Towards Fully-Local 2d Chiral CFTs from Conformal Nets: Bicommutant Categories and Fusion of Their Modules

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A thesis submitted for the degree of

DPhil in Theoretical Physics

Hilary 2026

Abstract

We give a new definition of a bicommutant category, a categorified analogue of a von Neumann algebra. Our definition is independent of a choice of faithful representation: as a bi-involutive W^*-tensor category, it is required to act on its own absorbing ideal, with the left and right actions being each other's commutants. The absorbing ideal thus plays the role that the standard form plays for a von Neumann algebra. We develop a string calculus that makes the coherences of this definition manifest. We define modules over bicommutant categories and the 'categorified' Connes fusion of such modules, and we propose a definition of the 3-category that bicommutant categories form.

We introduce left- and right-localised endomorphisms of the algebras that a conformal net $\mathcal{A}$ assigns to multi-intervals, generalising the Doplicher–Haag–Roberts endomorphisms, and prove that these form a monoidal category equivalent to the category $\mathrm{Sol}(\mathcal{A})$ of solitonic representations of the net. Using this equivalence, we prove that $\mathrm{Sol}(\mathcal{A})$ is a bicommutant category for every conformal net, assuming neither finite-index nor strong additivity. As a corollary, its Drinfeld centre is the representation category $\mathrm{Rep}(\mathcal{A})$.

We partially develop a fully-local Segal-style functorial framework for two-dimensional chiral conformal field theories arising from conformal nets. We construct the dimension-0 and dimension-1 components, assigning a bicommutant category to a point with a one-dimensional collar and appropriate bimodule W^*-categories to 1-dimensional cobordisms, and we establish compatibility of the fusion of module categories with the gluing of 1-dimensional cobordisms. The circle is assigned $\mathrm{Rep}(\mathcal{A})^{\mathrm{abs}}$, which agrees with the Drinfeld centre of the value on the point up to passing to absorbing ideal.

Acknowledgements

I would like to thank Merton College for the generous support throughout my degree.

I would like to thank my supervisor, André, for all of your support and guidance, both mathematical and otherwise. Our interactions have always been a genuine pleasure; from sitting on the floor in your office to the tops of snow-covered mountains, meetings with you never failed to lift me out of a low mood or a moment of doubt, and you have a remarkable ability to make anyone feel genuinely excited about mathematics. For reasons I will never quite understand, you had faith in my mathematical abilities, and for that I am immensely grateful. Thank you for always making time for my questions and helping me build intuition for the mathematics that I came to love. I feel extraordinarily lucky to have had you as an advisor and even more so for the way you managed to be both a wonderful supervisor and a friend.

I would like to thank Graeme. It is itself quite extraordinary that I got to talk to the person who laid the foundation for my field of research. Thank you for meeting with me at All Souls, for telling me about the important papers from the past and for being patient enough to teach me concepts from geometry and homotopy theory. Thank you also for thinking with me about open questions and showing me what being passionate about 'understanding things deeply' feels like. I value all the interactions with you greatly.

Thank you to all the amazing people in my field, who showed excitement for my work and offered great insights. Theo, Mayuko, David, Claudia, Konrad, Chris, Matthias, Dan, Ingo, and Christoph, just to mention a few names.

Thank you to Severin for being a wonderful collaborator and putting our project on hold while I finished my thesis. Thanks to Lucas and Thomas for their interest and their patient reads of early drafts of the thesis.

Thanks to my maths friends Adri, Glen, Filippos, and Thibault for making life fun and putting up with me when I was being a bit of a princess sometimes.

Thanks to Chetan for being the only person other than André to be invested in the results of my thesis as much as I was. I would not have been where I am without your unconditional support. Thanks for being my friend for the last eight years.

Thanks to my parents and family for their love and trust.

Finally, thanks to everyone who loved and supported me in this journey, from long walks to staying up with me to making me dinner; for making sure I felt loved and cared for at all times.

Contents

Notation and Conventions

- We shall use the term manifold to mean a compact oriented smooth manifold with (possibly empty) boundary unless specified otherwise.
- An interval is a connected 1-manifold with nonempty boundary.
- A circle is a connected 1-manifold with empty boundary.
  - The standard circle $S^1 \subsetneq \mathbb{C}$ consists of complex numbers of unit norm.
  - We denote the subinterval of S^1 consisting of complex numbers of non-negative (non-positive) imaginary part by S_+ (S_-).
  - We choose the counterclockwise orientation.
- A multi-interval is a disjoint union of intervals.
- Let M be a manifold. Then
  - its orientation allows us to split ∂M into incoming and outgoing boundaries, which we denote as $\partial_- M$ and $\partial_+ M$ respectively.
  - we write $\mathrm{Diff}^+(M)$ for the group of orientation preserving smooth diffeomorphisms of M.
  - we denote M with the opposite orientation by $\overline{M}$.
  - given an $N \subseteq M$, we write $\mathrm{cl}(N)$ for its closure in M.
  - given an $N \subseteq M$, we write $M \setminus^c N$ for the closure of $M \setminus N$ in M.
- The standard interval is $[0,1]$ considered as a smooth manifold with $\partial_-[0,1] = \{0\}$ and $\partial_+[0,1] = \{1\}$.

- Let κ denote an infinite cardinal. We call a Hilbert space κ-separable if it admits a basis of cardinality $< \kappa$, and we call a von Neumann algebra κ-separable if it admits a faithful representation on a κ-separable Hilbert space. All Hilbert spaces that appear in this document are assumed to be $\aleph_1$-separable (i.e. usual notion of separability for Hilbert spaces). Further, all von Neumann algebras will also be $\aleph_1$-separable.

- Given a von Neumann algebra A and a subset $X \subseteq A$, we denote its commutant by $[X]'_A$.

- We denote the category of von Neumann algebras and homomorphisms by vNAlg and the bicategory of von Neumann algebras, bimodules and intertwiners as $\mathrm{vNAlg}^{\mathrm{bimod}}$.

- Given an embedding between manifolds $i\colon N \hookrightarrow M$ we will treat N to be a subset of M by identifying N and $i(N)$. Similarly, given an injective homomorphism of von Neumann algebras, we treat it as a subalgebra.

- All inner products appearing in this thesis are antilinear in the first variable and linear in the second.

- The term conformal net by default means a chiral conformal net throughout the text.

Chapter 1

Exposition and Introduction

1.1 A Brief History of QFT

Quantum field theory (QFT) lies at the heart of modern theoretical physics. Physicists have made revolutionary progress in the subject starting in the early days of quantum mechanics in the first quarter of the 20$^{\text{th}}$ century, and this progress continues today. Substantial mathematics has arisen from the methods and principles of QFT. However, we are still far from an entirely mathematically rigorous foundation and understanding of this subject.

One of the first attempts to axiomatise quantum field theory came in the form of the **Wightman axioms**, first introduced in [WG64] (see [SW16] for exposition) which aim to capture the behaviour of quantum fields on Minkowski spacetime M. Here, quantum fields are operator-valued distributions and have a rich algebraic structure. The data of the QFT consists of:

- a complex Hilbert space H (space of states).
- a vacuum vector $\Omega \in H$ of unit norm.
- a collection of field operators (operator valued distributions) Φ_a for $a \in \mathcal{I}$ (some indexing set) defined on M, so that each Φ_a sends a test function f on M to an operator $\Phi_a(f)$, such that they share a common domain D which is dense in H and $\Omega \in D$.
- a projective unitary representation $U\colon \mathrm{Sym} \to PU(H)$ of the relevant symmetry group of the theory. There is usually a subgroup $\mathrm{Trans} \subset \mathrm{Sym}$

of translations, and we require that the restriction of U to this subgroup lifts to a strongly continuous unitary representation of Trans $\cong M$. There is a self-adjoint generator of translations, P called the energy-momentum operator, which lives in $\mathrm{Hom}(D, H) \otimes M^*$, characterised by

$$U(x) = \exp\bigl(i\langle P, x\rangle\bigr), \qquad x \in \mathrm{Trans}.$$

The group Sym can be taken to be the group of isometries for a general Riemannian QFT, the group of conformal transformations for a CFT, the group of diffeomorphisms for a topological QFT.

With the above data, there are the following axioms:

- Covariance: The dense domain D is invariant under the action of Sym as well as the actions of the fields and the fields satisfy the following for any test function f and any $g \in \mathrm{Sym}$

$$U(g)\Phi_a(f)\,U(g)^* = \Phi_a\left(f \circ g^{-1}\right)$$

- Locality: For any two test functions f_1, f_2 whose supports are spacelike separated, and any two fields Φ_{a_1}, Φ_{a_2}, we have $[\Phi_{a_1}(f_1), \Phi_{a_2}(f_2)] = 0$.

- Spectrum condition: The energy operator H is the generator of time translations, and the momentum operators $P^1, \ldots, P^{d-1}$ are the generators of space translations. We require that the joint spectrum of $(H, P^1, \ldots, P^{d-1})$ lies in the closed forward light cone; equivalently, the operators H and $H^2 - \sum_i (P^i)^2$ are both positive.

- The only translation invariant vectors are multiples of Ω.

- Ω is cyclic under the actions of $U(g)$ and $\Phi_a(f)$.

In the case of conformal field theories, the chiral fields, axiomatised into vertex operator algebras (VOAs) by Borcherds [Bor86], naturally arise as Wightman fields.

Bounded operators are extracted from compactly smeared field operators (which are typically unbounded) through spectral projections and other methods. The von Neumann algebra generated by these operators is called the algebra of observables. These observables correspond to physically measurable quantities, and, hence, it is natural to consider an axiomatisation where the algebra of observables takes centre stage. An early axiomatisation of the algebras of observables was done by Haag and Kastler [HK64]. A related, more recent axiomatisation of observables is the theory of factorisation algebras [BD04, CG16, CG21].

The **Haag-Kastler axioms** for a QFT on a Lorentzian manifold M in terms of the algebras of observables are as follows:

- To every causally closed subset U of M, we assign a C^*-algebra $\mathcal{O}(U)$.
- To every inclusion $U_1 \subseteq U_2$, we should get an inclusion $\mathcal{O}(U_1) \hookrightarrow \mathcal{O}(U_2)$. Whenever we have $U_1 \subseteq U_2 \subseteq U_3$ the following triangle should commute

$$\begin{tikzcd} \mathcal{O}(U_1) \arrow[r] \arrow[d] & \mathcal{O}(U_2) \arrow[dl] \\ \mathcal{O}(U_3) & \end{tikzcd}$$

- Whenever $U_1, U_2 \subseteq U$ are space-like separated, in $\mathcal{O}(U)$ we have

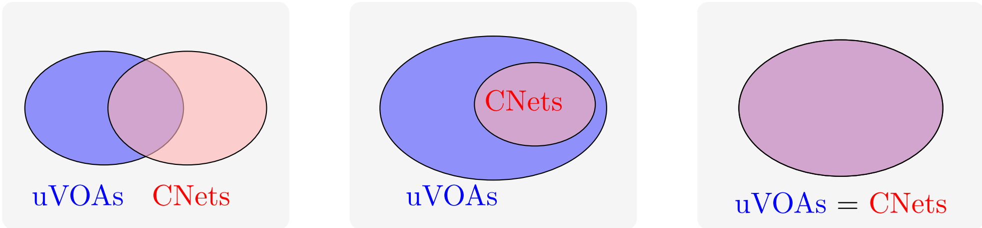
$$[\mathcal{O}(U_1), \mathcal{O}(U_2)] = 0.$$

Often it is required that these algebras act on a Hilbert space of states and there is a cyclic vacuum vector.

The two formalisms, Wightman axioms and Haag-Kastler axioms, capture the Heisenberg approach to Quantum Field Theory, where the algebraic properties of fields (and observables) are studied. These approaches are together referred to as Algebraic Quantum Field Theory (AQFT).

In the case of chiral conformal field theories, the Haag-Kastler axioms on a cylinder give rise to the notion of a conformal net introduced in [BMT88, BGL93, GF93], (see Definition 3.2.1), which is shown to be a factorisation algebra on a circle in [Hen16].

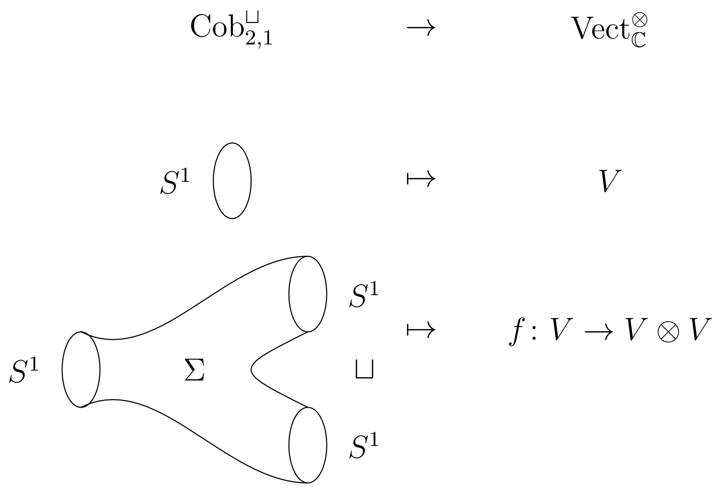
It is *conjectured* that unitary vertex operator algebras and conformal nets are in a bijective correspondence. In [CKLW18] it was shown that every unitary vertex operator algebra satisfying a certain technical assumption called strong locality yields a conformal net, and that it is possible to recover the starting VOA (up to isomorphism) from the constructed net. The work of Henriques and Tener [HT25] shows that every conformal net has an associated unitary VOA. The following illustration (adapted from [HT25]) summarises the current understanding of the relationship between uVOAs and conformal nets.



by [CKLW18] by [HT25] conjecturally

In the work that follows, we will take Conformal Nets to be the algebraic quantum field theoretic descriptions of 2-dimensional chiral CFTs.

In 1988-89, another axiomatic approach to QFT was proposed by Atiyah [Ati88] and Segal [Seg88, Seg04], which is now studied broadly as Functorial Quantum Field Theory (FQFT). FQFT axiomatises the properties of a path integral, which is one of the central tools for physicists to study QFT. In this approach, a space of states is assigned to space slices, and propagators are assigned to spacetime interpolating them. More precisely, it is a symmetric monoidal functor from $\mathrm{Cob}_{2,1}^{\sqcup}$, the category of manifolds and cobordisms with disjoint union as the monoidal structure to $\mathrm{Vect}_{\mathbb{C}}^{\otimes}$, the category of vector spaces and linear maps with tensor product as the monoidal structure. Functoriality captures the fact that correlators are consistent with the cutting and gluing of spacetimes; this is sometimes also referred to as locality. The monoidality of the functor captures the nature of independent physical systems, that is, disjoint systems are assigned a tensor product of state spaces. The quantum nature of the theory is also reflected in the non-cartesian monoidal structure of the target category.



This formalism has been the most fruitful in the study of topological quantum field theories. Following the introduction of extended topological field theories as first discussed in [Law93] and [Fre93], the field was transformed by the Cobordism Hypothesis, proposed in [BD95] and a detailed outline of a proof was given in [Lur09]. The Cobordism hypothesis is a statement about the classification of fully-local or fully-extended topological quantum field theories in terms of fully-dualisable objects in the chosen algebraic target category. Fully-local implies that the field theory is not just a functor between categories but a higher functor between higher categories and a d-dimensional field theory will assign algebraic data to all k-manifolds with boundaries and corners for $0 \leq k \leq d$. For topological field theories, it can be heuristically said that the theory is fully determined by what it assigns to the point.

The relation between algebraic quantum field theory and functorial quantum field theory has been made explicit in the topological setting in [Sch14], and is also evident in the theory of factorisation homology [AF15]. In particular, fully-extended functorial TQFT can be constructed from locally constant factorisation algebras on $\mathbb{R}^n$, E_n-algebras which are interpreted as the local algebras of observables for topological quantum field theories.

In this thesis, we take a similar approach, for theories more general than TQFTs. We aim to bridge AQFT and FQFT in the chiral CFT setting. We take the data of a conformal net as input and aim to give a partial assignment for the fully-extended functorial formalism for a two-dimensional chiral conformal field theory. We will assign algebraic data to a point and to 1-dimensional cobordisms.

1.2 Chiral Conformal Field Theory

Two-dimensional conformal field theories (CFT) are a very special class of field theories which sit at the sweet spot between topological field theories, which are relatively well understood, and general quantum field theories, for which we are far from any rigorous understanding non-perturbatively. The qualifier 'conformal' refers to the dependence of the theories on only the conformal structure of the spacetime manifold. A conformal structure is an equivalence class of metrics with respect to local rescaling. When working with CFTs, it is important to emphasise the distinction between full CFTs and chiral CFTs.

A **full CFT** is a symmetric-monoidal projective functor from the category of conformal cobordisms to a chosen category of topological vector spaces. For unitary theories, the target category can be chosen to be the category of Hilbert spaces and bounded linear maps.

A full CFT can be thought of as built from a chiral and an anti-chiral half coupled together [Seg88] corresponding to the holomorphic (left-moving) and anti-holomorphic (right-moving) parts.

Many 2d **chiral CFTs** studied in physics arise as gapless boundary conditions for a 3d TQFT [Wit89, EMSS89]. Chiral CFTs admit three independent axiomatisations. The first is the data of a vertex operator algebra, together with its representations and conformal blocks [FBZ04]. The second is that of a conformal net [GF93], and the third is the Segal, or functorial, formulation [Seg88, Seg04].

In the following, we give the Segal definition of a chiral CFT as a functor out of the conformal cobordism category. Conjecturally, part of the data in this definition can be reformulated in terms of a modular functor; for the definition of modular functor, see [Seg04]. To the best of our knowledge, the formulation given below does not appear elsewhere in the literature in this form and is due to Henriques [Hen25]. We reformulate it and record it here in detail for expositional completeness.

Following [Seg04] and [Hen25], 2d chiral CFT consists of the following:

- A strong monoidal functor of bicategories Z_χ^{top} from $\mathrm{Cob}^{\mathrm{conf,or}}_{2+\epsilon,2,1}$ to 2-(Top)Vect. The domain bicategory $\mathrm{Cob}^{\mathrm{conf,or}}_{2+\epsilon,2,1}$ is the bicategory of oriented conformal cobordisms whose objects are compact closed oriented 1-manifolds, whose 1-morphisms are complex cobordisms (which are Riemann surfaces with boundary equipped with a decomposition of the boundary components into incoming and outgoing), and whose top level morphisms are biholomorphic diffeomorphisms that preserve boundary. Since the top level morphisms are only the invertible morphisms, we use the notation '$2+\epsilon$'. The target bicategory 2-(Top)Vect can be modelled by a topological version of LinCat, the bicategory of linear categories, functors and natural transformations. An important property is that $\mathrm{End}_{\text{2-(Top)Vect}}(\mathbf{1}) \simeq \text{(Top)Vect}$ where $\mathbf{1}$ denotes the monoidal unit.

This functor assigns a linear category $\mathcal{C}_S := Z_\chi^{\mathrm{top}}(S)$ to every closed (compact, smooth, oriented) 1-manifold S, and to every complex cobordism Σ with incoming and outgoing boundaries S_{in} and S_{out} respectively, a linear functor, $Z_\chi^{\mathrm{top}}(\Sigma) := F_\Sigma$, that is $F_\Sigma : \mathcal{C}_{S_{\mathrm{in}}} \to \mathcal{C}_{S_{\mathrm{out}}}$ compatible with disjoint unions, identity cobordisms, and composition (gluing) of cobordisms.

- A lax-natural transformation Z_χ^{hol} between the strong monoidal bi-functors Z_χ^{top} as defined above and $\mathrm{const}_{\mathbf{1}}$ which is the constant functor mapping every object in the domain to the monoidal unit $\mathbf{1}$ of the target bi-category and every morphism to identity on the monoidal unit. This natural transformation is required to be pointwise faithful.

  For every closed 1-manifold S, we get a faithful functor $U_S := Z_\chi^{\mathrm{hol}}(S)$, $U_S\colon \mathcal{C}_S \to \text{(Top)Vect}$ that equips $\mathcal{C}_S$ with the structure of a concrete linear category. For every complex cobordism Σ from S_{in} to S_{out}, we get a natural transformation as follows:

$$\begin{tikzcd} \mathcal{C}_{S_{\mathrm{in}}} \arrow[r, "U_{S_{\mathrm{in}}}"] \arrow[d, "F_\Sigma"'] \arrow[dr, phantom, "\swarrow"] & \text{(Top)Vect} \arrow[d, "\mathrm{id}_{\text{(Top)Vect}}"] \\ \mathcal{C}_{S_{\mathrm{out}}} \arrow[r, "U_{S_{\mathrm{out}}}"'] & \text{(Top)Vect} \end{tikzcd}$$

  In other words, for every object $X \in \mathcal{C}_{S_{\mathrm{in}}}$, the assignment $Z_\chi^{\mathrm{hol}}(\Sigma)$ determines a continuous linear map $U_{S_{\mathrm{in}}}(X) \to U_{S_{\mathrm{out}}}\big(F_\Sigma(X)\big)$.

  Equivalently, the data of Z_χ^{top} and Z_χ^{hol} can be packaged into a single functor

$$Z_\chi : \mathrm{Cob}_{2+\epsilon,2,1}^{\mathrm{conf,or}} \to \text{(Top)LinCat}_{\mathrm{concrete}}$$

valued in (Top)LinCat$_{\text{concrete}} \simeq \text{TopAlg}_*^{\text{bimod}}$, where (Top)LinCat$_{\text{concrete}}$ denotes the category of concrete linear categories and concrete functors and $\text{TopAlg}_*^{\text{bimod}}$ denotes the category of topological algebras, pointed bimodules, and pointed bimodule maps. A concrete linear category is a pair $(\mathcal{C}, U_{\mathcal{C}})$, where $\mathcal{C}$ is a linear category and $U_{\mathcal{C}} \colon \mathcal{C} \to$ (Top)Vect is a faithful functor. A concrete functor is a pair (F, η) of a linear functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural transformation $\eta : U_{\mathcal{C}} \Rightarrow U_{\mathcal{D}} \circ F$.

- Let $\text{Ann}\,(S^1)$ be the endomorphisms of the standard circle S^1 in the conformal bordism category and $\widetilde{\text{Ann}}_c(S^1)$ be a central extension by the group $\mathbb{C}^\times \times \mathbb{Z}$ as defined in [Hen25, Equation 12] and constructed in detail in [HT24, Section 5]. $B\text{Ann}\,(S^1)$ and $B\widetilde{\text{Ann}}_c(S^1)$ are deloopings which form one-object bi-categories that include into $\text{Cob}_{2+\epsilon,2,1}^{\text{conf,or}}$.

  Let $\text{const}_{Z_\chi^{\text{top}}(S^1)}$ be a functor that assigns $\mathcal{C}_{S^1}$ to the object of $B\widetilde{\text{Ann}}_c(S^1)$ and $\text{id}_{\mathcal{C}_{S^\infty}}$ to every $\tilde{A} \in \widetilde{\text{Ann}}_c(S^1)$.

  The data of a chiral CFT includes a natural isomorphism

$$T \colon \text{const}_{Z_\chi^{\text{top}}(S^1)} \Rightarrow Z_\chi^{\text{top}}$$

  corresponding to trivialisation of annuli, that is, an isomorphism $F_A(X) \xrightarrow{\simeq} X$ for every object $X \in \mathcal{C}_{S^1}$, every annulus $A \in \text{Ann}(S^1)$ and every lift to the central extension $\tilde{A} \in \widetilde{\text{Ann}}_c(S^1)$.

- Finally, we consider the whiskering of natural transformations $Z_\chi^{\text{hol}} \circ T$ as in the following square,

$$\begin{tikzcd}
S^1 \arrow[r, mapsto] \arrow[d, "\tilde{A}"'] & \mathcal{C}_{S^1} \arrow[r, equal] \arrow[d, equal, "\text{id}"] \arrow[dr, phantom, "\overset{T}{\swarrow}\simeq"] & \mathcal{C}_{S^1} \arrow[r, "U_{S^1}"] \arrow[d, "F_A"] \arrow[dr, phantom, "\swarrow Z_\chi^{\text{hol}}"] & \text{(Top)Vect} \arrow[d, equal, "\text{id}"] \\
S^1 \arrow[r, mapsto] & \mathcal{C}_{S^1} \arrow[r, equal] & \mathcal{C}_{S^1} \arrow[r, "U_{S^1}"'] & \text{(Top)Vect}
\end{tikzcd}$$

  For every object $X \in \mathcal{C}_{S^1}$, the above diagram induces a map

$$\tilde{A} \mapsto \Big(U_{S^1}(X) \longrightarrow U_{S^1}(F_A(X)) \xrightarrow{\simeq} U_{S^1}(X) \Big)$$

We require that the map from $\widetilde{\mathrm{Ann}}_c(S^1) \to \mathrm{End}(U_{S^1}(X))$ is *holomorphic*. More details of this holomorphicity condition can be found in [Hen25].

We finally summarise the definition of a chiral CFT in the following diagram:

$$\begin{tikzcd}
B\widetilde{\mathrm{Ann}}_c\left(S^1\right) \arrow[r] \arrow[rrr, bend left=30, "\mathrm{const}_{Z_{\mathcal{X}}^{\mathrm{top}}(S^1)}"] & B\mathrm{Ann}\left(S^1\right) \arrow[r] & \mathrm{Cob}_{2+\epsilon,2,1}^{\mathrm{conf,or}} \arrow[u, phantom, "\sim\Downarrow T"] \arrow[r, "Z_{\mathcal{X}}^{\mathrm{top}}"] \arrow[r, bend right=50, "\mathrm{const}_1"'] \arrow[r, phantom, shift right=4, "\Downarrow Z_{\mathcal{X}}^{\mathrm{hol}}"] & \text{2-(Top)Vect}
\end{tikzcd} \tag{1.1}$$

In the thesis that follows, we solely work with unitary chiral conformal field theories; we fix the topological vector spaces to be Hilbert spaces and the category $(\mathrm{Top})\mathrm{LinCat}^{\mathrm{concrete}}$ to be concrete complete W^*-categories, denoted $W^*\text{-}\mathrm{Cat}_{\mathrm{concrete}} \simeq \mathrm{vNAlg}_*^{\mathrm{bimod}}$ developed in [HNP24]. Importantly, to a closed compact 1-manifold, we assign a von Neumann algebra or equivalently a complete W^*-category with a faithful functor to Hilb; and to a closed Riemann surface, we assign a pointed Hilbert space.

2d chiral CFTs are very closely related to 3d Topological Quantum Field Theories. They form a bulk/boundary system. The holographic principle states that the space of states over a 2-dimensional surface for a 3d TQFT can be identified with the space of conformal blocks for the boundary 2d chiral CFT. More physically, as explained in [SF00], this vector space is the space of (gauge equivalence classes of) boundary conditions for the fields appearing in the path integral of the 3d theory.

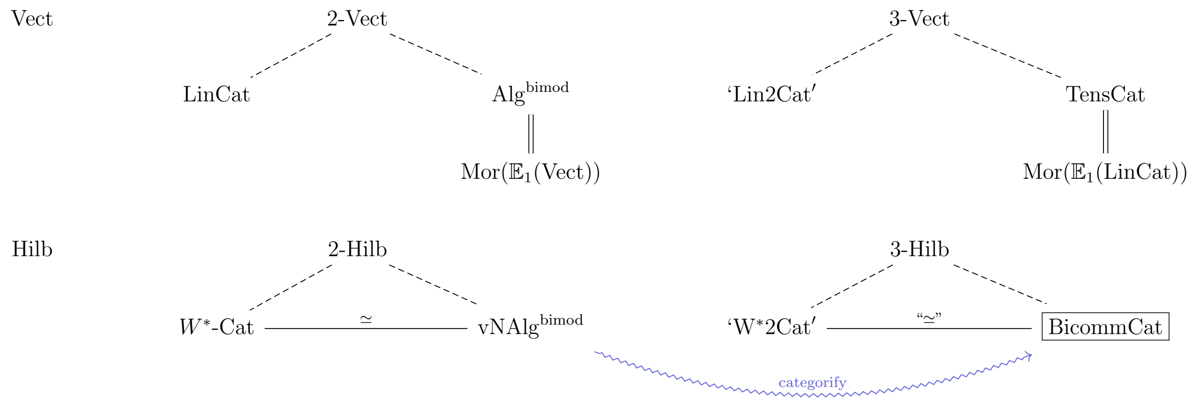
From the Baez-Dolan-Lurie Cobordism hypothesis, we know that it is often insightful to consider fully-extended theories; that is, theories that are representations of cobordism categories extending all the way down to a point. It is then a natural question to extend a functorial chiral CFT down to a point, that is, to build it as a functor out of a fully-extended conformal cobordism category $\mathrm{Cob}_{2+\epsilon,2,1,0}^{\mathrm{conf,or}}$.

For a fixed target category, the object assigned to the point by a functorial TQFT is a fully-dualisable object. Therefore, it is important to characterise fully-dualisable objects of typically used target categories. It is easy to check that the fully-dualisable objects of Vect and Hilb are finite dimensional vector spaces and Hilbert spaces, respectively. In the bicategory of algebras, the fully-dualisable objects are separable algebras [Lur09] but for von Neumann algebras, a fully-dualisable von Neumann algebra is necessarily a finite direct sum of type I factors [BDH22]. Finally, the fully-dualisable objects in the Morita category of tensor categories are fusion categories [DSPS20].

Previously, the 3-category of conformal nets constructed in [BDH18] was proposed to be the target category for twisted 2-dimensional quantum field theories of which 2d chiral CFTs are an instance. This was proposed as a symmetric monoidal 3-category that deloops the bicategory of von Neumann algebras, bimodules, and intertwiners, that is, $\mathrm{Hom}_{\mathrm{CNet}}(\mathbf{1},\mathbf{1}) = \mathrm{vNAlg}^{\mathrm{bimod}}$. They show that the collection of conformal nets, defects, sectors, and intertwiners, equipped with the fusion of defects and fusion of sectors, forms a symmetric monoidal 3-category. The finite-index conformal nets are fully dualisable objects in this 3-category of conformal nets [BDH22]. This 3-category a priori does not look analogous to the standard 3-category used as a target for 3-dimensional theories, TensCat, the 3-category of tensor categories, bimodule categories, equivariant functors and natural transformations.

As we enter the realm of functional analytic categories, the usual Morita constructions do not generalise. A von Neumann algebra is not, in general, an associative algebra object (or $\mathbb{E}_1$ object) in the symmetric monoidal category of Hilbert spaces and bounded linear maps because the underlying vector space of a von Neumann algebra is not a Hilbert space to start with. But, there is strong evidence to believe that complete W^*-categories are categorified Hilbert spaces (see the table of analogies in the beginning of [HNP24]). For a model for 3-Hilb, we would like it to be analogous to TensCat, where we replace tensor categories by functional analytic tensor categories that ‘categorify’ von Neumann algebras. These tensor categories are called bicommutant categories and have previously been introduced in [Hen17a, Hen17c, HP23]. The Morita category of bicommutant categories is expected to deloop W^*-Cat.

In the following chart, we summarise the typically used algebraic target categories:



1.3 Overview

The thesis can be roughly divided into three parts.

Part One: Definition of Bicommutant Category

Bicommutant categories are higher categorical analogues of von Neumann algebras, introduced by Henriques in [Hen17c] and further developed jointly with Penneys [HP17] and in [Hen17a]. A von Neumann algebra can be characterised concretely as a unital $*$-subalgebra $A \subset B(H)$ satisfying $A'' = A$, where the bicommutant A'' is taken inside $B(H)$. The categorical analogue replaces the ambient algebra $B(H)$ by the tensor category $\mathrm{Bim}(R)$ of bimodules over a hyperfinite factor R, equipped with the monoidal structure given by Connes' fusion. The category $\mathrm{Bim}(R)$ admits two involutions: an antilinear involution at the level of objects (the conjugate bimodule) and an involution at the level of morphisms (the adjoint of a bounded linear map of Hilbert spaces). A tensor category equipped with both such involutions is called a *bi-involutive tensor category*.

Definition (Older definition from [Hen17a])**.** A **bicommutant category** is a bi-involutive tensor category $\mathcal{T}$ for which there exists a hyperfinite factor R and a fully faithful bi-involutive tensor functor $\mathcal{T} \to \mathrm{Bim}(R)$ such that the natural inclusion $\mathcal{T} \hookrightarrow \mathcal{T}''$ is an equivalence of bi-involutive tensor categories, where $\mathcal{T}''$ denotes the double commutant of $\mathcal{T}$ inside $\mathrm{Bim}(R)$.

Here, we write $\mathcal{T}'$ for the commutant of the tensor category $\mathcal{T}$ in the tensor category $\mathrm{Bim}(R)$. It is the category whose objects are pairs (Y, e) with $Y \in$

$\mathrm{Bim}(R)$ and $e = \{e_X : X \otimes Y \to Y \otimes X\}_{X \in \mathcal{T}}$ a half-braiding with all the elements of $\mathcal{T}$ (which here we identify with their image in $\mathrm{Bim}(R)$), abusing the notation. The bicommutant $\mathcal{T}'' := (\mathcal{T}')'$ is equipped with a natural 'inclusion' functor $\mathcal{T} \to \mathcal{T}''$.

This definition categorifies the von Neumann bicommutant theorem: just as a $*$-algebra $A \subset B(H)$ is a von Neumann algebra if and only if $A = A''$, a bi-involutive tensor category is a bicommutant category if and only if it coincides with its own bicommutant inside $\mathrm{Bim}(R)$.

Every von Neumann algebra admits a canonical faithful representation called the Haagerup Standard form [Haa75] or the L^2-space of the von Neumann algebra, on which the algebra acts from the left and right, and the commutant of the left action is the right action, and vice versa.

Correspondingly, one is led to establish a similar notion for bicommutant categories which does not depend on any choice. The first part of the thesis is dedicated to presenting a modified definition of a bicommutant category. We have also worked towards developing a string calculus that makes the coherences in the definition of a bicommutant category manifest and easy to interpret. The passage from the abstract notion that we introduce in this thesis to the concrete one previously existing is immediate. The converse direction, however, remains open. We also define modules over bicommutant categories and give a definition of 'categorified' Connes fusion.

Part Two: Solitonic Representations of Conformal Nets

The primary examples of bicommutant categories arise from unitary fusion categories [HP17] and from conformal nets [Hen17a]: in particular, the category of solitonic representations of a finite-index conformal net is a bicommutant category [Hen17a], which is the main result that this part of the thesis generalises.

We review key definitions and properties of conformal nets in detail in the thesis. Briefly, a conformal net $\mathcal{A}$ is a mathematical axiomatisation of two-dimensional chiral conformal field theory in the operator-algebraic framework of Haag and Kastler [HK64]. It consists of a Hilbert space H_0 (the vacuum sector), a vacuum vector $\Omega \in H_0$, and an assignment

$$I \mapsto \mathcal{A}(I)$$

that associates a von Neumann algebra of observables to every interval $I \subset S^1$. This assignment is required to be isotonous ($I \subset J$ implies $\mathcal{A}(I) \subset \mathcal{A}(J)$) and local (algebras associated to disjoint intervals commute). The structure is further equipped with a projective action of $\mathrm{Diff}^+(S^1)$ on H_0 encoding conformal symmetry, and the generator of rotations L_0 is required to have positive spectrum. A key consequence of these axioms is Haag duality,

$$\mathcal{A}(I') = \mathcal{A}(I)',$$

which asserts that the algebra assigned to the complement of an interval is precisely the commutant [GF93, BGL93].

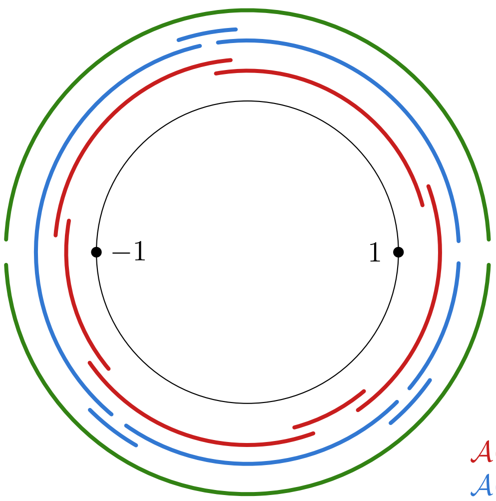
The representation theory of conformal nets is most naturally described using the Doplicher–Haag–Roberts framework [DHR71, DHR74], where localised endomorphisms provide the basic language for analysing superselection sectors. Concretely, a superselection sector is given by a representation of the observable algebra that is locally equivalent to the vacuum, and the DHR analysis shows that such sectors can be described in terms of endomorphisms ρ that are localised in some interval, in the sense that ρ acts trivially on algebras associated to disjoint intervals.

These localised endomorphisms form a category, with tensor product given by composition, and with a natural braiding coming from the statistics operators studied by Fredenhagen, Rehren, and Schroer [FRS89]. In the setting of conformal nets, the relation between representations and DHR endomorphisms can be made precise in the work of Longo and Rehren [LR95]. A modern presentation of the result can also be found in Section 6 of [Gui21]. Moreover, under suitable finiteness conditions (for instance, complete rationality), this category becomes a modular tensor category, as shown by Kawahigashi, Longo, and Müger [KLM01].

The key insight for this part of the thesis is that, analogous to the known equivalence between the categories of endomorphisms and bimodules for type III factors, as well as the equivalence between DHR endomorphisms and representations of the net, we establish similar equivalences for solitonic representations. We introduce left- and right-localised endomorphisms and prove that the categories of these are monoidally equivalent to the category of solitonic representations of the net.

In the figures below we illustrate the intervals on S^1 to show that the actions of algebras associated to these intervals by the conformal net give rise to the relevant representation-theoretic structures: a general endomorphism of the algebra associated to upper semicircle S_+ determines an $(\mathcal{A}(S_+), \mathcal{A}(S_+))$-bimodule (here we are using Haag duality which tells us $\mathcal{A}(S_+) \simeq \mathcal{A}(S_-)^{\mathrm{op}}$) or a Hilbert space that has action of algebras associated to all $\{\mathcal{A}(I)\}_{I \subseteq S_+}$ and $\{\mathcal{A}(I)\}_{I \subseteq S_-}$ but not of those intervals that cross the boundary points. A DHR (bi-localised) endomorphism, whose support is compactly contained away from both 1 and -1, determines a DHR representation of the conformal net; and a left- or right-localised endomorphism, that is supported close to one of the boundary points 1 or -1, determines a solitonic representation of the net.

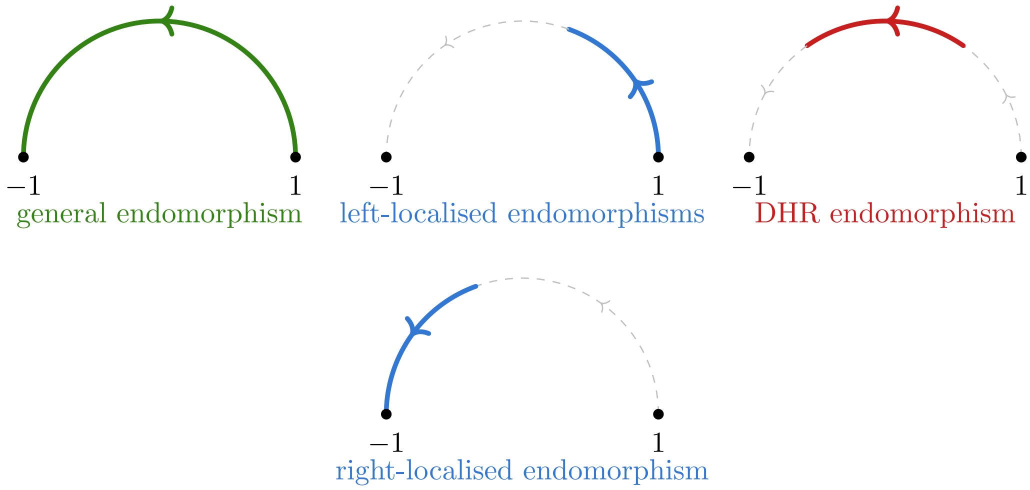
We discover that for different pursuits of interest, either the model with endomorphisms or the model with Hilbert spaces with actions is easy to work with. To give an instance, the monoidal structure is easy to describe for the endomorphism categories since it is just composition of endomorphisms whereas the involutions on the category are hidden. This is easy to describe in the Hilbert space with actions perspective since we have complex conjugation on Hilbert spaces.



$\mathcal{A}(I)\ \ \forall\, I \subset S^1$ acts: Representation of conformal net $\mathcal{A}$

$\mathcal{A}(I)\ \ \forall\, I \subset S^1 \mid 1 \notin \mathring{I}$ acts: Solitonic Representation of $\mathcal{A}$

$\mathcal{A}(I)\ \ \forall\, I \subseteq S_+ \ \wedge\ \forall\, I \subseteq S_-$ acts: $\mathcal{A}(S_+) \otimes \mathcal{A}(S_-)$module



We prove that the category of solitonic representations of a conformal net is a bicommutant category. We prove this statement without any assumption on the conformal net. Importantly, we discard the previously important finite-index assumption as well as a weaker condition called strong additivity, which some physical examples of interest do not possess. In this generality, we also show that the Drinfeld centre of the category of solitonic representations is the category of representations of the net.

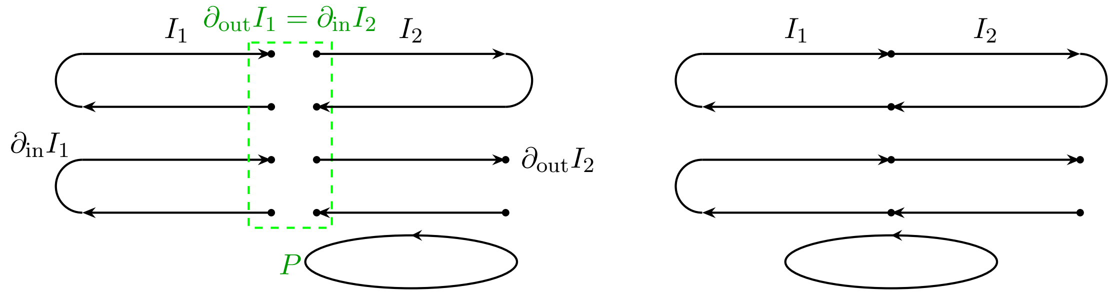
Part Three: Assignment to Cobordisms

After introducing the candidate target 3-category BicommCat and constructing what we can potentially assign to a point, namely, the category of solitonic representations, as previously proposed in [Hen17c] for WZW models, our final aim is to fully-extend the functorial description of general 2d chiral CFTs.

Cob	Algebraic Assignment
Point P + 1-dim collar	A Bicommutant category $\mathcal{W}_P$
1-cobordism $I\colon P_{\text{in}} \to P_{\text{out}}$	$\mathcal{W}_{\text{in}}$-$\mathcal{W}_{\text{out}}$-bimodule W*-category ${}_{\mathcal{W}_{\text{in}}}\mathcal{M}_{\mathcal{W}_{\text{out}}}$ realised as A-Mod for a von Neumann algebra A
2-cobordism $\Sigma\colon M_{\text{in}} \to M_{\text{out}}$	A concrete equivariant W^*-functor F : A_{in}-Mod $\to$ A_{out}-Mod realised by a pointed A_{out}-A_{in}-bimodule

A closed 1-cobordism S has $\mathcal{W}_{\text{in}} = \mathcal{W}_{\text{out}} = \mathbf{1} = \text{Hilb}$, so the bimodule W*-category is a plain W*-category. It has the structure of a (braided) monoidal (often non-unital) category. A closed 2-cobordism has $A_{\mathrm{in}} = A_{\mathrm{out}} = \mathbb{C}$, so the pointed bimodule reduces to a pointed Hilbert space $(\mathcal{H}, \Omega \in \mathcal{H})$, this can be seen from the fact that every W^*-functor **Hilb** $\to$ **Hilb** is necessarily of the form $- \boxtimes_{\mathbb{C}} H$ for some Hilbert space H [HNP24].

We fix a conformal net $\mathcal{A}$. In this thesis, we only construct the assignment to dimension 0 and 1. The gluing of 1-manifolds corresponds to 'categorified' Connes fusion of modules over the bicommutant categories assigned to the gluing boundary.



We show that,

$$\mathcal{W}_{\partial_{\mathrm{in}} I_1} \circlearrowright \mathcal{M}_{I_1} \boxtimes_{\mathcal{W}_P} \mathcal{M}_{I_2} \circlearrowleft \mathcal{W}_{\partial_{\mathrm{out}} I_2} \simeq \mathcal{W}_{\partial_{\mathrm{in}} I_1} \circlearrowright \mathcal{M}_{I_1 \cup_P I_2} \circlearrowleft \mathcal{W}_{\partial_{\mathrm{out}} I_2}$$

and, this is consistent with the pointings of the bimodule categories.

Chapter 2

Bicommutant Categories

Bicommutant categories are higher categorical analogues of von Neumann algebras. They were first introduced in [Hen17c] and further explored in [HP17] and [Hen17a].

Theorem 2.0.1 (von Neumann Bicommutant Theorem [vN30] (also Section II.3 of [Tak79]))**.** *Let H be a Hilbert space and $A \subseteq B(H)$ be a unital $*$-algebra. The following conditions are equivalent:*

(i) $A = [[A]'_{B(H)}]'_{B(H)}$.

(ii) *A is closed in any one of the* strong, weak, ultrastrong, or ultraweak *topologies on $B(H)$.*

We notice that condition (i) is more algebraic in flavour than (ii). It is therefore more straightforward to analogously define *'categorified'* von Neumann algebras using the property of being closed under taking a double commutant.

This definition requires representing A faithfully on a Hilbert space H, although the condition is independent of this choice of representation. There is, in fact, a canonical Hilbert space L^2A on which A acts faithfully called the standard form or the non-commutative L^2-space of A. This is an (A, A)-bimodule and the images in $B(L^2A)$ of the left and right actions of A on L^2A are each other's commutants.

We define a bicommutant category $\mathcal{T}$ as a bi-involutive W^*-tensor category such that its left and right actions on a specific W^*-category, namely $\mathcal{T}^{\text{abs}}$, are each other's commutants. The existence of this module is part of the definition of a bicommutant category. A detailed analysis of complete W^*-categories as being analogous to Hilbert spaces is found in [HNP24].

In this chapter, we give a new and detailed definition of a bicommutant category. We introduce a string calculus that helps the reader unravel the definition. We also define modules over bicommutant categories and their fusion. This fusion is the categorified analogue of the 'Connes fusion' of von Neumann algebra modules.

2.1 Hilbert Spaces and von Neumann Algebras

In this section, we recall without proof basic results and definitions about Hilbert spaces and von Neumann algebras. Throughout the thesis, we assume general familiarity with the structure and theory of von Neumann algebras. A pre-Hilbert space is a vector space equipped with a positive definite sesquilinear form.

Lemma 2.1.1. *A pre-Hilbert space H is a Hilbert space (i.e. H is complete) if and only if the map*

$$H \to \{f : \overline{H} \to \mathbb{C} \,|\, f \text{ is bounded and linear}\} : \xi \mapsto \langle -, \xi \rangle$$

is an isomorphism.

If H is a Hilbert space, we write $B(H)$ for the $*$-*algebra of bounded operators* on H.

Definition 2.1.2. A *von Neumann algebra* on H is a unital $*$-subalgebra of $B(H)$ such that $A = [[A]'_{B(H)}]'_{B(H)}$.

Given two von Neumann algebras $A \subset B(H)$ and $B \subset B(K)$, their *spatial tensor product* $A \bar{\otimes} B$ is the von Neumann algebra generated by the algebraic tensor product $A \otimes^{alg} B$ acting on the Hilbert space tensor product $H \otimes K$.

A $*$-algebra homomorphism $f : A \to B$ between von Neumann algebras is called *normal* if $f(\sup a_i) = \sup f(a_i)$ for every bounded increasing net a_i of positive elements of A.

Definition 2.1.3. A *(von Neumann algebra) homomorphism* between two von Neumann algebras is a normal unital $*$-algebra homomorphism.

Homomorphisms between von Neumann algebras behave much more rigidly than homomorphisms between algebras. For example:

Lemma 2.1.4. *If $f : A \to B$ is a von Neumann algebra homomorphism, then A splits as a von Neumann algebra direct sum $A = A_0 \oplus A_1$, where $A_0 = \ker(f)$ and $A_1 \cong \mathrm{im}(f)$.* □

So every homomorphism $A \to B$ between von Neumann algebras admits a canonical factorisation as a projection onto a direct summand, followed by an injective homomorphism.

Definition 2.1.5. A left (right) module over a von Neumann algebra A is a Hilbert space H equipped with a homomorphism $A \to B(H)$ ($A^{\mathrm{op}} \to B(H)$). An A-B-*bimodule* is a Hilbert space H equipped with two homomorphisms $A \to B(H)$, $B^{\mathrm{op}} \to B(H)$ whose images commute in $B(H)$.

We write ${}_AH$ (H_A) to indicate that H is a left (right) module, and ${}_AH_B$ to indicate that H is an A-B-bimodule.

The following is a fundamental result about the representation theory of von Neumann algebras:

Proposition 2.1.6. *Let A be a von Neumann algebra, and H a faithful A-module. Then for every non-zero A-module K, $\mathrm{Hom}(K, H) \neq 0$ [HNP24, Proposition 2.6].*

By polar decomposition and an application of Zorn's Lemma, we then have:

Corollary 2.1.7. *Let A be a von Neumann algebra, and H a faithful A-module. Then for every A-module K there exists a set I such that K is isomorphic to a submodule of $H^{\oplus I}$. We may furthermore arrange for the inclusion $K \hookrightarrow H^{\oplus I}$ to be of the form*

$$K = \bigoplus_{i \in I} K_i \xhookrightarrow{\oplus_i f_i} \bigoplus_{i \in I} H$$

where each $f_i : K_i \hookrightarrow H$ is an isometric inclusion.

An abstract von Neumann algebra, also known as a W^*-*algebra*, is a $*$-algebra isomorphic to a von Neumann algebra on some Hilbert space. By the celebrated work of Haagerup [Haa75], every abstract von Neumann algebra A admits a canonically associated Hilbert space L^2A on which it acts, called its *standard* *form*, or non-commutative L^2-space. The space L^2A is in fact an A-A-bimodule, and comes equipped with a canonical antiunitary involution

$$J : L^2A \to L^2A \tag{2.1}$$

called *modular conjugation*. The modular conjugation satisfies $J(a\xi b) = b^*J(\xi)a^*$ for all $a, b \in A$ and $\xi \in L^2A$. Importantly, the left and right actions of A on L^2A are each other's commutants.

The L^2 construction is compatible with the operation of taking corners:

Lemma 2.1.8. *[Haa75, Lemma 2.6] If $p \in A$ is a projection, then there is a canonical unitary $pL^2(A)p \simeq L^2(pAp)$, compatible with the modular conjugations, and with left and right actions of pAp.*

Lemma 2.1.9. *If $p, q \in A$ are projections, then $pL^2(A)q = 0$ if and only if $pAq = 0$.*

The construction $A \mapsto L^2A$ is functorial for isomorphisms of von Neumann algebras, but not for general morphisms. It satisfies:

Lemma 2.1.10. *If $u \in A$ is a unitary, then the operators $L^2(\mathrm{Ad}(u))$ and $\xi \mapsto u\xi u^*$ on $L^2(A)$ agree.*

We write $\lambda : A \to B(L^2A)$ and $\rho : A^{\mathrm{op}} \to B(L^2A)$ for the two actions of A on L^2A.

Definition 2.1.11 ([Sau83, Con94])**.** If H is a right A-module, and K is a left A-module, their *Connes fusion* $H \boxtimes_A K$ is the completion of the vector space

$$\mathrm{Hom}_{A\text{-Mod}}(L^2A_A, H_A) \otimes_A L^2A \otimes_A \mathrm{Hom}_{A\text{-Mod}}({}_AL^2A, {}_AK)$$

with respect to the inner product $\langle \varphi_1 \otimes \xi_1 \otimes \psi_1, \varphi_2 \otimes \xi_2 \otimes \psi_2 \rangle := \langle \xi_1, \lambda^{-1}(\varphi_1^* \circ \varphi_2)\xi_2\rho^{-1}(\psi_1^* \circ \psi_2)\rangle$.

Remark 2.1.12. *The Connes fusion of H_A and ${}_AK$ also admits two asymmetric descriptions, as completions of $H \otimes_A \mathrm{Hom}_A(L^2A, K)$ and of $\mathrm{Hom}_A(L^2A, H) \otimes_A K$, respectively.*

Proposition 2.1.13. *For $f\colon A \to B$ a von Neumann algebra isomorphism, we have that the following diagram commutes:*

$$\begin{tikzcd}
L^2A \boxtimes_A L^2A \arrow[r, "\text{unitor}"] \arrow[d, "L^2f \boxtimes_A L^2f"'] & L^2A \arrow[d, "L^2f"] \\
L^2B \boxtimes_B L^2B \arrow[r, "\text{unitor}"'] & L^2B
\end{tikzcd}$$

The collection of all von Neumann algebras and all bimodules for von Neumann algebras forms the objects and 1-morphisms of a bicategory. Composition of 1-morphisms is given by Connes fusion, and the unit 1-morphism on the object A is the bimodule ${}_AL^2A_A$. As part of the data of this bicategory, there exist unitary associators and left and right unitors (see [KW24, Proposition A.2.5]):

$$\begin{gathered}
{}_A(H \boxtimes_B K) \boxtimes_C L_D \cong {}_AH \boxtimes_B (K \boxtimes_C L)_D, \\
{}_AL^2A \boxtimes_A H_B \cong {}_AH_B, \quad \text{and} \quad {}_AH \boxtimes_B L^2B_B \cong {}_AH_B.
\end{gathered} \tag{2.2}$$

We refer the reader to [Bro03, BLM04, KW24, CPJP22] for more details about this bicategory.

2.2 Bicommutant Categories

Bi-involutive W*-tensor categories

In this section, following [HNP24], we review the notions of $*$-category, W^*-category, W^*-functor, W^*-tensor category, and bi-involutive W^*-tensor category. We define the unitary commutant of a W^*-tensor functor, the absorbing ideal of a tensor category, and introduce bicommutant categories.

In [HP17], [Hen17a], and [Hen17c], a notion of bicommutant category was introduced, that depends on a choice of module category for the bicommutant category (in much the same way as the standard definition of a von Neumann algebra depends on a choice of representing Hilbert space). Here, we present an alternative definition of bicommutant category that does not depend on such a choice. We instead introduce a canonical module on which the category acts faithfully from the left and right. This is analogous to the standard form or L^2-space for a von Neumann algebra.

Definition 2.2.1. A $*$-category ($\mathbb{C}$-linear *dagger category*) $\mathcal{W}$ is a $\mathbb{C}$-linear category equipped with antilinear functions

$$(-)^{\dagger} : \mathrm{Hom}_{\mathcal{W}}(X,Y) \to \mathrm{Hom}_{\mathcal{W}}(Y,X)$$

for all pairs of objects X, Y in $\mathcal{W}$ such that, for any object X, $\mathrm{id}_X^{\dagger} = \mathrm{id}_X$, for any morphism $f \colon X \to Y$, $\left(f^{\dagger}\right)^{\dagger} = f$ and for two composable morphisms f and g we have $(g \circ f)^{\dagger} = f^{\dagger} \circ g^{\dagger}$.

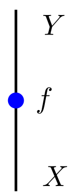
We note that a dagger category is a general notion where the Hom-spaces need not be enriched in vector spaces, but the map $(-)^{\dagger}$ exists. In this chapter we restrict ourselves to $\mathbb{C}$-linear dagger categories.

Given a $\mathbb{C}$-linear dagger category $\mathcal{W}$, we write $\mathcal{W}^{\oplus}$ for the category whose objects are formal finite sums of objects of $\mathcal{W}$, and whose morphisms are given by $\mathrm{Hom}(\oplus X_i, \oplus Y_j) := \oplus_{i,j} \mathrm{Hom}(X_i, Y_j)$.

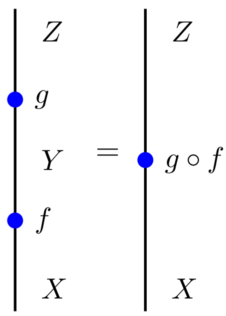
Graphical calculus is a very useful tool for various categorical operations. Here we shall introduce the graphical notation for dagger categories. These have previously been developed and well-studied (see [Sel10] for a survey).

For a category $\mathcal{W}$, an object is represented by a labelled wire and a morphism is represented by a labelled bullet. All diagrams will be read from bottom to top.

