

A LIPSCHITZ METRIC FOR THE CAMASSA–HOLM EQUATION

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Abstract

We analyze stability of conservative solutions of the Cauchy problem on the line for the Camassa–Holm (CH) equation. Generically, the solutions of the CH equation develop singularities with steep gradients while preserving continuity of the solution itself. In order to obtain uniqueness, one is required to augment the equation itself by a measure that represents the associated energy, and the breakdown of the solution is associated with a complicated interplay where the measure becomes singular. The main result in this paper is the construction of a Lipschitz metric that compares two solutions of the CH equation with the respective initial data. The Lipschitz metric is based on the use of the Wasserstein metric.

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1. Introduction

We study the Cauchy problem for weak conservative solutions of the Camassa–Holm (CH) equation, which reads as

$$u_t + uu_x + p_x = 0, \quad (1.1a)$$

$$\mu_t + (u\mu)_x = (u^3 - 2pu)_x, \quad (1.1b)$$

where $p(t, x)$ is given by

$$p(t, x) = \frac{1}{4} \int_{\mathbb{R}} e^{-|x-y|} u^2(t, y) dy + \frac{1}{4} \int_{\mathbb{R}} e^{-|x-y|} d\mu(t, y), \quad (1.2)$$

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with initial data $(u, \mu)|_{t=0} = (u_0, \mu_0)$. In this case, the natural solution space consists of all pairs (u, μ) such that

$$u(t, \cdot) \in H^1(\mathbb{R}), \quad \mu(t, \cdot) \in \mathcal{M}_+(\mathbb{R}), \quad \text{and} \quad d\mu_{\text{ac}} = (u^2 + u_x^2) dx,$$

where $\mathcal{M}_+(\mathbb{R})$ denotes the set of all positive and finite Radon measures on \mathbb{R} . Our main goal is to prove the existence of a metric d such that

$$d((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))) \leq \alpha(t) d((u_{1,0}, \mu_{1,0}), (u_{2,0}, \mu_{2,0})),$$

for two weak conservative solutions $(u_j(t), \mu_j(t))$ $j = 1, 2$ of (1.1) with initial data $(u_{j,0}, \mu_{j,0})$ $j = 1, 2$. Here $\alpha(t)$ depends on the total energy of the solutions and $\alpha(0) = 1$.

The CH equation has been introduced in the seminal paper [10]; see also [11]. Originally derived in the context of models for shallow water (see also [27, 58]), it also turns up in models for hyperelastic rods [24, 28, 55]. Since it captures the nonlinear effects that give insight into important phenomena such as breaking waves and breaking rods, the CH equation has been intensively studied. The intricate behavior of solutions of the Cauchy problem will be the focus of this paper. We will not discuss further properties of the CH equation, for example, the fact that the equation is completely integrable and allows for a geometric interpretation. With the latter, we mean that the CH equation is a re-expression of a geodesic flow on the diffeomorphism group of the line or the circle [26, 31, 60]. Several extensions and generalizations exist, but we will focus on (1.1).

The intriguing aspect of solutions to the Cauchy problem is the generic development of singularities in finite time, irrespective of the smoothness of the initial data. A solution may develop steep gradients, but in contrast to, say, hyperbolic conservation laws, the solution itself remains continuous. A finer analysis reveals that the energy density $u^2 + u_x^2$ develops singularities, and at breakdown, which is often referred to as wave breaking, energy concentrates on sets of measure zero. Thus it becomes useful to introduce a measure, here denoted by μ , that encodes the energy, and away from breakdowns this measure should coincide with the energy density $u^2 + u_x^2$. In technical terms, we consider a nonnegative Radon measure μ with an absolutely continuous part $d\mu_{\text{ac}} = (u^2 + u_x^2) dx$. This measure μ satisfies equation (1.1b) (and as can easily be verified, $\mu = u^2 + u_x^2$ will satisfy the same equation in the case of smooth solutions). An illustrating example of how intricate the structure of the points of wave breaking may be can be found in [40]. The behavior in the proximity of the point of wave breaking, and, in particular, the prolongation of the solution past wave breaking, has been extensively studied. See, for example, [6, 7, 21, 22, 25, 32, 33, 39, 43–54, 56, 57] and references therein. The key point here is that past wave breaking

uniqueness fails, and there is a continuum of distinct solutions [47], with two extreme cases called dissipative and conservation solutions. To understand this conundrum, it turns out to be advantageous to rewrite the equation in a different set of variables where the solution remains smooth.

The explicit peakon–antipeakon solution [41], which illustrates this problem, is given by

$$u(t, x) = \begin{cases} -\alpha(t)e^x, & x \leq -\gamma(t), \\ \beta(t) \sinh(x), & -\gamma(t) < x < \gamma(t), \\ \alpha(t)e^{-x}, & \gamma(t) \leq x, \end{cases} \quad (1.3)$$

where

$$\alpha(t) = \frac{E}{2} \sinh\left(\frac{E}{2}t\right), \quad \beta(t) = E \frac{1}{\sinh(\frac{E}{2}t)}, \quad \gamma(t) = \ln\left(\cosh\left(\frac{E}{2}t\right)\right),$$

where $E = \|u(t)\|_{H^1}$ for all $t \neq 0$. This function is a weak conservative solution which consists of a ‘peak’ moving to the right and an ‘antipeak’ moving to the left; see Figure B.3. At $t = 0$ the ‘peak’ and ‘antipeak’ collide, and the solution vanishes, yet the solution is highly nontrivial before and after the collision time. Clearly the trivial solution will coincide with this solution at $t = 0$, yet the trivial solution and (1.3) are very different at any other time. Thus it is not clear how to derive a metric comparing two solutions, that is stable under the time evolution. This is the task of the present work.

To be more precise, we are here presenting a metric d with the property that

$$d((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))) \leq \alpha(t)d((u_{1,0}, \mu_{1,0}), (u_{2,0}, \mu_{2,0})),$$

for two weak conservative solutions $(u_j(t), \mu_j(t))$, $j = 1, 2$, of (1.1) with initial data $(u_{j,0}, \mu_{j,0})$, $j = 1, 2$. Here $\alpha(t)$ is a continuous function with $\alpha(0) = 1$, which may depend on the total energies involved, but not on the particular solutions. We stress that no standard Sobolev norm nor Lebesgue space norm will work. There exist alternative metrics for solutions of the CH equation; see [8, 42, 43]. In [8] the periodic case is treated by approximating the solution by multipeakons. The metric is defined by optimizing over a class of functions. The approach in [42, 43] depends on a reformulation of the CH equation in terms of Lagrangian variables. An intrinsic problem in this formulation is that of *relabeling*, where there will be many different parametrizations in Lagrangian coordinates, corresponding to one and the same solution $(u(t), \mu(t))$ in Eulerian variables. Thus one has to compute the distance between equivalence classes, which is not transparent. In the present approach, the key idea is to introduce a new set of variables, where one variable plays a role similar to a characteristic, while the remaining variables are linked to (u, μ) with the help of the ‘characteristic’. As we will outline next, there is no

need to resort to equivalence classes or to optimize over classes of functions, and in spite of this proof being longer, we consider this approach to be more natural.

Our approach is based on the fact that a natural metric for measuring distances between Radon measures (with the same total mass) is given through the Wasserstein (or Monge–Kantorovich) distance d_W , which in one dimension is defined with the help of pseudoinverses; see [67]. Given a nonnegative measure μ of finite mass $M > 0$, we define the cumulative distribution function associated to μ as

$$F(x) = \mu((-\infty, x)),$$

which is a nondecreasing function from \mathbb{R} onto $[0, M]$, left continuous and with limit from the right at any point $x \in \mathbb{R}$. The pseudoinverse associated to μ denoted by \mathcal{X} is the function from $[0, M]$ onto \mathbb{R} given by

$$\mathcal{X}(\eta) = \sup\{x \mid F(x) < \eta\}.$$

The pseudoinverse of F is a nondecreasing function from $[0, M]$ onto \mathbb{R} , left continuous and with limit from the right (caglad) at any point $x \in \mathbb{R}$. Notice the different convention adopted here with respect to the usual one in probability theory defining cumulative distribution functions continuous from the right and with limits from the left (cadlag) at every point. We prefer to have caglad instead of cadlag functions due to the use of the methods from [54] developed under the present convention. Wasserstein distances between nonnegative measures with the same mass can be defined via L^p -norms of the difference between their associated pseudoinverses; see [19, 20, 61, 67] and the references therein.

The approach of using Wasserstein distances to control the expansion of solutions of evolutionary PDEs leading to curves of probability measures goes back to the proofs of the mean-field limit of McKean–Vlasov and Vlasov equations in the late seventies and eighties of the last century. We refer to the classical references [5, 29, 63–65] proving these large particle limits by means of the bounded Lipschitz distance and the coupling method. See the recent results and surveys in [3, 12, 13, 35, 36]. The optimal transport viewpoint for one-dimensional models was developed using pseudoinverse distributions for nonlinear aggregation and diffusion equations in [15, 19, 20, 61, 66] and the references therein, showing the contractivity of the Wasserstein distance in one dimension without the heavy machinery of optimal transport developed for general gradient flows in [1]. More recently, these metrics have been used with success to show uniqueness past the blow-up time for multidimensional aggregation equations [14] using gradient flow solutions. It is also interesting to point out that gradient flow solutions of the aggregation equation in one dimension with particular potentials are equivalent to entropy solutions of the Burgers

equation as proven in [4]. Another strategy using unbalanced optimal transport tools has been recently analyzed in [34] with the objective of understanding the relation between the incompressible Euler equations and the CH equations.

Finally, it is worth mentioning that there have been several works [2, 17, 18, 30, 37, 38, 59, 62, 68] making use of this change of variables to produce numerical schemes capable of going over blow-up of solutions to nonlinear aggregations and being able to capture the blow-up of solutions of aggregation–diffusion models in one dimension such as toy versions of the Keller–Segel model for chemotaxis. It is a nice avenue of research to use this approach to produce numerical schemes for conservative solutions of the CH equation; see [23] for related particle methods.

In the present work, we will adapt this strategy of defining suitable distances between measures to the present problem of finding good metrics for solutions of the CH equation. Let $(u(t, \cdot), \mu(t, \cdot))$ be a weak conservative solution to the CH equation with total energy $\mu(t, \mathbb{R}) = C > 0$ (which for simplicity here is assumed to be smooth). Let

$$F(t, x) = \mu(t, (-\infty, x)) = \int_{-\infty}^x d\mu(t)$$

due to the smoothness, and introduce the basic quantity

$$G(t, x) = \int_{-\infty}^x (2p - u^2)(t, y) dy + F(t, x) = 2p_x(t, x) + 2F(t, x).$$

The key function here is the (spatial) inverse of the strictly increasing function G for fixed time t . To that end, we define

$$\mathcal{Y}(t, \eta) = \sup\{x \mid G(t, x) < \eta\}.$$

Formally we have that $G(t, \mathcal{Y}(t, \eta)) = \eta$ for all $\eta \in (0, 2C)$ and $\mathcal{Y}(t, G(t, x)) = x$ for all $x \in \mathbb{R}$. Here it is important to note that the domain of \mathcal{Y} depends on the total energy C . Next, we want to determine the time evolution of \mathcal{Y} . Direct formal calculations yield that

$$\mathcal{Y}_t(t, G(t, x)) + \mathcal{Y}_\eta(t, G(t, x))G_t(t, x) = 0, \quad (1.4a)$$

$$\mathcal{Y}_\eta(t, G(t, x))G_x(t, x) = 1. \quad (1.4b)$$

Thus we need to compute the time evolution of $G(t, x)$ before being able to compute the time evolution of $\mathcal{Y}(t, \eta)$. To that end, we find, after some computations, that

$$G_t(t, x) + uG_x(t, x) = \frac{2}{3}u^3(t, x) + S(t, x), \quad (1.5)$$

where

$$S(t, x) = \int_{\mathbb{R}} e^{-|x-y|} \left(\frac{2}{3} u^3 - u_x p_x - 2pu \right) (t, y) dy. \quad (1.6)$$

Introducing $\eta = G(t, x)$, $\mathcal{S}(t, \eta) = S(t, \mathcal{Y}(t, \eta))$, and

$$\mathcal{U}(t, \eta) = u(t, \mathcal{Y}(t, \eta)), \quad (1.7)$$

we find by combining (1.4) and (1.5) that

$$\mathcal{Y}_t(t, \eta) + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) (t, \eta) \mathcal{Y}_\eta(t, \eta) = \mathcal{U}(t, \eta),$$

where we used that $\mathcal{Y}(t, G(t, x)) = x$ for all $x \in \mathbb{R}$. As far as the time evolution of $\mathcal{U}(t, \eta)$ is concerned, we find

$$\mathcal{U}_t(t, \eta) = -\mathcal{Q}(t, \eta) - \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{U}_\eta(t, \eta),$$

where we introduced $\mathcal{Q}(t, \eta) = p_x(t, \mathcal{Y}(t, \eta))$. Thus, formally we end up with the system

$$\begin{aligned} \mathcal{Y}_t(t, \eta) + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{Y}_\eta(t, \eta) &= \mathcal{U}(t, \eta), \\ \mathcal{U}_t(t, \eta) + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{U}_\eta(t, \eta) &= -\mathcal{Q}(t, \eta). \end{aligned}$$

However, this system is not closed, and we need to introduce the function

$$\mathcal{P}(t, \eta) = p(t, \mathcal{Y}(t, \eta)), \quad (1.8)$$

and determine its time evolution. We find, after some computations, that

$$\mathcal{P}_t(t, \eta) + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{P}_\eta(t, \eta) = \mathcal{Q}\mathcal{U}(t, \eta) + \mathcal{R}(t, \eta),$$

where

$$\begin{aligned} \mathcal{R}(t, \eta) &= \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta + \mathcal{U} \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U} \mathcal{Q} \mathcal{Y}_\eta(t, \theta) d\theta. \end{aligned}$$

To summarize, we have established the following system of differential equations

$$\mathcal{Y}_t(t, \eta) + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{Y}_\eta(t, \eta) = \mathcal{U}(t, \eta), \quad (1.9a)$$

$$\mathcal{U}_t(t, \eta) + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{U}_\eta(t, \eta) = -\mathcal{Q}(t, \eta), \quad (1.9b)$$

$$\mathcal{P}_t(t, \eta) + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{P}_\eta(t, \eta) = \mathcal{Q}\mathcal{U}(t, \eta) + \mathcal{R}(t, \eta), \quad (1.9c)$$

where

$$\mathcal{Q}(t, \eta) = -\frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) d\theta, \quad (1.10a)$$

$$\mathcal{S}(t, \eta) = \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - \mathcal{U}_\eta \mathcal{Q} - 2\mathcal{P} \mathcal{U} \mathcal{Y}_\eta \right) (t, \theta) d\theta, \quad (1.10b)$$

$$\begin{aligned} \mathcal{R}(t, \eta) &= \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta + \mathcal{U} \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U} \mathcal{Q} \mathcal{Y}_\eta(t, \theta) d\theta. \end{aligned} \quad (1.10c)$$

Derived under assumptions of smoothness of the functions involved, the same system is valid also in the general case of weak conservation solutions. However, that requires considerable analysis, and Section 3 is devoted to that. The next step is to estimate the time evolution of these quantities $(\mathcal{Y}, \mathcal{U}, \mathcal{P})$. It turns out that the natural functional space is the space of square integrable functions for the unknowns $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$. For this reason, we prefer to work with $\mathcal{P}^{1/2}$ rather than \mathcal{P} . Section 3 focuses on the first qualitative properties of the time evolution of the solutions of (1.9) for weak conservative solutions of the CH equation (1.1) as well as the propagation in time of the L^2 -norm of the unknowns.

The main aim of our work is to identify the right distance between two general conservative solutions of the CH equation (1.1), or equivalently, between two general L^2 solutions $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ of system (1.9) with possibly different energies. In order to compare solutions with different energies, we need to rescale the solutions of (1.9) in such a way that they are defined on the same interval. Since the natural functional space for our unknowns $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ was identified as the L^2 -functional space, it seems natural to do a scaling conserving the L^2 -norms of the unknowns $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$, but leading to the domain being independent of the total energy C .

Let us define the scaled unknowns $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$ associated to a conservative solution $(u(t), \mu(t))$ with energy $C = \mu(t, \mathbb{R})$ of the CH equation (1.1) as $\tilde{\mathcal{Y}}(t, \eta) = \sqrt{2C} \mathcal{Y}(t, 2C\eta)$, $\tilde{\mathcal{U}}(t, \eta) = \sqrt{2C} \mathcal{U}(t, 2C\eta)$, and $\tilde{\mathcal{P}}^{1/2}(t, \eta) = \sqrt{2C} \mathcal{P}^{1/2}(t, 2C\eta)$, where $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ is the solution of (1.9). This scaling allows also for the zero solution to (1.1) to be included in our considerations, as outlined in Section 4. A similar system to (1.9) can be written for $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$, but this is postponed to Section 4. With this new set of unknowns in place, we can now define a metric to compare two general conservative solutions (u_i, μ_i) , $i = 1, 2$, of (1.1) with total

energy $C_i = \mu_i(\mathbb{R})$. We define it as

$$d((u_1, \mu_1), (u_2, \mu_2)) = \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|_{L^2([0,1])} + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|_{L^2([0,1])} \\ + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|_{L^2([0,1])} + |\sqrt{2C_1} - \sqrt{2C_2}|.$$

Our main result reads as follows.

THEOREM 1.1. *Consider initial data $u_{i,0} \in H^1(\mathbb{R})$, $\mu_{i,0} \in \mathcal{M}_+(\mathbb{R})$ such that $d(\mu_{ac})_{i,0} = (u_i^2 + u_{i,x}^2) dx$ and $C_i = \mu_i(\mathbb{R})$, and let (u_i, μ_i) for $i = 1, 2$ denote the corresponding weak conservative solutions of the CH equation (1.1). Then we have that*

$$d((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))) \leq e^{\mathcal{O}(1)t} d((u_{1,0}, \mu_{1,0}), (u_{2,0}, \mu_{2,0})),$$

where $\mathcal{O}(1)$ denotes a constant depending only on $\max_j(C_j)$ remaining bounded as $\max_j(C_j) \rightarrow 0$.

The main core of this work lies in estimating the Lipschitz property of the right-hand side of the equivalent system to (1.9) in the L^2 -sense for the unknowns $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$. This is much easier in case we compare to the zero solution as it coincides with the propagation of the L^2 -norms of the unknowns. Due to the intricate nonlinearities of the right-hand sides of (1.9), this leads in the general case to long detailed technical estimates that are displayed in full in Subsections 5.1, 5.2, and 5.3. In the case of peakon–antipeakon solutions, as solution (1.3) is denoted, all quantities described in this paper can be computed explicitly. The details are to be found in Appendix B.

A notational comment is in order. We decided to denote by $\mathcal{O}(1)$ constants depending on $\max_j(C_j)$ that may change from line to line along the proofs, but remain bounded as $\max_j(C_j) \rightarrow 0$. Explicit tracking of the constants could be possible but it is highly cumbersome and avoided for the sake of the reader.

2. Formal ideas: transformations with smoothness

Let us start by explaining all the mathematical details for the transformation in the case of smooth solutions as outlined in the introduction. Let $(u(t, \cdot), \mu(t, \cdot))$ be a weak conservative solution to the CH equation with total energy $\mu(t, \mathbb{R}) = C > 0$. We assume that $F(t, x)$, given by

$$F(t, x) = \int_{-\infty}^x d\mu(t), \quad (2.1)$$

is increasing and smooth, and, in particular, that $\mu = \mu_{ac} = (u^2 + u_x^2) dx$ for all t . Introduce the function

$$\begin{aligned} G(t, x) &= \int_{-\infty}^x (2p - u^2)(t, y) dy + F(t, x) \\ &= 2p_x(t, x) + 2F(t, x), \end{aligned}$$

where we used integration by parts and (1.2). First of all, note that the function $G(t, x)$ satisfies

$$\lim_{x \rightarrow -\infty} G(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} G(x) = 2C,$$

since $|p_x(t, x)| \leq p(t, x)$ and p is an H^1 function on the line due to (1.2). Moreover, the function $(2p - u^2)(t, x) \geq 0$ for all $(t, x) \in \mathbb{R}^2$ as the following computation shows,

$$\begin{aligned} (2p - u^2)(t, x) &= \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (u^2 + F_x)(t, y) dy - u^2(t, x) \\ &= \frac{1}{2} \int_{-\infty}^x e^{y-x} u^2(t, y) dy + \frac{1}{2} \int_x^{\infty} e^{x-y} u^2(t, y) dy \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} F_x(t, y) dy - u^2(t, x) \\ &= \frac{1}{2} u^2(t, x) - \frac{1}{2} \int_{-\infty}^x e^{y-x} 2uu_x(t, y) dy \\ &\quad + \frac{1}{2} u^2(t, x) + \frac{1}{2} \int_x^{\infty} e^{x-y} 2uu_x(t, y) dy \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} F_x(t, y) dy - u^2(t, x) \\ &\geq \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (F_x(t, y) - 2|uu_x|(t, y)) dy \\ &= \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (|u(t, y)| - |u_x(t, y)|)^2 dy \geq 0. \end{aligned} \quad (2.2)$$

Thus the function $G(t, x)$ is nondecreasing and, in our case, since the function $F(t, x)$ is smooth, also $G(t, x)$ is smooth.

REMARK 2.1. Estimate (2.2), that is, $2p - u^2 \geq 0$, remains valid also in the case where the functions are nonsmooth.

Last but not least, we want to make sure that $G(t, x)$ is strictly increasing, so that its pseudoinverse will have no jumps. $F(t, x)$ is constant if and only if $d\mu$,

u , and u_x are equal to zero. Therefore assume that there exists (for fixed t) some interval $[b, c]$ such that $d\mu(t, x) = u(t, x) = u_x(t, x) = 0$ for all $x \in [b, c]$. Then the only term that can save us is $p(t, x)$, which in general satisfies $p(t, x) \geq 0$ for all $(t, x) \in \mathbb{R}^2$. However, whenever $\mu(t, \mathbb{R}) \neq 0$, one has by its definition in (1.2) that $p(t, x) > 0$ and the claim follows.

Thus the function $G(t, x)$ is strictly increasing and continuous, and we can consider its pseudoinverse $\mathcal{Y}: [0, 2C] \rightarrow \mathbb{R}$, which in this case coincides with its inverse and which is given by

$$\mathcal{Y}(t, \eta) = \sup\{x \mid G(t, x) < \eta\}. \quad (2.3)$$

Since $G(t, x)$ is strictly increasing and continuous, we have that $G(t, \mathcal{Y}(t, \eta)) = \eta$ for all $\eta \in (0, 2C)$ and $\mathcal{Y}(t, G(t, x)) = x$ for all $x \in \mathbb{R}$. By the smoothness assumption on F , direct calculations yield that

$$\mathcal{Y}_t(t, G(t, x)) + \mathcal{Y}_\eta(t, G(t, x))G_t(t, x) = 0, \quad (2.4a)$$

$$\mathcal{Y}_\eta(t, G(t, x))G_x(t, x) = 1. \quad (2.4b)$$

Thus we need to compute the time evolution of $G(t, x)$ before being able to compute the time evolution of $\mathcal{Y}(t, \eta)$. The following calculations are only valid in the case of smooth solutions, but we will show in the next section how to overcome this issue for weak conservative solutions. Since $e^{-|x-y|}/2$ is the integral kernel of $(-\partial_x^2 + 1)^{-1}$, we observe from (1.2) that p is the solution to

$$p - p_{xx} = \frac{1}{2}u^2 + \frac{1}{2}\mu$$

and hence

$$\begin{aligned} p_t - p_{txx} &= uu_t + \frac{1}{2}F_{xt} \\ &= -u^2u_x - up_x - \frac{1}{2}(uF_x)_x + \frac{1}{2}(u^3)_x - (pu)_x \\ &= \frac{1}{6}(u^3)_x - (pu)_x - \frac{1}{2}(uF_x)_x - up_x, \end{aligned}$$

where we used the abbreviation $\mu = F_x$. Thus we end up with

$$\begin{aligned} p_t(t, x) &= -\frac{1}{2} \int_{\mathbb{R}} \text{sign}(x-y) e^{-|x-y|} \left(\frac{1}{6}u^3 - pu - \frac{1}{2}uF_x \right)(t, y) dy \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} up_x(t, y) dy. \end{aligned}$$

Similar calculations yield that

$$p_{xt}(t, x) = -\frac{1}{6}u^3 + pu + \frac{1}{2}uF_x + \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} \left(\frac{2}{3}u^3 - u_xp_x - 2pu \right)(t, y) dy.$$

Thus we get for the time evolution of $G(t, x)$ that

$$G_t(t, x) + uG_x(t, x) = \frac{2}{3}u^3(t, x) + S(t, x), \quad (2.5)$$

where

$$S(t, x) = \int_{\mathbb{R}} e^{-|x-y|} \left(\frac{2}{3}u^3 - u_x p_x - 2pu \right)(t, y) dy.$$

Combining (2.4) and (2.5), we end up with

$$\mathcal{Y}_t(t, G(t, x)) + (\frac{2}{3}u^3(t, x) + S(t, x))\mathcal{Y}_\eta(t, G(t, x)) = u(t, x).$$

Introducing $\eta = G(t, x)$, we deduce

$$\mathcal{Y}_t(t, \eta) + (\frac{2}{3}u^3 + S)(t, \mathcal{Y}(t, \eta))\mathcal{Y}_\eta(t, \eta) = u(t, \mathcal{Y}(t, \eta)),$$

where we used that $\mathcal{Y}(t, G(t, x)) = x$ for all $x \in \mathbb{R}$. As far as the time evolution of

$$\mathcal{U}(t, \eta) = u(t, \mathcal{Y}(t, \eta)) \quad (2.6)$$

is concerned, we have

$$\begin{aligned} \mathcal{U}_t(t, \eta) &= u_t(t, \mathcal{Y}(t, \eta)) + u_x(t, \mathcal{Y}(t, \eta))\mathcal{Y}_t(t, \eta) \\ &= u_t(t, \mathcal{Y}(t, \eta)) + uu_x(t, \mathcal{Y}(t, \eta)) - (\frac{2}{3}u^3 + S)u_x(t, \mathcal{Y}(t, \eta))\mathcal{Y}_\eta(t, \eta) \\ &= -p_x(t, \mathcal{Y}(t, \eta)) - (\frac{2}{3}u^3 + S)u_x(t, \mathcal{Y}(t, \eta))\mathcal{Y}_\eta(t, \eta) \\ &= -\mathcal{Q}(t, \eta) - (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{U}_\eta(t, \eta), \end{aligned}$$

where we introduced $\mathcal{Q}(t, \eta) = p_x(t, \mathcal{Y}(t, \eta))$ and $\mathcal{S}(t, \eta) = S(t, \mathcal{Y}(t, \eta))$. Thus, formally we end up with the system

$$\mathcal{Y}_t(t, \eta) + (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{Y}_\eta(t, \eta) = \mathcal{U}(t, \eta), \quad (2.7a)$$

$$\mathcal{U}_t(t, \eta) + (\frac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{U}_\eta(t, \eta) = -\mathcal{Q}(t, \eta), \quad (2.7b)$$

where $\mathcal{Q}(t, \eta)$ and $\mathcal{S}(t, \eta)$ can be written as

$$\begin{aligned} \mathcal{Q}(t, \eta) &= -\frac{1}{4} \int_{\mathbb{R}} \text{sign}(\mathcal{Y}(t, \eta) - y) e^{-|\mathcal{Y}(t, \eta) - y|} (u^2(t, y) + F_x(t, y)) dy \\ &= -\frac{1}{4} \int_{\mathbb{R}} \text{sign}(\mathcal{Y}(t, \eta) - y) e^{-|\mathcal{Y}(t, \eta) - y|} (2(u^2(t, y) - p(t, y)) + G_x(t, y)) dy \\ &= -\frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) d\theta, \end{aligned}$$

and

$$\begin{aligned}\mathcal{S}(t, \eta) &= \int_{\mathbb{R}} e^{-|\mathcal{Y}(t, \eta) - y|} \left(\frac{2}{3} u^3 - u_x p_x - 2pu \right) (t, y) dy \\ &= \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - \mathcal{U}_\eta \mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta \right) (t, \theta) d\theta,\end{aligned}$$

with

$$\mathcal{P}(t, \eta) = p(t, \mathcal{Y}(t, \eta)). \quad (2.8)$$

It is then natural, in order to close system (2.7), that besides the quantities u and μ , p in the new variables must be considered. One main reason is that these three quantities turn up in the definition of G . We already computed before that

$$\begin{aligned}p_t(t, x) &= -\frac{1}{2} \int_{\mathbb{R}} \text{sign}(x - y) e^{-|x - y|} \left(\frac{1}{6} u^3 - pu - \frac{1}{2} u F_x \right) (t, y) dy \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} e^{-|x - y|} u p_x(t, y) dy \\ &= \frac{1}{2} \int_{\mathbb{R}} \text{sign}(x - y) e^{-|x - y|} \left(\frac{1}{3} u^3 + \frac{1}{2} u G_x \right) (t, y) dy \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} e^{-|x - y|} u p_x(t, y) dy,\end{aligned}$$

where we used that

$$G_x(t, x) = 2p(t, x) - u^2(t, x) + F_x(t, x).$$

Thus direct computations yield the additional equation

$$\mathcal{P}_t(t, \eta) + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{P}_\eta(t, \eta) = \mathcal{Q}\mathcal{U}(t, \eta) + \mathcal{R}(t, \eta),$$

where

$$\begin{aligned}\mathcal{R}(t, \eta) &= \frac{1}{4} \int_{\mathbb{R}} \text{sign}(\mathcal{Y}(t, \eta) - y) e^{-|\mathcal{Y}(t, \eta) - y|} \left(\frac{2}{3} u^3 + u G_x \right) (t, y) dy \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} e^{-|\mathcal{Y}(t, \eta) - y|} u p_x(t, y) dy \\ &= \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta + \mathcal{U} \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U} \mathcal{Q} \mathcal{Y}_\eta(t, \theta) d\theta.\end{aligned}$$

We summarize the result in the following proposition.

PROPOSITION 2.2. *Let (u, μ) denote a smooth solution of (1.1). Define \mathcal{Y} by (2.3), \mathcal{U} by (2.6), and \mathcal{P} by (2.8). Then the following system of differential equations holds:*

$$\mathcal{Y}_t(t, \eta) + (\tfrac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{Y}_\eta(t, \eta) = \mathcal{U}(t, \eta), \quad (2.9a)$$

$$\mathcal{U}_t(t, \eta) + (\tfrac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{U}_\eta(t, \eta) = -\mathcal{Q}(t, \eta), \quad (2.9b)$$

$$\mathcal{P}_t(t, \eta) + (\tfrac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{P}_\eta(t, \eta) = \mathcal{Q}\mathcal{U}(t, \eta) + \mathcal{R}(t, \eta), \quad (2.9c)$$

where

$$\mathcal{Q}(t, \eta) = -\frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) d\theta, \quad (2.10a)$$

$$\mathcal{S}(t, \eta) = \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - \mathcal{U}_\eta\mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta \right) (t, \theta) d\theta, \quad (2.10b)$$

$$\begin{aligned} \mathcal{R}(t, \eta) &= \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta + \mathcal{U} \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U}\mathcal{Q}\mathcal{Y}_\eta(t, \theta) d\theta. \end{aligned} \quad (2.10c)$$

In the next section we will derive this system of equations also in the general case without assuming smoothness of the quantities involved; see (3.28).

Let us finish this section by checking some properties of system (2.9), which will also hold in the case of weak conservative solutions as we will see in the next section.

The quantity $\frac{2}{3}\mathcal{U}^3 + \mathcal{S}$ is the velocity field of the three equations in (2.9). Instead of applying a characteristic method to estimate the solutions to this system, we will perform integration by parts by which the η -derivative of this quantity will naturally appear.

LEMMA 2.3. *Given (u, μ) a smooth solution of (1.1), then the solution to (2.9) satisfies*

$$|(\tfrac{2}{3}\mathcal{U}^3 + \mathcal{S})_\eta| \leq \mathcal{O}(1).$$

Proof. Since $G(t, \mathcal{Y}(t, \eta)) = 2p_x(t, \mathcal{Y}(t, \eta)) + 2F(t, \mathcal{Y}(t, \eta)) = \eta$, we have due to the smoothness that

$$G_x(t, \mathcal{Y}(t, \eta))\mathcal{Y}_\eta(t, \eta) = (2p - u^2 + F_x)(t, \mathcal{Y}(t, \eta))\mathcal{Y}_\eta(t, \eta) = 1.$$

Since $(2p - u^2)(t, x) \geq 0$ due to (2.2) and $(F_x - u^2)(t, x) \geq 0$ due to (2.1), we have that

$$\begin{aligned} 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) &\leq (2p - u^2 + F_x)(t, \mathcal{Y})\mathcal{Y}_\eta(t, \eta) = 1, \\ \mathcal{U}^2\mathcal{Y}_\eta(t, \eta) &\leq 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) \leq 1, \\ 2|\mathcal{U}\mathcal{U}_\eta(t, \eta)| &= 2|uu_x(t, \mathcal{Y})\mathcal{Y}_\eta(t, \eta)| \leq (u^2 + u_x^2)(t, \mathcal{Y})\mathcal{Y}_\eta(t, \eta) \\ &= F_x(t, \mathcal{Y})\mathcal{Y}_\eta(t, \eta) \leq (2p - u^2 + F_x)(t, \mathcal{Y})\mathcal{Y}_\eta(t, \eta) = 1. \end{aligned}$$

We conclude that

$$\mathcal{P}\mathcal{Y}_\eta(t, \eta) \leq \frac{1}{2}, \quad |\mathcal{U}\mathcal{U}_\eta(t, \eta)| \leq \frac{1}{2}, \quad \text{and} \quad \mathcal{U}^2\mathcal{Y}_\eta(t, \eta) \leq 1. \quad (2.11)$$

From the fact that the energy is conserved, it follows that $u(t) \in H^1(\mathbb{R})$ for all $t \geq 0$ and

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \sqrt{C}.$$

Therefore, the term $\mathcal{U}^2\mathcal{U}_\eta$ is bounded by $\mathcal{O}(1)$.

Furthermore, $\mathcal{S}_\eta(t, \eta)$ is bounded. In particular, one can establish that

$$\mathcal{S}_\eta(t, \eta) \leq \mathcal{O}(1)\mathcal{P}\mathcal{Y}_\eta(t, \eta),$$

which is going to play a key role. Indeed, by definition one has

$$S(t, x) = \int_{\mathbb{R}} e^{-|x-y|} \left(\frac{2}{3}u^3 - u_x p_x - 2pu \right) (t, y) dy,$$

and hence

$$S_x(t, x) = - \int_{\mathbb{R}} \text{sign}(x - y) e^{-|x-y|} \left(\frac{2}{3}u^3 - u_x p_x - 2pu \right) (t, y) dy.$$

Our aim is to show that

$$S_x(t, x) \leq \mathcal{O}(1)p(t, x).$$

First of all, note that we have

$$\begin{aligned} \left| \int_{\mathbb{R}} \text{sign}(x - y) e^{-|x-y|} \frac{2}{3}u^3(t, y) dy \right| &\leq \frac{8}{3} \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \frac{1}{4} \int_{\mathbb{R}} e^{-|x-y|} u^2(t, y) dy \\ &\leq \mathcal{O}(1)p(t, x). \end{aligned}$$

Moreover,

$$\begin{aligned} & \left| \int_{\mathbb{R}} \text{sign}(x-y) e^{-|x-y|} (u_x p_x + 2pu)(t, y) dy \right| \\ & \leq \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (2u^2 + u_x^2 + 3p^2)(t, y) dy \\ & \leq 2p(t, x) + \frac{3}{2} \int_{\mathbb{R}} e^{-|x-y|} p^2(t, y) dy, \end{aligned}$$

since $|p_x(t, x)| \leq p(t, x)$. Thus it remains to show that the last term can be bounded by a multiple of $p(t, x)$. Our reasoning will be based on integration by parts and the fact that

$$p(t, x) - p_{xx}(t, x) = \frac{1}{2}u^2(t, x) + \frac{1}{2}F_x(t, x).$$

Indeed, first direct computations yield

$$\begin{aligned} & \int_{\mathbb{R}} e^{-|x-y|} p^2(t, y) dy \\ & = \int_{\mathbb{R}} e^{-|x-y|} p \left(\frac{1}{2}u^2 + \frac{1}{2}F_x \right) (t, y) dy + \int_{\mathbb{R}} e^{-|x-y|} pp_{xx}(t, y) dy \\ & = I_1(t, x) + I_2(t, x). \end{aligned}$$

Since $p(t, x) \leq \frac{1}{2} \int_{\mathbb{R}} F_x(t, y) dy \leq \frac{1}{2}C$, we have that

$$I_1(t, x) \leq \|p(t, \cdot)\|_{L^\infty(\mathbb{R})} \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (u^2 + F_x)(t, y) dy \leq 2\|p(t, x)\|_{L^\infty(\mathbb{R})} p(t, x).$$

As far as I_2 is concerned, we have

$$\begin{aligned} & \int_{-\infty}^x e^{y-x} pp_{xx}(t, y) dy \\ & = pp_x(t, x) - \int_{-\infty}^x e^{y-x} (pp_x + p_x^2)(t, y) dy \\ & = pp_x(t, x) - \frac{1}{2}p^2(t, x) + \int_{-\infty}^x e^{y-x} \left(\frac{1}{2}p^2 - p_x^2 \right) (t, y) dy, \end{aligned}$$

and

$$\begin{aligned} & \int_x^\infty e^{x-y} pp_{xx}(t, y) dy \\ & = -pp_x(t, x) - \int_x^\infty e^{x-y} (p_x^2 - pp_x)(t, y) dy \\ & = -pp_x(t, x) - \frac{1}{2}p^2(t, x) + \int_x^\infty e^{x-y} \left(\frac{1}{2}p^2 - p_x^2 \right) (t, y) dy. \end{aligned}$$

Thus

$$I_2(t, x) = -p^2(t, x) + \int_{\mathbb{R}} e^{-|x-y|} \left(\frac{1}{2} p^2 - p_x^2 \right) (t, y) dy$$

and subsequently

$$\begin{aligned} & \int_{\mathbb{R}} e^{-|x-y|} p^2(t, y) dy \\ & \leq 2 \|p(t, \cdot)\|_{L^\infty(\mathbb{R})} p(t, x) - p^2(t, x) + \int_{\mathbb{R}} e^{-|x-y|} \left(\frac{1}{2} p^2 - p_x^2 \right) (t, y) dy \\ & \leq 2 \|p(t, \cdot)\|_{L^\infty(\mathbb{R})} p(t, x) + \int_{\mathbb{R}} e^{-|x-y|} \left(\frac{1}{2} p^2 - p_x^2 \right) (t, y) dy. \end{aligned}$$

Reshuffling the terms, we end up with

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} p^2(t, y) dy & \leq \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} (p^2 + p_x^2)(t, y) dy \\ & \leq 2 \|p(t, \cdot)\|_{L^\infty(\mathbb{R})} p(t, x) \leq Cp(t, x), \end{aligned} \quad (2.12)$$

showing the desired estimate. \square

Next, we show that all properties seen in this section for smooth solutions remain true for weak conservative solutions to (1.1).

3. Rigorous transformation: weak conservative solutions

To accommodate for the wave breaking of the solutions, it has turned out to be advantageous to rewrite the CH equation from the original Eulerian variables into Lagrangian variables; see [6, 54]. We will show that the system of equations obtained in Proposition 2.2 holds for the weak conservative solutions introduced in [54]. With this aim in mind let us start by summarizing their approach, which uses the different adopted convention followed in this work for cumulative distribution functions to be continuous from the left and with limit from the right (caglad) at all points $x \in \mathbb{R}$. Therefore, both $F(t, \cdot)$ and $G(t, \cdot)$ are nondecreasing and caglad functions.

Given some initial data (u_0, μ_0) , the corresponding initial data in Lagrangian coordinates is then given by

$$y(0, \xi) = \sup\{x \mid x + F_0(x) < \xi\}, \quad (3.1a)$$

$$H(0, \xi) = \xi - y(0, \xi), \quad (3.1b)$$

$$U(0, \xi) = u(0, y(0, \xi)), \quad (3.1c)$$

and $(y(t, \cdot), U(t, \cdot), H(t, \cdot))$ are the solutions of

$$y_t(t, \xi) = U(t, \xi), \quad (3.2a)$$

$$U_t(t, \xi) = -Q(t, \xi), \quad (3.2b)$$

$$H_t(t, \xi) = (U^3 - 2PU)(t, \xi), \quad (3.2c)$$

where

$$P(t, \xi) = \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma, \quad (3.3a)$$

$$Q(t, \xi) = -\frac{1}{4} \int_{\mathbb{R}} \text{sign}(\xi - \sigma) e^{-|y(t, \xi) - y(t, \sigma)|} (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma. \quad (3.3b)$$

Moreover, the relation between P and Q is given by

$$P_{\xi}(t, \xi) = Q(t, \xi) y_{\xi}(t, \xi), \quad (3.4a)$$

$$Q_{\xi}(t, \xi) = (P - \frac{1}{2}U^2) y_{\xi}(t, \xi) - \frac{1}{2} H_{\xi}(t, \xi). \quad (3.4b)$$

Introduce the function

$$\begin{aligned} I(t, \xi) &= \int_{-\infty}^{\xi} (2P - U^2) y_{\xi}(t, \sigma) d\sigma = \int_{-\infty}^{\xi} (2Q_{\xi} + H_{\xi})(t, \sigma) d\sigma \\ &= 2Q(t, \xi) + H(t, \xi), \end{aligned}$$

where we used that

$$\lim_{\xi \rightarrow -\infty} Q(t, \xi) = 0 = \lim_{\xi \rightarrow -\infty} H(t, \xi),$$

which follows from the definition of $H(t, \xi)$. The relation between H and F is given by

$$F(t, y(t, \xi)) \leq H(t, \xi) \leq F(t, y(t, \xi) +).$$

Here we have introduced the common notation

$$\Phi(x \pm) = \lim_{\epsilon \downarrow 0} \Phi(x \pm \epsilon). \quad (3.5)$$

Notice that $I(t, \xi) + H(t, \xi)$ is the Lagrangian counterpart to the function $G(t, x)$. To convince oneself that this is really the case, one should take a quick look back first. The function $G(t, x)$ was defined as

$$G(t, x) = \int_{-\infty}^x (2p - u^2)(t, y) dy + F(t, x).$$

Thus, whenever $F(t, x)$ has a jump of height α at a point \bar{x} , that is, $\mu(t, \{\bar{x}\}) = \alpha$, then also $G(t, x)$ has a jump of height α at \bar{x} , since the function $2p - u^2$ is continuous. Furthermore, the point \bar{x} in Eulerian coordinates is mapped to some maximal interval $[\xi_l, \xi_r]$ in Lagrangian coordinates, on which $y_\xi(t, \xi) = 0$ and $H_\xi(t, \xi) = 1$ for a specific choice of a relabeling function. In fact, $H_\xi(t, \xi) > 0$ for all $\xi \in [\xi_l, \xi_r]$ as proven in [54, Theorem 4.2] and [54, Definition 2.6]. Thus a close look at $Q(t, \xi)$ reveals that $Q(t, \xi) = Q(t, \xi_l) - \frac{1}{2} \int_{\xi_l}^{\xi} H_\xi(t, \sigma) d\sigma$ for all $\xi \in [\xi_l, \xi_r]$ and hence

$$\begin{aligned} I(t, \xi) &= 2Q(t, \xi) + H(t, \xi) \\ &= 2Q(t, \xi_l) - \int_{\xi_l}^{\xi} H_\xi(t, \sigma) d\sigma + H(t, \xi_l) + \int_{\xi_l}^{\xi} H_\xi(t, \sigma) d\sigma \\ &= 2Q(t, \xi_l) + H(t, \xi_l) = I(t, \xi_l). \end{aligned}$$

In short, we have that $I(t, \xi) = I(t, \xi_l)$ for all $\xi \in [\xi_l, \xi_r]$. This allows us now to follow a similar approach as for the Hunter–Saxton (HS) equation in [9, 16]. Therefore introduce

$$J(t, \xi) = I(t, \xi) + H(t, \xi) = \int_{-\infty}^{\xi} (2P - U^2)y_\xi(t, \sigma) d\sigma + H(t, \xi), \quad (3.6)$$

and observe that for all solutions except the zero solution $J(t, \xi)$ is strictly increasing and continuous. In more detail, one has for all solutions except the zero solution that $P(t, \xi) \neq 0$ for all $\xi \in \mathbb{R}$, since $y_\xi + H_\xi > 0$ almost everywhere due to [54, Definition 2.6]. Moreover, if $H_\xi(t, \bar{\xi}) = 0$ for some $\bar{\xi}$ one has that $U(t, \bar{\xi}) = 0$ and $y_\xi(t, \bar{\xi}) \neq 0$, since $y_\xi H_\xi(t, \bar{\xi}) = U^2 y_\xi(t, \bar{\xi}) + U_\xi^2(t, \bar{\xi})$ almost everywhere due to [54, Definition 2.6], and hence the ξ -derivative of $J(t, \xi)$ is strictly positive at the point $\bar{\xi}$.

LEMMA 3.1. *Given $\mathcal{Y}(t, \eta) = \sup\{x \mid G(t, x) < \eta\}$, then*

$$\mathcal{Y}(t, \eta) = y(t, l(t, \eta)), \quad (3.7)$$

where we have introduced $l(t, \cdot): [0, 2C] \rightarrow \mathbb{R}$ by

$$l(t, \eta) = \sup\{\xi \mid J(t, \xi) < \eta\}. \quad (3.8)$$

Proof. For each time t we have

$$y(t, \xi) = \sup\{x \mid x + F(t, x) < y(t, \xi) + H(t, \xi)\},$$

which implies that

$$y(t, \xi) + F(t, y(t, \xi)) \leq y(t, \xi) + H(t, \xi) \leq y(t, \xi) + F(t, y(t, \xi) +).$$

Moreover, one has that $G(t, x) - F(t, x)$ is continuous, and, in particular,

$$\begin{aligned} G(t, y(t, \xi)) - F(t, y(t, \xi)) &= \int_{-\infty}^{y(t, \xi)} (2p - u^2)(t, y) dy \\ &= \int_{-\infty}^{\xi} (2P - U^2)y_{\xi}(t, \sigma) d\sigma = I(t, \xi). \end{aligned}$$

Thus one has

$$y(t, \xi) + G(t, y(t, \xi)) \leq y(t, \xi) + J(t, \xi) \leq y(t, \xi) + G(t, y(t, \xi) +).$$

Subtracting $y(t, \xi)$ in the above inequality, we end up with

$$G(t, y(t, \xi)) \leq J(t, \xi) \leq G(t, y(t, \xi) +) \quad \text{for all } \xi \in \mathbb{R}.$$

Comparing the last equation and (3.8), we have

$$G(t, y(t, l(t, \eta))) \leq J(t, l(t, \eta)) = \eta \leq G(t, y(t, l(t, \eta) +)).$$

Since $y(t, \cdot)$ is surjective and nondecreasing, we end up with

$$\mathcal{V}(t, \eta) = \sup\{x \mid G(t, x) < \eta\} = y(t, l(t, \eta)), \quad (3.9)$$

thereby proving (3.7). \square

In the next step we want to establish rigorously the corresponding system of differential equations. Hence we first have to establish that the function $l(t, \eta)$ is differentiable, both with respect to time and space.

3.1. The differentiability of $Q(t, \xi)$. The differentiability of $Q(t, \xi)$ with respect to ξ has been proven in [54] and $Q_{\xi}(t, \xi)$ is given by (3.4b). Thus it is left to establish the differentiability with respect to time of $Q(t, \xi)$. To be more precise, we are going to establish the Lipschitz continuity of $Q(t, \xi)$ with respect to time.

Let us recall that since $U(t, \cdot) \in H^1(\mathbb{R})$, one has in particular that $U(t, \cdot) \in L^{\infty}(\mathbb{R})$. Moreover, direct calculations yield

$$\begin{aligned} U^2(t, \xi) &= U^2(t, \xi) - U^2(t, -\infty) = \int_{-\infty}^{\xi} 2UU_{\xi}(t, \sigma) d\sigma \\ &\leq \int_{-\infty}^{\xi} H_{\xi}(t, \sigma) d\sigma \leq H(t, \infty) = C, \end{aligned} \quad (3.10)$$

and we end up with

$$\|U(t, \cdot)\|_{L^{\infty}(\mathbb{R})} \leq \sqrt{C} \quad \text{for all } t \in \mathbb{R}.$$

Here we used *Xavier's relation* from [54] which asserts that

$$U^2 y_\xi^2(t, \xi) + U_\xi^2(t, \xi) = y_\xi H_\xi(t, \xi), \quad (3.11)$$

and hence

$$|U_\xi(t, \xi)| \leq \sqrt{y_\xi H_\xi}(t, \xi) \leq \frac{1}{2}(y_\xi + H_\xi)(t, \xi), \quad (3.12a)$$

$$|U y_\xi(t, \xi)| \leq \sqrt{y_\xi H_\xi}(t, \xi) \leq \frac{1}{2}(y_\xi + H_\xi)(t, \xi), \quad (3.12b)$$

$$|U U_\xi(t, \xi)| \leq \frac{1}{2} H_\xi(t, \xi). \quad (3.12c)$$

Similar considerations apply for $P(t, \xi)$. Indeed one has

$$\begin{aligned} 0 \leq P(t, \xi) &= \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t, \sigma) - y(t, \xi)|} (U^2 y_\xi + H_\xi)(t, \sigma) d\sigma \\ &\leq \frac{1}{4} \int_{\mathbb{R}} 2H_\xi(t, \sigma) d\sigma = \frac{1}{2} C. \end{aligned} \quad (3.13)$$

Since $|Q(t, \xi)| \leq P(t, \xi)$, we end up with

$$\|Q(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2} C, \quad \|P(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2} C \quad \text{for all } t \in \mathbb{R}. \quad (3.14)$$

Direct calculations, using (3.11) and (3.12), yield

$$\begin{aligned} &-\mathcal{O}(1)(y_\xi(t, \xi) + H_\xi(t, \xi)) \\ &\leq (y_\xi(t, \xi) + H_\xi(t, \xi))_t \\ &= U_\xi(t, \xi) + 3U^2 U_\xi(t, \xi) - 2Q U y_\xi(t, \xi) - 2P U_\xi(t, \xi) \\ &\leq \mathcal{O}(1)(y_\xi(t, \xi) + H_\xi(t, \xi)), \end{aligned}$$

since $U(t, \xi)$, $P(t, \xi)$, and $Q(t, \xi)$ are uniformly bounded with respect to both space and time due to (3.10), (3.13), and (3.14). Thus, we have for $s < t$ that

$$\begin{aligned} &(y_\xi(s, \xi) + H_\xi(s, \xi))e^{-\mathcal{O}(1)(t-s)} \\ &\leq y_\xi(t, \xi) + H_\xi(t, \xi) \leq (y_\xi(s, \xi) + H_\xi(s, \xi))e^{\mathcal{O}(1)(t-s)} \end{aligned}$$

or equivalently

$$\begin{aligned} (y_\xi(t, \xi) + H_\xi(t, \xi))e^{-\mathcal{O}(1)(t-s)} &\leq (y_\xi(s, \xi) + H_\xi(s, \xi)) \\ &\leq (y_\xi(t, \xi) + H_\xi(t, \xi))e^{\mathcal{O}(1)(t-s)}. \end{aligned} \quad (3.15)$$

Recall that

$$\begin{aligned} Q(t, \xi) &= -\frac{1}{4} \int_{-\infty}^{\xi} e^{y(t, \sigma) - y(t, \xi)} (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma \\ &\quad + \frac{1}{4} \int_{\xi}^{\infty} e^{y(t, \xi) - y(t, \sigma)} (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma \\ &=: \frac{1}{4} Q_1(t, \xi) + \frac{1}{4} Q_2(t, \xi). \end{aligned}$$

We are only establishing the Lipschitz continuity with respect to time for $Q_1(t, \xi)$, since the argument for $Q_2(t, \xi)$ follows the same lines. Let $t < \hat{t}$. Then one has

$$\begin{aligned} &|Q_1(t, \xi) - Q_1(\hat{t}, \xi)| \\ &\leq \int_{-\infty}^{\xi} |e^{y(t, \sigma) - y(t, \xi)} - e^{y(\hat{t}, \sigma) - y(\hat{t}, \xi)}| (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma \\ &\quad + \int_{-\infty}^{\xi} e^{y(\hat{t}, \sigma) - y(\hat{t}, \xi)} |(U^2 y_{\xi} + H_{\xi})(t, \sigma) - (U^2 y_{\xi} + H_{\xi})(\hat{t}, \sigma)| d\sigma \\ &=: I_1 + I_2. \end{aligned}$$

As far as I_1 is concerned, we observe, using Lemma A.1(i), that

$$\begin{aligned} |e^{y(t, \sigma) - y(t, \xi)} - e^{y(\hat{t}, \sigma) - y(\hat{t}, \xi)}| &\leq |y(t, \sigma) - y(\hat{t}, \sigma)| + |y(t, \xi) - y(\hat{t}, \xi)| \\ &\leq \int_t^{\hat{t}} (|U(s, \sigma)| + |U(s, \xi)|) ds \leq \mathcal{O}(1)|t - \hat{t}|, \end{aligned}$$

where we used that both $e^{y(\hat{t}, \sigma) - y(\hat{t}, \xi)}$ and $e^{y(t, \sigma) - y(t, \xi)}$ are bounded from above by one, and that U can be uniformly bounded both with respect to space and time due to (3.10). Thus we end up with

$$\begin{aligned} I_1 &= \int_{-\infty}^{\xi} |e^{y(t, \sigma) - y(t, \xi)} - e^{y(\hat{t}, \sigma) - y(\hat{t}, \xi)}| (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma \\ &\leq \mathcal{O}(1)|t - \hat{t}| \int_{-\infty}^{\xi} 2H_{\xi}(t, \sigma) d\sigma \leq \mathcal{O}(1)|t - \hat{t}|, \end{aligned}$$

where we used that $U^2 y_{\xi} \leq H_{\xi}$.

Thus it remains to establish a similar estimate for I_2 . The idea is to use a similar strategy combined with (3.12a) and (3.15). We have

$$\begin{aligned} I_2 &= \int_{-\infty}^{\xi} e^{y(\hat{t}, \sigma) - y(\hat{t}, \xi)} |(U^2 y_{\xi} + H_{\xi})(t, \sigma) - (U^2 y_{\xi} + H_{\xi})(\hat{t}, \sigma)| d\sigma \\ &\leq \int_{-\infty}^{\xi} e^{y(\hat{t}, \sigma) - y(\hat{t}, \xi)} \int_t^{\hat{t}} |4U^2 U_{\xi} - 4QU y_{\xi} - 2PU_{\xi}|(s, \sigma) ds d\sigma \end{aligned}$$

$$\begin{aligned}
&\leq \int_{-\infty}^{\xi} e^{y(\hat{t}, \sigma) - y(\hat{t}, \xi)} \mathcal{O}(1) \int_t^{\hat{t}} (y_{\xi} + H_{\xi})(s, \sigma) ds d\sigma \\
&\leq \mathcal{O}(1) \int_{-\infty}^{\xi} e^{y(\hat{t}, \sigma) - y(\hat{t}, \xi)} \int_t^{\hat{t}} e^{\mathcal{O}(1)(\hat{t}-s)} (y_{\xi} + H_{\xi})(\hat{t}, \sigma) ds d\sigma \\
&\leq \mathcal{O}(1) \int_{-\infty}^{\xi} e^{y(\hat{t}, \sigma) - y(\hat{t}, \xi)} (y_{\xi} + H_{\xi})(\hat{t}, \sigma) \int_t^{\hat{t}} e^{\mathcal{O}(1)(\hat{t}-s)} ds d\sigma \\
&\leq \mathcal{O}(1) e^{\mathcal{O}(1)(\hat{t}-t)} |\hat{t} - t| \leq \mathcal{O}(1) |\hat{t} - t|,
\end{aligned}$$

under the assumption that $|\hat{t} - t| \leq 1$ (or in general bounded). Thus we established that

$$|Q_1(t, \xi) - Q_1(\hat{t}, \xi)| \leq \mathcal{O}(1) |\hat{t} - t|$$

for all pairs t, \hat{t} , such that $|\hat{t} - t| \leq 1$. Furthermore, one has

$$|Q(t, \xi) - Q(\hat{t}, \xi)| \leq \mathcal{O}(1) |\hat{t} - t|$$

for all pairs t, \hat{t} , such that $|\hat{t} - t| \leq 1$. Thus for fixed ξ , the function Q is locally Lipschitz with respect to time, and hence differentiable almost everywhere by Rademacher's theorem.

