Using variation tasks to investigate learners' attention to structure in the area of proving

Jason Bentley

DPhil Student

Department of Education

University of Oxford

St Stephen's House

Supervisors:

Prof. Gabriel Stylianides

Prof. Jenni Ingram

Thesis submitted for the degree of Doctor of Philosophy in Education

October 2025

Word count: 73224

Acknowledgements

I wish to thank my supervisors, Professor Gabriel Stylianides (Lead) and Professor Jenni Ingram, for their outstanding support, resourcefulness, and critical insights throughout my DPhil research. Their constructive feedback and attention to detail have been of great influence to my thesis. My thanks also go to the many departmental lecturers whose specialised teachings have shaped my work; and to the administrative staff whose essential assistance I could always rely upon. Finally, my sincere gratitude goes to my family for their flexibility and patience during my DPhil journey. Their sacrifices, silent as they often were, will always be deeply appreciated.

Abstract

Proving is fundamental to the learning of mathematics because the ability to prove involves understanding, identifying, and expressing generalities. Working with generalities, i.e., generalising, lies at the heart of learning, and hence teaching, mathematics. Diversity across the designs and demands of proving tasks makes identifying cross-task structural and algorithmic generalities difficult. Consequently, no overall learning or teaching approach can be adopted. This pedagogical challenge might explain why, in secondary schools, proof is often taught in isolation to other mathematical activities, i.e., on a proof- or task-specific basis. These isolated teaching approaches can provide poor preparation for proof production because they do not first address the kinds of mathematical actions that proving can involve.

In this study I posit ways to alleviate two key problems that can arise from the challenges of proof pedagogy: (1) insufficient scaffolding, and (2) indistinct characterisation of learners' proving actions. To address insufficient scaffolding, I designed sequences of slightly varied (consecutive number) proof arguments and accompanying prompts, which I call *proof variation tasks*. I used the variation tasks to encourage learners to perform the kinds of mathematical actions that can underpin the construction of proofs. I designed an isomorphic pair of variation tasks in each of two distinct domains: number and geometry. Each domain-pair of variation tasks comprised an *initial* task and a conceptually similar but more challenging *transfer* task. To address the indistinct characterisation of learners' proving actions, I formulated the GOLDEN framework. The acronym GOLDEN captures the six broad actions that learners similarly performed on each domain-pair of variation tasks: *Generalising*, *Organising*, *Localising*, *Deconstructing*, *Extending* and *Networking*.

I gave the variation tasks to three Year 8 learners and three Year 10 learners (12- to 13-year-olds and 14- to 15-year-olds, respectively) all of whom were new to any formal notion of proof and argumentation. The learners engaged with the variation tasks in videoed semi-structured interviews that I conducted in the dual role of teacher/researcher. I used the GOLDEN framework to describe and compare learners' actions on the initial and transfer variation tasks within and across each of the two domains.

I provide evidence that consecutive-number variation tasks which feature structurally similar arguments can be used to encourage learners to perform a range of proving actions, thereby supporting their development in the area of proof. I also introduce the GOLDEN framework as a task-generic analytical tool for describing and identifying learners' proving actions. Suggestions for future research, and implications for advancing pedagogy are discussed.

Keywords: argumentation; learning transfer; proof; proving; structure; variation theory.

Contents

List of Figures

List of Tables

Chapter 1: Introduction

In this opening chapter I argue that proving is fundamentally important for the learning and teaching of mathematics because the kinds of activities involved in producing a proof (e.g., generalising) can find ubiquitous expression across the pedagogy of the discipline. I then raise some of the persistent problems in the learning and teaching of proof that have been reported in the literature. I follow this by introducing the key theoretical concepts that I drew on to formulate my aims and research questions. My aims centre on generating resources to alleviate two resultant challenges of the pedagogical problems that I raise. My research questions narrowed my focus and motivated my design of: (1) nuanced tasks in each of two distinct domains (number and geometry); and (2) an analytical framework for describing learners' mathematical actions as they worked on the nuanced tasks. I used the tasks and the analytical framework that I designed to respond to four research questions, which I state at the end of this chapter.

1.1 The importance and challenges of proof pedagogy

Proof is a key construct in mathematics curricula because the range of activities required to produce proofs, i.e., to prove, pervade the pedagogy of mathematics in general. There is, though, a longstanding consensus that proof is hard-to-teach and hard-to-learn (Knuth et al., 2009, p. 153; Küchemann & Hoyles, 2009, p. 171; Stylianides & Stylianides, 2017, p. 121). Key factors that contribute to this viewpoint include: (1) teachers' often limited knowledge of how to develop learners' ability to prove (Harel & Sowder, 2007, pp. 836-837; Stylianides & Stylianides, 2018, p. 110); (2) delaying the teaching of proof until the latter years of secondary school (Knuth, 2002, p. 61; Stylianides & Stylianides, 2009a, pp. 237–238); (3) learners put in situations where they are suddenly expected to prove (Ball et al., 2002, pp. 207–208), i.e., without teachers' prior consideration of processes that can precede proof production; and (4) teaching proof in isolation from other mathematical activities (Stylianides & Stylianides, 2006a, p. 202). While these restrictions on proof pedagogy are not mutually exclusive, I delineate them here as a backdrop and prelude to my overarching goal of suggesting ways in which they might each be eased. To make progress on easing these restrictions, I drew on three key constructs: (1) learners' attention to mathematical structure (attention to structure), (2) the role of variation in mathematical task design (variation theory), and (3) the transfer of learning from a one situation to a subsequent situation (transfer theory). I next summarise each of these constructs and explain their relevance to my study.

1.2 Summary of key constructs

Attention to structure, i.e., learners' tasked-based actions (for example, testing a value, rearranging an equation, and representing algebraically). I use the term *action* as a proxim for a learner's attention to mathematical structure (hereafter 'structure'). With 'action' I refer to a mathematical act that a learner performs; whereas, 'attention to structure' refers to both an action (the attention paid) and the notation or diagram on which an action is performed (the structure). For me, learners' attention to structure finds expression in two often entwinned yet distinct types of actions: acts of mental reasoning (e.g., a verbal explanation of an identified numerical relationship) and acts of physical inscription (e.g., the written work that a learner produces as they engage with a task).

In my study I described the range of actions that learners performed as they worked on the proof variation tasks that I designed. Categorising learners' actions led to my formulation of an attention-to-structure framework. The range of actions in the framework, perhaps partially, can be used to aid teachers' articulation, identification, and analysis of learners' attention to structure on any argument- or proof-based task. Hence, the framework has potential to provide teachers with the kind of knowledge needed to help develop learners' ability to prove. The framework can also be used to discern and characterise the different proving actions that a learner performs. Characterising differences in learners' proving actions on a task is important because different actions can lead to different learning trajectories (paths of progression) which may or may not produce a successful end result. Further, knowledge of learners' proving actions can inform the design of tasks aimed at reinforcing and advancing learners' understandings of proving processes and, by extension, their production of proofs. Tasks that are designed with opportunities for learners to identify generalities, e.g., recursive patterns, are particularly valuable because they can prompt learners to make conjectures (Stylianides & Stylianides, 2006a, p. 202) which can form the backbone of a resolution (Mason et al., 2010, p. 62). Relatedly, the tasks that I designed for the learners in my study featured sequences of slightly varied argument examples with several inherent patterns that afforded opportunities for sense-making and courses of action. I based the design of the tasks on variation theory, which is the second key construct on which I drew.

Variation theory is a set of principles that relate to learning and to task design. The theory's chief principle is that difference (and sameness) are necessary conditions for noticing (awareness) and learning because learners use what they see as (in)variant to make sense of a situation (Marton, 2015, p. 40; Runesson, 2006, p. 402). In turn, apprehending invariance can lead to generalising (Mason et al., 2010, p. 232), and generalising is a key process of proving (Stylianides & Stylianides, 2009a, p. 241). Hence, I considered variation theory to be an appropriate basis on which to design the number and geometry tasks in my study. The tasks that I designed featured given iterations of arguments (and accompanying propositions) with (in)variant features. With this (in)variance, I aimed to afford learners ranging opportunities to make sense of and act on the argument structures that I provided. In each of the two domains, I designed a pair of variation tasks (an *initial* variation task and a 'follow-up' variation task). I designed each follow-up variation task with transfer theory in mind. Hence, I refer to each follow-up variation task as the *transfer* task. Variation theory and transfer theory can be seen as complementary because both involve a focus on learners' attention to sameness and difference (Marton, 2006, p. 512). Transfer theory is the third key construct that I drew on.

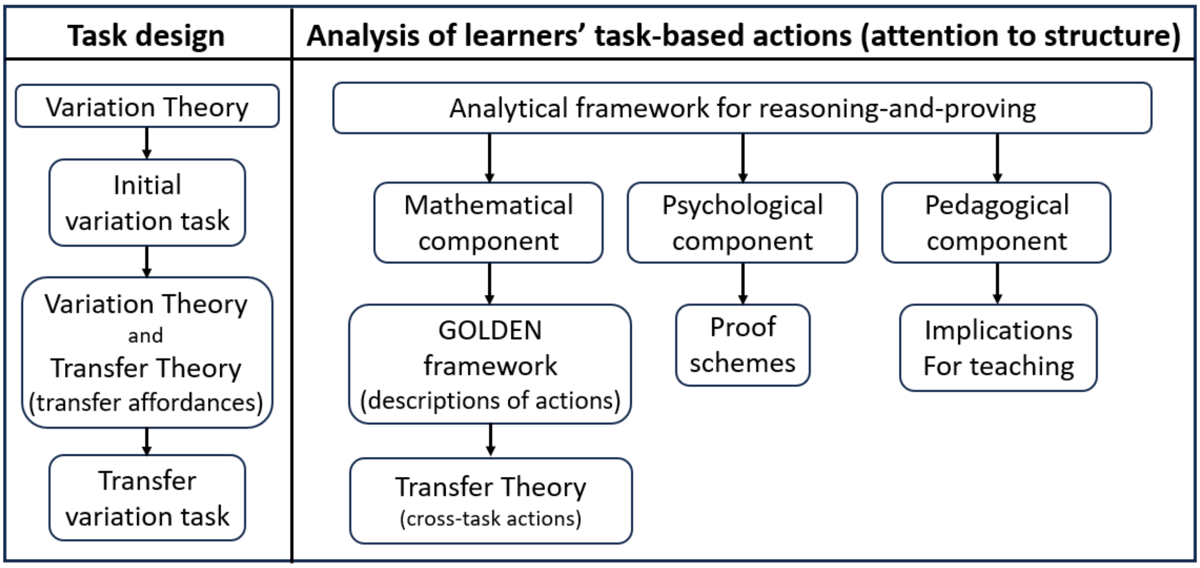
Transfer theory is a general term that refers to a range of perspectives on how prior experience and learning affects what is capable of being done in a subsequent situation (Marton, 2006, p. 499). Learning transfer is traditionally thought of as 'taking what is learnt in one context and using it in another context' (Royer, 1979, p. 57; Singley & Anderson, 1989, p. 5). Alternative views of learning transfer have been proposed, e.g., near (similar) transfer and far (dissimilar) transfer (Royer, 1979, p. 55), transfer as the influence of prior learning experiences (Marton, 2006, p. 499), and transfer as the generalisation of learning (Lobato, 2006, p. 436). The 'object of transfer', i.e., what is transferred, has been variously conceived as: exact procedures (Novick, 1988, p. 458), adaptive/analogous reasoning (Nokes, 2009, p. 3), and what learners notice (Lobato et al., 2012, p. 433). In my study, I saw the object of transfer as learners' actions. Specifically, in each respective domain, I took a learner's transferred actions to be the actions that they performed on the *initial* variation task and similarly performed on the *transfer* variation task. I was interested in transferred actions because the learners had no formal prior experience of proof argumentation, and the roles of any actions transferred across a domain-pair of variations tasks were potentially of particular importance. However, the characterisation of transferred actions was only a subset of my analysis of learners' actions. My main focus was the characterisation of learners' actions on the respective initial and transfer variation tasks in each of the two domains. It was all of the learners' actions combined that I used to formulate the components of my attention-to-structure framework, which I describe in Section 2.3. I positioned the framework within Stylianides' (2008) broad (overarching) framework for reasoning-and-proving, which I next introduce.

1.3 Overview of theoretical framework

In Figure 1.1, the left-hand column reflects how my consideration of variation theory and transfer theory preceded my designs of the initial and the transfer variation tasks (in each domain). The right-hand column shows how I positioned descriptions of learners' actions and transfer theory under the mathematical component of Stylianides' (2008) overarching framework for reasoning and proving. Stylianides' (2008) framework comprised three components (mathematical, psychological, and pedagogical) that serve as broad lenses through which learners' reasoning and proving can be analysed. The *mathematical* component refers to processes of proof, the *psychological* component refers to a learner's perceptions of aspects of proving, and the *pedagogical* component refers to how a learner's actions and perceptions can be more closely aligned with conventional thinking. In my study I viewed the actions that learners performed on the variation tasks, including transferred actions, as expressions of the mathematical component.

As part of my later discussion of learners' task responses, I consider the implications that learners' actions had for the development of their *proof schemes*. A proof scheme is a personally held (psychological) perception that influences the way an individual organises, makes sense of, and methodises a course of action when proving (Harel & Sowder, 2007, p. 808). Hence, a learner's proof scheme can mediate their attention to proof-related structures (Lee, 2016, p. 28) such as the arguments in the tasks that I gave to learners in my study. I position the construct of proof schemes under the psychological component of Stylianides' (2008, p. 10) framework for reasoning-and-proving.

Figure 1.1: Overview of theoretical framework.



I next expand on my aims and state the four questions that guided my research. I follow this, in Chapter 2, by elaborating on processes of proof and on the constructs that I summarised earlier. Namely, Stylianides' (2008, p. 10) analytical framework for reasoning-and-proving, learners' attention to structure, variation theory, transfer theory, and proof schemes.

1.4 Aims

To address my goal of generating resources that could be used to alleviate the problems of proof pedagogy that I discussed in Section 1.1, I aimed to formulate a framework for describing learners' proving actions on tasks in different domains. I then aimed to apply the framework to identify actions that aided learners' progression through the variation tasks that I designed.

As I mentioned in Section 1.2, all of the learners in my study were new to any formal notion of proof argumentation. Thus, another aim of my study was to initiate learners into performing a range of proving actions on formal proof arguments. To meet this aim, I designed for the learners domain-pairs of variation tasks that featured given iterations of purposefully designed (consecutive number) proof arguments (see Appendices A and B). With each pair of variation tasks, I prompted learners to identify elements of (in)variance. My intention with this prompt was to give learners a starting point from which to make sense of, and attend to, the argument structures that I provided. By characterising learners' attention to structure on the proof variation tasks, I aimed to: (1) reveal the kinds of actions that can lead to progression on a task (including successful task completion), and (2) highlight how features of my variation tasks can prompt learners to perform proving actions. In the same way, these two insights are available to other educators wishing to try similar tasks to gain similar understandings about their learners.

I could not find any consecutive-number argument variation tasks of the kind that I designed and used in my study. Hence, variation tasks that feature arguments (and accompanying propositions) with respective consecutive numbers may be a hitherto untried style of resource for familiarising learners and teachers with processes of proof. With the tasks that I designed, I aimed to give the learners enough information to write their 'own' cases that were based on the arguments that I provided. Thus, the argument variations that I provided served as examples for learners to compare to when, in a later part of the tasks, I asked learners to write a case for a proposition that was similar to the given propositions. The variation tasks that I designed are therefore a potentially helpful way to prepare secondary school learners (of any age) for formal proof production. Further, the consecutive-number argument variations that I provided can be seen as pre-performed empirical tests that might encourage learners to generalise rather than, or as well as, engage in the well-known act of testing individual values (Harel & Sowder, 2007, p. 809; Küchemann & Hoyles, 2009, p. 171; Stylianides & Stylianides, 2020, p. S87).

1.5 Research questions

An attention-to-structure framework, as I previously justified, can provide teachers with a more comprehensive and distinct means of characterising and analysing learners' proving actions. This, in turn, can inform teachers of where and how learners can next be developed. Such a framework is also potentially beneficial to the mathematics education research community. As research on attention to structure grows, the current lack of shared understanding regarding use of the term *structure* and, by extension, what it means to *attend to* structure (Kieran, 2018, p. 80; Venkat et al., 2019, p. 13) stands to increase. This is a major concern. It is, however, hoped that my attention-to-structure framework can help to unify a range of researcher perspectives on what is meant by structure, and how it can be attended to.

My first research question related to the potential composition of an attention-to-structure framework. My focus was then guided by three further questions that drew on the application of such a framework to characterise learners' actions on variation tasks in the domains of number and geometry that I mentioned earlier.

Research question 1: What might comprise a framework for describing learners' attention to structure?

Research question 2: What characterises Y8 and Y10 learners' attention to structure on an initial and a transfer proof variation task in the respective domains of number and geometry?

Research question 3: What characterises Y8 and Y10 learners' attention to structure across a pair of (initial and transfer) proof variation tasks in the respective domains of number and geometry?

Research question 4: How do Y8 and Y10 learners' attention to structure compare across both domains?

I articulated my responses to each of the four research questions in terms of the actions that learners performed on the consecutive-number proof variation tasks that I designed. As I outlined earlier, the design of the tasks on which learners performed their actions (and hence the actions themselves) were underpinned by several theoretical concepts, each of which warrant further explanation. I next elaborate on the concepts; in doing so, I draw on the related literature and I explain its relevance to my study. Where I focus on a particular conceptualisation of a construct, I provide justification for this.

Chapter 2: Theoretical Concepts

Any discussion about processes of proving requires some explanation of what is meant by a *proof*; not least because conceptions of proof vary within and across communities of mathematicians, researchers, teachers, and learners (Cadwallader-Olsker, 2011; Healy & Hoyles, 2000; Knuth, 2002). In this chapter, I first discuss different perspectives on the definition and function of a 'proof', then I convey my own conceptions in the context of my study. I follow this by elaborating on the theoretical concepts that I outlined in Chapter 1. I then explain how I characterised learners' actions by moving back and forth between my analysis of learners' task responses and the literature. I give an overview of how the learners' actions that I characterised led me to identify broad categories of actions that comprised the main components of the framework that I formulated. I identified six broad categories of actions (Generalising, Organising, Localising, Deconstructing, Extending and Networking); hence, I refer to my framework as the GOLDEN framework. To provide a broad conceptual backdrop for the framework's six main actions, I describe and exemplify a wide range of mathematical situations in which each of the six actions can be performed. I follow each broad conceptual description by briefly describing how the tasks in my study afforded learners' performance of each action. In Chapter 3, I provide a more detailed methodological account of how I coded all of the actions (including the sub-actions) that comprised the GOLDEN framework.

2.1 Proof and processes of proof

Processes of proof e.g., conjecturing, experimenting, verifying, and communicating (Stylianides & Stylianides, 2006a) are important because they can aid learners' ability to give convincing logical explanations and help to develop high order reasoning in all aspects of school mathematics curricula (Ball et al., 2002, p. 907; Herizal et al., 2019, p. 1; Stylianides & Stylianides, 2017, p. 121). As Harel and Sowder (2007, p. 806) put it, 'no one questions the importance of proof in mathematics, and in school mathematics'. But what *is* a proof, and what is the purpose of a proof? There is no consensus on what is meant by a mathematical proof (Weber, 2008, p. 433). The meaning and acceptance of a proof can differ between individuals and across different demographic communities. To a professional mathematician, a proof may be a means by which to convince the mathematical community of the irrefutability of a theorem. This perception can be seen as different from that of a mathematics teacher who might prioritise, or solely hold, an *educational* view of proof (e.g., a topic to be taught)(Knuth, 2002, p. 84). This view may differ still from that of a school-age learner who might view proof as an area in which to acquire enough competence to respond fully and accurately to proof tasks. For the purposes of this study, and as a teacher/researcher, I primarily hold an educational view of proof. In step with Stylianides (2009, p. 265), I take a proof to be a valid argument (sequence of assertions) made for or against a mathematical proposition. *Valid* refers to the coherent sequencing of assertions based on accepted mathematical conventions and truths (ibid).

The ability to prove, i.e., produce a proof, requires the prover to reason in a way that links the specificity of a given proposition to accepted mathematical generalities such as conventional algebraic representations and numeric and geometric properties. For example, an algebraic proof for the proposition: *doubling any integer will always result in an even integer* might be: *for 2n (where n is any integer), 2 is always a factor, so 2n will always be even*. Relating the specific to the general, and vice versa, is important because much of mathematics is learnt and done by first seeing an instance of an object (i.e., a symbol or diagram, or any combination of these seen as a single entity) (Mason et al., 2009, p. 10) as representative of a generality (Mason & Pimm, 1984).

Provers must also be able to communicate valid (Stylianides, 2009, p. 265) argumentation (reasoning) (Stylianides, 2007, p. 292) and be conclusive by removing any doubt about the truth of their reasoning (Harel, 2013, p. 124). However, it is well documented that, in the context of proof development, these activities are difficult for learners (Küchemann & Hoyles, 2009, p. 150; Stylianides, 2016, p. 78). A widely reported consequence of such difficulties is that learners often see empirical arguments, i.e., evidence from testing specific values, as conclusive proofs (Harel & Sowder, 1998; Healy & Hoyles, 2000; Küchemann & Hoyles, 2005; Stylianides & Stylianides, 2009b). It naturally follows that these difficulties present challenges for teachers (Knuth et al., 2009, p. 153; Stylianides & Stylianides, 2017). A further problem is that while learners' difficulties in learning to prove have been widely reported over many decades, the number of studies designed to initiate learners into the processes of proof (particularly studies that break new theoretical and methodological ground) remain sparce (Stylianides & Stylianides, 2018). Consequently, significant problems of students' learning in the area of proof remain without solutions (ibid). This situation has led to a call for much-needed further research to provide greater insights into how proof and proving can be effectively learnt and taught (Stylianides & Stylianides, 2017, p. 124). I provided one such insight with my study. Specifically, I characterised the nature of Y8 and Y10 learners' actions as they worked on specially designed variations of proof arguments in their first formal encounter with proving.

Since all of the learners were new to any formal notion of proof, my study was not an intervention. Rather, it was intended to demonstrate the potential role and value of employing variation tasks and identifying learners' actions in the pedagogy of proof. However, the kind of variation tasks that I designed, and the framework that I formulated, can be used to underpin the creation of interventions aimed at alleviating the aforementioned problems that learners can experience in the area of proving. Hence, and as shown in Figure 1, my study concerned proof from two, often entwinned, perspectives: *mathematical* and *pedagogical* (Stylianides, 2008). Mathematical because my proof tasks were designed, in part, to encourage learners to explain (convince) and verify (demonstrate), both of which are key proving processes (Harel & Sowder, 2007) and functions of proof (de Villiers, 1990; Stylianides, 2008). Pedagogical because my variation tasks offer a nuanced and accessible way for learners to initially engage in processes of proof; and, my framework provides educators with a tool for analysing learners' task engagement.

2.2 Elaboration of theoretical concepts

2.2.1 Analytical framework for reasoning-and-proving

The three components in Stylianides' (2008, p. 10) reasoning-and-proving framework (mathematical, psychological and pedagogical) provided me with initial lenses with which to consider learners' proving actions. Each of the components offers a distinct analytical perspective on the broad proving activities of: identifying patterns, making conjectures, argumentation, and providing proofs. I next summarise the framework's three components and explain how they each form an analytical category under which I located key concepts of my study.

The *mathematical* component focuses on analysis of many well-known activities that can precede proof production, e.g., testing values, generalising, and generating cases, and encompasses examination of these activities in an integrated way (p. 9). Since the main focus of my study was the characterisation of learners' actions on argument/proving tasks, I positioned these actions, i.e., my attention-to-structure framework, under the mathematical component.

The *psychological* component centres on how a learner perceives the role and status of aspects of reasoning-and-proving (such as an argument or a pattern) (p. 12). I included in the psychological component how a learner perceives their actions on structure. In my study, I took this psycho-analytical approach by considering learners' responses to questions that probed their reasoning as they worked on the variation tasks, e.g., 'What are you thinking now?'. I also included in the psychological component what a learner's actions on the tasks suggested about the development of their (task-specific) proof scheme(s).

The *pedagogical* component involves a combination of the mathematical and psychological components. The term 'pedagogical' refers to how the, perhaps naïve, nature of a learner's perceptions and actions regarding a proof (or element of a proof) compare to those of a more knowledgeable other; together with, identifying where and how a learner's understandings can be advanced (p. 13). I relate to the pedagogical component when, in Section 6.7, I discuss the practical implications that emerged from the findings of my study.

2.2.2 Structure: seeing, appreciating and attending to

As I mentioned earlier, all of the learners in my study were new to any formal conceptions of proof. I did not, therefore, expect any of the learners to produce a complete proof in response to the tasks that I designed (although there was scope in the tasks' requirements for learners to do this). Rather, I focused on the ways in which learners engaged with the argument *structures* given in the tasks.

I refer to the individual elements that comprise a mathematical structure as *objects*. However, all of the objects that comprise a structure can collectively be seen as a single object if perceived as such by the beholder (Tall et al., 1999); e.g., directly seeing 10 × 10 as 100, or n + n as 2n. To briefly restate, I take an *object* to be 'a symbol or diagram, or any combination of these seen as a single entity' (Mason et al., 2009, p. 10). In this definition, the dependent clause 'seen as a single entity' is important because it discerns an object seen as a single (composite) structure from an object seen as an element of a structure. I contextualise my use of the term *object* when in the later chapters I discuss how learners attended to the argument structures given in the tasks that I designed.

A structure, then, can be seen as any combination of mathematical objects e.g., integers in a number sentence, symbols in an algebraic expression, or elements of a geometric diagram (see, e.g., Kieran, 2018, p. 80). It is also possible to see structure by "looking through" the specificity of a construction (Mason & Pimm, 1984); e.g., by seeing 4 × 5 as a commutative object rather than seeing three individual objects (symbols) that are calculable. Seeing structure by looking through a construction is considered more sophisticated than the compositional view of structure because the former requires awareness of an object as a "transparent" instance of a generality, e.g., a property or class (Mason et al., 2009, p. 23; Venkat et al., 2019, p. 14). I next use 4^2 to compare the two foresaid ways of seeing structure.

The compositional view: i.e., seeing a structure as the objects from which it can be composed, or into which it can be decomposed (Hoch & Dreyfus, 2004, p. 50) e.g., seeing 4^2 as 2 × 2 × 2 × 2.

The transparency view: i.e., seeing a structure as a specific instantiation of a generality (Mason et al., 2009). For example, seeing 4^2 as an even integer because the index and base are both even.

These two ways of seeing structure can also apply to an individual number if it is not perceived merely in terms of its cardinality. For example, from the compositional view, and using

Venkat et al's (2019) architectural analogy, the number 9 can be seen as constructible from the multiplicative and additive architecture $2 \times 2 \times 2 + 1$, which is one of many architectural permutations. Alternatively, from the transparent view, the number 9 can be perceived as belonging to the general class of square numbers. Before I leave my conceptualisations of structure, I wish to point out that some structures (e.g., proof arguments with several assertions) can be seen to comprise sub-structures (e.g., individual assertions) which, in turn, can be seen to comprise individual objects.

Since all of the learners in my study were new to any formal notion of proof, I expected them to see a given argument structure predominately in terms of its compositional elements rather than as a single object. As I explained earlier, I was not only interested in what structure learners *saw*, but with what structure learners *acted on*. Acting on structure requires something more than simply seeing structure; namely, *appreciation* (Mason et al., 2009). Appreciating structure, as distinct from seeing structure, requires at some level an awareness of a manipulatable element (p. 15). For example, appreciating that the multiplicative architecture of the natural number 12 can comprise a variety of combinations of its factors. Another example, at a higher level of awareness, is appreciating that the number 12 is *even* because its multiplicative architecture must involve at least one integer factor that is itself even. Taken together, *seeing* and *appreciating* afford permissible 'actions on' or 'attention to' structure. With *actions*, I refer to any transformation of objects into other objects (Cottrill et al., 1996) (e.g., calculating a result or rearranging an equation) and any non-transformational act of reasoning (e.g., explaining a pattern) or organising (e.g., classifying or discerning invariance). Hence, I take a learner's *attention to* structure to be any *action on* structure. As I mentioned earlier, I focused on characterising learners' actions on specially designed activities that I called 'consecutive-number proof variation tasks'; which, as the name suggests, I based on the variation theory of learning (Kullberg et al., 2017).

2.2.3 Variation theory: Implications for learning and for task design

Despite its name, variation 'theory' is widely regarded as a set of principles that relate to awareness and learning (Marton & Booth, 1997), task design (Watson, 2017, p. 87), and evaluating learners' perceptions and understandings (Watson & Mason, 2006, p. 95). The main principle of variation is that difference, i.e., comparative contrast, is a necessary condition for learning (Marton & Booth, 1997).

When an individual experiences altered versions of 'the same' situation, they are afforded an opportunity to discern sameness and difference, i.e., (in)variant features. Becoming aware of (noticing) sameness and difference is a precursor to learning (Kullberg et al., 2017, p. 560;

Runesson, 2006, p. 403), and differences in what features are discerned can lead to differences in what is learnt (Marton et al., 2004, p. 9; Mason & Pimm, 1984, p. 287). Yet, every teacher of mathematics uses variation in their teaching, so what is it about the notion of variation that makes it one of NCETM's 'five big ideas' in teaching for mastery (NCETM, 2015)? Essentially, the answer lies in how conscious and systematic teachers are about the way they vary and keep constant particular features of the examples they use when they are teaching. In the same way, the principles of variation fit well with mathematical task design because tasks can be created that feature iterations of structures in which a particular feature has been systematically varied. In doing so, the varied feature is made available to notice, and the effect of the variation is made available to learn. Teachers can use such variations as examples with the intention of orienting learners' attention to a particular feature and to an associated learning goal. While there is no guarantee that any specifically-intended learning will be 'lived' (acquired) by the learner (Marton et al., 2004, p. 5), varying a particular feature (dimension) across an otherwise invariant string of examples can orient learners' attention to *critical features*, i.e., those that make the difference between whether the subject matter is mastered or not (Pang & Marton, 2007, p. 4); which, in turn, can cause learners to compare, notice, conjecture, and understand (Marton, 2015, p. 56; Marton & Pang, 2006, p. 5). Hence, drawing on the principles of variation when designing or selecting educative examples and tasks has potentially powerful pedagogic implications. Take, for instance, the exercise 12 ÷ 4 =, 12 ÷ 2 =, 12 ÷ 1 =, 12 ÷ 0.5 =, in which the divisors have been systematically varied in such a way as to recursively vary the result. By noticing the pattern of variation in the divisors, a learner might next become aware of a co-varying or 'dependency' relationship. Then, by extension, they may see an underlying generality and use it to reason why the situation co-varies in the way that it does. Such reasoning might be a teacher's planned learning goal (an intended object of learning) (Marton & Pang, 2006, p. 210); however, as I mentioned earlier, what is grasped by a learner (the lived object of learning) (Marton et al., 2004, p. 5), may differ (Marton & Pang, 2006, p. 200). In other words, variation is in the eye of the beholder. This might explain why some researchers in mathematics education suggest that it is the *in*variant features rather than the variant features across iterated structures that primarily trigger noticing and learning (see, e.g., Mason, 2017; Watson & Mason, 2006).

Because combinations of variance and invariance can occur in an unlimited number of ways (Marton, 2015, p. 248), problems can arise if care is not taken about how many features are simultaneously varied and kept constant. Even when care is taken, a confluence of teacher-intended and student-lived objects of learning may not be achieved. Yet, such a mismatch need not be limiting. An adept teacher can see this disparity as an opportunity to link, and so shift, a learner's attention to the intended object of learning. An intended learning goal might require a learner to simultaneously see at least two aspects of variance, what Marton, 2015, (p. 51-52) calls *fusion*. There is, of course, much potential value in using given variations more openly, i.e., by giving learners greater control of what they discern and learn from the (in)variance that they see. Even in situations where learners are given greater exploratory freedom on a variation task, an intended learning goal may still be in play, though learners' attention may not be as directly oriented towards it.

It is often prudent to think of given variations in terms of the range of concepts and procedures that are available to learn. When designing a variation task with a learning goal in mind, we can ask three key questions: (1) which features (dimensions) can be discerned that might move learners towards the intended learning goal? (2) in what ways can (in)variance be controlled to afford discernment of those dimensions? and, (3) in what ways might discernment of those dimensions move learners toward the intended learning goal? We can further and more broadly ask, as did Ingram (2014) and Lobato et al. (2013), what stimuli besides task design can influence what students notice and learn? An answer common to both of their findings is the role and management of the social interaction that accompanies a variation task, e.g., how a task is represented and talked about by a teacher. Further, shifts and differences in how and what students noticed and learnt were found to be influenced by social norms between learners, and between teachers and learners; e.g., differences in language use, gesturing (Ingram, 2014; Lobato et al., 2013), and turn taking (Ingram, 2014). These findings had implications for the management of my own behaviour when, in the role of teacher-researcher, I conducted task-based interviews with the learners in my study. In Section 3.8, I discuss the influence that these implications had on my task-based interactions with learners.

The variation tasks that I gave to learners each featured three slightly different variations of proof arguments. When I designed each respective initial and transfer variation task, I deliberately controlled the variance of a dimension that was in a corresponding position in each of the given arguments. I did this to aid identification of sameness, difference, and patterns of co-variance across the given structures; i.e., to afford sense-making opportunities. Tasks designed on the basis of variation theory have been deemed useful for *initial* sense-making (Watson & Mason, 2006), so I considered the use of variation tasks to be particularly apt for the learners in my study, all of whom were new to any formal notion of proof. As I mentioned earlier, I designed domain-pairs of variation tasks in each of two distinct domains (number and geometry). The second variation task in each domain-pair was a 'transfer' task. The transfer tasks were important because they afforded opportunities to identify how any consistent actions performed across a domain-pair of variation tasks aided (or hindered) a learner's understanding of, and progress on, a task. I next discuss some conceptions of transfer, factors that can mediate transfer, and their relevance to my study.

2.2.4 Learning transfer

The transfer of learning is traditionally viewed as the taking of experience gained in one situation and using it in another situation (Lobato, 2006, p. 431). Learning transfer is therefore a core aim of education (Bruner, 1996, p. 129; Carraher & Schliemann, 2002, p. 1). I investigated learning transfer as part of my analysis of learners' actions in my study. To do this, I looked for the actions that each learner similarly performed on both the initial and the transfer variation task within (and across) each domain. My characterisation of transferred actions supplemented my broader characterisation of learners' (non-transferred) actions. I characterised learners' actions because they potentially offered valuable insights into which actions, and sequences of actions, aided or hindered learners' responses to the prompts on the variation tasks; e.g., when asked to generate a new case based on a set of given arguments. Further, as I mentioned earlier, the role of transferred actions can be perceived as especially valuable due to their repeated use in different situations. I am not suggesting that learners' transferred actions were newly learnt while working on an initial variation task. All of the learners in my study were, for example, already familiar with the operational processes of adding, subtracting, multiplying, and dividing. However, in each respective domain, I deliberately designed the initial and the transfer variation tasks so that they featured seven common elements. I did this to provide, in each separate domain, a distinct but conceptually-related pair of tasks that afforded opportunities for learners to act on similar objects and structures, and to evoke similar cross-task thinking. I further explain my rationale for the design of the tasks in Section 3.7.

Learning transfer (hereafter 'transfer') has been conceptualised in alternative ways (Cox, 1997; Lobato & Siebert, 2002; Rebello et al., 2007). Two prevalent commonalities exist across these conceptions, the first of which constitutes a definition of transfer: (1) the application of prior experience to a subsequent situation, and (2) processes of generalisation that underlie the application of experience from a prior situation. Differences in conceptions of transfer are starkest over time. For example, classical views of transfer prominently reference a reliance on stimulus (behaviourist) responses triggered by the presence of identical surface features in different situations (Thorndike, 1906). Whereas, subsequent conceptions of transfer, in the cognitive domain, have focused on: schema mapping (Gick & Holyoak, 1983; Reed, 1993); "horizontal" (i.e., near or similar) and "vertical" (i.e., far or dissimilar) applications of knowledge (Rebello et al., 2007, p. 228; Royer, 1979, p. 54); procedural (rule-based) similarities (Nokes, 2009; Reed et al., 1974); and, socio-cultural situatedness (Engle, 2006; Lave, 1988; Nemirovsky, 2011). Yet, in contexts that respectively afforded each of the foresaid conceptualisations, what some researchers have reported is a *lack* of transfer (e.g., Carraher & Schliemann, 2002, p. 3; McKeough et al., 1995; Rebello et al., 2007, p. 223). However, as Lobato (2008, p. 298) pointed out, a failure to evidence transfer may simply be the result of a mismatch between a learner's cross-situation activities and an observer's own preconceptions of what counts as transfer. Relatedly, more recent conceptions of transfer have prioritised how learners, rather than researchers, see situations as similar (Lobato, 2012; Lobato & Siebert, 2002). In doing so, these researchers have embraced subtler, less direct, notions of connection-making across situations. These connections include reasoning about underlying causes (Lobato, 2008, p. 291) and the influence of discerning sameness and difference (Marton, 2006, p. 512). I include these two subtler conceptions of transfer in the non-transformational aspect of my definition of an action that I gave in Section 2.2.2.

The conceptual of broadening of transfer to incorporate the influence of discerning (in)variation means that transfer can also be shaped by the act of noticing difference across situations. Indeed, as Watson and Mason (2005, p. 108) pointed out, it is against a backdrop of difference that sameness and similarity can be perceived. The influence of noticing difference across situations can potentially result in transfer expressed as diversity in an individual's cross-situational actions, rather than similarity. That is, by how experience in one situation influences what a learner does in another situation, 'even if the influence results in differential behaviour' (Marton, 2006, p. 528). For example, where a learner leaves off in one task, they may continue on from in another task; or, when a learner gives a counterexample of prior learning. Differential transfer might involve procedural adaptation, structural alteration or in-situ (re)construction of reasoning. All of which are activities that I was sensitive to when I identified learners' transferred actions in my study.

The idea that acts of transfer can be influenced by noticing sameness and difference suggests that *everything* is transfer. Alternatively put, learning always depends in some way on what is already known (Carraher & Schliemann, 2002, p. 2). To overcome this dilemma, I took transferred actions to be actions that a learner performed on a transfer task and that I could justifiably map to their actions on the corresponding initial task. I confined these justifications to the nuanced sub-actions that I identified within each main action in the framework. On the other hand, one could argue that *nothing* is transferred, i.e., that new experiences are simply assimilated into existing schemas (Piaget, 1970, p. 707). However, and as an example of my own transfer, my rejoinder is that assimilation can be seen as a form of *retrospective* transfer, which Marton (2006, p. 512) describes as how a new experience modifies a prior experience.

The *object* of transfer, i.e., 'what transfers', has also been conceived in a variety of ways, e.g., as content knowledge, holistic conceptualisation, and noticing (Lobato, 2012, p. 237). As I described in Section 1.2, in my study, I saw the object of transfer as a learner's *actions*. I next elaborate on the concept of proof schemes, the implications for which I considered as part of my reflections on learners' actions in Chapters 5 and 6.

2.2.5 Proof schemes

The notion of *proof schemes* was introduced by Harel and Sowder (1998). A proof scheme is a personally held perception that influences the way an individual organises, makes sense of and methodises a course of action in order to develop or validate a proof or part of a proof (Harel & Sowder, 2007, p. 808; Lee, 2016). Proof schemes are context-specific and can serve as lenses for conceptualising the way a learner perceives and engages with a proof-based task. I supplemented my characterisation of learners' proving actions by discussing the implications that their actions have for the development of their proof schemes. For example, the actions that a learner performed on an initial variation task may hint at, or be a clear manifestation of, a specific type of proof scheme; which, may or may not be reinforced by the learner's actions on a corresponding transfer task.

Proof schemes, which can also be seen as modes of justification (Stylianides and Stylianides, 2009b), sources of conviction (Buchbinder, 2018, p. 260), or mediators for attending to structure, have been classified in several ways (Balacheff, 1988; Harel & Sowder, 2007). Three proof schemes are particularly prevalent in the literature:

External conviction proof schemes refer to authority or received convention, e.g., seeking and/or following a teacher's didactics or a procedure illustrated in a textbook (Cartiglia et al., 2004, p. 288).

Empirical proof schemes feature example use including the substitution of specific numbers, e.g., to test the truth or truth domain of a proposition (Cadwallader-Olsker, 2011, p. 43; Harel & Sowder, 2007, p. 809).

Analytical proof schemes are expressed by deductive reasoning (Buchbinder, 2018, p. 260; Harel & Sowder, 2007, p. 809) and are initiated either heuristically or due to a perceived intellectual need (Harel, 2013; Stylianides and Stylianides, 2009b), e.g., to find and state an exhaustive generalisation.

In the context of my study, I interpretated proof schemes in terms of how a learners' actions might mediate their perceived approaches to the variation tasks. Hence, as I explained in Section 1.3, I positioned proof schemes under the psychological component of Stylianides' (2008, p. 10) analytical framework for reasoning-and-proving. However, my consideration of learners' proof schemes only supplemented my principal means of characterising learners' task responses, which was in terms of the actions in the GOLDEN framework. It is to an overview of the framework that I next turn.

2.3 The GOLDEN framework for attention to structure

In this section, I mainly focus on conceptualising the six main actions in the framework. In later chapters, I give growing attention to the framework's sub-actions. In this way, I first broadly introduce the framework, then elaborate on its sub-actions as the thesis unfolds.

I raised earlier four well recognised restrictions on learners' development of proof: (1) teachers' often limited knowledge of how to develop learners' ability to prove (Harel & Sowder, 2007, p. 836-837; Stylianides & Stylianides, 2018, p. 110); (2) delaying the teaching of proof until the latter years of secondary school (Knuth, 2002, p. 61; Stylianides & Stylianides, 2009a, pp. 237-238); (3) learners put in situations where they are suddenly expected to prove (Ball et al., 2002, pp. 207–208) and, (4) teaching proof in isolation from other mathematical activities (Stylianides & Stylianides, 2006a, p. 202). I then described how the four restrictions can, singly or collectively, lead to two further restrictions on the learning and teaching of proof (insufficient scaffolding and indistinct characterisation of learners' proving processes). My hope is that proof variation tasks of the kind I designed for my study, together with my framework for describing learners' actions on the tasks, can be used to alleviate these problems. As an analytical tool, the uses of my attention-to-structure framework are at least sixfold. The descriptions of learners' proving actions that comprise the framework can:

(1) Provide teachers and researchers with a formal and adaptable lens and language with which to analyse and articulate learners' proving processes.
(2) Aid the identification of patterns and relationships in learners' proving actions within and across tasks.
(3) Respond to the call for analytical frameworks that allow for the study of tasks that can support the learning of proof (Stylianides & Stylianides, 2018, p. 100).
(4) Assist the study of how actions (and combinations of actions) can lead to the production of valid arguments and proofs by learners who are new to formal processes of proving.
(5) Inform instructional support and task design aimed at advancing learners' proof-related understandings and abilities.
(6) Elicit greater clarity and unity in regard to researchers' use of the terms 'structure' and 'attention to structure'(see, e.g., Venkat et al., 2019, p. 13).

Regarding use (6), different researchers have used the term ‘structure’ with diverse and even opposing meanings (Hoch & Dreyfus, 2004, p. 50; Venkat et al., 2019, p. 13), or have used the term assumptively (Kieran, 2018, p. 80) and so sidestepped any need for a definition. This ambiguity of meaning is compounded by the wide variety of mathematical contexts in which ‘structure’ has been used. These contexts include: transitions to algebraic representation (Warren, 2003); spatial layouts of task responses (Knuth, 2002); increasing and reducing abstraction (Hazzan & Zazkis, 2005); generality-based reasoning (Küchemann & Hoyles, 2005); logic statements (Selden et al., 2018); how contextualised problems relate to relevant calculations (Dougherty et al., 2015); and, using examples to ‘see’ structure (Ozgur et al., 2017). Within such contexts, what is meant by ‘structure’ is further obscured by the range of sematic and lexical fields with which the term has been associated. As Venkat et al. (2019, p. 13) pointed out, the term ‘structure’ is frequently enmeshed within a set of terms that are sometimes juxtaposed but seen as distinct, and at other times, seen as synonymous. This might explain why the meaning of the term is often undefined or assumed (Kieran, 2018, p. 80). Obviously, such lack of clarity has implications for what it can mean to ‘attend to’ structure. However, my framework provides a single source for describing a broad range of ways in which learners can act on, i.e., attend to, structure. The six main actions in the framework, individually or collectively, are broad enough to potentially describe, to some extent, learners’ engagement on any mathematical task. The sub-actions in the framework can be adapted to the nuanced ways in which learners engage with a task’s specificity. Hence, my framework has potential as a tool for the analysis of learners’ actions on mathematical tasks beyond the area of proving. Use of the framework, it is hoped, will stimulate greater shared understanding in regard to the characterisation of ‘structure’ and ‘attention to structure’.

As I mentioned earlier, I formulated the framework from actions that, across the learners, were common to responses to tasks in the distinct domains of number and geometry. I wanted to characterise learners’ actions because different actions can lead to different avenues of structural reasoning and manipulation, and hence different learning trajectories and outcomes. I derived the actions in my framework through a synthesis of the literature and my analysis of learners’ responses to the pair of variation tasks in each domain. My trawl of the literature and analysis of learners’ task responses yielded six main actions. Namely, *Generalising*, *Organising*, *Localising*, *Deconstructing*, *Extending* and *Networking* (acronymically, GOLDEN). For each main action, I identified three sub-actions (see Appendix E).

To summarise from earlier, with *action* I refer to any physical or mental transformation of objects into other objects (Cottrill et al., 1996) and any non-transformational act of reasoning or organising. A transformational action can be as simple as combining a physical (written) object, e.g., the number 5, with the mentally evoked objects '+' and '6', thus transforming the number 5 into the calculable structure '5 + 6'. Calculating the result of 5 + 6 is a further action because it transforms 5 + 6 into 11. In the context of my study, I did not directly include such isolated operations in the framework. However, such operations were inherent in some of the actions that did comprise components of the framework, e.g., dividing (when simplifying a fraction). My conception of *action* included transformational acts of organising (e.g., rearranging an equation) as well as non-transformational acts of organising (e.g., foregrounding a dimension of sameness while simultaneously backgrounding difference, i.e., discerning variance and invariance); my conception of action further included non-transformational acts of reasoning (e.g., explaining an identified consistency or pattern). I grouped all of the learners' respective actions into six categories which formed the six main actions (components) in the GOLDEN framework. All of the learners' actions were rooted in noticing. The notion of noticing is non-trivial. As Mason (2003, 2018) pointed out in his work on forms of noticing and structuring attention, what a learner notices is dependent on how much of a mathematical structure they are holding in mind in the moment. When I characterised the learners' actions, I therefore inherently took into consideration what a learner noticed. Hence, I considered both a learner's attention to structure and structure of attention. The GOLDEN framework's six main components can be summarised thus: *Generalising* involved reasoning about identified consistencies; *Organising* described acts of prioritising (foregrounding and backgrounding); *Localising* comprised adjusting and testing values; *Deconstructing* concerned shortening (reducing), inversing, and reverse engineering; *Extending* related to continuing an identified pattern; and, *Networking* involved integrating knowledge from discrete curriculum areas.

Before I elaborate on each of the main actions in the GOLDEN framework, it is worth briefly discussing the hierarchal nature of structure. In any structure-based mathematical context, structures can be seen to exist within structures. Attention to which, at any hierarchical level, can involve seeing a structure as a single object. For example, when calculating the scale factors of two mathematically similar triangles that share two sides, both triangles concurrently (or just one of their shared sides) might be treated as a single object. A structure can also take the form of a process seen as a single object, e.g., treating the additive process (b + c) as a single object when dividing a(b + c) by (b + c), or by directly apprehending 10^2 as 100 (Sfard, 1991; Tall et al., 1999). Learners' awareness of different levels of encompassed structure is akin to Mason's (1998, p. 248, 2003) 'forms of attention', which he described as recognising, stressing, and ignoring properties and relationships. The structure that a learner perceives obviously has implications for what structure they subsequently attend to. For, what you do not notice, you cannot act upon (Mason, 2002, p. 7). In other words, noticing triggers actions (Mason, 2017). There is, though, an alternative view, that noticing is itself an action, i.e., an act of organising by foregrounding and backgrounding elements of structure when multiple information is completing for one's attention (Lobato et al., 2013, p. 809). I adopted this view of noticing in my study, hence my reason for mapping learners' discernment of (in)variance to the organising component of my framework.

I developed the GOLDEN framework by alternating between the literature on mathematical actions and my observations of learners' attention to structure on the variation tasks. In doing so, I asked 'how well does the literature explain what I am observing?' I anticipated that some, if not all, of the learners would generalise and specialise (act on local structure) because these actions are central to mathematical learning (e.g., Ellis, 2007; Mason et al., 2010; Park and Kim, 2017; Umierski and Tiedemann, 2019); and, I deliberately designed the variation tasks to encourage generalising and specialising. Although I anticipated that learners would generalise and specialise, I aimed to minimise any confirmation bias by acting with reflexivity when characterising learners' actions. Further, generalising and specialising are generic terms and the nuanced ways that learners performed these actions were much less predictable.

I next provide a ranging conceptual backdrop for each of the main actions in GOLDEN framework. To do this, I move beyond the context of my study to describe and exemplify a range of situations in which each action can be performed. I follow each ranging conceptual description with an overview of each main action as afforded by the tasks in my study. In the next chapter, I discuss the design of the variation tasks and how I developed and applied the GOLDEN framework as an analytical tool. In Chapter 4 I draw on the results of my analysis of learners' actions, and sub-actions, to answer each of my research questions.

Generalising

Generalising actions are commonly associated with searching for, identifying and using a rule, axiom, or property-based relationship, e.g., the associative and commutative properties of addition and multiplication (Boero et al., 1999; Buchbinder & Zaslavsky, 2011, p. 269; Ching & Nunes, 2017, p. 67; Küchemann, 2008; Venkat et al., 2019, p. 14). Generalising actions include abstracting common properties and procedural routines (Warren, 2003), reasoning and representing algebraically (Kaput et al., 2008; Küchemann & Hoyles, 2005), similarity-making (Ellis, 2007, p. 225), and acts of a consciously consistent type repeatedly performed across tasks of a similar nature (Jurow, 2004). The latter is conceptually analogous to traditional processes of learning transfer (Ellis, 2007, p. 222). The generalising component focuses on how and for what reason a learner performs a generalising action. Generalities take different situation-dependent forms and exist on hierarchical levels (Marton, 2015, p. 226; Marton & Ko, 2004, p. 48). A learner's task responses can often reveal generalities that they readily recognise, e.g., canonical properties or algebraic conventions, and those that they arrive at by the generation of particular examples, i.e., by inductive reasoning. A learner may also generalise from only one particular case, if they see that case as representative of, i.e., an instance of, the general case (Mason & Pimm, 1984).

Generalising actions include producing conjectures and justifications based on sensed regularities conceived as patterns (Mason et al., 2010, p. 9). Conceptions of patterns typically centre on processes or on results. Harel and Soto (2017) describe these conceptions, respectively, as *process pattern generalisation* (PPG), e.g., justifying how and why graphs of the form $y = ax^2$ can be symmetrical for any variable *a*, and *result pattern generalisation* (RPG), e.g., justifying why the results of a family of calculations such as -1^2, - 1, -2^2, – 2, -3^2, - 3, … cannot be negative or odd. Hence, PPG and RPG should not be interpreted as merely finding at face value a 'next term or figure in a pattern', which is a process that Küchemann (2010) claimed can actually confound learners' ability to form structural generalisations. One explanation for Küchemann's claim is that a recognised pattern may be only plausible (inconclusive) rather than definite (conclusive) (Stylianides, 2016, p. 82). Pattern-based generalising actions can also be triggered by discerning sameness and difference across provided iterations of procedures illustrated in variation tasks (Watson & Mason, 2006, p. 106) of the kind that I used in my study (see Chapter 3, Section 3.7).

Generalising affordances of the proof variation tasks

I designed the proof variation tasks with ranging opportunities for the learners to generalise at varying levels of complexity. For example, learners could somewhat superficially generalise by adhering to the consistent format of the given argument structures when generating their own cases. Such fidelity can be seen as a form of process pattern generalisation (Harel & Soto, 2017). Learners might also generalise by reasoning about an identified consistency within the processes of given arguments, e.g., the influence that an invariant divisor has on a next given step. Such generalising would also constitute a form of process pattern generalisation (ibid). Alternatively, learners may generalise by comparing each of the given arguments and then reason about a consistent relationship between a value in a process and a value in the accompanying propositions, i.e., an argument's resultant value. For example, how the values of correspondingly-positioned addends consistently co-vary with correspondingly-positioned values in the arguments' accompanying propositions. Attending to structure in this way can be seen as a form of result pattern generalisation (ibid).

Organising

Organising actions can involve arranging objects to compose or rearrange a structure. These actions, which inherently imbue representing, include: compiling symbols when constructing a number sentence; ordering objects into columns or rows (e.g., when making the largest possible number with six given digits); connecting a sequence of assertions in a proof argument (Stylianides, 2009, p. 265); and, rearranging objects for the purposes of solving, changing the subject, or simplifying. In my study, I discerned simplifying conceived as rearranging (e.g., when solving an equation with an unknown on both sides) from the kinds of simplifying that exposes architectural or structural factors (e.g., when simplifying a fraction, ratio, or surd). I mapped the latter conception of simplifying to the deconstructing component.

Organising actions also include processes of representation such as the spatial positioning objects during geometric and statistical diagram-construction (Lobato et al., 2014); laying out answers and responses (e.g., sequencing multiple written calculations when problem solving); forming equations; and, mental or physical acts of 'parking' in which a structural element is held in abeyance then later (re)introduced into a situation (Mason, 2018, p. 326). In addition, and as I explained earlier, organising actions include foregrounding and backgrounding part of a task when multiple elements of information are competing for a one's attention (Lobato et al., 2013, p. 809). For example, prioritising part of a task when deciding on an initial place of entry or attack point, or rejecting one idea in favour of another idea. Some organising actions can occur near-simultaneously with generalising actions, e.g., classifying-and-sorting activities. In Section 3.9.2, I discuss how I viewed and dealt with the notion of learners' co-occurring actions. Organising actions further include mentally simulating the (re)arrangement of objects through consideration of what moves are possible and useful to perform. Rearranging objects is different from replacing (varying) an object, the latter of which I map to the Localising component of the framework. Discerning between possible and useful moves requires what Hoch and Dreyfus (2004) called 'structure sense', and involves anticipating the outcome, or possible outcome, of a prospective move, i.e., a structure's 'future state' (Maciejewski & Star, 2019). Anticipatory structure sense can trigger accompanying justifications. For example, justifying why rearranging rather than factorising or solving an equation might be seen as the best next step in a course of action.

Anticipating the effect that an action will have on a structure is, of course, a notion applicable to all of the actions in the GOLDEN framework. The possible and useful actions that a learner calls to mind are examples of how experience from a past situation can influence, perhaps indirectly, what is done or is possible to do in a future situation. This transfer-related concept is a supplementary aspect of my analysis of learners' actions, and one to which I return in Chapter 3.

Organising affordances of the proof variation tasks

In each respective domain (number and geometry) the variation tasks that I designed featured sequences of three similar arguments. With one of the prompts in each domain I asked learners what they saw as the same and as different when looking across the given arguments (see, prompt 2a in Appendices A and B, respectively). To respond to this prompt, a learner must mentally organise the multiple elements of information that, in the given argument variations, are competing for their attention (see, e.g., Lobato et al., 2013, p. 809). A learner might organise this information by foregrounding (while backgrounding) an invariant feature, e.g., a coefficient that is consistent in its position and value in all of the arguments. On the other hand, a learner may organise the information in the argument variations by giving priority to a variant feature. Alternatively, a learner might foreground invariance and variance at the same time, e.g., by simultaneously seeing a rule for a recursive pattern as invariant and the terms in the pattern as variant. For me, organising information by foregrounding invariance is different to generalising because discerning sameness is merely a tacit act of noticing, whereas generalising actions are the outcome of, and are accompanied by, explanatory reasoning.

Localising

I use the term *localising* to cover a variety of actions that include and go beyond those that Mason et al. (2010, p. 21) called *specialising*. The term *localising*, in comparison to *specialising*, is also more indicative of the kind of structure that a learner is acting on. Localising actions, as is the case with specialising actions, involve reducing the complexity of a situation for the purpose of sense-making (2010, p. 231), e.g., by testing examples and using simpler numbers. Local attention to structure is situation-bound and takes little or no account of a whole task. Attention is paid, and actions are made, only in relation to objects/structure within a task. That is, without consideration of any generalities or any concepts, objects, or processes that could be introduced from other mathematical topics or domains (see *Networking*). Hence, any conceived axioms are only local evocations rather than instances of, say, canonical properties. Local attention to structure would, for example, lead to discovering by measurement that the exterior angle sum of an irregular hexagon is 360° (as opposed to immediate evocation of '360°' as the exterior angle sum of any polygon).

Localising includes acting on interior references and internal similarities when reasoning about a structure. In the context of proving, interior references refer to when part of a proof calls on anything stated earlier in that proof. For example, . . . Let $x \in A$ . . . $A \subseteq B$ . . . Since $x \in A$ then $x \in B$ . . . (Savic, 2017, p. 159). 'Similarity in proof' refers to when an assertion in a proof is repeated for another part of the same proof (p. 159). Learners can localise by empirically adjusting part of a given proof to achieve a desired effect elsewhere within that proof; e.g., when aiming to generate a specific numerical result or parity of result. This kind of action has been called 'experimenting' (Cuoco et al., 1996) because it involves adjusting and testing what Mason (2021, p. 4) called a 'dimension of possible variation' (an alterable feature) in a task. For example, adjusting a value in the structure $\frac{3+5}{2} = 4$ to produce a result of 7; or, in the context of proving, the well-known action of empirically testing a single case (Dahlberg & Housman, 1997; Stylianides & Stylianides, 2009b). While it is possible to see the general in the particular (Mason & Pimm, 1984), testing more than one case, especially systematically, can cause an underlying (generic) structure to emerge (Venkat et al, 2019, p. 14). Systematic local attention of this kind is akin to 'abstraction conceived as decontextualization' (Lobato et al., 2012, p. 435; Singley & Anderson, 1989, p. 33) and, if enough evidence is gathered across tested cases, can lead to identifying a generality (Mason et al., 2010, p. 9). Such a process might be termed 'generalisation by genesis', but I use the term 'locally-induced generalisation' because it reflects two of the actions in the GOLDEN framework. Localising also incorporates provers' derivation and use of *local counterexamples*; coined by Lakatos (1976, p. 10) and employed by Komatsu et al., (2017) in their study of learners' validation of proofs, a *local counterexample* is an example that rejects only a specific step within a proof (Lakatos, 1976, p. 10).

Localising affordances of the proof variation tasks

The variation tasks that I designed make available a range of opportunities for learners to act on local structure, not least because all of the learners were new to any formal conceptions of argumentation and proof. A learner may, for example, adjust one of the values in a given argument with the aim of achieving a required end result. If, after making an adjustment, a required end result is not achieved, a learner may then make further local adjustments based on how their previous adjustment appeared to influence their previous result. For instance, on the number initial variation task, a learner may empirically (re)adjust then (re)test a given addend value; similarly, on the geometry transfer variation task, a learner might empirically test different x and y coefficient values. Indeed, Localising by empirically testing a range of values is a well-documented act (Dahlberg & Housman, 1997; Stylianides & Stylianides, 2009b) that learners new to proof often perform.

Deconstructing

Deconstructing includes partitioning, factorising, and other de-compositional actions that expose a permutation of a structure's architecture. For example, the number 10 notated as $5\sqrt{4}$, $(8 - 3)\sqrt{4}$, or $8\sqrt{4} - 3\sqrt{4}$; or, $6a^2$ deconstructed to $2a \times 3a$ or to $a^2(2 \times 3)$. Deconstructing actions can include reductive reframing or reformatting (Venkat et al., 2019). For example, expressing $2n + 1$ or $n + (n + 1)$ as 'odd' or $2(n + 1) - 2$ as 'even'; and, narrowing a range or domain of applicability, e.g., expressing $4 \geq 3$ as $3 < 4$. Further deconstructing actions can include simplifying conceived as exposing architectural (structural) factors. For example, simplifying $\frac{27}{45}$ to $\frac{9}{15}$ then to $\frac{3}{5}$, 27 : 45 to 3 : 5, $\sqrt{45}$ to $\sqrt{9} \times \sqrt{5}$ then to $3\sqrt{5}$; and, $\frac{27x - 45}{9}$ to $3x - 5$. The last example also decreases the number of objects (terms) in the initial (fractional) structure, so can be seen to deconstruct by having a dismantling effect. This dismantling view of deconstructing is different to decomposing, and in the context of asking learners to write cases for given arguments (as in my study), can have implications for the transparency and 'explanatory power' (Stylianides et al., 2016) of a structure. Deconstructing actions also include spatially separating compound and overlapping geometric objects to facilitate or clarify one's reasoning.

Recall that my definition of action includes mental (imagined) transformations of objects into other objects (Cottrill et al., 1996). Thus, deconstructing actions also include verbally conveying a mentally pictured future state of a structure following a deconstructive act, e.g., a resultant structure after performing an inverse operation.

Deconstructing affordances of the proof variation tasks

I deliberately made available in both domain-pairs of variation tasks a range of structural relationships (connections) that learners might reason about and manipulate. For example, most of the assertions (steps) in the given arguments are in the form of equations, and the argument structures feature co-varying relationships and operations that are subsequently inversed. Therefore, when engaging with the tasks, learners might reason in a deconstructive way by thinking in terms of "reversing" or "doing and undoing". As well as verbalising their mental reasoning, learners may, of course, act on the given argument structures by way of their written productions; indeed, the case-generation prompts in the tasks directly encouraged the learners to do so. The design of the tasks also affords opportunities for learners to deconstruct by 'cancelling out' or by truncating (shortening) the given structures. For example, when prompted to write a new case that is similar to the given arguments, some learners may shorten a given step if it eases *cognitive load* (Paas & Van Merriënboer, 1994), aids understanding, expediates their response, or they feel that they are still responding thoroughly enough. In some instances, a learner may reduce their own case by one or more given assertations/steps so that they can then focus their attention on (attend to) a local structure (Localising). Learners may also deconstruct by 'reverse engineering', e.g., working backwards through given steps to identify the origin or reason for a given value.