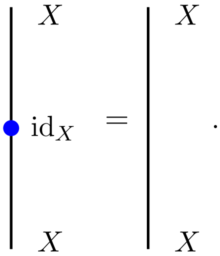
A morphism $f \colon X \to Y$ in $\mathcal{W}$ is denoted as:



The composition of $f \colon X \to Y$ and $g \colon Y \to Z$ is denoted by stacking the morphism tokens vertically,

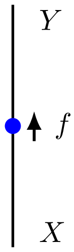


The identity morphism is suppressed graphically,



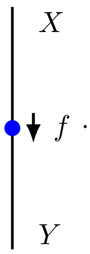
We are now able to represent a chain of the type $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n$ by choosing $n-1$ distinct points in the interior of the interval $[0,1]$, ordered increasingly, and labelling them by the morphisms $f_1, \ldots, f_{n-1}$. The interval itself represents the underlying string, read from 0 (bottom) to 1 (top), and the segments of the string are labelled by objects X_i; the endpoints being the objects X_1 and X_n respectively. All chains obtained from this chain on inserting identities or composing adjacent morphisms are equivalent in the graphical calculus.

Now, let $\mathcal{W}$ be a dagger category. The graphical notation now includes a local orientation of the morphism. We denote $f\colon X \to Y$ as



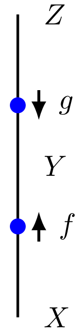
(2.3)

and $f^\dagger$ can be viewed as f reflected about an axis orthogonal to the string, that is, $f^\dagger\colon Y \to X$ is



The composition is stacking morphisms as before, for instance, the diagram for

$X \xrightarrow{f} Y \xrightarrow{g^\dagger} Z$ is given by



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W^*-categories were first introduced in [GLR85]. We will review the key definitions and theorems and we refer the reader to [GLR85] and [HNP24] for details.

Definition 2.2.2. A W*-*category* is a $*$-category $\mathcal{W}$ such that for every object $X \in \mathcal{W}^{\oplus}$ the $*$-algebra $\mathrm{End}_{\mathcal{W}^{\oplus}}(X)$ is a von Neumann algebra. All W^*-categories considered in this thesis will be assumed to admit a set of generators as defined in [HNP24, Definition 3.5].

A W^*-category is said to be $\aleph_1$-*complete* if it is idempotent complete [HNP24, Definition 3.3] and it admits all countable direct sums [HNP24, Definition 3.7]. We shall often just abbreviate this to '*complete*'.

It is called *locally* $\aleph_1$-*small* if for every object $X \in \mathcal{W}$, the von Neumann algebra $\mathrm{End}(X)$ is separable (here, a von Neumann algebra is called separable if it can be made to act faithfully on a separable Hilbert space).

All W^*-categories considered in this thesis will be locally $\aleph_1$-small, and $\aleph_1$-complete. By [HNP24, Proposition 6.11], these are exactly the W^*-categories of the form

$$\{H \in A\text{-Mod} \mid H \text{ is separable}\}$$

where A is a von Neumann algebra which can be written as a direct sum of (possibly uncountably many) separable von Neumann algebras.

We denote by $\overline{\mathcal{W}}$ the category with the same objects as $\mathcal{W}$, and complex conjugate hom-spaces.

Definition 2.2.3. A $*$-functor $F\colon \mathcal{W}_1 \to \mathcal{W}_2$ between W^*-categories is called a *functor of* W^*-*categories* if for every object $A \in \mathcal{W}_1$ the induced map $\mathrm{End}(A) \to \mathrm{End}(F(A))$ is a normal homomorphism of von Neumann algebras.

We now define the Hilb-valued inner product for a complete $W*$-category. This was introduced in [HNP24] as a categorified analogue of the inner product on Hilbert spaces.

Definition 2.2.4. Every W^*-category admits a canonical Hilb-valued sesquilinear inner product denoted $\langle -,- \rangle : \overline{\mathcal{W}} \times \mathcal{W} \to \mathrm{Hilb}$, defined by

$$\langle X, Y \rangle : = \mathrm{proj}_{2,1}\left(L^2\left(\mathrm{End}\left(X \oplus Y\right)\right)\right)$$

for all pairs of objects $X, Y \in \mathcal{W}$, where $\mathrm{proj}_{2,1} \in \mathrm{End}(X \oplus Y)$ is the projection $\xi \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_Y \end{smallmatrix}\right)\xi\left(\begin{smallmatrix} 1_X & 0 \\ 0 & 0 \end{smallmatrix}\right)$. We call this the $(2,1)$-corner of $L^2\mathrm{End}(X \oplus Y)$ [HNP24, Definition 4.1]

The map $\langle X_1, Y_1 \rangle \to \langle X_2, Y_2 \rangle$ induced by a pair of morphisms $f \colon X_1 \to X_2$, $g \colon Y_1 \to Y_2$ is defined in [HNP24, Definition 4.1]. There is also a unitary isomorphism,

$$J_{X,Y} \colon \langle X, Y \rangle \xrightarrow{\simeq} \overline{\langle Y, X \rangle} \tag{2.4}$$

natural in X, Y given by the restriction of the modular conjugation map on $L^2\mathrm{End}\left(X \oplus Y\right)$. It is involutive in the sense that the following diagram commutes:

$$\begin{tikzcd} \langle X, Y \rangle \arrow[r, "J_{X,Y}"] \arrow[d] & \overline{\langle Y, X \rangle} \arrow[dl, "\overline{J_{Y,X}}"] \\ \overline{\overline{\langle X, Y \rangle}} & \end{tikzcd}$$

Further, $J_{X,X}$ agrees with the modular conjugation on

$$\langle X, X \rangle = L^2(\mathrm{End}(X)). \tag{2.5}$$

Definition 2.2.5. Given a functor $F \colon \mathcal{W}_1 \to \mathcal{W}_2$ between complete W^*-categories, its *unitary adjoint* is defined as the composite

$$F^\dagger : \mathcal{W}_2 \xrightarrow{\simeq} \mathrm{Func}(\overline{\mathcal{W}_2}, \mathrm{Hilb}) \xrightarrow{-\circ \overline{F}} \mathrm{Func}(\overline{\mathcal{W}_1}, \mathrm{Hilb}) \xrightarrow{\simeq} \mathcal{W}_1$$

Equivalently, the functor $F^\dagger : \mathcal{W}_2 \to \mathcal{W}_1$ is specified by the requirement that

$$\langle X, F^\dagger(Y) \rangle \;\cong\; \langle F(X), Y \rangle \tag{2.6}$$

naturally for $X \in \mathcal{W}_1$ and $Y \in \mathcal{W}_2$.

The equivalences $\mathcal{W} \simeq \mathrm{Func}(\overline{\mathcal{W}}, \mathrm{Hilb})$ for W^*-categories used in the definition of unitary adjoint above are discussed in [HNP24, Proposition 4.12].

Definition 2.2.6. A *W^*-tensor category* $(\mathcal{T}, \otimes, 1, \alpha, l, r)$ is a W^*-category equipped with a monoidal structure which is compatible with the W^*-structure in the sense that $\otimes\colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a bilinear functor of W^*-categories, and the associator α and left and right unitors l, r are unitary.

A *tensor functor* between two W^*-tensor categories $\mathcal{T}_1$ and $\mathcal{T}_2$, is a triple (F, μ, i), where $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ is a functor of W^* categories, and

$$\mu_{A,B}\colon F(A) \otimes_{\mathcal{T}_2} F(B) \to F(A \otimes_{\mathcal{T}_1} B) \qquad \text{and} \qquad i\colon 1_{\mathcal{T}_2} \to F(1_{\mathcal{T}_1}) \tag{2.7}$$

are unitary natural transformations, satisfying $\mu_{A,B\otimes C} \circ (\mathrm{id}_{F(A)} \otimes \mu_{B,C}) = \mu_{A\otimes B,C} \circ (\mu_{A,B} \otimes \mathrm{id}_{F(C)})$ and $\mu_{1,A} \circ (i \otimes \mathrm{id}_{F(A)}) = \mathrm{id}_{F(A)} = \mu_{A,1} \circ (\mathrm{id}_{F(A)} \otimes i)$ (suppressing associators).

If a W^*-tensor category $\mathcal{T}$ comes equipped with an action of $\mathbb{Z}/2$ in which the non-trivial element of $\mathbb{Z}/2$ acts antilinearly and anti-monoidally, then $\mathcal{T}$ is called a bi-involutive W^*-tensor category. Alternatively, we give the definition below.

Definition 2.2.7. A *bi-involutive W^*-tensor category*[1] is a W^*-tensor category $\mathcal{T}$ equipped with a covariant antilinear, anti-tensor dagger functor

$$\overline{} : \mathcal{T} \to \mathcal{T}^{\mathrm{mop}}$$

called the conjugate. $\mathcal{T}^{\mathrm{mop}}$ represents $\mathcal{T}$ with the opposite tensor product.

The functor $\overline{}$ is involutive, meaning that for every $A \in \mathcal{T}$, we are given unitary natural isomorphisms $\varphi_A : A \to \overline{\overline{A}}$ satisfying $\varphi_{\overline{A}} = \overline{\varphi_A}$.

The structure data of this anti-tensor functor are denoted

$$\nu_{A,B} : \overline{A} \otimes \overline{B} \xrightarrow{\simeq} \overline{B \otimes A} \qquad \text{and} \qquad j : 1 \to \overline{1}$$

[1] A remark on terminology: the name 'bi-involutive W^*-tensor category' is slightly redundant. The two involutions referred to by 'bi-involutive' are the dagger on morphisms and the conjugation on objects, but the dagger is already part of the W^*-structure. More accurate names would therefore be 'involutive W^*-tensor category' (the conjugation being the one additional involution) or 'bi-involutive tensor category' (with no W^* prefix). We nevertheless keep the established terminology.

and satisfy

$$\begin{tikzcd}
\overline{A}\otimes(\overline{B}\otimes\overline{C}) \arrow[r, "\mathrm{id}_{\overline{A}}\otimes\nu_{B,C}"] & \overline{A}\otimes\overline{C\otimes B} \arrow[d, "\nu_{A,C\otimes B}"] \\
(\overline{A}\otimes\overline{B})\otimes\overline{C} \arrow[u, "\alpha_{\overline{A},\overline{B},\overline{C}}"] \arrow[d, "\nu_{A,B}\otimes\mathrm{id}_{\overline{C}}"'] & \overline{(C\otimes B)\otimes A} \arrow[d, "\overline{\alpha_{C,B,A}}"] \\
\overline{B\otimes A}\otimes\overline{C} \arrow[r, "\nu_{B\otimes A,C}"] & \overline{C\otimes(B\otimes A)}
\end{tikzcd}$$

$$\begin{tikzcd}
1\otimes\overline{A} \arrow[r, "j\otimes\mathrm{id}_{\overline{A}}"] \arrow[d, "l_{\overline{A}}"'] & \overline{1}\otimes\overline{A} \arrow[d, "\nu_{1,A}"] \\
\overline{A} & \overline{A\otimes 1} \arrow[l, "\overline{r_A}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{A}\otimes 1 \arrow[r, "\mathrm{id}_{\overline{A}}\otimes j"] \arrow[d, "r_{\overline{A}}"'] & \overline{A}\otimes\overline{1} \arrow[d, "\nu_{A,1}"] \\
\overline{A} & \overline{1\otimes A} \arrow[l, "\overline{l_A}"]
\end{tikzcd}$$

Finally, we require the compatibility conditions $\varphi_1 = \overline{j}\circ j$ and $\varphi_{A\otimes B} = \overline{\nu_{B,A}}\circ \nu_{\overline{A},\overline{B}}\circ(\varphi_A\otimes\varphi_B)$.

Definition 2.2.8. We define a natural map $f_{X,Y}\colon \overline{\langle X,Y\rangle}\to\langle\overline{X},\overline{Y}\rangle$ by

$$\begin{aligned}
\langle\overline{X},\overline{Y}\rangle &= p_{\overline{Y}}L^2(\mathrm{End}(\overline{X}\oplus\overline{Y}))p_{\overline{X}} \\
&\simeq p_{\overline{Y}}L^2(\overline{\mathrm{End}(X\oplus Y)})p_{\overline{X}} \\
&\simeq p_{\overline{Y}}\overline{L^2(\mathrm{End}(X\oplus Y))}p_{\overline{X}} \\
&= \overline{p_Y L^2(\mathrm{End}(X\oplus Y))p_X} \\
&= \overline{\langle X,Y\rangle}
\end{aligned}$$

Lemma 2.2.9. *For any two objects $X,Y\in\mathcal{T}$, the following diagram commutes:*

$$\begin{tikzcd}
\langle X,Y\rangle \arrow[r] \arrow[d, "\langle\varphi_X,\varphi_Y\rangle"'] & \overline{\overline{\langle X,Y\rangle}} \arrow[d, "\overline{f_{X,Y}}"] \\
\langle\overline{\overline{X}},\overline{\overline{Y}}\rangle & \overline{\langle\overline{X},\overline{Y}\rangle} \arrow[l, "f_{\overline{X},\overline{Y}}"]
\end{tikzcd}$$

Proof. This follows from [HNP24, Lemma 4.7]. ☐

Definition 2.2.10. We define $c_{X,Y}\colon\langle\overline{X},Y\rangle\to\langle\overline{Y},X\rangle$ to be the composition

$$\langle\overline{X},Y\rangle \xrightarrow{J_{\overline{X},Y}} \overline{\langle Y,\overline{X}\rangle} \xrightarrow{f_{Y,\overline{X}}} \langle\overline{Y},\overline{\overline{X}}\rangle \xrightarrow{\langle\overline{\mathrm{id}_Y},\varphi_X^{-1}\rangle} \langle\overline{Y},X\rangle \tag{2.8}$$

Lemma 2.2.11. *For any two objects $X,Y\in\mathcal{T}$, we have $c_{X,Y}\circ c_{Y,X}=\mathrm{id}_{\langle\overline{X},Y\rangle}$.*

Proof. This statement is equivalent to asking for the following hexagon to commute

$$\begin{tikzcd}
\langle \overline{X}, Y\rangle \arrow[r, "J_{\overline{X},Y}"] & \overline{\langle Y, \overline{X}\rangle} \arrow[r, "f_{Y,\overline{X}}"] & \langle \overline{Y}, \overline{\overline{X}}\rangle \arrow[d, "\langle \mathrm{id}_{\overline{Y}}, \varphi_X^{-1}\rangle"] \\
\langle \overline{X}, \overline{\overline{Y}}\rangle \arrow[u, "\langle \mathrm{id}_{\overline{X}}, \varphi_Y^{-1}\rangle"] & \overline{\langle X, \overline{Y}\rangle} \arrow[l, "f_{X,\overline{Y}}"'] & \langle \overline{Y}, X\rangle \arrow[l, "J_{\overline{Y},X}"']
\end{tikzcd}$$

However, this hexagon is just the $(2,1)$-corner of the following hexagon:

$$\begin{tikzcd}
L^2\mathrm{End}(\overline{X} \oplus Y) \arrow[r, "J", "\boxed{1}"'] & \overline{L^2\mathrm{End}(Y \oplus \overline{X})} \arrow[r, "\boxed{2}"'] & L^2\mathrm{End}(\overline{Y} \oplus \overline{\overline{X}}) \arrow[d, "L^2((\mathrm{id}_{\overline{Y}} \oplus \varphi_X^{\dagger}) \circ - \circ (\mathrm{id}_{\overline{Y}} \oplus \varphi_X))", "\boxed{3}"'] \\
L^2\mathrm{End}(\overline{X} \oplus \overline{\overline{Y}}) \arrow[u, "L^2((\mathrm{id}_{\overline{X}} \oplus \varphi_Y^{\dagger}) \circ - \circ (\mathrm{id}_{\overline{X}} \oplus \varphi_Y))", "\boxed{6}"'] & \overline{L^2\mathrm{End}(X \oplus \overline{Y})} \arrow[l, "\boxed{5}"] & L^2\mathrm{End}(\overline{Y} \oplus X) \arrow[l, "\boxed{4}", "J"']
\end{tikzcd}$$

Elements of the form $a \cdot \Omega$ where $a \in \mathrm{End}(\overline{X} \oplus Y)$ and $\Omega \in L^2\mathrm{End}(\overline{X} \oplus Y)$ is a cyclic and separating element are dense in $L^2\mathrm{End}(\overline{X} \oplus Y)$. So we pick such an $a \cdot \Omega$ and trace the hexagon

$$\begin{tikzcd}
a \cdot \Omega \arrow[d, "\boxed{1}"] \\
\Omega \cdot a^{\dagger} \arrow[d, "\boxed{2}"] \\
\Omega \cdot \overline{a^{\dagger}} \arrow[d, "\boxed{3}"] \\
\Omega \cdot ((\varphi_X^{-1} \oplus \mathrm{id}_{\overline{Y}}) \circ \overline{a^{\dagger}} \circ (\varphi_X \oplus \mathrm{id}_{\overline{Y}})) \arrow[d, "\boxed{4}"'] \\
((\varphi_X^{-1} \oplus \mathrm{id}_{\overline{Y}}) \circ \overline{a} \circ (\varphi_X \oplus \mathrm{id}_{\overline{Y}})) \cdot \Omega \arrow[d, "\boxed{5}"'] \\
((\overline{\varphi_X^{-1}} \oplus \mathrm{id}_{\overline{\overline{Y}}}) \circ \overline{\overline{a}} \circ (\overline{\varphi_X} \oplus \mathrm{id}_{\overline{\overline{Y}}})) \cdot \Omega \arrow[d, "\boxed{6}"'] \\
((\varphi_{\overline{X}}^{-1} \oplus \varphi_Y^{-1}) \circ \overline{\overline{a}} \circ (\varphi_{\overline{X}} \oplus \varphi_Y)) \cdot \Omega
\end{tikzcd}$$

In $\boxed{4}$ we used $\overline{a^\dagger}^\dagger = \overline{a}$ and in $\boxed{6}$ we used $\overline{\varphi_X} = \varphi_{\overline{X}}$. We have also consistently used that $\varphi_X^\dagger = \varphi_X^{-1}$. The above hexagon commuting follows from the naturality of φ, that is

$$\begin{tikzcd}
\overline{X} \oplus Y \arrow[r, "a"] \arrow[d, "\varphi_{\overline{X}} \oplus \varphi_Y"'] & \overline{X} \oplus Y \arrow[d, "\varphi_{\overline{X}} \oplus \varphi_Y"] \\
\overline{\overline{\overline{X}}} \oplus \overline{\overline{Y}} \arrow[r, "\overline{\overline{a}}"'] & \overline{\overline{\overline{X}}} \oplus \overline{\overline{Y}}
\end{tikzcd}$$

$\square$

Lemma 2.2.12. *For any two objects $X, Y \in \mathcal{T}$, the following diagram commutes*

$$\begin{tikzcd}
\langle \overline{X}, Y \rangle \arrow[r, "J_{\overline{X},Y}"] \arrow[d, "c_{X,Y}"'] & \overline{\langle Y, \overline{X} \rangle} \arrow[r, "\overline{\langle \varphi_Y, \mathrm{id}_{\overline{X}} \rangle}"] & \overline{\langle \overline{\overline{Y}}, \overline{X} \rangle} \arrow[d, "\overline{c_{\overline{Y},\overline{X}}}"] \\
\langle \overline{Y}, X \rangle \arrow[r, "J_{\overline{Y},X}"'] & \overline{\langle X, \overline{Y} \rangle} \arrow[r, "\overline{\langle \varphi_X, \mathrm{id}_{\overline{Y}} \rangle}"'] & \overline{\langle \overline{\overline{X}}, \overline{Y} \rangle}
\end{tikzcd}$$

Proof. We need to show the following diagram commutes

$$\begin{tikzcd}
\langle \overline{X}, Y \rangle \arrow[r, "J_{\overline{X},Y}"] \arrow[d, "J_{\overline{X},Y}"'] & \overline{\langle Y, \overline{X} \rangle} \arrow[r, "\overline{\langle \varphi_Y, \mathrm{id}_{\overline{X}} \rangle}"] & \overline{\langle \overline{\overline{Y}}, \overline{X} \rangle} \arrow[d, "\overline{J_{\overline{\overline{Y}},\overline{X}}}"] \\
\overline{\langle Y, \overline{X} \rangle} \arrow[d, "f_{Y,\overline{X}}"'] & & \overline{\overline{\langle \overline{X}, \overline{\overline{Y}} \rangle}} \arrow[d, "\overline{f_{\overline{X},\overline{\overline{Y}}}}"] \\
\langle \overline{Y}, \overline{\overline{X}} \rangle \arrow[d, "\langle \mathrm{id}_Y, \varphi_X^{-1} \rangle"'] & & \overline{\langle \overline{\overline{X}}, \overline{\overline{\overline{Y}}} \rangle} \arrow[d, "\overline{\langle \mathrm{id}_{\overline{\overline{X}}}, \varphi_{\overline{Y}}^{-1} \rangle}"] \\
\langle \overline{Y}, X \rangle \arrow[r, "J_{\overline{Y},X}"'] & \overline{\langle X, \overline{Y} \rangle} \arrow[r, "\overline{\langle \varphi_X, \mathrm{id}_{\overline{Y}} \rangle}"'] & \overline{\langle \overline{\overline{X}}, \overline{Y} \rangle}
\end{tikzcd}$$

This is the $(2,1)$-corner of

$$\begin{tikzcd}
L^2\mathrm{End}(\overline{X}\oplus Y) \arrow[r, "J", "\boxed{1}"'] \arrow[d, "J"'] & \overline{L^2\mathrm{End}(Y\oplus\overline{X})} \arrow[r, "\overline{L^2\mathrm{Ad}_{\varphi_Y\oplus\mathrm{id}_{\overline{X}}}}", "\boxed{2}"'] & \overline{L^2\mathrm{End}(\overline{\overline{Y}}\oplus\overline{X})} \arrow[d, "\overline{J}", "\boxed{3}"'] \\
\overline{L^2\mathrm{End}(Y\oplus\overline{X})} \arrow[d] & & \overline{\overline{L^2\mathrm{End}(\overline{X}\oplus\overline{\overline{Y}})}} \arrow[d, "\boxed{4}"'] \\
L^2\mathrm{End}(\overline{Y}\oplus\overline{\overline{X}}) \arrow[d, "L^2\mathrm{Ad}_{\mathrm{id}_{\overline{Y}}\oplus\varphi_X^{-1}}"'] & & \overline{L^2\mathrm{End}(\overline{\overline{X}}\oplus\overline{\overline{\overline{Y}}})} \arrow[d, "\overline{L^2\mathrm{Ad}_{\mathrm{id}_{\overline{\overline{X}}}\oplus\varphi_{\overline{Y}}^{-1}}}", "\boxed{5}"'] \\
L^2\mathrm{End}(\overline{Y}\oplus X) \arrow[r, "J"'] & \overline{L^2\mathrm{End}(X\oplus\overline{Y})} \arrow[r, "\overline{L^2\mathrm{Ad}_{\varphi_X\oplus\mathrm{id}_{\overline{Y}}}}"'] & \overline{L^2\mathrm{End}(\overline{\overline{X}}\oplus\overline{Y})}
\end{tikzcd}$$

We pick $a\cdot\Omega \in L^2\mathrm{End}(\overline{X}\oplus Y)$ where $a \in \mathrm{End}(\overline{X}\oplus Y)$ and $\Omega \in L^2\mathrm{End}(\overline{X}\oplus Y)$ cyclic as in the proof of Lemma 2.2.11. We use that Ω is in the positive cone so J acts on it trivially, and trace the numbered path.

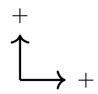
$$\begin{tikzcd}
a\cdot\Omega \arrow[d, "\boxed{1}"] \\
\Omega\cdot a^\dagger \arrow[d, "\boxed{2}"] \\
\Omega\cdot((\mathrm{id}_{\overline{X}}\oplus\varphi_Y)\circ a^\dagger\circ(\mathrm{id}_{\overline{X}}\oplus\varphi_Y^{-1})) \arrow[d, "\boxed{3}"] \\
((\mathrm{id}_{\overline{X}}\oplus\varphi_Y)\circ a\circ(\mathrm{id}_{\overline{X}}\oplus\varphi_Y^{-1}))\cdot\Omega \arrow[d, "\boxed{4}"] \\
((\mathrm{id}_{\overline{\overline{X}}}\oplus\varphi_{\overline{Y}})\circ \overline{a}\circ(\mathrm{id}_{\overline{\overline{X}}}\oplus\varphi_{\overline{Y}}^{-1}))\cdot\Omega \arrow[d, "\boxed{5}"] \\
\overline{a}\cdot\Omega
\end{tikzcd}$$

Tracing the other path similarly gives the map $a\cdot\Omega \mapsto \overline{a}\cdot\Omega$, thus the square commutes. $\square$

String Calculus for Bi-involutive Tensor Categories

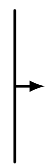
We shall now recall the graphical calculus for a tensor category and subsequently extend it to bi-involutive tensor categories. The definitions and coherences of a bi-involutive W*-tensor category involve a substantial amount of structure data, the dagger on morphisms, the conjugation on objects, and the natural isomorphisms ν, φ, and j together with their coherence conditions. The graphical calculus we develop here encodes this structure, making the coherences manifest. We enrich the standard string diagram notation in two stages: first, following [Sel10], morphisms in a dagger category are given a local orientation so that reflection about a horizontal axis implements the dagger; second, objects in a bi-involutive tensor category are equipped with a co-orientation (a normal direction to the string) so that reflection about a vertical axis implements the conjugation functor $\overline{(\cdot)}$. With these conventions, the coherence isomorphisms $\nu_{A,B}$, φ_A, and j are absorbed into the notation. We then further extend the calculus to accommodate the Hilb-valued inner product by placing string diagrams on a cylinder with two marked lines; this cylinder notation will be used throughout the remainder of the chapter, in particular when spelling out the coherences in the definition of a bicommutant category in Section 2.3.

We use the convention that tensor is read left to right and composition is read bottom to top, that is



The monoidal unit **1** is suppressed in the graphical notation.

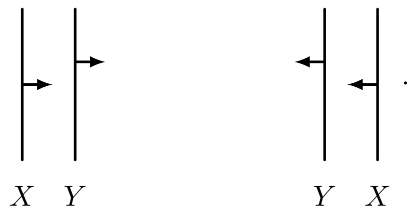
For a bi-involutive tensor category, in addition to the local orientation of the morphisms as in a dagger category, as in Equation 2.3, the objects (strings) and morphisms are also given a co-orientation (normal), that is, the object X is represented as,



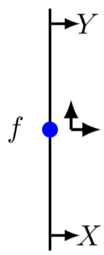
X

and $\overline{X}$ is drawn by reversing the co-orientation of the strings or reflecting along the vertical axis, that is, the object $\overline{X}$ is represented as,

This definition makes ν and φ as in definition 2.2.7 implicit. The unit object is transparent, so j is also built into the graphical notation. The following strings illustrate $X \otimes Y$ and $\overline{Y} \otimes \overline{X} \simeq \overline{X \otimes Y}$.



A morphism $f \colon X \to Y$ is denoted as

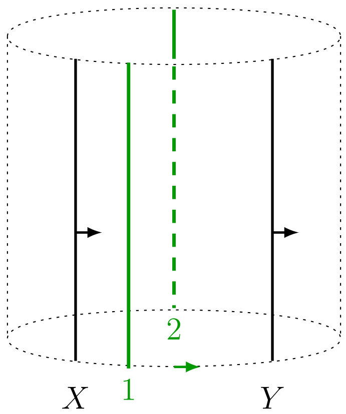


(2.9)

and $\bar{f} \colon \overline{X} \to \overline{Y}$ is obtained by reflecting the diagram along the vertical axis, and as before, $f^\dagger \colon Y \to X$ is obtained by reflecting the diagram along the horizontal axis.

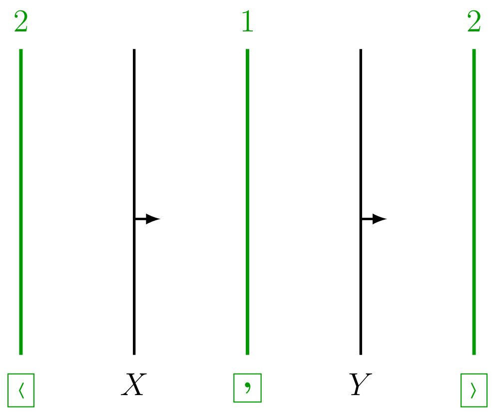
Further, for any bi-involutive W^*-tensor category $\mathcal{T}$, we introduce a graphical notation to compute the Hilb-valued inner product. The graphical notation builds on the notation established for bi-involutive tensor categories and consistently assigns a Hilbert space to any diagram of parallel strands on a tube (cylinder) with two marked lines.

For two objects X, Y, we now describe how to interpret the following diagram:



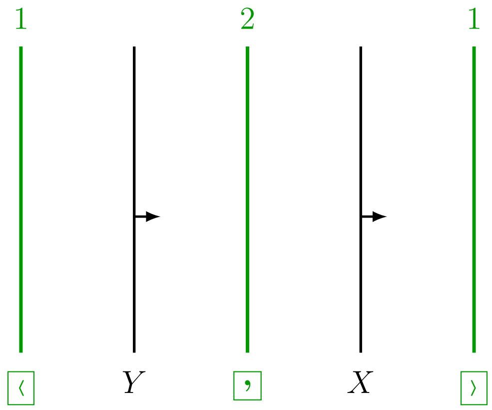
(2.10)

We first choose one of the green lines and cut open the cylinder (for example, cutting open line 2):

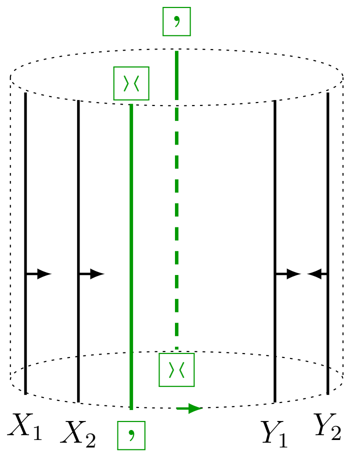


We interpret both planes using the ordinary string calculus (we read in the direction of the orientation of the circle which has been cut open in each plane), and put the left hand plane in the first input of the inner product after an overall $\overline{-}$, and the right in the second input, that is, as $\langle \overline{X}, Y \rangle$. So line labelled 1 represents the '**,**' and the two lines obtained from cutting open at the line labelled 2 represent '$\langle$' and '$\rangle$'. If we had cut along line 1 instead, we would have obtained the isomorphic Hilbert space $\langle \overline{Y}, X \rangle$ (the isomorphism given by $c_{X,Y}$).

This is illustrated in the diagram below:



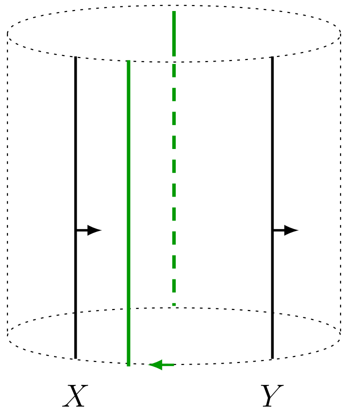
For a more complex example, to the diagram below, we assign the Hilbert space $\langle \overline{X_1 \otimes X_2}, Y_1 \otimes \overline{Y_2} \rangle \simeq \langle \overline{Y_1 \otimes \overline{Y_2}}, X_1 \otimes X_2 \rangle$ by Equation 2.8.



(2.11)

Remark 2.2.13. *Though the choice of which line we open to interpret the diagram does not matter (as the two choices give isomorphic results), in our diagrams we shall often label a line by either a comma or brackets to indicate the choice we are making. For example, in the previous figure, at the bottom we interpret the diagram as* $\langle \overline{X_1 \otimes X_2}, Y_1 \otimes \overline{Y_2} \rangle$ *and at the top as* $\langle \overline{Y_1 \otimes \overline{Y_2}}, X_1 \otimes X_2 \rangle$ *by suppressing the* c *isomorphism.*

If $\langle \overline{X}, Y \rangle$ is represented by Equation 2.10, then we represent $\overline{\langle \overline{X}, Y \rangle}$ by reversing the orientation of the circle, that is, by the diagram



We note that this diagram can also be read as $\langle\overline{\overline{\overline{Y}}}, \overline{X}\rangle \simeq \langle Y, \overline{X}\rangle$. This thus shows that our diagrammatic calculus implements $J_{\overline{X},Y} \colon \overline{\langle\overline{X}, Y\rangle} \simeq \langle Y, \overline{X}\rangle$

Remark 2.2.14. *We claim that any equation between morphisms in a bi-involutive tensor category or between inner products of objects follows from the axioms of a bi-involutive tensor category if and only if it holds in the graphical calculus up to isotopy. To prove this we would use techniques similar to [Sel10], which is beyond the scope of this thesis.*

Bi-involutive tensor functors

A dagger tensor functor $F \colon \mathcal{T}_1 \to \mathcal{T}_2$ between bi-involutive tensor categories with natural coherence maps $\mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ and unitor $i : \mathbf{1}_{\mathcal{T}_2} \to F(\mathbf{1}_{\mathcal{T}_1})$ is called *a bi-involutive W^*-tensor functor* if it comes equipped with a unitary natural transformation

$$\gamma_A : F(\overline{A}) \to \overline{F(A)}$$

satisfying the following coherences:

$$
\begin{tikzcd}
F(\overline{A \otimes B}) \arrow[d, "\gamma_{A\otimes B}"] & F(\overline{B} \otimes \overline{A}) \arrow[l, "F(\nu^{\mathcal{T}_1}_{B,A})"'] & F(\overline{B}) \otimes F(\overline{A}) \arrow[l, "\mu_{\overline{B},\overline{A}}"'] \arrow[d, "\gamma_B \otimes \gamma_A"] \\
\overline{F(A \otimes B)} & \overline{F(A) \otimes F(B)} \arrow[l, "\overline{\mu_{A,B}}"] & \overline{F(B)} \otimes \overline{F(A)} \arrow[l, "\nu^{\mathcal{T}_2}_{F(B),F(A)}"]
\end{tikzcd}
\tag{2.12}
$$

$$
\begin{tikzcd}
F(\overline{\overline{A}}) \arrow[d, "\gamma_{\overline{A}}"] & F(A) \arrow[l, "F(\varphi^{\mathcal{T}_1}_A)"] \arrow[d, "\varphi^{\mathcal{T}_2}_{F(A)}"'] \\
\overline{F(\overline{A})} \arrow[r, "\overline{\gamma_A}"'] & \overline{\overline{F(A)}}
\end{tikzcd}
\tag{2.13}
$$

The coherence $\gamma_1 = \overline{i} \circ j^{\mathcal{T}_2} \circ i^{-1} \circ F(j^{\mathcal{T}_1})^{-1}$ for the unit follows from the coherence for $\gamma_{A\otimes B}$. Finally, a bi-involutive natural transformation is a monoidal natural transformation that intertwines the coherences of the two tensor functors.

If $\mathcal{W}$ is a complete W^*-category, then the category $\mathrm{End}(\mathcal{W})$ of endofunctors of $\mathcal{W}$ is a bi-involutive W^*-tensor category, with involution on objects provided by the operation $F \to F^\dagger$ and monoidal structure given by composition. We can give natural unitary isomorphisms $\varphi_F : F \to F^{\dagger\dagger}$ and $\nu_{F,G} : F^\dagger \circ G^\dagger \to (G \circ F)^\dagger$ as follows:

- φ_F is defined by the requirement that $\langle (\varphi_F)_X, Y\rangle$ is equal to the composite

  $$\langle F(X), Y\rangle \cong \langle X, F^\dagger(Y)\rangle \xrightarrow{J} \overline{\langle F^\dagger(X), Y\rangle} \cong \overline{\langle Y, F^{\dagger\dagger}(X)\rangle} \xrightarrow{J^{-1}} \langle F^{\dagger\dagger}(X), Y\rangle.$$

  where the inner product is the Hilb-valued inner product on $\mathcal{W}$.

- $\nu_{F,G}$ is defined similarly, by asking that $\langle X, (\nu_{F,G})_Y\rangle$ is equal to the composite

  $$\langle X, F^\dagger \circ G^\dagger(Y)\rangle \xrightarrow{\simeq} \langle F(X), G^\dagger(Y)\rangle \xrightarrow{\simeq} \langle (G\circ F)(X), Y\rangle \xrightarrow{\simeq} \langle X, (G\circ F)^\dagger(Y)\rangle.$$

  These isomorphisms φ and ν satisfy the required coherences as shown in [HNP24, Lemma 4.16].

The commutant of a tensor category

From [HP17, Section 2.3], we review the definition of the commutant of a category. The unitary *Drinfeld centre* $\mathcal{Z}(\mathcal{T})$ of a W^*-tensor category $\mathcal{T}$ is the W^*-category whose objects are pairs $(X, \{e_X\})$, where X is an object of $\mathcal{T}$ and $\{e_X\} = (e_{X,Y} \colon X \otimes Y \xrightarrow{\cong} Y \otimes X)_{Y \in \mathcal{T}}$ is a family of unitary isomorphisms called a *half-braiding*. The half-braiding is required to be natural in Y, and to make the following diagram commute for every $Y, Z \in \mathcal{T}$, (the associators are suppressed):

$$\begin{tikzcd}
& Y \otimes X \otimes Z \arrow[dr, "\mathrm{id}_Y \otimes e_{X,Z}"] & \\
X \otimes Y \otimes Z \arrow[ur, "e_{X,Y} \otimes \mathrm{id}_Z"] \arrow[rr, "e_{X,Y\otimes Z}"'] & & Y \otimes Z \otimes X
\end{tikzcd}$$

A morphism $(X_1, \{e_{X_1}\}) \to (X_2, \{e_{X_2}\})$ in $\mathcal{Z}(\mathcal{T})$ is a morphism $f : X_1 \to X_2$ in $\mathcal{T}$ such that $(\mathrm{id}_Y \otimes f) \circ e_{X_1,Y} = e_{X_2,Y} \circ (f \otimes \mathrm{id}_Y)$ for every $Y \in \mathcal{T}$.

The tensor product of objects of $\mathcal{Z}(\mathcal{T})$ is given by

$$(X_1, \{e_{X_1}\}) \otimes (X_2, \{e_{X_2}\}) := (X_1 \otimes X_2, \{e_{X_1 \otimes X_2}\})$$

where

$$e_{X_1 \otimes X_2, Y} := (e_{X_1,Y} \otimes \mathrm{id}_{X_2}) \circ (\mathrm{id}_{X_1} \otimes e_{X_2,Y})$$

.

Finally, $\mathcal{Z}(\mathcal{T})$ is equipped with a unitary braiding

$$\beta : (X_1 \otimes X_2, \{e_{X_1 \otimes X_2}\}) \xrightarrow{\cong} (X_2 \otimes X_1, \{e_{X_2 \otimes X_1}\})$$

given by e_{X_1,X_2}.

A notion of *relative centre* can be defined when $\mathcal{T}$ is a subcategory of some bigger W^*-tensor category $\mathcal{E}$ or, more generally, when $\mathcal{T}$ is equipped with a faithful tensor functor $\iota \colon \mathcal{T} \to \mathcal{E}$.

Definition 2.2.15. [HP17, Definition 2.9] Let $\iota : \mathcal{T} \to \mathcal{E}$ be a tensor functor between W^*-tensor categories. The *unitary commutant* or *relative centre* $\mathcal{Z}(\iota \colon \mathcal{T} \to \mathcal{E})$ of $\mathcal{T}$ inside $\mathcal{E}$ (denoted, $\mathcal{Z}(\iota)$) is the category whose objects are pairs $(X, \{e_X\})$, where X is an object of $\mathcal{E}$ and

$$\{e_X\} = (e_{X,Y} : X \otimes \iota Y \to \iota Y \otimes X)_{Y \in \mathcal{T}} \tag{2.14}$$

is a collection of unitary isomorphisms, called a half-braiding. The half-braiding is required to be natural in Y, and to make the following diagram commute for every $Y, Z \in \mathcal{T}$:

$$
\begin{tikzcd}
(X \otimes \iota Y) \otimes \iota Z \arrow[r, "e_{X,Y} \otimes \mathrm{id}_{\iota Z}"] \arrow[d, "\alpha_{X,\iota Y,\iota Z}"'] & (\iota Y \otimes X) \otimes \iota Z \arrow[d, "\alpha_{\iota Y, X, \iota Z}"] \\
X \otimes (\iota Y \otimes \iota Z) \arrow[d, "\mathrm{id}_X \otimes \mu_{Y,Z}"'] & \iota Y \otimes (X \otimes \iota Z) \arrow[d, "\mathrm{id}_{\iota Y} \otimes e_{X,Z}"] \\
X \otimes \iota (Y \otimes Z) \arrow[d, "e_{X, Y \otimes Z}"'] & \iota Y \otimes (\iota Z \otimes X) \\
\iota (Y \otimes Z) \otimes X & (\iota Y \otimes \iota Z) \otimes X \arrow[l, "\mu_{Y,Z} \otimes \mathrm{id}_X"] \arrow[u, "\alpha_{\iota Y, \iota Z, X}"']
\end{tikzcd}
$$

A morphism $(X_1, \{e_{X_1}\}) \to (X_2, \{e_{X_2}\})$ in $\mathcal{Z}(\iota)$ is a morphism $f : X_1 \to X_2$ in $\mathcal{E}$ satisfying $(\mathrm{id}_{\iota Y} \otimes f) \circ e_{X_1,Y} = e_{X_2,Y} \circ (f \otimes \mathrm{id}_{\iota Y})$ for every $Y \in \mathcal{T}$.

The tensor product in $\mathcal{Z}(\iota)$ is given by the same formula as for the Drinfeld centre discussed above.

Finally, there is a tensor functor $\mathcal{Z}(\iota) \to \mathcal{E}$ given by $(X, \{e_X\}) \mapsto X$.

Remark 2.2.16. *The unitary Drinfeld centre $\mathcal{Z}(\mathcal{T})$ of a W^*-tensor category $\mathcal{T}$ is the commutant of $\mathcal{T}$ inside itself: $\mathcal{Z}(\mathcal{T}) = \mathcal{Z}(\mathrm{id}_{\mathcal{T}})$.*

In specific situations, when $\iota\colon \mathcal{T} \to \mathcal{E}$ is obvious from the context, it is convenient to use the notation $\mathcal{T}'$ for $\mathcal{Z}(\iota\colon \mathcal{T} \to \mathcal{E})$.

Lemma 2.2.17 ([Müg03, Lemma 3.2])**.** *Let $(X, \{e_X\}) \in \mathcal{Z}(\iota)$ be an object, and let $\mathbf{1} \in \mathcal{T}$ be the unit object. Then $e_{X,\mathbf{1}} : X \otimes \iota\mathbf{1} \to \iota\mathbf{1} \otimes X$ is the following composite of unit isomorphisms: $X \otimes \iota\mathbf{1} \cong X \otimes \mathbf{1} \cong X \cong \mathbf{1} \otimes X \cong \iota\mathbf{1} \otimes X$.*

If $\mathcal{T} \to \mathcal{E}$ is a bi-involutive tensor functor between bi-involutive W^*-tensor categories, then $\mathcal{Z}(\iota)$ is a bi-involutive W^*-tensor category. The $*$-structure is inherited from $\mathcal{E}$, and the conjugate of an object $(X, \{e_X\}) \in \mathcal{Z}(\iota)$ is given by $(\overline{X}, \{e_{\overline{X}}\})$, with $e_{\overline{X},Y}$ defined as,

$$e_{\overline{X},Y} : \overline{X} \otimes \iota Y \xrightarrow{\mathrm{id}_{\overline{X}} \otimes \varphi_{\iota Y}} \overline{X} \otimes \overline{\overline{\iota Y}} \xrightarrow{\nu_{X,\overline{\iota Y}}} \overline{\overline{\iota Y} \otimes X} \xrightarrow{\overline{e_{X,\overline{Y}}}^{-1}} \overline{X \otimes \overline{\iota Y}} \xrightarrow{\nu^{-1}_{\overline{\iota Y},X}} \overline{\overline{\iota Y}} \otimes \overline{X} \xrightarrow{\varphi^{-1}_{\iota Y} \otimes \mathrm{id}_{\overline{X}}} \iota Y \otimes \overline{X}.$$

A key role in what follows is played by *absorbing* objects of a W^*-tensor category. Defining them requires the auxiliary notion of a non-zero-divisor.

Definition 2.2.18. Let $\mathcal{T}$ be a monoidal category. A *left-non-zero-divisor* is an object $X \in \mathcal{T}$ such that the functor $X \otimes - : \mathcal{T} \to \mathcal{T}$ is faithful. Similarly, an object $X \in \mathcal{T}$ is a *right-non-zero-divisor* if $- \otimes X : \mathcal{T} \to \mathcal{T}$ is faithful.

Definition 2.2.19. An object $\Omega \in \mathcal{T}$ is *weakly left absorbing* if it is a left-non-zero-divisor and for every left-non-zero-divisor $X \in \mathcal{T}$ we have $\Omega \otimes X \cong \bigoplus_{\text{countable}} \Omega$. An object $\Omega \in \mathcal{T}$ is *left absorbing* if it is a left-non-zero-divisor and for every left-non-zero-divisor $X \in \mathcal{T}$ we have $\Omega \otimes X \cong \Omega$. The notion of a *(weakly) right absorbing* object is defined analogously.

An object is called *(weakly) absorbing* if it is both (weakly) left absorbing and (weakly) right absorbing.

Remark 2.2.20. *We note that an object Ω in a bi-involutive tensor category is a left absorbing object iff its conjugate $\overline{\Omega}$ is right absorbing.*

Lemma 2.2.21. *(i) If Ω_1 and Ω_2 are absorbing objects, then $\Omega_1 \cong \Omega_2$.*

(ii) If Ω_1 is left absorbing, Ω_2 is right absorbing, and both Ω_1 and Ω_2 are non-zero-divisors (on both sides), then $\Omega_1 \simeq \Omega_2$ and both are absorbing. In particular, a left or right absorbing object in a bi-involutive tensor category is absorbing.

(iii) If Ω is weakly (right, left, or both) absorbing, then $\bigoplus_{\mathbb{N}} \Omega$ is (right, left, or both) absorbing.

Proof. The first part of the lemma follows from the second. The second part follows from the computation

$$\Omega_1 \simeq \Omega_1 \otimes \Omega_2 \simeq \Omega_2$$

where we use the left-absorbingness of Ω_1 in the first equivalence and the right-absorbingness of Ω_2 in the second. Now since $\Omega_1 \simeq \Omega_2$, it is also right-absorbing, so it absorbs from both sides.

For the third part, we compute

$$(\bigoplus_{\mathbb{N}} \Omega) \otimes X \simeq \bigoplus_{\mathbb{N}} (\Omega \otimes X) = \bigoplus_{\mathbb{N}} \bigoplus_{\text{countable}} \Omega \simeq \bigoplus_{\mathbb{N}} \Omega$$

We compute similarly on the left. $\square$

Definition 2.2.22. If $\mathcal{T}$ is a complete W^*-tensor category which admits absorbing objects, we define its *absorbing ideal* $\mathcal{T}^{\mathrm{abs}} \subset \mathcal{T}$ to be the idempotent completion of the full subcategory on the absorbing objects of $\mathcal{T}$.

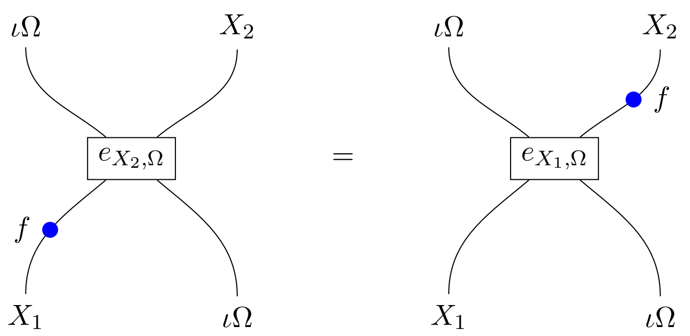
Remark 2.2.23. *Let $X \in \mathcal{T}$ and $\Omega \in \mathcal{T}$ be absorbing. Then X is either a zero-divisor or not. If it is not, then $X \otimes \Omega \simeq \Omega \in \mathcal{T}^{\mathrm{abs}}$. If X is a zero-divisor, then $X' = \mathbb{1} \oplus X$ is not, so $\Omega \oplus X \otimes \Omega \simeq X' \otimes \Omega \simeq \Omega$, so $X \otimes \Omega$ is a summand of Ω, so it is in $\mathcal{T}^{\mathrm{abs}}$. Thus, $\mathcal{T}^{\mathrm{abs}}$ is an ideal.*

Let $\iota\colon \mathcal{T} \to \mathcal{E}$ be a tensor functor between complete W^*-tensor categories. Extending Definition 2.2.15 to the context of non-unital tensor categories, we can define the relative commutant $Z(\iota|_{\mathcal{T}^{\mathrm{abs}}})$ of the non-unital tensor functor $\iota|_{\mathcal{T}^{\mathrm{abs}}} : \mathcal{T}^{\mathrm{abs}} \to \mathcal{E}$. Restriction of half-braidings provides a natural tensor functor $\mathcal{Z}(\iota) \to \mathcal{Z}(\iota|_{\mathcal{T}^{\mathrm{abs}}})$.

Proposition 2.2.24. *Let $\mathcal{T}$ be a complete W^*-tensor category, and $\Omega \in \mathcal{T}^{\mathrm{abs}}$ an absorbing object. Let $\iota\colon \mathcal{T} \to \mathcal{E}$ be a tensor functor such that $\iota(\Omega)$ is non-zero-divisor. Then the functor $\mathcal{Z}(\iota) \to \mathcal{Z}(\iota|_{\mathcal{T}^{\mathrm{abs}}})$ is fully faithful.*

Proof. Faithfulness is trivial, since a morphism in $\mathcal{Z}(\iota)$ is just a morphism in $\mathcal{E}$ satisfying a certain condition, and the same for $\mathcal{Z}(\iota|_{\mathcal{T}^{\mathrm{abs}}})$.

To show fullness, let $(X_1, \{e_{X_1}\})$ and $(X_2, \{e_{X_2}\})$ be two objects in $\mathcal{Z}(\iota)$ and f a morphism $(X_1, \{e_{X_1}\}|_{\mathcal{T}^{\mathrm{abs}}}) \to (X_2, \{e_{X_2}\}|_{\mathcal{T}^{\mathrm{abs}}})$ in $\mathcal{Z}(\iota|_{\mathcal{T}^{\mathrm{abs}}})$. By definition, this means that the following equality



holds true for every object $\Omega \in \mathcal{T}^{\mathrm{abs}}$.

We need to show that f is a morphism in $\mathcal{Z}(\iota)$. Namely, we need to show that for all $Y \in \mathcal{T}$, the equation

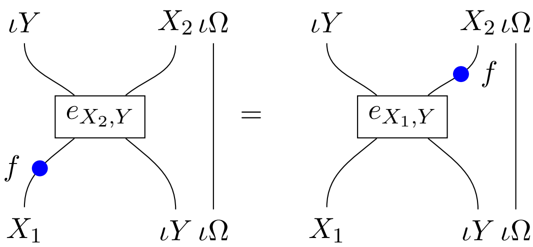
$$(\mathrm{id}_{\iota Y} \otimes f) \circ e_{X_1,Y} = e_{X_2,Y} \circ (f \otimes \mathrm{id}_{\iota Y}) \tag{2.15}$$

holds. If Equation 2.15 holds true for $Y = Y_1 \oplus Y_2$, then it also holds true for Y_1 and Y_2 separately, for we can derive the equation for Y_i from the equation of Y by composing with the inclusions and projections. It is therefore enough to prove Equation 2.15 for objects $Y \in \mathcal{T}$ which are non-zero-divisors (as any object is a direct summand of non-zero-divisors, note if A is zero-divisor then $\mathbb{1} \oplus A$ is not).

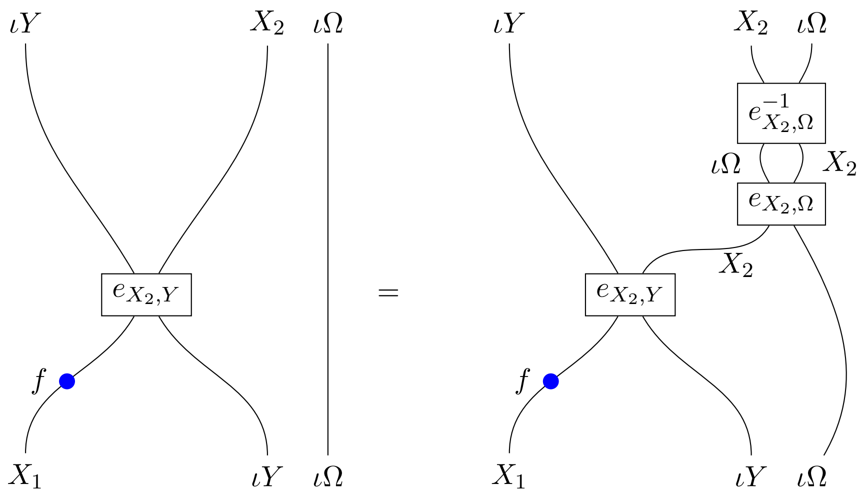
Let $Y \in \mathcal{T}$ be a non-zero-divisor. Given an absorbing object $\Omega \in \mathcal{T}^{\text{abs}}$, tensoring with $\iota\Omega$ is a faithful functor since $\iota\Omega$ is a non-zero-divisor. So it is enough to prove the equation 2.15 after tensoring with $\mathrm{id}_{\iota\Omega}$.