A close look at the above argument reveals that we cannot only differentiate $Q(t, \xi)$ with respect to time, but also that we can apply the dominated convergence theorem, which yields

$$\begin{aligned}
Q_t(t, \xi) &= -\frac{1}{4} \int_{\mathbb{R}} \frac{d}{dt} (\text{sign}(\xi - \sigma) e^{-|y(t, \xi) - y(t, \sigma)|} (U^2 y_{\xi} + H_{\xi})(t, \sigma)) d\sigma \\
&= -\frac{2}{3} U^3(t, \xi) + 2PU(t, \xi) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} \left(\frac{2}{3} U^3 y_{\xi} - 2PU y_{\xi} - QU_{\xi} \right)(t, \sigma) d\sigma \quad (3.16)
\end{aligned}$$

after some integrations by parts.

Next we want to show that $Q_t(t, \xi)$ can be uniformly bounded by a constant $\mathcal{O}(1)$. Observe that (3.10) and (3.13) imply

$$\begin{aligned}
&\left| \int_{\mathbb{R}} e^{-|y(t, \sigma) - y(t, \xi)|} \left(\frac{2}{3} U^3 y_{\xi} - 2PU y_{\xi} - QU y_{\xi} \right)(t, \sigma) d\sigma \right| \\
&\leq \mathcal{O}(1) \left(\int_{\mathbb{R}} e^{-|y(t, \sigma) - y(t, \xi)|} y_{\xi}(t, \sigma) d\sigma \right) \leq \mathcal{O}(1). \quad (3.17)
\end{aligned}$$

Thus recalling (3.16) and combining (3.10), (3.13), and (3.17), we finally end up with

$$\|Q_t(t, \cdot)\|_{L^{\infty}(\mathbb{R})} \leq \mathcal{O}(1) \quad \text{for all } t \in \mathbb{R}. \quad (3.18)$$

This completes the argument regarding the differentiability of Q . Notice that a direct application of the dominated convergence theorem using formula (3.16) shows that Q_t is a continuous function in time, under the assumption that P_t exists and satisfies an estimate similar to (3.18).

Next we focus on the differentiability with respect to time of P . An analogous argument to the one for Q leads to the differentiability in time of P . Let us show that the derivative of P with respect to time exists by applying the dominated convergence theorem to

$$\lim_{s \rightarrow t} \frac{P(t, \xi) - P(s, \xi)}{t - s}, \quad (3.19)$$

where ξ is chosen such that y is differentiable with respect to time. Since

$$\frac{d}{dt}(e^{-|y(t, \xi) - y(t, \sigma)|}(U^2 y_\xi + H_\xi)(t, \sigma))$$

exists almost everywhere, it is left to show that we can find a function in $L^1(\mathbb{R})$, which bounds the integrand of (3.19) uniformly in s . Therefore observe that we can write, using (3.2), for $s < t$, that

$$\begin{aligned} P(t, \xi) - P(s, \xi) &= \frac{1}{4} \int_{-\infty}^{\xi} \int_s^t e^{y(l, \sigma) - y(l, \xi)} (U(l, \sigma) - U(l, \xi))(U^2 y_\xi + H_\xi)(l, \sigma) dl d\sigma \\ &\quad + \frac{1}{4} \int_{\xi}^{\infty} \int_s^t e^{y(l, \xi) - y(l, \sigma)} (U(l, \xi) - U(l, \sigma))(U^2 y_\xi + H_\xi)(l, \sigma) dl d\sigma \\ &\quad + \frac{1}{4} \int_{\mathbb{R}} \int_s^t e^{-|y(l, \sigma) - y(l, \xi)|} (4U^2 U_\xi - 4U Q y_\xi - 2P U_\xi)(l, \sigma) dl d\sigma \\ &= I_1 + I_2 + I_3. \end{aligned}$$

As far as the first term is concerned, observe that (3.15) implies that

$$\begin{aligned} &\left| \int_s^t e^{y(l, \sigma) - y(l, \xi)} (U(l, \sigma) - U(l, \xi))(U^2 y_\xi + H_\xi)(l, \sigma) dl \right| \\ &\leq \mathcal{O}(1) \int_s^t e^{y(l, \sigma) - y(l, \xi)} |(U^2 y_\xi + H_\xi)(l, \sigma)| dl \\ &\leq \mathcal{O}(1) e^{\mathcal{O}(1)|t-s|} e^{y(t, \sigma) - y(t, \xi)} (y_\xi + H_\xi)(t, \sigma) |t - s| \\ &\leq \mathcal{O}(1) e^{y(t, \sigma) - y(t, \xi)} (y_\xi + H_\xi)(t, \sigma) |t - s|, \end{aligned}$$

if we assume that $|t - s| \leq 1$ or any other fixed constant. Moreover, the function in the last line belongs to $L^1(\mathbb{R})$. The remaining two terms can be bounded by a function, which is of the same form and belongs to $L^1(\mathbb{R})$ uniformly in s .

Thus all assumptions of the dominated convergence theorem are fulfilled, and we end up with

$$\begin{aligned}
 P_t(t, \xi) &= U Q(t, \xi) + \frac{1}{4} \int_{\mathbb{R}} \text{sign}(\xi - \sigma) e^{-|y(t, \xi) - y(t, \sigma)|} U (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma \\
 &\quad + \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} (4U^2 U_{\xi} - 4U Q y_{\xi} - 2P U_{\xi})(t, \sigma) d\sigma \\
 &= U Q(t, \xi) + \frac{1}{2} \int_{\mathbb{R}} \text{sign}(\xi - \sigma) e^{-|y(t, \xi) - y(t, \sigma)|} \\
 &\quad \times U \left(\frac{1}{3} U^2 y_{\xi} + Q_{\xi} + H_{\xi} \right)(t, \sigma) d\sigma \\
 &\quad - \frac{1}{2} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} U Q y_{\xi}(t, \sigma) d\sigma,
 \end{aligned} \tag{3.20}$$

where we used integration by parts together with (3.4b) in the last step. Moreover, we have that

$$\begin{aligned}
 |P_t(t, \xi)| &\leq |U Q(t, \xi)| + \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} |U| (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma \\
 &\quad + \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} (4U^2 |U_{\xi}| + 4|U Q| y_{\xi} + 2P |U_{\xi}|)(t, \sigma) d\sigma \\
 &\leq \|U(t, \cdot)\|_{L^{\infty}} P(t, \xi) \\
 &\quad + \frac{1}{4} \|U(t, \cdot)\|_{L^{\infty}} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} (U^2 y_{\xi} + H_{\xi})(t, \sigma) d\sigma \\
 &\quad + \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} (2U^4 y_{\xi} + 3H_{\xi} + 2U^2 y_{\xi} + 3P^2 y_{\xi})(t, \sigma) d\sigma \\
 &\leq \mathcal{O}(1) P(t, \xi) + \frac{3}{4} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} P^2 y_{\xi}(t, \sigma) d\sigma \\
 &\quad + \frac{1}{4} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} (2\|U(t, \cdot)\|_{L^{\infty}}^2 U^2 y_{\xi} + 3H_{\xi} + 2U^2 y_{\xi})(t, \sigma) d\sigma \\
 &\leq \mathcal{O}(1) P(t, \xi) + \frac{3}{4} \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} P^2 y_{\xi}(t, \sigma) d\sigma \\
 &\leq \mathcal{O}(1) \left(P(t, \xi) + \int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} P^2 y_{\xi}(t, \sigma) d\sigma \right)
 \end{aligned}$$

due to (3.10). It remains to show that

$$\int_{\mathbb{R}} e^{-|y(t, \xi) - y(t, \sigma)|} P^2 y_{\xi}(t, \sigma) d\eta \leq \mathcal{O}(1) P(t, \xi).$$

The proof follows the same lines as the one of (2.12) in Eulerian coordinates.

3.2. The differentiability of $l(t, \eta)$. Finally, we can start thinking about the differentiability with respect to space and time of $l(t, \eta)$. Therefore, recall that the function $l(t, \eta)$ is defined as

$$l(t, \eta) = \sup\{\xi \in \mathbb{R} \mid J(t, \xi) < \eta\}.$$

Differentiability with respect to η : For every $t \in \mathbb{R}$, the function $J(t, \cdot) = 2Q(t, \cdot) + 2H(t, \cdot)$ is strictly increasing and continuous, and hence differentiable almost everywhere with respect to ξ . Its inverse $J^{-1}(t, \cdot): [0, 2C] \rightarrow \mathbb{R}$ is therefore also strictly increasing and continuous, and hence differentiable almost everywhere. Since we have by definition that $J(t, l(t, \eta)) = \eta$ for all η , it follows immediately that $J(t, l(t, \cdot))$ is Lipschitz continuous with Lipschitz constant one and hence differentiable almost everywhere with respect to η . Since

$$J^{-1}(t, J(t, l(t, \eta))) = l(t, \eta) \quad \text{for all } \eta,$$

we can finally conclude that $l(t, \eta)$ is differentiable almost everywhere with respect to η . In particular, one has (cf. (3.6)) that

$$J_\xi(t, l(t, \eta))l_\eta(t, \eta) = 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) - \mathcal{U}^2\mathcal{Y}_\eta(t, \eta) + \mathcal{H}_\eta(t, \eta) = 1, \quad (3.21)$$

with

$$\begin{aligned} \mathcal{H}(t, \eta) &= H(t, l(t, \eta)), & \mathcal{P}(t, \eta) &= P(t, l(t, \eta)), \\ \text{and } \mathcal{U}(t, \eta) &= U(t, l(t, \eta)). \end{aligned}$$

These identities follow from equality (3.9) since $\mathcal{P}(t, \eta) = p(t, \mathcal{Y}(t, \eta))$ and $P(t, \xi) = p(t, y(t, \xi))$, and analogously $\mathcal{U}(t, \eta) = u(t, \mathcal{Y}(t, \eta))$ and $U(t, \xi) = u(t, y(t, \xi))$.

Following the same argument as in the smooth case (see (2.11)), we end up with

$$0 \leq 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) \leq 1, \quad 2|\mathcal{U}\mathcal{U}_\eta(t, \eta)| \leq 1, \quad \text{and} \quad 0 \leq \mathcal{U}^2\mathcal{Y}_\eta(t, \eta) \leq 1. \quad (3.22)$$

Differentiability with respect to t : As far as the differentiability of $l(t, \eta)$ with respect to time is concerned, the argument is a bit more involved. First of all, note that for any time t we have $J(t, l(t, \eta)) = \eta$ and hence

$$J(t, l(t, \eta)) = J(\tilde{t}, l(\tilde{t}, \eta)) \quad \text{for all } \eta \text{ and } \tilde{t}.$$

In particular, we can conclude that

$$\begin{aligned} J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta)) &= J(\tilde{t}, l(\tilde{t}, \eta)) - J(t, l(\tilde{t}, \eta)) \\ &= \int_t^{\tilde{t}} J_t(s, l(\tilde{t}, \eta)) ds \\ &= 2 \int_t^{\tilde{t}} (Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) ds. \end{aligned}$$

We already showed that $Q_t(t, \xi)$ exists and is given by (3.16). However, a close look reveals that $Q_t(t, \xi)$ is uniformly bounded both with respect to space and time and $Q_t(t, \xi)$ is continuous with respect to space for fixed t . The same holds for

$$H_t(t, \xi) = U^3(t, \xi) - 2PU(t, \xi).$$

Thus we have that

$$\lim_{\tilde{t} \rightarrow t} \frac{J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta))}{t - \tilde{t}} = \lim_{\tilde{t} \rightarrow t} 2 \int_t^{\tilde{t}} \frac{Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))}{t - \tilde{t}} ds \quad (3.23)$$

from a formal point of view. Thus it is left to establish that the limit exists and is finite.

Since $J_t(t, \xi) = 2Q_t(t, \xi) + 2H_t(t, \xi)$ and both $Q_t(t, \xi)$ and $H_t(t, \xi)$ can be bounded uniformly both in space and time by a constant only dependent on the total energy C , it follows that there exists a constant $\mathcal{O}(1)$ such that

$$|J_t(t, \xi)| \leq \mathcal{O}(1) \quad \text{for all } t \text{ and } \xi.$$

Thus it follows that for all $s \in [t - |t - \tilde{t}|, t + |t - \tilde{t}|]$, we have

$$J(s, \xi) \in [J(t, \xi) - \mathcal{O}(1)|\tilde{t} - t|, J(t, \xi) + \mathcal{O}(1)|\tilde{t} - t|].$$

Accordingly we can conclude, since $J(t, l(t, \eta)) = \eta$ for all η , that

$$J(s, l(t, \eta)) \in [\eta - \mathcal{O}(1)|\tilde{t} - t|, \eta + \mathcal{O}(1)|\tilde{t} - t|],$$

and hence

$$l(t, \eta) = l(s, \eta(s)) \quad \text{for some } \eta(s) \in [\eta - \mathcal{O}(1)|\tilde{t} - t|, \eta + \mathcal{O}(1)|\tilde{t} - t|], \quad (3.24)$$

and, in particular,

$$|\eta - \eta(s)| \leq \mathcal{O}(1)|\tilde{t} - t| \quad \text{for all } s \in [t - |t - \tilde{t}|, t + |t - \tilde{t}|].$$

Similar considerations yield that for all $s \in [\tilde{t} - |t - \tilde{t}|, \tilde{t} + |t - \tilde{t}|]$, we can find $\tilde{\eta}(s)$ such that

$$l(\tilde{t}, \eta) = l(s, \tilde{\eta}(s)) \quad \text{and} \quad |\eta - \tilde{\eta}(s)| \leq \mathcal{O}(1)|t - \tilde{t}|.$$

Let us return to the integral we are actually interested in. Namely,

$$\begin{aligned} \int_t^{\tilde{t}} J_t(s, l(\tilde{t}, \eta)) ds &= 2 \int_t^{\tilde{t}} (Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) ds \\ &= 2 \int_t^{\tilde{t}} (Q_t(s, l(t, \eta)) + H_t(s, l(t, \eta))) ds \\ &\quad + 2 \int_t^{\tilde{t}} ((Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) \\ &\quad - (Q_t(s, l(t, \eta)) + H_t(s, l(t, \eta)))) ds \\ &= \tilde{I}_1 + \tilde{I}_2. \end{aligned}$$

What we are hoping for is that the second term \tilde{I}_2 is of order $o(|t - \tilde{t}|)$, and hence plays no role when computing (3.23). Therefore observe that

$$\begin{aligned} H_{t\xi}(t, \xi) &= 3U^2U_\xi(t, \xi) - 2QUy_\xi(t, \xi) - 2PU_\xi(t, \xi), \\ Q_{t\xi}(t, \xi) &= -2U^2U_\xi(t, \xi) + 2QUy_\xi(t, \xi) + 2PU_\xi(t, \xi) \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} \text{sign}(\xi - \sigma) e^{-|y(t, \xi) - y(t, \sigma)|} \\ &\quad \times \left(\frac{2}{3} U^3 y_\xi - 2PUy_\xi - QU_\xi \right) (t, \sigma) d\sigma y_\xi(t, \xi). \end{aligned}$$

Notice that the second derivative $Q_{t\xi}$ is well defined by a dominated convergence argument as before for Q_t . Recalling (3.24), we have

$$\begin{aligned} \tilde{I}_2 &= \int_t^{\tilde{t}} J_t(s, l(\tilde{t}, \eta)) - J_t(s, l(t, \eta)) ds \\ &= 2 \int_t^{\tilde{t}} ((Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) - (Q_t(s, l(t, \eta)) + H_t(s, l(t, \eta)))) ds \\ &= 2 \int_t^{\tilde{t}} \int_{l(t, \eta)}^{l(\tilde{t}, \eta)} (Q_{t, \xi} + H_{t, \xi})(s, z) dz ds \\ &= 2 \int_t^{\tilde{t}} \int_{l(t, \eta)}^{l(\tilde{t}, \eta)} \left[U^2 U_\xi(s, z) \right. \\ &\quad \left. - \frac{1}{2} \int_{\mathbb{R}} \text{sign}(z - \sigma) e^{-|y(s, z) - y(s, \sigma)|} \right. \end{aligned}$$

$$\begin{aligned}
 & \times \left(\frac{2}{3} U^3 y_\xi - 2PUy_\xi - QU_\xi \right) (s, \sigma) d\sigma y_\xi(s, z) \Big] dz ds \\
 &= 2 \int_t^{\tilde{t}} \int_{l(s, \eta(s))}^{l(s, \tilde{\eta}(s))} \left[U^2 U_\xi(s, z) \right. \\
 & \quad \left. - \frac{1}{2} \int_{\mathbb{R}} \text{sign}(z - \sigma) e^{-|y(s, z) - y(s, \sigma)|} \right. \\
 & \quad \times \left. \left(\frac{2}{3} U^3 y_\xi - 2PUy_\xi - QU_\xi \right) (s, \sigma) d\sigma y_\xi(s, z) \right] dz ds \\
 &= 2 \int_t^{\tilde{t}} \int_{\eta(s)}^{\tilde{\eta}(s)} \mathcal{U}^2 \mathcal{U}_\eta(s, m) dm ds \\
 & \quad - \int_t^{\tilde{t}} \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} \text{sign}(m - n) e^{-|\mathcal{Y}(s, m) - \mathcal{Y}(s, n)|} \\
 & \quad \times \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta \right) (s, n) dn \mathcal{Y}_\eta(s, m) dm ds,
 \end{aligned}$$

where η , $\eta(s)$, and $\tilde{\eta}(s)$ satisfy

$$l(t, \eta) = l(s, \eta(s)) \quad \text{and} \quad l(\tilde{t}, \eta) = l(s, \tilde{\eta}(s)),$$

and

$$|\eta - \eta(s)| \leq \mathcal{O}(1)|t - \tilde{t}| \quad \text{and} \quad |\eta - \tilde{\eta}(s)| \leq \mathcal{O}(1)|t - \tilde{t}|. \quad (3.25)$$

As far as the first term is concerned, we can apply (3.10), (3.22) and (3.25) as follows:

$$\begin{aligned}
 & \left| 2 \int_t^{\tilde{t}} \int_{\eta(s)}^{\tilde{\eta}(s)} \mathcal{U}^2 \mathcal{U}_\eta(s, m) dm ds \right| \\
 & \leq \int_t^{\tilde{t}} \|\mathcal{U}(s, \cdot)\|_{L^\infty([0, 2C])} |\eta(s) - \tilde{\eta}(s)| ds \leq \mathcal{O}(1)|t - \tilde{t}|^2. \quad (3.26)
 \end{aligned}$$

As far as the second and last term is concerned, we want to apply the Cauchy–Schwarz inequality to the integral term at first. Indeed,

$$\begin{aligned}
 & \left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} \text{sign}(m - n) e^{-|\mathcal{Y}(s, m) - \mathcal{Y}(s, n)|} \right. \\
 & \quad \times \left. \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta \right) (s, n) dn \mathcal{Y}_\eta(s, m) dm \right| \\
 & \leq \left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s, m) - \mathcal{Y}(s, n)|} \left[\frac{2}{3} \mathcal{U}^2 \sqrt{\mathcal{Y}_\eta} (\mathcal{U}^2 \mathcal{Y}_\eta)^{1/2} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& + 2\mathcal{P}\sqrt{\mathcal{Y}_\eta}(\mathcal{U}^2\mathcal{Y}_\eta)^{1/2} + \mathcal{P}\sqrt{\mathcal{Y}_\eta}\sqrt{\mathcal{H}_\eta} \Big] (s, n) \, dn \, \mathcal{Y}_\eta(s, m) \, dm \Big| \\
& \leq \left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \left(\frac{2}{3}\mathcal{U}^2\sqrt{\mathcal{Y}_\eta} + 3\mathcal{P}\sqrt{\mathcal{Y}_\eta} \right) \sqrt{\mathcal{H}_\eta}(s, n) \, dn \, \mathcal{Y}_\eta(s, m) \, dm \right| \\
& \leq \int_{\eta(s)}^{\tilde{\eta}(s)} \left(\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \left(\frac{2}{3}\mathcal{U}^2 + 3\mathcal{P} \right)^2 \mathcal{Y}_\eta(s, n) \, dn \right)^{1/2} \\
& \quad \times \left(\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \mathcal{H}_\eta(s, n) \, dn \right)^{1/2} \mathcal{Y}_\eta(s, m) \, dm \\
& \leq \int_{\eta(s)}^{\tilde{\eta}(s)} \left(\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \left(\frac{8}{9}\mathcal{U}^4 + 18\mathcal{P}^2 \right) \mathcal{Y}_\eta(s, n) \, dn \right)^{1/2} \\
& \quad \times \left(\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \mathcal{H}_\eta(s, n) \, dn \right)^{1/2} \mathcal{Y}_\eta(s, m) \, dm \\
& \leq \mathcal{O}(1) \left(\int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) \, dn \mathcal{Y}_\eta(s, m) \, dm \right)^{1/2} \\
& \quad \times \left(\int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \mathcal{H}_\eta(s, n) \, dn \mathcal{Y}_\eta(s, m) \, dm \right)^{1/2} \\
& \leq \mathcal{O}(1) \left(\int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) \, dn \mathcal{Y}_\eta(s, m) \, dm \right)^{1/2} \\
& \quad \times \left(\int_{\eta(s)}^{\tilde{\eta}(s)} 4\mathcal{P} \mathcal{Y}_\eta(s, m) \, dm \right)^{1/2} \\
& \leq \mathcal{O}(1) \left(\int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) \, dn \mathcal{Y}_\eta(s, m) \, dm \right)^{1/2} \\
& \quad \times \sqrt{|\eta(s) - \tilde{\eta}(s)|} \\
& \leq \mathcal{O}(1) \left(|t - \tilde{t}| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) \, dn \mathcal{Y}_\eta(s, m) \, dm \right)^{1/2}.
\end{aligned}$$

Thus it is left to show that

$$\left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) \, dn \mathcal{Y}_\eta(s, m) \, dm \right|$$

is bounded uniformly with respect to both space and time. Therefore observe that, since all the involved terms are positive, we have

$$\begin{aligned}
& \left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) \, dn \mathcal{Y}_\eta(s, m) \, dm \right| \\
& \leq \int_0^{2C} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) \, dn \mathcal{Y}_\eta(s, m) \, dm
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{2C} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \mathcal{Y}_\eta(s, m) dm (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) dn \\
&\leq 2 \int_0^{2C} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) dn \leq 6C,
\end{aligned}$$

where we used (3.22) in the last step. Thus we obtain

$$\begin{aligned}
&\left| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} \text{sign}(m-n) e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \right. \\
&\quad \times \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta \right)(s, n) dn \mathcal{Y}_\eta(s, m) dm \Big| \\
&\leq \mathcal{O}(1) \left(|t - \tilde{t}| \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P}) \mathcal{Y}_\eta(s, n) dn \mathcal{Y}_\eta(s, m) dm \right)^{1/2} \\
&\leq \mathcal{O}(1) |t - \tilde{t}|^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_t^{\tilde{t}} \int_{\eta(s)}^{\tilde{\eta}(s)} \int_0^{2C} \text{sign}(m-n) e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \right. \\
&\quad \times \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta \right)(s, n) dn \mathcal{Y}_\eta(s, m) dm ds \Big| \\
&\leq \mathcal{O}(1) |t - \tilde{t}|^{3/2}.
\end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27), we end up with

$$\begin{aligned}
|\tilde{I}_2| &= \left| 2 \int_t^{\tilde{t}} (Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) - (Q_t(s, l(t, \eta)) + H_t(s, l(t, \eta))) ds \right| \\
&\leq \mathcal{O}(1) (|t - \tilde{t}|^{3/2} + |t - \tilde{t}|^2).
\end{aligned}$$

As far as \tilde{I}_1 is concerned, we would like to apply the mean-value theorem. Note therefore that we showed before that the function $(Q_t + H_t)(t, \xi)$ is continuous with respect to time, and hence the function $(Q_t + H_t)(s, l(t, \eta))$, considered as a function of s , is continuous with respect to s . Thus we end up with

$$\tilde{I}_1 = 2 \int_t^{\tilde{t}} (Q_t + H_t)(s, l(t, \eta)) ds = 2(\tilde{t} - t)(Q_t + H_t)(\tilde{s}, l(t, \eta))$$

for some \tilde{s} between t and \tilde{t} .

Last but not least, we therefore have

$$\begin{aligned} 2 \int_t^{\tilde{t}} (Q_t + H_t)(s, l(\tilde{t}, \eta)) ds &= \tilde{I}_1 + \tilde{I}_2 \\ &= 2(\tilde{t} - t)(Q_t + H_t)(\tilde{s}, l(t, \eta)) + \mathcal{O}(1)|\tilde{t} - t|^{3/2} \\ &\quad + \mathcal{O}(1)|\tilde{t} - t|^2, \end{aligned}$$

for some \tilde{s} between t and \tilde{t} , which implies (cf. (3.23)) that

$$\begin{aligned} \lim_{\tilde{t} \rightarrow t} \frac{J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta))}{t - \tilde{t}} &= \lim_{\tilde{t} \rightarrow t} 2 \int_t^{\tilde{t}} \frac{(Q_t + H_t)(s, l(\tilde{t}, \eta))}{t - \tilde{t}} ds \\ &= 2(Q_t + H_t)(t, l(t, \eta)). \end{aligned}$$

Recalling that $J(t, l(t, \eta)) = \eta$ for all t and η , we have that

$$J^{-1}(t, J(t, l(t, \eta))) = l(t, \eta),$$

if we, as before, denote the inverse to $J(t, \cdot)$ by $J^{-1}(t, \cdot)$. Thus we have

$$\begin{aligned} l_t(t, \eta) &= \lim_{\tilde{t} \rightarrow t} \frac{l(t, \eta) - l(\tilde{t}, \eta)}{t - \tilde{t}} \\ &= \lim_{\tilde{t} \rightarrow t} \frac{J^{-1}(t, J(t, l(t, \eta))) - J^{-1}(t, J(t, l(\tilde{t}, \eta)))}{t - \tilde{t}} \\ &= \lim_{\tilde{t} \rightarrow t} \frac{J^{-1}(t, J(t, l(t, \eta))) - J^{-1}(t, J(t, l(\tilde{t}, \eta)))}{J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta))} \\ &\quad \times \lim_{\tilde{t} \rightarrow t} \frac{J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta))}{t - \tilde{t}}. \end{aligned}$$

We have established that the last limit on the right-hand side exists and the existence of the first one follows from the fact that $J^{-1}(t, \cdot)$ is differentiable almost everywhere.

REMARK 3.2. The above argument relies heavily on the fact that the function $l(t, \cdot): [0, 2C] \rightarrow \mathbb{R}$ is continuous and strictly increasing, the reason being that $J(t, \cdot)$ is continuous and strictly increasing. This is in contrast to the HS equation in [16], where the function $H(t, \cdot)$ is increasing, but not necessarily strictly increasing and its inverse may have jumps. In the HS equations, if $H(t, \cdot)$ is constant on some interval I , then $H(s, \cdot)$ will be constant on I for any s , since H is independent of time. Furthermore, this means that the jumps in the new coordinates might change in height with respect to time, but never change their position. This is the essential difference to the CH equation where the jumps would turn up and disappear again immediately, which motivates the choice of $G(t, x)$ and $J(t, \xi)$.

3.3. New system: the right quantities. Since we have shown that $l(t, \eta)$ is differentiable almost everywhere with respect to both space and time, we can now establish rigorously the system of differential equations in our new coordinates. Recall that $J(t, l(t, \eta)) = \eta$ and hence direct calculations yield

$$\begin{aligned} J_t(t, l(t, \eta)) + J_\xi(t, l(t, \eta))l_t(t, \eta) &= 0, \\ J_\xi(t, l(t, \eta))l_\eta(t, \eta) &= 1. \end{aligned}$$

According to the time evolution in Lagrangian coordinates, we thus end up with

$$\begin{aligned} l_\eta(t, \eta) &= \frac{1}{J_\xi(t, l(t, \eta))}, \\ l_t(t, \eta) &= -\frac{J_t(t, l(t, \eta))}{J_\xi(t, l(t, \eta))} = -J_t(t, l(t, \eta))l_\eta(t, \eta). \end{aligned}$$

As far as the new coordinates are concerned, we recall that

$$\begin{aligned} \mathcal{Y}(t, \eta) &= y(t, l(t, \eta)), \quad \mathcal{U}(t, \eta) = U(t, l(t, \eta)), \\ \text{and } \mathcal{P}(t, \eta) &= P(t, l(t, \eta)), \end{aligned}$$

and direct calculations using the differentiabilitys proved for Q , P , and l in Subsections 3.1 and 3.2 yield the following differential equations for the unknowns $(\mathcal{Y}, \mathcal{U}, \mathcal{P})$.

THEOREM 3.3. *Let (u, μ) denote a weak conservative solution of (1.1). Define \mathcal{Y} by (2.3), \mathcal{U} by (2.6), and \mathcal{P} by (2.8). Then the following system of differential equations holds:*

$$\mathcal{Y}_t(t, \eta) + \left(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\right)\mathcal{Y}_\eta(t, \eta) = \mathcal{U}(t, \eta), \quad (3.28a)$$

$$\mathcal{U}_t(t, \eta) + \left(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\right)\mathcal{U}_\eta(t, \eta) = -\mathcal{Q}(t, \eta), \quad (3.28b)$$

$$\mathcal{P}_t(t, \eta) + \left(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\right)\mathcal{P}_\eta(t, \eta) = \mathcal{Q}\mathcal{U}(t, \eta) + \mathcal{R}(t, \eta), \quad (3.28c)$$

where

$$\mathcal{Q}(t, \eta) = -\frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) d\theta, \quad (3.29a)$$

$$\mathcal{S}(t, \eta) = \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - \mathcal{U}_\eta\mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta \right)(t, \theta) d\theta, \quad (3.29b)$$

$$\begin{aligned}\mathcal{R}(t, \eta) &= \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta + \mathcal{U} \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U} \mathcal{Q} \mathcal{Y}_\eta(t, \theta) d\theta.\end{aligned}\quad (3.29c)$$

REMARK 3.4. This system is identical to the system derived in the smooth case (cf. (2.9)) as expected. We also remind the reader that $\mathcal{Q}(t, \eta) = \mathcal{Q}(t, l(t, \eta))$, $\mathcal{S}(t, \eta) = J_t(t, l(t, \eta)) - \frac{2}{3} \mathcal{U}^3(t, \eta)$, and $\mathcal{R}(t, \eta) = P_t(t, l(t, \eta)) - \mathcal{Q} \mathcal{U}(t, \eta)$.

3.4. Functional setting: consistency of the new coordinates. The next main question is which functional space we should work in such that the right-hand side of system (3.28) can be regarded as a Lipschitz function of the chosen unknowns. For instance, it is difficult or rather impossible to establish the Lipschitz continuity of \mathcal{Q} with respect to \mathcal{P} , \mathcal{U} , and \mathcal{Y} . However, we will see that one can establish that \mathcal{Q} is Lipschitz continuous with respect to $\mathcal{P}^{1/2}$, \mathcal{U} , and \mathcal{Y} . At first sight this seems to be surprising, but on the other hand we established in Eulerian coordinates that $2p(t, x) \geq u^2(t, x)$, which rewrites in our new coordinates as

$$2\mathcal{P}(t, \eta) \geq \mathcal{U}^2(t, \eta). \quad (3.30)$$

Thus it seems somehow natural that $\mathcal{P}(t, \eta)^{1/2}$ and $|\mathcal{U}(t, \eta)|$ will behave similarly.

The aim of this subsection is to derive the system of differential equations for \mathcal{Y} , \mathcal{U} , and $\mathcal{P}^{1/2}$ and subsequently to establish that all terms turning up in this system are well defined by assuming that each of the new variables $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ is in $L^2([0, 2C])$.

Replacing the equation for \mathcal{P} with the corresponding one for $\mathcal{P}^{1/2}$, we find that system (3.28) reads as

$$\mathcal{Y}_t + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{Y}_\eta = \mathcal{U}, \quad (3.31a)$$

$$\mathcal{U}_t + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) \mathcal{U}_\eta = -\mathcal{Q}, \quad (3.31b)$$

$$(\mathcal{P}^{1/2})_t + \left(\frac{2}{3} \mathcal{U}^3 + \mathcal{S} \right) (\mathcal{P}^{1/2})_\eta = \frac{\mathcal{Q} \mathcal{U}}{2\mathcal{P}^{1/2}} + \frac{\mathcal{R}}{2\mathcal{P}^{1/2}}, \quad (3.31c)$$

where \mathcal{Q} , \mathcal{S} , and \mathcal{R} are given by (3.29).

Assume for the moment that $\mathcal{P}^{1/2}$, \mathcal{U} , and \mathcal{Y} belong to $L^2([0, 2C])$. Then we want to show that all terms appearing in the above system also belong to $L^2([0, 2C])$. Therefore it is important to keep in mind, in addition to (3.30), that

$$2\mathcal{P} \mathcal{Y}_\eta(t, \eta) - \mathcal{U}^2 \mathcal{Y}_\eta(t, \eta) + \mathcal{H}_\eta(t, \eta) = 1 \quad (3.32)$$

and, in particular (cf. (3.12), (3.21), and (3.22)),

$$2|\mathcal{U}\mathcal{U}_\eta(t, \eta)| \leq \mathcal{H}_\eta(t, \eta) \leq 1, \quad (3.33a)$$

$$\mathcal{U}^2\mathcal{Y}_\eta(t, \eta) \leq \mathcal{H}_\eta(t, \eta) \leq 1, \quad (3.33b)$$

$$2|\mathcal{Q}|\mathcal{Y}_\eta(t, \eta) \leq 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) \leq 1. \quad (3.33c)$$

As an immediate consequence we obtain that

$$\mathcal{U}^3\mathcal{Y}_\eta(t, \eta) \quad \text{and} \quad \mathcal{U}^3\mathcal{U}_\eta(t, \eta)$$

are uniformly bounded. The last term of this form can be estimated as follows:

$$|\mathcal{U}^3(\mathcal{P}^{1/2})_\eta(t, \eta)| = \left| \frac{\mathcal{U}^3\mathcal{Q}\mathcal{Y}_\eta}{2\mathcal{P}^{1/2}}(t, \eta) \right| \leq \left| \frac{\mathcal{U}^3}{2\mathcal{P}^{1/2}}(t, \eta) \right| \leq \mathcal{U}^2(t, \eta),$$

and is therefore uniformly bounded. Here we used (3.30).

As far as \mathcal{S} is concerned, we want to establish that

$$|\mathcal{S}(t, \eta)| \leq \mathcal{O}(1)\mathcal{P}(t, \eta),$$

implying that

$$|\mathcal{S}\mathcal{Y}_\eta(t, \eta)| \leq \mathcal{O}(1)\mathcal{P}\mathcal{Y}_\eta(t, \eta) \leq \mathcal{O}(1),$$

$$|\mathcal{S}\mathcal{U}_\eta(t, \eta)| \leq \mathcal{S}^2\mathcal{Y}_\eta(t, \eta) + \mathcal{H}_\eta(t, \eta) \leq \mathcal{O}(1),$$

$$|\mathcal{S}(\mathcal{P}^{1/2})_\eta(t, \eta)| \leq \mathcal{O}(1)\mathcal{P}^{3/2}\mathcal{Y}_\eta(t, \eta) \leq \mathcal{O}(1).$$

Indeed, by the definition of \mathcal{S} and since $|\mathcal{Q}| \leq \mathcal{P}$ and (cf. (3.12a))

$$\mathcal{U}_\eta^2(t, \eta) \leq \mathcal{H}_\eta\mathcal{Y}_\eta(t, \eta),$$

we have

$$\begin{aligned} |\mathcal{S}(t, \eta)| &= \left| \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - \mathcal{U}_\eta\mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta \right)(t, \theta) d\theta \right| \\ &\leq \|\mathcal{U}(t, \cdot)\|_{L^\infty([0, 2C])} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U}^2\mathcal{Y}_\eta(t, \theta) d\theta \\ &\quad + 2 \left(\int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{P}^2\mathcal{Y}_\eta(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left(\int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} \mathcal{U}^2\mathcal{Y}_\eta(t, \theta) d\theta \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \mathcal{H}_\eta(t, \theta) d\theta \right)^{1/2} \\
& \leq 4\sqrt{C} \mathcal{P}(t, \eta) + 6 \left(\int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta \right)^{1/2} \mathcal{P}^{1/2}(t, \eta).
\end{aligned}$$

Thus it is left to show that

$$\int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta \leq \mathcal{O}(1) \mathcal{P}(t, \eta). \quad (3.34)$$

As several times before, the main tool will be integration by parts together with

$$\mathcal{P}(t, \eta) = \frac{1}{4} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} (2(\mathcal{U}^2 - \mathcal{P}) \mathcal{Y}_\eta(t, \theta) + 1) d\theta.$$

Direct computations yield

$$\begin{aligned}
& \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta \\
& = \mathcal{P}^2(t, \eta) - \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} 2\mathcal{P} \mathcal{Q} \mathcal{Y}_\eta(t, \theta) d\theta \\
& = \mathcal{P}^2(t, \eta) - 2\mathcal{P} \mathcal{Q}(t, \eta) \\
& \quad + \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} (2\mathcal{Q}^2 \mathcal{Y}_\eta + 2\mathcal{P} \mathcal{Q}_\eta)(t, \theta) d\theta \\
& = \mathcal{P}^2(t, \eta) - 2\mathcal{P} \mathcal{Q}(t, \eta) \\
& \quad + \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} 2(\mathcal{P}^2 + \mathcal{Q}^2) \mathcal{Y}_\eta(t, \theta) d\theta \\
& \quad + \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} (2(\mathcal{P} - \mathcal{U}^2) \mathcal{Y}_\eta(t, \theta) - 1) \mathcal{P}(t, \theta) d\theta. \quad (3.35)
\end{aligned}$$

Here we have used (3.4b) and (3.32). Thus, rearranging the terms, we end up with

$$\begin{aligned}
& \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta \\
& \leq \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} (\mathcal{P}^2 + 2\mathcal{Q}^2) \mathcal{Y}_\eta(t, \theta) d\theta \\
& = 2\mathcal{P} \mathcal{Q}(t, \eta) - \mathcal{P}^2(t, \eta)
\end{aligned}$$

$$\begin{aligned}
& + \int_0^\eta e^{\mathcal{Y}(t,\theta) - \mathcal{Y}(t,\eta)} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) \mathcal{P}(t, \theta) d\theta \\
& \leq 2\mathcal{P}^2(t, \eta) + 4\|\mathcal{P}(t, \cdot)\|_{L^\infty([0, 2C])} \mathcal{P}(t, \eta) \\
& \leq 6\|\mathcal{P}(t, \cdot)\|_{L^\infty([0, 2C])} \mathcal{P}(t, \eta).
\end{aligned} \tag{3.36}$$

Similar computations yield that

$$\begin{aligned}
\int_\eta^{2C} e^{\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)} \mathcal{P}^2 \mathcal{Y}_\eta(t, \theta) d\theta & \leq \int_\eta^{2C} e^{\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)} (\mathcal{P}^2 + 2\mathcal{Q}^2) \mathcal{Y}_\eta(t, \theta) d\theta \\
& = -\mathcal{P}^2(t, \eta) - 2\mathcal{P}\mathcal{Q}(t, \eta) \\
& \quad + \int_\eta^{2C} e^{\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)} \\
& \quad \times (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) \mathcal{P}(t, \theta) d\theta \\
& \leq 2\mathcal{P}^2(t, \eta) + 4\|\mathcal{P}(t, \cdot)\|_{L^\infty([0, 2C])} \mathcal{P}(t, \eta) \\
& \leq 6\|\mathcal{P}(t, \cdot)\|_{L^\infty([0, 2C])} \mathcal{P}(t, \eta).
\end{aligned}$$

This proves (3.34) due to (3.13).

There are two terms left to investigate. Namely, the term $\mathcal{Q}\mathcal{U}/\mathcal{P}^{1/2}$, and we easily find that

$$\left| \frac{\mathcal{Q}\mathcal{U}}{\sqrt{2}\mathcal{P}^{1/2}}(t, \eta) \right| \leq \mathcal{P}(t, \eta),$$

and hence it is uniformly bounded. The last term $\mathcal{R}/\mathcal{P}^{1/2}$ is a bit more involved. We have

$$\begin{aligned}
& |\mathcal{R}(t, \eta)| \\
& \leq \left| \frac{1}{4} \int_0^{2C} \text{sign}(\eta - \theta) e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta + \mathcal{U} \right)(t, \theta) d\theta \right| \\
& \quad + \left| \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} \mathcal{U} \mathcal{Q} \mathcal{Y}_\eta(t, \theta) d\theta \right| \\
& \leq \frac{1}{4} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} (2(\mathcal{U}^2 - \mathcal{P})\mathcal{Y}_\eta(t, \theta) + 1) |\mathcal{U}|(t, \theta) d\theta \\
& \quad + \frac{1}{4} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} 2\mathcal{P} |\mathcal{U}| \mathcal{Y}_\eta(t, \theta) d\theta \\
& \quad + \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} |\mathcal{U} \mathcal{Q}| \mathcal{Y}_\eta(t, \theta) d\theta \\
& \leq \|\mathcal{U}(t, \cdot)\|_{L^\infty([0, 2C])} \mathcal{P}(t, \eta) + \frac{1}{2} \int_0^{2C} e^{-|\mathcal{Y}(t,\eta) - \mathcal{Y}(t,\theta)|} (\mathcal{P}^2 + \mathcal{U}^2) \mathcal{Y}_\eta(t, \theta) d\theta \\
& \leq \mathcal{O}(1) \mathcal{P}(t, \eta).
\end{aligned} \tag{3.37}$$

Thus

$$\left| \frac{\mathcal{R}}{\mathcal{P}^{1/2}(t, \eta)} \right| \leq \mathcal{O}(1) \mathcal{P}^{1/2}(t, \eta),$$

and hence belongs to $L^2([0, 2C])$.

Later on, we will need in some of our estimates (cf. (3.3a)) that

$$\mathcal{P}(t, \eta) = \frac{1}{4} \int_0^{2C} e^{-|\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta)|} (\mathcal{U}^2 \mathcal{Y}_\eta + \mathcal{H}_\eta)(t, \theta) d\theta,$$

which implies the estimate

$$\int_0^\eta e^{-(\mathcal{Y}(t, \eta) - \mathcal{Y}(t, \theta))} (\mathcal{U}^2 \mathcal{Y}_\eta + \mathcal{H}_\eta)(t, \theta) d\theta \leq 4\mathcal{P}(t, \eta). \quad (3.38)$$

3.5. The choice of the distance. In order to motivate the choice of the distance between two solutions with the same energy and to outline the strategy for proving the Lipschitz continuity, let us come back to system (3.31). We observe that the unknowns $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ are advected by the same velocity field in the η variable. Moreover, since all terms make sense under the assumption that the unknowns belong to $L^2([0, 2C])$, as checked in the previous subsection, we consider a toy problem of the form

$$f_t + \mathcal{A}(f) f_\eta = \mathcal{B}(f),$$

where f might be a vector valued function in $L^2([0, 2C])$. This toy problem resembles our situation with the complicated operator dependencies on the unknowns. Within this functional framework, it is natural to look at the evolution of the L^2 -norm of the unknown f . Given two solutions f_1 and f_2 to $f_t + \mathcal{A}(f) f_\eta = \mathcal{B}(f)$, direct computations yield

$$\begin{aligned} \frac{d}{dt} \|f_1 - f_2\|_{L^2([0, 2C])}^2 &= 2 \int_0^{2C} (f_1 - f_2)(f_{1,t} - f_{2,t})(t, \theta) d\theta \\ &= -2 \int_0^{2C} (\mathcal{A}(f_1) f_{1,\eta} - \mathcal{A}(f_2) f_{2,\eta})(f_1 - f_2)(t, \theta) d\theta \\ &\quad + 2 \int_0^{2C} (\mathcal{B}(f_1) - \mathcal{B}(f_2))(f_1 - f_2)(t, \theta) d\theta \\ &= -2 \int_0^{2C} \mathcal{A}(f_1)(f_{1,\eta} - f_{2,\eta})(f_1 - f_2)(t, \theta) d\theta \\ &\quad - 2 \int_0^{2C} f_{2,\eta}(\mathcal{A}(f_1) - \mathcal{A}(f_2))(f_1 - f_2)(t, \theta) d\theta \end{aligned}$$

$$\begin{aligned}
& + 2 \int_0^{2C} (\mathcal{B}(f_1) - \mathcal{B}(f_2))(f_1 - f_2)(t, \theta) d\theta \\
& = \mathcal{A}(f_1)(f_1 - f_2)^2(t, 0) - \mathcal{A}(f_1)(f_1 - f_2)^2(t, 2C) \\
& \quad + \int_0^{2C} \mathcal{A}'(f_1) f_{1,\eta} (f_1 - f_2)^2(t, \theta) d\theta \\
& \quad - 2 \int_0^{2C} f_{2,\eta} (\mathcal{A}(f_1) - \mathcal{A}(f_2))(f_1 - f_2)(t, \theta) d\theta \\
& \quad + 2 \int_0^{2C} (\mathcal{B}(f_1) - \mathcal{B}(f_2))(f_1 - f_2)(t, \theta) d\theta,
\end{aligned}$$

where the first two terms in the last line are going to vanish if we impose the correct boundary conditions or rather if we have a good behavior of the solutions at both boundaries. Under this assumption, we can then use norm estimates if we know that $\mathcal{A}'(f_1) f_{1,\eta}$ and $f_{2,\eta}$ are uniformly bounded by a constant $\mathcal{O}(1)$ and that \mathcal{A} and \mathcal{B} are Lipschitz continuous with Lipschitz constant $\mathcal{O}(1)$ with respect to f to conclude that

$$\frac{d}{dt} \|f_1 - f_2\|_{L^2([0, 2C])}^2 \leq \mathcal{O}(1) \|f_1 - f_2\|_{L^2([0, 2C])}^2,$$

and Gronwall's lemma then implies

$$\|(f_1 - f_2)(t)\|_{L^2([0, 2C])} \leq \|(f_1 - f_2)(0)\|_{L^2([0, 2C])} e^{\mathcal{O}(1)t}.$$

Hence the strategy consists in proving first a propagation in time for the L^2 -norm of f in order to get bounds on the unknowns and check the validity of the approach. Then the second, and the most important and technical point, is to establish the Lipschitz estimates. We will come back to this point in Section 4, where we will also fix the following problem in the definition of our metric: the domain of definition of our unknowns depends on the total energy. This is unsatisfactory, if we want to compare solutions with different total energies.

3.6. Propagation of L^2 bounds: moment conditions. Consider (u, μ) a weak conservative solution of (1.1) and $f = (\mathcal{V}, \mathcal{U}, \mathcal{P}^{1/2})$, defined by (2.3), (2.6), and (2.8), the solution to system (3.31). We want to show that the L^2 -norm of f is propagated in time. Since both \mathcal{U} and \mathcal{P} are bounded functions due to (3.10) and (3.13), we are reduced to show the propagation of the L^2 -norm of \mathcal{V} . First of all, note that we have

$$\mathcal{V}(t, G(t, x)) = x \quad \text{for all } x \in \mathbb{R},$$

where $G(t, x)$ is given by

$$G(t, x) = \int_{-\infty}^x (2p - u^2)(t, y) dy + F(t, x).$$

Notice that the x -distributional derivative of G is a measure that we denoted by $\nu(t, \cdot) \in \mathcal{M}_+(\mathbb{R})$ given by

$$d\nu(t, x) = (2p - u^2)(t, x) + d\mu(t, x).$$

Since $\mathcal{Y}(t, \eta)$ is the pseudoinverse to $G(t, x)$, then \mathcal{Y} pushes forward the uniform distribution on $[0, 2C]$ to ν (see [67]) and then

$$\int_{\mathbb{R}} x^2 d\nu = \int_0^{2C} \mathcal{Y}^2(t, \eta) d\eta.$$

Thus $\mathcal{Y} \in L^2([0, 2C])$ is equivalent to $\nu(t, \cdot)$ having a finite second moment.

PROPOSITION 3.5. *Let (u, μ) denote a weak conservative solution of (1.1), and by $f = (\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ the solution to system (3.31); then*

$$\int_0^{2C} \mathcal{Y}^2(t, \eta) d\eta \leq e^{\mathcal{O}(1)t} \left(1 + \int_0^{2C} \mathcal{Y}^2(0, \eta) d\eta \right).$$

Proof. We show the propagation of the second moment in time using the Lagrangian coordinates. Note first that

$$\begin{aligned} \int_{\mathbb{R}} x^2 d\mu(t, x) &= \int_{\mathbb{R}} y^2 H_{\xi}(t, \xi) d\xi, \\ \int_{\mathbb{R}} x^2 u^2(t, x) dx &= \int_{\mathbb{R}} y^2 U^2 y_{\xi}(t, \xi) d\xi, \\ \int_{\mathbb{R}} x^2 p(t, x) dx &= \int_{\mathbb{R}} y^2 P y_{\xi}(t, \xi) d\xi. \end{aligned}$$

We start by computing the derivatives of $y^2 H_{\xi}$, $y^2 U^2 y_{\xi}$, and $y^2 P y_{\xi}$ with respect to time. Direct computations using the formulas in Section 3.1 then yield

$$\begin{aligned} |(y^2 H_{\xi})_t| &= |2yU H_{\xi} + y^2(3U^2 U_{\xi} - 2PU_{\xi} - 2QU y_{\xi})| \\ &\leq y^2 H_{\xi} + U^2 H_{\xi} + 3y^2 |U| H_{\xi} + 2y^2 P^2 y_{\xi} + 2y^2 H_{\xi} \\ &\leq \mathcal{O}(1)(y^2 H_{\xi} + y^2 P y_{\xi} + H_{\xi}) \end{aligned}$$

and

$$\begin{aligned} |(y^2 P y_\xi)_t| &= |2y U P y_\xi + y^2 P_t y_\xi + y^2 P U_\xi| \\ &\leq y^2 P^2 y_\xi + U^2 y_\xi + y^2 |P_t| y_\xi + y^2 P (y_\xi + H_\xi) \\ &\leq \mathcal{O}(1)(y^2 H_\xi + y^2 P y_\xi + H_\xi), \end{aligned}$$

where we used (3.20). Thus

$$\frac{d}{dt}(\|y^2 P y_\xi\|_{L^1} + \|y^2 H_\xi\|_{L^1}) \leq \mathcal{O}(1)(\|y^2 P y_\xi\|_{L^1} + \|y^2 H_\xi\|_{L^1} + C)$$

and

$$\int_{\mathbb{R}} (y^2 P y_\xi + y^2 H_\xi)(t, \xi) d\xi \leq e^{\mathcal{O}(1)t} \left(\int_{\mathbb{R}} (y^2 P y_\xi + y^2 H_\xi)(0, \xi) d\xi + C \right).$$

Since $y^2 U^2 y_\xi(t, \xi) \leq y^2 H_\xi(t, \xi)$, it follows that also

$$\int_{\mathbb{R}} y^2 U^2 y_\xi(t, \xi) d\xi$$

remains finite for all times, which proves the desired estimate since

$$\int_{\mathbb{R}} x^2 dv(t, x) = \int_{\mathbb{R}} x^2 (2p - u^2)(t, x) dx + \int_{\mathbb{R}} x^2 d\mu(t, x). \quad \square$$

The propagation of moments not only implies the feasibility of the strategy illustrated in Section 3.5 but also gives us a control on the ‘boundary terms’. In fact, since $(u^2 + u_x^2)(t, x) \leq d\mu(t, x)$ and $(2p - u^2)(t, x) \geq 0$, the previous result implies that

$$\int_{\mathbb{R}} x^2 p(t, x) dx, \quad \int_{\mathbb{R}} x^2 u^2(t, x) dx, \quad \int_{\mathbb{R}} x^2 u_x^2(t, x) dx, \quad \text{and} \quad \int_{\mathbb{R}} x^2 d\mu(t, x) \quad (3.39)$$

are all finite. We show next that (3.39) implies that

$$xu(t, x) \rightarrow 0 \quad \text{and} \quad x^2 p(t, x) \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty. \quad (3.40)$$

Indeed, (3.39) and $u \in H^1(\mathbb{R})$ imply that

$$xu(t, x) \in L^2(\mathbb{R}) \quad \text{and} \quad (xu(t, x))_x = u(t, x) + xu_x(t, x) \in L^2(\mathbb{R}).$$

Hence $xu(t, x)$ belongs to $H^1(\mathbb{R})$ for any fixed t and in particular $xu(t, x) \rightarrow 0$ as $x \rightarrow \pm\infty$ for any fixed t .

The argument for $x p(t, x)$ is a bit more involved, but follows the same lines. To be more precise, $p(t, x) \in L^2(\mathbb{R})$, $|p_x(t, x)| \leq p(t, x)$, and

$$\int_{\mathbb{R}} x^2 p^2(t, x) dx \leq \mathcal{O}(1) \int_{\mathbb{R}} x^2 p(t, x) dx < \infty,$$

imply that

$$x p(t, x) \in L^2(\mathbb{R}) \quad \text{and} \quad (x p(t, x))_x = p(t, x) + x p_x(t, x) \in L^2(\mathbb{R}).$$

Hence $x p(t, x)$ belongs to $H^1(\mathbb{R})$ for any fixed t and in particular $x p(t, x) \rightarrow 0$ as $x \rightarrow \pm\infty$.

Finally, it is left to show that $x \sqrt{p(t, x)} \rightarrow 0$ as $x \rightarrow \pm\infty$. Therefore note that $p(t, x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$|(x \sqrt{p(t, x)})_x| = \left| \sqrt{p(t, x)} + x \frac{p_x(t, x)}{2\sqrt{p(t, x)}} \right| \leq \sqrt{p(t, x)} + |x| \sqrt{p(t, x)},$$

which together with (3.39) implies that

$$x \sqrt{p(t, x)} \in L^2(\mathbb{R}) \quad \text{and} \quad (x \sqrt{p(t, x)})_x \in L^2(\mathbb{R}).$$

Hence $x \sqrt{p(t, x)}$ belongs to $H^1(\mathbb{R})$ for any fixed t and in particular $x^2 p(t, x) \rightarrow 0$ as $x \rightarrow \pm\infty$. These properties will allow us to carry out several integrations by parts when deriving the anticipated Lipschitz estimate.

4. The right metric: solutions with different energies

For the remaining part of this paper, we will consider two distinct solutions (u_j, μ_j) for $j = 1, 2$ of the CH equation (1.1) and estimate their difference using a carefully selected norm based on our new variables $(\mathcal{Y}_j, \mathcal{U}_j, \mathcal{P}_j^{1/2})$ for $j = 1, 2$ that yields a Lipschitz metric. The main idea remains to a large extent the one from the HS equation in [16], where a Lipschitz metric for measuring the distance between solutions with nonzero energy has been constructed.

We want to define a Lipschitz metric, which can measure the distance between solutions with different total energy based on our new coordinates. At first sight this seems to be impossible for two reasons:

1. The support of the new variables depends on the total energy since

$$\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2} : [0, 2C] \rightarrow \mathbb{R}.$$

2. In the case of the zero solution, that is, $(u, \mu) = (0, 0)$, the function \mathcal{Y} is not defined and the same applies to the new variables $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$.