Extending

Extending actions include reasoning about objects that are not present by considering the objects that are present. Attending to structure by extending includes actions triggered by the previous identification of a pattern (generalising). In such cases (in my view), a generalising action and an extending action may be near-simultaneous but not co-occurring. Extending actions are not limited to finding a next (sequential) term, figure, or result in a pattern. For instance, structures that involve patterns of consecutive numbers can be non-consecutively extended. Extending actions, i.e., the products of extending actions, can enrich a learner's sense of a concept by expanding what Sinclair et al. (2011) called their 'personal example space'. A learner's example space has been defined as the scope of objects and construction techniques that a learner can call to mind in response to an exemplification task (Watson & Mason, 2005). It therefore follows that expanding a learner's example space has the potential to increase the flexibility of their thinking.

Pattern identification and extension, as Küchemann (2010) pointed out, can result from a direct request for a next term/figure, or from engagement with a task without any such request; (I see questions such as 'what is the same and what is different about …? ' as occupying the middle ground). The source of a pattern conceived by a learner can influence their subsequent course of action. Extending a *process*-based pattern can lead to a different learning trajectory than extending, in the same situation, a *result*-based pattern (Harel & Soto, 2017). A pattern can also be started and extended from a single structure, e.g., in the case of $2n$, substituting for n a pattern of odd or consecutive integers and listing the evaluated results. In fact, any object or structure can be extended by systematically altering, in the form of a pattern, any available and noticed dimension of variation (Watson & Mason, 2005). Extending in this way, i.e., systematically, can develop the level of one's reasoning (Henningsen & Stein, 1997, p. 538) because it can prefigure the generalising action of conjecturing (Mason et al., 2010, p. 9). Extending is different from the act of lengthening a string of objects when constructing a number sentence, which I see as a composing/arranging action and so map to the Organising component.

The extending view of attention to structure also includes lengthening an object, or objects, present in geometric diagrams, e.g., extending a line segment (see, e.g., Harel & Soto, 2017, p. 240; Komatsu et al., 2014, p. 40) or arc to make a task appear more familiar (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Palatnik & Dreyfus, 2018); which, in turn, can make available further opportunities for reasoning and manipulation.

Extending affordances of the proof variation tasks

In the given argument structures (and accompanying propositions) in my variation tasks there are several patterns that are available for learners to notice, reason about, and extend. These patterns include recursively increasing addend values (in the number task) and recursively increasing coefficient values (in the geometry task). In both domains, the initial and the transfer variation tasks each require learners to write an argument, i.e., a new case, for a proposition that is non-consecutive to the given propositions/arguments. Some learners might therefore extend a conceived pattern as part of writing their own cases. Opportunities exist for learners to extend patterns identified solely within the given arguments' assertations (a process pattern), or patterns identified in relationships between a process value and a proposition value, i.e., an argument's resultant value (a process-result pattern).

Networking

Networking actions involve integrating a concept from one domain (or context) with a concept from another domain (or context). In secondary mathematics in England, it is the norm to think of 'domains' as number, algebra, geometry, and statistics; and 'concepts' as topics segregated broadly at the level of curriculum-aligned textbooks (see, e.g., Capewell et al., 2004; Heylings et al., 2010). Differences in networked concepts might be perceived as relatively diverse, e.g., drawing on the (number) concept of division to prove (in geometry) that all circles are mathematically similar; or, perceived as relatively slight, e.g., drawing on the (number) concept of multiplying fractions to calculate a consecutive event probability (in statistics). A networked concept can also be introduced from a different context within the same domain. For example, in the number domain, a learner might decide to convert decimal values to fractions before multiplying. Differences in the diversity of networked concepts can be aligned with the notion of near and far (Clements & Sarama, 2009, p. 62) or horizontal and vertical (Rebello et al., 2007, p. 228) transfer. However, networking actions involve the direct integration of juxtaposed concepts and so can differ from some of the subtler conceptions of transfer. For example, transfer conceived as noticing (Lobato et al., 2012) or as the influence of noticing contrast (Marton, 2006).

Networking actions can be thought of as 'asset-based' actions because they require recourse to concepts conceived as helpful that are in one's repertoire. Another way to conceptualise networking is by contrast to localising. Take, for example, interpretation of the number sequence 3, 6, 12; when localising, one might focus only on 3 and 6 then reason about a multiplicative or additive relationship. Whereas, a networking action might result in an interpretation involving the number 9 (if one networks by introducing the concept of range). A further form of networking involves calling into play numbers of different classes to those that are present in a task. For example, testing if a given pattern involving only natural numbers holds for fractional values. Relatedly, networking can be thought of in terms of one's *concept image*, i.e., the network of cognitive connections that one makes in response to apprehending a mathematical concept (Tall & Vinner, 1981). For example, the range of concepts and procedures that one brings to mind in relation to the notion of 'angle sums', or to the notion of 'simplifying'. Networking actions therefore require an appreciable level of disjunctive thinking.

Networking affordances of the proof variation tasks

The concepts and relationships imbued in a task have implications for what concepts from other domains or contexts a learner might integrate, or contemplate integrating. For example, inherent in the variation tasks in my study are the concepts of dividing and of co-varying relationships, which might cause a learner to network by calling into play the respective concepts of fractions and proportion. The geometry task, for instance, features proportional relationships between given coefficients, which might cause a learner to feel (not unjustly) that bringing their knowledge of ratio to bear on the situation might be helpful. Since the tasks in both domains feature only natural numbers, further networking affordances include introducing decimal or negative values.

While the six main actions in the GOLDEN framework are not completely separable, I have justified their delineation by giving qualitatively different descriptions for each of them. These differences are also reflected in the three, more specific, sub-actions that I positioned under each of the six main actions. I characterise the sub-actions in more detail in Chapter 3 and Chapter 4. I next discuss my methodological approach to my study, including the design of the tasks and the procedures that I used to collect and analyse learners' task responses.

Chapter 3: Methodology

In this chapter I provide an overview of how I responded to each of my four research questions. I explain how my lived experience served to align my philosophical standpoint with the tenets of pragmatism. I then justify the qualitative approach that I took and my rationale for the design of the tasks. I describe how I recruited the participants and I recount the methodological adaptations that I made in light of my pilot study. I follow this by justifying my choice and use of task-based interviews, and I detail how my method of data analysis led to my formulation of the GOLDEN framework. In justifying my methodological choices, I explain why I chose a particular method or instrument in preference to alternative methods and instruments.

I then discuss how I applied the GOLDEN framework to characterise and compare learners' actions within and across the pair of variation tasks in each domain. I follow this with an account of my ethical considerations and some of the methodological limitations of my study.

3.1 Introduction

In my study I primarily focused on characterising and describing links between learners' actions on pairs of variation tasks within and across the domains of number and geometry. As I stated in Chapter 1, four research questions provided and guided my focus. I next give a broad and concise description of how I responded to each of the four research questions:

(1) What might comprise a framework for describing learners' attention to structure?

I addressed this question by formulating the GOLDEN framework (see Appendix E) which I derived from a combination of the literature and by thematically coding the actions that learners performed on the variation tasks that I designed.

(2) What characterises Y8 and Y10 learners' attention to structure on an initial and a transfer proof variation task in the respective domains of number and geometry?

To respond to this question, in each domain separately, I used the GOLDEN framework to describe Y8's (then Y10's) actions on each respective variation task. I then used these descriptions to compare Y8's and Y10's actions (see Chapter 4, Section 4.2).

(3) What characterises Y8 and Y10 learners' attention to structure *across* an initial and a transfer proof variation task in the respective domains of number and geometry?

To address this question, in each domain separately, I used the GOLDEN framework to describe patterns and relationships in Y8's (then Y10's) actions across each respective domain-pair of variation tasks. I then used these descriptions to compare Y8's and Y10's actions (see, Chapter 4, Section 4.3).

(4) How do Y8 and Y10 learners' attention to structure compare across both domains?
To respond to RQ4, I first compared Y8's actions on the pair of number variation tasks to their actions on the pair of geometry variation tasks. I then repeated the comparison for Y10. I followed this by comparing Y8's and Y10's cross-domain actions (see, Chapter 4, Section 4.4).

My four research questions together with my philosophical standpoint influenced, indeed directed, my methodology and choice of methods. I next discuss the key aspects of my methodology, task design, and data collection and analysis. An explanation of my philosophical standpoint sets the scene.

3.2 Standpoint and philosophical position

Although the notions of *standpoint* and *philosophical position* are not mutually exclusive, I make a key distinction between the two terms. For me, a standpoint is a personally held research-based perspective imbued with the inherent core assumptions that one holds as a result of their lived experience. A researcher's standpoint influences, perhaps not very consciously, their perception of how their field of study is investigated and advanced; whereas, 'position' denotes for me a *chosen* or *adopted* philosophy that underpins one's methodological and reporting approaches. I see a philosophical position as a judiciously held stance based on a critical awareness of a position's strengths and limitations. I next justify my philosophical position in terms of its ontological and epistemological tenets.

My many years of experience as a teacher-researcher have resulted in me holding a post-positive standpoint. I hold this standpoint because I have come to appreciate that even highly quantitative (positivist) research is inevitably shaped by a range of subjective choices; e.g., what data to focus on and what calculations to use. At the same time, I value the use of descriptive and inferential statistics that support the narrative reporting of identified trends in largely qualitative research. Hence, my post-positivist standpoint is not a radical one. Rather, I accommodate qualitative and quantitative paradigms through an emphasis on mixed methods. Indeed, describing a study only in a delineated way (as either qualitative or as quantitative) is often too reductive. As Creswell (2003) pointed out, the best that can be said is that studies tend to be more quantitative or qualitative in nature. While post-positivist approaches such as mixed methods have done much to integrate paradigms (quantitative data can support qualitative data and vice-versa) (Sammons & Davis, 2016, p. 491), my own research has hitherto involved small sample sizes in which statistical 'support' for my qualitative findings has meant little. This led me to adopt the philosophical position of *pragmatism*, an ideology that prioritises the *usefulness* of research (Ling & Ling, 2020, p. 18; Scott, 2017, p. 243) and so allays any concerns that I have over conducting studies with a small *N*; and, simultaneously avoids the need to reconcile any qualitative subjectivity and quantitative objectivity (Bryman, 2006, p. 116).

To convey my ontological perspective, I next describe a scenario based on my educational view of proof that I described in Chapter 2. A norm in the educational view of proof is that a proof which a learner is asked to produce already exists and is known to their teacher. If a learner is unsuccessful in their attempt(s) to produce a proof, we can adopt the learner's reality that, for them, the proof does not exist because they do not have knowledge of it[1]. Alternatively, we can adopt a broader reality and say that the proof *does* exist but only to those who have knowledge of it. In the pedagogy of mathematics, holding the latter (dualistic) ontological perspective has been described as: *the acceptance of the objective existence of axiomatic truths while also accepting that, to unaware individuals, such truths do not exist* (see, e.g., Ernest, 1991, p. 17). As a teacher-researcher, this dualistic view of reality reflects my personal ontological belief that a teacher can reify (bring into existence) knowledge for their learners, in the same way that a researcher can reify knowledge for their research community. Epistemologically, I position myself at the intersection of these dual (learner and teacher) viewpoints because, as a teacher-researcher, my aim is to achieve intersubjectivity by bringing together these two realities. This aim involves seeing the learning of mathematics from the perspective of the learner. Hence, my personal epistemological position holds two key interpretivist assumptions: (1) I am accurately experiencing what the learner is experiencing, and (2) my findings will be interpreted by others in the way that I intended (see, e.g., Barnes and Bloor, 1982, p. 22). The notion of *interpretivism* (see, e.g., Scott, 2017, p. 257) therefore also applied to how I construed learners' actions on the tasks in my study, and is a key tenet of my pragmatist position.

3.3 Research type

I adopted a predominately qualitative (interpretivist) paradigm for my study. I took an analytical approach (in a largely non-statistical sense) to characterising, categorising and comparing learners' tasked-based actions. On occasion, I supplemented my characterisations of learners' actions with salient frequencies and comparative modal values, e.g., when distinctive patterns emerged in learners' cross-task actions. In doing so, I appropriated Sammons and Davis's (2016, p. 477) rationale that 'collected data can be qualitatively and quantitatively analysed and used to co-supportively link findings and reveal patterns'.

---

[1] A learner may also accept the existence of a proof despite being incapable of its production, which is similarly an indication of a need for the development of the learner's understanding.

I primarily identified, characterised and categorised learners' actions inductively, i.e., in a largely emergent way. I did this by analysing learners' task responses in an a largely emergent way. To aid my inductive characterisation of learners' actions, and as I explained earlier, I moved back and forth between learners' response data and the literature, while asking 'how well does this theory explain what I am observing?' I wanted to distance myself from any personally held preconceptions of what counts as an action and what actions learners would perform. However, I have long been familiar with two actions that are ubiquitous in the literature on mathematics education (generalising and specialising) and so to some extent I was expecting learners to perform expressions of these actions in my study. While I remained acutely aware of avoiding any circularity (confirmation bias) in my characterisation of learners' actions, this expectation nonetheless lent a convergent (deductive) (Clement, 2000, p. 558) quality to my otherwise grounded analytical approach. Indeed, few researchers can claim that they enter a field tabula rasa, unencumbered by notions of the phenomena they seek to understand (Flinders & Mills, 1993, p. xi; Y. Gu, 2014, p. 79).

In part, my study was also phenomenographical (Marton, 1986). Variation theory, on which I based the design of the proof variation tasks, is rooted in phenomenology because the theory primarily concerns *personal perceptions* of objects (phenomena) (ibid) brought to one's awareness by patterns of sameness and difference (Marton & Trigwell, 2000; Pang et al., 2017, p. 63). Hence, my analysis and categorisation of learners' discernment of sameness and difference, which one of my prompts in each domain-pair of variation tasks directly asks for, was phenomenographical.

3.4 Research design

My research design fit that of an empirical case study (Bassey, 1999, p. 58). Thus, it involved space- and time-bound (Tight, 2016, p. 385) small-scale instances (Yin, 2018, p. 53) of an investigated situation, i.e., learners' actions on proof-variation tasks. I chose a case study design because I wanted to micro-analyse the actions of a small number of participants ($n = 12$). Although the number of participants was somewhat modest, they nonetheless had potential to perform a broad enough range of actions across the number and geometry variation tasks to formulate an attention-to-structure framework. Further, because my focus was on developing and testing the applicability of a new framework, a larger sample size was, arguably, not necessary. While my sample size could not generate any statistical power, my adoption of a small-scale case study design meant that I could be particularly fastidious in regard to my analysis of the actions that each learner performed. Close and rich analysis of participant-specific data is, of course, a key affordance of a case study research design (Tight, 2016, p. 385).

To aid my comparative analysis of learners' actions I deliberately kept consistent five methodological dimensions of my study: (1) the recruitment procedure; (2) the design of the tasks that I gave to the learners; (3) the type and extent of my interaction with the learners when I conducted the task-based interviews; (4) an interview duration limit of 45 minutes; and, (5) three sub-actions for each of the six main actions in the GOLDEN framework. My rationale for the last consistency was that when I identified more than three sub-actions per main action, these additional sub-actions were yet further refinements of a previously identified sub-action. I limited to three the number of sub-actions per main action because I wanted the components of my framework to strike a balance between their relevance to my study and their adaptability for similar studies. A framework that is too context-specific may have little or no external applicability. On the other hand, a framework that is too general may have little or no analytical value.

My study focused on formulating and applying the GOLDEN framework which I based on the actions that learners performed on the variation tasks that I designed. Thus, my study was not an intervention, e.g., it did not involve cycles of fieldwork or introduce a change as does action research (Kemmis et al., 2014, p. 19). Rather, I offer my variation tasks and the GOLDEN framework as adaptable resources with which researchers and teachers can design their own interventions and analyse learners' responses.

The small size of my sample meant that taking an experimental approach to my study, e.g., by performing a randomised control test with comparative treatments, was not appropriate because the validity of any claims that I made on my findings would be weak. Nor did I consider survey research or any other non-interview research design that can severely limit the ability to interactively investigate a context (Yin, 2018, p. 50). Instead, I chose to conduct clinical task-based interviews (Ginsburg, 1981; Miller et al., 2014, p. 2) because I am formulating (and testing the applicability of) a new framework to describe learners' actions on the tasks that I designed. Further, as learners worked on the tasks, I wanted to ask them 'think aloud' questions (see, e.g., Leighton, 2017, p. 13) to reveal their mathematical minds in action, i.e., to gain access to their reasoning actions. I chose to audio/video-record and transcribe the task-based interviews so that I could: (1) (re)analyse learners' actions at a more practical speed and in greater depth (Ingram & Elliott, 2019, p. 192), (2) draw on a combination of learners' utterances and their accompanying written productions (ibid), and (3) respond more comprehensively to all of my research questions.

3.4.1 Pilot study

To determine if the design of the variation tasks encouraged actions that were diverse enough for my formulation of an attention-to-structure framework, I first tested the tasks in pilot interviews. Eight learners took part in the pilot interviews (two pairs of Year 8 learners and two pairs of Year 10 learners). The pilot study also served to reveal two unanticipated impediments, i.e., any improvements that I could make, to the design of the interviews. I describe later in this section the two improvements that I made to the interviews in light of my pilot study. I next explain how I recruited the learners for the pilot study. Incidentally, I recruited all of the learners for the pilot study and for the main study at the same time, i.e., at one recruitment meeting. In this section, I focus on the pilot study and its implications for the design of the main study. I discuss my recruitment of participants more broadly in Section 3.6 ‘Sampling strategy’.

I conducted my study in the dual role of teacher-researcher. All of the learners that I invited to take part the pilot study (and main study) were from a Y8 group and a Y10 group that I taught for four 1-hour lessons per week in my role as a mathematics teacher at the participant school. Approximately half of the learners that I invited were in the Y8 group and approximately half were in the Y10 group. I made all of the learners in each group aware of my research at the end of one of their respective mathematics lessons. At this time, I also informed the learners about a voluntary ‘research information and recruitment’ meeting that would take place the following day during the second half of their lunchtime. I instructed all of the learners who wished to attend the meeting to eat their lunch beforehand and, as each attendee arrived at the meeting, I gained verbal confirmation from them that they had done so. I also applied this eating stipulation to all of the learners in the pilot and main interviews, all of which took place during learners' lunchtimes after they had eaten their lunch (see Section 3.10 ‘Ethical considerations’ for further details). At the research information meeting I recruited the desired number of participants, i.e., two pairs of Y8 learners and two pairs of Y10 learners for the pilot study, and three pairs of Y8 learners and three pairs of Y10 learners for the main study. I recruited each pair of learners on a voluntary first-come basis regardless of their prior attainment. All of the learners that I recruited for the pilot study were different to those that I recruited for the main study.

At the research information meeting I explained to the attendees the purpose and nature of my study, the voluntary conditions, their right to withdraw at any point without being viewed negatively, and the anonymised ways in which I would collect and use their task response data. I then requested (and gained) verbal assent to participate from each of the learners that wished to take part in the research. At this point, I again reassured the learners that they were in no way obliged to participate and that they could withdraw at any time without needing to give an explanation. Once I had recruited the desired number of participants, and to mitigate against any feelings of rejection, I explained politely to any further learners who were willing to take part that I now had enough participants, and I thanked them for their interest. I recruited learners in pairs in the hope that duos would promote more fruitful discussions in the task-based interviews than learners working independently. However, the pilot interviews revealed two key insights that led me to interview learners individually in the main study: (1) one learner in each pair tended to assume dominance over the tasks and talk (despite my efforts as interviewer to include both learners), and (2) on the few occasions when both learners interacted with each other, these often led to simultaneous utterances that, at their point of co-occurrence, proved difficult to separately discern and probe. Further, in moments when both learners were silently contemplating a task, I found myself torn between who to probe with 'think aloud' questions (for a list of these questions, see Appendix C). I discuss the ethical implications of adapting the number of interviewees from two to one per sitting later in Section 3.10.1.

The second adaptation that I made in light of the pilot study concerned the number of interviews that I asked each learner to participate in. In the pilot study, I asked each pair of learners to participate in two interviews (one for the number task and the other for the geometry task) but timetable constraints in the participating school meant that these two interviews took place across different intervening durations that were sometimes as far as seven days apart. I viewed these differences in between-interview durations as an extraneous variable. I also felt that these intervening durations posed a potential obstacle to a learner's continuity of thought across the number and geometry tasks, and so in the main study I gave each learner both tasks in one interview.

A key reason that I conducted the pilot interviews was to determine if the tasks that I designed encouraged the learners to collectively perform a range of actions that could be used to formulate an attention-to-structure framework. At this point I felt that my control of (in)variance in the design of the variation tasks paid off, because the answer was in the affirmative. In each domain-pair of variation tasks, my deliberate inclusion of invariant and variant features, e.g., recursive patterns and dependency relationships, resulted in the pilot learners performing a range of actions that were broadly comparable in both domains. These actions gave me an early indication of some viable main themes for an attention-to-structure framework, e.g., specialising, generalising, and deconstructing actions. I added further actions, as well as sub-actions, when I analysed learners' attention to structure in the main study. I discuss the development of the framework in more detail in Section 3.9.

3.5 Extenuating circumstances

For my main study, and as I described earlier, I intended to collect task-based interview data from 12 participants (six Y8 learners and six Y10 learners). My timeframe for the collection of this interview data was six weeks, i.e., two interviews per week. However, nearly halfway through the 12 interviews, an unexpected and imminent pandemic-related school closure meant that the final number of participants in my main study was reduced to six (three Y8 learners and three Y10 learners). Nonetheless, I was still able to achieve the main aims of my study which were to formulate and illustrate the utility of an attention-to-structure framework. In this sense, sample size becomes less important.

3.6 Sampling strategy

I conducted the study in a state secondary school in England, where I have worked as a mathematics teacher for the past six years. Six learners that I regularly taught (originally 12) participated in the main study; these comprised three Year 8 'set two' learners (12- to 13-year-olds) and three Year 10 'set two' learners (14- to 15-year-olds). As I explained earlier, I recruited the learners for the main study at the same time and in the same way as I recruited the learners for the pilot study. For the reasons that I described earlier, I recruited learners in pairs both for the pilot study and for the main study. However, following the paired interviews in pilot study, I chose to interview learners individually in the main study. This prompted a concern that some learners may longer wish to participate in the main study because they were expecting to be interviewed in pairs. In anticipation of this possibility, I explained to the learners in the main study my reasons for changing from paired to individual interviews. I then offered these learners three options: (1) they could withdraw if they no longer wished to participate; (2) they could invite a non-participating friend to accompany them in the interview (class)room, or (3) they could continue to participate and be interviewed individually if they were happy to do so. All of the learners that I recruited for the main study chose to continue their participation in light of the change to individual interviews, and only two learners elected to be accompanied by a non-participating friend.

I considered all of the participants to be an opportunistic sample rather than a convenience sample because they were all learners that I regularly taught rather than learners that were the most immediately available to invite. My sample was also non-random and non-representative because my recruitment strategy involved explaining to learners that I required volunteers who are willing to be anonymously videoed as they discuss their task-based thinking with me. These conditions may have attracted learners with higher levels of self-efficacy than the learners who did not volunteer to participate. Yet, this potential bias can be seen as largely incidental because my focus was on the actions that learners performed rather than their own sense of self-proficiency. My study was also cross-sectional (Marvasti & Freie, 2016) because I collected all of my main data, from each respective interviewee, at one timepoint rather than at multiple points over time.

I chose Year 8 and Year 10 participant groups because I wanted my sample to include learners with different durations of past secondary education experience. I avoided inviting Year 7 learners to participate because they were still relatively new to secondary school, and I avoided Year 11 learners so as not to interrupt their preparations for their GCSE examinations. I originally considered using for my final data set only the actions of learners that I deemed to provide particularly comprehensive combinations of utterances and written productions. Thus, my originally intended sampling method had a purposive bias. However, each learner in the main interviews performed a rich array of actions and so, with further impetus from the pandemic-related curtailment, I decided to use the data generated by all six main participants.

3.7 Task design: Rationale and formulation

As I outlined Section 1.2, I designed a pair of variation tasks in each of two distinct domains, i.e., number and geometry (the tasks in both domains involved algebra). I discuss in this section my reasons for drawing on the constructs of variation (Marton & Booth, 1997) and transfer (see, e.g., Lobato, 2012) when I designed the number and geometry tasks. I then narrow my focus and explain my rationale for the design of each respective prompt in each of the tasks.

Broadly, my design of the tasks was influenced by a confluence of the theoretical constructs that I drew on and my intention to create situations that encouraged learners new to formal notions of proof to perform a range of proving actions. I could then use the actions that learners performed to answer my research questions. Namely, by formulating an attention-to-structure framework and by characterising links in learners' actions within and across the tasks in both domains.

The design of the tasks was also influenced by my two decades' experience as a secondary school teacher that, perhaps inevitably, have led to my view of the purpose of proof as primarily educational. That is, as an area of mathematics, the development of which strengthens learners' understanding of the subject. Accordingly, my view of the nature of proof has been shaped by the style and demand of proof tasks that I have seen in secondary school curriculum resources. Albeit indirectly, I therefore expect that to some extent my exposure to curriculum resources further influenced my design of the proof tasks.

3.7.1 Rationale for drawing on variation

Variation tasks can help learners to make meaningful connections (Peng et al., 2017, p. 122) and mathematical proficiency can be manifested by engagement in varying problems (Gu et al., 2004, p. 322). Yet, some researchers have cautioned that control of variation does not necessarily lead to the development of skills (Mok et al., 2008; Peng et al., 2017). This ambiguity is echoed by Cai and Nie's (2007) claim that teaching with variation makes sense to foster students' learning but lacks enough empirical studies to verify it. I contribute to clarifying this ambiguity with my use of variation tasks in my study. In doing so, I simultaneously respond to Stylianides and Stylianides' (2017, p. 124) call for further insights into how proving can be effectively learnt and taught.

Another reason for my use of variation tasks is their potential to remedy a major problem identified by Stylianides (2008). The problem concerns how proof at secondary school level is often taught formally and without adequate scaffolding (p. 9). My use of variation tasks deliberately and directly addresses both of these concerns. Not only are my variation tasks intended to be accessible to learners at Key Stage 3, they also provide iterations of ready-made proof procedures (arguments) that afford the identification of underlying structural relationships, thereby offering a form of scaffolding. Used in this way, variation tasks can open a window on proof argumentation that existing curriculum resources often leave closed. Indeed, little is known about how the use of variation tasks can facilitate learners' development of proving. Further justification for my use of variation tasks lies in the ranging opportunities that they afford to characterise learners' attention to structure (which all of my research questions focus on). For, variation tasks, when purposefully designed, make available a range of mathematical features and relationships for learners to potentially discern, attend to, and reify (see, e.g., Runesson, 2006). All or any of which might influence the way that a learner engages with a task that they perceive as conceptually similar to a task that they have previously met (as was my intention with the transfer task that I designed in each domain-pair of variation tasks).

Each of the initial and transfer variation tasks that I designed illustrated a sequence of arguments (and accompanying propositions) that featured consecutive numbers – see prompts 2a and 3 in Appendices D and E, respectively. I rooted the design of the argument sequences in the construct of variation because the control of (in)variance, when used strategically, affords learners opportunities to make sense of new situations (Marton & Pang, 2006; Watson & Mason, 2006, p. 92), and so they were well suited to the learners in my study, all of whom were new to proof arguments. Variation tasks afford sense-making opportunities because, when designed and employed judiciously, they provide learners with the means to compare provided variations and to discern difference against a backdrop of consistency, or consistency against a backdrop of difference (Mason, 2017a). In other words, if learners are offered several examples, they can start making sense of them by doing what comes naturally, that is, identifying what is the same and what is different about the constructions (Watson & Mason, 2005, p. 106). The regulation of sameness and difference in the design of a variation task can encourage a learner's engagement because the control of (in)variance can trigger noticing (Kullberg et al., 2017, p. 567), and noticing can trigger further interaction with a task (Mason, 2002, p. 7). I held these affordances of variation in mind when I designed the sequences of iterated arguments in my tasks. The argument iterations can also be seen as pre-performed empirical tests that might encourage learners to generalise about the arguments as a whole, rather than simply engaging in the well-documented activity of testing individual values (see, e.g., Küchemann & Hoyles, 2009; Stylianides & Stylianides, 2009b).

When I designed the variation tasks, I also considered what kind of mathematical actions are likely to support learners' proving actions. To do this, I asked myself 'what in the variation tasks could I make available to notice, act on, and learn in order to engage learners in processes of proof?' As Watson (2010, p. 138) pointed out, to learn new abstract ideas, it is the structures of, qualities of, and relations within generalisations that have to be identified and compared. When I designed the variation tasks, I drew on a synthesis of Watson's focus on generalisations and on Mariotti and Pedemonte's (2019, p. 759) assertion that the approach to proof is more accessible to learners when the problem requires the construction of a conjecture. My variation tasks were also influenced by Stylianides and Stylianides' (2006a, p. 203) observation that patterns can give rise to conjecturing actions, which in turn motivate the development and validation of arguments that may or may not lead to a proof. With these three interrelated suggestions in mind, I designed for learners sequences of argument variations that, in each domain, afforded the discernment of a general conjecture. To create these affordances, I controlled the (in)variance in each sequence of propositions/arguments in such a way as to provide learners with the opportunity to discern *critical features*. With critical features I mean the features that make the difference between whether the subject matter is mastered or not (Pang & Marton, 2007, p. 4). For example, in the number initial variation task, I considered the critical feature to be the consistent relationship between the predicted answer and the addend in each respective argument. Specifically, the predicted answer is always half the value of the addend in each respective argument (see Appendix A, prompt 2a). Discerning such critical relationships or patterns might, I hoped, prompt learners to then reason about and manipulate, i.e., act on, what they discerned. More specifically, what a learner discerns from the arguments in each respective initial variation task could serve as a starting point from which they might: (1) develop their comprehension of the provided arguments; (2) explain the logic underlying the truth of the accompanying propositions; (3) write a further (non-consecutive) iteration of the provided arguments; and (4) write or complete more complex proof-related structures in the transfer tasks. All of which involve performing a range of actions that I intended to characterise.

While the four points in the aforementioned trajectory represent my desired learning outcomes, i.e., my *intended objects of learning* (Marton & Pang, 2006), I did not assume that learners would necessarily perform the kind of helpful actions that I hoped they would perform. Although I included in the design of the tasks several critical features that learners might act on, I remained acutely aware throughout the task design process that these features and actions were personally conceived and so may differ from what the learners might act on.

Despite the powerful potential that the principles of variation hold for the design of mathematical tasks (see, e.g., Runesson, 2005), I could not find any proof-based tasks that provide learners with numerically consecutive arguments, as do the variation tasks in my study. As I described earlier, the design of a variation task, i.e., the control of variant and invariant features, can influence what features of mathematical structure a learner discerns and acts on. Which, in turn, can influence what actions the learner might then perform on, i.e., transfer to, a conceptually-similar task (Marton, 2006, p. 499). The construct of *learning transfer* influenced my design of the transfer variation task in each domain, discussion of which I next turn.

3.7.2 Rationale for drawing on transfer theory

As part of my characterisation of learners' attention to structure I wanted to identify, within and across both domains, the actions that learners performed on the more structured, hence easier, initial variation tasks, and then also performed on the less structured, hence harder, transfer variation tasks. More specifically, my aim was to describe how actions performed on the initial variation tasks might contribute to success on the transfer variation tasks. The premise of this aim is that what is experienced in one situation can influence what is possible in a subsequent situation, which is how Marton (2006) broadly frames the notion of transfer. In step with Gu et al's (2004, p. 322) comment that mathematical proficiency can be manifested by engagement in transferring strategies, I wanted to characterise helpful transferred actions because they might hold particular importance for advancing learners in the area of proof and proving.

Conceptions of transfer encompass a range of key perspectives. Namely, transfer seen as: *direct transportation* (Thorndike, 1906) e.g., an identically repeated action triggered by an identical situation, such as always substituting a same value for x; *transformation* (Greeno et al., 1993) e.g., adjusting a known computational process, altering a requirement of a task, or reformatting a structure; *incrementally cued actions* (Wagner, 2010) e.g., using a known method to expand a bracket then spotting and performing a resultant simplification; *production of sameness* (Lobato, 2003) e.g., generating examples of a known procedure or representing the same concept in different ways; and, *underlying influence* (Marton, 2006) e.g., a secondary effect of noticing sameness or difference across situations. Transfer can, then, be directly linked to outcomes such as successful task completion or fruitful or efficient actions. I identified these links as part of my characterisation of learners' actions in my study because they provided valuable insights into how learners' proving processes might next be advanced.

To restate from earlier, I considered learners' actions to be the 'objects of transfer', i.e., what is transferred. However, I am not suggesting that the actions that a learner performed on a transfer variation task were newly acquired by them on the immediately prior initial variation task, although this may sometimes have been the case. Rather, I considered a transferred action to be any action performed on an initial variation task that was qualitatively replicated (or of discernible influence) on a transfer variation task. I elaborate on my design of the transfer tasks in the next section, where I explain my rationale for each of the prompts in the number and geometry tasks.

3.7.3 Rationale for the design of the prompts

Five key stimuli shaped my design of the number and geometry tasks: (1) the call for activities that encourage and support learners' engagement in processes of proving (e.g., Stylianides & Stylianides, 2017, p. 124); (2) the need to provide the learners with examples of ready-made proof arguments (since all of the learners were new to any formal concept of proof); (3) the utility of variation for initiating sense-making in new situations (Marton & Booth, 1997); (4) the affordances that conceptually similar tasks have for transfer; and, (5) my own experience of the nature of proof tasks as represented in secondary school curriculum resources.

In each respective domain, I designed an isomorphic sequence of three prompts aimed at revealing four aspects of learners' responses: (1) their competence in underlying concepts relevant to a subsequent domain-specific pair of variation tasks (prompt 1); (2) their discernment of sameness and difference across the provided argument variations (prompt 2a); their actions when asked to explain and show the truth of the given propositions, and when asked to write a case that was conceptually similar to those provided (prompts 2b to 2d); and, (4) evidence of transfer (prompt 3).

I first designed the sequence of three prompts for the number task, then I designed an isomorphic sequence for the geometry task. I chose to design tasks that were situated in two different domains for the purposes of comparative analysis and because I wanted to avoid the domain restriction of learners working only on the number task. To aid my comparative analysis of learners' actions across the number and geometry tasks, I kept consistent three key elements across in the sequence of prompts in each domain: (1) an introductory prompt featuring a given five-step argument designed to orient learners to concepts in the following pair of variation tasks; (2) an initial variation task that featured three consecutive-number iterations of a five-step argument, in relation to which I asked learners to write a similar (non-consecutive) case; and, (3) a transfer variation task that was conceptually similar to the initial variation task, in relation to which I again asked learners to write a similar (non-consecutive) case.

I next explain in more detail my rationale for designing and employing each of the three prompts in each domain. Namely, the introductory prompt (prompt 1), the initial variation task (prompt 2) and the transfer variation task (prompt 3). I follow this by elaborating on my rationale for the consistencies in the isomorphic design of the prompts in both domains.

3.7.3.1 Number introductory task

My intention with the introductory task in each domain was to orient and prepare the learners for the concepts in the subsequent domain-pair of variation tasks. Responses to the introductory task in each domain also provided me with an additional source of data, i.e., a contextual backdrop, against which I could compare learners' actions on the variation tasks.

The number introductory task featured two sub-prompts (prompts 1a and 1b). The first sub-prompt featured a proposition and an accompanying argument comprising five steps (assertions) that, when followed, always resulted in the number 3 for any initially chosen value. I then asked learners, still in relation to prompt 1a, to show that the given argument will always be true, i.e., will result in 3 for any initially chosen value (see, Figure 3.1). I used prompt 1a because I wanted to assess the extent to which learners might generalise, i.e., represent algebraically the steps in the provided argument (which was conceptually similar to the arguments featured in the following initial variation task). I suspected that some, if not all, of the learners would know how to represent the five steps algebraically but, since they were new to proof arguments, would not think to do so. In anticipation of this, I used a further (conditional) prompt (see, Figure 3.1, prompt 1b) that directly asked learners who did not represent algebraically the five steps in the argument, to do so.

Figure 3.1: The number introductory task.

1a) Jack says that if you pick any number and do the following steps, the answer will always be 3.

Step 1: Choose any number
Step 2: Multiply it by 2
Step 3: Add 6
Step 4: Divide by 2
Step 5: Subtract the chosen number

Try to show that Jack's statement is always true.

1b) If you haven't already, try to write the above 5 steps as algebra statements.

My reason, in prompt 1b, for directly asking learners to express algebraically the steps in the argument was to ready them for the kind of algebraic thinking that would be helpful in the following variation tasks.

3.7.3.2 Number initial variation task

The number initial variation task consisted of four sub-prompts (prompts 2a to 2d). Prompt 2a featured three consecutive-number iterations of a proposition and a five-step argument that were similar to the proposition and argument in the introductory prompt. For any initially chosen number, the three arguments always resulted, respectively, in 3, 4 and 5 (see, Figure 3.2). I then asked learners to compare the three given arguments and identify any elements of sameness and of difference. Probing learners' perceptions of sameness and difference across the three arguments enabled me to see what each learner saw, i.e., foregrounded, when they initially looked at the argument variations. These phenomenological (Marton, 1986; Marton & Neuman, 1996, p. 317) insights gave me an additional perspective with which to contextualise and characterise learners' actions. For example, some of the actions that were performed on the variation tasks were common only to learners who foregrounded an element of sameness across the three arguments.

With prompt 2b, I elicited learners' reasoning actions by asking them to explain why the predicted results of 3, 4 and 5 are always true for the respective given arguments. To facilitate learners' reasoning, I controlled the (in)variance of features (dimensions) in the arguments to make available to discern and learn a range of structural elements and concepts, e.g., recursive patterns, bivariate relationships, and the notion of 'doing and undoing' (see, Figure 3.2, prompt 2b). I also wanted learners to write a new case for a proposition similar to the given propositions. To do this, with prompt 2c, I asked learners to show how the steps in the three previously given arguments could be used to always get an answer of 7 for initially chosen number. I chose 7 as the target answer because it was non-consecutive to the results of 3, 4 and 5 in the given arguments, and so might discourage learners from thinking merely in terms of a superficial next pattern.

Figure 3.2: The number initial variation task.

2a) Sam says that if you pick any number and follow the steps on the left, the answer will always be 3, if you follow the steps in the middle the answer will always be 4, and if you follow the steps on the right the answer will always be 5.

Answer always 3	Answer always 4	Answer always 5
Step 1: Choose any number (n)	**Step 1:** Choose any number (n)	**Step 1:** Choose any number (n)
Step 2: $2n$	**Step 2:** $2n$	**Step 2:** $2n$
Step 3: $2n + 6$	**Step 3:** $2n + 8$	**Step 3:** $2n + 10$
Step 4: $\frac{2n+6}{2}$	**Step 4:** $\frac{2n+8}{2}$	**Step 4:** $\frac{2n+10}{2}$
Step 5: $\frac{2n+6}{2} - n$	**Step 5:** $\frac{2n+8}{2} - n$	**Step 5:** $\frac{2n+10}{2} - n$

What is the same and what is different about these three methods?

2b) Try to *explain* why Sam's predicted answers always come true.

2c) Try to use Sam's steps to *show* how you can always get an answer of 7.

2d) Try again to *explain* why Sam's predicted answers always come true.

Prompt 2d was the same as prompt 2b, i.e., I again asked learners to explain why the predicted results of 3, 4 and 5 are always true for the given arguments. My purpose in repeating this prompt was to reveal if and how a learner's interim attempt to write a case that always resulted in 7 (prompt 2c) had augmented their prior reasoning about the truth of the propositions that accompanied the given arguments.

3.7.3.3 Number transfer variation task

To afford the transfer of actions that learners performed on the initial variation task (prompt 2a) I designed a conceptually similar 'transfer variation task' (prompt 3). As with the initial variation task, the transfer task featured iterated arguments that produced a consistent predicted result for any initially chosen number. On the initial variation task, one of the given consecutive-number arguments featured a procedure for always obtaining a result of 5 (see Figure 3.2); whereas, the transfer variation task featured three examples of a slightly different procedure for always obtaining a result of 5 (see Figure 3.3). The transfer task also featured an algebraic proof for the exemplified procedures (see, Figure 3.3). The transfer task was (deliberately) more challenging than the initial task because generalities were more difficult to identify. I then asked learners to use the given examples and the given proof to show algebraically how to always get an answer of 7, which was the same target answer that I used in the initial task.

Figure 3.3: The number transfer variation task.

3) Luka looks at Sam's way of getting an answer that is always 5, and says that he knows a slightly similar way to always get an answer of 5 (shown below).

Answer always 5	Example A
Step 1: Choose any number	6
Step 2: Add the next number	6 + 7 = 13
Step 3: Add 9	13 + 9 = 22
Step 4: Divide by 2	$\frac{22}{2} = 11$
Step 5: Subtract the chosen number	11 – 6 = 5

Answer always 5	Example B
Step 1: Choose any number	7
Step 2: Add the next number	7 + 8 = 15
Step 3: Add 9	15 + 9 = 24
Step 4: Divide by 2	$\frac{24}{2} = 12$
Step 5: Subtract the chosen number	12 – 7 = 5

Answer always 5	Algebraic Proof
Step 1: Choose any number	n
Step 2: Add the next number	$n + (n + 1) = 2n + 1$
Step 3: Add 9	$2n + 1 + 9 = 2n + 10$
Step 4: Divide by 2	$\frac{2n + 10}{2} = n + 5$
Step 5: Subtract the chosen number	$n + 5 - n = 5$

Try to show *using algebra* how Luka's method can be used to always get an answer of 7.

As with the initial variation task, I asked learners to show a procedure that always resulted in 7 because it was non-consecutive to the given examples (that always resulted in 5). Again, I felt that this non-consecutiveness might discourage learners from thinking only in terms of superficial patterns. To further afford the transfer of actions, I deliberately included in the design of the transfer task several conceptual and structural elements that were present in the initial variation task. Consistencies in the initial and transfer variation tasks included: (1) illustrations of at least two arguments, i.e., worked examples; (2) given iterations of a five-step argument, of which the first step was to choose any number; (3) a given proposition and five-step arguments that always resulted in a predicted answer regardless of the initially chosen number (of which at least one always produced a result of 5); (4) argument structures with an addend that co-varied with the predicted result in the proposition; (5) a divisor of 2 (in step 4) that leaves a final step of subtracting the initially chosen number; (6) an element of constructing and deconstructing ('doing and undoing'); (7) several other consistent numerical relationships across the sequences of given arguments; and, (8) a requirement to use the given structures to write a non-consecutive iteration that always results in 7.

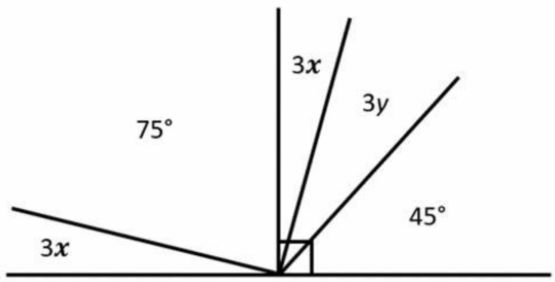
3.7.3.4 Geometry introductory task

As I mentioned earlier, I designed the geometry task with a sequence of three prompts that were isomorphic to the three prompts in the number task. Namely, an introductory task, an initial variation task, and a transfer variation task. In parallel with the number introductory task, the geometry introductory task featured a given five-step argument (prompt 1) intended to orient and prepare learners for concepts that were inherent in the following initial and transfer variation tasks. The geometry introductory task featured a diagram comprising a series of five angles on a horizontal line segment. Of the five angles, two were numeric and three were algebraic. Of the three algebraic angles, two were in x and one was in y (see, Figure 3.4).

Figure 3.4: The geometry introductory task.

1) Maddison says that for the diagram below she can show that **$y = 2x$**.

(Not to scale)



Maddison's method

Step 1: The angle sum to the left of the vertical line is 90°, so set up the equation $3x + 75° = 90°$

Step 2: Solve $3x + 75° = 90°$ by doing 90° - 75° = 15°, then do 15° ÷ 3 to give **$x = 5°$**

Step 3: The angle sum to the right of the vertical line is 90°, so using $x = 5°$, set up the equation $3y + 15° + 45° = 90°$

Step 4: Solve $3y + 15° + 45° = 90°$ by doing 90° - 45° - 15° = 30°, then do 30° ÷ 3 to give **$y = 10°$**

Step 5: State that **$x = 5°$** and **$y = 10°$**, so **$y = 2x$**

When you consider the diagram and steps shown above, which mathematical facts and ideas come to mind?

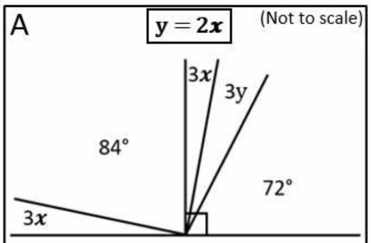
The accompanying five-step argument involved geometric and algebraic reasoning and showed a valid procedure for the proposition $y = 2x$. I then prompted learners to look at the given diagram and argument and say what associated facts and ideas came to mind. Alternately put, I designed the introductory task to encourage each learner's evocation of a situated concept image (Tall & Vinner, 1981) associated with the diagram and procedure that I provided. A learner might, for example, bring to mind relevant angle properties or situated equations that they may find useful when working on the following initial and transfer variation tasks.

3.7.3.5 Geometry initial variation task

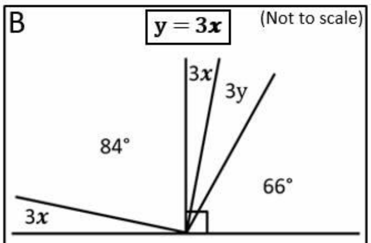
In parallel to the number task, the initial variation task in the geometry domain consisted of four sub-prompts (prompts 2a to 2d) (see, Figure 3.5). Prompt 2a, again in parallel to the number task, featured three consecutive-number iterations of a five-step argument that were similar to the argument given in the introductory prompt. These three arguments showed, respectively, that $y = 2x$, $y = 3x$ and $y = 4x$. Across these argument structures, I asked learners to identify sameness and difference. To encourage reasoning actions, with prompt 2b I then asked learners to verbally explain what a diagram (similar to the diagrams that accompanied the given arguments) would look like that could be used to show that $y = 6x$. For the same reason that I gave for the corresponding number task, I deliberately asked learners for an outcome with a numerical value that was not consecutive to those in the outcomes in the given iterations. With prompt 2c, I asked learners to use the steps in the given arguments to show what the steps would look like for a diagram in which $y = 6x$. With prompt 2d, I again asked learners to describe what a diagram (similar to the diagrams that accompanied the given arguments) would look like that could be used to show that $y = 6x$.

Figure 3.5: The geometry initial variation task.

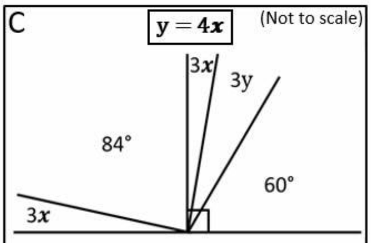
2a) Eve looks at Maddison's method, and says that by using similar steps she can show for diagram A that $\mathbf{y = 2x}$, for diagram B that $\mathbf{y = 3x}$, and for diagram C that $\mathbf{y = 4x}$.



Step 1: $3x + 84° = 90°$
Step 2: $90° - 84° = 6°$, then do $6° \div 3$ to give $\mathbf{x = 2°}$
Step 3: $3y + 6° + 72° = 90°$
Step 4: $90° - 72° - 6° = 12°$, then do $12° \div 3 =$ to give $\mathbf{y = 4°}$
Step 5: state that $\mathbf{x = 2°}$ and $\mathbf{y = 4°}$ so $\mathbf{y = 2x}$



Step 1: $3x + 84° = 90°$
Step 2: $90° - 84° = 6°$, then do $6° \div 3$ to give $\mathbf{x = 2°}$
Step 3: $3y + 6° + 66° = 90°$
Step 4: $90° - 66° - 6° = 18°$, then do $18° \div 3 =$ to give $\mathbf{y = 6°}$
Step 5: state that $\mathbf{x = 2°}$ and $\mathbf{y = 6°}$ so $\mathbf{y = 3x}$



Step 1: $3x + 84° = 90°$
Step 2: $90° - 84° = 6°$, then do $6° \div 3$ to give $\mathbf{x = 2°}$
Step 3: $3y + 6° + 60° = 90°$
Step 4: $90° - 60° - 6° = 24°$, then do $24° \div 3 =$ to give $\mathbf{y = 8°}$
Step 5: state that $\mathbf{x = 2°}$ and $\mathbf{y = 8°}$ so $\mathbf{y = 4x}$

What is the same and what is different about illustrations A, B and C?

2b) Try to describe what a similar diagram would look like that could be used to show $y = 6x$.

2c) Considering your response to **2b**, try to use Eve's steps to show that $y = 6x$.

2d) Try again to describe what a similar diagram would look like that could be used to show $y = 6x$.

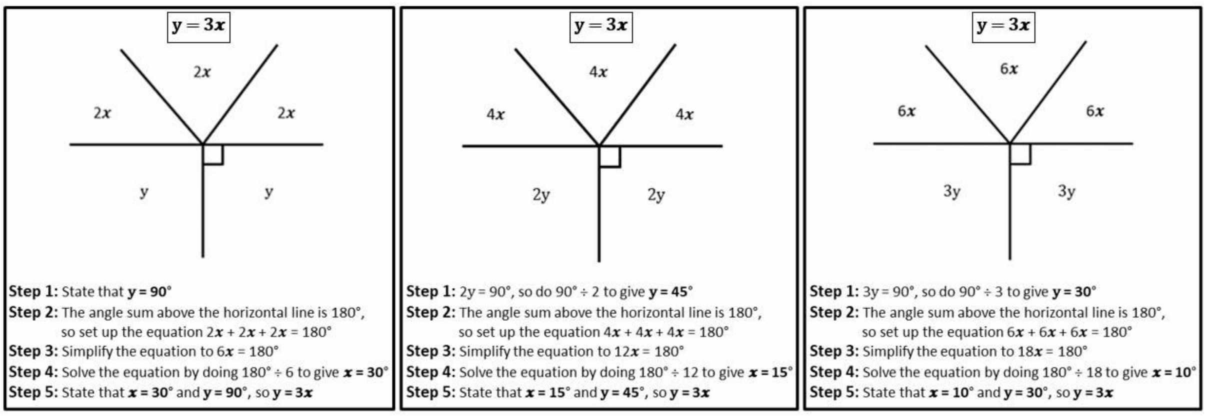
As in the number task, prompt 2d was a repetition of prompt 2b because I wanted to reveal if and how a learner's interim actions had augmented their prior reasoning. In the case of the geometry task, my reason for this repetition was to discover if a learner's interim attempt to show the steps for a diagram in which $y = 6x$ (prompt 2c) had advanced their reasoning about what a diagram would look like that could be used to show that $y = 6x$.

3.7.3.6 Geometry transfer variation task

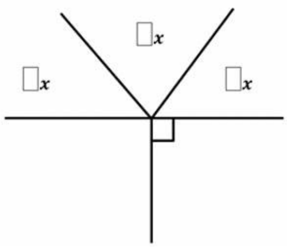
Again in parallel to the number transfer task, my rationale for the design of the geometry transfer task (prompt 3) was to afford the transfer of actions performed on the initial variation task (prompt 2). To create this affordance, and still in parallel with the number task, I designed a transfer task that was conceptually similar to the initial variation task. On the initial variation task, one of the three given arguments featured a procedure for showing that $y = 3x$ (see Figure 3.5); whereas, the transfer variation task featured three examples of a slightly different diagram and procedure, all of which showed that $y = 3x$ (see Figure 3.6). As in the number domain, the transfer task was (deliberately) more challenging than the initial task because generalities were more difficult to identify. With the transfer task, as I did with prompts 2b and 2d in the initial task, I asked the learners to use the given diagrams to illustrate what a similar diagram would look like that could be used to show that $y = 6x$. To assist learners' responses to the geometry transfer task, I included a template of the diagrams illustrated in the given variations.

Figure 3.6: The geometry transfer variation task.

3) Hannah used Eve's way of showing that $\mathbf{y = 3x}$ to show, in a similar way, that $\mathbf{y = 3x}$ for each of the diagrams below.



Can you use the diagram below to show that $\mathbf{y = 6x}$? (You can use Hannah's steps to help you, if you like).



As with the number task, to further afford the transfer of actions, I deliberately included in the design of the transfer task several conceptual and structural elements that were present in the initial variation task. Consistencies in the initial and transfer variation tasks included: (1) three iterations of a geometric diagram and a five-step argument; (2) arguments that satisfy a given proposition in the form of an equation in y and x (of which one involves the proposition that $y = 3x$); (3) argument structures that involve reasoning about angle sums; (4) an element of constructing and deconstructing, i.e., forming and solving equations in x and in y, respectively; (5) consecutive numbers, i.e., a value that is incremented by 1, across the three given iterations; (6) several other consistent numeric and algebraic relationships across the sequences of given arguments; and, (7) a requirement to use the given argument structures to write a non-consecutive argument for the proposition that $y = 6x$.

3.7.4 Task design consistencies across domains

In addition to comparing learners' actions on the pair of variation tasks within each respective domain, I also compared learners' actions on the pairs of variation tasks *across* domains. To facilitate my cross-domain analysis of learners' actions, and as I described earlier, I designed in each domain a sequence of three parallel prompts (an introductory task, an initial variation task, and a transfer variation task). I next elaborate on the cross-domain consistencies in the design of each pair of parallel prompts.

The introductory task in each domain (shown in Figures 3.1 and 3.4, respectively) featured a five-step argument designed to encourage the learners to recall concepts that were relevant to the subsequent domain-pair of variation tasks. As I further explain in Section 3.9.1, responses to the introductory tasks also provided me with a backdrop against which to compare learners' actions on the variation tasks. (In Sections 3.9.1 and 3.9.2, I elaborate on how I analysed learners' responses to all of the tasks in my study).

As with the introductory tasks, I used a consistent rationale for the design of both domain-pairs of variation tasks. Cross-domain consistencies in my design of the variation tasks involved: (1) control of (in)variant features to make available to discern and learn a range of structural elements and concepts (Marton, 2015), e.g., recursive patterns, dependency relationships, and co-variance; (2) consideration of the actions that learners might perform (see, e.g., Kieran et al., 2015, p. 47; Vale et al., 2018) as a result of discerning structural elements e.g., identifying a pattern in the given arguments might lead to an extending action or to reasoning about a conjecture (Stylianides & Stylianides, 2006a, p. 203); and, (3) conceptual similarity in the respective domain-pairs of variations tasks to afford the capture and characterisation of learners' commonly performed, i.e., transferred, actions. The latter rationale is consistent with Zawojewski and Silver's (1998) claim that variation can be used as a tool to scaffold the construction of different tasks that are conceptually related.

To aid my close cross-domain comparative analysis of learners' actions, I kept consistent four key structural features in both domain-pairs of variation tasks: (1) three worked examples in the initial variation tasks, and at least two worked examples in the transfer variation tasks; (2) initial variation tasks that featured consecutive-number iterations of a five-step argument; (3) an element of constructing and deconstructing in the initial tasks and in the transfer tasks; and (4) argument iterations in the respective initial variation tasks that each produced a different (consecutive number) result (see Figures 3.2 and 3.5); and, argument iterations in the respective transfer tasks that each produced the same result (see Figures 3.3 and 3.6). With the respective transfer tasks, my reason for featuring different arguments that each produced that same result was that this narrower framing incrementally moved learners towards the generality of a formal proof (compared to each of the different propositions and outcomes shown in each of the different iterations in the initial variation tasks).

As I explained earlier, any anticipated actions afforded by my control of (in)variant features in the argument sequences, were only personally conceived. So, I remained open to identifying and characterising any alternative actions that were triggered by a learner's discernment of alternative (in)variant features, i.e., features that were learner- rather than researcher-conceived. Without consideration of what Lobato (2003, 2008) called the actor-oriented (learner) perspective, my analysis risked staying solely within my personal frame of reference.

3.7.5 Further task design considerations

To support the development of learners' comprehension of the number and geometry tasks, I sequenced the three main prompts in each domain so that they incrementally built in complexity. Since all of the learners were new to any formal notion of proof, I avoided in the prompts use of the term 'prove'. Instead, I asked learners to 'explain' and to 'show'[2] which are common proxims for 'prove' in secondary school curriculum resources in England. My use of these proxims align with Tall et al.'s (2011, p. 18) observation that various degrees of proof are suggested in school mathematics by the terms 'explain', 'justify' and 'show'. Because it was unreasonable to expect the learners to immediately write their own proof arguments, I

[2] In my experience, tasks with the term 'show how' are more likely to feature the mathematical structure that needs to be manipulated (compared to tasks with the term 'prove').

deliberately designed tasks that provided examples of arguments, i.e., argument variations. Learners could then act directly on (or base their actions on) the ready-made argument structures. I also avoided use a fully formed assertion-based argument format with logical connectives, e.g., the "if then" $(p \longrightarrow q)$ modus ponens style. Instead, I used an argument format presented as a series of numbered procedural steps, which I read aloud and explained in a consistent way to each learner during their interview. The presented steps in all of the variation tasks featured procedural notation. The steps in the initial variation tasks featured only procedural notation (see Figures 3.2 and 3.5), whereas the steps in the (more demanding) transfer tasks also featured written descriptions of the procedural notation (see Figures 3.3 and 3.6). Again, with these design choices I aimed to avoid the common yet unrealistic expectation that pupils in secondary schools are suddenly required to understand and write proofs (Ball et al., 2002, pp. 207–208). In addition, I avoided use of two-column proofs because in my experience of over 20 years as a mathematics teacher in secondary schools in England, I have not seen this proof format in any Key Stage 3 or Key Stage 4 curriculum resources or examinations.

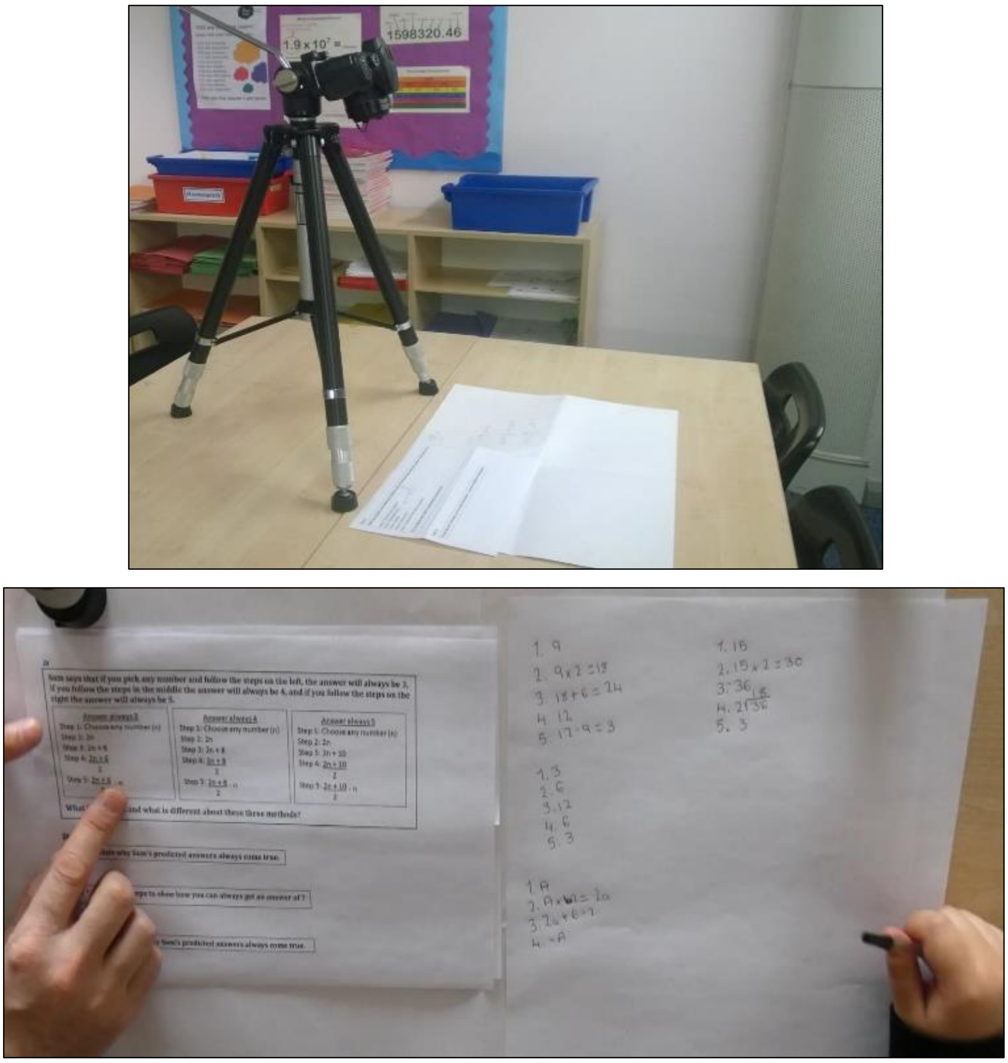
Further, I did not expect or require learners to write 'structural proofs' (Küchemann, 2008) that involve a recognised instantiation of a property or that necessitate invoking a (non-present) proof axiom, e.g., $2n$ and $2n + 1$ to represent, respectively, even and odd integers. Another key aspect of my task design was my repeated use of the generality-imbued term 'always'. For example, in the prompt: *Try to explain why Sam's predicted answers always come true*. Since the learners were new to proof, I did not expect them to have a sense of the irrefutability of proofs. I therefore made repeated use of the term 'always' because I felt that it had a formative quality with potential to lay a foundation for learners' later, more formal, conceptualisation of a proof's conclusiveness. This conclusiveness is expressed in Harel and Sowder's (2007) definition of proving as 'the removal of any doubt about the truth of an argument' (p. 808). However, in the foundational context of my study, I suspected that learners' use of the word 'always' might relate only to regularities that they identified in the given argument variations rather than to a complete (learner-generated) proof - although the variation tasks provided scope for learners to construct complete, i.e., conclusive, proofs. Nevertheless, I designed all of the tasks to be suitable for secondary school learners of any age. Hence, my tasks offer a means to overcome the reported delay in the teaching of proof until the latter years of secondary school (Knuth, 2002, p. 61; Stylianides and Stylianides, 2009a, pp. 237–238);

In sum, the sequences of number and geometry argument variations that I designed each involved providing learners with a careful mixture of opportunities. Namely, to search, identify, question, manipulate, test, justify, and revisit aspects of a problem. All of which, according to Maher (2009), can lead young students to comprehend and act on structural relationships. The tasks therefore provided me with a means to characterise learners' proving actions, thus generating the data that I needed to respond to each of my research questions.

3.8 Data collection method: Semi-structured task-based interviews

The learners worked on the tasks in videoed semi-structured interviews that I conducted and recorded in my role as teacher/researcher. The interviews generated two anonymised forms of data: (1) audio-video recordings of learners as they responded to the tasks and interacted verbally with the interviewer, and (2) learners' written responses to the tasks. To anonymise the interviews, I trained the video camera downwards and locked it in a fixed position that captured only the task sheet and the learner's hand as they wrote their responses to the tasks (see Figure 3.7). (Later in this chapter, in Section 3.10.3, I discuss in more detail how I anonymised the learners and the data that I collected).