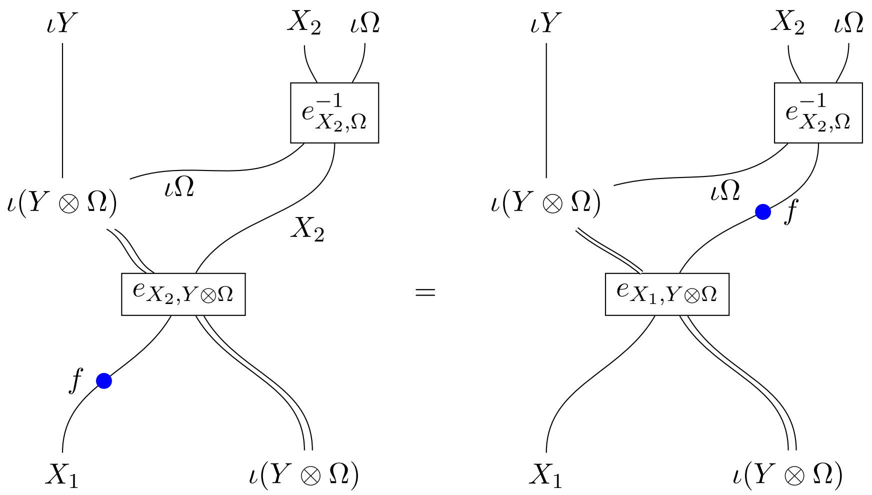
$$((\mathrm{id}_{\iota Y} \otimes f) \circ e_{X_1,Y}) \otimes \mathrm{id}_{\iota\Omega} = (e_{X_2,Y} \circ (f \otimes \mathrm{id}_{\iota Y})) \otimes \mathrm{id}_{\iota\Omega} \tag{2.16}$$

In diagrams, this is:

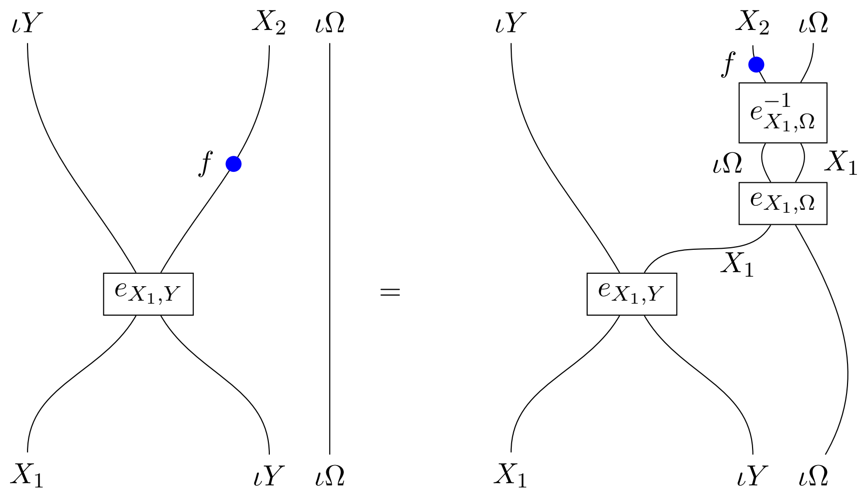


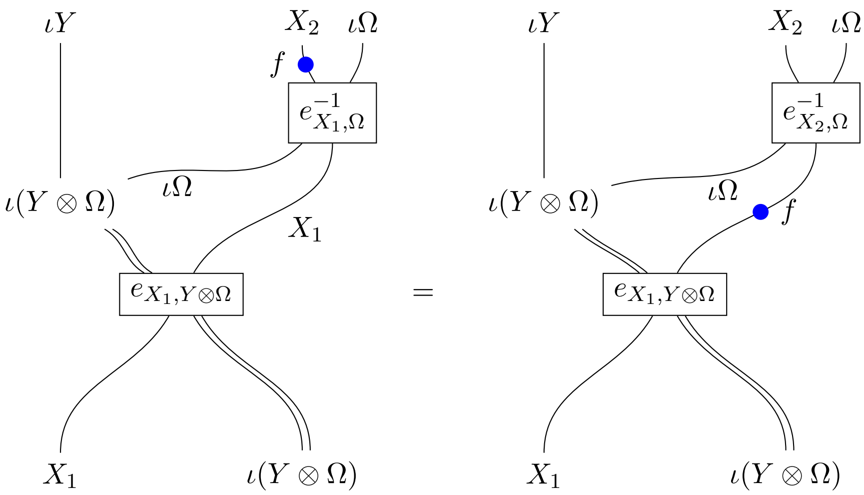
Using that $\iota Y \otimes \iota\Omega \cong \iota(Y \otimes \Omega)$ is absorbing, and starting with the left-hand side of equation 2.16, we compute:





Also, starting from the right-hand side of equation 2.16 we compute,





which together gives us the equality in equation 2.16 and, further, using that tensoring with $\iota\Omega$ is faithful, we get equation 2.15 for any object $Y \in \mathcal{T}$. $\square$

Corollary 2.2.25. *Let $\iota\colon \mathcal{T} \to \mathcal{E}$ and $\Omega \in \mathcal{T}^{\mathrm{abs}}$ be as in Proposition 2.2.24. Then for $(X, \{e_X\}) \in \mathcal{Z}(\iota)$, the half-braiding $e_{X,Y}$ is completely determined by $e_{X,\Omega}$.*

Proof. Let $(X, e_X), (X, e'_X) \in \mathcal{Z}(\iota)$ be two objects such that $e_{X,\Omega} = e'_{X,\Omega}$. By naturality of half-braidings with respect to inclusions of sub-objects, we then have $(X, \{e_X\}|_{\mathcal{T}^{\mathrm{abs}}}) = (X, \{e'_X\}|_{\mathcal{T}^{\mathrm{abs}}})$ in $\mathcal{Z}(\iota|_{\mathcal{T}^{\mathrm{abs}}})$. By Proposition 2.2.24, since $\mathrm{id}_X : (X, \{e_X\}|_{\mathcal{T}^{\mathrm{abs}}}) \to (X, \{e'_X\}|_{\mathcal{T}^{\mathrm{abs}}})$ is a morphism in $\mathcal{Z}(\iota|_{\mathcal{T}^{\mathrm{abs}}})$, it is also a morphism $(X, e_X) \to (X, e'_X)$ in $\mathcal{Z}(\iota)$. The result follows. $\square$

Bicommutant Categories

Before giving the definition of a bicommutant category, we recall the picture that it is meant to categorify. A von Neumann algebra A acts on its standard form L^2A by commuting left and right actions λ and ρ, and these actions are each other's commutants. Both of these actions are compatible with the inner product:

$$\langle a\xi, \eta\rangle = \langle \xi, a^*\eta\rangle, \qquad \langle \xi a, \eta\rangle = \langle \xi, \eta a^*\rangle, \qquad a \in A,\ \xi, \eta \in L^2A.$$

and are related as $\rho(a) = J\lambda(a^*)J$, with J the modular conjugation.

Our definition axiomatises a categorified analogue of this picture, with no ambient category $\mathrm{Bim}(R)$ and no choice of representation. The absorbing ideal $\mathcal{T}^{\mathrm{abs}}$, with its Hilb-valued inner product, plays the role of L^2A (its existence replaces the existence of the standard form; see also the comparison with Hilbert–Schmidt operators at the end of Section 2.3). There will be functors L and R that play the role of λ and ρ, and the requirement that the two functors in 2.17 be equivalences is the categorified bicommutant relation. Finally, the data γ^L and γ^R categorify the two displayed identities: combined with 2.18, they yield isomorphisms $\langle X \otimes \Omega_1, \Omega_2 \rangle \cong \langle \Omega_1, \overline{X} \otimes \Omega_2 \rangle$, which move an object of $\mathcal{T}$ across the inner product of $\mathcal{T}^{\mathrm{abs}}$ at the cost of an antilinear involution. For a von Neumann algebra these identities are *properties* of the actions; after categorification the isomorphisms witnessing them are *data*, and we must record the coherences they satisfy. Diagrams 2.20–2.24 express compatibility with the tensor structure and the involutions; Diagram 2.28 says that, just as ρ is determined by λ through J, the datum γ^R is determined by γ^L through the symmetry c of Definition 2.2.10; and diagram 2.26 is the categorified statement that the inner product on $\mathcal{T}^{\mathrm{abs}}$ is cyclic, a trace-like property.

Definition 2.2.26. Let $\mathcal{T}$ be a complete bi-involutive W^*-tensor category that admits absorbing objects. This gives us two commuting W^*-tensor functors:

$$\begin{aligned} L\colon \quad \mathcal{T} &\to \mathrm{End}(\mathcal{T}^{\mathrm{abs}}) : X \mapsto X \otimes - \\ R\colon \mathcal{T}^{\mathrm{mop}} &\to \mathrm{End}(\mathcal{T}^{\mathrm{abs}}) : X \mapsto - \otimes X, \end{aligned}$$

equivalently,

$$\begin{aligned} \mathcal{T} &\to \mathcal{Z}(R\colon \mathcal{T}^{\mathrm{mop}} \to \mathrm{End}(\mathcal{T}^{\mathrm{abs}})) \\ \mathcal{T}^{\mathrm{mop}} &\to \mathcal{Z}(L\colon \mathcal{T} \to \mathrm{End}(\mathcal{T}^{\mathrm{abs}})). \end{aligned} \tag{2.17}$$

Assume $\mathcal{T}$ is further equipped with unitary natural isomorphisms $\gamma^L\colon L(\overline{}) \Rightarrow L()^\dagger$ and $\gamma^R\colon R(\overline{}) \Rightarrow R()^\dagger$ that upgrade L and R to commuting bi-involutive W^*-tensor functors (meaning the diagrams 2.20, 2.21, 2.22, 2.23, and 2.24 drawn below are required to commute).

A *bicommutant category* is a bi-involutive W*-tensor category $\mathcal{T}$ as above, equipped with $\gamma^{L,R}$ as above, with the property that the two functors in 2.17 are equivalences of bi-involutive W*-tensor categories, and that the diagrams 2.25

(enforcing $\gamma^{L,R}$ determine each other) and 2.26 (enforcing a 'cyclicity of trace' like condition) commute.

We now take the time to spell out the coherence diagrams that appear in the above definition.

We first observe that for all $X \in \mathcal{T}$, the functor $L(X) : \mathcal{T}^{\text{abs}} \to \mathcal{T}^{\text{abs}}$ admits its unitary adjoint $L(X)^\dagger$ which is defined by the equation

$$\langle X \otimes \Omega_1, \Omega_2 \rangle \;\cong\; \langle \Omega_1, L(X)^\dagger(\Omega_2) \rangle \tag{2.18}$$

for all $\Omega_1, \Omega_2 \in \mathcal{T}^{\text{abs}}$ and this isomorphism is natural in $\Omega_1, \Omega_2 \in \mathcal{T}^{\text{abs}}$ and $X \in \mathcal{T}$. The natural isomorphisms $\gamma^L_X \colon L(\overline{X}) \to L(X)^\dagger$ can be used to induce a map from $\langle X \otimes \Omega_1, \Omega_2 \rangle \to \langle \Omega_1, \overline{X} \otimes \Omega_2 \rangle$.

For ease of interpretation of the coherences that follow, we use an absorbing object of the form $\overline{\Omega_1}$ (this loses no generality as $\overline{(-)}$ is an involution) and the map induced by $\gamma^L_{\overline{X}}$ as above to get the isomorphism, which we also denote γ^L_X:

$$\gamma^L_X \colon = \langle \overline{\Omega_1 \otimes X}, \Omega_2 \rangle \xrightarrow[\simeq]{\nu^{-1}_{\Omega_1, X}} \langle \overline{X} \otimes \overline{\Omega_1}, \Omega_2 \rangle \to \langle \overline{\Omega_1}, \overline{\overline{X}} \otimes \Omega_2 \rangle \xrightarrow[\simeq]{\varphi_X} \langle \overline{\Omega_1}, X \otimes \Omega_2 \rangle$$

.

Similarly, $R(X)^\dagger$ is defined by the equation

$$\langle \Omega_1 \otimes X, \Omega_2 \rangle \;\cong\; \langle \Omega_1, R(X)^\dagger(\Omega_2) \rangle \tag{2.19}$$

for all $\Omega_1, \Omega_2 \in \mathcal{T}^{\text{abs}}$. The natural isomorphisms $\gamma^R_X \colon R(\overline{X}) \to R(X)^\dagger$ can be used to induce a map from $\langle \Omega_1 \otimes X, \Omega_2 \rangle \to \langle \Omega_1, \Omega_2 \otimes \overline{X} \rangle$.

As before, we use the absorbing object $\overline{\Omega_1}$ and the map induced by $\gamma^R_{\overline{X}}$ to obtain the following isomorphism, which we also denote γ^R_X:

$$\gamma^R_X \colon = \langle \overline{X \otimes \Omega_1}, \Omega_2 \rangle \xrightarrow[\simeq]{\nu^{-1}_{X, \Omega_1}} \langle \overline{\Omega_1} \otimes \overline{X}, \Omega_2 \rangle \to \langle \overline{\Omega_1}, \Omega_2 \otimes \overline{\overline{X}} \rangle \xrightarrow[\simeq]{\varphi_X} \langle \overline{\Omega_1}, \Omega_2 \otimes X \rangle.$$

The above defined isomorphisms

$$\begin{aligned} \gamma^L_X &: \langle \overline{\Omega_1 \otimes X}, \Omega_2 \rangle \to \langle \overline{\Omega_1}, X \otimes \Omega_2 \rangle \\ \gamma^R_X &: \langle \overline{X \otimes \Omega_1}, \Omega_2 \rangle \to \langle \overline{\Omega_1}, \Omega_2 \otimes X \rangle. \end{aligned}$$

satisfy the following coherences:

The coherence 2.12 for L and R being bi-involutive functors simplifies to give the following:

$$\begin{tikzcd}
\langle \overline{\Omega_1 \otimes Y \otimes X}, \Omega_2 \rangle \arrow[r, "\gamma^L_X"] \arrow[dr, "\gamma^L_{Y\otimes X}"'] & \langle \overline{\Omega_1 \otimes Y}, X \otimes \Omega_2 \rangle \arrow[d, "\gamma^L_Y"] \\
& \langle \overline{\Omega_1}, Y \otimes X \otimes \Omega_2 \rangle
\end{tikzcd} \tag{2.20}$$

and

$$\begin{tikzcd}
\langle \overline{Y \otimes X \otimes \Omega_1}, \Omega_2 \rangle \arrow[r, "\gamma^R_Y"] \arrow[dr, "\gamma^R_{Y\otimes X}"'] & \langle \overline{X \otimes \Omega_1}, \Omega_2 \otimes Y \rangle \arrow[d, "\gamma^R_X"] \\
& \langle \overline{\Omega_1}, \Omega_2 \otimes Y \otimes X \rangle
\end{tikzcd} \tag{2.21}$$

and the coherence 2.13 can be modified to read:

$$\begin{tikzcd}
\langle \overline{\Omega_1 \otimes X}, \Omega_2 \rangle \arrow[r, "\gamma^L_{\Omega_1,X,\Omega_2}"] \arrow[d, "J_{\overline{\Omega_1\otimes X},\Omega_2}"] & \langle \overline{\Omega_1}, X \otimes \Omega_2 \rangle \arrow[d, "J_{\overline{\Omega_1},X\otimes\Omega_2}"] & \\
\overline{\langle \Omega_2, \overline{\Omega_1 \otimes X} \rangle} \arrow[d, "\overline{\langle \varphi_{\Omega_2}, \nu_{\Omega_1,X}{}^{-1} \rangle}"] & \overline{\langle X \otimes \Omega_2, \overline{\Omega_1} \rangle} \arrow[r, "\overline{\langle \varphi_{X\otimes\Omega_2}, \mathrm{id}_{\overline{\Omega_1}} \rangle}"] & \overline{\langle \overline{\overline{X \otimes \Omega_2}}, \overline{\Omega_1} \rangle} \arrow[d, "\overline{\langle \overline{\nu^{-1}_{\overline{X},\overline{\Omega_2}}}, \mathrm{id}_{\overline{\Omega_1}} \rangle}"] \\
\overline{\langle \overline{\overline{\Omega_2}}, \overline{X} \otimes \overline{\Omega_1} \rangle} & & \overline{\langle \overline{\overline{\Omega_2} \otimes \overline{X}}, \overline{\Omega_1} \rangle} \arrow[ll, "\overline{\gamma^L_{\overline{\Omega_2},\overline{X},\overline{\Omega_1}}}"]
\end{tikzcd} \tag{2.22}$$

and

$$\begin{tikzcd}
\langle \overline{X \otimes \Omega_1}, \Omega_2 \rangle \arrow[r, "\gamma^R_{X,\Omega_1,\Omega_2}"] \arrow[d, "J_{\overline{X\otimes\Omega_1},\Omega_2}"] & \langle \overline{\Omega_1}, \Omega_2 \otimes X \rangle \arrow[d, "J_{\overline{\Omega_1},\Omega_2\otimes X}"] & \\
\overline{\langle \Omega_2, \overline{X \otimes \Omega_1} \rangle} & \overline{\langle \Omega_2 \otimes X, \overline{\Omega_1} \rangle} \arrow[r, "\overline{\langle \varphi_{\Omega_2\otimes X}, \mathrm{id}_{\overline{\Omega_1}} \rangle}"] & \overline{\langle \overline{\overline{\Omega_2 \otimes X}}, \overline{\Omega_1} \rangle} \arrow[d, "\overline{\langle \overline{\nu^{-1}_{\overline{X},\overline{\Omega_2}}}, \mathrm{id}_{\overline{\Omega_1}} \rangle}"] \\
\overline{\langle \overline{\overline{\Omega_2}}, \overline{\Omega_1} \otimes \overline{X} \rangle} \arrow[u, "\overline{\langle \varphi^{-1}_{\Omega_2}, \nu_{\Omega_1,X} \rangle}"'] & & \overline{\langle \overline{\overline{X} \otimes \overline{\Omega_2}}, \overline{\Omega_1} \rangle} \arrow[ll, "\overline{\gamma^R_{\overline{X},\overline{\Omega_2},\overline{\Omega_1}}}"]
\end{tikzcd} \tag{2.23}$$

The following coherence ensures that γ^L and γ^R commute:

$$\begin{tikzcd}
\langle \overline{Y \otimes \Omega_1 \otimes X}, \Omega_2 \rangle \arrow[r, "\gamma^L_X"] \arrow[d, "\gamma^R_Y"] & \langle \overline{Y \otimes \Omega_1}, X \otimes \Omega_2 \rangle \arrow[d, "\gamma^R_Y"] \\
\langle \overline{\Omega_1 \otimes X}, \Omega_2 \otimes Y \rangle \arrow[r, "\gamma^{\mathrm{L}}_{\mathrm{X}}"'] & \langle \overline{\Omega_1}, X \otimes \Omega_2 \otimes Y \rangle.
\end{tikzcd} \tag{2.24}$$

The next requirement states that γ^L and γ^R determine each other:

$$\begin{tikzcd}
\langle \overline{X \otimes \Omega_1}, \Omega_2 \rangle \arrow[r, "\gamma^R_X"] \arrow[d, "c_{X\otimes\Omega_1,\Omega_2}"'] & \langle \overline{\Omega_1}, \Omega_2 \otimes X \rangle \\
\langle \overline{\Omega_2}, X \otimes \Omega_1 \rangle \arrow[r, "(\gamma^L_X)^{-1}"'] & \langle \overline{\Omega_2 \otimes X}, \Omega_1 \rangle. \arrow[u, "c_{\Omega_2\otimes X,\Omega_1}"']
\end{tikzcd} \tag{2.25}$$

Finally, the last diagram in the definition is a cyclic invariance property for every triple of absorbing objects $\Omega_1, \Omega_2, \Omega_3 \in \mathcal{T}^{\mathrm{abs}}$:

$$\begin{tikzcd}
\langle \overline{\Omega_1}, \Omega_3 \otimes \Omega_2 \rangle \arrow[r, "{\gamma^R_{\Omega_2}}^{-1}"] \arrow[d, "c_{\Omega_1,\Omega_2\otimes\Omega_3}"'] & \langle \overline{\Omega_2 \otimes \Omega_1}, \Omega_3 \rangle \arrow[d, "\gamma^L_{\Omega_1}"] \\
\langle \overline{\Omega_3 \otimes \Omega_2}, \Omega_1 \rangle & \langle \overline{\Omega_2}, \Omega_1 \otimes \Omega_3 \rangle. \arrow[l, "{\gamma^R_{\Omega_3}}^{-1}"]
\end{tikzcd} \tag{2.26}$$

Remark 2.2.27. *Given absorbing $\Omega, \Omega_1, \Omega_2 \in \mathcal{T}^{\mathrm{abs}}$, a unitary $u \colon \Omega_1 \otimes \Omega \to \Omega_1$,* *and $X \in \mathcal{T}$, the following diagram commutes:*

$$\begin{tikzcd}
\langle \overline{\Omega_1 \otimes X}, \Omega_2 \rangle \arrow[r, "\gamma^L_{\Omega_1, X, \Omega_2}"] \arrow[d, "\langle \overline{u^{-1} \otimes \mathrm{id}_X}, \mathrm{id}_{\Omega_2} \rangle"'] & \langle \overline{\Omega_1}, X \otimes \Omega_2 \rangle \arrow[d, "\langle \overline{u^{-1}}, \mathrm{id}_X \otimes \mathrm{id}_{\Omega_2} \rangle"] \\
\langle \overline{\Omega_1 \otimes \Omega \otimes X}, \Omega_2 \rangle \arrow[d, "\gamma^L_{\Omega_1, \Omega \otimes X, \Omega_2}"'] & \langle \overline{\Omega_1 \otimes \Omega}, X \otimes \Omega_2 \rangle \\
\langle \overline{\Omega_1}, \Omega \otimes X \otimes \Omega_2 \rangle \arrow[ur, "\gamma^L_{\Omega_1, \Omega, X \otimes \Omega_2}"'] &
\end{tikzcd}$$

Note all the arrows of this pentagon except for the top horizontal one are of the form $\gamma^L_{\Omega_i, \Omega_j, \Omega_k}$ where $\Omega_i, \Omega_j, \Omega_k \in \mathcal{T}^{\mathrm{abs}}$. So γ^L is fully determined by its values on the absorbing ideal. The same is true for γ^R.

Lemma 2.2.28. *Consider the following diagram:*

$$\begin{tikzcd}
\overline{\langle \overline{\Omega_1 \otimes X}, \Omega_2 \rangle} \arrow[r, "\overline{\gamma^L_{\Omega_1, X, \Omega_2}}"] \arrow[d, "f_{\overline{\Omega_1 \otimes X}, \Omega_2}"'] & \overline{\langle \overline{\Omega_1}, X \otimes \Omega_2 \rangle} \arrow[d, "f_{\overline{\Omega_1}, X \otimes \Omega_2}"] \\
\langle \overline{\overline{\Omega_1 \otimes X}}, \overline{\Omega_2} \rangle \arrow[d, "\langle \overline{\nu^{-1}_{X, \Omega_2}}, \mathrm{id}_{\overline{\Omega_2}} \rangle"'] & \langle \overline{\overline{\Omega_1}}, \overline{X \otimes \Omega_2} \rangle \arrow[d, "\langle \mathrm{id}_{\overline{\overline{\Omega_1}}}, \nu^{-1}_{\Omega_2, X} \rangle"] \\
\langle \overline{\overline{X} \otimes \overline{\Omega_1}}, \overline{\Omega_2} \rangle \arrow[r, "\gamma^R_{\overline{X}, \overline{\Omega_1}, \overline{\Omega_2}}"'] & \langle \overline{\overline{\Omega_1}}, \overline{\Omega_2} \otimes \overline{X} \rangle
\end{tikzcd}$$

If any two out of the above diagram, Equation 2.22, or Equation 2.25 commute, so does the third.

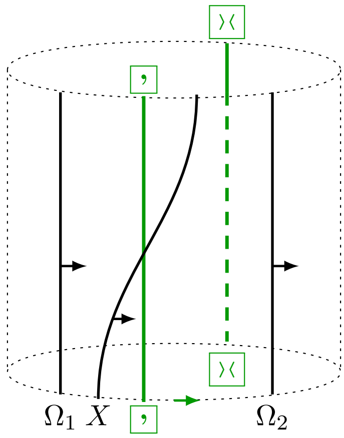
Proof. This follows from simply unpacking the definition of c in terms of f and J and using the coherences of ν and φ. $\square$

2.3 String Calculus for Bicommutant Categories

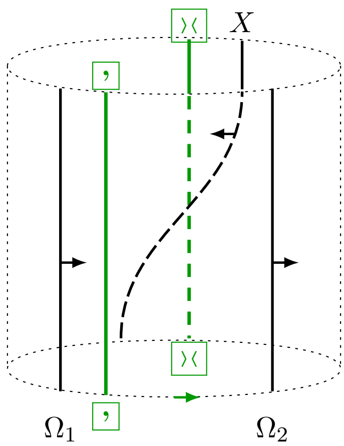
We now interpret the structure isomorphisms and coherences that appear in the definition of a bicommutant category using the string diagrams for the Hilb-valued inner product introduced above in Equation 2.10. Throughout this subsection, as before, solid black strings (dashed when they are at the back) represent objects of $\mathcal{T}$. We fix two such strings on the cylinder, and they represent a pair of absorbing objects $\Omega_1, \Omega_2 \in \mathcal{T}^{\mathrm{abs}}$.

The two basic structure isomorphisms γ^L and γ^R are depicted as a string crossing the green cut line of the cylinder. γ^L is crossing the line labelled ‘,’ and γ^R is depicted crossing with the line labelled by ‘$\rangle\,\langle$’.

Diagram 2.27 represents $\gamma^L_X\colon \langle\overline{\Omega_1\otimes X},\Omega_2\rangle \to \langle\overline{\Omega_1}, X\otimes\Omega_2\rangle$. Intuitively, the right action of X on Ω_1 is being ‘unravelled’ and interpreted as the left action of X on Ω_2. Diagram 2.28 represents $\gamma^R_X\colon \langle\overline{X\otimes\Omega_1},\Omega_2\rangle \to \langle\overline{\Omega_1},\Omega_2\otimes X\rangle$. Intuitively, the left action of X on Ω_1 is being ‘unravelled’ and interpreted as the right action of X on Ω_2.



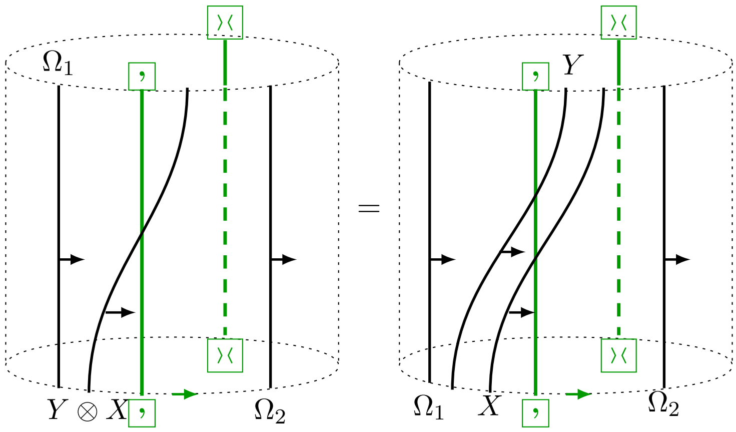
(2.27)



(2.28)

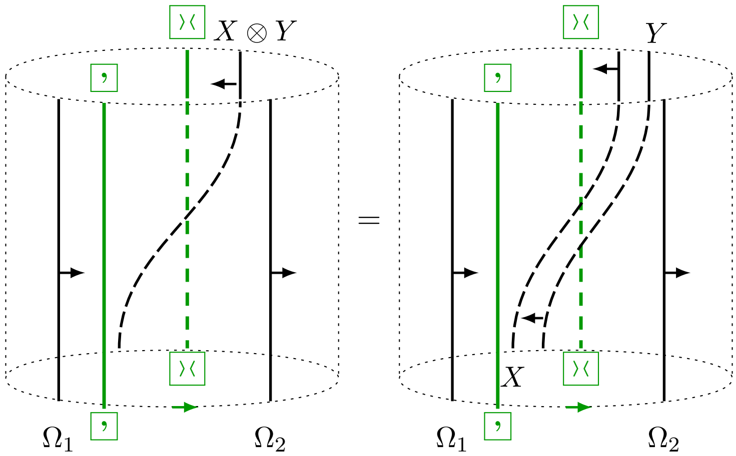
The coherences in Equations 2.20 and 2.21 express the compatibility of γ^L and γ^R with respect to the tensor product. Equation 2.20 states that moving $Y\otimes X$ through the cut in a single step is the same as first moving X and then moving Y; diagrammatically, a pair of parallel solid strings can be slid through the separator one at a time or together. Similarly, Equation 2.21 is the same statement for γ^R.

Diagrammatically, Equation 2.20 and Equation 2.21 give the following equations:



$$(2.29)$$

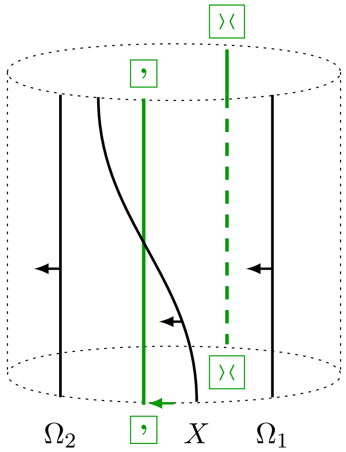
and



$$(2.30)$$

We can interpret Diagram 2.31 in two ways, as $\langle \overline{\Omega_1 \otimes X}, \Omega_2 \rangle \xrightarrow{\gamma^L_{\Omega_1, X, \Omega_2}} \langle \overline{\Omega_1}, X \otimes \Omega_2 \rangle$ and as $\overline{\langle \overline{\overline{\Omega_2}}, \overline{X} \otimes \overline{\Omega_1} \rangle} \xrightarrow{\overline{\gamma^L_{\overline{\Omega_2}, \overline{X}, \overline{\Omega_1}}}^{-1}} \overline{\langle \overline{\overline{\Omega_2} \otimes \overline{X}}, \overline{\Omega_1} \rangle}$. Coherence 2.22 tells us that these are equivalent. Coherence 2.23 is similar but for γ^R instead. This is the diagrammatic content of the compatibility between γ^L, γ^R and the involution $\overline{(\cdot)}$

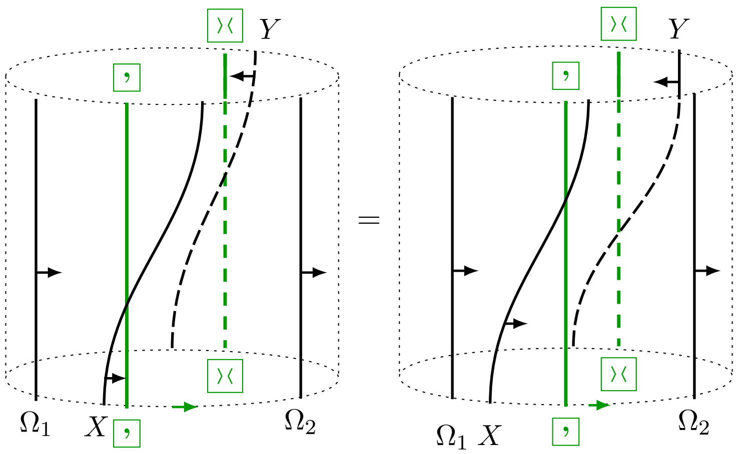
on Hilb.



(2.31)

Coherence 2.24 asserts that the left and right actions commute: applying γ^L_X and γ^R_Y in either order produces the same map $\langle \overline{Y \otimes \Omega_1 \otimes X}, \Omega_2\rangle \to \langle \overline{\Omega_1}, X \otimes \Omega_2 \otimes Y\rangle$. Diagrammatically, this is the statement that string X and string Y can be slid past separators independently.

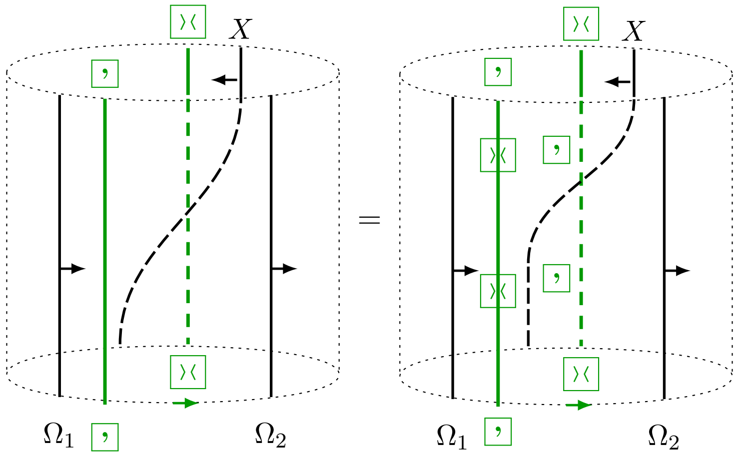
Coherence 2.24 gives the following equation:



(2.32)

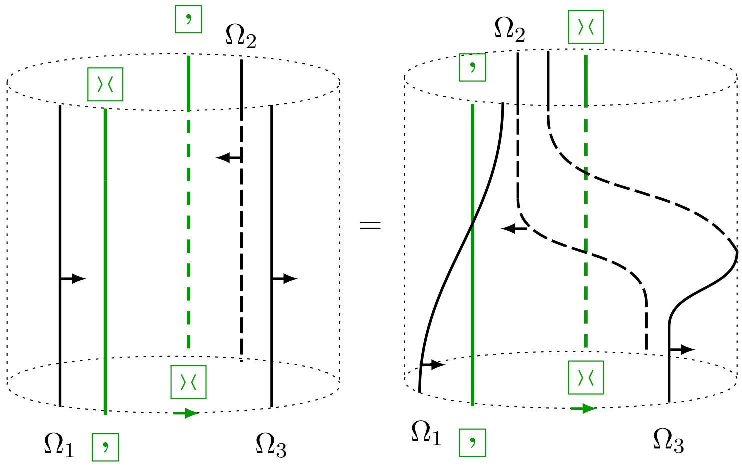
Coherence 2.25, which asserts that γ^R is determined by γ^L via the symmetry c

that flips the labels of the separators.



(2.33)

Finally, the coherence with cyclicity of absorbing objects 2.26 graphically gives the following equation:



(2.34)

This coherence encodes the sense in which the Hilb-valued inner product on $\mathcal{T}^{\text{abs}}$ is 'cyclic'.

We would like to remark here about the analogy between von Neumann algebras and bicommutant categories. We notice that $\mathcal{T}^{\text{abs}}$ behaves a lot like the Hilbert-Schmidt operators. Recall that a bounded operator $\xi \in B(H)$ on a Hilbert space H is called Hilbert-Schmidt if $\|\xi\|_2^2 := \mathrm{Tr}(\xi^*\xi) < \infty$, equivalently if $\sum_n \|\xi e_n\|^2 < \infty$ for some (hence any) orthonormal basis (e_n) of H. Importantly, the Hilbert-Schmidt operators form a two-sided $*$-ideal $\mathcal{L}^2(H) \subset \mathcal{B}(H)$ which is a Hilbert space with inner product $\langle \xi_1, \xi_2 \rangle = \mathrm{Tr}(\xi_2^*\xi_1)$. Moreover, $\mathcal{L}^2(H)$ admits a canonical identification with the Hilbert tensor product $H \otimes \overline{H}$. One should think of $\mathcal{T}^{\mathrm{abs}}$ as the two-sided $*$-ideal (bi-involutive ideal) of $\mathcal{T}$ which is a 'categorified' von Neumann algebra (analogous to $B(H)$). Notably, $\mathcal{T}^{\mathrm{abs}}$ is a complete W^*-category ('categorified Hilbert space') that admits a 'categorified' trace analogous to $\mathcal{L}^2(H)$. The L^2-space of $\mathcal{B}(H)$ is precisely the space of Hilbert-Schmidt operators.

Conjecture 2.3.1. *The coherences that appear in Definition 2.2.26 of a bicommutant category $\mathcal{T}$ imply that a Hilbert space can be consistently assigned to a diagram of parallel strings on a cylinder labelled by objects of $\mathcal{T}^{\mathrm{abs}}$ and $\mathcal{T}$ as long as there are at least two strings labelled by objects of $\mathcal{T}^{\mathrm{abs}}$.*

Remark 2.3.2. *This is part of upcoming work and out of scope of this thesis. The proof strategy is very similar to the one used in [BDH17]. We build a 2-cell complex from choices, structure maps and coherences and prove it is connected and simply connected.*

2.4 Modules over Bicommutant Categories

Just as a von Neumann algebra is best studied through its modules, a bicommutant category should act on W*-categories. In this section we define such modules and their 'categorified Connes' fusion.

Let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbf{1}, \alpha, l, r, ^{-}, \nu, j, \varphi, \gamma^L, \gamma^R)$[2] be a bicommutant category.

Definition 2.4.1. A left module over $\mathcal{T}$ is a W^*-category $\mathcal{M}$ equipped with a bi-involutive W^*-tensor functor $\mathcal{T} \to \mathrm{End}(\mathcal{M})$ such that the functor $\mathcal{M} \simeq \mathrm{Hom}_{\mathcal{T}}(\mathcal{T}, \mathcal{M}) \to \mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{\mathrm{abs}}, \mathcal{M})$ induced by the inclusion $\mathcal{T}^{\mathrm{abs}} \to \mathcal{T}$ is fully faithful. We can similarly define right modules.

Remark 2.4.2. *The definition of a module might evolve in the future; the correct definition should ask for the action to be a morphism of bicommutant categories, for which we do not yet have a complete definition. In addition, the 2-category of modules over a bicommutant category should be a W^*2-category, a notion that does not exist yet.*

[2] We note that by Equation 2.25, γ^R is determined by γ^L.

Hilb is a bicommutant category with absorbing object $\ell^2\mathbb{N}$, and $\mathrm{Hilb}^{\mathrm{abs}} = \mathrm{Hilb}$ (it is easy to verify all the coherences). Every complete W^*-category is canonically a Hilb-module. The tensor action of Hilb on any complete W^*-category is discussed in [HNP24, Equation 7]. The extra condition on restriction of the action to the absorbing ideal is trivially true since $\mathrm{Hilb}^{\mathrm{abs}} \simeq \mathrm{Hilb}$. For a W*-category $\mathcal{M}$, we provide the map $\gamma^L_H \colon \langle \overline{H} \otimes M_1, M_2 \rangle \to \langle M_1, H \otimes M_2 \rangle$. Without loss of generality, we can take $\mathcal{M} = A$-Mod for a von Neumann algebra A [HNP24, Proposition 4.8], then γ^L is given by:

$$\begin{aligned}
\langle \overline{H} \otimes M_1, M_2 \rangle &= \overline{\overline{H} \boxtimes_{\mathbb{C}} M_1} \boxtimes_A M_2 \\
&\simeq (\overline{M_1} \boxtimes_{\mathbb{C}} \overline{\overline{H}}) \boxtimes_A M_2 \\
&\simeq (\overline{M_1} \boxtimes_{\mathbb{C}} H) \boxtimes_A M_2 \\
&\simeq (H \boxtimes_{\mathbb{C}} \overline{M_1}) \boxtimes_A M_2 \\
&\simeq H \boxtimes_{\mathbb{C}} (\overline{M_1} \boxtimes_A M_2) \\
&\simeq (\overline{M_1} \boxtimes_A M_2) \boxtimes_{\mathbb{C}} H \\
&\simeq \overline{M_1} \boxtimes_A (M_2 \boxtimes_{\mathbb{C}} H) \\
&\simeq \overline{M_1} \boxtimes_A (H \boxtimes_{\mathbb{C}} M_2) \\
&= \langle M_1, H \otimes M_2 \rangle
\end{aligned}$$

Conjecture 2.4.3. *Given a bicommutant category $\mathcal{T}$ and a left $\mathcal{T}$-module $\mathcal{M}$, $\mathrm{End}_{\mathcal{T}\text{-Mod}}(\mathcal{M})$ is a bicommutant category.*

It is worth mentioning two special cases of this conjecture. If $\mathcal{T} = \mathrm{Hilb}$ and $\mathcal{M} = A$-Mod for some von Neumann algebra A, $\mathrm{End}_{\mathcal{T}\text{-Mod}}(\mathcal{M}) = \mathrm{Bim}(A)$ is a bicommutant category. If $\mathcal{T} = \mathrm{Hilb}\,[G]$ is the category of G-graded Hilbert spaces for a possibly-infinite discrete group G, and $\mathcal{M} = \mathrm{Hilb}$, then $\mathrm{End}_{\mathcal{T}\text{-Mod}}(\mathcal{M}) = \mathrm{Rep}(G)$ is a bicommutant category. These special cases with their absorbing ideals and bicommutant property are discussed in [Hen26].

Definition 2.4.4. Given $\mathcal{M}$ and $\mathcal{N}$ right and left $\mathcal{T}$-modules respectively, we can define their fusion as

$$\mathcal{M} \boxtimes_{\mathcal{T}} \mathcal{N} = \mathrm{proj}_{2,1}\left(\mathrm{End}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}} \oplus \mathcal{N}\right)^{\mathrm{abs}}\right)$$

where $\mathrm{proj}_{2,1}(\xi) := \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{id}_{\mathcal{N}} \end{smallmatrix}\right) \xi \left(\begin{smallmatrix} \mathrm{id}_{\mathcal{M}} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ as before.

An object of $\mathrm{End}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}} \oplus \mathcal{N}\right)$ is of the form

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \in \begin{pmatrix} \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}}, \overline{\mathcal{M}}\right) & \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\mathcal{N}, \overline{\mathcal{M}}\right) \\ \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}}, \mathcal{N}\right) & \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\mathcal{N}, \mathcal{N}\right) \end{pmatrix}$$

Lemma 2.4.5. *An object*

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

is absorbing if and only if for any non-zero-divisors $F_{11} \in \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}}, \overline{\mathcal{M}}\right)$ *and* $F_{22} \in \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\mathcal{N}, \mathcal{N}\right)$ *we have* $F_{ii} \circ \Omega_{ij} \circ F_{jj} \simeq \Omega_{ij}$

Proof. Only if: Suppose $F_{11} \in \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}}, \overline{\mathcal{M}}\right)$ and $F_{22} \in \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\mathcal{N}, \mathcal{N}\right)$ are not zero-divisors and Ω_{ij} are functors as above such that the matrix in the statement of the lemma is absorbing. Then

$$\begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix} \in \begin{pmatrix} \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}}, \overline{\mathcal{M}}\right) & \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\mathcal{N}, \overline{\mathcal{M}}\right) \\ \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}}, \mathcal{N}\right) & \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\mathcal{N}, \mathcal{N}\right) \end{pmatrix}$$

is clearly not a zero-divisor. Then we have

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \circ \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix} = \begin{pmatrix} \Omega_{11} \circ F_{11} & \Omega_{12} \circ F_{22} \\ \Omega_{21} \circ F_{11} & \Omega_{22} \circ F_{22} \end{pmatrix} \simeq \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

Writing similar equations from the left, we obtain the desired conditions on Ω_{ij}.

If: Suppose we have Ω_{ij} satisfying the conditions. We compute

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \circ \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} \Omega_{11} \circ F_{11} \oplus \Omega_{12} \circ F_{21} & \Omega_{11} \circ F_{12} \oplus \Omega_{12} \circ F_{22} \\ \Omega_{21} \circ F_{11} \oplus \Omega_{22} \circ F_{21} & \Omega_{21} \circ F_{12} \oplus \Omega_{22} \circ F_{22} \end{pmatrix}$$

Given $A_{11} \in \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}}, \overline{\mathcal{M}}\right)$ a non-zero-divisor, we have

$$A_{11} \circ (\Omega_{11} \circ F_{11} \oplus \Omega_{12} \circ F_{21}) = A_{11} \circ \Omega_{11} \circ F_{11} \oplus A_{11} \circ \Omega_{12} \circ F_{21} \simeq \Omega_{11} \circ F_{11} \oplus \Omega_{12} \circ F_{21}$$

Thus $\Omega_{11} \circ F_{11} \oplus \Omega_{12} \circ F_{21}$ is left absorbing, but since $\mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}}, \overline{\mathcal{M}}\right)$ is a bi-involutive tensor category, it is also right absorbing by Lemma 2.2.21(ii), and since any two absorbing objects are isomorphic, we obtain $\Omega_{11} \circ F_{11} \oplus \Omega_{12} \circ F_{21} \simeq \Omega_{11}$. Similarly $\Omega_{21} \circ F_{12} \oplus \Omega_{22} \circ F_{22} \simeq \Omega_{22}$.

Similarly, consider $\Omega_{21} \circ \Omega_{21}^{\dagger} \circ (\Omega_{21} \circ F_{11} \oplus \Omega_{22} \circ F_{21})$. Since Ω_{21} absorbs objects of $\mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}}, \overline{\mathcal{M}}\right)$ from the right, this is isomorphic to Ω_{21}. And since Ω_{21} and Ω_{22} absorb objects of $\mathrm{Hom}_{\mathcal{T}\text{-Mod}}(\mathcal{N}, \mathcal{N})$ from the left, this expression is also equivalent to $\Omega_{21} \circ F_{11} \oplus \Omega_{22} \circ F_{21}$. Thus $\Omega_{21} \circ F_{11} \oplus \Omega_{22} \circ F_{21} \simeq \Omega_{21}$. We can make a similar argument for Ω_{12} too. This shows that

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

is absorbing. □

Corollary 2.4.6.

$$\mathcal{M} \boxtimes_{\mathcal{T}} \mathcal{N} \simeq \mathrm{End}_{\mathrm{Hom}_{\mathcal{T}\text{-Mod}}(\overline{\mathcal{M}}, \mathcal{N})}(\Omega_{21})^{\mathrm{op}}\text{-Mod}$$

where Ω_{21} *satisfies* $\Omega_{22} \circ \Omega_{21} \circ \Omega_{11} \simeq \Omega_{21}$ *and where* Ω_{11} *is an absorbing object of* $\mathrm{Hom}_{\mathcal{T}\text{-Mod}}(\overline{\mathcal{M}}, \overline{\mathcal{M}})$ *and* Ω_{22} *is an absorbing object of* $\mathrm{Hom}_{\mathcal{T}\text{-Mod}}(\mathcal{N}, \mathcal{N})$.

Proof. The fusion $\mathcal{M} \boxtimes_{\mathcal{T}} \mathcal{N}$ is a subcategory of the absorbing ideal of $\mathrm{End}_{\mathcal{T}\text{-Mod}}(\overline{\mathcal{M}} \oplus \mathcal{N})$. By Lemma 2.4.5, this ideal is generated by

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{21}^{\dagger} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

By [HNP24, Proposition 3.18], we have

$$\mathrm{End}_{\mathcal{T}\text{-Mod}}(\overline{\mathcal{M}} \oplus \mathcal{N})^{\mathrm{abs}} \simeq \mathrm{End}_{\mathrm{End}_{\mathcal{T}\text{-Mod}}(\overline{\mathcal{M}} \oplus \mathcal{N})}(\Omega)^{\mathrm{op}}\text{-Mod}$$

Since morphisms are given component-wise in

$$\mathrm{End}_{\mathcal{T}\text{-Mod}}(\overline{\mathcal{M}} \oplus \mathcal{N}) = \begin{pmatrix} \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}}, \overline{\mathcal{M}}\right) & \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\mathcal{N}, \overline{\mathcal{M}}\right) \\ \mathrm{Hom}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{M}}, \mathcal{N}\right) & \mathrm{Hom}_{\mathcal{T}\text{-Mod}}(\mathcal{N}, \mathcal{N}) \end{pmatrix}$$

we obtain the desired equivalence. □

Further, if Ω_{21} satisfies the weaker condition that $\Omega_{22} \circ \Omega_{21} \simeq \bigoplus_{\text{countable}} \Omega_{21}$ and $\Omega_{21} \circ \Omega_{11} \simeq \bigoplus_{\text{countable}} \Omega_{21}$, then, clearly, $\Omega_{21}' = \ell^2\mathbb{N} \otimes \Omega_{21}$ absorbs Ω_{11} and Ω_{22} as $\bigoplus_{\text{countable}} \ell^2\mathbb{N} \simeq \ell^2\mathbb{N}$. The product of a Hilbert space and an object of a complete W^*-category is defined in [HNP24, Equation 7]. Further, we have

$$\mathrm{End}(\Omega_{21}') = \mathrm{End}(\ell^2\mathbb{N} \otimes \Omega_{21}) \simeq B(\ell^2\mathbb{N})\overline{\otimes}\mathrm{End}(\Omega_{21})$$

But $B(\ell^2\mathbb{N})$ is Morita equivalent to $\mathbb{C}$, so we obtain the following corollary.

Corollary 2.4.7.

$$\mathcal{M} \boxtimes_{\mathcal{T}} \mathcal{N} \simeq \mathrm{End}_{\mathrm{Hom}_{\mathcal{T}\text{-Mod}}(\overline{\mathcal{M}},\mathcal{N})} (\Omega_{21})^{\mathrm{op}}\text{-Mod}$$

where Ω_{21} satisfies $\Omega_{22} \circ \Omega_{21} \simeq \bigoplus_{\mathrm{countable}} \Omega_{21}$ and $\Omega_{21} \circ \Omega_{11} \simeq \bigoplus_{\mathrm{countable}} \Omega_{21}$ and where Ω_{11} is a weakly absorbing object of $\mathrm{Hom}_{\mathcal{T}\text{-Mod}}(\overline{\mathcal{M}},\overline{\mathcal{M}})$ and Ω_{22} is a weakly absorbing object of $\mathrm{Hom}_{\mathcal{T}\text{-Mod}}(\mathcal{N},\mathcal{N})$.

Definition 2.4.8. Given pointings $M\colon \mathrm{Hilb} \to \mathcal{M}$ and $N\colon \mathrm{Hilb} \to \mathcal{N}$, we define the pointing on $\mathcal{M} \boxtimes_{\mathcal{T}} \mathcal{N}$ to be the dagger of the functor

$$\begin{aligned}\mathrm{Hom}_{T}\left(\overline{\mathcal{M}},\mathcal{N}\right)^{\mathrm{abs}} \to &\mathrm{Hilb}\\ &F \mapsto \left(N^\dagger \circ F \circ \overline{M} \circ \dagger_{\mathrm{Hilb}}\right)(\mathbb{C})\end{aligned}$$

where $\dagger_{\mathrm{Hilb}}\colon \mathrm{Hilb} \to \overline{\mathrm{Hilb}}$ is the dagger on Hilb.

Definition 2.4.9. A functor $F\colon W_1 \to W_2$ of W*-categories is called dominant if every object of W_2 is a direct summand of an object in the essential image of W_1.

Proposition 2.4.10. *Let $\mathcal{T}$ be a bicommutant category. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{T}$-module categories. Let Ω be an absorbing object of $\mathcal{T}$. If the action functors $\Omega \cdot_{\mathcal{N}} -\colon \mathcal{N} \to \mathcal{N}$ and $\Omega \cdot_{\mathcal{M}} -\colon \mathcal{M} \to \mathcal{M}$ are faithful and dominant respectively, then $\mathrm{Res}\colon \mathrm{Hom}_{\mathcal{T}\text{-Mod}}(\mathcal{M},\mathcal{N}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{T}^{\mathrm{abs}}\text{-Mod}}(\mathcal{M},\mathcal{N})$ is an equivalence of categories.*

Proof. Res is clearly faithful. We can show Res is full using an argument similar to Proposition 2.2.24.

We now show Res is essentially surjective. That is, given a functor $F\colon \mathcal{M} \to \mathcal{N}$, equipped with linearity data for the action of Ω, we construct linearity data for F with an arbitrary object $X \in \mathcal{T}$. Concretely, we must construct a unitary $u_{X,m}\colon X \cdot_{\mathcal{N}} F(m) \to F(X \cdot_{\mathcal{M}} m)$ for every $X \in \mathcal{T}$ and every $m \in \mathcal{M}$.

Uniqueness of $u_{X,m}$: The map $\mathrm{id}_{\Omega} \cdot_{\mathcal{N}} u_{X,m}$ must be equal to the composite

$$\begin{tikzcd}
\Omega \cdot_{\mathcal{N}} (X \cdot_{\mathcal{N}} F(m)) \arrow[r] & (\Omega \otimes X) \cdot_{\mathcal{N}} F(m) \arrow[r, "u_{\Omega \otimes X, m}"] & F((\Omega \otimes X) \cdot_{\mathcal{M}} m) \arrow[d] \\
& \Omega \cdot_{\mathcal{N}} F(X \cdot_{\mathcal{M}} m) & F(\Omega \cdot_{\mathcal{M}} (X \cdot_{\mathcal{M}} m)) \arrow[l]
\end{tikzcd}$$

Since $\Omega \cdot -$ is faithful, this uniquely determines $u_{X,m}$.

Existence of $u_{X,m}$: Since $\Omega \cdot_{\mathcal{M}} -$ is dominant, we can write m as (a direct summand of) $\Omega \cdot_{\mathcal{M}} m'$.
$u_{X,\Omega \cdot_{\mathcal{M}} m'}$ must be equal to the composite

$$\begin{tikzcd}
X \cdot_{\mathcal{N}} F(\Omega \cdot_{\mathcal{M}} m') \arrow[r, "\mathrm{id}_X \cdot_{\mathcal{N}} u^{-1}_{\Omega, m'}"] & X \cdot_{\mathcal{N}} (\Omega \cdot_{\mathcal{N}} F(m') \arrow[r] & (X \otimes \Omega) \cdot_{\mathcal{N}} F(m')) \arrow[d, "u_{X \otimes \Omega, m'}"] \\
& F(X \cdot_{\mathcal{M}} (\Omega \cdot_{\mathcal{M}} m')) & F((X \otimes \Omega) \cdot_{\mathcal{M}} m') \arrow[l]
\end{tikzcd}$$

One can see that $u_{X,m}$ satisfies all the necessary coherences from the coherences satisfied by it when X is absorbing. $\square$

Proposition 2.4.11. *Let $\mathcal{T}$ be a bicommutant category, then*

$$\mathcal{T}^{\mathrm{abs}} \boxtimes_{\mathcal{T}} \mathcal{T}^{\mathrm{abs}} \simeq \mathcal{T}^{\mathrm{abs}}$$

Proof. Using the definition of fusion,

$$\begin{aligned}
\mathcal{T}^{\mathrm{abs}} \boxtimes_{\mathcal{T}} \mathcal{T}^{\mathrm{abs}} &= \mathrm{proj}_{2,1}\left(\mathrm{End}_{\mathcal{T}\text{-Mod}}\left(\overline{\mathcal{T}^{\mathrm{abs}}} \oplus \mathcal{T}^{\mathrm{abs}}\right)^{\mathrm{abs}}\right) \\
&\simeq \mathrm{End}_{\mathcal{T}\text{-Mod}}\left(\mathcal{T}^{\mathrm{abs}}\right)^{\mathrm{abs}} \\
&\simeq \mathcal{T}^{\mathrm{abs}}
\end{aligned}$$

The last equivalence uses that $\mathcal{T} \simeq \mathrm{End}_{\mathcal{T}\text{-Mod}}\left(\mathcal{T}^{\mathrm{abs}}\right)$, which follows from the bicommutant property. $\square$

Conjecture 2.4.12. *Let $\mathcal{M}$ be a $\mathcal{T}$-module category,*

$$\mathcal{M} \boxtimes_{\mathcal{T}} \mathcal{T}^{\mathrm{abs}} \simeq \mathcal{M}$$

Conjecture 2.4.13. *Given a right $\mathcal{T}_1$-module $\mathcal{M}_1$, a $(\mathcal{T}_1, \mathcal{T}_2)$-bimodule $\mathcal{N}$, and a left $\mathcal{T}_2$-module $\mathcal{M}_2$, we have*

$$(\mathcal{M}_1 \boxtimes_{\mathcal{T}_1} \mathcal{N}) \boxtimes_{\mathcal{T}_2} \mathcal{M}_2 \simeq \mathcal{M}_1 \boxtimes_{\mathcal{T}_1} (\mathcal{N} \boxtimes_{\mathcal{T}_2} \mathcal{M}_2)$$

The tensor product of complete W*-categories was defined in [HNP24, Definition 3.33].

Proposition 2.4.14. *For complete* W**-categories $\mathcal{M}$ and $\mathcal{N}$, we have an equivalence $\mathcal{M} \boxtimes_{\mathrm{Hilb}} \mathcal{N} \simeq \mathcal{M}\widehat{\otimes}\mathcal{N}$.*

Proof. Without loss of generality, we assume $\mathcal{M} = A$-Mod and $\mathcal{N} = B$-Mod for von Neumann algebras A and B. We have

$$A\text{-Mod} \boxtimes_{\mathrm{Hilb}} B\text{-Mod} = \mathrm{Hom}(\overline{A\text{-Mod}}, B\text{-Mod})^{\mathrm{abs}}$$

By [HNP24, Proposition 3.28], we see that this category is generated by (B, A^{op})-bimodules that absorb (B, B)-bimodules on Connes fusing over B and $(A^{\mathrm{op}}, A^{\mathrm{op}})$-bimodules on Connes fusing over A^{op}.

If $B = A^{\mathrm{op}}$, it is shown in [HP17, Example 5.7], that such an absorbing object is given by ${}_BL^2B\boxtimes_{\mathbb{C}}\ell^2\mathbb{N}\boxtimes_{\mathbb{C}}L^2B_B$. The absorbing ideal generated by this is exactly (B, B)-bimodules where the $B \otimes B^{\mathrm{op}}$-action extends to a $B\overline{\otimes}B^{\mathrm{op}}$-action, that is, the ideal is equivalent to $(B\overline{\otimes}B^{\mathrm{op}})$-Mod.