The key idea is to use a rescaling; more precisely, given $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$, where $\mathcal{Y} : [0, 2C] \rightarrow \mathbb{R}$ with $C \neq 0$. Then we can introduce

$$\begin{aligned}\tilde{\mathcal{Y}}(t, \eta) &= A\mathcal{Y}(t, A^2\eta), & \tilde{\mathcal{U}}(t, \eta) &= A\mathcal{U}(t, A^2\eta), \\ \tilde{\mathcal{P}}^{1/2}(t, \eta) &= A\mathcal{P}^{1/2}(t, A^2\eta), & \tilde{\mathcal{H}}(t, \eta) &= A^3\mathcal{H}(t, A^2\eta),\end{aligned}\quad (4.1)$$

with

$$A = \sqrt{2C} \quad (4.2)$$

for notational purposes.

This rescaling has three main properties that motivate our choice:

1. The support of the variables $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$ is independent of the energy, which allows us to compare arbitrary solutions with nonzero energy.
2. In the last section, we discussed that the natural solution space for the unknowns $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ is $L^2([0, 2C])$. Since our rescaling preserves the L^2 -norm, the natural solution space for the unknowns $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$ is $L^2([0, 1])$.
3. The most important property is that this rescaling allows us to set

$$(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2}) = (0, 0, 0) \quad \text{for } (u, \mu) = (0, 0).$$

A close look at the definition of $\tilde{\mathcal{Y}}(t, \eta)$ reveals that $\tilde{\mathcal{Y}}(t, \eta)$ is the pseudoinverse to $\tilde{G}(t, x)$, which is given by

$$\tilde{G}(t, x) = \frac{1}{A^2} G\left(t, \frac{x}{A}\right) \quad \text{for } A \neq 0.$$

If one compares $\tilde{G}(t, x)$ with $G(t, x)$ for small A , then the graph of $G(t, x)$ is squeezed in the x direction, but stretched in the y direction (cf. Figure 1). Taking a sequence of functions $G_n(t, x)$ with A_n tending to zero, then the corresponding sequence of functions $\tilde{G}_n(t, x)$ tends to the Heaviside function, which has as pseudoinverse $\tilde{\mathcal{Y}}(t, \eta) = 0$ (cf. Figure 2).

Direct calculation yields the following theorem.

THEOREM 4.1. *Let (u, μ) denote a weak conservative solution of (1.1). Define $\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}$ and $\tilde{\mathcal{P}}^{1/2}$ by (4.1). Then the following system of differential equations holds:*

$$\tilde{\mathcal{Y}}_t + \left(\frac{2}{3} \frac{1}{A^5} \tilde{\mathcal{U}}^3 + \frac{1}{A^6} \tilde{\mathcal{S}} \right) \tilde{\mathcal{Y}}_\eta = \tilde{\mathcal{U}}, \quad (4.3a)$$

$$\tilde{\mathcal{U}}_t + \left(\frac{2}{3} \frac{1}{A^5} \tilde{\mathcal{U}}^3 + \frac{1}{A^6} \tilde{\mathcal{S}} \right) \tilde{\mathcal{U}}_\eta = -\frac{1}{A^2} \tilde{\mathcal{Q}}, \quad (4.3b)$$

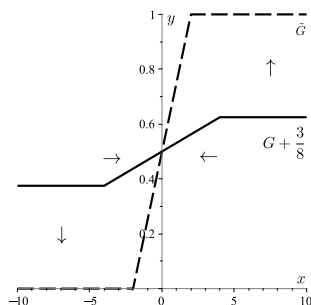


Figure 1. An example of how the rescaling changes the graph of a function $G : \mathbb{R} \rightarrow \frac{1}{4}$. The picture shows the function $G + \frac{3}{8}$ (solid) and its rescaled version \tilde{G} (dash).

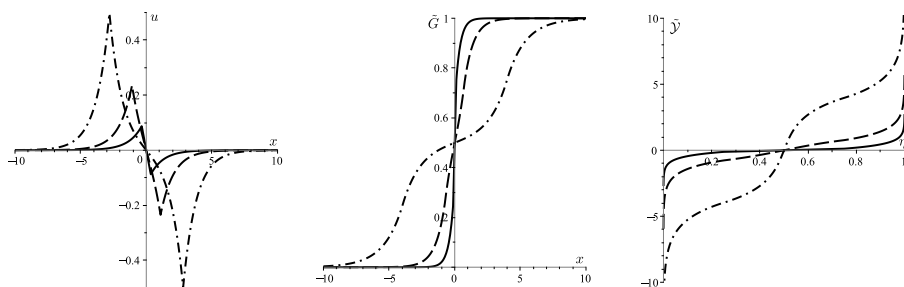


Figure 2. Three different peakon–antipeakon solutions u (left) with total energy 1 (dash–dot), 0.5 (dash), and 0.25 (solid) and the corresponding functions G (middle) and \tilde{Y} (right).

$$(\tilde{\mathcal{P}}^{1/2})_t + \left(\frac{2}{3} \frac{1}{A^5} \tilde{\mathcal{U}}^3 + \frac{1}{A^6} \tilde{\mathcal{S}} \right) (\tilde{\mathcal{P}}^{1/2})_\eta = \frac{1}{2A^2} \frac{\tilde{\mathcal{Q}}\tilde{\mathcal{U}}}{\tilde{\mathcal{P}}^{1/2}} + \frac{1}{2A^3} \frac{\tilde{\mathcal{R}}}{\tilde{\mathcal{P}}^{1/2}}, \quad (4.3c)$$

where

$$\begin{aligned} \tilde{\mathcal{Q}}(t, \eta) &= A^3 \mathcal{Q}(t, A^2 \eta) \\ &= -\frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{A} |\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta)|} (2(\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \theta) + A^5) d\theta, \end{aligned} \quad (4.4a)$$

$$\begin{aligned} \tilde{\mathcal{S}}(t, \eta) &= A^4 \mathcal{S}(t, A^2 \eta) \\ &= \int_0^1 e^{-\frac{1}{A} |\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta)|} \left(\frac{2}{3} \tilde{\mathcal{U}}^3 \tilde{\mathcal{Y}}_\eta - \tilde{\mathcal{U}}_\eta \tilde{\mathcal{Q}} - 2\tilde{\mathcal{P}} \tilde{\mathcal{U}} \tilde{\mathcal{Y}}_\eta \right) (t, \theta) d\theta, \end{aligned} \quad (4.4b)$$

$$\begin{aligned}\tilde{\mathcal{R}}(t, \eta) &= A^5 \mathcal{R}(t, A^2 \eta) \\ &= \frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{A} |\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta)|} \left(\frac{2}{3} A \tilde{\mathcal{U}}^3 \tilde{\mathcal{Y}}_\eta + A^6 \tilde{\mathcal{U}} \right) (t, \theta) \\ &\quad - \frac{1}{2} \int_0^1 e^{-\frac{1}{A} |\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta)|} \tilde{\mathcal{U}} \tilde{\mathcal{Q}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta.\end{aligned}\quad (4.4c)$$

Now, we can define the metric between two general conservative solutions to (1.1). To simplify the notation we will from now on assume that all norms are on $L^2([0, 1])$, unless otherwise indicated, and write

$$\|\Psi\| = \|\Psi\|_{L^2([0, 1])},$$

for any function Ψ . We will not always explicitly indicate the time dependence in all quantities and write $\|\Psi\|$ rather than $\|\Psi(t)\|$.

DEFINITION 4.2. Let (u_i, μ_i) for $i = 1, 2$ denote two conservative solutions of (1.1). We define the distance between them as

$$\begin{aligned}d((u_1, \mu_1), (u_2, \mu_2)) &= \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \\ &\quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| + |A_1 - A_2|,\end{aligned}$$

where $A_i = \sqrt{2C_i}$ and $C_i = \mu_i(\mathbb{R})$, $i = 1, 2$, and $(\tilde{\mathcal{Y}}_i, \tilde{\mathcal{U}}_i, \tilde{\mathcal{P}}_i^{1/2})$ are given by (4.1).

REMARK 4.3. Notice that in the case of one of the two solutions being the trivial solution, the distance reduces to the L^2 -norm of the solution in the right functional space, that is,

$$d((u, \mu), (0, 0)) = \|\tilde{\mathcal{Y}}\| + \|\tilde{\mathcal{U}}\| + \|\tilde{\mathcal{P}}^{1/2}\| + \sqrt{2C},$$

where $C = \mu(\mathbb{R})$, that we already estimated in Section 3.6. It remains to show that the right-hand side of system (4.3) is Lipschitz continuous with respect to the new unknowns $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$.

The main result of this work can finally be stated. The proof consists in showing the corresponding Lipschitz estimates in each of the components of the distance in Definition 4.2. This is done in the next section in Lemmas 5.3–5.5. Collecting these results leads to our main theorem due to Gronwall's lemma.

THEOREM 4.4. Consider initial data $u_{i,0} \in H^1(\mathbb{R})$, $\mu_{i,0} \in \mathcal{M}_+(\mathbb{R})$ such that $d(\mu_{ac})_{i,0} = (u_i^2 + u_{i,x}^2) dx$ and $C_i = \mu_i(\mathbb{R})$, and let (u_i, μ_i) for $i = 1, 2$ denote

the corresponding conservative solutions of the CH equation (1.1). Then we have that

$$d((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))) \leq e^{\mathcal{O}(1)t} d((u_{1,0}, \mu_{1,0}), (u_{2,0}, \mu_{2,0})),$$

where $\mathcal{O}(1)$ denotes a constant depending only on $\max_j(C_j)$ remaining bounded as $\max_j(C_j) \rightarrow 0$.

Now, let us start to do some preparatory work for the next section by collecting the main estimates we need in the new variables. Introduce

$$\begin{aligned} \tilde{\mathcal{D}}(t, \eta) &= \int_0^\eta e^{\frac{1}{A}(\tilde{\mathcal{Y}}(t, \theta) - \tilde{\mathcal{Y}}(t, \eta))} \left((\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \theta) + \frac{1}{2} A^5 \right) d\theta \\ &= \frac{1}{2} \int_0^\eta e^{\frac{1}{A}(\tilde{\mathcal{Y}}(t, \theta) - \tilde{\mathcal{Y}}(t, \eta))} (\tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta + \tilde{\mathcal{H}}_\eta)(t, \theta) d\theta, \end{aligned} \quad (4.5a)$$

$$\begin{aligned} \tilde{\mathcal{E}}(t, \eta) &= \int_\eta^1 e^{\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \left((\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \theta) + \frac{1}{2} A^5 \right) d\theta \\ &= \frac{1}{2} \int_\eta^1 e^{\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} (\tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta + \tilde{\mathcal{H}}_\eta)(t, \theta) d\theta, \end{aligned} \quad (4.5b)$$

so that we can write

$$\tilde{\mathcal{P}}(t, \eta) = \frac{1}{2A} (\tilde{\mathcal{D}}(t, \eta) + \tilde{\mathcal{E}}(t, \eta)) \quad \text{and} \quad \tilde{\mathcal{Q}}(t, \eta) = \frac{1}{2} (-\tilde{\mathcal{D}}(t, \eta) + \tilde{\mathcal{E}}(t, \eta)) \quad (4.6)$$

and

$$\tilde{\mathcal{Q}}(t, \eta) = -\tilde{\mathcal{D}}(t, \eta) + A\tilde{\mathcal{P}}(t, \eta). \quad (4.7)$$

Note that both $\tilde{\mathcal{D}}(t, \eta)$ and $\tilde{\mathcal{E}}(t, \eta)$ have some very nice properties. Namely,

$$0 \leq \tilde{\mathcal{D}}(t, \eta) \leq 2A\tilde{\mathcal{P}}(t, \eta) \quad \text{and} \quad 0 \leq \tilde{\mathcal{E}}(t, \eta) \leq 2A\tilde{\mathcal{P}}(t, \eta), \quad (4.8)$$

and

$$\left| \frac{d}{d\eta} \tilde{\mathcal{D}}(t, \eta) \right| = \left| (\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \eta) + \frac{1}{2} A^5 - \frac{1}{A} \tilde{\mathcal{D}} \tilde{\mathcal{Y}}_\eta(t, \eta) \right| \leq \mathcal{O}(1) A^5, \quad (4.9)$$

$$\left| \frac{d}{d\eta} \tilde{\mathcal{E}}(t, \eta) \right| = \left| -(\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \eta) - \frac{1}{2} A^5 + \frac{1}{A} \tilde{\mathcal{E}} \tilde{\mathcal{Y}}_\eta(t, \eta) \right| \leq \mathcal{O}(1) A^5. \quad (4.10)$$

The scaling leads to some changes in the standard estimates. Some of the problematic terms depending on $\frac{1}{A}$ in system (4.3) can be controlled since

$$\begin{aligned} 2\tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \eta) &\leq A^5, \quad 2|\tilde{\mathcal{U}} \tilde{\mathcal{U}}_\eta(t, \eta)| \leq A^4, \quad \tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta(t, \eta) \leq A^5, \\ \text{and} \quad \tilde{\mathcal{H}}_\eta(t, \eta) &\leq A^5, \end{aligned} \quad (4.11)$$

from (3.33) and (4.1), as well as

$$A^2 \tilde{\mathcal{U}}_\eta^2(t, \eta) \leq \tilde{\mathcal{H}}_\eta \tilde{\mathcal{Y}}_\eta(t, \eta) \leq A^5 \tilde{\mathcal{Y}}_\eta(t, \eta). \quad (4.12)$$

In addition, equation (3.32) becomes

$$2\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t, \eta) - \tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta(t, \eta) + \tilde{\mathcal{H}}_\eta(t, \eta) = A^5. \quad (4.13)$$

Moreover, we can show that

$$\begin{aligned} 4\|\tilde{\mathcal{P}}(t, \cdot)\|_{L^\infty} &= 4\|A^2\mathcal{P}(t, \cdot)\|_{L^\infty} \leq A^4 \\ 2\|\tilde{\mathcal{U}}^2(t, \cdot)\|_{L^\infty} &= 2\|A^2\mathcal{U}^2(t, \cdot)\|_{L^\infty} \leq A^4. \end{aligned} \quad (4.14)$$

We now collect all the estimates we will need on our solutions in the new variables with the explicit dependence on A . This will be repeatedly used in the next section.

$$0 \leq 4\tilde{\mathcal{P}} \leq A^4, \quad [\text{from (4.14) and (3.13)}], \quad (4.15a)$$

$$\sqrt{2}|\tilde{\mathcal{U}}| \leq A^2, \quad [\text{from (4.14) and (3.10)}], \quad (4.15b)$$

$$|\tilde{\mathcal{Q}}| \leq A\tilde{\mathcal{P}}, \quad [\text{from common sense}], \quad (4.15c)$$

$$\tilde{\mathcal{U}}^2 \leq 2\tilde{\mathcal{P}}, \quad [\text{from (3.30)}], \quad (4.15d)$$

$$2\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta \leq A^5, \quad [\text{from (4.11)}], \quad (4.15e)$$

$$2|\tilde{\mathcal{U}}\tilde{\mathcal{U}}_\eta| \leq A^4, \quad [\text{from (4.11)}], \quad (4.15f)$$

$$0 \leq \tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta \leq A^5, \quad [\text{from (4.11)}], \quad (4.15g)$$

$$\tilde{\mathcal{U}}_\eta^2 \leq A^3\tilde{\mathcal{Y}}_\eta, \quad [\text{from (4.12) and (3.12a)}], \quad (4.15h)$$

$$\sqrt{2}|\tilde{\mathcal{U}}|\tilde{\mathcal{P}}^{1/2}\tilde{\mathcal{Y}}_\eta \leq A^5, \quad [\text{from (4.15e) and (4.15g)}], \quad (4.15i)$$

$$|\tilde{\mathcal{R}}| \leq \mathcal{O}(1)A^3\tilde{\mathcal{P}}, \quad [\text{from (3.37)}], \quad (4.15j)$$

$$0 \leq \tilde{\mathcal{H}}_\eta \leq A^5, \quad [\text{from (4.11)}], \quad (4.15k)$$

$$0 \leq \tilde{\mathcal{Y}}_\eta, \quad [\text{from the definition}], \quad (4.15l)$$

$$A^2\tilde{\mathcal{U}}_\eta^2 \leq \tilde{\mathcal{H}}_\eta\tilde{\mathcal{Y}}_\eta, \quad [\text{from (4.12) and (3.12a)}], \quad (4.15m)$$

$$0 \leq \tilde{\mathcal{D}} \leq 2A\tilde{\mathcal{P}}, \quad [\text{from (4.8)}], \quad (4.15n)$$

$$0 \leq \tilde{\mathcal{E}} \leq 2A\tilde{\mathcal{P}}, \quad [\text{from (4.8)}], \quad (4.15o)$$

$$2\sqrt{2}\tilde{\mathcal{P}}\tilde{\mathcal{U}}_\eta \leq A^6, \quad [\text{from (4.15a), (4.15e), and (4.15h)}], \quad (4.15p)$$

$$2\tilde{\mathcal{P}}\tilde{\mathcal{U}}_\eta^2 \leq A^8, \quad [\text{from (4.15e) and (4.15h)}], \quad (4.15q)$$

$$4\tilde{\mathcal{P}}\tilde{\mathcal{U}}_\eta^2 \leq A^7\tilde{\mathcal{Y}}_\eta, \quad [\text{from (4.15a) and (4.15h)}]. \quad (4.15r)$$

In addition, we have several integrals that appear frequently:

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{U}}^2(t, \theta) d\theta \leq 6\tilde{\mathcal{P}}(t, \eta), \quad [\text{from (4.16g)}], \quad (4.16a)$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq \frac{3}{2} A^5 \tilde{\mathcal{P}}(t, \eta),$$

[from (3.36) and (4.15a)], (4.16b)

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 4A\tilde{\mathcal{P}}(t, \eta), \quad [\text{from (3.38)}], \quad (4.16c)$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{H}}_\eta(t, \theta) d\theta \leq 4A\tilde{\mathcal{P}}(t, \eta), \quad [\text{from (3.38)}], \quad (4.16d)$$

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 2A\tilde{\mathcal{P}}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16e)

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{H}}_\eta(t, \theta) d\theta \leq 4A\tilde{\mathcal{P}}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16f)

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{U}}^2(t, \theta) d\theta \leq 6\tilde{\mathcal{P}}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16g)

$$\int_0^\eta e^{-\frac{5}{4A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 4A\tilde{\mathcal{P}}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16h)

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}}(t, \theta) d\theta \leq 7\tilde{\mathcal{P}}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16i)

$$\int_0^\eta e^{-\frac{1}{2A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq A\mathcal{O}(1)\tilde{\mathcal{P}}^{1/2}(t, \eta), \quad [\text{from Lemma A.10}],$$

(4.16j)

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{Y}(t,\eta)-\tilde{Y}(t,\theta))} \tilde{\mathcal{P}}^{1+\beta} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 3\frac{1+\beta}{\beta} \frac{A^{1+4\beta}}{4^\beta} \tilde{\mathcal{P}}(t, \eta), \quad \beta > 0,$$

[from Lemma A.10]. (4.16k)

As for derivatives of the quantity $\tilde{\mathcal{P}}^{1/2}$ we have

$$|(\tilde{\mathcal{P}}^{1/2})_\eta| \leq \frac{1}{2A} \tilde{\mathcal{P}}^{1/2} \tilde{\mathcal{Y}}_\eta, \quad (4.17a)$$

$$|\tilde{\mathcal{U}}(\tilde{\mathcal{P}}^{1/2})_\eta| \leq \frac{3}{8} A^4, \quad (4.17b)$$

$$((\tilde{\mathcal{P}}^{1/2})_\eta)^2 \leq \frac{1}{8} A^3 \tilde{\mathcal{Y}}_\eta, \quad (4.17c)$$

$$((\tilde{\mathcal{P}}^{1/2})_\eta)^2 \leq \frac{1}{16} A^2 \tilde{\mathcal{Y}}_\eta^2. \quad (4.17d)$$

5. Lipschitz estimates in the new metric

We will repeatedly use the following elementary identity and estimate.

LEMMA 5.1. *Let $a_j, b_j \in \mathbb{R}$ for $j = 1, 2$. Then we have*

$$a_2 b_2 - a_1 b_1 = (b_2 - b_1)(a_2 \mathbb{1}_{b_1 < b_2} + a_1 \mathbb{1}_{b_1 \geq b_2}) + \min(b_1, b_2)(a_2 - a_1).$$

Here

$$\mathbb{1}_{\mathcal{K}} = \begin{cases} 1, & \mathcal{K} \text{ is true,} \\ 0, & \mathcal{K} \text{ is false.} \end{cases}$$

We also use $\mathbb{1}_{\mathcal{K}}$ to denote the characteristic (indicator) function of a set \mathcal{K} . Furthermore, we have the estimate

$$|\min(a_1, b_1) - \min(a_2, b_2)| \leq \max(|a_1 - a_2|, |b_1 - b_2|).$$

LEMMA 5.2. *Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be two Lipschitz continuous functions with Lipschitz constants c_f and c_g , respectively. Then the function $h: \mathbb{R} \rightarrow \mathbb{R}$ defined by $h = \min(f, g)$ is a Lipschitz continuous function with Lipschitz constant bounded by $\max(c_f, c_g)$.*

Proof. Use the previous lemma. □

Introduce for any function Φ the functions

$$\Phi^- = \min(0, \Phi), \quad \Phi^+ = \max(0, \Phi). \quad (5.1)$$

We then have

$$\Phi^- \leq 0 \leq \Phi^+, \quad \Phi^- \Phi^+ = 0, \quad \Phi = \Phi^+ + \Phi^-, \quad |\Phi| = \Phi^+ - \Phi^-. \quad (5.2)$$

For two functions Φ, Ψ we have

$$|\Phi^\pm - \Psi^\pm| \leq |\Phi - \Psi|. \quad (5.3)$$

Frequently, we will have to estimate quantities like

$$\int_0^1 e^{-\frac{1}{A} |\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta)|} (\dots)(t, \theta) d\theta.$$

We will rewrite it as follows:

$$\begin{aligned}
 & \int_0^1 e^{-\frac{1}{A}|\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta)|}(\dots)(t,\theta) d\theta \\
 &= \left(\int_0^\eta + \int_\eta^1 \right) e^{-\frac{1}{A}|\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta)|}(\dots)(t,\theta) d\theta \\
 &= \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}(\dots)(t,\theta) d\theta \\
 &\quad + \int_\eta^1 e^{\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}(\dots)(t,\theta) d\theta.
 \end{aligned} \tag{5.4}$$

Both integrals can be estimated in the same manner, using that the argument in the exponential is negative in both cases. Eliminating the absolute value will allow us to perform integration by parts.

5.1. Lipschitz estimates for $\tilde{\mathcal{Y}}$. From the system of differential equations, we have

$$\tilde{\mathcal{Y}}_{i,t} + \left(\frac{2}{3} \frac{1}{A_i^5} \tilde{\mathcal{U}}_i^3 + \frac{1}{A_i^6} \tilde{\mathcal{S}}_i \right) \tilde{\mathcal{Y}}_{i,\eta} = \tilde{\mathcal{U}}_i, \tag{5.5}$$

where

$$\begin{aligned}
 \tilde{\mathcal{P}}_i(t,\eta) &= \frac{1}{4A_i} \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|} (2(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t,\theta) + A_i^5) d\theta, \\
 \tilde{\mathcal{Q}}_i(t,\eta) &= -\frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|} (2(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t,\theta) + A_i^5) d\theta, \\
 \tilde{\mathcal{S}}_i(t,\eta) &= \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|} \left(\frac{2}{3} \tilde{\mathcal{U}}_i^3 \tilde{\mathcal{Y}}_{i,\eta} - \tilde{\mathcal{Q}}_i \tilde{\mathcal{U}}_{i,\eta} - 2\tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta} \right) (t,\theta) d\theta.
 \end{aligned}$$

Thus we have

$$\begin{aligned}
 & \frac{d}{dt} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\eta) d\eta \\
 &= 2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{1,t} - \tilde{\mathcal{Y}}_{2,t})(t,\eta) d\eta \\
 &= 2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta) d\eta \\
 &\quad + \frac{4}{3} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \left(\frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} - \frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} \right) (t,\eta) d\eta
 \end{aligned}$$

$$\begin{aligned}
& + 2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \left(\frac{1}{A_2^6} \tilde{\mathcal{S}}_2 \tilde{\mathcal{Y}}_{2,\eta} - \frac{1}{A_1^6} \tilde{\mathcal{S}}_1 \tilde{\mathcal{Y}}_{1,\eta} \right) (t, \eta) d\eta \\
& = I_1 + I_2 + I_3.
\end{aligned} \tag{5.6}$$

The strategy is to use integration by parts for the last two integrals I_2 and I_3 , while we want to use straightforward estimates for I_1 , which will finally yield that

$$\begin{aligned}
& \frac{d}{dt} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
\end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant which depends on $A = \max_j(A_j)$ and which remains bounded as $A \rightarrow 0$.

The first integral I_1 : Note that

$$\begin{aligned}
|I_1| & = \left| 2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (t, \eta) d\eta \right| \\
& \leq \int_0^1 ((\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 + (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2) (t, \eta) d\eta \\
& = \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2.
\end{aligned}$$

The second integral I_2 : Note that

$$\begin{aligned}
\frac{3}{4} |I_2| & = \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \left(\frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} - \frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} \right) (t, \eta) d\eta \right| \\
& \leq \frac{1}{(\max_j(A_j))^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}) (t, \eta) d\eta \right| \\
& \quad + \frac{|A_1^5 - A_2^5|}{A_1^5 A_2^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{A_2 < A_1}) (t, \eta) d\eta \right| \\
& \leq \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (t, \eta) d\eta \right| \\
& \quad + \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) [\tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^2 \leq \tilde{\mathcal{U}}_1^2} + \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^2 < \tilde{\mathcal{U}}_2^2}] \right. \\
& \quad \times (\tilde{\mathcal{U}}_2 + \tilde{\mathcal{U}}_1) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) (t, \eta) d\eta \left. \right| \\
& \quad + \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta}) (t, \eta) d\eta \right|
\end{aligned}$$

$$+ \frac{|A_1^5 - A_2^5|}{A_1^5 A_2^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{A_2 < A_1})(t, \eta) d\eta \right|$$

$$= J_1 + J_2 + J_3 + J_4.$$

We will write

$$A = \max_j (A_j), \quad a = \min_j (A_j). \quad (5.7)$$

We commence with J_1 : Since $\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta}(t, \eta) \leq A_i^5 \leq A^5$ for all t and η , we have

$$J_1 = \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right|$$

$$\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2.$$

Next term is J_2 : Using once more that $\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta}(t, \eta) \leq A^5$ for all t and η , we get

$$J_2 = \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) [\tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^2 \leq \tilde{\mathcal{U}}_1^2} + \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^2 < \tilde{\mathcal{U}}_2^2}] \right.$$

$$\times (\tilde{\mathcal{U}}_2 + \tilde{\mathcal{U}}_1) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)(t, \eta) d\eta \left| \right.$$

$$\leq \frac{1}{A^5} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| (2\tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^2 \leq \tilde{\mathcal{U}}_1^2} + 2\tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^2 < \tilde{\mathcal{U}}_2^2})(t, \eta) d\eta$$

$$\leq 2(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2).$$

Next, it is J_3 : Here integration by parts plays an essential role. Thus we have to determine first whether or not the function $\eta \mapsto \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{U}}_j^2)(t, \eta)$ is differentiable, and, if so, if its derivative is bounded. Recall from Lemma A.2 (ii) that the function $\eta \mapsto \min_j (\tilde{\mathcal{U}}_j^2)(t, \eta)$ is Lipschitz continuous with Lipschitz constant at most A^4 . Thus

$$\left| \frac{d}{d\eta} \left(\tilde{\mathcal{U}}_1(t, \eta) \min_j (\tilde{\mathcal{U}}_j^2(t, \eta)) \right) \right| \leq \frac{3}{2} A^4 \|\tilde{\mathcal{U}}_1\|_{L^\infty} \leq \mathcal{O}(1) A^6, \quad (5.8)$$

and integration by parts together with (3.40) yields

$$J_3 = \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{U}}_j^2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) d\eta \right|$$

$$= \left| -\frac{1}{2A^5} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{U}}_j^2)(t, \eta) \right|_{\eta=0}^1$$

$$\begin{aligned}
& + \frac{1}{2A^5} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta} (\tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2))(t, \eta) d\eta \Big| \\
& = \frac{1}{2A^5} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta} (\tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2))(t, \eta) d\eta \right| \\
& \leq \frac{3}{4A} \|\tilde{\mathcal{U}}_1(t, \cdot)\|_{L^\infty} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
& \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
\end{aligned}$$

Finally, we consider J_4 : Direct calculations yield

$$\begin{aligned}
J_4 & \leq \frac{|A_1^5 - A_2^5|}{A_1^5 A_2^5} \left(\mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \right| \right. \\
& \quad \left. + \mathbb{1}_{A_2 < A_1} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right| \right) \\
& \leq \frac{|A_1^5 - A_2^5|}{A_1^5 A_2^5} (\mathbb{1}_{A_1 \leq A_2} A_1^7 + \mathbb{1}_{A_2 < A_1} A_2^7) \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) d\eta \\
& \leq \frac{|A_1^5 - A_2^5|}{A^3} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
& \leq 5A|A_1 - A_2| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2),
\end{aligned}$$

where we again used that $\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta} \leq A_i^5$ and $\|\tilde{\mathcal{U}}_i\|_{L^\infty} \leq A_i^2$.

The third integral I_3 : We will consider several smaller parts of I_3 by inserting the definition of $\tilde{\mathcal{S}}_i$, and combine them in the end. We write

$$I_3 = \frac{4}{3} I_{31} - 4 I_{32} - 2 I_{33}, \quad (5.9)$$

where

$$\begin{aligned}
I_{31} & = \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
& \quad \times \left(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta,
\end{aligned}$$

$$\begin{aligned}
 I_{32} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta, \\
 I_{33} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta.
 \end{aligned}$$

Recall definitions (5.1). Then we have

$$\begin{aligned}
 I_{31} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} (\tilde{\mathcal{U}}_2^-)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &\quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{K}^- + \tilde{K}^+)(t, \eta) d\eta.
 \end{aligned}$$

Since both integrals have the same structure, it suffices to consider the second integral. Furthermore, applying the device described in (5.4), it suffices to study the terms, denoted by K^\pm , with the upper limit of the inner integral replaced by η .

Therefore observe that we can write

$$\begin{aligned}
 K^+(t, \eta) &= \frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 &\quad - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 &= \frac{1}{(\max_j(A_j))^6} \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
 &\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 \leq A_2} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 &\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 < A_1} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 &= \frac{1}{A_6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) \\
 &\quad + \frac{1}{A_6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \right) \\
 &\quad + \frac{1}{A_6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad - \frac{1}{A_6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
 &\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 \leq A_2} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 &\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 < A_1} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 &= \frac{1}{A_6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right.
 \end{aligned}$$

$$\begin{aligned}
& \times ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \Big) \\
& + \frac{1}{A^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \right. \\
& \times ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \Big) \\
& + \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& + \frac{1}{A^6} \mathbb{1}_{A_2 < A_1} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
& \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
& + \frac{1}{A^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) \\
& - \frac{1}{A^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left(\int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \Big) \\
& + \frac{1}{A^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) \\
& - \frac{1}{A^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left(\int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) \\
& + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 \leq A_2} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 < A_1} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& = (J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9 + J_{10})(t, \eta). \tag{5.10}
\end{aligned}$$

Here

$$B(\eta) = \{(t, \theta) \mid e^{\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_1(t, \eta)} \leq e^{\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, \eta)}\} \quad (5.11)$$

which means especially that $B(\eta)$ depends heavily on η ! Observe that the set $B(\eta)$ is scale invariant in the sense that $(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2)$ and $(c\tilde{\mathcal{Y}}_1, c\tilde{\mathcal{Y}}_2)$ define the same set $B(\eta)$ for any constant c .

As far as the first two terms J_1 and J_2 are concerned, they have the same structure, and hence we only consider the term J_1 , that is,

$$\begin{aligned} & \int_0^1 J_1(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta. \end{aligned}$$

The main ingredients are the following observations:

$$|(\tilde{\mathcal{U}}_1^+)^3(t, \eta) - (\tilde{\mathcal{U}}_2^+)^3(t, \eta)| \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+} \leq 3\tilde{\mathcal{U}}_2^2(t, \eta) |\tilde{\mathcal{U}}_1(t, \eta) - \tilde{\mathcal{U}}_2(t, \eta)|,$$

and (4.16c).

Thus we have

$$\begin{aligned} & \left| \int_0^1 J_1(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \right| \\ &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \right. \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \left. \right| \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}^2(t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ & \quad + \frac{9}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right) d\eta \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{36}{A^{11}} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right) d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& + \frac{18}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta d\eta \\
& = \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& - \frac{18A_2}{A^6} \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \Big|_{\eta=0}^1 \\
& + \frac{18A_2}{A^6} \int_0^1 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + 18 \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2,
\end{aligned}$$

where in the last step we used (4.15b) and (4.16c), which imply that

$$\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \leq 8A^5 \tilde{\mathcal{P}}_2(t, \eta), \quad (5.12)$$

and hence the left-hand side of (5.12) tends to 0 as η to 0 or 1 according to (3.40). Thus the second term above vanishes.

As far as the third and fourth terms J_3 and J_4 are concerned, they again have the same structure, and hence we only consider the integral corresponding to J_3 , that is,

$$\begin{aligned}
& \int_0^1 J_3(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \\
& = \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta.
\end{aligned}$$

Recall Lemma A.1(ii). Direct computations yield

$$\begin{aligned}
& \left| \int_0^1 J_3(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \right| \\
& = \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \right. \\
& \quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \Big| \\
& \leq \frac{4|A_1 - A_2|}{aA^6 e} \mathbb{1}_{A_1 \leq A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{16|A_1 - A_2|^2}{a^2 A^{12} e^2} \mathbb{1}_{A_1 \leq A_2} \\
& \quad \times \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& \quad + \frac{16a^4 |A_1 - A_2|^2}{a^2 A^{12} e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& \quad + \frac{64|A_1 - A_2|^2}{A^9 e^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& \quad + \frac{32|A_1 - A_2|^2}{A^4 e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{64A_2 |A_1 - A_2|^2}{A^4 e^2} \\
& \quad \times \left(- \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big|_{\eta=0} + \int_0^1 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right) \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{128A^2}{e^2} |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1)(\| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + |A_1 - A_2|^2).
\end{aligned}$$

As far as the terms J_5 and J_6 are concerned, they again have the same structure, and hence we only consider the integral corresponding to J_5 , that is,

$$\begin{aligned}
\int_0^1 J_5(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
&\quad \times \left. \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta.
\end{aligned}$$

The main ingredient is the estimate

$$\begin{aligned} & |e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}| \\ & \leq \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (|\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_1(t,\theta)| + |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_1(t,\eta)|) \\ & \text{for all } (t, \theta) \in B(\eta), \end{aligned} \quad (5.13)$$

which follows from Lemma A.1 (i).

Direct computations yield

$$\begin{aligned} & \left| \int_0^1 J_5(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \right| \\ & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \right. \right. \\ & \quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) d\eta \Big| \\ & \leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ & \quad \times \left(\int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \right. \\ & \quad \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\ & \leq \frac{1}{aA^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ & \quad + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{a^2 A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ & \quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)| e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\ & \leq \frac{4}{A^4} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ & \quad + \frac{1}{a^2 A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ & \quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \end{aligned}$$

$$\begin{aligned}
 &\leq (2A + 1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4}{a^2 A^{11}} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{2}{a^2 A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{2}{a^2 A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\eta)} \\
 &\quad \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\theta)} \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \\
 &= \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad - \frac{2}{a A^6} \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big|_{\eta=0}^1 \\
 &\quad + \frac{2}{a A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
 \end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant which only depends on A and which remains bounded as $A \rightarrow 0$. We used (cf. (4.16c)) that

$$\begin{aligned}
 \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta &\leq \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
 &\leq 4A_2 \tilde{\mathcal{P}}_2(t, \eta).
 \end{aligned}$$

In the last step we used (4.15b), (4.15g), and finally that

$$\begin{aligned}
 &\left| \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right| \\
 &\leq (\|\tilde{\mathcal{Y}}_1 \tilde{\mathcal{U}}_1\|_{L^\infty}^2 + \|\tilde{\mathcal{Y}}_2 \tilde{\mathcal{U}}_2\|_{L^\infty}^2) 8A \tilde{\mathcal{P}}_2(t, \eta).
 \end{aligned}$$

Since $\tilde{\mathcal{P}}_i(t, \eta)$ tends to 0 as η tends to 0 and 1, the term on the left-hand side tends to zero as η tends to 0 and 1, respectively (cf. (3.40)).

Consider next the terms J_7 and J_8 , that is,

$$\begin{aligned} (J_7 + J_8)(t, \eta) &= \frac{1}{A^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ &\quad - \frac{1}{A^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right). \end{aligned}$$

Here we have to be a bit more careful. Introducing

$$E = \left\{ (t, \eta) \mid \int_0^\eta \Omega(t, \eta, \theta) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \leq \int_0^\eta \Omega(t, \eta, \theta) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right\}, \quad (5.14)$$

with

$$\Omega(t, \eta, \theta) = \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3(t, \theta),$$

we can write

$$\begin{aligned} &(J_7 + J_8)(t, \eta) \\ &= \frac{1}{A^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ &\quad - \frac{1}{A^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\ &= \frac{1}{A^6} (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \\ &\quad \times \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] \\ &\quad + \frac{1}{A^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\ &\quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \mathbb{1}_E(t, \eta) \\ &\quad + \frac{1}{A^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \end{aligned}$$

$$\begin{aligned} & \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \mathbb{1}_{E^c}(t, \eta) \\ & = (L_1 + L_2 + L_3)(t, \eta). \end{aligned}$$

As far as the first term L_1 is concerned, the corresponding integral can be estimated as follows (we use (3.40)):

$$\begin{aligned} & \left| \int_0^1 L_1(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \right| \\ & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \right. \\ & \quad \times \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] d\eta \left. \right| \\ & = \left| -\frac{1}{2A^6} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \right. \\ & \quad \times \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] \Big|_{\eta=0}^1 \\ & \quad + \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \\ & \quad \times \frac{d}{d\eta} \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] d\eta \left. \right| \\ & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2, \end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant only depending on A , which remains bounded as $A \rightarrow 0$, since the derivative

$$\frac{d}{d\eta} \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right]$$

exists and is uniformly bounded; see Lemma A.8.

As far as the last term L_3 (a similar argument works for L_2) is concerned, the corresponding integral can be estimated as follows:

$$\begin{aligned} & \int_0^1 L_3(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \\ & = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\ & \quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \mathbb{1}_{E^c}(t, \eta) d\eta \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \\
&\quad \times \left[(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right]_{\theta=0}^{\eta} \\
&\quad - \int_0^{\eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \\
&\quad \times \left[\left(\frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right. \\
&\quad \left. + \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \left(\frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3 \right)(t, \theta) \right] d\theta \Big] d\eta \\
&= -\frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \\
&\quad \times \int_0^{\eta} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \\
&\quad \times \left[\left(\frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right. \\
&\quad \left. + \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \left(\frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3 \right)(t, \theta) \right] d\theta d\eta \\
&= L_{31} + L_{32}.
\end{aligned}$$

As far as the first term L_{31} is concerned, we have, since

$$\min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{i,\eta}(t, \eta) \leq A^5 \min_j(\tilde{\mathcal{U}}_j^+) \leq \frac{1}{\sqrt{2}} A^7,$$

that

$$\begin{aligned}
|L_{31}| &\leq \left| \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) d\eta \right| \\
&\leq \frac{A}{\sqrt{2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 = \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned}$$

where $\mathcal{O}(1)$ denotes a constant, which only depends on A and which remains bounded as $A \rightarrow 0$.

The second term L_{32} , on the other hand, is a bit more demanding. We start by considering the first part of L_{32} . From Lemma A.2 we have that

$$\left| \frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right| \leq \frac{1}{a} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \max_j(\tilde{\mathcal{Y}}_{j,\eta}(t, \theta)).$$

This implies that

$$\begin{aligned}
& \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\
& \quad \times \left(\frac{d}{d\theta} \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min(\tilde{\mathcal{U}}_j^+)^3(t, \theta) d\theta d\eta \right| \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\
& \quad \times \left(\frac{d}{d\theta} \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min(\tilde{\mathcal{U}}_j^+)^3(t, \theta) d\theta \right)^2 d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{1}{a^2 A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
& \quad \times \min(\tilde{\mathcal{U}}_j^+)^3 \max(\tilde{\mathcal{Y}}_{j,\eta})(t, \theta) d\theta \Big)^2 d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{1}{a^2 A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right)^2 d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{a^2}{2A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\
& \quad \times \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) d\theta \Big)^2 d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& \quad + \frac{1}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \\
& \quad \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} d\theta \right) d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& \quad + \frac{1}{2A^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \\
& \quad \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \frac{1}{2A^5} \int_0^1 8A\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + 2A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)} \int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + 4AA_2 \left(- \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
&\quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
\end{aligned}$$

Here we used (4.13) and (4.16e). Furthermore, we use (4.16f).

Next, we turn to the second half of L_{32} . Recall first (4.16g).

From Lemma A.2 we have that

$$\left| \frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right| \leq 2A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta).$$

Thus we can conclude as before

$$\begin{aligned}
&\frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
&\quad \times \left. \left(\frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3 \right)(t, \theta) d\theta d\eta \right| \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
&\quad \times \left. \left(\frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3 \right)(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
&\quad \times \left. \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \frac{4}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \frac{24}{A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + 12A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A}\tilde{\mathcal{Y}}_2(t, \eta)} \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{\frac{1}{A}\tilde{\mathcal{Y}}_2(t, \theta)} d\theta \right) d\eta \\
&= \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + 12aA \left(- \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
&\quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
\end{aligned}$$

We conclude that

$$|L_{32}| \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.$$

Finally, we have a look at J_9 (the argument for J_{10} follows the same lines). We have

$$\begin{aligned}
&\left| \int_0^1 J_9(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) d\eta \right| \\
&= \left| \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \right. \\
&\quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \right| \\
&\leq \frac{A_2^6 - A_1^6}{\sqrt{2} A_1^6 A_2^6} A_1^2 \mathbb{1}_{A_1 \leq A_2} \\
&\quad \times \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&\leq 4 \frac{A_2^6 - A_1^6}{\sqrt{2} A_1^6 A_2^6} A_1^3 \mathbb{1}_{A_1 \leq A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
&\leq 2 \frac{A_2^6 - A_1^6}{\sqrt{2} A_1^6 A_2^6} A_1^8 \mathbb{1}_{A_1 \leq A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) d\eta \\
&\leq 6\sqrt{2} A(A_2 - A_1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\leq 6A(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

We now turn our attention to I_{32} , that is,

$$I_{32} = \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta.$$

As before, we are only going to establish the estimates for one part of it, since the other parts can be treated similarly. Let

$$\begin{aligned} \tilde{I}_{32} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &= \frac{1}{A_6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &= \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3. \end{aligned} \tag{5.15}$$

Direct calculations yield for \tilde{K}_2 (and similarly for \tilde{K}_3) that

$$\begin{aligned} |\tilde{K}_2| &\leq \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 \leq A_2} \\ &\quad \times \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &\leq \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\quad \times \left(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^2 d\eta \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \left(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta d\eta \right)^{1/2} \\
&\quad \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta d\eta \Big)^{1/2} \\
&\leq \sqrt{6} \frac{A_2^6 - A_1^6}{A_1^3 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \left(\int_0^1 \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \eta) d\eta \right)^{1/2} \\
&\leq \frac{\sqrt{6}}{2} \frac{A_2^6 - A_1^6}{A^4} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_2 - A_1|^2),
\end{aligned}$$

where we used (4.16b), (4.16c), and $A_1 \leq A_2$.

On the other hand, the term \tilde{K}_1 needs to be rewritten a bit more. Namely,

$$\begin{aligned}
\tilde{K}_1 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
&\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right) d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \theta) d\theta \right) d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \right) d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
&\quad \times \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right.
\end{aligned}$$

$$\begin{aligned}
 & - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
 & = \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 + \tilde{J}_5.
 \end{aligned}$$

We start by having a close look at \tilde{J}_1 (\tilde{J}_2 can be handled similarly). One has

$$\begin{aligned}
 |\tilde{J}_1| & \leq \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \right. \\
 & \quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right) d\eta \right| \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 & \quad \times \left. (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})(\tilde{\mathcal{P}}_2^{1/2} + \tilde{\mathcal{P}}_1^{1/2}) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 & \quad \times \left(2 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}) \tilde{\mathcal{P}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{16}{A^{11}} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{8}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{8}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \\
 & \quad \times \left(\int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2
 \end{aligned}$$

$$\begin{aligned}
& + \frac{8A_2}{A^6} \left(- \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
& + \int_0^1 (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2),
\end{aligned}$$

where we used that

$$\begin{aligned}
0 & \leq \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2})^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
& \leq A^4 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& \leq A^4 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \leq A^5 \mathcal{O}(1) \tilde{\mathcal{P}}_2^{1/2}(t, \eta),
\end{aligned}$$

where the very last term tends to 0 as $\eta \rightarrow 0, 1$. In the last step we used (4.16b).

Next, we investigate \tilde{J}_3 (\tilde{J}_4 can be handled in a similar way). The argument is a bit more involved and hence we start with some preliminary estimates. One has

$$\begin{aligned}
& \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& \leq 2A^4 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& \leq 2A^4 \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \leq A^5 \mathcal{O}(1) \tilde{\mathcal{P}}_2^{1/4}(t, \eta),
\end{aligned}$$

where the term on the right-hand side tends to 0 for $\eta \rightarrow 0, 1$. In the last step we used (4.16j); cf. (3.35) (and the derivative $\tilde{\mathcal{Q}}_{2,\eta}$ can be found from (4.7) and (4.9)).

With these estimates in mind, we end up with, recalling (4.16e),

$$\begin{aligned}
 |\tilde{J}_3| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \right. \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \Big| \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{2}{A^{11}} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{1}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t, \eta)} \left(\int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t, \theta)} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &= \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{1}{A^6} \left(-2A_2 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big|_{\eta=0}^1 \right. \\
 &\quad \left. + 2A_2 \int_0^1 (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right) \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2).
 \end{aligned}$$

As far as \tilde{J}_5 is concerned, we have to rewrite it a bit more. Namely,

$$\begin{aligned}
 \tilde{J}_5 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
 &\quad \times \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right.
 \end{aligned}$$

$$\begin{aligned}
& - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
& = \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\
& \quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
& \quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
& \quad \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
& \quad \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
& \quad \times \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
& \quad \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& \quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right.
\end{aligned}$$

$$\begin{aligned} & \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\ &= \tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 + \tilde{L}_4 + \tilde{L}_5, \end{aligned}$$

where $B(\eta)$ is given by (5.11).

Both \tilde{L}_1 and \tilde{L}_2 can be handled in much the same way, and therefore we only consider \tilde{L}_1 . One has

$$\begin{aligned} |\tilde{L}_1| &= \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \right. \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\ &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \Big| \\ &\leq \frac{4}{aA^6 e} \mathbb{1}_{A_1 \leq A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{16}{a^2 A^{12} e^2} \mathbb{1}_{A_1 \leq A_2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta |A_1 - A_2|^2 \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{8}{A^{10} e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta |A_1 - A_2|^2 \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{8}{A^{10} e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{5}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{32}{A^9 e^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{16}{A^4 e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{4A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \\ &\quad \times \int_0^\eta e^{\frac{1}{4A_2} \tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta |A_1 - A_2|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{64}{A^3 e^2} \left(- \int_0^\eta e^{-\frac{1}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
&\quad + \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \Big| A_1 - A_2 \Big|^2 \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2),
\end{aligned}$$

where we used (4.16h).

Next we turn our attention to \tilde{L}_3 and \tilde{L}_4 , which can be handled in much the same way, and therefore we only consider \tilde{L}_3 . One has

$$\begin{aligned}
|\tilde{L}_3| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \right. \\
&\quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
&\quad \times \left. \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \Big| \\
&\leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta (|\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)|) \right. \\
&\quad \times \left. e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{aA^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
&\quad + \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)| \right. \\
&\quad \times \left. e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
&\leq \frac{1}{\sqrt{2}A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{3/4} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
&\quad + \frac{1}{\sqrt{2}A^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)| e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{3/4} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\sqrt{2}A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{3/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
&\quad + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{2A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2^{3/4} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \frac{3}{A^4} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \frac{1}{2A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2^{3/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \frac{1}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2^{3/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \frac{1}{A^5} \left(- \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2^{3/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
&\quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2^{3/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned}$$

where we used (4.16k) for $\beta = \frac{1}{2}$.

Finally, we can turn our attention to \tilde{L}_5 , which we need to split into several parts. We write

$$\begin{aligned}
\tilde{L}_5 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
&\quad \times \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
&\quad \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
&\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \right.
\end{aligned}$$

$$\begin{aligned}
& \times \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \\
& \quad \times \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \right. \\
& \quad \times \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \Big) d\eta \\
& \quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
& \quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \right. \\
& \quad \times \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \Big) d\eta \\
& \quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \mathbb{1}_{\tilde{E}}(t, \eta) \\
& \quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \right. \\
& \quad \times \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \Big) d\eta \\
&= \tilde{L}_{51} + \tilde{L}_{52} + \tilde{L}_{53},
\end{aligned}$$

where \tilde{E} is given by

$$\begin{aligned}
\tilde{E} &= \left\{ (t, \eta) \mid \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. \leq \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right\}. \quad (5.16)
\end{aligned}$$

As far as the first term \tilde{L}_{51} is concerned, it can be estimated as follows (we use (3.40)):

$$\begin{aligned}
|\tilde{L}_{51}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \right. \\
& \quad \times \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \right. \\
& \quad \times \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \Big) d\eta \Big|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2A^6} \left| -(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \right. \\
&\quad \times \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
&\quad \times \left. \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
&\quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \\
&\quad \times \frac{d}{d\eta} \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
&\quad \times \left. \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right) d\eta \Big| \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned} \tag{5.17}$$

where $\mathcal{O}(1)$ denotes some constant only depending on A , which remains bounded as $A \rightarrow 0$, provided we can show that the derivative in the latter integral exists and is uniformly bounded. In fact, from Lemma A.6 we have that

$$\frac{d}{d\eta} \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right)$$

exists and is uniformly bounded.

As far as the term \tilde{L}_{52} (a similar argument works for \tilde{L}_{53}) is concerned, the integral can be estimated as follows:

$$\begin{aligned}
\tilde{L}_{52} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
&\quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
&\quad \times \left. \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
&\quad \times \left[(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \right. \\
&\quad \times \left. \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \right] \Big|_{\theta=0}^\eta
\end{aligned}$$

$$\begin{aligned}
& - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \\
& \times \left[\left(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) \right. \\
& \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \left(\frac{d}{d\theta} \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \right)(t, \theta) d\theta \right] d\eta \\
& = -\frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) d\eta \\
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
& \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \\
& \times \left[\left(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) \right. \\
& \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \left(\frac{d}{d\theta} \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \right)(t, \theta) d\theta \right] d\eta \\
& = M_1 + M_2.
\end{aligned}$$

As far as the first term M_1 is concerned, we have since

$$2 \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{i, \eta}(t, \eta) \leq A^5 \min_j (\tilde{\mathcal{U}}_j^+)(t, \eta) \leq \frac{A^7}{\sqrt{2}},$$

that

$$\begin{aligned}
|M_1| & \leq \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) d\eta \\
& \leq \frac{A}{2\sqrt{2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 = \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant, which only depends on A and which remains bounded as $A \rightarrow 0$.

The second term M_2 , on the other hand, is a bit more demanding. (i): First of all, recall (A.4), that is,

$$\left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right| \leq \frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \max_j (\tilde{\mathcal{Y}}_{j, \eta})(t, \theta),$$

which implies that

$$\begin{aligned}
& \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\
& \quad \times \left(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\
& \quad \times \left(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big)^2 d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{1}{4a^2 A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \quad \times \left(\int_0^\eta | \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 | (t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right)^2 d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \frac{a^2}{8A^2 A_2^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \quad \times \left(\int_0^\eta | \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 | (t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})^{1/2}(t, \theta) d\theta \right)^2 d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& \quad + \frac{1}{8A_2^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} d\theta \right) \\
& \quad \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right) d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& \quad + \frac{1}{A_2^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} d\theta \right) d\eta \\
& \leq \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 \\
& \quad + \frac{A}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t, \eta)} \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t, \theta)} d\theta \right) d\eta \\
& = \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 - AA_2 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \Big|_{\eta=0}^1 \\
& \quad + AA_2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
& \leq \mathcal{O}(1) \| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2,
\end{aligned}$$

where we used (4.16e) and (4.16f).

(ii): First of all, we have to establish that $\min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta)$ is Lipschitz continuous with a uniformly bounded Lipschitz constant. More precisely, in Lemma A.3 we show that

$$\left| \frac{d}{d\theta} (\min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+))(t, \theta) \right| \leq 2A^4 (\min_j(\tilde{\mathcal{P}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta).$$

We are now ready to establish a Lipschitz estimate for the second part of M_2 . Indeed,

$$\begin{aligned} & \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \right. \\ & \quad \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\ & \quad \times \left(\frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right)(t, \theta) d\theta d\eta \left. \right| \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\ & \quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\ & \quad \times \left. \left(\frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right)(t, \theta) d\theta \right)^2 d\eta \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ & \quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\ & \quad \times \left. |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| (\min_j(\tilde{\mathcal{P}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right)^2 d\eta \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4(1 + \sqrt{2})^2}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right)^2 d\eta \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ & \quad + \frac{36}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \end{aligned}$$

$$\begin{aligned}
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \frac{252}{A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + 126A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + 126A \left(-2A_2 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
&\quad + \int_0^1 2A_2 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned}$$

where we used (4.16i). Thus we find that

$$|\tilde{L}_{52}| + |\tilde{L}_{53}| \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.$$

We now turn our attention to I_{33} , that is,

$$\begin{aligned}
I_{33} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
&\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta.
\end{aligned}$$

As before, we apply (5.4). Thus it suffices to study the following term.

$$\begin{aligned}
\bar{I}_{33} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
&\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
&\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&\quad + \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& = \bar{K}_1 + \bar{K}_2 + \bar{K}_3.
\end{aligned}$$

Direct calculations yield for \bar{K}_2 (and similarly for \bar{K}_3) that

$$\begin{aligned}
|\bar{K}_2| & \leq \frac{A_2^6 - A_1^6}{A_1^5 A_2^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 |\tilde{\mathcal{U}}_{1,\eta}|(t, \theta) d\theta \right) d\eta \\
& \leq \frac{A_2^6 - A_1^6}{A_1^5 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
& \quad \times \left(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 |\tilde{\mathcal{U}}_{1,\eta}|(t, \theta) d\theta \right)^2 d\eta \right)^{1/2} \\
& \leq \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \left(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \right. \\
& \quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{H}}_{1,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
& \leq \sqrt{6} \frac{A_2^6 - A_1^6}{A_1^3 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \left(\int_0^1 \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \eta) d\eta \right)^{1/2} \\
& \leq \frac{\sqrt{6}}{2} \frac{A_2^6 - A_1^6}{A^4} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

Here we used (4.15m), (4.16b), and (4.16d).

