Chapter VI, and therefore we assume that it is satisfied here. It then follows that the right hand side of (A4.1) is $O(\delta^2)$, and on changing the variables to x and ξ, the stability criterion may be written

$$\sigma \geq 0 \text{ if } \int_V [\alpha\xi^{n+1}\phi - \phi_\xi^{\,2}]\,dV \geq O(\delta^2)\ . \qquad (A4.4)$$

Substituting the expression for ϕ from (A4.3) into (A4.4) and performing the integration, we obtain

$$\sigma \geq 0 \text{ if } \alpha \geq \frac{(n+4)(n+5)(n+6)}{2\int_{x_0}^{x_s} H^{n+6}\,dx}\left[\frac{1}{3}\int_{x_0}^{x_s} H^3\,dx + O(\delta^2)\right]. \qquad (A4.5)$$

Thus an upper bound for the critical stability parameter α_c is given approximately by

$$\alpha_c \leq \frac{(n+4)(n+5)(n+6)\int_{x_0}^{x_s} H^3\,dx}{6\int_{x_0}^{x_s} H^{n+6}\,dx}\ . \qquad (A4.6)$$

It is evident that from a practical point of view, this value is far too high and gives no useful information beyond (perhaps) indicating that for $\alpha \sim 1$, the solution is likely to be stable. We might also surmise that the relatively high value of n is a stabilising feature, but that increasing the depth would be destabilising.

In point of fact, we can show that the estimate (A4.5) is quite useless, assuming the initial perturbation is made from the datum profile H_0. For in that case, if we define

$$H_{max} = \max_{x_0 < x < x_s} H^* , \qquad (A4.7)$$

then $\chi_2'(H_{max}) > 0$ (see the discussion on page 141), and hence from (7.66)

$$\alpha H_{max}^{\;n+3} < n + 3\ ; \qquad (A4.8)$$

but then

$$\frac{\int_{x_0}^{x_s} H^3\,dx}{\int_{x_0}^{x_s} H^{n+6}\,dx} \geq \frac{1}{H_{max}^{\;n+3}} > \frac{\alpha}{n+3}$$

from (A4.8), and so (A4.5) can only hold if

$$1 > \frac{(n+4)(n+5)(n+6)}{6(n+3)} ,$$

which is plainly false.

The point is, however, that even if the steady solutions are essentially stable, we have demonstrated a mechanism whereby instability can set in if α is large enough. This is in contrast to the attitude of previous authors, who have been content to discuss instabilities from a phenomenological point of view: 'many glaciers surge, therefore the steady state must be unstable'.

APPENDIX 5

Existence and Uniqueness of the Outer Flow Solution when $\beta_2 \ll 1$

Consider the differential equation posed in (8.31) and (8.32) on page 160,

$$\Psi_{\xi\xi} = \xi^n G(\Psi) \tag{A5.1}$$

with boundary conditions

$$\begin{aligned} \Psi &= s \text{ on } \xi = 0 , \\ \Psi &= 0 , \ \Psi_\xi = -u_b \text{ on } \xi = H , \end{aligned} \tag{A5.2}$$

where

$$\begin{aligned} u_b &= 0 , \ x < x_Q , \\ u_b &= u_b(H) , \ x_Z < x < x_M , \end{aligned} \tag{A5.3}$$

and H is to be determined as part of the solution. G is a monotone increasing function: we can suppose without any loss of generality that $\max(s) = 1$, so that $0 < s < 1$ (and hence $0 < \Psi < 1$), and also that $\chi = 1$ when $\xi = 0$ and $s = 1$ (this makes x_T and x_E the same point); then $\chi_0 < G < 1$, and we may for convenience extend the definition of G beyond (0,1) by putting $G = \chi_0$ in $\Psi < 0$, $G = 1$ in $\Psi > 1$.

In this appendix we shall prove the following result: the problem posed in (A5.1) and (A5.2) has a unique solution, with the property that H is an increasing function of s (and hence of x in the accumulation area) as long as the sliding law $u_b(H)$ is monotone increasing.