Using a very similar argument, we can show that when A and B are distinct we have

$$\mathrm{Hom}(\overline{A\text{-Mod}}, B\text{-Mod})^{\mathrm{abs}} \simeq (A\overline{\otimes}B)\text{-Mod}$$

So, we have

$$A\text{-Mod} \boxtimes_{\mathrm{Hilb}} B\text{-Mod} \simeq (A\overline{\otimes}B)\text{-Mod} \simeq A\text{-Mod}\widehat{\otimes}B\text{-Mod}$$

□

2.5 Towards the symmetric monoidal 3-category of Bicommutant Categories

We do not build the 3-category BicommCat in as much detail as [BDH18] does CNet, the 3-category of conformal nets. In this section, we spell out in detail the bicategory $\mathrm{Hom}(\mathcal{T}_1, \mathcal{T}_2)$ for any two objects. Following the style of [BJS21], we give a proposed definition:

Conjecture 2.5.1. *There exists a* 3*-category* BicommCat*, whose:*

- *Objects are bicommutant categories.*
- 1*-morphisms are bimodule categories between such.*
- 2*-morphisms are bimodule functors between such.*
- 3*-morphisms are natural transformations between such.*

The monoidal structure on BicommCat *will be the subject of study in future work. Similar to the spatial tensor product of von Neumann algebras, we expect there to exist the spatial tensor product of bicommutant categories that makes this* 3*-category a symmetric monoidal* 3*-category.*

The composition of 1*-morphisms is 'categorified Connes fusion'. Compositions of* 2*- and* 3*-morphisms are composition of functors and of natural transformations in the usual sense.*

We use Section 2.2.2 in [DSPS20] as a reference for module categories, functors, and transformations for tensor categories. For any two bicommutant categories $\mathcal{T}_1, \mathcal{T}_2$, we define the bicategory $\mathcal{T}_1$-Mod-$\mathcal{T}_2$ to be the strict 2-category whose

- objects are tuples $(\mathcal{M}, L, R, \beta)$ where $\mathcal{M}$ is a W^*-category, $L\colon \mathcal{T}_1 \to \mathrm{End}(\mathcal{M})$ and $R\colon \mathcal{T}_2^{\mathrm{mop}} \to \mathrm{End}(\mathcal{M})$ are bi-involutive tensor functors (with their usual structure maps, $\mu^L, \mu^R, \gamma^L, \gamma^R$ with coherences) giving $\mathcal{M}$ the structure of a left $\mathcal{T}_1$-module and right $\mathcal{T}_2$-module respectively. We denote $L(X)(m)$ and $R(X)(m)$ as $X \cdot m$ and $m \cdot X$ respectively. β is a bi-involutive natural isomorphism

$$\beta_{X,Y}\colon R(Y) \circ L(X) \xrightarrow{\simeq} L(X) \circ R(Y), \qquad X \in \mathcal{T}_1,\ Y \in \mathcal{T}_2,$$

natural in both arguments, such that the following hold in addition to the standard coherences for module categories:

(i) two additional pentagon axioms hold:

$$\begin{tikzcd}
((X \otimes X') \cdot m) \cdot Y \arrow[rr, "\beta_{X \otimes X', Y, m}"] & & (X \otimes X') \cdot (m \cdot Y) \\
(X \cdot (X' \cdot m)) \cdot Y \arrow[u, "\mu^L_{X, X', m} \cdot \mathrm{id}_Y"] \arrow[dr, "\beta_{X, Y, X' \cdot m}"'] & & X \cdot (X' \cdot (m \cdot Y)) \arrow[u, "\mu^L_{X, X', m \cdot Y}"'] \\
& X \cdot ((X' \cdot m) \cdot Y) \arrow[ur, "\mathrm{id}_X \cdot \beta_{X', Y, m}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(X \cdot m) \cdot (Y \otimes Y') \arrow[rr, "\beta_{X, Y \otimes Y', m}"] & & X \cdot (m \cdot (Y \otimes Y')) \\
((X \cdot m) \cdot Y) \cdot Y' \arrow[u, "\mu^R_{Y, Y', X \cdot m}"] \arrow[dr, "\beta_{X, Y, m} \cdot \mathrm{id}_{Y'}"'] & & X \cdot ((m \cdot Y) \cdot Y') \arrow[u, "\mathrm{id}_X \cdot \mu^R_{Y, Y', m}"'] \\
& (X \cdot (m \cdot Y)) \cdot Y' \arrow[ur, "\beta_{X, Y', m \cdot Y}"'] &
\end{tikzcd}$$

(ii) two additional triangle axioms hold for $m \in \mathcal{M}$:

$$\begin{tikzcd}
(\mathbf{1} \cdot m) \cdot Y \arrow[rr, "\beta_{\mathbf{1}, Y, m}"] \arrow[dr] & & \mathbf{1} \cdot (m \cdot Y) \arrow[dl] \\
& m \cdot Y &
\end{tikzcd}
\quad
\begin{tikzcd}
(\mathbf{1} \cdot m) \cdot \mathbf{1} \arrow[rr, "\beta_{\mathbf{1}, \mathbf{1}, m}"] \arrow[d] & & \mathbf{1} \cdot (m \cdot \mathbf{1}) \arrow[d] \\
\mathbf{1} \cdot m \arrow[r] & m & m \cdot \mathbf{1} \arrow[l]
\end{tikzcd}
\quad
\begin{tikzcd}
(X \cdot m) \cdot \mathbf{1} \arrow[rr, "\beta_{X, \mathbf{1}, m}"] \arrow[dr] & & X \cdot (m \cdot \mathbf{1}) \arrow[dl] \\
& X \cdot m &
\end{tikzcd}$$

(iii) coherence for compatibility between the bi-involutive structure and β holds: Similar to the condition that appeared in the definition of a bicommutant category in Equation 2.24, we get the following diagram for all $m, n \in \mathcal{M}$ where the inner product is the Hilb-valued inner product on $\mathcal{M}$ (here we do not write the maps explicitly with all indices but indicate the maps used to construct each arrow):

$$
\begin{tikzcd}[column sep=small]
& \langle \overline{X} \cdot (m \cdot \overline{Y}), n \rangle \arrow[dl, "\beta^{-1}_{\overline{X},\overline{Y}}"'] \arrow[dr, "\gamma^L_X"] & \\
\langle (\overline{X} \cdot m) \cdot \overline{Y}, n \rangle \arrow[d, "\gamma^R_Y"'] & & \langle \overline{Y} \cdot m, X \cdot n \rangle \arrow[d, "\gamma^R_Y"] \\
\langle \overline{X} \cdot m, n \cdot Y \rangle \arrow[dr, "\gamma^L_X"'] & & \langle m, (X \cdot n) \cdot Y \rangle \arrow[dl, "\beta_{X,Y}"] \\
& \langle m, X \cdot (n \cdot Y) \rangle &
\end{tikzcd}
\tag{2.35}
$$

and finally the faithfulness condition holds: the functor $\mathrm{Hom}_{\mathcal{T}_1}(\mathcal{T}_1, \mathcal{M}) \to \mathrm{Hom}_{\mathcal{T}_1}(\mathcal{T}_1^{\mathrm{abs}}, \mathcal{M})$ induced by the inclusion $\mathcal{T}_1^{\mathrm{abs}} \hookrightarrow \mathcal{T}_1$ is fully faithful, and similarly for $\mathcal{T}_2$.

- 1-morphisms from $(\mathcal{M}, L_{\mathcal{M}}, R_{\mathcal{M}}, \beta^{\mathcal{M}})$ to $(\mathcal{N}, L_{\mathcal{N}}, R_{\mathcal{N}}, \beta^{\mathcal{N}})$ are pairs (F, f) where $F \colon \mathcal{M} \to \mathcal{N}$ is a W*-functor and f consists of unitary natural isomorphisms

$$
f^L_X \colon F \circ L_{\mathcal{M}}(X) \xrightarrow{\simeq} L_{\mathcal{N}}(X) \circ F, \qquad f^R_Y \colon F \circ R_{\mathcal{M}}(Y) \xrightarrow{\simeq} R_{\mathcal{N}}(Y) \circ F,
$$

natural in $X \in \mathcal{T}_1$ and $Y \in \mathcal{T}_2$ respectively, such that

  (i) monoidal compatibility: The evident pentagon relation for both f^L and f^R holds, and the triangle relations, namely $f^L_{\mathbf{1}} = \mathrm{id}_F$ and $f^R_{\mathbf{1}} = \mathrm{id}_F$ hold up to unitors.

  (ii) bi-involutive compatibility: f^L and f^R are morphisms of bi-involutive functors, i.e. the squares relating $f^L_{\overline{X}}$ to γ^L and $f^R_{\overline{Y}}$ to γ^R (for both actions) commute.

(iii) β-compatibility: The following diagram commutes

$$\begin{tikzcd}
F \circ L_{\mathcal{M}}(X) \circ R_{\mathcal{M}}(Y) \arrow[r, "f^L_X \circ \mathrm{id}"] & L_{\mathcal{N}}(X) \circ F \circ R_{\mathcal{M}}(Y) \arrow[d, "\mathrm{id} \circ f^R_Y"] \\
F \circ R_{\mathcal{M}}(Y) \circ L_{\mathcal{M}}(X) \arrow[u, "\mathrm{id} \circ \beta^{\mathcal{M}}_{X,Y}"] \arrow[d, "f^R_Y \circ \mathrm{id}"'] & L_{\mathcal{N}}(X) \circ R_{\mathcal{N}}(Y) \circ F \\
R_{\mathcal{N}}(Y) \circ F \circ L_{\mathcal{M}}(X) \arrow[r, "\mathrm{id} \circ f^L_X"'] & R_{\mathcal{N}}(Y) \circ L_{\mathcal{N}}(X) \circ F \arrow[u, "\beta^{\mathcal{N}}_{X,Y} \circ \mathrm{id}"']
\end{tikzcd}$$

Composition of 1-morphisms $(G, g) \circ (F, f) := (G \circ F,\ g \cdot f)$ is given by

$$(g \cdot f)^L_X := (g^L_X \circ \mathrm{id}_F) \circ (\mathrm{id}_G \circ f^L_X) \colon G \circ F \circ L_{\mathcal{M}}(X) \xrightarrow{\simeq} L_{\mathcal{N}}(X) \circ G \circ F,$$

and similarly for $(g \cdot f)^R_Y$. The identity 1-morphism on $(\mathcal{M}, L, R, \beta)$ is $(\mathrm{Id}_{\mathcal{M}}, \mathrm{id})$.

- 2-morphisms from (F, f) to (G, g) (two 1-morphisms $(\mathcal{M}, L_{\mathcal{M}}, R_{\mathcal{M}}, \beta^{\mathcal{M}}) \to (\mathcal{N}, L_{\mathcal{N}}, R_{\mathcal{N}}, \beta^{\mathcal{N}})$) are unitary natural transformations $\eta \colon F \Rightarrow G$ such that the squares

$$\begin{tikzcd}
F \circ L_{\mathcal{M}}(X) \arrow[r, "\eta \circ \mathrm{id}"] \arrow[d, "f^L_X"'] & G \circ L_{\mathcal{M}}(X) \arrow[d, "g^L_X"] \\
L_{\mathcal{N}}(X) \circ F \arrow[r, "\mathrm{id} \circ \eta"'] & L_{\mathcal{N}}(X) \circ G
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
F \circ R_{\mathcal{M}}(Y) \arrow[r, "\eta \circ \mathrm{id}"] \arrow[d, "f^R_Y"'] & G \circ R_{\mathcal{M}}(Y) \arrow[d, "g^R_Y"] \\
R_{\mathcal{N}}(Y) \circ F \arrow[r, "\mathrm{id} \circ \eta"'] & R_{\mathcal{N}}(Y) \circ G
\end{tikzcd}$$

commute for all $X \in \mathcal{T}_1$ and $Y \in \mathcal{T}_2$. Vertical composition is composition of natural transformations. Horizontal composition is the standard horizontal composition of natural transformations.

Chapter 3

Solitonic Endomorphisms

Given a von Neumann algebra, one can consider a category of endomorphisms of the algebra, where the objects are endomorphisms and the morphisms are intertwiners. We can also consider the category of bimodules over the von Neumann algebra. Both of these categories are W*-tensor categories, and in general there is a fully faithful monoidal functor from endomorphisms to bimodules, given by twisting the action of the algebra on the L^2-space of the algebra. When the algebra is a type III factor, this functor becomes an equivalence of categories after adding a 0 object. We then introduce the definition of conformal nets and recall the main results about conformal nets that play a key role in this work. Next, we define 'left- (or right-) localised' endomorphisms as a generalisation of DHR (Doplicher-Haag-Roberts) endomorphisms, which are fully localised. Just as DHR endomorphisms correspond to representations of conformal nets, the 'left- (or right-) localised' endomorphisms will correspond to solitonic representations.

3.1 Bimodules and Algebra Endomorphisms

Given a von Neumann algebra A, a left A-module is a Hilbert space H equipped with a normal homomorphism $\alpha\colon A \to B(H)$. A morphism of left A-modules is a bounded linear map of Hilbert spaces that intertwines the two actions. This defines the W*-category A-Mod of left A-modules. Similarly, we have the category Mod-A of right A-modules.

Finally, given two von Neumann algebras A, B, we write A-Mod-B for the W^*-category of A-B-bimodules.

For $M \in A$-Mod-B, we write $\overline{M} \in B$-Mod-A for the complex conjugate Hilbert space, equipped with the action $b\overline{\xi}a := \overline{a^*\xi b^*}$.

Definition 3.1.1. Given $M \in$ Mod-A and $N \in A$-Mod, we define their Connes fusion as

$$M \boxtimes_A N \colon = \langle \overline{M}, N \rangle = \mathrm{proj}_{2,1} L^2\mathrm{End}_{A\text{-Mod}}(\overline{M} \oplus N)$$

This defines a functor

$$\boxtimes \colon A\text{-Mod-}B \times B\text{-Mod-}C \to A\text{-Mod-}C$$

and in particular, defines a binary operation on A-Mod-A which makes A-Mod-A a W^*-tensor category.

Remark 3.1.2. *This definition of Connes fusion matches with the standard definition. To see this we note that, by [HNP24, Corollary 3.15], it is enough to compare the outputs of the two definitions for $M = N = L^2A$. That is, we need to check that, using the above definition, we have $L^2A \boxtimes_A L^2A \cong L^2A$, and that this isomorphism is natural with respect to the algebras* $\mathrm{End}(L^2A_A)$ *and* $\mathrm{End}({}_AL^2A)$. *We can compute* $\mathrm{proj}_{2,1}L^2\mathrm{End} = L^2A$. *The required naturality can be easily verified using the definition of the actions of* $\mathrm{End}(M_A)$ *and* $\mathrm{End}({}_AN)$ *on* $\langle \overline{M}, N \rangle$ *([HNP24, Definition 4.1]).*

The functor $\overline{(\cdot)}$ makes the category A-Mod-A of A-A-bimodules into a bi-involutive W^*-tensor category. The unitary isomorphism $\varphi_M \colon M \to \overline{\overline{M}}$ is identity, and the map $\nu_{M,N} \colon \overline{M} \boxtimes_A \overline{N} \to \overline{N \boxtimes_A M}$ is as in [HP23, Section 4.2]. We let $\mathrm{Bim}\,(A)$ denote this bi-involutive W^*-tensor category.

Definition 3.1.3. Given von Neumann algebras A and B, we define $\underline{\mathrm{Hom}}(A, B)$ to be the dagger category whose objects are von Neumann algebra homomorphisms $\rho \colon A \to B$, and whose morphisms are given by

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}(A,B)}(\rho, \rho') = \{b \in B \mid \forall a \in A \colon b\rho(a) = \rho'(a)b\}.$$

A morphism $\rho \to \rho'$ is denoted $b_{\rho,\rho'}$ (the subscripts indicate the source and target of the morphism). The composition of morphisms is given by $(b_2)_{\rho_2,\rho_3} \circ (b_1)_{\rho_1,\rho_2} :=$

$(b_2b_1)_{\rho_1,\rho_3}$. The identity morphism of an object ρ is $1_{\rho,\rho}$, and the dagger operation is given by $(a_{\rho,\rho'})^\dagger := (a^*)_{\rho',\rho}$.

The composition of homomorphisms produces a W^*-functor

$$\otimes : \underline{\mathrm{Hom}}(A,B) \times \underline{\mathrm{Hom}}(B,C) \to \underline{\mathrm{Hom}}(A,C)$$

given by $f \otimes g := g \circ f$, and we define $\underline{\mathrm{End}}\,(A)$ to be the strict monoidal W^*-category $\underline{\mathrm{Hom}}(A,A)$.

Remark 3.1.4. *The W^*-category $\underline{\mathrm{Hom}}(A,B)$ is not complete. In particular, it is missing a 0 object.*

There is a functor $\mathcal{F}\colon \underline{\mathrm{Hom}}(A,B) \to A\text{-Mod-}B$ from the above category to the category of A-B-bimodules. It sends $(\rho\colon A \to B) \in \underline{\mathrm{Hom}}(A,B)$ to the bimodule

$$\mathcal{F}(\rho) = {}^\rho L^2 B$$

where ${}^\rho L^2B$ denotes the standard form of B equipped with its usual right action of B, and the left action of A given by the composite of ρ with the left action of B. At the level of morphisms, given $b_{\rho,\sigma}\colon \rho \to \sigma$, we have $\mathcal{F}(b_{\rho,\sigma}) = L_b$, where $L_b \in \mathrm{End}(L^2B)$ denotes the left action of $b \in B$.

The functors $\mathcal{F}$ are fully faithful, and compatible with composition in the sense that there exist unitary natural transformations filling the commutative diagrams

$$\begin{array}{rccc} \otimes : & \underline{\mathrm{Hom}}(A,B) \times \underline{\mathrm{Hom}}(B,C) & \to & \underline{\mathrm{Hom}}(A,C) \\ & {\scriptstyle \mathcal{F}\times\mathcal{F}}\downarrow & \Rrightarrow & \downarrow{\scriptstyle \mathcal{F}} \\ \boxtimes : & A\text{-Mod-}B \times B\text{-Mod-}C & \to & A\text{-Mod-}C, \end{array} \tag{3.1}$$

and subject to suitable coherence isomorphisms. Here, the unitary filling the diagram (3.1) is given by

$$\begin{array}{l} {}^{\rho_1}L^2B \boxtimes_B {}^{\rho_2}L^2C \supset {}^{\rho_1}L^2B \otimes_B \mathrm{Hom}_{B\text{-Mod}}(L^2B, {}^{\rho_2}L^2C) \longrightarrow {}^{\rho_2\circ\rho_1}L^2C. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \xi \otimes f \ \mapsto \ f(\xi) \end{array}$$

Specialising to the case $A = B$, we get a fully faithful monoidal functor

$$\mathcal{F}\colon \underline{\mathrm{End}}(A) \to \mathrm{Bim}(A). \tag{3.2}$$

Definition 3.1.5. For any category $\mathcal{C}$, we define $\mathcal{C}^{\cup 0}$ to be its 0 object completion. The objects of $\mathcal{C}^{\cup 0}$ are those of $\mathcal{C}$ together with a new object called 0 which has a unique map (the zero map) to and from each of the other objects.

Proposition 3.1.6. *If A is a type III factor, then the functor (3.2) defined above extends to an equivalence of monoidal categories*

$$\mathcal{F}^{\cup 0} : \underline{\mathrm{End}}\,(A)^{\cup 0} \to \mathrm{Bim}\,(A)\,.$$

Proof. Since $\mathcal{F}$ is monoidal and fully faithful, the same holds for $\mathcal{F}^{\cup 0}$. It remains to prove that $\mathcal{F}^{\cup 0}$ is essentially surjective.

Let ${}_AX_A$ be an A-A-bimodule. If $X = 0$, then it's in the image of the zero object of $\underline{\mathrm{End}}\,(A)^{\cup 0}$. If $X \neq 0$, since A is a type III factor, it has a unique (separable) right module up to isomorphism. So, we may pick a right A-module isomorphism between X and L^2A, and thus, we assume without loss of generality that $X_A = L^2A_A$. The only maps that commute with the right action of A on L^2A are of the form L_x for some $x \in A$. The left action of $a \in A$ on X is therefore given by $L_{\rho(a)}$ for some map $\rho\colon A \to A$. Using that $x \mapsto L_x$ is injective, one then checks that ρ is a normal $*$-homomorphism. □

3.2 Conformal Nets

Definitions

Definition 3.2.1 ([BMT88, BGL93, GF93])**.** We define a conformal net on S^1 to be the data of a Hilbert space H_0 (the vacuum sector) equipped with a strongly continuous projective unitary representation U of $\mathrm{Diff}^+(S^1)$ on H_0 and a vector $\Omega \in H_0$ called the vacuum vector, as well as the data of a von Neumann algebra $\mathcal{A}\,(I) \subseteq B\,(H_0)$ for every subinterval $I \subsetneq S^1$. These are required to satisfy the following properties:

- **Isotony:** For any two subintervals $I, I' \subset S^1$, if $I \subseteq I'$, then $\mathcal{A}\,(I) \subseteq \mathcal{A}\,(I')$.
- **Locality:** For any two subintervals $I, I' \subset S^1$, if $\mathring{I} \cap \mathring{I}' = \emptyset$, then $[\mathcal{A}\,(I)\,, \mathcal{A}\,(I')] = 0$. Here, $\mathring{I}$ denotes the interior of I.

- **Covariance:** For any $g \in \mathrm{Diff}^+(S^1)$ and any interval $I \subset S^1$, we have $\mathrm{Ad}_{U(g)}(\mathcal{A}(I)) = \mathcal{A}(g(I))$. If $f \in \mathrm{Diff}^+(S^1)$ fixes I pointwise, then $\mathrm{Ad}_{U(f)}$ fixes $\mathcal{A}(I)$ pointwise.

- **Vacuum:** For any $g \in \text{Möb} \subset \mathrm{Diff}^+(S^1)$, Ω is preserved by $U(g)$ (up to a non-zero scalar multiple) and Ω is cyclic for $\bigvee_{I\subset S^1} \mathcal{A}(I)$. Further, Ω is the unique Möb-invariant element of H_0 up to scalars.

- **Positivity of the energy:** The restriction of U to the rotation subgroup admits a lift to a representation, and its generator has positive spectrum.

The following properties hold for any conformal net:

- **Haag duality:** The commutant of $\mathcal{A}(I)$ in $B(H_0)$ is $\mathcal{A}(S^1 \setminus \mathring{I})$, [GF93, Theorem 2.19(ii)].

- **Reeh-Schlieder:** The vacuum vector Ω is cyclic and separating for each local algebra $\mathcal{A}(I)$. See [GF93, Corollary 2.8], and [RS61], [Bor68] for more details.

- **Bisognano-Wichmann:** This was shown originally in [BW75, BW76] and in [BGL93] for conformal nets. Let $S_+ := \{z \in S^1 \,|\, \mathrm{Im}(z) \geq 0\}$. Let $\Lambda(t)$ be the one-parameter subgroup of Möbius transformations that fix the points $\pm 1 \in S^1$, normalised so that the derivative at 1 of $\Lambda(t)$ is e^{-t}. And let $\vartheta : S^1 \to S^1$ be the map $z \mapsto \bar{z}$. Then $U(\Lambda(2\pi t)) = \Delta^{it}$, where Δ is the modular operator associated to the von Neumann algebra $\mathcal{A}(S_+)$ and the vector $\Omega \in H_0$. Moreover, the modular conjugation, denoted $\Theta\colon H_0 \to H_0$, satisfies $\mathrm{Ad}_\Theta(\mathcal{A}(I)) = \mathcal{A}(\vartheta(I))$ for every $I \subset S^1$. See [GF93, Theorem 2.19], and also [Lon08, Theorem 6.2.3], [Kaw15, Theorem 3.3] and references therein. Said differently, there exists a canonical unitary isomorphism
$$BW\colon H_0 \to L^2\mathcal{A}(S_+)$$
intertwining the operator $U(\Lambda(2\pi t))$ with the modular flow Δ^{it}, and the operator $\Theta\colon H_0 \to H_0$ (defined above) with the modular conjugation $J\colon L^2\mathcal{A}(S_+) \to L^2\mathcal{A}(S_+)$.

  More generally, for every interval $I \subsetneq S^1$, letting $\Lambda_I(t) \subset \text{Möb}$ be the one-parameter subgroup that fixes the boundary points of I, we have

$U(\Lambda_I(2\pi t)) = \Delta_I^{it}$, where Δ_I is the modular operator associated to $\mathcal{A}(I)$ and $\Omega \in H_0$. Moreover, the modular conjugation Θ_I for $\mathcal{A}(I)$ implements the unique order two element $\vartheta_I \in \text{Möb} \rtimes \mathbb{Z}/2$ that fixes ∂I in the sense that it satisfies $\mathrm{Ad}_{\Theta_I}(\mathcal{A}(K)) = \mathcal{A}(\vartheta_I(K))$ for every $K \subset S^1$.

- **Factoriality:** Each local algebra $\mathcal{A}(I)$ is a type III_1 factor, [GF93, Theorem 2.13].
- **Additivity:** If $\mathcal{I} = \{I_i\}$ is a collection of intervals such that $\mathring{I} \subset \bigcup_i \mathring{I}_i$, then $\mathcal{A}(I) \subset \bigvee_{I_i \in \mathcal{I}} \mathcal{A}(I_i)$, [Kaw15, Theorem 3.5], [Lon08, Corollary 6.2.6].
- **Action of piecewise smooth diffeomorphisms:** Let $\mathrm{Diff}_+^{1,\mathrm{ps}}(S^1)$ be the group of orientation-preserving homeomorphisms of S^1 that are C^1 and piecewise smooth. Then the projective action of $\mathrm{Diff}_+(S^1)$ extends to a projective action of the larger group $\mathrm{Diff}_+^{1,\mathrm{ps}}(S^1)$ [DVIT20].

Let INT denote the category whose objects are oriented abstract intervals, and whose morphisms are smooth orientation preserving embeddings. And let vNAlg be the category of von Neumann algebras and algebra homomorphisms. It was observed in [BDH15, Construction 4.2] that from the data of a conformal net $\mathcal{A}$ on S^1 one can build a covariant functor

$$\mathcal{A}\colon \mathrm{INT} \to \mathrm{vNAlg}.$$

We call this functor the *coordinate-free net associated to* $\mathcal{A}$, and represent it by the same symbol $\mathcal{A}$.

The construction of this functor goes as follows. Given an interval $I \in \mathrm{INT}$ consider the contractible groupoid Emb_I whose objects are smooth embeddings of I into S^1, and whose every hom-set contains exactly one element.

We first define a functor $\mathcal{A}_I\colon \mathrm{Emb}_I \to \mathrm{vNAlg}$. On objects, $\mathcal{A}_I(\iota) = \mathcal{A}(\iota(I))$. To define $\mathcal{A}_I$ on morphisms, we need to provide an isomorphism $\mathcal{A}_I(\iota) \to \mathcal{A}_I(\iota')$ for every pair of embeddings $\iota, \iota' \in \mathrm{Emb}_I$. Given ι, ι', pick a diffeomorphism $\varphi\colon S^1 \to S^1$ that extends $\iota' \circ \iota^{-1}\colon \iota(I) \to \iota'(I)$. Then, $a \mapsto \mathrm{Ad}_{U(\varphi)} a$ defines an isomorphism from $\mathcal{A}(\iota(I)) \to \mathcal{A}(\iota'(I))$. This isomorphism is independent of the choice of extension φ: given another extension $\widetilde{\varphi}$, then $\varphi^{-1} \circ \widetilde{\varphi}$ is identity on $\iota(I)$, therefore, $\mathrm{Ad}_{U(\varphi^{-1}\circ\widetilde{\varphi})}$ is the identity on $\mathcal{A}(\iota(I))$, implying the assignment is independent of φ. We then define

$$\mathcal{A}(I) := \lim_{\iota \in \mathrm{Emb}_I} \mathcal{A}(\iota(I)).$$

An element $a \in \mathcal{A}(I)$ is a collection of operators $\{a_\iota \in \mathcal{A}(\iota(I))\}_{\iota \in \mathrm{Emb}_I}$ such that $\mathrm{Ad}_{U(\varphi)} a_\iota = a_{\varphi \circ \iota}$ for each embedding ι and each diffeomorphism φ of the standard circle S^1.

Remark 3.2.2. *A conformal net $\mathcal{A}$ is called* strongly additive *if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ whenever $I_1 \cup I_2 = I$, and $I_1 \cap I_2$ is a single point. The conformal nets considered in this thesis are not required to satisfy that condition. Though [BDH15] only works with strongly additive nets, we note that the above construction did not use strong additivity. This will be true for all further constructions we import from [BDH15, BDH17] unless stated otherwise.*

Proposition 3.2.3. *[Lon08, Lemma 3.6.3] Let $\mathcal{A}$ be a conformal net on the standard circle S^1. Then the following are equivalent:*

- *Haag duality on abstract intervals, that is, $\{a \in \mathcal{A}(I) | ab = ba \ \forall b \in \mathcal{A}(\iota(J))\} = \mathcal{A}(I \setminus^c \iota(J))$ for every embedding $\iota\colon J \to I$ such that $I \setminus^c \iota(J)$ is connected.*
- *$\mathcal{A}$ on S^1 is strongly additive.*

Conformal nets satisfy a weaker version of Haag duality on intervals, even in the absence of strong additivity:

Proposition 3.2.4. *Let I be an interval and let $J \subsetneq I$ and $K \subsetneq I$ be such that $J \cup K = I$ and such that the intersection $J \cap K$ has nonempty interior. Then, writing $\mathrm{cl}(I \setminus J)$ for the closure of $I \setminus J$, we have*

$$\mathcal{A}(K) \cap [\mathcal{A}(J)]'_{\mathcal{A}(I)} = \{a \in \mathcal{A}(K) | ab = ba \ \forall b \in \mathcal{A}(J)\} = \mathcal{A}(\mathrm{cl}(I \setminus J))$$

Proof. We may assume without loss of generality that the intervals I, J and K are embedded in the standard circle S^1. This allows us to view all our algebras as subalgebras of $B(H_0)$. We then have:

$$\begin{aligned}
\mathcal{A}(K) \cap [\mathcal{A}(J)]'_{\mathcal{A}(I)} &= \mathcal{A}(K) \cap \mathcal{A}(J)' \\
&= \mathcal{A}(S^1 \setminus \mathring{K})' \cap \mathcal{A}(J)' \text{ by Haag duality} \\
&= (\mathcal{A}(S^1 \setminus \mathring{K}) \vee \mathcal{A}(J))' \\
&= \mathcal{A}((S^1 \setminus \mathring{K}) \cup J)' \qquad \text{by Additivity} \\
&= \mathcal{A}(S^1 \setminus \mathrm{cl}(I \setminus J)^{\circ})' \\
&= \mathcal{A}(\mathrm{cl}(I \setminus J)) \qquad\qquad \text{by Haag duality.}
\end{aligned}$$

□

Lemma 3.2.5 (Inner Covariance). *Given an interval I and a diffeomorphism $\phi\colon I \to I$ which is identity in a neighbourhood of ∂I, there is a unitary $u \in \mathcal{A}(I)^{\times}$ such that $\mathrm{Ad}_u = \mathcal{A}(\phi)$.*

Proof. It is enough to show this when $I \subsetneq S^1$. We extend ϕ to a diffeomorphism $\widetilde{\phi}\colon S^1 \to S^1$ by identity. Pick a unitary $u\colon H_0 \to H_0$ that represents $U(\widetilde{\phi})$. Since $\widetilde{\phi}$ is identity of $S^1 \setminus^c I$, we have $u\mathcal{A}(S^1 \setminus^c I)u^* = \mathcal{A}(S^1 \setminus^c I)$, so $u \in [\mathcal{A}(S^1 \setminus^c I)]'_{B(H_0)} = \mathcal{A}(I)$ (where we have used Haag duality). □

Representations

Definition 3.2.6. A representation of a conformal net $\mathcal{A}$ on the standard circle S^1, also called a sector of $\mathcal{A}$, is a Hilbert space H equipped with actions $\alpha_I\colon \mathcal{A}(I) \to B(H)$ for all intervals $I \subsetneq S^1$ such that for all inclusions $J \subseteq I \subseteq S^1$ we have

$$\alpha_I|_{\mathcal{A}(J)} = \alpha_J.$$

A morphism $f\colon (H_1, \{\alpha_I\}_{I\subset S^1}) \to (H_2, \{\beta_I\}_{I\subset S^1})$ between representations is a linear map $f\colon H_1 \to H_2$ of Hilbert spaces such that $f \circ \alpha_I(a) = \beta_I(a) \circ f$ for all $I \subsetneq S^1$ and all $a \in \mathcal{A}(I)$. The category of representations of $\mathcal{A}$ is denoted $\mathrm{Rep}_{S^1}(\mathcal{A})$ or simply $\mathrm{Rep}(\mathcal{A})$.

Given a diffeomorphism $\phi\colon S_1 \to S_2$, we get an equivalence of categories

$$\begin{aligned}
\phi^*\colon \mathrm{Rep}_{S_2}(\mathcal{A}) &\xrightarrow{\sim} \mathrm{Rep}_{S_1}(\mathcal{A}) \\
\left(H, \{\alpha_{I_2}\}_{I_2\subset S_2}\right) &\mapsto \left(H, \left\{\alpha_{\phi(I_1)} \circ \mathcal{A}\left(\phi|_{I_1}\right)\right\}_{I_1\subset S_1}\right)
\end{aligned}$$

3.3 Left-Localised Endomorphisms

We fix a conformal net $\mathcal{A}$. The main objects of study in this chapter are endomorphisms of the von Neumann algebras $\mathcal{A}(I)$ associated to multi-intervals I that are 'localised' near one boundary component of I. A special class of such endomorphisms arises naturally in the DHR (Doplicher-Haag-Roberts) framework for superselection sectors, where endomorphisms are localised in a bounded region. Here we adapt that framework to the setting of algebras assigned by conformal nets to intervals, working with left- and right-localised endomorphisms rather than compactly localised ones. The key results of this chapter are: a characterisation of left-localised endomorphisms (Lemma 3.3.7), a canonical form theorem showing any left-localised endomorphism is unitarily equivalent to one supported in any prescribed neighbourhood of the left boundary (Lemma 3.3.13), and the fact that the functor $\mathcal{T}^+_{\mathcal{A}}(i)$ induced by a left-localised embedding is a monoidal equivalence (Theorem 3.3.19). In the following chapter, these categories are shown to be monoidally equivalent to solitonic representations after 0 object completion (Theorem 4.5.4).

Definition 3.3.1. If I is a multi-interval, define $\mathcal{A}(I)$ to be the spatial tensor product of $\mathcal{A}(I_i)$ where I_i are the connected components of I.

Remark 3.3.2. *We will give a definition of $\mathcal{A}(I)$ for an arbitrary compact 1-manifold I in Definition 4.6.5. This will be compatible with Definition 3.3.1 due to Equation 4.9.*

Lemma 3.3.3. *Given a multi-interval I and a diffeomorphism $\phi\colon I \to I$ which is identity near ∂I, there is a $u \in \mathcal{A}(I)$ such that $\mathcal{A}(\phi) = \mathrm{Ad}_u$.*

Proof. Let I_i be the connected components of I. Then ϕ induces diffeomorphisms $\phi_i\colon I_i \to I_i$ each of which are identity near ∂I_i. So we have $u_i \in \mathcal{A}(I_i)$ such that $\mathcal{A}(\phi_i) = \mathrm{Ad}_{u_i}$ by inner covariance. Now, note that

$$\phi = (\phi_1 \sqcup \mathrm{id}_{I_2} \sqcup \cdots \sqcup \mathrm{id}_{I_n}) \circ (\mathrm{id}_{I_1} \sqcup \phi_2 \sqcup \cdots \sqcup \mathrm{id}_{I_n}) \circ \cdots \circ (\mathrm{id}_{I_1} \sqcup \mathrm{id}_{I_2} \sqcup \cdots \sqcup \phi_n)$$

Thus $\mathcal{A}(\phi) = \mathrm{Ad}_{\widehat{u_1}} \circ \cdots \circ \mathrm{Ad}_{\widehat{u_n}} = \mathrm{Ad}_{\widehat{u_1}\cdots\widehat{u_n}}$ where $\widehat{u_i}$ are the images of u_i under the map $\mathcal{A}(I_i) \to \mathcal{A}(I_1)\overline{\otimes}\cdots\overline{\otimes}\mathcal{A}(I_n) \simeq \mathcal{A}(I)$. $\square$

We also have Haag duality for multi-intervals.

Proposition 3.3.4. *Let I be a multi-interval and let $J \subsetneq I$ and $K \subsetneq I$ be multi-intervals such that $\partial_+ J = \partial_+ I$, $\partial_- K = \partial_- I$, and $J \cup K = I$, and the intersection $J \cap K$ has nonempty interior. Then we have*

$$\mathcal{A}(K) \cap [\mathcal{A}(J)]'_{\mathcal{A}(I)} := \{a \in \mathcal{A}(K) | ab = ba \; \forall b \in \mathcal{A}(J)\} = \mathcal{A}(I \setminus^c J)$$

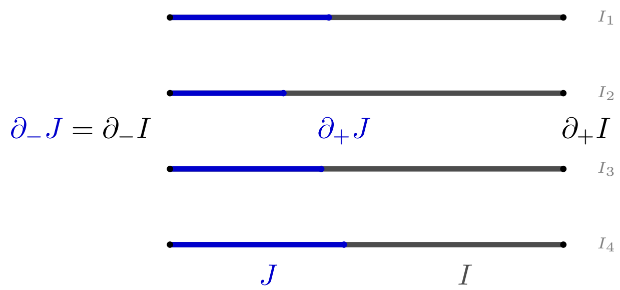
Proof. This follows from Equation 4.9, Proposition 3.2.4, and [Tak79, Chapter IV, Theorem 5.9] which tells us that given disjoint subintervals $I_1, I_2 \subsetneq S^1$, we have

$$[\mathcal{A}(I_1)\overline{\otimes}\mathcal{A}(I_2)]'_{B(H_0 \boxtimes_{\mathbb{C}} H_0)} \simeq [\mathcal{A}(I_1)]'_{B(H_0)}\overline{\otimes}[\mathcal{A}(I_2)]'_{B(H_0)}.$$

$\square$

Definition 3.3.5. Let I be a multi-interval and let J be a multi-interval with $i\colon J \hookrightarrow I$, an embedding. We say an endomorphism $\rho\colon \mathcal{A}(I) \to \mathcal{A}(I)$ is identity on $\mathcal{A}(J)$ if $\rho(x) = x$ for all $x \in \mathcal{A}(J)$.

Definition 3.3.6. Let I be a multi-interval. We say a sub-multi-interval J is left-localised if $\partial_- J = \partial_- I$ and right-localised if $\partial_+ J = \partial_+ I$. Given a multi-interval I' and an embedding $i\colon I' \hookrightarrow I$ such that $i(I') \subseteq I$ is left-localised, we say i is a left-localised embedding. We denote the category of multi-intervals and left-localised embeddings by $\mathrm{MInt}_{\mathrm{ll}}$ where the subscript 'll' stands for left-localised.



The following lemma gives four equivalent ways to express the extendability, which we will use interchangeably.

Lemma 3.3.7. *Let I be a multi-interval, let J be a right-localised sub-multi-interval of I, and let $\rho\colon \mathcal{A}(I) \to \mathcal{A}(I)$ be a homomorphism which is identity on $\mathcal{A}(J)$. Then the following are equivalent:*

(i) There exists a multi-interval I', a left-localised embedding $I \hookrightarrow I'$, and a homomorphism $\rho' \colon \mathcal{A}(I') \to \mathcal{A}(I')$ which is identity on $\mathcal{A}(I' \setminus^c (I \setminus^c J))$ and $\rho'(x) = \rho(x)$ for $x \in \mathcal{A}(I)$. We say ρ' extends ρ by identity.

(ii) For every left-localised sub-multi-interval $K \subseteq I$ such that ρ is identity on $\mathcal{A}(I \setminus^c K)$, we have that $\rho(\mathcal{A}(K)) \subseteq \mathcal{A}(K)$.

(iii) There exists a left-localised sub-multi-interval $K \subseteq I$ such that $I \setminus^c J \subsetneq K$ and $\rho(\mathcal{A}(K)) \subseteq \mathcal{A}(K)$.

(iv) For every multi-interval I' and left-localised embedding $I \hookrightarrow I'$ there is a unique homomorphism $\rho' \colon \mathcal{A}(I') \to \mathcal{A}(I')$ which extends ρ by identity.

Proof. Clearly $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (i)$. We show $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$.
<u>$\mathbf{(i)} \Rightarrow \mathbf{(ii)}$:</u> We have

$$\rho(\mathcal{A}(K)) = \rho'(\mathcal{A}(K)) \subseteq [\rho'(\mathcal{A}(I' \setminus^c K))]'_{\mathcal{A}(I')} = [\mathcal{A}(I' \setminus^c K)]'_{\mathcal{A}(I')}$$

The first equality is true because ρ' is an extension of ρ by identity, the next inclusion is because $\mathcal{A}(K)$ and $\mathcal{A}(I' \setminus^c K)$ commute in $\mathcal{A}(I')$ and ρ' is an $*$-algebra homomorphism, and the last equality is because ρ' is identity on $\mathcal{A}(I' \setminus^c K)$. Since $\rho(\mathcal{A}(K)) \subseteq \mathcal{A}(I)$, we have

$$\rho(\mathcal{A}(K)) \subseteq \mathcal{A}(I) \cap [\mathcal{A}(I' \setminus^c K)]'_{\mathcal{A}(I')} = \mathcal{A}(K)$$

The last equality follows from Proposition 3.3.4.
<u>$\mathbf{(iii)} \Rightarrow \mathbf{(iv)}$:</u> Let $J' = I' \setminus^c (I \setminus^c J)$. J' and I are sub-multi-intervals of I' such that $J' \cup I = I'$ and $J' \cap I = J$. By additivity, $\mathcal{A}(J')$ and $\mathcal{A}(I)$ generate $\mathcal{A}(I')$. Therefore, if a ρ' exists, it is unique, as it is fully determined by its values on $\mathcal{A}(J')$ and $\mathcal{A}(I)$ which are specified to be $\mathrm{id}_{\mathcal{A}(J')}$ and ρ respectively. So we only need to show that such a ρ' exists.

Choose a diffeomorphism $f \colon I \xrightarrow{\sim} I'$ such that $f(p) = p$ for $p \in K$. Define $\rho' = \mathcal{A}(f) \circ \rho \circ \mathcal{A}(f^{-1})$. We show ρ' defined this way extends ρ by identity.

Let $x \in \mathcal{A}(J')$. Then $\mathcal{A}(f^{-1})(x) \in \mathcal{A}(J)$. Thus,

$$\rho'(x) = \mathcal{A}(f)(\rho(\mathcal{A}(f^{-1})(x))) = \mathcal{A}(f)(\mathcal{A}(f^{-1})(x)) = x$$

where in the second equality we used that ρ is identity on $\mathcal{A}(J)$.

Let $x \in \mathcal{A}(K)$. Then

$$\rho'(x) = \mathcal{A}(f)(\rho(\mathcal{A}(f^{-1})(x))) = \mathcal{A}(f)(\rho(x)) = \rho(x)$$

where in the second equality we used that $\mathcal{A}(f)$ is identity on $\mathcal{A}(K)$ and in the third equality we used the same fact along with $\rho(\mathcal{A}(K)) \subseteq \mathcal{A}(K)$. ρ' matches ρ on $\mathcal{A}(I)$ as $\mathcal{A}(K)$ and $\mathcal{A}(J)$ generate $\mathcal{A}(I)$ and ρ' and ρ are identical on both of these subalgebras of $\mathcal{A}(I)$. $\square$

Definition 3.3.8. Given a multi-interval I, we say a von Neumann algebra homomorphism $\rho\colon \mathcal{A}(I) \to \mathcal{A}(I)$ is left-localised if there are sub-multi-intervals $J, K \subsetneq I$ such that J is right-localised, K is left-localised, $I \setminus^c J \subsetneq K$, ρ is identity on $\mathcal{A}(J)$, and $\rho(\mathcal{A}(K)) \subseteq \mathcal{A}(K)$.

Remark 3.3.9. *We could have used any of the four equivalent statements in Lemma 3.3.7 in Definition 3.3.8.*

Definition 3.3.10. Given a multi-interval I, we say $a \in \mathcal{A}(I)$ is a left-localised element if there exists a left-localised sub-multi-interval $K \subsetneq I$ such that $a \in \mathcal{A}(K)$.

Definition 3.3.11. For a left-localised endomorphism $\rho \in \underline{\mathrm{End}}(\mathcal{A}(I))$, define its *left support* as

$$\operatorname{supp} \rho := I \setminus^c J$$

where J is the closure of the union of all right-localised sub-multi-intervals J' such that ρ is identity on $\mathcal{A}(J')$. We similarly define the right support of a right-localised endomorphism.

Note that, by additivity, we have ρ is identity on $\mathcal{A}(I \setminus^c \operatorname{supp} \rho)$.

Remark 3.3.12. *If ρ is both left-localised and right-localised, the left support of ρ need not be the same as its right support, and the notation* $\operatorname{supp} \rho$ *is therefore potentially ambiguous. In that case, the support is defined to be the intersection of the left support and the right support.*

Lemma 3.3.13. *Let I be a multi-interval and $\rho\colon \mathcal{A}(I) \to \mathcal{A}(I)$ be a left-localised endomorphism. For any right-localised sub-multi-interval $J \subsetneq I$ such that $J \cap$* $\operatorname{supp} \rho$ *has nonempty interior and as many connected components as I, there is a unitary $u \in \mathcal{A}(\operatorname{supp} \rho)^\times$ such that for all $x \in \mathcal{A}(J)$ we have $\rho(x) = uxu^*$.*

Proof. Let $J' = I \setminus^c \operatorname{supp}\rho$. Consider a diffeomorphism $\phi\colon I \to I$ such that there exists an open neighbourhood U of ∂I such that $\phi(p) = p$ for $p \in U$ and $J \subseteq \phi(J')$. By Lemma 3.3.3, there is a unitary $v \in \mathcal{A}(I)^\times$ such that for all $x \in \mathcal{A}(I)$ we have $\mathcal{A}(\phi)(x) = vxv^*$. Define $u = \rho(v)v^*$. For an $x \in \mathcal{A}(J)$, we compute

$$\begin{aligned}
uxu^* &= \rho(v)v^*x(\rho(v)v^*)^* \\
&= \rho(v)v^*xv\rho(v^*) \\
&= \rho(v)\mathcal{A}(\phi^{-1})(x)\rho(v^*) \\
&\overset{(*)}{=} \rho(v)\rho(\mathcal{A}(\phi^{-1})(x))\rho(v^*) \\
&= \rho(v\mathcal{A}(\phi^{-1})(x)v^*) \\
&= \rho(\mathcal{A}(\phi)(\mathcal{A}(\phi^{-1})(x))) \\
&= \rho(x)
\end{aligned}$$

where in $(*)$ we have used that $\mathcal{A}(\phi^{-1})(\mathcal{A}(J)) \subseteq \mathcal{A}(J')$ and ρ is identity on $\mathcal{A}(J')$.

Let $J'' \subsetneq U$ be a right-localised sub-multi-interval of I such that ρ is identity on $\mathcal{A}(J'')$. Then $\phi|_{I\setminus^c J''}\colon I \setminus^c J'' \to I \setminus^c J''$ is a diffeomorphism which is identity on an open neighbourhood of $\partial(I \setminus^c J'')$. Thus, we can choose $v \in \mathcal{A}(I \setminus^c J'')^\times \subseteq \mathcal{A}(I)^\times$. Such a v still satisfies $\mathrm{Ad}_V = \mathcal{A}(\phi)$ on all of $\mathcal{A}(I)$ as ϕ is identity on J''. Also, since ρ is identity on $\mathcal{A}(J'')$, we have $\rho(\mathcal{A}(I \setminus^c J'')) \subseteq \mathcal{A}(I \setminus^c J'')$, so $\rho(v) \in \mathcal{A}(I \setminus^c J'')$. Therefore $u = \rho(v)v^* \in \mathcal{A}(I \setminus^c J'')$.

For $x \in \mathcal{A}(I \setminus^c \operatorname{supp}\rho)$, we have $\rho(x) = x = uxu^*$, so $u \in [\mathcal{A}(I \setminus^c \operatorname{supp}\rho)]'_{\mathcal{A}(I)}$. Thus

$$u \in [\mathcal{A}(I \setminus^c \operatorname{supp}\rho)]'_{\mathcal{A}(I)} \cap \mathcal{A}(I \setminus^c J'') = \mathcal{A}(\operatorname{supp}\rho)$$

where we have used Proposition 3.3.4. □

Definition 3.3.14. For any multi-interval I, define the category of left-localised endomorphisms $\mathcal{T}^+_{\mathcal{A}}(I)$ to be the subcategory of $\operatorname{End}(\mathcal{A}(I))$ on left-localised endomorphisms and morphisms represented by left-localised elements of $\mathcal{A}(I)$. We give this the induced monoidal structure. Namely, the tensor product of objects is given by $\rho \otimes \sigma\colon = \sigma \circ \rho$, and the tensor product of morphisms $a : \rho \to \rho'$ and $b : \sigma \to \sigma'$ is $a \otimes b := \sigma'(a)b = b\sigma(a)$. The †-structure is given by $a^\dagger := a^*$.

We can similarly define the category of right-localised endomorphisms as $\mathcal{T}^-_{\mathcal{A}}(I) = \mathcal{T}^+_{\mathcal{A}}(\overline{I})$.

Definition 3.3.15. Let ρ, σ be objects in $\mathcal{T}_{\mathcal{A}}^{+}(I)$ and $a\colon \rho \to \sigma$ be a morphism in $\mathcal{T}_{\mathcal{A}}^{+}(I)$. Define $\operatorname{supp} a = \operatorname{supp} \sigma \cup \operatorname{supp} \rho$, this is the union of the bigger of the two sub-multi-intervals in the supports of ρ and σ in each component of I. Clearly, $\operatorname{supp} a \subsetneq I$ is left-localised.

Lemma 3.3.16. *Let ρ, σ be objects in $\mathcal{T}_{\mathcal{A}}^{+}(I)$ and $a\colon \rho \to \sigma$ be a morphism in $\mathcal{T}_{\mathcal{A}}^{+}(I)$. Then $a \in \mathcal{A}(\operatorname{supp} a)$.*

Proof. For $x \in \mathcal{A}(I \setminus^{c} \operatorname{supp} a)$, we have $ax = a\rho(x) = \sigma(x)a = xa$. So $a \in [\mathcal{A}(I \setminus^{c} \operatorname{supp} a)]'_{\mathcal{A}(I)}$. Since a is a left-localised element of $\mathcal{A}(I)$, there is a left-localised sub-multi-interval $K \subsetneq I$ such that $a \in \mathcal{A}(K)$. Without loss of generality, we can assume $\operatorname{supp} a \subsetneq K$. Therefore, $a \in \mathcal{A}(K) \cap [\mathcal{A}(I \setminus^{c} \operatorname{supp} a)]'_{\mathcal{A}(I)} = \mathcal{A}(\operatorname{supp} a)$. □

The pictorial intuition for the above result is that intertwiners between localised endomorphisms are themselves localised.

Lemma 3.3.17. *Let I be a multi-interval and $\phi\colon I \to I$ be a non-surjective embedding which is identity on a neighbourhood of $\partial_{+} I$. Then $\mathcal{A}(\phi)$ is left-absorbing in $\mathcal{T}_{\mathcal{A}}^{+}(I)$. Further, $\operatorname{End}_{\mathcal{T}_{\mathcal{A}}^{+}(I)}(\mathcal{A}(\phi)) = \mathcal{A}(I \setminus^{c} \operatorname{im}(\phi))$.*

Proof. Clearly, $\mathcal{A}(\phi)$ is a left-localised endomorphism. For any left-localised $\rho\colon \mathcal{A}(I) \to \mathcal{A}(I)$, by Lemma 3.3.13, we can find a unitary u such that $\rho' := \operatorname{Ad}_u \circ \rho$ is localised to the left of $\operatorname{im}(\phi)$. Note u supplies an isomorphism $\rho \to \rho'$.

Then we have

$$\mathcal{A}(\phi) \otimes \rho = \rho \circ \mathcal{A}(\phi) \underset{u}{\xrightarrow{\sim}} \rho' \circ \mathcal{A}(\phi) = \mathcal{A}(\phi)$$

A morphism $a\colon \mathcal{A}(\phi) \to \mathcal{A}(\phi)$ in $\mathcal{T}_{\mathcal{A}}^{+}(I)$ is a left-localised element $a \in \mathcal{A}(I)$ such that for all $x \in \mathcal{A}(I)$ we have $a\mathcal{A}(\phi)(x) = \mathcal{A}(\phi)(x)a$. In other words, $a \in [\mathcal{A}(\operatorname{im}(\phi))]'_{\mathcal{A}(I)}$. Since a is left-localised, there is a left-localised sub-multi-interval $K \subsetneq I$ such that $a \in \mathcal{A}(K)$ and $K \cap \operatorname{im}(\phi)$ is a sub-multi-interval of I. Thus $K \cup \operatorname{im}(\phi) = I$, and so we have

$$\operatorname{End}_{\mathcal{T}_{\mathcal{A}}^{+}(I)}(\mathcal{A}(\phi)) = \mathcal{A}(K) \cap [\mathcal{A}(\operatorname{im}(\phi))]'_{\mathcal{A}(I)} = \mathcal{A}(I \setminus^{c} \operatorname{im}(\phi))$$

□

Definition 3.3.18. Let I, I' be multi-intervals and $i\colon I \hookrightarrow I'$ be an embedding that makes I a left-localised sub-multi-interval of I'. Define the strict monoidal functor $\mathcal{T}^+_{\mathcal{A}}(i) : \mathcal{T}^+_{\mathcal{A}}(I) \to \mathcal{T}^+_{\mathcal{A}}(I')$ on

- *objects as*: Given a left-localised endomorphism $\rho\colon \mathcal{A}(I) \to \mathcal{A}(I)$, define

$$\mathcal{T}^+_{\mathcal{A}}(i)(\rho)\colon \mathcal{A}(I') \to \mathcal{A}(I')$$

  to be the extension of ρ by identity using Lemma 3.3.7. We shall also denote this by ρ^{ext}.

- *morphisms as*: For $a_{\rho,\sigma}\colon \rho \to \sigma$ a morphism in $\mathcal{T}^+_{\mathcal{A}}(I)$, define

$$\mathcal{T}^+_{\mathcal{A}}(i)(a_{\rho,\sigma}) = a_{\mathcal{T}^+_{\mathcal{A}}(i)(\rho),\mathcal{T}^+_{\mathcal{A}}(i)(\sigma)}$$

Proof of functoriality. It is clear that $\mathcal{T}^+_{\mathcal{A}}(i)$ preserves identity morphisms and composition. We show that

- $\mathcal{T}^+_{\mathcal{A}}(i)(\rho)$ is an object of $\mathcal{T}^+_{\mathcal{A}}(I')$: Let $J \subseteq I$ be a right-localised sub-multi-interval such that ρ is identity on $\mathcal{A}(J)$. Since $\mathcal{T}^+_{\mathcal{A}}(i)(\rho)$ is an extension of ρ by identity, we have that it is identity on $\mathcal{A}(I' \setminus^c (I \setminus^c J))$ and $\mathcal{T}^+_{\mathcal{A}}(i)(\rho)(\mathcal{A}(I \setminus^c J)) = \rho(\mathcal{A}(I \setminus^c J)) \subseteq \mathcal{A}(I \setminus^c J)$. Therefore, $\mathcal{T}^+_{\mathcal{A}}(i)(\rho)$ is left-localised.

- $\mathcal{T}^+_{\mathcal{A}}(i)(a_{\rho,\sigma})$ is a morphism in $\mathcal{T}^+_{\mathcal{A}}(I')$: Since $a \in \mathcal{A}(\operatorname{supp} a)$, it is a left-localised element of $\mathcal{A}(I')$. To show that $a\, \mathcal{T}^+_{\mathcal{A}}(i)(\rho)(x) = \mathcal{T}^+_{\mathcal{A}}(i)(\sigma)(x)a$ for all $x \in \mathcal{A}(I')$, it is enough to check for $x \in \mathcal{A}(I)$ and $x \in \mathcal{A}(I' \setminus^c \operatorname{supp} a)$. This reduces to

$$\begin{aligned} a\rho(x) =& \sigma(x)a \text{ for } x \in \mathcal{A}(I) \\ ax =& xa \qquad \text{for } x \in \mathcal{A}(I' \setminus^c \operatorname{supp} a) \end{aligned}$$

  The first equation is true as a is a morphism from ρ to σ and the second is true as $a \in \mathcal{A}(\operatorname{supp} a)$.

- Strict monoidality Given two objects ρ, ρ', the objects $\mathcal{T}^+_{\mathcal{A}}(i)(\rho) \otimes \mathcal{T}^+_{\mathcal{A}}(i)(\rho') = \mathcal{T}^+_{\mathcal{A}}(i)(\rho') \circ \mathcal{T}^+_{\mathcal{A}}(i)(\rho)$ and $\mathcal{T}^+_{\mathcal{A}}(i)(\rho \otimes \rho') = \mathcal{T}^+_{\mathcal{A}}(i)(\rho' \circ \rho)$ are extensions of $\rho \otimes \rho' = \rho' \circ \rho$ by identity, and thus must be equal as extensions by identity are unique. It is also clear that $\mathcal{T}^+_{\mathcal{A}}(i)(\mathrm{id}_{\mathcal{A}(I)}) = \mathrm{id}_{\mathcal{A}(I')}$.

□

Theorem 3.3.19. *Let I, I' be multi-intervals and $i \colon I \hookrightarrow I'$ be a left-localised embedding. Then $\mathcal{T}^+_{\mathcal{A}}(i)$ is an equivalence of monoidal categories.*

Proof. Since $\mathcal{T}^+_{\mathcal{A}}(i)$ is a monoidal functor, it is enough to prove it is an equivalence, for an inverse will automatically be monoidal. $\mathcal{T}^+_{\mathcal{A}}(i)$ is clearly faithful.