\bar{K}_1 , on the other hand, needs to be rewritten a bit more. Recall (4.7). Then we can write

$$\begin{aligned}
\bar{K}_1 & = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
& = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(A_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - A_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right. \\
& \left. - \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right) d\eta. \tag{5.18}
\end{aligned}$$

In the next step we are applying integration by parts, so that we get rid of $\tilde{\mathcal{U}}_{j,\eta}(t, \theta)$ in the integrands and hence can therefore have a splitting into positive and negative parts. Therefore note that for $i = 1, 2$, we have

$$\begin{aligned}
& \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_{i,\eta}(t, \theta) d\theta \\
& = e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i(t, \theta) \Big|_{\theta=0}^\eta \\
& \quad - \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \left(\frac{1}{A_i} \tilde{\mathcal{P}}_i + \frac{1}{A_i^2} \tilde{\mathcal{Q}}_i \right) \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) d\theta \\
& = \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i(t, \eta) \\
& \quad - \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \left(\frac{2}{A_i} \tilde{\mathcal{P}}_i - \frac{1}{A_i^2} \tilde{\mathcal{D}}_i \right) \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) d\theta \\
& = \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i(t, \eta) \\
& \quad - \frac{2}{A_i} \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) d\theta \\
& \quad + \frac{1}{A_i^2} \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \tilde{\mathcal{D}}_i \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) d\theta,
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \tilde{\mathcal{D}}_i \tilde{\mathcal{U}}_{i,\eta}(t, \theta) d\theta \\
& = \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \\
& \quad \times \left(\int_0^\theta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\theta) - \tilde{\mathcal{Y}}_i(t,l))} \left((\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t, l) + \frac{1}{2} A_i^5 \right) dl \right) \tilde{\mathcal{U}}_{i,\eta}(t, \theta) d\theta \\
& = \int_0^\eta \left(\int_0^\theta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,l))} \left((\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t, l) + \frac{1}{2} A_i^5 \right) dl \right) \tilde{\mathcal{U}}_{i,\eta}(t, \theta) d\theta \\
& = \left(\int_0^\theta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,l))} \left((\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t, l) + \frac{1}{2} A_i^5 \right) dl \right) \tilde{\mathcal{U}}_i(t, \theta) \Big|_{\theta=0}^\eta \\
& \quad - \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \left((\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) + \frac{1}{2} A_i^5 \right) \tilde{\mathcal{U}}_i(t, \theta) d\theta
\end{aligned}$$

$$= \tilde{\mathcal{D}}_i \tilde{\mathcal{U}}_i(t, \eta) - \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t, \eta) - \tilde{\mathcal{Y}}_i(t, \theta))} \left(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i \right) \tilde{\mathcal{Y}}_{i, \eta}(t, \theta) + \frac{1}{2} A_i^5 \tilde{\mathcal{U}}_i(t, \theta) d\theta.$$

Here we used that for $\theta \leq \eta$,

$$\begin{aligned} 0 &\leq \int_0^\theta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t, \eta) - \tilde{\mathcal{Y}}_i(t, l))} \left(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i \right) \tilde{\mathcal{Y}}_{i, \eta}(t, l) + \frac{1}{2} A_i^5 dl \\ &\leq \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t, \eta) - \tilde{\mathcal{Y}}_i(t, l))} \left(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i \right) \tilde{\mathcal{Y}}_{i, \eta}(t, l) + \frac{1}{2} A_i^5 dl \\ &\leq \tilde{\mathcal{D}}_i(t, \eta) \leq 2A_i \tilde{\mathcal{P}}_i(t, \eta). \end{aligned}$$

We finally end up with

$$\begin{aligned} \bar{K}_1 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(A_2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2, \eta} - A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1, \eta})(t, \eta) d\eta \\ &+ \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1, \eta} - \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2, \eta})(t, \eta) d\eta \\ &+ \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\ &\times \left(\tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\ &- \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\ &+ \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right. \\ &- \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \Big) d\eta \\ &+ \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\ &- \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\ &+ \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2^5 \tilde{\mathcal{U}}_2(t, \theta) d\theta \right. \\ &- \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} A_1^5 \tilde{\mathcal{U}}_1(t, \theta) d\theta \Big) d\eta \\ &= \bar{K}_{11} + \bar{K}_{12} + \bar{K}_{13} + \bar{K}_{14} + \bar{K}_{15} + \bar{K}_{16}. \end{aligned} \tag{5.19}$$

We start by considering \bar{K}_{11} , which can be further split into

$$\begin{aligned}\bar{K}_{11} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(A_2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta} - A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) d\eta \\ &= \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} (A_2 - A_1) \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\ &\quad + \frac{1}{A^6} \mathbb{1}_{A_2 < A_1} (A_2 - A_1) \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\ &\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\ &\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) d\eta \\ &\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \\ &\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\ &= \bar{B}_{11} + \bar{B}_{12} + \bar{B}_{13} + \bar{B}_{14} + \bar{B}_{15} + \bar{B}_{16}.\end{aligned}$$

For \bar{B}_{11} we have (and similarly for B_{12}) that

$$\begin{aligned}|\bar{B}_{11}| &= \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} (A_2 - A_1) \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right| \\ &\leq \frac{A}{2\sqrt{2}} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|).\end{aligned}$$

Recalling (4.15i), we have for \bar{B}_{13} (and similarly for \bar{B}_{14}) that

$$\begin{aligned}|\bar{B}_{13}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \right| \\ &\leq \frac{2}{A^5} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\ &\leq \sqrt{2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}|(t, \eta) d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}\|^2).\end{aligned}$$

As far as \bar{B}_{15} is concerned, we want to use integration by parts. Therefore it is important to recall (3.40) and Lemma A.3 (ii), which imply that

$$\begin{aligned}
 |\bar{B}_{15}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta}) \min_j (\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \right| \\
 &= \frac{a}{2A^6} \left| (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j (\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) \right|_{\eta=0}^1 \\
 &\quad - \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta} (\min_j (\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \Big| \\
 &= \frac{a}{2A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta} (\min_j (\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
 &\leq \frac{A}{2\sqrt{2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
 \end{aligned} \tag{5.20}$$

\bar{B}_{16} is straightforward. Indeed, one has using (4.15e) that

$$\begin{aligned}
 |\bar{B}_{16}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \min_j (\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \right| \\
 &\leq \frac{1}{2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|.
 \end{aligned}$$

We continue with \bar{K}_{12} . We can split \bar{K}_{12} as follows:

$$\begin{aligned}
 \bar{K}_{12} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta})(t, \eta) d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2) \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \eta) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2) \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \eta) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \min_j (\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta}) \min_j (\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \\
 &= \bar{B}_{21} + \bar{B}_{22} + \bar{B}_{23} + \bar{B}_{24}.
 \end{aligned}$$

As the first step we need to establish an estimate for $(\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2)(t, \eta)$. Note that $|\tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta}(t, \eta)| \leq A^{5/2} \sqrt{\tilde{\mathcal{Y}}_{i,\eta}(t, \eta)}$, and hence it is in general unbounded. Thus our estimate for $(\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2)(t, \eta)$ must take care of this problem.

Applying Lemma A.9 finally yields

$$\begin{aligned}
 |\bar{\mathcal{B}}_{21}| &\leq \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2| |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \eta) d\eta \\
 &\leq \frac{2}{A^{9/2}} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 (\tilde{\mathcal{U}}_1^2 + \tilde{\mathcal{P}}_1) |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\quad + \frac{2\sqrt{2}}{A^{9/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\quad + \frac{4}{A^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{\lambda}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\quad + \frac{2\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{\lambda}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\quad + \frac{3}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\quad + \frac{12\sqrt{2}}{\sqrt{3}eA^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\quad + \frac{3}{2A^2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \right) |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 &\quad + \frac{6}{A^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} d\theta \right) |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 & \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{16}{A} \int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) e^{-\frac{1}{A} \tilde{\mathcal{Y}}_1(t,\eta)} \\
 & \quad \times \left(\int_0^\eta e^{\frac{1}{A} \tilde{\mathcal{Y}}_1(t,\theta)} ((\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 + (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 + A(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2)(t, \theta) d\theta \right) d\eta \\
 & \quad + 12A^{1/2} |A_1 - A_2| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 & \quad \times \left(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) e^{-\frac{3}{4A} \tilde{\mathcal{Y}}_1(t,\eta)} \left(\int_0^\eta e^{\frac{3}{4A} \tilde{\mathcal{Y}}_1(t,\theta)} d\theta \right) d\eta \right)^{1/2} \\
 & \quad + \frac{3A^{1/2}}{2} \left(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) e^{-\frac{2}{A} \tilde{\mathcal{Y}}_1(t,\eta)} \left(\int_0^\eta e^{\frac{2}{A} \tilde{\mathcal{Y}}_1(t,\theta)} d\theta \right) d\eta \right)^{1/2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 & \quad + 6A^{1/2} |A_1 - A_2| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
 & \quad \times \left(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) e^{-\frac{3}{4A} \tilde{\mathcal{Y}}_1(t,\eta)} \left(\int_0^\eta e^{\frac{3}{4A} \tilde{\mathcal{Y}}_1(t,\theta)} d\theta \right) d\eta \right)^{1/2} \\
 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Following the same lines, one obtains

$$|\bar{B}_{22}| \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).$$

The estimate for \bar{B}_{23} is straightforward. Namely,

$$\begin{aligned}
 |\bar{B}_{23}| & \leq \frac{2}{A^5} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
 & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2.
 \end{aligned}$$

As far as \bar{B}_{24} is concerned, recall from Lemma A.4 (ii) that

$$\left| \frac{d}{d\eta} (\min_j (\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta)) \right| \leq \mathcal{O}(1) A^7,$$

which yields, together with (3.40), that

$$\begin{aligned}
 |\bar{B}_{24}| & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta}) \min_j (\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \right| \\
 & = \frac{1}{2A^6} \left| (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j (\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) \right|_{\eta=0}^1 \\
 & \quad - \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \frac{d}{d\eta} (\min_j (\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta)) d\eta \Big|
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \frac{d}{d\eta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
\end{aligned}$$

Next, we have a look at \bar{K}_{13} , which can be rewritten as follows:

$$\begin{aligned}
\bar{K}_{13} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
&\quad \times \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
&\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
&\quad \times \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
&\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
&\quad \times \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^- \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
&\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^- \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&= \bar{K}_{13}^+ + \bar{K}_{13}^-.
\end{aligned}$$

Note that both \bar{K}_{13}^+ and \bar{K}_{13}^- have the same structure and hence we are only going to present the details for \bar{K}_{13}^+ , which needs to be rewritten a bit more. Namely,

$$\begin{aligned}
\bar{K}_{13}^+ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \\
&\quad \times \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
&\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&= \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
& + \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_2 < \tilde{\mathcal{D}}_1}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
& \times \left. \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
& \times \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
& \times \left. \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta \left(\min_j \left(e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) - \min_j \left(e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \right) \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \mathbb{1}_{A_2 < A_1} \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
& \times \left(\int_0^\eta \left(\min_j \left(e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) - \min_j \left(e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \right) \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \\
& \times \min_k \left(\int_0^\eta \min_j \left(e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \\
& \times \left(\int_0^\eta \min_j \left(e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_D(t, \eta) \\
& \times \left(\int_0^\eta \min_j \left(e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \Big) d\eta \\
& = \hat{B}_{31} + \hat{B}_{32} + \bar{B}_{31} + \bar{B}_{32} + \bar{B}_{33} + \bar{B}_{34} \\
& + \bar{B}_{35} + \bar{B}_{36} + \hat{B}_{35} + \hat{B}_{36} + \bar{B}_{37} + \bar{B}_{38} + \bar{B}_{39},
\end{aligned}$$

where $B(\eta)$ by (5.11) and

$$D = \left\{ (t, \eta) \mid \int_0^\eta \gamma(t, \eta, \theta) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \leq \int_0^\eta \gamma(t, \eta, \theta) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right\}, \quad (5.21)$$

where

$$\Upsilon(t, \eta, \theta) = \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+).$$

We start by estimating \hat{B}_{31} (a similar argument works for \hat{B}_{32}). Direct calculations yield

$$\begin{aligned} |\hat{B}_{31}| &= \mathbb{1}_{A_1 \leq A_2} \frac{1}{aA^7} |A_2 - A_1| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &\leq \frac{2}{A^7} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &\leq \frac{2}{A^7} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right)^{1/2} d\eta \\ &\leq \frac{2\sqrt{6}}{A^4} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) d\eta \\ &\leq \sqrt{6}A |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2). \end{aligned}$$

For \bar{B}_{31} (a similar argument works for \bar{B}_{32}) we would like to apply some of the estimates established when investigating \bar{K}_1 . Thus we split \bar{B}_{31} into even smaller parts, that is,

$$\begin{aligned} |\bar{B}_{31}| &\leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\bar{d}_{11} + \bar{d}_{12}) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{\bar{\mathcal{D}}_1 \leq \bar{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \bar{T}_1 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \bar{T}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \bar{T}_3 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
& = B_{311} + B_{312} + B_{313} + B_{314}.
\end{aligned}$$

By (A.12) and (A.13), we have that

$$(\bar{d}_{11} + \bar{d}_{12})(t, \theta) \leq 2A^{3/2} \tilde{\mathcal{D}}_2^{1/2} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) + 2\sqrt{2} A^{3/2} \tilde{\mathcal{D}}_2^{1/2}(t, \theta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|,$$

and hence

$$\begin{aligned}
B_{311} & \leq \frac{2}{A^{11/2}} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{D}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{2\sqrt{2}}{A^{11/2}} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{4}{A^{11}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{D}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{2\sqrt{2}}{A^{11/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \times \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \right)^{1/2} \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{4}{A^{11}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{2\sqrt{2}}{A^{11/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \right. \\
& \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{8}{A^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{4}{A^{5/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \times \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{8A_2}{A^5} \left[- \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right]_{\eta=0}^1 \\
& + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \Big] \\
& + 4A^{1/2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta \right)^{1/2} \\
& \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
\end{aligned}$$

Recalling the estimate for \bar{T}_1 , we have that

$$\begin{aligned}
B_{312} & \leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{U}}_2|^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{4}{A^4} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, l) dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{8\sqrt{2}}{\sqrt{3}eA^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta |A_1 - A_2| \\
& \leq 5\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{U}}_2|^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{16}{A^8} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big)^2 d\eta \\
& + \frac{2}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big)^2 d\eta \\
& + \frac{1}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, l) dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big)^2 d\eta \\
& + \frac{128}{3e^2A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big)^2 d\eta |A_1 - A_2|^2 \\
& \leq 5\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^4 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{16}{A^8} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{2}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, l) dl \right)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{128}{3e^2 A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{2}{A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
& + \frac{64}{A^7} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{8}{A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{4}{A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \left(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{512}{3e^2 A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta |A_1 - A_2|^2 \\
& \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
& + \frac{32}{A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \\
& \times \left(\int_0^\theta e^{-(\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A} \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, l) dl \right) d\theta \Big) d\eta \\
& + 4 \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \\
& \times \left(\int_0^\theta e^{-(\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) d\theta \Big) d\eta \\
& + 2 \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \\
& \times \left(\int_0^\theta e^{-(\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,l))} dl \right) d\theta \Big) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{256}{3e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{4A} \tilde{\mathcal{Y}}_2(t,\theta)} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\theta e^{-(\frac{1}{2A} \tilde{\mathcal{Y}}_2(t, \eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t, l))} dl \right) d\theta \Big) d\eta |A_1 - A_2|^2 \\
& \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + A \left[-A_2 \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
& \quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \Big] \\
& \quad + \frac{32}{A} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \left[-2 \int_0^\theta e^{\frac{1}{A} \tilde{\mathcal{Y}}_2(t, l) - \frac{1}{2A} (\tilde{\mathcal{Y}}_2(t, \eta) + \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, l) dl \right]_{\theta=0}^\eta \\
& \quad + 2 \int_0^\eta e^{\frac{1}{2A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, \eta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \Big] d\eta \\
& \quad + 4A_2 \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \\
& \quad \times \left[-2 \int_0^\theta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t, l) - \frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t, \eta) + \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right]_{\theta=0}^\eta \\
& \quad + 2 \int_0^\eta e^{\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, \eta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \Big] d\eta \\
& \quad + 2A_2 \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \left[-2 \int_0^\theta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t, l) - \frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t, \theta) + \tilde{\mathcal{Y}}_2(t, \eta))} dl \right]_{\theta=0}^\eta \\
& \quad + 2 \int_0^\eta e^{\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, \eta))} d\theta \Big] d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \quad + \frac{256A}{3e^2} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \left[-4 \int_0^\theta e^{\frac{3}{4A} \tilde{\mathcal{Y}}_2(t, l) - (\frac{1}{4A} \tilde{\mathcal{Y}}_2(t, \theta) + \frac{1}{2A} \tilde{\mathcal{Y}}_2(t, \eta))} \right]_{\theta=0}^\eta \\
& \quad + 4 \int_0^\eta e^{\frac{1}{2A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, \eta))} d\theta \Big] d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \quad + \frac{64}{A} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) e^{-\frac{1}{2A} \tilde{\mathcal{Y}}_2(t, \eta)} \left(\int_0^\eta e^{\frac{1}{2A} \tilde{\mathcal{Y}}_2(t, \theta)} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right) d\eta \\
& \quad + 8A \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t, \eta)} \left(\int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t, \theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
& \quad + 4A \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t, \eta)} \left(\int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t, \theta)} d\theta \right) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{1024A}{3e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A} \tilde{\mathcal{Y}}_2(t, \eta)} \left(\int_0^\eta e^{\frac{1}{2A} \tilde{\mathcal{Y}}_2(t, \theta)} d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + 64 \left[-2 \int_0^\eta e^{\frac{1}{2A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, \eta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
& + 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) d\eta \Big] \\
& + 8AA_2 \left[-2 \int_0^\eta e^{\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, \eta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
& + 2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \Big] \\
& + 4AA_2 \left[-2 \int_0^\eta e^{\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, \eta))} d\theta \right]_{\eta=0}^1 + 2 \int_0^1 d\eta \Big] \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{1024A^2}{3e^2} \left[-2 \int_0^\eta e^{\frac{1}{2A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, \eta))} d\theta \right]_{\eta=0}^1 + 2 \int_0^1 d\eta \Big] |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

Similar calculations yield for B_{313} ,

$$\begin{aligned}
B_{313} & \leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{2\sqrt{2}}{A^4} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{\sqrt{2}A^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{2A^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, l) dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{4\sqrt{2}}{\sqrt{3}eA^3} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
& \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{8}{A^8} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{1}{2A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{1}{4A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, l) dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{32}{3e^2A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta |A_1 - A_2|^2 \\
& \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{8}{A^8} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{2A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{4A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, l) dl \right)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{32}{3e^2 A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta |A_1 - A_2|^2 \\
& \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{1}{2A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
& + \frac{32}{A^7} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{2}{A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \left(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{128}{3e^2 A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{A}{4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
& + \frac{16}{A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \\
& \times \left. \left(\int_0^\theta e^{-(\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A} \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) d\theta \right) d\eta \\
& + \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \\
& \times \left. \left(\int_0^\theta e^{-(\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) d\theta \right) d\eta \\
& + \frac{1}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \\
& \times \left. \left(\int_0^\theta e^{-(\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2} \tilde{\mathcal{Y}}_2(t,l))} dl \right) d\theta \right) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{64}{3e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{4A} \tilde{\mathcal{Y}}_2(t,\theta)} \right. \\
& \times \left. \left(\int_0^\theta e^{-(\frac{1}{2A} \tilde{\mathcal{Y}}_2(t,\eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t,l))} dl \right) d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq 5 \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{A}{4} \left[-A_2 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
& + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \Big] \\
& + \frac{16}{A} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta)
\end{aligned}$$

$$\begin{aligned}
& \times \left[-2 \int_0^\theta e^{\frac{1}{A} \tilde{Y}_2(t,l) - \frac{1}{2A} (\tilde{Y}_2(t,\eta) + \tilde{Y}_2(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right]_{\theta=0}^\eta \\
& + 2 \int_0^\eta e^{\frac{1}{A} (\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \Big] d\eta \\
& + A_2 \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) \\
& \times \left[-2 \int_0^\theta e^{\frac{1}{A_2} \tilde{Y}_2(t,l) - \frac{1}{2A_2} (\tilde{Y}_2(t,\eta) + \tilde{Y}_2(t,\theta))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, l) dl \right]_{\theta=0}^\eta \\
& + 2 \int_0^\eta e^{\frac{1}{A_2} (\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, \theta) d\theta \Big] d\eta \\
& + \frac{A_2}{2} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) \left[-2 \int_0^\theta e^{\frac{1}{A_2} \tilde{Y}_2(t,l) - \frac{1}{2A_2} (\tilde{Y}_2(t,\theta) + \tilde{Y}_2(t,\eta))} dl \right]_{\theta=0}^\eta \\
& + 2 \int_0^\eta e^{\frac{1}{A_2} (\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} d\theta \Big] d\eta \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
& + \frac{64A}{3e^2} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) \left[-4 \int_0^\theta e^{\frac{3}{4A} \tilde{Y}_2(t,l) - (\frac{1}{4A} \tilde{Y}_2(t,\theta) + \frac{1}{2A} \tilde{Y}_2(t,\eta))} \right]_{\theta=0}^\eta \\
& + 4 \int_0^\eta e^{\frac{1}{2A} (\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} d\theta \Big] d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1) \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
& + \frac{32}{A} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) e^{-\frac{1}{2A} \tilde{Y}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{2A} \tilde{Y}_2(t,\theta)} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right) d\eta \\
& + 2A \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{Y}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{2A_2} \tilde{Y}_2(t,\theta)} (\tilde{Y}_1 - \tilde{Y}_2)^2(t, \theta) d\theta \right) d\eta \\
& + A \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{Y}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{2A_2} \tilde{Y}_2(t,\theta)} d\theta \right) d\eta \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
& + \frac{256A}{3e^2} \int_0^1 \tilde{Y}_{2,\eta}(t, \eta) e^{-\frac{1}{2A} \tilde{Y}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{2A} \tilde{Y}_2(t,\theta)} d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1) \|\tilde{Y}_1 - \tilde{Y}_2\|^2 \\
& + 32 \left[-2 \int_0^\eta e^{\frac{1}{A} (\tilde{Y}_2(t,\theta) - \tilde{Y}_2(t,\eta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
& + 2 \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \eta) d\eta \Big]
\end{aligned}$$

$$\begin{aligned}
& + 2AA_2 \left[-2 \int_0^\eta e^{\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,\eta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
& + 2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \Big] \\
& + AA_2 \left[-2 \int_0^\eta e^{\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,\eta))} d\theta \right]_{\eta=0}^1 + 2 \int_0^1 d\eta \Big] \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{256A^2}{3e^2} \left[-2 \int_0^\eta e^{\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,\eta))} d\theta \right]_{\eta=0}^1 + 2 \int_0^1 d\eta \Big] |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

Last but not least, direct calculations for B_{314} yield

$$\begin{aligned}
B_{314} & \leq \frac{6}{A^3} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& \leq \frac{6}{A^3} |A_1 - A_2| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
& \quad \times \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \right. \\
& \quad \times \left. \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \right)^{1/2} \\
& \leq \frac{6}{A^3} |A_1 - A_2| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
& \quad \times \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \right. \\
& \quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
& \leq \frac{12}{A^{5/2}} |A_1 - A_2| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
& \quad \times \left(\int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \right. \\
& \quad \times \left. \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
& \leq 6\sqrt{2} |A_1 - A_2| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|
\end{aligned}$$

$$\begin{aligned} & \times \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) e^{-\frac{1}{4A} \tilde{\mathcal{Y}}_2(t, \theta)} \right. \right. \\ & \times \left. \left. \left(\int_0^\theta e^{\frac{3}{4A} \tilde{\mathcal{Y}}_2(t, l) - \frac{1}{2A} \tilde{\mathcal{Y}}_2(t, \eta)} dl \right) d\theta \right) d\eta \right)^{1/2} \\ & \leq \mathcal{O}(1) |A_1 - A_2| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|. \end{aligned}$$

Collecting the estimates we have obtained, we conclude

$$|\bar{B}_{31}| \leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).$$

Recalling (4.16e), a direct computation yields for \bar{B}_{33} (and in much the same way for \bar{B}_{34}) that

$$\begin{aligned} |\bar{B}_{33}| & \leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ & \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ & \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \right)^2 d\eta \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ & + \frac{4}{A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{2A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ & \times \left(\int_0^\eta e^{-\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right) d\eta \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ & + \frac{4}{A^6} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right) d\eta \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ & + \frac{2}{A} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t, \eta)} \left(\int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t, \theta)} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right) d\eta \\ & \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ & + \frac{2A_2}{A} \left[-2 \int_0^\eta e^{-\frac{1}{2A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right]_{\eta=0}^1 \end{aligned}$$

$$+ 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) d\eta \Big] \\ \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2).$$

Direct calculations yield for \bar{B}_{35} (and in much the same way for \bar{B}_{36}) that

$$\begin{aligned} |\bar{B}_{35}| &\leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\ &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) d\eta \\ &\leq \frac{1}{A^7 A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta (|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) + |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta)) \right. \\ &\quad \times e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\ &\leq \frac{1}{A^7 A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^7 A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \right) d\eta \\ &\leq \frac{2}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta \\ &\quad + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ &\quad + \frac{4}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \end{aligned}$$

$$\begin{aligned}
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{2\sqrt{6}}{A^4} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
& \quad + \frac{16}{A^{13}} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
& \leq (1 + \sqrt{6}A) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \quad + A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t, \eta)} \left(\int_0^\eta e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t, \theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
& = (1 + \sqrt{6}A) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \quad + AA_2 \left[- \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right]_{\eta=0}^1 \\
& \quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \Big] \\
& \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
\end{aligned} \tag{5.22}$$

Direct calculations yield for \hat{B}_{35} (and in much the same way for \hat{B}_{36}) that

$$\begin{aligned}
|\hat{B}_{35}| & \leq \mathbb{1}_{A_1 \leq A_2} \frac{4|A_1 - A_2|}{aA^7 e} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta \min_j (e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& \leq \frac{8}{aA^6 e} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
& \leq \frac{16}{aA^{11/2} e} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2^{1/2} \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{256}{a^2 A^{11} e^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& + \frac{16A}{e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

As far as the term \bar{B}_{37} is concerned, it can be estimated as follows (we use (3.40)):

$$\begin{aligned}
|\bar{B}_{37}| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta})(t, \eta) \right. \\
& \times \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
& \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \Big) d\eta \Big| \\
&= \frac{1}{2A^7} \left| (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
& \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \Big) \Big|_{\eta=0}^1 \\
& - \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \frac{d}{d\eta} \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
& \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \Big) d\eta \Big| \\
&= \frac{1}{2A^7} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \frac{d}{d\eta} \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
& \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \Big) d\eta \Big| \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned} \tag{5.23}$$

where $\mathcal{O}(1)$ denotes some constant only depending on A , which remains bounded as $A \rightarrow 0$, provided we can show that the derivative in the latter integral exists and is uniformly bounded; see Lemma A.7.

As far as the term \bar{B}_{38} (a similar argument works for \bar{B}_{39}) is concerned, the integral can be estimated as follows:

$$\begin{aligned}
 \bar{B}_{38} &= \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \Big) d\eta \\
 &= \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \left[(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \Big|_{\theta=0}^\eta \\
 &\quad - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \left[\left(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \\
 &\quad \left. \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left(\frac{d}{d\theta} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta \right] d\eta \right] \\
 &= -\frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) d\eta \\
 &\quad + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{D^c}(t, \eta) \\
 &\quad \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \left[\left(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \\
 &\quad \left. \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left(\frac{d}{d\theta} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta \right] d\eta \\
 &= \bar{M}_1 + \bar{M}_2.
 \end{aligned}$$

As far as the first term \bar{M}_1 is concerned, we have since

$$\begin{aligned}
 \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{i,\eta}(t, \eta) &\leq 2 \min_j(A_j \tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{i,\eta}(t, \eta) \\
 &\leq A^6 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) \leq \frac{A^8}{\sqrt{2}},
 \end{aligned}$$

that

$$\begin{aligned} |\bar{M}_1| &\leq \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\ &\leq \frac{A}{\sqrt{2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 = \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2, \end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant, which only depends on A and which remains bounded as $A \rightarrow 0$.

The second term \bar{M}_2 , on the other hand, is a bit more demanding.

(i): First of all, recall (A.4), that is,

$$\left| \frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| \leq \frac{1}{a} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j(\tilde{\mathcal{Y}}_{j,\eta})(t, \theta),$$

which implies that

$$\begin{aligned} &\frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \mathbb{1}_{D^c}(t, \eta) \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right. \\ &\quad \times \left(\frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta d\eta \Big| \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \mathbb{1}_{D^c}(t, \eta) \\ &\quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \left(\frac{d}{d\theta} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \right. \\ &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big)^2 d\eta \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{a^2 A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\ &\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{2A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right)^2 d\eta \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ &\quad + \frac{a^2}{2A^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{2A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right)^2 d\eta \\ &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\ &\quad + \frac{1}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{2A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) d\eta \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \quad + \frac{1}{A_2^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) \\
& \quad \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right) d\eta \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \quad + \frac{6}{A_2^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) d\eta \\
& \leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \quad + 3A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta \\
& = \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 - 6AA_2 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \Big|_{\eta=0}^1 \\
& \quad + 6AA_2 \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
& \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned}$$

where we used (4.13), (4.16e), and (4.16f). Again, $\mathcal{O}(1)$ denotes some constant only depending on A , which remains bounded as $A \rightarrow 0$.

(ii): First of all, we have to establish that $\min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta)$ is Lipschitz continuous with a uniformly bounded Lipschitz constant. More precisely, in Lemma A.4 we show that

$$\left| \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+))(t, \theta) \right| \leq \mathcal{O}(1) \sqrt{A} A^4 (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta).$$

We are now ready to establish a Lipschitz estimate for the second part of \bar{M}_2 . Indeed,

$$\begin{aligned}
& \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
& \quad \times \left. \left(\frac{d}{d\theta} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right)(t, \theta) d\theta d\eta \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2 \mathbb{1}_{D^c}(t, \eta) \\
&\quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
&\quad \times \left. \left(\frac{d}{d\theta} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right)(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \right. \\
&\quad \times \left. (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \mathcal{O}(1) A_2 \left(-2 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
&\quad + \int_0^1 2(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned}$$

where we used (4.16i). Thus we conclude that

$$|\bar{B}_{38}| + |\bar{B}_{39}| \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.$$

Next we have a look at \bar{K}_{14} , which can be rewritten as follows:

$$\begin{aligned} \bar{K}_{14} &= \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &= \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^- \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^- \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &= \bar{K}_{14}^+ + \bar{K}_{14}^-. \end{aligned}$$

Note that both \bar{K}_{14}^+ and \bar{K}_{14}^- have the same structure. Moreover, having a close look at \bar{K}_{14}^+ one has

$$\bar{K}_{14}^+ = -3\tilde{K}_1,$$

where \tilde{K}_1 is defined in (5.15). Thus we can immediately conclude that

$$|\bar{K}_{14}^+| \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|).$$

Next, we have a look at \bar{K}_{15} , which can be rewritten as follows:

$$\begin{aligned} \bar{K}_{15} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^-)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \end{aligned}$$

$$\begin{aligned}
& - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
& = \bar{K}_{15}^+ + \bar{K}_{15}^-.
\end{aligned}$$

Note that both \bar{K}_{15}^+ and \bar{K}_{15}^- have the same structure. Having a close look at \bar{K}_{15}^+ one has

$$\bar{K}_{15}^+ = \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8)(t, \eta) d\eta,$$

where J_1, \dots, J_8 are defined in (5.10). Thus we can conclude immediately that

$$\begin{aligned}
|\bar{K}_{15}^+| & \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
& \quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

Finally, we have a look at \bar{K}_{16} , which can be rewritten as follows:

$$\begin{aligned}
\bar{K}_{16} & = \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \eta) \left(\tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 \tilde{\mathcal{U}}_2(t, \theta) d\theta \right. \\
& \quad \left. - \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} A_1^5 \tilde{\mathcal{U}}_1(t, \theta) d\theta \right) d\eta \\
& = \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2(t, \theta) d\theta \right) d\eta \\
& \quad + \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1(t, \theta) d\theta \right) d\eta \\
& \quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\
& \quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_1 \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \mathbb{1}_{A_1 \leq A_2} \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta \left(\min_j \left(e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \right. \right. \\
& \quad \left. \left. - \min_j \left(e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \right) \tilde{\mathcal{U}}_2(t, \theta) d\theta \right) d\eta \\
& + \mathbb{1}_{A_2 < A_1} \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \left(\int_0^\eta \left(\min_j \left(e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \right. \right. \\
& \quad \left. \left. - \min_j \left(e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \right) \tilde{\mathcal{U}}_1(t, \theta) d\theta \right) d\eta \\
& + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta \min_j \left(e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \\
& + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta \min_j \left(e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) (\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-) \mathbb{1}_{\tilde{\mathcal{U}}_2^- \leq \tilde{\mathcal{U}}_1^-}(t, \theta) d\theta \right) d\eta \\
& + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta \min_j \left(e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ \leq \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \right) d\eta \\
& + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta \min_j \left(e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) (\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-) \mathbb{1}_{\tilde{\mathcal{U}}_1^- \leq \tilde{\mathcal{U}}_2^-}(t, \theta) d\theta \right) d\eta \\
& + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \\
& \quad \times \left(\int_0^\eta \min_j \left(e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\
& + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \eta) \\
& \quad \times \left(\int_0^\eta \min_j \left(e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} \right) \max_j (\tilde{\mathcal{U}}_j^-)(t, \theta) d\theta \right) d\eta \\
& = \bar{B}_{61} + \bar{B}_{62} + \bar{B}_{63} + \bar{B}_{64} + \bar{B}_{65} + \bar{B}_{66} \\
& \quad + \bar{B}_{67}^+ + \bar{B}_{67}^- + \bar{B}_{68}^+ + \bar{B}_{68}^- + \bar{B}_{69}^+ + \bar{B}_{69}^-.
\end{aligned}$$

The key observation, which rescues the whole paper, is again

$$A_i^5 \leq 2(\tilde{\mathcal{P}}_i \tilde{\mathcal{Y}}_{i,\eta} + \tilde{\mathcal{H}}_{i,\eta})(t, \eta),$$

which yields for \bar{B}_{61} (and similarly for \bar{B}_{62}) that

$$\begin{aligned} |\bar{B}_{61}| &\leq \frac{A_2^5 - A_1^5}{2A^{11}} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta \\ &\leq \frac{5}{2A^7} |A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\quad \times \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right)^2 d\eta \right)^{1/2} \\ &\leq \frac{10}{A^7} |A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\quad \times \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left[\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 \right. \right. \\ &\quad \left. \left. + \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2| \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right)^2 \right] d\eta \right)^{1/2} \\ &\leq \frac{10}{A^7} |A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \right. \\ &\quad \times \left[\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \right. \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ &\quad \left. \left. + \|\tilde{\mathcal{U}}_2(t, \cdot)\|_{L^\infty}^2 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right)^2 \right] d\eta \right)^{1/2} \\ &\leq \frac{10}{A^7} |A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\quad \times \left(\int_0^1 (6A^6 + 16A^2 \|\tilde{\mathcal{U}}_2(t, \cdot)\|_{L^\infty}^2) \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) d\eta \right)^{1/2} \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_2 - A_1|^2). \end{aligned}$$

Next, we have a close look at \bar{B}_{63} (and similarly for \bar{B}_{64}). Recalling the definition of $B(\eta)$ (5.11), we have

$$\begin{aligned}
 |\bar{B}_{63}| &\leq \frac{a^5}{2A^6A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta (|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) + |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta)) \right. \\
 &\quad \times \left. e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{a^5}{2A^6A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{a^5}{2A^6A_2} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{a^5}{A^6A_2^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta}) |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{a^5}{A^6A_2^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta}) |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{1}{A^6A_2^2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
 &\quad \times \left(\frac{1}{A_2} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} + A_2 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} + \|\tilde{\mathcal{U}}_2(t, \cdot)\|_{L^\infty} \tilde{\mathcal{H}}_{2,\eta} \right) (t, \theta) d\theta \\
 &\quad + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{1}{A^{12}A_2^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
 &\quad \times \left. (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta}) |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \\
 &\quad + \frac{2}{A^{12}A_2^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left[\left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 \right.
 \end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2| \tilde{\mathcal{H}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta \\
& \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{2}{A^{12} A_2^2} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}^2(t, \eta) \\
& \quad \times \left[\left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \right. \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
& \quad + \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2, \eta}(t, \theta) d\theta \right) \\
& \quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2, \eta}(t, \theta) d\theta \right) \right] d\eta \\
& \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \quad + \frac{8}{A^{12} A_2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}^2(t, \eta) \\
& \quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2, \eta})(t, \theta) d\theta \right) d\eta \\
& \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4}{A^8} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) e^{-\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t, \eta)} \\
& \quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 e^{\frac{1}{A_2} \tilde{\mathcal{Y}}_2(t, \theta)} (\tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2, \eta})(t, \theta) d\theta \right) d\eta \\
& = \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{4A_2}{A^8} \left(- \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \quad \times (\tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2, \eta})(t, \theta) d\theta \Big|_{\eta=0}^1 \\
& \quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 (\tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2, \eta})(t, \eta) d\eta \Big) \\
& \leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
\end{aligned}$$

Next, we have a look at \bar{B}_{65} (a similar argument works for \bar{B}_{66}). Direct calculations yield

$$\begin{aligned}
|\bar{B}_{65}| & \leq \frac{a^5}{2A^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \\
& \quad \times \left(\frac{4}{ae} \int_0^\eta \min_j (e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta |A_1 - A_2|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{A^2 e} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
&\leq \frac{4}{A^2 A_2^3 e} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta}) |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
&\leq |A_1 - A_2|^2 + \frac{32}{A^4 A_2^{10} e^2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
&\quad \times \left[\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \right. \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
&\quad + \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) \\
&\quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) \right] d\eta \\
&\leq |A_1 - A_2|^2 + \frac{128}{A^4 A_2^9 e^2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right) d\eta \\
&\leq |A_1 - A_2|^2 + \frac{64}{A^4 A_2^4 e^2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad \times \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{2A_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right) d\eta \\
&\leq |A_1 - A_2|^2 + \frac{128}{A^4 A_2^3 e^2} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad \times \left[- \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \right]_{\eta=0}^1 \\
&\quad + \int_0^1 (\tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{H}}_{2,\eta})(t, \eta) d\eta \Big] \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

Next, we have a look at \bar{B}_{67}^+ (a similar argument works for \bar{B}_{67}^- and \bar{B}_{68}^\pm). Direct calculations yield

$$\begin{aligned}
 |\bar{B}_{67}^+| &\leq \frac{a^5}{2A^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{\Lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+|(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{A_2^5}{4A^7} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{\Lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1|(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{2A^7} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{\Lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta}) |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1|(t, \theta) d\theta \right)^2 d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{1}{A^7} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left[\left(\int_0^\eta e^{-\frac{1}{\Lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1|(t, \theta) d\theta \right)^2 \right. \\
 &\quad \left. + \left(\int_0^\eta e^{-\frac{1}{\Lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2,\eta} |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1|(t, \theta) d\theta \right)^2 \right] d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^7} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{2\Lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{2\Lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{1}{A^7} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{\Lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{\Lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{2}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{2\Lambda_2} \tilde{\mathcal{Y}}_2(t, \eta)} \left(\int_0^\eta e^{\frac{1}{2\Lambda_2} \tilde{\mathcal{Y}}_2(t, \theta)} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{4}{A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{\Lambda_2} \tilde{\mathcal{Y}}_2(t, \eta)} \left(\int_0^\eta e^{\frac{1}{\Lambda_2} \tilde{\mathcal{Y}}_2(t, \theta)} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
 &\quad + \frac{2A_2}{A^6} \left(-2 \int_0^\eta e^{-\frac{1}{2\Lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
 &\quad + 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta
 \end{aligned}$$

$$\begin{aligned}
& + \frac{4A_2}{A^6} \left(- \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) \Big|_{\eta=0}^1 \\
& + \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{H}}_{2,\eta}(t, \eta) d\eta \Big) \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2).
\end{aligned} \tag{5.24}$$

Finally, we consider \bar{B}_{69}^+ (a similar argument works for \bar{B}_{69}^-). Here integration by parts will play the main role. Indeed, we have

$$\begin{aligned}
|\bar{B}_{69}^+| &= \frac{a^5}{2A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta})(t, \eta) \right. \\
&\quad \times \left. \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \right| \\
&= \frac{a^5}{4A^6} \left| (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \right. \\
&\quad \times \left. \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right|_{\eta=0}^1 \\
&\quad - \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \\
&\quad \times \frac{d}{d\eta} \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \Big| \\
&\leq \frac{a^5}{4A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \eta) \right. \\
&\quad \times \left. \frac{d}{d\eta} \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \right| \\
&\leq \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|,
\end{aligned} \tag{5.25}$$

where $\mathcal{O}(1)$ denotes some constant depending on A , which remains bounded as $A \rightarrow 0$, provided that we can show that both

$$a^5 \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \leq \mathcal{O}(1)A^2 \min(\tilde{\mathcal{P}}_j)(t, \eta)$$

and the derivative

$$a \frac{d}{d\eta} \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \tag{5.26}$$

exist and are uniformly bounded.

Direct computations yield

$$\begin{aligned}
 & a^5 \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\
 & \leq \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} A_i^5 |\tilde{\mathcal{U}}_i|(t, \theta) d\theta \\
 & \leq 2 \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} (\tilde{\mathcal{P}}_i \tilde{\mathcal{Y}}_{i,\eta} + \tilde{\mathcal{H}}_{i,\eta}) |\tilde{\mathcal{U}}_i|(t, \theta) d\theta \\
 & \leq 2 \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta))} \left(\frac{1}{A_i} \tilde{\mathcal{P}}_i^2 \tilde{\mathcal{Y}}_{i,\eta} + A_i \tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta} + \|\tilde{\mathcal{U}}_i(t, \cdot)\|_{L^\infty} \tilde{\mathcal{H}}_{i,\eta} \right) (t, \theta) d\theta \\
 & \leq \mathcal{O}(1) A_i^2 \tilde{\mathcal{P}}_i(t, \eta).
 \end{aligned}$$

The result for (5.26) is contained in Lemma A.5.

LEMMA 5.3. Let $\tilde{\mathcal{Y}}_i$ denote two solutions of (5.5). Then we have

$$\begin{aligned}
 \frac{d}{dt} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 & \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
 & \quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
 \end{aligned} \tag{5.27}$$

where $\mathcal{O}(1)$ denotes some constant which depends on $A = \max_j(A_j)$, which remains bounded as $A \rightarrow 0$.

5.2. Lipschitz estimates for $\tilde{\mathcal{U}}$. From the system of differential equations, we have

$$\tilde{\mathcal{U}}_{i,t} + \left(\frac{2}{3} \frac{1}{A_i^5} \tilde{\mathcal{U}}_i^3 + \frac{1}{A_i^6} \tilde{\mathcal{S}}_i \right) \tilde{\mathcal{U}}_{i,\eta} = -\frac{1}{A_i^2} \tilde{\mathcal{Q}}_i, \tag{5.28}$$

where

$$\begin{aligned}
 \tilde{\mathcal{Q}}_i(t, \eta) &= -\frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta)|} (2(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) + A_i^5) d\theta, \\
 \tilde{\mathcal{S}}_i(t, \eta) &= \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta) - \tilde{\mathcal{Y}}_i(t,\theta)|} \left(\frac{2}{3} \tilde{\mathcal{U}}_i^3 \tilde{\mathcal{Y}}_{i,\eta} - \tilde{\mathcal{Q}}_i \tilde{\mathcal{U}}_{i,\eta} - 2\tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta} \right) (t, \theta) d\theta.
 \end{aligned}$$

Thus we have

$$\begin{aligned}
 & \frac{d}{dt} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) d\eta \\
 &= 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_{1,t} - \tilde{\mathcal{U}}_{2,t})(t, \eta) d\eta
 \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \left(\frac{1}{A_2^2} \tilde{\mathcal{Q}}_2 - \frac{1}{A_1^2} \tilde{\mathcal{Q}}_1 \right) (t, \eta) d\eta \\
&\quad + \frac{4}{3} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \left(\frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{U}}_{2,\eta} - \frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{U}}_{1,\eta} \right) (t, \eta) d\eta \\
&\quad + 2 \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \left(\frac{1}{A_2^6} \tilde{\mathcal{S}}_2 \tilde{\mathcal{U}}_{2,\eta} - \frac{1}{A_1^6} \tilde{\mathcal{S}}_1 \tilde{\mathcal{U}}_{1,\eta} \right) (t, \eta) d\eta \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

The strategy is to use integration by parts for the last two integrals I_2 and I_3 , while we want to use straightforward estimates for I_1 , which will finally yield that

$$\begin{aligned}
&\frac{d}{dt} \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
\end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant which only depends on $A = \max_j(A_j)$ and which remains bounded as $A \rightarrow 0$.

The first integral I_1 : Note that we can split I_1 as follows:

$$\begin{aligned}
I_1 &= \mathbb{1}_{A_1 \leq A_2} 2 \frac{A_1^2 - A_2^2}{A_1^2 A_2^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{Q}}_1(t, \eta) d\eta \\
&\quad + \mathbb{1}_{A_2 < A_1} 2 \frac{A_1^2 - A_2^2}{A_1^2 A_2^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{Q}}_2(t, \eta) d\eta \\
&\quad + 2 \frac{1}{A^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{Q}}_2 - \tilde{\mathcal{Q}}_1)(t, \eta) d\eta \\
&= I_{11} + I_{12} + I_{13}.
\end{aligned}$$

As far as I_{11} is concerned (a similar argument works for I_{12}), we have

$$\begin{aligned}
|I_{11}| &\leq \mathbb{1}_{A_1 \leq A_2} 4 \frac{|A_1 - A_2|}{A_1 A_2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_1(t, \eta) d\eta \\
&\leq A^2 |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

Note that

$$\begin{aligned} |I_{13}| &= \left| 2 \frac{1}{A^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{Q}}_1 - \tilde{\mathcal{Q}}_2)(t, \eta) d\eta \right| \\ &\leq \int_0^1 \left((\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 + \frac{1}{A^4} (\tilde{\mathcal{Q}}_1 - \tilde{\mathcal{Q}}_2)^2 \right)(t, \eta) d\eta, \end{aligned}$$

and hence it suffices to show that

$$\begin{aligned} \|\tilde{\mathcal{Q}}_1 - \tilde{\mathcal{Q}}_2\| &\leq \mathcal{O}(1)A^2(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \\ &\quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| + |A_1 - A_2|), \end{aligned} \quad (5.29)$$

which is equivalent to

$$\begin{aligned} \|\tilde{\mathcal{Q}}_1 - \tilde{\mathcal{Q}}_2\|^2 &\leq \mathcal{O}(1)A^4(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\ &\quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

To begin with, we observe that we can write

$$\begin{aligned} (\tilde{\mathcal{Q}}_1 - \tilde{\mathcal{Q}}_2)(t, \eta) &= (A_1\tilde{\mathcal{P}}_1 - A_2\tilde{\mathcal{P}}_2)(t, \eta) \\ &\quad + (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1)(t, \eta) \\ &= (A_1 - A_2)\tilde{\mathcal{P}}_1 \\ &\quad + A_2(\tilde{\mathcal{P}}_1^{1/2} + \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad + (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1)(t, \eta) \\ &= K_1(t, \eta) + K_2(t, \eta) + K_3(t, \eta). \end{aligned}$$

As far as $K_1(t, \eta)$ is concerned, we have

$$|A_1 - A_2| \|\tilde{\mathcal{P}}_1\|_{L^\infty} \leq \frac{A^4}{4} |A_1 - A_2|.$$

As far as $K_2(t, \eta)$ is concerned, we have

$$\|\tilde{\mathcal{P}}_1 - \tilde{\mathcal{P}}_2\| \leq A^2 \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|,$$

since $\|\tilde{\mathcal{P}}_i^{1/2}\|_{L^\infty}$ can be bounded by a constant, which only depends on A .

As far as $K_3(t, \eta)$ is concerned, Lemma A.9 implies immediately that

$$\begin{aligned} \|\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2\| &\leq \mathcal{O}(1)A^2(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \\ &\quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| + |A_1 - A_2|). \end{aligned}$$

This finishes the proof of (5.29).

The second integral I_2 : Note that we can write

$$\begin{aligned}
 \frac{3}{4}I_2 &= \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \left(\frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{U}}_{2,\eta} - \frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{U}}_{1,\eta} \right) (t, \eta) d\eta \\
 &= \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_2^3 \tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{U}}_{1,\eta}) (t, \eta) d\eta \\
 &\quad + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_1^3 \tilde{\mathcal{U}}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{U}}_{2,\eta} \mathbb{1}_{A_2 < A_1}) (t, \eta) d\eta \\
 &= \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{U}}_{2,\eta} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_2^2 \tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{U}}_{1,\eta}) \tilde{\mathcal{U}}_1 (t, \eta) d\eta \\
 &\quad + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_1^3 \tilde{\mathcal{U}}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{U}}_{2,\eta} \mathbb{1}_{A_2 < A_1}) (t, \eta) d\eta \\
 &= \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{U}}_{2,\eta} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{U}}_1^2) \tilde{\mathcal{U}}_{2,\eta} \tilde{\mathcal{U}}_1 (t, \eta) \mathbb{1}_{\tilde{\mathcal{U}}_1^2 \leq \tilde{\mathcal{U}}_2^2} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{U}}_1^2) \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta} (t, \eta) \mathbb{1}_{\tilde{\mathcal{U}}_2^2 < \tilde{\mathcal{U}}_1^2} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_{1,\eta}) \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{U}}_j^2) (t, \eta) d\eta \\
 &\quad + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_1^3 \tilde{\mathcal{U}}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{U}}_{2,\eta} \mathbb{1}_{A_2 < A_1}) (t, \eta) d\eta \\
 &= \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{U}}_{2,\eta} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{U}}_1^2) \tilde{\mathcal{U}}_{2,\eta} \tilde{\mathcal{U}}_1 (t, \eta) \mathbb{1}_{\tilde{\mathcal{U}}_1^2 \leq \tilde{\mathcal{U}}_2^2} (t, \eta) d\eta \\
 &\quad + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{U}}_1^2) \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta} (t, \eta) \mathbb{1}_{\tilde{\mathcal{U}}_2^2 < \tilde{\mathcal{U}}_1^2} (t, \eta) d\eta \\
 &\quad + \frac{1}{2A^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \frac{d}{d\eta} (\tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{U}}_j^2)) (t, \eta) d\eta \\
 &\quad + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_1^3 \tilde{\mathcal{U}}_{1,\eta} \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{U}}_{2,\eta} \mathbb{1}_{A_2 < A_1}) (t, \eta) d\eta,
 \end{aligned}$$

where we used integration by parts in the last step together with $\tilde{\mathcal{U}}_i(t, \eta) \rightarrow 0$ as $\eta \rightarrow 0, 1$. As far as the derivative in the last integral is concerned, observe that

$$|\tilde{\mathcal{U}}_{1,\eta} \min_j (\tilde{\mathcal{U}}_j^2)(t, \eta)| \leq |\tilde{\mathcal{U}}_1^2 \tilde{\mathcal{U}}_{1,\eta}(t, \eta)| \leq \frac{A^4}{2} \|\tilde{\mathcal{U}}_1\|_{L^\infty} \leq \frac{A^6}{2\sqrt{2}}, \quad (5.30)$$

since $2|\tilde{\mathcal{U}}_i \tilde{\mathcal{U}}_{i,\eta}(t, \eta)| \leq A_i^4 \leq A^4$ for all t and η . Furthermore, we established before that the function $\theta \mapsto \min_j (\tilde{\mathcal{U}}_j^2(t, \eta))$ is Lipschitz continuous with Lipschitz constant at most A^4 ; see (5.8). Thus

$$\left| \frac{d}{d\eta} \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{U}}_j^2)(t, \eta) \right| \leq \frac{3}{2} A^4 \|\tilde{\mathcal{U}}_1\|_{L^\infty} \leq \frac{3}{2\sqrt{2}} A^6 \quad (5.31)$$

and

$$\left| \frac{1}{A_1^5 A_2^5} \tilde{\mathcal{U}}_i^3 \tilde{\mathcal{U}}_{i,n} \mathbb{1}_{A_i=a} \right| \leq \frac{1}{A_1^5 A_2^5} \frac{1}{4} A_i^8 \mathbb{1}_{A_i=a} \leq \frac{1}{A^2}.$$

Finally, we get that

$$\begin{aligned} \frac{3}{4} |I_2| &= \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \left(\frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{U}}_{2,\eta} - \frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{U}}_{1,\eta} \right) (t, \eta) d\eta \right| \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{2}{A^2} |A_1^5 - A_2^5| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) d\eta \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2). \end{aligned}$$

The third integral I_3 : We will split I_3 into several terms that we treat separately, and combine them in the end.

Recall that we introduced the functions $\tilde{\mathcal{U}}_i^- = \min(0, \tilde{\mathcal{U}}_i)$ and $\tilde{\mathcal{U}}_i^+ = \max(0, \tilde{\mathcal{U}}_i)$ with properties (5.2) and (5.3). We write

$$I_3 = \frac{4}{3} I_{31} - 4 I_{32} - 2 I_{33},$$

where

$$\begin{aligned} I_{31} &= \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta, \quad (5.32a) \\ I_{32} &= \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \end{aligned}$$

$$- \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta, \quad (5.32b)$$

$$I_{33} = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta. \quad (5.32c)$$

Thus we have

$$I_{31} = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ \times \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ \times \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} (\tilde{\mathcal{U}}_2^-)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ + \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ \times \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta.$$

Since both inner integrals have the same structure, it suffices to consider the second integral. Recall rewrite (5.4). Thus we need to estimate the following term:

$$\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\ - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ = \frac{1}{A_6} \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right.$$

$$\begin{aligned}
& -\tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& + \mathbb{1}_{A_1 \leq A_2} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left(\tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
& + \mathbb{1}_{A_2 < A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& = \frac{1}{A_6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \\
& + \frac{1}{A_6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
& + \frac{1}{A_6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& - \frac{1}{A_6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& + \mathbb{1}_{A_1 \leq A_2} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left(\tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
& + \mathbb{1}_{A_2 < A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& = \frac{1}{A_6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \\
& + \frac{1}{A_6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
& + \frac{1}{A_6} \mathbb{1}_{A_1 \leq A_2} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \\
& \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& + \frac{1}{A_6} \mathbb{1}_{A_2 < A_1} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \\
& \times \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& + \frac{1}{A_6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \\
& \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \\
& - \frac{1}{A_6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \\
& \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& - \frac{1}{A^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& + \mathbb{1}_{A_1 \leq A_2} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left(\tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
& + \mathbb{1}_{A_2 < A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \hat{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& = (J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9 + J_{10})(t, \eta), \tag{5.33}
\end{aligned}$$

where $B(\eta)$ is given by (5.11).

As far as the integral that contains J_1 (a similar argument works for J_2) is concerned, we have

$$\begin{aligned}
& \left| \int_0^1 J_1(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
& \leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 + \int_0^1 J_1^2(t, \eta) d\eta \\
& \leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|_{L^2}^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\
& \leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 \\
& \quad + \frac{1}{A^9} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} 3(\tilde{\mathcal{U}}_2^+)^2 |\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+| \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& \leq \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 \\
& \quad + \frac{9}{A^4} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 \\
& \leq \left(1 + \frac{36}{A^3} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \right) \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2 \\
& \leq \mathcal{O}(1) \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2, \tag{5.34}
\end{aligned}$$

where we used (4.15e), (4.15g), and (4.15h).

