

# AN ISOMORPHISM OF MOTIVIC GALOIS GROUPS

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ABSTRACT. In characteristic 0 there are essentially two approaches to the conjectural theory of mixed motives, one due to Nori and the other one due to, independently, Hanamura, Levine, and Voevodsky. Although these approaches are a priori quite different it is expected that ultimately they can be reduced to one another. In this article we provide some evidence for this belief by proving that their associated motivic Galois groups are canonically isomorphic.

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## 1. INTRODUCTION

**Motives and motivic Galois groups.** Let  $k$  be a field of characteristic 0. Following Beilinson, Deligne, Grothendieck among others, there should be a  $\mathbb{Q}$ -linear abelian monoidal category  $\mathcal{MM}(k)$  of (*mixed*) *motives* over  $k$  together with a monoidal functor  $M : (\mathrm{Var}/k)^{\mathrm{op}} \rightarrow \mathcal{MM}(k)$ , associating to each variety  $X/k$  its

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motive  $M(X)$ , the universal cohomological invariant of  $X$ . Every cohomology theory  $h : (\text{Var}/k)^{\text{op}} \rightarrow \mathcal{A}$  for varieties over  $k$  should factor through a realization functor  $R_h : \mathcal{MM}(k) \rightarrow \mathcal{A}$ , i.e.  $h(X) = R_h(M(X))$ . For some cohomology theories one would expect this realization functor to present  $\mathcal{MM}(k)$  as a neutral Tannakian category with Tannakian dual  $\mathcal{G}(k)$ , a pro-algebraic group called the *motivic Galois group* of  $k$ . One of the main practical advantages of the Tannakian description of motives is that it would allow the translation of arithmetic and geometric questions about  $k$ -varieties into questions about (pro-)algebraic groups and their representations. Moreover, the maximal pro-reductive quotient of this group is supposed to coincide with what was classically known as the motivic Galois group, namely the group associated to the Tannakian subcategory of *pure* motives over  $k$  (i.e. the universal cohomology theory for *smooth projective* varieties; see [32] for the philosophy underlying this smaller group).

Although this picture is still conjectural, there are candidates for these objects and related constructions. Assume there is an embedding  $\sigma : k \hookrightarrow \mathbb{C}$ . In this situation there are essentially two existing approaches to motives, one due to Nori and another due to several mathematicians, including Voevodsky. Nori constructed a diagram of pairs of varieties together with the Betti representation into finite dimensional  $\mathbb{Q}$ -vector spaces, and applied to it his theory of Tannaka duality for diagrams. It yields a universal factorization for the Betti representation through a  $\mathbb{Q}$ -linear abelian category  $\mathbf{HM}(k)$  with a faithful exact  $\mathbb{Q}$ -linear functor  $\mathcal{O}_{\text{Bti}} : \mathbf{HM}(k) \rightarrow \mathbf{Mod}^f(\mathbb{Q})$  to finite dimensional  $\mathbb{Q}$ -vector spaces.  $\mathbf{HM}(k)$  is a neutral Tannakian category with fiber functor  $\mathcal{O}_{\text{Bti}}$ , whose Tannakian dual  $\mathcal{G}_N(k)$  is Nori's motivic Galois group.

On the other hand, there is the better known construction of  $\mathbf{DM}(k)$ , the triangulated category of Voevodsky motives. This is a candidate not for the category of motives but its derived category, and from which the former should be obtained as the heart of a t-structure.<sup>1</sup> Ayoub constructed a Betti realization functor  $\text{Bti}^* : \mathbf{DM}(k) \rightarrow \mathbf{D}(\mathbb{Q})$  to the category of graded  $\mathbb{Q}$ -vector spaces. He also proved that this functor together with its right adjoint  $\text{Bti}_*$  satisfies the assumptions of his weak Tannakian formalism which in the case at hand endows  $\text{Bti}^* \text{Bti}_* \mathbb{Q}$  with the structure of a Hopf algebra. And finally, he established that its homology is concentrated in non-negative degrees, hence the Hopf algebra structure passes to its homology in degree 0. Ayoub's motivic Galois group  $\mathcal{G}_A(k)$  is the spectrum of this Hopf algebra.

**Main result.** Our main goal is to prove that the two motivic Galois groups just described are isomorphic, thus answering a question of Ayoub in [4].

**Theorem (instance of 9.1).** There is an isomorphism of affine pro-algebraic groups over  $\text{Spec}(\mathbb{Q})$ :

$$\mathcal{G}_A(k) \cong \mathcal{G}_N(k).$$

Let us try to put this result into perspective. As explained in [3], the difference between the two approaches to motives is extreme. Nori's construction relies on transcendental data in an essential way whereas  $\mathbf{DM}(k)$  is defined purely in terms of algebro-geometric data. This has the effect that while morphisms in  $\mathbf{DM}(k)$  can be related to previously known algebro-geometric invariants of varieties, morphisms

<sup>1</sup>More precisely, its full subcategory of geometric motives (i.e. compact objects) is a candidate for the bounded derived category of  $\mathcal{MM}(k)$ .

and extensions in  $\mathbf{HM}(k)$  are intractable. On the other hand, the universal property defining  $\mathbf{HM}(k)$  would be enough to characterize the true category of motives (if the latter exists), whereas in Voevodsky's approach it is difficult even to extract an abelian category. One of the ultimate goals in the theory of motives therefore is to create a bridge connecting these two approaches. This goal is considered to be far out of reach at the moment, but the result above can be seen as providing a weak link while sidestepping the more difficult and deep issues. Moreover, it suggests breaking up the goal into two subgoals: understanding the relation between (compact) Voevodsky motives and comodules over  $\mathrm{Bti}^*\mathrm{Bti}_*\mathbb{Q}$  on the one hand, and proving that this Hopf algebra is homologically concentrated in degree 0 on the other hand; see [3] and [4, §2.4] for further discussion.

Even if the link we provide here is a weak one, it can still be seen as evidence for the “correctness” of the two approaches to motives. Moreover, although both constructions of motivic Galois groups are based on some form of Tannaka duality, the precise form is quite different in the two cases (cf. [4, Introduction]); therefore the isomorphism in the theorem can be seen as a surprising phenomenon. Finally, the identification of the two groups allows for transfer of techniques and results, not easily available on both sides without the identification. We plan to use this fact in the future to give a more elementary description of the Kontsevich-Zagier period algebra with fewer generators and relations. More precisely, we intend to show that the algebra considered in [6, §2.2] is canonically isomorphic to the Kontsevich-Zagier period algebra, as was claimed in *loc. cit.*

We would like to remark that a conditional proof of our main result has been given independently by Jon Pridham. In [27, Exa. 3.20] he sketches how the existence of a motivic  $t$ -structure (which renders the Betti realization  $t$ -exact) would imply the isomorphism of motivic Galois groups. The argument uses the theory of Tannaka duality for dg categories developed in *loc. cit.*

**About the proof.** As one would expect from the relation between motives and their associated Galois groups, proving our main result involves “comparing” Nori motives with Voevodsky motives. As we remarked above, this is a non-trivial task and we can hope to relate these two categories only indirectly:

- We construct a realization of Nori motives in the category of linear representations of Ayoub's Galois group:

$$\mathbf{Rep}(\mathcal{G}_N(k)) \rightarrow \mathbf{Rep}(\mathcal{G}_A(k)).$$

The main ingredients used in this construction are the six functors formalism for motives without transfers due to Ayoub and Voevodsky, and its compatibility with the Betti realization, proved by Ayoub.

- We construct a realization of motives without transfers in the category of graded  $\mathcal{O}(\mathcal{G}_N(k))$ -comodules:

$$\mathbf{DA}(k) \rightarrow \mathbf{coMod}^{\mathbb{Z}}(\mathcal{O}(\mathcal{G}_N(k))).$$

In this construction the main tool used is the Basic Lemma due to Nori and, independently, to Beilinson.

In fact, we work throughout with arbitrary principal ideal domains as coefficients, not only  $\mathbb{Q}$ . Since we also use extensively the six functors formalism, we are forced to work with  $\mathbf{DA}(k)$ , motives without transfers, instead of  $\mathbf{DM}(k)$ . In any case,

the motivic Galois group of Ayoub does not see the difference between these two categories.

The two realizations will induce morphisms between the two Galois groups, and the hard part is to prove that these are inverses to each other. In one direction, we rely heavily on one of the main results of Ayoub's approach to motivic Galois groups, namely a specific model he has given for the object in  $\mathbf{DA}(k)$  representing Betti cohomology. Analysing this model closely we can show that the coordinate ring of  $\mathcal{G}_A(k)$  as a  $\mathcal{G}_A(k)$ -representation is generated by  $\mathcal{G}_N(k)$ -representations  $H_{\text{Betti}}^i(X, Z; \mathbb{Q}(j))$ . This will allow us to prove the morphism  $\mathcal{G}_A(k) \rightarrow \mathcal{G}_N(k)$  a closed immersion. For the other direction we will prove that  $\mathcal{G}_N(k) \rightarrow \mathcal{G}_A(k)$  is a section to  $\mathcal{G}_A(k) \rightarrow \mathcal{G}_N(k)$ , and here the idea is to reduce all verifications to a class of pairs of varieties whose relative motive in  $\mathbf{DA}(k)$  (and its effective version) are easier to handle. We have found that the pairs  $(X, Z)$  where  $X$  is smooth and  $Z$  a simple normal crossings divisor work well for our purposes, and we study their motives without transfers in detail.

**Outline of the article.** We now give a more detailed account of the article. In §2 we recall the construction and basic properties of Nori motives and the associated Galois group. We also state a monoidal version of the universal property of his category of motives the proof of which is given in appendix A. In §3 we recall the construction and basic properties of Morel-Voevodsky motives (or motives without transfer) and the Betti realization. We also explain in detail in which sense the functor  $\text{Bti}^*$  is a Betti realization. We briefly recall the construction of Ayoub's Galois group for Morel-Voevodsky motives in §4.

In §5 we construct motives  $\mathcal{R}_A(X, Z, n)$  in  $\mathbf{DA}(k)$  for  $X$  a variety,  $Z \subset X$  a closed subvariety and  $n$  a non-negative integer. These motives are defined in terms of the six functors, and have the property that  $H_0(\text{Bti}^* \mathcal{R}_A(X, Z, n)) \cong H_n(X(\mathbb{C}), Z(\mathbb{C}))$  naturally. Here is where our decision to use the six functors formalism pays off as its compatibility with the Betti realization immediately reduces us to prove the existence of a natural isomorphism between sheaf cohomology and singular cohomology of pairs of (locally compact) topological spaces. We were not able to find the required proofs for this last comparison in the literature, and we therefore decided to provide them in a separate appendix (to wit, appendix B). We end this section by showing how this construction yields a morphism of Hopf algebras  $\varphi_A : \mathcal{O}(\mathcal{G}_N(k)) \rightarrow \mathcal{O}(\mathcal{G}_A(k))$ .

The following two sections 6 and 7 are devoted to defining a morphism in the other direction, at least on the "effective" bialgebras (the Hopf algebras are obtained from these effective bialgebras by inverting a certain element). For this, in §6, we recall Nori's version of the Basic Lemma, and explain how it leads to algebraic cellular decompositions of the singular homology of affine varieties. As an application we obtain a functor from smooth affine schemes to the derived category of effective Nori motives. In §7.1 we show how Kan extensions in the context of dg categories allow us to extend it to a functor  $LC^*$  defined on the category of effective Morel-Voevodsky motives.  $LC^*$  is then shown to give rise to the sought after morphism of bialgebras (§7.2).