To do this, let us consider the initial value problem (A5.1) with the conditions

$$\Psi(0) = s , \ \Psi'(0) = -u_s < 0 . \tag{A5.4}$$

Since $G(\Psi) \leq 1$, we find on integrating twice that

$$\Psi < s - u_s\xi + \frac{\xi^{n+2}}{(n+1)(n+2)} \qquad (A5.5)$$

while $\Psi_\xi < 0$, and since by choosing u_s sufficiently large the right hand side has a zero for some positive ξ, it follows that for u_s sufficiently large, Ψ also has a zero for positive ξ (in fact two, since $\Psi \to \infty$ as $\xi \to \infty$).

Now consider

$$\Psi_1'' = \xi^n G(\Psi_1) , \qquad (A5.6)$$

$$\Psi_2'' = \xi^n G(\Psi_2) , \qquad (A5.7)$$

$$\Psi_1(0) = \Psi_2(0) = s , \qquad (A5.8)$$

$$0 > \Psi_1'(0) > \Psi_2'(0) . \qquad (A5.9)$$

At $\xi = 0$, $\Psi_1' - \Psi_2' > 0$, hence $\Psi_1 > \Psi_2$ in a neighbourhood of the origin. Within this neighbourhood $G(\Psi_1) > G(\Psi_2)$ (while $0 < \Psi_1 < 1$), so $\Psi_1'' > \Psi_2''$ and $\Psi_1' - \Psi_2'$ increases. It is positive at zero, and therefore $\Psi_1' - \Psi_2' > 0$ throughout the neighbourhood, i.e. $\Psi_1 - \Psi_2$ increases. Since we remain in the neighbourhood while $\Psi_1 > \Psi_2$, the above shows that $\Psi_1 > \Psi_2$ for all $\xi > 0$. Thus solutions of (A5.1) and (A5.4) are monotone increasing functions of $\Psi'(0) = -u_s$ for fixed s. We now show that they are continuous functions of u_s as well.

Consider Ψ_1 and Ψ_2 as defined by (A5.6)-(A5.8), and

$$\Psi_1'(0) = \Psi_2'(0) + \delta , \ \delta > 0 . \qquad (A5.10)$$

If we suppose that G' is bounded, then using the mean value theorem we have

$$\Psi_1'' - \Psi_2'' = \xi^n[G(\Psi_1) - G(\Psi_2)] < M(\Psi_1 - \Psi_2) , \qquad (A5.11)$$

provided we restrict our attention to a finite range of ξ. By comparison with $\Psi_1'' - \Psi_2'' = M(\Psi_1 - \Psi_2)$ subject to the boundary conditions (A5.8) and (A5.10), it follows from (A5.11) that

$$0 < \Psi_1 - \Psi_2 < \delta \sinh\sqrt{M}\xi , \qquad (A5.12)$$

and therefore by taking δ small enough we have that Ψ is continuously dependent on u_s (but not uniformly at $\xi = \infty$).

Now if we denote the first zero of Ψ by $\xi = H$, and call the slope there $\Psi'(H) = - u_b$, then it follows from the arguments above that u_b and H are continuous functions of $\Psi'(0)$, and furthermore one easily shows in the same manner that $\Psi'(H)$ and H are monotone increasing functions of $\Psi'(0)$: this is obvious from graphical considerations (Figure A5.1): thus u_b given by the solution of the initial value problem is a decreasing function of H. It is zero when H is a double zero of Ψ, and increases to ∞ as $u_s \to \infty$, as is easily seen from a consideration of (A5.5), for example. Thus if the sliding law $u_b(H)$ is an increasing function of H, there exists a unique solution to (A5.1) and (A5.2) given in $x_Z < x < x_M$ by the intersection of the curves, and in $x < x_Q$ by the double zero of Ψ where $u_b = 0$. These are illustrated in Figure A5.2. If the sliding law is multivalued (Chapter IV), the interesting possibility exists of multiple steady states: these are unlikely to occur unless $\Psi'(H)$ has sufficiently small slope (Figure A5.3). There are of necessity an odd number of intersections: we should then expect the odd-numbered ones to be stable and the even-numbered ones to be unstable, as shown.