As far as the third and fourth terms J_3 and J_4 are concerned, they again have the same structure, and hence we only consider the integral corresponding to J_3 .

$$\begin{aligned}
& \left| \int_0^1 J_3(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
& = \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right|
\end{aligned}$$

$$\begin{aligned}
& \times \int_0^\eta \left(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \Big| \\
& \leq \frac{2\sqrt{2}a}{A^6 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \quad \times \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta |A_1 - A_2| \\
& \leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{8}{A^{10} e^2} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{32}{Ae^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

As far as the third and fourth terms J_5 and J_6 are concerned, they again have the same structure, and hence we only consider the integral corresponding to J_5 . Thus, using (5.13),

$$\begin{aligned}
& \left| \int_0^1 J_5(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
& = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \left(\int_0^\eta \left(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \right) \right. \right. \\
& \quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) d\eta \Big| \\
& \leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \quad \times \left(\int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \right. \\
& \quad \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& \leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \\
& \quad + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{a^2 A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta)
\end{aligned}$$

$$\begin{aligned}
 & \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)| e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \frac{4a}{A^5} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_{2, \eta}|(t, \eta) d\eta + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
 & \quad + \frac{1}{a^2 A^9} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
 & \quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
 & \leq (\sqrt{2}A^2 + 1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2) + \frac{2}{A^6} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \\
 & \quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2). \tag{5.35}
 \end{aligned}$$

Next are the terms J_7 and J_8 . Therefore recall (5.14), which implies

$$\begin{aligned}
 J_7 + J_8 &= \frac{1}{A^6} \tilde{\mathcal{U}}_{2, \eta}(t, \eta) \\
 & \quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
 & \quad - \frac{1}{A^6} \tilde{\mathcal{U}}_{1, \eta}(t, \eta) \\
 & \quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) \\
 &= \frac{1}{A^6} (\tilde{\mathcal{U}}_{2, \eta} - \tilde{\mathcal{U}}_{1, \eta})(t, \eta) \\
 & \quad \times \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right] \\
 & \quad + \frac{1}{A^6} \tilde{\mathcal{U}}_{1, \eta}(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
 & \quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \mathbb{1}_E(t, \eta) \\
 & \quad + \frac{1}{A^6} \tilde{\mathcal{U}}_{2, \eta}(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
 & \quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \mathbb{1}_{E^c}(t, \eta) \\
 &= L_1 + L_2 + L_3.
 \end{aligned}$$

As far as the first term L_1 is concerned, the corresponding integral can be estimated as follows, using (A.11),

$$\begin{aligned}
& \left| \int_0^1 L_1(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
&= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_{1,\eta})(t, \eta) \right. \\
&\quad \times \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] d\eta \Big| \\
&= \left| -\frac{1}{2A^6} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) \right. \\
&\quad \times \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] \Big|_{\eta=0}^1 \\
&\quad + \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) \\
&\quad \times \frac{d}{d\eta} \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] d\eta \Big| \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2, \tag{5.36}
\end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant only depending on A , which remains bounded as $A \rightarrow 0$.

As far as the last term L_3 (a similar argument works for L_2) is concerned, the corresponding integral can be estimated as follows:

$$\begin{aligned}
& \left| \int_0^1 L_3(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
&= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
&\quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \mathbb{1}_{E^c}(t, \eta) d\eta \Big| \\
&= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \right. \\
&\quad \times \left[(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right]_{\theta=0}^\eta \\
&\quad - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \left[\left(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j (\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right. \\
&\quad \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left(\frac{d}{d\theta} \min_j (\tilde{\mathcal{U}}_j^+)^3 \right)(t, \theta) \right] d\theta \Big| \mathbb{1}_{E^c}(t, \eta) d\eta \Big|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{A^6} \left| - \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) d\eta \right. \\
&\quad + \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{E^c}(t, \eta) \\
&\quad \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \left[\left(\frac{d}{d\theta} \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \min_j (\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right. \\
&\quad \left. + \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \left(\frac{d}{d\theta} \min_j (\tilde{\mathcal{U}}_j^+)^3 \right)(t, \theta) \right] d\theta d\eta \Big| \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2) + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) \right. \\
&\quad \times \left[\frac{1}{a} \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \max_j (\tilde{\mathcal{Y}}_{j,\eta})(t, \theta) \right. \\
&\quad \left. \left. + 2A^4 \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) \right] d\theta \right)^2 d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2) + \frac{54}{A} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2), \tag{5.37}
\end{aligned}$$

where we used (A.4), (4.16a), and (A.6).

Finally, we have a look at the integral, which contains J_9 (a similar argument works for J_{10}), where we can assume $A_1 \leq A_2$. Thus

$$\begin{aligned}
&\left| \int_0^1 J_9(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
&= \left(\frac{1}{A_1^6} - \frac{1}{A_2^6} \right) \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \right. \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \Big| \\
&= \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \right. \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \Big| \\
&\leq 3 \frac{A_2 - A_1}{A_1^2 A_2} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \right. \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \Big|
\end{aligned}$$

$$\begin{aligned}
&\leq |A_1 - A_2|^2 + \frac{9}{A_1^4 A_2^2} \int_0^1 (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 \tilde{\mathcal{U}}_{1,\eta}^2(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^2 d\eta \\
&\leq |A_1 - A_2|^2 + \frac{9}{A_1 A_2^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&\leq |A_1 - A_2|^2 + \frac{9}{A_2^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&\leq |A_1 - A_2|^2 + \frac{36}{A} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \eta) d\eta \\
&\leq |A_1 - A_2|^2 + 18A^4 \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2.
\end{aligned} \tag{5.38}$$

We now turn to the integral I_{32} . Recall equation (5.32b), namely,

$$\begin{aligned}
I_{32} = &\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
&\left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta.
\end{aligned}$$

By first using the decomposition $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$, and then involving (5.4), we need to estimate terms of the type

$$\begin{aligned}
\tilde{I}_{32} = &\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
&\left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta.
\end{aligned}$$

We invoke Lemma 5.1 and find

$$\begin{aligned}
\tilde{I}_{32} = &\frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
&\left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 \leq A_1} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 < A_2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
& = N_1 + N_2 + N_3.
\end{aligned}$$

We consider first N_1 , where we get

$$\begin{aligned}
N_1 & = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left[\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right] d\eta \\
& = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\
& \quad \times \left[\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right. \\
& \quad + \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \theta) d\theta \\
& \quad + \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \\
& \quad + \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
& \quad + \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& \quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right] d\eta \\
& = N_{11} + N_{12} + N_{13} + N_{14} + N_{15} + N_{16}.
\end{aligned} \tag{5.39}$$

The terms N_{11} and N_{12} can be treated similarly. To that end, we find

$$\begin{aligned}
|N_{11}| & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \right. \\
& \quad \times \left[\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right. \\
& \quad \left. \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \theta) d\theta \right] d\eta \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
&\quad + \frac{4}{A^9} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}|^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{16}{A^8} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}|^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2),
\end{aligned}$$

where we have used (4.15e), (4.15h), and (4.16c).

The terms N_{13} and N_{14} follow the same estimates. More precisely,

$$\begin{aligned}
|N_{13}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
&\quad \times \left. \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \right| \\
&\leq \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) |\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+| \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) |\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+| \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
&\quad + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned} &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{2}{A^{11}} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2, \end{aligned}$$

by applying (4.15e), (4.15q), and (4.16e).

The term $N_{15} + N_{16}$ can be estimated as follows:

$$\begin{aligned} N_{15} + N_{16} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ &\quad \times \left[\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right] d\eta \\ &= \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\ &\quad \left. \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\ &\quad \left. \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\ &\quad \left. \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \end{aligned}$$

$$\begin{aligned}
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left[\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big] d\eta \\
& = N_{151} + N_{152} + N_{153} + N_{154} + N_{155}.
\end{aligned}$$

Unfortunately, each of these terms needs a special treatment.

The terms N_{151} and N_{152} can be handled as follows:

$$\begin{aligned}
|N_{151}| &= \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
& \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \Big| \\
&\leq \frac{4}{A^6 A_1 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{16}{A^{12} A_1^2 e^2} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta |A_1 - A_2|^2 \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{8}{A^4 e^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) d\eta |A_1 - A_2|^2 \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2),
\end{aligned}$$

where we used (4.16e).

The terms N_{153} and N_{154} can be handled as follows:

$$\begin{aligned}
|N_{153}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
& \times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))})
\end{aligned}$$

$$\begin{aligned}
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta d\eta \Big| \\
& \leq \frac{1}{A^6 a} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \quad \times \int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \\
& \quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \\
& = \frac{1}{A^6 a} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \\
& \quad + \frac{1}{A^6 a} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \\
& \leq \frac{1}{A^6 a} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{U}}_j^+)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
& \quad + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{A^{12} a^2} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2 \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
& \leq \frac{\sqrt{3}}{2A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) d\eta + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
& \quad + \frac{9}{4A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

where we used (4.16k). We consider now N_{155} :

$$\begin{aligned}
 N_{155} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\
 &\quad \times \left[\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
 &\quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right] d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)_\eta(t, \eta) \\
 &\quad \times \min \left[\int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
 &\quad \times \left(\int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t, \theta) d\theta \right) d\eta \\
 &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \mathbb{1}_{\tilde{E}}(t, \eta) \\
 &\quad \times \left(\int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t, \theta) d\theta \right) d\eta \\
 &= N_{1551} + N_{1552} + N_{1553},
 \end{aligned}$$

which, yet again, requires separate treatment. Here \tilde{E} is defined in (5.16).

The term N_{1551} can be handled as the term \tilde{L}_{31} (cf. (5.17)),

$$|N_{1551}| \leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2.$$

The terms N_{1552} and N_{1553} can be treated in the same manner:

$$\begin{aligned}
 |N_{1552}| &\leq \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \right. \\
 &\quad \times \left(\int_0^\eta \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t, \theta) d\theta \right) d\eta \Big| \\
 &\leq \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) |\tilde{\mathcal{U}}_{2,\eta}| \mathbb{1}_{\tilde{E}^c}(t, \eta) d\eta \\
 &\quad + \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \right. \\
 &\quad \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \left[\frac{d}{d\theta} \left(\min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min(\tilde{\mathcal{P}}_j) \min(\tilde{\mathcal{U}}_j^+) \right) \right.
 \end{aligned}$$

$$+ \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \frac{d}{d\theta} (\min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)) \Big] (t, \theta) d\theta d\eta |$$

$$\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2).$$

The term N_2 (and also N_3) can be treated as follows, keeping in mind that $A_2 \leq A_1$.

$$|N_2| \leq \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2, \eta}| (t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2}$$

$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta$$

$$\leq \sqrt{6} \frac{A_1^6 - A_2^6}{A^6 A_2^3} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{P}}_2| |\tilde{\mathcal{U}}_{2, \eta}| (t, \eta) d\eta$$

$$\leq 3\sqrt{3} A^2 |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1| (t, \eta) d\eta$$

$$\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).$$

Finally, we now turn to the integral I_{33} . Recall equation (5.32c), namely,

$$I_{33} = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta \right.$$

$$\left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1, \eta}(t, \theta) d\theta \right) d\eta.$$

By first involving (5.4) and then Lemma 5.1, we see that it suffices to estimate

$$\tilde{I}_{33} = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta \right.$$

$$\left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1, \eta}(t, \theta) d\theta \right) d\eta$$

$$= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\tilde{\mathcal{U}}_{2, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta \right.$$

$$\left. - \tilde{\mathcal{U}}_{1, \eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1, \eta}(t, \theta) d\theta \right) d\eta$$

$$+ \mathbb{1}_{A_2 \leq A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2, \eta}(t, \eta)$$

$$\times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta d\eta$$

$$\begin{aligned}
& + \mathbb{1}_{A_1 < A_2} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta d\eta \\
& = M_1 + M_2 + M_3.
\end{aligned}$$

To our dismay, the estimate for M_1 is rather involved. We estimate, using first that $\tilde{\mathcal{Q}}_i = A_i \tilde{\mathcal{P}}_i - \tilde{\mathcal{D}}_i$ (cf. (4.7)):

$$\begin{aligned}
M_1 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
& \quad - \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (A_2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta} - A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta})(t, \eta) d\eta \\
& \quad - \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta})(t, \eta) d\eta \\
& \quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\
& \quad \times \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
& \quad - \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\
& \quad \times \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right.
\end{aligned}$$

$$\begin{aligned}
& -\tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \Big(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \Big(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 \tilde{\mathcal{U}}_2(t, \theta) d\theta \\
& - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} A_1^5 \tilde{\mathcal{U}}_1(t, \theta) d\theta \Big) d\eta \\
& = W_1 + W_2 + W_3 + W_4 + W_5 + W_6.
\end{aligned}$$

Here we have used the rewrite employed when manipulating the term \bar{K}_1 from the expression (5.18) to (5.19). We start by considering the term W_1 :

$$\begin{aligned}
W_1 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(A_2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta} - A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta})(t, \eta) d\eta \\
&= \frac{1}{A^6} \mathbb{1}_{A_1 \leq A_2} (A_2 - A_1) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \eta) d\eta \\
&\quad + \frac{1}{A^6} \mathbb{1}_{A_2 < A_1} (A_2 - A_1) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \eta) d\eta \\
&\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\
&\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) d\eta \\
&\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_{1,\eta}) \min_j (\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \\
&\quad - \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \min_j (\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) d\eta \\
&= W_{11} + W_{12} + W_{13} + W_{14} + W_{15} + W_{16}.
\end{aligned}$$

For W_{11} we have (and similarly for W_{12}) that

$$\begin{aligned}
|W_{11}| &\leq \frac{A^2}{8} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

The terms W_{13} and W_{14} are similar:

$$\begin{aligned} |W_{11}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \right| \\ &\leq \frac{2a}{A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta}|(t, \eta) d\eta \\ &\leq \frac{A}{2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}| d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}\|^2), \end{aligned}$$

using (4.15a) and (4.15f).

The term W_{15} goes as follows:

$$\begin{aligned} |W_{15}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_{1,\eta}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \right| \\ &= \left| \frac{a}{2A^6} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) \right|_{\eta=0}^1 \\ &\quad - \frac{a}{2A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \Big| \\ &\leq \frac{a}{2A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2) \right|(t, \eta) d\eta \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2; \end{aligned}$$

see the estimates for \bar{B}_{15} (cf. (5.20)). As for the term W_{16} , we get

$$\begin{aligned} |W_{16}| &= \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \min_j(\tilde{\mathcal{P}}_j) |\tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2, \end{aligned}$$

using (4.15p).

As for the term W_2 , we find

$$\begin{aligned} -W_2 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta})(t, \eta) d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \eta) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \eta) d\eta \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_{1,\eta}) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \\
& - \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) d\eta \\
& = W_{21} + W_{22} + W_{23} + W_{24}.
\end{aligned}$$

The terms W_{21} and W_{22} can be treated similarly. We need to estimate $\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1$. Applying Lemma A.9, we have

$$\begin{aligned}
|W_{21}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \eta) d\eta \right| \\
&\leq \frac{2}{A^{9/2}} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| (\tilde{\mathcal{U}}_1^2 + \tilde{\mathcal{P}}_1) |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta \\
&\quad + \frac{2\sqrt{2}}{A^{9/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta \\
&\quad + \frac{4}{A^3} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta \\
&\quad + \frac{2\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta \\
&\quad + \frac{3}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta \\
&\quad + \frac{12\sqrt{2}}{\sqrt{3}eA^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \\
&\quad \times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta |A_1 - A_2| \\
&\quad + \frac{3}{2A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \right) |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \frac{6}{A^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} d\theta \right) |\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta \\
& \leq \frac{2}{A^{9/2}} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \frac{A^{13/2}}{2\sqrt{2}} d\eta \\
& + \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \frac{3A^8}{8} d\eta \\
& + \frac{2\sqrt{2}}{A^{9/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \frac{A^{13/2}}{2\sqrt{2}} d\eta \\
& + \frac{4}{A^3} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2} d\eta \\
& + \frac{2\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2} d\eta \\
& + \frac{3}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2} d\eta \\
& + \frac{12\sqrt{2}}{\sqrt{3}eA^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \frac{A^4}{2} d\eta |A_1 - A_2| \\
& + \frac{3}{2A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \right) \frac{A^4}{2} d\eta \\
& + \frac{6}{A^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \eta) \frac{A^4}{2} d\eta \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2),
\end{aligned}$$

where we have used estimates (4.15a), (4.15b), (4.15f), and (4.15n). The term W_{23} goes as follows:

$$\begin{aligned}
|W_{23}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_{1,\eta}) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \right| \\
&\leq \left| \frac{1}{2A^6} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) \right|_{\eta=0}^1 \\
&\quad - \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \Big| \\
&\leq \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \left| \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2) \right|(t, \eta) d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2,
\end{aligned}$$

by applying Lemma A.4 (ii).

The term W_{24} goes as follows:

$$\begin{aligned} |W_{24}| &\leq \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \min_j (\tilde{\mathcal{D}}_j) |\tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2, \end{aligned}$$

using

$$\min_j (\tilde{\mathcal{D}}_j) |\tilde{\mathcal{U}}_{1,\eta}| \leq 2A_1 \tilde{\mathcal{P}}_1 |\tilde{\mathcal{U}}_{1,\eta}| \leq \frac{A^7}{\sqrt{2}},$$

from (4.15n) and (4.15p).

Next, we turn to the term W_3 :

$$\begin{aligned} W_3 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ &\quad \times \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ &\quad \times \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ &\quad \times \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^- \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^- \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &= W_{31} + W_{32}. \end{aligned}$$

These terms can be treated similarly:

$$\begin{aligned} W_{31} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \\ &\quad \times \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \end{aligned}$$

$$\begin{aligned}
& -\tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
& = \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
& \quad + \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& \quad + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
& \quad + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_2 < \tilde{\mathcal{D}}_1}(t, \theta) d\theta \right) d\eta \\
& \quad + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \\
& \quad + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \right) d\eta \\
& \quad + \frac{1}{A^7} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\
& \quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& \quad + \frac{1}{A^7} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \right.
\end{aligned}$$

$$\begin{aligned}
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \Big) d\eta \\
& - \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t, \eta) \\
& \times \min_k \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta} \mathbb{1}_{D^c}(t, \eta) \\
& \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta} \mathbb{1}_D(t, \eta) \\
& \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta \\
& = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + Z_8 + Z_9 + Z_{10} + Z_{11} + Z_{12} + Z_{13},
\end{aligned}$$

where the set D is defined by (5.21). At this point, it cannot come as a surprise that we have to treat these terms separately.

Let us start with Z_1 (Z_2 is similar):

$$\begin{aligned}
|Z_1| & \leq \frac{2}{A^7} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} d\eta
\end{aligned}$$

$$\begin{aligned} &\leq \frac{2\sqrt{6}}{A^4} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_1 |\tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2). \end{aligned}$$

The terms Z_3 and Z_4 go as follows:

$$\begin{aligned} |Z_3| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\ &\quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \right| \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \end{aligned}$$

by Lemma A.9. The terms Z_5 and Z_6 can be treated similarly:

$$\begin{aligned} |Z_5| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\ &\quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \right| \\ &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{A^{14}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\ &\leq \mathcal{O}(1)\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2, \end{aligned}$$

by applying (4.16e), estimating $\tilde{\mathcal{D}}_2 \leq 2A_2\tilde{\mathcal{P}}_2$ (cf. (4.15n)), and subsequently $2\sqrt{2}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_{2,\eta}^2 \leq A_2^6$ (cf. (4.15q)). The terms Z_7 and Z_8 follow this pattern:

$$\begin{aligned} |Z_8| &\leq \frac{4}{A^7 a e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{3}{4A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} 2A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\ &\leq \frac{4\sqrt{2}}{A^5 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{1,\eta}|(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_2(t,\eta))} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \end{aligned}$$

$$\begin{aligned} &\leq \frac{4\sqrt{6}}{A^2 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_1^{1/2} |\tilde{\mathcal{U}}_{1,\eta}|(t, \eta) d\eta |A_1 - A_2| \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2). \end{aligned}$$

The terms Z_9 and Z_{10} follow this pattern:

$$\begin{aligned} |Z_9| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \right. \\ &\quad \left. \left. \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \right| \\ &\leq \frac{a}{\sqrt{2}A^7} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\ &\quad \times \left(\int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \right. \\ &\quad \left. \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\leq \frac{1}{\sqrt{2}A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{1}{\sqrt{2}A^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) d\theta \right) d\eta \\ &\leq \frac{\sqrt{2}}{A^5} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta \\ &\quad + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{2}{A^{10}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|^2(t, \theta) d\theta \right) d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2). \end{aligned}$$

The term Z_{11} :

$$\begin{aligned} |Z_{11}| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t, \eta) \right. \\ &\quad \times \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right) d\eta \Big| \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|_2^2, \end{aligned}$$

following the estimates employed for the term \bar{B}_{37} ; see (5.23).

The terms Z_{12} and Z_{13} may be treated as follows:

$$\begin{aligned} |Z_{12}| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2, \eta} \mathbb{1}_{D^c}(t, \eta) \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \right. \\ &\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \Big) d\eta \Big| \\ &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2, \eta} \mathbb{1}_{D^c}(t, \eta) \right. \\ &\quad \times \left[\left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \right) \Big|_{\theta=0}^\eta \right. \\ &\quad \left. \left. - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} (\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+)) (t, \theta) d\theta \right] d\eta \right| \\ &\leq \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2, \eta} \mathbb{1}_{D^c} \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \eta) d\eta \right| \\ &\quad + \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2, \eta} \mathbb{1}_{D^c}(t, \eta) \right. \\ &\quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} (\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+)) (t, \theta) d\theta \right) d\eta \Big| \\ &\leq \frac{A^2}{4} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) d\eta \\ &\quad + \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2, \eta}|(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \left| \frac{d}{d\theta} (\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \right. \right. \\ &\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+)) \Big| (t, \theta) d\theta \Big) d\eta \\ &= \tilde{M}_1 + \tilde{M}_2. \end{aligned}$$

Here we find

$$\tilde{M}_1 \leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),$$

while \tilde{M}_2 requires more care:

$$\begin{aligned}
 \tilde{M}_2 &= \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \left| \frac{d}{d\theta} (\min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta)))}) \right. \right. \\
 &\quad \times \min(\tilde{\mathcal{D}}_j) \min(\tilde{\mathcal{U}}_j^+)(t, \theta) \Big| d\theta \Big) d\eta \\
 &\leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \right. \\
 &\quad \times \left| \frac{d}{d\theta} (\min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta)))}) \min(\tilde{\mathcal{D}}_j) \min(\tilde{\mathcal{U}}_j^+) \right. \\
 &\quad \left. \left. + \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta)))}) \frac{d}{d\theta} (\min(\tilde{\mathcal{D}}_j) \min(\tilde{\mathcal{U}}_j^+)) \right| (t, \theta) d\theta \right) d\eta \\
 &\leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \left| \frac{d}{d\theta} (\min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta)))}) \right. \right. \\
 &\quad \times \min(\tilde{\mathcal{D}}_j) \min(\tilde{\mathcal{U}}_j^+)(t, \theta) \Big| d\theta \Big) d\eta \\
 &\quad + \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \left| \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta)))}) \right. \right. \\
 &\quad \times \frac{d}{d\theta} (\min(\tilde{\mathcal{D}}_j) \min(\tilde{\mathcal{U}}_j^+)(t, \theta)) \Big| d\theta \Big) d\eta \\
 &\leq \frac{1}{aA^7} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta)))}) \right. \\
 &\quad \times \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min(\tilde{\mathcal{D}}_j) \min(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big) d\eta \\
 &\quad + \mathcal{O}(1) \frac{1}{A^{5/2}} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 &\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta)))}) (\min(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right) d\eta \\
 &= \tilde{M}_{21} + \tilde{M}_{22},
 \end{aligned}$$

where estimates for the derivatives come from Lemmas A.2 and A.4. We find for the term \tilde{M}_{21} that

$$\begin{aligned}
 \tilde{M}_{21} &= \frac{1}{aA^7} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta)))}) \right. \\
 &\quad \times \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min(\tilde{\mathcal{D}}_j) \min(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big) d\eta
 \end{aligned}$$

$$\begin{aligned}
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{a^2 A^{14}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
&\quad \times \max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big)^2 d\eta \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{2} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
&\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{A^3}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
&\quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{A^3}{2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\theta)} d\theta \right) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

where we used the estimate

$$\max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (\tilde{\mathcal{D}}_j) \leq 2 \max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (A_j \tilde{\mathcal{P}}_j) \leq 2A \max_j (\tilde{\mathcal{P}}_j \tilde{\mathcal{Y}}_{j,\eta}) \leq A^6,$$

using (4.15e) and (4.15n), as well as (4.15q) and (4.16a). The term \tilde{M}_{22} reads as

$$\begin{aligned}
\tilde{M}_{22} &= \mathcal{O}(1) \frac{1}{A^{5/2}} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
&\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\min_j (\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{2A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A^2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \mathcal{O}(1) \frac{1}{A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta)
\end{aligned}$$

$$\begin{aligned} & \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\ & \leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2). \end{aligned}$$

Here we employed

$$\min_j (\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2| \leq \sqrt{2A_2} \tilde{\mathcal{P}}_2^{1/2} + \sqrt{2} \tilde{\mathcal{P}}_2^{1/2} \leq \sqrt{2}(1 + \sqrt{A_2}) \tilde{\mathcal{P}}_2^{1/2},$$

as well as (4.15q) and (4.16i).

Next, we turn to the term W_4 :

$$\begin{aligned} -W_4 &= \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ & \quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &= W_4^+ + W_4^-, \end{aligned}$$

with

$$\begin{aligned} W_4^\pm &= \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^\pm \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ & \quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^\pm \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2), \end{aligned}$$

using that $W_4^+ = 3N_1$ (see (5.39)) and similarly for W_4^- .

Next, we turn to the term W_5 :

$$\begin{aligned} W_5 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ & \quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta. \end{aligned}$$

We take positive and negative parts of the term $\tilde{\mathcal{U}}_j^3$, and thus it suffices to study the term

$$\begin{aligned} W_5^+ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \left[\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \\ & \quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right] d\eta. \end{aligned}$$

Having a close look at W_5^+ one has

$$W_5^+ = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8)(t, \eta) d\eta,$$

where J_1, \dots, J_6 are defined in (5.33). Thus we can conclude immediately that

$$\begin{aligned} |W_5^+| &\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\ &\quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

Next, we turn to the term W_6 :

$$\begin{aligned} 2W_6 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \left(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2^5 \tilde{\mathcal{U}}_2(t, \theta) d\theta \right. \\ &\quad \left. - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} A_1^5 \tilde{\mathcal{U}}_1(t, \theta) d\theta \right) d\eta \\ &= \frac{A_2^5 - A_1^5}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\ &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2(t, \theta) d\theta d\eta \\ &\quad + \frac{A_2^5 - A_1^5}{A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\ &\quad \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1(t, \theta) d\theta d\eta \\ &\quad + \frac{a^5}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \tilde{\mathcal{U}}_2(t, \theta) d\theta d\eta \\ &\quad + \frac{a^5}{A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \tilde{\mathcal{U}}_1(t, \theta) d\theta d\eta \\ &\quad + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t, \theta) d\theta d\eta \\ &\quad + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \end{aligned}$$

$$\begin{aligned}
& \times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_1 \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta d\eta \\
& + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \\
& + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\
& \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-) \mathbb{1}_{\tilde{\mathcal{U}}_2^- \leq \tilde{\mathcal{U}}_1^-}(t, \theta) d\theta d\eta \\
& + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta d\eta \\
& + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\
& \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-) \mathbb{1}_{\tilde{\mathcal{U}}_1^- < \tilde{\mathcal{U}}_2^-}(t, \theta) d\theta d\eta \\
& - \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t, \eta) \\
& \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta d\eta \\
& - \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t, \eta) \\
& \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \max_j (\tilde{\mathcal{U}}_j^-)(t, \theta) d\theta d\eta \\
& = W_{61} + W_{62} + W_{63} + W_{64} + W_{65} + W_{66} + W_{67}^\pm + W_{68}^\pm + W_{69}^\pm.
\end{aligned}$$

Here we go again! The terms W_{61} and W_{62} :

$$\begin{aligned}
|W_{61}| &= \frac{A_2^5 - A_1^5}{A^6} \mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
&\quad \times \left. \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2(t, \theta) d\theta d\eta \right| \\
&\leq \frac{A_2^5 - A_1^5}{A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right)^{1/2} d\eta \\
& \leq \frac{\sqrt{65}}{A^2} |A_2 - A_1| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) d\eta \\
& \leq \sqrt{35} A^2 |A_2 - A_1| \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \\
& \leq \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2),
\end{aligned}$$

using (4.15q) and (4.16a).

The terms W_{63} and W_{64} :

$$\begin{aligned}
|W_{63}| & \leq \frac{4a^4}{A^6 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \times \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta |A_1 - A_2| \\
& \leq \frac{4}{A^2 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
& \leq \frac{4\sqrt{6}}{A^2 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) d\eta |A_1 - A_2| \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

Next comes the terms W_{65} and W_{66} :

$$\begin{aligned}
|W_{65}| & = \frac{a^5}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
& \times \left. \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t, \theta) d\theta d\eta \right| \\
& \leq \frac{1}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
& \times \int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \\
& \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta \\
&\quad + \frac{1}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta \\
&\leq \frac{1}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta \\
&\quad + \frac{1}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\leq \frac{\sqrt{6}}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) d\eta \\
&\quad + \frac{\sqrt{6}}{A^2} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

using (4.15q), (4.16a), and (5.13).

The terms W_{67}^\pm and W_{68}^\pm have a similar structure:

$$\begin{aligned}
|W_{67}^+| &= \frac{a^5}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
&\quad \times \left. \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \right| \\
&\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{A^2} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)(t, \theta) d\theta \right)^2 d\eta
\end{aligned}$$

$$\begin{aligned}
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \right) d\eta \\
 &\leq \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
 &\quad + A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) e^{-\frac{1}{\lambda_2} \tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{\lambda_2} \tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2,
 \end{aligned}$$

using (4.15h).

Finally, the terms W_{69}^\pm :

$$\begin{aligned}
 |W_{69}^+| &= \frac{a^5}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t, \eta) \right. \\
 &\quad \times \left. \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta d\eta \right| \\
 &\leq \frac{1}{2A} \left| \left((\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right|_{\eta=0}^1 \\
 &\quad - \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \eta) \frac{d}{d\eta} \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta d\eta \Big| \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2;
 \end{aligned}$$

see estimates for \tilde{B}_{67}^\pm (cf. (5.25)) and Lemma A.5.

The terms M_2 and M_3 can be treated similarly. More precisely,

$$\begin{aligned}
 |M_2| &\leq \mathbb{1}_{A_2 \leq A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \right. \\
 &\quad \times \left. \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta d\eta \right| \\
 &\leq \mathbb{1}_{A_2 \leq A_1} \frac{|A_1^6 - A_2^6|}{A^6 A_2^5} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| |\tilde{\mathcal{U}}_{2,\eta}|(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_{2,\eta}|(t, \theta) d\theta d\eta \\
 &\leq \mathbb{1}_{A_2 \leq A_1} 6 \frac{|A_1 - A_2|}{A A_2^5} \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \\
 &\quad \times \left(\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_{2,\eta}|(t, \theta) d\theta \right)^2 d\eta \right)^{1/2}
 \end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{1}_{A_2 \leq A_1} 6 \frac{|A_1 - A_2|}{AA_2^6} \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \\
&\quad \times \left(\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \right. \\
&\quad \left. \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
&\leq \mathbb{1}_{A_2 \leq A_1} 6\sqrt{6} \frac{|A_1 - A_2|}{AA_2^3} \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \left(\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2 \tilde{\mathcal{P}}_2^2(t, \eta) d\eta \right)^{1/2} \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

Here we used

$$|\tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}| \leq \frac{1}{A_2} \tilde{\mathcal{P}}_2 \sqrt{\tilde{\mathcal{Y}}_{2,\eta} \tilde{\mathcal{H}}_{2,\eta}}$$

(cf. (4.15m)) as well as (4.15a) and (4.15q). In addition, we applied (4.16b) and (4.16d).

We have shown the anticipated result.

LEMMA 5.4. *Let $\tilde{\mathcal{U}}_i$ be two solutions of (5.28). Then we have*

$$\begin{aligned}
&\frac{d}{dt} \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
\end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant which only depends on $A = \max_j(A_j)$ and which remains bounded as $A \rightarrow 0$.

5.3. Lipschitz estimates for $\tilde{\mathcal{P}}$ (or $\tilde{\mathcal{P}}^{1/2}$). From the system of differential equations we recall

$$(\tilde{\mathcal{P}}_i^{1/2})_t + \left(\frac{2}{3} \frac{1}{A_i^5} \tilde{\mathcal{U}}_i^3 + \frac{1}{A_i^6} \tilde{\mathcal{S}}_i \right) (\tilde{\mathcal{P}}_i^{1/2})_\eta = \frac{1}{2A_i^2} \frac{\tilde{\mathcal{Q}}_i \tilde{\mathcal{U}}_i}{\tilde{\mathcal{P}}_i^{1/2}} + \frac{1}{2A_i^3} \frac{\tilde{\mathcal{R}}_i}{\tilde{\mathcal{P}}_i^{1/2}}, \quad (5.40)$$

where

$$\tilde{\mathcal{Q}}_i(t, \eta) = -\frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{A_i} |\tilde{\mathcal{Y}}_i(t, \eta) - \tilde{\mathcal{Y}}_i(t, \theta)|} (2(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) + A_i^5) d\theta, \quad (5.41a)$$

$$\tilde{\mathcal{S}}_i(t, \eta) = \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t, \eta) - \tilde{\mathcal{Y}}_i(t, \theta)|} \left(\frac{2}{3} \tilde{\mathcal{U}}_i^3 \tilde{\mathcal{Y}}_{i, \eta} - \tilde{\mathcal{Q}}_i \tilde{\mathcal{U}}_{i, \eta} - 2 \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i, \eta} \right) (t, \theta) d\theta, \quad (5.41b)$$

$$\begin{aligned} \tilde{\mathcal{R}}_i(t, \eta) &= \frac{1}{4} \int_0^1 \text{sign}(\eta - \theta) e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t, \eta) - \tilde{\mathcal{Y}}_i(t, \theta)|} \left(\frac{2}{3} A_i \tilde{\mathcal{U}}_i^3 \tilde{\mathcal{Y}}_{i, \eta} + A_i^6 \tilde{\mathcal{U}}_i \right) (t, \theta) d\theta \\ &\quad - \frac{1}{2} \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t, \eta) - \tilde{\mathcal{Y}}_i(t, \theta)|} \tilde{\mathcal{U}}_i \tilde{\mathcal{Q}}_i \tilde{\mathcal{Y}}_{i, \eta} (t, \theta) d\theta. \end{aligned} \quad (5.41c)$$

Thus we have

$$\begin{aligned} &\frac{d}{dt} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \eta) d\eta \\ &= 2 \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_1^{1/2})_t - (\tilde{\mathcal{P}}_2^{1/2})_t)(t, \eta) d\eta \\ &= -\frac{2}{3} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left(\frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta - \frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \right) (t, \eta) d\eta \\ &\quad - \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left(\frac{1}{A_1^6} \tilde{\mathcal{S}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta - \frac{1}{A_2^6} \tilde{\mathcal{S}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta \right) (t, \eta) d\eta \\ &\quad + \frac{1}{2} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left(\frac{1}{A_1^2} \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^2} \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right) (t, \eta) d\eta \\ &\quad + \frac{1}{2} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left(\frac{1}{A_1^3} \frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^3} \frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right) (t, \eta) d\eta \\ &= \frac{2}{3} I_1 + I_2 + \frac{1}{2} I_3 + \frac{1}{2} I_4. \end{aligned}$$

We will estimate each of these terms, yielding that

$$\begin{aligned} &\frac{d}{dt} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant which only depends on $A = \max_j (A_j)$ and which remains bounded as $A \rightarrow 0$.

The term I_1 : Here we do as follows:

$$\begin{aligned} I_1 &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left(\frac{1}{A_2^5} \tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta - \frac{1}{A_1^5} \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \right) (t, \eta) d\eta \\ &= \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta - \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta) (t, \eta) d\eta \end{aligned}$$

$$\begin{aligned}
& + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \times (\tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{A_2 < A_1} + \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{A_1 \leq A_2})(t, \eta) d\eta \\
& = \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{U}}_2^2 (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) d\eta \\
& + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2^2 (\tilde{\mathcal{P}}_2^{1/2})_\eta - \tilde{\mathcal{U}}_1^2 (\tilde{\mathcal{P}}_1^{1/2})_\eta) \tilde{\mathcal{U}}_1(t, \eta) d\eta \\
& + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \times (\tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{A_2 < A_1} + \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{A_1 \leq A_2})(t, \eta) d\eta \\
& = \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \tilde{\mathcal{U}}_2^2 (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) d\eta \\
& + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{U}}_1^2) (\tilde{\mathcal{P}}_2^{1/2})_\eta \tilde{\mathcal{U}}_1 \mathbb{1}_{\tilde{\mathcal{U}}_1^2 \leq \tilde{\mathcal{U}}_2^2}(t, \eta) d\eta \\
& + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{U}}_1^2) (\tilde{\mathcal{P}}_1^{1/2})_\eta \tilde{\mathcal{U}}_1 \mathbb{1}_{\tilde{\mathcal{U}}_2^2 < \tilde{\mathcal{U}}_1^2}(t, \eta) d\eta \\
& + \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{U}}_j^2)(t, \eta) d\eta \\
& + \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \times (\tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{A_2 < A_1})(t, \eta) d\eta \\
& = I_{11} + I_{12} + I_{13} + I_{14} + I_{15}.
\end{aligned}$$

We first estimate

$$|(\tilde{\mathcal{P}}_i^{1/2})_\eta| = \frac{|\tilde{\mathcal{P}}_{i,\eta}|}{2\tilde{\mathcal{P}}_i^{1/2}} = \frac{|\tilde{\mathcal{Q}}_i \tilde{\mathcal{Y}}_{i,\eta}|}{2A_i^2 \tilde{\mathcal{P}}_i^{1/2}} \leq \frac{1}{2A_i} \tilde{\mathcal{P}}_i^{1/2} \tilde{\mathcal{Y}}_{i,\eta},$$

and thus

$$|\tilde{\mathcal{U}}_i \tilde{\mathcal{P}}_i^{1/2} \tilde{\mathcal{Y}}_{i,\eta}| \leq \frac{1}{2} (\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta} + \tilde{\mathcal{P}}_i \tilde{\mathcal{Y}}_{i,\eta}) \leq \frac{3}{4} A_i^5.$$

We start with the term I_{11} :

$$\begin{aligned}
|I_{11}| & \leq \mathcal{O}(1) \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1|(t, \eta) d\eta \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2).
\end{aligned}$$

Subsequently (and similarly for I_{13}),

$$\begin{aligned} |I_{12}| &\leq \frac{1}{A^5} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1| |\tilde{\mathcal{U}}_2 + \tilde{\mathcal{U}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta \tilde{\mathcal{U}}_1| \mathbb{1}_{\tilde{\mathcal{U}}_1^2 \leq \tilde{\mathcal{U}}_2^2}(t, \eta) d\eta \\ &\leq \frac{2}{A^5} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1| \tilde{\mathcal{U}}_2^2 |(\tilde{\mathcal{P}}_2^{1/2})_\eta| (t, \eta) d\eta \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2). \end{aligned}$$

The term I_{14} requires more estimates (see (5.31)):

$$\begin{aligned} |I_{14}| &= \frac{1}{A^5} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2)(t, \eta) d\eta \right| \\ &= \left| \frac{1}{2A^5} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2)(t, \eta) \right|_{\eta=0}^1 \\ &\quad - \frac{1}{2A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \left(\frac{d}{d\eta} \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2) \right) (t, \eta) d\eta \Big| \\ &= \frac{1}{2A^5} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \left(\frac{d}{d\eta} \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2) \right) (t, \eta) d\eta \right| \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2. \end{aligned}$$

The last term can be estimated as follows:

$$\begin{aligned} |I_{15}| &\leq \left| \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \right. \\ &\quad \times (\tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{A_1 \leq A_2} + \tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{A_2 < A_1})(t, \eta) d\eta \Big| \\ &\leq 5 \frac{|A_1 - A_2| A^4}{A_1^5 A_2^5} \\ &\quad \times \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (|\tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta| \mathbb{1}_{A_1 \leq A_2} + |\tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{A_2 < A_1})(t, \eta) d\eta. \end{aligned}$$

We consider, in particular, the term

$$\frac{A^4}{A_1^5 A_2^5} |\tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta| \mathbb{1}_{A_1 \leq A_2} \leq \frac{A^4}{A_1^5 A_2^5} \frac{3}{16} A_1^8 \mathbb{1}_{A_1 \leq A_2} \leq \frac{3}{16} A^2.$$

Thus

$$|I_{15}| \leq \frac{30}{16} A^2 |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta \\ \leq \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2).$$

The term I_2 : Recall that the term reads as follows:

$$I_2 = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left(\frac{1}{A_2^6} \tilde{\mathcal{S}}_2(\tilde{\mathcal{P}}_2^{1/2})_\eta - \frac{1}{A_1^6} \tilde{\mathcal{S}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta \right) (t, \eta) d\eta.$$

Here we can follow the estimates of the term I_3 ; cf. (5.6) and (5.9). Here we go. Define

$$I_{21} = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta, \\ I_{22} = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ \times \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta, \\ I_{23} = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ \times \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta \right. \\ \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1, \eta}(t, \theta) d\theta \right) d\eta.$$

Using the definition (5.41b) of $\tilde{\mathcal{S}}$, we can write

$$I_2 = \frac{2}{3} I_{21} - 2 I_{22} - I_{23}.$$

We commence with the estimates for I_{21} :

$$I_{21} = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta)$$

$$\begin{aligned}
& \times \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
& = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} (\tilde{\mathcal{U}}_2^-)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
& \quad + \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta.
\end{aligned}$$

The two terms are similar, and we only discuss the second one. Next, we use trick (5.4). Thus we have to consider the term

$$\begin{aligned}
& \frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
& \quad - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
& = \frac{1}{A^6} \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) \\
& \quad + \mathbb{1}_{A_1 \leq A_2} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
& \quad + \mathbb{1}_{A_2 < A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
& = \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
& \quad \times \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
& \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
& + \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
& - \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
& + \mathbb{1}_{A_1 \leq A_2} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
& + \mathbb{1}_{A_2 < A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
& = \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
& \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \\
& + \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
& \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
& + \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
& \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta \\
& + \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
& \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta d\eta \\
& + \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \left(\int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right. \\
& \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) \\
& - \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \left(\int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \right. \\
& \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \Big) \\
& + \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \left(\int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{A^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) \\
& + \mathbb{1}_{A_1 \leq A_2} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
& + \mathbb{1}_{A_2 < A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
& = (I_{211} + I_{212} + I_{213} + I_{214} + I_{215} + I_{216} + I_{217} + I_{218} + I_{219} + I_{220})(t, \eta).
\end{aligned} \tag{5.42}$$

The same estimate works for I_{211} and I_{212} (cf. (5.34)):

$$\left| \int_0^1 I_{211}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \int_0^1 I_{211}^2(t, \eta) d\eta,$$

which shows that it suffices to estimate

$$\begin{aligned}
\int_0^1 I_{211}^2(t, \eta) d\eta & \leq \frac{1}{A^{12}} \int_0^1 (\tilde{\mathcal{P}}_2^{1/2})_\eta^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \quad \times (\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \Big)^2 d\eta \\
& \leq \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2,
\end{aligned}$$

where we used that $((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2 \leq \frac{1}{8} A_2^3 \tilde{\mathcal{Y}}_{2, \eta}$.

Regarding the term I_{213} (and I_{214}), we find

$$\begin{aligned}
& \left| \int_0^1 I_{213}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \\
& = \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
& \quad \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta \Big| \\
& \leq \frac{2\sqrt{2}a}{A^6 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
& \quad \times \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta |A_1 - A_2| \\
& \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
& \quad + \frac{8}{A^{10} e^2} \int_0^1 (\tilde{\mathcal{P}}_2^{1/2})_\eta^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)
\end{aligned}$$

$$\begin{aligned} & \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\ & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{4}{Ae^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) d\eta |A_1 - A_2|^2 \\ & \leq \mathcal{O}(1)(\| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + |A_1 - A_2|^2). \end{aligned}$$

Regarding the term I_{215} (and I_{216}), we find

$$\begin{aligned} & \left| \int_0^1 I_{215}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \\ &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\ & \quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\ & \quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \left. \right) d\eta \left. \right| \\ & \leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\ & \quad \times \left(\int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \right. \\ & \quad \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \left. \right) d\eta \\ & \leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\ & \quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \\ & \quad + \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{1}{a^2 A^{12}} \int_0^1 (\tilde{\mathcal{P}}_2^{1/2})_\eta^2(t, \eta) \\ & \quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)| e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\ & \leq \mathcal{O}(1)(\| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2), \end{aligned}$$

using the estimates in (5.35) and $((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2 \leq \frac{1}{8} A^3 \tilde{\mathcal{Y}}_{2,\eta}$.

The next terms are $I_{217} + I_{218}$:

$$I_{217} + I_{218} = \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta)$$

$$\begin{aligned}
& \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& - \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
& \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \\
& = \frac{1}{A^6} ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \\
& \times \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] \\
& + \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\
& \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \mathbb{1}_E(t, \eta) \\
& + \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\
& \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \mathbb{1}_{E^c}(t, \eta) \\
& = L_{211} + L_{212} + L_{213}.
\end{aligned}$$

The term L_{211} can be estimated as in (5.36), yielding the result

$$\left| \int_0^1 L_{211}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2.$$

The terms L_{212} and L_{213} can be treated similarly; we list L_{213} :

$$\left| \int_0^1 L_{213}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2);$$

see (5.37). The terms I_{219} and I_{220} share the same behavior. We find, when $A_1 \leq A_2$ (otherwise I_{219} vanishes; see (5.38)), that

$$\begin{aligned}
& \left| \int_0^1 I_{219}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) d\eta \right| \\
& = \left(\frac{1}{A_1^6} - \frac{1}{A_2^6} \right) \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \right. \\
& \quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \right|
\end{aligned}$$

$$\leq \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2).$$

Next, we consider the term I_{22} :

$$\begin{aligned} I_{22} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad \times \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\ &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta. \end{aligned}$$

We start by replacing the integrals $\int_0^1 (\dots) d\theta$ with $\int_0^\eta (\dots) d\theta$; cf. (5.4). Furthermore, we write $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$, and hence it suffices to consider

$$\begin{aligned} \tilde{I}_{22} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad \times \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\ &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad \times \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\ &\quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 \leq A_1} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 < A_2} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &= N_1 + N_2 + N_3. \end{aligned}$$

We consider first N_1 , where we get

$$\begin{aligned}
 N_1 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 &\quad \times \left[(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
 &\quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right] d\eta \\
 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 &\quad \times \left[(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right. \\
 &\quad + (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta} \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \theta) d\theta \\
 &\quad + (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \\
 &\quad + (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1, \eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
 &\quad + (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
 &\quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right] d\eta \\
 &= N_{11} + N_{12} + N_{13} + N_{14} + N_{15} + N_{16}.
 \end{aligned} \tag{5.43}$$

The terms N_{11} and N_{12} can be treated similarly. To that end, we find

$$\begin{aligned}
 |N_{11}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right) d\eta \Big| \\
 &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^{12}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 \\
 & \quad + \frac{1}{2A^9} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} | \tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2} |^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{2}{A^8} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) \\
 & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} | \tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2} |^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
 & \leq \mathcal{O}(1) \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2,
 \end{aligned}$$

where we have used (4.15e) and (4.17c). The terms N_{13} and N_{14} follow the same lines. More precisely,

$$\begin{aligned}
 |N_{13}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
 & \quad \times \left. \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \right| \\
 & \leq \frac{1}{A^6} \int_0^1 | \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} | | (\tilde{\mathcal{P}}_2^{1/2})_\eta | (t, \eta) \\
 & \quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) | \tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+ | \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta d\eta \\
 & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{1}{A^{12}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\
 & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) | \tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+ | \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\
 & \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{1}{A^{12}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\
 & \quad \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
 & \quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta
 \end{aligned}$$

$$\begin{aligned} &\leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \frac{2}{A^{11}} \int_0^1 \tilde{\mathcal{P}}_2 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\ &\leq \mathcal{O}(1) (\| \tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2} \|^2 + \| \tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1 \|^2), \end{aligned}$$

by applying (4.15e), (4.16e), and (4.17c). The term $N_{15} + N_{16}$ can be estimated as follows:

$$\begin{aligned} N_{15} + N_{16} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad \times \left[(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\ &\quad - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \\ &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \left. \right] d\eta \\ &= \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \right. \\ &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \left. \right) d\eta \\ &\quad + \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right. \\ &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \left. \right) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\ &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \end{aligned}$$

$$\begin{aligned}
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{B^c(\eta)}(t, \theta) d\theta \\
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left[(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big] d\eta \\
& = N_{151} + N_{152} + N_{153} + N_{154} + N_{155},
\end{aligned}$$

which all, unfortunately, need a special treatment.

The terms N_{151} and N_{152} can be handled as follows:

$$\begin{aligned}
|N_{151}| & \leq \mathbb{1}_{A_1 \leq A_2} \frac{2\sqrt{2}a}{A^6 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
& \leq \frac{2\sqrt{2}a}{A^6 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
& \leq \frac{2\sqrt{2}a}{A^3 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \tilde{\mathcal{P}}_2^{1/2}(t, \eta) d\eta |A_1 - A_2| \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

The terms N_{153} and N_{154} can be handled as follows:

$$\begin{aligned}
|N_{153}| & = \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
& \times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta d\eta \Big| \\
& \leq \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta)
\end{aligned}$$

$$\begin{aligned}
& \times \int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \\
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta \\
& = \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \\
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta \\
& + \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \\
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta \\
& \leq \frac{1}{\sqrt{2}A^5} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
& + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{2A^{10}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta \\
& \leq \frac{\sqrt{3}}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2^{1/2} |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) d\eta + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
& + \frac{1}{2A^{10}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{2A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{2a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

using (4.15e), (4.16b), (4.16e), and (4.17a). We consider now N_{155} :

$$\begin{aligned}
N_{155} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \times \left[(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right] d\eta
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
&\quad \times \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right] d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
&\quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t, \theta) d\theta \right) d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \mathbb{1}_{\tilde{E}}(t, \eta) \\
&\quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t, \theta) d\theta \right) d\eta \\
&= N_{1551} + N_{1552} + N_{1553},
\end{aligned}$$

which, yet again, requires separate treatment. Here \tilde{E} is defined in (5.16). The term N_{1551} can be handled as the term \tilde{L}_{31} (cf. (5.17)),

$$|N_{1551}| \leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2.$$

The terms N_{1552} and N_{1553} can be treated in the same manner:

$$\begin{aligned}
|N_{1552}| &\leq \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \mathbb{1}_{\tilde{E}^c}(t, \eta) \\
&\quad \times \left| \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t, \theta) d\theta \right| d\eta \\
&\leq \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) |(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
&\quad \times \left| \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \left[\frac{d}{d\theta} (\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))})) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \right. \right. \\
&\quad \left. \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \frac{d}{d\theta} (\min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)) \right] (t, \theta) d\theta \right| d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2).
\end{aligned}$$

The term N_2 (and also N_3) can be treated as follows:

$$\begin{aligned} |N_2| &\leq \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta \\ &\leq \sqrt{6} \frac{A_1^6 - A_2^6}{A^6 A_2^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{P}}_2| (\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) d\eta \\ &\leq \mathcal{O}(1) |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

This concludes the discussion of the term I_{22} .

The term I_{23} is handled as follows:

$$\begin{aligned} I_{23} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ &\quad \times \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta \right. \\ &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1, \eta}(t, \theta) d\theta \right) d\eta. \end{aligned}$$

We start by replacing the integrals $\int_0^1 (\dots) d\theta$ with $\int_0^\eta (\dots) d\theta$; cf. (5.4). Thus

$$\begin{aligned} \tilde{I}_{23} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta \right. \\ &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta \right. \\ &\quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\ &\quad + \mathbb{1}_{A_2 \leq A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \end{aligned}$$

$$\begin{aligned}
& \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} \tilde{Q}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta d\eta \\
& + \mathbb{1}_{A_1 < A_2} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
& \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta)-\tilde{Y}_1(t,\theta))} \tilde{Q}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta d\eta \\
& = M_1 + M_2 + M_3.
\end{aligned}$$

We estimate, using first that $\tilde{Q}_i = A_i \tilde{\mathcal{P}}_i - \tilde{\mathcal{D}}_i$ (cf. (4.7)):

$$\begin{aligned}
M_1 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} \tilde{Q}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta)-\tilde{Y}_1(t,\theta))} \tilde{Q}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left(A_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - A_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta)-\tilde{Y}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
& \quad - \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta)-\tilde{Y}_1(t,\theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (A_2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta - A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) d\eta \\
& \quad - \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta - \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) d\eta \\
& \quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \quad \times \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right. \\
& \quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta)-\tilde{Y}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 \tilde{Y}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
& \quad - \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \quad \times \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta)-\tilde{Y}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{Y}_{2,\eta}(t, \theta) d\theta \right.
\end{aligned}$$

$$\begin{aligned}
& - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
& - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2^5 \tilde{\mathcal{U}}_2(t, \theta) d\theta \\
& - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} A_1^5 \tilde{\mathcal{U}}_1(t, \theta) d\theta \Big) d\eta \\
& = W_1 + W_2 + W_3 + W_4 + W_5 + W_6.
\end{aligned}$$

Here we have used the rewrite employed when manipulating the term \bar{K}_1 from the expression (5.18) to (5.19). We start by considering the term W_1 :

$$\begin{aligned}
W_1 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (A_2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta - A_1 \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) d\eta \\
&= \mathbb{1}_{A_1 \leq A_2} \frac{A_2 - A_1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) d\eta \\
&\quad + \mathbb{1}_{A_2 < A_1} \frac{A_2 - A_1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) d\eta \\
&\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\
&\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) d\eta \\
&\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \min_j (\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \\
&\quad + \frac{a}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1) \min_j (\tilde{\mathcal{P}}_j) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) d\eta \\
&= W_{11} + W_{12} + W_{13} + W_{14} + W_{15} + W_{16}.
\end{aligned}$$

Using (4.15a), (4.15b), (4.15e), (4.17a), and (4.17b), we find that

$$\begin{aligned}
& W_{11} + W_{12} + W_{13} + W_{14} + W_{16} \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

Regarding the term W_{15} :

$$\begin{aligned} |W_{15}| &= \frac{a}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \right| \\ &= \left| \frac{a}{2A^6} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2(t, \eta) \right|_{\eta=0}^1 \\ &\quad - \frac{a}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \Big| \\ &= \frac{a}{2A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2, \end{aligned}$$

using the same estimates as for \bar{B}_{13} (cf. (5.20)) and Lemma A.3 (ii).

As for the term W_2 , we find

$$\begin{aligned} -W_2 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta - \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta) (t, \eta) d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1} (t, \eta) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2} (t, \eta) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \\ &\quad - \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{P}}_1^{1/2})_\eta (t, \eta) d\eta \\ &= W_{21} + W_{22} + W_{23} + W_{24}. \end{aligned}$$

The terms W_{21} and W_{22} can be treated similarly. We need to estimate $\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1$. Applying Lemma A.9, we have

$$\begin{aligned} |W_{21}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta (t, \eta) \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1} (t, \eta) d\eta \right| \\ &\leq \frac{2}{A^{9/2}} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta| (t, \eta) d\eta \\ &\quad + \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| (\tilde{\mathcal{U}}_1^2 + \tilde{\mathcal{P}}_1) |\tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta| (t, \eta) d\eta \\ &\quad + \frac{2\sqrt{2}}{A^{9/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{D}}_1^{1/2} |\tilde{\mathcal{U}}_1 (\tilde{\mathcal{P}}_1^{1/2})_\eta| (t, \eta) d\eta \end{aligned}$$

$$\begin{aligned}
& + \frac{4}{A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
& \times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
& + \frac{2\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
& \times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
& + \frac{3}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
& \times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
& + \frac{3}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
& \times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \right) |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
& + \frac{6}{A^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
& \times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} d\theta \right) |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
& + \frac{12\sqrt{2}}{\sqrt{3}eA^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \\
& \times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} d\theta \right)^{1/2} |\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t, \eta) d\eta \\
& \leq \frac{2}{A^{9/2}} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \frac{A^{13/2}}{4} d\eta \\
& + \frac{1}{A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \frac{3A^8}{8\sqrt{2}} d\eta \\
& + \frac{2\sqrt{2}}{A^{9/2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \frac{A^{13/2}}{4} d\eta \\
& + \frac{4}{A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2\sqrt{2}} d\eta \\
& + \frac{2\sqrt{2}}{A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2\sqrt{2}} d\eta
\end{aligned}$$

$$\begin{aligned}
& + \frac{3}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \frac{A^4}{2\sqrt{2}} d\eta \\
& + \frac{3}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \right) \frac{A^4}{2\sqrt{2}} d\eta \\
& + \frac{6}{A^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \frac{A^4}{2\sqrt{2}} d\eta \\
& + \frac{12\sqrt{2}}{\sqrt{3}eA^2} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \frac{A^4}{2\sqrt{2}} d\eta \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2),
\end{aligned}$$

where we have used estimates (4.15a), (4.15b), (4.15n), and (4.17b). Furthermore,

$$\begin{aligned}
|W_{23}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) d\eta \right| \\
&\leq \left| \frac{1}{2A^6} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2(t, \eta) \right|_{\eta=0}^1 \\
&\quad - \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \Big| \\
&= \frac{1}{2A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_2)(t, \eta) d\eta \right| \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2,
\end{aligned}$$

by applying Lemma A.4 (ii). The term W_{24} goes as follows:

$$\begin{aligned}
|W_{24}| &= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) d\eta \right| \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2)
\end{aligned}$$

using

$$\min_j(\tilde{\mathcal{D}}_j) |(\tilde{\mathcal{P}}_1^{1/2})_\eta| \leq 2A_1 \tilde{\mathcal{P}}_1 |(\tilde{\mathcal{P}}_1^{1/2})_\eta| \leq \tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta} \leq \frac{A^7}{4}$$

from (4.15a), (4.15e), (4.15n), and (4.17a).