To see that $\mathcal{T}^+_{\mathcal{A}}(i)$ is essentially surjective, we note that any object of $\mathcal{T}^+_{\mathcal{A}}(I')$ is isomorphic to an object localised within I using a unitary supplied by Lemma 3.3.13.

We now show fullness. Let ρ, σ be two objects of $\mathcal{T}^+_{\mathcal{A}}(I)$ and $a \colon \mathcal{T}^+_{\mathcal{A}}(i)(\rho) \to \mathcal{T}^+_{\mathcal{A}}(i)(\sigma)$ be a morphism in $\mathcal{T}^+_{\mathcal{A}}(I')$. So we have $a\mathcal{T}^+_{\mathcal{A}}(i)(\rho)(x) = \mathcal{T}^+_{\mathcal{A}}(i)(\sigma)(x)a$ for all $x \in \mathcal{A}(I')$. For $x \in \mathcal{A}(I)$, this reduces to $a\rho(x) = \sigma(x)a$.

Now we need to check that a is a left-localised element of $\mathcal{A}(I)$. Note that $\operatorname{supp} \mathcal{T}^+_{\mathcal{A}}(i)(\rho) = \operatorname{supp} \rho \subsetneq I$, and similarly for σ, so $\operatorname{supp} a \subsetneq I$. Since $a \in \mathcal{A}(\operatorname{supp} a)$, a is a left-localised element of $\mathcal{A}(I)$. □

Remark 3.3.20. *Though $\mathcal{T}^+_{\mathcal{A}}(i)$ is a strict monoidal functor, it is not a strict equivalence of strict monoidal categories, for it is merely essentially surjective. In particular, an inverse to $\mathcal{T}^+_{\mathcal{A}}(i)$ would not be a strict monoidal functor.*

Definition 3.3.21. For a multi-interval I, define $\mathrm{DHR}_{\mathcal{A}}(I)$ to be the subcategory of $\underline{\mathrm{End}}(\mathcal{A}(I))$ on objects and morphisms which are in both $\mathcal{T}^+_{\mathcal{A}}(I)$ and $\mathcal{T}^-_{\mathcal{A}}(I)$.

Remark 3.3.22. *Versions of all results in this chapter apply to $\mathrm{DHR}_{\mathcal{A}}(I)$ too. In particular, given $\rho \in \mathrm{DHR}_{\mathcal{A}}(I)$ and any multi-interval $J \subsetneq I$ (need not be localised to any side), there is a ρ' isomorphic to ρ in $\mathrm{DHR}_{\mathcal{A}}(I)$ which is identity on $\mathcal{A}(I \setminus^c J)$.*

Definition 3.3.23. Let I be a multi-interval and $\Pi \subseteq \pi_0(I)$. We define the category $\mathcal{T}_{\mathcal{A}}(I, \Pi)$ to be the subcategory of $\mathcal{T}^+_{\mathcal{A}}(I)$ on

- objects ρ such that for every connected component I_i of I which is in Π, there is a left-localised subinterval $K_i \subsetneq I_i$ and there is a sub-multi-interval $J \subsetneq I$ such that ρ is identity on every $\mathcal{A}(K_i)$, $J \cap I_i$ does not intersect ∂I_i and for connected components $I_{i'} \notin \Pi$, $J \cap I_{i'} \subsetneq I_{i'}$ is left-localised, and $\rho(\mathcal{A}(J)) \subseteq \mathcal{A}(J)$.
- morphisms are localised similarly.

Remark 3.3.24. *All results proved in this section apply to this more general set-up.*

Chapter 4

Solitonic Sectors

The previous chapter developed the algebraic machinery of left-localised endomorphisms, defining for each multi-interval I the category $\mathcal{T}^+_{\mathcal{A}}(I)$ of endomorphisms of $\mathcal{A}(I)$ that are the identity on a right-localised subinterval. The present chapter develops the complementary *Hilbert-space* picture of the same objects: the category of *solitonic representations* of a conformal net $\mathcal{A}$. On the circle S^1, one is led to consider representations of all local algebras $\mathcal{A}(J)$ for proper subintervals $J \subsetneq S^1$ whose interior does not contain a fixed marked point $1 \in S^1$, with no compatibility condition imposed across that point. The resulting category, $\mathrm{Sol}(\mathcal{A}) = \mathcal{R}_{\mathcal{A}}(S^1, \{1\})$, is the central object of study in this chapter.

One of the main results (Theorem 4.5.4) is an equivalence of dagger monoidal categories

$$\mathrm{Sol}(\mathcal{A}) \;\simeq\; \mathcal{T}^+_{\mathcal{A}}(S_+)^{\cup 0},$$

so that every solitonic representation arises as the vacuum Hilbert space H_0 with its $\mathcal{A}(S_+)$-action twisted by a left-localised endomorphism, and vice versa. The monoidal structure on $\mathrm{Sol}(\mathcal{A})$ is Connes fusion $M \boxtimes_{\mathcal{A}(S_+)} N$, and corresponds under the equivalence to composition of left-localised endomorphisms. We show that the category of solitonic representations is a bicommutant category according to Definition 2.2.26. In particular, the bicommutant property of $\mathrm{Sol}(\mathcal{A})$ is established, yielding as a corollary that the Drinfeld centre of $\mathrm{Sol}(\mathcal{A})$ is $\mathrm{Rep}(\mathcal{A})$. This is a generalisation of the result in [Hen17a] which only establishes the theorem under the assumption of rationality. Also, we prove our result without the assumption of strong additivity.

4.1 A little note on the term 'soliton'

The word *soliton* entered physics in 1965. Studying solitary-wave solutions of the Korteweg-de Vries equation numerically, Zabusky and Kruskal [ZK65] observed that these waves retained their shape and speed after collision. This particle-like behaviour is what led them to coin the term. The suffix '-on' was chosen deliberately, by analogy with electrons and photons, to signal the authors' belief that these objects behaved like particles. [Tao09] is a survey on the stability theory of solitons in non-linear dispersive PDEs for those who wish to go down that path.

Solitonic solutions of classical field equations have long been a central object of study in quantum field theory. In this setting, [Frö76] (and references therein) investigated the existence of new superselection sectors and states whose field-operator expectation values resemble those of a classical soliton solution. Such states were termed 'soliton configurations': their field expectation values at spatial infinity differ from those of the vacuum, and consequently they do not lie in the vacuum sector but instead generate a distinct superselection sector of the theory. In particular, Section 6 of [Frö76] (see Definition 9) develops a general, model-independent theory of *soliton automorphisms*, which provides an early foundational reference for the material of the present chapter.

In the specific context of conformal nets, solitonic representations were studied in [Müg99, LX04, KLX05]. Their relationship to symmetry-breaking boundary conditions was explored by Fuchs and Schweigert [FS00, SF00]. If we temporarily assume that the concepts of a chiral algebra and conformal net are interchangeable, a symmetry-breaking boundary condition only preserves a subalgebra $\mathcal{B} \subseteq \mathcal{A}$ of the chiral symmetry algebra, and symmetry-breaking boundary conditions are in bijection with solitonic representations of $\mathcal{A}$ relative to $\mathcal{B}$. When $\mathcal{B} = \mathcal{A}^G$ for a finite group G, this reduces to the orbifold setting, studied in [KLX05, Müg05, Xu00]. The solitonic representations considered in this thesis are the extreme case in which the preserved subalgebra is trivial, that is, $\mathcal{B} = \mathbb{C}$.

4.2 Solitonic Representations

Definition 4.2.1. Let I be a 1-manifold and let $P \subsetneq I$ be a finite subset. We define the category $\mathcal{R}_{\mathcal{A}}(I, P)$ to be the category whose

- objects are tuples $(H, \alpha_J \colon \mathcal{A}(J) \to B(H))$ where H is a Hilbert space, $J \subsetneq I$ are intervals such that $P \cap J = P \cap \partial J$ (that is, P is never in the interior of J), and α_J are homomorphisms of von Neumann algebras. Given two subintervals $J_1, J_2 \subsetneq I$, if $J_1 \subseteq J_2$ we require that $\alpha_{J_2}|_{\mathcal{A}(J_1)} = \alpha_{J_1}$.
- morphisms are maps of Hilbert spaces that intertwine the actions of $\mathcal{A}(J)$ for every subinterval $J \subsetneq I$.

We give it a dagger structure using the Hermitian conjugate of Hilbert space maps.

Intuitively, the marked points in the previous definition are the locations where no compatibility between local actions is imposed.

Definition 4.2.2. Given two 1-manifolds I, I', an embedding $i \colon I \hookrightarrow I'$, and finite subsets $P \subsetneq I$ and $P' \subsetneq I'$ such that $P' \cap i(I) \subseteq i(P)$, we define the following functor

$$\begin{aligned} i^* \colon \mathcal{R}_{\mathcal{A}}(I', P') &\to \mathcal{R}_{\mathcal{A}}(I, P) \\ \left(H, \{\alpha_{J'}\}_{J' \subsetneq I'}\right) &\mapsto \left(H, \{\alpha_{i(J)} \circ \mathcal{A}\left(i|_J\right)\}_{J \subsetneq I}\right) \end{aligned}$$

Definition 4.2.3. We call the category $\mathcal{R}_{\mathcal{A}}(S, \emptyset)$ the category of representations of $\mathcal{A}$ on S or the category of S-sectors of $\mathcal{A}$ and denote it $\mathrm{Rep}_S(\mathcal{A})$. If $S = S^1$, we call them just representations and denote this category $\mathrm{Rep}(\mathcal{A})$.

Definition 4.2.4. We call the category $\mathcal{R}_{\mathcal{A}}(S^1, \{1\})$ the category of solitonic representations of $\mathcal{A}$ and denote it $\mathrm{Sol}(\mathcal{A})$.

Lemma 4.2.5. *Let $p = 1 \in S^1$ be our fixed base point, and let $\{I_i \subset S^1\}_{i \in \mathcal{I}}$ be a collection of intervals that satisfy*

$$\bigcup_{i \in \mathcal{I}} I_i = S^1 \qquad \text{and} \qquad \bigcup_{i \in \mathcal{I}} \mathring{I}_i = S^1 \setminus \{p\}.$$

Let H be a Hilbert space equipped with actions $\rho_i : \mathcal{A}(I_i) \to B(H)$ for $i \in \mathcal{I}$ which are compatible in the sense that:

- $\rho_i|_{\mathcal{A}(I_i \cap I_j)} = \rho_j|_{\mathcal{A}(I_i \cap I_j)} \colon \mathcal{A}(I_i \cap I_j) \to B(H)$.

- *For every $j, k \in \mathcal{I}$ and every intervals $J \subset I_j$, $K \subset I_k$ with disjoint interiors, the algebras $\rho_j(\mathcal{A}(J))$ and $\rho_k(\mathcal{A}(K))$ commute.*

Then for every interval $I \subsetneq S^1$, the actions

$$\rho_i|_{\mathcal{A}(I \cap I_i)} : \mathcal{A}(I \cap I_i) \to B(H)$$

extend uniquely to an action of $\mathcal{A}(I)$ on H, and collectively these actions equip H with the structure of a solitonic representation of the conformal net $\mathcal{A}$.

Proof. This is a special case of [Hen16, Lemma 4], applied to the manifold $S^1_{\text{cut}} := [0, 2\pi]$ (with the identification between $(0, 2\pi)$ and $S^1 \setminus \{p\}$ provided by the exponential map). □

Remark 4.2.6. *Although strong additivity is appealed to in the proof of [Hen16, Lemma 4], it is not needed here: the covers occurring in the argument consist of intervals whose interiors overlap, so additivity already generates the algebra of the covered interval. Strong additivity would be required only for covers by finitely many closed subintervals overlapping in endpoints alone, and no such cover occurs. The implementing unitaries, moreover, are localised by diffeomorphism covariance alone.*

The definition of $\mathrm{Sol}(\mathcal{A})$ is the conformal-net counterpart of Fröhlich's notion of a soliton sector [Frö76]. In Fröhlich's framework, a soliton automorphism of the observable algebra is one that acts as the identity in the left asymptotic region and as a global symmetry in the right asymptotic region. The single marked point $1 \in S^1$ in our definition plays the role of an asymptotic interpolation locus. Intervals not containing 1 in their interior correspond to observables localised away from the interpolation point, and on these the representation is required to be consistent (isotonous); but no condition is imposed at 1 itself. To give an example, combining results from [Hen17b, Hen17a, Hen17c], the category of solitons for the conformal net $\mathcal{A}_{G,k}$ for a compact, connected, simply connected group G and $k \in H^4_+(BG, \mathbb{Z})$ is the category $\mathrm{Rep}_k(\Omega G)$ of locally normal representations of the based loop group at level k [Hen17c, Main Definition]. The based loop group is a subgroup of the free loop group LG, $\Omega G = \{\text{continuous } f : S^1 \to G \mid f(1) = e_G\}$.

Definition 4.2.7. Let S be a disjoint union of n copies of the standard circle S^1 and let $\Pi \subseteq \{1, \ldots, n\} \simeq \pi_0(S)$ index a collection of circles in S. Define $\mathcal{S}_{\mathcal{A}}(n, \Pi)$ to be $\mathcal{R}_{\mathcal{A}}(S, \{1_i\}_{i \in \pi_0(S) \setminus \Pi})$. Here the circles in Π are unmarked and contribute to representations, and the others are marked at their copy of 1 and contribute solitonic representations.

Remark 4.2.8. *In what follows, we shall often only prove statements for intervals and for the standard circle (rather than multi-intervals and disjoint unions of standard circles) for simplicity; however, the proofs extend straightforwardly.*

4.3 The Vacuum Sector

Recall the vacuum sector H_0. This is an object of $\mathrm{Rep}(\mathcal{A})$.

Definition 4.3.1. We define the map $C\colon H_0 \boxtimes_{\mathcal{A}(S_+)} H_0 \to H_0$ to be

$$H_0 \boxtimes_{\mathcal{A}(S_+)} H_0 \xrightarrow[\sim]{\mathrm{BW} \boxtimes \mathrm{BW}} L^2\mathcal{A}(S_+) \boxtimes_{\mathcal{A}(S_+)} L^2\mathcal{A}(S_+) \xrightarrow[\sim]{\text{unitor}} L^2\mathcal{A}(S_+) \xrightarrow[\sim]{\mathrm{BW}^{-1}} H_0$$

Here the right action of $\mathcal{A}(S_+)$ on H_0 comes from the left action of $\mathcal{A}(S_-)$ and the fact that $\mathcal{A}(S_+) \simeq \mathcal{A}(\overline{S_-}) \simeq \mathcal{A}(S_-)^{\mathrm{op}}$ under the isomorphism given by Ad_Θ. This map is clearly linear with respect to the left and right $\mathcal{A}(S_+)$-actions on $H_0 \boxtimes_{\mathcal{A}(S_+)} H_0$ and H_0.

Definition 4.3.2. Given two circles S_1 and S_2, and two sectors $\left(H_i, (\alpha_I^i)_{I \subset S_i}\right) \in \mathrm{Rep}_{S_i}(\mathcal{A})$ for $i = 1, 2$, a diffeomorphism $\phi\colon S_1 \to S_2$ and a unitary $u\colon H_1 \to H_2$, we say u implements ϕ if the following diagram commutes for all intervals $I \subsetneq S_1$

$$\begin{tikzcd}
\mathcal{A}(I) \arrow[r, "\mathcal{A}(\phi|_I)"] \arrow[d, "\alpha_I^1"'] & \mathcal{A}(\phi(I)) \arrow[d, "\alpha_{\phi(I)}^2"] \\
B(H_1) \arrow[r, "\mathrm{Ad}_u"'] & B(H_2).
\end{tikzcd}$$

Equivalently, u implements ϕ if $u : H_1 \to \phi^*(H_2)$ is a homomorphism of S_1-sectors.

Lemma 4.3.3. *Let $\psi_1, \psi_2 \colon S^1 \to S^1$ be orientation-preserving piecewise-smooth C^1 diffeomorphisms that fix $1, -1$ and satisfy*

$$\begin{tikzcd} S_- \arrow[r, hook] \arrow[d, "\vartheta"'] & S^1 \arrow[r, "\psi_1"] & S^1 \arrow[d, "\vartheta"] \\ S_+ \arrow[r, hook] & S^1 \arrow[r, "\psi_2"'] & S^1. \end{tikzcd} \tag{4.1}$$

That is, $\psi_1(z)^* = \psi_2(z^*)$ for $z \in S_-$. Then define

$$\psi_\cup(z) := \begin{cases} \psi_1(z) & \text{if } z \in S_+ \\ \psi_2(z) & \text{if } z \in S_-. \end{cases}$$

and let $v_1, v_2, v_\cup$ implement $\psi_1, \psi_2, \psi_\cup$. Then the following diagram commutes up to a phase

$$\begin{tikzcd} H_0 \boxtimes_{\mathcal{A}(S_+)} H_0 \arrow[r, "C"] \arrow[d, "v_1 \boxtimes v_2"'] & H_0 \arrow[d, "v_\cup"] \\ H_0 \boxtimes_{\mathcal{A}(S_+)} H_0 \arrow[r, "C"'] & H_0. \end{tikzcd} \tag{4.2}$$

Proof. We define a group G of piecewise smooth C^1 diffeomorphisms as follows:

$$\begin{aligned} G &:= \{(\psi_1, \psi_2) \text{ satisfying Equation 4.1}\} \\ &= \{(\varphi_1, \varphi_2, \varphi_3) \colon \text{piecewise smooth } C^1 \text{ diffeomorphisms } S_+ \to S_+ \text{ s.t.} \\ &\qquad \varphi_1 \cup (\vartheta \circ \varphi_2 \circ \vartheta) \text{ and } \varphi_2 \cup (\vartheta \circ \varphi_3 \circ \vartheta) \text{ are } C^1\} \end{aligned}$$

The equivalence between the two descriptions is given by $\varphi_1 = \psi_1|_{S_+}$, $\varphi_2 = \psi_2|_{S_+} = \vartheta \circ (\psi_1|_{S_-}) \circ \vartheta$, and $\varphi_3 = \vartheta \circ (\psi_2|_{S_-}) \circ \vartheta$.

For every $(\varphi_1, \varphi_2, \varphi_3) \in G$, we must show that

$$\begin{aligned} &\forall\ v_1 \in U(H_0) \text{ implementing } \varphi_1 \cup (\vartheta \circ \varphi_2 \circ \vartheta), \\ &\quad v_2 \in U(H_0) \text{ implementing } \varphi_2 \cup (\vartheta \circ \varphi_3 \circ \vartheta), \\ &\quad v_\cup \in U(H_0) \text{ implementing } \varphi_1 \cup (\vartheta \circ \varphi_3 \circ \vartheta), \text{ Equation 4.2} \\ &\text{holds.} \end{aligned} \tag{4.3}$$

Consider the following subgroups of G:

$$\begin{aligned} G_1 &:= \{(\varphi, \mathrm{id}_{S_+}, \mathrm{id}_{S_+})\} \\ G_2 &:= \{(\mathrm{id}_{S_+}, \mathrm{id}_{S_+}, \varphi)\} \\ G_3 &:= \{(\varphi, \varphi, \varphi)\}. \end{aligned}$$

These subgroups generate G, as any element $(\varphi_1, \varphi_2, \varphi_3) \in G$ can be written as

$$(\varphi_1, \varphi_2, \varphi_3) = (\varphi_2, \varphi_2, \varphi_2) \circ \left(\varphi_2^{-1} \circ \varphi_1, \mathrm{id}_{S_+}, \mathrm{id}_{S_+}\right) \circ \left(\mathrm{id}_{S_+}, \mathrm{id}_{S_+}, \varphi_2^{-1} \circ \varphi_3\right)$$

Therefore, it is enough to check Equation 4.3 for elements of G_1, G_2, G_3.

We first show that elements of G_1 satisfy Equation 4.3. In that case, $\psi_2 = \mathrm{id}_{S^1}$, $\psi_1|_{S_-} = \mathrm{id}_{S_-}$, and $\psi_\cup = \psi_1$, we may take $v_1 = v_\cup$, and Equation 4.2 reduces to

$$\begin{tikzcd} H_0 \boxtimes_{\mathcal{A}(S_+)} H_0 \arrow[r, "C"] \arrow[d, "v_1 \boxtimes \mathrm{id}_{H_0}"'] & H_0 \arrow[d, "v_1"] \\ H_0 \boxtimes_{\mathcal{A}(S_+)} H_0 \arrow[r, "C"'] & H_0, \end{tikzcd}$$

which commutes by naturality of unitors. The case of G_2 is similar.

It remains to show that elements of G_3 satisfy Equation 4.3. In that case, $\psi := \psi_1 = \psi_2 = \psi_\cup = \varphi \cup (\vartheta \circ \varphi \circ \vartheta)$, and we may take $v_1 = v_2 = v_\cup$. Upon conjugating by the Bisognano-Wichmann isomorphism, Equation 4.2 reads:

$$\begin{tikzcd} L^2(\mathcal{A}(S_+)) \boxtimes_{\mathcal{A}(S_+)} L^2(\mathcal{A}(S_+)) \arrow[r, "\text{unitor}"] \arrow[d, "w \boxtimes w"'] & L^2(\mathcal{A}(S_+)) \arrow[d, "w"] \\ L^2(\mathcal{A}(S_+)) \boxtimes_{\mathcal{A}(S_+)} L^2(\mathcal{A}(S_+)) \arrow[r, "\text{unitor}"'] & L^2(\mathcal{A}(S_+)) \end{tikzcd} \tag{4.4}$$

where $w := BW \circ v_1 \circ BW^*$.

We claim that w is equal to $L^2\mathcal{A}(\varphi)$ up to a phase, and that Equation 4.4 therefore follows from Proposition 2.1.13. To finish the proof, we are thus reduced to showing that $u = L^2\mathcal{A}(\varphi) \circ w^*$ is a multiple of the identity. Indeed, it is easy to check that u is equivariant with respect to both the left and right $\mathcal{A}(S_+)$ actions on $L^2\mathcal{A}(S_+)$. The result follows since $L^2\mathcal{A}(S_+)$ is an irreducible bimodule. $\square$

Corollary 4.3.4. *If $\psi_1 = \mathrm{id}_{S^1}$ in Lemma 4.3.3, we can choose $v_1 = \mathrm{id}_{H_0}$ and $v_\cup = v_2$, and then Equation 4.2 commutes on the nose.*

4.4 The Monoidal Structure on Solitonic Representations

Let I_+, I_- be subintervals of S^1 such that $S_- \subsetneq I_-$, $S_+ \subsetneq I_+$, and $\partial_- I_+ = \partial_+ I_- = \{1\}$. Let $I_\cup = (I_- \setminus^c S_-) \cup (I_+ \setminus^c S_+)$. We wish to define a bilinear functor

$$\begin{aligned} \hat{\boxtimes}\colon \mathcal{A}(I_-)\text{-Mod} \times \mathcal{A}(I_+)\text{-Mod} &\to \mathcal{A}(I_\cup)\text{-Mod} \\ (M, N) &\mapsto M \boxtimes_{\mathcal{A}(S_+)} N \end{aligned} \tag{4.5}$$

By [HNP24, Corollary 3.15], in order to define a functor $F : \mathcal{A}(I_-)\text{-Mod} \times \mathcal{A}(I_+)\text{-Mod} \to \mathcal{A}(I_\cup)\text{-Mod}$, it is enough to specify it on generators of $\mathcal{A}(I_-)$-Mod and $\mathcal{A}(I_+)$-Mod. The vacuum sector H_0 with its standard $\mathcal{A}(I_-)$-action is a generator of $\mathcal{A}(I_-)$-Mod. Similarly, H_0 with its standard $\mathcal{A}(I_+)$-action is a generator of $\mathcal{A}(I_+)$-Mod. On those generators, we define $F(H_0, H_0) := H_0$, where the latter H_0 is equipped with its standard $\mathcal{A}(I_\cup)$-action. To construct F as a functor, we still need to define it at the level of morphisms: for $a \in \mathrm{End}_{A(I_-)}(H_0)$ and $b \in \mathrm{End}_{A(I_+)}(H_0)$, we simply declare $F(a, b) := ab \in \mathrm{End}_{A(I_\cup)}(H_0)$.

When M and N are both equal to H_0, the functor F agrees (at the level of underlying Hilbert spaces) with the functor $(M, N) \mapsto M \boxtimes_{\mathcal{A}(S_+)} N$. By [HNP24, Corollary 3.15], the formula given in Equation 4.5 is therefore a valid formula for the functor F, and we may reinterpret our construction above as having endowed $M \boxtimes_{\mathcal{A}(S_+)} N$ with the structure of an $\mathcal{A}(I_\cup)$-module.

Definition 4.4.1. The monoidal structure on the category of solitonic representations of a conformal net $\mathcal{A}$ is given by:

$$\begin{aligned} \boxtimes\colon \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) &\to \mathrm{Sol}(\mathcal{A}) \\ M, N &\mapsto M \boxtimes_{\mathcal{A}(S_+)} N \end{aligned}$$

If $I \subseteq S^1$ is an interval that is contained in either S_- or S_+, then it is clear how $\mathcal{A}(I)$ acts on $M \boxtimes_{\mathcal{A}(S_+)} N$. If $-1 \in \mathring{I}$ and $1 \notin I$, then we can define the action of $\mathcal{A}(I)$ on $M \boxtimes_{\mathcal{A}(S_+)} N$ via Equation 4.5 with $I_- := I \cup S_-$ and $I_+ := I \cup S_+$. These actions are compatible with each other. By Lemma 4.2.5, they therefore equip $M \boxtimes_{\mathcal{A}(S_+)} N$ with the structure of a solitonic representation.

We give $\mathrm{Sol}(\mathcal{A})$ the structure of a bi-involutive tensor category (Definition 2.2.7) inherited from $(\mathcal{A}(S_+), \mathcal{A}(S_+))$-bimodules (see [HP23, Section 4.2]). The only thing that requires checking is that the associators and unitors are morphisms in $\mathrm{Sol}(\mathcal{A})$.

To check that the left unitor $\lambda : H_0 \boxtimes M \to M$ is a morphism in $\mathrm{Sol}(\mathcal{A})$, i.e., that we have a natural transformation

$$\begin{tikzcd}
& \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) \arrow[dr, "\boxtimes"] \arrow[d, phantom, "\lambda \Downarrow"] & \\
\mathrm{Sol}(\mathcal{A}) \arrow[ur, "H_0 \times \mathrm{id}_{\mathrm{Sol}(\mathcal{A})}"] \arrow[rr, "="'] & & \mathrm{Sol}(\mathcal{A}),
\end{tikzcd}$$

it is enough to check that for every interval I_+ satisfying $S_+ \subsetneq I_+$ and $\partial_- I_+ = \{1\}$, the whiskering

$$\begin{tikzcd}
& \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) \arrow[dr, "\boxtimes"] \arrow[d, phantom, "\lambda \Downarrow"] & & \\
\mathrm{Sol}(\mathcal{A}) \arrow[ur, "H_0 \times \mathrm{id}_{\mathrm{Sol}(\mathcal{A})}"] \arrow[rr, "="'] & & \mathrm{Sol}(\mathcal{A}) \arrow[r, "\mathrm{forget}"] & \mathcal{A}(I_+)\text{-Mod}
\end{tikzcd} \tag{4.6}$$

is a natural transformation between two functors from $\mathrm{Sol}(\mathcal{A})$ to $\mathcal{A}(I_+)$-Mod. We may rewrite Equation 4.6 as

$$\begin{tikzcd}
& & \mathrm{Sol}(\mathcal{A}) \times \mathcal{A}(I_+)\text{-Mod} \arrow[dr, "\boxtimes"] \arrow[d, phantom, "\lambda \Downarrow"] & \\
\mathrm{Sol}(\mathcal{A}) \arrow[r, "\mathrm{forget}"] & \mathcal{A}(I_+)\text{-Mod} \arrow[ur, "H_0 \times \mathrm{id}_{\mathcal{A}(I_+)\text{-Mod}}"] \arrow[rr, "="'] & & \mathcal{A}(I_+)\text{-Mod},
\end{tikzcd}$$

so it is enough to show that

$$\begin{tikzcd}
& \mathrm{Sol}(\mathcal{A}) \times \mathcal{A}(I_+)\text{-Mod} \arrow[dr, "\boxtimes"] \arrow[d, phantom, "\lambda \Downarrow"] & \\
\mathcal{A}(I_+)\text{-Mod} \arrow[ur, "H_0 \times \mathrm{id}_{\mathrm{Sol}(\mathcal{A})}"] \arrow[rr, "="'] & & \mathcal{A}(I_+)\text{-Mod}
\end{tikzcd}$$

is a natural transformation of functors from $\mathcal{A}(I_+)$-Mod to $\mathcal{A}(I_+)$-Mod. For the latter, it is sufficient to perform the verification on a generator of $\mathcal{A}(I_+)$-Mod. The vacuum sector $H_0 \in \mathcal{A}(I_+)$-Mod is a generator, and the unitor map $\lambda : H_0 \boxtimes H_0 \to H_0$ (Definition 4.3.1) is $\mathcal{A}(I_+)$-equivariant by the construction Equation 4.5 of the action. This finishes the verification that the left unitor is a morphism in $\mathrm{Sol}(\mathcal{A})$. The case of right unitors is similar.

We now address the case of the associators. We need to check that the associator $\alpha : (M\boxtimes N)\boxtimes P \to M\boxtimes(N\boxtimes P)$ is a morphism in $\mathrm{Sol}(\mathcal{A})$. Equivalently, we need to check that α provides a natural transformation

$$\begin{tikzcd} & \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) \arrow[dr] \arrow[dd, Rightarrow, "\alpha"] & \\ \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) \arrow[ur] \arrow[dr] & & \mathrm{Sol}(\mathcal{A}). \\ & \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) \arrow[ur] & \end{tikzcd}$$

It is enough to check that for all intervals I_+ and I_- satisfying $S_+ \subsetneq I_+$, $S_- \subsetneq I_-$ and $\partial_- I_+ = \partial_+ I_- = \{1\}$, the whiskering

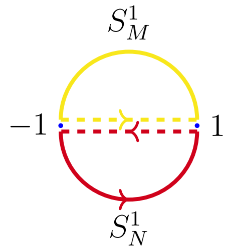
$$\begin{tikzcd} & \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) \arrow[dr] \arrow[dd, Rightarrow, "\alpha"] & & \\ \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) \arrow[ur] \arrow[dr] & & \mathrm{Sol}(\mathcal{A}) \arrow[r, "\text{forget}"] & \mathcal{A}(I_\cup)\text{-Mod} \\ & \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) \arrow[ur] & & \end{tikzcd}$$

is a natural transformation between functors from $\mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A}) \times \mathrm{Sol}(\mathcal{A})$ to $\mathcal{A}(I_\cup)$-Mod. We may rewrite the above as

$$\begin{tikzcd} & & \mathcal{A}(I_-)\text{-Mod} \times \mathcal{A}(I_+)\text{-Mod} \arrow[dr] \arrow[dd, Rightarrow, "\alpha"] & \\ \mathrm{Sol}(\mathcal{A})^3 \arrow[r] & \mathcal{A}(I_-)\text{-Mod} \times \mathrm{Sol}(\mathcal{A}) \times \mathcal{A}(I_+)\text{-Mod} \arrow[ur] \arrow[dr] & & \mathcal{A}(I_\cup)\text{-Mod} \\ & & \mathcal{A}(I_-)\text{-Mod} \times \mathcal{A}(I_+)\text{-Mod} \arrow[ur] & \end{tikzcd}$$

So it is enough to check α is a natural transformation. The latter can be checked on generators of $\mathcal{A}(I_-)$-Mod and of $\mathcal{A}(I_+)$-Mod, so all we need to check is that for every soliton $M \in \mathrm{Sol}(\mathcal{A})$ the map $\alpha : (H_0 \boxtimes M) \boxtimes H_0 \to H_0 \boxtimes (M \boxtimes H_0)$ is a morphism in $\mathrm{Sol}(\mathcal{A})$. This map can be rewritten as a composition of unitors, all of which have been shown already to be morphisms in $\mathrm{Sol}(\mathcal{A})$. This finishes the verification that the associator is a morphism in $\mathrm{Sol}(\mathcal{A})$.

We visualise this fusion as two copies of S^1 associated to M and N (let them be called S^1_M and S^1_N) being joined over $(S^1_N)_+$ and $(S^1_M)_-$ to form another copy of S^1, which we call $(S^1)_{M\boxtimes N}$. We illustrate this in the following diagram:



Remark 4.4.2. *In the case of $\mathcal{S}_{\mathcal{A}}(n,\Pi)$, the role of H_0 is played by $H_0 \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} H_0$ instead, and we can give it a monoidal structure the same way.*

4.5 Equivalence with Left-Localised Endomorphisms

Definition 4.5.1. Given an object $\rho \in \mathcal{T}^+_{\mathcal{A}}(S_+)$, we define a solitonic representation ${}^{\rho}H_0$ to have underlying Hilbert space H_0 and the action of

- $\mathcal{A}(I)$ where $I \subsetneq S^1$ is an interval such that $S_+ \subsetneq I$ and $\partial_- I = \{1\}$ is given by
$$\mathcal{A}(I) \xrightarrow{\rho'} \mathcal{A}(I) \to B(H_0)$$
where ρ' is the extension of ρ by identity.

- $\mathcal{A}(I)$ where $I \subsetneq S^1$ is an interval such that $S_- \subsetneq I$ and $\partial_+ I = \{1\}$ is given by
$$\mathcal{A}(I) \to B(H_0) \xrightarrow{\mathrm{Ad}_u} B(H_0)$$
where $u \in \mathcal{A}(\operatorname{supp}\rho) \subseteq B(H_0)$ is a unitary such that $\rho(x) = uxu^*$ for $x \in \mathcal{A}(I \cap S_+)$.

To see that the second point does not depend on choices, note that two unitaries u, u' satisfying the given conditions will differ only by a unitary in $\mathcal{A}(\operatorname{supp}\rho)$ commuting with all of $\mathcal{A}(I \cap S_+)$, so they only differ by a scalar. Thus Ad_u is well-defined.

Every other interval $I \subsetneq S^1$ whose interior does not contain 1 is contained in one of the above and thus acts on ${}^{\rho}H_0$. These actions are clearly compatible with each other, hence by Lemma 4.2.5 equip H_0 with the structure of a solitonic representation.

Definition 4.5.2. We define a functor $R\colon \mathcal{T}^+_{\mathcal{A}}(S_+) \to \mathrm{Sol}(\mathcal{A})$ on

- objects by mapping $\rho \in \mathcal{T}^+_{\mathcal{A}}(S_+)$ to ${}^{\rho}H_0$.
- morphisms by mapping a morphism $a\colon \rho \to \rho'$ in $\mathcal{T}^+_{\mathcal{A}}(S_+)$ to $a \in \mathcal{A}(S_+) \subseteq B(H_0)$ viewing it as map $H_0 \to H_0$.

We give it the data of a monoidal functor using the map

$$ {}^{\rho}H_0 \boxtimes {}^{\rho'}H_0 = {}^{\rho}(H_0 \boxtimes {}^{\rho'}H_0) \xrightarrow{C} {}^{\rho}({}^{\rho'}H_0) = {}^{\rho'\circ\rho}H_0 $$

We extend this to a monoidal functor $R\colon \mathcal{T}^+_{\mathcal{A}}(S_+)^{\cup 0} \to \mathrm{Sol}(\mathcal{A})$ by mapping the 0 object to the 0 object. It is easy to see that C satisfies the necessary coherences from Corollary 4.3.4.

Lemma 4.5.3. *Given $H \in \mathrm{Sol}(\mathcal{A})$, the images of actions of $\mathcal{A}(S_+) \to B(H)$ and $\mathcal{A}(S_-) \to B(H)$ commute.*

Proof. Choose a countable collection of intervals $\{I_i \subsetneq S_+\}_{i\in\mathbb{N}}$ such that $1 \notin I_i$ for any $i \in \mathbb{N}$ and

$$ \bigcup_{i\in\mathbb{N}} \mathring{I}_i = \mathring{S}_+ $$

By additivity, the algebra generated by all the $\mathcal{A}(I_i)$ in $\mathcal{A}(S_+)$ is dense. So it is sufficient to prove that the image of each $\mathcal{A}(I_i) \to B(H)$ commutes with $\mathcal{A}(S_-) \to B(H)$ for every $i \in \mathbb{N}$.

Let $J_i \subsetneq S_+$ be the subinterval of S^1 whose boundaries are 1 and $\partial_- I_i$ and which does not intersect the interior of I_i. This is always an interval as I_i is a proper subinterval of S_+ which does not contain 1. S_- and I_i are both subintervals of $S^1 \setminus^c J_i$, this is a subinterval of S^1 whose interior does not contain 1. Thus, the actions of $\mathcal{A}(S_-)$ and $\mathcal{A}(I_i)$ on H are restrictions of the action of $\mathcal{A}(S^1 \setminus^c J_i)$. Since $I_i \subsetneq S_+$, I_i and S_- have disjoint interiors, so by locality, we have that $\mathcal{A}(S_-)$ and $\mathcal{A}(I_i)$ commute as subalgebras of $\mathcal{A}(S^1 \setminus^c J_i)$. Therefore, the images of $\mathcal{A}(S_-) \to B(H)$ and $\mathcal{A}(I_i) \to B(H)$ commute. □

Theorem 4.5.4. *The functor $R\colon \mathcal{T}^+_{\mathcal{A}}(S_+)^{\cup 0} \to \mathrm{Sol}(\mathcal{A})$ is an equivalence of dagger monoidal categories.*

Proof. It is enough to show it is an equivalence. It is clearly faithful and dagger.

Fullness: Let $\rho, \rho' \in \mathcal{T}_{\mathcal{A}}^{+}(S_+)$. Let $f\colon {}^{\rho}H_0 \to {}^{\rho'}H_0$ be a map of solitonic representations. Define $J = \operatorname{supp}\rho \cup \operatorname{supp}\rho' \subsetneq S_+$. Since f is $\mathcal{A}(S^1 \setminus^c J)$-linear and the actions of $\mathcal{A}(S^1 \setminus^c J)$ on the domain and codomain of f are just its actions on H_0, we have $f \in [\mathcal{A}(S^1 \setminus^c J)]'_{B(H_0)} = \mathcal{A}(J)$. Therefore $f \in \mathcal{A}(S_+)$ is a left-localised element. The $\mathcal{A}(S_+)$-linearity of f gives us that $f\rho(x) = \rho'(x)f$ for all $x \in \mathcal{A}(S_+)$. Thus, $f\colon \rho \to \rho'$ is a morphism in $\mathcal{T}_{\mathcal{A}}^{+}(S_+)$. Since maps to and from a 0 object are unique, R is full on hom-sets to or from 0 as well.

Essential surjectivity: Let $M \in \mathrm{Sol}(\mathcal{A})$. If M is 0, it has a preimage. So we consider the case where M is not 0. Then, M is isomorphic as an $\mathcal{A}(S_-)$-module to $L^2\mathcal{A}(S_-)$ which is equivalent to H_0 as an $\mathcal{A}(S_-)$-module by the Bisognano-Wichmann theorem. We assume M is of this form. Since the commutant of $\mathcal{A}(S_-)$ in $B(H_0)$ is $\mathcal{A}(S_+)$, the action of $\mathcal{A}(S_+)$ on M has to factor through the standard action of $\mathcal{A}(S_+)$ on M by Lemma 4.5.3, therefore M is isomorphic to ${}^{\rho}H_0$ for some $\rho\colon \mathcal{A}(S_+) \to \mathcal{A}(S_+)$. Let $I \subsetneq S^1$ be an interval such that $S_+ \subsetneq I$ and $\partial_- I = \{1\}$. Then, as the action of $\mathcal{A}(S^1 \setminus^c I)$ on ${}^{\rho}H_0$ is just the standard action of $\mathcal{A}(S^1 \setminus^c I)$, by Haag duality, we have that the $\mathcal{A}(I)$-action on ${}^{\rho}H_0$ factors through the canonical action of $\mathcal{A}(I)$ on H_0. This gives us a map $\rho'\colon \mathcal{A}(I) \to \mathcal{A}(I)$ which is ρ on $\mathcal{A}(S_+)$ and identity on $\mathcal{A}(I \setminus^c S_+)$. Choosing an I' bigger than I, we see this even admits an extension by identity. Therefore, $\rho' \in \mathcal{T}_{\mathcal{A}}^{+}(I)$ by Lemma 3.3.7. However, this category is equivalent to $\mathcal{T}_{\mathcal{A}}^{+}(S_+)$, so it must have a preimage (up to isomorphism) ρ'', and we have ${}^{\rho}H_0 \simeq {}^{\rho''}H_0$, which is in the image of R. $\square$

This allows us to import the bi-involutive tensor structure on $\mathrm{Sol}(\mathcal{A})$ to $\mathcal{T}_{\mathcal{A}}^{+}(S_+)^{\cup 0}$ up to a contractible choice (a choice of inverse to the above equivalence). They are also both manifestly W*-categories.

Corollary 4.5.5. *For any interval $I \subsetneq S^1$ such that $\partial_- I = \{1\}$, $\mathcal{T}_{\mathcal{A}}^{+}(I)^{\cup 0} \simeq \mathrm{Sol}(A)$.*

Corollary 4.5.6. *Let $I \subsetneq S^1$ be a subinterval such that 1 is in the interior of I and -1 is not in the interior of I. Then we have a canonical equivalence $\mathcal{T}_{\mathcal{A}}^{+}(I \cap S_+) \simeq \mathcal{T}_{\mathcal{A}}^{-}(I \cap S_-)^{\mathrm{mop}}$.*

Proof. This follows from Corollary 4.5.5 and the fact that $\overline{}\colon \mathrm{Sol}(\mathcal{A}) \to \mathrm{Sol}(\mathcal{A})^{\mathrm{mop}}$ is an equivalence and this functor is ϑ^* where $\vartheta\colon S^1 \to S^1$ is the complex conjugation map which maps S_+ to $\overline{S_-}$. $\square$

Lemma 4.5.7. *Let S be a disjoint union of n copies of the standard circle S^1 and $\Pi \subseteq \pi_0(S)$. Let I be a sub-multi-interval of S such that the inclusion $I \hookrightarrow S$ is an isomorphism on connected components, and for components $I_i \subsetneq S^1_i$ such that $S^1_i \notin \Pi$, $\partial_- I_i = \{1_i\} \subseteq S_i$. Then $\mathcal{T}_{\mathcal{A}}(I, \Pi)^{\cup 0}$ is equivalent to $\mathcal{S}_{\mathcal{A}}(n, \Pi)$.*

In particular, for any interval $I \subsetneq S^1$, we have $\mathrm{DHR}_{\mathcal{A}}(I)^{\cup 0} \simeq \mathrm{Rep}(\mathcal{A})$.

Proof. The proof of this is identical to that of Corollary 4.5.5. □

Corollary 4.5.8. *Let I be a multi-interval and $P \subsetneq I$ a 0-dimensional submanifold such that the inclusion map $P \hookrightarrow I$ is an isomorphism on the set of connected components. Let I_1 and I_2 be the sub-multi-intervals of I to the right and left of P, where right and left are defined using the orientation of I. Then there is a canonical equivalence $\mathcal{T}^+_{\mathcal{A}}(I_1) \simeq \mathcal{T}^-_{\mathcal{A}}(I_2)^{\mathrm{mop}}$.*

Proof. Similar to Corollary 4.5.6. □

4.6 The Assignment of a von Neumann Algebra to an arbitrary 1-Manifold by a Conformal Net

Definition 4.6.1. Given a circle S and a diffeomorphism $\phi\colon S \to S^1$, define $H_0(S, \phi) = \phi^*(H_0)$. We also define $H_0(\bigsqcup_n S^1) = H_0^{\boxtimes n}$, and for a non-standard circle, we can pull the actions back along diffeomorphisms similar to the case of a single circle. We will also use the notation $H_0^{\phi}(S)$ for $H_0(S, \phi)$.

Lemma 4.6.2. *Given a circle S and two diffeomorphisms $\phi_1, \phi_2\colon S \to S^1$ there is an isomorphism $H_0^{\phi_1}(S) \xrightarrow{\sim} H_0^{\phi_2}(S)$ in $\mathcal{R}_{\mathcal{A}}(S, \emptyset)$ which is well-defined up to a phase.*

Proof. Such an isomorphism is supplied by a $u\colon H_0 \to H_0$ that implements $\phi_2 \circ \phi_1^{-1}$, which is unique up to a phase. □

Remark 4.6.3. *In the previous lemma, if $\phi_2 \circ \phi_1^{-1} \in$ Möb, then there is a canonical u that implements $\phi_2 \circ \phi_1^{-1}$.*

Given a complex disc $D \subsetneq \mathbb{C}$, there is a unique holomorphic isomorphism $\psi\colon D \xrightarrow{\sim} \mathbb{D}$ to the standard disc up to a Möbius transformation. We define $H_0(D) = H_0(\partial D, \psi|_{\partial D})$ and this is defined canonically with no phase ambiguity.

Remark 4.6.4. *We often do not specify an isomorphism $\phi \colon S \to S^1$ when writing $H_0(S)$ for they all produce canonically equivalent choices.*

The following assignment was first described in [BDH17, Theorem 1.3].

Definition 4.6.5. Given a 1-manifold M, we pick a finite cover $\mathcal{I} = \{I_1, I_2, \ldots, I_n\}$ of M by intervals with disjoint interiors. This gives us a cover of the surface $M \times [0,1]$ by rectangles $I_i \times [0,1]$. We get circles $S_i := \partial(I_i \times [0,1])$, with the orientation that agrees with that of $M \times \{0\}$ on $I_i \times \{0\}$. Let $p_1, \ldots, p_m \in M$ be points where two intervals in $\mathcal{I}$ meet. Let $A_i := \mathcal{A}(\{p_i\} \times [0,1])$. We consider the following Connes fusion

$$H_M := \boxtimes_{\{A_i\}} \{H_0(S_j)\}_{1 \leq i \leq m, 1 \leq j \leq n} \tag{4.7}$$

Define $\mathcal{A}(M)$ as

$$\mathcal{A}(M) := \bigvee_{I_i \in \mathcal{I}} \mathcal{A}(I_i) \text{ as a subalgebra of } B(H_M). \tag{4.8}$$

[BDH17, Lemma 1.9] shows that the Hilbert space 4.7 is independent of the choice of cover up to non-canonical unitary isomorphism but [BDH17, Lemma 1.12] shows that the algebra 4.8 is independent of the choice of cover up to a canonical isomorphism. [BDH17, Theorem 1.3] tells us that

$$\mathcal{A}(I_1 \sqcup I_2) = \mathcal{A}(I_1) \bar{\otimes} \mathcal{A}(I_2) \tag{4.9}$$

Lemma 4.6.6. *Given circles S_1 and S_2 and embeddings $\overline{J} \hookrightarrow S_1$ and $J \hookrightarrow S_2$, let $S_\cup$ be the circle $(S_1 \setminus^c \overline{J}) \sqcup_{\partial J} (S_2 \setminus^c J)$ with a smooth structure compatible with the smooth structures on $S_1 \setminus^c \overline{J}$ and $S_2 \setminus^c J$. There is an isomorphism $C_J \colon H_0(S_1) \boxtimes_{\mathcal{A}(J)} H_0(S_2) \xrightarrow{\sim} H_0(S_\cup)$ which is canonical up to a phase.*

Proof. Choose isomorphisms $\phi_1 \colon S_1 \to S^1$ and $\phi_2 \colon S_2 \to S^1$ such that $\phi_1(\overline{J}) = S_-$ and $\phi_2(J) = S_+$ and the following diagram commutes

$$\begin{tikzcd}
\overline{J} \arrow[r, "\phi_1|_{\overline{J}}"] \arrow[d, "\mathrm{id}_J"'] & S_- \arrow[r, "S^1"] & S^1 \arrow[d, "\vartheta"] \\
J \arrow[r, "\phi_2|_J"'] & S_+ \arrow[r, "S^1"] & S^1
\end{tikzcd}$$

Define the map $\phi_\cup \colon S^\cup \to S^1$ as

$$\phi_\cup(p) = \begin{cases} \phi_1(p) \text{ if } p \in S_1 \setminus^c \overline{J} \\ \phi_2(p) \text{ if } p \in S_2 \setminus^c J \end{cases}$$

We use this to give a smooth structure on $S_\cup$. Clearly,

$$(\phi_\cup^{-1})^*(H_0(S_1,\phi_1) \boxtimes_{\mathcal{A}(J)} H_0(S_2,\phi_2)) = H_0 \boxtimes_{\mathcal{A}(S_+)} H_0 \xrightarrow[\sim]{C} H_0$$

Therefore, $\phi_\cup^*(C) \colon H_0(S_1,\phi_1) \boxtimes_{\mathcal{A}(J)} H_0(S_2,\phi_2) \xrightarrow{\sim} H_0(S_\cup,\phi_\cup)$ is an isomorphism. If we choose different ϕ_1, ϕ_2, this map would only change by a phase, by Lemma 4.3.3. By Lemma 4.6.2, any other choices of $H_0(S_1), H_0(S_2), H_0(S_\cup)$ are canonically equivalent to our choices up to a phase. Since a circle has a unique smooth structure up to non-canonical isomorphism, choosing a distinct smooth structure on $S_\cup$ would amount to choosing distinct ϕ_1, ϕ_2, which would only change our choices by a phase. □

4.7 Absorbing Objects

Lemma 4.7.1. *Given a multi-interval I and a non-surjective embedding $\phi \colon I \to I$ which is identity on a neighbourhood of $\partial_+ I$, we have that $\mathcal{A}(\phi)$ is an absorbing object of $\mathcal{T}^+_{\mathcal{A}}(I)$.*

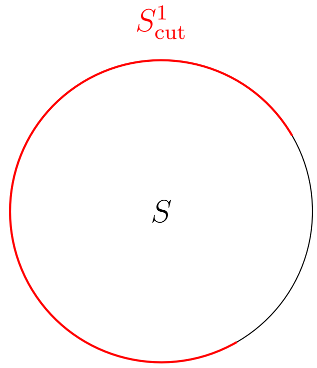
Proof. This follows from Lemma 3.3.17, Remark 2.2.20, and Lemma 2.2.21. □

Lemma 4.7.2. *Let S be a circle with an embedding of $S^1_{\text{cut}} \colon = [0, 2\pi]$. Every subinterval $I \subsetneq S^1$ which does not contain 1 in its interior has a unique preimage in S^1_{cut} under the map $S^1_{\text{cut}} \to S^1$ given by $\theta \mapsto e^{i\theta}$.*

Consider $H_0(S)$ as an object of $\text{Sol}(\mathcal{A})$ by defining the action of an interval $I \subsetneq S^1$ as

$$\mathcal{A}(I) \to \mathcal{A}(S^1_{\text{cut}}) \to B(H_0(S))$$

Then this is an absorbing object.



Proof. It is enough to prove that $H_0(S, \phi)$ is absorbing for some $\phi\colon S \to S^1$.

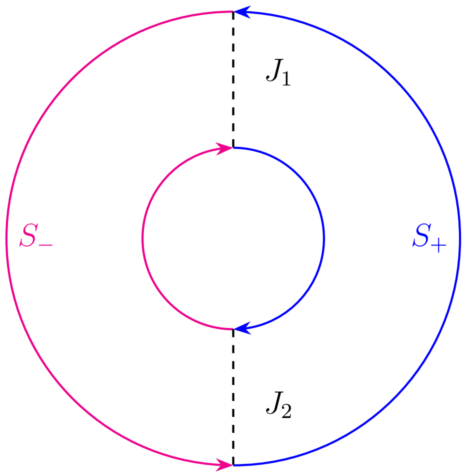
Choose $\phi\colon S \to S^1$ such that there is an $a \in (0, \pi)$ such that the following diagram commutes

$$\begin{tikzcd}
{[a, 2\pi]} \arrow[r, hook] \arrow[d, hook] & {[0, 2\pi]} \arrow[r, hook] & S \arrow[d, "\phi"] \\
{[0, 2\pi]} \arrow[rr, two heads, "\theta \mapsto e^{i\theta}"] & & S^1
\end{tikzcd}$$

By the above condition, $\phi([0, \pi]) \subsetneq S^+$ is a right-localised subinterval, and the map $\varphi\colon S_+ \simeq [0, \pi] \simeq \phi([0, \pi]) \hookrightarrow S_+$ is a non-surjective embedding such that it is identity on the neighbourhood of $\partial_+ S_+$ given by the image of $[a, \pi]$ under the exponential map. The $\mathcal{A}(S_-)$-action on $H_0(S, \phi)$ is just the $\mathcal{A}(S_-)$-action on H_0, the $\mathcal{A}(S_+)$-action is given by $\mathcal{A}(S_+) \xrightarrow{\mathcal{A}(\varphi)} \mathcal{A}(S_+) \to B(H_0)$. Therefore, $H_0(S, \phi) \simeq {}^{\mathcal{A}(\varphi)}H_0$. By Lemma 4.7.1, $\mathcal{A}(\varphi)$ is absorbing in $\mathcal{T}^+_{\mathcal{A}}(S_+)$, and therefore by Theorem 4.5.4, ${}^{\mathcal{A}(\varphi)}H_0$ is an absorbing object of $\mathrm{Sol}(\mathcal{A})$. $\square$

Remark 4.7.3. *We can think of the circle S above as a triangle using the obvious map $(S_+ \sqcup_{\{-1\}} S_-) \sqcup_{\partial S^1_{\text{cut}}} (S \setminus^c S^1_{\text{cut}}) \to S$. The three sides are S_+, $\overline{S_+} \simeq S_-$, and $S \setminus^c S^1_{\text{cut}}$. We can identify $S \setminus^c S_{\text{cut}}$ with $\{1\} \times [0, 1]$. So this triangle can be thought of as S^1 but the point 1 has been stretched out into an interval.*

Lemma 4.7.4. *Let $J_1 \subsetneq S_+$ be a left-localised subinterval and let $J_2 \subsetneq S_+$ be a right-localised subinterval such that $J_1 \cap J_2 = \emptyset$. Then $H_0 \boxtimes_{\mathcal{A}(J_1 \cup J_2)} H_0$ can be given the structure of a representation using methods of Section 4.4. It is an absorbing object in* $\mathrm{Rep}(\mathcal{A})$.



Proof. By Lemma 4.5.7, every non-zero $\mathcal{A}$-representation is of the form ${}^{\rho}H_0$ for a $\rho \in \mathrm{DHR}_{\mathcal{A}}(S_+)$. We compute

$$ {}^{\rho}H_0 \boxtimes_{\mathcal{A}(S_+)} (H_0 \boxtimes_{\mathcal{A}(J_1\cup J_2)} H_0) \simeq ({}^{\rho}H_0 \boxtimes_{\mathcal{A}(S_+)} H_0) \boxtimes_{\mathcal{A}(J_1\cup J_2)} H_0 \simeq {}^{\rho}H_0 \boxtimes_{\mathcal{A}(J_1\cup J_2)} H_0 $$

Let $I \subsetneq S^1$ be a subinterval such that $S_+ \subsetneq I$ and $S_- \setminus^c \overline{J_1 \cup J_2} \subsetneq I$. The action of $\mathcal{A}(I)$ on ${}^{\rho}H_0$ is given by $\mathcal{A}(I) \xrightarrow{\rho^{\mathrm{ext}}} \mathcal{A}(I) \to B(H_0)$. By Remark 3.3.22, there is $\rho' \in \mathrm{DHR}_{\mathcal{A}}(I)$ such that $\operatorname{supp}\rho' \subsetneq S_- \setminus^c \overline{J_1 \cup J_2}$ and there is a $u\colon \rho' \simeq \rho^{\mathrm{ext}}$. Since $\operatorname{supp}\rho' \subsetneq S_-$, $\rho'(\mathcal{A}(S_-)) \subseteq \mathcal{A}(S_-)$, so we obtain a map $\rho'' \in \mathrm{DHR}_{\mathcal{A}}(S_-)$ such that ρ' is an extension of ρ'' by identity. Let $H_0^{\rho''}$ be the object of $\mathrm{Rep}(\mathcal{A})$ whose $\mathcal{A}(S_-)$-action is twisted by ρ'' (as in Definition 4.5.1).