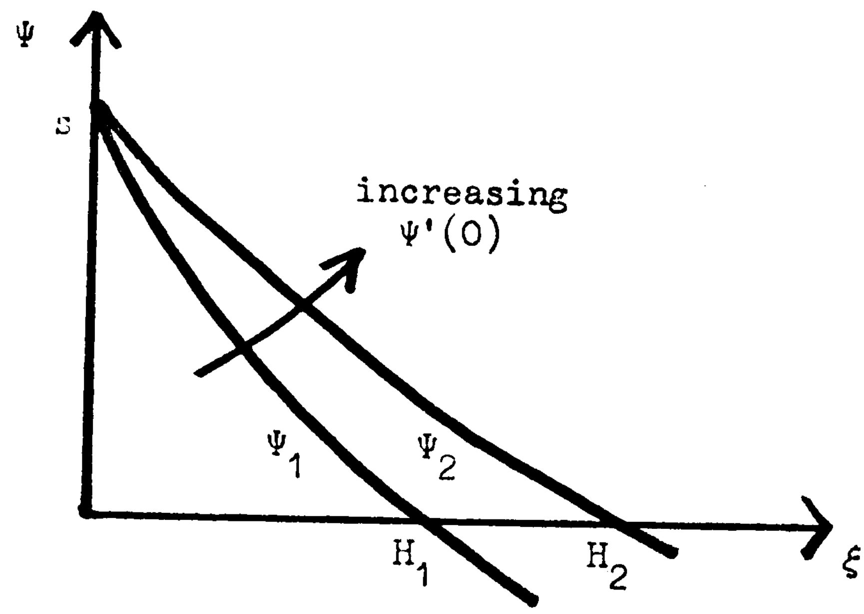


Figure A5.1

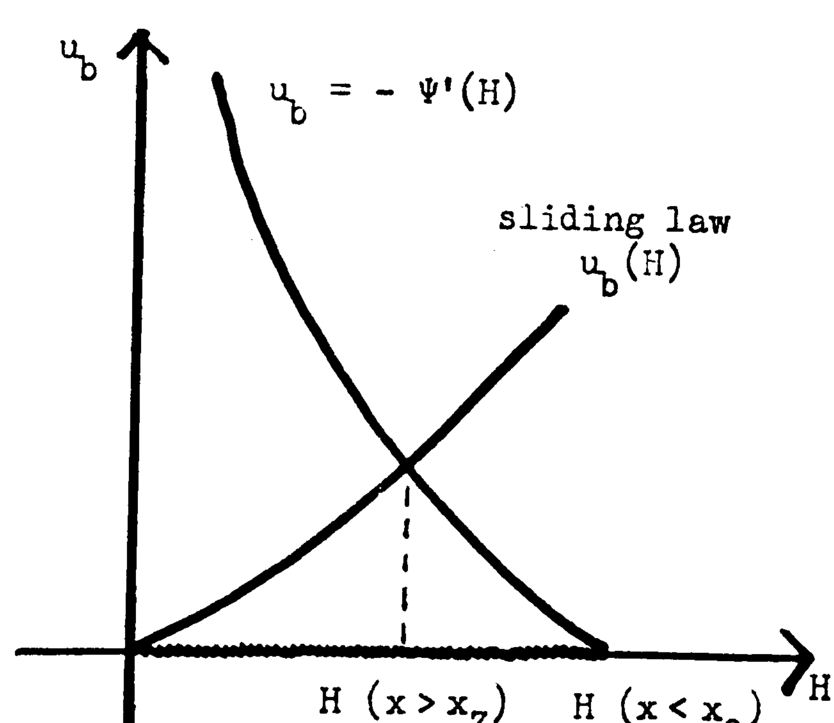


Figure A5.2

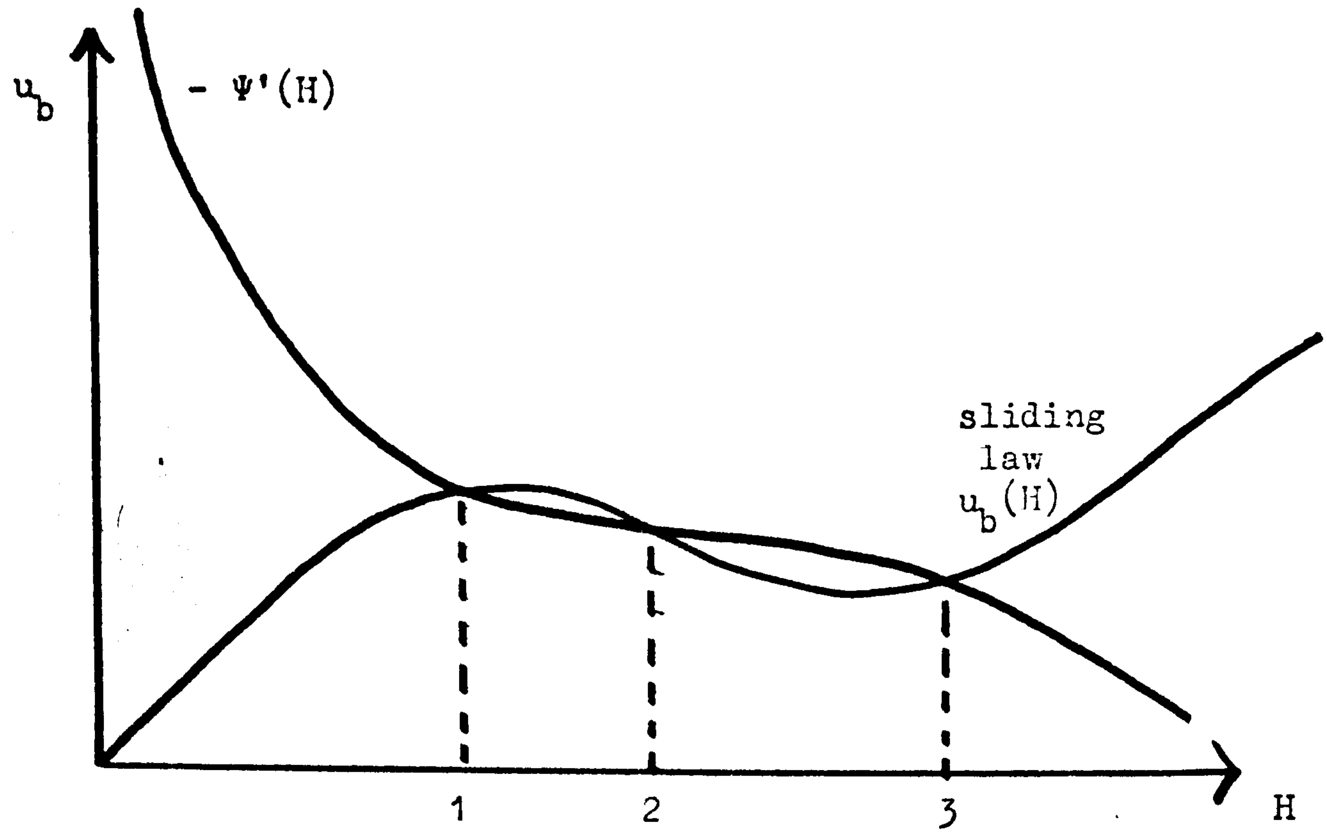


Figure A5.3

Finally we examine the effect of varying s. Suppose we have two solutions (Ψ_i, s_i, H_i) of (A5.1) and (A5.2) with $H_1 > H_2$. Put $Z = H_i - \xi$ and consider Ψ_i as functions of Z, so that

$$\begin{aligned} \Psi_1'' &= (H_1 - Z)^n G(\Psi_1) , \\ \Psi_2'' &= (H_2 - Z)^n G(\Psi_2) , \\ \Psi_1(0) &= \Psi_2(0) = 0 , \\ \Psi_1'(0) &\geq \Psi_2'(0) \geq 0 , \end{aligned} \tag{A5.13}$$

the last equation being inequality or equality depending on whether $x_Z < x < x_M$ or $x < x_Q$, respectively. At $Z = 0$, $\Psi_1 - \Psi_2 = 0$, $\Psi_1' - \Psi_2' \geq 0$ and $\Psi_1'' - \Psi_2'' = (H_1^n - H_2^n)\chi_0 > 0$; therefore $\Psi_1' - \Psi_2'$ is initially positive and so $\Psi_1 - \Psi_2$ is positive in a neighbourhood of $Z = 0$: but then $G(\Psi_1) > G(\Psi_2)$ and so $\Psi_1'' - \Psi_2''$ remains positive, and as before $\Psi_1 - \Psi_2$ is positive in $0 < Z < H_2$. Since Ψ_1 is increasing, it follows that $s_1 > \Psi_1(H_2) > \Psi_2(H_2) = s_2$, and therefore s is a monotone increasing function of H: hence also H increases with s in $x < x_Q$ and $x_Z < x < x_M$.

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