We also collect additional results on realizing (Morel-)Voevodsky motives in Nori motives which are not strictly necessary for our main theorem but, we believe, of independent interest. In §7.3, we extend our constructions to take into account correspondences thus obtaining a variant of  $LC^*$  for effective Voevodsky motives

(i.e. effective motives with transfers). Then, in §7.4, we also prove that these realizations pass to the stable categories of motives (with and without transfers). From this we finally deduce mixed Hodge realizations on motives with and without transfers.

The next section is all about explicit computations involving Morel-Voevodsky motives associated to pairs of schemes. The recurrent theme is that these computations are feasible if one restricts to the pairs  $(X, Z)$  where  $X$  is smooth and  $Z$  is a simple normal crossings divisor. We call these almost smooth pairs, and resolution of singularities implies that there are enough of them. This allows us to reduce computations for general pairs to these more manageable ones. In §8.1 we give models for the latter on the effective level, and determine their image under the functor  $LC^*$  explicitly. This allows us to compare their comodule structure (with respect to Nori's effective bialgebra mentioned above) to the one of the Betti homology of the pair. As a corollary, we see that the morphism of bialgebras passes to the Hopf algebras  $\varphi_N : \mathcal{O}(\mathcal{G}_A(k)) \rightarrow \mathcal{O}(\mathcal{G}_N(k))$ . In §8.2 we give good models for  $\mathcal{R}_A(X, Z, n)$  and their duals on the stable level, when  $(X, Z)$  is almost smooth, and we describe their Betti realization.

§9 is the heart of the article. Ayoub has given a “singular” model for the object in  $\mathbf{DA}(k)$  representing Betti cohomology. Using our description of  $\mathcal{R}_A(X, Z, n)$  and performing a close analysis of Ayoub's model we establish that the Hopf algebra  $\mathcal{O}(\mathcal{G}_A(k))$  as a comodule over itself is a filtered colimit of Nori motives  $H^i(X(\mathbb{C}), Z(\mathbb{C}); \mathbb{Q}(j))$ , where  $(X, Z)$  is almost smooth and  $i, j \in \mathbb{Z}$ . This will be seen to imply surjectivity of  $\varphi_A$ , while on the other hand we also prove that  $\varphi_N \varphi_A$  is the identity by proving that it is so on motives of almost smooth pairs.

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**Notation and conventions.** We fix a field  $k$  of characteristic 0 together with an embedding  $\sigma : k \hookrightarrow \mathbb{C}$ . By a scheme we mean a quasi-projective scheme over  $k$ . A variety is a reduced scheme. Rings are always assumed commutative and unital. Monoidal categories (resp. functors, transformations) are assumed symmetric and unitary if not stated otherwise. Algebras and coalgebras in monoidal categories (also known as monoids and comonoids, respectively) are assumed unitary resp. counitary.

$\Lambda$  throughout denotes a fixed ring, assumed noetherian if not stated otherwise. The symbol  $\mathbf{Cpl}(\Lambda)$  denotes the category of (unbounded) complexes of  $\Lambda$ -modules. Our conventions are homological, i.e. the differentials decrease the indices, and the shift operator satisfies  $(A[p])_n = A_{p+n}$ . For an abelian category  $\mathcal{A}$ ,  $\mathbf{D}(\mathcal{A})$  denotes

its derived category, and  $\mathbf{D}(\Lambda) := \mathbf{D}(\mathbf{Cpl}(\Lambda))$ . Also,  $\text{Ind } \mathcal{A}$  denotes the category of ind objects in  $\mathcal{A}$ .

## 2. NORI'S GALOIS GROUP

We begin by recalling the construction of Nori motives and the associated motivic Galois group (cf. [26, 20, 19]). We also describe the universal property of the category of Nori motives in a monoidal setting (Theorem 2.3).

**2.1. Effective Nori motives.** A theory of motives, at the very least, should assign to any pair of varieties  $(X, Z)$  objects  $H_n(X, Z)$  ( $n \in \mathbb{N}$ ), the  $n$ th relative (homological) motive of the pair. Moreover, this assignment should be functorial in the pair, and come with a “boundary” morphism  $H_n(X, Z) \rightarrow H_{n-1}(Z)$ . We can formalize this using the diagram<sup>2</sup>  $\mathcal{D}_N$ :

- its vertices are triples  $(X, Z, n)$  where  $X$  is a variety,  $Z$  is a closed subvariety of  $X$ , and  $n$  is an integer;
- there are two types of edges: a single edge from  $(X, Z, n)$  to  $(Z, W, n-1)$  for any triple  $X \supset Z \supset W$ , and edges  $(X, Z, n) \rightarrow (X', Z', n)$  indexed by morphisms  $f : X \rightarrow X'$  which restrict to  $f : Z \rightarrow Z'$ .

A representation of such a diagram is simply a morphism of directed graphs  $T : \mathcal{D}_N \rightarrow \mathcal{C}$  into a category  $\mathcal{C}$ .

Instead of specifying explicitly what axioms such a representation should satisfy, Nori's idea was to impose *all* axioms satisfied by a fixed homology theory. This is made precise by his Tannakian theory for diagrams which asserts that associated to a representation  $T : \mathcal{D} \rightarrow \mathbf{Mod}^f(\Lambda)$  of any diagram  $\mathcal{D}$  in the category of finitely generated  $\Lambda$ -modules, there is a  $\Lambda$ -linear abelian category  $\mathcal{C}(T)$  together with a factorization

$$\mathcal{D} \xrightarrow{\tilde{T}} \mathcal{C}(T) \xrightarrow{o} \mathbf{Mod}^f(\Lambda)$$

where  $o$  is a faithful exact  $\Lambda$ -linear functor. Moreover, this category is universal for such a factorization. This is applied to the singular homology representation  $H_\bullet : \mathcal{D}_N \rightarrow \mathbf{Mod}^f(\Lambda)$  which takes a vertex  $(X, Z, n)$  to the relative singular homology  $\Lambda$ -module  $H_n(X^{\text{an}}, Z^{\text{an}}; \Lambda)$  of the associated topological spaces on the  $\mathbb{C}$ -points (this uses  $\sigma : k \hookrightarrow \mathbb{C}$ ). To the single edge  $(X, Z, n) \rightarrow (Z, W, n-1)$  it associates the boundary map of the long exact sequence of a triple  $H_n(X^{\text{an}}, Z^{\text{an}}) \rightarrow H_{n-1}(Z^{\text{an}}, W^{\text{an}})$ , and to an edge  $(X, Z, n) \rightarrow (X', Z', n)$  corresponding to  $f : X \rightarrow X'$  it associates the morphism in homology induced by  $f^{\text{an}} : X^{\text{an}} \rightarrow X'^{\text{an}}$ .<sup>3</sup>

**Definition 2.1.** The category  $\mathcal{C}(H_\bullet)$  is denoted by  $\mathbf{HM}^{\text{eff}}$ . It is the category of *effective (homological) Nori motives*.

**Remark 2.2.** The construction of  $\mathcal{C}(T)$  is easy to describe. A finite (full) subdiagram  $\mathcal{F} \subset \mathcal{D}$  gives rise to a  $\Lambda$ -algebra  $\text{End}(T|_{\mathcal{F}}) \subset \prod_v \text{End}_\Lambda(T(v))$  of families of compatible (with respect to the edges) endomorphisms indexed over the vertices of  $\mathcal{F}$ . We then set

$$\mathcal{C}(T) = \varinjlim_{\mathcal{F} \subset \mathcal{D}} \mathbf{Mod}^f(\text{End}(T|_{\mathcal{F}})),$$

this filtered 2-colimit being indexed by the finite subdiagrams ordered by inclusion.

<sup>2</sup>By a diagram we mean a directed graph.

<sup>3</sup>Here, and in the sequel we refrain from writing the coefficients in the homology when these can be guessed from the context. Also, we sometimes write  $H_\bullet(X, Z)$  instead of  $H_\bullet(X^{\text{an}}, Z^{\text{an}})$ .

In case  $\Lambda$  is a principal ideal domain and  $T$  takes values in finitely generated free  $\Lambda$ -modules, the dual  $\mathcal{A}(T|_{\mathcal{F}}) = \text{End}(T|_{\mathcal{F}})^{\vee}$  carries a canonical coalgebra structure for any finite subdiagram  $\mathcal{F} \subset \mathcal{D}$ . Moreover,  $\mathcal{C}(T)$  can then be described as the category  $\mathbf{coMod}^f(\mathcal{A}(T))$  of  $\mathcal{A}(T)$ -comodules in  $\mathbf{Mod}^f(\Lambda)$ ,<sup>4</sup> where

$$\mathcal{A}(T) = \varinjlim_{\mathcal{F} \subset \mathcal{D}} \mathcal{A}(T|_{\mathcal{F}}).$$

**2.2. Stabilization.** In order to talk about Tate twists and to stabilize the category of effective Nori motives we will need a monoidal version of Nori's Tannakian theory for diagrams. The problem one faces is that the tensor product of relative motives  $H_n(X, Z) \otimes H_{n'}(X', Z')$  is typically not equal to  $H_{n+n'}((X, Z) \times (X', Z'))$  but only related to the latter via the Künneth spectral sequence. It is therefore natural to restrict to pairs whose homology is concentrated in a single degree.

Consider the full subdiagram  $\mathcal{D}_N^g$  of  $\mathcal{D}_N$  consisting of *good pairs*, i.e. vertices  $(X, Z, n)$  with  $X \setminus Z$  smooth and  $H_{\bullet}(X, Z; \mathbb{Z})$  a free abelian group concentrated in degree  $n$ . It follows essentially from the Basic Lemma (recalled in §6) that  $\mathcal{C}(H_{\bullet}|_{\mathcal{D}_N^g})$  is canonically equivalent to  $\mathbf{HM}^{\text{eff}}$  (see [26, Pro. 3.2] or [20, Cor. 1.7] for a proof). Moreover, on  $\mathcal{D}_N^g$  there is a “commutative product structure with unit” in the sense of [20] induced by the cartesian product of varieties, and  $H_{\bullet}|_{\mathcal{D}_N^g}$  is canonically a u. g. m. representation (see appendix A for a recollection on these notions). This endows  $\mathbf{HM}^{\text{eff}}$  with a monoidal structure such that the functor  $o$  mapping to  $\mathbf{Mod}^f(\Lambda)$  is monoidal ([26, Thm. 4.1], [20, Pro. B.16]). As in the non-monoidal case it has a universal property which we state in the following instance of Nori's Tannaka duality theorem in the monoidal setting A.1 (cf. also [9, 20, 19]).