Next comes W_3 , namely,

$$\begin{aligned}
W_3 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
&\quad \times \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A^2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right.
\end{aligned}$$

$$\begin{aligned}
& - \left(\tilde{\mathcal{P}}_1^{1/2} \right)_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
& = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \quad \times \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
& \quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \quad \times \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^- \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^- \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
& = W_{31} + W_{32},
\end{aligned}$$

and the two terms can be treated in the same manner. Thus

$$\begin{aligned}
W_{31} & = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \quad \times \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
& = \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^6} \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
& \quad + \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& \quad + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{D}}_2 < \tilde{\mathcal{D}}_1}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \Big) d\eta \\
& + \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
& \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \Big) d\eta \\
& + \mathbb{1}_{A_2 < A_1} \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
& \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
& \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
& \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right. \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \Big) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \\
& \times \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}(t, \eta) \\
& \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_e(t, \eta) \\
& \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \right) d\eta \\
& = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + Z_8 + Z_9 + Z_{10} + Z_{11} + Z_{12} + Z_{13},
\end{aligned}$$

where the set e is given by (5.21). Here there is no alternative but to treat these terms more or less separately. For the terms Z_1 and Z_2 , we find

$$\begin{aligned}
|Z_1| & \leq \frac{2}{A^7} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_1^{1/2})_\eta| \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

For the terms Z_3 and Z_4 , we find

$$\begin{aligned}
|Z_3| & = \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
& \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{D}}_1 \leq \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \right| \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + \|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}\|^2 + |A_2 - A_1|^2)
\end{aligned}$$

by Lemma A.9.

For the terms Z_5 and Z_6 , we find

$$\begin{aligned}
|Z_5| & = \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
& \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \right|
\end{aligned}$$

$$\begin{aligned} &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^{14}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{D}}_j)(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2), \end{aligned}$$

by applying (4.16e), estimating $\tilde{\mathcal{D}}_2 \leq 2A_2\tilde{\mathcal{P}}_1$ (cf. (4.15n)), and subsequently (4.17a).

For the terms Z_7 and Z_8 , we find

$$\begin{aligned} |Z_7| &= \mathbb{1}_{A_1 \leq A_2} \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \right. \\ &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \left. \right) d\eta \Big| \\ &\leq \frac{4}{A_1 A^7 e} \left| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \right. \\ &\quad \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{D}}_2 |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \Big| |A_1 - A_2| \\ &\leq \frac{4\sqrt{2}a}{A^6 e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left(\int_0^\eta \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

For the terms Z_9 and Z_{10} , we find

$$\begin{aligned} |Z_9| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right. \\ &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \left. \right) d\eta \Big| \\ &\leq \frac{1}{aA^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta (|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) + |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta)) \right. \\
& \times e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{D}}_2 \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \Big) d\eta \\
& \leq \frac{1}{\sqrt{2}A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{\sqrt{2}A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& \leq \frac{\sqrt{2}}{A^5} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
& + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
& + \frac{2}{A^{10}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}^2(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right) d\eta \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

applying the same set of estimates applied when studying \bar{B}_{35} ; see (5.22). For the term Z_{11} , we find

$$\begin{aligned}
|Z_{11}| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \right. \\
&\quad \times \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right) d\eta \Big| \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2;
\end{aligned}$$

see estimates for \bar{B}_{37} (cf. (5.23)).

For the terms Z_{12} and Z_{13} , we find

$$\begin{aligned}
 |Z_{12}| &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}(t, \eta) \right. \\
 &\quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \right) d\eta \Big| \\
 &= \frac{1}{A^7} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}(t, \eta) \right. \\
 &\quad \times \left[\left(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \right) \Big|_{\theta=0}^\eta \right. \\
 &\quad \left. - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \left(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right] d\eta \Big| \\
 &= \left| \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \eta) d\eta \right. \\
 &\quad - \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \right. \\
 &\quad \times \left. \frac{d}{d\theta} \left(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right) d\eta \Big| \\
 &\leq \frac{A^2}{4\sqrt{2}} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) d\eta \\
 &\quad + \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \right. \\
 &\quad \times \left. \left| \frac{d}{d\theta} \left(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right| \right) d\eta \\
 &= \tilde{M}_1 + \tilde{M}_2.
 \end{aligned}$$

We find directly

$$|\tilde{M}_1| \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2).$$

For the other term \tilde{M}_2 , we proceed as follows:

$$\begin{aligned}
 |\tilde{M}_2| &\leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \right. \\
 &\quad \times \left. \left| \frac{d}{d\theta} \left(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right| \right) d\eta \\
 &\leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta)
 \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \left[\left| \left(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \right| \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right. \right. \\
& \left. \left. + \left| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| \left| \frac{d}{d\theta} (\min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)) \right| \right] (t, \theta) d\theta \right) d\eta \\
& \leq \frac{1}{A^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) \right. \\
& \quad \times \left[\frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) \right. \\
& \quad \left. \left. + \mathcal{O}(1) A^{9/2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) \right] d\theta \right) d\eta \\
& = \tilde{M}_{21} + \tilde{M}_{22},
\end{aligned}$$

by using Lemmas A.2 and A.4. For \tilde{M}_{21} , we find

$$\begin{aligned}
\tilde{M}_{21} &= \frac{1}{aA^7} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta) \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
& \quad \times |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big) d\eta \\
&\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{a^2 A^{14}} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
& \quad \times |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big)^2 d\eta \\
&\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
& \quad + \frac{1}{2} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
& \quad + \frac{1}{8A^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \mathcal{O}(1) \int_0^1 \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),
\end{aligned}$$

using

$$\max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(\tilde{\mathcal{D}}_j) \leq 2 \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(A_i \tilde{\mathcal{P}}_j) \leq 2 \max_j(A_i \tilde{\mathcal{P}}_j \tilde{\mathcal{Y}}_{j,\eta}) \leq A^6,$$

and (4.15e), (4.16a), and (4.17c). For \tilde{M}_{22} , we find

$$\begin{aligned} \tilde{M}_{22} &= \frac{\mathcal{O}(1)}{A^{5/2}} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta| \mathbb{1}_{D^c}(t, \eta) \\ &\quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right) d\eta \\ &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{\mathcal{O}(1)}{A^5} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\ &\quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right)^2 d\eta \\ &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{\mathcal{O}(1)}{A^5} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right)^2 d\eta \\ &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\ &\quad + \frac{\mathcal{O}(1)}{A^7} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) d\theta \right) d\eta \\ &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\ &\quad + \frac{\mathcal{O}(1)}{A^7} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right) d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2), \end{aligned}$$

using that (cf. (4.15d) and (4.15n))

$$\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2| \leq \sqrt{2A_2} \tilde{\mathcal{P}}_2^{1/2} + |\tilde{\mathcal{U}}_2| \leq \sqrt{2}(1 + \sqrt{A_2}) \tilde{\mathcal{P}}_2^{1/2},$$

and (4.15e), (4.16i), and (4.17a).

We now consider the term W_4 :

$$\begin{aligned} W_4 &= -\frac{3}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\ &\quad \times \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right. \end{aligned}$$

$$\begin{aligned}
& - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
& = -\frac{3}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \quad \times \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
& \quad - \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
& \quad \times \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^- \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^- \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
& = W_4^+ + W_4^-.
\end{aligned}$$

Since $W_4^+ = -3N_1$ (see (5.43)), we find that

$$|W_4| \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + |A_1 - A_2|^2).$$

The next term W_5 would be laborious:

$$\begin{aligned}
W_5 & = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
& = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
& \quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^-)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \quad - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \Big) d\eta \\
& = W_5^+ + W_5^-.
\end{aligned}$$

Unfortunately, having a close look at W_5^+ one has

$$W_5^+ = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(I_{211} + I_{212} + I_{213} + I_{214} + I_{215} + I_{216} + I_{217} + I_{218})(t, \eta) d\eta,$$

where I_{211}, \dots, I_{218} are defined in (5.42). Thus we can conclude immediately that

$$|W_5^+| \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).$$

For the term W_6 , we find

$$\begin{aligned} W_6 &= \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2^5 \tilde{\mathcal{U}}_2(t, \theta) d\theta \right. \\ &\quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} A_1^5 \tilde{\mathcal{U}}_1(t, \theta) d\theta \right) d\eta \\ &= \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_1 \leq A_2} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2(t, \theta) d\theta d\eta \\ &\quad + \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_1(t, \theta) d\theta d\eta \\ &\quad + \mathbb{1}_{A_1 \leq A_2} \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \tilde{\mathcal{U}}_2(t, \theta) d\theta \right) d\eta \\ &\quad + \mathbb{1}_{A_2 < A_1} \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \tilde{\mathcal{U}}_1(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\ &\quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \tilde{\mathcal{U}}_1 \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \right) d\eta \\ &\quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \\
& + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
& \times \left(\int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-) \mathbb{1}_{\tilde{\mathcal{U}}_2^- \leq \tilde{\mathcal{U}}_1^-}(t, \theta) d\theta \right) d\eta \\
& + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
& \times \left(\int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \right) d\eta \\
& + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t, \eta) \\
& \times \left(\int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-) \mathbb{1}_{\tilde{\mathcal{U}}_1^- < \tilde{\mathcal{U}}_2^-}(t, \theta) d\theta \right) d\eta \\
& + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \\
& \times \left(\int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\
& + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \\
& \times \left(\int_0^\eta \max_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^-)(t, \theta) d\theta \right) d\eta \\
& = W_{61} + W_{62} + W_{63} + W_{64} + W_{65} + W_{66} \\
& + W_{67}^+ + W_{67}^- + W_{68}^+ + W_{68}^- + W_{69}^+ + W_{69}^-.
\end{aligned}$$

The terms W_{61} and W_{62} :

$$\begin{aligned}
|W_{61}| &= \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_1 \leq A_2} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
&\quad \times \left. \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2(t, \theta) d\theta d\eta \right| \\
&\leq 5 \frac{A_2 - A_1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} d\theta \right)^{1/2} d\eta \\
& \leq 5\sqrt{6} \frac{A_2 - A_1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) d\eta \\
& \leq \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2),
\end{aligned}$$

using (4.15e), (4.16a), and (4.17a).

The terms W_{63} and W_{64} :

$$\begin{aligned}
|W_{63}| &= \mathbb{1}_{A_1 \leq A_2} \frac{a^5}{2A^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
&\quad \times \left| \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \tilde{\mathcal{U}}_2(t, \theta) d\theta \right| d\eta \\
&\leq \frac{4a^5}{2aA^6e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
&\leq \frac{2}{A^2e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
&\leq \frac{2\sqrt{6}}{A^2e} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) d\eta |A_1 - A_2| \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

The terms W_{65} and W_{66} :

$$\begin{aligned}
|W_{65}| &= \frac{a^5}{2A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
&\quad \times \left. \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t, \theta) d\theta d\eta \right| \\
&\leq \frac{1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \\
&\quad \times \int_0^\eta (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \\
&\quad \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta \\
&\quad + \frac{1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{U}}_2|(t, \theta) d\theta d\eta \\
&\leq \frac{1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta \\
&\quad + \frac{1}{2A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\leq \frac{\sqrt{3}}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| |\tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) d\eta \\
&\quad + \frac{\sqrt{3}}{\sqrt{2}A^2} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{P}}_2^{1/2}| |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t, \eta) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

using (4.15e), (4.16a), (4.17a), and (5.13).

The terms W_{67}^\pm and W_{68}^\pm :

$$\begin{aligned}
|W_{67}^+| &= \frac{a^5}{2A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
&\quad \times \left. \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \right| \\
&\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{4A^2} \int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \\
&\quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{16A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_2^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \right) d\eta \\
& \leq \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 \\
& + \frac{A}{32} \int_0^1 \tilde{\mathcal{Y}}_{2, \eta}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \right) d\eta \\
& \leq \mathcal{O}(1) (\| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2 + \| \tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2 \|^2)
\end{aligned}$$

following the estimates used for \bar{B}_{65}^+ ; see (5.24). The term W_{69}^\pm :

$$\begin{aligned}
|W_{69}^+| &= \frac{a^5}{2A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t, \eta) \right. \\
&\quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \Big| \\
&= \frac{a^5}{4A^6} \left| \left((\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right|_{\eta=0}^1 \\
&\quad - \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{d}{d\eta} \left(\int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \Big| \\
&\leq \frac{a^5}{4A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \\
&\quad \times \left| \frac{d}{d\eta} \left(\int_0^\eta \min_j(e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right| d\eta \\
&\leq \mathcal{O}(1) \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2;
\end{aligned}$$

see estimates employed for \bar{B}_{67} (cf. (5.25)) and Lemma A.5.

The terms M_2 and M_3 can be treated similarly. More precisely,

$$\begin{aligned}
|M_2| &\leq \mathbb{1}_{A_2 \leq A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t, \eta) \right. \\
&\quad \times \left. \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2, \eta}(t, \theta) d\theta d\eta \right| \\
&\leq \mathbb{1}_{A_2 \leq A_1} \frac{|A_1^6 - A_2^6|}{A^6 A_2^5} \int_0^1 | \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} | | (\tilde{\mathcal{P}}_2^{1/2})_\eta | (t, \eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 | \tilde{\mathcal{U}}_{2, \eta} | (t, \theta) d\theta d\eta \\
&\leq \mathbb{1}_{A_2 \leq A_1} 6 \frac{|A_1 - A_2|}{A A_2^5} \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \| \\
&\quad \times \left(\int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 | \tilde{\mathcal{U}}_{2, \eta} | (t, \theta) d\theta \right)^2 d\eta \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{1}_{A_2 \leq A_1} 6 \frac{|A_1 - A_2|}{AA_2^6} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad \times \left(\int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \right. \\
&\quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2, \eta}(t, \theta) d\theta \right) d\eta \right)^{1/2} \\
&\leq \mathbb{1}_{A_2 \leq A_1} 6\sqrt{6} \frac{|A_1 - A_2|}{AA_2^3} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \left(\int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2 \tilde{\mathcal{P}}_2^2(t, \eta) d\eta \right)^{1/2} \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

Here we used

$$|\tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_{2, \eta}| \leq \frac{1}{A_2} \tilde{\mathcal{P}}_2 \sqrt{\tilde{\mathcal{Y}}_{2, \eta} \tilde{\mathcal{H}}_{2, \eta}},$$

cf. (4.15m), as well as (4.15c), (4.15e), and (4.17d). In addition, we applied (4.16b) and (4.16d).

The term I_3 : We have the following estimates:

$$\begin{aligned}
|I_3| &\leq \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \left| \frac{1}{A_1^2} \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^2} \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right| (t, \eta) d\eta \\
&\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \left\| \frac{1}{A_1^2} \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^2} \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2.
\end{aligned}$$

We consider the latter term and find

$$\begin{aligned}
&\left\| \frac{1}{A_1^2} \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^2} \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2 \\
&\leq 3 \mathbb{1}_{A_1 \leq A_2} \left| \frac{1}{A_1^2} - \frac{1}{A_2^2} \right|^2 \left\| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} \right\|^2 + 3 \mathbb{1}_{A_2 < A_1} \left| \frac{1}{A_1^2} - \frac{1}{A_2^2} \right|^2 \left\| \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2 \\
&\quad + 3 \frac{1}{A^4} \left\| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2 \\
&\leq \mathcal{O}(1) |A_1 - A_2|^2 + 3 \frac{1}{A^4} \left\| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2,
\end{aligned}$$

where we have used that

$$\left| \frac{\tilde{\mathcal{Q}}_i \tilde{\mathcal{U}}_i}{\tilde{\mathcal{P}}_i^{1/2}} \right| \leq \frac{A_i \tilde{\mathcal{P}}_i \tilde{\mathcal{U}}_i}{\tilde{\mathcal{P}}_i^{1/2}} = A_i \tilde{\mathcal{P}}_i^{1/2} |\tilde{\mathcal{U}}_i| \leq \mathcal{O}(1) A_i^5.$$

We consider the latter term and find

$$\begin{aligned}
& \frac{1}{A^4} \left\| \frac{\tilde{Q}_1 \tilde{U}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{\tilde{Q}_2 \tilde{U}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2 \\
&= \frac{1}{A^4} \int_0^1 \left| \frac{\tilde{Q}_1 \tilde{U}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{\tilde{Q}_2 \tilde{U}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right|^2 (\mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} + \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1})(t, \eta) d\eta \\
&\leq \frac{2}{A^4} \int_0^1 \left| \frac{1}{\tilde{\mathcal{P}}_2^{1/2}} (\tilde{Q}_1 \tilde{U}_1 - \tilde{Q}_2 \tilde{U}_2) + \left(\frac{1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{\tilde{\mathcal{P}}_2^{1/2}} \right) \tilde{Q}_1 \tilde{U}_1 \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\
&\quad + \frac{2}{A^4} \int_0^1 \left| \frac{1}{\tilde{\mathcal{P}}_1^{1/2}} (\tilde{Q}_1 \tilde{U}_1 - \tilde{Q}_2 \tilde{U}_2) + \left(\frac{1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{\tilde{\mathcal{P}}_2^{1/2}} \right) \tilde{Q}_2 \tilde{U}_2 \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) d\eta \\
&\leq \frac{4}{A^4} \int_0^1 \left| \frac{1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} (\tilde{Q}_1(\tilde{U}_1 - \tilde{U}_2) + \tilde{U}_2(\tilde{Q}_1 - \tilde{Q}_2)) \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\
&\quad + \frac{4}{A^4} \int_0^1 \left| \frac{\tilde{Q}_1 \tilde{U}_1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}) \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\
&\quad + \frac{4}{A^4} \int_0^1 \left| \frac{1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} (\tilde{Q}_1(\tilde{U}_1 - \tilde{U}_2) + \tilde{U}_2(\tilde{Q}_1 - \tilde{Q}_2)) \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) d\eta \\
&\quad + \frac{4}{A^4} \int_0^1 \left| \frac{\tilde{Q}_2 \tilde{U}_2}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}) \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) d\eta \\
&= 4(I_{31} + I_{32} + I_{33} + I_{34}).
\end{aligned}$$

We treat terms I_{31} and I_{32} ; the others are similar.

$$\begin{aligned}
I_{31} &\leq \frac{2}{A^4} \int_0^1 \left| \left(\frac{\tilde{Q}_1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} \right)^2 |\tilde{U}_1 - \tilde{U}_2|^2 \right. \\
&\quad \left. + \left(\frac{\tilde{U}_2}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} \right)^2 |\tilde{Q}_1 - \tilde{Q}_2|^2 \right| \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{U}_1 - \tilde{U}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
\end{aligned}$$

where we have used (5.29), that

$$\begin{aligned}
\left| \frac{\tilde{Q}_1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} \right| \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} &\leq \frac{A_1 \tilde{\mathcal{P}}_1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} \\
&\leq A \frac{\max_i(\tilde{\mathcal{P}}_i)}{\max_i(\tilde{\mathcal{P}}_i^{1/2})} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} \leq A \max_i(\tilde{\mathcal{P}}_i^{1/2}) \leq \frac{A^3}{2},
\end{aligned}$$

and

$$\left| \frac{\tilde{\mathcal{U}}_2}{\max_i (\tilde{\mathcal{P}}_i^{1/2})} \right| \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} \leq \frac{|\tilde{\mathcal{U}}_2|}{\tilde{\mathcal{P}}_2^{1/2}} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} \leq \sqrt{2}.$$

Next follows

$$\begin{aligned} I_{32} &\leq \frac{1}{A^4} \int_0^1 \left| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} (\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}) \right|^2 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta \\ &\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}\|^2, \end{aligned}$$

using that

$$\left| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} \right| \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2} \leq \left| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1} \right| \leq |A_1 \tilde{\mathcal{U}}_1| \leq \frac{A^3}{\sqrt{2}}.$$

This proves that

$$\begin{aligned} &\frac{1}{A^4} \left\| \frac{\tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{\tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right\|^2 \\ &\leq \mathcal{O}(1) (\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \end{aligned}$$

and thus I_3 has the right form.

The term I_4 :

$$\begin{aligned} I_4 &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left(\frac{1}{A_1^3} \frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^3} \frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}} \right) (t, \eta) d\eta \\ &= \mathbb{1}_{A_1 \leq A_2} \left(\frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}} (t, \eta) d\eta \\ &\quad + \mathbb{1}_{A_2 < A_1} \left(\frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}} (t, \eta) d\eta \\ &\quad + \frac{1}{A^3} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left(\frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2) \right. \\ &\quad \left. - \left(\frac{1}{\tilde{\mathcal{P}}_2^{1/2}} - \frac{1}{\tilde{\mathcal{P}}_1^{1/2}} \right) (\tilde{\mathcal{R}}_2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1} + \tilde{\mathcal{R}}_1 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}) \right) (t, \eta) d\eta \\ &= \mathbb{1}_{A_1 \leq A_2} \left(\frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}} (t, \eta) d\eta \end{aligned}$$

$$\begin{aligned}
& + \mathbb{1}_{A_2 < A_1} \left(\frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}}(t, \eta) d\eta \\
& + \frac{1}{A^3} \int_0^1 \left[\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2) - (\tilde{\mathcal{P}}_1^{1/2} \right. \\
& \quad \left. - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} (\tilde{\mathcal{R}}_2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1} + \tilde{\mathcal{R}}_1 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}) \right] (t, \eta) d\eta \\
& = J_1 + J_2 + J_3 - J_4 - J_5,
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \mathbb{1}_{A_1 \leq A_2} \left(\frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}}(t, \eta) d\eta, \\
J_2 &= \mathbb{1}_{A_2 < A_1} \left(\frac{1}{A_1^3} - \frac{1}{A_2^3} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}}(t, \eta) d\eta, \\
J_3 &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2)(t, \eta) d\eta, \\
J_4 &= \frac{1}{A^3} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} \tilde{\mathcal{R}}_2 \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t, \eta) d\eta, \\
J_5 &= \frac{1}{A^3} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2 \frac{1}{\tilde{\mathcal{P}}_1^{1/2} \tilde{\mathcal{P}}_2^{1/2}} \tilde{\mathcal{R}}_1 \mathbb{1}_{\tilde{\mathcal{P}}_1 \leq \tilde{\mathcal{P}}_2}(t, \eta) d\eta.
\end{aligned}$$

For J_1 (and similarly for J_2), we find

$$\begin{aligned}
|J_1| &\leq \frac{A_2^3 - A_1^3}{a^3 A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \mathcal{O}(1) a^5 d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
\end{aligned}$$

where we used that

$$\left| \frac{\tilde{\mathcal{R}}_i}{\tilde{\mathcal{P}}_i^{1/2}} \right| \leq \mathcal{O}(1) A_i^3 \tilde{\mathcal{P}}_i^{1/2} \leq \mathcal{O}(1) A_i^5.$$

For J_3 , we find

$$\begin{aligned}
J_3 &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) (\tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2)(t, \eta) d\eta \\
&= \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left(\int_0^1 \text{sign}(\eta - \theta) \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} A_1 \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} - e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} A_2 \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta} \right)(t, \theta) d\theta \Big) d\eta \\
& + \frac{1}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \times \left(\int_0^1 \text{sign}(\eta - \theta) (e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} A_1^6 \tilde{\mathcal{U}}_1 \right. \\
& \left. - e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} A_2^6 \tilde{\mathcal{U}}_2) (t, \theta) d\theta \right) d\eta \\
& + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \times \left(\int_0^1 (e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{U}}_2 \tilde{\mathcal{Q}}_2 \tilde{\mathcal{Y}}_{2,\eta} \right. \\
& \left. - e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{U}}_1 \tilde{\mathcal{Q}}_1 \tilde{\mathcal{Y}}_{1,\eta}) (t, \theta) d\theta \right) d\eta \\
& = J_{11} + J_{12} + J_{13}.
\end{aligned}$$

Next we replace all the inner integrals $\int_0^1 \cdots d\theta$ by $(\int_0^\eta + \int_\eta^1) \cdots d\theta$ (cf. (5.4)), consider only the terms with $\int_0^\eta \cdots d\theta$, and call the corresponding quantities \tilde{J}_{11} , \tilde{J}_{12} , \tilde{J}_{13} . Thus

$$\begin{aligned}
\tilde{J}_{11} = & \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} A_1 \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta} \right. \\
& \left. - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} A_2 \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}) (t, \theta) d\theta \right) d\eta,
\end{aligned} \tag{5.44}$$

$$\begin{aligned}
\tilde{J}_{12} = & \frac{1}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} A_1^6 \tilde{\mathcal{U}}_1 \right. \\
& \left. - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} A_2^6 \tilde{\mathcal{U}}_2) (t, \theta) d\theta \right) d\eta,
\end{aligned} \tag{5.45}$$

$$\begin{aligned}
\tilde{J}_{13} = & \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2 \tilde{\mathcal{Q}}_2 \tilde{\mathcal{Y}}_{2,\eta} \right. \\
& \left. - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1 \tilde{\mathcal{Q}}_1 \tilde{\mathcal{Y}}_{1,\eta}) (t, \theta) d\theta \right) d\eta.
\end{aligned} \tag{5.46}$$

For the term \tilde{J}_{11} we write $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$, collect terms, and study the positive part and the negative part separately.

With a slight abuse of notation, we need to consider the term

$$\begin{aligned}
\tilde{J}_{11} &= \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
&\quad \times \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} A_1 (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta} \right. \\
&\quad \left. - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2 (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta})(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
&\quad \times \left(\mathbb{1}_{A_2 < A_1} (A_1 - A_2) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right. \\
&\quad + \mathbb{1}_{A_1 \leq A_2} (A_1 - A_2) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
&\quad + a \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} ((\tilde{\mathcal{U}}_1^+)^3 - (\tilde{\mathcal{U}}_2^+)^3) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+} \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad + a \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((\tilde{\mathcal{U}}_1^+)^3 - (\tilde{\mathcal{U}}_2^+)^3) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leq \tilde{\mathcal{U}}_2^+} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
&\quad - a \mathbb{1}_{A_2 < A_1} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\
&\quad \times \min_j ((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad - a \mathbb{1}_{A_1 \leq A_2} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\
&\quad \times \min_j ((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
&\quad - a \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\
&\quad \times \min_j ((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \\
&\quad - a \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\
&\quad \times \min_j ((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{1, \eta} \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta
\end{aligned}$$

$$\begin{aligned}
& - a \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
& + a \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \Big) d\eta \\
& = K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8 + K_9 + K_{10}.
\end{aligned}$$

It never stops, but we need to study the terms in groups.

The term K_1 can be estimated as follows (K_2 is similar):

$$\begin{aligned}
|K_1| &= \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left| \mathbb{1}_{A_2 < A_1} (A_1 - A_2) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right| d\eta \\
&\leq \frac{\sqrt{2}}{3} \int_0^1 \frac{\tilde{\mathcal{P}}_1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta |A_1 - A_2| \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

The term K_3 can be estimated as follows (K_4 is similar):

$$\begin{aligned}
|K_3| &= \frac{a}{6A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} ((\tilde{\mathcal{U}}_1^+)^3 - (\tilde{\mathcal{U}}_2^+)^3) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \Big| \\
&\leq \frac{1}{2A^2} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^2 |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+| \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\
&\leq \frac{1}{2} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{2A^4} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})^2} \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^2 |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+| \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \frac{1}{2} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
&\quad + \frac{1}{2A^4} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})^2} \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)
\end{aligned}$$

$$\begin{aligned} & \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+|^2 (\tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta \\ & \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2), \end{aligned}$$

using (4.15g), (4.16c), and (5.3).

As for K_5 (K_6 is similar), we traverse the following path:

$$\begin{aligned} |K_5| &= \frac{1}{6A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\ & \quad \times \left(-a \mathbb{1}_{A_2 < A_1} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))}) \right. \\ & \quad \times \min_j((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \left. \right) d\eta \Big| \\ & \leq \frac{2}{3A^3 e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{3}{4A_1}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))} \min_j((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\ & \leq \frac{2\sqrt{2}A^2}{3} \int_0^1 \frac{\tilde{\mathcal{P}}_1^{1/2}}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta |A_1 - A_2| \\ & \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

As for K_7 (K_8 is similar), we traverse the following path:

$$\begin{aligned} |K_7| &= \frac{a}{6A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\ & \quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{Y}_1(t,\eta) - \tilde{Y}_1(t,\theta))}) \right. \\ & \quad \times \min_j((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \left. \right) d\eta \Big| \\ & \leq \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} \right. \\ & \quad \times (|\tilde{Y}_2(t, \eta) - \tilde{Y}_1(t, \eta)| + |\tilde{Y}_2(t, \theta) - \tilde{Y}_1(t, \theta)|) (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \left. \right) d\eta \\ & \leq \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{Y}_2 - \tilde{Y}_1|(t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{Y}_2(t,\eta) - \tilde{Y}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& \leq \frac{\sqrt{2}}{3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2(t, \eta) d\eta \\
& + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
& + \frac{1}{36A^6} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})^2}(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) \\
& + \frac{1}{36A^6} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})^2}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_2^+)^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|^2 (\tilde{\mathcal{U}}_2^+)^4 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) \\
& + \frac{1}{9A^5} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})^2} \tilde{\mathcal{P}}_2(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|^2 (\tilde{\mathcal{U}}_2^+)^4 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2).
\end{aligned}$$

Here we have applied (4.15b), (4.15g), (4.16c), and (5.13). Lo and behold, we can do K_9 and K_{10} in one sweep:

$$\begin{aligned}
K_9 + K_{10} & = -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
& \left. - \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j((\tilde{\mathcal{U}}_j^+)^3) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) d\eta \\
& = -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \right) d\eta \\
& = -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left[\left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \right) \Big|_{\theta=0}^\eta \right. \\
& \quad \left. - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \right) (t, \theta) d\theta \right] d\eta \\
& = -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t, \eta) d\eta \\
& \quad + \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (t, \theta) d\theta \right) d\eta \\
& \quad + \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t, \theta) d\theta \right) d\eta \\
& = L_1 + L_2 + L_3,
\end{aligned}$$

where

$$\begin{aligned}
L_1 & = -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t, \eta) d\eta, \\
L_2 & = \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (t, \theta) d\theta \right) d\eta, \\
L_3 & = \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t, \theta) d\theta \right) d\eta.
\end{aligned}$$

We easily find

$$\begin{aligned} |L_1| &\leq \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min((\tilde{\mathcal{U}}_j^+)^3)(t, \eta) d\eta \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2), \end{aligned}$$

using (4.15b) and (4.15d). Furthermore, applying Lemma A.2,

$$\begin{aligned} |L_2| &\leq \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ &\quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \right. \\ &\quad \times \min((\tilde{\mathcal{U}}_j^+)^3) \max_j(\tilde{\mathcal{Y}}_{j, \eta})(t, \theta) d\theta \Big) d\eta \\ &\leq \frac{1}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ &\quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{A_j}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{U}}_j^2(t, \theta) d\theta \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|^2 \max_j(\tilde{\mathcal{U}}_j^4 \tilde{\mathcal{Y}}_{j, \eta})^2(t, \theta) d\theta \right)^{\frac{1}{2}} d\eta \\ &\leq \mathcal{O}(1) \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2), \end{aligned}$$

using (4.15g), (4.16a), and

$$\min_j((\tilde{\mathcal{U}}_j^+)^3) \max_j(\tilde{\mathcal{Y}}_{j, \eta}) \leq \min_j(\tilde{\mathcal{U}}_j^+) \max_j(\tilde{\mathcal{U}}_j^2 \tilde{\mathcal{Y}}_{j, \eta}) \leq \frac{A^7}{\sqrt{2}}.$$

Another application of Lemma A.2 yields

$$\begin{aligned} |L_3| &\leq \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ &\quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \left| \frac{d}{d\theta} \min((\tilde{\mathcal{U}}_j^+)^3) \right|(t, \theta) d\theta \right) d\eta \\ &\leq \frac{A^2}{3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\
& \leq \frac{A^2}{3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \\
& \quad \times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|^2(t, \theta) d\theta \right)^{1/2} d\eta \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),
\end{aligned}$$

using (4.16a). We find

$$\begin{aligned}
|K_9 + K_{10}| & \leq |L_1| + |L_2| + |L_3| \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2).
\end{aligned}$$

Thus we can sum up the estimates for \tilde{J}_{11} , and we find

$$|\tilde{J}_{11}| \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + |A_1 - A_2|^2).$$

We now proceed to the next term from (5.45):

$$\begin{aligned}
\tilde{J}_{12} &= \frac{1}{4A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} A_1^6 \tilde{\mathcal{U}}_1 - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^6 \tilde{\mathcal{U}}_2)(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{4A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \quad \times \left((A_1^6 - A_2^6) \mathbb{1}_{A_1 < A_2} \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2(t, \theta) d\theta \right. \\
& \quad + (A_1^6 - A_2^6) \mathbb{1}_{A_2 \leq A_1} \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1(t, \theta) d\theta \\
& \quad \left. + a^6 \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1 - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2)(t, \theta) d\theta \right) d\eta \\
&= M_1 + M_2 + M_3.
\end{aligned}$$

The term M_1 (M_2 is similar) can be estimated as follows:

$$\begin{aligned}
 |M_1| &\leq \frac{1}{4A^3} |A_1^6 - A_2^6| \mathbb{1}_{A_1 < A_2} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2|(t, \theta) d\theta \right) d\eta \\
 &\leq \frac{3A^2}{2} |A_1 - A_2| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} d\theta \right)^{1/2} d\eta \\
 &\leq \frac{3\sqrt{3}A^2}{\sqrt{2}} |A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) d\eta \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
 \end{aligned}$$

using (4.16a). As for the next term M_3 , we first write $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$, collect terms, and study the positive part and the negative part separately. In the interest of the reader, we do not change the notation. Thus

$$\begin{aligned}
 M_3 &= \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 &\quad \times \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+)(t, \theta) d\theta d\eta \\
 &= \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+))(t, \theta) d\theta \right. \\
 &\quad + \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\mathbb{1}_{\tilde{\mathcal{U}}_2^+ \leq \tilde{\mathcal{U}}_1^+} (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+))(t, \theta) d\theta \\
 &\quad \left. + \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\
 &= M_{31} + M_{32} + M_{33}.
 \end{aligned}$$

The terms M_{31} and M_{32} can be treated similarly. Thus

$$\begin{aligned}
 |M_{31}| &= \frac{a^6}{4A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
 &\quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+))(t, \theta) d\theta \right) d\eta \right|
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4A^2} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2^5 |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+|(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_2^{1/2}(t, \eta) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| d\eta \\
&\leq \mathcal{O}(1) \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2),
\end{aligned}$$

using

$$\begin{aligned}
&a^5 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+|(t, \theta) d\theta \\
&\leq \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2^5 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
&= \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} ((2\tilde{\mathcal{P}}_2 - \tilde{\mathcal{U}}_2^2) \tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{H}}_{2, \eta}) |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
&= 2 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
&\quad + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2, \eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
&\quad - \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
&\leq 2 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
&\quad + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2, \eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
&\leq 2 \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
&\quad + \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{H}}_{2, \eta}(t, \theta) d\theta \right)^{1/2}
\end{aligned}$$

$$\begin{aligned} & \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{H}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\ & \leq \mathcal{O}(1) A^3 \tilde{\mathcal{P}}_2^{1/2}(t, \eta) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|, \end{aligned}$$

using (4.13), (4.15e), (4.15k), (4.16d), and (4.16e). The term M_{33} goes as follows:

$$\begin{aligned} M_{33} &= \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ & \quad \times \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\ & \quad \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big) d\eta \\ &= \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ & \quad \times \mathbb{1}_{A_1 \leq A_2} \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\ & \quad \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big) d\eta \\ & \quad + \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ & \quad \times \mathbb{1}_{A_2 < A_1} \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\ & \quad \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big) d\eta \\ & \quad + \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ & \quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \mathbb{1}_{B(\eta)} \right. \\ & \quad \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big) d\eta \\ & \quad + \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\ & \quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \mathbb{1}_{B(\eta)^c} \right. \end{aligned}$$

$$\begin{aligned} & \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big) d\eta \\ & = M_{331} + M_{332} + M_{333} + M_{334}, \end{aligned}$$

where $B(\eta)$ is defined by (5.11). The terms M_{331} and M_{332} can be treated in the same manner. More specifically,

$$\begin{aligned} |M_{331}| & \leq \frac{A^2}{e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\ & \leq \frac{A^2}{e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\ & \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

The terms M_{333} and M_{334} can be treated in a similar manner. More specifically,

$$\begin{aligned} |M_{333}| & \leq \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left(\int_0^\eta |e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}| \mathbb{1}_{B(\eta)} \right. \\ & \quad \times \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big) d\eta \\ & \leq \frac{A^2}{4} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\ & \quad \times (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \Big) d\eta \\ & \leq \frac{A^2}{4} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \mathbb{1}_{B(\eta)} \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \\ & \quad + \frac{A^2}{4} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) d\eta \end{aligned}$$

$$\begin{aligned}
&\leq \frac{A^2}{4} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} d\theta \right)^{1/2} d\eta \\
&\quad + \frac{A^2}{4} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2(t, \theta) d\theta \right)^{1/2} \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right)^{1/2} d\eta \\
&\leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \tilde{\mathcal{P}}_2^{1/2}(t, \eta) d\eta \\
&\quad + \mathcal{O}(1) \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\| \tilde{\mathcal{P}}_2^{1/2}(t, \eta) d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),
\end{aligned}$$

using (4.16a) and (5.13).

And the final term from J_3 (cf. (5.46)) can be estimated as follows:

$$\begin{aligned}
\tilde{J}_{13} = & \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2 \tilde{\mathcal{Q}}_2 \tilde{\mathcal{Y}}_{2, \eta} \right. \\
& \left. - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1 \tilde{\mathcal{Q}}_1 \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \right) d\eta.
\end{aligned}$$

We introduce the positive and negative parts of $\tilde{\mathcal{U}}_j$, that is, $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$ (see (5.1)) and introduce $\tilde{\mathcal{Q}}_j = A_j \tilde{\mathcal{P}}_j - \tilde{\mathcal{D}}_j$. We study the term with $\tilde{\mathcal{P}}_j$ first:

$$\begin{aligned}
\tilde{J}_{131} = & \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
& \times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} A_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta} \right. \\
& \left. - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} A_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
&\quad \times \left[\mathbb{1}_{A_1 \leq A_2} (A_2 - A_1) \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right. \\
&\quad + \mathbb{1}_{A_2 < A_1} (A_2 - A_1) \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad + a \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \mathbb{1}_{\tilde{\mathcal{P}}_1 < \tilde{\mathcal{P}}_2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
&\quad + a \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \mathbb{1}_{\tilde{\mathcal{P}}_2 \leq \tilde{\mathcal{P}}_1} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad + a \int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
&\quad + a \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j (\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ \leq \tilde{\mathcal{U}}_1^+} \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad + a \mathbb{1}_{A_1 \leq A_2} \int_0^\eta (e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \\
&\quad \times \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
&\quad + a \mathbb{1}_{A_2 < A_1} \int_0^\eta (e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\
&\quad \times \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad + a \int_0^\eta (e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \mathbb{1}_{B(\eta)} \\
&\quad \times \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
&\quad + a \int_0^\eta (e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \mathbb{1}_{B(\eta)^c} \\
&\quad \times \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad + a \int_0^\eta \min_j (e^{-\frac{1}{a} (\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
&\quad \times \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \Big] d\eta \\
&= K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8 + K_9 + K_{10} + K_{11}.
\end{aligned}$$

The terms K_1 and K_2 can be treated similarly. Thus

$$\begin{aligned}
 |K_1| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
 &\quad \times \left(\mathbb{1}_{A_1 \leq A_2} (A_2 - A_1) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \Big| \\
 &\leq \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
 &\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

The terms K_3 and K_4 can be treated similarly. Thus

$$\begin{aligned}
 |K_3| &= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_1 - \tilde{\mathcal{P}}_2) \mathbb{1}_{\tilde{\mathcal{P}}_1 < \tilde{\mathcal{P}}_2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \Big| \\
 &\leq \frac{1}{A^2} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \right. \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_2^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \Big| \\
 &\leq \frac{1}{A^2} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}^2(t, \theta) d\theta \right)^{1/2} d\eta \\
 &\leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_2^{1/2}(t, \eta)
 \end{aligned}$$

$$\begin{aligned} & \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} d\eta \\ & \leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2, \end{aligned}$$

using (4.15e) and (4.16c). As for K_5 and K_6 , we find

$$\begin{aligned} |K_5| &= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\ & \quad \left. + \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \right| \\ & \leq \frac{1}{2A^2} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \quad + \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+| \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\ & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{4A^4} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) |\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+| \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^2 d\eta \\ & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\ & \quad + \frac{1}{4A^4} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+)^2(t, \theta) d\theta \right) d\eta \\ & \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{2A^3} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} \tilde{\mathcal{P}}_2(t, \eta) \\ & \quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+)^2(t, \theta) d\theta \right) d\eta \\ & \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2), \end{aligned}$$

using (4.15e) and (4.16e).

The terms K_7 and K_8 allow for the following estimates:

$$\begin{aligned}
 |K_7| &= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
 &\quad \times \mathbb{1}_{A_1 \leq A_2} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \\
 &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta \Big| \\
 &\leq \frac{2}{A^3 e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta |A_1 - A_2| \\
 &\leq \frac{2}{A^3 e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left(\int_0^1 e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left(\int_0^1 e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

The terms K_9 and K_{10} allow for the following estimates:

$$\begin{aligned}
 |K_9| &= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
 &\quad + \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \mathbb{1}_{B(\eta)} \\
 &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta \Big| \\
 &\leq \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \\
 &\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta d\eta
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
&\quad + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| (t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
&\leq \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
&\quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{4A^6} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) + \frac{1}{4A^6} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2 \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),
\end{aligned}$$

using (4.15a), (4.15e), and (4.16c). The last term K_{11} receives special treatment

$$\begin{aligned}
|K_{11}| &= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \right. \\
&\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{1, \eta} - \tilde{\mathcal{Y}}_{2, \eta})(t, \theta) d\theta \Big) d\eta \Big| \\
&= \frac{a}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left[\left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) \right) \right]_{\theta=0}^1 \\
& - \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \\
& \times \frac{d}{d\theta} \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (t, \theta) d\theta \right) d\eta \Big| \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) \\
& + \frac{a}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \times \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \left| \left(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right. \\
& \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left(\frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right) \right| (t, \theta) d\theta d\eta \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2) \\
& + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \times \int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j(\tilde{\mathcal{Y}}_{j,\eta}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \right. \\
& \left. + 2aA^4 \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\min_j(\tilde{\mathcal{P}}_j^{1/2}) + |\tilde{\mathcal{U}}_2|) \right) (t, \theta) d\theta d\eta \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2) \\
& + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{1,\eta} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+(t, \theta) d\theta \right) d\eta \\
& + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+(t, \theta) d\theta \right) d\eta \\
& + A^2 \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta)
\end{aligned}$$

$$\begin{aligned} & \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) (\tilde{\mathcal{P}}_2^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right) d\eta \\ & = \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2) + K_{71} + K_{72} + K_{73}, \end{aligned}$$

using Lemmas A.2 and A.3, and

$$\begin{aligned} & \min_j (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta) \\ & \leq e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+(t, \theta) + e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+(t, \theta). \end{aligned}$$

Here K_{71} and K_{72} allow for the same treatment:

$$\begin{aligned} 2K_{71} &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+(t, \theta) d\theta \right) d\eta \\ & \leq \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \\ & \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta}^2 \tilde{\mathcal{P}}_1^2 (\tilde{\mathcal{U}}_1^+)^2(t, \theta) d\theta \right)^{1/2} d\eta \\ & \leq A^2 \sqrt{2} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(\tilde{\mathcal{P}}_1^{1/2}(t, \eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| d\eta \\ & \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2), \end{aligned}$$

using (4.15g) and (4.16b). Furthermore,

$$\begin{aligned} K_{73} &= A^2 \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) (\tilde{\mathcal{P}}_2^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right) d\eta \\ & \leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ & \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right) d\eta \end{aligned}$$

$$\begin{aligned}
&\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \\
&\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{\tilde{\mathcal{A}}_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right)^2 d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
&\quad + \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{1}{2\tilde{\mathcal{A}}_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) \\
&\quad \times \left(\int_0^\eta e^{-\frac{3}{2\tilde{\mathcal{A}}_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1) \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \mathcal{O}(1) \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} \tilde{\mathcal{P}}_2(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2\tilde{\mathcal{A}}_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

using (4.15d) and (4.16i). This proves that

$$\tilde{J}_{131} \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).$$

It remains to consider \tilde{J}_{132} :

$$\begin{aligned}
-\tilde{J}_{132} &= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
&\quad \times \left(\int_0^\eta (e^{-\frac{1}{\tilde{\mathcal{A}}_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2, \eta} \right. \\
&\quad \left. - e^{-\frac{1}{\tilde{\mathcal{A}}_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{D}}_1 \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \right) d\eta \\
&= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
&\quad \times \left[\int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right. \\
&\quad + \int_0^\eta e^{-\frac{1}{\tilde{\mathcal{A}}_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \mathbb{1}_{\tilde{\mathcal{D}}_2 \leq \tilde{\mathcal{D}}_1}(t, \theta) d\theta \\
&\quad \left. + \int_0^\eta (e^{-\frac{1}{\tilde{\mathcal{A}}_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} - e^{-\frac{1}{\tilde{\mathcal{A}}_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}) \right.
\end{aligned}$$

$$\times \min_j (\tilde{\mathcal{D}}_j)(t, \theta) d\theta \Big] d\eta \\ = \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3.$$

The terms \tilde{A}_1 and \tilde{A}_2 allow for the same treatment. Specifically, (see Lemma A.9)

$$\begin{aligned} |\tilde{A}_1| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\ &\quad \times \left[\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right] d\eta \Big| \\ &\leq \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ &\quad \times \left[\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} (\bar{d}_{11} + \bar{d}_{12}) \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right. \\ &\quad + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} T_1 \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \\ &\quad + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} T_2 \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \\ &\quad \left. + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} T_3 \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right] d\eta \\ &= \frac{1}{2} (A_{11} + A_{12} + A_{13} + A_{14}). \end{aligned}$$

Again we are forced to consider individual terms,

$$\begin{aligned} A_{11} &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ &\quad \times \left[\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} (\bar{d}_{11} + \bar{d}_{12}) \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right] d\eta \\ &\leq \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\ &\quad \times \left[\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} (2A^{3/2} \tilde{\mathcal{D}}_2^{1/2} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \right. \\ &\quad \left. + 2\sqrt{2} A^{3/2} \tilde{\mathcal{D}}_2^{1/2} \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|)(t, \theta) d\theta \right] d\eta \\ &\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{4}{A^3} \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \tilde{\mathcal{D}}_2^{1/2} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) d\theta \right)^2 d\eta \\
& + \frac{2\sqrt{2}}{A^{3/2}} \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \left(\int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \right. \\
& \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \tilde{\mathcal{D}}_2^{1/2} (t, \theta) d\theta \right)^2 d\eta \right)^{1/2} \\
& \leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
& + \frac{4}{A^3} \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} (t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|^2(t, \theta) d\theta \right) d\eta \\
& + \frac{2\sqrt{2}}{A^3} \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \left[\int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \right. \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} (t, \theta) d\theta \right) \\
& \times \left. \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} 2A \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (t, \theta) d\theta \right) d\eta \right]^{1/2} \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),
\end{aligned}$$

using (4.15e), (4.15n), and (4.16c). Next we find (see (A.14))

$$\begin{aligned}
A_{12} &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} T_1 \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
& \leq \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2|^3 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) d\theta \right) d\eta \\
& + 4 \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} (t, \theta) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \sqrt{2}A \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left(\int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \Big) d\eta \\
& + A \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left(\int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, l) dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \Big) d\eta \\
& + \frac{8\sqrt{2}A}{\sqrt{3}e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left(\int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right)^{1/2} |A_1 - A_2| \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \Big) d\eta \\
& \leq 5 \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^6} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \\
& \times \left(\int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{U}}_2|^3 \tilde{\mathcal{Y}}_{2, \eta} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) d\theta \right)^2 d\eta \\
& + 16 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \Big)^2 d\eta \\
& + 2A^2 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \Big)^2 d\eta \\
& + A^2 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, l) dl \right) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \Big)^2 d\eta \\
& + \frac{128A^2}{3e^2} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} \left(\int_0^\eta e^{-\frac{1}{\lambda_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right)^{1/2} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \Big)^2 d\eta |A_1 - A_2|^2 \\
& \leq 5 \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A^6} \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} (t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^4 \tilde{\mathcal{Y}}_{2, \eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2 (t, \theta) d\theta \right) d\eta \\
& + 16 \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 (t, l) dl \right) \tilde{\mathcal{Y}}_{2, \eta} (t, \theta) d\theta \right) d\eta \\
& + 2A^2 \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2 (t, l) dl \right) \tilde{\mathcal{Y}}_{2, \eta} (t, \theta) d\theta \right) d\eta \\
& + A^2 \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} (t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (t, l) dl \right)^2 \tilde{\mathcal{Y}}_{2, \eta} (t, \theta) d\theta \right) d\eta \\
& + \frac{128A^2}{3e^2} \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} (t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{3}{4A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right) \tilde{\mathcal{Y}}_{2, \eta} (t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) + \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 (t, l) dl \right) \tilde{\mathcal{Y}}_{2, \eta} (t, \theta) d\theta \right) d\eta \\
& + \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \Big) d\eta \\
& + \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) + B_1 + B_2 + B_3,
\end{aligned}$$

using (4.15b), (4.15g), and (4.16c). As for B_1 , we find

$$\begin{aligned}
B_1 &= \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
&= \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
& \times \left. \left(\int_0^\theta e^{-\left(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A}\tilde{\mathcal{Y}}_2(t,l)\right)} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right) d\theta \right) d\eta \\
&= \mathcal{O}(1) \int_0^1 \left[\left(-2Ae^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)} \right. \right. \\
& \times \left. \left. \left(\int_0^\theta e^{-\left(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A}\tilde{\mathcal{Y}}_2(t,l)\right)} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, l) dl \right) \right) \right]_{\theta=0}^\eta \\
& \quad + 2A \int_0^\eta e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)} e^{-\left(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A}\tilde{\mathcal{Y}}_2(t,\theta)\right)} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \Big] d\eta \\
&= -2A\mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta d\eta \\
& \quad + 2A\mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta d\eta \\
&\leq \mathcal{O}(1)\|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2,
\end{aligned}$$

while for B_2 , we estimate as follows:

$$\begin{aligned}
 B_2 &= \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 &\quad \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t,l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) d\theta \right) d\eta \\
 &\leq \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 &\quad \times \left. \left(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t,l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) d\theta \right) d\eta \\
 &= \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) \right. \\
 &\quad \times \left. \left(\int_0^\theta e^{-(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t,l) dl \right) d\theta \right) d\eta \\
 &= \mathcal{O}(1) \int_0^1 \left[\left(-2A_2 e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)} \right. \right. \\
 &\quad \times \left. \left(\int_0^\theta e^{-(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t,l) dl \right) \right) \Big|_{\theta=0}^\eta \\
 &\quad \left. + 2A_2 \int_0^\eta e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)} e^{-(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t,\theta) d\theta \right] d\eta \\
 &= -2A_2 \mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t,\theta) d\theta d\eta \\
 &\quad + 2A_2 \mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t,\theta) d\theta d\eta \\
 &\leq \mathcal{O}(1) \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2,
 \end{aligned}$$

while for B_3 , we estimate as follows:

$$\begin{aligned}
 B_3 &= \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 &\quad \times \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) d\theta \right) d\eta |A_1 - A_2|^2 \\
 &\leq \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{3}{8A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
 &\quad \times \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) d\theta \right) d\eta |A_1 - A_2|^2
 \end{aligned}$$

$$\begin{aligned}
&= \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \theta)} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \right. \\
&\quad \times \left. \left(\int_0^\theta e^{-(\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t, l))} dl \right) d\theta \right) d\eta |A_1 - A_2|^2 \\
&= \mathcal{O}(1) \int_0^1 \left[\left(-\frac{8}{3} A e^{-\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \theta)} \right. \right. \\
&\quad \times \left. \left. \left(\int_0^\theta e^{-(\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t, l))} dl \right) \right) \right]_{\theta=0}^\eta \\
&\quad + \frac{8}{3} A \int_0^\eta e^{-\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \theta)} e^{-(\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t, \theta))} d\theta \Big] d\eta |A_1 - A_2|^2 \\
&= -\frac{8}{3} A \mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{3}{4A} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} d\theta d\eta |A_1 - A_2|^2 \\
&\quad + \frac{8}{3} A \mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{3}{8A} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} d\theta d\eta |A_1 - A_2|^2 \\
&\leq \mathcal{O}(1) |A_1 - A_2|^2.
\end{aligned}$$

Thus we find that

$$A_{12} \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2 + |A_1 - A_2|^2).$$

Next we find (see (A.15))

$$\begin{aligned}
A_{13} &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} T_2 \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
&\leq \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \left(\tilde{\mathcal{P}}_j |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) \right. \right. \\
&\quad + 2\sqrt{2} A^3 \left(\int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right)^{1/2} \\
&\quad + \frac{A^4}{\sqrt{2}} \left(\int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right)^{1/2} \\
&\quad \left. \left. + \frac{A^4}{2} \left(\int_0^\theta e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, l) dl \right) \right) \right) d\theta \right) d\eta
\end{aligned}$$

$$\begin{aligned}
& + \frac{4\sqrt{2}A^4}{\sqrt{3}e} \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right)^{1/2} |A_1 - A_2| \Big) d\theta \Big) d\eta \\
& \leq 5 \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^6} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \tilde{\mathcal{P}}_j |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \theta) d\theta \right)^2 d\eta \\
& \quad + 8 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
& \quad \times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right)^{1/2} d\theta \Big)^2 d\eta \\
& \quad + \frac{A^2}{2} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
& \quad \times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right)^{1/2} d\theta \Big)^2 d\eta \\
& \quad + \frac{A^2}{4} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
& \quad \times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, l) dl \right) d\theta \Big)^2 d\eta \\
& \quad + \frac{32A^2}{3e^2} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
& \quad \times \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right)^{1/2} d\theta \Big)^2 d\eta |A_1 - A_2|^2 \\
& \leq 5 \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
& \quad + \frac{1}{A^6} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right) d\eta \\
& \quad + 8 \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) d\theta \Big) d\eta \\
& + \frac{A^2}{2} \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, l) dl \right) d\theta \right) d\eta \\
& + \frac{A^2}{4} \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, l) dl \right)^2 d\theta \right) d\eta \\
& + \frac{32A^2}{3e^2} \int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2}(t, \eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) d\theta \right) d\eta |A_1 - A_2|^2 \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) \\
& + \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta \\
& + \mathcal{O}(1) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\
& \times \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2|^2
\end{aligned}$$

$$\begin{aligned} &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2) + B_4 + B_2 + B_3 \\ &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + |A_1 - A_2|^2), \end{aligned}$$

following the approach used for A_{12} . As for B_4 , we find

$$\begin{aligned} B_4 &= \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\ &\quad \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) d\theta \right) d\eta \\ &\leq \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right. \\ &\quad \times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,l) dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) d\theta \right) d\eta \\ &= \mathcal{O}(1) \int_0^1 \left(\int_0^\eta e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) \right. \\ &\quad \times \left. \left(\int_0^\theta e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A}\tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,l) dl \right) d\theta \right) d\eta \\ &= \mathcal{O}(1) \int_0^1 \left[\left(-2Ae^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)} \right. \right. \\ &\quad \times \left. \left. \left(\int_0^\theta e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A}\tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,l) dl \right) \right) \right]_{\theta=0}^\eta \\ &\quad + 2A \int_0^\eta e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)} e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{1}{A}\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,\theta) d\theta \Big] d\eta \\ &= -2A\mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,\theta) d\theta d\eta \\ &\quad + 2A\mathcal{O}(1) \int_0^1 \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,\theta) d\theta d\eta \\ &\leq \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2. \end{aligned}$$

Thus we find that

$$A_{12} \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2 + |A_1 - A_2|^2).$$

Next we find (see (A.16))

$$\begin{aligned}
A_{14} &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A^2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} T_3 \mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t, \theta) d\theta \right) d\eta \\
&\leq 12A|A_1 - A_2| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A^2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \left(\int_0^\theta e^{-\frac{3}{4A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right) d\theta \right) d\eta \\
&\leq 12A|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \left(\int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \right. \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A^2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \right. \\
&\quad \times \left. \left. \left(\int_0^\theta e^{-\frac{3}{4A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right) d\theta \right)^2 d\eta \right)^{1/2} \\
&\leq 12A|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad \times \left(\int_0^1 \frac{1}{(\max_j (\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \left(\int_0^\eta e^{-\frac{1}{A^2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right) \right. \\
&\quad \times \left. \left(\int_0^\eta e^{-\frac{1}{A^2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \left(\int_0^\theta e^{-\frac{3}{4A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right)^2 d\theta \right) d\eta \right)^{1/2} \\
&\leq \mathcal{O}(1)|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \left(\int_0^1 \left(\int_0^\eta e^{-\frac{1}{A^2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \right. \right. \\
&\quad \times \left. \left. \left(\int_0^\theta e^{-\frac{3}{4A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right)^2 d\theta \right) d\eta \right)^{1/2} \\
&\leq \mathcal{O}(1)|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad \times \left(\int_0^1 \left(\int_0^\eta e^{-\frac{3}{8A} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \right. \right. \\
&\quad \times \left. \left. \left(\int_0^\theta e^{-\frac{3}{4A} (\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, l))} dl \right) d\theta \right) d\eta \right)^{1/2} \\
&\leq \mathcal{O}(1)|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad \times \left(\int_0^1 \left(\int_0^\eta e^{-\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \theta)} \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) \right. \right. \\
&\quad \times \left. \left. \left(\int_0^\theta e^{-(\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t, l))} dl \right) d\theta \right) d\eta \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq \mathcal{O}(1)|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad \times \left(\int_0^1 \left[\left(-\frac{8}{3} A e^{-\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \theta)} \int_0^\theta e^{-(\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t, l))} dl \right) \right]_{\theta=0}^\eta \right. \\
&\quad \left. + \frac{8}{3} A \int_0^\eta e^{-\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \theta)} e^{-(\frac{3}{8A} \tilde{\mathcal{Y}}_2(t, \eta) - \frac{3}{4A} \tilde{\mathcal{Y}}_2(t, \theta))} d\theta \right] d\eta \Big)^{1/2} \\
&\leq \mathcal{O}(1)|A_1 - A_2| \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad \times \left(\int_0^1 \left[-\frac{8}{3} A \int_0^\eta e^{-\frac{3}{4A} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} d\theta \right. \right. \\
&\quad \left. \left. + \frac{8}{3} A \int_0^\eta e^{-\frac{3}{8A} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} d\theta \right] d\eta \right)^{1/2} \\
&\leq \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2).
\end{aligned}$$

Thus we have shown that

$$\tilde{A}_1 + \tilde{A}_2 \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2 + |A_1 - A_2|^2).$$

We next consider the term \tilde{A}_3 :

$$\begin{aligned}
\tilde{A}_3 &= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \int_0^\eta (e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2, \eta} \\
&\quad - e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1, \eta}) \min_j (\tilde{\mathcal{D}}_j)(t, \theta) d\theta d\eta \\
&= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \\
&\quad \times \left[\int_0^\eta e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2, \eta} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right. \\
&\quad + \int_0^\eta e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{1, \eta} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ \leq \tilde{\mathcal{U}}_1^+}(t, \theta) d\theta \\
&\quad + \mathbb{1}_{A_1 \leq A_2} \int_0^\eta (e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))}) \\
&\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \\
&\quad + \mathbb{1}_{A_2 < A_1} \int_0^\eta (e^{-\frac{1}{A_2} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{A_1} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \\
&\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad \left. + \int_0^\eta (e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{A} (\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}) \right]
\end{aligned}$$

$$\begin{aligned}
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \\
& + \int_0^\eta (e^{-\frac{1}{\alpha}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{\alpha}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \\
& \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \\
& + \int_0^\eta \min_j(e^{-\frac{1}{\alpha}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t, \theta) d\theta \Big] d\eta \\
& = A_{31} + A_{32} + A_{33} + A_{34} + A_{35} + A_{36} + A_{37}.
\end{aligned} \tag{5.47}$$

Terms A_{31} and A_{32} allow for the same treatment:

$$\begin{aligned}
|A_{31}| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
&\quad \times \left. \left(\int_0^\eta e^{-\frac{1}{\Lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+}(t, \theta) d\theta \right) d\eta \right| \\
&\leq \frac{1}{A^2} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{\Lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+|(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^4} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{\Lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+|(t, \theta) d\theta \right)^2 d\eta \\
&\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{A^4} \int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2} (t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{3}{2\Lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2\Lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \right) d\eta \\
&\leq \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\
&\quad + \frac{2}{A^3} \int_0^1 \left(\int_0^\eta e^{-\frac{1}{\Lambda_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2),
\end{aligned}$$

using (4.15e), (4.15n), and (4.16e).