Figure 3.7: Fixed position video recorder and an example of the anonymising frame.



I chose to conduct semi-structured interviews because, compared to less interactive instruments of data collection, they afforded my in-the-moment probing of learners' mathematical minds in action. Probing learners' thinking as they engage with tasks fits Ginsburg's (1981) description of clinical 'think aloud' interviews, and the data that I collected by questioning learners in this way supplemented the data from learners' written responses and other utterances. With these complementary and re-analysable forms of data I was able to identify learners' actions (RQ1) and, by extension, compare learners' actions (RQs 2 to 4).

All six learners participated individually in their interview (two learners elected to be accompanied by a non-participating friend). Each interview was of approximately 45-minute duration and each learner attempted first the number task and then the geometry task. I conducted all of the interviews during regular school hours. As I mentioned earlier, I designed the interviews using a semi-structured format (see, e.g., Ginsburg, 1981; Miller et al., 2014, p. 2). The structured aspect of the interviews involved giving each learner the same tasks in the same order and giving no learner any instructional directions on how to successfully complete the tasks. To supplement the structured aspect of the interviews, I also read aloud each of the prompts and gave a brief explanation of the given steps in the argument variations. The semi-structured aspect of the interviews involved moments when I responded verbally to any clarification-seeking questions that learners asked, and when I questioned a learner's action or prolonged silence. When these moments occurred, I asked learners probing questions, e.g., What are you thinking now? and Why did you do that? My rationale for asking the latter question was that asking for such justifications can cause learners to notice, act on, and even create, structure (Maher, 2009). (See Appendix C for a list of in-the-moment questions that I devised and used in the interviews).

I next describe my method of data analysis and the coding strategy that led to my formulation of a framework for describing learners' proving actions, i.e., the GOLDEN framework (RQ1). I discuss my findings in relation to each of my research questions in the next chapter.

3.9 Methods of data analysis and formulation of the GOLDEN framework

First, I will describe how I analysed learners' responses to the introductory tasks in the number and geometry domains, respectively. I will follow this by explaining how I analysed learners' responses to the initial and the transfer variation tasks in each domain. As I explained in Section 3.7.3.1, learners' responses the introductory tasks served only as a comparative backdrop to their actions on the variation tasks. I did not, therefore, use learners' actions on the introductory tasks to develop the GOLDEN framework. Rather, I developed the framework by analysing the actions that learners performed on the domain-pairs of variation tasks that I specially designed to encourage a broader range of actions.

3.9.1 Analysis of responses to the introductory tasks

I analysed learners' engagement with each respective introductory task by looking for and categorising a binary difference in their responses. For example, learners who transitioned to (and did not transition to) algebraic representation. Due to the sample size, I limited to two the number of response categories in each domain, and across both year groups combined I aimed for an equal number of learners in each category. The two categories of responses set up a comparative context for identifying any links with learners' subsequent actions on the variation tasks. The duration of learners' engagement with the introductory tasks was short enough for me to identify binary categories of responses directly from the video-recordings of their task-based interviews.

<u>Number introductory task</u>

With the number introductory task, I asked learners to show that the given argument would result in 3 for any initially chosen number, i.e., that the given proposition must (always) be true (see, Figure 3.1, prompt 1a). I first sought a binary difference in the responses of learners who transitioned, and did not transition, to algebraic representation, i.e., without being prompted to do so. However, and as I suspected, none of the learners transitioned to algebraic representation (on prompt 1a) until they were directly asked to do so on prompt 1b. I then re-analysed all learners' responses to prompt 1a and I identified three potentially alternative binary categories. Namely, learners who:

(1) Were (and were not) convinced of the proposition's irrefutably by empirical testing alone.

(2) Referred to (and did not refer to) the process of inversing or 'doing and undoing'.

(3) Tested (and did not test) different classes of number. (With the term 'tested different classes of number' I refer to learners who made a specific verbal reference to, and tested, a "negative number" as well as a natural number, or a "square number" as well as a number that is not square. With the term 'did not test different classes of number' I refer to learners who made no verbal distinctions between number classes and whose tests were limited to natural numbers).

Learners' responses in relation to (1) and (2) did not fall evenly into each category. Hence, it was binary categorisation (3) (shown in Table 3.1) that I used as a backdrop against which to compare learners' subsequent actions on the number variation tasks.

Table 3.1: Binary categorisation of responses to the number introductory task.

Learner	Number introductory task responses (prompt 1a) Tested different classes of integer (e.g., a negative integer and a natural number)	Did not test different classes of number.
8A	•	
8B		•
8C	•	
10D	•	
10E		•
10F		•

I also considered categorising responses to the number introductory task by differences in learners' presentational format when, on prompt 1b, they were asked to represent algebraically the steps given in prompt 1a. However, this variable was too diverse to use for comparison purposes because each learner's presentational format was in some way unique. As I mentioned earlier, none of the learners transitioned to algebraic representation until they were directly prompted to do so (on prompt 1b). I suspected learners did not represent algebraically until directly prompted to do so because they lacked an appreciation of "algebra-ising" as a form of generalising. Which, in turn, may be a consequence of being taught a curriculum that involved treating algebra as a series of distinct topics, e.g., simplifying expressions, forming and solving equations, and changing the subject of a formula. It was also likely that learners did not immediately transition to algebra because they were new to formal processes of proof and proving.

Geometry introductory task

As with the number introductory task, my intention with the geometry introductory task was to set up a binary backdrop of learners' responses to which I could compare learners' actions in the variation tasks. To do this, I asked learners (in prompt 1) what facts and ideas came to mind in relation to a given geometric diagram that featured angles in x and in y, and an accompanying argument for $y = 2x$ (see Figure 3.4). As with the number introductory task, I analysed learners' engagement with the geometry introductory task by looking for and categorising a binary difference in their responses. It was readily evident that, in each respective year group, learners either referred only to geometric facts, e.g., angles on a line add to 180°; or, also compared angles in the given diagram and reasoned about a process, e.g., by stating that 90° - 75° "is the same as" $3x = 15°$. As with the two chosen categories in the number introductory task, an equal number of learners fell into each category. Hence, it was this difference in responses (shown in Table 3.2) that I used as a backdrop against which to compare learners' subsequent actions on the geometry variation tasks.

Table 3.2: Binary categorisation of responses to the geometry introductory task.

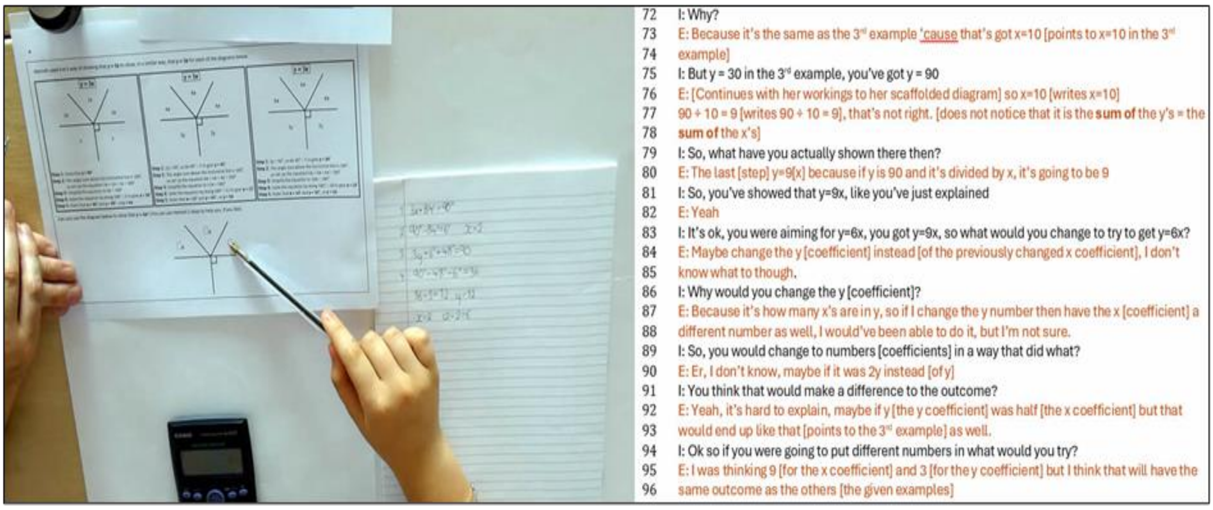
	Geometry introductory task responses	
Learner	Referred only to a geometric fact (e.g., angles on a line add to 180°)	Referred to a geometric fact, compared the given angles, and reasoned about a process. (e.g., 90° - 75° "is the same as" $3x$ = 15°)
8A		•
8B	•	
8C		•
10D	•	
10E		•
10F	•	

Later, in Section 5.5, I discuss links between responses to the introductory tasks and learners' actions on the domain-pairs of variation tasks. I will next describe how I analysed learners' responses to the initial and transfer variation tasks. I follow this by explaining how I formulated the GOLDEN framework.

3.9.2 Analysis of responses to variation tasks and deriving the GOLDEN framework

Learners' responses to the variation tasks were considerably more protracted and comprehensive than their responses to the introductory tasks. To facilitate my direct video-based analysis of learners' responses to the initial and transfer variation tasks, I hand-typed and used verbatim transcriptions (Ingram & Elliott, 2019, p. 193) of the videos. I typed all of the transcriptions alongside the corresponding video footage, using a split screen format (see Figure 3.8). To provide further context for my analysis, I included in the transcriptions any salient pointing gestures that learners made as they spoke (see, e.g., Figure 3.8, line 73). I chose not to use a more nuanced transcription method, e.g., the Jefferson system (2004) and its associated symbolism, because I was primarily interested in identifying learners' actions (i.e., process coding) rather than conversational aspects such as pace, pitch, and intonation. I structured each transcript in three chronological sections that mirrored the order in which the three main prompts appeared on the task sheets, i.e., (1) the introductory task, (2) the initial variation task, and (3) the transfer variation task.

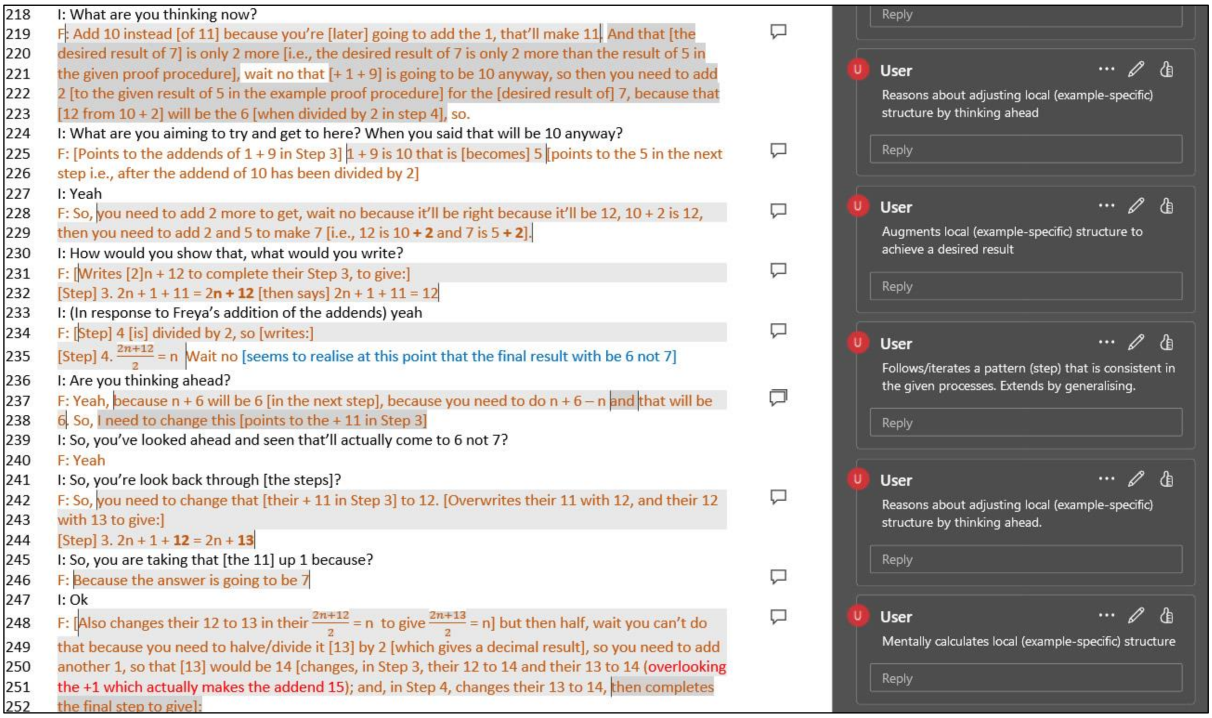
Figure 3.8: The split screen format used to transcribe responses to the variation tasks.



I initially attempted, in a single analytical pass of the transcripts, to categorise learners' actions directly into literature-informed themes in a process akin to what Saldaña (2013, p. 262) called eclectic coding. However, this approach had several limitations: it was largely ungrounded, it did not involve the merging of any groups of actions, and it comprised only three broad main actions with no sub-actions. Hence, this direct coding approach resulted in a framework with categories that were too general and lacked any close analytical insights. In light of this, and after experimenting with and deciding against the use of NVIVO software, I moved to a more traditional and grounded triple-pass coding (Richards, 2015) approach. Analysing learners' responses using triple-pass coding was a more intimate and inductive approach and proved beneficial for three key reasons: (1) it resulted in my identification of six main actions, (2) it led me to discern and define three sub-actions per main action, and (3) it facilitated my justifications for each sub-action's exclusivity.

For the first of my three passes, and with RQ1 in mind, I used line-by-line open analysis (Charmaz, 2006) of my transcriptions of all learners' responses to the variation tasks in both domains. In this initial pass I identified and briefly described as many actions (codes) as I could find. To do this, I used Microsoft Word to highlight and write a short accompanying description of each action that I identified (see Figure 3.9).

Figure 3.9: Example of identifying and describing a learner's actions in a first coding pass.



According to Charmaz, line-by-line analysis (coding) is particularly appropriate for interview data because this data is often less familiar to the researcher than, say, data from personally written field notes. For me, such lack of familiarity was largely a result of the extent and intricacies of learners' responses.

In my second pass, I used more generic 'axial' codes (Saldaña, 2013, p. 51) to characterise each of my descriptions of the actions that I identified in my first pass. I used a single-word gerund to respectively characterise a total six axial codes, and these constituted my framework's six main themes, i.e., actions. Namely, Generalising, Organising, Localising, Deconstructing, Extending, and Networking. At this point I was looking at my data through the lens of the six main actions and so I had a clearer sense of what my final framework would comprise.

In my third pass, I discerned (for consistency) three sub-actions for each of the six main actions. To do this, I used the nuances in the narratives of how each main theme evolved. In doing so, I ensured that my description of each sub-action was broad enough to apply to task responses in both domains. This sometimes required carefully genericising the specificity of a sub-action's description just enough for it to become applicable to both domains. For example, my initial description for one of the localising sub-actions was, '*selecting an initial value and starting from step 1 to test a previously adjusted value in a subsequent step*'. However, when I cross-checked if this action was relevant to both domains, it was apparent that an aspect of the description (*selecting an initial value*) was applicable only to the number task. To resolve this, I slightly genericised the description so that it similarly applied to what learners did when they performed this sub-action in both domains. To do this, I revised my initial description to '*re-adjusting a previously adjusted value and testing the re-adjustment*'. Conversely, it was sometimes necessary to refine, i.e., increase the specificity of, a sub-action's description to delineate it from a similar sub-action. When this was necessary, I checked back through learners' associated actions and ensured that, where relevant, any actions that I had previously mapped to an unrefined sub-action were re-mapped appropriately. On occasion, I found it practical to merge two sub-actions. For example, I initially coded one of the localising sub-actions as '*adjusting a given value*', which I later merged with a similar code '*testing an adjusted value by ignoring one or more of the given steps*'; thus generating the resultant code: '*adjusting a value in a given argument and testing the adjustment by ignoring one or more of the given steps*'. At this point, I defined the criteria for structuring the framework: main actions must be performed in each separate domain; whereas, the sub-actions for each main action could be distributed across both domains.

My identification of sub-actions also led me to discern actions that I initially took to be co-occurring as near-simultaneous rather than simultaneous. For example, I initially saw the acts of generalising and of extending a pattern as co-occurring, but on closer consideration I discerned that one cannot extend a pattern without first identifying the pattern with a generalising act. Only the organising sub-actions of foregrounding sameness and foregrounding difference did I treat as possible to co-occur. For example, when a numerical sequence that increased by a constant value was seen both as an instance of sameness and of difference. Yet, even in such cases, it could be argued that one must marginally perceive either sameness or difference before seeing the two as occurring together. Treating the foregrounding of sameness and difference as a possible co-occurrence generated a third *organising* sub-action. Part of my analysis involved comparing each learner's *organising* sub-action to the actions that they subsequently performed on the variation tasks because I wanted to identify any potential relationships. For example, learners who foregrounded only a dimension of sameness might initiate their further engagement in the variation tasks by performing actions that are different to learners who foregrounded only a dimension of difference. Because my focus was on characterising learners' actions primarily as gerunds, I did not discern between (directly) prompted and unprompted actions. During the task-based interviews I spoke only to read out and explain the demand of the tasks and to probe learners' thinking. At no point did I directly prompt any of the learners by suggesting to them a way to respond to a task.

To minimise potential bias (circularity) from any pre-conceptions I had about the actions that learners might perform, I derived my descriptions of learners' actions by moving back and forth between the task response data and the literature. In doing so, I asked *how well does this literary* *account describe what I am observing?* When descriptions of an action in the literature did not adequately capture my observations of a learner's actions in my study, the literary descriptions sometimes led me to: (1) define a new action or new category of action, and (2) refine my initial characterisation of an action or of a category of an action. Moving between my data and the literature to formulate descriptions of learners' actions , i.e., inductive and deductive 'hybrid' coding, meant that the components of my framework were not limited to my personal pre-conceptions of actions, nor to actions previously defined by researchers. For example, I found descriptions of acts of specialising (see, e.g., Mason et al., 2010) to be too narrow to describe the range of similar actions that learners performed in my study. This led me to use instead the broader term *localising*. I nonetheless acknowledge that any literature-informed preconceptions I had about the ways in which learners can attend to structure will have inevitably shaped, to some extent, what I took to be learners' actions.

As I described earlier, my initial coding approach involved directly mapping each identified action to one of two largely preconceived main themes. While my initial approach was predominately ungrounded and resulted in a framework that was too course-grained to be of descriptive and analytical use, I saw it as a valuable step in moving to a more systematic and grounded coding method. That is to say, a coding method which resulted in a framework that more closely reflected the nuanced actions that the learners performed. Namely, the GOLDEN framework.

My analysis of learners' responses to the variation tasks resulted in the formulation (then application) of the GOLDEN framework. By formulating the framework, I responded to RQ1. By applying the framework as an analytical tool to characterise and compare learners' actions on the variation tasks, I responded to RQ2 to RQ4. RQ2 and RQ3 involved only characterising learners' actions, whereas RQ4 involved comparing learners' actions. I characterised and compared learners' actions on the variation tasks in each respective domain and for each year group separately.

3.9.3 Application of the GOLDEN framework

My first application of the framework as an analytical tool was for the binary identification of each action that each learner performed and did not perform. My second application of the framework was to identify the chronological sequences of actions that each learner performed. Having binary knowledge of each learner's (in)actions meant that I could compare learners' successful and unsuccessful completion of the variation tasks to the actions they did and did not perform. Binary knowledge of learners' actions also provided a useful basis for my discussion on the role and affordance of each performed action when seen in the context of a sequence of actions. I performed sequential (chronological) analysis of learners' actions for two related reasons: (1) binary analysis 'froze in time' a learner's actions, and (2) I wanted to characterise the actions that comprised learners' fruitful trajectories on the variation tasks. Taken together, my analysis of all learners' responses to all of the prompts in both domains involved five key stages:

Stage 1. Transcription of the video-recorded task-based interviews (necessary for analysis).

Stage 2. Analysis of responses to the introductory tasks:

- Identification of learners who, on the number task, tested and did not test numbers with different properties.
- Identification of learners who, on the geometry task, referred only to a geometric fact, and who referred to a fact and a process.

Stage 3. Coding of actions performed on the variation tasks (i.e., formulation of the GOLDEN framework):

- *First pass*: identification and highlighting of all learners' actions.
- *Second pass*: axial coding and colour-coded categorisation of each axially coded action into a theme, i.e., main action.
- *Third pass*: discernment of three sub-actions per main action.

Stage 4. Application of the GOLDEN framework as an analytical tool:

- (Preparation) tabulation of each learner's actions.
- Chronological coding, and tabulation of each learner's sequences of actions on the variation tasks (see Appendix F).

Stage 5. Characterisation and comparison of actions across year groups and domains.

It is my systematic and grounded analysis that I see as justification for the appropriateness of the GOLDEN framework as a valid lens for describing learners' actions in my study. Nonetheless, few researchers, as I previously mentioned, can claim to enter a field tabula rasa, uninfluenced by prior perceptions of what they seek to understand (Flinders & Mills, 1993, p. xi; Y. Gu, 2014, p. 79). Hence, strictly speaking, I used a mixed coding approach (Miles & Huberman, 1994) because two of the six main categories (generalising and localising) were somewhat prefigured by my existing knowledge of the literature, while the other four categories (organising, deconstructing, extending and networking) were induced using open coding from grounded theory (Strauss, 1987). The latter four actions can also be found in the literature but I did not commence the data analysis holding any preconceptions about the likelihood of finding any of these actions, as I did with generalising and localising actions. As Strauss and Corbin (1990) explained, coding influenced by existing knowledge of the literature is appropriate in the use of grounded theory if codes can be justified, and if pre-existing knowledge from "received theories" does not excessively restrict the categorisation process.

3.10 Ethical considerations

As I mentioned earlier, I recruited all of the learners and conducted all of the interviews as the regular teacher of all the participants, hence I acted in the dual role of teacher-researcher. The research took place in the learners' and teacher/researcher's regular secondary school setting. The pilot and the main study combined involved a total of 14 participants comprising seven Y8 learners (12- to 13-year-olds) and seven Y10 learners (14- to 15-year-olds). All of the fieldwork took place during normal school hours and received the full consent of the school at senior and middle management levels. CUREC 1A ethical approval was granted.

As I described in Section 3.4.1 'Pilot study', I asked any learners that were interested in participating to attend a voluntary 'project information' meeting in my classroom during one lunchtime (after they had eaten their lunch). At the voluntary meeting, I gave the learners further details about the nature of my study. These details included the data that I would collect and the anonymous way in which it would be handled and disseminated. I also explained to the learners their right to withdraw at any point without the need to give an explanation. I then requested and gained verbal assent from all of the learners that wanted to participate. Next, I explained that I needed, on a first come basis, 10 learners from each year group (a pandemic-related school closure meant that seven learners from each year group actually participated). At this point, I again reassured the learners that they were in no way obliged to participate and that they could withdraw at any time. The learners then lined up at my desk and I recruited the first 10 learners from each year group, i.e., eight learners for the pilot study and 12 learners for the main study.

The nature of the research participants (i.e., school-age learners) and school setting necessitated consideration of four specific ethical areas, which I next discuss in turn: (1) positionality, i.e., power relations and participants' sensitivities, (2) consent and assent, (3) anonymity and confidentiality, and (4) institutional requirements.

3.10.1 Positionality: Power relations and participants' sensitivities

At the time of the fieldwork, I had regularly taught all of the participants for approximately seven weeks. Any teacher-researcher researching their own students is faced with a number of ethical considerations, particularly around the 'asymmetry of the power relationship' (Farrimond, 2016, p. 81). Thus, I reassured the learners that they were in no way obliged to take part, and that should they decide to participate and at any point wish to withdraw, they could do so without the need to give an explanation. I further reassured the learners that should they chose to withdraw I would not view them in any way less favourably. I made clear this reassurance in case any learners felt obliged to participate due to any social pressure (Farrimond, 2016, p. 81). For example, wanting to please me (as their teacher) or through a concern that I have authority over them and that I mark their work and can make decisions that could adversely affect them. I also restated this reassurance when, in light of the pilot study, I chose to conduct the interviews in the main study with individual learners rather than with pairs of learners as I originally planned. As I described earlier, following my decision to conduct individual rather than paired interviews, I offered learners the option to bring along a non-participating friend. All of the learners that I originally recruited for the main study chose to continue their participation, and two of these learners elected to be accompanied in their interview by a non-participating friend. As was the case with all of the interviewees, non-participating friends' attendance to their regular timetabled lessons was not compromised.

3.10.2 Consent and assent

In the participant-school it is common practice for classroom-based teaching situations and interviews with learners to be video-recorded. This school norm, for which learners are rarely anonymised, is usually for the purpose of weekly in-school professional development sessions. For example, staff training that involves lesson study or students' opinions, i.e., 'student voice'. The senior management team at the participant-school also promote the use of classroom-based video recordings for the purpose of teachers' self-reflection and self-evaluation. Hence, the video recordings that I made for my current study did not represent a departure from the common practices of the school. Because I fully anonymised my study and my goal was a school norm, i.e., to better support my learners' mathematical development, parental consent for the process was not necessary and so was not sought. I did, however, obtain informed consent from the participant-school's Vice Principal as well as from the school's Director of Mathematics. As I detailed earlier, I also gained verbal informed assent from all of the interested learners prior to their participation.

3.10.3 Anonymity and confidentiality

I maintained the anonymity of all the learners' identities throughout all aspects of my study. During the task-based interviews, I positioned the video recorder to visually frame and capture only each learner's writing hand and written work (the video data also included audio recordings of learners' utterances). To pseudonymise the learners, I allocated monosyllabic alphabetical names to the Y8 learners (Ash, Beth, and Carl) and disyllabic alphabetical names to the Y10 learners (Dylan, Emma, and Freya). I stored all of the video recordings and linkage information regarding learners' identities on an encrypted and password protected USB flash drive which I kept in a locked cupboard in my classroom at the participant-school. When all of the electronic and physical data that I collected for my study is no longer required, I will securely and permanently destroy it.

3.10.4 Institutional requirements

I did not collect any data for my study until I obtained ethical approval from my university's Departmental Research Ethics Committee (DREC). I sought ethical approval by submitting to the DREC a completed and gatekeeper-signed copy of each form required for a CUREC 1A research application. Once I was granted ethical approval, I remained aware that should any additional ethical implications, planned or otherwise, arise in relation to my recruitment process or to my methods of data collection, then I should inform the DREC before proceeding. No additional ethical implications arose that warranted informing the DREC, and so this was not necessary.

3.11 Methodological limitations

During the development of my study, I identified some methodological limitations. I next describe these limitations and, where relevant, how I countered them. I also justify why the limitations did not dimmish the value that my study has for educators and the research community.

While my sample of six learners limited any external generalisability and had no statistical power, it afforded the microanalysis of learners' actions necessary for me to formulate an attention-to-structure framework. As I mentioned earlier, my focus was on developing a new framework and testing its applicability, and so to this extent a larger sample size was not imperative. By microanalysing learners' actions, I was able to identify 18 sub-actions, some of which, with a larger sample, I may have overlooked. A sample size of six also meant that I could give particularly detailed narrative accounts of learners' actions, including instances of actions that exemplified my broader observations.

When I looked for learners' actions that could contribute to an attention to structure framework, I found it useful to set aside lenses of 'correctness' such as 'right', 'wrong' and 'most efficient'. Instead, I described learners' actions in terms of gerunds; and, I included actions regardless of whether or not they resulted in an immediate move towards successful completion of a task. It could therefore be argued that my framework includes inefficient actions. My justification for including these actions was that less efficient actions sometimes helped learners to see and perform alternative, more fruitful, actions. By considering *combinations* of learners' actions in my descriptive narratives, I aimed to avoid the potential limitation of 'freezing a learner's individual actions in time', this also allowed me to characterise the genesis and trajectory of a learner's actions, and to attribute these to any productive advancement through a task.

While the initial variation tasks deliberately afforded learners opportunities to identify and use a range of patterns of varying complexity, it could be argued that the tasks can be successfully completed by focusing on relatively surface syntactic structures. Yet, as Watson and Mason (2006, p. 92-93) suggested, even in a highly structured situation, different learners may have different engagement experiences in which their use of a surface pattern may advance their understanding a particular concept. Further, and still in relation to the initial variation tasks, to encourage more sophisticated reasoning about an identified surface pattern (or any identified pattern) I asked learners to explain how the steps in the sequences of given arguments always validated the corresponding propositions. In addition, to mitigate against the use of any identified surface patterns in any of the variation tasks, I asked the learners to write non-consecutive iterations of the given arguments.

At the time of the fieldwork, I had been the regular teacher of the learners for seven weeks. Because of this, I made a conscious effort not to let my prior experience of the learners influence my characterisation of their actions. In doing so, I aimed to remain 'professionally estranged' while conducting the task-based interviews. In other words, I wanted to avoid confirming any pre-existing understandings that I had of learners' mathematical thinking because I did not want to 'meet myself in the results of the data analysis'. However, as I was the sole analyser of learners' task responses, this was to some extent inevitable. In an effort to minimise my (interpretivist) first person authority when describing learners' actions, I asked two of my teaching colleagues to also describe any actions that the learners performed. However, beyond comments such as "they used a pattern" and "they tried different numbers" both colleagues struggled to articulate more precisely the actions that learners performed. While my teaching colleagues' interpretations were too limited to be contributory, the superficiality of their responses reflected the reported gap in teachers' familiarity with processes of proof (Harel & Sowder, 2007, p. 836-837; Stylianides & Stylianides, 2018, p. 110) - a gap that my framework contributes to filling. Although my colleagues' vague interpretations of learners' actions meant that I did not perform an inter-rater reliability check, I aimed to mitigate this through a grounded observation- and literature-based analytical approach (Miles & Huberman, 1994) that left little scope for researcher bias.

3.12 Conclusion

I chose a qualitative case study approach because it afforded close and descriptively rich analysis of learners' response data necessary to answer my research questions. My relatively small sample ($n = 6$) further influenced my choice of a case study approach. My research questions set the criteria for my data analysis. I responded to my first research question *What might comprise a framework for describing learners' attention to structure?* by compiling the GOLDEN framework from learners' diverse range of individual actions. I then applied the GOLDEN framework as an analytical tool to respond to RQs 2 to 4, which involved comparing learners' actions within and across the variation tasks in both domains and year groups. To afford learners opportunities to perform a variety of proving actions, I designed in two domains isomorphic pairs of tasks based on the principles of variation (Marton, 2015; Marton & Booth, 1997). I chose variation tasks because they fit well with inducting learners into 'new' mathematical situations. That is, if the tasks feature systematic iterations with judiciously controlled (in)variance across which learners can discern sameness and difference (Kullberg et al., 2017; Marton & Pang, 2006). In turn, such discernment aids (or perhaps *is*) sense-making.

Each domain-pair of variation tasks comprised an initial task and a conceptually-similar follow-on task that served as a transfer task. Whereby transfer, or more specifically the objects of transfer, were conceived as the actions that learners performed as they worked on the tasks. Since it has been some time since I mentioned my definition of 'action', I will restate it again here: with *actions*, I refer to any transformation of objects into other objects (Cottrill et al., 1996), or any act of non-transformational reasoning or organising.

I formulated the GOLDEN framework by thematically coding the actions that learners performed in semi-structured task-based clinical interviews (Ginsburg, 1981; Miller et al., 2014, p. 2), or what Willis (2005) calls cognitive interviews. During the interviews I probed learners' cognition by asking 'think aloud' questions (Leighton, 2017) to elicit justifications for their salient actions and to reveal their thoughts during prolonged moments of apparent contemplation. Although I suspected that learners would in some way perform the broad ubiquitous actions of generalising and specialising, I deliberately distanced myself from any pre-defined (personally-conceived) actions that a learner 'must perform' in order for their attention to structure to count as an 'action'. Rather, I formulated the GOLDEN framework in an emergent way by characterising and categorising the nuanced actions that emanated from the learners as they worked on the number and geometry variation tasks that I designed. Because I was researching my own learners, and I was acting in the dual role of teacher/researcher, I was continuously aware of - and aimed to minimise as far as possible - the inevitable power imbalance. To aid this minimisation, I informed and reminded all of the participants (and prospective participants) of the voluntary nature of my study, and of their right to withdraw at any time without explanation or consequence.

Chapter 4: Results of data analysis

In this chapter, I will report directly the results of my analysis of learners' attention to structure on the number and geometry tasks. My analysis resulted in two overarching outcomes: (1) the characterisation of Year 8 and Year 10 learners' actions on the tasks in both domains, and (2) seven key findings. To report my respective characterisations of learners' actions I first take a collective (year group) approach, rather than an individual learner approach (e.g., by reporting Year 8 learners' actions on the number initial variation task). Then, to exemplify my collective characterisations, I illustrate representative actions performed by individual learners. (I take a more interpretive approach to my results in the Discussion chapter). By characterising learners' actions, I responded to all four of my research questions that I presented in Section 1.5. Accordingly, I will structure this chapter by my responses to each of my respective research questions.

To address RQ1, I formulated the GOLDEN framework by thematically characterising all learners' actions on the initial and the transfer variation tasks in both domains (Section 4.1). I then applied the GOLDEN framework as an analytical tool to address each of my subsequent research questions, all of which concern the characterisation and comparison of learners' actions within and across the two domains and year groups.

To convey my results for RQ2, and in alignment with my research paradigm, I characterised learners' actions on each respective variation task in each of the two domains (Section 4.2). I focused my characterisations on salient actions, e.g., the fruitful and repeated actions that learners performed. For each domain separately, I organised my discussion of salient actions by first characterising Year 8 learners' actions on the initial variation task (Sections 4.2.1 and 4.2.6) followed by the transfer variation task (Sections 4.2.2 and 4.2.7). In the same way (i.e., for each domain separately), I first characterised Year 10 learners' actions on the initial variation task (Sections 4.2.3 and 4.2.8) followed by the transfer variation task (Sections 4.2.4 and 4.29). I then compare, again in each respective domain, the actions of the Year 8 learners to the actions of the Year 10 learners (Sections 4.2.5 and 4.2.10). I sequenced my comparative discussions of learners' actions in the same order that the prompts appeared on the task sheets in each domain. Specifically, I compared Y8 and Y10 learners' actions first on the introductory task, then on the initial variation task, and then on the transfer variation task, respectively in each domain.

To communicate my results for RQ3, I characterised learners' actions *across* the initial and the transfer variation tasks in each of the two respective domains (Section 4.3). To do this, I first compared the actions that Year 8 learners performed on the initial variation task to the actions that they performed on the transfer variation task (Sections 4.3.1. and 4.3.4). In the same way, I then compared the actions that Year 10 learners performed on the initial variation task to the actions that they performed on the transfer variation task (Sections 4.3.2. and 4.3.5).

The results that I report in relation to the first three research questions have been limited only to comparisons of learners' actions across *year groups*. To extend this, and to impart my results for RQ4, I comparatively characterise learners' actions across *domains* (Section 4.4). To do this, for each year group separately, I again describe the salient actions that I identified, including fruitful sequences of actions that learners similarly performed on the tasks in both domains (Section 4.4.1 to 4.4.3).

4.1 First research question: Results

RQ1: *What might comprise a framework for describing learners' attention to structure?*

To respond to my first research question, I formulated the GOLDEN framework for describing learners' actions on the variation tasks that I designed in two domains (number and geometry). The GOLDEN framework is the central contribution of my research. I applied the GOLDEN framework as an analytical tool to respond to RQ2-RQ4, which relate to characterising and comparing learners' actions on the variation tasks within and across both domains and year groups. As I discussed in Chapter 3 (Methodology), the framework describes six main actions (each comprising three sub-actions) that learners similarly performed on the variation tasks in each domain. I will discuss each of the framework's six main actions, along with their respective sub-actions, later in this section.

To restate from earlier, with *action* I refer to any physical or mental transformation of objects into other objects (Cottrill et al., 1996), or any act of non-transformational reasoning or organising. The actions in the GOLDEN framework captured how learners, none of whom had previously been introduced to any formal concepts of proof argumentation, attended to given argument structures when tasked to: (1) identify (in)variance; (2) explain the truth of propositions that accompanied given arguments, and (3) write arguments that were conceptually similar to given arguments. As I described earlier, my formulation of the GOLDEN framework not only responded (directly) to my first research question, it was also a prerequisite for responding to each of my subsequent research questions; all of which are predicated on the application of the framework as a tool with which to characterise and compare learners' proving actions. As I elaborate later (Chapter 5), beyond my study, I offer the GOLDEN framework as an adaptable tool for other researchers and teachers who wish to analyse and articulate learners' proving actions.

To restate from Chapter 3 (Section 3.9.2), the framework that I initially derived from my first attempt at coding resulted in a limited range of categories that did not adequately capture the scope of learners' actions. My subsequent move to a more grounded triple-pass coding approach yielded six main categories of actions: Generalising, Organising, Localising, Deconstructing, Extending and Networking (hence the acronymic name 'GOLDEN' framework). In each of the six main categories, I identified three sub-actions. I organised the three sub-actions in each category so that, arguably, the first sub-action was the least sophisticated and the third sub-action was the most sophisticated. In organising my discussion of each main category, I aimed to broadly follow the same order of increasingly sophisticated sub-actions. I refer to the three sub-actions in each main action using an abbreviated format, e.g., I refer to the three Generalising sub-actions as, respectively, G1, G2, and G3, and to the three Organising sub-actions as, respectively, O1, O2, and O3.

One reason why proof is hard-to-teach and hard-to-learn (Stylianides & Stylianides, 2017) is that proving tasks can span different mathematical domains and be vastly diverse in nature. This diversity can make identifying structural and algorithmic generalities across proof tasks difficult, if not impossible. However, the GOLDEN framework offers a set of generalities in the form of learners' actions. The framework can be used to describe learners' actions on any proof variation task that features given structures (as well as on a potentially broader range of mathematical tasks, see Section 6.7). The GOLDEN framework has inherent generic applicability to proof variation tasks featuring given structures because I derived it from actions that learners consistently performed on such tasks in different domains (number and geometry). The framework does not describe learners' actions in terms of mere operations (calculations) that can detract from thinking more structurally (Küchemann, 2008, p. 8.5) and, individually, are unlikely to be applicable to all proof variation tasks. I found that for the actions in the framework to be task-generic, they needed to 'subtler' than calculating by attending to underlying structure. My identification of a task-generic set of actions represents my first finding**.**

Finding 1

Learners' attention to structure can be described in terms of the actions in the GOLDEN framework.

I elaborate on the utility of the GOLDEN framework as a pedagogical tool in more detail in Section 6.7. As part of my elaboration, as I mentioned earlier, I discuss the potential applicability of the framework beyond proof variation tasks.

Generalising actions

The Generalising actions that learners performed revealed a range or 'hierarchy' of generalities of varying sophistication. Some learners generalised in a superficial way by using, in the writing of their own cases, consistencies in the presentational format of the given arguments. However, I saw this kind of generalising as having little or no underlying reasoning and so I did not include it in my framework. Instead, I comprised the Generalising component of three process-oriented generalising actions, all of which learners performed with inherent explanatory reasoning.

G1 generalising actions featured reasoning about a process that involved one identified generality, e.g., explaining how, across a set of given arguments, the value of correspondingly positioned addends consistently increased. Some generalising acts involved reasoning by simultaneously attending to two process consistencies (G2). For example, describing a common effect (across a given set of arguments) of dividing both terms in a numerator, or explaining a consistent relationship between coefficients of x and y. With the generalising actions that I have so far discussed, learners focused their inherent reasoning solely on a process (G1) or processes (G2), termed 'process pattern generalisations' by Harel and Soto (2017). In addition to solely process-focused generalising actions, learners sometimes generalised by reasoning (across a set of given arguments) about a consistent relationship between corresponding processes and their corresponding results (G3). For example, how, across a set of given arguments, the value of correspondingly positioned addends consistently co-varied with the values of the given arguments' end results. The identification of such consistencies is similar to what Harel and Soto (2017) called 'result pattern generalisation' because they both involve reasoning about identified regularity across a set of results. Yet, G3 generalisations differ from result pattern generalisations because the latter focuses solely on patterns identified across results, whereas learners' G3 actions involved identification of a consistent relationship between corresponding processes and corresponding results. I termed identified consistencies of this kind *process-result* pattern generalisations and they contribute a third form of generalisation to Harel and Soto's (2017) *process* pattern generalisation and *result* pattern generalisation. I elaborate on the implications of learners' process-result pattern generalisations in Chapter 5, Section 5.1.2.

Organising actions

The three sub-actions in the Organising component involved acts of foregrounding and backgrounding elements of structure when multiple sources of information, in the given arguments, competed for learners' attention. The foregrounding and backgrounding actions associated with Organising are akin to Lobato et al's (2013) notions of 'noticing' and of ‘information selection’. When I prompted learners to identify sameness and difference across the arguments provided in the initial variation tasks, some learners organised the given information (argument structures) by initially foregrounding only an element of sameness (O1), e.g., by identifying invariance in a correspondingly positioned coefficients or angle values. In response to the same prompt, other learners organised the provided information by giving priority to only an element of difference (O2), e.g., by identifying variance in the values of correspondingly positioned addends. There were also cases where learners simultaneously foregrounded sameness and difference (O3), e.g., by seeing a rule for a noticed pattern as something that was ‘the same’, while concurrently seeing the values in the pattern as something that was ‘different’. Organising by foregrounding sameness is different to Generalising because the former is, for me, solely an act of identifying a noticed dimension, whereas Generalising requires accompanying explanatory underpinning reasoning.

Localising actions

When reasoning about the truth of the given propositions or when writing an argument similar to the given arguments, learners did not always (initially at least) see or treat a given argument or diagram as a single object. In the words of Mason (2003), learners’ attention did not always ‘hold a whole’ structure. Instead, learners often acted on local structure by adjusting and testing a value in one of the steps in an argument. I broadly framed such focused attention as Localising. Learners’ Localising actions tended to take the form of adjusting and testing an individual value in an attempt to achieve a desired end result.

Akin to the notion of *empirical arguments* (Stylianides and Stylianides, 2009b), learners tested their adjusted values either by using only part of a given argument (L1) or by using all of a given argument (L2). Learners who tested their adjusted value by using only part of a given argument tended to perform their test from the adjusted step onwards. In situations where a learner’s tested adjustment did not result in a desired outcome, they sometimes re-adjusted (and re-tested) a value (L3). In such cases, learners commonly based their re-adjusted value on the result of their most recent test.

Deconstructing actions

The fourth main component in the GOLDEN framework I described as ‘Deconstructing’. I deemed a learner to perform a deconstructing act when, in writing their own case, they reduced the number of terms in a given step or they reasoned in relation to the notion of inversing. Some learners reduced (shortened) the length of the notation in a given step (D1) by omitting terms, i.e., by jumping to and working with only the result of a step. Deconstructing by shortening a given step (D1) is different to ignoring entire steps when testing an adjusted value (L1) because, in the latter action, a learner's focus of attention is on the result or influence of an adjustment. The second of the three deconstructing sub-actions concerned learners who attended to the argument structures by 'reverse engineering', i.e., by working backwards through all or part of a given step, or by working from a given step to a prior given step (D2). For example, starting from a given end result and working backwards through the elements of the preceding steps in order to identify (the position of) a value that could be adjusted to produce a different, i.e., desired, end result; and, to justify the magnitude of a possible alternative value. A further example of a D2 action occurred when, in relation to a given argument or accompanying diagram, a learner reasoned about which terms sum to 90° or to 180°. The final sub-action that I mapped to the deconstructing component involved reasoning about an identified relationship in terms of inversing, 'opposites' and 'doing and undoing', including acts of 'cancelling out' objects (D3). Inverse reasoning D3 actions reference a deconstructing effect, and were often preceded (near-simultaneously) by G2 (two-process) generalisations. For example, explaining (when generalising about the given number arguments) that 'n is multiplied by 2 then divided by 2' (G2) and that this will 'leave the number that you multiplied' (D3).

Extending actions

As I mentioned in Chapter 2 (Section 2.3), I deliberately designed the provided argument variations so that they afforded learners several opportunities to continue an identified pattern. Indeed, when prompted to write a non-consecutive argument that was similar to the provided (consecutive) arguments, several learners extended an identified pattern of some kind. For some learners, the thought of extending an identified pattern seemed to give them the impetus, or confidence, to commence writing their own (non-consecutive but conceptually similar) argument; i.e., a new case for a similar proposition. In addition to repeating a pattern to generate a new case, the act of Extending a pattern sometimes enabled a learner to further reason about, or to see and explain (justify) the truth of an argument's accompanying proposition. I identified three types of pattern extensions that learners performed. The first type of extension was of a pattern in a process involving one generality (E1). For example, continuing a pattern in the increment of correspondingly positioned addends or coefficients. The second type of extension was of a pattern in a process involving two generalities (E2). Extending patterns of this kind involved simultaneously attending to two varying dimensions. For instance, continuing a pattern that was simultaneously identified in the increment of correspondingly positioned addends *and* coefficients. The third type of extension involved repeating a process-result pattern (E3) (Harel & Soto, 2017). For example, repeating for the generation of a new case a consistent link identified between correspondingly positioned addends an end result. I discerned Generalising from the act of extending a pattern because the former prefigured the latter and involved justifying (giving underlying reasoning).

Networking actions

I identified Networking actions when a learner referred to, or introduced and applied, a concept from a qualitatively different curriculum topic. I found learners' Networking actions promising for two key reasons: (1) they evidence the usefulness of flexible (disjunctive) thinking, i.e., the kind of thinking that can be stymied when mathematical topics are taught in isolation from each other; and (2) they resonate with the kind of 'external' thinking that is required to see a specific structure as an instance of a generality (Mason et al., 2009), which is a key tenet of proving. Learners networked in relation to a limited but somewhat diverse range of concepts from discrete curriculum topics, e.g., fractions, ratio and proportion, and BIDMAS. I framed the first Networking sub-action in terms of learners who reasoned about an argument by referencing, but not applying, a qualitatively discrete concept (N1). For example, when a learner referred to but did not use the concept of fractions in relation to semi-circles that he drew on the geometry transfer task. The second Networking sub-action that I formulated involved directly applying to an argument a discrete concept (N2). A learner performed a N2 action when, for instance, they introduced the concept of proportion while reasoning about an identified co-varying relationship between two given values. The final Networking sub-action concerned reasoning that involved applying to an argument a value other than a positive integer (N3). For example, referring to testing a negative value, rather than only positive integers, when attempting to support (or refute) a given proposition. A further example of a N3 action was reasoning about the use of a decimal value, rather than only integer values, to achieve a desired result.

As I mentioned earlier, by describing acts of generalising, organising, localising, deconstructing, extending and networking, the GOLDEN framework provides a means for educators to analyse and articulate the ways in which learners can attend to structure when tasked to write arguments that are conceptually similar to provided arguments. I next apply the GOLDEN framework as an analytical tool to address each of my subsequent research questions (RQ2 to RQ4), all of which are predicated on characterising and comparing Y8 and Y10 learners' actions on the variation tasks within and across both domains.

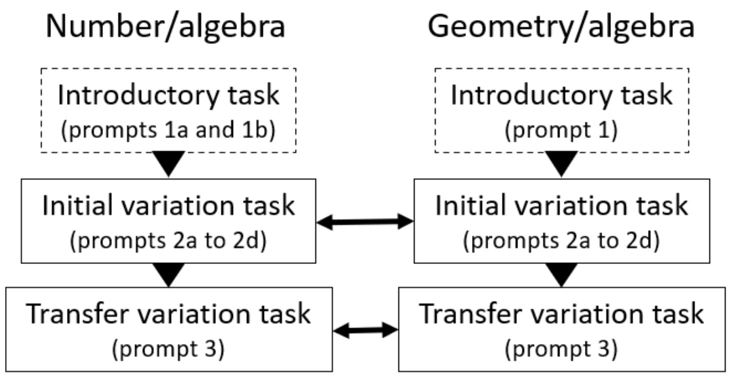
4.2 Second research question: Results

RQ2: *What characterises Y8 and Y10 learners' attention to structure on an initial and transfer proof variation task in the domains of number and geometry?*

I structured my response to RQ2 by first describing Year 8 learners' actions on the respective number initial and transfer variation tasks (Sections 4.2.1 and 4.2.2). I then describe Year 10 learners' actions on these same tasks (Sections 4.2.3 and 4.2.4). I follow this by comparing the actions of both year groups, again on these same tasks (Section 4.2.5). I then repeat this reporting structure for learners' actions on the geometry initial and transfer variation tasks (Sections 4.2.6 to 4.2.10).

As I mentioned in Section 3.7.3, I kept consistent the prompt structures in the number and geometry variation tasks. The consistent prompt structures, illustrated in Figure 4.1, aided my comparative reporting of learners' actions across domains.

Figure 4.1: The isomorphic prompt structures of the variation tasks in both domains.



Note. In the number introductory task, prompt 1b was only applicable to any learners who, in response to prompt 1a, did not transition to algebraic representation. While both introductory tasks respectively probed learners' knowledge of the concepts inherent in the subsequent variation tasks, I did not deem a follow-up prompt necessary for the introductory task in the geometry domain.

As discussed in Section 4.1, I identified and described learners' actions on the variation tasks by applying the GOLDEN framework as an analytical tool. For each prompt in each variation task, I sequenced my reports of learners' actions, where present[3], in the same order as they appear in the GOLDEN framework, i.e., generalising, organising, localising, deconstructing,

[3] Learners did not, individually or collectively, perform all six actions in their response(s) to any singular prompt.

extending, and networking. As described in Section 4.1, each of the six main actions comprised three sub-actions which I refer to with consistent abbreviations (e.g., G1, G2, and G3 for the Generalising sub-actions). At relevant points in my descriptions, I draw on learner-specific excerpts to illustrate salient actions.

4.2.1 Y8's actions on the number *initial* variation task

The number task initial variation task comprised four prompts (2a to 2d). I will describe learners' actions by discussing separately each of the four prompts. With prompt 2a, shown in Figure 4.2, I gave learners three argument variations and I asked them a compound question: *What is the same and what is different about the given arguments?* I asked a compound question because I wanted to introduce the dimensions of sameness and difference simultaneously (or as near simultaneously as possible). With this simultaneity I created a situation for each learner to freely identify dimensions of sameness and difference in the order that they personally perceived them in. More specifically, I wanted to reveal how each learner would organise, i.e., foreground and background, the information in the given argument variations by prioritising a dimension of sameness or of difference.

Figure 4.2: The first prompt on the number initial variation task.

2a) Sam says that if you pick any number and follow the steps on the left, the answer will always be 3, if you follow the steps in the middle the answer will always be 4, and if you follow the steps on the right the answer will always be 5.

Answer always 3	Answer always 4	Answer always 5
Step 1: Choose any number (n)	**Step 1:** Choose any number (n)	**Step 1:** Choose any number (n)
Step 2: $2n$	**Step 2:** $2n$	**Step 2:** $2n$
Step 3: $2n + 6$	**Step 3:** $2n + 8$	**Step 3:** $2n + 10$
Step 4: $\frac{2n + 6}{2}$	**Step 4:** $\frac{2n + 8}{2}$	**Step 4:** $\frac{2n + 10}{2}$
Step 5: $\frac{2n + 6}{2} - n$	**Step 5:** $\frac{2n + 8}{2} - n$	**Step 5:** $\frac{2n + 10}{2} - n$

What is the same and what is different about these three methods?

Across Y8 learners' responses to prompt 2a, I identified one Generalising action and five Organising actions. The Generalising action involved reasoning about a single process in the given arguments (G1). The five Organising actions involved all three sub-actions: prioritising a dimension of sameness (O1), prioritising a dimension of difference (O2), and simultaneously foregrounding a dimension of sameness and of difference (O3). As I described in Section 4.1, for me, prioritising sameness (O1) is different to Generalising because the former is solely an act of identifying a noticed dimension, whereas Generalising involves an accompanying element of underlying explanatory reasoning. In response to prompt 2a, one Y8 learner, Ash, followed his act of foregrounding a difference (O2) with a near-simultaneous Generalising action; i.e., by first identifying that "[each respective] *step three and four is different*" then explaining the reason for the difference, "*it's adding 2 each time*" (G1). Across both year groups and domains, I identified three such instances of near-simultaneous Organising and Generalising and actions when learners were prioritising sameness and/or difference in response to prompt 2a. Each of the three instances of near-simultaneity occurred only in relation to an Organising action that involved foregrounding a dimension of difference (O2). One explanation for this consistency is that identifying a noticed difference can trigger an underlying explanation for the difference; whereas, for a noticed dimension of sameness, only identification may be deemed necessary. Hence, asking learners to identify dimensions of difference in the context of mathematical variation tasks can encourage them to justify, and to communicate their justifications. Both of which are hallmarks of proving (Harel & Sowder, 2007; Stylianides & Stylianides, 2009b) and so are valuable activities for the development of learners' proof productions. I elaborate on learners' propensity to justify their noticing of difference in Section 4.2.8.

With prompt 2b (which I later repeated with prompt 2d) I asked learners to verbally explain why the predicted results for the three given arguments, shown in Figure 4.2, are always true. My reason for repeating prompt 2b was to evaluate if learners' responses to the interim prompt (2c), for which learners wrote a case similar to the given arguments, would increase the sophistication of their reasoning about the truth of the propositions. Figure 4.3 illustrates prompts 2b to 2d (all of which were isomorphic to prompts 2b to 2d in the geometry task).

Figure 4.3: Prompts 2b to 2d on the number initial variation task.

2b) Try to *explain* why Sam's predicted answers always come true.

2c) Try to use Sam's steps to *show* how you can always get an answer of 7.

2d) Try again to *explain* why Sam's predicted answers always come true.

Across all three Y8 learners' responses to 2b (presented in Table 4.1), I identified three Generalising actions; specifically, one G2 action and two G3 actions. I also identified one Networking action (N2).

Table 4.1: Y8's sub-actions on prompt 2b of the number task.

	Sub-actions performed (frequencies are shown in parentheses)
Ash	No actions performed
Belle	G3 (1)
Carl	G2 (1), G3 (1), N2 (1)

Note. Actions are presented in the order that they appear in the GOLDEN framework which, in the case of Y8 (prompt 2b), is also the chronological order in which the actions were performed. In Chapter 5, I discuss in more detail the chronologies of learners' actions.

On prompt 2b, Ash was unable to explain why the predicted answers must be true, he simply responded "*don't know*" and did not perform any actions. However, Ash was later able to perform two different actions in response to the iteration of prompt 2b (prompt 2d), which I describe later in this section. The G2 generalising action, which was performed by Carl, involved simultaneously reasoning about two consistencies that he identified in the processes in the given arguments, and is an example of what Harel and Soto (2017) called process-pattern generalisation (PPG). Carl followed his G2 action by reasoning about a consistent relationship between the result and a process in each of the three given arguments (G3): "*he always divides the added number by 2 to get the answer; 6 divided by 2 is 3, 8 divided by 2 is 4, 10 divided by 2 is 5.* As I suggested in Section 4.1, generalisation of this kind, i.e., 'process-result pattern generalisation' (which I refer to as PRPG) contributes a further form of generalisation to Harel and Soto's (2017) process pattern generalisation (PPG) and result pattern generalisation (RPG). Carl also made a brief follow-up comment: "*That's the basic way of doing it, not the higher way of doing it*", this suggested that, at this point, Carl was perhaps (becoming) aware that there was a more sophisticated relational explanation for the truth of the given propositions. Carl went on to perform a G3 generalising action in each of the subsequent prompts in the number task. Belle, also performed a G3 action on Prompt 2b which she later repeated on a subsequent prompt. In Chapter 5 (Section 5.1.2), I elaborate on how Generalising actions facilitated learners' reasoning and progression through the tasks.

With prompt 2c, I asked learners to write a case that was conceptually similar but non-consecutive to the sequence of three given arguments (see Figure 4.3), i.e., one that would produce a result of 7 for any initially chosen number. Analysis of Y8 learners' responses to prompt 2c yielded a total of 19 actions comprising five of the six main actions in the GOLDEN framework, i.e., all except any Organising actions. The number of actions performed on a prompt often reflected comparative (in)efficiencies between different learners' trajectories to a successful end result. For example, as clearly shown in Table 4.2, all of the Y8 learners were successful on prompt 2c but Ash and Belle achieved success with a considerably lower variety and frequency of actions than Carl. In Section 4.4.3, I say more about the implications of the number of actions that a learner performed on a prompt. As also shown in Table 4.2, almost half of the learners' actions on prompt 2c were a form of Generalising, and no individual sub-action was commonly performed by all three learners.

Table 4.2: Y8's sub-actions on prompt 2c of the number task.

	Sub-actions performed (frequencies are shown in parentheses)	Successful (S)/ Unsuccessful (U)
Ash	G1 (1), D1 (1), E1 (1)	S
Belle	G3 (1), L1 (1), E3 (1)	S
Carl	G1 (4), G3 (2), L2 (1), D1 (1), E1 (1), E2 (1), N2 (3)	S

Note. Actions are presented in the same order as they appear in the GOLDEN framework. I discuss the chronologies of learners' actions in Chapter 5.

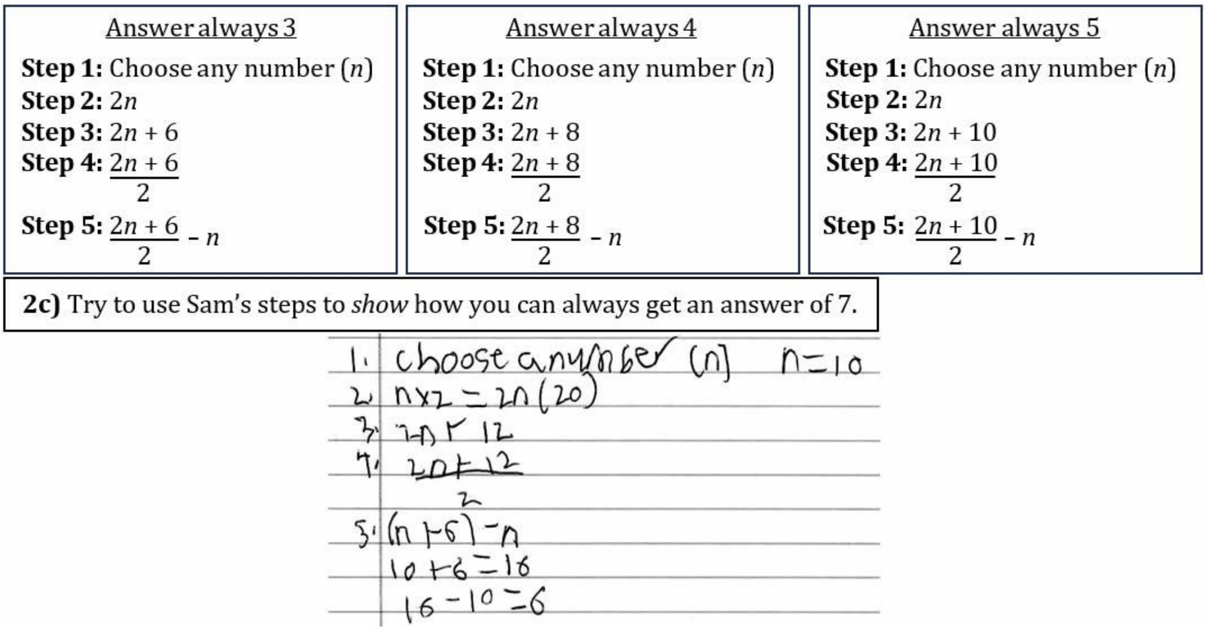
To write a case that consistently produced a result of 7, all of the learners used patterns that they identified in the given arguments, which explains the prevalence of Generalising actions. It was likely that no Organising actions were performed because learners had previously foregrounded and/or backgrounded the information presented in the given arguments in their response to prompt 2a.

Where learners performed Localising actions, these involved adjusting and testing a value by using part or all of the steps in the given arguments (L1 and L2, respectively). All of the Y8 learners were successful in writing an argument that consistently produced a result of 7 without the need to re-adjust any of their previously adjusted values (L3). It was evident that using a PRPG (G3), e.g., Belle's reasoning that "*you are always adding double what the answer is going to be*", played a key role in learners' successes. Although, one learner, Carl, moved back and forth between different Generalising actions four times before achieving a successful outcome.

In writing their responses to prompt 2c, two of the Y8 learners used D1 Deconstructing actions by, respectively, reducing the arguments to (and working on) only the final two steps; and, by deliberately ignoring the division of 2n in step 4 and focusing only on division of the numerical terms. While such D1 Deconstructing actions can be seen as 'personally efficient' ways of working to produce a desired result, the omissions in their responses suggested that these two learners did not, at this point, adequately appreciate how adhering to the format of the given arguments can contribute to convincing others of the validity of their own cases.

Every Y8 learner performed an Extending action on prompt 2c that was catalysed by their earlier pattern generalisations. While all three learners used generalisations, only Carl (in his effort to write a case that consistently resulted in 7) extended the sequence of three given arguments by first writing a 'bridging' case; i.e., one that consistently resulted in 6 before writing a case that consistently resulted in 7 (presented in Figure 4.4).

Figure 4.4: Carl's 'bridging' case that produced a result of 6 for any initially chosen number.



Some learners performed their Extending action(s) near-simultaneously with an associated preceding Generalising action. Even when learners Extended by 'immediately' jumping to and stating a next value in a pattern, they still reasoned, albeit fleetingly, about the general underlying structure of the pattern.

Among Y8 learners' responses to prompt 2c in the number task, only Carl performed any Networking actions, i.e., three N2 actions. N2 actions involved the application, or attempted application, of a concept or procedure from curriculum area qualitatively discrete to those inherent in a given task. All three of Carl's N2 actions aided, in different ways, the writing of his own case. With his first two N2 actions he applied the concepts of proportion and ratio to describe a previously identified consistent relationship between the addends in the given arguments and the predicted results. Carl's third N2 action, involved the introduction of brackets (see Figure 4.4, step 5), which he followed with the comment "*I will now use BIDMAS*". This sequence of events suggested that Carl was perhaps unaware that the structure of his step 5 already featured the correct order of operations. Carl's introduction of brackets and explicit mention of 'BIDMAS' helped him to test if his adjustment of a given addend value would produce a desired end result of 6 for his own 'bridging' case (L2). I elaborate on each main action and salient sequences of actions in Chapter 5: Discussion.

With prompt 2d, as with prompt 2b, I again asked learners to explain why the predicted results are always true. In comparison to prompt 2b, learners' responses to prompt 2d broadly involved more accurate and precise (efficient) reasoning; and, in some cases, more sophisticated sub-actions. For example, Ash and Belle's D3 actions (presented in Table 4.3) revealed their recognition of inverse operations inherent in the given arguments, which were key to explaining the truth of the propositions.