**Theorem 2.3.** *Let  $\Lambda$  be a principal ideal domain. Suppose we are given a right exact monoidal abelian  $\Lambda$ -linear category  $\mathcal{A}$ <sup>5</sup> together with a monoidal faithful exact  $\Lambda$ -linear functor  $o : \mathcal{A} \rightarrow \mathbf{Mod}^f(\Lambda)$  and a u. g. m. representation  $T : \mathcal{D}_N^g \rightarrow \mathcal{A}$  such that the following diagram of solid arrows commutes.*

$$\begin{array}{ccc} \mathcal{D}_N^g & \xrightarrow{T} & \mathcal{A} \\ \tilde{H}_{\bullet} \downarrow & \nearrow & \downarrow o \\ \mathbf{HM}^{\text{eff}} & \xrightarrow{o} & \mathbf{Mod}^f(\Lambda) \end{array}$$

*Then there exists a monoidal functor  $\mathbf{HM}^{\text{eff}} \rightarrow \mathcal{A}$  (unique up to unique monoidal isomorphism), represented by the dotted arrow in the diagram rendering the two triangles commutative (up to monoidal isomorphism).*

*Moreover, this functor is faithful exact  $\Lambda$ -linear.*

Assume now that  $\Lambda$  is a principal ideal domain. By the discussion preceding the Theorem and Remark 2.2,  $\mathbf{HM}^{\text{eff}}$  can be identified with the category of comodules over the coalgebra  $\mathcal{A}(H_{\bullet}|_{\mathcal{D}_N^g})$ . The monoidal structure on  $\mathbf{HM}^{\text{eff}}$  endows this coalgebra with the structure of a commutative algebra turning it into a (commutative) bialgebra ([26, §4.2], [20, Pro. B.16]).

<sup>4</sup>For our conventions regarding comodules see appendix C.

<sup>5</sup>Hence  $(\mathcal{A}, \otimes)$  is a monoidal abelian  $\Lambda$ -linear category such that  $\otimes$  is right exact  $\Lambda$ -linear in each variable.



- Definition 2.4.** (1) We denote by  $\mathbf{H}_N^{\text{eff}} := \mathbf{H}_{N,\Lambda}^{\text{eff}} := \mathcal{A}(\mathbf{H}_\bullet |_{\mathcal{D}_N^g})$  *Nori's effective motivic bialgebra*.
- (2) *Nori's motivic Hopf algebra*  $\mathbf{H}_N := \mathbf{H}_{N,\Lambda}$  is the commutative Hopf algebra obtained from  $\mathbf{H}_N^{\text{eff}}$  by localizing (as an algebra) with respect to an element  $s_N \in \mathbf{H}_N^{\text{eff}}$  described below (see also [26, p. 13]).
- (3) *Nori's motivic Galois group*  $\mathcal{G}_N := \mathcal{G}_{N,\Lambda}$  is the spectrum of  $\mathbf{H}_N$ . Thus it is a pro-group scheme over  $\text{Spec}(\Lambda)$ .
- (4) The category  $\mathbf{HM} := \mathbf{HM}_\Lambda := \mathbf{coMod}^f(\mathbf{H}_{N,\Lambda})$  is the category of (*homological*) *Nori motives*.

It remains to describe the element  $s_N \in \mathbf{H}_N^{\text{eff}}$  corresponding to the Tate twist. Choose an isomorphism

$$H_1(\mathbb{G}_m, \{1\}) \xrightarrow{\sim} \Lambda. \quad (2.5)$$

Then  $s_N \in \mathbf{H}_N^{\text{eff}}$  is the image of  $1 \in \Lambda$  under the composition

$$\Lambda \xleftarrow[\sim]{(2.5)} H_1(\mathbb{G}_m, \{1\}) \xrightarrow{\text{ca}} \mathbf{H}_N^{\text{eff}} \otimes H_1(\mathbb{G}_m, \{1\}) \xrightarrow[\sim]{(2.5)} \mathbf{H}_N^{\text{eff}} \otimes \Lambda \cong \mathbf{H}_N^{\text{eff}},$$

where  $\text{ca}$  denotes the coaction of  $\mathbf{H}_N^{\text{eff}}$  on  $H_1(\mathbb{G}_m, \{1\})$ . Clearly,  $s_N$  does not depend on the choice of (2.5).

### 3. BETTI REALIZATION FOR MOREL-VOEVODSKY MOTIVES

As explained in the introduction, we will work with motives without transfers in order to use the six functors formalism. Based on Voevodsky's original construction of the triangulated category of motives with transfers  $\mathbf{DM}(k)$ , and following an insight by Morel, this formalism has been worked out by Ayoub in [1]. In this section we briefly recall the construction of this category of motives, and the associated Betti realization from [2] and [4]. We also prove a few results not stated there explicitly. In particular we give a dg model for the Betti realization.

**3.1. Effective Morel-Voevodsky motives.** For the category of effective motives without transfers, and just as with Nori motives, we will start with a certain category of varieties. Instead of imposing axioms satisfied by the Betti realization however, Morel and Voevodsky give the axioms explicitly (Mayer-Vietoris and homotopy invariance). The precise construction may profitably be viewed from the perspective of *universal model dg categories* in the sense of [10].

We start with a small category  $\mathcal{C}$  with finite products, endowed with a Grothendieck topology  $\tau$  and  $I \in \mathcal{C}$  an “object parametrizing homotopies”; also fix any ring  $\Lambda$ . The category  $\mathbf{UC} = \mathbf{Psh}(\mathcal{C}, \mathbf{Cpl}(\Lambda))$  of presheaves on  $\mathcal{C}$  with values in complexes of  $\Lambda$ -modules can be endowed with three model structures (among others):

- The *projective model structure* whose fibrations (resp. weak equivalences) are objectwise epimorphisms (resp. quasi-isomorphisms). This defines the model category underlying the universal model dg category associated to  $\mathcal{C}$ . Its homotopy category is just the derived category  $\mathbf{D}(\mathbf{UC})$ .
- The *projective  $\tau$ -local model structure* arises from the projective model structure by Bousfield localization with respect to  $\tau$ -hypercovers. This defines the model category underlying the universal  $\tau$ -local model dg category associated to  $\mathcal{C}$ . Its homotopy category is equivalent to the derived category of  $\tau$ -sheaves on  $\mathcal{C}$ .



- The *projective*  $(I, \tau)$ -local model structure arises as a further Bousfield localization with respect to arrows  $\Lambda(I \times Y)[i] \rightarrow \Lambda(Y)[i]$ , where  $\Lambda : \mathcal{C} \rightarrow \mathbf{UC}$  denotes the “Yoneda embedding”, and  $Y \in \mathcal{C}$  and  $i \in \mathbb{Z}$  are arbitrary. This defines the model category underlying the universal  $(\tau, I)$ -local model dg category associated to  $\mathcal{C}$ . Its homotopy category is a  $\Lambda$ -linear unstable (or “effective”)  $I$ -homotopy theory of  $(\mathcal{C}, \tau)$ .

In each case, the model category is stable and monoidal (for the objectwise tensor product) hence the homotopy categories are triangulated monoidal. The following examples will be of interest to us (notation is explained subsequently):

$\mathcal{C}$	$\tau$	$I$	$\Lambda$ -linear unstable $I$ -homotopy theory
$\mathbf{Sm}/X$	Nis or ét	$\mathbb{A}_X^1$	$\mathbf{DA}^{\mathrm{eff}}(X) = \mathbf{DA}_{\Lambda}^{\mathrm{eff}}(X)$
$\mathbf{SmAff}/X$	Nis or ét	$\mathbb{A}_X^1$	$\mathbf{DA}_{\mathrm{aff}}^{\mathrm{eff}}(X) = \mathbf{DA}_{\mathrm{aff}, \Lambda}^{\mathrm{eff}}(X)$
$\mathbf{AnSm}/X$	usu	$\mathbb{D}_X^1$	$\mathbf{AnDA}^{\mathrm{eff}}(X) = \mathbf{AnDA}_{\Lambda}^{\mathrm{eff}}(X)$
$\mathbf{Open}/X$	usu	$X$	$\mathbf{D}(X, \Lambda)$

Here, in the first two examples,  $X$  is a scheme, and  $\mathbf{Sm}/X$  (resp.  $\mathbf{SmAff}/X$ ) denotes the category of smooth schemes over  $X$  (resp. which are affine in the absolute sense) endowed with the Nisnevich or the étale topology. In case  $X = \mathrm{Spec}(k)$ , we denote this category by  $\mathbf{Sm}$  (resp.  $\mathbf{SmAff}$ ).

**Definition 3.1.**  $\mathbf{DA}^{\mathrm{eff}}(X)$  is the category of *effective Morel-Voevodsky  $X$ -motives*.

In the few instances where the topology chosen plays any role, we will make this explicit. Also, if  $X = \mathrm{Spec}(k)$  then we simply write  $\mathbf{DA}^{\mathrm{eff}}$ . Since every scheme is covered by affine open subschemes, we obtain the following easy fact (see [10, Cor. 5.16] for a proof).

**Lemma 3.2.** *The canonical inclusion  $\mathbf{SmAff}/X \rightarrow \mathbf{Sm}/X$  induces a triangulated monoidal equivalence  $\mathbf{DA}_{\mathrm{aff}}^{\mathrm{eff}}(X) \xrightarrow{\sim} \mathbf{DA}^{\mathrm{eff}}(X)$ .*

In the third example above,  $X$  is a complex analytic space, i. e. a “complex space” in the sense of [14] which is supposed to be denumerable at infinity, and  $\mathbf{AnSm}/X$  denotes the category of complex analytic spaces smooth over  $X$  with the topology usu given by open covers. If  $X$  is the terminal object pt, then  $\mathbf{AnSm}/X$  is denoted simply by  $\mathbf{Man}_{\mathbb{C}}$ .  $\mathbb{D}^1$  denotes the open unit disk considered as a complex analytic space. As above, the  $\mathbb{D}^1$ -homotopy theory is denoted by  $\mathbf{AnDA}^{\mathrm{eff}}$  in case  $X = \mathrm{pt}$ .

Finally, in the fourth example  $X$  denotes a topological space,  $\mathbf{Open}/X$  the category associated to the preorder of open subsets of  $X$ . It is endowed with the topology usu given by open covers. The  $(X, \mathrm{usu})$ -local and the usu-local model structures evidently agree, and their homotopy category  $\mathbf{D}(X) = \mathbf{D}(X, \Lambda)$  is (canonically identified with) the derived category of sheaves on  $X$ .

**3.2. Effective Betti realization.** The Betti realization due to Ayoub in [2] will now link the categories just introduced, as follows. For a complex analytic space  $X$  there is an obvious inclusion  $\iota_X : \mathbf{Open}/X \rightarrow \mathbf{AnSm}/X$  which defines a morphism of sites and induces a Quillen equivalence ([2, Thm. 1.8])

$$(\iota_X^*, \iota_{X*}) : \mathbf{U}(\mathbf{Open}/X)/\mathrm{usu} \rightarrow \mathbf{U}(\mathbf{AnSm}/X)/(\mathbb{D}_X^1, \mathrm{usu}).$$

If  $X = \mathrm{Spec}(k)$ , the left adjoint takes a complex to the associated constant presheaf and is denoted by  $(\bullet)_{\mathrm{cst}}$ , while the right adjoint is the global sections functor and accordingly denoted by  $\Gamma$ .

Any scheme  $Y$  gives rise to a complex analytic space  $Y^{\text{an}}$ , namely the topological space  $(Y \times_{k,\sigma} \mathbb{C})(\mathbb{C})$  with the natural complex analytic structure. We obtain an analytification functor  $\text{An}_X : \text{Sm}/X \rightarrow \text{AnSm}/X^{\text{an}}$  which induces Quillen adjunctions

$$(\text{An}_X^*, \text{An}_{X*}) : \mathbf{U}(\text{Sm}/X) \rightarrow \mathbf{U}(\text{AnSm}/X^{\text{an}})$$

for the corresponding model structures considered above. The left adjoint  $\text{An}_X^*$  in fact preserves  $(I, \tau)$ -local weak equivalences (see [4, Rem. 2.57]).