The terms A_{33} and A_{34} can be treated as follows:

$$\begin{aligned}
 |A_{33}| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \right. \\
 &\quad \times \mathbb{1}_{A_1 \leq A_2} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta d\eta \Big| \\
 &\leq \frac{\sqrt{2}a}{A^3 e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) d\eta |A_1 - A_2| \\
 &\leq \frac{2\sqrt{2}}{A^2 e} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left(\int_0^\eta \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right)^{1/2} d\eta |A_1 - A_2| \\
 &\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
 \end{aligned}$$

Furthermore, the terms A_{35} and A_{36} can be estimated like this:

$$\begin{aligned}
 |A_{35}| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
 &\quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) d\eta \Big| \\
 &\leq \frac{1}{2aA^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
 &\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (|\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_1(t, \eta)| + |\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta)|) \right. \\
 &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \Big) d\eta
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{a}{2\sqrt{2}A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\
&\quad + \frac{a}{2\sqrt{2}A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2, \eta} \mathbb{1}_{B(\eta)}(t, \theta) d\theta \right) d\eta \\
&\leq \frac{1}{\sqrt{2}A} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} d\eta \\
&\quad + \frac{1}{\sqrt{2}A} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right)^{1/2} d\eta \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2),
\end{aligned}$$

using (4.15e), (4.15n), and (4.16e). As for A_{37} , we follow this path:

$$\begin{aligned}
|A_{37}| &= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \right. \\
&\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2, \eta} - \tilde{\mathcal{Y}}_{1, \eta})(t, \theta) d\theta \Big) d\eta \Big| \\
&= \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \right. \\
&\quad \times \left[\left(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \theta) \right) \Big|_{\theta=0}^\eta \right. \\
&\quad \left. \left. - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \right] d\eta \right|
\end{aligned}$$

$$\begin{aligned}
& \times \frac{d}{d\theta} \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta \Bigg] d\eta \Bigg| \\
& \leq \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t, \eta) d\eta \right| \\
& \quad + \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \right. \right. \\
& \quad \times \frac{d}{d\theta} \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \right) (t, \theta) d\theta \Bigg) d\eta \Bigg| \\
& \leq \frac{1}{A^2} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| (t, \eta) d\eta \\
& \quad + \frac{1}{2A^3} \left| \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t, \eta) \left(\int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \right. \right. \\
& \quad \times \left(\left(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \right. \\
& \quad \left. \left. + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \left(\frac{d}{d\theta} \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \right) \right) (t, \theta) d\theta \right) d\eta \Bigg| \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2) \\
& \quad + \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| (t, \eta) \\
& \quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \frac{1}{a} \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
& \quad \times \max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \\
& \quad \left. \left. + \mathcal{O}(1) A^5 \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\min_j (\tilde{\mathcal{D}}_j^{1/2}) + |\tilde{\mathcal{U}}_2|) \right) (t, \theta) d\theta \right) d\eta \\
& \leq \mathcal{O}(1) (\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2) + \frac{1}{2} (A_{371} + A_{372}),
\end{aligned}$$

using (4.15a), (4.15b), (4.15n), and Lemmas A.2 and A.4. Here

$$\begin{aligned}
A_{371} &= \frac{1}{aA^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| (t, \eta) \\
& \quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
& \quad \times \max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) (t, \theta) d\theta \Bigg) d\eta
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\sqrt{2}a}{A^2} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \left(\int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \right. \\
&\quad \times \left. (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta} + e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta})(t, \theta) d\theta \right) d\eta \\
&\leq \frac{\sqrt{2}}{A} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left[\left(\int_0^\eta e^{-\frac{3}{2A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right)^{1/2} \right. \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_1}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right)^{1/2} \\
&\quad + \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta \right)^{1/2} \\
&\quad \times \left. \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right)^{1/2} \right] d\eta \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

using $\max_j(a_j) \leq a_1 + a_2$ and $\min_j(b_j) \leq b_k$. Then

$$\begin{aligned}
A_{372} &= \mathcal{O}(1) \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) (\min_j(\tilde{\mathcal{D}}_j^{1/2}) + |\tilde{\mathcal{U}}_2|)(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1| e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2^{1/2}(t, \theta) d\theta \right) d\eta \\
&\leq \mathcal{O}(1) \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{P}}_2(t, \theta) d\theta \right)^{1/2} \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t, \theta) d\theta \right)^{1/2} d\eta \\
&\leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

using (4.15d), (4.15n), and (4.16i). This completes the estimate for \tilde{A}_3 (see (5.47)):

$$\tilde{A}_3 \leq \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).$$

For J_4 (and similarly for J_5) the estimates read as

$$\begin{aligned} |J_4| &\leq \frac{1}{A^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|^2 \frac{|\tilde{\mathcal{R}}_2|}{\tilde{\mathcal{P}}_2}(t, \eta) d\eta \\ &\leq \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2, \end{aligned}$$

using (4.15j).

We have shown the anticipated result.

LEMMA 5.5. Let $\tilde{\mathcal{P}}_i^{1/2}$ be two solutions of (5.40) for $i = 1, 2$. Then we have

$$\begin{aligned} \frac{d}{dt} \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\ \leq \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant which only depends on $A = \max_j(A_j)$ and which remains bounded as $A \rightarrow 0$.

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Appendix A. Lipschitz continuity and uniformly bounded Lipschitz constants

We need to establish that a number of complicated functions are Lipschitz continuous with uniformly bounded Lipschitz constants. In a desperate attempt to ease the readability of the main estimates, we collect these results here together with some other estimates that are essential in Section 5.

LEMMA A.1. (i) *We have*

$$|e^{a_1-a_2} - e^{b_1-b_2}| \leq \max(e^{a_1-a_2}, e^{b_1-b_2})(|b_1 - a_1| + |b_2 - a_2|) \quad (\text{A.1})$$

$$\leq |b_1 - a_1| + |b_2 - a_2|, \quad (\text{A.2})$$

$$a_1 < a_2, \quad b_1 < b_2.$$

(ii) *Let $0 < a \leq A_j \leq A$, $j = 1, 2$. Then we have*

$$|e^{-\frac{1}{A_2}x} - e^{-\frac{1}{A_1}x}| \leq \frac{4}{ae} e^{-\frac{3}{4A}x} |A_2 - A_1|, \quad x \in [0, \infty). \quad (\text{A.3})$$

Proof. (i) The result follows from the elementary inequality

$$|e^a - e^b| = \left| \int_b^a e^x dx \right| \leq e^b \left| \int_b^a dx \right| \leq e^b |b - a|, \quad a < b.$$

(ii) For $x \in [0, \infty)$ one can write

$$\begin{aligned} |e^{-\frac{1}{A_2}x} - e^{-\frac{1}{A_1}x}| &= \left| \int_{A_1}^{A_2} \frac{1}{s^2} x e^{-\frac{1}{s}x} ds \right| \\ &\leq e^{-\frac{3}{4A}x} \left| \int_{A_1}^{A_2} \frac{1}{s^2} x e^{-\frac{1}{4s}x} ds \right| \\ &\leq \frac{4}{ae} e^{-\frac{3}{4A}x} |A_2 - A_1|. \end{aligned}$$

Here we used in the last step that for $s \in [a, A]$, the function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(x) = \frac{1}{s^2} x e^{-\frac{1}{4s}x}$ attains its maximum at $x = 4s$ and

$$0 \leq f(4s) = \frac{4}{se} \leq \frac{4}{ae}. \quad \square$$

LEMMA A.2. (i) *The function $\theta \mapsto \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))})$ is nondecreasing for almost every η and thus differentiable almost everywhere. We have that*

$$\left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right| \leq \frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \max_j (\tilde{\mathcal{Y}}_{j, \eta}(t, \theta)). \quad (\text{A.4})$$

(ii) The function $\theta \mapsto \min_j (\tilde{\mathcal{U}}_j^2)(t, \theta)$ is differentiable almost everywhere with

$$\left| \frac{d}{d\theta} \min_j (\tilde{\mathcal{U}}_j^2)(t, \theta) \right| \leq A^4. \quad (\text{A.5})$$

(iii) The function $\theta \mapsto \min_j (\tilde{\mathcal{U}}_j^+)^3(t, \theta)$ is differentiable almost everywhere with

$$\left| \frac{d}{d\theta} \min_j (\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right| \leq 2A^4 \min_j (\tilde{\mathcal{U}}_j^+)(t, \theta). \quad (\text{A.6})$$

Proof. (i): First of all, note that the function $\theta \mapsto \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))})$ is nondecreasing and hence differentiable almost everywhere. Consider in the following $\theta < \eta$. Assume that for fixed θ , such that the given function is differentiable, we have

$$\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \quad (\text{A.7})$$

and that there exists a sequence $\theta_n \uparrow \theta$ such that

$$\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta_n))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta_n))} \quad \text{for all } n.$$

Then we have

$$\begin{aligned} & \left| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta_n))}) \right| \\ &= e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta_n))} \\ &\leq \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} (\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta_n)), \end{aligned}$$

since $\tilde{\mathcal{Y}}_i(t, \cdot)$ is nondecreasing.

$$\begin{aligned} \left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right| &\leq \lim_{\theta_n \uparrow \theta} \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \frac{\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_1(t, \theta_n)}{\theta - \theta_n} \\ &= \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \\ &= \frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta). \end{aligned}$$

Assume, on the other hand, that for fixed θ we have

$$\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))}$$

and that there exists a sequence $\theta_n \downarrow \theta$ such that

$$\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta_n))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta_n))} \quad \text{for all } n.$$

Then we have

$$\begin{aligned} & \left| \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta_n))}) \right| \\ &= e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta_n))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \\ &\leq \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta_n))} (\tilde{\mathcal{Y}}_1(t, \theta_n) - \tilde{\mathcal{Y}}_1(t, \theta)), \end{aligned}$$

since $\tilde{\mathcal{Y}}_i(t, \cdot)$ is nondecreasing. Thus

$$\begin{aligned} \left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right| &\leq \lim_{\theta_n \downarrow \theta} \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta_n))} \frac{\tilde{\mathcal{Y}}_1(t, \theta_n) - \tilde{\mathcal{Y}}_1(t, \theta)}{\theta_n - \theta} \\ &= \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \\ &= \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta). \end{aligned}$$

Thus in case (A.7) we find

$$\left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right| \leq \frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta).$$

If we instead of (A.7) assume

$$\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))},$$

a similar argument yields

$$\left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right| \leq \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta).$$

Thus we conclude that in general we have

$$\left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right| \leq \frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \max_j (\tilde{\mathcal{Y}}_{j, \eta})(t, \theta).$$

(ii): We have (cf. Lemma 5.1) that

$$\begin{aligned} & \left| \min_j (\tilde{\mathcal{U}}_j^2(t, \eta)) - \min_j (\tilde{\mathcal{U}}_j^2(t, \theta)) \right| \\ &\leq \max(|\tilde{\mathcal{U}}_1^2(t, \eta) - \tilde{\mathcal{U}}_1^2(t, \theta)|, |\tilde{\mathcal{U}}_2^2(t, \eta) - \tilde{\mathcal{U}}_2^2(t, \theta)|) \\ &\leq 2 \max_j \|\tilde{\mathcal{U}}_j \tilde{\mathcal{U}}_{j, \eta}\|_{\infty} |\eta - \theta| \leq A^4 |\eta - \theta|, \end{aligned} \tag{A.8}$$

using (4.15f).

(iii): Note that one has that for any positive function $m(x)$,

$$\begin{aligned} |m^3(x) - m^3(y)| &= (m^2(x) + m(x)m(y) + m^2(y))|m(x) - m(y)| \\ &\leq (m^2(x) + 2m(x)m(y) + m^2(y))|m(x) - m(y)| \\ &= (m(x) + m(y))^2|m(x) - m(y)| \\ &= (m(x) + m(y))|m^2(x) - m^2(y)|. \end{aligned}$$

If we replace $m(x)$ by $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta)$, we have

$$\begin{aligned} &|\min_j(\tilde{\mathcal{U}}_j^+)^3(t, \eta) - \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta)| \\ &\leq (\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) + \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta))|\min_j(\tilde{\mathcal{U}}_j^+)^2(t, \eta) - \min_j(\tilde{\mathcal{U}}_j^+)^2(t, \theta)| \\ &\leq (\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) + \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta)) \\ &\quad \times \max(|(\tilde{\mathcal{U}}_1^+)^2(t, \eta) - (\tilde{\mathcal{U}}_1^+)^2(t, \theta)|, |(\tilde{\mathcal{U}}_2^+)^2(t, \eta) - (\tilde{\mathcal{U}}_2^+)^2(t, \theta)|) \\ &\leq A^4(\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) + \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta))|\eta - \theta| \end{aligned}$$

(see (A.8)), and hence $\min_j(\tilde{\mathcal{U}}_j^+)^3(t, \eta)$ is differentiable almost everywhere with

$$\left| \frac{d}{d\theta} \min_j(\tilde{\mathcal{U}}_j^+)^3(t, \theta) \right| \leq 2A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta). \quad \square$$

LEMMA A.3. (i) *The function $\eta \mapsto \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta)$ is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+))(t, \eta) \right| \leq 2A^4(\min_j(\tilde{\mathcal{P}}_j))^{1/2} + |\tilde{\mathcal{U}}_k|)(t, \eta), \quad k = 1, 2.$$

(ii) *The function $\eta \mapsto \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_k(t, \eta)$, $k = 1, 2$ is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_k)(t, \eta) \right| \leq \frac{1}{\sqrt{2}} A^6.$$

Proof. (i) We only present the proof for the case $k = 2$, since the case $k = 1$ is similar.

Given $0 \leq \eta_1 < \eta_2 \leq 1$, we assume without loss of generality that

$$0 \leq \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) - \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1).$$

To ease the notation, we introduce the function

$$d(t, \eta) = \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta).$$

We will distinguish several cases:

(1): If $d(t, \eta_2) = 0$, then $d(t, \eta_1) = 0$ and one has

$$\begin{aligned} 0 &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \left| \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) \right| \\ &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) (|\tilde{\mathcal{P}}_1(t, \eta_2) - \tilde{\mathcal{P}}_1(t, \eta_1)| + |\tilde{\mathcal{P}}_2(t, \eta_2) - \tilde{\mathcal{P}}_2(t, \eta_1)|) \\ &\leq A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1|. \end{aligned}$$

(2): If $d(t, \eta_2) > 0$, then $\min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) > 0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) > 0$ or equivalently

$$\tilde{\mathcal{P}}_1(t, \eta_2) > 0, \quad \tilde{\mathcal{P}}_2(t, \eta_2) > 0, \quad \tilde{\mathcal{U}}_1(t, \eta_2) > 0, \quad \text{and} \quad \tilde{\mathcal{U}}_2(t, \eta_2) > 0.$$

(2a): Assume that $d(t, \eta_1) = 0$ and $\min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) = 0$. Then

$$\begin{aligned} 0 &\leq d(t, \eta_2) - d(t, \eta_1) \\ &\leq (\min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \min_j(\tilde{\mathcal{P}}_j)(t, \eta_1)) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \\ &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) (|\tilde{\mathcal{P}}_1(t, \eta_2) - \tilde{\mathcal{P}}_1(t, \eta_1)| + |\tilde{\mathcal{P}}_2(t, \eta_2) - \tilde{\mathcal{P}}_2(t, \eta_1)|) \\ &\leq A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1|. \end{aligned}$$

(2b): Assume that $d(t, \eta_1) = 0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = 0$. Then one either has that

(i): $\tilde{\mathcal{U}}_2(t, \eta_1) \leq \tilde{\mathcal{U}}_2^+(t, \eta_1) = \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1)$, and we can write

$$\begin{aligned} 0 &\leq d(t, \eta_2) - d(t, \eta_1) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) \\ &\leq \int_{\eta_1}^{\eta_2} \left(\tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \right) (t, s) ds \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\eta_1}^{\eta_2} \left(|\tilde{\mathcal{U}}_{2,\eta}| \min_j(\tilde{\mathcal{P}}_j) + |\tilde{\mathcal{U}}_2| \left(\frac{1}{A_1} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta} + \frac{1}{A_2} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} \right) \right) (t, s) ds \\
&\leq \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\
&\leq \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|,
\end{aligned}$$

or

(ii): $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = \tilde{\mathcal{U}}_1^+(t, \eta_1)$. Then there exists a maximal interval $[\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$ for all $s \in [\eta_1, a)$ and $\tilde{\mathcal{U}}_1^+(t, a) = \tilde{\mathcal{U}}_2^+(t, a)$. Moreover, there exists a maximal interval $[b, a] \subset [\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t, s) > 0$ for all $s \in (b, a]$ and $\tilde{\mathcal{U}}_1^+(t, b) = 0$. Hence we can write

$$\begin{aligned}
0 &\leq d(t, \eta_2) - d(t, \eta_1) \\
&\leq d(t, \eta_2) - d(t, a) + d(t, a) - d(t, b) \\
&\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, a) \\
&\quad + \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, a) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, b) \\
&\leq \int_a^{\eta_2} \left(\tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right) (t, s) ds \\
&\quad + \int_b^a \left(\tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_1 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right) (t, s) ds \\
&\leq \int_a^{\eta_2} \left(\frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_2| \right) (t, s) ds \\
&\quad + \int_b^a \left(\frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_1| \right) (t, s) ds \\
&\leq \frac{A^4}{\sqrt{2}} \int_b^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_b^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\
&\leq \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\
&\leq \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|.
\end{aligned}$$

Note, in the case that $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$ for all $s \in [\eta_1, \eta_2]$, the estimate starts with

$$0 \leq d(t, \eta_2) - d(t, \eta_1) \leq \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, b),$$

where $(b, \eta_2]$ denotes the maximal interval such that $\tilde{\mathcal{U}}_1^+(t, s) > 0$ for all $s \in (b, \eta_2]$.

(2c): Assume that $d(t, \eta_1) > 0$; then $\min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) > 0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = \min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) > 0$. Then one either has that

(i): $\min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) = \tilde{\mathcal{U}}_2(t, \eta_1)$, and we have (as before)

$$\begin{aligned} 0 &\leq d(t, \eta_2) - d(t, \eta_1) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) \\ &\leq \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|, \end{aligned}$$

or

(ii): $\min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) = \tilde{\mathcal{U}}_1(t, \eta_1)$. Then there exists a maximal interval $[\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$ for all $s \in [\eta_1, a)$ and $\tilde{\mathcal{U}}_1^+(t, a) = \tilde{\mathcal{U}}_2^+(t, a)$. Moreover, there exists a maximal interval $[b, a] \subset [\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t, s) > 0$ for all $s \in (b, a]$ and $\tilde{\mathcal{U}}_1^+(t, b) = 0$. Hence we can write

$$\begin{aligned} 0 &\leq d(t, \eta_2) - d(t, \eta_1) \\ &\leq d(t, \eta_2) - d(t, a) + d(t, a) - d(t, b) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, a) \\ &\quad + \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, a) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t, b) \\ &\leq \int_a^{\eta_2} \left(\tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right)(t, s) ds \\ &\quad + \int_b^a \left(\tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_1 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right)(t, s) ds \\ &\leq \int_a^{\eta_2} \left(\frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_2| \right)(t, s) ds \\ &\quad + \int_b^a \left(\frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_1| \right)(t, s) ds \\ &\leq \frac{A^4}{\sqrt{2}} \int_b^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_b^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \end{aligned}$$

$$\begin{aligned} &\leq \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j (\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|. \end{aligned}$$

Note, in the case that $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$ for all $s \in [\eta_1, \eta_2]$, the estimate starts with

$$0 \leq d(t, \eta_2) - d(t, \eta_1) \leq \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{P}}_j)(t, b),$$

where $(b, \eta_2]$ denotes the maximal interval such that $\tilde{\mathcal{U}}_1^+(t, s) > 0$ for all $s \in (b, \eta_2]$.

Thus we showed that

$$|d(t, \eta_2) - d(t, \eta_1)| \leq \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|,$$

or, in other words, $d(t, \cdot) = \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t, \cdot)$ is Lipschitz continuous with Lipschitz constant $\frac{3}{2\sqrt{2}} A^6$, which is independent of time, and thus differentiable almost everywhere. Moreover, a close look reveals that

$$\begin{aligned} |d(t, \eta_2) - d(t, \eta_1)| &\leq A^4 \min_j (\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1| \\ &\quad + \frac{A^4}{\sqrt{2}} \left| \int_{\eta_1}^{\eta_2} \min_j (\tilde{\mathcal{P}}_j)^{1/2}(t, s) ds \right| + A^4 \left| \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \right|. \end{aligned}$$

Since both $|\tilde{\mathcal{U}}_2|(t, \cdot)$ and $\min_j (\tilde{\mathcal{P}}_j)^{1/2}(t, \cdot)$ are continuous, the fundamental theorem of calculus implies that

$$\begin{aligned} \left| \frac{d(t, \eta_2) - d(t, \eta_1)}{\eta_2 - \eta_1} \right| &\leq A^4 \min_j (\tilde{\mathcal{U}}_j^+)(t, \eta_2) \\ &\quad + \frac{A^4}{\sqrt{2}} \min_j (\tilde{\mathcal{P}}_j)^{1/2}(t, \tilde{\eta}) + A^4 |\tilde{\mathcal{U}}_2|(t, \tilde{\eta}) \end{aligned}$$

for some $\tilde{\eta}$ between η_1 and η_2 . Letting $\eta_2 \rightarrow \eta_1$ we thus obtain for almost every η that

$$\begin{aligned} \left| \frac{d}{d\eta} \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t, \eta) \right| &= \left| \frac{d}{d\eta} d(t, \eta) \right| \\ &\leq 2A^4 \left(\min_j (\tilde{\mathcal{P}}_j)^{1/2} + |\tilde{\mathcal{U}}_2| \right)(t, \eta). \end{aligned}$$

(ii) We have that

$$\begin{aligned} & \left| \min_j(\tilde{\mathcal{P}}_j)(t, \eta) - \min_j(\tilde{\mathcal{P}}_j)(t, \tilde{\eta}) \right| \\ & \leq \max(|\tilde{\mathcal{P}}_1(t, \eta) - \tilde{\mathcal{P}}_1(t, \tilde{\eta})|, |\tilde{\mathcal{P}}_2(t, \eta) - \tilde{\mathcal{P}}_2(t, \tilde{\eta})|) \\ & \leq \frac{A^4}{2} |\eta - \tilde{\eta}|. \end{aligned}$$

Thus, for almost every η ,

$$\begin{aligned} \left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_k)(t, \eta) \right| &= \left| \left(\frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right) \tilde{\mathcal{U}}_k(t, \eta) + \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{U}}_{k,\eta}(t, \eta) \right| \\ &\leq \frac{A^4}{2} \|\tilde{\mathcal{U}}_k\|_\infty + \|\tilde{\mathcal{P}}_k \tilde{\mathcal{U}}_{k,\eta}\|_\infty \\ &\leq \frac{1}{\sqrt{2}} A^6. \end{aligned} \quad \square$$

LEMMA A.4. (i) *The function $\eta \mapsto \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta)$ is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\begin{aligned} & \left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+))(t, \eta) \right| \\ & \leq \mathcal{O}(1) \sqrt{A} A^4 (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_k|)(t, \eta), \quad k = 1, 2. \end{aligned}$$

(ii) *The function $\eta \mapsto \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_k(t, \eta)$, $k = 1, 2$, is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{U}}_k)(t, \eta) \right| \leq \mathcal{O}(1) A^7.$$

Proof. (i) We only present the proof for the case $k = 2$, since the case $k = 1$ is similar.

Given $0 \leq \eta_1 < \eta_2 \leq 1$, we assume without loss of generality that

$$0 \leq \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) - \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1).$$

To ease the notation, we introduce the function

$$\tilde{d}(t, \eta) = \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta).$$

We will distinguish several cases:

(1): If $\bar{d}(t, \eta_2) = 0$, then $\bar{d}(t, \eta_1) = 0$ and one has

$$\begin{aligned} 0 &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \min_j(\tilde{\mathcal{D}}_j)(t, \eta_1)| \\ &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) (|\tilde{\mathcal{D}}_1(t, \eta_2) - \tilde{\mathcal{D}}_1(t, \eta_1)| + |\tilde{\mathcal{D}}_2(t, \eta_2) - \tilde{\mathcal{D}}_2(t, \eta_1)|) \\ &\leq \mathcal{O}(1) A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1|. \end{aligned}$$

(2): If $\bar{d}(t, \eta_2) > 0$, then $\min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) > 0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) > 0$ or equivalently

$$\tilde{\mathcal{D}}_1(t, \eta_2) > 0, \quad \tilde{\mathcal{D}}_2(t, \eta_2) > 0, \quad \tilde{\mathcal{U}}_1(t, \eta_2) > 0, \quad \text{and} \quad \tilde{\mathcal{U}}_2(t, \eta_2) > 0.$$

(2a): Assume that $\bar{d}(t, \eta_1) = 0$ and $\min_j(\tilde{\mathcal{D}}_j)(t, \eta_1) = 0$. Then

$$\begin{aligned} 0 &\leq \bar{d}(t, \eta_2) - \bar{d}(t, \eta_1) \\ &\leq (\min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \min_j(\tilde{\mathcal{D}}_j)(t, \eta_1)) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \\ &\leq \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) (|\tilde{\mathcal{D}}_1(t, \eta_2) - \tilde{\mathcal{D}}_1(t, \eta_1)| + |\tilde{\mathcal{D}}_2(t, \eta_2) - \tilde{\mathcal{D}}_2(t, \eta_1)|) \\ &\leq \mathcal{O}(1) A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1|. \end{aligned}$$

(2b): Assume that $\bar{d}(t, \eta_1) = 0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = 0$. Then one either has that

(i): $\tilde{\mathcal{U}}_2(t, \eta_1) \leq \tilde{\mathcal{U}}_2^+(t, \eta_1) = \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1)$ and we can write

$$\begin{aligned} 0 &\leq d(t, \eta_2) - d(t, \eta_1) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_1) \\ &\leq \int_{\eta_1}^{\eta_2} \tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{D}}_j)(t, s) + \tilde{\mathcal{U}}_2 \frac{d}{d\theta} \min_j(\tilde{\mathcal{D}}_j)(t, s) ds \\ &\leq \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_{2,\eta}| \min_j(\tilde{\mathcal{D}}_j)(t, s) ds \\ &\quad + \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2| \left((\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1,\eta} - \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{Y}}_{1,\eta} + \frac{1}{2} A_1^5 \right) \\ &\quad + \left((\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2,\eta} - \frac{1}{A_2} \tilde{\mathcal{D}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \frac{1}{2} A_2^5 \right) (t, s) ds \end{aligned}$$

$$\begin{aligned} &\leq \sqrt{A}A^4 \int_{\eta_1}^{\eta_2} \min_j (\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds + \mathcal{O}(1)A^5 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \mathcal{O}(1)A^7 |\eta_2 - \eta_1| \end{aligned}$$

or

(ii): $\min_j (\tilde{\mathcal{U}}_j^+)(t, \eta_1) = \tilde{\mathcal{U}}_1^+(t, \eta_1)$. Then there exists a maximal interval $[\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$ for all $s \in [\eta_1, a)$ and $\tilde{\mathcal{U}}_1^+(t, a) = \tilde{\mathcal{U}}_2^+(t, a)$. Moreover, there exists a maximal interval $[b, a] \subset [\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t, s) > 0$ for all $s \in (b, a]$ and $\tilde{\mathcal{U}}_1^+(t, b) = 0$. Hence we can write

$$\begin{aligned} 0 &\leq \bar{d}(t, \eta_2) - \bar{d}(t, \eta_1) \\ &\leq \bar{d}(t, \eta_2) - \bar{d}(t, a) + \bar{d}(t, a) - \bar{d}(t, b) \\ &\leq \tilde{\mathcal{U}}_2 \min_j (\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j (\tilde{\mathcal{D}}_j)(t, a) \\ &\quad + \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{D}}_j)(t, a) - \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{D}}_j)(t, b) \\ &\leq \int_a^{\eta_2} \left(\tilde{\mathcal{U}}_{2,\eta} \min_j (\tilde{\mathcal{D}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\eta} \min_j (\tilde{\mathcal{D}}_j) \right)(t, s) ds \\ &\quad + \int_b^a \left(\tilde{\mathcal{U}}_{1,\eta} \min_j (\tilde{\mathcal{D}}_j) + \tilde{\mathcal{U}}_1 \frac{d}{d\eta} \min_j (\tilde{\mathcal{D}}_j) \right)(t, s) ds \\ &\leq \int_a^{\eta_2} (\sqrt{A}A^4 \min_j (\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1)A^5 |\tilde{\mathcal{U}}_2|)(t, s) ds \\ &\quad + \int_b^a (\sqrt{A}A^4 \min_j (\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1)A^5 |\tilde{\mathcal{U}}_1|)(t, s) ds \\ &\leq \sqrt{A}A^4 \int_b^{\eta_2} \min_j (\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds + \mathcal{O}(1)A^5 \int_b^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \sqrt{A}A^4 \int_{\eta_1}^{\eta_2} \min_j (\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds + \mathcal{O}(1)A^5 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \mathcal{O}(1)A^7 |\eta_2 - \eta_1|. \end{aligned}$$

Note, in the case that $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$ for all $s \in [\eta_1, \eta_2]$, the estimate starts with

$$0 \leq d(t, \eta_2) - d(t, \eta_1) \leq \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{D}}_j)(t, b),$$

where $(b, \eta_2]$ denotes the maximal interval such that $\tilde{\mathcal{U}}_1^+(t, s) > 0$ for all $s \in (b, \eta_2]$.

(2c): Assume that $\bar{d}(t, \eta_1) > 0$; then $\min_j(\tilde{\mathcal{D}}_j)(t, \eta_1) > 0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = \min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) > 0$. Then one either has that

(i): $\min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) = \tilde{\mathcal{U}}_2(t, \eta_1)$, and we have (as before)

$$\begin{aligned} 0 &\leq \bar{d}(t, \eta_2) - \bar{d}(t, \eta_1) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_1) \\ &\leq \int_{\eta_1}^{\eta_2} \left(\frac{1}{2} \sqrt{A} A^6 \min_j(\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1) A^5 |\tilde{\mathcal{U}}_2| \right) (t, s) ds \\ &\leq \mathcal{O}(1) A^7 |\eta_2 - \eta_1|, \end{aligned}$$

or

(ii): $\min_j(\tilde{\mathcal{U}}_j)(t, \eta_1) = \tilde{\mathcal{U}}_1(t, \eta_1)$. Then there exists a maximal interval $[\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$ for all $s \in [\eta_1, a)$ and $\tilde{\mathcal{U}}_1^+(t, a) = \tilde{\mathcal{U}}_2^+(t, a)$. Moreover, there exists a maximal interval $[b, a] \subset [\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t, s) > 0$ for all $s \in (b, a]$ and $\tilde{\mathcal{U}}_1^+(t, b) = 0$. Hence we can write

$$\begin{aligned} 0 &\leq \bar{d}(t, \eta_2) - \bar{d}(t, \eta_1) \\ &\leq \bar{d}(t, \eta_2) - \bar{d}(t, a) + \bar{d}(t, a) - \bar{d}(t, b) \\ &\leq \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t, a) \\ &\quad + \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, a) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, b) \\ &\leq \int_a^{\eta_2} \left(\tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{D}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j) \right) (t, s) ds \\ &\quad + \int_b^a \left(\tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{D}}_j) + \tilde{\mathcal{U}}_1 \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j) \right) (t, s) ds \\ &\leq \int_a^{\eta_2} \left(\frac{1}{2} \sqrt{A} A^6 \min_j(\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1) A^5 |\tilde{\mathcal{U}}_2| \right) (t, s) ds \\ &\quad + \int_b^a \left(\frac{1}{2} \sqrt{A} A^6 \min_j(\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1) A^5 |\tilde{\mathcal{U}}_1| \right) (t, s) ds \\ &\leq \sqrt{A} A^4 \int_b^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds + \mathcal{O}(1) A^5 \int_b^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \sqrt{A} A^4 \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds + \mathcal{O}(1) A^5 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \\ &\leq \mathcal{O}(1) A^7 |\eta_2 - \eta_1|. \end{aligned}$$

Note, in the case that $\tilde{\mathcal{U}}_1^+(t, s) < \tilde{\mathcal{U}}_2^+(t, s)$ for all $s \in [\eta_1, \eta_2]$, the estimate starts with

$$0 \leq d(t, \eta_2) - d(t, \eta_1) \leq \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t, b),$$

where $(b, \eta_2]$ denotes the maximal interval such that $\tilde{\mathcal{U}}_1^+(t, s) > 0$ for all $s \in (b, \eta_2]$.

Thus we showed that

$$|\bar{d}(t, \eta_2) - \bar{d}(t, \eta_1)| \leq \mathcal{O}(1)A^7|\eta_2 - \eta_1|,$$

or, in other words, $\bar{d}(t, \cdot) = \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \cdot)$ is Lipschitz continuous with Lipschitz constant $\mathcal{O}(1)A^7$, which is independent of time, and thus differentiable almost everywhere. Moreover, a close look reveals that

$$\begin{aligned} & |\bar{d}(t, \eta_2) - \bar{d}(t, \eta_1)| \\ & \leq \mathcal{O}(1)A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2)|\eta_2 - \eta_1| \\ & \quad + \sqrt{A}A^4 \left| \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, s) ds \right| + \mathcal{O}(1)A^5 \left| \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t, s) ds \right|. \end{aligned}$$

Since both $|\tilde{\mathcal{U}}_2|(t, \cdot)$ and $\min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, \cdot)$ are continuous, the fundamental theorem of calculus implies that

$$\begin{aligned} \left| \frac{\bar{d}(t, \eta_2) - \bar{d}(t, \eta_1)}{\eta_2 - \eta_1} \right| & \leq \mathcal{O}(1)A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \\ & \quad + \sqrt{A}A^4 \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t, \tilde{\eta}) + \mathcal{O}(1)A^5 |\tilde{\mathcal{U}}_2|(t, \tilde{\eta}) \end{aligned}$$

for some $\tilde{\eta}$ between η_1 and η_2 . Letting $\eta_2 \rightarrow \eta_1$ we thus obtain for almost every η that

$$\begin{aligned} \left| \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta) \right| & = \left| \frac{d}{d\eta} \bar{d}(t, \eta) \right| \\ & \leq \mathcal{O}(1)\sqrt{A}A^4 (\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|)(t, \eta). \end{aligned}$$

(ii) We have that

$$\begin{aligned} & |\min_j(\tilde{\mathcal{D}}_j)(t, \eta) - \min_j(\tilde{\mathcal{D}}_j)(t, \tilde{\eta})| \\ & \leq \max(|\tilde{\mathcal{D}}_1(t, \eta) - \tilde{\mathcal{D}}_1(t, \tilde{\eta})|, |\tilde{\mathcal{D}}_2(t, \eta) - \tilde{\mathcal{D}}_2(t, \tilde{\eta})|) \\ & \leq \mathcal{O}(1)A^5 |\eta - \tilde{\eta}|, \end{aligned}$$

and hence, for almost every η ,

$$\left| \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j)(t, \eta) \right| \leq \mathcal{O}(1)A^5.$$

This implies, for almost every η ,

$$\begin{aligned} \left| \frac{d}{d\eta} (\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_k)(t, \eta) \right| &= \left| \left(\frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j) \right) \tilde{\mathcal{U}}_k(t, \eta) + \min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_{k,\eta}(t, \eta) \right| \\ &\leq \mathcal{O}(1)A^7 + 2A\|\tilde{\mathcal{P}}_k\|\tilde{\mathcal{U}}_{k,\eta} \\ &\leq \mathcal{O}(1)A^7. \end{aligned}$$

□

LEMMA A.5. *The function*

$$\eta \mapsto a \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta$$

is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with

$$\left| \frac{d}{d\eta} \left(a \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right) \right| \leq \mathcal{O}(1)A^2.$$

Proof. To prove the existence and boundedness of the derivative, we will prove Lipschitz continuity. Let $0 \leq \eta_1 < \eta_2 \leq 1$; then

$$\begin{aligned} &a \left| \int_0^{\eta_1} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right. \\ &\quad \left. - \int_0^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \right| \\ &\leq a \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\ &\quad + a \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\ &\quad - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))})) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\ &\leq \frac{aA^2}{\sqrt{2}} |\eta_2 - \eta_1| \\ &\quad + a \int_0^{\eta_1} (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta_1) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta_2) - \tilde{\mathcal{Y}}_1(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \end{aligned}$$

$$\begin{aligned}
& + a \int_0^{\eta_1} (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta_1) - \tilde{\mathcal{Y}}_2(t, \theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \eta_2) - \tilde{\mathcal{Y}}_2(t, \theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\
& \leq \frac{aA^2}{\sqrt{2}} |\eta_2 - \eta_1| \\
& + \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1, \eta}(t, s) ds \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\
& + \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, s) ds \min_j(\tilde{\mathcal{U}}_j^+)(t, \theta) d\theta \\
& \leq \frac{aA^2}{\sqrt{2}} |\eta_2 - \eta_1| \\
& + \int_{\eta_1}^{\eta_2} \left(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{U}}_1^+(t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{1, \eta}(t, s) ds \\
& + \int_{\eta_1}^{\eta_2} \left(\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{U}}_2^+(t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{2, \eta}(t, s) ds \\
& \leq \frac{aA^2}{\sqrt{2}} |\eta_2 - \eta_1| + \frac{1}{A_1^5} \int_{\eta_1}^{\eta_2} \left(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \right. \\
& \times \left(\frac{1}{A_1} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1, \eta} + A_1 \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1, \eta} + \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{H}}_{1, \eta} \right) (t, \theta) d\theta \Big) \tilde{\mathcal{Y}}_{1, \eta}(t, s) ds \\
& + \frac{1}{A_2^5} \int_{\eta_1}^{\eta_2} \left(\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \right. \\
& \times \left(\frac{1}{A_2} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + A_2 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta} + \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{H}}_{2, \eta} \right) (t, \theta) d\theta \Big) \tilde{\mathcal{Y}}_{2, \eta}(t, s) ds \\
& \leq \frac{aA^2}{\sqrt{2}} |\eta_2 - \eta_1| \\
& + \mathcal{O}(1) \frac{1}{A_1^3} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, s) ds \\
& + \mathcal{O}(1) \frac{1}{A_2^3} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, s) ds \\
& \leq \mathcal{O}(1) A^2 |\eta_2 - \eta_1|.
\end{aligned}$$

□

LEMMA A.6. *The function*

$$\eta \mapsto \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right)$$

is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere. The derivative satisfies,

$$\left| \frac{d}{d\eta} \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) \right| \leq \mathcal{O}(1)A^7. \quad (\text{A.9})$$

Proof. Introduce

$$\begin{aligned} \tilde{a}(t, \eta) &= \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) \\ \tilde{b}(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ \tilde{c}(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta. \end{aligned}$$

Thus

$$\tilde{a}(t, \eta) = \min(\tilde{b}, \tilde{c})(t, \eta).$$

Then we have to show that $\tilde{a}(t, \cdot)$ is Lipschitz continuous with a Lipschitz constant, which only depends on A . Clearly, we have that

$$|\tilde{a}(t, \eta_1) - \tilde{a}(t, \eta_2)| \leq \max(|\tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2)|, |\tilde{c}(t, \eta_1) - \tilde{c}(t, \eta_2)|),$$

and it suffices to show that both $\tilde{b}(t, \cdot)$ and $\tilde{c}(t, \cdot)$ are Lipschitz continuous with a Lipschitz constant, which only depends on A . We are only going to establish the Lipschitz continuity for \tilde{b} , since the argument for \tilde{c} follows the same lines.

Let $0 \leq \eta_1 < \eta_2 \leq 1$. Then we have to consider two cases:

$$0 \leq \tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2) \quad \text{and} \quad 0 \leq \tilde{b}(t, \eta_2) - \tilde{b}(t, \eta_1).$$

(i): $0 \leq \tilde{b}(t, \eta_2) - \tilde{b}(t, \eta_1)$: By definition, we have

$$\begin{aligned} \tilde{b}(t, \eta_2) - \tilde{b}(t, \eta_1) &= \int_0^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \\ &\quad \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &\quad - \int_0^{\eta_1} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \\ &\quad \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &= \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \end{aligned}$$

$$\begin{aligned}
& \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& + \int_0^{\eta_1} \left(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
& \quad \left. - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \\
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& \leq \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \\
& \quad \times \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta,
\end{aligned}$$

where we used in the last step that $0 \leq \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \eta)$ and that $\tilde{\mathcal{Y}}_1(t, \eta)$ is increasing, which implies

$$\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \geq \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}).$$

Moreover, note that $\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \leq 1$, for $0 \leq \eta_1 \leq \theta \leq \eta_2$, and that

$$0 \leq 2\sqrt{2} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \leq A_1^7 \leq A^7,$$

by (4.15b) and (4.15e). Thus

$$\begin{aligned}
0 & \leq \tilde{b}(t, \eta_2) - \tilde{b}(t, \eta_1) \\
& \leq \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& \leq \frac{A^7}{2\sqrt{2}} |\eta_2 - \eta_1|.
\end{aligned}$$

(ii): $0 \leq \tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2)$: By definition, we have

$$\begin{aligned}
0 & \leq \tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2) \\
& = \int_0^{\eta_1} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& \quad - \int_0^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& = \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}))
\end{aligned}$$

$$\begin{aligned}
& \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& - \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& \leq \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))})) \\
& \quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta.
\end{aligned}$$

Now we have to be much more careful than before. Namely, we have (as before)

$$\begin{aligned}
0 & \leq \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
& \leq \frac{1}{a} \int_{\eta_1}^{\eta_2} (e^{\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_1(t, s))} \tilde{\mathcal{Y}}_{1,\eta}(t, s) + e^{\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, s))} \tilde{\mathcal{Y}}_{2,\eta}(t, s)) ds.
\end{aligned}$$

Hence

$$\begin{aligned}
& \tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2) \\
& \leq \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))})) \\
& \quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& \leq \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& \quad + \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2,\eta}(t, s) ds \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
& = \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left(\int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
& \quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left(\int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
& \leq \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
& \quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
& = \int_{\eta_1}^{\eta_2} \tilde{B}_1(t, s) ds + \int_{\eta_1}^{\eta_2} \tilde{B}_2(t, s) ds.
\end{aligned}$$

As far as $\tilde{B}_1(t, s)$ is concerned, we have

$$\begin{aligned} & \int_{\eta_1}^{\eta_2} \tilde{B}_1(t, s) ds \\ &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \left(\int_0^s e^{-\frac{1}{a}(\tilde{Y}_1(t,s) - \tilde{Y}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\ &\leq \frac{1}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \left(\int_0^s e^{-\frac{1}{A_1}(\tilde{Y}_1(t,s) - \tilde{Y}_1(t,\theta))} \tilde{\mathcal{P}}_1^{3/4} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\ &\leq \frac{1}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \left(\int_0^s e^{-\frac{1}{A_1}(\tilde{Y}_1(t,s) - \tilde{Y}_1(t,\theta))} \tilde{\mathcal{P}}_1^{3/2} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \\ &\quad \times \left(\int_0^s e^{-\frac{1}{A_1}(\tilde{Y}_1(t,s) - \tilde{Y}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\ &\leq 3A^2 \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\ &\leq \frac{3}{2} A^7 |\eta_2 - \eta_1|, \end{aligned}$$

using (4.16k).

As far as $\tilde{B}_2(t, s)$ is concerned, we have to be more careful. Therefore recall that we have (cf. (4.13)) that

$$A_2^5 = 2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) - \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta) \leq 2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta).$$

Therefore we can write

$$\begin{aligned} & \frac{1}{a} \int_0^s e^{-\frac{1}{a}(\tilde{Y}_2(t,s) - \tilde{Y}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &\leq \frac{\sqrt{2}}{a} \int_0^s e^{-\frac{1}{A_2}(\tilde{Y}_2(t,s) - \tilde{Y}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j)^{3/2} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &\leq \int_0^s e^{-\frac{1}{A_2}(\tilde{Y}_2(t,s) - \tilde{Y}_2(t,\theta))} \tilde{\mathcal{P}}_2^{1/4} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &\leq \frac{A_1^5}{2A_2^5} \int_0^s e^{-\frac{1}{A_2}(\tilde{Y}_2(t,s) - \tilde{Y}_2(t,\theta))} \tilde{\mathcal{P}}_2^{1/4} (2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \\ &\leq \frac{A_1^5}{2A_2^5} \int_0^s e^{-\frac{1}{A_2}(\tilde{Y}_2(t,s) - \tilde{Y}_2(t,\theta))} \left(2\tilde{\mathcal{P}}_2^{5/4} \tilde{\mathcal{Y}}_{2,\eta} + \frac{A_2}{\sqrt{2}} \tilde{\mathcal{H}}_{2,\eta} \right)(t, \theta) d\theta \\ &\leq \frac{A_1^5}{2A_2^5} \left(2\sqrt{2} A_2^2 \tilde{\mathcal{P}}_2(t, s) + 2 \int_0^s e^{-\frac{1}{A_2}(\tilde{Y}_2(t,s) - \tilde{Y}_2(t,\theta))} \tilde{\mathcal{P}}_2^{5/4} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right). \end{aligned}$$

Note that the integral term can be bounded by $\mathcal{O}(1)\tilde{\mathcal{P}}_2(t, s)$ since

$$\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{5/4} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \leq \frac{15}{\sqrt{2}} A_2^5 \tilde{\mathcal{P}}_2(t, s)$$

by (4.16k). We end up with

$$\begin{aligned} \int_{\eta_1}^{\eta_2} \tilde{B}_2(t, s) ds &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \\ &\quad \times \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\ &\leq \frac{17}{\sqrt{2}} \frac{A_1^5}{A_2^3} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, s) ds \\ &\leq \frac{17}{2\sqrt{2}} A^7 |\eta_2 - \eta_1|. \end{aligned}$$

Moreover,

$$\begin{aligned} \tilde{b}(t, \eta_1) - \tilde{b}(t, \eta_2) &\leq \int_{\eta_1}^{\eta_2} \tilde{B}_1(t, s) ds + \int_{\eta_1}^{\eta_2} \tilde{B}_2(t, s) ds \\ &\leq \mathcal{O}(1) A^7 |\eta_2 - \eta_1|, \end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant, which only depends on A and which remains bounded as $A \rightarrow 0$.

Finally combining both cases yields that there exists a constant $\mathcal{O}(1)$, which only depends on A and which remains bounded as $A \rightarrow 0$, such that

$$|\tilde{b}(t, \eta_2) - \tilde{b}(t, \eta_1)| \leq \mathcal{O}(1) A^7 |\eta_2 - \eta_1|$$

and subsequently

$$|\tilde{a}(t, \eta_2) - \tilde{a}(t, \eta_1)| \leq \mathcal{O}(1) A^7 |\eta_2 - \eta_1|.$$

This proves that the derivative exists for almost every η and is bounded by (A.9). \square

LEMMA A.7. *The function*

$$\eta \mapsto \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right)$$

is Lipschitz continuous with uniformly bounded Lipschitz constant and thus differentiable almost everywhere. The derivative satisfies,

$$\left| \frac{d}{d\eta} \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right) \right| \leq \mathcal{O}(1) A^8. \quad (\text{A.10})$$

Proof. We present the following argument. Introduce

$$\begin{aligned}\bar{a}(t, \eta) &= \min_k \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k, \eta}(t, \theta) d\theta \right), \\ \bar{b}(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta, \\ \bar{c}(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) d\theta.\end{aligned}$$

Thus

$$\bar{a}(t, \eta) = \min(\bar{b}, \bar{c})(t, \eta).$$

We have to show that $\bar{a}(t, \cdot)$ is Lipschitz continuous with a Lipschitz constant, which only depends on A . Clearly, we have that

$$|\bar{a}(t, \eta_1) - \bar{a}(t, \eta_2)| \leq \max(|\bar{b}(t, \eta_1) - \bar{b}(t, \eta_2)|, |\bar{c}(t, \eta_1) - \bar{c}(t, \eta_2)|),$$

and it suffices to show that both $\bar{b}(t, \cdot)$ and $\bar{c}(t, \cdot)$ are Lipschitz continuous with a Lipschitz constant, which only depends on A . We are only going to establish the Lipschitz continuity for \bar{b} since the argument for \bar{c} follows the same lines. Let $0 \leq \eta_1 < \eta_2 \leq 1$. Then we have to consider two cases:

$$0 \leq \bar{b}(t, \eta_1) - \bar{b}(t, \eta_2) \quad \text{and} \quad 0 \leq \bar{b}(t, \eta_2) - \bar{b}(t, \eta_1).$$

(i): $0 \leq \bar{b}(t, \eta_2) - \bar{b}(t, \eta_1)$: By definition, we have

$$\begin{aligned}\bar{b}(t, \eta_2) - \bar{b}(t, \eta_1) &= \int_0^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\quad - \int_0^{\eta_1} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &= \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\ &\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\quad + \int_0^{\eta_1} (\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\ &\quad - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))})) \\ &\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta\end{aligned}$$

$$\begin{aligned} &\leq \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\ &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta, \end{aligned}$$

where we used in the last step that $0 \leq \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \eta)$ and that $\tilde{\mathcal{Y}}_i(t, \eta)$ is increasing, which implies

$$\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \geq \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}).$$

Moreover, note that $\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \leq 1$ for $0 \leq \eta_1 \leq \theta \leq \eta_2$ and that

$$0 \leq \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \leq \frac{1}{\sqrt{2}} A_1^8 \leq \frac{1}{\sqrt{2}} A^8$$

by (4.15b), (4.15e), and (4.15n). Thus

$$0 \leq \bar{b}(t, \eta_2) - \bar{b}(t, \eta_1) \leq \frac{1}{\sqrt{2}} A^8 |\eta_2 - \eta_1|.$$

(ii) $0 \leq \bar{b}(t, \eta_1) - \bar{b}(t, \eta_2)$: By definition, we have

$$\begin{aligned} &0 \leq \bar{b}(t, \eta_1) - \bar{b}(t, \eta_2) \\ &= \int_0^{\eta_1} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\quad - \int_0^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &= \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))})) \\ &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\quad - \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\leq \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))})) \\ &\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta. \end{aligned}$$

Now we have to be much more careful than before. Namely, we have (as before)

$$\begin{aligned} 0 &\leq \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\ &\leq \frac{1}{a} \int_{\eta_1}^{\eta_2} (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1, \eta}(t, s) + e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, s)) ds. \end{aligned}$$

Hence

$$\begin{aligned} &\bar{b}(t, \eta_1) - \bar{b}(t, \eta_2) \\ &\leq \int_0^{\eta_1} (\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))})) \\ &\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\leq \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1, \eta}(t, s) ds \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &\quad + \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, s) ds \\ &\quad \times \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\ &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1, \eta}(t, s) \\ &\quad \times \left(\int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) ds \\ &\quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2, \eta}(t, s) \\ &\quad \times \left(\int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) ds \\ &\leq \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1, \eta}(t, s) \\ &\quad \times \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) ds \\ &\quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2, \eta}(t, s) \\ &\quad \times \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \min_j (\tilde{\mathcal{D}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \right) ds \\ &= \int_{\eta_1}^{\eta_2} \bar{B}_1(t, s) ds + \int_{\eta_1}^{\eta_2} \bar{B}_2(t, s) ds. \end{aligned}$$

As far as $\bar{B}_1(t, s)$ is concerned, we have

$$\begin{aligned}
 & \int_{\eta_1}^{\eta_2} \bar{B}_1(t, s) ds \\
 &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\
 &\leq \sqrt{2}A \int_{\eta_1}^{\eta_2} \left(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1^{3/4} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\
 &\leq \sqrt{2}A \int_{\eta_1}^{\eta_2} \left(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1^{3/2} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \\
 &\quad \times \left(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right)^{1/2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\
 &\leq 6A^3 \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\
 &\leq 3A^8 |\eta_2 - \eta_1|,
 \end{aligned}$$

where we used (4.15n).