Table 4.3: Y8's sub-actions before and after the case-generation prompt on the number task.

	Sub-actions performed on 2b		Sub-actions performed on 2d
Ash	No actions performed	Prompt 2c involved writing a case for a proposition similar to the given propositions.	G2 (1), D3 (1)
Belle	G3 (1)		D3 (1)
Carl	G2 (1), G3 (1), N2 (1)		G3 (1)

Note. Frequencies are shown in parentheses. Actions are presented in the same order as they appear in the GOLDEN framework. I discuss the chronologies of learners' actions in Chapter 5.

As presented in the right-hand column of Table 4.3, the Y8 learners performed on prompt 2d a total of two Generalising actions (G2 and G3) and two Deconstructing (D3) actions. On the earlier iteration of prompt 2d (prompt 2b), Ash did not perform any actions. However, insights gained by writing his own case for the intervening prompt (2c) seemed to lead him to reason simultaneously about two process generalities in the given arguments (G2); which, in turn, led Ash to successfully explain the truth of the propositions in terms of "doing and undoing", i.e., inverse operations (D3). Belle made a similar shift, at this point, by seeing and reasoning in terms of inversing: "*you are working it out then going backwards, back to the number [predicted answer]*" (D3). Ash and Belle's shifts suggested that they were now able to see and explain the 'why' of the arguments. Carl's response to the earlier iteration of prompt 2d was somewhat protracted and, at times, convoluted. Yet, his response to 2d was markedly more precise. While Carl did not reason in terms of inversing (which was arguably the most insightful way of explaining the truth of the propositions), he was able to articulate his explanation with a single, more precise, PRPG reasoning action (G3). The aforementioned advances in learners' responses constituted my second finding.

Finding 2

After writing a case for a proposition that was similar to the given propositions, learners attended to more complex structure and shifted to more precise and sophisticated reasoning.

Because these shifts in learners' reasoning actions were concerned with explaining the truth of propositions (as prompted to do so), they reflected advances in what Watson et al. (2016) termed learners' 'explanatory proving'. I elaborate on finding 2 and other developments in learners' actions in Chapter 5: Discussion.

4.2.2 Y8's actions on the number *transfer* variation task

As I described in Section 3.7.3, the number transfer variation task (prompt 3) was conceptually similar to the initial variation task. My aim with the conceptual similarity was to aid my identification of transferred actions, i.e., actions that were similarly performed on the initial variation task and the transfer variation task. In this section, I report specifically on the actions that learners performed on the transfer task, i.e., in isolation to their actions on the initial task. I discuss learners' actions in terms of transfer across the variation tasks in Sections 4.3.1 to 4.3.6.

As with the initial variation task, the transfer task featured three given arguments that each produced the same predicted result for any initially chosen number. However, in the transfer task, two of the arguments were specific numerical examples and the third argument was a proof represented algebraically. In addition, the steps in the transfer arguments were different to those in the initial variation task, and all of the argument variations produced a predicted result of 5 only. With the transfer task, shown in Figure 4.5, I asked learners to use the three given argument variations to write, using algebra, a similar case that would consistently produce a result of 7 for any initially chosen value.

Figure 4.5: The number transfer variation task.

3) Luka looks at Sam's way of getting an answer that is always 5, and says that he knows a slightly similar way to always get an answer of 5 (shown below).

Answer always 5	Example A
Step 1: Choose any number	6
Step 2: Add the next number	$6 + 7 = 13$
Step 3: Add 9	$13 + 9 = 22$
Step 4: Divide by 2	$\frac{22}{2} = 11$
Step 5: Subtract the chosen number	$11 - 6 = 5$

Answer always 5	Example B
Step 1: Choose any number	7
Step 2: Add the next number	$7 + 8 = 15$
Step 3: Add 9	$15 + 9 = 24$
Step 4: Divide by 2	$\frac{24}{2} = 12$
Step 5: Subtract the chosen number	$12 - 7 = 5$

Answer always 5	Algebraic Proof
Step 1: Choose any number	n
Step 2: Add the next number	$n + (n + 1) = 2n + 1$
Step 3: Add 9	$2n + 1 + 9 = 2n + 10$
Step 4: Divide by 2	$\frac{2n + 10}{2} = n + 5$
Step 5: Subtract the chosen number	$n + 5 - n = 5$

Try to show *using algebra* how Luka's method can be used to always get an answer of 7.

I identified in the Y8 learners' responses a total of 14 sub-actions comprising all six main actions except Organising (presented in Table 4.4). Each of the Y8 learners performed at least one Localising action, and all except Belle performed Generalising and Deconstructing actions. Table 4.4 also shows if learners worked with the numerical examples and/or the proof as well as if they successfully completed the task.

Table 4.4: Y8's sub-actions on the number transfer variation task.

	Sub-actions performed (frequencies are shown in parentheses)	Numerical examples/ proof/both	Successful/ Unsuccessful
Ash	G1 (1), G3 (1), L2 (1), D1 (1), E1 (1)	Proof	Successful
Belle	L2 (2)	Numerical examples	Unsuccessful
Carl	G1 (2), G3 (1), L1 (1), L2 (1), D1 (1), N2 (1)	Proof	Successful

Note. Actions are presented in the same order as they appear in the GOLDEN framework. I discuss the chronologies of learners' actions in Chapter 5.

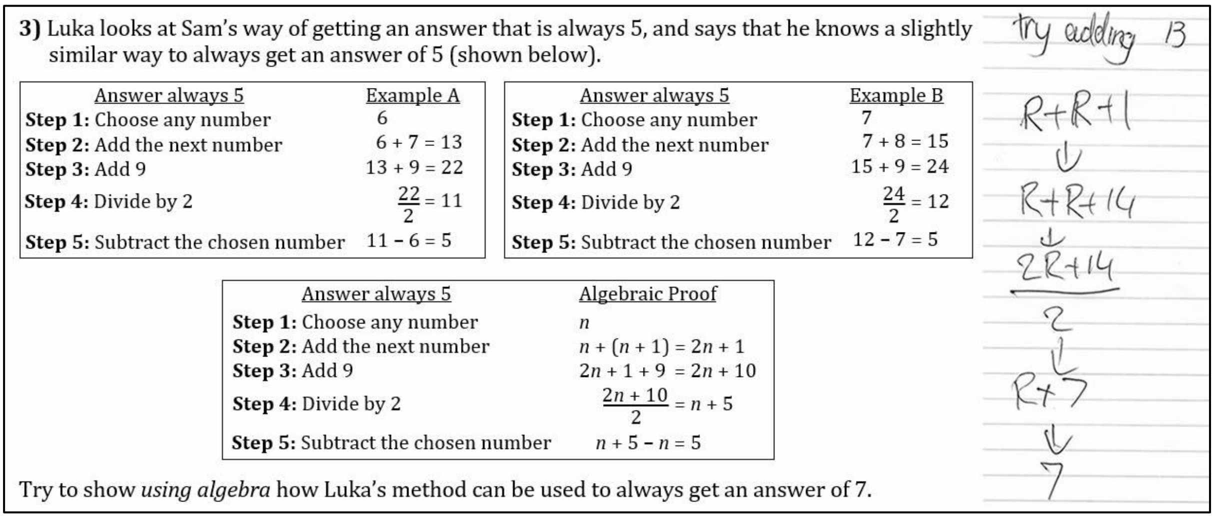
Ash and Carl performed G1 (single-process) Generalisations and, later, performed more sophisticated (G3) actions. The latter of which were the consequence of reasoning about a process-result pattern generality (PRPG). The foresaid shifts provided evidence that consecutive-number variation tasks can encourage and support learners' use of generalising actions; which, as I explained in Section 2.3, was one of the intended affordances of my task design.

Localising actions involved increasing the value of a given addend based on various conceptions of what would, or might, consistently produce a result of 7 for any initially chosen number. Ash and Carl responded successfully to the task by reasoning about a PRPG and by making local adjustments to the given algebraic proof. Whereas Belle, who was unsuccessful, did not generalise and localised only by tinkering with addend values given in one of the numerical arguments. Specifically, Ash and Carl based the value of their local adjustments on a combination of the outcome of an identified process generality and the value of the predicted result. Whereas, Belle based her addend adjustment on a conjecture that 'increasing a given addend to two more than its augend might result in a successful outcome because [the required result of] 7 is 2 more than [the given result of] 5'. This suggested that reasoning in terms of a process-result pattern generality (PRPG) can provide a more robust basis for making local adjustments than localised conjecturing alone. Nonetheless, the outcomes of Belle's different local adjustments moved her towards the territory of 'trial and improvement', which can lead to discovering a helpful generality (Mason et al., 2010) such as a consistent process-result relationship.

Ash and Carl both used D1 Deconstructing actions but in different ways. Ash, in the writing of his own case, shortened all of the equations in the given algebraic proof (shown in Figure 4.6). Carl Deconstructed by crossing out an algebraic term in the given proof to help him focus on a numerical aspect of a calculation in the writing of his own case. As Carl crossed out this algebraic term, he explained that, "*To make it [his case] a bit simpler, I'm going to get rid of that [2n in step 3] for now*". In a subsequent step, Carl reintroduced the algebraic term in its correct position. Deconstructing by crossing out, truncating, or 'parking' structure (D1) is different to foregrounding/backgrounding when Organising structure; in my study, the former refers to the removal of structure (decreasing the number of objects) so has a dismantling quality, whereas the latter is concerned with information processing that results in noticing sameness and difference.

An Extending action (E1) was performed by Ash which was a result of reasoning that the consistent addend value of 9 (in step 3 of the given arguments) needed to be increased by 4 to obtain a result of 7 for any initially chosen number. Ash used this reasoning to extend the pattern of consistent addend values from 9 to 13 (shown in Figure 4.6).

Figure 4.6: Ash's response to the number transfer variation task.


3) Luka looks at Sam's way of getting an answer that is always 5, and says that he knows a slightly similar way to always get an answer of 5 (shown below).

Answer always 5	Example A
Step 1: Choose any number	6
Step 2: Add the next number	6 + 7 = 13
Step 3: Add 9	13 + 9 = 22
Step 4: Divide by 2	$\frac{22}{2} = 11$
Step 5: Subtract the chosen number	11 − 6 = 5

Answer always 5	Example B
Step 1: Choose any number	7
Step 2: Add the next number	7 + 8 = 15
Step 3: Add 9	15 + 9 = 24
Step 4: Divide by 2	$\frac{24}{2} = 12$
Step 5: Subtract the chosen number	12 − 7 = 5

Answer always 5	Algebraic Proof
Step 1: Choose any number	n
Step 2: Add the next number	$n + (n + 1) = 2n + 1$
Step 3: Add 9	$2n + 1 + 9 = 2n + 10$
Step 4: Divide by 2	$\frac{2n + 10}{2} = n + 5$
Step 5: Subtract the chosen number	$n + 5 - n = 5$

Try to show *using algebra* how Luka's method can be used to always get an answer of 7.

Only Carl used a Networking action (N2) on the transfer task. With his N2 action, Carl explained that he was again going to use the 2 : 1 (addend total : result) ratio that he identified in the initial variation task.

4.2.3 Y10's actions on the number *initial* variation task

Since the first prompt (2a) in the number initial variation task (shown in Figure 4.2) asked 'what is the same and what is different about the given arguments?', I suspected that Y10 learners would only perform Organising actions, i.e., by foregrounding and backgrounding aspects of the given information. While this was largely the case, some Y10 learners also performed

Generalising actions. In total, I identified two Generalising actions, one that involved reasoning about a single process (G1), and one that focused on a process-result relationship (G3); and, nine Organising actions comprising six that prioritised a dimension of sameness (O1), and three that prioritised a dimension of difference (O2).

Y10 learners' responses varied markedly in their level of detail. In regard to prioritising a dimension of sameness (O1), Freya gave the vaguest response, "*the method [in the three given arguments] is the same*" (O1), but she preceded this by specifically identifying differences in the addends (O2). Dylan and Emma both identified dimensions of sameness followed by a dimension of difference. Dylan and Emma's identification of difference in the given addends was followed by a generalisation that involved justifying the underlying reason for the difference, as was the case with Ash in Y8. Yet, Emma's justification can be seen as more sophisticated than Ash's and Dylan's because Emma reasoned about a general PRPG relationship between respective addends and respective results: "*it [the addend] is always double the number that the number [predicted answer] always is*" (G3); whereas, Ash and Dylan's reasoning for the same identified difference in addends focused only on the addends, "*it's adding/they go up by two each time*" (G1).

On prompt 2b, which asked for an explanation for why the predicted answers are always true and was later iterated with prompt 2d, Y10 learners performed a total of 10 actions. These 10 actions (presented in Table 4.5) comprised Generalising, Organising and Deconstructing.

Table 4.5: Y10's sub-actions on prompt 2b of the number task.

	Sub-actions performed (frequencies are shown in parentheses)
Dylan	G2 (1), G3 (1)
Emma	G2 (1), G3 (1)
Freya	G1 (1), G2 (3), O1 (1), D3 (1)

Note. Actions are presented in the order that they appear in the GOLDEN framework. In Chapter 5, I discuss in more detail the chronologies of learners' actions.

Across the Y10 learners, the Generalising actions involved all three sub-actions, i.e., reasoning about one, and two, consistencies in the processes (G1 and G2, respectively), and reasoning about a consistent process-result (PRPG) relationship (G3). On the prior prompt (2a), Dylan reasoned only about one process generality (G1) but on prompt 2b Dylan's focus of attention and reasoning grew to encompass a generalisation involving two processes (G2) and a process-result PRPG generalisation (G3). Dylan evidenced the latter by reasoning: "*it [each respective addend] is half of whatever the amount you get [the predicted result]*". While Dylan's G3 reasoning did not explain the truth of the propositions in terms of inverse operations, it nonetheless represented a conceptual shift that involved attending to a growing amount of the argument structures. It was not until Dylan wrote his own case on prompt 2c that he started to reason (on prompt 2d) in terms of inversing: "*you divide [by 2] to get rid of [cancel] the [coefficient of] 2*". Freya was the only Y10 learner to perform Organising and Deconstructing actions (O1 and D3) which, together with her G1 and G2 Generalising actions, seemed to fluctuate in their sophistication but eventually culminated in an albeit tentative and indirectly reasoned move towards the notion of inversing: "*you're 'adding' then dividing* (G2) *so you'll go back*" (D3).

As I explained in Section 4.2.1, with prompt 2c (presented in Figure 4.3) I asked learners to write their own case that was conceptually similar to the three given arguments but one that would result in 7 for any initially chosen number. Analysis of Y10 learners' responses yielded a total of 11 actions comprising four of the six main actions in the GOLDEN framework. As shown in Table 4.6, and similarly to the Y8's responses, almost half of the Y10 learners' actions on prompt 2c were a form of Generalising. However, Y8's responses to prompt 2c did not involve any commonly performed sub-actions across all three learners; whereas, all of the Y10 learners reasoned in terms of an identified process-result pattern (G3) then extended their identified process-result pattern (E3). Table 4.6 also shows if learners responded successfully to prompt 2c.

Table 4.6: Y10's sub-actions on prompt 2c of the number initial variation task.

	Sub-actions performed (frequencies are shown in parentheses)	Successful/ Unsuccessful
Dylan	G1 (1), G2 (1), G3 (1), E1 (1), E3 (1)	Successful
Emma	G3 (1), E3 (1), D1 (1)	Successful
Freya	G3 (1), E3 (1), N2 (1)	Successful

Note. Actions are presented in the same order as they appear in the GOLDEN framework. I discuss the chronologies of learners' actions in Chapter 5.

Identifying (then extending) a process-result pattern appeared to be a critical element of all the Y10 learners' successes in writing an argument that consistently produced a result of 7. Dylan reached his G3 generalisation by first reasoning about a pattern in the addends (G1) and then reasoning about a pattern in the operations in step 4 (G2). The other two Y10 learners, Emma and Freya, performed their G3 generalisations on prompt 2c without any precursory G1 or G2 actions. Emma performed her G3 action immediately, which suggested that she was applying existing knowledge of this (process-result) generality gained from her growing recognition of this relationship while working on the earlier prompts (2a and 2b). Freya's first action briefly involved Networking (which I discuss later in this section) following which she suddenly recognised and reasoned about the G3 process-result relationship: "*Oh wait hang on, what you're adding each time is double the number [predicted result]*". Apart from Freya's initial and brief

Networking action, all of the Y10 learners soon identified in the given argument variations a generality of some kind in response to prompt 2c. Such efficient identification of generalities might explain why none of the Y10 learners performed any Localising actions which, as with Mason et al's (2010) description of *specialising* actions, can lead to seeing generalities. As I did with my analysis of Y8's responses to prompt 2c, I inferred that the absence of any Y10 Organising actions was due to their prior foregrounding and/or backgrounding of the information presented in the given arguments in their earlier response to prompt 2a.

Only Emma used a Deconstructing action which she performed by omitting the first four given steps in the writing of her own 'argument'. Emma's omitted steps meant that she essentially reduced into a single (final) step all five steps in the given arguments (D1). While Emma did not precede her single written step of $\frac{2n+14}{2} - n$ with the steps in its construction, it nonetheless consistently produced a result of 7 for any substituted value for n. Emma also uttered aloud each term as she was writing the single step, which gave the impression that she was perhaps thinking about each individual step while at the same time reducing the full argument structure. Such reductions in written structure are useful because they provide classroom discussion points for what constitutes an 'argument'. Conversations of this kind can include the 'explanatory' and 'convincing' elements of argumentation. I elaborate on the implications of reducing structure in Chapter 5, Section 5.1.5.

As I mentioned earlier, on prompt 2c, all of Y10's Extending actions immediately followed corresponding Generalising actions; i.e., Dylan's E1 action (extending a single process) immediately followed his single-process generalisation (G1), and all three learners' E3 actions (extending a process-result action) immediately followed their process-result generalisations (G3). Only Freya performed a Networking action (N2) which, in this instance, was unfruitful because she (briefly) saw it appropriate to directly substitute for n the given predicted result value of 7. Freya's momentary conflation of algebraic concepts might be explained by her prior experience of lessons in which the sole requirement was to substitute given values into given expressions then calculate the result.

On prompt 2d, i.e., the iteration of prompt 2b, I again asked learners to explain why the predicted results are always true. Y10 learners performed on prompt 2d a total of nine actions (six Generalising and three Deconstructing). As presented in Table 4.7, collectively, Y10's Generalising actions comprised all three sub-actions. Each of the three learners performed a Generalising action and reasoned in terms of (the effect of) inversing (D3). All of their inversing actions followed, near simultaneously, one of their G2 (process) or G3 (process-result) Generalising actions.

Table 4.7: Y10's sub-actions before and after the case-generation prompt on the number task.

	Sub-actions performed on 2b	Prompt 2c involved writing a case for a proposition similar to the given propositions.	Sub-actions performed on 2d
Dylan	G2 (1), G3 (1)		G2 (1), D3 (1)
Emma	G3 (1)		G1 (1), G2 (1), G3 (1), D3 (1)
Freya	G1 (1), G2 (3), O1 (1), D3 (1)		G2 (1), G3 (1), D3 (1)

Note. Frequencies are shown in parentheses. Actions are presented in the same order as they appear in the GOLDEN framework. I discuss the chronologies of learners' actions in Chapter 5.

As with the Y8 learners, Y10's explanatory reasoning actions on prompt 2d were somewhat more formal and precise, i.e., sophisticated and convincing, than on the earlier iteration (prompt 2b). For example, Freya's explanation for the truth of the propositions on prompt 2b began: "*Is it dividing? ... 2n divided by 2? ... I'm not ... they're all even numbers*"; whereas, after writing her own case, Freya's explanation became more somewhat more focused, descriptive, and convincing, "*You're doubling the number [predicted result] then halving [it], so it's back to the number [predicted result]*". In the same way, Emma's explanation for the truth of the propositions on prompt 2b began: "*When you add 6, or add 8, or add 10, I think that's the main part of it because it's connected to the answer*"; but for prompt 2d Emma's reasoning evolved to become: "*You're timesing by 2 then dividing by 2* (G2), *and the number [addend of 14] divided by 2 is 7, so it's always the same* (D3)". As I described earlier, Dylan's explanations in response to prompts 2b and 2c also demonstrated a similar shift in formality and precision.

The aforementioned advances in explanatory reasoning (from prompt 2b to 2d) is evidence for the idea that providing heavily scaffolded support for learners to write their own cases can broaden what structure they notice and reason about, and can contribute to increasing the precision of their related explanations. In Section 4.2.5, I compare in more detail the actions of both year groups on the number variation tasks.

4.2.4 Y10's actions on the number *transfer* variation task

As I described earlier, the number transfer variation task (presented in Figure 4.5) featured three arguments that were conceptually similar to those given in the initial variation task. To briefly restate, three aspects of the arguments given in transfer task were different to those in the initial task: (1) the procedures (structures); (2) the first two were specific numerical examples and the third one was an algebraic representation of the numerical examples; and (3) for any initially chosen number, all three produced a predicted result of 5. With the transfer task, I asked learners to write, using algebra, an argument that was similar to the given algebraic argument, but one that would result in 7 for any initially chosen number.

As presented in Table 4.8, I identified in Y10 learners' responses a total of 12 actions comprising five of the six main actions in the framework, i.e., all except Extending actions. I designed the transfer task so that it was harder than on the initial task to see any process-result patterns across the given arguments. Hence, it was likely that no Extending actions were performed because no learner first generalised by identifying, across the given arguments, a definite process-result pattern that they could continue. Only Emma performed a G3 (process-result) Generalising action and this was assumptive in nature because she generalised on the basis of an invariant addend value across the given arguments (or from a single argument) rather than from a co-varying relationship: "*if for an answer of 5 it [the addend in step 3] is 9 then for an answer of 7 it [the addend in step 3] will be 11*". Emma made this utterance in a tentative manner which suggested that she was experimenting, or at least not certain of the outcome. When Emma tested her suggested addend value of 11, she saw that it produced an end result of 6 rather than the required end result of 7 (shown in Figure 4.7 along with the given proof). Emma's G3 action, which after testing led her to consider alternative addend values, was the only Generalising action performed by any of the Y10 learners on the transfer task.

Figure 4.7: Emma's test of a conjectured process-result generality that produced a result of 6.

Answer always 5	Algebraic Proof
Step 1: Choose any number	n
Step 2: Add the next number	$n + (n + 1) = 2n + 1$
Step 3: Add 9	$2n + 1 + 9 = 2n + 10$
Step 4: Divide by 2	$\frac{2n + 10}{2} = n + 5$
Step 5: Subtract the chosen number	$n + 5 - n = 5$

Try to show *using algebra* how Luka's method can be used to always get an answer of 7.

n

$n + (n+1) = 2n+1$

$2n+1+11 = 2n+12$

$\frac{2n+12}{2} = n+6$

$n+6-n = 6$

The only Organising action was a foregrounded dimension of sameness (O1) identified by Dylan and performed as his initial action: "*it might be something to do with the [addend of] 9 because that's in all of them*". It was possible that Dylan used this dimension of sameness as an entry point because it was potentially part of a (useful) general relationship. However, Dylan did not go on to reason about a generality that involved his O1 observation; this was perhaps because his O1 observation was not part of an obvious co-varying relationship. Instead, Dylan Localised by adjusting and testing different addend values (L2 and L3). Indeed, the lack of obvious co-varying general relationships in the given argument structures seemed to 'force' all of the Y10 learners to, at some point, Localise using L2 and L3 actions. As shown in Table 4.8, Localising, i.e., (re)adjusting and (re)testing given addend values (L2 and L3), was the most common action and accounted for just over half of all Y10's actions on the transfer task. Table 4.8 also shows if learners worked with the numerical examples and/or the proof as well as if they successfully completed the task.

Table 4.8: Y10's sub-actions on the number transfer variation task.

	Sub-actions performed (frequencies are shown in parentheses)	Numerical examples/ proof/both	Successful/ Unsuccessful
Dylan	O1 (1), L2 (1), L3 (1), N3 (1)	Numerical examples	Unsuccessful
Emma	G3 (1), L2 (1), L3(1), D2 (1)	Proof	Unsuccessful
Freya	L2 (2), L3 (1), D2 (1)	Proof	Successful

Note. Actions are presented in the same order as they appear in the GOLDEN framework. In Chapter 5, I discuss in more detail the chronologies of learners' actions.

The D2 Deconstructing action, performed by Emma, occurred after her test of an adjusted addend proved unsuccessful, and seemed to be an alternative approach to searching for a relationship that she could use in the writing of her own case. Emma's D2 action involved starting from one of the given results and working backwards through the steps while looking for a critical relationship: "*the answer is odd, so maybe what you add in step 3, after adding the next number in step 2, should [also] be odd*". Emma did not seem to pursue this conjecture but it may have influenced her next action which was to adjust her previous addend adjustment from 11 to 13 (L3).

The N3 Networking action (introducing a value other than a positive integer) was performed by Dylan and was a heuristic strategy to overcome his previously tested addend adjustment that resulted in 6.5, rather than the desired result of 7. Dylan chose to work on one of the numerical example arguments and suggested keeping his previously adjusted addend value and augmenting his case with a final step of 'adding 0.5'. Dylan then explained that if the added value in his additional step worked for his numerical argument, he would then make the same adjustment to the given algebraic argument.

4.2.5 Comparison of Y8 and Y10's actions on the number variation tasks

Here, I compare to each other both year groups' actions on each of the respective number variation tasks. To do this, I first compare both year groups' actions on the *initial* variation task. I follow this by comparing both year groups' actions on the *transfer* variation task. Later, in Sections 4.3 to 4.3.6, for each domain separately, I characterise Y8's actions *across* the initial and transfer tasks, then I characterise Y10's actions across these same tasks. I follow this by comparing both year groups' foresaid cross-task actions.

Initial variation task

As presented in Table 4.9, across all four combined prompts (2a-2d) on the initial variation task, Y8s collectively performed a total of 32 actions and Y10s collectively performed at total of 40 actions. The most common main (superordinate) action in each respective year group was Generalising, which was approximately half of each year group's sub-actions. For Y8, the joint most common Generalising sub-actions were G1 (single process) generalisations and G3 (process-result) generalisations; whereas, in Y10, the most common Generalising sub-action was G2 (two-process) generalisations with an almost equal frequency of G3 (process-result) generalisations. Each year group's most common Generalising sub-action(s) were also each group's most common sub-action across all six main actions combined. Although not shown in the Table 4.9, in each respective year group, a Generalising action was the most common first action across all of the prompts in the initial variation task. The most common sub-action(s) performed by each year group on each prompt on the initial variation task are presented in Table 4.9.

As I described Section 4.1, I organised the three sub-actions in each main action so that, arguably, the first sub-action was the least sophisticated and the third sub-action was the most sophisticated. A comparison of both year groups' sub-group 3 actions (i.e., G3, O3, L3, D3, E3, and N3 actions) is presented in Table 4.9. Across all four combined prompts on the initial variation task, Y8s collectively performed a total of nine sub-group 3 actions; whereas, collectively, Y10s performed a total of 14 of these same actions.

Table 4.9: Comparison of Y8 and Y10's actions on the number initial variation task.

Number initial variation task		
	Y8	Y10
Number of sub-actions	32	40
Most common main action	G	G
Most common sub-action(s)	G1/G3	G2
Prompt 2a: most common sub-action(s)	O1/O2	O1
Prompt 2b: most common sub-action	G3	G2
Prompt 2c: most common sub-action(s)	G1	G3/E3
Prompt 2d: most common sub-action(s)	D3	G2/D3
Number of sub-group 3 actions	9	14
Most common consecutive main actions	G, E/G, N	G, D
Unperformed main actions	-	L

Note. G (Generalising); O (Organising); L (Localising); D (Deconstructing); E (Extending); N (Networking).

In Y8, across all four respective prompts in the initial task, the joint most common sequence of consecutive main actions were Generalising actions followed by Extending actions (with an equal number of Generalising actions followed by Networking actions); whereas, for Y10, this same sequence comprised Generalising actions followed by Deconstructing actions. Also presented in Table 4.9 are any main actions that each respective year group did not perform. None of the Y10 learners performed any Localising actions, which was the only main action not performed by either year group on the number initial variation task.

<u>Transfer variation task</u>

On the number transfer task (prompt 3), and as presented in Table 4.10, Y8s collectively performed a total of 14 actions and Y10s collectively performed at total of 12 actions. For Y8, the joint most common main actions were Generalising and Localising which, respectively, were each approximately one third of all their sub-actions; whereas, Y10's most common main action was Localising, which were approximately half of all their sub-actions. With Y8's joint most common main actions (Generalising and Localising), the most frequent sub-action was L2 (adjusting/testing a given value using all five steps); whereas, with Y10's most common main action, the most frequent sub-action was L3 (re-adjusting a previously adjusted value and testing the re-adjustment). The aforementioned most common sub-actions were, for each respective year group, also the most frequent sub-actions across all six main actions. Although not shown in Table 4.10, in each respective year group, as with the initial task, a Generalising action was the most common first action on the transfer task. Each year group's most common sub-action on each prompt in the transfer task is presented in Table 4.10.

On the transfer variation task, Y8s collectively performed only two sub-group 3 actions, both of which were G3 (process-result) generalisations; whereas, collectively, Y10s performed six sub-group 3 actions, four of which were L3 (re-adjusting a previously adjusted value and testing the re-adjustment). Y8's sub-group 3 actions were nearly one quarter of their collective actions, whereas Y10's sub-group 3 actions were exactly half of all their collective actions.

Table 4.10: Comparison of Y8 and Y10's actions on the number transfer variation task.

Number transfer variation task		
	Y8	Y10
Number of sub-actions	14	12
Most common main action	G/L	L
Most common sub-action	L2	L3
Number of sub-group 3 actions	2	6
Most common consecutive main actions	L, D	D, D
Unperformed main actions	O	O

Note. G (Generalising); O (Organising); L (Localising); D (Deconstructing).

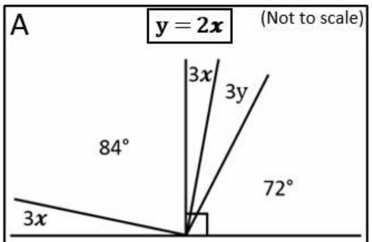
In Y8, the most common sequence of consecutive main actions was Localising followed by Deconstructing; whereas, in Y10, it was repeated Localising actions that mostly comprised adjusting and testing then re-adjusting and re-testing a given value. I also presented in Table 4.10 any main actions that each respective year group did not perform; none of the learners performed any Organising actions, which was the only main action not performed by either year group on the number transfer variation task.

4.2.6 Y8's actions on the geometry *initial* variation task

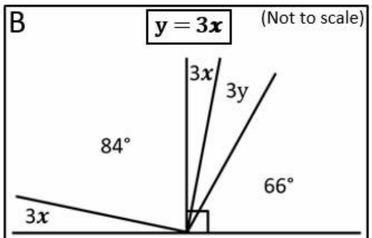
As I illustrated in Figure 4.1, the geometry initial variation task comprised four prompts (2a to 2d) that were isomorphic to these same prompts in the number initial variation task. I next describe learners' actions on each of these four prompts in the geometry domain. With prompt 2a, as with this same prompt in the number domain, I provided three argument variations featuring consecutive numbers, and I asked the learners what is the same and what is different about them (for ease of reference, I again present prompt 2a, in Figure 4.8). My reason for asking this question was the same as in the number initial task, i.e., I sought to reveal how each learner would organise (foreground and background) the information in the arguments by prioritising a dimension of sameness or difference.

Figure 4.8: The first prompt (2a) on the geometry initial variation task.

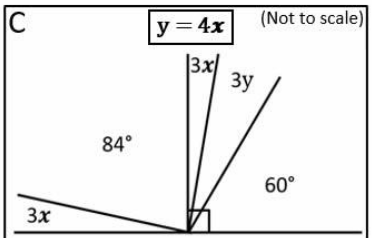
2a) Eve looks at Maddison's method, and says that by using similar steps she can show for diagram A that $y = 2x$, for diagram B that $y = 3x$, and for diagram C that $y = 4x$.



Step 1: $3x + 84° = 90°$
Step 2: $90° - 84° = 6°$, then do $6° \div 3$ to give $\boldsymbol{x = 2°}$
Step 3: $3y + 6° + 72° = 90°$
Step 4: $90° - 72° - 6° = 12°$, then do $12° \div 3 =$ to give $\mathbf{y = 4°}$
Step 5: state that $\boldsymbol{x = 2°}$ and $\mathbf{y = 4°}$ so $\mathbf{y = 2}\boldsymbol{x}$



Step 1: $3x + 84° = 90°$
Step 2: $90° - 84° = 6°$, then do $6° \div 3$ to give $\boldsymbol{x = 2°}$
Step 3: $3y + 6° + 66° = 90°$
Step 4: $90° - 66° - 6° = 18°$, then do $18° \div 3 =$ to give $\mathbf{y = 6°}$
Step 5: state that $\boldsymbol{x = 2°}$ and $\mathbf{y = 6°}$ so $\mathbf{y = 3}\boldsymbol{x}$



Step 1: $3x + 84° = 90°$
Step 2: $90° - 84° = 6°$, then do $6° \div 3$ to give $\boldsymbol{x = 2°}$
Step 3: $3y + 6° + 60° = 90°$
Step 4: $90° - 60° - 6° = 24°$, then do $24° \div 3 =$ to give $\mathbf{y = 8°}$
Step 5: state that $\boldsymbol{x = 2°}$ and $\mathbf{y = 8°}$ so $\mathbf{y = 4}\boldsymbol{x}$

What is the same and what is different about illustrations A, B and C?

Collectively, Y8 learners' responses to prompt 2a involved only Organising actions. In total, seven Organising actions were performed, of which three involved prioritising a dimension of sameness (O1), and four involved prioritising a dimension of difference (O2). Belle and Carl first foregrounded at least one dimension of sameness (O1) followed by a dimension of difference (O2), whereas Ash performed only O2 actions. All three learners initially attended to the diagrams rather than the accompanying steps. Ash and Carl focused exclusively on the diagrams, whereas the Belle's final Organising action was performed on the accompanying steps. When identifying sameness and difference, learners' dominant attention to the diagrams rather than to the accompanying processes (steps) might be due to easier direct visual apprehension, i.e., a reduction in *cognitive load* (Paas & Van Merriënboer, 1994).

With prompt 2b, which I later repeated with prompt 2d, I asked learners to verbally explain what a diagram similar to the given diagrams would look like to show that $y = 6x$. As with the number task, my reason for repeating prompt 2b was to evaluate if responses to the interim prompt (2c), for which learners produced a written argument similar to the given arguments, would increase the sophistication of their reasoning, this time in regard to their descriptions of a diagram that would show that $y = 6x$. Figure 4.9 shows prompts 2b to 2d, which are isomorphic to prompts 2b to 2d on the number task.

Figure 4.9: Prompts 2b to 2d on the geometry initial variation task.

2b) Try to describe what a similar diagram would look like that could be used to show $y = 6\boldsymbol{x}$.

2c) Considering your response to **2b**, try to use Eve's steps to show that $y = 6\boldsymbol{x}$.

2d) Try again to describe what a similar diagram would look like that could be used to show $y = 6\boldsymbol{x}$.

In response to prompt 2b, all three Y8 learners performed the same three actions in the same order (presented in Table 4.11). Specifically, each Y8 learner first reasoned about a single-process generalisation (G1) by explaining that in each subsequent given diagram, the right-hand numerical angle decreased by 6° (see Figure 4.8). Each learner then combined their G1 generalisation with a generality in the given results, i.e., a consistent increase of 1 in the x coefficient (G3). Finally, each learner extended their G1 generality to two further consecutive cases, which indicated that they were concurrently extending a process pattern (subtracting 6°) and a result pattern ($y = 2x$, $y = 3x$ and $y = 4x$ to $y = 6x$)(E3).

Table 4.11: Y8's sub-actions on prompt 2b of the geometry task.

	Sub-actions performed (frequencies are shown in parentheses)
Ash	G1 (1), G3 (1), E3 (1)
Belle	G1 (1), G3 (1), E3 (1)
Carl	G1 (1), G3 (1), E3 (1)

Note. Actions are presented in the order that they appear in the GOLDEN framework which, in the case of Table 4.11, are also the chronological order in which the actions were performed. In Chapter 5, I discuss in more detail the chronologies of learners' actions.

With prompt 2c, I asked learners to write a case that was conceptually similar to the given arguments; i.e., one for which $y = 6x$. As presented in Table 4.12, I identified in Y8 learners' responses to prompt 2c a total of eight actions comprising four of the six main actions in the GOLDEN framework, i.e., all except Localising and Networking actions. Two of the learners began by foregrounding a dimension of sameness (O1), one learner reduced by omission part of the steps in the given structures (D1), and all three learners performed Extending actions. Table 4.12 also shows if learners responded successfully to prompt 2c.

Table 4.12: Y8's sub-actions on prompt 2c of the geometry task.

	Sub-actions performed (frequencies are shown in parentheses)	Successful (S)/ Unsuccessful (U)
Ash	O1 (1), D1 (1), E1 (1)	S
Belle	E1 (2)	S
Carl	G3 (4), O1 (1), E3 (1)	U

Note. Actions are presented in the same order as they appear in the GOLDEN framework. I discuss the chronologies of learners' actions in Chapter 5.

Ash and Belle successfully produced an argument for $y = 6x$, but they both appeared to largely depend on systematically substituting into the given steps their extended pattern value of 48°. For example, Belle commented that, "*Where 'they' changed it, I changed it*" (see the variations in Figure 4.8). Hence, Belle's attention was oriented to a dimension of variation (against a backdrop of invariance) in the design of given structures. While Extending by substitution provided learners with a method of writing their 'own' case, it perhaps reduced their sense of simultaneously testing a value and seeing why $y = 6x$. Nonetheless, the control of (in)variance in the design of the three given arguments enabled these two learners to successfully write a similar five-step case of their own within approximately 5-mins of seeing the (unfamiliar) task. Thus, case construction by Extending, i.e., systematic substitution, is a potentially valuable introductory form of attending to structure. Carl did not successfully write a case for $y = 6x$ because he chose to focus only on drawing the, albeit correct, diagram. Perhaps Carl did not write an accompanying argument because he was convinced that his diagram adequately showed that $y = 6x$. If so, this raises a question akin to that asked by Stylianides et al. (2016); specifically, what are the conditions at secondary school level for an argument to be explanatory? The obvious answer is that an explanatory argument comprises a valid sequence of logically linked assertions that satisfy a proposition. However, arguments can be visual rather than symbolic. Further, GCSE examination board marking schemes sometimes credit responses to argument- and proof-related tasks if adequate thinking can be inferred from a student's response which, in some cases, may be a written augmentation of a given diagram. I discuss some of the advantages and pitfalls of learners deviating from the fidelity of the given argument structures in the next chapter (Section 5.1.4)

With prompt 2d, which was a repetition of prompt 2b, I again asked learners to describe what a diagram similar to the given diagrams would look like to show that $y = 6x$. As I explained earlier in this section, I repeated prompt 2b to assess if the interim act of writing an iteration of the given arguments would increase the sophistication of learners' descriptions and underlying reasoning for the appearance of an argument for $y = 6x$. In comparison to prompt 2b, learners' responses to prompt 2d were slightly more explanatory. Ash and Carl gave somewhat more emphasis to a relationship inherent in a process-result pattern (G3), which was reflected in their more detailed explanations of how far to extend the pattern (E3); i.e., by subtracting 6° from the right-hand numerical angle in each subsequent given argument variation. Specifically, Ash explained that, "*if for 4 [the 4 in $y = 4x$] it [the right-hand numerical angle] is 60°, then for 6 [the 6 in $y = 6x$] it [the right-hand numerical angle] is 48 degrees*". While Belle's response was rather limited, she demonstrated a promising shift of attention away from sole reliance on the diagrams to a focus on differences in the accompanying argument structures (O2).

Table 4.13 is a comparison of the sub-actions that Y8 learners performed on prompts 2b and 2d, i.e., before and after writing their own case. From a component (coding) perspective, the actions of Ash and Carl were the same on prompts 2b and 2d. Yet, as I explained, the actions inherent in their descriptions involved slightly more explanatory detail.

Table 4.13: Y8's sub-actions before and after the case-generation prompt on the geometry task.

	Sub-actions performed on 2b	Prompt 2c involved writing a case for a proposition similar to the given propositions.	Sub-actions performed on 2d
Ash	G1 (1), G3 (1), E3 (1)		G1 (1), G3 (1), E3 (1)
Belle	G1 (1), G3 (1), E3 (1)		O2 (1)
Carl	G1 (1), G3 (1), E3 (1)		G1 (1), G3 (1), E3 (1)

Note. Frequencies are shown in parentheses. Actions are presented in the same order as they appear in the GOLDEN framework which, in Table 4.13, are also the same order in which they were chronologically performed. In Chapter 5, I discuss in more detail the chronologies of learners' actions.

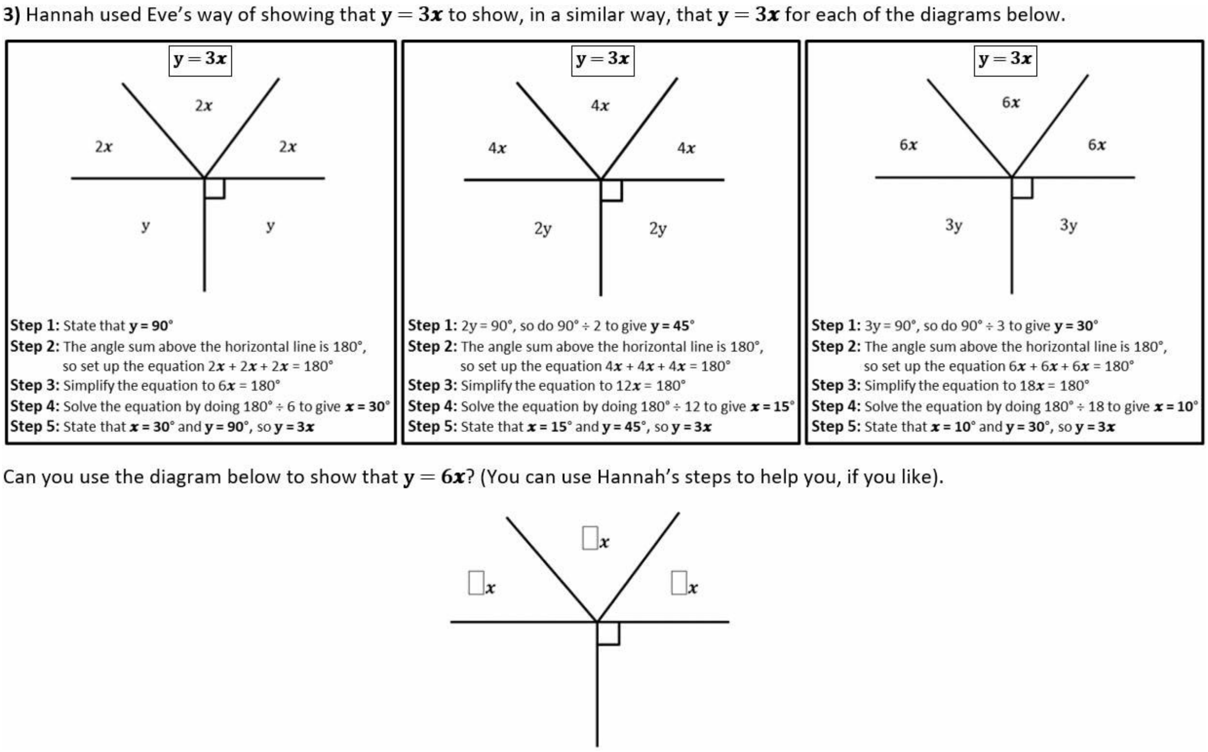
The increases in the sophistication of Y8's responses (from prompt 2b to 2d) were arguably less extensive than in their responses to the same prompts in the number domain. Yet, Y8's moderate advances in explanatory detail on prompt 2d on the geometry task provided some further evidence that encouraging learners to write a case based on a given sequence of similar arguments can aid the articulation and extent of their descriptions and explanations of such arguments. I elaborate on these advances in Y8 learners' actions in Chapter 5, Section 5.2.

4.2.7 Y8's actions on the geometry *transfer* variation task

In Section 3.7.3.6 I explained how I designed the geometry transfer task (prompt 3) to be conceptually similar to the initial variation task in the same domain. I also explained that my aim with this conceptual similarity was to aid my identification of transferred actions and their roles (which relates to RQ3). I discuss learners' actions in terms of transfer in Sections 4.3 and 4.4, respectively. In this section, I first address RQ2 by reporting on the *individual* actions that learners performed in response to the geometry transfer task.

As with the initial variation task, the transfer task featured a sequence of three geometric diagrams and accompanying five-step arguments. However, this time, the arguments did not result in coefficients of x that were consecutive numbers. Instead, the sequence of argument variations all resulted in x coefficients of 3, i.e., $y = 3x$ (see, Figure 4.10). The transfer task also featured a template of the given geometric diagrams, and which I asked each learner to use in their response. With the transfer task, I asked the learners to use the given argument variations and the diagram template to show that $y = 6x$.

Figure 4.10: The geometry transfer variation task.



As presented in Table 4.14, I identified in Y8 learners' responses a total of 12 sub-actions comprising all six main actions except Organising. All of the Y8 learners performed Generalising and Deconstructing actions, and all except for Belle performed Extending actions. Table 4.14 also indicates if a learner, prior to writing their own case, reasoned using the given diagrams and/or the accompanying arguments, as well as if they were successful or unsuccessful in completing the task.

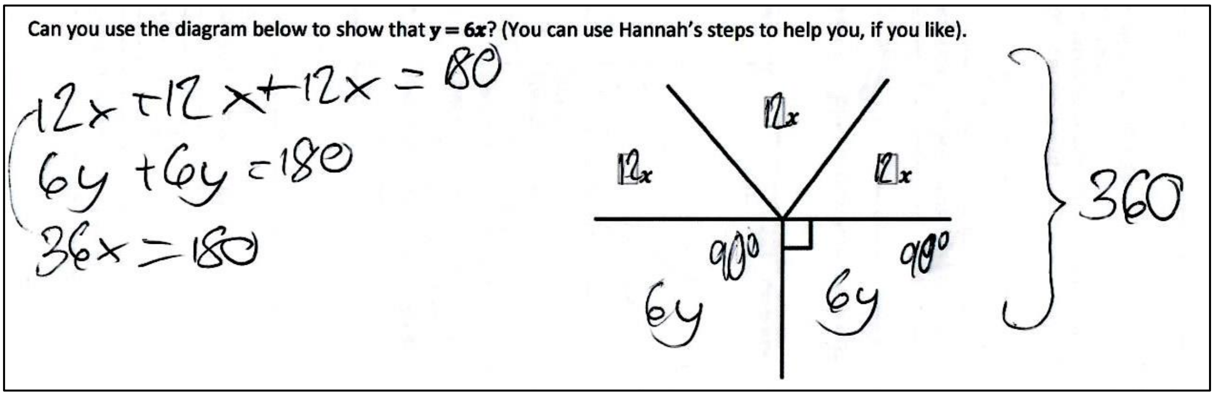
Table 4.14: Y8's sub-actions on the geometry transfer variation task.

	Sub-actions performed (frequencies are shown in parentheses)	Diagrams/Argument/ Both	Successful/ Unsuccessful
Ash	G3 (1), D1 (1), E1 (1)	Diagram	Unsuccessful
Belle	G2 (1), L2 (1), D1 (1)	Diagram	Unsuccessful
Carl	G2 (1), G3 (1), D1 (1), E3 (1), N1 (1), N2 (1)	Diagram	Unsuccessful

Note. Actions are presented in the same order as they appear in the GOLDEN framework. I discuss the chronologies of learners' actions in Chapter 5.

Each learner's first action was a form of Generalising. Belle and Carl first performed a G2 (two-process) generalisation by reasoning about a consistent relationship between the x and y coefficients in each given diagram. Belle described this consistent relationship as the x coefficient being 'double' the y coefficient, and Carl explained the same relationship using the concepts of proportion and ratio, "*for every one y there are two x's, one to two*". Ash made a process-result generalisation (G3) but generalised only from the third given example by assumptively and fallaciously reasoning that "*for [the given result of] y = 3x they [the given respective x and y coefficients] are 6, 6, 6 and 3, 3, so for [a result of] y = 6x they [the respective x and y coefficients] will be 12, 12, 12 and 6, 6*". It was possible that Ash based his reasoning on doubling all of the coefficients in the third example because its given result was $y = 3x$, and the task required a result of $y = 6x$, i.e., a coefficient of x that was double the coefficient of x in the given result. Ash then used his conjectured coefficient values in his own case (see Figure 4.11) and subsequently saw that his reasoning was fallacious.

Figure 4.11: Ash's response to the geometry transfer variation task.



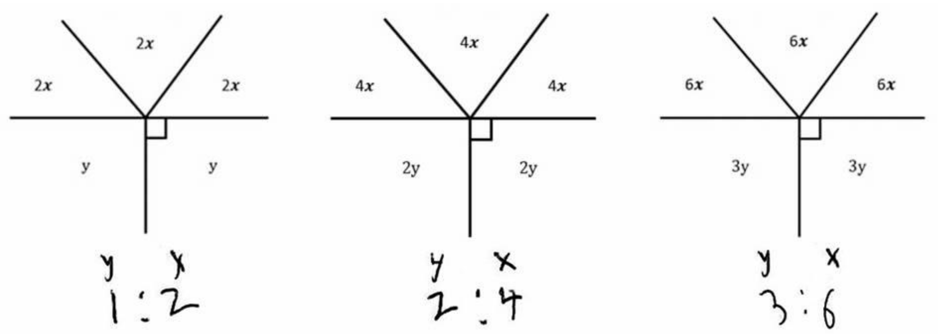
Note. Ash followed his inscriptions shown in Figure 4.11 with use of a calculator to work out that $x = 5$ and $y = 15$, i.e., that $y = 3x$ rather than the required result of $y = 6x$.

While Generalising (then Extending) from a single example can be a useful way to provoke mathematical thinking (see, e.g., Bills et al., 2004, p. 18), variation tasks that feature a sequence of structurally similar examples can eliminate the need for such potentially fallacious conjectures. Indeed, an important affordance of variation tasks that feature structurally similar examples is the opportunity for learners to reason on the basis of generalities that they identify *across* the given examples. Yet, unsuccessful outcomes that are the result of reasoning from a single example can have a steering, or at least a focus-shifting, effect because they can potentially influence a learner's choice of alternative test values and avenues of reasoning.

Carl also extended an assumed generality identified by attending only to the third given example. One explanation for these two learners generalising from a single example is that the transfer task, in comparison to the initial task, was (deliberately) somewhat harder to generalise from; specifically, in the initial task there was a pattern in the results of the given arguments, whereas in the transfer task, all of the given arguments resulted in the same outcome of $y = 3x$. Ash and Carl's conjectures highlighted how learners who appeared to search for but did not identify a general pattern sometimes moved to Generalising from a single example; and as suggested by Ash's fallacious generality, perhaps without appreciating that such conjectures might lead to an unsuccessful end result. I elaborate on the learners' Generalising actions in Chapter 5, Section 5.1.2.

Belle performed the only Localising action by testing, using all of the given steps, an x coefficient of 6 (L2). While Belle used all of the given steps to write her test argument, she sometimes shortened part of the given structure in a step. As presented in Table 4.14, all three Y8 learners at some point in the writing of their own case shortened part of a given step. I describe such shortening of structure as a D1 Deconstructing action. In regard to Networking actions, Carl referred to but did not use the concept of fractions (N1), yet found useful the application of ratio (summarised in Figure 4.12) to describe the relationship of the x and y coefficients in each respective given example (N2).

Figure 4.12: Carl's use of ratio to describe the relationship of the x and y coefficients.



Although all of the Y8 learners responded unsuccessfully to the transfer task they each performed valuable actions that are central to developing the ability to identify and produce a proof. For example, searching for, identifying, and reasoning about relationships (including generalities) in a progressive way that involved attending to a growing amount of structure. In Section 5.2, I discuss in more detail the value of such progressive actions.

4.2.8 Y10's actions on the geometry *initial* variation task

Across all of the Year 10's responses to prompt 2a, I identified two Generalising actions and eight Organising actions. The two Generalising actions comprised a G2 action (reasoning about a generality involving two-processes) and a G3 action (reasoning about a process-result generality). Emma performed the G2 action by explaining that with the three angles on the right-hand side of the vertical line segment in each consecutive diagram, the sum of $3x$ and $3y$ must be higher than in the preceding diagram because the remaining numerical angle decreases in each diagram. Emma's G2 Generalising action immediately followed her indication of what she saw as a dimension of difference across the given examples (i.e., the decreasing numerical angle) and was further evidence for my third finding.

Finding 3

When learners identify a dimension of difference across given argument variations, it can prompt them to justify and communicate the reason for their identified difference.

I later elaborate on learners' spontaneous verbal reasoning as a result of foregrounding difference (in Chapter 5, Section 5.1.3). Dylan's G2 (process-result) Generalising action was an explanation of how in the given diagrams the right-hand numerical angles decreased as the coefficients in the results increased. Of the eight Organising actions, five involved prioritising a dimension of sameness (O1) and three involved prioritising a dimension of difference (O2). Of the five O1 Organising actions (foregrounding sameness), Dylan performed one and Freya performed four. Dylan's O1 action referred to the fact that on all of the given angle diagrams the coefficient for all of the variables was 3. All of Freya's O1 actions were focused on consistencies in the arguments rather than on the diagrams. For example, "*Step one is the same*", "*step two is the same*", and "*step three always starts with $3y$*". The only sub-action performed by all three Y10 learners on prompt 2a on the geometry initial variation task was O2 (foregrounding a difference). Dylan and Emma identified the right-hand numerical angles in the given diagrams (72°, 66°, and 60°, respectively) as a difference, and Freya indicated a difference in the coefficients in the three given results.

In response to prompt 2b which, as I mentioned earlier, I later repeated with prompt 2d, the Y8 learners collectively performed three main actions (Generalising, Organising, and Extending). As presented in Table 4.15, learners performed a total of nine actions involving two sub-actions in each of the three main actions. Specifically, all three learners reasoned about the consistent decrease in the numerical angle on the right-hand side of each given diagram (72°, 66° and 60°, see Figure 4.8), which was a single-process generalisation (G1). Only Dylan reasoned about a process-result generalisation (G3), which he did by explaining that, "*every time the [x] coefficient [in the respective given results of $y = 2x$, $y = 3x$ and $y = 4x$] goes up, the [numerical] angle on the right side [of each respective diagram] is going down by minus six*". Freya performed the only two Organising actions, these involved separately foregrounding a dimension of sameness

(O1) and a dimension of difference (O2). Freya explained that, to achieve a result of $y = 6x$, she would focus on "*the angle that is different [i.e., the only given angle that varied across the given examples]" because "all the other [respective given] angles were the same*". The Organisation of structural information inherent in Freya's comment captures succinctly an example of how controlling (in)variance when designing a task can orient a learner's attention.

All of the learners' Extending actions immediately followed their Generalising actions, and did so at the same level of sophistication. Specifically, E1 extending actions followed G1 Generalising actions, and E3 Extending actions followed G3 Generalising actions. I elaborate on this common sequence of actions in Chapter 5, Section 5.1.6.

Table 4.15: Y10's sub-actions on prompt 2b of the geometry task.

	Sub-actions performed (frequencies are shown in parentheses)
Dylan	G1 (1), G3 (1), E3 (1)
Emma	G1 (1), E1 (1)
Freya	G1 (1), O1 (1), O2 (1), E1 (1)

Note. Actions are presented in the order that they appear in the GOLDEN framework which. In Chapter 5, I discuss in more detail the chronologies of learners' actions.

Prompt 2c involved writing an argument that was similar to the given arguments, i.e., one for which $y = 6x$. As presented in Table 4.16, all three Y10 learners performed the same three actions in the same order. Specifically, these actions first involved systematically foregrounding sameness (O1) then, separately, difference (O2) across the given arguments. Each learner followed these two actions by Extending a single process (E1) pattern that they identified on the preceding prompt (2b). All of the Y10 learners appeared to perform their Organising actions for two reasons: (1) to maintain fidelity with the written (notational) format of the given steps, and (2) to identify where the given argument structures varied, i.e., where to substitute their decreasing pattern extension value of 48° that they identified on the preceding prompt. Dylan exemplified the latter reason by explaining that, "*I kept looking back down the steps [in the given examples] for what numbers are the same and what numbers are different, and if they are different, you add [substitute] the numbers for the new one [argument] in the place of the different ones*". By substituting, in the writing of their own case, a value based on a previously identified pattern, each learner was extending a pattern in given arguments (E1). Because the generality that learners extended was identified on the previous prompt (2b), I did not consider learners to be actively Generalising on prompt 2c. Rather, I saw the generality on which learners based their (prompt 2c) Extending actions to be, at this point, established, i.e., implicit, in their thinking.

Table 4.16: Y10's sub-actions on prompt 2c of the geometry task.

	Sub-actions performed (frequencies are shown in parentheses)	Successful/ Unsuccessful
Dylan	O1 (1), O2 (1), E1 (1)	Successful
Emma	O1 (1), O2 (1), E1 (1)	Successful
Freya	O1 (1), O2 (1), E1 (1)	Successful

Note. Frequencies are shown in parentheses. Actions are presented in the same order as they appear in the GOLDEN framework which, in Table 4.16, are also the same order in which they were chronologically performed. In Chapter 5, I discuss in more detail the chronologies of learners' actions.

With Prompt 2d I again asked learners to describe what a diagram similar to the given diagrams would look like to show that y = 6x. As I explained earlier in this section, prompt 2d was a repetition of prompt 2b and enabled me to assess if learners' interim act of writing their own case would refine their descriptions of an argument for $y = 6x$. Although learners' responses to prompt 2d were conceptually similar to their responses to prompt 2b (see, Table 4.17), they were somewhat more efficient and succinct, and so seemed more confident. For example, Freya's response to prompt 2b involved searching for and identifying dimensions of (in)variance, whereas on prompt 2d she focused only on explaining the underlying reasons for the dimensions of invariance that she identified.

Table 4.17: Y10's sub-actions before and after the case-generation prompt on the geometry task.

	Sub-actions performed on 2b	Prompt 2c involved writing a case for a proposition similar to the given propositions.	Sub-actions performed on 2d
Dylan	G1 (1), G3 (1), E3 (1)		G1 (1), G3 (1), E3 (1)
Emma	G1 (1), E1 (1)		G1 (1), E1 (1)
Freya	G1 (1), O1 (1), O2 (1), E1 (1)		G1 (1), O1 (1), O2 (1)

Note. Frequencies are shown in parentheses. Actions are presented in the same order as they appear in the GOLDEN framework. In Chapter 5, I discuss in more detail the chronologies of learners' actions.

4.2.9 Y10's actions on the geometry transfer variation task

As presented in Table 4.18, I identified in Y10 learners' responses to the geometry transfer variation task a total of 15 sub-actions. As with Y8's responses to this same task, Y10's responses comprised all six main actions except Organising. The only main action that was performed by all of the Y10 learners was Localising. One explanation for this common action is that generalities were not easy to identify, so learners tended to act on local structure by (re)adjusting and (re)testing values based on conjectures or on the outcome of a prior test, rather than by extending an identified pattern. Dylan and Freya made generalisations that involved two processes (G2) but only Dylan – the only learner who responded successfully - made the 'critical' generalisation by comparing (on each respective given argument) the sum of the y coefficients to the sum of the x coefficients and relating this to the given result of $y = 3x$ (G3). To identify this critical pattern, Dylan moved from a G2 (two-process) generalisation to a G3 (process-result generalisation), which was a shift that exemplified my fourth finding.

Finding 4

Learners' sequences of actions involved working with increasingly complex structure and progressively identifying relationships.

In Chapter 5 (Section 5.2), I discuss in more detail the implications of learners progressively attending to more complex structure. Deconstructing actions only occurred when learners shortened part of a given step in the writing of their own case (D1). Two modes of Extending sub-actions (E2 and E3) were performed, these were always extensions of preceding and corresponding modes of Generalising, i.e., E2 actions always followed G2 actions, and E3 actions always followed G3 actions. Emma performed the only Networking action by speculating and describing as wrong any result comprising a x coefficient that is a recurring decimal number.

I also indicated in Table 4.18 if a learner, prior to writing their own case, reasoned using the given diagrams and/or the accompanying arguments, as well as if they were successful or unsuccessful in completing the task.

Table 4.18: Y10's sub-actions on the geometry transfer variation task.

	Sub-actions performed (frequencies are shown in parentheses)	Diagrams/Arguments/Both	Successful/ Unsuccessful
Dylan	G2 (1), G3 (2), L3 (2), E3 (2), D1 (1)	Diagrams	Successful
Emma	L2 (1), L3 (1), D1 (1), N3 (1)	Diagrams	Unsuccessful
Freya	G2 (1), L2 (1), E2 (1)	Both	Unsuccessful

Note. Actions are presented in the same order as they appear in the GOLDEN framework. I discuss the chronologies of learners' actions in Chapter 5.

4.2.10 Comparison of Y8 and Y10's actions on the geometry variation tasks

Here, as with my earlier comparison of both year groups' actions in the number domain (Section 4.2.5), I first compare both year groups' actions on the geometry initial variation task. I follow this by comparing the year groups' actions on the transfer variation task in the same domain. Later, in Section 4.3.6, and as part of my response to RQ3, I compare both year groups' actions across the pair of geometry variation tasks.

Initial variation task

As presented in Table 4.19, across all four combined prompts (2a-2d) on the initial variation task, Y8s collectively performed a total of 31 actions (which was one fewer than on this same task in the number domain); Y10s collectively performed at total of 36 actions (which was four fewer than on this same task in the number domain). The most commonly performed main (superordinate) action performed across Y8 was Generalising, whereas the most common main action performed across Y10 was Organising. Y8 performed 11 Generalising actions of which five were G1 (single process) generalisations and six were G3 (process-result) generalisations. Y10 performed ten Organising actions of which five involved foregrounding a dimension of sameness (O1) and five involved foregrounding a dimension of difference (O2). Although not shown in the Table 4.19, In both year groups respectively, the most common first main action across all of the prompts in the initial variation task was Organising. The most common sub-action(s) performed by each year group on each respective prompt in the initial variation task are presented in Table 4.19. Across all four combined prompts on the initial variation task, Y8s collectively performed a total of nine sub-group 12 actions; whereas, collectively, Y10s performed a total of 5 of these same actions.

Table 4.19: Comparison of Y8 and Y10's actions on the geometry initial variation task.

Geometry initial variation task		
	Y8	Y10
Number of sub-actions	31	36
Most common main action	G	O
Most common sub-action(s)	G3/E3	O1
Prompt 2a: most common sub-action(s)	O2	O1
Prompt 2b: most common sub-action	G1/G3/E3	G1
Prompt 2c: most common sub-action(s)	E1	O1/O2/E1
Prompt 2d: most common sub-action(s)	G1/G3/E3	G1
Number of sub-group 3 actions	12	5
Most common consecutive main actions	G(1), G(3), E	O(1), O(2)/O(2), O(1)
Unperformed main actions	L and N	L, D and N

Note. G (Generalising); O (Organising); L (Localising); D (Deconstructing); E (Extending); N (Networking).