**Definition 3.3.** The *effective Betti realization* is the composition

$$\text{Bti}^{\text{eff},*} : \mathbf{DA}^{\text{eff}}(X) \xrightarrow{\text{An}_X^*} \mathbf{AnDA}^{\text{eff}}(X) \xrightarrow[\sim]{R\iota_{X*}} \mathbf{D}(X).$$

By construction, this is a triangulated monoidal functor.

**3.3. Stabilization.** Motives will be obtained from effective motives by a stabilization process which we again describe in the abstract setting first. Let  $\mathcal{M}$  be a cellular left-proper monoidal model category and  $T \in \mathcal{M}$  a cofibrant object. The category  $\mathbf{Spt}_T^\Sigma \mathcal{M}$  of symmetric  $T$ -spectra in  $\mathcal{M}$  admits the following two model structures (among others):

- The *projective unstable model structure* whose fibrations (resp. weak equivalences) are levelwise fibrations (resp. weak equivalences).
- The *projective stable model structure* arises from the unstable one by Bousfield localization with respect to morphisms  $\text{Sus}_T^{n+1}(T \otimes K) \rightarrow \text{Sus}_T^n(K)$  for cofibrant objects  $K \in \mathcal{M}$ .

Here,  $(\text{Sus}_T^i, \text{Ev}_i) : \mathbf{UC} \rightarrow \mathbf{Spt}_T^\Sigma \mathbf{UC}$  denotes the canonical adjunction,  $\text{Ev}_i$  being evaluation at level  $i$ . For the details (also concerning the existence of the model structures) we refer to [18]. Again, the model categories are both monoidal, and if  $\mathcal{M}$  was stable then so is  $\mathbf{Spt}_T^\Sigma \mathcal{M}$ . If not mentioned explicitly otherwise, when we refer to *the* model structure on  $\mathbf{Spt}_T^\Sigma \mathcal{M}$  we mean the stable one. In  $\mathbf{Spt}_T^\Sigma \mathcal{M}$ , tensoring with  $T$  becomes a Quillen equivalence, and it should be thought of as the universal such model category although this is not quite true in the obvious sense (cf. [18, §9]; see also [28] for the  $(\infty, 1)$ -categorical version of the picture, and where such a universal property *can* be proven). Still, we call it the  $T$ -stabilization of  $\mathcal{M}$ .

In the algebraic geometric examples above we choose  $T_X$  to be a cofibrant replacement of  $\Lambda(\mathbb{A}_X^1)/\Lambda(\mathbb{G}_{m,X})$ .

**Definition 3.4.** The resulting  $T_X$ -stable  $\mathbb{A}^1$ -homotopy theory of  $(\text{Sm}/X, \tau)$ , denoted by  $\mathbf{DA}(X) = \mathbf{DA}_\Lambda(X)$ , is the category of *Morel-Voevodsky  $X$ -motives*.

As before (Lemma 3.2), the affine version  $\mathbf{DA}_{\text{aff}}(X)$  is canonically equivalent. Again, we leave the topology implicit most of the time, and in case  $X = k$  we also write  $\mathbf{DA}$ . There is the notion of a compact motive, namely an object in  $\mathbf{DA}$  which is compact in the sense of triangulated categories, and the full subcategory of compact motives forms a thick triangulated subcategory.

In the analytic setting, stabilization is performed with respect to a cofibrant replacement  $T_X$  of the quotient presheaf  $\Lambda(\mathbb{A}_X^{1,\text{an}})/\Lambda(\mathbb{G}_{m,X}^{\text{an}})$ . The resulting homotopy category is denoted by  $\mathbf{AnDA}(X) = \mathbf{AnDA}_\Lambda(X)$  (and again simply  $\mathbf{AnDA}$  in case  $X = \text{pt}$ ). [2, Lem. 1.10] together with [18, Thm. 9.1] show that the adjunction

$$(\text{Sus}_{T_X}^0, \text{Ev}_0) : \mathbf{U}(\text{AnSm}/X)/(\mathbb{D}_X^1, \text{usu}) \rightarrow \mathbf{Spt}_{T_X}^\Sigma \mathbf{U}(\text{AnSm}/X)/(\mathbb{D}_X^1, \text{usu})$$

defines a Quillen equivalence. Moreover, the analytification functor passes to the level of symmetric spectra and preserves stable  $(I, \tau)$ -local equivalences.

**Definition 3.5.** The *Betti realization* is the composition

$$\mathrm{Bti}^* : \mathbf{DA}(X) \xrightarrow{\mathrm{An}_X^*} \mathbf{AnDA}(X) \xrightarrow[\sim]{\mathrm{R}\ell_X * \mathrm{REv}_0} \mathbf{D}(X).$$

Again, it is a triangulated monoidal functor.

**3.4. Six functor formalism.** We recall that the six functors constitute a formalism on the categories  $\mathbf{DA}(X)$  for schemes  $X$  which associates to any morphism of schemes  $f : X \rightarrow Y$  adjunctions

$$(\mathrm{L}f^*, \mathrm{R}f_*) : \mathbf{DA}(Y) \rightarrow \mathbf{DA}(X), \quad (\mathrm{L}f_!, f^!) : \mathbf{DA}(X) \rightarrow \mathbf{DA}(Y), \quad (3.6)$$

and which endows  $\mathbf{DA}(X)$  with a closed monoidal structure

$$(\otimes^{\mathrm{L}}, \mathrm{R}\underline{\mathrm{Hom}})^6.$$

All these functors are triangulated. The formalism governs the relation between them, e.g. under what conditions two of these functors can be identified or when they commute. Some of these relations are given explicitly in [1, Sch. 1.4.2]. We will also heavily use the part concerning duality. Recall that on compact motives there is a contravariant autoequivalence  $(\bullet)^{\vee}$  which exchanges the two adjunctions in (3.6) so that for example  $(\mathrm{R}f_* M)^{\vee} \cong \mathrm{L}f_! M^{\vee}$  for any compact motive  $M$  (see [1, Thm. 2.3.75]).

The same formalism is available in the analytic (see [2]) and in the topological setting (at least if the topological space is locally compact, see e.g. [22]). The main result of Ayoub in [2] is that the Betti realization is compatible with these, at least if one restricts to compact motives.

**3.5. dg enhancement.** In the remainder of the section we will exhibit the (effective) Betti realization as the derived functor of a left Quillen dg functor (Proposition 3.7). This will be used in §7 to construct a motivic realization  $\mathbf{DA}^{\mathrm{eff}} \rightarrow \mathbf{D}(\mathrm{Ind}\mathbf{HM}^{\mathrm{eff}})$ . Our argument here relies on our discussion of left dg Kan extensions in [10], and we will frequently refer to that paper.

Let  $X$  be a complex analytic space. Denote by  $\mathrm{Sg}(X)$  the complex of singular chains in  $X$  (with  $\Lambda$ -coefficients). This extends to a lax monoidal functor  $\mathrm{Sg}$  on topological spaces in virtue of the Eilenberg-Zilber map (cf. [12, VI, 12]). Its “left dg Kan extension” (rather, the functor underlying the left dg Kan extension of [10, Fact 2.1]) is denoted by

$$\mathrm{Sg}^* : \mathbf{U}(\mathrm{Man}_{\mathbb{C}}) \rightarrow \mathbf{Cpl}(\Lambda).$$

It possesses an induced lax-monoidal structure, by [10, Lemma 2.2]. Moreover, for each complex manifold  $X$ ,  $\mathrm{Sg}(X)$  is projective cofibrant hence  $\bullet \otimes \mathrm{Sg}(X)$  is a left Quillen functor. It follows from [10, Lemma 2.5] that  $\mathrm{Sg}^*$  is also a left Quillen functor with respect to the projective model structures.

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<sup>6</sup>In the literature the symbols  $\mathrm{L}$  and  $\mathrm{R}$  indicating that some left or right derivation takes place are often dropped from the notation. For us however, the distinction between the derived and underived functors will be important which is why we stick to the clumsier notation.

**Proposition 3.7.**  $\mathrm{LSg}^*$  takes  $(\mathbb{D}^1, \mathrm{usu})$ -local equivalences to quasi-isomorphisms and the induced functor

$$\mathbf{DA}^{\mathrm{eff}} \xrightarrow{\mathrm{An}^*} \mathbf{AnDA}^{\mathrm{eff}} \xrightarrow{\mathrm{LSg}^*} \mathbf{D}(\Lambda)$$

is isomorphic to  $\mathrm{Bti}^{\mathrm{eff},*}$  as monoidal triangulated functor. In particular, the following triangle commutes up to a monoidal isomorphism:

$$\begin{array}{ccc} \mathrm{Sm} & \xrightarrow{\Lambda(\bullet)} & \mathbf{DA}^{\mathrm{eff}} \\ & \searrow \mathrm{Sg} \circ \mathrm{An} & \downarrow \mathrm{Bti}^{\mathrm{eff},*} \\ & & \mathbf{D}(\Lambda) \end{array}$$

*Proof.* In fact, we will deduce the first statement from the second.

For this let us recall the “singular analytic complexes” constructed in [4, §2.2.1]. We denote by  $\mathbb{D}^1(r)$  the open disk of radius  $r$  centered at the origin (thus  $\mathbb{D}^1 = \mathbb{D}^1(1)$ ) and by  $\mathbb{D}^n(r)$  the  $n$ -fold cartesian product ( $n \geq 0$ ). Letting  $r > 1$  vary we obtain pro-complex manifolds  $\overline{\mathbb{D}}^n = (\mathbb{D}^n(r))_{r>1}$ . There is an obvious way to endow the family  $(\overline{\mathbb{D}}^n)_{n \geq 0}$  with the structure of a cocubical object in the category of pro-complex manifolds (see [4, Déf. 2.19]). For any complex manifold  $X$  one then deduces a cubical  $\Lambda$ -module  $\underline{\mathrm{hom}}(\overline{\mathbb{D}}^\bullet, X)$ , where the latter in degree  $n$  is given by  $\varinjlim_{r>1} \Lambda \mathrm{Man}_{\mathbb{C}}(\mathbb{D}^n(r), X)$ . The associated simple complex (see [4, Déf. A.4]) is called the singular analytic complex associated to  $X$ , and is denoted by  $\mathrm{Sg}^{\mathbb{D}}(X)$ . It clearly extends to a functor

$$\mathrm{Sg}^{\mathbb{D}} : \mathrm{Man}_{\mathbb{C}} \rightarrow \mathbf{Cpl}(\Lambda),$$

and admits a natural lax monoidal structure induced by the association

$$(a : \mathbb{D}^m(r) \rightarrow X, b : \mathbb{D}^n(r) \rightarrow Y) \mapsto (a \times b : \mathbb{D}^{m+n}(r) \rightarrow X \times Y).$$

We would now like to prove that  $\mathrm{Sg}^{\mathbb{D}}$  and  $\mathrm{Sg}$  are monoidally quasi-isomorphic, and for this we need a third, intermediate singular complex.

For any real number  $r > 1$ , denote by  $\mathbb{I}^1(r)$  the open interval  $(-r, r)$ . Set  $\mathbb{I}^1 = \mathbb{I}^1(1)$ . There is an obvious analytic embedding  $\mathbb{I}^1(r) \rightarrow \mathbb{D}^1(r)$  of real analytic manifolds. Denote by  $\mathbb{I}^n(r)$  the  $n$ -fold cartesian product of  $\mathbb{I}^1(r)$ . Letting  $r > 1$  vary we obtain a pro-real analytic manifold  $\overline{\mathbb{I}}^n$ . There is an obvious embedding of pro-real analytic manifolds  $\overline{\mathbb{I}}^n \rightarrow \overline{\mathbb{D}}^n$  for each  $n$ , and by restriction this induces the structure of a cocubical object in pro-real analytic manifolds on  $\overline{\mathbb{I}}^\bullet$ .