We have an isomorphism of $\mathcal{A}$-representations

$$ {}^{\rho}H_0 \boxtimes_{\mathcal{A}(J_1\cup J_2)} H_0 \xrightarrow{u \boxtimes_{\mathcal{A}(J_1\cup J_2)} \mathrm{id}_{H_0}} H_0^{\rho''} \boxtimes_{\mathcal{A}(J_1\cup J_2)} H_0 \xrightarrow{\mathrm{id}} H_0 \boxtimes_{\mathcal{A}(J_1\cup J_2)} H_0 $$

The final map is a map of representations as the $\mathcal{A}(S_+ \cup J_1 \cup J_2 \cup \overline{J_1} \cup \overline{J_2})$-action on $H_0^{\rho''}$ is the same as the $\mathcal{A}(S_+ \cup J_1 \cup J_2 \cup \overline{J_1} \cup \overline{J_2})$-action on H_0. Thus, $H_0 \boxtimes_{\mathcal{A}(J_1\cup J_2)} H_0$ is left-absorbing. Since $\mathrm{Rep}(\mathcal{A})$ is a bi-involutive tensor category, $H_0 \boxtimes_{\mathcal{A}(J_1\cup J_2)} H_0$ absorbs from both sides. □

Lemma 4.7.5. *For a circle S, there is a canonical equivalence $\mathcal{A}(S)$-Mod $\simeq$ $\mathrm{Rep}_S(\mathcal{A})^{\mathrm{abs}}$. In particular, $L^2\mathcal{A}(S)$ is isomorphic to the constructed absorbing object.*

Proof. It is enough to prove this for the standard circle S^1, and show that this equivalence is natural with respect to diffeomorphisms $S^1 \to S^1$.

We consider the object

$$\Omega\colon = H_0(\partial(S_- \times [0,1])) \boxtimes_{\mathcal{A}(\overline{[0,1]}\sqcup[0,1])} H_0(\partial(S_+ \times [0,1]))$$

By Lemma 4.7.4, this is an absorbing object of $\mathrm{Rep}_{S^1\times\{0\}}(\mathcal{A})$. $S^1 \times \{1\}$ then has the opposite orientation to $S^1 \times \{0\}$, and is thus canonically $\overline{S^1 \times \{0\}}$.

We have $\mathrm{End}_{\mathrm{Rep}(\mathcal{A})}(\Omega) = \mathcal{A}(S^1 \times \{1\}) = \mathcal{A}(S^1 \times \{0\})^{\mathrm{op}}$ by Definition 4.6.5. Since Ω is a generator of $\mathrm{Rep}_{S^1\times\{0\}}(\mathcal{A})^{\mathrm{abs}}$, we have

$$\mathrm{Rep}_{S^1\times\{0\}}(\mathcal{A})^{\mathrm{abs}} \simeq \mathrm{End}(\Omega)^{\mathrm{op}}\text{-Mod} \simeq \mathcal{A}(S^1 \times \{0\})\text{-Mod}$$

This construction is clearly natural with respect to diffeomorphisms of S^1. $\square$

Lemma 4.7.6. *Given an absorbing object $\rho_0 \in \mathrm{DHR}_{\mathcal{A}}(S_+)$, there is a subinterval $I \subsetneq S_+$ such that $I \cap \partial S_+ = \emptyset$ and $\mathrm{im}(\rho_0) \cap \mathcal{A}(I) = \mathbb{C}$.*

Proof. Let $J_1 \subsetneq S_+$ be a left-localised subinterval and J_2 be a right-localised subinterval. By Lemma 4.7.4, $H_0 \boxtimes_{\mathcal{A}(J_1\cup J_2)} H_0$ is an absorbing object of $\mathrm{Rep}(\mathcal{A})$. Let $\rho \in \mathrm{DHR}_{\mathcal{A}}(S_+)$ be such that it is identity on $\mathcal{A}(J_1 \cup J_2)$. Then

$$H_0 \boxtimes_{\mathcal{A}(J_1\cup J_2)} H_0 \boxtimes_{\mathcal{A}(S_+)} {}^{\rho}H_0 \xrightarrow{\text{unitor}} H_0 \boxtimes_{\mathcal{A}(J_1\cup J_2)} {}^{\rho}H_0 \xrightarrow{\text{id}} H_0 \boxtimes_{\mathcal{A}(J_1\cup J_2)} H_0$$

Under the equivalence of $\mathrm{DHR}_{\mathcal{A}}(S_+)$ and $\mathrm{Rep}(\mathcal{A})$, unitors map to identity in $\mathrm{DHR}_{\mathcal{A}}(S_+)$ as they are part of the coherence data for the monoidal structure, and $\mathrm{DHR}_{\mathcal{A}}(S_+)$ is strict monoidal. Let ρ_0 be a preimage of the absorbing object under the equivalence, so we have

$$\rho_0 \otimes \rho = \rho_0 \implies \rho \circ \rho_0 = \rho_0$$

So every ρ localised inside $I := S_+ \setminus^c (J_1 \cup J_2)$ is identity on $\mathrm{im}(\rho_0)$, this is possible only if $\mathrm{im}(\rho_0) \cap \mathcal{A}(I) = \mathbb{C}$. $\square$

Lemma 4.7.7. *Let S be a disjoint union of n copies of the standard circle S^1 and let $\Pi \subseteq \pi_0(S)$. For each copy $S^1_i \in \Pi$, choose an absorbing object $\Omega_i \in \mathrm{Rep}(\mathcal{A})$ and for each copy $S^1_{i'} \notin \Pi$ choose an absorbing object $\Omega_{i'} \in \mathrm{Sol}(\mathcal{A})$. Then $\Omega_1 \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} \Omega_n$ is absorbing in $\mathcal{S}_{\mathcal{A}}(n, \Pi)$.*

Proof. For every $S_i \in \pi_0(S) \setminus \Pi$ let $\phi_i \colon S_+ \to S_+$ $(i \notin \Pi)$ be non-surjective embeddings which are identity in a neighbourhood of $\partial_+ S_+$. For every $i \in \Pi$, let

$\rho_i\colon \mathcal{A}(S_+) \to \mathcal{A}(S_+)$ be an absorbing object of $\mathrm{DHR}_{\mathcal{A}}(S_+)$. Further, let $I_i \subsetneq S_+$ be intervals such that $\mathrm{im}(\rho_i) \cap \mathcal{A}(I_i) = \mathbb{C}$. For $i \in \pi_0(S) \setminus \Pi$, define $\rho_i := \mathcal{A}(\phi_i)$, which is an absorbing object of $\mathcal{T}^+_{\mathcal{A}}(S_+)$.

Choose isomorphisms of $\mathcal{A}$-representations $u_i\colon \Omega_i \xrightarrow{\sim} {}^{\rho_i}H_0$ for $i \in \Pi$ and isomorphisms of $\mathcal{A}$-solitons $u_i\colon \Omega_i \xrightarrow{\sim} {}^{\rho_i}H_0$ for $i \notin \Pi$. These always exist as any two absorbing objects of a category are equivalent. Let $\rho\colon \mathcal{A}(\bigsqcup_n S_+) \to \mathcal{A}(\bigsqcup_n S_+)$ be defined as

$$\mathcal{A}(S_+)\overline{\otimes}\cdots\overline{\otimes}\mathcal{A}(S_+) \xrightarrow{\rho_1\overline{\otimes}\cdots\overline{\otimes}\rho_n} \mathcal{A}(S_+)\overline{\otimes}\cdots\overline{\otimes}\mathcal{A}(S_+)$$

where we have used $\mathcal{A}(\bigsqcup_n S_+) \simeq \mathcal{A}(S_+)\overline{\otimes}\cdots\overline{\otimes}\mathcal{A}(S_+)$.

Now, since the map $\mathcal{T}_{\mathcal{A}}(\bigsqcup_n S_+, \Pi) \to \mathcal{S}_{\mathcal{A}}(n, \Pi)$ given by $\sigma \mapsto {}^{\sigma}(H_0 \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} H_0)$ is an equivalence on non-zero objects and

$${}^{\rho_1\overline{\otimes}\cdots\overline{\otimes}\rho_n}(H_0 \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} H_0) \simeq {}^{\rho_1}H_0 \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} {}^{\rho_n}H_0 \xrightarrow{u_1\boxtimes_{\mathbb{C}}\cdots\boxtimes_{\mathbb{C}}u_n} \Omega_1 \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} \Omega_n$$

it is enough to show that $\sigma \circ \rho \simeq \rho$. This follows from an argument similar to Lemma 3.3.17 where we choose a $\sigma' \simeq \sigma$ such that σ' is identity on $\mathrm{im}(\rho)$. $\square$

4.8 $\mathcal{T}^+_{\mathcal{A}}(I)^{\mathrm{abs}}$ as a bi-involutive tensor bimodule over $\mathcal{T}^+_{\mathcal{A}}(I)$

The goal of this section is to equip $\mathcal{T}^+_{\mathcal{A}}(I)^{\mathrm{abs}}$ with the structure needed in Definition 2.2.26: we construct the maps γ^L and γ^R that exhibit $\mathcal{T}^+_{\mathcal{A}}(I)^{\mathrm{abs}}$ as a bi-involutive tensor bimodule over $\mathcal{T}^+_{\mathcal{A}}(I)$.

Fixing an absorbing object $\Omega = \mathcal{A}(\phi)$, the Hilb-valued inner product $\langle \Omega, \Omega \rangle$ is identified with the vacuum sector H_0 via the Bisognano-Wichmann isomorphism. Under this identification, the maps γ^L and γ^R, which move a tensor factor from one slot of the inner product to the other, should be implemented by diffeomorphisms of the circle: the 'doubling' maps. However, unitaries that implement diffeomorphisms are only defined up to a phase. We circumvent this by defining piecewise smooth doubling maps that live in the Thompson group, which acts (in a non-obvious way, and non-canonically) smoothly on the circle, such that when we pull back the Bott-Thurston cocycle onto it, it vanishes, so we lose our phase ambiguities. This enables us to define γ^L and γ^R and verify all the coherences. We also show that the choices made to construct them (for example to give the action of the Thompson group of S^1) give the same results for γ^L and γ^R.

Definition 4.8.1. [CFP96] Let the Thompson group Th be defined as the subgroup of $\mathrm{Homeo}\,(\mathbb{R}/\mathbb{Z})$ consisting of g that satisfy the following:

- g is orientation preserving.
- g is piecewise affine.
- g maps dyadic numbers (numbers of the form $p/2^q$) to dyadic numbers.
- g has finitely many points of discontinuous derivative.
- At every point, the left and right derivatives of g are a power of 2.
- All points where g has discontinuous derivative are dyadic.
- $g(0)$ is a dyadic number.

We consider the Thompson group as a subgroup of $\mathrm{Homeo}\,(S^1)$ under the isomorphism $\mathbb{R}/\mathbb{Z} \to S^1$ given by $x \mapsto e^{2\pi i x}$.

Construction 4.8.2. *Let $f\colon [0,1] \to [0,2]$ be a smooth diffeomorphism such that $f(0) = 0$, $f(1) = 2$, and $f'(x) = 1$ in some neighbourhood of both 0 and 1. From f, a faithful continuous action $\varphi_f\colon \mathrm{Th} \to \mathrm{Diff}(S^1)$ of Th on S^1 was constructed in [GS87, Section III.1, Lemmas 1.1-1.8].*

Definition 4.8.3. For each $c \in \mathbb{R}$, there is a central extension of $\mathrm{Diff}^+(S^1)$ by $U(1)$, where the multiplication on $\widetilde{\mathrm{Diff}^+}(S^1) \simeq U(1) \times \mathrm{Diff}^+(S^1)$ is given by:

$$(z_1, \phi_1) \cdot (z_2, \phi_2) \mapsto (z_1 \cdot z_2 \cdot B_c(\phi_1, \phi_2), \phi_1 \circ \phi_2)$$

where $B_c\colon \mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1) \to S^1$ is the Bott-Thurston cocycle [Bot77]:

$$B_c(\phi_1, \phi_2) := \exp\left(-\frac{ic}{48\pi} \int_{S^1} \log((\phi_1 \circ \phi_2)')\mathrm{d}(\log(\phi_2'))\right). \tag{4.10}$$

Theorem 4.8.4. *[GS87, Théorème F] The pullback of the Bott-Thurston cocycle along φ_f as in Construction 4.8.2 is trivial in $H^2(\mathrm{Th}, S^1)$. Also,* $\mathrm{Hom}(\mathrm{Th}, \mathrm{U}(1)) = 0$ *[GS87, Corollaire C].*

Corollary 4.8.5. *Given a projective representation u of* $\mathrm{Diff}^+(S^1)$ *on H with 2-cocycle cohomologous to B_c, we thus have a unique $\widetilde{u}_f$ such that the following diagram commutes*

$$\begin{tikzcd} \mathrm{Th} \arrow[r, "\varphi_f"] \arrow[d, "\widetilde{u}_f"'] & \mathrm{Diff}^+(S^1) \arrow[d, "u"] \\ \mathrm{U}(H) \arrow[r, two heads] & \mathbb{P}\mathrm{U}(H) \end{tikzcd}$$

Let $I \subsetneq S^1$ be an interval such that $S_+ \subsetneq I$ and $\partial_- I = \{1\}$. Let $\phi\colon I \to I$ be a non-surjective embedding such that it is identity on a neighbourhood of $\partial_+ I$ and $I \setminus^c \mathrm{im}(\phi) = S_+$. Then, $\Omega = \mathcal{A}(\phi)$ is an absorbing object of $\mathcal{T}_{\mathcal{A}}^+(I)$ by Lemma 3.3.17. Further, we have that $\mathrm{End}(\Omega) \simeq \mathcal{A}(S_+)$.

In the rest of this section we demonstrate how to give a bi-involutive tensor action of $\mathcal{T}_{\mathcal{A}}^+(I)$ on $\mathcal{T}_{\mathcal{A}}^+(I)^{\mathrm{abs}}$, that is, we construct the maps γ^L, γ^R from Definition 2.2.26.

Definition 4.8.6. Let $S_+ \cup S_+$ be the topological manifold $S_+ \sqcup S_+/(\partial_- \iota_2(S_+) \sim \partial_+ \iota_1(S_+))$ where $\iota_1, \iota_2\colon S_+ \to S_+ \sqcup S_+$ are the inclusions of S_+ as the two parts of $S_+ \sqcup S_+$ respectively. Let $i_1, i_2\colon S_+ \to S_+ \cup S_+$ be the compositions of ι_1, ι_2 with the quotient map $S_+ \sqcup S_+ \to S_+ \cup S_+$. We give $S_+ \cup S_+$ a smooth structure by transporting the smooth structure on $I \setminus \mathrm{im}(\phi \circ \phi)$ along the homeomorphism

$$\begin{aligned} S_+ \cup S_+ &\to I \setminus \mathrm{im}(\phi \circ \phi) \\ x &\mapsto \begin{cases} i_1^{-1}(x) & \text{if } x \in \mathrm{im}(i_1) \\ \phi(i_2^{-1}(x)) & \text{if } x \in \mathrm{im}(i_2) \end{cases} \end{aligned}$$

Note also that the complex conjugation map $\vartheta\colon S^1 \to S^1$ induces an orientation reversing diffeomorphism $r\colon S_+ \simeq S_-$.

Definition 4.8.7. Let $d\colon [0,1] \to [0,2]$ be an orientation-preserving homeomorphism and $D\colon S_+ \to S_+ \cup S_+$ be the induced map under the identification

$[0,1] \xrightarrow[x \mapsto \exp(i\pi x)]{\sim} S_+$. Define $\alpha^L, \alpha^R \colon S^1 \xrightarrow{\sim} S^1$ as

$$\alpha^L(p) = \begin{cases} (D^{-1} \circ i_2)(p) & \text{if } p \in S_+ \\ (D^{-1} \circ i_1 \circ i_2^{-1} \circ D \circ \vartheta)(p) & \text{if } p \in S_- \text{ and } D(\vartheta(p)) \in \operatorname{im}(i_2) \\ (\vartheta \circ D \circ \vartheta)(p) & \text{if } p \in S_- \text{ and } D(\vartheta(p)) \in \operatorname{im}(i_1) \end{cases}$$

$$\alpha^R(p) = \begin{cases} (D^{-1} \circ i_1)(p) & \text{if } p \in S_+ \\ (D^{-1} \circ i_2 \circ i_1^{-1} \circ D \circ \vartheta)(p) & \text{if } p \in S_- \text{ and } D(\vartheta(p)) \in \operatorname{im}(i_1) \\ (\vartheta \circ D \circ \vartheta)(p) & \text{if } p \in S_- \text{ and } D(\vartheta(p)) \in \operatorname{im}(i_2) \end{cases}$$

Define $d_p \colon S^1 \to S^1$ to be the map induced by the following map under the equivalence $\mathbb{R}/\mathbb{Z} \simeq S^1$

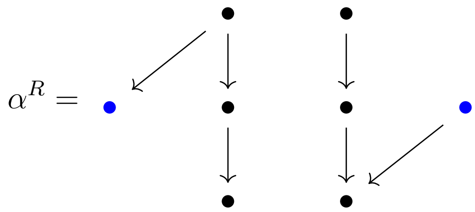
$$\begin{aligned} \mathbb{R}/\mathbb{Z} &\to \mathbb{R}/\mathbb{Z} \\ x &\mapsto d(x \bmod 1)/2 \end{aligned}$$

We require d to be such that $d_p, \alpha^L, \alpha^R \in \mathrm{Th}$.

Notation. We introduce a notation for diffeomorphisms of S^1 built from D. We choose an identification $r \colon S_+ \simeq S_-$. We represent S^1 by two dots, representing the two copies of S_+. The right dot represents S_+ and the left S_-. In general, n dots represent a circle built by gluing n copies of S_+. And a straight line between two dots represents the identity map between those two copies of S_+. If two lines come out of a point to two copies of S_+, that represents it being doubled by D, and two lines coming into a point represents the sources of those lines being mapped to a single S_+ using D^{-1}. Clearly, doubling (and anti-doubling) can only happen between neighbouring dots. We give some examples:

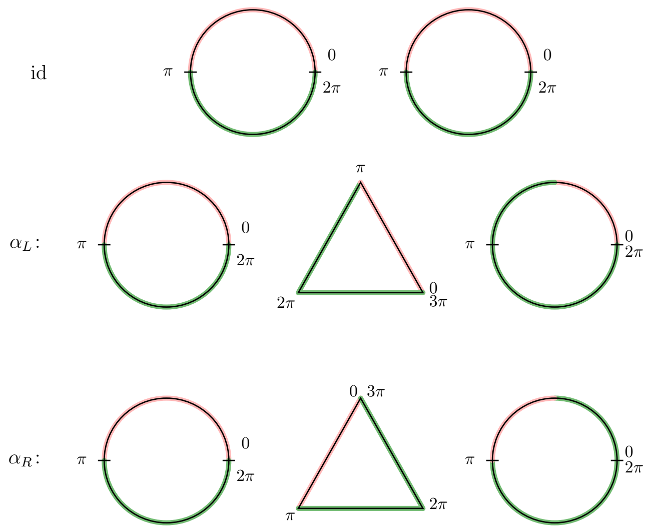
$$\mathrm{id}_{S^1} = \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[d] \\ \bullet & \bullet \end{tikzcd}$$

$$\alpha^L = \begin{tikzcd} & \bullet \arrow[dl] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[dr] & \bullet \arrow[dr] & \bullet \arrow[d] \\ & \bullet & \bullet \end{tikzcd}$$

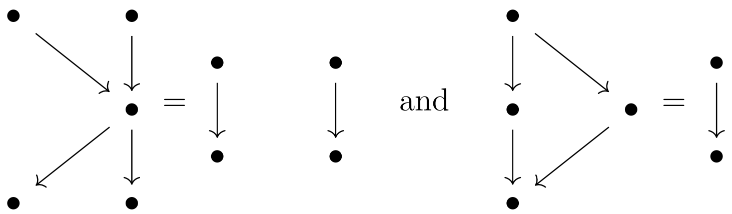


In this representation of α^R, we have merged the first copy of S_+ in the middle row with the last copy. This is possible as the middle three S_+-s constitute a circle, so those two points are actually adjacent. That is, the two blue circles represent the same copy of S_+.

To illustrate what these maps look like on S^1, we also present the following interpretation:



The following two simplifications can be used when computing using these diagrams:



$\square$

Definition 4.8.8. Choose an f as in Construction 4.8.2. We define a map

$[0,1) \to [0,1)$ by the following diagram

$$\begin{tikzcd}
S^1 & \mathbb{R}/\mathbb{Z} \arrow[l] & {[0,1)} \arrow[l] \arrow[d, "d_c^f"] \\
S^1 \arrow[u, leftarrow, "\varphi_f(d_p)"'] & \mathbb{R}/\mathbb{Z} \arrow[l] & {[0,1)} \arrow[l]
\end{tikzcd}$$

Define $d^f\colon [0,1] \to [0,2]$ to be the smooth diffeomorphism given by $d^f(1) = 2$ and for $x \in [0,1)$, $d^f(x) = 2d_c^f(x)$. Finally, define $D^f\colon S_+ \to S_+ \cup S_+$ to be the diffeomorphism induced by d^f under the identification $[0,1] \simeq S_+$.

Definition 4.8.9. Let $\widetilde{\psi}\colon I \to I$ be defined as

$$\widetilde{\psi}(p) = \begin{cases} \phi(p) & \text{if } p \in \mathrm{im}\,(\phi) \\ \phi(D^f(p)) & \text{if } p \in S_+ \text{ and } D^f(p) \in \mathrm{im}(i_2) \\ D^f(p) & \text{if } p \in S_+ \text{ and } D^f(p) \in \mathrm{im}(i_1) \end{cases}$$

Note, $\widetilde{\psi}|_{\mathrm{im}(\phi)} = \phi|_{\mathrm{im}(\phi)}$, therefore, $\widetilde{\psi} \circ \phi = \phi \circ \phi$. Let $\psi \in \mathcal{A}(I)^\times$ be a unitary that implements $\widetilde{\psi}$ (this exists by piecewise smooth inner covariance [DVIT20]), so, $\mathcal{A}(\widetilde{\psi}) = \mathrm{Ad}_\psi$, and thus we have $\mathrm{Ad}_\psi \circ \mathcal{A}(\phi) = \mathcal{A}(\phi) \circ \mathcal{A}(\phi)$. So we can view ψ as an isomorphism $\psi\colon \Omega \to \Omega \otimes \Omega$.

Definition 4.8.10. Note that the identification $r\colon S_+ \simeq S_-$ gives us an isomorphism $\overline{\Omega} \simeq \Omega$. Define $t\colon \langle \overline{\Omega}, \Omega \rangle \xrightarrow{\sim} H_0$ to be the composite

$$\langle \overline{\Omega}, \Omega \rangle \simeq \langle \Omega, \Omega \rangle \xrightarrow{\sim} L^2\mathrm{End}(\Omega) \xrightarrow{\sim} L^2\mathcal{A}(S_+) \xrightarrow[\mathrm{BW}^{-1}]{\sim} H_0$$

Definition 4.8.11. We define the maps $\gamma^L_{\Omega,\Omega,\Omega}, \gamma^R_{\Omega,\Omega,\Omega}\colon \langle \overline{\Omega \otimes \Omega}, \Omega \rangle \to \langle \overline{\Omega}, \Omega \otimes \Omega \rangle$ by the following diagrams

$$\begin{tikzcd}
\langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, dotted, "\gamma^L_{\Omega,\Omega,\Omega}"] \arrow[d, "\langle \overline{\psi^{-1}}, \mathrm{id}_\Omega \rangle"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \\
\langle \overline{\Omega}, \Omega \rangle \arrow[d, "t"'] & \langle \overline{\Omega}, \Omega \rangle \arrow[u, "\langle \overline{\mathrm{id}_\Omega}, \psi \rangle"'] \\
H_0 \arrow[r, "\tilde{u}_f(\alpha^L)"'] & H_0 \arrow[u, "t^{-1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, dotted, "\gamma^R_{\Omega,\Omega,\Omega}"] \arrow[d, "\langle \overline{\psi^{-1}}, \mathrm{id}_\Omega \rangle"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \\
\langle \overline{\Omega}, \Omega \rangle \arrow[d, "t"'] & \langle \overline{\Omega}, \Omega \rangle \arrow[u, "\langle \overline{\mathrm{id}_\Omega}, \psi \rangle"'] \\
H_0 \arrow[r, "\tilde{u}_f(\alpha^R)"'] & H_0 \arrow[u, "t^{-1}"']
\end{tikzcd}$$

Lemma 4.8.12. *The maps $\gamma^L_{\Omega,\Omega,\Omega}$ and $\gamma^R_{\Omega,\Omega,\Omega}$ are* $\mathrm{End}(\Omega) \times \mathrm{End}(\Omega) \times \mathrm{End}(\Omega)$*-linear.*

Proof. We show the statement for $\gamma^L_{\Omega,\Omega,\Omega}$, (the argument for $\gamma^R_{\Omega,\Omega,\Omega}$ is similar). For $a_1, a_2, a_3 \in \mathrm{End}(\Omega)$, we need to show that

$$\begin{tikzcd}
\langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\gamma^L_{\Omega,\Omega,\Omega}"] \arrow[d, "\langle \overline{a_1 \otimes a_2}, a_3 \rangle"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \arrow[d, "\langle \overline{a_1}, a_2 \otimes a_3 \rangle"] \\
\langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\gamma^L_{\Omega,\Omega,\Omega}"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle
\end{tikzcd}$$

Expanding out the definition of $\gamma^L_{\Omega,\Omega,\Omega}$, we have

$$\begin{tikzcd}
\langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[d, "\langle \overline{a_1 \otimes a_2}, a_3 \rangle"'] & \langle \overline{\Omega}, \Omega \rangle \arrow[l, "\langle \overline{\psi}, \mathrm{id}_\Omega \rangle"'] & H_0 \arrow[l, "t^{-1}"'] \arrow[d] & H_0 \arrow[l, "\widetilde{u}_f(\alpha^L)^{-1}"'] \arrow[r, "t^{-1}"] \arrow[d] & \langle \overline{\Omega}, \Omega \rangle \arrow[r, "\langle \overline{\mathrm{id}_\Omega}, \psi \rangle"] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \arrow[d, "\langle \overline{a_1}, a_2 \otimes a_3 \rangle"] \\
\langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\langle \overline{\psi^{-1}}, \mathrm{id}_\Omega \rangle"'] & \langle \overline{\Omega}, \Omega \rangle \arrow[r, "t"'] & H_0 \arrow[r, "\widetilde{u}_f(\alpha^L)"'] & H_0 & \langle \overline{\Omega}, \Omega \rangle \arrow[l, "t"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \arrow[l, "\langle \overline{\mathrm{id}_\Omega}, \psi^{-1} \rangle"']
\end{tikzcd}$$

The red path evaluates to $\mathrm{Ad}_t(\langle \mathrm{Ad}_{\psi^{-1}}(a_1 \otimes a_2), a_3 \rangle)$. Since ψ implements $\tilde{\psi}$ which is defined in terms of D^f, under conjugation by t, we see that this action of $\mathrm{End}(\Omega) \times \mathrm{End}(\Omega) \times \mathrm{End}(\Omega)$ on $\langle \overline{\Omega \otimes \Omega}, \Omega \rangle$ becomes the actions of the three algebras $\mathcal{A}(\overline{(D^f)^{-1}(\mathrm{im}(i_1))})$, $\mathcal{A}(\overline{(D^f)^{-1}(\mathrm{im}(i_2))})$, and $\mathcal{A}(S_+)$ on H_0.

Similarly, the blue path turns the action of $\mathrm{End}(\Omega) \times \mathrm{End}(\Omega) \times \mathrm{End}(\Omega)$ on $\langle \overline{\Omega}, \Omega \otimes \Omega \rangle$ into the actions of the three algebras $\mathcal{A}(S_-)$, $\mathcal{A}((D^f)^{-1}(\mathrm{im}(i_1)))$, and $\mathcal{A}((D^f)^{-1}(\mathrm{im}(i_2)))$ on H_0.

Let $\widetilde{\alpha}^L_f = \varphi_f(\alpha^L)$. It is given by

$$\widetilde{\alpha}^L_f(p) = \begin{cases} ((D^f)^{-1} \circ i_2)(p) & \text{if } p \in S_+ \\ ((D^f)^{-1} \circ i_1 \circ i_2^{-1} \circ D^f \circ \vartheta)(p) & \text{if } p \in S_- \text{ and } D^f(\vartheta(p)) \in \mathrm{im}(i_2) \\ (\vartheta \circ D^f \circ \vartheta)(p) & \text{if } p \in S_- \text{ and } D^f(\vartheta(p)) \in \mathrm{im}(i_1) \end{cases}$$

Note that for any interval $K \subsetneq S^1$, we have

$$\mathrm{Ad}_{\widetilde{u}_f(\alpha^L)}(\mathcal{A}(K)) = \mathrm{Ad}_{u(\varphi_f(\alpha^L))}(\mathcal{A}(K)) = \mathcal{A}(\widetilde{\alpha}^L_f(K))$$

Thus, to show the linearity square commutes, we have to show that under $\widetilde{\alpha}^L_f$, the three intervals

$$\overline{(D^f)^{-1}(\mathrm{im}(i_1))}, \overline{(D^f)^{-1}(\mathrm{im}(i_2))}, S_+ \subsetneq S^1$$

map to the three intervals

$$S_-, (D^f)^{-1}(\mathrm{im}(i_1)), (D^f)^{-1}(\mathrm{im}(i_2)) \subsetneq S^1$$

This is verified easily by the definition of $\widetilde{\alpha}^L_f$. $\square$

Lemma 4.8.13. *The map $\gamma^L_{\Omega,\Omega,\Omega}$ is independent of the choice of f and d up to a phase.*

Proof. We choose another pair of f', d' and obtain the representation $\widetilde{u}_{f'}\colon \mathrm{Th} \to \mathrm{U}(H_0)$ and the maps $\psi', \alpha^{L\prime}, \gamma^{L\prime}_{\Omega,\Omega,\Omega}$, etc as described above. To show $\gamma^{L\prime}_{\Omega,\Omega,\Omega} \propto \gamma^L_{\Omega,\Omega,\Omega}$, we need to show the following diagram commutes up to a phase:

$$\begin{tikzcd}
H_0 \arrow[r, "\widetilde{u}_{f'}(\alpha^{L\prime})"] \arrow[d, "t^{-1}"'] & H_0 \\
\langle \overline{\Omega}, \Omega \rangle \arrow[d, "\langle \overline{\psi'}, \mathrm{id}_\Omega \rangle"'] & \langle \overline{\Omega}, \Omega \rangle \arrow[u, "t"'] \\
\langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[d, "\langle \overline{\psi^{-1}}, \mathrm{id}_\Omega \rangle"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \arrow[u, "\langle \overline{\mathrm{id}_\Omega}, (\psi')^{-1} \rangle"'] \\
\langle \overline{\Omega}, \Omega \rangle \arrow[d, "t"'] & \langle \overline{\Omega}, \Omega \rangle \arrow[u, "\langle \overline{\mathrm{id}_\Omega}, \psi \rangle"'] \\
H_0 \arrow[r, "\widetilde{u}_f(\alpha^L)"] & H_0 \arrow[u, "t^{-1}"']
\end{tikzcd}$$

The left and right vertical compositions represent $u(\widetilde{C}_L), u(\widetilde{C}_R) \in \mathbb{P}\mathrm{U}(H)$ where $\widetilde{C}_L, \widetilde{C}_R\colon S^1 \to S^1$ are the following diffeomorphisms:

$$\widetilde{C}_L(x) = \begin{cases} x & \text{if } x \in S_+ \\ (\vartheta \circ (D^f)^{-1} \circ D'^{f'} \circ \vartheta)(x) & \text{if } x \in S_- \end{cases}$$

$$\widetilde{C}_R(x) = \begin{cases} ((D'^{f'})^{-1} \circ D^f)(x) & \text{if } x \in S_+ \\ x & \text{if } x \in S_- \end{cases}$$

Further, the two horizontal maps represent $u(\widetilde{\alpha}^L_f)$ and $u(\widetilde{\alpha}^{L\prime}_{f'})$. Thus, since being equal up to a phase in $U(H_0)$ is the same as being equal in $\mathbb{P}\mathrm{U}(H_0)$, to show the above diagram commutes up to a phase, it is enough to show the following diagram commutes (only up to a phase):

$$\begin{tikzcd} H_0 \arrow[r, "u(\widetilde{\alpha}^{L\prime})"] \arrow[d, "u(\widetilde{C}_L)"'] & H_0 \\ H_0 \arrow[r, "u(\widetilde{\alpha}^L)"'] & H_0 \arrow[u, "u(\widetilde{C}_R)"'] \end{tikzcd}$$

This commutes because $\widetilde{C}_R \circ \widetilde{\alpha}^L_f \circ \widetilde{C}_L = \widetilde{\alpha}^{L\prime}_{f'}$. □

Definition 4.8.14. For any three absorbing objects $\Omega_1, \Omega_2, \Omega_3$, choose isomorphisms $\phi_i \colon \Omega_i \to \Omega$ $(i = 1, 2, 3)$. Then define $\gamma^L_{\Omega_1,\Omega_2,\Omega_3}$ by the following diagram

$$\begin{tikzcd} \langle \overline{\Omega_1 \otimes \Omega_2}, \Omega_3 \rangle \arrow[r, "\gamma^L_{\Omega_1,\Omega_2,\Omega_3}"] \arrow[d, "\langle \overline{\phi_1 \otimes \phi_2}, \phi_3 \rangle"'] & \langle \overline{\Omega_1}, \Omega_2 \otimes \Omega_3 \rangle \arrow[d, "\langle \overline{\phi_1}, \phi_2 \otimes \phi_3 \rangle"] \\ \langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\gamma^L_{\Omega,\Omega,\Omega}"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \end{tikzcd}$$

This definition is independent of the choices of ϕ_i due to Lemma 4.8.12, and is in fact natural in all three components by the same lemma. We can similarly define $\gamma^R_{\Omega_1,\Omega_2,\Omega_3} \colon \langle \overline{\Omega_1 \otimes \Omega_2}, \Omega_3 \rangle \to \langle \overline{\Omega_2}, \Omega_3 \otimes \Omega_1 \rangle$.

Lemma 4.8.15. *The following square commutes:*

$$\begin{tikzcd} \langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\gamma^R_{\Omega,\Omega,\Omega}"] \arrow[d, "c_{\Omega \otimes \Omega,\Omega}"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \\ \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \arrow[r, "\left(\gamma^L_{\Omega,\Omega,\Omega}\right)^{-1}"'] & \langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[u, "c_{\Omega \otimes \Omega,\Omega}"'] \end{tikzcd}$$

Remark 4.8.16. *This shows Equation 2.25 holds when all three objects are absorbing. It also shows that $\gamma^R_{\Omega,\Omega,\Omega}$ is uniquely determined by $\gamma^L_{\Omega,\Omega,\Omega}$ and is thus natural and does not depend on the choice of f up to a phase.*

Proof. We note that the map $\overline{\langle \overline{\Omega}, \Omega \rangle} \xrightarrow{f_{\overline{\Omega},\Omega}} \langle \overline{\overline{\Omega}}, \overline{\Omega} \rangle \xrightarrow{\langle \varphi_\Omega^{-1}, \mathrm{id}_{\overline{\Omega}} \rangle} \langle \Omega, \overline{\Omega} \rangle$ is just the antilinear identity map $\overline{H_0(S^1)} \to H_0(\overline{S^1})$. Consider the following square:

$$\begin{tikzcd}
\langle \overline{\Omega\otimes\Omega}, \Omega\rangle \arrow[r, "c_{\Omega\otimes\Omega,\Omega}"] \arrow[d, "\langle \overline{\psi^{-1}}{,}\mathrm{id}_\Omega\rangle"'] & \langle \overline{\Omega}, \Omega\otimes\Omega\rangle \\
\langle \overline{\Omega}, \Omega\rangle \arrow[r, "c_{\Omega,\Omega}"'] \arrow[d, "t"'] & \langle \overline{\Omega}, \Omega\rangle \arrow[u, "\langle \overline{\mathrm{id}_\Omega}{,}\psi\rangle"'] \\
H_0 \arrow[r, "H_0(\vartheta)"'] & H_0 \arrow[u, "t^{-1}"']
\end{tikzcd} \tag{4.11}$$

where $H_0(\vartheta)$ is J under the Bisognano-Wichmann map. The top square commutes by naturality of c and the second square is essentially the definition of c and using the fact that $f_{\overline{\Omega},\Omega}$ is identity.

We compute

$$(\alpha^L)^{-1}(p) = \begin{cases} (\vartheta \circ D^{-1} \circ i_1 \circ \vartheta)(p) & \text{if } p \in S_- \\ (\vartheta \circ D^{-1} \circ i_2 \circ i_1^{-1} \circ D)(p) & \text{if } p \in S_+ \text{ and } D(p) \in \mathrm{im}(i_1) \\ D(p) & \text{if } p \in S_+ \text{ and } D(p) \in \mathrm{im}(i_2) \end{cases}$$

Thus, the commuting of the square in the statement of the lemma amounts to checking $\alpha^R = \vartheta \circ (\alpha^L)^{-1} \circ \vartheta$. This is easily verified from the definitions of α^R, α^L, and $(\alpha^L)^{-1}$ above. □

Lemma 4.8.17. *The following diagram commutes:*

$$\begin{tikzcd}
\langle \overline{\Omega\otimes\Omega}, \Omega\rangle \arrow[r, "\gamma^L_{\Omega,\Omega,\Omega}"] \arrow[d, "J_{\overline{\Omega\otimes\Omega},\Omega}"] & \langle \overline{\Omega}, \Omega\otimes\Omega\rangle \arrow[d, "J_{\overline{\Omega},\Omega\otimes\Omega}"] & \\
\overline{\langle \Omega, \overline{\Omega\otimes\Omega}\rangle} \arrow[d, "\overline{\langle \varphi_\Omega{,}\nu_{\Omega,\Omega}{}^{-1}\rangle}"] & \overline{\langle \Omega\otimes\Omega, \overline{\Omega}\rangle} \arrow[r, "\overline{\langle \varphi_{\Omega\otimes\Omega}{,}\mathrm{id}_{\overline{\Omega}}\rangle}"] & \overline{\langle \overline{\overline{\Omega\otimes\Omega}}, \overline{\Omega}\rangle} \arrow[d, "\overline{\langle \nu^{-1}_{\overline{\Omega},\overline{\Omega}}{,}\mathrm{id}_{\overline{\Omega}}\rangle}"] \\
\overline{\langle \overline{\overline{\Omega}}, \overline{\Omega}\otimes\overline{\Omega}\rangle} & & \overline{\langle \overline{\overline{\Omega}\otimes\overline{\Omega}}, \overline{\Omega}\rangle} \arrow[ll, "\overline{\gamma^L_{\overline{\Omega},\overline{\Omega},\overline{\Omega}}}"]
\end{tikzcd}$$

Proof. By Lemma 2.2.28 and noting that $f_{\overline{\Omega},\Omega}$ is the antilinear identity as shown in the proof of Lemma 4.8.15, it suffices to show that $\overline{\gamma^L_{\Omega,\Omega,\Omega}}$ and $\gamma^R_{\Omega,\Omega,\Omega}$ both arise as $\widetilde{u}_f$ of the same diffeomorphism. This is clear from the definition of α^L and α^R, for α^R α^L map to each other under reversing orientation. $\square$

Lemma 4.8.18. *The following square commutes:*

$$\begin{tikzcd}
\langle \overline{\Omega \otimes \Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\gamma^L_{\Omega\otimes\Omega,\Omega,\Omega}"] \arrow[d, "\gamma^R_{\Omega,\Omega\otimes\Omega,\Omega}"'] & \langle \overline{\Omega \otimes \Omega}, \Omega \otimes \Omega \rangle \arrow[d, "\gamma^R_{\Omega,\Omega,\Omega\otimes\Omega}"] \\
\langle \overline{\Omega \otimes \Omega}, \Omega \otimes \Omega \rangle \arrow[r, "\gamma^L_{\Omega,\Omega,\Omega\otimes\Omega}"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \otimes \Omega \rangle
\end{tikzcd}$$

Remark 4.8.19. *This shows Equation 2.24 holds when all four objects are absorbing.*

Proof. We suppress the isomorphism $\langle \overline{\Omega}, \Omega \rangle \simeq H_0$ in the following. By the naturality and the definitions of $\gamma^L_{\Omega_1,\Omega_2,\Omega_3}$ and $\gamma^R_{\Omega_1,\Omega_2,\Omega_3}$ we construct the outer squares on our diagram

$$\begin{tikzcd}
 & \langle \overline{\Omega}, \Omega \rangle \arrow[r, "\widetilde{u}(\alpha^L)"] \arrow[d, red, "\langle \overline{\psi}{,}\mathrm{id}_\Omega \rangle"'] & \langle \overline{\Omega}, \Omega \rangle \arrow[d, blue, "\langle \overline{\mathrm{id}_\Omega}{,}\psi \rangle"] & \\
 & \langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\gamma^L_{\Omega,\Omega,\Omega}"] \arrow[d, red, "\langle \overline{\psi\otimes\mathrm{id}_\Omega}{,}\mathrm{id}_\Omega \rangle"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \arrow[d, blue, "\langle \overline{\psi}{,}\mathrm{id}_\Omega\otimes\mathrm{id}_\Omega \rangle"] & \\
\langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[d, red, "\langle \overline{\psi^{-1}}{,}\mathrm{id}_\Omega \rangle"] \arrow[ddd, bend right=60, "\gamma^R_{\Omega,\Omega,\Omega}"'] & \langle \overline{\Omega \otimes \Omega \otimes \Omega}, \Omega \rangle \arrow[l, red, "\langle \overline{\mathrm{id}_\Omega\otimes\psi^{-1}}{,}\mathrm{id}_\Omega \rangle"'] \arrow[r, "\gamma^L_{\Omega\otimes\Omega,\Omega,\Omega}"] \arrow[ddd, "\gamma^R_{\Omega,\Omega\otimes\Omega,\Omega}"] & \langle \overline{\Omega \otimes \Omega}, \Omega \otimes \Omega \rangle \arrow[r, blue, "\langle \overline{\mathrm{id}_\Omega\otimes\mathrm{id}_\Omega}{,}\psi^{-1} \rangle"] \arrow[ddd, "\gamma^R_{\Omega,\Omega,\Omega\otimes\Omega}"'] & \langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[d, blue, "\langle \overline{\psi^{-1}}{,}\mathrm{id}_\Omega \rangle"'] \arrow[ddd, bend left=60, "\gamma^R_{\Omega,\Omega,\Omega}"] \\
\langle \overline{\Omega}, \Omega \rangle \arrow[d, "\widetilde{u}(\alpha^R)"] & & & \langle \overline{\Omega}, \Omega \rangle \arrow[d, "\widetilde{u}(\alpha^R)"'] \\
\langle \overline{\Omega}, \Omega \rangle \arrow[d, teal, "\langle \overline{\mathrm{id}_\Omega}{,}\psi \rangle"] & & & \langle \overline{\Omega}, \Omega \rangle \arrow[d, magenta, "\langle \overline{\mathrm{id}_\Omega}{,}\psi \rangle"'] \\
\langle \overline{\Omega}, \Omega \otimes \Omega \rangle \arrow[r, teal, "\langle \overline{\psi}{,}\mathrm{id}_\Omega\otimes\mathrm{id}_\Omega \rangle"'] & \langle \overline{\Omega \otimes \Omega}, \Omega \otimes \Omega \rangle \arrow[r, "\gamma^L_{\Omega,\Omega,\Omega\otimes\Omega}"'] \arrow[d, teal, "\langle \overline{\mathrm{id}_\Omega\otimes\mathrm{id}_\Omega}{,}\psi^{-1} \rangle"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \otimes \Omega \rangle \arrow[d, magenta, "\langle \overline{\mathrm{id}_\Omega}{,}\mathrm{id}_\Omega\otimes\psi^{-1} \rangle"] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \arrow[l, magenta, "\langle \overline{\mathrm{id}_\Omega}{,}\psi\otimes\mathrm{id}_\Omega \rangle"'] \\
 & \langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\gamma^L_{\Omega,\Omega,\Omega}"'] \arrow[d, teal, "\langle \overline{\psi^{-1}}{,}\mathrm{id}_\Omega \rangle"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle \arrow[d, magenta, "\langle \overline{\mathrm{id}_\Omega}{,}\psi^{-1} \rangle"] & \\
 & \langle \overline{\Omega}, \Omega \rangle \arrow[r, "\widetilde{u}(\alpha^L)"'] & \langle \overline{\Omega}, \Omega \rangle &
\end{tikzcd}$$

We show the central square commutes by following the coloured path. The blue and teal paths are clearly identity. We represent the composites of the red and magenta paths by $\widetilde{R}$ and $\widetilde{M}$ respectively.

Since ψ implements $\widetilde{\psi}$ which is defined in terms of D^f, we have $\widetilde{R} = \widetilde{u}_f(R)$ and $\widetilde{M} = \widetilde{u}_f(M)$ where R and M are given by

$$R = \begin{tikzcd}
 & & \bullet \arrow[dl, red] \arrow[d, red] & \bullet \arrow[d, red] \\
 & \bullet \arrow[dl, red] \arrow[d, red] & \bullet \arrow[d, red] & \bullet \arrow[d, red] \\
\bullet \arrow[dr, red] & \bullet \arrow[dr, red] & \bullet \arrow[d, red] & \bullet \arrow[d, red] \\
 & \bullet \arrow[dr, red] & \bullet \arrow[d, red] & \bullet \arrow[d, red] \\
 & & \bullet & \bullet
\end{tikzcd} \tag{4.12}$$

$$M = \begin{tikzcd}
\bullet \arrow[d, magenta] & \bullet \arrow[d, magenta] \arrow[dr, magenta] & & \\
\bullet \arrow[d, magenta] & \bullet \arrow[d, magenta] \arrow[dr, magenta] & \bullet \arrow[dr, magenta] & \\
\bullet \arrow[d, magenta] & \bullet \arrow[d, magenta] & \bullet \arrow[d, magenta] & \bullet \arrow[dl, magenta] \\
\bullet \arrow[d, magenta] & \bullet \arrow[d, magenta] & \bullet \arrow[dl, magenta] & \\
\bullet & \bullet & &
\end{tikzcd}$$

Given this, it is clear that

$$\alpha^L \circ \alpha^R \circ R = M \circ \alpha^R \circ \alpha^L$$

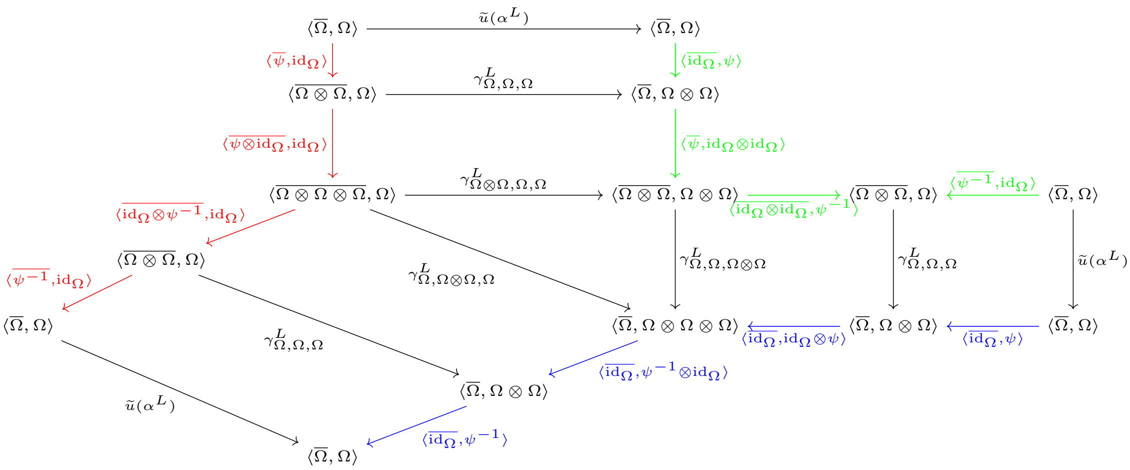
The square we need to commute is just $\widetilde{u}_f$ of this equation, and thus commutes. $\square$

Lemma 4.8.20. *The following triangle commutes:*

$$\begin{tikzcd}
\langle \overline{\Omega \otimes \Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\gamma^L_{\overline{\Omega \otimes \Omega}, \Omega, \Omega}"] \arrow[dr, swap, "\gamma^L_{\overline{\Omega}, \Omega \otimes \Omega, \Omega}"] & \langle \overline{\Omega \otimes \Omega}, \Omega \otimes \Omega \rangle \arrow[d, "\gamma^L_{\overline{\Omega}, \Omega, \Omega \otimes \Omega}"] \\
 & \langle \overline{\Omega}, \Omega \otimes \Omega \otimes \Omega \rangle
\end{tikzcd}$$

Remark 4.8.21. *This shows that Equation 2.20 commutes.*

Proof. We construct the outer squares on each edge of the triangle using the naturality and definition of $\gamma^L_{\Omega_1,\Omega_2,\Omega_3}$

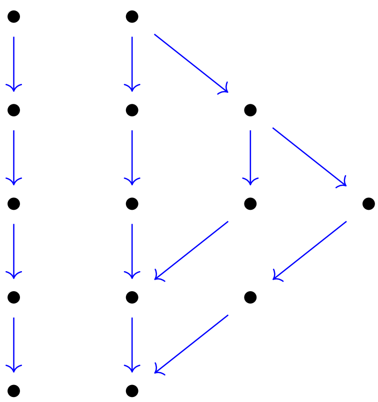


(4.13)

Clearly, the green path composes to identity. Let the composites of the red and blue paths be $\widetilde{R}$ and $\widetilde{B}$. Then the commuting of the central triangle is equivalent to showing

$$\widetilde{B}\circ\widetilde{u}_f(\alpha^L)\circ\widetilde{u}_f(\alpha^L)=\widetilde{u}_f(\alpha^L)\circ\widetilde{R}$$

We have $\widetilde{R}=\widetilde{u}_f(R)$ and $\widetilde{B}=\widetilde{u}_f(B)$ where R is as in Equation 4.12 and B is the diffeomorphism given by



It is now easy to compute that

$$B\circ\alpha^L\circ\alpha^L=\alpha^L\circ R$$

Thus, the given triangle commutes. □

Lemma 4.8.22. $\gamma^L_{\Omega_1,\Omega_2,\Omega_3}$ *is independent of the choice of* f *and* d*.*

Proof. Pick an f, d and f', d' and construct $\gamma^L_{\Omega_1,\Omega_2,\Omega_3}$ and $\gamma^{L'}_{\Omega_1,\Omega_2,\Omega_3}$ as above. From Lemma 4.8.13, we know that $\gamma^L_{\Omega_1,\Omega_2,\Omega_3} = x(\Omega_1,\Omega_2,\Omega_3)\gamma^{L'}_{\Omega_1,\Omega_2,\Omega_3}$ where $x(\Omega_1,\Omega_2,\Omega_3) \in \mathrm{U}(1)$. By naturality of γ^L and $\gamma^{L'}$, for any three isomorphisms $\phi_i \colon \Omega_i \to \Omega$ $(i = 1, 2, 3)$, we have the following diagrams

$$\begin{tikzcd}
\langle \overline{\Omega_1 \otimes \Omega_2}, \Omega_3 \rangle \arrow[r, "\gamma^L_{\Omega_1,\Omega_2,\Omega_3}"] \arrow[d, "\langle \overline{\phi_1 \otimes \phi_2}, \phi_3 \rangle"'] & \langle \overline{\Omega_1}, \Omega_2 \otimes \Omega_3 \rangle \arrow[d, "\langle \overline{\phi_1}, \phi_2 \otimes \phi_3 \rangle"] \\
\langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\gamma^L_{\Omega,\Omega,\Omega}"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle \overline{\Omega_1 \otimes \Omega_2}, \Omega_3 \rangle \arrow[r, "\gamma^{L'}_{\Omega_1,\Omega_2,\Omega_3}"] \arrow[d, "\langle \overline{\phi_1 \otimes \phi_2}, \phi_3 \rangle"'] & \langle \overline{\Omega_1}, \Omega_2 \otimes \Omega_3 \rangle \arrow[d, "\langle \overline{\phi_1}, \phi_2 \otimes \phi_3 \rangle"] \\
\langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\gamma^{L'}_{\Omega,\Omega,\Omega}"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle
\end{tikzcd}$$

Using these equations and the fact that $\gamma^L_{\Omega_1,\Omega_2,\Omega_3} = x(\Omega_1,\Omega_2,\Omega_3)\gamma^{L'}_{\Omega_1,\Omega_2,\Omega_3}$ and $\gamma^L_{\Omega,\Omega,\Omega} = x(\Omega,\Omega,\Omega)\gamma^{L'}_{\Omega,\Omega,\Omega}$, we see that $x(\Omega_1,\Omega_2,\Omega_3) = x(\Omega,\Omega,\Omega)$.

Now, since both γ^L and $\gamma^{L'}$ satisfy Lemma 4.8.20, we have that $x(\Omega,\Omega,\Omega)^2 = x(\Omega,\Omega,\Omega)$, and thus, $x(\Omega,\Omega,\Omega) = 1$. □

Lemma 4.8.23. *The following diagram commutes:*

$$\begin{tikzcd}
\langle \overline{\Omega}, \Omega \otimes \Omega \rangle \arrow[d, "c_{\Omega,\Omega\otimes\Omega}"'] & \langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[l, "\gamma^R_{\Omega,\Omega,\Omega}"'] \arrow[d, "\gamma^L_{\Omega,\Omega,\Omega}"] \\
\langle \overline{\Omega \otimes \Omega}, \Omega \rangle \arrow[r, "\gamma^R_{\Omega,\Omega,\Omega}"'] & \langle \overline{\Omega}, \Omega \otimes \Omega \rangle
\end{tikzcd}$$

Remark 4.8.24. *This shows that Equation 2.26 commutes.*

Proof. On substituting the definitions of $\gamma^L_{\Omega,\Omega,\Omega}$ and $\gamma^R_{\Omega,\Omega,\Omega}$ from Definition 4.8.11 and using Equation 4.11, the diagram in the statement of the lemma reduces to the square

$$\begin{tikzcd}
H_0 \arrow[d, "H_0(\vartheta)"'] & H_0 \arrow[l, "\widetilde{u}_f(\alpha^R)"'] \arrow[d, "\widetilde{u}_f(\alpha^L)"] \\
H_0 \arrow[r, "\widetilde{u}_f(\alpha^R)"'] & H_0
\end{tikzcd}$$

It is easy to compute that $\alpha^L = \alpha^R \circ \vartheta \circ \alpha^R$, thus the above square commutes (note we have suppressed the anti-linear identity map f which appears in this computation, see the proof of Lemma 4.8.15). □

Lemma 4.8.25. *Let X be a non-zero-divisor. The map*

$$\widetilde{\gamma'}_{\Omega,\Omega,X,\Omega}\colon \langle\overline{\Omega\otimes\Omega\otimes X},\Omega\rangle \xrightarrow{\tilde{\gamma}^L_{\Omega,\Omega\otimes X,\Omega}} \langle\overline{\Omega},\Omega\otimes X\otimes\Omega\rangle \xrightarrow{(\tilde{\gamma}^L_{\Omega,\Omega,X\otimes\Omega})^{-1}} \langle\overline{\Omega\otimes\Omega},X\otimes\Omega\rangle$$

is $\mathrm{End}(\Omega\otimes\Omega)\times\mathrm{End}(X)\times\mathrm{End}(\Omega)$*-linear.*

Proof. Since X is not a zero-divisor, $X\otimes\Omega$ is an absorbing object. Given $f\in\mathrm{End}(X)$, using the naturality of $\tilde{\gamma}^L_{\Omega,\Omega\otimes X,\Omega}$ with respect to $\mathrm{id}_\Omega\otimes f$ in the second component and the naturality of $\tilde{\gamma}^L_{\Omega,\Omega,X\otimes\Omega}$ with respect to $f\otimes\mathrm{id}_\Omega$ in the third component, we see the given map is $\mathrm{End}(X)$-linear. We can see it is $\mathrm{End}(\Omega)$-linear similarly.