As far as $\bar{B}_2(t, s)$ is concerned, we have to be more careful. Therefore recall that we have (cf. (4.13)) that

$$A_2^5 = 2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) - \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta) \leq 2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta).$$

Therefore we can write

$$\begin{aligned}
 & \frac{1}{a} \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 &\leq \frac{2\sqrt{2}A}{a} \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j)^{3/2} \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 &\leq 2A \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{1/4} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
 &\leq A \frac{A_1^5}{A_2^5} \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{1/4} (2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \\
 &\leq A \frac{A_1^5}{A_2^5} \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \left(2\tilde{\mathcal{P}}_2^{5/4} \tilde{\mathcal{Y}}_{2,\eta} + \frac{A_2}{\sqrt{2}} \tilde{\mathcal{H}}_{2,\eta} \right)(t, \theta) d\theta \\
 &\leq A \frac{A_1^5}{A_2^5} \left(2\sqrt{2}A_2^2 \tilde{\mathcal{P}}_2(t, s) + 2 \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{5/4} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right).
 \end{aligned}$$

Note that the integral term can be bounded by $\mathcal{O}(1)\tilde{\mathcal{P}}_2(t, s)$ since

$$\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^{5/4} \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \leq \frac{15}{\sqrt{2}} A_2^2 \tilde{\mathcal{P}}_2(t, s)$$

by (4.16k). We end up with

$$\begin{aligned} & \int_{\eta_1}^{\eta_2} \bar{B}_2(t, s) ds \\ &= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\ &\leq \frac{17\sqrt{2}A^6}{A_2^3} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, s) ds \\ &\leq \frac{17A^8}{\sqrt{2}} |\eta_2 - \eta_1|. \end{aligned}$$

Moreover,

$$\begin{aligned} \bar{b}(t, \eta_1) - \bar{b}(t, \eta_2) &\leq \int_{\eta_1}^{\eta_2} \bar{B}_1(t, s) ds + \int_{\eta_1}^{\eta_2} \bar{B}_2(t, s) ds \\ &\leq \mathcal{O}(1)A^8 |\eta_2 - \eta_1|, \end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant, which only depends on A and which remains bounded as $A \rightarrow 0$.

Finally, combining both cases yields that there exists a constant $\mathcal{O}(1)$, which only depends on A and which remains bounded as $A \rightarrow 0$, such that

$$|\bar{b}(t, \eta_2) - \bar{b}(t, \eta_1)| \leq \mathcal{O}(1)A^8 |\eta_2 - \eta_1|$$

and subsequently

$$|\bar{a}(t, \eta_2) - \bar{a}(t, \eta_1)| \leq \mathcal{O}(1)A^8 |\eta_2 - \eta_1|.$$

This proves that the derivative exists for almost every η and is bounded by (A.10). \square

LEMMA A.8. *The function*

$$\eta \mapsto \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right]$$

is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere. The derivative satisfies

$$\left| \frac{d}{d\eta} \min_k \left[\int_0^\eta \min_j (e^{-(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right] \right| \leq \mathcal{O}(1)A^6, \quad (\text{A.11})$$

where $\mathcal{O}(1)$ denotes a constant, which only depends on A and which remains bounded as $A \rightarrow 0$.

Proof. To that end, we present the following argument. Introduce

$$\begin{aligned} a(t, \eta) &= \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t, \theta) d\theta \right], \\ b(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta, \\ c(t, \eta) &= \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta. \end{aligned}$$

Thus

$$a(t, \eta) = \min(b, c)(t, \eta).$$

Then we have to show that $a(t, \cdot)$ is Lipschitz continuous with a Lipschitz constant, which only depends on A and hence on C . Clearly, we have that

$$|a(t, \eta_1) - a(t, \eta_2)| \leq \max(|b(t, \eta_1) - b(t, \eta_2)|, |c(t, \eta_1) - c(t, \eta_2)|),$$

and it suffices to show that both $b(t, \cdot)$ and $c(t, \cdot)$ are Lipschitz continuous with a Lipschitz constant, which only depends on A . We are only going to establish the Lipschitz continuity for b , since the argument for c follows the same lines. Let $0 \leq \eta_1 < \eta_2 \leq 1$. Then we have to consider two cases:

$$0 \leq b(t, \eta_1) - b(t, \eta_2) \quad \text{and} \quad 0 \leq b(t, \eta_2) - b(t, \eta_1).$$

(i): $0 \leq b(t, \eta_2) - b(t, \eta_1)$: By definition, we have

$$\begin{aligned} b(t, \eta_2) - b(t, \eta_1) &= \int_0^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &\quad - \int_0^{\eta_1} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\ &= \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \end{aligned}$$

$$\begin{aligned}
& + \int_0^{\eta_1} \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
& \quad \left. - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \\
& \quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
& \leq \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta,
\end{aligned}$$

where we used in the last step that $0 \leq \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \eta)$ and that $\tilde{\mathcal{Y}}_1(t, \cdot)$ is increasing, which implies

$$\begin{aligned}
& \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
& = \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \eta_2) + \tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
& \geq \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \eta_2))}) \\
& \geq \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}).
\end{aligned}$$

Moreover, note that $\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \leq 1$ since $\eta_1 \leq \theta \leq \eta_2$, and that

$$0 \leq \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \leq |\tilde{\mathcal{U}}_1^3| \tilde{\mathcal{Y}}_{1, \eta}(t, \eta) \leq \frac{1}{\sqrt{2}} A_1^7 \leq \frac{1}{\sqrt{2}} A^7$$

by (4.15b) and (4.15g). Thus

$$\begin{aligned}
& 0 \leq b(t, \eta_2) - b(t, \eta_1) \\
& \leq \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
& \leq \frac{1}{\sqrt{2}} A^7 |\eta_2 - \eta_1|.
\end{aligned}$$

(ii): $0 \leq b(t, \eta_1) - b(t, \eta_2)$: By definition, we have

$$\begin{aligned}
& b(t, \eta_1) - b(t, \eta_2) \\
& = \int_0^{\eta_1} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
& \quad - \int_0^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{\eta_1} \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
&\quad \left. - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \\
&\quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad - \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\leq \int_0^{\eta_1} \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
&\quad \left. - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \\
&\quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta.
\end{aligned}$$

Now we have to be much more careful than before. Namely, if $\min_j (\tilde{\mathcal{Y}}_j(t, \theta) - \tilde{\mathcal{Y}}_j(t, \eta_2)) = \tilde{\mathcal{Y}}_i(t, \theta) - \tilde{\mathcal{Y}}_i(t, \eta_2)$, we have

$$\begin{aligned}
0 &\leq \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \\
&= \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) - e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t, \theta) - \tilde{\mathcal{Y}}_i(t, \eta_2))} \\
&\leq e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t, \theta) - \tilde{\mathcal{Y}}_i(t, \eta_1))} - e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t, \theta) - \tilde{\mathcal{Y}}_i(t, \eta_2))} \\
&= \frac{1}{a} \int_{\eta_1}^{\eta_2} e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t, \theta) - \tilde{\mathcal{Y}}_i(t, s))} \tilde{\mathcal{Y}}_{i, \eta}(t, s) ds \\
&\leq \frac{1}{a} \int_{\eta_1}^{\eta_2} (e^{\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_1(t, s))} \tilde{\mathcal{Y}}_{1, \eta}(t, s) + e^{\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, \theta) - \tilde{\mathcal{Y}}_2(t, s))} \tilde{\mathcal{Y}}_{2, \eta}(t, s)) ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&b(t, \eta_1) - b(t, \eta_2) \\
&\leq \int_0^{\eta_1} \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_1) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right. \\
&\quad \left. - \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t, \eta_2) - \tilde{\mathcal{Y}}_j(t, \theta))}) \right) \\
&\quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\leq \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t, s) - \tilde{\mathcal{Y}}_1(t, \theta))} \tilde{\mathcal{Y}}_{1, \eta}(t, s) ds \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta \\
&\quad + \frac{1}{a} \int_0^{\eta_1} \int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t, s) - \tilde{\mathcal{Y}}_2(t, \theta))} \tilde{\mathcal{Y}}_{2, \eta}(t, s) ds \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) d\theta
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left(\int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
&\quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left(\int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
&\leq \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
&\quad + \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
&= \int_{\eta_1}^{\eta_2} B_1(t, s) ds + \int_{\eta_1}^{\eta_2} B_2(t, s) ds.
\end{aligned}$$

As far as $B_1(t, s)$ is concerned, recall that

$$\begin{aligned}
\tilde{\mathcal{P}}_i(t, s) &= \frac{1}{4A_i} \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,s) - \tilde{\mathcal{Y}}_i(t,\theta)|} (2(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i) \tilde{\mathcal{Y}}_{i,\eta}(t, \theta) + A_i^5) d\theta \\
&= \frac{1}{4A_i} \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,s) - \tilde{\mathcal{Y}}_i(t,\theta)|} (\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta} + \tilde{\mathcal{H}}_{i,\eta})(t, \theta) d\theta,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_{\eta_1}^{\eta_2} B_1(t, s) ds \\
&= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
&\leq \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
&\leq \frac{a}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t, s) \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
&\leq \frac{4}{\sqrt{2}} A^2 \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, s) ds \\
&\leq \sqrt{2} A^7 |\eta_2 - \eta_1|.
\end{aligned}$$

As far as $B_2(t, s)$ is concerned, we have to be more careful. Therefore recall that we have

$$A_2^5 = 2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) - \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta) \leq 2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \eta) + \tilde{\mathcal{H}}_{2,\eta}(t, \eta).$$

Therefore we can write

$$\frac{1}{a} \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta$$

$$\begin{aligned}
&\leq \frac{1}{a} \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2| \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \\
&\leq \frac{A_1^5}{aA_2^5} \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2| (2\tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} + \tilde{\mathcal{H}}_{2,\eta})(t, \theta) d\theta \\
&\leq \frac{A_1^5}{aA_2^5} \int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \left(A_2 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} + \frac{1}{A_2} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} + A_2^2 \tilde{\mathcal{H}}_{2,\eta} \right) (t, \theta) d\theta \\
&\leq \frac{A_1^5}{aA_2^5} \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta)|} \left(A_2 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} + A_2^2 \tilde{\mathcal{H}}_{2,\eta} + \frac{1}{A_2} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta} \right) (t, \theta) d\theta \\
&\leq \frac{A_1^5}{aA_2^5} \left(4A_2^2 \tilde{\mathcal{P}}_2(t, s) + 4A_2^3 \tilde{\mathcal{P}}_2(t, s) \right. \\
&\quad \left. + \frac{1}{A_2} \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right).
\end{aligned}$$

Note that the integral term can be bounded by $\mathcal{O}(1)\tilde{\mathcal{P}}_2(t, s)$ since

$$\begin{aligned}
&\int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \\
&= A_2^7 \int_0^1 e^{-|\mathcal{Y}_2(t, A_2^2 s) - \mathcal{Y}_2(t, A_2^2 \theta)|} \mathcal{P}_2^2 \mathcal{Y}_{2,\eta}(t, A_2^2 \theta) d\theta \\
&= A_2^5 \int_0^{A_2^2} e^{-|\mathcal{Y}_2(t, A_2^2 s) - \mathcal{Y}_2(t, \theta)|} \mathcal{P}_2^2 \mathcal{Y}_{2,\eta}(t, \theta) d\theta \\
&\leq \mathcal{O}(1) A_2^5 \mathcal{P}_2(t, A_2^2 s) = \mathcal{O}(1) A_2^3 \tilde{\mathcal{P}}_2(t, s)
\end{aligned}$$

by (3.34). We end up with

$$\begin{aligned}
&\int_{\eta_1}^{\eta_2} B_2(t, s) ds \\
&= \frac{1}{a} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right) ds \\
&\leq \frac{A_1^5}{aA_2^5} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t, s) \\
&\quad \times \left(A_2^2 \mathcal{O}(A) \tilde{\mathcal{P}}_2(t, s) + \frac{1}{A_2} \int_0^1 e^{-\frac{1}{A_2} |\tilde{\mathcal{Y}}_2(t,s) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right) ds \\
&\leq \frac{\mathcal{O}(1) A_1^5 A_2^2}{aA_2^5} \int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, s) ds \\
&\leq \mathcal{O}(1) A^6 |\eta_2 - \eta_1|.
\end{aligned}$$

Moreover,

$$\begin{aligned} b(t, \eta_1) - b(t, \eta_2) &\leq \int_{\eta_1}^{\eta_2} B_1(t, s) ds + \int_{\eta_1}^{\eta_2} B_2(t, s) ds \\ &\leq \mathcal{O}(1)A^6|\eta_2 - \eta_1|, \end{aligned}$$

where $\mathcal{O}(1)$ only depends on A and hence on $C = \frac{A^2}{2}$, and remains bounded as $A \rightarrow 0$.

Finally, combining both cases yields that

$$|b(t, \eta_2) - b(t, \eta_1)| \leq \mathcal{O}(1)A^6|\eta_2 - \eta_1|$$

and subsequently

$$|a(t, \eta_2) - a(t, \eta_1)| \leq \mathcal{O}(1)A^6|\eta_2 - \eta_1|,$$

where $\mathcal{O}(1)$ remains bounded as $A \rightarrow 0$. This proves that the derivative exists for almost every η and is bounded by (A.11). \square

We need to estimate the pointwise difference between two functions $\tilde{\mathcal{D}}_j$, $j = 1, 2$. This is the content of the following lemma.

LEMMA A.9. *We have that*

$$\begin{aligned} |\tilde{\mathcal{D}}_1(t, \eta) - \tilde{\mathcal{D}}_2(t, \eta)| &\leq 2A^{3/2} \max_j (\tilde{\mathcal{D}}_j^{1/2})(t, \eta) |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| \\ &\quad + 2\sqrt{2}A^{3/2} \max_j (\tilde{\mathcal{D}}_j^{1/2})(t, \eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\quad + 4A^3 \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \\ &\quad + 2\sqrt{2}A^3 \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\ &\quad + \frac{3A^4}{\sqrt{2}} \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \\ &\quad + \frac{12\sqrt{2}A^4}{\sqrt{3}e} \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} d\theta \right)^{1/2} |A_1 - A_2| \\ &\quad + \tilde{\mathcal{U}}_j^2 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\ &\quad + \tilde{\mathcal{P}}_j |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\ &\quad + \frac{3A^4}{2} \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \end{aligned}$$

$$+ 6A^4 \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta |A_1 - A_2|,$$

for any value of $j = 1, 2$.

Proof. Direct calculations yield

$$\begin{aligned} & |\tilde{\mathcal{D}}_1(t, \eta) - \tilde{\mathcal{D}}_2(t, \eta)| \\ &= \left| \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \left((\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) + \frac{1}{2} A_1^5 \right) d\theta \right. \\ &\quad \left. - \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \left((\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) + \frac{1}{2} A_2^5 \right) d\theta \right| \\ &\leq \left| \int_0^\eta \left(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right) \right. \\ &\quad \times \left((\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) + \frac{1}{2} A_1^5 \right) \mathbb{1}_{B(\eta)^c}(t, \theta) d\theta \left| \right. \\ &\quad + \left| \int_0^\eta \left(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \right) \right. \\ &\quad \times \left((\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) + \frac{1}{2} A_2^5 \right) \mathbb{1}_{B(\eta)}(t, \theta) d\theta \left| \right. \\ &\quad + \left| \int_0^\eta \min_j \left(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \left((\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) + \frac{1}{2} A_1^5 \right) d\theta \right. \\ &\quad \left. - \int_0^\eta \min_j \left(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \left((\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) + \frac{1}{2} A_2^5 \right) d\theta \right| \\ &= \bar{d}_{11}(t, \eta) + \bar{d}_{12}(t, \eta) + \bar{d}_{13}(t, \eta), \end{aligned}$$

where $B(\eta)$ is defined in (5.11).

For $\bar{d}_{11}(t, \eta)$ we immediately obtain

$$\begin{aligned} \bar{d}_{11}(t, \eta) &\leq \frac{1}{A_1} \int_0^\eta (|\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)|) \\ &\quad \times e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \left((\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) + \frac{1}{2} A_1^5 \right) d\theta \\ &\leq \frac{\tilde{\mathcal{D}}_1(t, \eta)}{A_1} |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + \sqrt{2} A_1^{3/2} \tilde{\mathcal{D}}_1^{1/2}(t, \eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\ &\leq A_1^{3/2} \tilde{\mathcal{D}}_1^{1/2}(t, \eta) |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + \sqrt{2} A_1^{3/2} \tilde{\mathcal{D}}_1^{1/2}(t, \eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|. \end{aligned} \tag{A.12}$$

Following the same lines, we end up with

$$\begin{aligned} \bar{d}_{12}(t, \eta) &\leq \frac{1}{A_2} \int_0^\eta (|\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + |\tilde{\mathcal{Y}}_1(t, \theta) - \tilde{\mathcal{Y}}_2(t, \theta)|) \\ &\quad \times e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t, \eta) - \tilde{\mathcal{Y}}_2(t, \theta))} \left| (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) + \frac{1}{2} A_2^5 \right| d\theta \\ &\leq A_2^{3/2} \tilde{\mathcal{D}}_2^{1/2}(t, \eta) |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| + \sqrt{2} A_2^{3/2} \tilde{\mathcal{D}}_2^{1/2}(t, \eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|. \end{aligned} \quad (\text{A.13})$$

The last term $\bar{d}_{13}(t, \eta)$ needs a bit more work. Indeed,

$$\begin{aligned} \bar{d}_{13}(t, \eta) &= \left| \int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \left((\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) + \frac{1}{2} A_1^5 \right) d\theta \right. \\ &\quad \left. - \int_0^\eta \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \left((\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2, \eta}(t, \theta) + \frac{1}{2} A_2^5 \right) d\theta \right| \\ &\leq \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta)) d\theta \right| \\ &\quad + \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta)) d\theta \right| \\ &\quad + \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \frac{1}{2} A_1^5 \right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \frac{1}{2} A_2^5) d\theta \right| \\ &= \bar{T}_1(t, \eta) + \bar{T}_2(t, \eta) + \bar{T}_3(t, \eta). \end{aligned}$$

To estimate $\bar{T}_1(t, \eta)$, recall (3.40), (A.4), (A.5), and Lemma A.1 (ii), which imply

$$\begin{aligned} \bar{T}_1(t, \eta) &= \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2, \eta}(t, \theta)) d\theta \right| \\ &\leq \left| \int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t, \eta) - \tilde{\mathcal{Y}}_j(t, \theta))}) (\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{U}}_2^2) \tilde{\mathcal{Y}}_{1, \eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^2 \leq \tilde{\mathcal{U}}_1^2}(t, \theta) d\theta \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^\eta \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{U}}_2^2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^2 < \tilde{\mathcal{U}}_2^2}(t, \theta) d\theta \right| \\
& + \mathbb{1}_{A_1 \leq A_2} \left| \int_0^\eta \left(\min_j (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
& \quad \left. \left. - \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j (\tilde{\mathcal{U}}_j^2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right| \\
& + \mathbb{1}_{A_2 < A_1} \left| \int_0^\eta \left(\min_j (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \right. \\
& \quad \left. \left. - \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right) \min_j (\tilde{\mathcal{U}}_j^2) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right| \\
& + \left| \int_0^\eta \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2) (\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta})(t, \theta) d\theta \right| \\
& \leq 2 \int_0^\eta \min_j (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{U}}_1| \tilde{\mathcal{Y}}_{1,\eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
& + 2 \int_0^\eta \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta} |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t, \theta) d\theta \\
& + \mathbb{1}_{A_1 \leq A_2} \frac{4}{ae} \left| \int_0^\eta \min_j (e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right| |A_1 - A_2| \\
& + \mathbb{1}_{A_2 < A_1} \frac{4}{ae} \left| \int_0^\eta \min_j (e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right| |A_1 - A_2| \\
& + \left| \min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2) (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \right|_{\theta=0}^\eta \\
& - \int_0^\eta \frac{d}{d\theta} \left(\min_j (e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^2) (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) d\theta \right) \\
& \leq 2 \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
& + 2 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
& + \frac{2\sqrt{2}A}{e} \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \quad \times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2|
\end{aligned}$$

$$\begin{aligned}
& + \frac{2\sqrt{2}A}{e} \left(\int_0^\eta e^{-\frac{3}{4A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
& + \tilde{\mathcal{U}}_j^2 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
& + \frac{1}{a} \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^2)(\tilde{\mathcal{Y}}_{1,\eta} + \tilde{\mathcal{Y}}_{2,\eta}) |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \\
& + A^4 \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \\
& \leq 4A^3 \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
& + \frac{8\sqrt{2}A^4}{\sqrt{3}e} \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
& + \tilde{\mathcal{U}}_j^2 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
& + \sqrt{2}A^4 \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
& + A^4 \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta. \tag{A.14}
\end{aligned}$$

Following the same lines, one has

$$\begin{aligned}
\bar{T}_2(t, \eta) & = \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \right. \\
& \quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t, \theta)) d\theta \right| \\
& \leq \left| \int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{P}}_1 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_2 \leq \tilde{\mathcal{P}}_1}(t, \theta) d\theta \right| \\
& \quad + \left| \int_0^\eta \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{P}}_1 - \tilde{\mathcal{P}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) \mathbb{1}_{\tilde{\mathcal{P}}_1 < \tilde{\mathcal{P}}_2}(t, \theta) d\theta \right| \\
& \quad + \mathbb{1}_{A_1 \leq A_2} \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
& \quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right| \\
& \quad + \mathbb{1}_{A_2 < A_1} \left| \int_0^\eta (\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
& \quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right|
\end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta})(t, \theta) d\theta \right| \\
& \leq 2 \int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{P}}_1^{1/2} |\tilde{\mathcal{Y}}_{1,\eta}| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \theta) d\theta \\
& + 2 \int_0^\eta \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) |\tilde{\mathcal{P}}_2^{1/2} |\tilde{\mathcal{Y}}_{2,\eta}| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t, \theta) d\theta \\
& + \mathbb{1}_{A_1 \leq A_2} \frac{4}{ae} \left| \int_0^\eta \min_j(e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{2,\eta}(t, \theta) d\theta \right| |A_1 - A_2| \\
& + \mathbb{1}_{A_2 < A_1} \frac{4}{ae} \left| \int_0^\eta \min_j(e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \tilde{\mathcal{Y}}_{1,\eta}(t, \theta) d\theta \right| |A_1 - A_2| \\
& + \left| \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \right|_{\theta=0}^\eta \\
& - \int_0^\eta \frac{d}{d\theta} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)) (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t, \theta) d\theta \Big| \\
& \leq 2 \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\
& + 2 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\
& + \frac{2A}{e} \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
& + \frac{2A}{e} \left(\int_0^\eta e^{-\frac{3}{4A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}^2(t, \theta) d\theta \right)^{1/2} \\
& \times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
& + \tilde{\mathcal{P}}_j |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
& + \frac{1}{a} \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{Y}}_{1,\eta} + \tilde{\mathcal{Y}}_{2,\eta}) |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \\
& + \frac{A^4}{2} \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta
\end{aligned}$$

$$\begin{aligned}
&\leq 2\sqrt{2}A^3 \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\
&\quad + \frac{4\sqrt{2}A^4}{\sqrt{3}e} \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
&\quad + \tilde{\mathcal{P}}_j |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
&\quad + \frac{A^4}{\sqrt{2}} \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
&\quad + \frac{A^4}{2} \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta. \tag{A.15}
\end{aligned}$$

For the last term $\bar{T}_3(t, \eta)$, we have

$$\begin{aligned}
2\bar{T}_3(t, \eta) &= \left| \int_0^\eta \left(\min_j \left(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) A_1^5 - \left(\min_j \left(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) A_2^5 \right) d\theta \right| \\
&\leq \mathbb{1}_{A_1 \leq A_2} \left| \int_0^\eta \left(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) (A_1^5 - A_2^5) d\theta \right| \\
&\quad + \mathbb{1}_{A_2 < A_1} \left| \int_0^\eta \left(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) (A_1^5 - A_2^5) d\theta \right| \\
&\quad + a^5 \int_0^\eta \left| \min_j \left(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) - \min_j \left(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} \right) \right| d\theta \\
&\leq 10A^4 \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta |A_1 - A_2| \\
&\quad + \frac{4a^4}{e} \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta |A_1 - A_2| \\
&\leq 12A^4 \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta |A_1 - A_2|. \tag{A.16}
\end{aligned}$$

Thus we end up with

$$\begin{aligned}
|\tilde{\mathcal{D}}_1(t, \eta) - \tilde{\mathcal{D}}_2(t, \eta)| &\leq 2A^{3/2} \max_j (\tilde{\mathcal{D}}_j^{1/2})(t, \eta) |\tilde{\mathcal{Y}}_1(t, \eta) - \tilde{\mathcal{Y}}_2(t, \eta)| \\
&\quad + 2\sqrt{2}A^{3/2} \max_j (\tilde{\mathcal{D}}_j^{1/2})(t, \eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\quad + 4A^3 \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t, \theta) d\theta \right)^{1/2} \\
&\quad + 2\sqrt{2}A^3 \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t, \theta) d\theta \right)^{1/2} \\
&\quad + \frac{3A^4}{\sqrt{2}} \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t, \theta) d\theta \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{12\sqrt{2}A^4}{\sqrt{3}e} \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta \right)^{1/2} |A_1 - A_2| \\
& + \tilde{\mathcal{U}}_j^2 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
& + \tilde{\mathcal{P}}_j |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \eta) \\
& + \frac{3A^4}{2} \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t, \theta) d\theta \\
& + 6A^4 \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))} d\theta |A_1 - A_2|,
\end{aligned}$$

which proves the lemma. \square

LEMMA A.10. *We have the following estimates:*

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 2A \tilde{\mathcal{P}}(t, \eta), \quad (\text{A.17})$$

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{H}}_\eta(t, \theta) d\theta \leq 4A \tilde{\mathcal{P}}(t, \eta), \quad (\text{A.18})$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{U}}^2(t, \theta) d\theta \leq 6\tilde{\mathcal{P}}(t, \eta), \quad (\text{A.19})$$

$$\int_0^\eta e^{-\frac{5}{4A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 4A \tilde{\mathcal{P}}(t, \eta), \quad (\text{A.20})$$

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}(t, \theta) d\theta \leq 7\tilde{\mathcal{P}}(t, \eta), \quad (\text{A.21})$$

$$\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq A\mathcal{O}(1) \tilde{\mathcal{P}}^{1/2}(t, \eta), \quad (\text{A.22})$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}^{1+\beta} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 3 \frac{1+\beta}{\beta} \frac{A^{1+4\beta}}{4^\beta} \tilde{\mathcal{P}}(t, \eta), \quad \beta > 0. \quad (\text{A.23})$$

Proof. The proof of (A.17) goes as follows:

$$\begin{aligned}
& \int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\
& = \frac{2}{3} A \tilde{\mathcal{P}}(t, \eta) - \frac{2}{3A} \int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{Q}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\
& \leq \frac{2}{3} A \tilde{\mathcal{P}}(t, \eta) + \frac{2}{3} \int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta,
\end{aligned}$$

which implies

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 2A \tilde{\mathcal{P}}(t, \eta).$$

Next, we use that

$$\begin{aligned} \int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{H}}_\eta(t, \theta) d\theta &\leq \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{H}}_\eta(t, \theta) d\theta \\ &\leq 4A \tilde{\mathcal{P}}(t, \eta) \end{aligned}$$

(see (4.16d)), showing (A.18).

For (A.19) we find

$$\begin{aligned} &\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{U}}^2(t, \theta) d\theta \\ &= \theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{U}}^2(t, \theta) \Big|_{\theta=0}^\eta \\ &\quad - \int_0^\eta \theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \left(\frac{1}{A} \tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta + 2\tilde{\mathcal{U}} \tilde{\mathcal{U}}_\eta \right) (t, \theta) d\theta \\ &= \eta \tilde{\mathcal{U}}^2(t, \eta) - \int_0^\eta \theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \left(\frac{1}{A} \tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta + 2\tilde{\mathcal{U}} \tilde{\mathcal{U}}_\eta \right) (t, \theta) d\theta \\ &\leq \tilde{\mathcal{U}}^2(t, \eta) + \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \left(\frac{1}{A} \tilde{\mathcal{U}}^2 \tilde{\mathcal{Y}}_\eta + 2|\tilde{\mathcal{U}} \tilde{\mathcal{U}}_\eta| \right) (t, \theta) d\theta \\ &\leq \tilde{\mathcal{U}}^2(t, \eta) + 4\tilde{\mathcal{P}}(t, \eta) \\ &\leq 6\tilde{\mathcal{P}}(t, \eta). \end{aligned}$$

The proof of (A.20),

$$\int_0^\eta e^{-\frac{5}{4A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \leq 4A \tilde{\mathcal{P}}(t, \eta),$$

follows in the same manner as (A.17).

Furthermore, for (A.21) we find

$$\begin{aligned} &\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}(t, \theta) d\theta \\ &= \theta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \tilde{\mathcal{P}}(t, \theta) \Big|_{\theta=0}^\eta \\ &\quad - \int_0^\eta \theta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))} \left(\frac{3}{2A} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta + \frac{1}{A^2} \tilde{\mathcal{Q}} \tilde{\mathcal{Y}}_\eta \right) (t, \theta) d\theta \end{aligned}$$

$$\begin{aligned} &\leq \eta \tilde{\mathcal{P}}(t, \eta) + \frac{3}{A} \int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &\leq 7\tilde{\mathcal{P}}(t, \eta), \end{aligned}$$

which follows from (A.17).

In order to prove (A.22), we do as follows:

$$\begin{aligned} &\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &= 2A\tilde{\mathcal{P}}^2(t, \eta) - 8\tilde{\mathcal{P}}\tilde{\mathcal{Q}}(t, \eta) \\ &\quad + 8 \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \left(\tilde{\mathcal{P}}^2 + \frac{1}{A^2} \tilde{\mathcal{Q}}^2 \right) \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &\quad + 8 \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \left((\tilde{\mathcal{P}} - \tilde{\mathcal{U}}^2) \tilde{\mathcal{Y}}_\eta(t, \theta) - \frac{1}{2} A^5 \right) \tilde{\mathcal{P}}(t, \theta) d\theta, \end{aligned}$$

and hence

$$\begin{aligned} &\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &\leq 8\tilde{\mathcal{P}}\tilde{\mathcal{Q}}(t, \eta) + 8 \int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \left((\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \theta) + \frac{1}{2} A^5 \right) \tilde{\mathcal{P}}(t, \theta) d\theta \\ &\leq 8A\tilde{\mathcal{P}}^2(t, \eta) + 8 \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \left((\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \theta) + \frac{1}{2} A^5 \right) d\theta \right)^{1/2} \\ &\quad \times \left(\int_0^\eta \left((\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}}) \tilde{\mathcal{Y}}_\eta(t, \theta) + \frac{1}{2} A^5 \right) \tilde{\mathcal{P}}^2(t, \theta) d\theta \right)^{1/2} \\ &\leq A\mathcal{O}(1)\tilde{\mathcal{P}}^{1/2}(t, \eta). \end{aligned}$$

As for the proof of (A.23), we proceed as follows:

$$\begin{aligned} &\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^{1+\beta} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &= A e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^{1+\beta}(t, \theta) \Big|_{\theta=0}^\eta \\ &\quad - A(1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^\beta \tilde{\mathcal{P}}_\eta(t, \theta) d\theta \\ &= A\tilde{\mathcal{P}}^{1+\beta}(t, \eta) - \frac{1}{A}(1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\ &= A\tilde{\mathcal{P}}^{1+\beta}(t, \eta) - (1 + \beta) e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}(t, \theta) \Big|_{\theta=0}^\eta \\ &\quad + (1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} (\beta \tilde{\mathcal{P}}^{\beta-1} \tilde{\mathcal{P}}_\eta \tilde{\mathcal{Q}} + \tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}_\eta)(t, \theta) d\theta \end{aligned}$$

$$\begin{aligned}
 &= A\tilde{\mathcal{P}}^{1+\beta}(t, \eta) - (1 + \beta)\tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}(t, \eta) \\
 &\quad + (1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \left(\beta \tilde{\mathcal{P}}^{\beta-1} \tilde{\mathcal{Q}}^2 \tilde{\mathcal{Y}}_\eta \frac{1}{A^2} + \tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}_\eta \right) (t, \theta) d\theta.
 \end{aligned} \tag{A.24}$$

Recall (4.6), (4.9), and (4.10), which together imply

$$\tilde{\mathcal{Q}}_\eta(t, \eta) = \frac{1}{2}(\tilde{\mathcal{E}}_\eta - \tilde{\mathcal{D}}_\eta)(t, \eta) = -(\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta(t, \eta) - \frac{1}{2}A^5 + \tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t, \eta).$$

Thus

$$\begin{aligned}
 &\beta \tilde{\mathcal{P}}^{\beta-1} \tilde{\mathcal{Q}}^2 \tilde{\mathcal{Y}}_\eta \frac{1}{A^2} + \tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}_\eta \\
 &= \frac{\beta}{A^2} \tilde{\mathcal{Q}}^2 \tilde{\mathcal{P}}^{\beta-1} \tilde{\mathcal{Y}}_\eta + \tilde{\mathcal{P}}^{\beta+1} \tilde{\mathcal{Y}}_\eta + \tilde{\mathcal{P}}^\beta \left(-(\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta - \frac{1}{2}A^5 \right).
 \end{aligned}$$

Inserting this expression in (A.24) and re-ordering the terms, we find

$$\begin{aligned}
 &\beta \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^{1+\beta} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\
 &\quad + \frac{1 + \beta}{A^2} \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^{\beta-1} \tilde{\mathcal{Q}}^2 \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\
 &= (1 + \beta)\tilde{\mathcal{P}}^\beta \tilde{\mathcal{Q}}(t, \eta) - A\tilde{\mathcal{P}}^{1+\beta}(t, \eta) \\
 &\quad + (1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^\beta \left((\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta + \frac{1}{2}A^5 \right) (t, \theta) d\theta.
 \end{aligned}$$

Estimating this, using (4.5a), we find

$$\begin{aligned}
 &\beta \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^{1+\beta} \tilde{\mathcal{Y}}_\eta(t, \theta) d\theta \\
 &\leq (1 + \beta)\tilde{\mathcal{P}}^\beta |\tilde{\mathcal{Q}}|(t, \eta) \\
 &\quad + (1 + \beta) \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \tilde{\mathcal{P}}^\beta \left((\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta + \frac{1}{2}A^5 \right) (t, \theta) d\theta \\
 &\leq (1 + \beta)\|\tilde{\mathcal{P}}\|_\infty^\beta A\tilde{\mathcal{P}}(t, \eta) \\
 &\quad + (1 + \beta)\|\tilde{\mathcal{P}}\|_\infty^\beta \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t, \eta) - \tilde{\mathcal{Y}}(t, \theta))} \left((\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta + \frac{1}{2}A^5 \right) (t, \theta) d\theta \\
 &\leq (1 + \beta)\|\tilde{\mathcal{P}}\|_\infty^\beta A\tilde{\mathcal{P}}(t, \eta) + (1 + \beta)\|\tilde{\mathcal{P}}\|_\infty^\beta \tilde{\mathcal{D}}(t, \eta) \\
 &\leq 3(1 + \beta) \left(\frac{A^4}{4} \right)^\beta A\tilde{\mathcal{P}}(t, \eta),
 \end{aligned}$$

where we used (4.8), (4.15a), and (4.15c). □

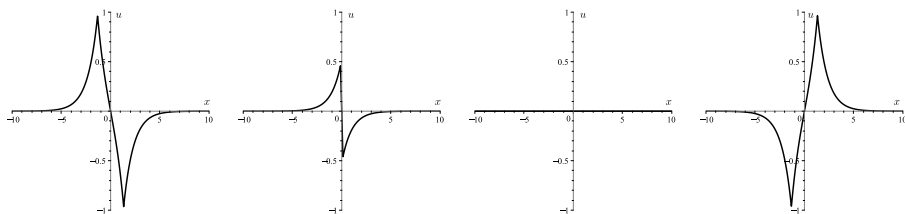


Figure B.3. Time evolution of $u(t, x)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

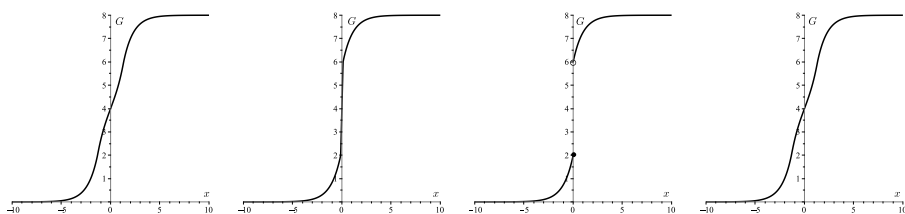


Figure B.4. Time evolution of $G(t, x)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

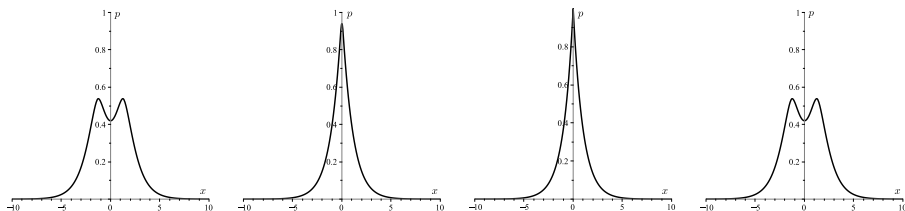


Figure B.5. Time evolution of $p(t, x)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

Appendix B. The antisymmetric peak–antipeakon example

Fortunately, one can compute explicitly the quantities described in this paper in the important case of an antisymmetric peak–antipeakon solution. The various functions are depicted on Figures B.3–B.14.

More precisely, consider the function [41]

$$u(t, x) = \begin{cases} \beta(t) \sinh(x), & |x| \leq \gamma(t), \\ \text{sign}(x)\alpha(t)e^{-|x|}, & |x| \geq \gamma(t), \end{cases}$$

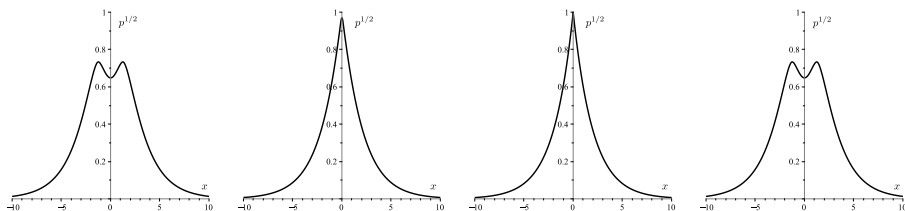


Figure B.6. Time evolution of $p^{1/2}(t, x)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

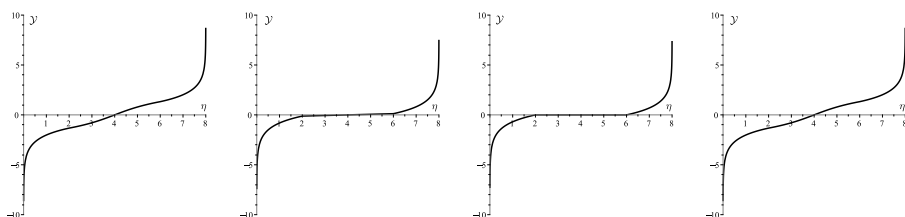


Figure B.7. Time evolution of $\mathcal{Y}(t, \eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

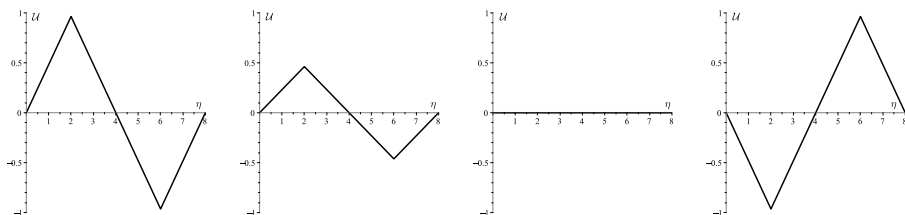


Figure B.8. Time evolution of $\mathcal{U}(t, \eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

$$= \begin{cases} -\alpha(t)e^x, & x \leq -\gamma(t), \\ \beta(t) \sinh(x), & -\gamma(t) \leq x \leq \gamma(t), \\ \alpha(t)e^{-x}, & \gamma(t) \leq x, \end{cases}$$

where

$$\alpha(t) = \frac{E}{2} \sinh\left(\frac{E}{2}(t - t_0)\right), \quad \beta(t) = E \frac{1}{\sinh\left(\frac{E}{2}(t - t_0)\right)},$$

$$\gamma(t) = \ln\left(\cosh\left(\frac{E}{2}(t - t_0)\right)\right).$$

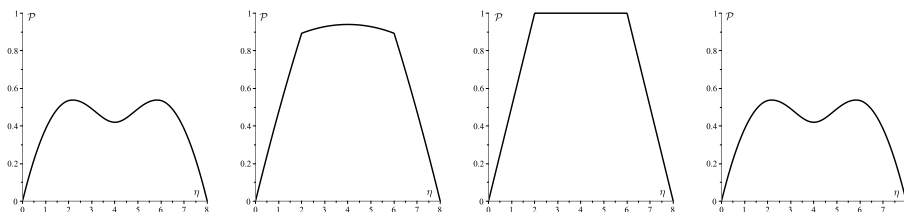


Figure B.9. Time evolution of $\mathcal{P}(t, \eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

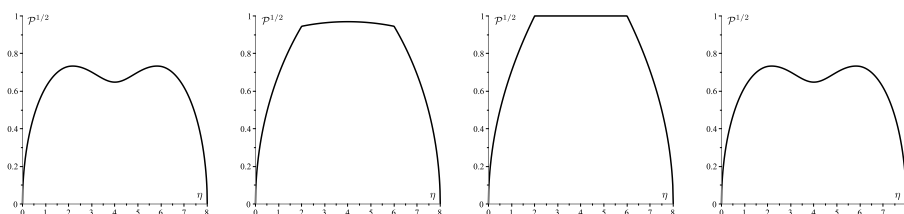


Figure B.10. Time evolution of $\mathcal{P}^{1/2}(t, \eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

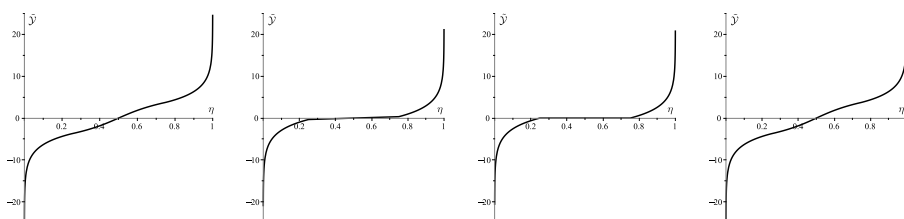


Figure B.11. Time evolution of $\tilde{\mathcal{Y}}(t, \eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

Here $E = \|u(t)\|_{H^1}$, $t \neq t_0$, denotes the total energy. The corresponding energy density is given by

$$\mu(t, x) = (u^2 + u_x^2)(t, x) = \begin{cases} 2\alpha^2(t)e^{2x}, & x \leq -\gamma(t), \\ \beta^2(t) \cosh(2x), & -\gamma(t) \leq x \leq \gamma(t), \\ 2\alpha^2(t)e^{-2x}, & \gamma(t) \leq x, \end{cases} \quad t \neq t_0,$$

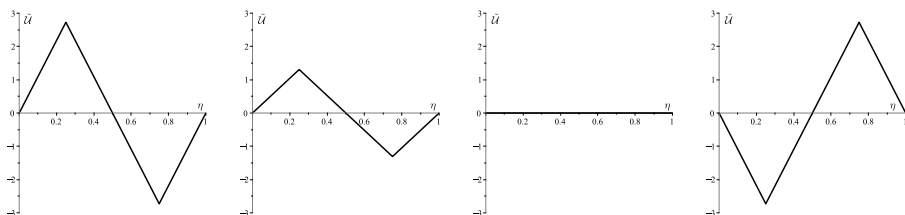


Figure B.12. Time evolution of $\tilde{U}(t, \eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

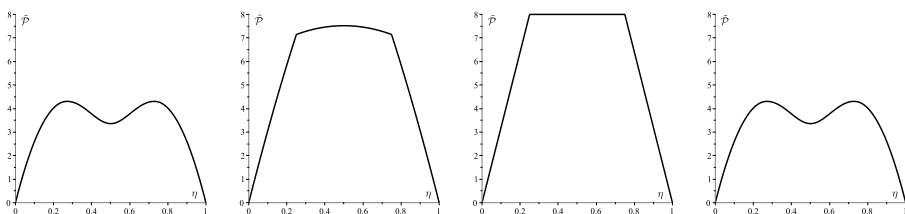


Figure B.13. Time evolution of $\tilde{P}(t, \eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

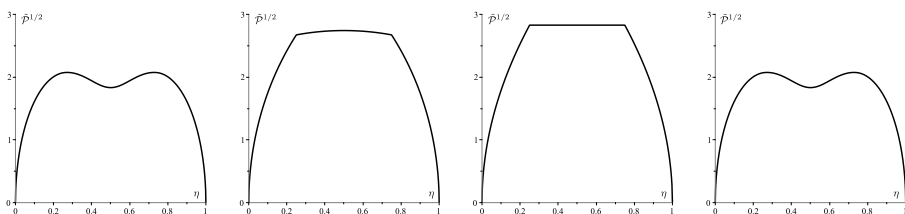


Figure B.14. Time evolution of $\tilde{P}^{1/2}(t, \eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0, 1.5, 2, 4$.

with $\mu(t_0, x) = E^2 \delta_0(x)$ for $t = t_0$. Hence $C = \mu(t, \mathbb{R}) = E^2$, and

$$F(t, x) = \begin{cases} \alpha(t)^2 e^{2x}, & x < -\gamma(t), \\ \frac{E^2}{4} \tanh^2\left(\frac{E}{2}(t - t_0)\right), & x = -\gamma(t), \\ \frac{1}{2} E^2 + \frac{1}{2} \beta(t)^2 \sinh(2x), & -\gamma(t) < x < \gamma(t), \\ E^2 - \frac{E^2}{4} \tanh^2\left(\frac{E}{2}(t - t_0)\right), & x = \gamma(t), \\ E^2 - \alpha(t)^2 e^{-2x}, & \gamma(t) < x. \end{cases}$$

In particular, this solution experiences wave breaking at time $t = t_0$, that is, $u_x(t, 0)$ tends to $-\infty$ as $t \rightarrow t_0^-$ and

$$F(t_0^-, x) = \begin{cases} 0, & x \leq 0, \\ E^2, & 0 < x. \end{cases}$$

The corresponding function $p_x(t, x)$, which can be computed using $p_x(t, x) = -u_t(t, x) - uu_x(t, x)$, is given by

$$p_x(t, x) = \begin{cases} \alpha'(t)e^x - \alpha(t)^2 e^{2x}, & x < -\gamma(t), \\ \frac{E^2}{4} - \frac{E^2}{4} \tanh^2\left(\frac{E}{2}(t - t_0)\right), & x = -\gamma(t), \\ -\beta'(t) \sinh(x) - \frac{1}{2}\beta(t)^2 \sinh(2x), & -\gamma(t) < x < \gamma(t), \\ -\frac{E^2}{4} + \frac{E^2}{4} \tanh^2\left(\frac{E}{2}(t - t_0)\right), & \gamma(t) = x, \\ -\alpha'(t)e^{-x} + \alpha(t)^2 e^{-2x}, & \gamma(t) < x. \end{cases}$$

In particular, one obtains at wave breaking time $t = t_0$ that

$$p_x(t_0^-, x) = \begin{cases} \frac{E^2}{4} e^x, & x < 0, \\ -\frac{E^2}{4} e^{-x}, & 0 < x. \end{cases}$$

Here it is important to note that $p_x(t_0^-, x)$ has a (negative) jump of height $-\frac{E^2}{2}$ at $x = 0$ at time $t = t_0$. For all other points $x \in \mathbb{R}$, the function $p_x(t_0^-, x)$ is continuous.

Thus the function $G(t, x) = 2p_x(t, x) + 2F(t, x)$ is given by

$$G(t, x) = \begin{cases} 2\alpha'(t)e^x, & x < -\gamma(t), \\ \frac{E^2}{2}, & x = -\gamma(t), \\ E^2 - 2\beta'(t) \sinh(x), & -\gamma(t) < x < \gamma(t), \\ \frac{3E^2}{2}, & x = \gamma(t), \\ 2E^2 - 2\alpha'(t)e^{-x}, & \gamma(t) < x. \end{cases}$$

In particular, one observes that $G_x(t, x) > 0$ for all $x \in \mathbb{R}$,

$$\lim_{x \rightarrow -\infty} G(t, x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} G(t, x) = 2E^2 = 2C.$$

Thus the limits at $\pm\infty$ are independent of time. Moreover, one has

$$G(t_0-, x) = \begin{cases} \frac{E^2}{2} e^x, & x \leq 0, \\ 2E^2 - \frac{E^2}{2} e^{-x}, & 0 < x. \end{cases}$$

Here it is important to note that $G(t_0-, x)$ has a jump of size E^2 due to the wave breaking at $t = t_0$, that is, $\mu(t_0, x) = E^2 \delta_0(x)$.

Direct computations using $p = p_{xx} + \frac{1}{2}u^2 + \frac{1}{2}d\mu$ yield

$$p(t, x) = \begin{cases} \alpha'(t)e^x - \frac{1}{2}\alpha(t)^2 e^{2x}, & x \leq -\gamma(t), \\ -\beta'(t) \cosh(x) - \frac{1}{2}\beta(t)^2 \cosh^2(x), & -\gamma(t) \leq x \leq \gamma(t), \\ \alpha'(t)e^{-x} - \frac{1}{2}\alpha(t)^2 e^{-2x}, & \gamma(t) \leq x. \end{cases}$$

In the new coordinates, using $G(t, \mathcal{Y}(t, \eta)) = \eta$, the solution reads as

$$\mathcal{Y}(t, \eta) = \begin{cases} \ln\left(\frac{\eta}{2\alpha'(t)}\right), & 0 < \eta \leq \frac{E^2}{2}, \\ \sinh^{-1}\left(\frac{E^2 - \eta}{2\beta'(t)}\right), & \frac{E^2}{2} \leq \eta \leq \frac{3E^2}{2}, \\ \ln\left(\frac{2\alpha'(t)}{2E^2 - \eta}\right), & \frac{3E^2}{2} \leq \eta < 2E^2, \end{cases}$$

and, applying $\mathcal{U}(t, \eta) = u(t, \mathcal{Y}(t, \eta))$,

$$\mathcal{U}(t, \eta) = \begin{cases} -\frac{\alpha(t)}{2\alpha'(t)}\eta, & 0 < \eta \leq \frac{E^2}{2}, \\ \frac{\beta(t)}{2\beta'(t)}(E^2 - \eta), & \frac{E^2}{2} \leq \eta \leq \frac{3E^2}{2}, \\ \frac{\alpha(t)}{2\alpha'(t)}(2E^2 - \eta), & \frac{3E^2}{2} \leq \eta < 2E^2, \end{cases}$$

and, by invoking $\mathcal{P}(t, \eta) = p(t, \mathcal{Y}(t, \eta))$,

$$\mathcal{P}(t, \eta) = \begin{cases} \frac{1}{2}\eta - \frac{\alpha(t)^2}{8\alpha'(t)^2}\eta^2, & 0 < \eta \leq \frac{E^2}{2}, \\ -\beta'(t)\sqrt{1 + \frac{(E^2 - \eta)^2}{4\beta'(t)^2}} \\ - \frac{1}{2}\beta(t)^2\left(1 + \frac{(E^2 - \eta)^2}{4\beta'(t)^2}\right), & \frac{E^2}{2} \leq \eta \leq \frac{3E^2}{2}, \\ \frac{1}{2}(2E^2 - \eta) - \frac{\alpha(t)^2}{8\alpha'(t)^2}(2E^2 - \eta)^2, & \frac{3E^2}{2} \leq \eta < 2E^2, \end{cases}$$

where we used the convention that $\sinh^{-1}(x)$ denotes the inverse of $\sinh(x)$.

For the scaled quantities we find, when we introduce $A = \sqrt{2C} = \sqrt{2}E$, that

$$\tilde{\mathcal{Y}}(t, \eta) = A\mathcal{Y}(t, A^2\eta) = \sqrt{2C} \begin{cases} \ln\left(\frac{C\eta}{\alpha'(t)}\right), & 0 < \eta \leq \frac{1}{4}, \\ \sinh^{-1}\left(\frac{C(1-2\eta)}{2\beta'(t)}\right), & \frac{1}{4} \leq \eta \leq \frac{3}{4}, \\ \ln\left(\frac{\alpha'(t)}{C(1-\eta)}\right), & \frac{3}{4} \leq \eta < 1, \end{cases}$$

$$\tilde{\mathcal{U}}(t, \eta) = A\mathcal{U}(t, A^2\eta) = \sqrt{2C} \begin{cases} -\frac{\alpha(t)}{\alpha'(t)}C\eta, & 0 < \eta \leq \frac{1}{4}, \\ \frac{\beta(t)}{2\beta'(t)}C(1-2\eta), & \frac{1}{4} \leq \eta \leq \frac{3}{4}, \\ \frac{\alpha(t)}{\alpha'(t)}C(1-\eta), & \frac{3}{4} \leq \eta < 1, \end{cases}$$

$$\tilde{\mathcal{P}}(t, \eta) = A^2\mathcal{P}(t, A^2\eta)$$

$$= 2C \begin{cases} C\eta - \frac{\alpha(t)^2}{2\alpha'(t)^2}C^2\eta^2, & 0 < \eta \leq \frac{1}{4}, \\ -\beta'(t)\sqrt{1 + \frac{C^2(1-2\eta)^2}{4\beta'(t)^2}} \\ - \frac{1}{2}\beta(t)^2\left(1 + \frac{C^2(1-2\eta)^2}{4\beta'(t)^2}\right), & \frac{1}{4} \leq \eta \leq \frac{3}{4}, \\ C(1-\eta) - \frac{\alpha(t)^2}{2\alpha'(t)^2}C^2(1-\eta)^2, & \frac{3}{4} \leq \eta < 1. \end{cases}$$

Conflict of Interest: None.

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