Denoting by  $i$  the inclusion  $\mathrm{Man}_{\mathbb{C}} \hookrightarrow \mathrm{Man}_{\mathbb{R}}^{\omega}$  we obtain a monoidal natural transformation

$$\mathrm{Sg}^{\mathbb{D}} \rightarrow \mathrm{Sg}^{\mathbb{I}} \circ i, \tag{3.8}$$

and it suffices to prove that this is sectionwise a quasi-isomorphism. Indeed, a similar argument as in [25, App. A, §2, Thm. 2.1] shows that the right hand side is monoidally quasi-isomorphic to the analogous functor of cubical complexes of *continuous* functions. And the latter is in turn monoidally quasi-isomorphic to  $\mathrm{Sg}$ , by [15, Thm. 5.1].

Following [4], we denote the “left dg Kan extension” of  $\mathrm{Sg}^{\mathbb{D}}$  again by the same symbol  $\mathrm{Sg}^{\mathbb{D}} : \mathbf{UMan}_{\mathbb{C}} \rightarrow \mathbf{Cpl}(\Lambda)$ . (That this indeed coincides with the functor in [4] follows from [10, Lemma 3.21]. Cocontinuity is a consequence of [4, Lem. A.3].) [4, Thm. 2.23] together with [4, Cor. 2.26, 2.27] show that  $\mathrm{Sg}^{\mathbb{D}}$  takes  $(\mathbb{D}^1, \mathrm{usu})$ -local

equivalences to quasi-isomorphisms. The same argument also shows that  $\mathrm{Sg}^{\mathbb{I}}$  takes  $(\mathbb{I}, \mathrm{usu})$ -local equivalences to quasi-isomorphisms. Now, let's start with a complex manifold  $X$ . We want to prove that (3.8) applied to  $X$  is a quasi-isomorphism. For this, choose a usu-hypercover  $X_{\bullet} \rightarrow X$  of complex manifolds such that each representable in each degree is contractible. This can also be considered as a usu-hypercover of real analytic manifolds, and by what we just discussed, the two horizontal arrows in the following commutative square are quasi-isomorphisms:

$$\begin{array}{ccc} \mathrm{Sg}^{\mathbb{D}}(X_{\bullet}) & \longrightarrow & \mathrm{Sg}^{\mathbb{D}}(X) \\ \downarrow & & \downarrow \\ \mathrm{Sg}^{\mathbb{I}}(i(X_{\bullet})) & \longrightarrow & \mathrm{Sg}^{\mathbb{I}}(i(X)) \end{array}$$

By [10, Lemma 4.2], we reduce to show that for any contractible complex manifold  $Y$ ,  $\mathrm{Sg}^{\mathbb{D}}(Y) \rightarrow \mathrm{Sg}^{\mathbb{I}}(i(Y))$  is a quasi-isomorphism, which is easy.

By [4, Cor. 2.26, Pro. 2.83],  $\mathrm{Bti}^{\mathrm{eff},*}$  is isomorphic to  $\mathrm{Sg}^{\mathbb{D}} \circ \mathrm{An}^*$  as triangulated monoidal functor hence the discussion above implies the second statement of the proposition. The first statement can now be deduced as follows. From the monoidal quasi-isomorphism  $\mathrm{Sg}^{\mathbb{D}} \sim \mathrm{Sg}$  we obtain triangulated monoidal isomorphisms (cf. [10, Lemma 2.2])

$$\mathrm{Sg}^{\mathbb{D}} \cong \mathrm{LSg}^{\mathbb{D}} \cong \mathrm{L}(\mathrm{Sg})^* : \mathbf{D}(\mathrm{UMan}_{\mathbb{C}}) \rightarrow \mathbf{D}(\Lambda).$$

Indeed, the second isomorphism can be checked on representables (these are compact generators of  $\mathbf{D}(\mathrm{UMan}_{\mathbb{C}})$  by [10, Lemma 3.20]) and these objects are cofibrant.  $\square$

**Remark 3.9.** Using the topological singular complex we can construct an explicit fibrant model for the unit spectrum in  $\mathbf{AnDA}$  as follows. Denote by  $\mathrm{Sg}^{\vee}$  the presheaf of complexes on  $\mathrm{Man}_{\mathbb{C}}$  which takes a complex manifold  $X$  to  $\mathrm{Sg}(X)^{\vee}$ . Let  $U = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta(\mathbb{P}^1)$ , and let  $u$  be a rational point of  $U$  over  $\mathbb{P}^1 \times \{\infty\}$ . As in [4, §2.3.1], we can use  $T^{\mathrm{an}} = \Lambda(U^{\mathrm{an}})/\Lambda(u^{\mathrm{an}})$  to form symmetric spectra (hence  $T^{\mathrm{an}}$  is a cofibrant replacement of  $\Lambda(\mathbb{A}^{1,\mathrm{an}})/\Lambda(\mathbb{G}_m^{\mathrm{an}})$ ). Fix  $\hat{\beta} \in \mathrm{Sg}_{-2}^{\vee}(U^{\mathrm{an}}, u^{\mathrm{an}}; \Lambda)$  whose class in  $H^2(U^{\mathrm{an}}, u^{\mathrm{an}}; \Lambda) \cong \Lambda$  is a generator. Define a symmetric  $T^{\mathrm{an}}$ -spectrum  $\mathbf{Sg}^{\vee}$  which in level  $n$  is  $\mathrm{Sg}^{\vee}[-2n]$  with the trivial  $\Sigma_n$ -action, and whose bonding maps are given by the adjoints of the quasi-isomorphism

$$\hat{\beta} \times \bullet : \mathrm{Sg}^{\vee}(X)[-2n] \rightarrow \mathrm{Sg}^{\vee}((U, u) \times X)[-2(n+1)]$$

for any complex manifold  $X$ .

The canonical morphism  $\Lambda_{\mathrm{cst}} \rightarrow \mathrm{Sg}^{\vee}$  induces by adjunction a morphism of symmetric spectra  $\mathrm{Sus}_{T^{\mathrm{an}}}^0 \Lambda_{\mathrm{cst}} \rightarrow \mathbf{Sg}^{\vee}$  which in level  $n$  is given by the composition

$$\begin{aligned} (T^{\mathrm{an}})^{\otimes n} \otimes \Lambda_{\mathrm{cst}} &\rightarrow (T^{\mathrm{an}})^{\otimes n} \otimes \mathrm{Sg}^{\vee} \\ &\xrightarrow{\mathrm{id} \otimes (\hat{\beta} \times \bullet)^n} (T^{\mathrm{an}})^{\otimes n} \otimes \underline{\mathrm{hom}}((T^{\mathrm{an}})^{\otimes n}, \mathrm{Sg}^{\vee}[-2n]) \\ &\xrightarrow{\mathrm{ev}} \mathrm{Sg}^{\vee}[-2n]. \end{aligned}$$

The first arrow is a usu-local equivalence, the second arrow is a quasi-isomorphism, and the third is a  $(\mathbb{D}^1, \mathrm{usu})$ -local equivalence since  $T^{\mathrm{an}}$  is invertible in  $\mathbf{AnDA}^{\mathrm{eff}}$ , by [2, Lem. 1.10]. It follows that  $\mathrm{Sus}_{T^{\mathrm{an}}}^0 \Lambda_{\mathrm{cst}} \rightarrow \mathbf{Sg}^{\vee}$  is a levelwise  $(\mathbb{D}^1, \mathrm{usu})$ -local

equivalence. Since the source is an  $\Omega$ -spectrum so is  $\mathbf{Sg}^\vee$ . Also, since  $\Lambda_{\text{cst}}$  is  $\mathbb{D}^1$ -local so is  $\mathbf{Sg}^\vee$  levelwise. Finally, for any usu-hypercover  $X_\bullet \rightarrow X$  of a complex manifold  $X$ ,  $\text{Sg}^\vee(X) \rightarrow \text{Sg}^\vee(X_\bullet)$  is a quasi-isomorphism which proves that  $\mathbf{Sg}^\vee$  is levelwise usu-fibrant.

Summing up, we have proved that  $\mathbf{Sg}^\vee$  is a projective stable  $(\mathbb{D}^1, \text{usu})$ -fibrant replacement of  $\text{Sus}_{T^{\text{an}}}^0 \Lambda_{\text{cst}}$ .

#### 4. AYOUB'S GALOIS GROUP

We recall here the construction of Ayoub's motivic Galois group in [4]. In section 1 of that paper he develops a weak Tannaka duality theory which allows to factor certain monoidal functors  $f : \mathcal{M} \rightarrow \mathcal{E}$  between monoidal categories universally as

$$\mathcal{M} \xrightarrow{\tilde{f}} \mathbf{coMod}(\mathcal{H}(f)) \xrightarrow{o} \mathcal{E} \quad (4.1)$$

for a commutative bialgebra  $\mathcal{H}(f) \in \mathcal{E}$ , where  $o$  is the forgetful functor, and where both functors in the factorization are monoidal. This was applied in [4] to the monoidal (effective) Betti realization functor

$$\text{Bti}^* : \mathbf{DA} \rightarrow \mathbf{D}(\Lambda) \quad (\text{resp. } \text{Bti}^{\text{eff},*} : \mathbf{DA}^{\text{eff}} \rightarrow \mathbf{D}(\Lambda)),$$

see Definitions 3.3 and 3.5.

**Definition 4.2.** (1) *Ayoub's effective motivic bialgebra* is  $\mathcal{H}_A^{\text{eff}} := \mathcal{H}_{A,\Lambda}^{\text{eff}} := \mathcal{H}(\text{Bti}^{\text{eff},*}) \in \mathbf{D}(\Lambda)$ .  
 (2) *Ayoub's motivic Hopf algebra* is  $\mathcal{H}_A := \mathcal{H}_{A,\Lambda} := \mathcal{H}(\text{Bti}^*) \in \mathbf{D}(\Lambda)$ . It is indeed a (commutative) Hopf algebra, as shown in [4].

The bialgebras do not depend (up to canonical isomorphism) on the topology chosen. Explicitly, as objects in  $\mathbf{D}(\Lambda)$  they are given by  $\mathcal{H}_A = \text{Bti}^* \text{Bti}_* \Lambda$  and  $\mathcal{H}_A^{\text{eff}} = \text{Bti}^{\text{eff},*} \text{Bti}_*^{\text{eff}} \Lambda$ .

We said above that these bialgebras enjoy a universal property; let us recall the precise statement for the effective case (an analogous statement holds in the stable situation but we will not use this).