Consider the diagram

$$\begin{tikzcd}
& \langle\overline{\Omega},\Omega\otimes\Omega\otimes X\otimes\Omega\rangle \arrow[dr, "(\tilde{\gamma}^L_{\Omega,\Omega\otimes\Omega,X\otimes\Omega})^{-1}"] & \\
\langle\overline{\Omega\otimes\Omega\otimes\Omega\otimes X},\Omega\rangle \arrow[ur, "\tilde{\gamma}_{\Omega,\Omega\otimes\Omega\otimes X,\Omega}"] \arrow[dr, "\tilde{\gamma}_{\Omega\otimes\Omega,\Omega\otimes X,\Omega}"'] & & \langle\overline{\Omega\otimes\Omega\otimes\Omega},X\otimes\Omega\rangle \\
& \langle\overline{\Omega\otimes\Omega},\Omega\otimes X\otimes\Omega\rangle \arrow[uu, "\tilde{\gamma}^L_{\Omega,\Omega,\Omega\otimes X\otimes\Omega}"'] \arrow[ur, "(\tilde{\gamma}^L_{\Omega\otimes\Omega,\Omega,X\otimes\Omega})^{-1}"'] &
\end{tikzcd}$$

Both triangles of this diagram commute by Lemma 4.8.20 and thus the entire diagram commutes.

By naturality of $\tilde{\gamma}^L_{\Omega_1,\Omega_2,\Omega_3}$, we see that the top path of this diagram is the conjugation of $\widetilde{\gamma'}_{\Omega,\Omega,X,\Omega}$ by $\langle\overline{\mathrm{id}_\Omega\otimes\psi\otimes\mathrm{id}_X},\mathrm{id}_\Omega\rangle$. Thus to show the $\mathrm{End}(\Omega\otimes\Omega)$-linearity of $\widetilde{\gamma'}_{\Omega,\Omega,X,\Omega}$ it suffices to show the $\mathrm{End}(\Omega\otimes\Omega\otimes\Omega)$-linearity of the top path of the diagram. The top path is manifestly $\mathrm{End}(\Omega)\times\mathrm{End}(\Omega\otimes\Omega\otimes\Omega)$-linear and the bottom path is $\mathrm{End}(\Omega\otimes\Omega\otimes\Omega)\times\mathrm{End}(\Omega)$-linear. Taking $\Omega=\mathcal{A}(\phi)$, we see that we have $\mathcal{A}(S_+\cup S_+)\times\mathcal{A}(S_+)$ and $\mathcal{A}(S_+)\times\mathcal{A}(S_+\cup S_+)$ linearity, so by additivity of the conformal net $\mathcal{A}$, this extends to $\mathcal{A}(S_+\cup S_+\cup S_+)$-linearity, which is what we need. $\square$

Definition 4.8.26. Let A_1, A_2 be objects of $\mathcal{T}^+_{\mathcal{A}}(I)^{\mathrm{abs}}$ and X be an object of $\mathcal{T}^+_{\mathcal{A}}(I)$. Choose maps $\iota_i\colon A_i\to\Omega$ and $p_i\colon\Omega\to A_i$ $(i=1,2)$ such that $p_i\circ\iota_i=\mathrm{id}_{A_i}$. Choose an isomorphism $\psi'\colon\Omega\to\Omega\otimes\Omega$ (this does not have to be the same as the previously chosen ψ). We define $\widetilde{\gamma}^L_{A_1,X,A_2}$ by the diagram

$$\begin{tikzcd}
\langle \overline{A_1 \otimes X}, A_2 \rangle \arrow[r, "\widetilde{\gamma}^L_{A_1,X,A_2}"] \arrow[d, "\langle \overline{\iota_1 \otimes \mathrm{id}_X}, \iota_2 \rangle"'] & \langle \overline{A_1}, X \otimes A_2 \rangle \\
\langle \overline{\Omega \otimes X}, \Omega \rangle \arrow[d, "\langle \overline{\psi' \otimes \mathrm{id}_X}, \mathrm{id}_\Omega \rangle"'] & \langle \overline{\Omega}, X \otimes \Omega \rangle \arrow[u, "\langle \overline{p_1}, \mathrm{id}_X \otimes p_2 \rangle"'] \\
\langle \overline{\Omega \otimes \Omega \otimes X}, \Omega \rangle \arrow[r, "\widetilde{\gamma}'_{\Omega,\Omega,X,\Omega}"'] & \langle \overline{\Omega \otimes \Omega}, X \otimes \Omega \rangle \arrow[u, "\langle \overline{(\psi')^{-1}}, \mathrm{id}_X \otimes \mathrm{id}_\Omega \rangle"']
\end{tikzcd}$$

Remark 4.8.27. *This does not depend on the choice of ψ' and the choices of ι_i, p_i by Lemma 4.8.25. It is also natural in all components. Further, the fact that it satisfies all the coherences of a bicommutant category follows from coherences of $\widetilde{\gamma}^L_{\Omega_1,\Omega_2,\Omega_3}$ which we have already shown, its naturality (also shown above), along with Remark 2.2.27.*

4.9 The Bicommutant Property

Definition 4.9.1. Let I be a multi-interval. Let $K \subsetneq I$ be a left-localised sub-multi-interval of I and $J \subsetneq I$ be a right-localised sub-multi-interval of I such that $J \cap K = \emptyset$. Define the functor $E\colon \mathcal{T}^-_{\mathcal{A}}(I) \to \mathcal{Z}(\mathcal{T}^+_{\mathcal{A}}(I) \hookrightarrow \underline{\mathrm{End}}(\mathcal{A}(I)))$ as the composite

$$\mathcal{T}^-_{\mathcal{A}}(I) \xleftarrow{\sim} \mathcal{T}^-_{\mathcal{A}}(J) \xrightarrow{E_{J,K}} \mathcal{Z}(\mathcal{T}^+_{\mathcal{A}}(K) \xrightarrow{\sim} \mathcal{T}^+_{\mathcal{A}}(I) \hookrightarrow \underline{\mathrm{End}}(\mathcal{A}(I))) \xleftarrow{\sim} \mathcal{Z}(\mathcal{T}^+_{\mathcal{A}}(I) \hookrightarrow \underline{\mathrm{End}}(\mathcal{A}(I)))$$

where $E_{J,K}$ is defined on

- objects by mapping $\rho \in \mathcal{T}^-_{\mathcal{A}}(J)$ to $(\rho^{\mathrm{ext}}, e_{\rho^{\mathrm{ext}},\tau^{\mathrm{ext}}} = 1)$ where ρ^{ext} means extending ρ to an endomorphism of $\mathcal{A}(I)$.

  This works as given any $\tau \in \mathcal{T}^+_{\mathcal{A}}(K)$ we have $\rho^{\mathrm{ext}} \otimes \tau^{\mathrm{ext}} = \tau^{\mathrm{ext}} \otimes \rho^{\mathrm{ext}}$. To see this, it is enough to check for $x \in \mathcal{A}(I \setminus^c J)$ and $x \in \mathcal{A}(I \setminus^c K)$ as these generate $\mathcal{A}(I)$. For $x \in \mathcal{A}(I \setminus^c K)$ we have $\tau^{\mathrm{ext}}(x) = x$ and $\rho^{\mathrm{ext}}(x) \in \mathcal{A}(I \setminus^c K)$ and thus $\tau^{\mathrm{ext}}(\rho^{\mathrm{ext}}(x)) = \rho^{\mathrm{ext}}(x)$, so

  $$(\rho^{\mathrm{ext}} \otimes \tau^{\mathrm{ext}})(x) = \tau^{\mathrm{ext}}(\rho^{\mathrm{ext}}(x)) = \rho^{\mathrm{ext}}(x) = \rho^{\mathrm{ext}}(\tau^{\mathrm{ext}}(x)) = (\tau^{\mathrm{ext}} \otimes \rho^{\mathrm{ext}})(x)$$

We can perform a similar computation for $x \in \mathcal{A}(I \setminus^c J)$.

- morphisms by mapping a morphism $a\colon \rho \to \rho'$ in $\mathcal{T}_{\mathcal{A}}^-(J)$ to the following map
$$a\colon (\rho^{\text{ext}}, e_{\rho^{\text{ext}},\mathcal{T}^{\text{ext}}} = 1) \to (\rho'^{\text{ext}}, e_{\rho'^{\text{ext}},\mathcal{T}^{\text{ext}}} = 1)$$
All the conditions to be a morphism in the relative centre are trivially satisfied as the half-braidings are identity.

Lemma 4.9.2. *E as defined in Definition 4.9.1 is independent of the choices of J, K.*

Proof. For any two left-localised sub-multi-intervals $K, K' \subsetneq I$, we have a canonical equivalence given by the composite $\mathcal{T}_{\mathcal{A}}^+(K) \xrightarrow{\sim} \mathcal{T}_{\mathcal{A}}^+(K \cup K') \xleftarrow{\sim} \mathcal{T}_{\mathcal{A}}^+(K')$. Similarly for any two right-localised sub-multi-intervals $J, J' \subsetneq I$ we have a canonical equivalence $\mathcal{T}_{\mathcal{A}}^-(J) \xrightarrow{\sim} \mathcal{T}_{\mathcal{A}}^-(J')$. Consider the following diagram

$$\begin{tikzcd}
\mathcal{T}_{\mathcal{A}}^-(I) & \mathcal{T}_{\mathcal{A}}^-(J) \arrow[l, "\sim"'] \arrow[r, "E_{J,K}"] \arrow[d, "\sim"'] & \mathcal{Z}(\mathcal{T}_{\mathcal{A}}^+(K) \xrightarrow{\sim} \mathcal{T}_{\mathcal{A}}^+(I) \hookrightarrow \underline{\text{End}}(\mathcal{A}(I))) \arrow[d, "\sim"] & \mathcal{Z}(\mathcal{T}_{\mathcal{A}}^+(I) \hookrightarrow \underline{\text{End}}(\mathcal{A}(I))) \arrow[l, "\sim"] \arrow[dl, "\sim"] \\
 & \mathcal{T}_{\mathcal{A}}^-(J') \arrow[ul, "\sim"] \arrow[r, "E_{J',K'}"] & \mathcal{Z}(\mathcal{T}_{\mathcal{A}}^+(K') \xrightarrow{\sim} \mathcal{T}_{\mathcal{A}}^+(I) \hookrightarrow \underline{\text{End}}(\mathcal{A}(I))) &
\end{tikzcd}$$

The central square clearly commutes strictly by the definition of $E_{J,K}$ while the two outer triangles commute strictly as extensions are unique. However, the top and bottom paths are precisely E defined using distinct intervals, so the definition of E does not depend on the choices of J and K. $\square$

Theorem 4.9.3. *E is an equivalence.*

Proof. It is enough to show that $E_{J,K}$ is an equivalence for a particular choice of J, K. We construct left and right inverses for it.

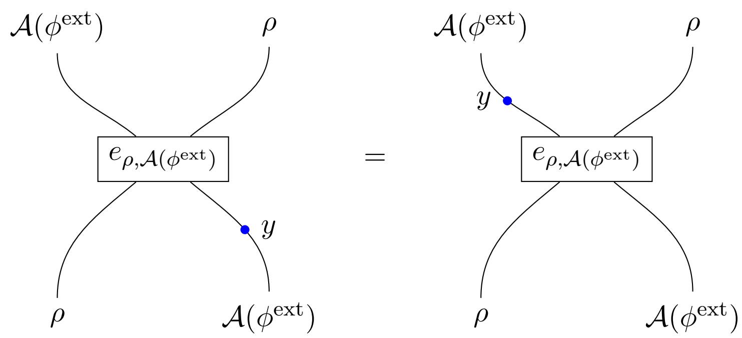
We pick a non-surjective embedding $\phi\colon K \to K$ which is identity in a neighbourhood of $\partial_+ K$. We denote by $\phi^{\text{ext}}\colon I \to I$ the map which is ϕ on $K \subsetneq I$ and $\text{id}_{I \setminus^c K}$ on $I \setminus^c K$. Clearly then $\mathcal{A}(\phi)^{\text{ext}} = \mathcal{A}(\phi^{\text{ext}})$. By Lemma 3.3.17, $\mathcal{A}(\phi)$ is an absorbing object in $\mathcal{T}_{\mathcal{A}}^+(K)$. Let $\mathcal{T}^a$ be the full subcategory of $\mathcal{T}_{\mathcal{A}}^+(K)$ on the single object $\mathcal{A}(\phi)$. We define the following forgetful functor as

$$\begin{aligned} U\colon \mathcal{Z}(\mathcal{T}_{\mathcal{A}}^+(K) \xrightarrow{\sim} \mathcal{T}_{\mathcal{A}}^+(I) \hookrightarrow \underline{\text{End}}(\mathcal{A}(I))) &\to \mathcal{Z}(\mathcal{T}^a \hookrightarrow \mathcal{T}_{\mathcal{A}}^+(K) \xrightarrow{\sim} \mathcal{T}_{\mathcal{A}}^+(I) \hookrightarrow \underline{\text{End}}(\mathcal{A}(I))) \\ (\rho, e_{\rho,\mathcal{T}^{\text{ext}}}) &\mapsto (\rho, e_{\rho,\mathcal{A}(\phi)}) \end{aligned}$$

and it does nothing on morphisms. By Proposition 2.2.24, U is fully faithful (Proposition 2.2.24 applies as $\mathcal{T}^a$ generates $\mathcal{T}^{\text{abs}}$ and it is enough to specify linearity data for the action of a generator).

Now we define a functor $E_2 \colon \mathcal{Z}(\mathcal{T}^a \hookrightarrow \mathcal{T}^+_{\mathcal{A}}(K) \xrightarrow{\sim} \mathcal{T}^+_{\mathcal{A}}(I) \hookrightarrow \underline{\mathrm{End}}(\mathcal{A}(I))) \to \mathcal{T}^-_{\mathcal{A}}(I)$ on

- objects by mapping $(\rho, e_{\rho,\mathcal{A}(\phi)})$ to $E_2(\rho, e_{\rho,\mathcal{A}(\phi)}) = \mathrm{Ad}_{e^*_{\rho,\mathcal{A}(\phi)}} \circ \rho \colon \mathcal{A}(I) \to \mathcal{A}(I)$. We check this is an object of $\mathcal{T}^-_{\mathcal{A}}(I)$. We first show that $E_2(\rho)$ is identity on $\mathcal{A}(I \setminus^c \mathrm{im}(\phi^{\text{ext}}))$. Let $y \in \mathcal{A}(I \setminus^c \mathrm{im}(\phi^{\text{ext}})) = \mathcal{A}(K \setminus^c \mathrm{im}(\phi))$, then by Lemma 3.3.17, $y \colon \mathcal{A}(\phi) \to \mathcal{A}(\phi)$ is a morphism in $\mathcal{T}^+_{\mathcal{A}}(K)$. The naturality of $e_{\rho,\mathcal{A}(\phi)}$ says



That is,

$$e_{\rho,\mathcal{A}(\phi^{\text{ext}})} y = \rho\,(y)\, e_{\rho,\mathcal{A}(\phi^{\text{ext}})} \implies \mathrm{Ad}_{e^*_{\rho,\mathcal{A}(\phi^{\text{ext}})}}\,(\rho\,(y)) = y$$

Next, we show $E_2(\rho)(\mathcal{A}(\mathrm{im}(\phi^{\text{ext}}))) \subseteq \mathcal{A}(\mathrm{im}(\phi^{\text{ext}}))$. Since $e_{\rho,\mathcal{A}(\phi)} \colon \rho \otimes \mathcal{A}(\phi^{\text{ext}}) \to \mathcal{A}(\phi^{\text{ext}}) \otimes \rho$ is a morphism in $\underline{\mathrm{End}}(\mathcal{A}(I))$, for all $x \in \mathcal{A}(I)$ we have

$$e_{\rho,\mathcal{A}(\phi)} \mathcal{A}(\phi^{\text{ext}})(\rho(x)) = \rho(\mathcal{A}(\phi^{\text{ext}})(x)) e_{\rho,\mathcal{A}(\phi)} \implies E_2(\rho)(\mathcal{A}(\phi^{\text{ext}})(x)) = \mathcal{A}(\phi^{\text{ext}})(\rho(x))$$

Therefore $E_2(\rho)$ is an object of $\mathcal{T}^-_{\mathcal{A}}(I)$.

- morphisms by mapping a morphism $a \colon (\rho, e_{\rho,\mathcal{A}(\phi)}) \to (\rho', e'_{\rho',\mathcal{A}(\phi)})$ to $e'^*_{\rho',\mathcal{A}(\phi)} a e_{\rho,\mathcal{A}(\phi)}$. We first check that $E_2(a)$ is a right-localised element of $\mathcal{A}(I)$. The following square commutes since a is morphism in a relative centre

$$\begin{tikzcd}
\rho \otimes \mathcal{A}(\phi) \arrow[r, "e"] \arrow[d, "a\otimes 1"] & \mathcal{A}(\phi) \otimes \rho \arrow[d, "1\otimes a"] \\
\rho' \otimes \mathcal{A}(\phi) \arrow[r, "e'"] & \mathcal{A}(\phi) \otimes \rho'
\end{tikzcd}$$

Thus, $ae = e'\mathcal{A}(\phi)(a)$. So $E_2(a) = \mathcal{A}(\phi)(a) \in \mathcal{A}(\mathrm{im}(\phi))$, and therefore, $E_2(a)$ is right-localised. Next we check that $E_2(a)$ is a morphism by computing

$$\begin{aligned}
E_2(a)E_2(\rho, e_{\rho,\mathcal{A}(\phi)})(x) &= e'^*_{\rho',\mathcal{A}(\phi)} a e_{\rho,\mathcal{A}(\phi)} e^*_{\rho,\mathcal{A}(\phi)} \rho(x) e_{\rho,\mathcal{A}(\phi)} \\
&= e'^*_{\rho',\mathcal{A}(\phi)} a \rho(x) e_{\rho,\mathcal{A}(\phi)} \\
&= e'^*_{\rho',\mathcal{A}(\phi)} \rho'(x) a e_{\rho,\mathcal{A}(\phi)} \\
&= e'^*_{\rho',\mathcal{A}(\phi)} \rho'(x) e'_{\rho',\mathcal{A}(\phi)} e'^*_{\rho',\mathcal{A}(\phi)} a e_{\rho,\mathcal{A}(\phi)} \\
&= E_2(\rho', e'_{\rho',\mathcal{A}(\phi)})(x) E_2(a)
\end{aligned}$$

We define one last functor. Let $G\colon \mathcal{T}^-_{\mathcal{A}}(I) \to \mathcal{T}^-_{\mathcal{A}}(J)$ be an inverse to $\mathcal{T}^-_{\mathcal{A}}(J) \xrightarrow{\sim} \mathcal{T}^-_{\mathcal{A}}(I)$. It comes with natural isomorphisms $\eta_\rho\colon \rho \to G(\rho^{\mathrm{ext}})$ for $\rho \in \mathcal{T}^-_{\mathcal{A}}(J)$ and $\epsilon_\sigma\colon G(\sigma)^{\mathrm{ext}} \to \sigma$ for $\sigma \in \mathcal{T}^-_{\mathcal{A}}(I)$. Therefore, we have

$$\epsilon_\sigma G(\sigma)^{\mathrm{ext}}(x) = \sigma(x)\epsilon_\sigma \text{ for } x \in \mathcal{A}(I) \implies G(\sigma)(x) = \epsilon^*_\sigma \sigma(x) \epsilon_\sigma \text{ for } x \in \mathcal{A}(J)$$

So $G(\sigma) = \mathrm{Ad}_{\epsilon^*_\sigma} \circ \sigma$.

It is easy to see that $E_2 \circ U \circ E_{J,K}\colon \mathcal{T}^-_{\mathcal{A}}(J) \to \mathcal{T}^-_{\mathcal{A}}(I)$ is just the canonical extension functor $\mathcal{T}^-_{\mathcal{A}}(J) \xrightarrow{\sim} \mathcal{T}^-_{\mathcal{A}}(I)$. Therefore, $G \circ E_2 \circ U \circ E_{J,K} \simeq \mathrm{id}_{\mathcal{T}^-_{\mathcal{A}}(J)}$. So $E_{J,K}$ has a left-inverse.

We compute

$$(U \circ E_{J,K} \circ G \circ E_2)(\rho, e_{\rho,\mathcal{A}(\phi)}) = (\mathrm{Ad}_{\epsilon^*_{E_2(\rho, e_{\rho,\mathcal{A}(\phi)})}} \circ \mathrm{Ad}_{e^*_{\rho,\mathcal{A}(\phi)}} \circ \rho, 1)$$

$\epsilon^*_{E_2(\rho,e_{\rho,\mathcal{A}(\phi)})} e^*_{\rho,\mathcal{A}(\phi)}$ supplies a natural isomorphism between $(U \circ E_{J,K} \circ G \circ E_2)(\rho, e_{\rho,\mathcal{A}(\phi)})$ and $(\rho, e_{\rho,\mathcal{A}(\phi)})$. So we have $U \circ E_{J,K} \circ G \circ E_2 \simeq \mathrm{id}$. We compose with U on the right, obtaining an equivalence $U \circ E_{J,K} \circ G \circ E_2 \circ U \simeq U$. Since U is fully faithful, it is left-cancellable, and we obtain an equivalence $E_{J,K} \circ G \circ E_2 \circ U \simeq \mathrm{id}$. So $E_{J,K}$ has a right-inverse.

Therefore, $E_{J,K}$ is an equivalence. $\square$

Corollary 4.9.4. *The Drinfeld centre of* $\mathcal{T}_{\mathcal{A}}^{+}(I)$ *is* $\mathrm{DHR}_{\mathcal{A}}(I)$.

Proof. The Drinfeld centre of $\mathcal{T}_{\mathcal{A}}^{+}(I)$ is the commutant of $\mathcal{T}_{\mathcal{A}}^{+}(I)$ in itself, which is just the intersection of the commutant of $\mathcal{T}_{\mathcal{A}}^{+}(I)$ in $\mathrm{End}(\mathcal{A}(I))$ with $\mathcal{T}_{\mathcal{A}}^{+}(I)$. This is thus the intersection of $\mathcal{T}_{\mathcal{A}}^{+}(I)$ and $\mathcal{T}_{\mathcal{A}}^{-}(I)$, and thus consists of endomorphisms which are identity near both boundaries. $\square$

Corollary 4.9.5. *The Drinfeld centre of* $\mathrm{Sol}(\mathcal{A})$ *is* $\mathrm{Rep}(\mathcal{A})$.

Remark 4.9.6. *This gives* $\mathrm{Rep}(\mathcal{A})$ *the structure of a braided monoidal category. This braiding is actually the same as the braiding on* $\mathrm{DHR}_{\mathcal{A}}(I)$ *which comes from restricting* E *to* $\mathrm{DHR}_{\mathcal{A}}(I)$.

Theorem 4.9.7. *Let* I *be a multi-interval.* $\mathcal{T}_{\mathcal{A}}^{+}(I)^{\cup 0}$ *is a bicommutant category.*

Proof. We verified all the coherences of the action of $\mathcal{T}_{\mathcal{A}}^{+}(I)$ on $\mathcal{T}_{\mathcal{A}}^{+}(I)^{\mathrm{abs}}$ in the previous section. Now we show the bicommutant property.

Choose a non-surjective embedding $\phi\colon I \to I$ which is identity on a neighbourhood of $\partial_{+}I$. Then $\Omega := \mathcal{A}(\phi)$ is an absorbing object of $\mathcal{T}_{\mathcal{A}}^{+}(I)$. We have the following commuting diagram:

$$\begin{tikzcd}
\mathcal{T}_{\mathcal{A}}^{+}(I) \arrow[r, "\rho\mapsto\rho\otimes-"] & \mathrm{Hom}_{\mathrm{completeW^*Cat}}(\mathcal{T}_{\mathcal{A}}^{+}(I)^{\mathrm{abs}}, \mathcal{T}_{\mathcal{A}}^{+}(I)^{\mathrm{abs}}) \arrow[d, "1"', "\simeq"] \\
 & \mathrm{Hom}_{\mathrm{W^*Cat}}(B\mathrm{End}(\Omega), \mathcal{T}_{\mathcal{A}}^{+}(I)^{\mathrm{abs}}) \\
 & \mathrm{Hom}_{\mathrm{W^*Cat}}(B\mathrm{End}(\Omega), B\mathrm{End}(\Omega))^{\cup 0} \arrow[u, "2"', "\simeq"] \arrow[d, "3"', "\simeq"] \\
 & \underline{\mathrm{End}}(\mathrm{End}(\Omega))^{\cup 0} \arrow[d, "4"', "\simeq"] \\
 & \mathrm{Bim}(\mathrm{End}(\Omega)) \arrow[d, "5"', "\simeq"] \\
\mathcal{T}_{\mathcal{A}}^{+}(I \setminus^{c} \mathrm{im}\,\phi) \arrow[r, "\rho\mapsto(X\mapsto{}^{\rho}X)"'] \arrow[uuuuu, "\simeq", "6"'] & \mathrm{Bim}(\mathcal{A}(I \setminus^{c} \mathrm{im}\,\phi))
\end{tikzcd}$$

These equivalences are

- ① uses the fact that $\mathcal{T}_{\mathcal{A}}^{+}(I)^{\mathrm{abs}}$ is generated by Ω.

- ② follows from the fact that $\mathrm{End}(\Omega)$ is a type III factor so $\mathcal{T}_{\mathcal{A}}^{+}(I)^{\mathrm{abs}} = (B\mathrm{End}(\Omega))^{\cup 0}$. Then any functor from $B\mathrm{End}(\Omega)$ would fall in the zero or non-zero piece.

- ③ is just by definition.

- ④ follows from Proposition 3.1.6.

- ⑤ follows from Lemma 3.3.17.

- ⑥ follows from Theorem 3.3.19.

From the above diagram, we see that verifying the bicommutant property for $\mathcal{T}_{\mathcal{A}}^{+}(I)$ follows from showing that, for any interval J, the double commutant of $\mathcal{T}_{\mathcal{A}}^{+}(J)$ in $\mathrm{Bim}(\mathcal{A}(J))$ is equivalent to $\mathcal{T}_{\mathcal{A}}^{+}(J)$. This follows from Theorem 4.9.3, as the commutant of $\mathcal{T}_{\mathcal{A}}^{+}(J)$ is $\mathcal{T}_{\mathcal{A}}^{-}(J)$, whose commutant is $\mathcal{T}_{\mathcal{A}}^{+}(J)$ again by a dual argument. □

Corollary 4.9.8. $\mathrm{Sol}(\mathcal{A})$ *is a bicommutant category.*

Remark 4.9.9. *We can similarly prove that* $\mathcal{S}_{\mathcal{A}}(n, \emptyset)$ *is a bicommutant category.*

Chapter 5

Assignment to 0- and 1-manifolds

In this chapter we assemble the results of Chapters 2–4 into the bottom two layers, that is, the values on points and on 1-manifolds of a fully-local, or fully-extended, functorial field theory associated to a chiral CFT. Fix a conformal net $\mathcal{A}$. To a 0-manifold equipped with a 1-dimensional collar we assign the category of solitonic representations of $\mathcal{A}$, which is a bicommutant category by Corollary 4.9.8; to a 1-dimensional cobordism I we assign the W*-category $\mathcal{A}(I)$-Mod, viewed as a bimodule category over the bicommutant categories attached to its incoming and outgoing boundaries.

The main result of the chapter is that this assignment is compatible with composition: given two gluable cobordisms, the bimodule associated to their gluing is equivalent to the categorified Connes fusion of the bimodules associated to the two gluable cobordisms, where the categorified Connes fusion is over the bicommutant category assigned to the boundary which is being glued over. We prove this unconditionally when every connected component of I_1 and I_2 meets the gluing boundary; the reduction of the general case to this one additionally assumes Conjecture 2.4.13, the associativity of categorified Connes fusion. Specialising to two half-circles glued into a circle, the procedure assigns to S^1 the category $\mathrm{Rep}(\mathcal{A})^{\mathrm{abs}} \simeq \mathcal{Z}(\mathrm{Sol}(\mathcal{A}))^{\mathrm{abs}}$, up to passing to the absorbing ideal, the Drinfeld centre of the value on the point, as expected of a fully-extended theory.

In the finite-index case $\mathrm{Rep}(\mathcal{A})^{\mathrm{abs}} \simeq \mathrm{Rep}(\mathcal{A})$, recovering the value that a non-fully-extended chiral CFT assigns to the circle.

5.1 Scope of the Construction

Given a $(2,1)$-category $\mathcal{C}$ and a 3-category $\mathcal{D}$, the data of a functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of the following:

- for every object $X \in \mathcal{C}$, an object $F(X) \in \mathcal{D}$
- for every 1-morphism $f\colon X \to Y$ in $\mathcal{C}$, a 1-morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$
- for every 2-morphism $\phi\colon f \to g$ in $\mathcal{C}$, a 2-morphism $F(\phi)\colon F(f) \to F(g)$ in $\mathcal{D}$
- for a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$, an invertible 2-morphism $\mu_{f,g}\colon F(g \circ f) \to F(g) \circ F(f)$ which is natural in f, g
- for every object X, an invertible 2-morphism $\Lambda_X\colon F(\mathrm{id}_X) \to \mathrm{id}_{F(X)}$ which is natural in X.
- a collection of 3-morphisms, for example, a 3-morphism $A_{f,g,h}$ of the type

$$\begin{tikzcd}
F((h \circ g) \circ f) \arrow[rr, "F(\alpha^{\mathcal{C}}_{h,g,f})"] \arrow[d, "\mu_{h\circ g,f}"'] & & F(h \circ (g \circ f)) \arrow[d, "\mu_{h,g\circ f}"] \\
F(h \circ g) \circ F(f) \arrow[d, "\mu_{h,g}\circ \mathrm{id}_{F(f)}"'] & \arrow[ur, phantom, "\Rightarrow A_{h,g,f}"] & F(h) \circ F(g \circ f) \arrow[d, "\mathrm{id}_{F(h)}\circ \mu_{g,f}"] \\
(F(h) \circ F(g)) \circ F(f) \arrow[rr, "\alpha^{\mathcal{D}}_{F(h),F(g),F(f)}"'] & & F(h) \circ (F(g) \circ F(f))
\end{tikzcd} \tag{5.1}$$

  which are required to satisfy various coherence conditions.

In this chapter, we will give a partial construction of a functor

$$Z^{\mathcal{A}}_{\chi}\colon \mathrm{Cob}^{\mathrm{or}}_1 \to \mathrm{BicommCat}$$

from the data of a conformal net $\mathcal{A}$. We give a sketch of the $(2,1)$-category $\mathrm{Cob}^{\mathrm{or}}_1$ in Section 5.2 and the 3-category BicommCat was sketched in Conjecture 2.5.1.

We give the data of assignments by $Z^{\mathcal{A}}_{\chi}$ to objects, 1-morphisms, and 2-morphisms in $\mathrm{Cob}^{\mathrm{or}}_1$ in Section 5.3. In Section 5.4 we provide the map μ (assuming Conjecture 2.4.13 on the associativity of the fusion of bimodule categories).

Since we have not constructed the associator of composition of 1-morphisms in BicommCat, we do not give the data of coherence 3-morphisms such as the one in Equation 5.1.

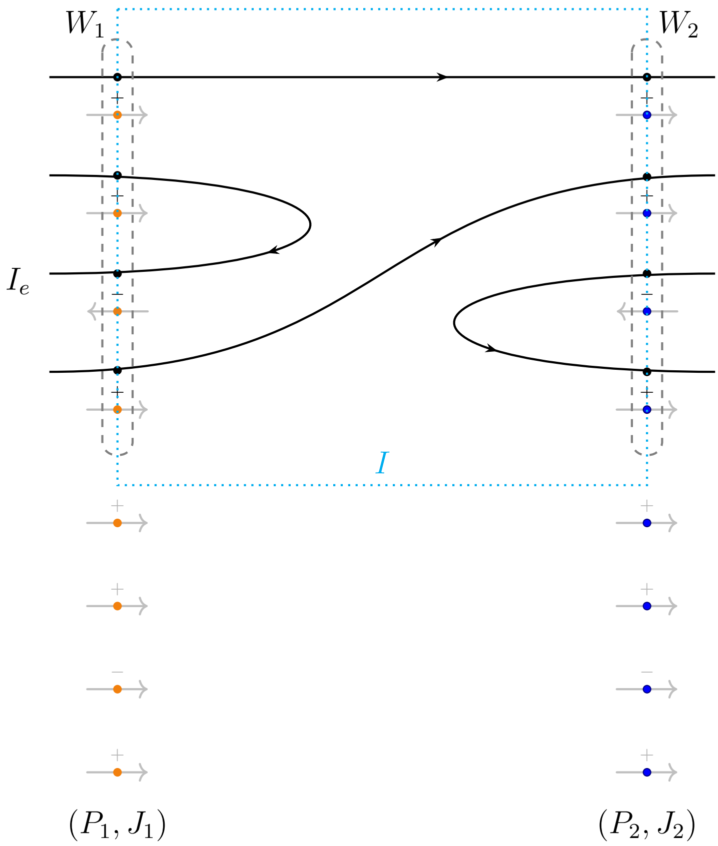
The functor $Z_\chi^{\mathcal{A}}$ is supposed to represent Z_χ^{top} of Section 1.2. We briefly comment on Z_χ^{hol} in Section 5.4.

5.2 The Cobordism Category

Definition 5.2.1. We sketch a definition of the $(2,1)$-category $\mathrm{Cob}_1^{\mathrm{or}}$ below. There are more rigorous treatments of this category, for instance in [LS21], however we isolate the objects, 1-morphisms, 2-morphisms, and composition which we shall need for our partial construction of $Z_\chi^{\mathcal{A}}$.

It is given by:

- objects are pairs (P, J) where
  - J is a multi-interval.
  - $P \subsetneq J$ is a 0-dimensional compact submanifold equipped with an orientation, that is, a finite collection of points in J, each equipped with a $+$ or $-$ label. We also require that the inclusion $P \hookrightarrow J$ is an isomorphism on connected components.
- A morphism from (P_1, J_1) to (P_2, J_2) is a tuple $(I_e, I, W_1, W_2, \iota_1, \iota_2)$ where
  - I_e is a 1-manifold.
  - $I \subsetneq I_e$ is a compact submanifold with boundary.
  - $W_i \subseteq J_i$ are sub-multi-intervals such that $P_i \subseteq W_i$ and the inclusions $W_i \hookrightarrow J_i$ are isomorphisms on connected components.
  - $\iota_i \colon W_i \hookrightarrow I_e$ are embeddings such that
    * if I is 1-dimensional, the images of P_1, P_2 fall on ∂I and we obtain an isomorphism $\partial I \simeq \overline{P_1} \sqcup P_2$. We say I is a cobordism of non-zero length.
    * if I is 0-dimensional, the images of P_1 and P_2 coincide with I. We say I is a cobordism of 0 length.



- Let $(I_e, I, W_1, W_2, \iota_1, \iota_2)$ and $(I'_e, I', W'_1, W'_2, \iota'_1, \iota'_2)$ be 1-morphisms from (P_1, J_1) to (P_2, J_2). Then a 2-morphism is a tuple (V_1, V_2, ϕ) where
  - $V_i \subseteq W_i \cap W'_i$ are multi-intervals such that $P_i \subsetneq V_i$ and the inclusions $P_i \hookrightarrow V_i$ are isomorphisms on connected components.
  - $\phi\colon \iota_1(V_1) \cup I \cup \iota_2(V_2) \to \iota'_1(V_1) \cup I' \cup \iota'_2(V_2)$ is a diffeomorphism such that
    * it maps I to I'.
    * there are neighbourhoods $V'_1 \supsetneq P_1$ and $V'_2 \supsetneq P_2$ such that $V'_i \subseteq V_i$ and $\phi \circ \iota_i|_{V'_i} = \iota'_i|_{V'_i}$.

  We set two 2-morphisms (V_1, V_2, ϕ) and $(\widetilde{V}_1, \widetilde{V}_2, \widetilde{\phi})$ to be equal if $\phi|_I = \widetilde{\phi}|_I$.

Given three objects $(P_1, J_1), (P_2, J_2), (P_3, J_3)$ and 1-morphisms

$$(I^1_e, I^1, W^1_1, W^1_2, \iota^1_1, \iota^1_2)\colon (P_1, J_1) \to (P_2, J_2)$$
$$(I^2_e, I^2, W^2_2, W^2_3, \iota^2_2, \iota^2_3)\colon (P_2, J_2) \to (P_3, J_3)$$

we define their composite to be $(I'_e, I', W^1_1, W^2_3, \iota^1_1, \iota^2_3)$ where

$$\begin{aligned} I'_e =& W^1_1 \sqcup_{W^1_1 \cap I^1} I^1 \sqcup_{W^1_2 \cap W^2_2 \cap I^1} (W^1_2 \cap W^2_2) \sqcup_{W^1_2 \cap W^2_2 \cap I^2} I^2 \sqcup_{W^2_3 \cap I^2} W^2_3 \\ I' =& I^1 \sqcup_{W^1_2 \cap W^2_2 \cap I^1} (W^1_2 \cap W^2_2) \sqcup_{W^1_2 \cap W^2_2 \cap I^2} I^2. \end{aligned}$$

We introduced collars in the definition exactly so that the above formulae for composition would give a smooth structure on the glued cobordism.

The identity morphism of (P, J) is simply $(J, P, J, J, \mathrm{id}_J, \mathrm{id}_J)$.

Definition 5.2.2. For any object (P, J) and any sub-multi-interval $J' \subseteq J$ such that $P \subsetneq J'$ and the inclusion $J' \hookrightarrow J$ is an isomorphism on the set of connected components, the cobordism $(J, P, J', J, J' \hookrightarrow J, \mathrm{id}_J)$ is an invertible 1-morphism from (P, J') to (P, J). We denote this cobordism $N_{J',J}$.

Definition 5.2.3. Given an object (P, J) in $\mathrm{Cob}_1^{\mathrm{or}}$, define the sub-multi-interval $J^+ \subsetneq J$ as

- if J is connected, then J^+ is the part of J to the right of P if P has positive orientation and the part of J to the left of P otherwise. The notions of left and right come from the orientation on J.
- if J is not connected, apply the previous construction to each connected component.

We define $J^- = J \setminus^c J^+$.

Given a cobordism of non-zero length $(I_e, I, W_1, W_2, \iota_1, \iota_2)\colon (P_1, J_1) \to (P_2, J_2)$, we note it is canonically equivalent to

$$(W_1 \sqcup_{W_1 \cap I} I \sqcup_{W_2 \cap I} W_2, I, W_1, W_2, \mathrm{id}, \mathrm{id})\colon (P_1, J_1) \to (P_2, J_2)$$

by the 2-morphism (W_1, W_2, id). We restrict to cobordisms of this form. Further we can precompose and postcompose this cobordism with N_{W_1,J_1} and N_{W_2,J_2} and this results in a cobordism

$$(W_1 \sqcup_{W_1 \cap I} I \sqcup_{W_2 \cap I} W_2, I, W_1, W_2, \mathrm{id}, \mathrm{id})\colon (P_1, W_1) \to (P_2, W_2)$$

Since every cobordism of non-zero length is of this form composed with cobordisms of 0 length, we shall only concentrate on cobordisms of non-zero length of the above form. That is, a cobordism of non-zero length from (P_1, J_1) to (P_2, J_2) consists of a 1-manifold I with embeddings $J_1^+ \hookrightarrow I$ and $J_2^- \hookrightarrow I$ that map P_1, P_2 to ∂I and supply an isomorphism $\partial I \simeq \overline{P_1} \sqcup P_2$ and J_1^+ and J_2^- are disjoint in I.

5.3 Assignment to objects, 1-morphisms, and 2-morphisms

Definition 5.3.1. Given an object (P, J) in $\mathrm{Cob}_1^{\mathrm{or}}$, we define J^{+p} as

- If J is connected, and P has positive orientation, then $J^{+p} = J^+$.
- If J is connected, and P has negative orientation, then $J^{+p} = \overline{J^+}$.
- If J has multiple components, J^{+p} is the disjoint union of J_i^{+p} where J_i are the connected components of J.

We can similarly define J^{-p}.

Definition 5.3.2. We define $Z_\chi^{\mathcal{A}}(P, J)$ to be the category $\mathcal{T}_{\mathcal{A}}^+(J^{+p})^{\cup 0}$.

Definition 5.3.3. We define $Z_\chi^{\mathcal{A}}(N_{J',J})$ to be the extension functor $\mathcal{T}_{\mathcal{A}}^+(J'^{+p})^{\cup 0} \xrightarrow{\sim} \mathcal{T}_{\mathcal{A}}^+(J^{+p})^{\cup 0}$.

Lemma 5.3.4. *Let (P_1, J_1) and (P_2, J_2) be objects of $\mathrm{Cob}_1^{\mathrm{or}}$ and I be a cobordism of non-zero length. That is, we have embeddings $\iota_1\colon J_1^+ \hookrightarrow I$ and $\iota_2\colon J_2^- \hookrightarrow I$ furnishing an isomorphism $\partial I \simeq \overline{P_1} \sqcup P_2$. Let $\rho\colon \mathcal{A}(J_1^{+p}) \to \mathcal{A}(J_1^{+p})$ be an object of $Z_\chi^{\mathcal{A}}(P_1, J_1)$. There exists a unique endomorphism $\rho^{\mathrm{ext}}\colon \mathcal{A}(I) \to \mathcal{A}(I)$ such that it is an extension of ρ by identity.*

Proof. Note that $\mathcal{A}(I \setminus^c \operatorname{supp} \rho)$ and $\mathcal{A}(J_1^+)$ generate $\mathcal{A}(I)$, and therefore it is enough to show such an extension exists, for any two extensions will be equal as they are equal on generating subalgebras.

Let I_c be the subset of I consisting of those connected components which intersect P_1 and let $I_n = I \setminus I_c$. Given a $\rho^c\colon \mathcal{A}(I_c) \to \mathcal{A}(I_c)$ which is an extension of ρ by identity, the $\rho^{\mathrm{ext}}\colon \mathcal{A}(I) \to \mathcal{A}(I)$ can be defined as

$$\mathcal{A}(I) = \mathcal{A}(I_c \sqcup I_n) \simeq \mathcal{A}(I_c)\overline{\otimes}\mathcal{A}(I_n) \xrightarrow{\rho^c \overline{\otimes} \mathrm{id}_{\mathcal{A}(I_n)}} \mathcal{A}(I_c)\overline{\otimes}\mathcal{A}(I_n) \simeq \mathcal{A}(I_c \sqcup I_n) = \mathcal{A}(I)$$

Therefore it is sufficient to prove the existence of ρ^c.

Define $I_{\text{ext}} = I_c \sqcup_{J_1^+} J_1$. Let $S_I = S^1 \times \pi_0(I_{\text{ext}})$ and $S_J = S^1 \times \pi_0(J_1)$ and let $i_I \colon I_{\text{ext}} \hookrightarrow S_I$ and $i_J \colon J_1 \hookrightarrow S_J$ be embeddings such that the following diagrams commute.

$$\begin{tikzcd} J_1 \arrow[r, hook, "i_J"] \arrow[d, two heads] & S_J \arrow[dl, two heads, "\mathrm{proj}_2"] \\ \pi_0(J_1) & \end{tikzcd} \qquad \begin{tikzcd} I_{\text{ext}} \arrow[r, hook, "i_I"] \arrow[d, two heads] & S_I \arrow[dl, two heads, "\mathrm{proj}_2"] \\ \pi_0(I_{\text{ext}}) & \end{tikzcd}$$

Since J_1 embeds into both S_I and S_J, we see that $H_0(S_I)$ and $H_0(S_J)$ have actions of $\mathcal{A}(J_1)$. $\mathcal{A}(J_1)$ is a type III factor as it a tensor product of type III factors, so any two non-zero modules are isomorphic. Thus, there exists a unitary $\mathcal{A}(J_1)$-linear map $u \colon H_0(S_I) \xrightarrow{\sim} H_0(S_J)$. Thus we obtain an isomorphism $\mathrm{Ad}_u \colon B(H_0(S_I)) \xrightarrow{\sim} B(H_0(S_J))$ such that

$$\begin{tikzcd} B(H_0(S_I)) \arrow[r, "\mathrm{Ad}_u", "\sim"'] & B(H_0(S_J)) \\ \mathcal{A}(J_1) \arrow[u, tail] \arrow[ur, tail] & \end{tikzcd} \tag{5.2}$$

Therefore, Ad_u maps $\mathcal{A}(i_I(J_1^-)) \subseteq B(H_0(S_I))$ to $\mathcal{A}(i_J(J_1^-)) \subseteq B(H_0(S_J))$. Define $\widehat{J} = S_J \setminus^c i_J(J_1^-)$. Since $\mathcal{A}(i_I(I_c))$ commutes with $\mathcal{A}(i_I(J_1^-))$ in $B(H_0(S_I))$, $\mathrm{Ad}_u(\mathcal{A}(i_I(I_c))) \subseteq [\mathcal{A}(i_J(J_1^-))]'_{B(H_0(S_J))} = \mathcal{A}(\widehat{J})$. Let $\epsilon \colon \mathcal{A}(I_c) \to \mathcal{A}(\widehat{J})$ be the following composition.

$$\mathcal{A}(I_c) \xrightarrow[\sim]{\mathcal{A}(i_I)|_{\mathcal{A}(I_c)}} \mathcal{A}(i_I(I_c)) \xrightarrow{\mathrm{Ad}_u|_{\mathcal{A}(i_I(I_c))}} \mathcal{A}(\widehat{J})$$

This map is injective.

Since $J_1^+ \subsetneq \widehat{J}$ is a sub-multi-interval localised to the boundary where ρ is not identity, there is a unique $\widehat{\rho} \colon \mathcal{A}(\widehat{J}) \to \mathcal{A}(\widehat{J})$ which extends ρ by identity.

Now, we show $\widehat{\rho}(\epsilon(\mathcal{A}(I_c))) \subseteq \operatorname{Im}\epsilon$. Since $\mathcal{A}(J_1^+)$ and $\mathcal{A}(I_c \setminus^c \operatorname{supp}\rho)$ topologically generate $\mathcal{A}(I_c)$, we just show that $\widehat{\rho}(\epsilon(\mathcal{A}(J_1^+))) \subseteq \operatorname{Im}\epsilon$ and $\widehat{\rho}(\epsilon(\mathcal{A}(I_c \setminus^c \operatorname{supp}\rho))) \subseteq \operatorname{Im}\epsilon$.

- We note that ϵ maps $\mathcal{A}(J_1^+) \subseteq \mathcal{A}(I_c)$ to $\mathcal{A}(J_1^+) \subseteq \mathcal{A}(\widehat{J})$ by Equation 5.2. Since $\widehat{\rho}|_{\mathcal{A}(J_1^+)} = \rho$, we have

$$\widehat{\rho}(\epsilon(\mathcal{A}(J_1^+))) \simeq \widehat{\rho}(\mathcal{A}(J_1^+)) = \rho(\mathcal{A}(J_1^+)) \subseteq \mathcal{A}(J_1^+) \simeq \epsilon(\mathcal{A}(J_1^+)) \subseteq \operatorname{im}(\epsilon)$$

- Since $\mathcal{A}(i_I(I_c \setminus^c \operatorname{supp}\rho))$ commutes with $\mathcal{A}(i_I(\operatorname{supp}\rho \cup J_1^-))$ in $B(H_0(S_I))$ and Equation 5.2, we have

  $$\epsilon(\mathcal{A}(I_c \setminus^c \operatorname{supp}\rho)) \subseteq [\mathcal{A}(\operatorname{supp}\rho \cup J_1^-)]'_{B(H_0(S_J))} = \mathcal{A}(\widehat{J} \setminus^c \operatorname{supp}\rho)$$

  Therefore $\widehat{\rho}$ is identity on $\epsilon(\mathcal{A}(I_c \setminus^c \operatorname{supp}\rho))$, so

  $$\widehat{\rho}(\epsilon(\mathcal{A}(I_c \setminus^c \operatorname{supp}\rho))) = \epsilon(\mathcal{A}(I_c \setminus^c \operatorname{supp}\rho)) \subseteq \operatorname{im}(\epsilon)$$

Since ϵ is injective, we define

$$\rho^c = (\epsilon|_{\operatorname{im}(\epsilon)})^{-1} \circ \widehat{\rho} \circ \epsilon \colon \mathcal{A}(I_c) \to \mathcal{A}(I_c)$$

The previous two bullet points show us that this map is indeed an extension of ρ. $\square$

Definition 5.3.5. Let (P_1, J_1) and (P_2, J_2) be objects of $\mathrm{Cob}_1^{\mathrm{or}}$ and I be a cobordism of non-zero length. That is, we have embeddings $\iota_1 \colon J_1^+ \hookrightarrow I$ and $\iota_2 \colon J_2^- \hookrightarrow I$ furnishing an isomorphism $\partial I \simeq \overline{P_1} \sqcup P_2$. Then define $Z_\chi^{\mathcal{A}}(I) = \mathcal{A}(I)\text{-Mod}$. We give it an action of $Z_\chi^{\mathcal{A}}(P_1, J_1)$ on the left as

$$\begin{aligned} \mathcal{T}_{\mathcal{A}}^+(J_1^{+p}) &\to \mathrm{Bim}(\mathcal{A}(I)) \xrightarrow{\sim} \mathrm{End}(\mathcal{A}(I)\text{-Mod}) \\ \rho &\mapsto {}^{\rho^{\mathrm{ext}}}L^2\mathcal{A}(I) \end{aligned}$$

where $\rho^{\mathrm{ext}} \colon \mathcal{A}(I) \to \mathcal{A}(I)$ was constructed in Lemma 5.3.4. We similarly give a right action of $Z_\chi^{\mathcal{A}}(P_2, J_2)$ as

$$\begin{aligned} \mathcal{T}_{\mathcal{A}}^+(J_2^{+p})^{\mathrm{mop}} \xrightarrow{\sim} \mathcal{T}_{\mathcal{A}}^-(J_2^{-p}) &\to \mathrm{Bim}(\mathcal{A}(I)) \xrightarrow{\sim} \mathrm{End}(\mathcal{A}(I)\text{-Mod}) \\ \rho &\mapsto {}^{\rho^{\mathrm{ext}}}L^2\mathcal{A}(I) \end{aligned}$$

where we have used Corollary 4.5.8. There is a canonical braiding between the two actions as $J_1^+ \cap J_2^- = \emptyset$ defined similarly to $E_{J,K}$ in Definition 4.9.1.

Remark 5.3.6. *We note that the action $\rho \cdot M \mapsto {}^{\rho^{\mathrm{ext}}}M$ is implemented in the picture of bimodules as ${}^{\rho^{\mathrm{ext}}}L^2\mathcal{A}(I) \boxtimes_{\mathcal{A}(I)} M$, but since ρ^{ext} is identity outside J_1^+, the action is only really modified over $\mathcal{A}(J_1^+)$, so this is equivalently ${}^{\rho}L^2\mathcal{A}(J_1^+) \boxtimes_{\mathcal{A}(J_1^+)} M$. The resulting $\mathcal{A}(J_1^+)$ and $\mathcal{A}(I \setminus^c J_1^+)$ actions glue together to form an $\mathcal{A}(I)$-action by arguments similar to Section 4.4.*

Lemma 5.3.7. *The action defined in Definition 5.3.5 makes* $\mathcal{A}(I)$-Mod *into a left* $Z_\chi^{\mathcal{A}}(P_1, J_1)$*-module.*

Proof. We need to prove that the following functor is fully faithful

$$\begin{aligned} \mathrm{res}_{\mathcal{A}(I)\text{-Mod}} \colon \mathcal{A}(I)\text{-Mod} \to & \mathrm{Hom}_{\mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})}(\mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})^{\mathrm{abs}}, \mathcal{A}(I)\text{-Mod}) \\ M \mapsto & F_M \colon \rho \mapsto {}^{\rho^{\mathrm{ext}}}M \end{aligned} \tag{5.3}$$

Note that given any $\rho, \sigma, \sigma' \in \mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})$ and a morphism $f \colon \sigma \to \sigma'$, $\mathrm{id}_\rho \otimes f \colon \rho \otimes \sigma \to \rho \otimes \sigma'$ is just $f \colon \sigma \circ \rho \to \sigma' \circ \rho$. Therefore $\rho \otimes - \colon \mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p}) \to \mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})$ is faithful. In particular, if ρ is absorbing, $\rho \otimes - \colon \mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})^{\mathrm{abs}} \to \mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})^{\mathrm{abs}}$ is faithful.

We restrict to the case where every connected component of I intersects P_1.

Next, for an absorbing $\rho \in \mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})$ and any $M \in \mathcal{A}(I)$-Mod, ${}^{\rho^{\mathrm{ext}}}M \simeq M$ as $\mathcal{A}(I)$ is a type III factor and any two non-zero modules are isomorphic. Thus, the action of $\rho \in \mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})$ is dominant.

By Proposition 2.4.10, we have an equivalence

$$\mathrm{Hom}_{\mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})}(\mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})^{\mathrm{abs}}, \mathcal{A}(I)\text{-Mod}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})^{\mathrm{abs}}}(\mathcal{T}_{\mathcal{A}}^{+}(J_1^{+p})^{\mathrm{abs}}, \mathcal{A}(I)\text{-Mod})$$

Let $S_I = I \sqcup_{\partial I} \overline{I}$ be a disjoint union of circles and equip them with a smooth structure compatible with I and $\overline{I}$ by choosing an isomorphism with $S^1 \times \pi_0(I)$. Then $H := H_0(S_I)$ is a left $\mathcal{A}(I)$-module, and as $\mathcal{A}(I)$ is type III, this is the only non-zero module up to isomorphism. We shall denote $F_{H_0(S_I)}$ as just F. Let $\beta \colon F \to F$ be a natural transformation of linear functors. That is, for every absorbing ρ, we have a map $\beta_\rho \colon {}^{\rho^{\mathrm{ext}}}H \to {}^{\rho^{\mathrm{ext}}}H$ such that

- **Naturality:** for every $f \colon \rho \to \rho'$, the following square commutes

$$\begin{tikzcd} {}^{\rho^{\mathrm{ext}}}H \arrow[r, "f"] \arrow[d, "\beta_\rho"'] & {}^{\rho'^{\mathrm{ext}}}H \arrow[d, "\beta_\rho"] \\ {}^{\rho^{\mathrm{ext}}}H \arrow[r, "f"'] & {}^{\rho'^{\mathrm{ext}}}H \end{tikzcd} \tag{5.4}$$

- **Commutes with linearity data of F:** for any ρ, ρ', the following square commutes

$$
\begin{tikzcd}
\rho'^{\text{ext}}(\rho^{\text{ext}}H) \arrow[r, "\beta_\rho"] \arrow[d, "\text{id}"'] & \rho'^{\text{ext}}(\rho^{\text{ext}}H) \arrow[d, "\text{id}"] \\
(\rho^{\text{ext}}\circ\rho'^{\text{ext}}H) \arrow[r, "\beta_{\rho\circ\rho'}"'] & (\rho^{\text{ext}}\circ\rho'^{\text{ext}}H)
\end{tikzcd}
\tag{5.5}
$$

Lastly, β_ρ being an $\mathcal{A}(I)$-linear endomorphism of ${}^{\rho^{\text{ext}}}H$ gives us that $\beta_\rho \in [\text{im}(\rho^{\text{ext}})]'_{B(H)}$. Equation 5.5 tells us that for any two absorbing objects ρ and ρ', β_ρ and β'_ρ are equal as maps of Hilbert spaces. We call this map β.

Now, an absorbing object $\rho \in \mathcal{T}^+_{\mathcal{A}}(J_1^{+p})$ is of the form $\mathcal{A}(\phi)$ where $\phi\colon J_1^{+p} \to J_1^{+p}$ is a non-surjective embedding which is identity on a neighbourhood of $\partial_+ J_1^{+p}$. ρ^{ext} is just $\mathcal{A}(\phi^{\text{ext}})$ which is the extension of ϕ to a map $\phi^{\text{ext}}\colon I \to I$ by identity. Taking $\rho = \rho' = \mathcal{A}(\phi)$ in Equation 5.4 and noting that $\text{End}(\mathcal{A}(\phi)) = \mathcal{A}(J_1^{+p} \setminus \text{im}(\phi))$ by Lemma 3.3.17, we obtain

$$\beta \in [\mathcal{A}(J_1^{+p} \setminus^c \text{im}(\phi))]'_{B(H)} = [\mathcal{A}(I \setminus^c \text{im}(\phi^{\text{ext}}))]'_{B(H)}$$

Since $\text{im}(\mathcal{A}(\phi^{\text{ext}})) = \mathcal{A}(\text{im}(\phi^{\text{ext}}))$, we have

$$\beta \in [\mathcal{A}(\text{im}(\phi^{\text{ext}}))]'_{B(H)}$$

However, since this is true for every choice of ϕ, we choose ϕ' such that $\text{im}(\phi'^{\text{ext}}) \cap (I \setminus^c \text{im}(\phi^{\text{ext}}))$ is a multi-interval. Further $\text{im}(\phi'^{\text{ext}}) \cup (I \setminus^c \text{im}(\phi^{\text{ext}})) = I$, so by additivity, $\mathcal{A}(\text{im}(\phi'^{\text{ext}}))$ and $\mathcal{A}(I \setminus^c \text{im}(\phi^{\text{ext}}))$ generate $\mathcal{A}(I)$. We compute:

$$
\begin{aligned}
\beta \in & [\mathcal{A}(\text{im}(\phi'^{\text{ext}}))]'_{B(H)} \cap [\mathcal{A}(I \setminus^c \text{im}(\phi^{\text{ext}}))]'_{B(H)} \\
&= [\mathcal{A}(\text{im}(\phi'^{\text{ext}})) \vee \mathcal{A}(I \setminus^c \text{im}(\phi^{\text{ext}}))]'_{B(H)} \\
&= [\mathcal{A}(I)]'_{B(H)} \\
&= \mathcal{A}(\overline{I}) \\
&= \text{End}_{\mathcal{A}(I)\text{-Mod}}(H)
\end{aligned}
$$

Therefore, the functor in Equation 5.3 is fully faithful when every connected component of I intersects P_1.