In Y8, across all four respective prompts in the initial task, the most common sequence of consecutive main actions were Generalising actions followed by Extending actions (i.e., G1, G3 and E3 actions, consecutively); whereas, across Y10, this same sequence comprised two Organising actions (i.e., sequences of O(1), O(2) and O(2), O(1) actions were both performed four times, respectively). Also presented in Table 4.19 are any main actions that each respective year group did not perform on the initial task. Both year groups did not perform any Localising or Networking actions. In addition, none of the Y10 learners performed any Deconstructing actions. However, as I stated earlier, each separate year group collectively performed all six main actions at some point on each domain-pair of variation tasks.

<u>Transfer variation task</u>

On the geometry transfer task (prompt 3), and as presented in Table 4.20, Y8s collectively performed a total of 12 actions and Y10s collectively performed at total of 15 actions. Across Y8, the most commonly performed main action was Generalising which were approximately one third of all their sub-actions; whereas, Y10's most common main action was Localising, which were one of all their sub-actions. Localising actions can lead to Generalising (Mason et al., 2010, p. 9), which might suggest that Y8's most common action was more sophisticated than Y10's. However, most of the Y8's based their generalisations on (single example) fallacious conjectures. The only successful response was Dylan's in Y10, which he achieved via repeated Localising acts of (re)adjusting and (re)testing given values. Of Y8's Generalising actions, the joint most common sub-actions were two-process generalisations (G2) and process-result generalisations (G3); whereas, with Y10's most common main action (Localising), the most common sub-action was L3 (re-adjusting a previously adjusted value and testing the re-adjustment). Although not shown in Table 4.20, in each respective year group, a Generalising action was the most common first action on the transfer task; this resonates with my fifth finding.

Finding 5
Generalising was the most common main action.

I will elaborate on the importance of this finding in Chapter 5, Section 5.1.2. Also presented in Table 4.20 is each year group's most common sub-action on each prompt in the transfer task.

On the transfer variation task, Y8s collectively performed four sub-group 3 actions; whereas, collectively, Y10s performed eight sub-group 3 actions. Y8's sub-group 3 actions were approximately one third of their collective actions, whereas Y10's sub-group 3 actions accounted for approximately half of all their collective actions.

Table 4.20: Comparison of Y8 and Y10's actions on the geometry transfer variation task.

Geometry transfer variation task		
	Y8	Y10
Number of sub-actions	12	15
Most common main action	G	L
Most common sub-action(s)	D1	L3
Number of sub-group 3 actions	4	8
Most common consecutive main actions	G, E, D	(G2, G3)/(G3, G3)/(G, E)/(L2, L3)/ (L3, L3)/(E, L)
Unperformed main actions	O	O

Note. G (Generalising); O (Organising); L (Localising); D (Deconstructing).

Across Y8, the most common sequence of consecutive main actions was Generalising, Localising and Deconstructing; whereas in Y10, six different combinations of sequences were most commonly performed (presented in Table 4.20). The only main action not performed by either year group on the geometry transfer variation task was Organising. I consider the implications of learners' actions, and sequences of actions, in more detail in the next chapter (Discussion).

4.3 Third research question: Results

RQ3: *What characterises Y8 and Y10 learners' attention to structure across a pair of (initial and transfer) proof variation tasks in the respective domains of number and geometry?*

In my earlier characterisation of learners' actions, I analysed the initial and transfer variation tasks in isolation from each other (RQ2). I next characterise learners' actions *across* the initial and transfer tasks in each respective domain (RQ3). As I described in Section 3.7.3, in each respective domain, the transfer variation task (prompt 3) was conceptually similar to the initial variation task. My aim with the conceptual similarity was to aid my identification of: (1) transferred actions, i.e., actions that a learner similarly performed on the initial and the transfer variation tasks within each respective domain, and (2) the role and affordances of transferred actions. As I explained in Section 2.2.3, the transfer tasks were important because they provided opportunities to identify how any consistent actions performed across a domain-pair of variation tasks aided (or hindered) a learner's understanding of, and progress through, a task. Therefore, in the current section, my analysis will include the characterisation of learners' transferred actions.

4.3.1 Y8's actions across the number initial and transfer variation tasks

As illustrated in Table 4.21, Y8 learners collectively performed less than half as many actions on the transfer task than they did on the initial task (33 versus 14, respectively). Generalising was the most common main action on the initial task and the joint most common main action on the transfer task (along with Localising). Yet, collectively, learners performed only one third as many Generalising actions on the transfer task than they did on the initial task (5 and 15, respectively). As I described in Section 4.2.2, in the number domain, the transfer task featured only one algebraic argument (and two specific numerical arguments), whereas with the initial task, all three arguments were algebraic. Hence, it was arguably harder to identify generalities across the given arguments on the transfer task than it was on the initial task.

While Y8 learners collectively performed an equal number of Generalising and Localising actions on the transfer task (five per action), the frequencies of these respective actions on the initial task (not fully shown in the Table 4.21) were 14 and 2. It was likely that the proportional differences in Generalising and Localising actions across the domain-pair variation tasks was due to the increased cognitive demand of the transfer task. Specifically, when learners seemed unable to identify a generality, they often shifted to performing a sequence of Localising actions. Such shifts to Localising contributed to L2 actions (testing a value using all given steps) being the most common sub-action on the transfer task; whereas on the initial task, the joint most common sub-actions where G1 (single process) and G3 (process-result) generalisations.

Learners' struggle on the transfer task was further implied by the comparative number of sub-group 3 actions (10 and 2 on the initial task and transfer task, respectively). Also compared in Table 4.21 are the most common consecutive main actions performed on each of the variation tasks, as well as any unperformed main actions.

Table 4.21: Comparison of Y8's actions the number initial and transfer variation tasks.

Number task (Y8)		
	Initial VT	Transfer VT
Number of sub-actions	33	14
Most common main action	G (14)	G (5)/L (5)
Most common sub-action(s)	G1 (6)/G3 (6)	L2 (4)
Number of sub-group 3 actions	10	2
Most common consecutive main actions	G, E (2)/G, N (2)	G, G (2)/L, D (2)
Unperformed main actions	-	O

Note. VT (variation task); G (Generalising); O (Organising); L (Localising); D (Deconstructing); E (Extending); N (Networking). Values in parentheses are frequencies.

I next turn to the actions that learners similarly performed on the initial task and the transfer task, i.e. transferred actions. I discuss the roles and affordances of learners' transferred actions in more detail in Chapter 5, Section 5.3.

Table 4.22 illustrates the sub-actions that each Y8 learner transferred. Only the learners who transferred sub-actions were successful in completing the transfer task (this was also the case in Y10 on the geometry transfer task). As illustrated in Table 4.22, Y8 learners collectively transferred sub-actions that involved all of the main actions except Organising. In total, 9 sub-actions were transferred, yet nearly all of these involved sub-group 1 and sub-group 2 actions. One possible explanation for the lack of sub-group 3 actions is that these actions were arguably the most sophisticated, and it was the learners first formal encounter with arguments and proofs represented algebraically. The most common transferred main/sub-action was Generalising a single process (G1).

Table 4.22: Y8's transferred actions across the number variation tasks.

Number task: Year 8		
Main action	Sub-action	Learner
Generalising	G1: single process generalisation	Ash (1), Carl (2)
Generalising	G3: process-result generalisation	Carl (1)
Localising	L2: adjusting and testing a value – using all steps	Carl (1)
Deconstructing	D1: reducing the structure of an individual step	Ash (1), Carl (1)
Extending	E1: extending a pattern involving a single process	Ash (1)
Networking	N2: applying a discrete curriculum concept	Carl (1)

Note. Values in parentheses are frequencies.

Slightly more than half of Ash's actions on the transfer task were transferred, as was the case with Carl. For example, Ash and Carl both performed a G1 (single process) generalisation on the initial variation task as well as on the transfer variation task. Specifically, Ash's first action both on the initial task and the transfer task was to generalise, across the given arguments, about how the addend in step 3 consistently increased. Carl, on the initial variation task, made the same G1 single process (addend) generalisation as Ash; but, on the transfer task, performed a G1 action that involved keeping constant a consistent addend value in step 2. Belle did not transfer any actions and was the only Y8 learner who did not respond successfully to the transfer task. I discuss further examples of learners' transferred actions in Section 5.3.

4.3.2 Y10's actions across the number initial and transfer variation tasks

As illustrated in Table 4.23, Y10 learners collectively performed less than one third as many actions on the transfer task than they did on the initial task (12 and 41, respectively). Generalising was the most common main action on the initial task, whereas on the transfer task, Localising was the most common main action (21 and 7, respectively). One explanation for this shift in the most common main action (a shift that was also present with the sub-actions) is that generalities were perhaps harder to identify on the transfer task due to the mix of numeric and algebraic structures (see Figure 4.5); and, as with Y8, when Y10 learners seemed unable to identify or confirm a conjectured generality, they often shifted to a sequence of Localising actions. (I elaborate on these shifts when I discuss Localising in Section 5.1.4 of the next chapter). With the Y10 learners who were not successful on the transfer task (Dylan and Emma), such shifts to Localising involved (re)adjusting given values (L3) in a way that moved them progressively closer to a successful end result; this led to my sixth finding.

Finding 6

Transferred actions aided learners' progress through a task, regardless of if a required end result was achieved.

I elaborate on this finding in Chapter 5, Section 5.3. I also compared in Table 4.23 the most common consecutive main actions performed on each of the variation tasks, as well as the unperformed main actions.

Table 4.23: Y10's actions across the number initial and transfer variation tasks.

Number task (Y10)		
	Initial VT	Transfer VT
Number of sub-actions	41	12
Most common main action	G (21)	L (7)
Most common sub-action	G2 (9)	L3 (4)
Number of sub-group 3 actions	14	6
Most common consecutive main actions	G, D (4)	D, D (3)
Unperformed main actions	L	E

Note. VT (variation task); G (Generalising); L (Localising); D (Deconstructing); E (Extending). Values in parentheses are frequencies.

Only two Y10 learners transferred actions (illustrated in Table 4.24). Specifically, Emma transferred one G3 (process-result generalisation) action, and Dylan transferred one O1 (foregrounding sameness) action. While Emma and Dylan did not achieve a successful end result on the transfer task, their transferred actions nonetheless took them close to doing so. Dylan's transferred Organising action helped him to make initial sense of the task; this led him to make a local adjustment to a value given in one of the numerical argument examples. Dylan's plan was to test his local adjustment on the numerical argument and, if successful, transfer his adjusted value to the corresponding position in the given algebraic proof. While Dylan chose a value that would have led to a successful outcome, a subsequent mental miscalculation prevented this from happening. Emma's transferred process-result Generalising action (as I described in Section 4.2.4) involved looking across the given arguments and reasoning that "*if for an answer of 5 it [the addend in step 3] is 9 then for an answer of 7 it [the addend in step 3] will be 11*". Emma then tested an addend of 11 (see Figure 4.7) and saw that the result was not 7. However, as the interview concluded, she remarked that, based on the result of testing an addend of 11, she would have next tested an addend of 13; which, had she conducted the test, would have produced a successful outcome.

Table 4.24: Y10's transferred actions across the number variation tasks.

Number task: Year 10		
Main action	Sub-action	Learner
Generalising	G3: process-result generalisation	Emma (1)
Organising	O1: foregrounding sameness	Dylan (1)

Note. Values in parentheses are frequencies.

4.3.3 Comparison of Y8 and Y10's actions across the number variation tasks

The data in Table 4.25 compares Y8's actions across the number initial and transfer variation tasks to Y10's actions across these same tasks. Across both variation tasks, the total number of actions for Y8 and Y10 was 47 and 53, respectively. On the initial task, the most common main action in each respective year group was Generalising. On the transfer task, across Y8, the joint most common main-actions were Generalising and Localising, whereas on the same task across Y10, the most common action was Localising. The joint most common Y8 sub-actions on the initial task were G1 (single process) generalisations and G3 (process-result) generalisations; whereas, Y10's action for this same measure on the same task was G2 (two-process) generalisations. On the transfer task, in both respective year groups, the most common sub-action was a form of Localising. Specifically, in Y8 this sub-action was L2 (adjusting a value in a given argument and testing the adjustment by using all of the given steps), while in Y10 it was an L3 sub-action (re-adjusting a previously adjusted value and testing the re-adjustment). Across both variation tasks, Y10 collectively performed nearly twice as many sub-group 3 actions then Y8 (12 and 20, respectively). Table 4.25 also illustrates the most common consecutive main actions as well as any unperformed main actions for each year group and for each respective variation task.

Table 4.25: Comparison of Y8 and Y10's actions across the number variation tasks.

	Number task			
	Y8	Y8	Y10	Y10
	IVT	TVT	IVT	TVT
Number of sub-actions	33	14	41	12
Most common main action	G (14)	G (5)/L (5)	G (21)	L (7)
Most common sub-action(s)	G1 (6)/G3 (6)	L2 (4)	G2 (9)	L3 (4)
Number of sub-group 3 actions	10	2	14	6
Most common consecutive main actions	G, E (2)/G, N (2)	G, G (2)/L, D (2)	G, D (4)	L, L (3)
Unperformed main actions	-	O	L	E

Note. IVT (initial variation task); TVT (transfer variation task). G (Generalising); O (Organising); L (Localising); D (Deconstructing); E (Extending); N (Networking).

On the number transfer task, the first sub-action performed by all of the learners in both year groups that exhibited transfer was a transferred action. This suggested that some of the actions performed on the initial task were not only useful enough to be worthy of transfer but were also given special importance, i.e. priority. I discuss the implications of learners' transferred actions in more detail in Chapter 5, Section 5.3.

None of the Y8 learners, except Carl, performed a Localising action on the initial task. Yet, on the transfer task, not only did each Y8 learner perform a L2 action (adjusting/testing a given value) but this was the most common sub-action for this year group. This increase in Localising actions across the initial and transfer variation tasks was even more evident in Y10 because, on the initial task, none of the year group performed any Localising actions; yet, on the transfer task they all performed L2 as well as L3 (re-adjusting and re-testing) Localising actions. Further, in Y10, L3 was the most commonly performed sub-action on the transfer task. The cross-task increase in Localising might be explained by comparative differences in the designs of the initial and the transfer tasks. Specifically, in the initial task, all of the given arguments were general (algebraic), whereas in the transfer task, only one of the given arguments was algebraic. Consequently, it was perhaps harder to identify generalities, e.g., co-varying relationships, on the transfer task; which may have 'forced' learners to instead perform local actions that can lead to finding generalities (see, e.g., Mason et al., 2010, p. 9). Across both year groups and all of the variation tasks, nearly all of the most common consecutive main actions began with a form of Generalising. The only exception involved Y10's consecutive Localising actions on the transfer task, which mostly involved adjusting and testing (then re-adjusting and re-testing) a given value in an effort to adapt a given argument so that it would generate a result of 7 for any initially chosen number. As I described earlier, two Y8 learners and one Y10 learner produced a successful procedure on the transfer task (illustrated in Table 4.4, Section 4.2.2 and Table 4.8, Section 4.2.4, respectively). In the next chapter, I elaborate on the implications of learners' actions.

4.3.4 Y8's actions across the geometry initial and transfer variation tasks

As illustrated in Table 4.26, on the geometry task, Y8 learners collectively performed approximately one third as many actions on the transfer task than they did on the initial task (31 and 12, respectively). In step with finding 5 (Section 4.2.10), Generalising was the most common main action on the initial task as well as on the transfer task. Yet, Y8s collectively performed almost one third as many Generalising actions on the transfer task than they did on the initial task (4 and 11, respectively). This dip in Y8's Generalising actions, which reflected their similar dip on the number task, may have been due to generalities on the transfer task being harder to identify than on the initial task. As with the initial task in both domains, identifying a process-result generality (G3) was perhaps critical to generating an argument for a similar proposition.

However, on the geometry transfer task, none of the Y8 learners identified a robust (non-fallacious) process-result generality, and none of them achieved a successful end result. I elaborate on the affordances of process-result generalities in Section 5.1.2.

Y8 learners performed one third as many, arguably more sophisticated, sub-group 3 actions on the transfer task than they did on the initial task (4 and 12, respectively), which further reflected their struggle on the transfer task. Also compared in Table 4.26 are the most common consecutive main actions performed on each of the variation tasks, as well as any unperformed main actions.

Table 4.26: Comparison of Y8's actions the geometry initial and transfer variation tasks.

Geometry task (Y8)		
	Initial VT	Transfer VT
Number of sub-actions	31	12
Most common main action	G (11)	G (4)
Most common sub-action(s)	G3 (6)/E3 (6)	D1 (3)
Number of sub-group 3 actions	12	4
Most common consecutive main actions	G, G, E (5)	G, E, D (2)
Unperformed main actions	L, D, N	O

Note. VT (variation task); G (Generalising); O (Organising); L (Localising); D (Deconstructing); E (Extending); N (Networking). Values in parentheses are frequencies.

Although not shown in Table 4.26, across both geometry variation tasks, all of the Y8 learners followed their G3 actions with corresponding E3 (process-result) Extending actions. Yet, none of the Y8s followed any of their G1 (single process) or G2 (two process) generalisations with any Extending actions. Because learners' E3 Extending actions were often a key aspect of generating a further case, the prevalence of consecutive G3 and E3 actions suggested further evidence for the usefulness of identifying or even conjecturing about a process-result (G3) generality.

Table 4.27 illustrates the sub-actions that each Y8 learner transferred. Only Generalising, Deconstructing, and Extending actions were transferred. Of the learners that evidenced transfer (Ash and Carl), the only common transferred actions were process-result generalisation (G3) and process-result extension (E3). While neither of these two learners successfully completed the transfer task, their commonly transferred sequence of process-result Generalising and Extending actions suggested that these actions were perceived as useful, even when conjecturing a generality from a single example. Incidentally, I chose to keep distinct Generalising and Extending actions because, across all of the data, learners did not always follow their G1, G2, and G3 actions by Extending.

Table 4.27: Y8's transferred actions across the geometry variation tasks.

Geometry task (Year 8)		
Main action	Sub-action	Learner
Generalising	G3: process-result generalisation	Ash (1), Carl (1)
Deconstructing	D1: reducing the structure of a given step	Ash (1)
Extending	E3: extending a process-result pattern	Ash (1), Carl (1)

Note. Values in parentheses are frequencies.

All three of Ash's actions on the transfer task were transferred, whereas of Carl's actions on the transfer task, less than half were transferred. Belle did not transfer any actions.

4.3.5 Y10's actions across the geometry initial and transfer variation tasks

As illustrated in Table 4.28, Y10 learners collectively performed slightly less than half as many actions on the transfer task than they did on the initial task (41 and 12, respectively). Organising was the most common main action on the initial task, whereas on the transfer task, Localising was the most common main action (18 and 5, respectively). An obvious explanation for this difference in the most common main actions is that on prompt 2a of the initial task learners were directly asked to discern (in)variance, i.e., Organise. However, the Y10 learners also performed several Organising actions when, on a later prompt on the initial task, they were asked to write their own case. Specifically, when asked to write a case similar to the given arguments (prompt 2c), all of the Y10 learners began with two Organising actions, i.e., by first foregrounding a dimension of sameness (O1) then a dimension of difference (O2). These two consistent Organising actions implied that learners found discerning (in)variance useful for making sense of, and initiating a response to, prompt 2c. This implication aligns with researchers' claims that discerning sameness and difference in variation can aid initial sense-making (e.g., Kullberg et al., 2017; Watson & Mason, 2006). Although the Y10 learners were, at this point, yet to work on the transfer task, their repeated use of Organising actions on prompts 2a and 2c of the initial task can be seen as a form of 'within task' transfer. Indeed, Freya's first two actions on each of the four prompts in the geometry initial variation task were forms of Organising. However, as I explained in Section 3.7.2, I was interested in actions that learners transferred from an initial task to a (conceptually similar) transfer task; hence, it was these actions that I counted as transferred in my study.

As I mentioned earlier, when Y10 learners moved from the initial task to the transfer task, their most common main action changed from Organising to Localising. Learners who were unsuccessful on the transfer task Localised by (re)adjusting argument values by conjecturing based on a process-only Generalisation or, in the case of Emma, on no prior generalisation at all. Dylan was the only learner to achieve success on the transfer task, and was the only learner that reasoned in terms of process-result generalities; this further supports the potential value of such generalisations. From the initial task to the transfer, Y10 learners' sub-group 3 actions increased from 5 to 8 and was the only increase of this kind by either year group in either domain. This increase was, in part, due learners' most common sub-group 3 action on the transfer task: re-adjusting tested values (L3), and reflected their struggles in the absence of robust generalisations. Also compared in Table 4.28 are the most common consecutive main actions performed on each of the variation tasks, as well as unperformed main actions.

Table 4.28: Y10's actions across the geometry initial and transfer variation tasks.

Geometry task (Y10)		
	Initial VT	Transfer VT
Number of sub-actions	36	15
Most common main action	O (18)	L (5)
Most common sub-action	O1 (10)	L3 (3)
Number of sub-group 3 actions	5	8
Most common consecutive main actions	O, O (9)	G, G (2)/E, L (2)/G, E (2)/ L, L, D (2)
Unperformed main actions	L, D, N	O

Note. VT (variation task); G (Generalising); O (Organising); L (Localising); D (Deconstructing); E (Extending); N (Networking). Values in parentheses are frequencies.

Only Dylan transferred any sub-actions (illustrated in Table 4.29) and, as I explained earlier, Dylan was the only learner to achieve success on the transfer task, and was the only learner on this same task that reasoned in terms of process-result generalisation. Indeed, process-result generalisations were part of Dylan's transferred actions, which he followed with associated E3 Extending actions that he also transferred. Dylan's most useful transferred process-result generalisation focused on how, in each respective given diagram, dividing the sum of the x

coefficients by the sum the y coefficients always equalled the x coefficient in the result; this relationship can be seen in Figure 4.10 and was critical to writing a similar case, i.e., successfully responding to the transfer task.

Table 4.29: Y10's transferred actions across the geometry variation tasks.

Geometry task (Y10)		
Main action	Sub-action	Learner
Generalising	G3: process-result generalisation	Dylan (2)
Extending	E3: extending a process-result pattern	Dylan (2)

Note. Values in parentheses are frequencies.

4.3.6 Comparison of Y8 and Y10's actions across the geometry variation tasks

The data in Table 4.30 compares Y8's actions across the geometry initial and transfer variation tasks to Y10's actions across these same tasks. Across both variation tasks, the total number of actions for Y8 and Y10 was 42 and 51, respectively. On the initial task, across Y8, the most common main action was Generalising, whereas on the same task across Y10, the most common action was Organising. On the transfer task, the most common main actions were Generalising and Localising, across Y8 and Y10, respectively. The joint most common Y8 sub-actions on the initial task were G3 (process-result) generalisations and E3 (process-result) extensions; whereas, Y10's action for this same measure on the same task was foregrounding a dimension of sameness (O1).

On the transfer task, the most common Y8 sub-action was D1 (shortening given steps), and across Y10 was L3 (testing a re-adjusted value). The number of sub-group 3 actions performed across both variation tasks, was 16 and 13, for Y8 and Y10 respectively. Table 4.30 also illustrates the most common consecutive main actions as well as any unperformed main actions for each year group and for each respective variation task. In the next chapter, I discuss the implications of learners' actions.

Table 4.30: Comparison of Y8 and Y10's actions across the geometry variation tasks.

	Geometry task			
	Y8	Y8	Y10	Y10
	IVT	TVT	IVT	TVT
Number of sub-actions	31	12	36	15
Most common main action	G (11)	G (4)	O (18)	L (5)
Most common sub-action(s)	G3 (6)/E3 (6)	D1 (3)	O1 (10)	L3 (3)
Number of sub-group 3 actions	12	4	5	8
Most common consecutive main actions	G, G, E (5)	G, E, D (2)	O, O (9)	G, G (2)/E, L (2) /G, E (2)/ L, L, D (2)
Unperformed main actions	L, D, N	O	L, D, N	O

Note. IVT (initial variation task); TVT (transfer variation task). G (Generalising); O (Organising); L (Localising); D (Deconstructing); E (Extending); N (Networking).

In Section 4.3.3, I explained that, on the number transfer task, the first sub-action performed by all of the learners who exhibited transfer was a transferred action. Yet, this trend was not evident on the geometry task. One explanation for this cross-domain disparity is that initial and transfer tasks were perhaps more conceptually diverse on the geometry task than they were on the number task. Such differences in conceptual diversity can be aligned with the comparative demands of near and far (Clements & Sarama, 2009, p. 62) or horizontal and vertical (Rebello et al., 2007, p. 228) transfer. Nonetheless, the actions that learners transferred after their first action on the geometry transfer task always facilitated their progress. For example, by transferring a D1 Deconstructing action, Ash, in Y8, was able to efficiently express (i.e., isolate), in the writing of his own case, his adaptations to the given arguments (despite being unsuccessful on the transfer task); and, Dylan, in Y10, transferred an E3 Extending action, the outcome of which prompted a shift to Localising which, in turn, moved him progressively closer to successful completion of the transfer task. These two examples provided further evidence for finding 6 (Section 4.3.2), i.e., that transferred actions aided learners' progress through a task, regardless of if a required end result was achieved.

4.4 Fourth research question: Results

RQ4: *How do Y8 and Y10 learners' attention to structure compare across both domains?*

I began this chapter by discussing actions that comprised my framework for describing learners' attention to structure on proof variation tasks (RQ1). Then, in each respective domain, I characterised learners' actions by analysing the initial and transfer variation tasks in isolation from each other (RQ2). I followed this by characterising learners' actions *across* the initial and transfer tasks in each respective domain (RQ3). My next, and final, level of analysis characterises learners' actions across the number and geometry domains (RQ4).

4.4.1 Cross-domain comparison of Y8's actions on the initial and transfer variation tasks

As illustrated in Table 4.31, across each respective domain-pair of variation tasks, Y8 learners performed more than double the number of actions on the initial task than they did on the transfer task. It is likely that this cross-domain consistency was due the initial tasks comprising four prompts and the transfer tasks comprising one prompt. Nonetheless, all of the prompts on the variation tasks in each domain were isomorphic, which afforded cross-domain comparisons of the number of actions that learners performed on the initial and transfer tasks; this, in turn, was suggestive of comparative (in)efficiencies between different learners' trajectories to successful end results. (In Section 4.4.3, I elaborate on the implications of the number of actions that a learner performed on a prompt). An alternative comparison of the frequencies of learners' actions involved their responses to the initial and transfer 'case-generation' prompts (prompt 2c on each initial task, and each prompt 3 which were the transfer tasks). The case-generation prompts were the prompts in each initial and transfer task which required learners to write a new case that was conceptually similar to three given arguments. There were no cross-domain consistencies in the total frequencies of actions on the case-generation prompts. Yet, across the domains, all three Y8 learners performed more actions on a transfer case-generation prompt than on the initial case-generation prompt in the same domain. All of these learners did so, in part, by Generalising more on the transfer case-generation prompt than they did on the initial case-generation prompt. Specifically, by performing more G2 and/or G3 actions. I elaborate on the implications of these sub-actions in Section 5.1.2.

Indeed, across Y8, in each respective domain, the most common (or joint most common) main action was a form of Generalising. This finding was somewhat expected because I deliberately designed the arguments in each given sequence of variations so that they featured several inherent structural similarities for learners to identify, reason about, and use. Further, the centrality of generalising in the learning and doing of mathematics is particularly well documented (e.g., Ellis et al., 2024; Mason et al., 2010; Sriraman, 2004). Indeed, as illustrated in Table 4.31, all of the most common, or joint most common, sequences of consecutive main actions on both variation tasks in both domains started with a form of Generalising.

On the initial variation task in each domain, Y8 learners' joint most common sub-actions featured G3 (process-result) generalisations; which, in the case of the number initial task, seemed to give learners enough conviction to write their own (successful) argument that was conceptually similar to the given arguments. Perhaps this experience of success motivated learners to make process-result conjectures in situations where they did not identify a robust process-result generality, i.e., on the subsequent geometry transfer task.

Table 4.31: Cross-domain comparison of Y8's actions on the initial and transfer tasks.

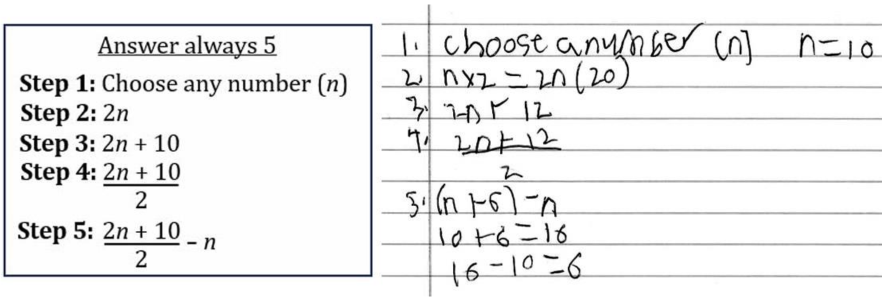
Year 8				
	Number		Geometry	
	IVT	TVT	IVT	TVT
Number of sub-actions	33	14	31	12
Most common main action	G (14)	G (5)/L (5)	G (11)	G (4)
Most common sub-action(s)	G1 (6)/G3 (6)	L2 (4)	G3 (6)/E3 (6)	D1 (3)
Number of sub-group 3 actions	10	2	12	4
Most common consecutive main actions	G, E/G, N	G, G/L, D	G, G, E	G, E, D
Unperformed main actions	-	O	L, D, N	O

Note. IVT (initial variation task); TVT (transfer variation task). G (Generalising); O (Organising); L (Localising); D (Deconstructing); E (Extending); N (Networking).

Table 4.32 illustrates a cross-domain comparison of Y8's transferred actions. Across both domains, two of the three Y8 learners (Ash and Carl) exhibited transfer. The main (superordinate) actions commonly transferred by both learners in both domains were Generalising and Extending. These two commonly transferred actions align with the notion that, for learners, Extending an identified generality was the essence of generating a further case, i.e., their own argument for a similar proposition. The sub-actions that both learners commonly transferred across domains were process-result generalisations (G3) and reducing the structure of a given step (D1). As I described earlier, on the number task, learners' own cases were nearly always an extension of a process-result generality that they previously identified across the given arguments; and, where such generalities were not found on the geometry transfer task, learners often conjectured about them.

In the writing of their own cases, Ash and Carl both performed (D1) Deconstructing actions. On step 2 of the number initial variation task, Carl performed a superficial decomposing action by exposing the multiplication sign in the given structure ‘$2n$’. Carl’s ‘spelling out’ of $2n$ as “$n \times 2$” (see Figure 4.13) was perhaps suggestive of his seemingly tentative confidence on the task. As shown on step 5 in Figure 4.13, before Carl substituted his chosen n value of 10, he reduced the number of objects given in step 5 by writing a simplified version, i.e., $(n + 6) - n$. While Carl’s simplification had a decomposing quality (it exposed 6 as a factor of 12), it also deconstructed a given structure by removing or ‘cancelling out’ some of its constituent objects. As I explained in Section 2.3, this ‘dismantling’ view of deconstructing is different to decomposing, and in the context of writing arguments similar to given arguments, can have implications for the transparency and ‘explanatory power’ (Stylianides et al., 2016) of a structure. I did not include decomposing structure in the Deconstructing component of the GOLDEN framework because such actions were not performed in both domains. However, when adapting the main components in the framework to other mathematical situations, decomposing actions such as partitioning and exposing structural factors might be relevant Deconstructing sub-actions.

Figure 4.13: Carl's response to the initial case-generation prompt (2c) on the number task.



While Carl’s version of step 5 sacrificed some of the structural fidelity of the given argument, it seemed to aid his in-the-moment focus on his objects of attention, while potentially easing his *cognitive load* (Paas & Van Merriënboer, 1994). In other words, writing again for step 5 the fraction in step 4 might have interrupted his thinking process. I elaborate on the implications of reducing the given structures in Section 5.1.5. Carl’s case in Figure 4.13 also showed that he conflated the writing of a general (algebraic) argument with the empirical testing of a specific value. Carl shifted in and out of these two modalities. Specifically, in steps 1 and 2 he substituted for n a value of 10, in steps 3 and 4 he represented algebraically (did not substitute), and in step 5 he represented algebraically as well as substituted. These shifts appeared to capture the moment that Carl transitioned, perhaps psychologically, from merely testing specific values to writing a general argument. Indeed, the next instance of a given structure that Carl generated (on the transfer task) was a fully algebraic general argument. This observation provided evidence that consecutive-number proof variation tasks of the kind that I used in my study can support this important transition.

Ash transferred three different sub-actions in each respective domain, whereas Carl transferred four different sub-actions on the number task and two different actions on the geometry task. Belle was the only Y8 learner who did not transfer any actions in any domain and was the only Y8 learner who did not achieve a successful result on both transfer tasks. I discuss learners' cross-domain transfer in further detail in Section 5.4.

Table 4.32: Cross-domain comparison of Y8's transferred actions.

	Year 8	
	Number/ algebra	Geometry
Transferred action	Learner	Learner
Generalising (G1)	Ash (1), Carl(2)	-
Generalising (G3)	Carl (1)	Ash (1), Carl (1)
Localising (L2)	Carl (1)	-
Deconstructing (D1)	Ash (1), Carl (1)	Ash (1)
Extending (E1)	Ash (1)	-
Extending (E3)	-	Ash (1), Carl (1)
Networking (N2)	Carl (1)	-

Note. G1 (single process generalisation); G3 (process-result generalisation); L2: (adjusting/testing a value, using all steps); D1 (reducing the structure of a given step); E1 (extending a single process pattern); E3 (extending a process-result pattern). Values in parentheses are frequencies.

4.4.2 Cross-domain comparison of Y10's actions on the initial and transfer variation tasks

As illustrated in Table 4.33, across each respective domain-pair of variation tasks, Y10s collectively performed more than double the number of actions on the initial task than they did on the transfer task. A comparison of the number of actions performed on the case-generation prompts showed that, in each respective domain, most of the Y10 learners performed more actions on the transfer case-generation prompt than they did on the initial task. This perhaps reflected the struggle (and resilience) that these learners experienced when writing their own case on each transfer task compared to doing so on each initial task. For example, if a learner did not generalise (or find a robust generality) they often shifted to consecutive Localising actions; and, when writing their own cases, all of the Y10 learners localised on both transfer tasks, whereas none of them did so on either of the, arguably easier, initial tasks.

Across the Y10 learners, and on across all of the variation tasks in both domains combined, the most common main action was a form of Generalising. On each separate initial task in each respective domain, Y10's most common main action was Generalising on the number task, and Organising on the geometry task. Generalising was valuable because it involved seeing and using underlying consistencies in given processes e.g., addends, and in process-result relationships, what Zazkis and Liljedahl (2002) called 'units of patterns'. Organising actions reflected how, in response to the arguably greater challenge presented by the geometry case-generation prompt on the initial task, all of the Y10 learners began with two different sense-making actions; i.e., by first foregrounding a dimension of sameness, followed by a dimension of difference. On the transfer task in both domains, and still in relation to the Y10 learners combined, the most common main action was Localising. This aligned with my previous observation that Y10's Localising actions reflected their struggles to find robust process-result generalities on both transfer tasks. As illustrated in Table 4.33, all of Y10's most common main actions on each initial and transfer task were reflected in their most commonly performed sub-actions on these same tasks.

On the geometry task, Y10 learners collectively performed more sub-group 3 actions on the transfer task than on the initial task; which was the only occurrence across the whole data set where the number of sub-group 3 actions on a transfer task exceeded those performed on the initial task in the same domain. Although not shown in Table 4.33, this difference in sub-group 3 actions on the geometry initial and transfer tasks was starkest when comparing these actions on the case-generation prompts. Specifically, none of the Y10 learners performed any sub-group 3 actions on the initial case-generation prompt (prompt 2c); whereas, on the transfer case-generation prompt, Emma performed non-consecutive Localising and Networking sub-group 3 actions, and Dylan performed a consecutive sequence of pairs of Generalising, Extending, and Localising sub-group 3 actions. As I stated earlier, Dylan was the only learner to achieve a successful result on the geometry transfer task.

Table 4.33: Cross-domain comparison of Y10's actions on the initial and transfer tasks.

	Year 10			
	Number		Geometry	
	IVT	TVT	IVT	TVT
Number of sub-actions	41	12	36	15
Most common main action	G (21)	L (7)	O (18)	L (5)
Most common sub-action(s)	G2 (9)	L3 (4)	O1 (10)	L3 (3)
Number of sub-group 3 actions	14	6	5	8
Most common consecutive main actions	G, D	D, D	O, O	G, G/E, L/G, E/ L, L, D
Unperformed main actions	L	E	L, D, N	O

Note. IVT (initial variation task); TVT (transfer variation task). G (Generalising); O (Organising); L (Localising); D (Deconstructing); E (Extending); N (Networking).

Presented in Table 4.34 is a cross-domain comparison of Y10's transferred actions. Across both domains, two of the three Y10 learners (Dylan and Emma) evidenced transfer. Across these learners, three main actions were transferred: Generalising, Organising, and Extending. The only main action commonly transferred by both learners in both domains was Generalising; which, for both learners, took the form of a G3 process-result sub-action. The transferred Organising action was performed by Dylan as his first action on the number transfer task. Performing an Organising action as a first action fit a consistent pattern across the whole data set. Specifically, all of the Organising actions on all of the case-generation prompts were performed as a first action or as first and second actions.

Dylan was the only Y10 learner to transfer an Extending action in any domain. Dylan's Extending took the form of an E3 process-result pattern extension which was the only E3 action performed by any Y10 on any transfer task. As indicated in Table 4.34, Dylan transferred two E3 actions on the geometry transfer task; while none of these extensions produced the desired result of y = 6x, they served to shift Dylan's attention a succession of Localising actions that eventually led to him being the only learner to respond successfully to the geometry transfer task. I elaborate on the implications of each action in Section 5.1.

Table 4.34: Cross-domain comparison of Y10's transferred actions.

	Number/ algebra	Geometry
Transferred action	Learner	Learner
Generalising (G3)	Emma (1)	Dylan (2)
Organising (O1)	Dylan (1)	-
Extending (E3)	-	Dylan (2)

Note. G3 (process-result generalisation); O1 (foregrounding sameness); E3: (extending a process-result pattern). Values in parentheses are frequencies.

4.4.3 Cross-domain comparison of Y8 and Y10's actions on the variation tasks

As I described in Sections 4.2.1 and 4.4.1, the number of actions that a learner performed on a prompt often suggested comparative (in)efficiencies between different learners' reasoning and trajectories to a successful end result. For example, on the number initial variation task, all of the Y8 and Y10 learners successfully wrote their own case, but Carl used more than double the number of actions than all of the other respective learners. While performing a comparatively high number of actions on a prompt was suggestive of a learner's inefficient attention to structure, it can also be seen as indicative of their resilience in searching for useful connections. Which, in turn, can contribute to achieving a successful end result. For example, on the geometry transfer task, Dylan performed at least twice as many actions as all of the other respective learners in both year groups, and was the only learner to successfully respond to this task.

<u>Initial variation tasks</u>

On the initial task in each respective domain, Y8's most common main action was Generalising; whereas, in Y10, this same measure on the same tasks was Generalising on the number initial task, and Organising on the transfer task. A cross-domain comparison of responses to the initial case-generation prompts (prompt 2c on each task) showed that nearly all of the Y8 learners who successfully wrote their own cases did so in less than four actions (of which a common main action was Extending). I expected Generalising to also be a common main action performed by the successful learners, but any 'lack' of Generalising was perhaps due to Extending an established generality that they reasoned about on a previous prompt (which seemed to be the case with Belle on the geometry task); or, due to tacit 'Generalising', for example, by previously foregrounding a dimension of sameness (which seemed to be the case with Ash on the geometry task). The notion of a learner's prior experience subtly affecting, or underlying, their engagement on a subsequent task was captured by Marton and Trigwell's (2000, p. 385) inclusion of unconscious influence as a form of transfer.

In Y10, the same cross-domain comparison of responses to the initial case-generation prompts showed that nearly all of the learners who successfully wrote their own cases did so in three actions (of which a common main action was again Extending). Where the forementioned successful Y10 learners did not generalise on an initial case-generation prompt, they consistently foregrounded first a dimension of sameness, then a dimension of difference. This suggests that, in some cases, merely discerning (in)variance rather than reasoning about the underlying structure of an identified pattern, i.e., Generalising, may be sufficient basis for an Extending action.

<u>Transfer variation tasks</u>

Across the transfer tasks in both domains, the most common main action was Generalising (in Y8) and Localising (in Y10). A possible explanation for this difference is that, across both domains, all of the Y8 learners started almost all of the transfer tasks by performing various G1, G2, and G3 generalisations, and broadly focused on conjecture-based, or experimental, G3 (process-result) generalisations. By comparison, Y10's broadly appeared to give more immediate focus to searching for a robust G3 (process-result) generalisation; either by performing a series of Localising actions, or by making local adaptations when their process-result generalisations resulted in an unsuccessful outcome.

Table 4.35 is a cross-domain comparison of Y8 and Y10's transferred actions. As illustrated in the table, Y8 and Y10 transferred 14 and 6 actions, respectively. Belle (Y8) and Freya (Y10) did not transfer any actions. While there was no overall pattern in the learners who transferred actions and the learners who achieved success on the transfer tasks, as I previously described, two salient transfer-related results emerged: (1) Belle was the only Y8 learner who did not transfer any actions on the number transfer task, and was the only Y8 learner that did not achieve success on this task; and (2) Dylan was the only Y10 learner who transferred any actions on the geometry transfer task, and was the only Y10 learner that achieved success on this task.

Collectively, Y8 transferred nearly double the number of actions on the number transfer task than they did on the geometry task; in Y10, this result was largely reversed. Across each respective year group, Generalising was the most common, or joint most common, transferred main action in each separate domain. Relatedly, across both domains, and across each year group separately, the most common, or joint most common transferred sub-actions were process-result (G3) generalisations. In the next chapter (Section 5.3), I discuss in more detail the implications of learners' transferred actions.

Table 4.35: Cross-domain comparison of Y8 and Y10's transferred actions.

	Year 8		Year 10	
	Number/ algebra	Geometry/ algebra	Number/ algebra	Geometry/ algebra
Transferred action	Learner		Learner	
Generalising (G1)	A (1), C (2)	-	-	-
Generalising (G3)	C (1)	A (1), C (1)	E (1)	D (2)
Organising (O1)	-	-	D (1)	-
Localising (L2)	C (1)	-	-	-
Deconstructing (D1)	A (1), C (1)	A (1)	-	-
Extending (E1)	A (1)	-	-	-
Extending (E3)	-	A (1), C (1)	-	D (2)
Networking (N2)	C (1)	-	-	-

Note. G1 (single process generalisation); G3 (process-result generalisation); L2: (adjusting/testing a value, using all steps); D1 (reducing the structure of a given step); E1 (extending a single process pattern); E3 (extending a process-result pattern); N2 (applying a discrete curriculum concept). A (Ash); C (Carl); D (Dylan); E (Emma). Values in parentheses are frequencies.

When taken together, my analysis of the whole set of response data revealed that all of the learners were immediately able to engage with all of my proof variation tasks, despite having no prior experience of proof argumentation. Across each respective year group, and on given argument variations in two different domains, learners performed a consistent range of actions that I captured in the GOLDEN framework. Some actions, e.g., process-result generalisations and transferred actions, appeared to have more salient implications for learners' task trajectories than other actions. Yet, all of the actions that learners performed on the variation tasks (regardless of if a successful end result was achieved) were activities that can underlie more formal proof production; this aggregated outcome led to my seventh and final finding.

Finding 7

Proof variation tasks can be used to encourage learners to perform actions that can support their ability to prove.

In Chapter 5, I discuss in more detail my responses to each of my research questions. In doing so, I elaborate on each of my findings. In Chapter 6, I discuss the implications that my findings have for advancing pedagogy as well as their contributions to the field of research.

Chapter 5: Discussion

In this chapter I will restate the problems that I set out to investigate, revisit my overarching aim, and discuss my responses to my research questions (Sections 5.1 to 5.4). In doing so, I elaborate on my findings and their implications for advancing learners' progression on the variation tasks in my study. I will also discuss the implications of any links between learners' responses to the introductory (familiarisation) tasks and their subsequent actions on the variation tasks (Section 5.5). In comparison to the Results chapter, my discussion in this chapter is broadly more interpretative and further draws on the literature and instances of learners' actions. I will also give further consideration to the roles and chronologies of learners' actions.

In Section 1.1, I cited four problematic restrictions on the teaching and learning of proof: (1) teachers' often limited knowledge of how to develop learners' proving abilities; (2) delaying the teaching of proof; (3) ill-prepared learners; and, (4) teaching proof in isolation from other mathematical activities. I then described how the four restrictions can lead to two further restrictions on proof pedagogy: insufficient scaffolding and indistinct characterisation of learners' proving processes. (In Section 6.7, and in light of my findings, I will explain how use of proof variation tasks and the GOLDEN framework can contribute to alleviating each of these restrictions). I have also described a problem that underlies and contributes to all of the foresaid restrictions: the sheer diversity in the nature of proof tasks which makes identifying cross-task structural and algorithmic generalities difficult, if not impossible. To alleviate this underlying problem, I identified a set of cross-task generalities in the form of learners' actions. Specifically, I formulated (and applied) the GOLDEN framework for describing actions that learners similarly performed on variation tasks in each of two respective domains: number and geometry. Indeed, formulation and application of an attention-to-structure framework was the main aim of my study. I will next discuss my responses to each of my research questions and elaborate on the corresponding findings.

5.1 Discussion in relation to RQ1

RQ1: *What might comprise a framework for describing learners' attention to structure?*

I responded to my first research question by formulating the GOLDEN framework. The framework was the main contribution of my study and the lens through which I articulated my responses to all of my subsequent research questions (RQ2 to RQ4). As I described in Section 3.9, the framework comprised six main actions that each consisted of three sub-actions (see Appendix E). The framework, which constituted finding 1, comprised actions that learners (across each year group separately) similarly performed on the number task and the geometry task, respectively. I derived all of the actions using a combination of the literature, open coding, and thematic analysis.

I next describe each action in the GOLDEN framework and give an overview, with data examples, of how it aided analysis of learners' progress on the variation tasks. I elaborate on the roles and affordances of the actions in Sections 5.2, 5.3 and 5.4, which respectively align with discussions of my responses to RQ2, RQ3 and RQ4.

5.1.2 Generalising actions

Learners performed three nuanced Generalising actions: reasoning about one process that was common to a set of given arguments (G1); reasoning by simultaneously attending to two processes that were common to a set of given arguments (G2); and, reasoning about a relationship between a process and a result that was common to a set of given arguments.

Across the entire data set, Generalising was the most common main action (finding 5) and accounted for slightly more than one third of all learners' actions across both tasks combined. In each separate year group, Generalising was also the most common, or joint most common, main action on the number task and the geometry task, respectively. Given that generalising is commonly considered a core and critical component of mathematics learning (Ellis, 2007; Mason et al., 2010; Zazkis & Liljedahl, 2002), the prevalence of this kind of action was perhaps unsurprising. Indeed, the tasks that I gave to learners deliberately featured structural variations, a key affordance of which was the opportunity to identify generalities (Kullberg et al., 2017). Similarly, the prompts that I designed were likely to influence the use of generalisations.

In Section 4.2.9, I reported finding 4: *learners' sequences of actions involved working with increasingly complex structure and progressively identifying relationships*. In step with finding 4, learners' process-only generalisations (G1 and G2) often led them to next make a further generalisation involving a further connection, i.e., a process-result relationship (G3). For example, Dylan started the geometry transfer task with a two-process generalisation (G2) after which he immediately noticed and reasoned about a process-result generality (G3). This sequence of generalisations led Dylan to compare on the given diagrams the sum of the x coefficients (above each horizontal line) to the sum of the y coefficients (below each horizontal line) which was critical to seeing the truth of the proposition. In other words, at the point of making his process-result generalisation, Dylan had attended to enough structural connections to see why $y = 3x$ for each given variation. Relatedly, Dylan's comparison of the sums of the respective x and y

coefficients might have enabled him to 'see the general through the particular' (Watson & Mason, 2005, p. 129); which, in the context of developing learners' proving abilities, is akin to, or can at least aid, seeing a case as an instance of a proof.

On prompt 2c of the number initial variation task, i.e., the initial case-generation prompt, it was possible for learners to write their own case (that resulted in 7) merely by continuing the process-only pattern in the given, correspondingly positioned, addends. Yet, on prompt 2c, process-result (G3) generalities often seemed pivotal in convincing learners that they had made enough consistent structural connections across the given arguments to write a similar argument, i.e., case, of their own. I made this inference because, on this task, when learners made a process-result generalisation, they commonly next extended the generality to the required new case. It was likely that the forementioned consecutive moves from process-result generalisation to case-generation were only popular on the number initial case-generation prompt because, on the same prompt on the geometry task, the addend values (in step 3, i.e., the varying angle) were not proportional to the values in the results (i.e., the resultant propositions). Similarly, on both transfer case-generation prompts, the results of the given arguments did not co-vary with a process. Hence, case-generation tasks that feature argument variations with a co-varying process-result generality might be particularly supportive introductory resources for learners yet to encounter formal proof argumentation.

The idea that process-result generalities were particularly supportive was further suggested by learners' responses to the transfer tasks. For example, on the number transfer task, Freya commented, "*I haven't spotted a connection because they [the given argument variations] all end [result] in [y =] 3x anyway*"; and, on the geometry transfer task, where Ash and Carl did not find a definitive process-result generality, they nonetheless conjectured about such generalities from a single given variation. In addition to the co-variation, learners appeared to see process-result generalities as particularly convincing (or worth conjecturing about) because they comprised a (possible) pattern that involved a result, and so had a self-checking quality that they could use to validate or refute their own cases. As I mentioned in Chapter 4, such process-result generalisations contribute a third form of generalisation to Harel and Soto's (2017) identification of process pattern generalisations (PPG) and result pattern generalisations (RPG).

Across the entire data set, not only was Generalising the most common main action, it was also the most common first main action performed on a prompt. These results suggested that variation tasks which draw attention to process and process-result generalities are particularly apt for introducing learners to proving; not least because generalising is a major activity in reasoning-and-proving exercises (Stylianides et al., 2013) and appears critical for students to develop their conception of proof (Harel & Fuller, 2009, p. 360).

5.1.3 Organising actions

Organising actions were so-called because they gave order to the multiple information competing for a learner's attention when they were presented with a set of argument structures with varying values. Three forms of Organising were exhibited: foregrounding a dimension of sameness (while simultaneously backgrounding difference) (O1); foregrounding a dimension of difference (while simultaneously backgrounding sameness) (O2); and, simultaneously prioritising dimensions of sameness and difference (O3). In other words, Organising actions involved discerning (in)variance, which is a key affordance of variation tasks (Kullberg et al., 2017). As I described in Section 4.1, for me, prioritising sameness (O1) is different to Generalising because the former is solely an act of identifying a noticed dimension, whereas Generalising involves an accompanying element of underlying explanatory reasoning. With the initial set of given argument variations in each domain, I first asked learners what was the same and what was different about the iterations. My rationale for first asking this question was that discerning (in)variance has not only been considered a starting point for making sense of data (Watson & Mason, 2006) but a necessary condition of learning (Marton, 2015; Marton & Booth, 1997). Although I suspected that learners would immediately start to discern (in)variance across the given argument variations without being so prompted, I specifically asked them to do so (with each prompt 2a) because I wanted them to directly reveal how they were Organising the information in the given arguments.

Across both domain-pairs of variation tasks, Organising actions were the most common main action after Generalising. Learners' Organising actions supported the notion that discerning (in)variance is a prelude to sense-making (Watson & Mason, 2006) because, across both domains, and across all prompts subsequent to each prompt 2a, learners performed 43 Organising actions of which 42 were first actions or first and second actions on a prompt. In addition, when learners did not find any computational relationships in the given arguments, they often shifted to 'lower order' reasoning about number parities; and, when reasoning about number parities proved unhelpful, these learners shifted to 'merely' foregrounding a dimension of sameness or difference. Such shifts, what Pirie and Kieren (1994) termed 'folding back to less sophisticated activity levels', further suggested that discerning (in)variance can be a starting point for sense-making.

Foregrounding sameness (O1) and foregrounding difference (O2) and combinations of these, were most commonly followed by (near-simultaneous) Generalising actions. (To restate from earlier, I discerned foregrounding sameness from Generalising because I took the former to be an act of indicating what is noticed, and the latter to require accompanying reasoning). The different combinations of Organising actions that preceded learners' Generalising actions made it difficult to identify if foregrounding sameness (O1) or foregrounding difference (O2) was more helpful to learners. My own view, in regard to mathematics pedagogy in general, is that foregrounding sameness is broadly more helpful than foregrounding difference because the former is more proximal to generalising than the latter; and generalising is at the heart of learning mathematics (Mason et al., 2010). In the task response data, there were clear instances of how foregrounding sameness (O1) and foregrounding difference (O2) aided learners in the writing of their own cases. For example, on the geometry transfer task, Freya confirmed that she looked for '*what was the same*' in the steps in the given arguments to guide how she wrote her own, conceptually similar, case. In regard to using an identified difference, the following interview excerpt reveals how, on the geometry initial task, Dylan leveraged the varying numbers in the given arguments to write a similar case of his own:

Interviewer:	What was most helpful about the given [argument] examples when you were doing your version [writing your own case] for $y = 6x$?
Dylan:	I kept looking back at the other three [given arguments] because there's that many numbers you can just blur it all into one when you are writing down the steps.
Interviewer:	When you said that you kept looking back at the three [given arguments], what aspect helped?
Dylan:	Which numbers are the same and what numbers are different in the three of them [given arguments]; then, if they [the numbers] are different, you add [position] the number for the new one [argument] in [the corresponding] place of [as] the different ones [numbers].
Interviewer:	Did you look at one [of the three given argument variations] in particular or did you look across them?
Dylan:	I looked at a number in [given argument] A, then looked at B, and if the same [correspondingly positioned] number was different, then I looked at C just to make sure that I was doing it right. Then replaced it [the varying number] with my number [48 degrees].

As I explained in Section 4.2.8, a further aspect of foregrounding difference led to my third finding: when learners identify a dimension of difference across given argument variations, it can encourage them to justify and communicate the reason for their identified difference. For example, when I asked Ash what was the same and different about the three given arguments on the number initial variation task (prompt 2a), he indicated that the respective addends were "*different*"; and, without being directly prompted to do so, justified this difference by explaining that "*steps 2 and 3 are adding 2 each time*". Dylan communicated a similar unprompted justification for the same observation on the same task: "*the difference is you are adding 2 each* *time on top of what is plused [each addend]*". Further, as I reported in Section 4.2.8, Emma, on prompt 2a of the geometry initial variation task, spontaneously justified an identified difference in correspondingly positioned numerical angle values. Emma did this by explaining that: *"[the sum of] 3x and 3y is going to get bigger each time [on each consecutive given diagram] because the other [remaining] angle keeps getting less, and that's why the [value of] y gets bigger [in each consecutive argument]*". Such spontaneous explanations can be seen and seized by teachers as preparation for more formal proof justifications. For, convincing oneself and persuading others through the use of explanations are essential subprocesses of proving (Harel & Sowder, 2007).

5.1.4 Localising actions

In the context of proving tasks, acting on local structure (what I call Localising) has been associated with empirically testing the truth of a proposition (e.g., Küchemann, 2008, p. 9.1). The foresaid association resonated directly with learners' actions on the number *introductory* task which required learners to show that a given proposition must (always) be true. I say more about responses to the introductory tasks in Section 5.5.

In regard to the domain-pairs of variation tasks, learners only performed Localising actions on the case-generation prompts (prompts 2c and 3 in each domain). Each case-generation prompt required the learners to write a case for a proposition that was slightly different to the propositions that accompanied a given set of arguments. For example, when given arguments for propositions of $y = 2x$, $y = 3x$, and of $y = 4x$, learners were asked to write a case for $y = 6x$ (for all of the case-generation task designs, see prompt 2c and prompt 3 in Appendices A and B, respectively). Thus, on the case-generation prompts, all of the Localising actions were associated with writing a new case for a similar proposition. All of the learners that localised did so by adjusting/testing a value within the given arguments in an attempt to achieve an end result that matched the new proposition. None of the learners who successfully achieved such an end result tested the truth of their case's proposition by empirically testing any further (alternative) initial values. This was perhaps to be expected on the geometry case-generation prompt, which only required learners to 'show that' $y = 6x$. However, this lack of further testing of initial values was also true of responses to the case-generation prompt on the number transfer task, which required learners to write a case that resulted in 7 for any initially chosen number. Perhaps learners who were successful on this task did not empirically test their own cases because they felt, not unreasonably, that they had fulfilled the didactic contract. Alternatively, learners may have appreciated the general (algebraic) nature of the arguments that they generated. Yet, my study was the first time that the learners had encountered formal argumentation, and none of the learners transitioned to algebraic representation on the number introductory task without being prompted to do so. Hence, the learners may not have seen an 'intellectual need' (Harel, 2013) to test the truth domain of their case's proposition because they were not conscious that different propositions can have different 'epistemic values' (Duval, 2007), i.e., they can be always true, sometimes true, or never true.

I identified three forms of Localising: adjusting a given value and testing the adjustment by ignoring one or more of the given steps (L1); adjusting a given value and testing the adjustment using all of the given steps (L2); and, re-adjusting a previously adjusted value and testing the re-adjustment (L3). When learners ignored a given step (L1), they also ignored some of the algorithmic and notational structure exemplified in the given arguments, as well as some of the connective logic between the steps. Only Y8 learners performed L1 actions. One possible explanation for this is that these Y8 learners did not always appreciate the sense of procedural direction afforded by following and using in their own cases all of the given steps; whereas, the Y10 learners had two additional years of curriculum experience and perhaps had a greater general appreciation of using each step in an exemplified structure. Yet, as I discuss later in this section, all of the Y8 learners used all of the given steps when writing their own case on the number transfer task. Ignoring one or more of the given steps when testing an adjusted value may also have been a focusing strategy that eased the *cognitive load* (Paas & Van Merriënboer, 1994) borne by attending to and writing all of the information in all of the steps.

By writing an adjusted argument using all of the given steps (L2), a learner replicated, or stayed close to, the algorithmic and notational structure of the given arguments. All of the learners performed L2 actions when writing their own case on the number transfer task, and half of the learners did so on the geometry transfer task. As I explained earlier, the transfer case-generation prompts were arguably more intellectually challenging than the initial case-generation prompts. Hence, on the transfer tasks, adhering to the structural format in all of the given steps may have supported some learners by providing a close sense of procedural direction. Further, systematically following and using all of the given steps when testing a substituted value potentially stood to aid: (1) awareness of the continuity and logic between successive steps, (2) conceptual understanding of a whole argument, and (3) appreciation of an argument's convincingness. In regard to the latter point, convincing others of an argument's logic, validity, and irrefutability are considered key functions of proof (de Villiers, 1990).

In Sections 4.3.2 and 4.4.2, I reported that learners found it harder to identify robust generalities across the given arguments on the transfer tasks than they did on initial tasks; and, when learners did not find a generality (which learners typically used to guide the writing of their own case) they often shifted to consecutive Localising actions. It seemed likely that learners who made these shifts to Localising did so, perhaps not very consciously, to establish a generality. For, acting on local structure is known to take place in situations where awareness of any generalities is yet to surface (Venkat et al., 2019), and can lead to generalising (Mason et al., 2010). In the context of my study, such Localising typically involved L2 and L3 (readjustment and retesting) actions. Specifically, acting on one of the given arguments by (re)adjusting/(re)testing a given addend (on the number transfer task), and (re)adjusting/(re)testing a given coefficient (on the geometry transfer task). For example, on the number transfer task, Belle did not Generalise and, in an effort to achieve a result of *7* for any initially chosen number, shifted to adjusting and testing addend values in, respectively, step 2 and step 3 of one of the given arguments. On the geometry transfer task, in an effort to achieve a result of $y = 6x$, Emma, who also did not generalise, tested x and y coefficients of *6* and *1*, respectively. However, because this resulted in $y = 9x$, she then tested alternative x and y coefficients of *9* and *3*, respectively. In Section 5.2, I further discuss the implications of learners' Localising actions.

5.1.5 Deconstructing actions

Learners' Deconstructing actions took three forms: reducing (shortening or truncating) the structure of a given step (D1); reverse engineering, i.e., working backwards through a given step (or part of a given step) or from one given step to an earlier given step (D2); and, reasoning that references the effect or outcome of inversing, cancelling out, or 'doing and undoing' (D3). Shortening given structure within a step (D1) seemed to concentrate a learner's focus on their current object of attention. For example, as I explained in Section 4.4.2, on the number transfer task, Carl crossed out (temporarily removed) an algebraic term in a given step and focused only on the non-algebraic terms. Carl did this to "*make it [the writing of his own case] a bit simpler … for now*". Carl then re-introduced the crossed out algebraic term in the writing of his next step. Hence, Carl held in abeyance an algebraic structure, which was an example of what Mason (2018) called 'parking'.

In Section 4.2.3, I described how Emma, in the writing of her own case on the number initial task, only wrote the final step of $\frac{2n+14}{2} - n$. Yet, in doing so, systematically referred (attended) to each individual given step. This form of D1 structure reduction was an example of *compression* (Bills et al., 2006) because the structure of the final step comprised (nested) all of the structures in all of the prior steps. However, this efficient way of working ignored the overall stepped structure of the given arguments. The stepped structures were important because they illustrated the systematic procedural construction of the final step and were akin to the *connected sequence of assertions* (Stylianides, 2009, p.265) in more formal proofs. Using all of the steps in the given structures can potentially facilitate learners in verifying their own cases and explaining to and convincing themselves and others of their cases' validity and truth. By extension, communicating the convincingness of cases can develop learners' appreciation of the *explanatory function* (Stylianides et al., 2016) of proofs. Further, when a learner systematically uses all of the given steps in the writing of their own case, it can potentially help them to see and treat a case as 'a whole' structure. This, in turn, can help a learner to see an argument's structure as an instantiation of a general structure, which is an important 'mapping comparison' for learners of proof to make.