**Fact 4.3** ([4, Pro. 1.55]). *Suppose we are given a commutative bialgebra  $K$  in  $\mathbf{D}(\Lambda)$  and a commutative diagram in the category of monoidal categories*

$$\begin{array}{ccc} \mathbf{DA}^{\text{eff}} & \xrightarrow{f} & \mathbf{coMod}(K) \\ & \searrow \text{Bti}^{\text{eff},*} & \downarrow o \\ & & \mathbf{D}(\Lambda) \end{array}$$

where  $o$  is the forgetful functor, such that  $f(A_{\text{cst}})$  is the trivial  $K$ -comodule associated to  $A$ , for any  $A \in \mathbf{D}(\Lambda)$ . Then there exists a unique morphism of bialgebras  $\mathcal{H}_A^{\text{eff}} \rightarrow K$  making the following diagram commutative:

$$\begin{array}{ccc} \mathbf{DA}^{\text{eff}} & \xrightarrow{f} & \mathbf{coMod}(K) \\ \downarrow \widetilde{\text{Bti}}^{\text{eff},*} & \nearrow & \downarrow o \\ \mathbf{coMod}(\mathcal{H}_A^{\text{eff}}) & \xrightarrow{o} & \mathbf{D}(\Lambda) \end{array}$$

Now, consider the functor  $H_0 : \mathbf{D}(\Lambda) \rightarrow \mathbf{Mod}(\Lambda)$  which associates to a complex its 0th homology. By [4, Cor. 2.105], the homology of  $\mathcal{H}_A^{\text{eff}}$  and  $\mathcal{H}_A$  is concentrated in non-negative degrees and it follows that the bialgebra (resp. Hopf algebra) structure descends to the 0th homology of  $\mathcal{H}_A^{\text{eff}}$  (resp.  $\mathcal{H}_A$ ).

**Definition 4.4.** (1) We set  $\mathbf{H}_A^{\text{eff}} := H_0(\mathcal{H}_A^{\text{eff}}) \in \mathbf{Mod}(\Lambda)$ . This is thus a bialgebra over  $\Lambda$ .  
 (2) We set  $\mathbf{H}_A := H_0(\mathcal{H}_A) \in \mathbf{Mod}(\Lambda)$ . This is thus a Hopf algebra over  $\Lambda$ .  
 (3) *Ayoub's motivic Galois group*  $\mathcal{G}_A := \mathcal{G}_{A,\Lambda}$  is the spectrum of  $\mathbf{H}_A$ . Thus it is a pro-group scheme over  $\text{Spec}(\Lambda)$ .

**Remark 4.5.** By [4, Thm. 2.14],  $\mathcal{H}_A$  (resp.  $\mathbf{H}_A$ ) is obtained by localization from  $\mathcal{H}_A^{\text{eff}}$  (resp.  $\mathbf{H}_A^{\text{eff}}$ ), as follows. Choose an isomorphism

$$\text{Bti}^{\text{eff},*}(T[2]) \xrightarrow{\sim} \Lambda. \quad (4.6)$$

We then let  $s_A \in \mathcal{H}_A^{\text{eff}}$  be the image of  $1 \in \Lambda$  under the composition

$$\Lambda \xleftarrow[\sim]{(4.6)} \text{Bti}^{\text{eff},*}(T[2]) \xrightarrow{\text{ca}} \mathcal{H}_A^{\text{eff}} \otimes \text{Bti}^{\text{eff},*}(T[2]) \xrightarrow[\sim]{(4.6)} \mathcal{H}_A^{\text{eff}} \otimes \Lambda \cong \mathcal{H}_A^{\text{eff}},$$

where ca denotes the coaction of  $\mathcal{H}_A^{\text{eff}}$  on  $\text{Bti}^{\text{eff},*}(T[2])$ . Clearly,  $s_A$  does not depend on the choice of the isomorphism (4.6). By [4, Thm. 2.14],  $\mathcal{H}_A$  is the sequential homotopy colimit of the diagram

$$\mathcal{H}_A^{\text{eff}} \xrightarrow{s_A \times \bullet} \mathcal{H}_A^{\text{eff}} \xrightarrow{s_A \times \bullet} \dots$$

Applying  $H_0$  we see that  $\mathbf{H}_A = \mathbf{H}_A^{\text{eff}}[s_A^{-1}]$  as an algebra.

In order to apply the results on the category of  $\mathbf{H}_A^{(\text{eff})}$ -comodules in appendix C we will need the following result.

**Lemma 4.7.** *Let  $\Lambda$  be a principal ideal domain. Then  $\mathbf{H}_A^{\text{eff}}$  and  $\mathbf{H}_A$  are flat  $\Lambda$ -modules.*

*Proof.* The proof is the same in both cases; we do it for  $\mathbf{H}_A^{\text{eff}}$ . By [5, Cor. 1.27],  $\mathcal{H}_A^{\text{eff}}$  sits in a distinguished triangle

$$C' \rightarrow \mathcal{H}_A^{\text{eff}} \rightarrow C \rightarrow C'[-1],$$

where  $C$  is a complex in  $\mathbf{D}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$ . (Explicitly,  $C' = C^0(\text{Gal}(\bar{k}, k), \Lambda)$ , the  $\Lambda$ -module of locally constant functions on the absolute Galois group of  $k$  with values in  $\Lambda$ , which maps canonically to  $\mathcal{H}_A^{\text{eff}}$ . Essentially due to the Rigidity Theorem of Suslin-Voevodsky, this map is a quasi-isomorphism for torsion coefficients.) Looking at the associated long exact sequence in homology one sees that all homologies of  $\mathcal{H}_A$  must be torsion-free thus flat.  $\square$

**Remark 4.8.** The Betti realization can be constructed in a similar way also for Voevodsky motives (see [5, §1.1.2]), and the same weak Tannakian formalism applies to give two bialgebras in  $\mathbf{D}(\Lambda)$ . It is proved in [5, Thm. 1.13] that they are canonically isomorphic to  $\mathcal{H}_A^{\text{eff}}$  and  $\mathcal{H}_A$ , respectively. In the case of the Hopf algebras (and this is the case we are chiefly interested in), this follows from the fact that the canonical functor

$$\mathbf{DA}^{\text{ét}}(k) \rightarrow \mathbf{DM}^{\text{ét}}(k)$$

is an equivalence.



## 5. MOTIVIC REPRESENTATION

The goal of this section is to factor the homology representation  $H_\bullet : \mathcal{D}_N \rightarrow \mathbf{Mod}^f(\Lambda)$  through the Betti realization  $H_0 \circ \mathrm{Bti}^* : \mathbf{DA} \rightarrow \mathbf{Mod}(\Lambda)$  (Propositions 5.3, 5.7) in order to obtain a morphism of bialgebras  $\varphi_\Lambda : \mathbf{H}_N \rightarrow \mathbf{H}_\Lambda$ . Let us see how to derive a solution  $\mathcal{R}_\Lambda : \mathcal{D}_N \rightarrow \mathbf{DA}$  to this task.

We saw in the previous section that for *smooth* schemes  $X$ ,  $\mathrm{Bti}^{\mathrm{eff},*}\Lambda(X)$  computes the Betti homology of  $X$ . A first guess might be that for *any* scheme  $X$ ,  $\mathrm{Bti}^{\mathrm{eff},*}\Lambda \mathrm{hom}(\bullet, X)$  also does. We don't know whether this is true. Instead we notice that, for  $X$  with smooth structure morphism  $\pi : X \rightarrow k$ , there are canonical isomorphisms

$$\mathrm{LSus}_T^0 \Lambda(X) \cong L\pi_{\#} \pi^* \Lambda \cong L\pi_! \pi^! \Lambda$$

in  $\mathbf{DA}$ . The last expression makes sense for *any* scheme  $X$ , and we will prove below that the Betti realization of this object indeed computes the Betti homology of  $X$ . We should remark that there is nothing original about this idea. The object  $L\pi_! \pi^! \Lambda$  (and not the presheaf  $\Lambda \mathrm{hom}(\bullet, X)$ ) is commonly considered to be the “correct” representation of  $X$  in  $\mathbf{DA}$ , and is therefore also called the (*homological*) *motive* of  $X$ . The six functors formalism also allows to naturally define a relative motive associated to a pair of schemes, and this will yield the representation  $\mathcal{R}_\Lambda$  we were looking for.

**5.1. Construction.** Let  $(X, Z, n)$  be a vertex in Nori's diagram of pairs. Fix the following notation:

$$Z \xrightarrow{i} X \xleftarrow{j} U, \quad \pi : X \rightarrow k,$$

where  $U = X \setminus Z$  is the open complement. Set

$$\mathcal{R}_\Lambda(X, Z, n) = L\pi_! Rj_* j^* \pi^! \Lambda[n] \in \mathbf{DA}.$$

This extends to a representation  $\mathcal{R}_\Lambda : \mathcal{D}_N \rightarrow \mathbf{DA}$  as follows:

- The first type of edge in  $\mathcal{D}_N$  is  $(X, Z, n) \rightarrow (Z, W, n-1)$ . We have the following distinguished (“localization”) triangle (of endofunctors) in  $\mathbf{DA}(X)$  (and similarly for the pair  $(Z, W)$ ):

$$i_! i^! \xrightarrow{\mathrm{adj}} \mathrm{id} \xrightarrow{\mathrm{adj}} Rj_* j^* \xrightarrow{\partial} i_! i^![-1]. \quad (5.1)$$

(Here, as in the sequel,  $\mathrm{adj}$  denotes the unit or counit of an adjunction.)

Applying  $L\pi_!$  and evaluating at  $\pi^! \Lambda[n]$ , we therefore obtain a morphism

$$\begin{aligned} \mathcal{R}_\Lambda(\partial) : \mathcal{R}_\Lambda(X, Z, n) &\xrightarrow{\partial} \mathcal{R}_\Lambda(Z, \emptyset, n-1) \\ &\xrightarrow{\mathrm{adj}} \mathcal{R}_\Lambda(Z, W, n-1). \end{aligned}$$

- The second type of edge  $(X, Z, n) \rightarrow (X', Z', n)$  is induced from a morphism of varieties  $f : X \rightarrow X'$  with  $f(Z) \subset Z'$ . We have the following commutative diagram of solid arrows in (endofunctors of)  $\mathbf{DA}(X')$ :

$$\begin{array}{ccccccc} Lf_! i_! i^! f^! & \longrightarrow & Lf_! f^! & \longrightarrow & Lf_! Rj_* j^* f^! & \longrightarrow & Lf_! i_! i^! f^![-1] \\ \downarrow \mathrm{adj} & & \downarrow \mathrm{adj} & & \downarrow \text{dotted} & & \downarrow \mathrm{adj} \\ i'_! i'^! & \longrightarrow & \mathrm{id} & \longrightarrow & Rj'_* j'^* & \longrightarrow & i'_! i'^![-1] \end{array}$$

where the rows are distinguished (“localization”) triangles, and where the dotted arrow is the unique morphism making the vertical arrows into a

morphism of triangles. (Uniqueness follows from the isomorphism  $Lf_!i_! \cong i'_!L(f|_Z)_!$  and the fact that there are no non-zero morphisms from  $i'_!$  to  $Rj'_*$ .) After applying  $L\pi'_!$ , shifting by  $n$ , and evaluating at  $\pi'^!\Lambda$  this dotted arrow gives the morphism  $\mathcal{R}_A(f) : \mathcal{R}_A(X, Z, n) \rightarrow \mathcal{R}_A(X', Z', n)$  associated to  $f$ .

We will prove in a moment that this representation has the expected properties. Before doing so we would like to recall the following classical result.