Now, consider the case when I has components not intersecting P_1. Let I^c

be the union of components that intersect P_1 and $I^n = I \setminus I^c$. So we have

$$\mathcal{A}(I)\text{-Mod} = \mathcal{A}(I^c \sqcup I^n)\text{-Mod} \simeq (\mathcal{A}(I^c)\overline{\otimes}\mathcal{A}(I^n))\text{-Mod} \simeq \mathcal{A}(I^c)\text{-Mod} \boxtimes \mathcal{A}(I^n)\text{-Mod}$$

Then the action of $\mathcal{T}^+_{\mathcal{A}}(J_1^{+p})$ on $\mathcal{A}(I)$-Mod is given by twisting by ρ^{ext} on the first half of the tensor product and by identity on the second part (for if we extended ρ to I^n, it would be identity on $\mathcal{A}(I^n)$). This gives us the vertical leg of the bottom triangle of the following diagram:

$$\begin{tikzcd}
\mathcal{A}(I)\text{-Mod} \arrow[r, "\mathrm{res}_{\mathcal{A}(I)\text{-Mod}}"] \arrow[d, "\sim"'] & \mathrm{Hom}_{\mathcal{T}^+_{\mathcal{A}}(J_1^{+p})}(\mathcal{T}^+_{\mathcal{A}}(J_1^{+p})^{\text{abs}}, \mathcal{A}(I)\text{-Mod}) \arrow[d, "\sim"] \\
\mathcal{A}(I^c)\text{-Mod} \boxtimes \mathcal{A}(I^n)\text{-Mod} \arrow[r] \arrow[dr, "\mathrm{res}_{\mathcal{A}(I^c)\text{-Mod}} \boxtimes \mathrm{id}_{\mathcal{A}(I^n)\text{-Mod}}"'] & \mathrm{Hom}_{\mathcal{T}^+_{\mathcal{A}}(J_1^{+p})}(\mathcal{T}^+_{\mathcal{A}}(J_1^{+p})^{\text{abs}}, \mathcal{A}(I^c)\text{-Mod} \boxtimes \mathcal{A}(I^n)\text{-Mod}) \arrow[d, "\sim"] \\
 & \mathrm{Hom}_{\mathcal{T}^+_{\mathcal{A}}(J_1^{+p})}(\mathcal{T}^+_{\mathcal{A}}(J_1^{+p})^{\text{abs}}, \mathcal{A}(I^c)\text{-Mod}) \boxtimes \mathcal{A}(I^n)\text{-Mod}
\end{tikzcd}$$

The top square commutes by naturality of res, and the bottom triangle commutes as the action of $\mathcal{T}^+_{\mathcal{A}}(J_1^{+p})$ on $\mathcal{A}(I^n)$-Mod is trivial. So $\mathrm{res}_{\mathcal{A}(I)\text{-Mod}}$ is a composite of equivalences and is thus an equivalence. □

Definition 5.3.8. Given a 2-morphism (V_1, V_2, ϕ) between cobordisms I and I', $Z^{\mathcal{A}}_{\chi}(V_1, V_2, \phi)$ is the functor $\mathcal{A}(\phi^{-1})^* \colon \mathcal{A}(I)\text{-Mod} \to \mathcal{A}(I')\text{-Mod}$.

5.4 Fusion of Module Categories associated to Intervals

Let $(P_1, J_1), (P, J), (P_3, J_3)$ be three objects and $I_1 \colon (P_1, J_1) \to (P, J)$ and $I_2 \colon (P, J) \to (P_3, J_3)$ be two cobordisms of non-zero length. In particular, this means that we have embeddings $J^+ \hookrightarrow I_2$ and $J^- \hookrightarrow I_1$ which take P to a subset of the boundary of I_1 and I_2 respectively. Let I be the composite cobordism, that is $I_1 \sqcup_J I_2$. Our goal in this section will be to give an equivalence showing the assignment $Z^{\mathcal{A}}_{\chi}$ we are defining preserves composition, i.e., an equivalence of categories

$$\mathcal{A}(I_1)\text{-Mod} \boxtimes_{Z^{\mathcal{A}}_{\chi}(P,J)} \mathcal{A}(I_2)\text{-Mod} \xrightarrow{\sim} \mathcal{A}(I)\text{-Mod}$$

Lemma 5.4.1. *Assuming Conjecture 2.4.13, such an equivalence can be built out of equivalences for the case when all connected components of I_1 and I_2 intersect P.*

Proof. Let I_1^n and I_2^n be the connected components of I_1 and I_2 that do not intersect P, and I_1^c and I_2^c be the connected components of them that do intersect P. Let $I^c = I_1^c \sqcup_J I_2^c$, then $I = I^c \sqcup I_1^n \sqcup I_2^n$. Let us assume we have an equivalence

$$\mathcal{A}(I_1^c)\text{-Mod} \boxtimes_{Z_\chi^{\mathcal{A}}(P,J)} \mathcal{A}(I_2^c)\text{-Mod} \xrightarrow{\sim} \mathcal{A}(I^c)\text{-Mod}$$

Then we can build an equivalence

$$\begin{aligned}
\mathcal{A}(I_1)\text{-Mod}\boxtimes_{Z_\chi^{\mathcal{A}}(P,J)}\mathcal{A}(I_2)\text{-Mod} = & \mathcal{A}(I_1^n \sqcup I_1^c)\text{-Mod} \boxtimes_{Z_\chi^{\mathcal{A}}(P,J)} \mathcal{A}(I_2^c \sqcup I_2^n)\text{-Mod} \\
\simeq & (\mathcal{A}(I_1^n)\overline{\otimes}\mathcal{A}(I_1^c))\text{-Mod} \boxtimes_{Z_\chi^{\mathcal{A}}(P,J)} (\mathcal{A}(I_2^c)\overline{\otimes}\mathcal{A}(I_2^n))\text{-Mod} \\
\simeq & (\mathcal{A}(I_1^n)\text{-Mod} \boxtimes_{\mathrm{Hilb}} \mathcal{A}(I_1^c)\text{-Mod}) \boxtimes_{Z_\chi^{\mathcal{A}}(P,J)} (\mathcal{A}(I_2^c)\text{-Mod} \boxtimes_{\mathrm{Hilb}} \mathcal{A}(I_2^n)\text{-Mod}) \\
\simeq & \mathcal{A}(I_1^n)\text{-Mod} \boxtimes_{\mathrm{Hilb}} (\mathcal{A}(I_1^c)\text{-Mod} \boxtimes_{Z_\chi^{\mathcal{A}}(P,J)} \mathcal{A}(I_2^c)\text{-Mod}) \boxtimes_{\mathrm{Hilb}} \mathcal{A}(I_2^n)\text{-Mod} \\
\simeq & \mathcal{A}(I_1^n)\text{-Mod} \boxtimes_{\mathrm{Hilb}} \mathcal{A}(I^c)\text{-Mod} \boxtimes_{\mathrm{Hilb}} \mathcal{A}(I_2^n)\text{-Mod} \\
\simeq & (\mathcal{A}(I_1^n)\overline{\otimes}\mathcal{A}(I^c)\overline{\otimes}\mathcal{A}(I_2^n))\text{-Mod} \\
\simeq & (\mathcal{A}(I_1^n \sqcup I^c \sqcup I_2^n))\text{-Mod} \\
= & \mathcal{A}(I)\text{-Mod}
\end{aligned}$$

where we used Proposition 2.4.14 that the fusion of Hilb-modules and the completed tensor product of W^*-categories are the same; and Conjecture 2.4.13 about the associativity of the fusion of modules over bicommutant categories. □

Therefore, we will concentrate on the case when every connected component of I_1 and I_2 intersects P. Our strategy will be to use Corollary 2.4.6. So we need to find $Z_\chi^{\mathcal{A}}(P,J)$-linear functors

$$\begin{aligned}
&\Omega_{11} \colon \mathcal{A}(\overline{I_1})\text{-Mod} \to \mathcal{A}(\overline{I_1})\text{-Mod absorbing in } \mathrm{Hom}_{Z_\chi^{\mathcal{A}}(P,J)}(\mathcal{A}(\overline{I_1})\text{-Mod}, \mathcal{A}(\overline{I_1})\text{-Mod}) \\
&\Omega_{22} \colon \mathcal{A}(I_2)\text{-Mod} \to \mathcal{A}(I_2)\text{-Mod absorbing in } \mathrm{Hom}_{Z_\chi^{\mathcal{A}}(P,J)}(\mathcal{A}(I_2)\text{-Mod}, \mathcal{A}(I_2)\text{-Mod}) \\
&\Omega_{21} \colon \mathcal{A}(\overline{I_1})\text{-Mod} \to \mathcal{A}(I_2)\text{-Mod such that } \Omega_{21} \circ \Omega_{11} \simeq \Omega_{21} \simeq \Omega_{22} \circ \Omega_{21}
\end{aligned} \tag{5.6}$$

where we have used that $\overline{\mathcal{A}(I_1)\text{-Mod}} \simeq \mathcal{A}(I_1)^{\mathrm{op}}\text{-Mod} \simeq \mathcal{A}(\overline{I_1})\text{-Mod}$. We want to find an Ω_{21} such that

$$\mathrm{End}_{\mathrm{Hom}_{Z_\chi^{\mathcal{A}}(P,J)}(\mathcal{A}(\overline{I_1})\text{-Mod},\mathcal{A}(I_2)\text{-Mod})}(\Omega_{21})^{\mathrm{op}} \simeq \mathcal{A}(I) \tag{5.7}$$

as then by Corollary 2.4.6, we have

$$\mathcal{A}(I_1)\text{-Mod} \boxtimes_{Z_\chi^{\mathcal{A}}(P,J)} \mathcal{A}(I_2)\text{-Mod} \xrightarrow{\sim} \mathcal{A}(I)\text{-Mod} \tag{5.8}$$

Definition 5.4.2. For any multi-interval I, define the family of circles $S_I = \overline{I} \sqcup_{\partial I} I$.

Lemma 5.4.3. *Choose a diffeomorphism between S_{I_2} and $S^1 \times \pi_0(S_{I_2})$ such that every point in P_3 maps to 1 in some circle and every point in P maps to -1 in some circle. Use this to give S_{I_2} a smooth structure. Then*

$$\mathrm{Hom}_{Z_\chi^{\mathcal{A}}(P,J)}(\mathcal{A}(I_2)\text{-Mod}, \mathcal{A}(I_2)\text{-Mod}) \simeq \mathcal{R}_{\mathcal{A}}(S_{I_2}, P_3)$$

Similarly, $\mathrm{Hom}_{Z_\chi^{\mathcal{A}}(P,J)}(\mathcal{A}(\overline{I_1})\text{-Mod}, \mathcal{A}(\overline{I_1})\text{-Mod}) \simeq \mathcal{R}_{\mathcal{A}}(S_{\overline{I_1}}, P_1)$.

Proof. We note

$$\begin{aligned}\mathrm{Hom}_{Z_\chi^{\mathcal{A}}(P,J)}(\mathcal{A}(I_2)\text{-Mod}, \mathcal{A}(I_2)\text{-Mod}) =& \mathcal{Z}(Z_\chi^{\mathcal{A}}(P,J) \to \mathrm{End}(\mathcal{A}(I_2)\text{-Mod})) \\ \simeq& \mathcal{Z}(Z_\chi^{\mathcal{A}}(P,J) \to \underline{\mathrm{End}}(\mathcal{A}(I_2)))\end{aligned}$$

By an argument similar to Theorem 4.9.3, this commutant is the subcategory of $\underline{\mathrm{End}}(\mathcal{A}(I_2))$ on endomorphisms which are identity on $\mathcal{A}(J^+)$ and the morphisms are $a \in \mathcal{A}(I_2 \setminus^c J^+)$ which intertwine endomorphisms. By Lemma 4.5.7, this is equivalent to $\mathcal{R}_{\mathcal{A}}(S_{I_2}, P_3)$, the equivalence is given by $\rho \mapsto {}^{\rho}H_0(S_{I_2})$. □

Lemma 5.4.4. *The forgetful functor*

$$\mathrm{Hom}_{Z_\chi^{\mathcal{A}}(P,J)^{\mathrm{abs}}}(\mathcal{A}(\overline{I_1})\text{-Mod}, \mathcal{A}(I_2)\text{-Mod}) \to \mathrm{Hom}_{Z_\chi^{\mathcal{A}}(P,J)}(\mathcal{A}(\overline{I_1})\text{-Mod}, \mathcal{A}(I_2)\text{-Mod})$$

is an equivalence.

Proof. Absorbing endomorphisms $\rho \in Z_\chi^{\mathcal{A}}(P,J)$ are injective, and therefore the action of ρ on $\mathcal{A}(I_2)$-Mod is faithful. This functor is also dominant as $L^2\mathcal{A}(I_2) \simeq {}^{\rho}L^2\mathcal{A}(I_2)$. Similarly the action on $\mathcal{A}(\overline{I_1})$-Mod is faithful and dominant. The result follows from Proposition 2.4.10. □

We can reduce to the case for when I is connected by the following computation:

Computation. Let $I^1, \ldots, I^n$ be the connected components of I. By Lemma 4.7.7, we can assume Ω_{22} to be of the form $\Omega_{22}^1 \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} \Omega_{22}^n$ where Ω_{22}^i is an absorbing object of $\mathcal{R}_{\mathcal{A}}(S_{I_2 \cap I^i}, P_3 \cap I^i)$ and similarly Ω_{11} is of the form $\Omega_{11}^1 \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} \Omega_{11}^n$ where Ω_{11}^i is an absorbing object of $\mathcal{R}_{\mathcal{A}}(S_{\overline{I_1 \cap I^i}}, P_1 \cap I^i)$.

Let J^i be the components of J which intersect I^i. Suppose we have functors

$$\Omega_{21}^i \in \mathrm{Hom}_{Z_\chi^{\mathcal{A}}(J^i, P \cap I^i)^{\mathrm{abs}}}(\mathcal{A}(\overline{I_1 \cap I^i})\text{-Mod}, \mathcal{A}(I_2 \cap I^i)\text{-Mod})$$

such that $\Omega_{22}^i \circ \Omega_{21}^i \simeq \Omega_{21}^i \simeq \Omega_{21}^i \circ \Omega_{11}^i$ and $\mathrm{End}(\Omega_{21}^i)^{\mathrm{op}} \simeq \mathcal{A}(I^i)$.

Define $\Omega_{21} = \Omega_{21}^1 \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} \Omega_{21}^n$. Clearly Ω_{21} absorbs Ω_{11} and Ω_{22}.

Let ω^i be an absorbing object in $Z_\chi^{\mathcal{A}}(J^i, P \cap I^i)$. Then we have linearity maps

$$t^i \colon \Omega_{21}^i(\omega^i \cdot -) \xrightarrow{\sim} \omega^i \cdot \Omega_{21}^i(-)$$

$\omega = \omega^1 \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} \omega^n$ is an absorbing object in $Z_\chi^{\mathcal{A}}(P, J)$ by Lemma 4.7.7. Each ω^i produces an object $\widetilde{\omega}^i$ in $Z_\chi^{\mathcal{A}}(P, J)$ defined as

$$\widetilde{\omega}^i = H_0(S_{(J^1)^+}) \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} \omega^i \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} H_0(S_{(J^n)^+})$$

where $S_{(J^i)^+} = (J^i)^+ \sqcup_{\partial (J^i)^+} \overline{(J^i)^+}$. We have

$$\omega = \widetilde{\omega}^1 \boxtimes_{\mathcal{A}(J^+)} \cdots \boxtimes_{\mathcal{A}(J^+)} \widetilde{\omega}^n$$

Now, we give Ω_{21} linearity data with respect to the action of ω as (we illustrate when there are only two connected components of I)

$$\begin{tikzcd}
(\Omega_{21}^1 \boxtimes \Omega_{21}^2)(\widetilde{\omega}^1 \boxtimes_{\mathcal{A}(J^+)} \widetilde{\omega}^2 \cdot -) \arrow[r] & (\Omega_{21}^1 \boxtimes \Omega_{21}^2)(\widetilde{\omega}^1 \cdot (\widetilde{\omega}^2 \cdot -)) \arrow[d, "t^1 \boxtimes_{\mathbb{C}} \text{unitors}"] \\
& \widetilde{\omega}^1 \cdot (\Omega_{21}^1 \boxtimes \Omega_{21}^2)(\widetilde{\omega}^2 \cdot -) \arrow[d, "\widetilde{\omega}^1(\text{unitors} \boxtimes_{\mathbb{C}} t^2)"] \\
(\widetilde{\omega}^1 \boxtimes_{\mathcal{A}(J^+)} \widetilde{\omega}^2) \cdot (\Omega_{21}^1 \boxtimes \Omega_{21}^2)(-) & \widetilde{\omega}^1 \cdot (\widetilde{\omega}^2 \cdot (\Omega_{21}^1 \boxtimes \Omega_{21}^2)(-)) \arrow[l]
\end{tikzcd}$$

We compute

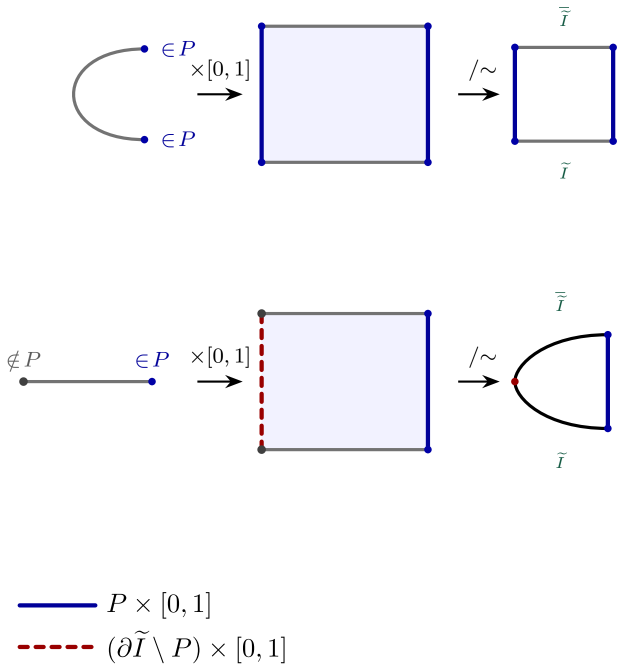
$$\mathrm{End}(\Omega_{21})^{\mathrm{op}} \simeq \overline{\bigotimes_i} \mathrm{End}(\Omega_{21}^i)^{\mathrm{op}} \simeq \overline{\bigotimes_i} \mathcal{A}(I^i) \simeq \mathcal{A}(\bigsqcup_i I^i) = \mathcal{A}(I)$$

Therefore, we can reduce to the case when I is connected. $\square$

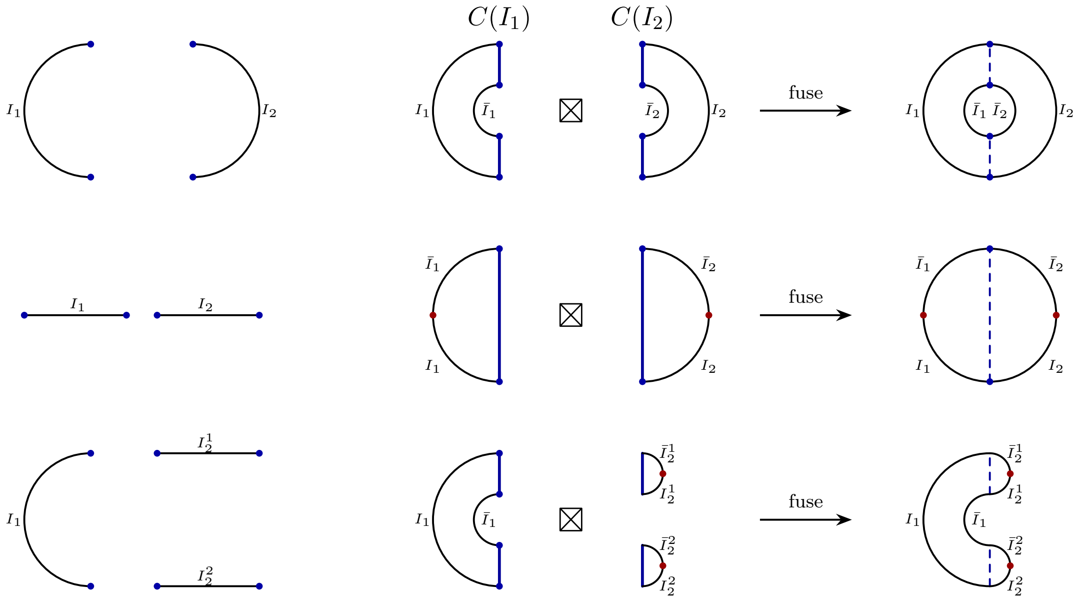
Definition 5.4.5. For any $\widetilde{I} \subseteq I$ being a union of connected components of I_1 or I_2, define

$$C(\widetilde{I}) = \partial(\widetilde{I} \times [0,1]) \sim \text{ where } (x,r) \sim (x',r') \text{ if } x = x' \in \partial\widetilde{I} \setminus P$$

We identify $\widetilde{I} \times \{0\} \subsetneq C(\widetilde{I})$ with $\widetilde{I}$ and $\widetilde{I} \times \{1\} \subsetneq C(\widetilde{I})$ with $\overline{\widetilde{I}}$.



We define Ω_{21} to be $H_0(C(I_1)) \boxtimes_{\mathcal{A}(P\times[0,1])} H_0(C(I_2))$. Below we illustrate three cases of the fusion construction $H_0(C(I_1)) \boxtimes_{\mathcal{A}(P\times[0,1])} H_0(C(I_2))$. In the first case I_1 and I_2 are each an elbow with P consisting of two points. In the second case both I_1 and I_2 are intervals glued over a single point. In the third case I_1 is an elbow and I_2 consists of two disjoint intervals glued over two points. All other cases can be obtained by permuting and iterating the cases illustrated below.



We use its left $\mathcal{A}(I)$-action as its right $\mathcal{A}(\overline{I_1})$-action, and it has a left $\mathcal{A}(I_2)$-action, thus making it an $(\mathcal{A}(I_2), \mathcal{A}(\overline{I_1}))$-bimodule, which we see as a functor $\mathcal{A}(\overline{I_1})$-Mod $\to \mathcal{A}(I_2)$-Mod.

Now, we give the linearity data. As before, it is enough to give linearity data for the action of $Z_\chi^{\mathcal{A}}(p, J_p)^{\text{abs}}$ for every $p \in P$ where J_p is the connected component of J containing p. There is exactly one connected component each of I_1 and I_2 containing p, call them $I_{1,p}$ and $I_{2,p}$. Then we have

$$\Omega_{21} \simeq \boxtimes_{\{\mathcal{A}(p\times[0,1])\}_{p\in P}} \{H_0(C(I_{1,p})), H_0(C(I_{2,p}))\}_{p\in P}$$

We note $\{H_0(C(I_{1,p})), H_0(C(I_{2,p}))\}_{p\in P}$ is a set, so duplicates are not overcounted. For a specific $p_0 \in P$, define

$$\begin{aligned}
\Omega_{21}^{n,p_0} &= \boxtimes_{\{\mathcal{A}(p\times[0,1])\}_{p\in P\setminus(\partial I_{1,p_0}\cup\partial I_{2,p_0})}} \left(\{H_0(C(I_{1,p})), H_0(C(I_{2,p}))\}_{p\in P} \setminus \{H_0(C(I_{1,p_0})), H_0(C(I_{2,p_0}))\}\right) \\
\Omega_{21}^{c,p_0} &= H_0(C(I_{1,p_0})) \boxtimes_{\mathcal{A}(p_0\times[0,1])} H_0(C(I_{2,p_0})) \simeq H_0(C(I_{1,p_0}) \sqcup_{p_0\times[0,1]} C(I_{2,p_0})) = H_0(C_I^{p_0})
\end{aligned}$$

where we have defined the circle $C_{p_0}^I = C(I_{1,p_0}) \sqcup_{p_0\times[0,1]} C(I_{2,p_0})$. We have an isomorphism

$$\Omega_{21} \simeq \Omega_{21}^{n,p_0} \boxtimes_{\mathcal{A}(((\partial I_{1,p_0}\cup\partial I_{2,p_0})\setminus\{p_0\})\times[0,1])} \Omega_{21}^{c,p_0}$$

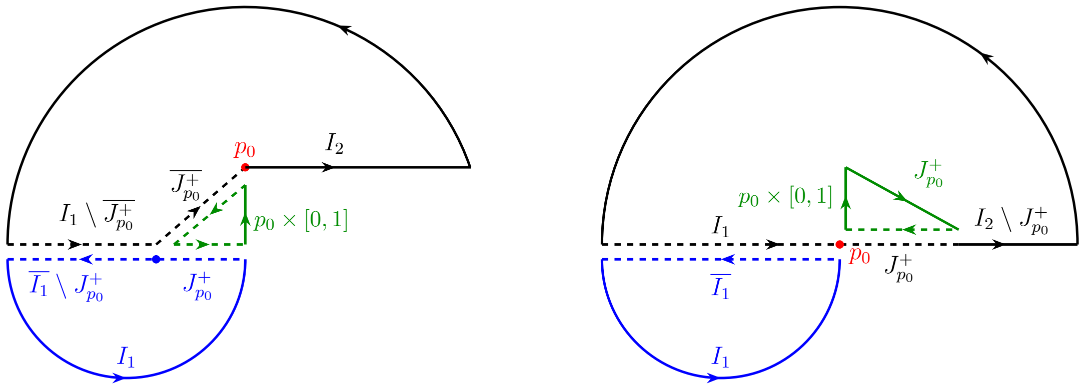
The entire $\mathcal{A}(I_{1,p_0})$ and $\mathcal{A}(I_{2,p_0})$ action is concentrated in Ω_{21}^{c,p_0}. Let the circle $S_{J_{p_0}}^{p_0}$ be defined as $(J_{p_0}^+ \sqcup_{\partial J_{p_0}\setminus\{p_0\}} \overline{J_{p_0}^+}) \sqcup_{p_0\sqcup\overline{p_0}} (p_0 \times [0,1])$. Then $H_0(S_{J_{p_0}}^{p_0})$ is absorbing in $Z_\chi^{\mathcal{A}}(p, J_p)$ by Remark 4.7.3. So to give linearity at p_0, it is enough to give a map

$$\begin{array}{c}\Omega_{21}^{c,p_0} \boxtimes_{\mathcal{A}(\overline{I_{1,p_0}})} (H_0(S^{p_0}_{J_{p_0}}) \boxtimes_{\mathcal{A}(J_p^+)} -) \\ \downarrow{\scriptstyle \eta_{p_0}} \\ H_0(S^{p_0}_{J_{p_0}}) \boxtimes_{\mathcal{A}(J_p^+)} (\Omega_{21}^{c,p_0} \boxtimes_{\mathcal{A}(\overline{I_{1,p_0}})} -)\end{array}$$

It is enough to specify η_{p_0} on a generator of $\mathcal{A}(\overline{I_{1,p_0}})$-Mod. We choose $H_0(S_{\overline{I_{1,p_0}}})$ as our generator. We choose an identification $\overline{J^+_{p_0}} \simeq J^-_{p_0}$, so $\mathcal{A}(J^+_{p_0})$ acts on this as $J^+_{p_0} \simeq \overline{J^-_{p_0}} \hookrightarrow \overline{I_{1,p_0}}$. η_{p_0} is given by the following diagram

$$\begin{tikzcd}
\begin{array}{c} H_0(C^I_{p_0}) \boxtimes_{\mathcal{A}(\overline{I_{1,p_0}})} \\ \Big(H_0(S^{p_0}_{J_{p_0}}) \boxtimes_{\mathcal{A}(J^+_{p_0})} H_0(S_{\overline{I_{1,p_0}}})\Big) \end{array} \arrow[r, "\eta_{p_0}"] \arrow[d, "\mathrm{id} \boxtimes C_{J^+_{p_0}}"'] & \begin{array}{c} H_0(S^{p_0}_{J_{p_0}}) \boxtimes_{\mathcal{A}(J^+_{p_0})} \\ \Big(H_0(C^I_{p_0}) \boxtimes_{\mathcal{A}(\overline{I_{1,p_0}})} H_0(S_{\overline{I_{1,p_0}}})\Big) \end{array} \\
\begin{array}{c} H_0(C^I_{p_0}) \boxtimes_{\mathcal{A}(\overline{I_1})} \\ H_0(S^{p_0}_{J_{p_0}} \sqcup_{J^+_{p_0}} S_{\overline{I_{1,p_0}}}) \end{array} \arrow[d, "C_{\overline{I_{1,p_0}}}"'] & \begin{array}{c} H_0(S^{p_0}_{J_{p_0}}) \boxtimes_{\mathcal{A}(J^+_{p_0})} \\ H_0(C^I_{p_0} \sqcup_{\overline{I_{1,p_0}}} S_{\overline{I_{1,p_0}}}) \end{array} \arrow[u, "\mathrm{id} \boxtimes \left(C_{\overline{I_1}}\right)^{-1}"'] \\
\begin{array}{c} H_0\Big(C^I_{p_0} \sqcup_{\overline{I_{1,p_0}}} \\ (S^{p_0}_{J_{p_0}} \sqcup_{J^+_{p_0}} S_{\overline{I_{1,p_0}}})\Big) \end{array} \arrow[r, "\mathrm{id}"] & \begin{array}{c} H_0\Big(S^{p_0}_{J_{p_0}} \sqcup_{J^+_{p_0}} \\ C^I_{p_0} \sqcup_{\overline{I_{1,p_0}}} S_{\overline{I_{1,p_0}}}\Big) \end{array} \arrow[u, "\left(C_{J^+}\right)^{-1}"']
\end{tikzcd}$$

Informally, η_{p_0} fuses an absorbing triangle in on one side of p_0 and undoes it on the other side of p_0 as illustrated below. In the illustration, the fusion at the level of the circles for the case of I_1 and I_2 being two intervals glued at a single point p_0 is shown, we observe that we get the same circle tracing along the solid lines.



The phase ambiguities in the definition of C cancel out in the definition of η_{p_0} as for each gluing interval, both C and C^{-1} are used. To see the naturality with respect to endomorphisms of $H_0(S^{p_0}_{J_0})$, we note that $\mathrm{End}_{Z^{\mathcal{A}}_X(J_{p_0},p_0)}(H_0(S^{p_0}_{J_0}))$ is given by the action of $\mathcal{A}(S^{p_0}_{J_0} \setminus (J_{p_0} \cup \overline{J_{p_0}}))$. Since this interval is never glued over, the naturality follows from the linearity of $C_{J^+_{p_0}}$ with respect to non-gluing actions. We argue similarly for naturality in $H_0(S_{\overline{I_{1,p_0}}})$.

Lemma 5.4.6. *Ω_{21} absorbs Ω_{11} and Ω_{22}.*

This shows that the Ω_{21} we have constructed does satisfy the conditions of Equation 5.6.

Proof. Let $I_2^1, \ldots, I_2^k$ be the connected components of I_2. By Lemma 5.4.3 and Lemma 4.7.7, we see Ω_{22} is of the form $\Omega^1_{22} \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} \Omega^k_{22}$ where Ω^i_{22} is an absorbing object in $\mathcal{R}_{\mathcal{A}}(S_{I_2^i}, P_3 \cap I_2^i)$. It is enough to show that Ω_{21} absorbs each Ω^i_{22}. Now, the action of $\mathcal{A}(I_2^i)$ on Ω_{21} is concentrated in $H_0(C(I_2^i))$. So it is enough to give an isomorphism

$$\Omega^i_{22} \boxtimes_{\mathcal{A}(I_2^i)} H_0(C(I_2^i)) \simeq H_0(C(I_2^i))$$

which is linear with respect to the $\mathcal{A}(C(I_2^i) \setminus^c (I_2^i \times \{1\}))$-action, for this is the $\mathcal{A}(I_2)$-action and the two actions used to fuse $H_0(C(I_2^i))$ with other circles to form Ω_{21}.

There are two cases, $\partial I_2^i \cap P_3 = \emptyset$ and $\partial I_2^i \cap P_3 \neq \emptyset$.

- If $\partial I_2^i \cap P_3 = \emptyset$, then $\mathcal{R}_{\mathcal{A}}(S_{I_2^i}, \emptyset) \simeq \mathrm{DHR}_{\mathcal{A}}(I_2^i)^{\cup 0}$ by Lemma 4.5.7. Given a $\rho \in \mathrm{DHR}_{\mathcal{A}}(I_2^i)$, we have

$$^{\rho}H_0(S_{I_2^i}) \boxtimes_{\mathcal{A}(I_2^i)} H_0(C(I_2^i)) \simeq {}^{\rho}H_0(C(I_2^i))$$

where we have modified the $\mathcal{A}(I_2)$-action on $H_0(C(I_2^i))$ by ρ. ${}^{\rho}H_0(C(I_2^i))$ still has actions of every subinterval because ρ is identity near the boundaries of ∂I_2^i, so the action continuously extends. Now we extend by identity and obtain a $\rho' \colon \mathcal{A}(I_2^i \cup (\partial_+ I_2^i \times [0,1]) \cup \overline{I_2^i}) \to \mathcal{A}(I_2^i \cup (\partial_+ I_2^i \times [0,1]) \cup \overline{I_2^i})$, and then we use Remark 3.3.22 and find a $\rho'' \colon \mathcal{A}(I_2^i \cup (\partial_+ I_2^i \times [0,1]) \cup \overline{I_2^i}) \to \mathcal{A}(I_2^i \cup (\partial_+ I_2^i \times [0,1]) \cup \overline{I_2^i})$ which is identity on $\mathcal{A}(I_2^i \cup (\partial_+ I_2^i \times [0,1]))$ and which is isomorphic to ρ'. We have isomorphisms

$$ {}^{\rho}H_0(C(I_2^i)) \simeq {}^{\rho'}H_0(C(I_2^i)) \simeq {}^{\rho''}H_0(C(I_2^i)) $$

which are linear with respect to the action of every subinterval. However, as $\mathcal{A}(C(I_2^i) \setminus^c (I_2^i \times \{1\})) = \mathcal{A}(C(I_2^i) \setminus^c \overline{I_2^i})$-modules, we have

$$ {}^{\rho''}H_0(C(I_2^i)) \simeq H_0(C(I_2^i)) $$

Since we proved this for any ρ, it holds for $\Omega_{22}^i = {}^{\rho}H_0(S_{I_2^i})$ where ρ is an absorbing object in $\mathrm{DHR}_{\mathcal{A}}(I_2^i)$. Such an Ω_{22}^i is absorbing.

- If $\partial I_2^i \cap P_3 \neq \emptyset$, then this intersection consists of a single point, call it p. Therefore $\mathcal{R}_{\mathcal{A}}(S_{I_2^i}, p)$ is a category of solitons. Let $q \in P$ be the other boundary point of I_2^i. By Remark 4.7.3, an absorbing object is $H_0((I_2^i \sqcup_q \overline{I_2^i}) \sqcup_{p \sqcup \overline{p}} (p \times [0,1]))$. In this case, $C(I_2^i) = (I_2^i \sqcup_p \overline{I_2^i}) \sqcup_{q \sqcup \overline{q}} (q \times [0,1])$. We compute

$$ \begin{aligned} & H_0((I_2^i \sqcup_q \overline{I_2^i}) \sqcup_{p \sqcup \overline{p}} (p \times [0,1])) \boxtimes_{\mathcal{A}(I_2^i)} H_0((I_2^i \sqcup_p \overline{I_2^i}) \sqcup_{q \sqcup \overline{q}} (q \times [0,1])) \\ & \simeq H_0(((I_2^i \sqcup_q \overline{I_2^i}) \sqcup_{p \sqcup \overline{p}} (p \times [0,1])) \sqcup_{I_2^i} ((I_2^i \sqcup_p \overline{I_2^i}) \sqcup_{q \sqcup \overline{q}} (q \times [0,1]))) \\ & \simeq H_0(\partial(I_2^i \times [0,1])) \end{aligned} $$

Choose a diffeomorphism $\phi \colon C(I_2^i) \xrightarrow{\sim} \partial(I_2^i \times [0,1])$ which is identity on $I_2^i \cup (q \times [0,1))$, then a u_ϕ which implements ϕ is a linear isomorphism of the required type.

$\square$

Therefore we have found an $\Omega_{21} \colon \mathcal{A}(\overline{I_1})$-Mod $\to \mathcal{A}(I_2)$-Mod which is $Z_\chi^{\mathcal{A}}(P,J)$-linear and absorbs non-zero functors of the type $\mathcal{A}(\overline{I_1})$-Mod $\to \mathcal{A}(\overline{I_1})$-Mod and $\mathcal{A}(I_2)$-Mod $\to \mathcal{A}(I_2)$-Mod on post-composing and pre-composing.

Lemma 5.4.7. $\mathrm{End}(\Omega_{21})^{\mathrm{op}} \simeq \mathcal{A}(I)$.

Here Ω_{21} is the object we have been constructing above and I is the core of gluing of the cobordism obtained by gluing I_1 and I_2 along (P, J). This lemma shows that Equation 5.7 is satisfied, thus giving us the desired equivalence Equation 5.8.

Proof. Let $\alpha\colon \Omega_{21} \to \Omega_{21}$ be an endomorphism of $(\mathcal{A}(I_2), \mathcal{A}(\overline{I_1}))$-bimodules which gives rise to a map of $Z_\chi^{\mathcal{A}}(P, J)$-linear functors. Then because α is a bimodule map, we see it has to be linear with respect to $\mathcal{A}(I_1), \mathcal{A}(I_2) \subseteq B(\Omega_{21})$.

Being a map of linear functors, for all $p \in P$, α needs to satisfy

$$(\mathrm{id}_{H_0(S^p_{J_p})} \boxtimes_{\mathcal{A}(J_p^+)} (\alpha \boxtimes_{\mathcal{A}(\overline{I_{1,p}})} \mathrm{id}_{H_0(S_{\overline{I_{1,p}}})})) \circ \eta'_p = \eta'_p \circ (\alpha \boxtimes_{\mathcal{A}(\overline{I_{1,p}})} (\mathrm{id}_{H_0(S^p_{J_p})} \boxtimes_{\mathcal{A}(J_p^+)} \mathrm{id}_{H_0(S_{\overline{I_{1,p}}})}))$$

where $\eta'_p\colon \Omega_{21} \boxtimes_{\mathcal{A}(\overline{I_{1,p}})} (H_0(S^p_{J_p}) \boxtimes_{\mathcal{A}(J_p^+)} H_0(S_{\overline{I_{1,p}}})) \to H_0(S^p_{J_p}) \boxtimes_{\mathcal{A}(J_p^+)} (\Omega_{21} \boxtimes_{\mathcal{A}(\overline{I_{1,p}})} H_0(S_{\overline{I_{1,p}}}))$ is induced by η_p.

This can only be satisfied if α is $\mathcal{A}(J_p)$-linear, as, in the two sides of the above equation, the J_p-actions are modified using those from the absorbing object $H_0(S^p_{J_p})$ in different parts of J_p. Since $J = \bigsqcup_p J_p$, we learn that α is $\mathcal{A}(J)$-linear.

Therefore, $\alpha \in [\mathcal{A}(I_1)]'_{B(\Omega_{21})} \cap [\mathcal{A}(I_2)]'_{B(\Omega_{21})} \cap [\mathcal{A}(J)]'_{B(\Omega_{21})}$. It is easy to see that any map which is $\mathcal{A}(I_1), \mathcal{A}(J), \mathcal{A}(I_2)$-linear produces a map of $Z_\chi^{\mathcal{A}}(P, J)$-linear functors. So we have

$$\mathrm{End}(\Omega_{21}) = [\mathcal{A}(I_1)]'_{B(\Omega_{21})} \cap [\mathcal{A}(I_2)]'_{B(\Omega_{21})} \cap [\mathcal{A}(J)]'_{B(\Omega_{21})} = [\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \vee \mathcal{A}(J)]'_{B(\Omega_{21})}$$

There are two cases, I can be an interval or a circle, as I is a connected 1-manifold.

- If I is an interval, $\Omega_{21} = H_0(S_I)$. So by Haag duality,

$$[\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \vee \mathcal{A}(J)]'_{B(\Omega_{21})} = \mathcal{A}(I \times \{1\})$$

- If I is a circle, Ω_{21} is the Hilbert space constructed in Definition 4.6.5. In this case we have the following by definition:

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \vee \mathcal{A}(J) = \mathcal{A}(I \times \{0\})$$

Therefore,

$$[\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \vee \mathcal{A}(J)]'_{B(\Omega_{21})} = \mathcal{A}(I \times \{1\})$$

Using $\mathcal{A}(I \times \{1\}) \simeq \mathcal{A}(\overline{I}) \simeq \mathcal{A}(I)^{\mathrm{op}}$, we obtain the desired result. $\square$

The category assigned to S^1

We can also do a direct computation of the fusion of S_+ and S_- to form S^1. Assume 1 and -1 have chosen intervals around them. In this situation, all three $\Omega_{11}, \Omega_{21}, \Omega_{22}$ are $Z_\chi^{\mathcal{A}}(P,J)$-linear functors from $\mathcal{A}(S_+)$-Mod $\to \mathcal{A}(S_+)$-Mod, where we have used $\overline{S_-} \simeq S_+$. So Ω_{21} is just the absorbing object in the category of $Z_\chi^{\mathcal{A}}(P,J)$-linear endomorphisms of $\mathcal{A}(S_+)$-Mod. Using the same argument as before, this is nothing but $\mathrm{DHR}_{\mathcal{A}}(S_+) \simeq \mathrm{Rep}(\mathcal{A})$. Thus we have that

$$Z_\chi^{\mathcal{A}}(S^1) \simeq \mathrm{Rep}(\mathcal{A})^{\mathrm{abs}} \simeq \mathcal{Z}(\mathrm{Sol}(\mathcal{A}))^{\mathrm{abs}} \tag{5.9}$$

where we have used Corollary 4.9.5.

For a functor valued in the Morita 3-category of tensor categories, the object assigned to S^1 is the Drinfeld centre of the object assigned to the point. In our setting, we have obtained $Z_\chi^{\mathcal{A}}(S^1) \simeq \mathcal{Z}(\mathrm{Sol}(\mathcal{A}))^{\mathrm{abs}}$, which matches this expectation up to taking $(-)^{\mathrm{abs}}$. It is a part of ongoing work to understand what finiteness assumptions ensure a bicommutant category is equivalent to its absorbing ideal.

In the finite index case, it was shown in [BDH17, Corollary 1.23], that $\mathrm{Rep}(\mathcal{A}) \simeq \mathcal{A}(S^1)$-Mod, so here we see that $\mathrm{Rep}(\mathcal{A}) = \mathrm{Rep}(\mathcal{A})^{\mathrm{abs}}$. For non-rational theories, the Hilbert space H_0 that the theory assigns to the unit disc $\mathbb{D}$ is not an object of $Z_\chi^{\mathcal{A}}(S^1)$.

Copointings

Definition 5.4.8. $Z_\chi^{\mathcal{A}}(P,J)$ is copointed by $Z_\chi^{\mathcal{A}}(P,J)^{\mathrm{abs}}$ viewed as a right $Z_\chi^{\mathcal{A}}(P,J)$-module. We also specify copointings $Z_\chi^{\mathrm{hol},\mathcal{A}}(I)\colon Z_\chi^{\mathcal{A}}(I) \to \mathrm{Hilb}$ given by the forgetful functor $\mathcal{A}(I)$-Mod $\to$ Hilb.

By taking daggers, we can turn copointings of $Z_\chi^{\mathcal{A}}(I)$to pointings. The forgetful functor $\mathcal{A}(I)$-Mod $\to$ Hilb corresponds to pointing $\mathcal{A}(I)$-Mod by $L^2\mathcal{A}(I)$.

Under the constructed equivalence

$$\mathcal{A}(I_1)\text{-Mod} \boxtimes_{Z_\chi^{\mathcal{A}}(P,J)} \mathcal{A}(I_2)\text{-Mod} \xrightarrow{\sim} \mathcal{A}(I)\text{-Mod}$$

these pointings are mapped to Ω_{21} viewed as an $\mathcal{A}(I)$-module. In the case I is an interval, this is equivalent to $L^2\mathcal{A}(I)$ using the BW map, and when I is a circle, this is $\mathrm{L}^2\mathcal{A}(I)$ by Lemma 4.7.5.

These (co)pointings will eventually be part of the data of Z_χ^{hol} of Section 1.2.

Chapter 6

Conclusion

6.1 Summary of Main Results

In this thesis, we introduced a refined definition of a bicommutant category, updating that of [Hen17c, HP17, Hen17a]. We defined modules over bicommutant categories and introduced a 'categorified' Connes fusion product for module categories over bicommutant categories. We reviewed the definition and properties of conformal nets and established the equivalence of categories between endomorphisms and bimodules over a type III von Neumann algebra.

We introduced left- (or right-) localised endomorphisms and established an equivalence of categories between solitonic representations of a conformal net and left- or right-localised endomorphisms of the algebra assigned to the upper half of the circle.

We proved that the category of solitonic representations of any conformal net is a bicommutant category. Crucially, this holds without assuming finite index or strong additivity, conditions previously required in all known results of this type. This substantially broadens the class of conformal nets to which our methods apply, and demonstrates that bicommutant behaviour is a fundamental feature of unitary representation theory (of conformal nets), more so than was previously understood.

Applying these tools, we constructed parts of the dimension-0 and dimension-1 components of a functorial two-dimensional chiral CFT from a conformal net. To 0-dimensional objects equipped with 1-dimensional collars, we assigned a bicommutant category; to 1-manifolds we assigned bimodule W^*-categories. We finally showed that the fusion of bimodules from composable cobordisms is the bimodule associated to their composite, contingent on Conjecture 2.4.13 on the associativity of categorified Connes Fusion.

6.2 Future Directions

The results of this thesis naturally lead to several directions for further research. We briefly outline some of the themes.

The most immediate extension of the present work is to complete the functorial assignment to 2-dimensional cobordisms, that is, to construct the value of the functor on Riemann surfaces with boundaries and corners, together with the various coherence data.

$$\begin{tikzcd} ? \arrow[r] \arrow[rrr, bend left=30, "?"] \arrow[rr, phantom, shift left=5ex, "\sim\Downarrow ?"] & ? \arrow[r] & \mathrm{Cob}^{\mathrm{conf,or}}_{2+\epsilon,2,1,0} \arrow[r, "Z_\chi^{\mathrm{top}}"] \arrow[r, bend right=50, "\mathrm{const}_1"'] \arrow[r, phantom, shift right=3ex, "\Downarrow Z_\chi^{\mathrm{hol}}"] & \text{3-Hilb} \end{tikzcd} \tag{6.1}$$

In Equation 6.1, the components constructed in part in this thesis are indicated in red, and the data that remains to be constructed in blue.

Our construction has treated the two layers Z_χ^{top} and Z_χ^{hol} of Section 1.2 asymmetrically: the substance of Chapters 4 and 5 lies in dimensions 0 and 1, where a manifold carries no local conformal invariants, so that the conformal cobordism category agrees with its topological counterpart and the (co)pointings of Definition 5.4.8 are canonical. At dimension 2 this reverses, and the pointings become the essential carrier of the chiral structure.

Indeed, as recalled in Section 1.2, Z_χ^{top} assigns to a complex cobordism Σ a functor $F_\Sigma\colon \mathcal{C}_{\partial_{\mathrm{in}}\Sigma} \to \mathcal{C}_{\partial_{\mathrm{out}}\Sigma}$. When $\partial\Sigma$ is closed, we know from Definition 5.3.5 that $\mathcal{C}_{\partial_{\mathrm{in}}\Sigma} = \mathcal{A}(\partial_{\mathrm{in}}\Sigma)$-Mod, and similarly for $\partial_{\mathrm{out}}\Sigma$; hence $F_\Sigma = H_\Sigma \boxtimes_{\mathcal{A}(\partial_{\mathrm{in}}\Sigma)} -$ for some bimodule ${}_{\mathcal{A}(\partial_{\mathrm{out}}\Sigma)}(H_\Sigma)_{\mathcal{A}(\partial_{\mathrm{in}}\Sigma)}$. The bimodule H_Σ is the space of conformal blocks of Σ, and changing the complex structure on Σ yields a non-canonically isomorphic space of conformal blocks [BDH17]. The transformation Z_χ^{hol}, by contrast, equips H_Σ with a pointing $\Omega_\Sigma \in H_\Sigma$, and it is this pointing that depends *holomorphically* on the complex structure of Σ, as sketched in Section 1.2. In the fully-extended setting, where Σ has corners and $\partial_{\text{in}}\Sigma$ is a 1-manifold with boundary, $\mathcal{C}_{\partial_{\text{in}}\Sigma}$ is instead one of the bimodule W*-categories of Chapter 5, but the division of labour between the two layers is the same.

Consequently, extending our assignment to 2-dimensional cobordisms with corners requires more than functors between the W*-categories constructed in Chapter 5, equipped with linearity data for the actions of the bicommutant categories assigned to the objects of Cob_1^{or}: one must also produce the pointings of those functors and prove that they vary holomorphically. It is only at that stage that the theory becomes chiral conformal rather than something close to topological. In particular, it remains to clarify how the holomorphicity condition of Section 1.2 should be formulated for a fully-local theory; we do not yet have the full picture of an extended version of Equation 1.1.

A central goal of the broader programme on bicommutant categories is to construct the symmetric monoidal 3-category BicommCat whose objects are bicommutant categories, 1-morphisms are bimodule categories, 2-morphisms are bimodule functors, and 3-morphisms are natural transformations. While we provided a detailed proposed definition of the bicategory $\mathcal{T}_1$-Mod-$\mathcal{T}_2$ for bicommutant categories $\mathcal{T}_1$ and $\mathcal{T}_2$, the completion of this 3-category requires constructing associativity isomorphisms for 'categorified' Connes fusion to establish composition of 1-morphisms. We were able to show that given two composable cobordisms, the bimodule category assigned to their composition is the categorified Connes fusion of the bimodules assigned to cobordisms. Since the operation of gluing is associative, we have some evidence for the associativity of the categorified Connes fusion. We have also not addressed the monoidal structure on this 3-category. The Morita category of von Neumann algebras is a symmetric monoidal bicategory with respect to the spatial tensor product of von Neumann algebras. The analogous notion for bicommutant categories has not yet been developed. In particular, it remains to show that our assignment is symmetric monoidal.

Given an inclusion of conformal nets $\mathcal{B} \subset \mathcal{A}$, one can consider the category $\text{Sol}_{\mathcal{B}}(\mathcal{A})$. These are solitonic representations of $\mathcal{A}$ that restrict to ordinary representations of the subnet $\mathcal{B}$ or in other words, solitons for $\mathcal{A}$ that are in the DHR sector for the subnet $\mathcal{B}$. We have the following series of forgetful functors for the inclusion of nets $\mathbb{C} \subset \text{Vir}_c \subset \mathcal{B} \subset \mathcal{A}$:

$$\text{Rep}(\mathcal{A}) := \text{Sol}_{\mathcal{A}}(\mathcal{A}) \to \text{Sol}_{\mathcal{B}}(\mathcal{A}) \to \text{Sol}_{\text{Vir}_c}(\mathcal{A}) \to \text{Sol}_{\mathbb{C}}(\mathcal{A}) =: \text{Sol}(\mathcal{A})$$

A natural generalisation of the present work would be to prove that $\mathrm{Sol}_{\mathcal{B}}(\mathcal{A})$ is a bicommutant category. An interesting result to seek would be to compute the Drinfeld centre, $\mathcal{Z}(\mathrm{Sol}_{\mathcal{B}}(\mathcal{A}))$. Consider the following two edge cases: $\mathrm{Sol}_{\mathbb{C}}(\mathcal{A}) = \mathrm{Sol}(\mathcal{A})$ and $\mathrm{Sol}_{\mathcal{A}}(\mathcal{A}) = \mathrm{Rep}(\mathcal{A})$. We showed that $\mathcal{Z}(\mathrm{Sol}(\mathcal{A})) \simeq \mathrm{Rep}(\mathcal{A}) \simeq \mathrm{Rep}(\mathcal{A}) \boxtimes \mathrm{Hilb}$. $\mathrm{Rep}(\mathcal{A})$ is a braided monoidal category, and when it is non-degenerate, $\mathcal{Z}(\mathrm{Rep}(\mathcal{A})) \simeq \mathrm{Rep}(\mathcal{A}) \boxtimes \mathrm{Rep}(\mathcal{A})^{\mathrm{rev}}$. This gives us a conjecture: $\mathcal{Z}(\mathrm{Sol}_{\mathcal{B}}(\mathcal{A})) \simeq \mathrm{Rep}(\mathcal{A}) \boxtimes \mathrm{Rep}(\mathcal{B})^{\mathrm{rev}}$. Of particular physical interest is the case $\mathcal{B} = \mathrm{Vir}_c$, the Virasoro subnet. $\mathrm{Sol}_{\mathrm{Vir}_c}(\mathcal{A})$ is expected to be the category of 'boundary topological line defects in the 2-dimensional chiral CFT' associated to $\mathcal{A}$, although this concept has not yet been formalised.

The central charge is an important invariant of a chiral CFT. In the setting of conformal nets, the central charge is captured by the projective action of $\mathrm{Diff}^+(S^1)$ on the vacuum Hilbert space H_0. Establishing an analogous projective action in the setting of bicommutant categories remains open.

Given a bicommutant category $\mathcal{T}$, an absorbing object $\Omega \in \mathcal{T}^{\mathrm{abs}}$, and a unitary $u\colon \Omega \otimes \Omega \to \Omega$, one can construct a net of von Neumann algebras on the circle: the algebra assigned to a dyadic interval is $\mathrm{End}(\Omega)$, and u supplies the inclusions associated to subdividing an interval into two halves. The resulting net is not $\mathrm{Diff}_+(S^1)$-covariant; it is instead covariant with respect to the Thompson group $\mathrm{Th} \subset \mathrm{Homeo}(S^1)$ of Definition 4.8.1. We call such an object a *Thompson net*. Making this construction precise, and identifying the conditions on the triple $(\mathcal{T}, \Omega, u)$ under which a Thompson net extends to a genuine conformal net, would provide a converse to the results of Chapter 4 where we get a bicommutant category from a conformal net. We note that a Thompson net cannot by itself detect the central charge as by Theorem 4.8.4, the Bott-Thurston cocycle trivialises on Th, so the projective actions arising this way admit honest lifts.

We are interested in finding a subgroup of $\mathrm{Diff}(S^1)$ that acts genuinely projectively on the solitonic category, and recovering central charge information from the cocycle.

Finally, we would like to highlight that the functional analytic categorical framework developed here calls for further foundational work. In [HNP24], we observed that the orthogonal direct sum completion and dagger idempotent completion of $B\mathbb{C}$ gives Hilb, but repeating this on BHilb only produces the subcategory of $\mathrm{vNAlg}^{\mathrm{bimod}}$ on type I algebras. We need a more functional analytic completion under which BHilb gives the entirety of $\mathrm{vNAlg}^{\mathrm{bimod}}$ (equivalently, the category of W^*-categories) as Hilb-Mod where Hilb is a bicommutant category. This will serve as the first step towards defining W^*2-categories. Ultimately, the goal is a complete and rigorous mathematical framework in which general functorial quantum field theories beyond TQFTs are realised.

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