I termed the second kind of Deconstructing sub-action *reverse engineering* (D2). Acts of reverse engineering occurred when a learner started from an assembled structure and worked backwards to identify what and how prior objects and operations combined to give the assembled structure that they started from. For example, in the structure $2n + 10$, identifying the underlying structure or 'reason for' the addend by recourse to the prior structure $2n + 1 + 9$. Hence, reverse engineering resonates with *decomposing* structure (see, e.g., Askew, 2015). Seeing the structure that underlies an assembled structure aided learners' connection-making, which broadened their understanding of an argument. For example, on the number transfer task, Emma correctly reasoned that "*the [predicted] answer is odd so the number you add [the addend in step 3] has to be an odd number as well*". This was true because, working further backwards still, the addend to which Emma referred was consistently preceded by $2n + 1$. Hence, the addition of a further odd addend always resulted in an addend sum that was even (which was necessary to avoid a decimal end result). Reverse engineering (D2) actions can be seen as thinking backwards to a prior state, i.e., the antithesis of what Maciejewski and Star (2019) referred to as thinking ahead to a future state. Used together, thinking ahead and thinking "in reverse" can lay foundations for what Matsuda and VanLehn (2005) called forward/backward approaches to (more formal) proof construction.

The third Deconstructing sub-action (D3) involved reasoning about an inversing process identified in the given arguments. Reasoning about inversing involves two processes ('doing and undoing') and two structural states (the 'done' state and the 'undone' state). I discerned D3 actions from two-process generalisations (G2) because the former specifically references the *effect* of inversing. For example, the relative clause in Freya's comment that '*n is multiplied by 2 then divided by 2, which takes you back to just n*'. Most D3 actions followed on, near-simultaneously, from a G2 Generalisation about an identified inverse relationship. Reasoning about the effect or outcome of inversing is a form of flexible thinking (Selter, 2009). On the number initial and transfer tasks, inversing actions were useful for seeing the overall critical feature (i.e., doing and undoing) in the given arguments; and hence for explaining the truth of the propositions. On the geometry task, when using a set of given steps to write a similar case, inversing actions were useful to solve, by elimination, equations in x and y. This, in turn, afforded comparison of the x and y values needed to confirm the required outcome of $y = 6x$. However, on the geometry task, while all of the learners followed the given steps when writing their own cases, they almost always reasoned solely in relation to the given diagrams.

5.1.6 Extending actions

I identified three kinds of Extending actions: extending a recognised or conjectured pattern in the given arguments that involved one process (E1); extending a recognised or conjectured pattern in the given arguments that involved two processes (E2); and, extending a recognised or conjectured pattern in the given arguments that involved a process and a result (E3). Most Extending actions were performed as a follow-on to, or consequence of, identifying a generality. As I reported in Section 4.2.8, Extending actions commonly followed Generalising actions at the same level of sophistication, i.e., E1 actions followed G1 actions, E2 actions followed G2 actions, and E3 actions followed G3 actions. Hence, Generalising appeared a necessary precursor to Extending; and, extending actions (in particular, continuing a process-result generality involving an increasing or decreasing value) appeared to play a pivotal role in generating a further case. Indeed, across both domains, and across all learners' actions combined, 15 of the 16 E3 (process-result) Extending actions followed a G3 generalisation. In some learning contexts, extending actions have been treated as a form of Generalising (see, e.g., Ellis, 2007). However, as I explained in Section 4.1, I discerned Extending from Generalising because with learners' actions on the tasks in my study, the latter discretely preceded the former. Moreover, Generalising actions were not always followed by Extending actions, which resonated with Reid's (2002) distinction that pattern identification does not necessarily extend beyond the domain that gave rise to it. I further discerned Generalising from Extending because, in my framework, Generalising involved giving an underlying reason (justifying), whereas Extending did not.

Examples of single-process (E1) Extending included continuing, in the writing of a case similar to the given arguments, an identified pattern in the increment of correspondingly positioned addends or coefficients; whereas, E2 actions involved concurrently continuing two such patterns. Across the whole data set, only two two-process (E2) Extending actions were performed, whereas the number of E1 and E3 actions was 13 and 16, respectively. The low number of E2 actions was perhaps explained by the comparative ease of E1 (single-process) extensions and the perceived value of continuing a pattern involving a result as well as a process (E3). For a pattern to be extended, one must have some sense of the underlying structure of the pattern. Relatedly, and in step with Stylianides' (2008) explanation that 'given structures can specify the process by which a pattern is extended', the given arguments variations served as templates for learners' generation of their own cases. It was possible for learners to see some of the patterns in the given arguments with only a superficial sense of underlying structure (e.g., correspondingly positioned addends that 'go up by 2 each time'). While continuing a pattern with such superficiality may have been 'going with the grain' (Watson, 2001, p. 472), it nonetheless provided learners with enough insight to write a new case for a proposition that was non-consecutive to the given propositions. Writing such a new case can be seen as an example of 'expanding the range of applicability of a concept' (Ellis, 2007); i.e., using (the concept of) the given argument structures to write a case for a proposition that was different to the given propositions. This can be said of all learners who generated a new case for a new proposition, regardless of if their new case produced a required end result. Mason (2017b, p. 110) makes the same point generically: 'extending informed by variation theory is just as vital as getting answers to tasks'.

5.1.7 Networking actions

Networking actions involved concept or topic integration akin to what Romberg (2016, p.301), in the context of problem solving, called 'domain integration'. I categorised three Networking sub-actions in learners' responses to the variation tasks: reasoning about an argument by referencing, but not applying, a concept from a qualitatively discrete curriculum area (N1); introducing and applying to an argument a concept from a qualitatively discrete curriculum area (N2); and, reasoning that involves applying to an argument a value other than a positive integer (N3).

Networking actions involved relational thinking that was broader than or 'outside' the concepts, procedures, objects, and structures presented within the given arguments. Such broad relational considerations were expressions of flexible thinking that drew on a learner's existing knowledge. As I mentioned in Section 2.3, Networking actions involved drawing on one's *concept image*, i.e., the network of cognitive connections that one makes in response to apprehending a mathematical concept (Tall & Vinner, 1981); or, more specifically, drawing on one's *evoked* concept image, i.e., the connections one brings to mind in relation to a particular mathematical situation/task (p. 152).

I identified a N1 action when a learner referred to, i.e., only considered, applying a concept from a discrete curriculum topic. For example, Carl referred to, but did not use, the concept of fractions (of circles) in relation to the given angle diagrams on the geometry transfer task. Carl abandoned his 'potentially relevant' networked concept because he seemed to feel, on reflection, that its application would not aid his task response. Alternatively, since he had not previously encountered formal argumentation, he may not have had the confidence to pursue using his networked concept. The latter possibility was also implied by Dylan when I asked him if the argument given in the number introductory task was always true, to which he hesitantly replied, "*not unless it works for fractions*". Dylan's response indicated that he had flexible thinking regarding different classes of number, but he perhaps did not have the confidence or feel an *intellectual need* (Harel, 2013) to apply, i.e., test, a fractional value. Had he done so, he would have seen that the given argument held true for initially chosen fractional values.

When learners directly applied to an argument a concept from a discrete curriculum area (N2), they usually did so to conceptualise an identified relationship. For example, on the geometry transfer task, Carl used the ratio 1 : 2 to describe a previously identified general relationship between the given x and y coefficients. By introducing the concept of ratio (which Carl also did to describe an addend-result relationship on the number task) he performed an act of *similarity-making* (Carraher & Schliemann, 2002), i.e., reframing a situation so that an existing form of knowledge can be applied to it. The application or mapping of a known concept onto a 'new' situation is, of course, a core aim of mathematics education. Yet, in my study, most of the learners did not perform any N2 actions. Incidentally, I did not treat 'substituting' as a form of Networking because it was an inherently essential action that all of the learners performed on both tasks, and no learner referred directly to substitution as a concept that they used or could use. Not performing N2 actions can perhaps be explained by learners *syntactically anchoring* (Tversky and Kahneman, 1974), i.e., staying within the cues and confines of an unfamiliar task. Inferring by extension, this suggests that such learners might benefit from being encouraged to think in a more mathematically integrated way.

A third form of Networking occurred when learners applied to an argument a value other than a positive integer (N3). All of the numbers in the given arguments were positive integers, hence calling into play a different class of number such as a negative, fractional, or decimal was a further expression of flexible, i.e., *disjunctive*, thinking (Toplak & Stanovich, 2002). On the number introductory task, learners' use of different classes of number was associated with testing the truth, or truth domain, of the given proposition (see Section 5.5). However, in the context of the case-generation prompts on the transfer variation tasks (on which all of the N3 actions where performed) such actions involved *compensating* (Askew, 2015) and 'rejecting'. As I explained in Section 4.2.1, Dylan performed a compensating action by adding 0.5 to the result of one of his own cases to achieve a desired end result of 7 for any initially chosen number. On the geometry transfer task, Emma reasoned in terms of decimals by indicating that she was aware that some test values would produce a decimal result, and that she would reject such values. Emma's identification and rejection of 'decimal-producing' test values can be seen as a precursor to the formal notion of proof refutation.

Learners' Networking actions, then, played important, or potentially important, roles in their attention to the argument structures. Flexible thinking that goes beyond the concepts, procedures, and objects presented within a given argument can aid a learner's *structure sense* (Hoch & Dreyfus, 2004) and populate their *example space* (i.e., increase the range of examples they can bring to mind) (Sinclair et al., 2011); which, in turn, can aid their inductive recognition of a case as an instance of a generality/proof. Yet, across all of the variation tasks combined, Networking was the least common main action. If Networking is not encouraged as part of learners' engagement with proof-related tasks it could perpetuate one of the problems of proof pedagogy identified by Stylianides and Stylianides (2006a); namely, proof taught in isolation to other mathematical activities (p. 202).

As I described earlier, the six main actions (Generalising, Organising, Localising, Deconstructing, Extending, and Networking) that each comprised three sub-actions, constituted finding 1: *Learners' attention to structure can be described in terms of the actions in the GOLDEN framework*. (See Appendix E for the full framework with further examples). By formulating the GOLDEN framework, I responded to RQ1. In the next three sections I discuss my responses to RQ2, RQ3, and RQ4, respectively. I organise these discussions around my RQ-related findings. For ease of reference, each of the seven findings that I presented in Chapter 4 are compiled in Table 5.1.

Table 5.1: The seven findings.

1. Learners' attention to structure can be described in terms of the actions in the GOLDEN framework.
2. After writing a case for a proposition that was similar to the given propositions, learners attended to more complex structure and shifted to more precise and sophisticated reasoning.
3. When learners identify a dimension of difference across given argument variations, it can prompt them to justify and communicate the reason for their identified difference.
4. Learners' sequences of actions involved working with increasingly complex structure and progressively identifying relationships
5. Generalising was the most common main action.
6. Transferred actions aided learners' progress through a task, regardless of if a required end result was achieved.
7. Proof variation tasks can be used to encourage learners to perform actions that can support their ability to prove.

5.2 Discussion in relation to RQ2

RQ2: *What characterises Y8 and Y10 learners' attention to structure on an initial and transfer variation task in the domains of number and geometry?*

To respond to RQ2, I used the GOLDEN framework to describe learners' actions on (but not across) a domain-pair of variation tasks in each of the two separate domains. These descriptions yielded findings 2 to 5 (see Table 5.1), which I will respectively discuss in this section. Where appropriate, I give particular emphasis to salient sequences of actions and their implications for learners' progress on the tasks. (I discuss learners' actions *across* each respective domain-pair of variation tasks in Section 5.3).

With finding 2 I reported that after writing a case for a proposition that was similar to the given propositions, learners attended to more complex structure and shifted to more precise and sophisticated reasoning. I identified this finding by comparing each respective learner's responses to a prompt that I positioned immediately before (and repeated immediately after) asking them to generate their own case that was similar to the given arguments on the initial variation task in each domain. On the number task, the repeated prompt asked learners to explain why the given arguments must be true. On the geometry task, the repeated prompt asked learners to describe what a diagram, similar to the given diagrams, would look like for a proposition of $y = 6x$ (as opposed to the given diagrams with respective propositions of $y = 2x$, $y = 3x$, and $y = 4x$). Learners' actions on the repeated prompt involved more structural connections and they more precisely articulated their reasoning. For example, after Ash and Belle wrote their own cases on the number task, they noticed and reasoned in terms of inverse operations that were present in the given arguments (which was key to explaining the truth of the propositions). Ash and Carl provided further evidence of finding 2 when, after writing their own cases on the geometry task, they gave more detailed explanations of a process-result relationship and how to extend it to generate an argument for $y = 6x$. As I explained in Section 4.2.3, these shifts in learners' explanatory reasoning suggested that asking learners to write a case based on given (heavily scaffolded) arguments can broaden their structural awareness and refine their related explanations.

Learners' case-generation and argument-related explanations (justifications) had implications for the development of their *proof schemes*. As I described in Section 1.3, a proof scheme is a personally held (psychological) perception that influences the way an individual organises, makes sense of, and methodises a course of action when proving (Harel & Sowder, 2007, p. 808). The values that learners tested when generating their own cases (L1 to L3) can be seen as an expression of the *empirical* proof scheme; which, according to Harel and Sowder (2007, p. 809), is characterised by example use and the substitution of specific numbers, e.g., to test the truth or truth domain of a proposition. Empirical proof schemes, i.e., repeated use of examples, might lead learners to reason recursively then prove by induction (Stylianides et al., 2016), which can enhance learners' conception of proof (Harel, 2001, p. 185). Indeed, the sequences of argument variations in the tasks that I designed supported such reasoning. Learners' argument-based explanations for the truth of the propositions similarly had potential implications for the development of their proof schemes. Specifically, such explanatory actions can be seen as the learner 'taking control'; e.g., by explaining 'their' truth for why a proposition or conjecture holds for all cases, or by seeing their justifications as removing, or aiming to remove, doubt. Such learner-centred actions move away from reliance on a teacher or textbook as the arbiter of what is right and wrong, i.e., mathematical truth. Hence, learners' argument-related explanations (justifications) can be seen as a departure from an *external conviction* proof scheme; which, Harel and Sowder (2007, p. 809) described as recourse to authority, e.g., seeking and/or following a teacher's didactics or a procedure illustrated in a task or textbook.

With finding 3 I described how learners who identified a dimension of difference across a given set of given argument variations also explained the reason for their identified difference without being prompted to do so. For example, rather than simply commenting that "*the added numbers are different*", learners explained the 'why' of the difference with remarks such as "*the added numbers are different because they are adding 2 each time*". Although rudimentary, such explanations have been seen as a form of justifying (see, e.g., Sowder & Harel, 1998, p. 670), which is an activity associated with, if not inherent in, proving (Harel & Sowder, 2007; Stylianides & Stylianides, 2009). Learners' explanations for their identified differences were obviously basic compared to justifying why a proof must be true. Yet, when learners give such basic explanations, teachers can seize the moment and draw learners' attention to the fact that, by giving a reason 'why', they are not simply accepting as true what it is they see. Hence, using argument variations to encourage the identification of difference can serve as a starting point for developing learners' awareness, appreciation, and competence of and in proof-related justifying.

With Finding 4 I conveyed how learners' sequences of actions involved working with increasingly complex structures and progressively identifying relationships. Three salient sequences of actions were shifts from:

(1) An Organising action (or sequence of Organising actions) to Generalising.

(2) A process-only generalisation (G1 or G2) to a process-result generalisation (G3).

(3) Generalising to Extending.

These three shifts, which I have presented in the chronological order that they were generally performed, captured the 'essential' or core action trajectory required for learners to generate their own cases. This trajectory can be broadly described as: making sense of the given argument variations (Organising), identifying a pattern across the given arguments (Generalising), and continuing a pattern to generate a required further case (Extending).

In regard to the first salient sequence, Organising actions can be described more superficially as acts of noticing. Most Organising actions occurred as a first action on a prompt. This was to be expected because, as Mason (2002, p.7) pointed out, one cannot act until one notices'. Where Organising (noticing) was not an identifiable first action on a prompt, it was possible that the learner had performed the necessary Organising action on a prior related prompt. Most of learners' Organising actions, or sequences of Organising actions, were followed by a Generalising action. The most common shift from Organising to Generalising was from an O1 (foregrounding sameness) action; which was unsurprising because foregrounding a dimension of sameness is a critical precursor to reasoning about a generality (Generalising).

With the second salient sequence, i.e., shifts from a process-only generalisation (G1 or G2) to a process-result generalisation (G3), learners' attention to structure broadened to incorporate the end results of the given arguments; i.e., the resultant values in the propositions that accompanied the arguments. On the number task initial variation task, all five of the learners who identified a consistent addend-result relationship in the given arguments (see Appendix A, prompt 2a) explained aloud the underlying reason for the generality. Indeed, Freya acted with a new surge of enthusiasm on suddenly identifying the foresaid relationship. As I mentioned in Section 5.1.6, identification of a process-result pattern seemed pivotal in providing learners with enough relational generality to commence writing their own, conceptually similar, case; presumably because an identified pattern across a set of given results (in addition to an accompanying process pattern) had a convincing (confirmatory) effect. I say more about learners' Generalising actions in my discussion of finding 5 later in this section.

The actions in the third salient sequence involved shifts from Generalising to Extending. As I explained in Section 5.1.6, I treated the two actions as distinct (i.e., near-simultaneous rather than co-occurring) because, in my framework, Generalising involved justification of a generality, and learners did not always immediately extend their generalisations. In comparison to the given arguments on the transfer variation tasks, the structure of the given arguments on the initial variation tasks were deliberately designed so that: (1) generalities were easier to identify, and (2) it was relatively easy to generate a similar case for a similar proposition merely by extending a process generality. The relative ease of writing a case simply by extending a process generality afforded learners an accessible starting point and an early experience of success. Further, cases written on the foresaid basis oriented learners' attention to further connections within the given structures (which allayed any concerns I had about such extensions being too superficial to give learners any relational understanding of the arguments). For example, on the number initial task, Ash wrote his own case simply by extending a previously identified addend pattern of + 6, + 8, + 10 in the given arguments. Ash's attention then broadened and he became aware of the inversing aspect, which was critical to explaining the 'why' of the propositions' truth.

While the three foresaid chronological sequences of actions were salient, learners task trajectories were not always so linear. As I mentioned in Sections 4.3.2 and 4.4.2, in situations where learners did not, or perhaps could not, identify a generality, they commonly shifted either to: (1) Localising by adapting a given value then testing their adaptation, or (2) Generalising by conjecture, i.e., from a single given argument and accompanying proposition. These two shifts were particularly present in responses to the respective transfer case-generation prompts, on which generalities were harder to identify. Almost half of all Localising actions on the transfer tasks were immediately followed by a further Localising action, the respective test values for which were based on the result of a prior tested value (akin to the concept of trial and improvement). Such repeated localised testing reflected learners' repeated attempts to adapt a given argument to satisfy a conjecture similar to the given conjectures; and hence create a situation (basis) for writing their own case. Regardless of if a successful end result is achieved on a task, Localising can help a learner to see why, or partially why, a given argument must be true. For example, on the given proof shown in Figure 5.1, seeing why an addend of 14 (rather than the addend of 10 shown in step 3) is needed to obtain an end result of 7. This can lead to seeing why the end result is 7 for *any* initially chosen number; which, in turn, is a context for a learner (with the support of a guiding teacher) to develop an appreciation of (ir)refutability.

Figure 5.1: The given proof on the number transfer variation task.

Answer always 5	Algebraic Proof
Step 1: Choose any number	n
Step 2: Add the next number	$n + (n + 1) = 2n + 1$
Step 3: Add 9	$2n + 1 + 9 = 2n + 10$
Step 4: Divide by 2	$\frac{2n + 10}{2} = n + 5$
Step 5: Subtract the chosen number	$n + 5 - n = 5$

With finding 5 I reported that, across the whole data set, Generalising was the most common main action (and accounted for more than one third of all learners' actions combined). As I explained in Section 5.1.2, in each year group separately, Generalising was the most common, or joint most common, main action on the number task and the geometry task, respectively. I also explained that, across the tasks in both domains combined, Generalising was the most common first main action performed on a prompt. Specifically, across all of the first main actions across all of the prompts, slightly more than half (30 out of 59) were a form of Generalising. The frequency of Generalising provided evidence that the design of my variation tasks succeeded in prompting learners to perform a kind of action that can be integral to proving. The importance given to generalising in the literature on proof pedagogy (see, e.g., Stylianides et al., 2013; Harel & Fuller, 2009, p. 360) is reflected in the pedagogy of mathematics generally (see, e.g., Ellis et al., 2024; Mason et al., 2010; Sriraman, 2004). Indeed, all of the main actions in the GOLDEN framework are relevant to other areas of mathematics. I say more about the broader applicability of the main actions in the GOLDEN framework in Section 6.7.

The proof variation tasks that I used featured close (consecutive number) iterations of arguments that served as 'worked examples'. The slight (systematic) variation across each set of given arguments afforded opportunities to see and explain structural patterns, and I directly encouraged this kind of engagement with the prompt: *what is the same and what is different about given arguments?* Searching for patterns, which can aid awareness of structure, is a key mathematical habit (Cuoco et al., 1996) but one that Mulligan et al. (2008) found did not come naturally to some learners. The three Generalising sub-actions that I identified (single process, two-process, and process-result) suggested an 'hierarchy' in terms of sophistication, i.e., the complexity of structure involved in the generalisations. Across the whole set of response data, this hierarchy was reflected on 13 of the 19 prompts on which learners performed more than one Generalising sub-action. Specifically, on these 13 prompts (that spanned five of the six learners), initial Generalising sub-actions were followed by Generalising sub-actions only of a more sophisticated nature. For example, successive (but not immediately consecutive) Generalising sub-action sequences of G1 then G2 then G3; or, G1 then G3 (see Appendix F). As I mentioned earlier, Generalising was the most common first action on a prompt, nearly half of which were G1 (single process) generalisations. This suggested that such sub-actions were particularly accessible to learners and made useful entry points on a task. The G2 generalisations that I identified demonstrated how some learners appeared to reason by simultaneously holding in mind two processes; whereas, G3 process-result generalisations seemed to give learners enough information and impetus to write their own case. Although learners' own cases were structurally close to the given arguments, the methodical process of writing the steps can potentially lead a learner to see a whole argument as a single object. This, in turn, can be seen as a step towards seeing a numerical case as an instance of an axiom, which has implications for developing a learner's conception of proof.

The importance of generalising was captured in three assertions made by Krutetskii (1976): 'good mathematicians have an awareness of structure by seeing and grasping generalities across several examples; they see global similarities in locally-produced examples; they see possible generalities even in single examples'. The latter assertion resonates with learners in my study who conjectured a generality from only one given argument. As I explained in Section 4.2.7, learners in my study shifted to conjecturing a generality from a single example when they did not, or appeared unable to, find a convincing pattern. While such conjectured Generalising usually involved fallacious reasoning, undesired and unexpected resultant outcomes can provide valuable opportunities, perhaps teacher-guided, for learners to reflect on why their approach did not work (Foster, 2019). Such reflections, can promote flexible thinking (see, e.g., Bills et al., 2004, p. 18), populate example spaces (Sinclair et al., 2011), and foster resilience-building habits of experimentation (Cuoco et al., 1996). All of the forementioned are desirable, if not inherent, aspects of advancing learners in the area of proving.

5.3 Discussion in relation to RQ3

RQ3: *What characterises Y8 and Y10 learners' attention to structure across a pair of (initial and transfer) proof variation tasks in the respective domains of number and geometry?*

In each respective domain, and for each year group separately, I responded to RQ3 by applying the GOLDEN framework to compare actions on the initial task to actions on the transfer task. In other words, I compared learners' actions across the domain-pair of variation tasks in each respective domain.

In each year group and domain, at least twice the number of actions were performed on the initial task than on the transfer task. It was possible that this consistency was influenced by: (1) the initial unfamiliar nature of the tasks, and (2) the greater number of prompts on the initial tasks than on the transfer tasks (although the additional prompts on the initial tasks oriented learners' attention in broadly similar ways as the transfer tasks). An arguably more equitable comparison was of actions on the initial and transfer *case-generation* prompts (i.e., in each separate domain, prompts 2c and 3, respectively – see Appendices A and B). This more focused comparison revealed that most learners performed more actions on a transfer case-generation prompt than they did on the corresponding initial case-generation prompt. The comparison of frequencies of actions on the case-generation prompts suggested that learners experienced greater intellectual challenge on the transfer task. Specifically, across all of the learners and across both domains, Localising actions were performed on only two of the initial case-generation prompts, yet such actions were performed on nearly all of the transfer case-generation prompts. I identified only two individual instances of Localising actions across the initial case-generation prompts, whereas 18 such actions were evident across the transfer case-generation prompts. As I mentioned in Section 5.2, Localising (along with conjecturing a generality) was associated with learners' struggles to adapt a given argument to satisfy a proposition similar to the propositions that accompanied the given arguments; thereby creating a structure for writing their own case. As I reported in Sections 4.3.2 and 4.4.2, learners found it harder to identify robust generalities across the given arguments on the transfer tasks than they did on initial tasks (partly due to the invariant propositions on the transfer tasks). Evidence of struggle on the transfer case-generation prompts was further suggested by comparison of the most common sub-group actions. Specifically, on the initial case-generation prompt in each respective domain, neither year group's most common sub-group action involved Localising; whereas, on the transfer case-generation prompt in each respective domain, a form of Localising was nearly always the most common sub-action. Further, across all Y10 learners combined, the most common sub-group 3 action on each respective transfer case-generation prompt was an L3 action, which reflected the *re*-adjusting then *re*-testing of given values.

Learners' Localising actions, i.e., (re)adjusting and (re)testing of given values, were expressions of *experimenting*, which Cuoco et al. (1996) posited as a beneficial habit to foster in the minds of mathematics learners (pp. 378-379). In my study, such local experimentation helped learners to identify dependency and co-varying relationships, e.g., through recognised patterns between tested values and undesired test results (akin to the concept of trial and improvement). Identification of such relationships increased the number of structural connections that learners attended to, which aided their understandings of the arguments. As with Mason et al's (2010) notion of 'specialising', becoming aware of more structural relationships by Localising can also afford, or increase the likelihood of, identifying generalities. As I explained in Section 2.3, I used the term *Localising* to cover a range of actions that include and go beyond those that Mason et al. (2010, p. 21) called *specialising*. The term *Localising*, in comparison to *specialising*, is also more indicative of the kind of structure that a learner is acting on. Learners' local adjustments, and tests based on the results of prior adjustments and tests, also stood to bolster their 'psychological resilience' (Johnston-Wilder & Lee, 2010) when a tested value did not produce a desired result. As Mason (2023) explained, 'resilience is developed through experience of modifying and correcting, not from trying to be perfect all of the time'. Further, experimenting by Localising can strengthen a learner's empirical proof scheme (Harel & Sowder, 2007), which can be helpful so long as a subsequent shift to seeing structure (rather than solely relying on examples) is encouraged (see, e.g., Küchemann, 2008).

My comparisons of learners' actions across each respective domain-pair of variation tasks also focused on transferred actions. i.e., sub-actions performed on an initial variation task that were also performed on (transferred to) a corresponding transfer task. My analysis of transfer yielded finding 6: *transferred actions aided learners' progress through a task, regardless of if a required end result was achieved*. Across all of the data, four of the six learners transferred a total of 20 sub-actions that collectively spanned all six main actions. Of the 20 transferred actions, 11 were performed on the number task and 9 were performed on the geometry task. In step with finding 5, Generalising was the most commonly transferred main action, and G3 (process-result) generalisations were the most commonly transferred sub-actions. As I suggested in Section 5.1.2, the prevalence of Generalising was perhaps unsurprising given that the tasks, in part, required learners to generate cases that were structurally similar to given arguments.

The six learners each responded to one transfer task in each of the two domains, which provided data on 12 individual transfer tasks. While transferred actions always aided a learner's progression on a task, successful responses were only achieved on one third of the transfer tasks. However, three of the four learners who exhibited transfer achieved a successful outcome on a transfer task in the same domain as their transferred action; and, the fourth learner who exhibited transfer (Emma) would also have been successful had she tested the x coefficient of 12, and y coefficient of 3; which, as an afterthought on concluding her interview, she said that she would have next tried. As I mentioned in Section 4.3.3, in Y8, on the number task, only learners who exhibited transfer successfully completed the transfer task (this was also the case in Y10 on the geometry transfer task). Indeed, across both year groups and domains, it was almost always the case that only learners who exhibited transfer successfully completed the transfer tasks. The forementioned observations implied that learners from both year groups found transferred actions particularly helpful. This implication was reinforced by a further observation: on the number task, in both year groups, all of the learners who demonstrated transfer performed as their first action on the transfer task a sub-action that they had previously performed on the initial task.

As I explained in Sections 3.7.3.3 and 3.7.3.6, in both domains, the transfer task was more intellectually challenging than the initial task; and, as I just mentioned, across both year groups and domains, it was almost always the case that only learners who exhibited transfer successfully completed the transfer tasks. Taken together, this suggested that the initial tasks could provide a context for learners to perform what I call transferable 'scaffolding actions'; i.e., actions that can support learners' progression on tasks that are more challenging than those on which the action was previously performed. 'Scaffolding actions' (which can be seen as a form of learner-generated or self-scaffolding) can augment the kind of 'templated' scaffolding provided by the given sets of argument structures. Hence, on the transfer tasks, two kinds of preparatory scaffolding were potentially in place and play. Seeing transferred actions as a form of scaffolding aligns with one of the factors that Henningsen and Stein (1997) found contributed to learners' development and maintenance of higher order mathematical thinking. Relatedly, the actions that learners transferred across a domain-pair of variation tasks in my study can be seen as expressions of 'far' or 'vertical' transfer (Rebello et al., 2007) if one considers 'near' or 'horizontal' transfer (Royer, 1979) to involve acting on structures that differ only by the magnitude of any numerical values.

I counted as transfer repeated sub-actions rather than repeated main actions. However, a case can be made for treating transfer at the (aggregated) main action level if, for example, one assumes that a G2 action on a transfer task was only performed as a consequence of previously performing a G1 action on the corresponding initial task. Such assumptions are expressions of what Marton (2006, p. 528) called 'transfer as influence'. Had I treated transfer at the main action level then, across both year groups and domains, transfer would have been evident on 10 of the 12 individual transfer tasks. However, I chose to treat transfer at the sub-group level to avoid making the foresaid assumptions. Further, the three sub-actions in each respective main action differed in their qualitative proximity to each other. My tabulation of each learner's coded sub-actions (see Appendix F) made identifying transferred actions easier. This ease was welcome because learners' utterances rarely conveyed that they were performing a transferred action. Only Carl provided overt verbal evidence of transfer, which he did in relation to conceptualising an identified numerical relationship in terms of ratio (Networking). Specifically, Carl did this by pointing back to his response to the initial number task and explaining that, "*I'm referring to this method [previously identified ratio relationship] again*".

How transferred actions aided learners' progress on the transfer tasks reflected the roles of each respective (sub)-action that I described in Sections 5.1.2 to 5.1.7 (see Table 4.35 for the sub-actions that each learner transferred in each domain). To broadly recap: transferred Generalising involved reasoning about identified or conjectured patterns that could potentially be extended in the writing of a learner's own case; transferred Organising helped to manage, or gave order to, the multiple information in the given variations, which aided initial sense-making; transferred Localising comprised adjusting then testing a given value in an effort to achieve an argument structure that met the conditions of a proposition similar to the given arguments' propositions; transferred Deconstructing helped learners to see relationships and abridge structure; transferred Extending contributed to generating a new case; and, transferred Networking aided interpretation of given structures through integration of 'external' concepts. The forementioned roles and affordances applied to transferred actions regardless of if a learner went on to achieve a successful end result on a transfer task.

5.4 Discussion in relation to RQ4

RQ4: *How do Y8 and Y10 learners' attention to structure compare across both domains?*

To respond to RQ4, I used the GOLDEN framework to compare Y8's actions on the number variation tasks to their actions on the geometry variation tasks. I then repeated the cross-domain comparison for Y10. As I explained in Section 4.1, the ranging diversity in the nature of proving tasks contributes to proof being hard-to-teach and hard-to-learn (Stylianides & Stylianides, 2017). Hence, my search for actions that learners commonly performed in each of the two different domains was potentially valuable. In Y8, in each separate domain, Generalising was the most common (or joint most common) main action on each respective initial and transfer task. This result aligned with finding 5, i.e., that Generalising was the overall most common main action. Yet, in Y10, a comparison of the most common actions on each respective initial and transfer task revealed that, across both domains combined, Localising was most popularly performed. Specifically, in Y10, in the number and geometry domains respectively, Localising was the most common action on the transfer task. This suggested that, in response to the intellectual challenge of the transfer tasks, the Y10 learners were more flexible thinkers than the Y8 learners because the Y10s more frequently used or shifted to Localising actions as well as (or instead of) testing conjectured generalities. Indeed, across the number and geometry transfer tasks, Y10 learners collectively performed exactly twice as many Localising actions than the collective Y8 learners (12 Localising actions and six Localising actions, respectively; see Appendix F).

In regard to cross-domain comparisons of transfer, as I showed in Table 4.35, none of the learners who transferred sub-actions within one domain did so within the other domain. This absence of cross-domain transfer was potentially due to conceptual (topical) differences in the number- and geometry-based tasks, as well as any cross-domain differences experienced in the demands of the tasks. The lack of cross-domain transfer reflected the difficulty that I described in Section 4.1 in regard to finding and using cross-domain generalities in the learning and teaching of proof. Yet, across each year group separately, learners performed all of the main actions in the GOLDEN framework on the number task and the geometry task, respectively. Further, all of the learners individually performed at least half of the main actions in each respective domain. These consistencies underlined the utility of the GOLDEN framework as a cross-domain tool for analysing and articulating learners' proving actions.

Across all of the transfer tasks, and across both year groups combined, the most common unperformed main action was Organising. One explanation for this lack of Organising actions was that, unlike the initial tasks, the transfer tasks did not directly ask learners to discern sameness and difference. Therefore, it was possible that Organising actions were sometimes performed but not made explicit; e.g., discerning sameness may have tacitly preceded a learner's first overt action on a transfer task; which, across all of the learners combined, was most commonly a form of Generalising. Only Dylan, on the number task, performed an overt Organising action on a transfer task. Dylan performed the Organising action as his first action on the transfer task which reflected how, on the initial tasks, learners' Organising actions were associated with early sense-making. Only one third of the learners in each respective year group successfully completed transfer tasks; whereas, almost all of the learners (in both year groups combined) successfully completed the initial tasks. Despite the challenge presented by some of the tasks to some of the learners, as I have repeatedly argued, the variation tasks nonetheless prompted all of the learners to perform a variety of actions that are requisite for proving. This perspective is consistent with Hanna's (2018, p.5) view that 'teaching proof is not solely concerned with crediting successful conclusions but also with attending to structure and seeing overall ideas'. Indeed, had all of the learners experienced little or no difficulty, and been successful on all of the tasks, it is possible that their variety of strategies (and the opportunities to characterise their inherent actions) may have been more limited; and hence less insightful. Learners' ranging task responses led to finding 7, i.e., *proof variation tasks can be used to encourage learners to perform actions that can support their ability to prove*.

As I described in Section 3.7, all of the given argument variations were presented as five-step procedures, each step of which I read aloud to each learner in their task-based interview. By reading aloud the stepped procedures I aimed to implicitly convey to the learners something of the explanatory quality of the arguments. Indeed, the stepped procedures, together with the prompts that encouraged learners to reason about the given arguments, provided a foundation for fostering learners' appreciation of the explanatory function of proofs. For, where a learning goal is to understand a proof proposition, a proof is most effective when it embodies explanation (Hanna, 2018, p. 5). The given arguments were essentially heavily scaffolded examples, and their conceptual and structural proximity to each other (within each domain) was deliberately intended to aid learners' cross-argument comparisons; further, by asking as a first prompt, *what is the same and what is different about the given arguments?* every learner could immediately engage with the structures.

As I explained in Section 2.2.3, structural (in)variance can be controlled to afford orienting learners' attention to discernment of a particular feature or relationship**.** For example, across the given propositions and arguments on the number initial task, I only (co)varied the predicted (propositional) results and one of the processes; which, afforded reasoning about a process-result generalisation (G3). Similarly, with the given variations on geometry transfer task, I systematically varied two procedural dimensions (the x and y coefficients), which afforded simultaneous reasoning about a two-process generalisation (G2). Yet, as Marton and Pang, (2006, p. 200) explained, what structural (in)variation a learner discerns and acts on may differ from that intended. Variation tasks that feature specific (numerical) examples of a given proof and accompanying conjecture, then ask for a proof for a slightly different conjecture provide a context for bridging between the specific and the general. For example, on the number transfer task, Dylan suggested that if his adaptations to a given specific example "*worked*", then he would "*do the same to [similarly adapt] the [given] proof*".

One possible criticism of the application of variation theory to mathematics pedagogy is that inductive reasoning cannot lead to a higher level of abstraction (Watson, 2017, p. 98). However, what constitutes a 'higher level' of abstraction is, like seeing variation, subjective. For instance, on the number initial variation task in my study, some learners shifted from identifying patterns across given arguments to partially explaining the truth of the arguments (as Freya did by reasoning that '*n is multiplied by 2 then divided by 2, which takes you back to just n*'). Learners also shifted from identifying patterns across given arguments to constructing their own case for a proposition that was different to the given propositions. For a learner new to formal processes of proving, generating such a new case might feel like a 'different' application, i.e., a form of higher abstraction. Yet, to a more knowledgeable other, the learner's case may be seen merely as a generalisation of mathematical relationships. I say more about the potential utility of my proof variation tasks in Section 6.7.

5.5 Introductory task responses and actions on the variation tasks

In this section, for each domain respectively, I discuss links between learners' responses to the introductory (orienting) task and their actions on the subsequent variation tasks. In Section 3.7.4, I explained that before the learners worked on each domain-pair of variation tasks, I gave them an introductory task (see Figures 3.1 and 3.4, respectively). As I explained in Section 3.9, learners' responses to the introductory tasks served only as a comparative backdrop to my characterisation of their actions on the variation tasks, and did not contribute to my formulation of the GOLDEN framework. In each domain, the introductory task featured a five-step argument that was conceptually and structurally similar to the subsequent variation tasks (particularly the initial variation task). Each introductory task was designed to orient learners by prompting them recall concepts that were relevant to the variation tasks that followed. As I described in Sections 3.9 to 3.9.1, I organised learners' responses to each introductory task into binary categories and compared these to their actions on the subsequent variation tasks.

With the number introductory task, I asked learners to show why the given proposition must (always) be true; this provided an opportunity to reveal if any learners transitioned to algebraic representation without being directly prompted to do so. Since none of the learners had any prior experience in proof argumentation, I anticipated (correctly) that none of the learners would make the transition. To mitigate this, I used a conditional follow-up prompt that asked learners who did not make the transition to represent each of the steps in the given argument 'using algebra'. In this way, I provided a context for potentially aiding learners' appreciation of how representing algebraically can be a useful way to generalise. The advice of Küchemann (2008, p. 9.1) succinctly captured my rationale: 'some proving tasks are difficult to solve without recourse to algebraic symbolisation and teachers need to be aware of this when choosing them. With such tasks, it is worth seeing whether there are ways of supporting learners' algebra'. All of the learners, when directly prompted to do so, were able to represent the given steps algebraically because they all had prior experience of 'mapping arithmetic operational structures to algebraic symbolism' (Warren, 2003). Hence, by directly prompting learners to represent the given argument algebraically, a provided a situation in which learners bridged a gap between knowing 'how to' and knowing 'when to'.

The two categories that I identified in the responses to the number introductory task were learners who tested (and did not test) different classes of number. Across both year groups combined, half of the six learners tested different classes of number; these learners collectively performed almost all of the sub-actions in the GOLDEN framework (17 out of 18); whereas, the other half of the learners (who did not test different classes of number) collectively performed a slightly lower number of different sub-actions (14 out of 18). Although somewhat tentative, this observation implied that learners who tested different classes of number were, perhaps unsurprisingly, more *disjunctive* thinkers (Toplak & Stanovich, 2002) than the learners who did not test different classes of number. In further regard to the number task, across the learners who tested different classes of number, sub-group 3 actions were evident in all six main actions in the framework; whereas, across those who did not test different number classes, sub-group 3 actions were evident only in four of the main actions. Since my sample was small, I did not include more refined comparisons of learners' responses to the introductory tasks and their actions on the variation tasks. Nonetheless, the two forementioned observations implied that the kind of flexible or 'adaptive' thinking (see, e.g., Selter, 2009) that led learners to test different number classes on the introductory task may also have prompted them to perform a greater range of actions on the variation tasks. This, in turn, potentially contributed to their higher number of more sophisticated sub-group 3 actions.

The two categories that I identified in the responses to the geometry introductory task were: learners who referred only to geometric facts (e.g., angles on a line add to 180°), and learners who also compared angles and reasoned about a process-based relationship (e.g., by stating that 90° - 75° "is the same as" $3x$ [= 15°]). Across both year groups combined, as with the number task, half of the learners fell into each category; (as I explained in Section 3.9.1, having an equal number of learners in each category was a key criterion for characterising the introductory tasks' response categories). The binary categories of responses to the geometry introductory task were suggestive of the distinction between declarative knowledge *of* … (Oswald et al., 2007), and relational knowledge of *why* … (Skemp, 1976). Across both year groups combined, learners who referred only to angle facts performed a slightly less diverse range of sub-actions than learners who also compared angles and reasoned about a process-based relationship (11 and 13 sub-actions, respectively). Further, across the learners who referred only to angle facts, no Networking actions were performed; whereas, across the learners who also compared angles and reasoned about a process-based relationship, all three Networking sub-actions were performed. As with the links between learners' responses to the number introductory task and their actions on the variation tasks, the theme of flexible thinking (Selter, 2009) again emerged. Specifically, the kind of flexible thinking (broader attention) required to think beyond angle facts to reasoning about process relationships was potentially useful for integrating curriculum topics on the variation tasks. In contrast, referring only to angle facts suggested more restricted 'static' or declarative' thinking (Oswald et al., 2007), which was further implied by this group's slightly narrower range of sub actions. Inferring by extension, encouraging learners to think in terms of processes might help them to become more flexible mathematical thinkers generally.

Chapter 6: Conclusion

In this final chapter, I will summarise the key aspects of my study. To do this, I will provide a précis of the challenges of proof pedagogy that I set out to alleviate, recap the appropriateness of the theoretical constructs and methodology, give synopses of my responses to the research questions, and revisit the two key contributions of my study (the GOLDEN framework and consecutive-number proof variation tasks). I will follow this with a discussion of the implications that my research has for advancing research and pedagogy. To conclude, I will consider the limitations. By discussing each of the foresaid aspects, my aim with this chapter is to provide an overall picture of the goal, process, discoveries, and implications of my study.

6.1 Alleviating persistent problems of proof pedagogy

Although proof pedagogy plays an important role in the learning and teaching of mathematics, further research is needed to alleviate enduring problems. Limitations reported in the literature have highlighted a need for research with three key characteristics: (1) a framework with details of how its application can aid learners' progress; (2) a narrow scope apt for efficient implementation and impact, and (3) the means to trigger and support conceptual change (Stylianides & Stylianides, 2018, p. 99). My study featured all three characteristics in combination. In regard to key characteristic (1), I formulated the GOLDEN framework to aid identification of proving actions, which can inform teachers of where and how learners can next be advanced. In regard to key characteristic (2), the highly scaffolded tasks that I used meant that all learners were able to test, reason about, and write their own cases within as little as 15 minutes. In regard to key characteristic (3), the consecutive-number argument variations that I used are an adaptable concept for introducing learners to proof argumentation, and the descriptions in the GOLDEN framework provide a language for conceptualising learners' actions.

I have argued that the diverse nature of proving tasks, both within and across domains, is a major factor underlying the notion that 'proof is hard-to-teach and hard-to-learn' (Stylianides & Stylianides, 2017). Specifically, I made the case that the diversity of proving tasks makes identifying structural and algorithmic generalities difficult; hence, no overall teaching approach can be adopted. However, in my study, I identified a set of cross-task/domain generalities in the form of actions that learners similarly performed on proving tasks in each of two different domains (number and geometry). I used these common learner actions to formulate the GOLDEN framework. Therefore, use of the framework has the potential to alleviate teachers' difficulties in identifying and distinctly describing the cross-task/domain proving actions that learners perform.

I also mentioned a further well-known challenge of proof pedagogy, i.e., learners often rely on example-based reasoning rather than structural reasoning. To contribute to alleviating this challenge, I used task prompts that asked learners to explain *why* propositions for given argument structures must be true. Further, by asking learners to identify (in)variance across close iterations of given propositions/arguments, all of the learners in both year groups were immediately able to engage with the structures.

The combined use of consecutive-number proof variation tasks and the GOLDEN framework affords: (1) augmenting teachers' knowledge of where and how to develop learners' ability to prove; (2) introducing the teaching of proof in the early years of secondary school; (3) acquainting learners with the kind of actions that can precede proof production; and, (4) cross-referencing and integrating proving actions with learners' actions in other mathematical areas. These affordances respectively relate to alleviation of the four restrictions on the teaching of proof that I mentioned in Section 1.1. Further, the given sets of argument variations in the tasks that I designed served as worked examples which could aid alleviation of the first resultant problem: *insufficient scaffolding*; and, the descriptions of the six main actions and 18 sub-actions in the GOLDEN framework could aid alleviation of the second resultant problem: *indistinct characterisation of learners' proving processes*.

6.2 Relevance of the theoretical concepts

Looking for and attending to argument structures is an important aspect of developing learners' appreciation of the truth, or truth domain, of a proof proposition; principally because 'many learners provide examples when asked to prove a universal statement' (Education Committee of the European Mathematical Society, 2011, p. 50). More specifically, many learners *rely solely* on examples. With some of the task prompts that I used, I deliberately encouraged learners to look for and reason about what was explanatory in the given argument structures, i.e., the 'why' of propositions'/arguments' truths. These prompts encouraged learners' actions *on* structure, and contributed to my development of the GOLDEN framework.

The principles of *variation theory* were critical to the design of the number and geometry tasks that I used. I controlled the amount of (in)variance across each set of given argument structures to afford initial sense-making. The close variations provided a situation in which I could ask '*what is the same and different about the given arguments?*' Asking this question created an immediate entry point at which all of the learners were able to subjectively engage with the given argument structures. Because this immediate engagement required discernment of (in)variance, it inherently involved foregrounding/backgrounding aspects of the multiple information in the given arguments (i.e., Organising actions). An especial nuance in the design of the variation tasks was the given propositions and argument iterations that featured consecutive-numbers. I could not find in the literature any consecutive-number proof variation tasks of the specific kind that I used in my study. The structural similarity of the given arguments was also intended to facilitate the writing of learners' own cases. To encourage further attention to the iterated structures, I deliberately asked learners to write a case that was non-consecutive to the given arguments. Even the learners who appeared to mechanically continue a superficial pattern subsequently (perhaps consequently) shifted their attention to conjecturing about increasingly sophisticated structural connections.

In regard to *learning transfer*, I identified a number of sub-actions that learners similarly performed on both of the variation tasks within a domain. A learner's transferred actions hinted at what cross-task moves they conceived as productive, or potential productive; especially when a transferred action was performed as an initial action on a (subsequent) task. My focus on transfer also revealed that no learner exhibited sub-action transfer in both domains; this observation lent further importance to my capture, in the GOLDEN framework, of actions that were common to learners' task responses in both domains.

The development of a learner's *proof scheme(s)* can play an important role in their evolving ability to prove. The case-generation tasks that I used prompted learners to adjust given values then test their adjustments, which provided a context for expressing the empirical proof scheme. While relying solely on examples is, with inexhaustive propositions, obviously not conclusive, application of the empirical proof scheme can lead to an inductive (generalising) step, and thus to use of the more sophisticated analytical proof scheme. Further, the case-generation tasks led learners to identify enough structural relationships in the given arguments to prompt the writing of their own cases, which can be seen as a move away from dependency on the authoritative proof scheme.

6.3 Appropriateness of the methodology

I adopted a qualitative case study methodology to enable a close, richly descriptive analysis of learners' task responses. I felt that a case study approach was well-suited to the context of my research, i.e., a sample of six learners (originally 12, pre-pandemic) and the depth of enquiry required to respond thoroughly to my research questions; all of which focused on characterising the variety of actions that learners performed on the tasks that I designed. Although the sample size was modest, I did not consider a larger participant-group essential because my primary aim was to devise and test a new framework. Further, across all of the learners' task responses, the range of actions performed was enough to generate 18 distinctive sub-actions, i.e., three sub-actions in each of six main actions. Wherewith, the six main actions were similarly performed on the tasks in each respective domain.

My pilot study resulted in two adaptations to the anonymised task-based interviews that I conducted. I increased the number of domain-specific tasks per sitting from one to two (due to timetable constraints at the participant school). In addition, I reduced from two to one the number of learners that I asked to participate in each interview (due to cross-talk or dominance of one of the learners in each initial pair).

After I coded learners' actions, to gain a degree of inter-rater reliability, I gave the task responses to two teaching colleagues and asked them to describe any learner actions. While the teachers' descriptions were too narrow to use, their limited interpretations further indicated a need for an attention-to-structure framework. To reduce my first-person authority, I aimed to distance myself from personally held pre-conceptions of what actions learners would perform. While I suspected that learners would generalise and 'specialise' (a term that I later expanded and called 'localising') I aimed to mitigate my pre-conceptions using a grounded observation-and literature-based analytical approach (Miles & Huberman, 1994). In Section 6.8, I say more about the steps that I took to reduce researcher bias.

Since I acted in the dual role of teacher/researcher, I reassured the learners that participation in the study was entirely voluntary. I further reassured the learners that they could withdraw at any stage without the need to provide a reason. I also explained that choosing not to participate, or deciding to withdraw, would not result in being viewed less favourably. This reassurance was particularly important due to the 'asymmetry of the power relationship' (Farrimond, 2016, p. 81).

6.4 Responses to the research questions

A unifying thread linked my four guiding research questions: the characterisation of learners' attention to structure, i.e., learners' actions. RQ1 led to my formulation of the GOLDEN framework, whereas RQ2-RQ4 involved applying the framework to describe learners' actions from respectively different analytical perspectives. I next summarise how I responded to each of the four questions.

Research question 1: *What might comprise a framework for describing learners' attention to structure?* To respond to the first research question, I looked for actions that learners similarly performed on proof variation tasks in the domains of number and geometry. I described learners' actions by reflexively moving between, and drawing on a combination of, the literature and my observations of the task responses. My thematic analysis of learners' collective task responses resulted in a framework that comprised six main actions (Generalising, Organising, Localising, Deconstructing, Extending, and Networking) each comprising three sub-actions (see Appendix E). As I described in Section 3.9.2., the emergent criteria that I used to structure the framework were twofold: the main actions were those that were common to learners' task responses in each respective domain; whereas, the sub-actions for each main action were distributed across both domains. The two structural criteria afford application of the framework to a broad range of contexts because the six main actions have cross-domain relevance for which nuanced (context-specific) sub-actions can be identified.

Research question 2: *What characterises Y8 and Y10 learners' attention to structure on an initial and a transfer proof variation task in the respective domains of number and geometry?* To respond to the second research question, in each year group and domain separately, I used the GOLDEN framework to describe and chronologically chart Y8 and Y10's sequences of actions on the initial variation task and transfer variation task, respectively (see Appendix F). I then used these descriptions to compare Y8's and Y10's actions (see Section 4.2).

Research question 3: *What characterises Y8 and Y10 learners' attention to structure across a pair of (initial and transfer) proof variation tasks in the respective domains of number and geometry?* To respond to the third research question, in each year group and domain separately, I used the GOLDEN framework to identify, describe, and compare patterns and relationships in actions across the domain-pair of variation tasks. In doing so, I gave particular focus to describing actions that were transferred across each respective domain-pair of variation tasks. I then used these descriptions to compare Y8's and Y10's actions (see Section 4.3).

Research question 4: *How do Y8 and Y10 learners' attention to structure compare across both domains?* My response to the fourth research question involved comparing each respective year group's actions on the number task to their actions on the geometry task. I followed this by comparing Y8's cross-domain actions to Y10's cross-domain actions (see Section 4.4).

As presented in Table 5.1, across my combined responses to the research questions, seven findings emerged. In response to RQ1, finding 1 emerged; in response to RQ2, findings 2 to 5 emerged; in response to RQ3, finding 6 emerged; and in response to RQ4, finding 7 emerged. Before I discuss the implications that the findings have for advancing research and pedagogy (Section 6.7), I summarise the two key contributions of my study.

6.5 Key contribution 1: The GOLDEN framework for proving actions

The GOLDEN framework is the main contribution of my research. The framework provides a lens and a language for conceptualising, identifying, and describing learners' proving actions. The framework can also aid researchers' unified use of the term 'structure', which I define as *the interrelated elements that comprise a mathematical object or objects*. Despite this concise definition, the term 'seeing structure' remains ambiguous because it can refer to seeing interrelated elements within a structure (Venkat et al., 2019) (i.e., the compositional view), or seeing a structure as a particular instance of a general structure (Mason & Pimm, 1984) (i.e., the transparency view). To remove this ambiguity, references to 'seeing structure' must make clear which of the two foresaid views, i.e., compositional or transparency, is intended. The compositional view can be alternatively described as the *makeup* of structure, whereas the transparency view can also be described as the *mapping* of structure.

I derived the main components of the framework from the actions that learners commonly performed on tasks in the distinct domains of number and geometry (both tasks involved algebra). Hence, the main actions in the framework offer teachers a set of domain-generic actions that can otherwise go undetected due to the sheer diversity in the nature of proving tasks. Knowledge of the GOLDEN actions can provide teachers with a conceptual language that can make them aware of and formalise what learners are doing when they attend to structure. By extension, this knowledge can equip teachers with an evaluative and diagnostic lens for interpreting learners' proving strategies. With such insights, teachers can be more closely poised to support learners in the moment. Further, the GOLDEN actions can be used as 'design anchors', i.e., foci in the design of tasks aimed at preparing learners for proof production. Such action-focused task designs have the potential to support a shift of attention from 'getting right answers' to attending to structure; which, is an example of the kind of conceptual shift that Stylianides and Stylianides (2018) highlighted as a key characteristic of needed research. I think learners should be encouraged to perform the GOLDEN actions not only to populate their repertoire of proving actions but because they permeate the doing of mathematics generally. Such encouragement could involve deliberately designing into school curricula and schemes of learning opportunities for learners' focused use of the actions.

6.6 Key contribution 2: Consecutive-number proof variation tasks

A major affordance of tasks that feature variations of mathematical structure is that a targeted dimension of variation across the given structures, e.g., a numerical value or relationship, can be isolated by controlling (in)variance. Hence, a targeted structural feature can potentially be brought to a learner's awareness by their discernment of contrast (Marton & Booth, 1997). Discerning (in)variance across a set of given structures also affords initial sense-making (Watson & Mason, 2006). Variations of structures that feature a consistent co-varying process-result aspect (as in my study) can afford progression from initial sense-making to more sophisticated reasoning and relational understanding, regardless of if an intended learning outcome is reached.

As I explained in Section 6.2, a novel and distinctive feature in the design of my variation tasks was the use of consecutive-number propositions and arguments. I could not find in the literature any proof variation tasks that featured propositions and arguments with consecutive numbers, and that required the generation of cases for non-consecutive propositions. The given argument variations served as highly scaffolded worked examples, and their numerical consecutiveness conveyed close structural and conceptual coherence. These design features provided an accessible context for seeing subtle shifts in structure. In turn, this promoted pattern spotting and reasoning about identified relationships, which are activities known to develop learners' sense of structure (see, e.g., Küchemann, 2010). The tasks in each domain involved a mix of given arguments, prompts to *show that* and *explain why* propositions must be true, and requests for learners to write their own cases. Taken together, the tasks prompted the 18 ranging sub-actions in the GOLDEN framework; and did so with learners all of whom had no formal experience proof argumentation. Further, the pairs of initial and transfer variation tasks that I used afforded identification of learners' transferred actions within and across domains. Transferred actions (and the frequencies of actions) provided additional insights into the kinds of mathematical moves that learners appeared to find particularly useful.

The two key contributions of my study responded to repeated calls for further research aimed at supporting proof pedagogy (e.g., Harel & Soto, 2017; Stylianides & Stylianides, 2018). I next discuss the implications that the key contributions and my findings have for advancing pedagogy.

6.7 Implications for advancing research and pedagogy

Research in mathematics education has, for decades, reported major challenges in the teaching and learning of proof (e.g., Harel & Sowder, 1998; Stylianides, 2008; Stylianides & Stylianides, 2025). These challenges have led to repeated calls for more research and resources aimed at addressing the reported difficulties that learners and teachers experience (e.g., Harel & Soto, 2017; Stylianides & Stylianides, 2018). In Section 1.1, I described how reported restrictions on the teaching and learning of proof can contribute to insufficient scaffolding and indistinct characterisation of learners' proving processes. I also raised a common underlying problem: the sheer diversity in the nature of proof tasks makes identifying cross-task procedural generalities (and hence the teaching and learning of proof) difficult. As I described in Sections 6.5 and 6.6, my study offers two key contributions that can alleviate the foresaid problems of practice. Namely, *consecutive-number proof variation tasks* as a scaffolded resource for introducing learners to argumentation and proof, and *the GOLDEN framework* as a task-generic set of characterised proving actions. The GOLDEN framework is different to several other action-based frameworks (e.g., Ellis et al., 2024; Miragliotta, 2022) because the scope of its six main components can describe actions in a broad range of mathematical situations; yet, its sub-actions reflect the context-specific nuances of learners more refined actions.

In regard to the consecutive-number proof variation tasks, other researchers' tasks that have provided learners with argument variations have featured considerably more distal arguments and prompts (see, e.g., Küchemann, 2008). Hence, the concept and design of my proof variation tasks further contribute to the research on proof pedagogy. Specifically, variation tasks of the kind that I employed in my study can be used to encourage learners to perform a wide range of actions that can support their ability to prove (finding 7). For example, with the number introductory task, I asked learners to show that a given proposition is 'always true'; to which learners responded by testing the truth domain. Then, on the initial variation task, I asked learners to explain *why* a set of given propositions must be true. By encouraging this move, the design of my variation tasks responded to a call for more research on shifting learners' attention from certainty to causality, thereby advancing their epistemological justifications (Harel & Soto, 2017, p. 241). I also asked learners to write their own cases that were conceptually similar to the given arguments; which, as I described in Section 4.2, advanced their structure sense and the sophistication and precision of their reasoning. These advances supported Pedemonte's (2008) claim that 'argument construction contributes to constructing a conjecture, whilst structural argumentation aids in justifying it'.

In regard to learning transfer, several researchers claim that lack of a robust schema in an initial situation impedes learners' abilities to apply their knowledge in a new situation (e.g., Rebello et al., 2007; Reed, 1993). Such impediments can be alleviated if, as in my study, an initial task is deliberately designed with several structural features that are also present in the design of a subsequent transfer task (even if the transfer task is more cognitively challenging). Further, the respective sets of argument variations in my tasks afforded learners' (re)affirmation of identified structural relationships and hence *cognitive unity* through repeated structure mapping (Boero et al. 1996, p. 119). Learners could then use such structure mapping to write a case for a proposition that was similar to the given propositions. I next suggest how teachers' use of the GOLDEN framework and proof variation tasks can alleviate: (1) each of the four restrictions that

I introduced in Section 1.1, and (2) the two resultant constraints of insufficient scaffolding and indistinct characterisation of learners' proving processes.

To alleviate the first restriction (teachers' often limited knowledge of how to develop learners' proving abilities) (Harel & Sowder, 2007) the GOLDEN framework can serve two main roles: (1) make visible what learners are doing when they respond to proof variation tasks, and (2) provide a lens and a language to discern, describe, and develop learners' proving actions. In regard to the latter role, a teacher might discern that a learner is attending only to one process and so orient the learners' attention to a further range of processes. Alternatively, a teacher might aim to develop a learner's sense of structure by prompting them to work backwards through several steps in a given argument to identify the origin or structural 'architecture' of an end result. From a broader perspective, the range of actions in the framework can help teachers to answer the question: *What tasks and teaching moves will I need to support learners' argumentation?*

To alleviate the second restriction (delaying the teaching of proof until the latter years of secondary school) (Knuth, 2002) I designed variation tasks that were appropriate for use with Y8 learners (12- to 13-year-olds). For example, all of the Y8 learners in my study responded successfully to the number initial variation task, and almost all of this same year group gave successful responses to the geometry initial variation task. Further, as I mentioned in Section 5.4, the first prompt on each of the initial variation tasks served as a particularly accessible 'hook'. Specifically, these first prompts asked the subjective question *what is the same and different about* a given set of structurally close (consecutive number) argument variations; this prompt meant that 'everyone could start' because essentially there were no 'right' or 'wrong' responses. Asking the foresaid question afforded orienting learners' attention to structural features 'available to be learnt' (Kullberg et al., 2017; Runesson, 2006) and used. For example, on the number initial variation task, co-varying relationships between procedural values in the given arguments and values in the accompanying propositions; and, on the geometry transfer task, co-varying relationships between the sums of the x coefficients and the sums of the y coefficients. Further, learners who identified a dimension of difference often communicated the reason for their identified difference without being prompted to do so (finding 3). I also found that once learners had discerned their dimension(s) of (in)variance, their attention then shifted, in response to the subsequent prompts, to identifying and acting on increasingly sophisticated structural features.

To alleviate the third restriction (ill-prepared learners) (Ball et al., 2002) the GOLDEN framework can aid teachers' awareness of ranging actions that can precede, and be inherent in, the production of proofs. Further, with the prompts in the number task, I asked learners to 'show that' as well as 'explain why' the given propositions must be true. These prompts can prepare learners by first encouraging them to consider truth domains (for which they may exhibit empirical proof schemes) then structure, i.e., what it is in an argument's steps that explains its proposition's conclusiveness. As Küchemann (2008) observed, students often generate examples instead of looking for structure, but students (and their teachers!) could be made more aware that it is also possible to see structure (or at least speculate about it) (p. 9.1).

To alleviate the fourth restriction (teaching proof in isolation from other mathematical activities) (Stylianides & Stylianides, 2006a) teachers can become attuned to where and when the six main actions in the GOLDEN framework occur across mathematics curricula. The six main actions (Generalising, Organising, Localising, Deconstructing, Extending, and Networking) are broad enough to be topic-generic (dependent on a task's design and demand). Hence, teachers can refer to the six actions (as infinitive verbs as well as gerunds) in a wide range of mathematical situations, thereby integrating their use.

Since the given arguments in the proof variation tasks are essentially sets of worked examples, they can alleviate the problem of insufficient scaffolding. The stepped structures and accompanying procedural prompts in the 'readymade' arguments can serve as templates for learners to write their own, conceptually similar, cases. The varying values across a set of given arguments also afford orienting learners' attention to 'dimensions available for change' (Watson & Mason, 2006), i.e., opportunities for structural alteration. Further, values in the steps can be adjusted by working directly on a given argument. Learners can then test their adjustments by following and using in the writing of their own case the presentational format of, and notation in, the given arguments. Dylan explained the use of this scaffolded process in relation to writing his own case on the geometry initial variation task: "*I kept looking back down the [given] steps for what numbers are the same and what numbers are different, and if they are different, you add [substitute] the numbers for the new one [case] in the place of the different ones*".