**Fact 5.2** ([17]). *Let  $(X, Z)$  be a pair of varieties. Then its analytification  $(X^{\text{an}}, Z^{\text{an}})$  is a locally finite CW-pair. In particular,  $X^{\text{an}}$  and  $Z^{\text{an}}$  are paracompact, locally contractible, and locally compact.*

**Proposition 5.3.** *Suppose that  $\Lambda$  is a principal ideal domain. Then there is an isomorphism of representations*

$$\begin{array}{ccc} \mathcal{D}_N & \xrightarrow{H_0 \widetilde{\text{Bti}}^* \mathcal{R}_A} & \mathbf{coMod}^f(\mathbf{H}_A) \\ & \searrow H_\bullet & \downarrow o \\ & & \mathbf{Mod}^f(\Lambda) \end{array}$$

*Proof.* By Fact 5.2 and B.3 the complex of relative singular cochains  $\text{Sg}(X^{\text{an}}, Z^{\text{an}}; \Lambda)^\vee$  provides a model for  $R\pi_*^{\text{an}} j_!^{\text{an}} \Lambda$  in  $\mathbf{D}(\Lambda)$ . We claim that the complex  $\text{Sg}(X^{\text{an}}, Z^{\text{an}})$  defines a strongly dualizable object in  $\mathbf{D}(\Lambda)$ . Indeed, using the distinguished triangle

$$\text{Sg}(Z^{\text{an}}) \rightarrow \text{Sg}(X^{\text{an}}) \rightarrow \text{Sg}(X^{\text{an}}, Z^{\text{an}}) \rightarrow \text{Sg}(Z^{\text{an}})[-1],$$

we reduce to prove it for  $\text{Sg}(X^{\text{an}})$ . As we will see in §6, this complex is quasi-isomorphic to a bounded complex of finitely generated free  $\Lambda$ -modules, thus it is a strongly dualizable object.<sup>7</sup>

Therefore the canonical map from  $\text{Sg}(X^{\text{an}}, Z^{\text{an}})$  to its double dual is a quasi-isomorphism, and we obtain the following sequence of isomorphisms in  $\mathbf{Mod}(\Lambda)$ , for every  $n \in \mathbb{Z}$ :

$$\begin{aligned} H_n(X^{\text{an}}, Z^{\text{an}}) &= H_n \text{Sg}(X^{\text{an}}, Z^{\text{an}}) \\ &\cong H_n(\text{Sg}(X^{\text{an}}, Z^{\text{an}})^{\vee\vee}) \\ &\cong H_n((R\pi_*^{\text{an}} Rj_!^{\text{an}} \Lambda)^\vee) && \text{see appendix B} \\ &\cong H_n \text{Bti}^*((R\pi_* j_! \Lambda)^\vee) && \text{by the main results of [2]} \\ &\cong H_n \text{Bti}^* L\pi_! Rj_* j^* \pi'^! \Lambda && \text{by duality} \\ &\cong H_0 \text{Bti}^* \mathcal{R}_A(X, Z, n) && \text{since Bti}^* \text{ is triangulated.} \end{aligned}$$

This defines the isomorphism in the proposition. We have to check that it is compatible with the two types of edges in  $\mathcal{D}_N$ .

Let  $f : (X, Z, n) \rightarrow (X', Z', n)$  be an edge in  $\mathcal{D}_N$ . Compatibility with respect to  $f$  will follow from the commutativity of the outer rectangle in the following diagram

<sup>7</sup>In the notation of section 8.1, this quasi-isomorphic complex is  $C^{\mathcal{Y}}(X, \emptyset)$  for a finite affine open cover  $\mathcal{Y}$  of  $X$ .

(namely, after applying  $H_n$  and noticing that  $\text{Bti}^*$  commutes with shifts):

$$\begin{array}{ccc}
\text{Sg}(X^{\text{an}}, Z^{\text{an}}) & \xrightarrow{\text{Sg}(f)} & \text{Sg}(X'^{\text{an}}, Z'^{\text{an}}) \\
\sim \downarrow & & \downarrow \sim \\
\text{Sg}(X^{\text{an}}, Z^{\text{an}})^{\vee\vee} & \xrightarrow{\text{Sg}(f)^{\vee\vee}} & \text{Sg}(X'^{\text{an}}, Z'^{\text{an}})^{\vee\vee} \\
\sim \downarrow & & \downarrow \sim \\
(\text{R}\pi_*^{\text{an}} j_!^{\text{an}} \Lambda)^{\vee} & \xrightarrow{\mathcal{R}_A^{\text{an}, \vee}(f)^{\vee}} & (\text{R}\pi_*'^{\text{an}} j_!'^{\text{an}} \Lambda)^{\vee} \\
\sim \downarrow & & \downarrow \sim \\
\text{Bti}^*(\text{R}\pi_* j_! \Lambda)^{\vee} & \xrightarrow{\mathcal{R}_A^{\vee}(f)^{\vee}} & \text{Bti}^*(\text{R}\pi_*' j_!'^{\vee} \Lambda)^{\vee} \\
\sim \downarrow & & \downarrow \sim \\
\text{Bti}^* \text{L}\pi_! \text{R} j_* j^* \pi^! \Lambda & \xrightarrow{\mathcal{R}_A(f)} & \text{Bti}^* \text{L}\pi_! \text{R} j_*' j'^* \pi^! \Lambda
\end{array}$$

We will describe the arrows as we go along proving each square commutative. Starting at the bottom,  $\mathcal{R}_A^{\vee}(f) : j_!^{\vee} \Lambda \rightarrow \text{R}f_* j_! \Lambda$  is defined dually to  $\mathcal{R}_A(f)$ . It is then clear that the bottom square commutes. The same definition (using the six functors formalism, that is) in the analytic setting gives rise to the arrow  $\mathcal{R}_A^{\text{an}, \vee}(f)$  in the third row. Again, by the main results of [2], the third square commutes as well. Commutativity of the first square is clear, while commutativity of the second square follows from Lemma B.4.

Let  $(X, Z, n)$  be a vertex in  $\mathcal{D}_N$  and  $W \subset Z$  a closed subvariety, giving rise to the edge  $(X, Z, n) \rightarrow (Z, W, n-1)$ . Compatibility with respect to this edge will follow from commutativity of the diagram

$$\begin{array}{ccccc}
H_n(X, Z) & \xrightarrow{\partial} & H_{n-1}(Z) & \xrightarrow{\quad} & H_{n-1}(Z, W) \\
\sim \downarrow & & \sim \downarrow & & \downarrow \sim \\
H_0 \text{Bti}^* \mathcal{R}_A(X, Z, n) & \xrightarrow{\partial} & H_0 \text{Bti}^* \mathcal{R}_A(Z, \emptyset, n-1) & \xrightarrow{\text{adj}} & H_0 \text{Bti}^* \mathcal{R}_A(Z, W, n-1)
\end{array}$$

where the vertical arrows are the isomorphisms constructed above, and where the horizontal arrows on the right are induced by  $(Z, \emptyset, n-1) \rightarrow (Z, W, n-1)$ . In particular, the right square commutes by what we have shown above, and we reduce to prove commutativity of the left square. It can be decomposed as follows (before applying  $H_n$ , using again that  $\text{Bti}^*$  commutes with shifts; the horizontal arrows will

be made explicit below):

$$\begin{array}{ccc}
 \mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}}) & \xrightarrow{\partial} & \mathrm{Sg}(Z^{\mathrm{an}})[-1] \\
 \sim \downarrow & & \downarrow \sim \\
 \mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}})^{\vee\vee} & \xrightarrow{\delta^\vee} & (\mathrm{Sg}(Z^{\mathrm{an}})[-1])^{\vee\vee} \\
 \sim \downarrow & & \downarrow \sim \\
 (\mathrm{R}\pi_*^{\mathrm{an}} j_!^{\mathrm{an}} \Lambda)^\vee & \xrightarrow{\delta^\vee} & (\mathrm{R}\pi_*^{\mathrm{an}} i_*^{\mathrm{an}} \Lambda[1])^\vee \\
 \sim \downarrow & & \downarrow \sim \\
 \mathrm{Bti}^*(\mathrm{R}\pi_* j_! \Lambda)^\vee & \xrightarrow{\delta^\vee} & \mathrm{Bti}^*(\mathrm{R}\pi_* i_* \Lambda[1])^\vee \\
 \sim \downarrow & & \downarrow \sim \\
 \mathrm{Bti}^* \mathrm{L}\pi_! \mathrm{R}j_* j^* \pi^! \Lambda & \xrightarrow{\mathcal{R}_\Lambda(\partial)} & \mathrm{Bti}^* \mathrm{L}\pi_! i_! i^! \pi^! \Lambda[-1]
 \end{array} \tag{5.4}$$

Starting at the bottom, the morphism  $\delta$  arises from the distinguished triangle of motives over  $X$ :

$$i_* \Lambda[1] \xrightarrow{\delta} j_! \Lambda \rightarrow \Lambda \rightarrow i_* \Lambda. \tag{5.5}$$

Taking the dual we obtain the other localization triangle (5.1). Thus commutativity of the bottom square follows.

We can consider the exact same distinguished triangle as (5.5) in the analytic setting. This gives rise to the arrow  $\delta^\vee$  in the third row of (5.4). Thus commutativity of the third square in (5.4) follows from the fact that the compatibility of the Betti realization with the six functors formalism is also compatible with the triangulations.

By Lemma B.6 in the appendix, the second square in (5.4) commutes if we take  $\delta^\vee$  in the second row to be induced by the short exact sequence of singular cochain complexes. We leave it as an exercise to prove that this renders the top square in (5.4) commutative after applying  $H_n$ .  $\square$

**5.2. Monoidality.** Our next goal is to prove that the isomorphism of the proposition preserves the u.g.m. structures of the two representations (restricted to  $\mathcal{D}_N^{\mathbb{K}}$ ; cf. appendix A), in other words that it is compatible with the Künneth isomorphism in Betti homology. But first we must define the u.g.m. structure on the representation  $H_0 \widetilde{\mathrm{Bti}}^* \mathcal{R}_\Lambda$ .

Let  $(X_1, Z_1)$  and  $(X_2, Z_2)$  be two pairs of varieties, and set  $\overline{X} = X_1 \times X_2$ ,  $\overline{Z}_1 = Z_1 \times X_2$ ,  $\overline{Z}_2 = X_1 \times Z_2$ ,  $\overline{Z} = \overline{Z}_1 \cup \overline{Z}_2$ . There is a canonical morphism (a motivic “cup product”)

$$\mathrm{R}\overline{\pi}_* \overline{j}_{1!} \Lambda \otimes^{\mathrm{L}} \mathrm{R}\overline{\pi}_* \overline{j}_{2!} \Lambda \rightarrow \mathrm{R}\overline{\pi}_* (\overline{j}_{1!} \Lambda \otimes^{\mathrm{L}} \overline{j}_{2!} \Lambda) \cong \mathrm{R}\overline{\pi}_* \overline{j}_! \Lambda \tag{5.6}$$

and we obtain (for  $\overline{n} = n_1 + n_2$ )

$$\begin{aligned}
 \tilde{\tau} : \mathcal{R}_\Lambda(\overline{X}, \overline{Z}, \overline{n}) &\xrightarrow{(5.6)^\vee} (\mathcal{R}_\Lambda(\overline{X}, \overline{Z}_1, 0) \otimes^{\mathrm{L}} \mathcal{R}_\Lambda(\overline{X}, \overline{Z}_2, 0))[\overline{n}] \xrightarrow{\gamma} \\
 &\mathcal{R}_\Lambda(\overline{X}, \overline{Z}_1, n_1) \otimes^{\mathrm{L}} \mathcal{R}_\Lambda(\overline{X}, \overline{Z}_2, n_2) \xrightarrow{\mathcal{R}_\Lambda(p_1) \otimes^{\mathrm{L}} \mathcal{R}_\Lambda(p_2)} \mathcal{R}_\Lambda(X_1, Z_1, n_1) \otimes^{\mathrm{L}} \mathcal{R}_\Lambda(X_2, Z_2, n_2)
 \end{aligned}$$

where  $p_i : \overline{X} \rightarrow X_i$  denotes the projection onto the  $i$ th factor. One word about the isomorphism  $\gamma$ : In the category of complexes there are two natural choices for  $\gamma$ , by following one of the two paths in the following square:

$$\begin{array}{ccc} (\bullet_1 \otimes \bullet_2)[\bar{n}] & \longrightarrow & (\bullet_1 \otimes \bullet_2[n_2])[n_1] \\ \downarrow & & \downarrow \\ (\bullet_1[n_1] \otimes \bullet_2)[n_2] & \longrightarrow & \bullet_1[n_1] \otimes \bullet_2[n_2] \end{array}$$

This square commutes up to the sign  $(-1)^{n_1 \cdot n_2}$ . We choose the  $\gamma$  which is the identity in degree 0. (Which of the two paths we choose thus depends on the sign conventions for the tensor product and shift in the category of chain complexes.)