The 18 sub-actions in the GOLDEN framework offer a language for teachers to articulate learners' attention to structure in ways that go beyond descriptions of learners' calculations and algebraic representations. Hence, teachers' use of the framework can alleviate the problem of indistinct characterisation of learners' proving processes. Specifically, teachers can improve their understanding of, and capacity to advance, learners' proving processes when they ground their analyses of learners' task responses in the language of the GOLDEN framework.

Different groups of learners may perform a different range of sub-actions on the tasks that I used. For example, I initially considered describing the Localising sub-actions in terms of learners' reasons for their choice of test values. I decided against this distinction because learners' reasons were somewhat superficial, e.g., 'random', 'easy', or 'close to the given number'. However, teachers wishing to use the tasks with more experienced learners may find that asking learners to give their reasons for their chosen test values might elicit a range of responses that can be used as Localising sub-actions. Indeed, the broad applicability of the GOLDEN framework lies in the adaptability of the sub-actions to the specific nuances of different learners' responses to tasks in different mathematical situations.

The proof variation tasks also have ranging scope for adaptation. However, with learners new to formal argumentation, I recommend retaining the close systematic use of consecutive-number proposition and argument variations (to aid pattern identification). As I mentioned earlier, identifying patterns across given argument examples can prepare learners for similar actions when they generate new cases of their own, including cases for non-consecutive propositions. For example, when learners' (re)adjusted and (re)tested values in identified patterns to generate their own cases in my study. Further, identifying patterns across cases can play a pivotal role in the production of proofs. For example, Ozgur et al. (2017) found that, in their study, successful provers searched for a pattern in a set of learner-generated examples. As I explained in Section 5.1.2, identification of process-result pattern generalities often appeared particularly pivotal in convincing learners that they had enough structural (relational) information to commence the writing of their own cases for propositions similar to the given propositions. Hence, teachers wishing to adapt the tasks may find that arguments with a co-varying process-result generality can be of particular support to learners yet to encounter formal proof argumentation.

Although I derived the GOLDEN framework in the context of my proof variation tasks, the six main actions can potentially be used to describe learners' actions on other mathematical tasks (as suggested by my broad descriptions of each main action in Section 2.3). The reason for the framework's broader applicability is that proving activities can find ubiquitous expression in the doing of mathematics generally. Stylianides and Stylianides (2006b, p. 211) made a similar point, 'proof can promote in students activities similar to those promoted in the discipline of mathematics'. For example, across all of the learners in my study, Generalising was the most common main action (finding 5), and it is widely accepted that generalising is central to the teaching and learning of mathematics (see, e.g., Ellis et al., 2024; Mason et al., 2010). Kieran (2018, p. 89) made a related point about Deconstructing activities by stating that such actions are 'central to the content area of number and numerical operations'. I next revisit my seven findings and summarise their implications for advancing research and pedagogy.

Finding 1: *Learners' attention to structure can be described in terms of the actions in the GOLDEN framework*. The GOLDEN framework augments the current research on proof pedagogy with a tool for conceptualising, analysing, and describing learners' proving actions. While I devised the framework in the context of consecutive-number proof variation tasks, its composite actions (particularly its main actions) are relevant to other proving tasks, and tasks in other mathematical areas. The actions described in the framework can help to unify researchers' ranging use of the term 'structure' (Venkat et al., 2019). The actions also provide a language that can make visible to teachers the mathematical moves that their learners are making. Hence, awareness of the GOLDEN actions can put teachers in a better position to support learners' proving abilities.

Finding 2: *After writing a case for a proposition that was similar to the given propositions, learners attended to more complex structure and shifted to more precise and sophisticated reasoning*. With part of the variation tasks, I asked learners to reason about the same set of given argument structures immediately before and after asking them to write a case similar to those in the given arguments. The process of writing a new case increased the number of structural relationships that learners incorporated into their reasoning, or improved the precision of their initial reasoning. Thus, asking learners to construct a case that is conceptually similar to a given set of consecutive-number argument variations can potentially improve their 'structure sense' (Hoch & Dreyfus, 2004).

Finding 3: *When learners identify a dimension of difference across given argument variations, it can prompt them to justify and communicate the reason for their identified difference*. Convincing others of irrefutability is a key function of a proof (Schifter, 2009, p. 80); and verbal justifications that accompany a written proof can supplement its 'explanatory power' (Stylianides et al., 2016). Relatedly, asking learners to identify dimensions of difference across given arguments variations can trigger learners' 'unprompted' verbal justifications, which can be seen as early preparation for more sophisticated verbal explanations.

Finding 4: *Learners' sequences of actions involved working with increasingly complex structure and progressively identifying relationships*. All of the learners were able to make progress on the tasks by incrementally attending to more complex structure. However, three sequences of actions led learners to attend to increasingly complex structure more efficiently than other sequences of actions. The three sequences involved shifts from: (1) an Organising action, or sequence of Organising actions, to Generalising; (2) a process-only generalisation to a process-result generalisation; and (3) Generalising to Extending. Taken together, these shifts captured the broad trajectory of actions required for learners to generate their own cases. Therefore, creating tasks that intentionally encourage learners to perform these shifts could support their proof production, e.g., through structure mapping, reasoning recursively, and induction (Stylianides et al., 2016).

Finding 5: *Generalising was the most common main action*. Indeed, Generalising featured in all three salient shifts that I just described in relation to finding 4. This was perhaps unsurprising because generalising processes are commonly seen as central to mathematics pedagogy (see, e.g., Ellis et al., 2024; Mason et al., 2010). In the context of the tasks in my study, learners used different 'levels' or complexities of generalities to: explain the truth of the propositions, predict an end result when adjusting a value, and write their own cases. The ranging usefulness of learners' Generalising actions provided strong evidence that tasks featuring consecutive-number proposition and argument variations can provide a context for performing a range of valuable activities associated with proving.

Finding 6: *Transferred actions aided learners' progress through a task, regardless of if a required end result was achieved*. My analysis of learners' task responses included descriptions of their transferred (repeated) actions across each respective domain-pair of variation tasks as well as across domains. This focus yielded several salient patterns. Learners who transferred sub-actions in one domain did not do so in the other domain; this underscored the challenge of identifying procedural generalities that span domains. In Year 8, on the number task, successful completion of the transfer task was limited only to learners who demonstrated transfer. This same pattern was present across Year 10's responses to the geometry task. Further, across both year groups and domains, it was almost always the case that only learners who demonstrated transfer successfully completed the transfer tasks. These patterns implied that learners in both year groups found the application of some of their previously performed actions useful on the transfer tasks. This implication was particularly evident in the number domain, where every learner who demonstrated transfer began the transfer task with a sub-action that they had previously performed on the initial task; thus signalling supporting evidence for the perceived value of learning transfer (see, e.g., Bruner, 1996, p. 129; Carraher & Schliemann, 2002, p. 1). Transferred actions were predominately forms of Generalising; however, across both year groups and domains, all six main actions in the GOLDEN framework were transferred. As finding 6 suggests, all transferred actions in some way aided learners' progress through a task, regardless of if a required end result was achieved (see Section 5.3 for specific details of the roles and affordances of learners' transferred actions).

For educators, the foresaid observations highlight the importance of: (1) designing sequences of tasks that encourage transferable actions, and (2) identifying and emphasising to learners transferred actions when and where they occur.

Finding 7: *Proof variation tasks can be used to encourage learners to perform actions that can support their ability to prove*. The tasks in both domains featured sequences of structurally similar (consecutive-number) propositions and arguments with systematic variations. Each sequence of variations afforded immediate comparison and sense-making, e.g., by identifying patterns such as consistent co-varying relationships across the given argument structures. The combined demands of the prompts elicited: conjecturing, empirical testing, reasoning, case-generation, and justifying; all of which are desirable and well-known processes of proving (see, e.g. Ball et al., 2002; Harel & Sowder, 2007; Mariotti, 2006). As shown in Appendix E (the GOLDEN framework), across both year groups and domains, 18 different proving actions, i.e., three sub-actions in each of six main actions, were performed in response to the variation tasks that I designed. Each of the seven findings reflected the key purposes of my study's two main contributions; i.e., the utility of the GOLDEN framework as an analytical tool, and the role of consecutive-number proof variation tasks as a resource for prompting proving actions.

My study suggested several avenues for future research. These avenues could involve exploring the GOLDEN framework's analytical, descriptive, and diagnostic potential in other learning contexts using the same, or similar, tasks as I used in my study. Other contexts might include different learner profiles (e.g., primary school learners or Y7 learners) and tasks that feature given variations of alternative arguments (e.g., involving number parity or the geometry of triangles). Investigating how nuances of GOLDEN sub-actions emerge across different age groups and task types can potentially refine understanding of the kinds of actions, and sequences of actions, that can support learners' development in the area of proving. Further research could also involve longitudinal studies that examine how reference to, and focused use of, the GOLDEN actions influence learners' capacity to reason and construct arguments and proofs over time. For example, over the course of a taught curriculum unit, school term, or school year.

Pedagogically, I think that proof variation tasks are a promising resource for introducing learners to argumentation and proving. In particular, variation tasks with propositions and arguments that both feature consecutive numbers, and hence afford learners' identification of co-varying process-result generalities. The highly scaffolded and structurally close design of the given arguments (particularly evident on the initial variation tasks) can exert a relatively light cognitive load, while affording opportunities to attend to structure by comparing (Kullberg et al., 2017), reasoning relationally (Skemp, 1976), and adapting (Prestage & Perks, 2001). Teachers could combine use of the GOLDEN framework and the variation tasks to draw attention to, and concretise, learners' proving actions; i.e., make visible what may have been previously invisible. Teachers can also use the variation tasks to foster classroom conversations about the truth of propositions, and root discussions about generating further cases for arguments in the language of the GOLDEN actions. To give learners more autotomy over a task, a practice linked with promoting engagement (Skilling et al., 2016), they can be asked to choose ways to further vary the sequences of propositions that accompany the given arguments. In doing so, learners' attention can be drawn to 'dimensions of possible variation' and 'ranges of permissible change'

(Watson & Mason, 2005). To promote groupwork, learners could then ask each other to write cases for each other's self-varied propositions. In such groupwork situations, with arguments that are essentially proofs represented algebraically (as in the number initial variation task), learners could empirically test each other arguments. Further, the framework can also inform the design of exercises that promote use of the GOLDEN actions in mathematical contexts other than proof, thus aiding curriculum integration.

When paired with the variation tasks, the GOLDEN framework could be used as a formative assessment instrument to help teachers to identify which proving actions are present, emerging, and need introducing. For greater cognitive challenge, given argument structures can be adapted to include fewer similarities so that generalities are more difficult to identify. Argument variations can also be given that involve proofs of theorems rooted in more sophisticated axioms. Argument variations that are conceptually similar (consecutive-number) numerical examples can potentially aid learners' inductive transition to generalised algebraic representation; particularly when learners are prompted to consider what it is in the given structures that explains the 'why', i.e., the truth of the accompanying propositions. Transitioning from numerical examples to algebraic representation can be seen as a form of *conceptual* transfer (Lobato et al., 2012) as well as *action* transfer. As I discussed in Section 5.5, the tasks in both domains involved algebraic symbolism which some learners may need initial support with. Such support is likely to involve introducing learners to, or recapping, relevant conventions of algebraic representation. For example, representing the doubling of an integer, and seeing as 'even', the term $2n$; and representing the addition of two consecutive integers, and seeing as 'odd', the expression $2n + 1$.

6.8 Limitations

The function of proof has been variously conceived, e.g., as explanatory (Hanna, 2018; Stylianides et al., 2016, p. 22) and as the removal of doubt about the truth of an assertion (Harel & Sowder, 2007, p. 808). In my study, I treat the function of proof as pedagogical; i.e., as a topic of study rather than a demonstrative tool devised and used by, say, professional mathematicians. My treatment of proof as pedagogical might be construed as promoting a narrow framing; e.g., that a learner's ability to prove can improve their performance on competency tests, and that proofs exist in only a limited range of formats. I implicitly took a pedagogical position on the function of proof due to the school-based setting and learners' unfamiliarity with argumentation. Nonetheless, I have described at relevant points how learners' task responses had implications for developing their appreciation of the function of proof as explanatory.

As I mentioned earlier, I aimed to minimise my subjective descriptions of learners' actions by asking two of my teaching colleagues to also describe the actions that the learners performed. However, my colleagues' interpretations were too vague to be contributory. While my colleagues' limited interpretations of learners' actions reflected the reported gap in teachers' familiarity with processes of proof (see, e.g., Harel & Sowder, 2007, p. 836-837; Stylianides & Stylianides, 2018, p. 110), this also meant that I did not perform an inter-rater reliability check. To mitigate my position as sole interpreter of learners' actions, I used a grounded observation- and literature-based analytical approach (Miles & Huberman, 1994) that left little scope for researcher bias. To do this, as I explained in Section 2.3, I moved with the deliberate reflexivity back and forth between learners' task responses and the action-related literature. In doing so, I repeatedly asked myself '*how well does the literature explain what I am observing?*' I nonetheless suspected that some, if not all, of the learners would generalise and specialise (i.e., act on local structure) because these two actions are particularly prevalent in the literature. Indeed, as I explained in Section 3.3, few researchers can claim to enter a field unencumbered by notions of the phenomena they seek to understand (Flinders & Mills, 1993, p. xi; Y. Gu, 2014, p. 79). However, I held no personal preconceptions of what might characterise any of the learners' sub-actions.

I developed the framework from the responses of six learners in two year groups, and more work is needed to determine how my findings might map to larger groups of learners, particularly from other year groups. (As I explained in Section 3.5, my sample was reduced due to the pandemic-related forced closure of the participant school). Although a major strength of the GOLDEN framework is that, individually or collectively, the main actions can be used to describe learners' actions on other mathematical tasks, such broader applicability is beyond the scope of this study.

6.9 Final remarks

Proving is fundamental to the learning of mathematics because the ability to prove involves performing a range of actions that underpin the learning of the subject as a whole. However, the diverse nature of proving tasks is a major factor of proof being 'hard-to-teach and hard-to-learn' (Knuth et al., 2009, p. 153; Küchemann & Hoyles, 2009, p. 171; Stylianides & Stylianides, 2017, p. 121) because no overall pedagogical approach can be adopted. With this study I introduced two resources (proof variation tasks and the GOLDEN framework) that contribute to alleviating some of the enduring challenges of proof pedagogy. I have generated evidence to show that learners who are yet to experience proof argumentation can perform a wide range of proving actions on variation tasks in the domains of number and geometry. The variation tasks featured two key elements: (1) sequences of structurally similar (consecutive-number) propositions and arguments, and (2) prompts with which I asked learners to: discern (in)variance, *show that* and *explain why* given propositions must be true, and write arguments similar to given arguments. From the collective responses to the tasks, I formulated the GOLDEN framework for describing learners' proving actions. I then used the framework to characterise learners' actions on the prompts within and across the two domains. These characterisations yielded seven findings that demonstrated the utility of my consecutive-number proof variation tasks and the GOLDEN framework.

Learners' absence of transfer across domains reflected a substantial problem in the pedagogy of proof: *identifying cross-domain procedural generalities*. Yet, across each year group separately, learners performed all of the main actions in the GOLDEN framework on the number task and the geometry task, respectively. Further, all of the learners individually performed at least half of the main actions in each respective domain. These consistencies exemplified the use of the GOLDEN framework as a tool for analysing, articulating, and assessing learners' cross-domain proving actions; indeed, learners' mathematical actions generally.

References

Askew, M. (2015). *Transforming primary mathematics: Understanding classrooms tasks, tools and talk* (2nd ed.). Routledge.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children.* (pp. 216–235). Hodder & Stoughton.

Ball, D. L., Hoyles, C., Jahnke, H. N., & Movshovitz-Hadar, N. (2002). The teaching of proof. In L. I. Tatsien (Ed.), *Proceedings of the International Congress of Mathematicians: Vol. III* (pp. 907–920). Higher Education Press.

Barnes, B., & Bloor, D. (1982). Relativism, rationalism and the sociology of knowledge. In M. Hollis & S. Lukes (Eds.), *Rationality and Relativism* (pp. 21–47). Basil Blackwell.

Bassey, M. (1999). *Case study research in educational settings*. Open University Press.

Bentley, J., & Stylianides, G. J. (2017). Drawing inferences from learners' examples and questions to inform task design and develop learners' spatial knowledge. *Journal of Mathematical Behavior*, *47*, 35–53. https://doi.org/10.1016/j.jmathb.2017.06.001

Bills, C., Bills, L., Watson, A., & Mason, J. (2004). *Thinkers: A collection of activities to provoke mathematical thinking*. Association of Teachers of Mathematics.

Bills, L., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). Exemplification in mathematics education. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 126–154). PME.

Bills, L., & Rowland, T. (1999). Examples, generalisation and proof. *Advances in Mathematics Education*, *1*(1), 103–116. https://doi.org/10.1080/14794809909461549

Bleiler-Baxter, S. K., & Pair, J. D. (2017). Engaging students in roles of proof. *Journal of Mathematical Behavior*, *47*, 16–34. https://doi.org/10.1016/j.jmathb.2017.05.005

Boero, P., Garuti, R., & Lemut, E. (1999). About the generation of conditionality of statements and its links with proving. *Proceedings of Th 23rd PME Conference*, *2*, 137–144.

Boero, P., Garuti, R., Mariotti M. A. (1996). Some dynamic mental processes underlying producing and proving conjectures. In L. Puig, & A. Gutierrez (Eds.), *Proceedings of the Twentieth Conference of the International Group for the Psychology of Mathematics Education PME-XX* (Vol. 2, pp. 121–128). PME.

Bruner, J. S. (1996). *The culture of education*. Harvard University Press.

Bryman, A. (2006). Paradigm Peace and the Implications for Quality. *International Journal of Social Research Methodology*, *9*(2), 111–126. https://doi.org/10.1080/13645570600595280

Buchbinder, O. (2018). Systematic exploration of examples as proof: Analysis with four theorectical frameworks. In G. Harel & A. J. Stylianides (Eds.), *Advances in Mathematics Education Research on Proof and Proving: An International Perspective* (pp. 253–268). Springer.

Buchbinder, O., & Zaslavsky, O. (2011). Is this a coincidence? The role of examples in fostering a need for proof. *ZDM - International Journal on Mathematics Education*, *43*(2), 269–281. https://doi.org/10.1007/s11858-011-0324-7

Cadwallader-Olsker, T. (2011). What do we mean by mathematical proof? *Journal of Humanistic Mathematics*, *1*(1), 33–60. https://doi.org/10.5642/jhummath.201101.04

Cai, J., & Nie, B. (2007). Problem solving in Chinese mathematics education: Research and practice. *ZDM Mathematics Education*, *39*, 459–475.

Callingham, R. (2005). A whole-school approach to developing mental computation strategies. In H. Chick & J. Vincent (Eds.), *Proceedings of the 29th conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 201–208). University of Melbourne.

Capewell, D., Kranat, J., & Mullarkey, P. (2004). *Framework maths. Revision: National curriculum*. Oxford University Press.

Carraher, D. W., & Schliemann, A. (2002). The transfer dilemma. *Journal of the Learning Sciences*, *11*(1), 1–24. https://doi.org/10.1207/S15327809JLS1101_1

Cartiglia, M., Furinghetti, F., & Paola, D. (2004). Patterns of reasoning in classroom. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for Psychology of Mathematics Education (PME).* (Vol. 2, Issue July, pp. 287–294). PME.

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative analysis. Thousand*. SAGE.

Ching, B. H., & Nunes, T. (2017). Children ' s understanding of the commutativity and complement principles: A latent profile analysis. *Learning and Instruction*, *47*, 65–79. https://doi.org/10.1016/j.learninstruc.2016.10.008

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547–589). Lawrence Erlbaum Associates.

Clements, D. H., & Sarama, J. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. Routledge.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *The Journal of Mathematical Behavior*, *15*(2), 167-192. doi:10.1016/S0732-3123(96)90015-2.

Cox, Brian. D. (1997). The rediscovery of the active learner in adaptive contexts: A developmental-historical analysis of transfer of training. *Educational Psychologist*, *32*(1), 41–55.

Creswell, J. W. (2003). *Research design: Qualitative, quantitative, and mixed methods approaches* (2nd edition). Sage.

Cuoco, A., Paul Goldenberg, E., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The Journal of Mathematical Behavior*, *15*(4), 375–402. https://doi.org/10.1016/S0732-3123(96)90023-1

Dahlberg, R. P., & Housman, D. L. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics*, *33*(3), 283–299.

de Villiers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras*, *24*, 17–24.

Dougherty, B. J., Bryant, D. P., Bryant, B. R., Darrough, R. L., & Pfannenstiel, K. H. (2015). Developing concepts and generalizations to build algebraic thinking: The reversibility, flexibility, and generalization approach. *Intervention in School and Clinic*, *50*(5), 273–281. https://doi.org/10.1177/1053451214560892

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Kluwer Academic Publishers.

Duval, R. (2007). Cognitive functioning and the understanding of mathematical processes of proof. In P. Boero (Ed.), *Theorems in school: From history, epistemology and cognition to classroom practice* (pp. 137–161). Sense Publishers.

Education Committee of the European Mathematical Society. (2011). Do theorems admit exceptions? Solid findings in mathematics education on empirical proof schemes. *EMS Newsletter*, *82*, 50–53.

Ellis, A. B. (2007). A taxonomy for categorizing generalizations: Generalizing actions and reflection generalizations. *Journal of the Learning Sciences*, *16*(2), 221–262. https://doi.org/10.1080/10508400701193705

Ellis, A. B., Waswa, A., Tasova, H. I., Hamilton, M., Moore, K. C., & Çelik, A. (2024). Classroom Supports for generalizing. *Journal for Research in Mathematics Education*, *55*(1), 7–30. https://doi.org/10.5951/jresematheduc-2022-0140

Engle, R. A. (2006). Framing interactions to foster generative learning: A situative explanation of transfer in a community of learners classroom. *Journal of the Learning Sciences*, *15*(4), 451–498. https://doi.org/10.1207/s15327809jls1504_2

Farrimond, H. (2016). The ethics of research. In D. Wyse, N. Selwyn, E. Smith, & L. E. Suter (Eds.), *The BERA/SAGE Handbook of Educational Research* (pp. 72–89). SAGE Publications Ltd.

Flinders, D. J., & Mills, G. E. (Eds.). (1993). *Theory and concepts in qualitative research: Perceptions from the field*. Teachers' College Press.

Foster, C. (2019). The fundamental problem with teaching problem solving. *Mathematics Teaching*, *265*, 8–10.

Gick, M. L., & Holyoak, K. J. (1983). Schema induction and analogical transfer. *Cognitive Psychology*, *15*(1), 1–38. https://doi.org/10.1016/0010-0285(83)90002-6

Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: Aims, rationales, techniques. *For the Learning of Mathematics*, *1*(3), 4–11.

Greeno, J. G., Smith, D. R., & Moore, J. L. (1993). Transfer of situated learning. In D. K. Detterman & R. J. Sternberg (Eds.), *Transfer on trial: intelligence, cognition, and instruction* (pp. 99–167). Ablex.

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309–347). World Scientific.

Gu, Y. (2014). To code or not to code: Dilemmas in analysing think-aloud protocols in learning strategies research. *System*, *43*(1), 74–81. https://doi.org/10.1016/j.system.2013.12.011

Hanna, G. (2018). Reflections on proof as explanation. In A. J. Stylianides & G. Harel (Eds.), *Advances in mathematics education research on proof and proving: An international perspective* (pp. 3–18). Springer International Publishing AG.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 185–212). Greenwood Publishing Group, Incorporated.

Harel, G. (2013). Intellectual need. In K. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 119–151). Springer.

Harel, G., & Fuller, E. (2009). Current contributions toward comprehensive perspectives on the learning and teaching proof. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proofs across the grades* (pp. 355–385).

Harel, G., & Soto, O. (2017). Structural reasoning. In *International Journal of Research in Undergraduate Mathematics Education* (Vol. 3, Issue 1, pp. 225–242). https://doi.org/10.1007/s40753-016-0041-2

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in college mathematics education III* (pp. 234–283). American Mathematical Society.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Information Age.

Hazzan, O., & Zazkis, R. (2005). Reducing abstraction: The case of school mathematics. *Educational Studies in Mathematics*, *58*(1), 101–119. https://doi.org/10.1007/s10649-005-3335-x

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*(4), 396–428. http://www.jstor.org/stable/749651

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, *28*(5), 524–549.

Herizal, H., Suhendra, S., & Nurlaelah, E. (2019). The ability of senior high school students in comprehending mathematical proofs. *Journal of Physics: Conference Series*, *1157*(2), online so give doi. https://doi.org/10.1088/1742-6596/1157/2/022123

Heylings, M. R., Appleton, M., Plass, C., Kranat, J., Rayner, D., & Tully, N. (2010). Oxford GCSE maths for OCR. Specification A: Higher. In *GCSE maths for OCR. Specification A higher*. Oxford University Press.

Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, *3*, 49–56.

Ingram, J. (2014). Shifting attention. *For the Learning of Mathematics*, *34*(3), 19–24.

Ingram, J., & Elliott, V. (2019). *Research methods for classroom discourse*. Bloomsbury Academic.

Jefferson, G. (2004). Glossary of transcript symbols with an introduction. In G. H. Lerner (Ed.), *Conversation Analysis: Studies from the first generation* (Vol. 125, pp. 13–31). John Benjamins.

Johnston-Wilder, S., & Lee, C. (2010). Mathematical Resilience. *Mathematics Teaching*, *218*, 38.

Jurow, A. S. (2004). Generalizing in interaction: Middle school mathematics students making mathematical generalizations in a population-modeling project. *Mind, Culture, and Activity*, *11*(4), 279–300. https://doi.org/10.1207/s15327884mca1104_4

Kanellos, I., Nardi, E., & Biza, I. (2018). Proof schemes combined: mapping secondary students' multi-faceted and evolving first encounters with mathematical proof. *Mathematical Thinking and Learning*, *20*(4), 277–294. https://doi.org/10.1080/10986065.2018.1509420

Kaput, J. J., Carraher, D. W., & Blanton, M. L. (Eds.). (2008). *Algebra in the early grades*. Lawrence Erlbaum Associates/National Council of Teachers of Mathematics.

Kemmis, S., McTaggart, R., & Nixon, R. (2014). *The action research planner: Doing critical participatory action research*. Springer.

Kieran, C. (2018). Seeking, using, and expressing structure in numbers and numerical operations: A fundamental path to developing early algebraic thinking. In C. Kieran (Ed.), *Teaching and Learning Algebraic Thinking with 5- to 12-Year-Olds: The Global Evolution of an Emerging Field of Research and Practice* (pp. 79–105). Springer.

Kieran, C., Doorman, M., & Ohtani, M. (2015). Frameworks and principles for task design. In A. Watson & M. Ohtani (Eds.), *Task Design In Mathematics Education, An ICMI Study 22* (pp. 47–81). Springer International Publishing. https://doi.org/10.1007/978-3-319-09629-2_2

Knuth, E. J. (2002). Teachers' conceptions of proofs in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, *5*(1), 61–88. https://doi.org/10.1023/A:1013838713648

Knuth, E. J., Choppin, J. M., & Bieda, K. N. (2009). Middle school students' production of mathematical justifications. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and Learning Proofs across the Grades* (pp. 153–170). Routledge.

Koichu, B., Zaslavsky, O., & Dolev, L. (2016). Effects of variations in task design on mathematics teachers' learning experiences: a case of a sorting task. *Journal of Mathematics Teacher Education*, *19*(4), 349–370. https://doi.org/10.1007/s10857-015-9302-2

Komatsu, K., Jones, K., Ikeda, T., & Narazaki, A. (2017). Proof validation and modification in secondary school geometry. *Journal of Mathematical Behavior*, *47*, 1–15. https://doi.org/10.1016/j.jmathb.2017.05.002

Komatsu, K., Tsujiyama, Y., Sakamaki, A., & Koike, N. (2014). Proof problems with diagrams: An opportunity for experiencing proofs and refutations. *For the Learning of Mathematics*, *34*(1), 36–42.

Krutetskii, V. A. (1976). (J. Teller, Trans.) In: J. Kilpatrick, & I. Wirszup (Eds.), *The psychology of mathematical abilities in school children*. University of Chicago Press.

Küchemann, D. (2008). *Looking for structure: A report of the proof materials project*. Dexter Graphics.

Küchemann, D. (2010). Using patterns generically to see structure. *Pedagogies: An International Journal*, *5*(3), 233–250. https://doi.org/10.1080/1554480X.2010.486147

Küchemann, D., & Hoyles, C. (2005). Pupils' awareness of structure on two number questions. In M. Bosch (Ed.), *Proceedings of the IV congress of the European society for research in mathematics education (CERME 4)* (pp. 438–447).

Küchemann, D., & Hoyles, C. (2009). From empirical to structural reasoning in mathematics: Tracking changes over time. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and Learning Proofs across the Grades* (pp. 171–191). Routledge.

Kullberg, A., Runesson Kempe, U., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM - Mathematics Education*, *49*(4), 559–569. https://doi.org/10.1007/s11858-017-0858-4

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge University Press.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge University Press.

Lee, K. S. (2016). Students' proof schemes for mathematical proving and disproving of propositions. *Journal of Mathematical Behavior*, *41*, 26–44. https://doi.org/10.1016/j.jmathb.2015.11.005

Leighton, J. P. (2017). *Using think-aloud interviews and cognitive labs in educational research*. New York: Oxford University Press.

Ling, L., & Ling, P. (2020). *Emerging methods and paradigms in scholarship and education research*. IGI Global, Information Science Reference. https://doi.org/10.4018/978-1-7998-1001-8

Lobato, J. (2003). How design experiments can inform a rethinking of transfer and vice versa. *Educational Researcher*, *32*(1), 17–20. https://doi.org/10.3102/0013189X032001017

Lobato, J. (2006). Alternative perspectives on the transfer of learning: History, issues, and challenges for future research. *Journal of the Learning Sciences*, *15*(4), 431–449. https://doi.org/10.1207/s15327809jls1504_1

Lobato, J. (2008). When students don't apply the knowledge you think they have, rethink your assumptions about transfer. In M. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics* (pp. 287–302).

Lobato, J. (2012). The actor-oriented transfer perspective and its contributions to educational research and practice. *Educational Psychologist*, *47*(3), 232–247. https://doi.org/10.1080/00461520.2012.693353

Lobato, J., Hohensee, C., & Diamond, J. (2014). What can we learn by comparing students' diagram-construction processes with the mathematical conceptions inferred from their explanations with completed diagrams? *Mathematics Education Research Journal*, *26*(3), 607–634. https://doi.org/10.1007/s13394-013-0106-3

Lobato, J., Hohensee, C., & Rhodehamel, B. (2013). Students' mathematical noticing. *Journal for Research in Mathematics Education*, *44*(5), 809–850.

Lobato, J., Rhodehamel, B., & Hohensee, C. (2012). "Noticing" as an alternative transfer of learning process. *Journal of the Learning Sciences*, *21*(3), 433–482. https://doi.org/10.1080/10508406.2012.682189

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behavior*, *21*(1), 87–116. https://doi.org/10.1016/S0732-3123(02)00105-0

Maciejewski, W., & Star, J. R. (2019). Justifications for choices made in procedures. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-019-09886-7

Maher, C. A. (2009). Children's reasoning: Discovering the idea of mathematical proof. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 120–132). Routledge.

Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: past, present and future* (pp. 173–204). Sense Publishers.

Mariotti, M. A., & Pedemonte, B. (2019). Intuition and proof in the solution of conjecturing problems'. *ZDM - Mathematics Education*, *51*(5), 759–777. https://doi.org/10.1007/s11858-019-01059-3

Marton, F. (1986). Phenomenography: A research approach to investigating different understandings of reality. *Journal of Thought*, *21*(3), 28.

Marton, F. (2006). Sameness and difference in transfer. *Journal of the Learning Sciences*, *15*(4), 499–535. https://doi.org/10.1207/s15327809jls1504_3

Marton, F. (2015). *Necessary conditions of learning*. Routledge.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Lawrence Erlbaum Associates.

Marton, F., & Ko, P. Y. (2004). Variation and the secret of the virtuoso. In F. Ton & A. B. M. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 43–62). L. Erlbaum Associates.

Marton, F., & Neuman, D. (1996). Phenomenography and children's experience of division. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 315–334). Lawrence Erlbaum Associates.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *Journal of the Learning Sciences*, *15*(2), 193–220. https://doi.org/10.1207/s15327809jls1502_2

Marton, F., Runesson, U., & Tsui, A. B. M. (2004). The space of learning. In F. Marton & A. B. M. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 3–40). L. Erlbaum Associates.

Marton, F., & Trigwell, K. (2000). Variatio est mater studiorum. *Higher Education Research & Development*, *19*(3), 381–395. https://doi.org/10.1080/07294360020021455

Marvasti, A., & Freie, C. (2016). Research interviews. In D. Wyse, N. Selwyn, E. Smith, & L. E. Suter (Eds.), *The BERA/SAGE Handbook of Educational Research* (1st ed., pp. 624–639). SAGE Publications Ltd.

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education*, *1*(3), 243–267. https://doi.org/10.1023/A:1009973717476

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge Falmer.

Mason, J. (2003). Structure of attention in the learning of mathematics. In J. Novotná (Ed.), *International Symposium on Elementary Mathematics Teaching* (pp. 9–16). Charles University.

Mason, J. (2017a). Issues in variation theory and how it could inform pedagogical choices. In R. Huang & Y. Li (Eds.), *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 407–438). Sense Publishers. https://doi.org/10.1007/978-94-6300-782-5_23

Mason, J. (2017b). Overcoming the algebra barrier: Being particular about the general, and generally looking beyond the particular, in homage to Mary Boole. In S. Stewart (Ed.), *And the Rest is Just Algebra* (pp. 97–117). Springer International Publishing. https://doi.org/10.1007/978-3-319-45053-7_6

Mason, J. (2018). Structuring structural awareness: A commentary on Chapter 13. In M. G. Bartolini Bussi & X. H. Sun (Eds.), *Building the Foundation: Whole Numbers in the Primary Grades: The 23rd ICMI Study* (pp. 325–340). Springer International Publishing. https://doi.org/10.1007/978-3-319-63555-2_14

Mason, J. (2021). How the theme of 'doing and undoing' applied to the action of exchange reveals overlooked core ideas in school mathematics. *MDPI*, *9*(13), Article 1530. https://doi.org/http://dx.doi.org/10.3390/math9131530

Mason, J. (2023). The Intimate Interplay Between Developing Teaching and Exploring Mathematics Through Reflecting in, on and for Teaching. In S. Stewart (Ed.), *Mathematicians' Reflections on Teaching: A Symbiosis with Mathematics Education Theories* (pp. 173–198). Springer International Publishing. https://doi.org/10.1007/978-3-031-34295-0_10

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd ed.). Pearson.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, *15*(3), 277–289. https://doi.org/10.1007/BF00312078

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal*, *21*(2), 10–32. https://doi.org/10.1007/BF03217543

Matsuda, N., & VanLehn, K. (2005). Advanced Geometry Tutor: An intelligent tutor that teaches proof-writing with construction. In C.-K. Looi, G. I. McCalla, B. Bredeweg, & J. Breuker (Eds.), *Proceedings ofthe 12th International Conference on Artificial Intelligence in Education* (pp. 443–450). Amsterdam: IOS Press.

McKeough, A., Lupart, J., & Marini, A. (Eds.). (1995). *Teaching for transfer: Fostering generalization in learning*. Routledge. https://doi.org/10.4324/9781315044736

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis (2nd ed.)*. Sage Publications, Inc.

Miller, K., Willson, S., Chepp, V., & Padilla, J.-L. (Eds.). (2014). *Cognitive interviewing methodology*. John Wiley & Sons, Inc.

Miragliotta, E. (2022). Geometric prediction: A framework to gain insight into solvers' geometrical reasoning. *The Journal of Mathematical Behavior*, *65*, 100927. https://doi.org/10.1016/j.jmathb.2021.100927

Mok, I. A. C., Cai, J., & Fung, A. F. (2008). Missing learning opportunities in classroom instruction: Evidence from an analysis of a well-structured lesson on comparing fractions. *The Mathematics Educator*, *11*, 111–126.

Mulligan, J., Mitchelmore, M., Marston, J., Highfield, K., & Kemp, C. (2008). Promoting mathematical pattern and structure in the first year of schooling: An intervention study. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepulveda (Eds.), *Proceedings of the Joint Meeting of PME 32 and PME-NA XXX* (Vol. 4, pp. 1–8). Cinvestav-UMSNH.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common Core State Standards for Mathematics. In *Common core state standards for mathematics*. Authors.

NCETM. (2019). *Five big ideas in teaching for mastery*. https://www.ncetm.org.uk/resources/50042

Nemirovsky, R. (2011). Episodic feelings and transfer of learning. *Journal of the Learning Sciences*, *20*(2), 308–337. https://doi.org/10.1080/10508406.2011.528316

Nokes, T. J. (2009). Mechanisms of knowledge transfer. *Thinking & Reasoning*, *15*(1), 1–36. https://doi.org/10.1080/13546780802490186

Novick, L. R. (1988). Analogical Transfer, Problem Similarity, and Expertise. *Journal of Experimental Psychology. Learning, Memory, and Cognition*, *14*(3), 510–520. https://doi.org/10.1037/0278-7393.14.3.510

Oswald, F. L., Hambrick, D. Z., & Jones, L. A. (2007). Keeping all the plates spinning: Understanding and predicting multitasking performance. In D. Jonassen (Ed.), *Learning to solve complex scientific problems* (pp. 77–96). Routledge.

Ozgur, Z., Ellis, A. B., Vinsonhaler, R., Dogan, M. F., & Knuth, E. J. (2017). From examples to proof: Purposes, strategies, and affordances of example use. *The Journal of Mathematical Behavior*, in press. https://doi.org/10.1016/J.JMATHB.2017.03.004

Paas, F. G. W. C., & Van Merrienboer, J. J. G. (1994). Variability of worked examples and transfer of geometrical problem-solving skills: A cognitive-load approach. *Journal of Educational Psychology*, *86*(1), 122–133. https://doi.org/10.1037/0022-0663.86.1.122

Palatnik, A., & Dreyfus, T. (2018). Students' reasons for introducing auxiliary lines in proving situations. *Journal of Mathematical Behavior*. https://doi.org/10.1016/j.jmathb.2018.10.004

Pang, M. F., Bao, J., & Ki, W. W. (2017). 'Bianshi' and the variation theory of learning: Illustrating two frameworks of variation and invariance in the teaching of mathematics. In R. Huang & Y. Li (Eds.), *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 43–67). Sense Publishers.

Pang, M. F., & Marton, F. (2007). On the paradox of pedagogy: The relative contribution of teachers and learners to learning. *Iskolakultúra*, *1*(1), 1–29.

Park, J. H., & Kim, D. W. (2017). How can students generalize examples? Focusing on the generalizing geometric properties. *Eurasia Journal of Mathematics, Science and Technology Education*, *13*(7), 3771–3800. https://doi.org/10.12973/eurasia.2017.00758a

Pedemonte, B. (2008). Argumentation and algebraic proof. *Zentralblatt fur Didaktik der Mathematik*, *40*(3), 385–400.

Peng, A., Li, J., Nie, B., & Li, Y. (2017). Characteristics of teaching mathematical problem solving in China: Analysis of a Lesson from the perspective of variation. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation: Confucian heritage meets western theories* (pp. 111–125). Sense Publishers.

Piaget, J. (1970). Piaget's theory. In P. H. Mussen (Ed.), *Manual of child psychology* (pp. 703–732). Wiley.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics*, *26*(2/3), 165–190. https://doi.org/10.1007/BF01273662

Prestage, S., & Perks, P. (2001). Adapting and Extending Secondary Mathematics Activities: New Tasks for Old. In *Adapting and Extending Secondary Mathematics Activities: New Tasks for Old* (1st ed.). Routledge. https://doi.org/10.4324/9780203462386

Rebello, N. S., Cui, L., Bennett, A. G., Zollman, D. A., & Ozimek, D. J. (2007). Transfer of learning in problem solving in the context of mathematics and physics. In *Learning to solve complex scientific problems* (pp. 223–246).

Reed, S. K. (1993). A schema-based theory of transfer. In D. K. Detterman & R. J. Sternberg (Eds.), *Transfer on trial: Intelligence, cognition, and instruction* (pp. 39–67). Ablex.

Reed, S. K., Ernst, G. W., & Banerji, R. (1974). The role of analogy in transfer between similar problem states. *Cognitive Psychology*, *6*(3), 436–450. https://doi.org/10.1016/0010-0285(74)90020-6

Reid, D. A. (2002). Conjectures and refutations in grade 5 mathematics, *Journal for Research in Mathematics Education, 33*(1), 5–29. https://doi.org/10.2307/749867

Richards, L. (2015). *Handling qualitative data: A practical guide (3rd ed.)*. Sage.

Romberg, T. A. (2016). Classroom instruction that fosters mathematical thinking and problem solving: Connections between theory and practice. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving*. Routledge.

Royer, J. M. (1979). Theories of the transfer of learning. *Educational Psychologist*, *14*(1), 53–69. https://doi.org/10.1080/00461527909529207

Runesson, U. (2005). Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. *Cambridge Journal of Education*, *35*(1), 69–87. https://doi.org/10.1080/0305764042000332506

Runesson, U. (2006). What is it possible to learn? On variation as a necessary condition for learning. *Scandinavian Journal of Educational Research*, *50*(4), 397–410. https://doi.org/10.1080/00313830600823753

Saldaña, J. (2013). *The coding manual for qualitative researchers* (Second Ed.). SAGE Publications Ltd.

Sammons, P., & Davis, S. (2016). Mixed methods approaches and their application in educational research. In D. Wyse, N. Selwyn, E. Smith, & L. E. Suter (Eds.), *The BERA/SAGE Handbook of Educational Research* (1st ed., pp. 477–504). SAGE Publications Ltd. https://doi.org/10.4135/9781473983953.n24

Savic, M. (2017). Does content matter in an introduction-to-proof course? *Journal of Humanistic Mathematics*, *7*(2), 149–160. https://doi.org/10.5642/jhummath.201702.07

Schifter, D. (2009). Representation-based Proof in the Elementary Grades. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades* (pp. 71–86). Routledge.

Scott, D. (2017). Interpretivism as a theory of knowledge. In D. Wyse, N. Selwyn, E. Smith, & L. E. Suter (Eds.), *The BERA/SAGE handbook of educational research* (Vol. 2, pp. 243–258). SAGE Publications, Limited. https://doi.org/10.4135/9781473983953.n12

Selden, A., Selden, J., & Benkhalti, A. (2018). Proof frameworks: A way to get started. *PRIMUS*, *28*(1), 31–45. https://doi.org/10.1080/10511970.2017.1355858

Selter, C. (2009). Creativity, flexibility, adaptivity, and strategy use in mathematics. *ZDM*, *41*(5), 619–625. https://doi.org/10.1007/s11858-009-0203-7

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1–36.

Sinclair, N., Watson, A., Zazkis, R., & Mason, J. (2011). The structuring of personal example spaces. *Journal of Mathematical Behavior*, *30*(4), 291-303. doi:10.1016/j.jmathb.2011.04.001.

Singley, M. K., & Anderson, J. R. (1989). *The transfer of cognitive skill*. Harvard University Press.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77,20–26.

Skilling, K., Bobis, J., Martin, A. J., Anderson, J., & Way, J. (2016). What secondary teachers think and do about student engagement in mathematics. *Mathematics Education Research Journal*, *28*, 545–566. https://doi.org/https://doi.org/10.1007/s13394-016-0179-x

Sowder, L., & Harel, G. (1998). Types of Students' Justifications. *The Mathematics Teacher*, *91*(8), 670–675. https://doi.org/10.5951/MT.91.8.0670

Sriraman, B. (2004). Reflective abstraction, uniframes and the formulation of generalizations. *The Journal of Mathematical Behavior*, *23*(2), 205–222. https://doi.org/10.1016/j.jmathb.2004.03.005

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research*. Sage Publications.

Strauss, A. L. (1987). *Qualitative analysis for social scientists*. Cambridge University Press. https://doi.org/10.1017/CBO9780511557842

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321. https://doi.org/10.2307/30034869

Stylianides, A. J., & Stylianides, G. J. (2006a). Content knowledge for mathematics teaching: the case of reasoning and proving. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education (Vol. 5)* (Vol. 5, pp. 201–208). PME.

Stylianides, A. J., & Stylianides, G. J. (2009a). Proof constructions and evaluations. *Educational Studies in Mathematics*, *72*(2), 237–253. https://doi.org/10.1007/s10649-009-9191-3

Stylianides, A. J., & Stylianides, G. J. (2018). Addressing key and persistent problems of students' learning: The case of proof. In G. Harel & A. J. Stylianides (Eds.), *Advances in Mathematics Education Research on Proof and Proving: An International Perspective* (pp. 99–133). Springer.

Stylianides, A. J., & Stylianides, G. J. (2025). Important mathematical competences for students in the age of generative artificial intelligence: proof comprehension and evaluation, and ChatGPT. *ΕΥΚΛΕΙΔΗΣ Γ΄ [Euclid III]*, *100*(1–2).

Stylianides, G. J. (2008). An analytic framework of reasoning-and-proving. *For the Learning of Mathematics*, *28*(1), 9–16.

Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical Thinking and Learning*, *11*(4), 258–288. https://doi.org/10.1080/10986060903253954

Stylianides, G. J. (2016). *Curricular resources and classroom use: The case of mathematics*. OUP.

Stylianides, G. J., Sandefur, J., & Watson, A. (2016). Conditions for proving by mathematical induction to be explanatory. *Journal of Mathematical Behavior*, *43*, 20-34. doi:10.1016/j.jmathb.2016.04.002.

Stylianides, G. J., & Stylianides, A. J. (2006b). "Making proof central to pre-high school mathematics is an appropriate instructional goal": Provable, refutable, or undecidable proposition? In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education (Vol. 5)* (Vol. 5, pp. 209–216). PME.

Stylianides, G. J., & Stylianides, A. J. (2009b). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education Journal for Research in Mathematics Education*, *40*(3), 314–352.

Stylianides, G. J., & Stylianides, A. J. (2017). Research-based interventions in the area of proof: The past, the present, and the future. *Educational Studies in Mathematics*, *96*(2), 119–127. https://doi.org/10.1007/s10649-017-9782-3

Stylianides, G. J., & Stylianides, A. J. (2020). Posing new researchable questions as a dynamic process in educational research. *International Journal of Science and Mathematics Education*.

Stylianides, G. J., Stylianides, A. J., & Moutsios-Rentzos, A. (2024). Proof and proving in school and university mathematics education research: A systematic review. *ZDM - International Journal on Mathematics Education*, *56*, 47–59. https://doi.org/https://doi.org/10.1007/s11858-023-01518-y

Stylianides, G. J., Stylianides, A. J., & Shilling-Traina, L. N. (2013). Prospective teachers' challenges in teaching reasoning-and-proving. *International Journal of Science and Mathematics Education*, *11*(6), 1463–1490. https://doi.org/https://doi.org/10.1007/s10763-013-9409-9

Tall, D., Thomas, M., Davis, G., Gray, E., & Simpson, A. (1999). What is the object of the encapsulation of a process? *The Journal of Mathematical Behavior*, *18*(2), 223–241. https://doi.org/10.1016/S0732-3123(99)00029-2

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169. https://doi.org/10.1007/BF00305619

Tall, D., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y. H. (2011). Cognitive Development of Proof. In G. Hanna & M. de Villiers (Eds.), *Proof and Proving in Mathematics Education: The 19th ICMI study* (Vol. 15, pp. 13–50). Springer. https://doi.org/doi.org/10.1007/978-94-007-2129-6_2

Thorndike, E. L. (1906). *The principles of teaching: Based on psychology*. New York: A.G. Seiler. https://babel.hathitrust.org/cgi/pt?id=uc1.$b58960&view=1up&seq=263

Tight, M. (2016). Case study research. In D. Wyse, N. Selwyn, E. Smith, & L. E. Suter (Eds.), *The BERA/SAGE Handbook of Educational Research* (1st ed., pp. 376–394). SAGE Publications Ltd.

Toplak, M. E., & Stanovich, K. E. (2002). The domain specificity and generality of disjunctive reasoning: Searching for a generalizable critical thinking skill. *Journal of Educational Psychology*, 94(1), 197–209.

Tversky, A., & Kahneman, D. (1974). Judgment under uncertainty: Heuristics and biases. *Science*, 185, 1124–1130.

Umierski, A., & Tiedemann, K. (2019). Generalizing distributive structures in primary school. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 1810–1817). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME. https://hal.archives-ouvertes.fr/hal-02435409/document

Vale, C., Widjaja, W., Doig, B., & Groves, S. (2018). Anticipating students' reasoning and planning prompts in structured problem-solving lessons. *Mathematics Education Research Journal*, *44*(5), 1–25. https://doi.org/10.1007/s13394-018-0239-5

Venkat, H., Askew, M., Watson, A., & Mason, J. (2019). Architecture of mathematical structure. *For the Learning of Mathematics*, *39*(1), 13–17.

Verschaffel, L., Greer, B., & Torbeyns, J. (2006). Numerical thinking. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: past, present and future* (pp. 51–81). Sense Publishers.

Wagner, J. F. (2010). A transfer-in-pieces consideration of the perception of structure in the transfer of learning. *Journal of the Learning Sciences*, *19*(4), 443–479. https://doi.org/10.1080/10508406.2010.505138

Warren, E. (2003). The role of arithmetic structure in the transition from arithmetic to algebra. *Mathematics Education Research Journal*, *15*(2), 122–137. https://doi.org/10.1007/BF03217374

Watson, A. (2001). Instances of mathematical thinking among low attaining students in an ordinary secondary classroom. *Journal of Mathematical Behavior*, *20*(4), 461–475. https://doi.org/10.1016/S0732-3123(02)00088-3

Watson, A. (2010). Shifts of mathematical thinking in adolescence. *Research in Mathematics Education*, *12*(2), 133–148. https://doi.org/10.1080/14794802.2010.496974

Watson, A. (2017). Pedagogy of variations: Synthesis of various notions of variation pedagogy. In R. Huang & Y. Li (Eds.), *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 85–103). Sense Publishers.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Lawrence Erlbaum Associates.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking & Learning: An International Journal*, *8*(2), 91–111. https://doi.org/10.1207/s15327833mtl0802_1

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education*, *39*(4), 431–459.

Willis, G. B. (2005). Cognitive interviewing. In *Cognitive interviewing*. United States of America, California, Thousand Oaks: SAGE Publications, Inc.

Yin, R. K. (2018). *Case study research and applications: Design and methods* (Sixth Ed.). Sage.

Zawojewski, J., & Silver, E. (1998). Assessing conceptual understanding. In G. Bright & J. Joyner (Eds.), *Classroom assessment in mathematics: Views from a National Science Foundation working conference* (pp. 287–294). University Press of America.

Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *An International Journal*, *49*(3), 379–402. https://doi.org/10.1023/A:1020291317178

Zodik, I., & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. *Educational Studies in Mathematics*, *69*(2), 165–182. https://doi.org/10.1007/s10649-008-9140-6

Appendices

Appendix A: Number task

1a) Jack says that if you pick any number and do the following steps, the answer will always be 3.

Step 1: Choose any number
Step 2: Multiply it by 2
Step 3: Add 6
Step 4: Divide by 2
Step 5: Subtract the chosen number

Try to show that Jack's statement is always true.

1b) If you haven't already, try to write the above 5 steps as algebra statements.

2a) Sam says that if you pick any number and follow the steps on the left, the answer will always be 3, if you follow the steps in the middle the answer will always be 4, and if you follow the steps on the right the answer will always be 5.

Answer always 3	Answer always 4	Answer always 5
Step 1: Choose any number (n)	**Step 1:** Choose any number (n)	**Step 1:** Choose any number (n)
Step 2: $2n$	**Step 2:** $2n$	**Step 2:** $2n$
Step 3: $2n + 6$	**Step 3:** $2n + 8$	**Step 3:** $2n + 10$
Step 4: $\frac{2n+6}{2}$	**Step 4:** $\frac{2n+8}{2}$	**Step 4:** $\frac{2n+10}{2}$
Step 5: $\frac{2n+6}{2} - n$	**Step 5:** $\frac{2n+8}{2} - n$	**Step 5:** $\frac{2n+10}{2} - n$

What is the same and what is different about these three methods?

2b) Try to *explain* why Sam's predicted answers always come true.

2c) Try to use Sam's steps to *show* how you can always get an answer of 7.

2d) Try again to *explain* why Sam's predicted answers always come true.

3) Luka looks at Sam's way of getting an answer that is always 5, and says that he knows a slightly similar way to always get an answer of 5 (shown below).

Answer always 5	Example A
Step 1: Choose any number	6
Step 2: Add the next number	$6 + 7 = 13$
Step 3: Add 9	$13 + 9 = 22$
Step 4: Divide by 2	$\frac{22}{2} = 11$
Step 5: Subtract the chosen number	$11 - 6 = 5$

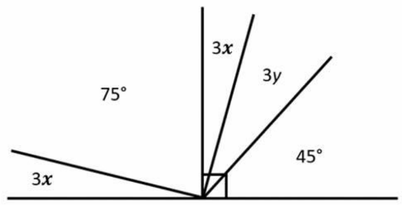
Answer always 5	Example B
Step 1: Choose any number	7
Step 2: Add the next number	$7 + 8 = 15$
Step 3: Add 9	$15 + 9 = 24$
Step 4: Divide by 2	$\frac{24}{2} = 12$
Step 5: Subtract the chosen number	$12 - 7 = 5$

Answer always 5	Algebraic Proof
Step 1: Choose any number	n
Step 2: Add the next number	$n + (n + 1) = 2n + 1$
Step 3: Add 9	$2n + 1 + 9 = 2n + 10$
Step 4: Divide by 2	$\frac{2n+10}{2} = n + 5$
Step 5: Subtract the chosen number	$n + 5 - n = 5$

Try to show *using algebra* how Luka's method can be used to always get an answer of 7.

Appendix B: Geometry task

1) Maddison says that for the diagram below she can show that **y = 2*x***.



Maddison's method

Step 1: The angle sum to the left of the vertical line is 90°, so set up the equation $3x + 75° = 90°$

Step 2: Solve $3x + 75° = 90°$ by doing 90° - 75° = 15°, then do 15° ÷ 3 to give $\boldsymbol{x = 5°}$

Step 3: The angle sum to the right of the vertical line is 90°, so using $x = 5°$, set up the equation 3y + 15° + 45° = 90°

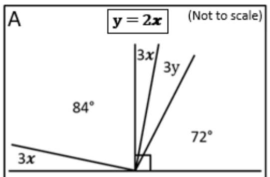
Step 4: Solve 3y + 15° + 45° = 90° by doing 90° - 45° - 15° = 30°, then do 30° ÷ 3 to give **y = 10°**

Step 5: State that $\boldsymbol{x = 5°}$ and **y = 10°**, so $\mathbf{y = 2}\boldsymbol{x}$

When you consider the diagram and steps shown above, which mathematical facts and ideas come to mind?

2a) Eve looks at Maddison's method, and says that by using similar steps she can show for diagram A that $\mathbf{y = 2}\boldsymbol{x}$, for diagram B that $\mathbf{y = 3}\boldsymbol{x}$, and for diagram C that $\mathbf{y = 4}\boldsymbol{x}$.

A



Step 1: $3x + 84° = 90°$

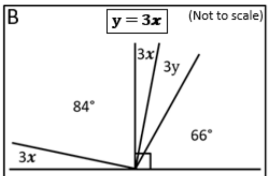
Step 2: 90° - 84° = 6°, then do 6° ÷ 3 to give $\boldsymbol{x = 2°}$

Step 3: 3y + 6° + 72° = 90°

Step 4: 90° - 72° - 6° = 12°, then do 12° ÷ 3 = to give **y = 4°**

Step 5: state that $\boldsymbol{x = 2°}$ and **y = 4°** so $\mathbf{y = 2}\boldsymbol{x}$

B



Step 1: $3x + 84° = 90°$

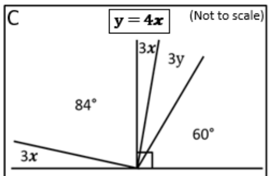
Step 2: 90° - 84° = 6°, then do 6° ÷ 3 to give $\boldsymbol{x = 2°}$

Step 3: 3y + 6° + 66° = 90°

Step 4: 90° - 66° - 6° = 18°, then do 18° ÷ 3 = to give **y = 6°**

Step 5: state that $\boldsymbol{x = 2°}$ and **y = 6°** so $\mathbf{y = 3}\boldsymbol{x}$

C



Step 1: $3x + 84° = 90°$

Step 2: 90° - 84° = 6°, then do 6° ÷ 3 to give $\boldsymbol{x = 2°}$

Step 3: 3y + 6° + 60° = 90°

Step 4: 90° - 60° - 6° = 24°, then do 24° ÷ 3 = to give **y = 8°**

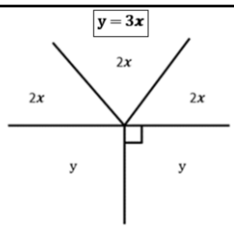
Step 5: state that $\boldsymbol{x = 2°}$ and **y = 8°** so $\mathbf{y = 4}\boldsymbol{x}$

What is the same and what is different about illustrations A, B and C?

2b) Try to describe what a similar diagram would look like that could be used to show $y = 6x$.

2c) Considering your response to **2b**, try to use Eve's steps to show that $y = 6x$.

3) Hannah used Eve's way of showing that $\mathbf{y = 3}\boldsymbol{x}$ to show, in a similar way, that $\mathbf{y = 3}\boldsymbol{x}$ for each of the diagrams below.



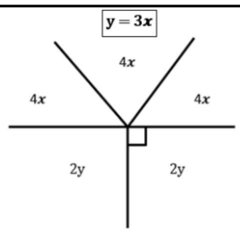
Step 1: State that **y = 90°**

Step 2: The angle sum above the horizontal line is 180°, so set up the equation $2x + 2x + 2x = 180°$

Step 3: Simplify the equation to $6x = 180°$

Step 4: Solve the equation by doing 180° ÷ 6 to give $\boldsymbol{x = 30°}$

Step 5: State that $\boldsymbol{x = 30°}$ and **y = 90°**, so $\mathbf{y = 3}\boldsymbol{x}$



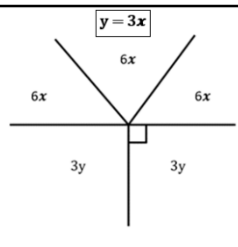
Step 1: 2y = 90°, so do 90° ÷ 2 to give **y = 45°**

Step 2: The angle sum above the horizontal line is 180°, so set up the equation $4x + 4x + 4x = 180°$

Step 3: Simplify the equation to $12x = 180°$

Step 4: Solve the equation by doing 180° ÷ 12 to give $\boldsymbol{x = 15°}$

Step 5: State that $\boldsymbol{x = 15°}$ and **y = 45°**, so $\mathbf{y = 3}\boldsymbol{x}$



Step 1: 3y = 90°, so do 90° ÷ 3 to give **y = 30°**

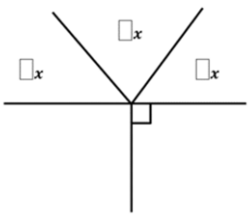
Step 2: The angle sum above the horizontal line is 180°, so set up the equation $6x + 6x + 6x = 180°$

Step 3: Simplify the equation to $18x = 180°$

Step 4: Solve the equation by doing 180° ÷ 18 to give $\boldsymbol{x = 10°}$

Step 5: State that $\boldsymbol{x = 10°}$ and **y = 30°**, so $\mathbf{y = 3}\boldsymbol{x}$

Can you use the diagram below to show that $\mathbf{y = 6}\boldsymbol{x}$? (You can use Hannah's steps to help you, if you like).



Appendix C: Spoken ad-hoc ‘think aloud’ interview prompts

At relevant points in the task-based interviews I used a question to probe a learner's thinking. A list of these questions is given below. I deliberately designed the questions to avoid giving learners the kind of instructional cues and options that could definitively channel their procedural thinking and actions.

What are you thinking now?
Why did you…?
How does that help?
Why did you do that?
Why did you choose that value?
How does that explain why the predicted answer always comes true?
Can you clarify what you mean by?
Are you stuck?
(If stuck) Can you explain how you are stuck?

Appendix D: Examples of task-based interview transcripts (both domains)

Learner 8C 'Carl': number transcript

Introductory task

I: (Showing and talking Carl through the 5 steps in Task 1) Jack says that if you pick any number and do the following [5] steps, the answer will always be 3. So the steps are: choose any number, you multiply it by 2, add 6, divide by 2, and then subtract the chosen number, and he's saying that the answer will always be 3 no matter which number you pick to start with.
C: Right
I: So, if you had to try to show that Jack's statement is always true, how would you try and show that Jack's statement is always true?
C: Well my initial thought is to take (points to the 'always be 3' part of Jack's statement and says) wait will it always be 3, so if we do (points to step 2, i.e., multiply by 2 and starts following the given steps but makes two errors at step 4): 3 x 2 that will be 6; (points to step 3) 6 + 6 = 12; (points to step 4 but divides by 6 not 2, then makes another error by giving the answer as 0) divide that [presumably 12 but maybe his was thinking of 6] by 6 that would be 0 so it can't be the initial number [3]. So, I will have to make, no way round it, 9
I: So did you say make 9?
C: So the initial number (points to the 'always be 3' part of Jack's statement and says) … if we do 9 take [subtract] something equals 3 (jumping to and thinking about trying to make the final step come to 3), it'll be ~~9~~ 12, minus 9.
I: So did you try a number to start with?
C: Yeah, 3, but it doesn't work because of the final step, it's logical but it's not the initial number [it does not give the predicted result of 3]
I: Okay just go back, which number did you try?
C: My initial number was 3
I: So you picked a number of 3?
C: (points to step 2) So then times by 2 which is 2
I: so *3* x 2?
C: oh 3 x 2 is 6, (points to step 3) plus 6 on to that [6] is 12, (points to step 4) divide that [12] by 2 which would have been 6, (points to step 5) subtract the chosen number it would have been 3 (points to the 'always be 3' part of Jack's statement and says), that is correct.
I: yeah
C: Yeah I backtracked on myself, I actually meant 6 [instead of his initial step 4 result of 0]
I: so you picked the number 3 for your any number?
C: yeah
I: you followed these steps and it came to 3?
C: Yes it did
I: so does that show that Jack's statement is *always* true?
C: No
I: Why not?
C: Because if you were to choose any number, it [the result] could be over [3] or under [3].
I: And what makes you say that?
C: So if for example I what to choose the number 1,000,000 (follows the 5 given steps using 1,000,000): so (step 1) 2,000,000; (step 2) 2,000,006; (step 3) 1,000,003; (step 4) subtract 1,000,000 that would be 3. That means that one [1,000,000] is correct [gives Jack's predicted result of 3] but not all numbers do [result in 3].
I: okay
C: If it [the initially chosen number] is even, *yes*. If it's odd, *no*.
I: okay, do you want to try an odd number?

C: okay I'll go 41 (again follows the five steps)
I: 41, ok, and follow the steps
C: (again follows the five steps) right so 41 x 2 = 82, that'll go to 88, 44, 41, ah actually yes it is **always** true … I messed up, it's true it's true.
I: No it's ok. Now you're saying it's *always* true? Is that [what you done] enough to convince you that it's always true.
C: Not really, it's not proven, because you could get one number that can make it [Jack's statement] automatically wrong.
I: And how would you do that?
C: Now I'm going to try the number 1, because this is the simplest one, (then again follows the five steps) 1 x 2 is 2; + 6 [8]; divide that by 2 [4]; then subtract it [3].
(Picks up his pen and says) so we know 1 = 3 (writes 1 = 3); if I choose 1,000,000 that will go to 3 (writes 1,000,000 = 3); and then 41 equals 3 (writes 41 = 3).
(Then writes out the final step for each number that he tried, and so writes):
4 – 1 = 3
1,000,003 – 1,000,000 = 3
44 – 41 = 3
I: okay just bringing you back to this statement here [i.e., try to show that Jack's statement is always true] does this [what you have written] show that it is always true?
C: Theoretically yes but we didn't try something, what if I went -1, -2, -3 etc etc?
I: Okay so do you want to try a negative number?
C: I'm going to choose -1, (writes -1 number) (then follows the remaining 4 steps) so (step 2) (concurrently writes and says) -1 x 2 = -2; (step 3) (concurrently writes and says) -2 + 6 = 4; (step 4) says [4] divided by 2 = 2 (writes 4 ÷ 2 = 2); (step 5) then if I do 2-1
I: For the final step you have to do 2 subtract the number that you chose.
C: so 2 - - 1?
I: yeah
C: okay, (writes) 2 - - 1 = 3, so we know a negative number can work, we do know that. So if this was an exam where you had about 1-hour, I wouldn't try negative numbers first. So if a question like this popped up, it would be that you have to *prove*.
I: okay so if you had all the time in the world, what would you do to try and prove that the answer always came to 3?
C: Well I wouldn't try every number in the world because that is not doable, I don't have all the time for it [testing every number] just in case [the statement isn't true]. So what I would have to try and do is do the first 10 numbers so I won't be her all day (writes 0,1,2,3,4,5,6,7,8,9,10), I go to -1,-2,-3,-4 (writes -1,-2,-3,-4,-5,-6,-7,-8,-9,10)
I: so you would try some negative numbers as well?
C: yeah
I: [Task 2] ok so next one, it says here (points to task 2) if you haven't already, try to write the above 5 steps as algebra statements
C: ok
I: so for step 1 choose any number, how would you write that using algebra?
C: So you would have to do it initially, this is what I was coming at earlier, I would create a letter, which could be for example n, I'm going to take the n from the example (writes n). So I'm going to put for my chosen number, for example, 10 (then writes = 10 to give n = 10).
I: Now by giving n a value then you've started with algebra the you're moving to numbers so let's see if we can stick just with algebra. So if we were doing this all as algebra.
C: All, so including the number 1 as well?
I: no, so step 1 is choose any number, so you've got n, so that's done. So step 2, how would you show that you are now multiplying that [n] by 2?
C: I would do my n without putting the number, times by 2 which will equal 2n (writes 2n)

I: ok, and how would you show that you are going to 6 to it?
C: says 2n+6 would equal, *something*, am I correct? (writes 2n+6=)
I: would we know what it equals because we are just writing these steps in algebra at the moment?
C: No, because an examiner would know what you've picked, so you have to make it logical or enquire or track back on yourself.
I: ok, let's move to step 4, thinking again about just writing these [steps] as algebra statements [quickly restates the 3 steps that Carl just wrote but ignores Carl's incorrectly written equals sign]; how would we show that we are dividing that [2n+6=] by 2?
C: Writes $\frac{2n+6}{2}$ and (as he is writing the fraction line says 'it could go wrong')
I: no that's good
C: I'll just put a bracket down (writes (÷) in brackets to the right of his $\frac{2n+6}{2}$)
I: and the final step is to subtract the chosen number, which you called n. So how would you show that you would subtract n from what you got there [points to his $\frac{2n+6}{2}$]?
C: Well we know that $\frac{2n+6}{2}$ will go down to n, straight away
I: Why do you know it will go down to n?
C: Because, this is what I'm thinking, I'm not an expert. Since I'm dividing by 2, I can knock that 2 (the 2 on the numerator) out and keep the n
I: What about the 6?
C: The 6 would turn to a 3. Now that 2 (the 2 on the denominator) is not usable [but he does not cancel it out] because I'm not able to divide a letter by a number. [i.e., performs partial doing/**undoing**]
I: okay but we didn't get to step 5, subtract the chosen number, so on your current workings how would you show that you are going to subtract your chosen number, which you called n?
C: so if I do n+3-n= [now correctly disregards the denominator of 2] should equal a possible number.
I: ok, do you want to see what would happen [points to the n+3-n=]?
C: should I?
I: you've got n+3-n
C: ok, what I'll have to do is, I'm going to use BIDMAS [networking], so I'll bracket that [puts brackets around the n+3 to give (n+3)-n and says] I don't know what n is yet because I don't know what taking this [points to the -n] away would be yet. So if I do n+3 [writes n+3] which in my example is 10 [points back to where he wrote n=10, line 89]. Judging by the fact that I've knocked the 2 off [points to his cancelled numerator of ~~2~~n+3] so that's [the cancelled 2 on the numerator] not mattering, so that would be 13 (writes =13 to gives n+3=13).
I: then take away n (points to the -n in his earlier written line of n+3-n)?
C: then if I do 13-n (my chosen number) = 3 (writes 13-n=3), what does the statement say [points back to the proposition], it [the result] will always be 3.

------------------- Start of **initial** variation task (comprising sub-tasks 2a to 2d) -------------------

I: **[<u>Task 2a</u>]:** ok great stuff. The next one, talks through the steps in prompt 3a
C: [points to the left-hand argument]I would automatically knock the 2 off straight away [on the numerator and denominator], like ratio I think you would do this [referring to the process of simplifying], if you do something to one number you would do it to the opposite side.
I: can you give an example?
C: writes 12:15 becomes 4:5
I: oh simplifying?
C: yeah
I: ok if I can bring you back to this [prompt 2a] then [answer always 2, answer always 4, answer always 5]. Looking across these three boxes, what is the same and what is different about these three methods? What do you notice is the same and is different?

C: I can figure out the same straight away, what they've done is, for all of them they used **the same** format, I'm talking about the step usage. What they've done to get +1 on the number [the consecutive predicted results] which is **different** [the consecutive predicted results], they've done +2 [on each addend] so 6+2=8, 8+2=10
I: are you seeing that as something that's the same or something that's different?
C: well a bit of both, (writes +1=+2), and says **the +2 is the same** but the +1 won't/will stay the same.
I: ok, points to and reads out **[Task 2b]** i.e., try to explain why Sam's predicted answers always come true. You can just do this verbally if you like. What is it in the steps that makes Sam's predicted answers always come true?
C: ok, I can see that it's the same method every time, and that he always divides the addend by 2 to get the predicted answer, 6 divided by 2=3, 8 divided by 2=4, 10 divided by 2=5.
I: ok
C: so it's [the addend] always +2 but the +1 stays the same.
I: how does that explain why the predicted answers always come true?
C: we can't prove that [Sam's statement] is always true unless we do some equations with them [this turned out to mean test out some numbers].
I: What is it in the steps that makes Sam's predicted answers always come true, always comes to 3, then 4, then 5?
C: It's what I said earlier, they've linked it here [dividing the addend by 2 to the predicted answer], that's the basic way of doing it not the higher way of doing it.
I: ok, let's look at this **[Task 2c]**, try to use Sam's steps to show how you can always get an answer of 7. So just as a reminder these are the steps for answer always 3, always 4, always 5. So what would the steps be for an answer of always, not 6, but 7?
C: Well to get it easier we should **always** find the formula [i.e., steps] for [always getting a result of] 6 first because if we go straight to jump [to 7] we wouldn't 100% know.
I: ok, so what would the steps be, so step 1?
C: so step 1 would be the same, choose your number (writes, 1. choose a number (n)), then it's (step 2) is n timesed by 2 which is your 2n (writes, 2. nx2= 2n). Now it's the tricky part, we have to figure out what this number [the addend] is. I put it [earlier] in this formula of +1=+2 so +1 for every 2 numbers [i.e., when the predicted result increases by 1, the addend increases by 2] so we are in the ratio of 1 and 2 [predicted answer: addend]. So, if I know that this [Step 4] always divides by 2, yes always.
I: ok, what is Step 3 going to look like?
C: So, you have to do the 2n plus by our number which is going to be 12 [because they are first aiming for a result of always 6, before aiming to get a result of always 7] because it's going to be the same here (points to where the addend is divided by 2) well it should always be the same but you can't prove (writes 3. 2n+)
I: Step 4 then?
C: so Step 4 would be 2n+12 divided by 2 that [dividing by 2] will always stay the same (writes 4. $\frac{2n+12}{2}$)
I: ok, Step 5?
C: Step 5 I will do in my [earlier] format [i.e., immediately cancelling the 2s on the numerator and denominator] (writes 5. n+6) now I'll use BIDMAS (writes (n+6)-n); then [rather than doing n-n] substitutes the value of 10 for n and says I will use the number 10 because it is the simplest (then writes 10+6=16 and 16-10=6)
I: ok. So what would you change/do differently to get an answer always of 7?
C: I would keep Steps 1 and 2 the same but this [points to the addend] I would change.
I: to?
C: 14

I: ok, reads **[Task 2d]**, try again to explain why Sam's predicted answers always come true. So has doing this [giving the steps for answer always 6 and 7] helped you to explain why Sam's predicted answers always come true?
C: (refers back to previous explanation) so I relate to Step 4 most of the time because this is where it comes into play. I'm going to knock off [cancel out] the 2s [on the numerator and denominator]. But it's here (points to the addend in the 2nd example) 8÷2=4, and in the next one 10÷2=5. Now Step 5, for me, changes I would knock it [the 2s on the numerator and denominator] off.
I: then follow what you got there (points to Step 5)
C: yeah
-------------------- End of **initial** variation task (comprising sub-tasks 2a to 2d) --------------------

-------------------- Start of **transfer** variation task --------------------
I: ok so last one [**prompt 3, puts Sam's and Luka's methods side by side for comparison purposes**], I know there's a lot of information on this so it's whatever you see.
C: whatever catches the eye?
I: yes. Luka looks at Sam's way of getting an answer that is always 5, and says that he knows a slightly similar way to always get an answer of 5. [then talks Carl through all of the steps in Luka's given two examples starting with n=6, then starting with n=7; then talks Carl through all of the Steps in Luka's given algebraic proof]. Try to show using algebra, so these [given] steps, how Luka's method can be used to always get an answer of 7. So not an answer of 5 but of 7. So looking through the steps.
C: one thing I noticed that could complicate students, its happened to me, [points to the algebra for Step 2: add the next number n+(n+1) and confuses the construction with n(n+1)]. Normally I would do n x n [multiply out the bracket] not +.
I: n x n would give you n^2 and take you in a different direction. [explains again to Carl why n + (n + 1) represent two consecutive numbers]. So could you try and show how Luka's steps could be used to always get an answer of 7? So the given steps always give an answer of 5, so what is it in the steps that you would?
C: I'm going to go by the writing [worded steps] because [in] the algebra he's done some bits [notation] that I wouldn't have done slash wouldn't really understand. That's the main bit that's complicating me that's circled (in Step 2: add the next number, draws a ring around the (n+1) and +1 in n+(n+1)=n+1). Because I wouldn't understand why he's done this. So I'm going to relate to this [points to the worded steps].
I: ok
C: So wait do they want [an answer always of] 7?
I: yeah
C: So it's the same as what I'm [I was] doing here to some extent [points to his previously written 'bridging' steps for Sam always getting an answer of 6, that he used to show/explain how Sam's steps could be used to always get an answer of 7, for prompt 3c].
I: ok
C: So [for Step 1] I'm going to keep n the same for my chosen number (writes n). My chosen number could be any number. [Then] I would do n + n (writes n + n) not [n] x n because I always thought (points back to his previously written steps for Prompt 2, i.e., *if you haven't already, try to write the above 5 steps as algebra statements*, and says) oh I did n *x 2* there, that's why, which still would be correct?
I: oh from the previous one [from Carl's written response to prompt 2]?
C: yeah
I: yeah, I can see why you would look back like that.
C: have I?
I: no, it's ok, these [points to the steps in prompt 4] are slightly different steps aren't they, some steps are the same and some are different.

C: He [Luka] has done it from a different standpoint.
I: Yeah, so what would you do with these steps to try and get an answer that is always 7, rather than always 5?
C: [shifts his focus from the given worded steps to the given algebraic proof] Well, if I'm going to go by the algebraic proof, since it's what's given to me, although it's a little bit more complicated. So I'm going to follow it by his footsteps, so [for Step 2] I would write + 1 and bracket that (writes +1 after his previously written n + n then puts a bracket around the n+1 to replicate n+(n+1) in Luka's Step 2). That [the n+1 in brackets] would be n+1 (writes n+1 under his n+(n+1)) because that would not change. To give:
n+(n+1)
n+1
without the brackets, then writes n+ in front of his previously written n+1 to give
n+n+1 I could leave this bit out for now (puts brackets around the n+n to give (n+n)+1)
I: ok
C: do you need to stop me?
I: no, so what would you do then to try and get this answer of 7, so so far Step 1 and Step 2 [in Luka's method] you've kept the same.
C: I'm going to follow it as if it was the algebraic proof because the examiners I'm pretty sure want you to link to what they've given. So I am going to follow his [Luka's] footsteps a bit, so I'll just get that bit [Step 2] finished (writes =2n+1 after his (n+n)+1 to give, for Step 2, (n+n)+1=2n+1). Now he [Luka] wants 7 if I'm correct (checks), yes he [Luka] does.
So I know, 2n will stay the same no matter what, so I will have my one [+1] (starts to write his notation for Step 3) 2n+1+ into what would be 7 x 2, I'm referring [back] to this method again where every one number is two to the secondary number [presumably meaning that the ratio of the predicted answer to the addend (total) is 2:1]. So I know that 7x2 is 14, so I'm going have to make 14 before I do anything.
I: Ok
C: so I'm going have to do +13 but leave this [the 2n] (writes +13 after his 2n+1 to give: 2n+1+13)
I: ok
C: Or, to make it a bit simpler, I'm going to get rid of that for now [crosses out the 2n to give ~~2n~~+1+13] because I can refer back to that [the crossed out 2n] straight after because of this previous step [Step 2]; so this [the sum of the addends in Step 3] =14 (writes =14 to give ~~2n~~+1+13=14).
I: what would you do next?
C: So I know 2n is nx2, so I'm not going to change it or else the provement [proof] will be incorrect (writes 2n under his previously written steps). So if I write 14 to that [his 2n] (writes +14 after his 2n to give 2n+14) that will be this step (points to Step 3) complete.
I: ok, next step, what would you do next?
C: I'm going to simple [simplify] this [2n+14] off. Writes, for Step 4, $\frac{2n+14}{2}$ then cancels to give n+7, then, for Step 5, subtracts n should give an answer of 7.
I: ok, great, thank you for that.

Learner 10D 'Dylan': number transcript

Introductory task

I: (Showing and talking Dylan through the 5 steps in Task 1) Jack says that if you pick any number and do the following [5] steps, the answer will always be 3. So the steps are: choose any number, you multiply it by 2, add 6, divide by 2, and then subtract the chosen number, and he's saying that the answer will always be 3 no matter which number you pick to start with. So, if you had to try to show that Jack's statement is always true, how would you try and show that Jack's statement is always true?
D: so what d'ya do just choose another number and do the same steps to get an answer of 3?
I: that's one way of doing it, it's up to you, what ideas come to mind to show that Jack's statement is always true?
D: (talks through the mental calculations) so if you chose 2 at the start, times that by 2 to get 4, then +6 gets you 10, then divide it by 2 to get 5, then that [his mental calculations] wouldn't get that [an answer of 3] would it?
I: the last step is to subtract the chosen number, what was your chosen number?
D: 2, oh yeah it does give 3
I: ok so you tried 2, any reason why you tried 2?
D: just' cause he [Jack] said the number [mentions 2 in the given steps]
I: ok. So you got an answer of 3, did what you did show that it [Jack's steps] are *always* true [always gives an answer of 3]? Like, what would you have to do to show it is always true?
D: try a different example, maybe a different multiplicational number.
I: different?
D: like a range
I: ok so what would you try
D: you could try 25 'cause it's a square number, to mix it up a bit. (Talks through testing 25 and again gets an answer 3).
I: ok, you tried 2 it worked, you tried 25 it worked, does that now show that it [Jack's statement] is always true?
D: Yeah. Well, you could try it with minus then it probably [always] be true wouldn't it?
I: ok, d'you want to try a negative number?
D: tries -6 (first mentally but loses track of his calculations so writes down):
-6
-12
-6
-3
[struggled slightly with the sign rules but eventually got](-3--6)=3
I: so it [Jack's statement] worked for a negative number
D: so it [Jack's statement] works for everything then doesn't it?
I: so you think it may work for everything?
D: not unless it does [for] fractions.
I: so you've tried a couple of different types of numbers, now do you think it's [always] true?
D: er yeah. Because it's true for negative and positive numbers.
I: ok. [moves to **Prompt 2**] If you haven't already, try to write the above 5 steps as algebra statements. So, for Step 1, how would you represent any number?
D: (writes x)
I: then multiply by 2, that's x^2 no 2x?
D: (writes 2x)
then + 6, that's 8x

I: well we want to add 6 to this [the 2x]
D: oh 2x+6 (writes 2x+6)
I: Then divide by 2, how would you show that you want to divide that [his 2x+6] by 2?
D: would it be x+3?
I: that's correct but rather than jumping to showing what the answer might look like, could you show the steps. How would you show that you are dividing all of this [2x+6] by 2.
D: writes ÷2 under his 2x and under his +6 to give:
2x + 6
÷2 ÷2
x + 3
I: Then Step 5, how would you show take away the x?
D: (under his prior notation writes -x)
(Then under his -x writes)
+3
(Then crosses out the + sign)
3

------------------- Start of **initial** variation task (comprising sub-tasks 2a to 2d) -------------------

I: ok. [moves to and talks through **Prompt 2a** but leaves all Dylan's prior working in sight]. Sam says that if you pick any number and follow the steps on the left, the answer will always be 3, if you follow the steps in the middle the answer will always be 4, and if you follow the steps on the right the answer will always be 5. What is the same and what is different about these three methods?
D: you've got 2n in all of them.
I: yeah
D: and at the end you [always] divide by 2 as well. Same
D: and the difference is it [the addend] goes up by 2 each time.
I: ok [Reads **Prompt 2b**] Try to explain why Sam's predicted answers always come true.
D: [refers to the addend] well it [the final result] is half of whatever the amount you get [for the addend]. So if you did 12 you'd get 2n+12, then ÷2, you'd get 6 as your answer. [does not mention the subtraction for Step 5]
I: How does that make Sam's statements *always* come true?
D: so he [Sam] has put the 3 options [always 3, always 4, always 5] so whichever number [is chosen] each one of them [Sam's options] is going to come true.
I: ok but what is it in the [Sam's] steps that's making it [always] come true?
D: it's diving by 2 isn't it? That's all I see.
I: when you say *it's* diving by 2
D: whatever number it is [the addend] so 2n + whatever number it is, then you divide by 2 and you get your [predicted] answer, so long as it's **always** 2n.
I: ok. [reads **Prompt 2c**] Try to use Sam's steps to show how you can always get an answer of 7. So let's have another look, here are Sam's steps for always 3, always 4, always 5. Can you show what the steps would be for always 7?
D: Step 1 you choose any number, Step 2 …
I: could you write then down?
D: (writes):
2n
2n+14
$\frac{2n + 14}{2}$
I: why 14?
D: [points mostly to Step 4] because if you know $\frac{2n+10}{2}$ is 5, you just add 4 onto that [the addend] to get 7 [when you later divide the addend by 2].

I: ok

D: [quickly writes in his missing n for Step 1] (then, for Step 5, writes)

$\frac{2n+14}{2}$ - n

I: ok. [reads **Prompt 2d**] Try again to explain why Sam's predicted answers always come true. Has the step of giving an example [of answers always 7] given you any extra thinking?

D: same as before [his responses to Prompts 3c and 3e] you just divide it [Step 3, i.e., the 2n+the addend] to get rid of the 2 [in 2n] then halve the plus [the addend] whatever the plus [addend] is. [again, does not mention the subtraction for Step 5].

-------------------- End of **initial** variation task (comprising sub-tasks 2a to 2d) --------------------

-------------------- Start of **transfer** variation task --------------------

I: ok, [leaves Dylan's prior working on show and shows and reads **Prompt 3**] Luka looks at Sam's way of getting an answer that is always 5, and says that he knows a slightly similar way to always get an answer of 5. Try to show using algebra how Luka's method can be used to always get an answer of 7. [gives a brief explanatory talk through all of Luka's prompt 4 steps, including the steps in the algebraic proof]. There's a lot of information here, so it's whatever you notice. Try to show using algebra how Luka's method can be used to always get an answer of 7. So bear in mind that the [three] given examples are for always 5, always 5, always 5.

D: It might be something to do with the +9

I: why the +9?

D: because the 9 is in all of them [all three of Luka's examples] isn't it?

I: yeah

D: obviously this step [Step 3]

I: ok, what would you do, what would you change, what would you do with the 9 or any aspects of it?

D: erm [15s pause] would it be, could you do, so if you look at [example] A [Step 3: Add 9: 13+9=22], could you do 13+**12** or something to get [an answer of] 7? So you'd get 25 [i.e., 13+12] then ÷ 2 = there's no one [number] is there? Oh yeah

I: do you want to just try that idea out [gestures to the paper to write down his thinking]. So just tell me again.

D: so if you want to try and make 7 on A

I: on A?

D: on example A

I: oh ok

D: if you want to try and make 7. I'm going to try and do it how they [Luka] did it all the way up to where they [Luka] have the 13 but you plus [add] another number on [for] 9 to get it [the result of Step 3] up [higher than 22]. Then 25 ÷ 2 to get it down, then do the division thing [÷ 2].

I: ok. And you're trying on example A, any particular reason why you are looking at example A?

D: it's just the first one.

I: ok, do you want to test that [idea of adjusting the addend] out?

D:(writes)

6+7=13 [given Step 2 in example A]

13+**12**=25 [changes the given addend of 9 to 12]

$\frac{25}{2}$ = 12.5 can you have decimals or not?

I: It's up to you

D: (writes) 12.5 – 6 = **6.5** and says if you did +13 [i.e., add a higher addend than his previous trial of 12, instead of the given 9 in Step 3].

I: so bear in mind that for example A the chosen number is 6, but what you're saying is you can adapting example A?

D: I was just thinking about changing the steps [where you] add 9 to 13 [Step 3] to get the answer higher because you only need 7 and I [just] got 6.5. If you add 13, a bigger number [in Step 3] you would have a bigger number on the fraction [in Step 4].
I: ok, if you had to have another go or do something different, what would you, thinking about this word *always* as well, that it [the answer] always comes to 7.
D: well if you did 13+13 and that got [=] 7 then you try if that [Example B] got 7 then last go on the last one, prove.
I: would that work if you started with a different starting number each time?
D: it might do. So … [mentally tries adding 13 which would indeed result in 7 for any initially chosen number i.e., changing Step 3 from Add 9 to Add 13 would always result in 7, but makes a calculation error and says] no it wouldn't it be 0.5 off.
I: ok any other thinking on that, especially around the word *always* why it would *always* come true.
D: not unless you added a sixth step in and said Add 0.5, you'd get it.
I: if this [Dylan's working that resulted in 6.5] did come to 7, would that show that it [Dyan's working] *always* comes true?
D: no, it's just a single one [test of one number].
I: so what would you have to do to show that it always comes true?
D: answer [try] another one, but in that one, Example B [this response indicates that they are still thinking in terms of empirical examples to "prove"], I'm 0.5 off again.
I: ok thank you

-------------------- End of **transfer** variation task --------------------

Learner 10E ‘Emma’: number transcript

Introductory task

I: (Showing and talking Emma through the 5 steps in Task 1) Jack says that if you pick any number and do the following [5] steps, the answer will always be 3. So the steps are: choose any number, you multiply it by 2, add 6, divide by 2, and then subtract the chosen number, and he’s saying that the answer will always be 3 no matter which number you pick to start with. So, if you had to try to show that Jack’s statement is always true, how would you try and show that Jack’s statement is always true?
E: so should I pick a number?
I: If that’s how you feel is the best way to approach it, yeah, it’s whatever comes to mind, so.
E: ok, let’s do 10
I: any particular reason why 10?
E: because it’s an even number so it’s a bit easier to do.
I: ok
E: (follows Jack’s steps by writing on one line the following) 10 x 2 = 20 + 6 =26 ÷ 2 = 13 – 10 = **3** so yeah he is right.
I: so you’re saying Jack’s right, what about this word *always*, have you shown that Jack’s statement is *always* true, that the answer will always be 3?
E: I think so
I: why?
E: because if you’re minusing the last, the number you came up with, then it’s going to go back to 3
I: ok, why do you think that it’s going to go back to 3, what’s going to make it go back to 3?
E: dividing it [26] by 2. When you add 6 that’s a multiple of 3 so it’s going to make the number [result] a multiple of 3.
I: I see what you’re doing, you’re looking for connections. Ok so **Prompt 1b** says if you haven’t already, try to write the above 5 steps as algebra statements. So if you had to pick any number how would you represent/show that as algebra?
E: n (writes n)
I: and if you are showing that you multiply that by 2?
E: writes x 2 after her n to give n x 2
I: and then add 6?
E: writes + 6 after her n x 2 to give n x 2 + 6
I: looking at what you’ve written there and algebra conventions, when you multiply a letter by 2, any other thoughts come to mind about how you might write that?
E: 2n
I: 2n yeah
E: so 2n + 6,
I: yeah
E: (away from her earlier written notation writes) 2n + 6
I: so you’ve got n times 2 then add 6, the next step is divide by 2
E: shall I put divide by 2?
I: ok
E: (draws a fraction line and a denominator of 2 to give) $\frac{2n+6}{2}$
I: yeah it’s divide all of it [2n+6] by 2, good. And the last step is subtract your chosen number, well you’re doing it with algebra now aren’t you so?
E: so am I subtracting n?
I: subtract n, yeah

E: (writes) $\frac{2n+6}{2} - n$

I: have you put your n on the top row or do you mean that to go?

E: it [the -n] goes with all of it [the fraction].

------------------- Start of **initial** variation task (comprising sub-tasks 2a to 2d) -------------------

I: ok **Prompt 2a** Sam says that if you pick any number and follow the steps on the left, the answer will always be 3, if you follow the steps in the middle the answer will always be 4, and if you follow the steps on the right the answer will always be 5. And actually, if you look at those steps [in Sam's steps to answer always 3] you might recognise something of what you've [already] got [written] down there.

E: yeah

I: so, looking across those three [given example] boxes there, what do you see as the same and what's different about those three?

E: you are **always** dividing it by 2, points to each respective denominators in Step 4

I: ok

E: and you **always** add it by a certain number, so these [points to the addends in Step 4] are all different numbers, 6, 8 and 10

I: ok, anything else that's the same or different?

E: you [**always**] take away the original number that you had (points to -n in Step 5 on the middle argument and on the right-hand argument)

I: are the added numbers [6,8,10] something that's different?

E: I think that's different, yeah

I: why do you think that?

E: because they [the addends] are different numbers. [then notices another connection] At the same time 5 [points to the predicted answer in the right-hand argument] is **always** double the number [points to the addend in the right-hand argument] that the answer **always** is.

I: right, so there's a connection going on there, good spot on that one. So **Prompt 2b** says, try to explain why Sam's predicted answers, always 3, always 4, always 5, always come true. So what is it in the steps that's making the answers always come true?

E: when you +6 or +8 or +10 I think that's the main part of it because it's connected to the [predicted] answer.

I: right, and how is that going to make that always 3, that always 4 and that always 5?

E: erm. 'Cause when you divide it [the addend] by 2 it's going to be the [predicted] answer.

I: Ok. **Prompt 2c** says, try to use Sam's steps to show how you can always get an answer of 7. So let's just look across these again, answer always 3, answer always 4, answer always 5.

E: so you'd do 2n after you chose a number (writes 2n); you would add 14 (writes + 14 after her 2n to give 2n + 14); then divide it by 2 (draws a fraction line and a denominator of 2 to give) $\frac{2n+14}{2}$; then take away the number you had (writes -n to give $\frac{2n+14}{2} - n$)

I: ok and **Prompt 2d** says, try again to explain why Sam's predicted answers always come true. So in other words, has doing that one of your own [answer always 7] helped you to see anything extra, anything more as to why there's always going to be an answer of 7 on yours and an answer of 5 there and a 4 there and a 3 there?

E: erm. I don't know, I think dividing it by 2 is a part of it because you're timesing it by 2 and then you divide it by 2. [Points to Step 5 in her case for 7] and it's the number [addend of 14] divided by 2 is 7 so it'll **always** be the same.

------------------- End of **initial** variation task (comprising sub-tasks 2a to 2d) -------------------

------------------- Start of **transfer** variation task -------------------

I: ok (introduces **Prompt 3** but leaves Emma's previous working on show). I know that there's lots going on here but it's just whatever you see that's all.

E: ok
I: Luka looks at Sam's way of getting an answer that is always 5, and says that he knows a slightly similar way to always get an answer of 5 (talks Emma through the steps in the two given examples and the algebraic proof). Try to show using algebra how Luka's method can be used to always get an answer of 7. So this [Luka's workings] show an answer always of 5, 5, 5. What could you change or adapt do you think [points at the algebraic proof] to always get an answer of 7?
E: I think it's what you add on at Step 3, so if for 5 it was [an addend of] 9, for 7 it's going to be [an addend of] 11 [i.e., 11 is 2 more than 9 because 7 is 2 more than 5]. So it's going to be (writes identically Step 2)
n+(n+1)=2n+1 (compares to given algebraic proof) then it's going to be
(Step 3) 2n+1+11, if I'm doing this right?
I: It's alright, I see what you've done, 11 is 2 more than 9 because 7 is 2 more than 5.
E: yeah
I: so you're over here now, this bit (points to the right-hand side of the equation in Step 3 of the given algebraic proof)
E: so equals 2n + 12 (writes = 2n + 12 after her 2n + 1 + 11 to give 2n + 1 + 11 = 2n + 12)
I: ok
E: (compares to given algebraic proof) 2n+12 divided by 2 (writes $\frac{2n+12}{2}$)
I: a-ha
E: (compares to given algebraic proof) and then equals n + 6 (writes n + 6 after her $\frac{2n+12}{2}$ to give $\frac{2n+12}{2} = n + 6$) did she get her 6 by halving 12 in her previous step or by cancelling the fraction in the current step?;
(compares to given algebraic proof) and then (for Step 5 speaks as she writes under her prior working) n + 6 - … I don't know if that's going to be right [appears to see at this point that the final result will be 6 not 7]
I: what have you realised?
E: I've done it wrong
I: are you thinking ahead?
E: yeah
I: ok, let's keep going and see what happens
E: (writes n = after her n + 6 – to give n + 6 – n =) well it's not gonna be if it's minus well six minus.
I: (talking through Step 5 in the given algebraic proof) well what you've got here is n + 5 – n just leaves the 5 behind, you've got n + *6* - n so what is that going to leave?
E: erm. 7 well it's supposed to be 7
I: ok and what will you have?
E: (pauses 9 secs) what do you mean?
I: (points to the right-hand side of the equation) what goes here?
E: 7 [I think this answer of 7 could have been what *should* go here rather than Emma's answer of 6, alternatively Emma may not at this point have seen that her n + 6 – n simplified to give 6].
I: so here [in her Step 5] you've got n + 6 – n so in these workings here that you've got
E: oh 6
I: yeah, so if you just finish that off
E: (writes 6 after her n + 6 - n to give **n + 6 - n = 6**)
I: Ok, well interesting stuff, the example was of answer always 5, you were aiming for an answer of 7, and what you've got there is in-between [an answer of 6].
E: yeah
I: if you had to make another change what do you think you could sort of adapt to …?
E: I think I somehow got wrong the number.

I: ok but you got some logic in that you made changes that you thought, you know, seemed logical, and it [you result] wasn't 7
E: I think it [points to the addend of 9 in Step 3] has to be an odd number maybe
I: why do you think odd?
E: because the answer is odd as well, and when I started with an even number (points to the + 12 in her Step 3) it ended (points to her final result of 6) with an even number {she actually added 11 to get the + 12 that she pointed to}
I: right, ok, so just keeping that thought what would you change if you had to run through the steps again
E: The number you add [points, in her Step 3, to the simplified addend of +12 in her 2n+1+11=2n**+12**]
I: And you what would you change that to?
E: 13
I: Based on the idea that that's odd, or is that? Sorry so why 13 again?
E: Because it's odd yeah, and **then** I think if it's add another 1 [to her previously indicated addend of 12 to make 13; alternatively, when Emma said "add another 1" she may have meant add another one to her 13 to give 14 which would then result in 7] it [the final result] is going to be 7. But I'm still not sure. Either way, in regard to adjusting the addend in Step 3, Emma seems to be getting closer to the necessary addend total of 14. Ideally, a further trial or trials would be needed.
I: yeah ok that's brilliant thanks for that.

-------------------- End of **transfer** variation task --------------------

Learner 8A ‘Ash’: geometry transcript

-------------------- Start of **initial** variation task (comprising sub-tasks 2a to 2d) --------------------

I: What do you see as the same and different about the [three given] illustrations?

A: **What’s different is** the degrees of angles in the left and right side where it says the number [only], not the number and letter.

I: So, did you say the left side is different?

A: No. the bits where the numbers are, where it says 72 and 84 degrees, every single one changes.

I: Ok, and are you looking at the diagrams are the steps?

A: The diagrams.

I: ok, anything else that you see as the same of different in those?

A: in A y is 2x, in B y is 3x and in C y is 4x

I: ok, and would you say that something that’s the same or different?

A: **different**

I: ok **next one [Task 2b]**, try to describe what a similar diagram [to the three given] would look like that could be used to show y = 6x

A: on the right side it [the numerical value] would 48°

I: where’ve you had your 48 from?

A: on [diagram] A it’s 72 [on the right] and goes does by 6, which is 66 on [diagram] B, and then goes down by another 6 on [diagram] C, so if we’re making it y=6x, we’ve got to take away 12 from 60 which is 48.

I: ok, good stuff. **[Task 2c]** Thinking about your answer to that, could you try to use Eve’s steps to show that y = 6x.

A: [Silent pause]

I: So we’ve got five steps for each given example, what would those 5 steps look like for not 2x, 3x, 4x but y=6x?

A: So step 1 would stay the same

I: Can you write them down

A: [follows the] writes 1 = same

I: why do you think that step 1 will stay the same?

A: Because it stayed the same for all 3 of them [the given examples]

I: ok

A: [checks across all 3 examples by pointing his pen at each, then writes]:

[step] 2 = same

[checks across examples A and B by pointing his pen at each then, adhering to the given notation, writes]:

[step] 3 = 90° - 48° - 6° = 36 then do

[step] 4 = 36° ÷ 3 = y[=]12

[step] 5 = state that x = 2 and y = 12

So y = 6x

I: ok, next one **[Task 2d],** try again to describe what a similar diagram would look like that could be used to show y = 6x

A: it would have 48 on the right.

-------------------- End of **initial** variation task (comprising sub-tasks 2a to 2d) --------------------

-------------------- Start of **transfer** variation task --------------------

I: Hannah used Eve’s way of showing that y=3x to show, in a similar way, that y=3x for each of the diagrams below. Can you use the diagram below to show that y = 6x? [interviewer explains each of the given steps]

A: They [each of the three angles above the line] are all 60°
I: yeah, ok
I: you can use the diagram below to help, you haven't just got one go, we can very quickly draw another one of the [scaffolded] diagrams, so any thoughts, what are you thinking at the moment?
A: I'm going to put [above the horizontal line] 12, 12, 12, and below put 6y, 6y
I: why 12 and 6?
A: Because here [in the 3rd example] for y=3x it's 6,6,6 [above the line] 3,3 [below the line], so if I go 12,12,12 [above the line] 6,6 [below the line] it'll be **double** that [above the line] and double [the co-efficient of] 3 is 6.
[writes 12s above the line and 6s below on the scaffolded diagram]
I: ok, what would you put for step 1?
A: [silently checks through the steps in the 3rd (right hand) example pointing his pen at each step, then writes 60° in each of the three angles above the line on the scaffolded diagram, and follows this by writing 90° in each of the two angles below the line. They then draw a long French bracket the entire height of the scaffolded diagram and write 360].
A: [then writes]
12x + 12x + 12x = 180° [above the line]
6y + 6y = 180 [below the line]
36x = 180°
A: [pauses]
I: what are you thinking now?
A: I'm confused on the last step
I: ok, so, were the given steps useful to do this [what you've written so far] or were you thinking about what you already know about angles?
A: don't know, I just looked around to see what I know.
I: ok, so you've got 36x=180°, so she [Hannah] finds out what 1x is worth and what 1y is worth, so any ideas how you would find what 1y or 1x is worth?
A: [pauses]
I: I mean you've got 36 x's there haven't you?
A: So would 1*x* be 3.6?
I: If you divided it [36] by 10 you'd get 3.6 wouldn't you?, but why would you think to do that?
A: don't know
I: what you've got is correct, 36x does = 180°; and 6y + 6y does =180°, but she's trying to show that you are comparing the x to the y, so she's found what 1y is worth and what 1x is worth, then compared what 1 of them is. So at the moment you've got 36 x's and for the y's you got 6y + 6y = 180, so any ideas where you'd go from there?
A: [pauses]
I: so what is it in these steps that is making a situation where you can find what 1x is and what 1y is?
A: [struggles and cannot solve to find x = 5° or to find y = 15°, which as per the given examples, stills shows that y = 3x not y = 6x as asked for, instead says] would 1*x* be 6?, no. Y would be 6x. would it be x is 36°?

-------------------- End of **transfer** variation task --------------------

Appendix E: The GOLDEN framework

Generalising	**G1:** Reasoning about **one process** that is common to a set of given arguments. E.g., explaining a recursive increase in correspondingly positioned addends or coefficients.
	G2: Reasoning by simultaneously attending to **two processes** that are common to a set of given arguments. E.g., explaining as a single process the (concurrent) division of two terms in a numerator, or explaining a relationship between coefficients of x and y.
	G3: Reasoning about a relationship between a **process and a result** that is common to a set of given arguments. E.g., explaining a relationship between the value of an addend and a predicted result, or between a coefficient and an end result.
Organising	**O1:** Prioritising a dimension of **sameness** when discerning sameness and difference across a set of given arguments. E.g., foregrounding an invariant divisor or an invariant angle position or angle value. (Prioritising a dimension of sameness is different to Generalising because the latter involves an element of explanatory reasoning, whereas the former is concerned merely with discerning by foregrounding and backgrounding).
	O2: Prioritising a dimension of **difference** when discerning sameness and difference across a set of given arguments. E.g., foregrounding a difference in the values of correspondingly positioned addends or coefficients.
	O3: Simultaneously foregrounding dimensions of **sameness and difference** when discerning sameness and difference across a set of given arguments. E.g., foregrounding addends or coefficients that consistently vary and are seen simultaneously as a dimension of 'sameness and difference'.
Localising	**L1:** Adjusting a value in a given argument and testing the adjustment by ignoring one or more of the given steps. E.g., adjusting and testing an addend or coefficient given in step 3 of a five-step argument, whereby the test is performed from the adjusted step onwards. (L1 actions can involve thinking ahead to achieve a specific type of outcome for an individual step or end result, e.g., an even number or a particular desired value).
	L2: Adjusting a value in a given argument and testing the adjustment by using all of the given steps. E.g., testing an adjustment of a value in an initial given step by systematically using the format of each subsequent given step; or, starting from an initial given step when testing an adjusted value elsewhere in an argument (whereby the test involves systematically performing all of the subsequent given steps). (As with L1 actions, L2 actions can involve thinking ahead to achieve a specific type of outcome for an individual step or end result, e.g., an even number or a particular desired value).
	L3: Re-adjusting a previously adjusted value and testing the re-adjustment. Reasoning for the re-adjusted value is influenced by an unsuccessful result of a previously adjusted and tested value. E.g., further increasing an addend or a coefficient to achieve a higher end result than was produced by a previous (lower value) increase.
Deconstructing	**D1:** Reducing the number of objects (structure) in a step of a given argument when reasoning about or writing a similar step for a similar case. E.g., truncating (shortening) a given step by ignoring a given term, or by writing only a simplified result of a step. (D1 step reductions are different to shortening an argument by an entire step when testing an adjusted value L1).
	D2. Reverse engineering, i.e., working backwards through part of a given step, or from one given step to an earlier given step. E.g., to identify the position and value of an adjustment (to achieve a desired result) by considering the effect of subsequent applied operations; or, when reasoning about which terms or angles sum to 90° or to 180°.
	D3: Reasoning that references the effect or outcome of inversing, cancelling out, or 'doing and undoing'. E.g., 'n is multiplied by 2 then divided by 2 which takes you back to just n'; or, solving an equation in x. (D3 actions are different to G2 actions that involve concurrently attending to two processes because the former makes specific reference to the effect of inversing. Hence, D3 actions can follow on, near-simultaneously, from a G2 Generalisation about an inverse relationship.
Extending	**E1:** Extending a recognised or conjectured pattern in the given arguments that involves **one process**. E.g., continuing, in the writing of a case similar to the given arguments, an identified pattern in the increment of correspondingly positioned addends or coefficients.
	E2: Extending a recognised or conjectured pattern in the given arguments that involves **two processes**. E.g., concurrently continuing, in the writing of a case similar to the given arguments, an identified pattern in each of two correspondingly positioned addends or coefficients.
	E3: Extending a recognised or conjectured pattern in the given arguments that involves a **process and a result**. E.g., continuing, in the writing of a case similar to the given arguments, an identified consistent link between an addend or a coefficient and a result.
Networking	**N1:** Reasoning about an argument by referencing, but not applying, a concept from a qualitatively discrete curriculum area . E.g., proportion or BIDMAS (order of operations).
	N2: Introducing and applying to an argument a concept from a qualitatively discrete curriculum area . E.g., fractions or ratio.
	N3: Reasoning that involves applying to an argument a value other than a positive integer. E.g., a negative integer or a decimal value.

Appendix F: Chronologically coded actions

		Initial variation task																												Transfer variation task							
		2a					2b						2c (argument-generation task)														2d				3 (argument-generation task)							
Number/algebra	Ash	O2	G1				no actions						G1	E1	D1											S	G2	D3			G1	E1	G3	L2	D1			
	Belle	O1	O2				G3						G3	E3	L1											S	D3				L2	L2						
	Carl	O1	O3				G2	G3	N2				E2	G1	G3	G1	N2	G1	G3	G1	L2	D1	N2	N2	E1	S	G3				G1	G1	G3	N2	L1	D1	L2	
	Dylan	O1	O1	O2	G1		G3	G2					G1	E1	G2	G3	E3									S	G2	D3			O1	L2	L3	N3				
	Emma	O1	O1	O2	O1	G3	G3	G2					G3	E3	D1											S	G1	G2	G3	D3	G3	L2	D2	L3				
	Freya	O2	O1				G2	G1	G2	O1	G2	D3	N2	G3	E3											S	G3	G2	D3		D2	L2	L3	L3				
Geometry/algebra	Ash	O2	O2				G1	G3	E3				O1	D1	E1											S	G1	G3	E3		G3	E3	D1					
	Belle	O1	O1	O2			G1	G3	E3				E1	E1												S	O2				G2	L2	D1					
	Carl	O1	O2				G1	G3	E3				O1	G3	E3											UNS	G1	G3	E3		N1	G2	N2	G3	E3	D1		
	Dylan	G3	O2	O1			G1	G3	E3				O1	O2	E1											S	G1	G3	E3		G2	G3	G3	E3	E3	L3	L3	D1
	Emma	O2	G2				G1	E1					O1	O2	E1											S	G1	E1			L2	L3	D1	N3				
	Freya	O1	O1	O2	O1	O1	O2	O1	G1	E1			O1	O2	E1											S	O2	O1	G1		G2	E2	L2					

Appendix G: Approved CUREC application

SECTION A: Filter for CUREC2 application This section determines whether your study raises more complex issues which require the completion of a full application for ethical review, known as the CUREC 2 application. **(Please mark 'X' in the Yes/No column as appropriate to indicate your response.)**		
1. Are research participants classed as **people whose ability to give free and informed consent is in question**? (This may include those under 18 (though see "competent youths"), prisoners, or adults "at risk".) Your attention is drawn to the University's **Safeguarding Code of Practice** and its implications for researchers involving children or adults at risk, including the need for the work to be risk assessed and for researchers to undertake related training. (**Note**: If any of your participants are aged 16 or under, please answer 'Yes' here **and** also answer question 5 below.)	Yes X	No
2. By taking part in the research, will participants be at serious risk of criminal prosecution (e.g. by providing information on drug abuse or child abuse)?	Yes	No X
3. Does the research involve the **deception** of participants?	Yes	No X
4. Does your research raise issues relevant to the **Counter-Terrorism and Security Act (the Prevent duty)**, which seeks to prevent people from being drawn into terrorism? Please see advice on this on our **Best Practice Guidance web page**.	Yes	No X
If you have answered 'No' to all of the questions above please go to **Section B**. If you have answered 'Yes' to any question above continue to question 5 below.		
5. Is your project covered by a CUREC **approved procedure** (formerly known as "CUREC Protocols")?	Yes X	No
If yes, please give research procedure number(s): IDREC 25 (Version 5.1)		
If you answered '**Yes'** to **ANY** of questions **1-4**, **and** answered '**No'** to question **5**, **please stop completing this checklist and do not submit it for ethical review**. Instead, please complete the CUREC 2 application form from the CUREC website. Then submit the CUREC 2 form for ethical review. If you answered 'Yes' to ANY of questions 1-3, **and** answered 'Yes' to question 5, please go on to **Section B**.		

SECTION B: Contact details and project description (NB: must be typed not handwritten)	
Contact details:	
1. **Principal investigator /** supervisor (if student research (title and full name):	Prof. Gabriel Stylianides and Dr Jenni Ingram
2. Name of student (if student research):	Jason Bentley
3. Degree programme, e.g. DPhil, BA, MPhil, BSc, MSc (if student research):	DPhil
4. Department or Institute name:	Department of Education
5. Address for correspondence (if different from above):	
6. University e-mail (**not** private email) and telephone:	jason.bentley@ssho.ox.ac.uk Phone: 07880540597

7. Name and status of others taking part in the project, e.g. third year undergraduate; postdoctoral research assistant:	n/a

SECTION B continued	
Project description:	
8. Title of research project:	Using variation tasks to investigate the relationship between learners' attention to structure and actions of proof transfer.
9. List of location(s) where project will be conducted:	Sir William Stanier School
10. If your research involves overseas travel or fieldwork and your department requires a travel risk assessment, will you have completed and returned a risk assessment form beforehand? (This has to be approved by your department before you travel. If you are travelling overseas, you are strongly advised to take out University travel insurance.)	Yes No Not required in this instance X
11. Anticipated duration of research project overall:	60 months or 5 years (maximum 5)
12. Anticipated start and end dates of the research project involving human participants:	From: (21/01/2018) as soon as possible To: (18/12/2023) Please note that you will need ethics approval **before** you start your research. CUREC 1As may take up to 30 days to process.
13. External organisation funding the research (if applicable): n/a	
14. Title and very brief and simple lay description of **research** (about 150 words), plus description (about 200 words) of the nature of participants.	
a) Title, brief description of research (150 words) in lay language. When describing the research, please include your methodology, how you are applying professional guidelines, and the use to which results/data will be put. **Please also declare any conflicts of interest here.**	

Title and description of the research:

Using variation tasks to investigate the relationship between learners' argument-based attention to structure and actions of proof transfer.

The goals are: (1) formulation of an attention-to-structure framework for characterising how learners can attend to mathematical structure, and (2) use this framework to relate the character of learners' proof-based attention to structure to their actions when engaging in tasks of proof transfer.

Research questions:

(1) Can a framework be formulated that is useful for describing attention to structure?

(2) In the context of an initial and a transfer proof-based variation task, where and how do Y8 and Y10 learners (new to proof) attend to structure?

(3) How can learners' attention to structure across the tasks in (2) be characterised?

(4) How does the character of learners' attention to structure from (3) vary across domains and year-groups?

SECTION B continued

b) Description of participants and **obtaining informed consent** (200 words). When describing participants, please include

- criteria for inclusion/exclusion
- method of recruitment
- processes for consent to participate

Please ensure you attach as separate documents (if applicable, in English translation):

- your recruitment and advertisement material e.g. a poster or brief invitation letter/ email
- information for participants to read (or hear) before they agree to take part e.g. written information sheets or (only if applicable) oral information scripts.
- a document to record informed consent. Templates for written consent forms and/or oral information scripts (in case of an oral consent process) are available from the CUREC website
- a guide to interview questions (this may be a list of questions to be asked, or a preliminary scope of questions), or a sample of other instruments (such as a sample questionnaire)
- (if relevant) debriefing document after participants have taken part

Data will be collected from semi-structured task-based one-to-one (teacher/learner) interviews. The principal investigator is the applicant, Jason Bentley, who as part of his DPhil project, will conduct the research in his role as mathematics teacher, with his own regularly timetabled students, at Sir William Stanier Academy School.

The pilot project will involve twelve participants: six year 7 students (11- to 12-year-olds) and six year 10 students (14- to 15-year-olds). The participants will be those who have prominently demonstrated a propensity for conversational interaction with me during their regular mathematics lessons. Hence, the pilot participants constitute a purposive sample.

At the end of one of the intended participants' regular mathematics lessons (i.e. as their class is exiting their mathematics classroom), I will briefly make the intended participants initially aware of the project. At this point, I will also inform them of a voluntary lunchtime meeting in which further details of the project will be explained. In this voluntary meeting, I will explain to the attendees the purpose and nature of the project, the voluntary aspect and the right to withdraw, and the anonymous ways in which data will be collected and used. I will also seek the number of interested participants.

15. What are the ethical issues connected with your research and what steps have you taken to address them? Please do not answer 'none'. The committee needs to see evidence that you have identified potential ethical issues with respect to your research and have taken steps to address them. These issues could relate to:

- your own physical and psychological safety as a researcher (please see the University's and Social Science Division's Safety in Fieldwork guidance
- participant burdens and/or risks, and
- data protection/ confidentiality (please also see section 18).

For more guidance on ethical issues, please see http://researchsupport.admin.ox.ac.uk/governance/ethics/resources

Permission for the project to take place will be sought from the Assistant Principal and from the Head of the Mathematics Faculty at Sir William Stanier Academy School.

Since I am the intended participants' regular mathematics teacher, it is possible that the intended participants may feel obliged to attend the voluntary lunchtime meeting, and to participate in the research. To alay this possibility, I will repeatedly emphasize the voluntary aspect of the research verbally, as well as display and discuss the points on the (oral) participant information sheet (version 3). To further allay any sense of obligation in potential participants, I will: (1) explain the nature of the research to the intended participants *as a group* (rather than on a personal one-to-one basis), (2) allow reasonable time (one week) for learners to decide on whether they would like to participate, and (3) explain that uninterested learners can convey their preference to not participate simply by not confirming a wish to participate within the one week consideration period.

The lunchtime 'project information' meeting will take place during the second half of intended participants' lunchtime. Intended participants will be advised to have their lunch before attending the meeting. This is to avoid the risk of these students missing their lunch by attending the meeting. This condition/precaution will also be applied when students participate in the task-based interviews, since the task-based interviews will also take place over a series of school lunchtimes.

During the lunchtime meeting, and in relation to the task-based interviews, all attendees will be made aware of the one-to-one student/teacher dynamic, and the fact that I will ask questions of them as they engage with the task(s). During my display and explanation of the points on the participant information sheet, attendees will be made aware of the anonymous way in which data will be collected and used, of the positioning and use of a video/audio recorder, and of their right to withdraw at any point.

During the task-based interviews (which will take place in my classroom during the second half of lunchtime), participants will be given the option of having a non-participatory friend in the classroom if they wish to. The questions that will be asked of participants are delibrately designed to avoid inclusion of instructional cues that will narrowly direct and channel their thinking.

To stay within the IDREC 25 (Version 5.1) protocol, the positioning of a video/audio recorder during the task-based interviews will be such that, in regard to video recordings, only the hands and the work of students will be captured. That is, the identity of all participants will remain anonymous for all aspects of the project. In accordance with university policy, when deemed no longer useful, all collected data will be securely and permenently destroyed.

16. Will you obtain informed consent according to CUREC guidelines and good practice in your discipline before participation?	Yes	No x

If you have marked '**No**', please give a brief explanation and justification for this decision here:

The videoing of teaching situations (even those in which all participants are positively identifiable) is considered to be an in-school norm for the purposes of professional teacher development e.g. for training sessions based on lesson study, and for teachers' self-reflection opportunities. Since the audio/video recordings for this project are fully anonymised and aimed at better supporting students' learning (by way of their regular mathematics teacher), only oral consent will be obtained from students. Written informed consent will however be obtained from the assistant principal and head of faculty, as well as the principal if required.

17. Will your research involve discussing sensitive issues? This could be information relating to race or ethnic origin, political opinions, religious beliefs, physical/mental health, trade union membership, sexual life or criminal activities.	Yes	No x

If you have marked '**Yes**', please make sure that you have included some **supporting information** (as directed in question 14 of this section) showing the range of questions covering these issues.

18. **Management and handling of personal and other research data**

Your management and handling of personal data and special category data of human participants, either directly or via a third party, will need to comply with the requirements of the General Data Protection Regulation (GDPR) and the new Data Protection Act, as set out in the University's Guidance on Data Protection and Research. In answering the questions below, please also consider the points raised in the Data Protection Checklist. For advice on research data management and security, please consult with the University's Research Data Team (**researchdata@ox.ac.uk)** and/or your local IT department and the University's web pages on research data management.

a) Will your research involve the collection of **records of consent** (e.g. written forms, audio-recorded, or other recorded consent)? **If 'Yes', these will be classed as fully identifiable personal data (directly linked to an individual).**	Yes	No x
b) Will your research involve the collection of **other personal data?** **If 'Yes'**, specify in what form(s) this will be stored:	Yes	No x
• Fully identifiable (directly linked to an individual)	Yes	No
• Pseudonymised (potentially identifiable as data may be attributed to an individual if linkage information can be accessed elsewhere by researchers)	Yes	No
• Fully anonymised (i.e. cannot be linked to an individual)	Yes	No
c) Will any of the personal data you collect classify as **special category** data? **If 'Yes',** in what form(s) will this be stored:	Yes	No x
• Fully identifiable (directly linked to an individual)	Yes	No
• Pseudonymised (potentially identifiable as data may be attributed to an individual if linkage information can be accessed elsewhere by researchers)	Yes	No
• Fully anonymised (i.e. cannot be linked to an individual)	Yes	No
d) How will any personally identifiable data be collected, transferred and backed up? Please describe the arrangements for any physical transfer of personal data (including paper records and data captured electronically via portable media) from where it is collected to local storage.		
Initially on a Micro USB card (used in the video camera), then fully transferred onto the principle investigator's encrypted USB pen drive.		
e) Where, and for how long, will participants' personally identifiable data be stored during and after the study? (Please outline the procedures for ensuring confidentiality, eg security arrangements, anonymisation or pseudonymisation of such data. Please distinguish between records of consent and other forms of personally identifiable data stored)		
Linkage information for the anonymisation of participants will be kept on the principle investigator's encrypted USB pen drive. Participants' anonymised work will be scanned and stored electronically, any paper-based documents will be stored in a locked storage cupboard in my classroom. It is expected that this data will be stored for up to 5 years after the pilot study.		
f) If storing pseudonymised data, please confirm that identifiers will be held separately from the research data and linked through a unique study number. Specify how and at what point the pseudonymisation will occur, how the linkage information will be stored and state whether or not (and when) the linkage will be destroyed.		

Participants will be referred to by consecutive letters of the alphabet. Their task sheets will be pre-marked as Student A, Student B, etc. Linkage information will be kept electronically on an encrypted USB pen drive which will be permanently formatted/erased (not merely deleted) when no longer required.
g) Who will have access to the personally identifiable data? If personally identifiable data is to be shared with another organisation, how will it be transferred/disclosed securely?
The principle investigator only, Jason Bentley.
h) When and how will personally identifiable data be destroyed? (NB. Personally identifiable data should be destroyed when no longer required.)
It is expected that this data will be stored for up to 5 years after the pilot study.
i) How, where and for how long will other research data be stored after the study has finished? For more information about University and research funder retention policies, please see the University's web pages on research data management.
It is expected that the work produced by participants (in scanned form), as well as the audio transcripts will be electronically stored on the principle investigator's encrypted USB stick for up to 5 years after the pilot study.

SECTION C: Methods and procedures to be used	
Method used: Please ensure you have addressed any potential ethical issues related to these methods in Section 14 and in your Participant Information Sheet	**Please mark 'X'**
1. Analysis of existing records	
2. Snowball sampling (recruiting through contacts of existing participants)	
3. Use of casual or local workers e.g. interpreters	
4. Participant observation	x
5. Covert observation	
6. Observation of specific organisational practices	
7. Participant completes questionnaire in hard copy	x
8. Participant completes online questionnaire or other online task	
9. Using social media	
10. Participant performs paper and pencil task	x
11. Participant performs verbal or aural task (e.g. for linguistic study)	x
12. Focus group	
13. Interview	x
14. Audio recording of participant (you will generally need specific consent from participants for this)	x
15. Video recording of participant (you will generally need specific consent from participants for this)	x
16. Photography of participant (you will generally need specific consent from participants for this)	
17. Others (please specify):	x
Video is of participants' work/hands only. Photographs of participants' work.	

SECTION D: Professional guidelines and training		
In this section, please mark 'X' against at least one of the following professional guidelines you aim to adhere to. You should use the principles listed in your chosen guideline(s) in conducting your own research. **Note**: this is not an exhaustive list.		**Please mark 'X'**
Research specialism/ methodology	**Association and guidance document**	
Anthropology	Association of Social Anthropologists of the UK and Commonwealth	
Criminology	http://www.britsoccrim.org/ethics/	
Education	British Educational Research Association Ethical Guidelines for Educational Research	x
Geography	Association of American Geographers Statement on Professional Ethics	
History	Oral History Society of the UK Ethical Guidelines	
Internet-based Research	British Psychological Society: Conducting Research on the Internet Association of Internet Researchers Ethics Guide Also see our Best Practice Guidance on internet-based research	
Law (Socio-Legal)	Socio-Legal Studies Association: Statement of Principles of Ethical Research	
Management	Academy of Management's Professional Code of Ethics	
Political Science	American Political Science Association (APSA) Guide to Professional Ethics in Political Science	
Politics	Political Studies Association. Guidelines for Good Professional Conduct	
Psychology	British Psychological Society Code of Ethics and Conduct	
Social Research	Social Research Association: Ethical Guidelines	
Sociology	The British Sociological Association: Statement of Ethical Practice	
Visual Research	ESRC National Centre for Research Methods Review Paper: Visual Ethics: Ethical Issues in Visual Research	
Other professional guidelines. Please specify the other guidelines used here*:*		
Please indicate what training in research ethics the researchers involved with this study have received, e.g. the title of the course and date completed (online training available at http://researchsupport.admin.ox.ac.uk/support/training/ethics). If no formal training has been undertaken, please indicate any discussions of research methodology between researchers and supervisors here.		
NIH Office of Extramural research training 'protecting human research participants'. Successful completion of training course. 21/01/13. Certificate no. 1336794. Research Integrity: Social and behavioural sciences (Online training/test). Score 20/20. Date: 28/07/17.		

SECTION E: Signatures (The SSH IDREC Secretariat accepts either option below. If you have a DREC, check which signature option it prefers.)

- **Option 1:** ‘Electronic signatures’, i.e. email confirmations from a University of Oxford email address, can be accepted. Separate emails should come from each of the relevant signatories as outlined below, indicating acceptance of the relevant responsibilities. **Pasted images of signatures cannot be accepted in the sections below.**
- **Option 2:** Handwritten (wet-ink) signatures. Please scan them and the rest of the checklist pages to create a single PDF document and email through.

Please ensure this checklist is signed by:

For staff research:	**For student research**:
1. Principal investigator	1. Principal investigator (project supervisor)
2. Head of Department (or nominee)	2. Head of Department (or nominee)
	3. Student researcher

1. **Principal investigator signature/supervisor signature** (if student research)

I understand my responsibilities as **principal investigator** as outlined in the CUREC glossary and guidance on the CUREC website.

I declare that the answers above accurately describe the research as presently designed, and that a new checklist will be submitted should the research design change in a way which would alter any of the above responses so as to require completion of CUREC 2 (involving full scrutiny by an IDREC). I will inform the relevant IDREC if I cease to be the principal investigator on this project and supply the name and contact details of my successor if appropriate.

Signature:

Date: 19th January 2019

Print name (block capitals): JASON BENTLEY

CUREC approval

[redacted]

Reply Reply all Forward

To: Gabriel Stylianides; Jenni Ingram

Tue 2019-04-30 12:48 PM

Cc: Jason Bentley; Education Research Office

Dear Prof. Stylianides and Dr Ingram,

Title and reference number: **Using Variation Tasks to Investigate the Relationship Between Learners’ Attention to Structure and Actions of Proof Transfer** - ED-CIA-19-093

The above application has now been considered on behalf of the Departmental Research Ethics Committee (DREC) in accordance with the procedures laid down by the University for ethical approval of all research involving human participants.

I am pleased to inform you that, on the basis of the information provided to DREC, the proposed research has been judged as meeting appropriate ethical standards, and accordingly, approval has been granted.

If your research involves participants whose ability to give free and informed consent is in question (this includes those under 18 and vulnerable adults), then it is advisable to read the following NSPCC professional reporting requirements for cases of suspected abuse http://www.nspcc.org.uk/Inform/research/questions/reporting_child_abuse_wda74908.html

Should there be any subsequent changes to the project which raise ethical issues not covered in the original application you should submit details to research.office@education.ox.ac.uk for consideration.

Good luck with your research study.

Yours sincerely,

[redacted]

Title [redacted]

[redacted]

Department of Education
15 Norham Gardens, Oxford OX2 6PY
E: [redacted]
W: [redacted]
T: [redacted]