We can now define the u. g. m. structure on  $H_0 \widetilde{\text{Bti}}^* \mathcal{R}_A$  as the following composition (for any  $v_1, v_2 \in \mathcal{D}_N$ ):

$$\begin{aligned} \tau_{(v_1, v_2)} : H_0 \widetilde{\text{Bti}}^* \mathcal{R}_A(v_1 \times v_2) &\xrightarrow{\tilde{\tau}} H_0 \widetilde{\text{Bti}}^* (\mathcal{R}_A(v_1) \otimes^L \mathcal{R}_A(v_2)) \\ &\xrightarrow{\sim} H_0 (\widetilde{\text{Bti}}^* \mathcal{R}_A(v_1) \otimes^L \widetilde{\text{Bti}}^* \mathcal{R}_A(v_2)) \rightarrow H_0 \widetilde{\text{Bti}}^* \mathcal{R}_A(v_1) \otimes H_0 \widetilde{\text{Bti}}^* \mathcal{R}_A(v_2). \end{aligned}$$

**Proposition 5.7.** *Assume that  $\Lambda$  is a principal ideal domain. Then:*

- (1) *The morphisms  $\tau_{(v_1, v_2)}$  define a u. g. m. structure on the representation  $H_0 \widetilde{\text{Bti}}^* \mathcal{R}_A : \mathcal{D}_N^g \rightarrow \mathbf{coMod}^f(\mathbf{H}_A)$ .*
- (2) *The isomorphism of the previous proposition is compatible with the u. g. m. structures, i. e. it induces an isomorphism of u. g. m. representations*

$$\begin{array}{ccc} \mathcal{D}_N^g & \xrightarrow{H_0 \widetilde{\text{Bti}}^* \mathcal{R}_A} & \mathbf{coMod}^f(\mathbf{H}_A) \\ & \searrow H_\bullet & \downarrow o \\ & & \mathbf{Mod}^f(\Lambda) \end{array}$$

*Proof.* For the first part we need to check that in  $\mathbf{coMod}^f(\mathbf{H}_A)$ , some morphisms are invertible and some diagrams commute. Both these properties can be checked after applying  $o : \mathbf{coMod}^f(\mathbf{H}_A) \rightarrow \mathbf{Mod}^f(\Lambda)$ . Since the corresponding properties are true for the representation  $H_\bullet$ , we see that to prove the proposition, it suffices to show that the isomorphism of the previous proposition takes  $o \circ \tau$  to the Künneth isomorphism.

Write  $v_1, v_2, \bar{v}$  for the motives  $\mathcal{R}_A(X_1, Z_1, 0)$ ,  $\mathcal{R}_A(X_2, Z_2, 0)$  and  $\mathcal{R}_A(\overline{X}, \overline{Z}, 0)$ , respectively. Consider then the following diagram:<sup>8</sup>

$$\begin{array}{ccccc} H_{\bar{n}} \text{Bti}^* \bar{v} & \xrightarrow{\gamma^{-1} \circ \tilde{\tau}} & H_{\bar{n}} (\text{Bti}^* v_1 \otimes^L \text{Bti}^* v_2) & \xrightarrow{\sim \gamma} & H_{n_1} \text{Bti}^* v_1 \otimes H_{n_2} \text{Bti}^* v_2 \\ \sim \downarrow & & \sim \downarrow & & \downarrow \sim \\ H_{\bar{n}}(\overline{X}, \overline{Z}) & \xrightarrow{\sim \text{"AW"}} & H_{\bar{n}}(\text{Sg}(X_1, Z_1) \otimes \text{Sg}(X_2, Z_2)) & \xrightarrow{\sim \gamma} & H_{n_1}(X_1, Z_1) \otimes H_{n_2}(X_2, Z_2) \end{array}$$

where we have written  $\bar{n}$  for the sum  $n_1 + n_2$ . The right square clearly commutes. The bottom horizontal arrow on the left is induced by the Alexander-Whitney map (it is really a zig-zag on the level of complexes) and  $\gamma$  in the bottom right induces the canonical isomorphism of the (algebraic) Künneth formula, hence it follows that

<sup>8</sup>Here, as in the sequel, we often write  $\text{Sg}(X, Z)$  for  $\text{Sg}(X^{\text{an}}, Z^{\text{an}})$ .

the composition of the arrows in the bottom row is nothing but the (topological) Künneth isomorphism. On the other hand, the composition of the arrows in the top row is  $\tau$ . Hence we are reduced to prove commutativity of the left square in the diagram above, and it suffices to do so before applying  $H_{\overline{n}}$ .

We now write  $\bar{v}_i$  for the motive  $\mathcal{R}_\Lambda(\bar{X}, \bar{Z}_i, 0)$ . Decompose  $\tilde{\tau}$  according to its definition, and use the fact that the Alexander-Whitney map admits a similar decomposition in  $\mathbf{D}(\Lambda)$ :

$$\begin{array}{ccccc}
 \text{Bti}^* \bar{v} & \xrightarrow{(5.6)^\vee} & \text{Bti}^*(\bar{v}_1 \otimes^L \bar{v}_2) & \xrightarrow{\mathcal{R}_\Lambda(p_1) \otimes^L \mathcal{R}_\Lambda(p_2)} & \text{Bti}^*(v_1 \otimes^L v_2) \\
 \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
 & & \text{Bti}^* \bar{v}_1 \otimes^L \text{Bti}^* \bar{v}_2 & \xrightarrow{\mathcal{R}_\Lambda(p_1) \otimes^L \mathcal{R}_\Lambda(p_2)} & \text{Bti}^* v_1 \otimes^L \text{Bti}^* v_2 \\
 \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
 \text{Sg}(\bar{X}, \bar{Z}) & \xrightarrow{\text{"AW-diag"}} & \text{Sg}(\bar{X}, \bar{Z}_1) \otimes \text{Sg}(\bar{X}, \bar{Z}_2) & \xrightarrow{\text{Sg}(p_1) \otimes \text{Sg}(p_2)} & \text{Sg}(X_1, Z_1) \otimes \text{Sg}(X_2, Z_2)
 \end{array}$$

We wrote “AW-diag” for the Alexander-Whitney diagonal approximation which is a zig-zag of morphisms of complexes (see below). It is clear that the upper right square commutes, as does the lower right square by the proof of the previous proposition. For the square on the left, notice that (5.6) equally defines a morphism in the category **AnDA**. Thus we now denote by  $\bar{v}$ ,  $\bar{v}_i$  the same expressions in terms of the four functors in **AnDA** instead of **DA**. Then the proof of the proposition will be complete if we can prove commutative the following diagram (in which all vertical arrows are the canonical invertible ones):

$$\begin{array}{ccc}
 \bar{v} & \xrightarrow{(5.6)^\vee} & \bar{v}_1 \otimes^L \bar{v}_2 \\
 \downarrow & & \downarrow \\
 (\pi_* \bar{j}_! \Lambda)^\vee & \xrightarrow{(5.6)^\vee} & (\pi_* \bar{j}_{1!} \Lambda \otimes^L \pi_* \bar{j}_{2!} \Lambda)^\vee \\
 \downarrow & & \downarrow \\
 \text{Sg}(\bar{X}, \bar{Z})^{\vee\vee} & \xleftarrow{\sim} \text{Sg}(\bar{X}, \bar{Z}_1 + \bar{Z}_2)^{\vee\vee} \xrightarrow{\sim^\vee} & (\text{Sg}(\bar{X}, \bar{Z}_1)^\vee \otimes \text{Sg}(\bar{X}, \bar{Z}_2)^\vee)^\vee \\
 \downarrow & & \downarrow \\
 \text{Sg}(\bar{X}, \bar{Z}) & \xleftarrow{\sim} \text{Sg}(\bar{X}, \bar{Z}_1 + \bar{Z}_2) \xrightarrow{\text{AW-diag}} & \text{Sg}(\bar{X}, \bar{Z}_1) \otimes \text{Sg}(\bar{X}, \bar{Z}_2)
 \end{array}$$

Here,  $\text{Sg}(\bar{X}, \bar{Z}_1 + \bar{Z}_2)$  denotes the free  $\Lambda$ -module on simplices in  $\bar{X}$  which are neither contained in  $\bar{Z}_1$  nor in  $\bar{Z}_2$ . The first rectangle clearly commutes, the second does so by Lemma B.8 (which may be applied because of Fact 5.2), the bottom right square is well-known to commute (see e.g. [12, VII, 8]), and the bottom left one obviously commutes as well.  $\square$

**5.3. Bialgebra morphism.** Using the proposition we obtain the following commutative (up to a u. g. m. isomorphism) rectangle of u. g. m. representations:

$$\begin{array}{ccc}
 \mathcal{D}_N^g & \xrightarrow{H_0 \widetilde{\text{Bti}}^* \mathcal{R}_\Lambda} & \mathbf{coMod}^f(\mathbf{H}_\Lambda) \\
 \tilde{\mathbf{H}}_\bullet \downarrow & \nearrow \text{dotted arrow} & \downarrow o \\
 \mathbf{coMod}^f(\mathbf{H}_N^{\text{eff}}) & \xrightarrow{o} & \mathbf{Mod}^f(\Lambda)
 \end{array} \tag{5.8}$$

Still assuming that  $\Lambda$  is a principal ideal domain we know, by Lemma 4.7 together with Fact C.1, that  $\mathbf{coMod}^f(\mathbf{H}_\Lambda)$  is an abelian category. Hence Theorem 2.3 yields a monoidal functor  $\overline{\varphi'_\Lambda}$  represented by the dotted arrow in the diagram (5.8), rendering it commutative (up to monoidal isomorphism). It then follows from [29, II, 3.3.1] that  $\overline{\varphi'_\Lambda}$  necessarily arises from a map of bialgebras  $\varphi'_\Lambda : \mathbf{H}_N^{\text{eff}} \rightarrow \mathbf{H}_\Lambda$ .

Recall (from §§2, 4) that the Hopf algebras  $\mathbf{H}_N$  and  $\mathbf{H}_\Lambda$  are obtained from  $\mathbf{H}_N^{\text{eff}}$  and  $\mathbf{H}_\Lambda^{\text{eff}}$  by localization with respect to elements  $s_N$  and  $s_\Lambda$  respectively. Using the commutativity of (5.8) one easily checks that  $\varphi'_\Lambda(s_N) = s_\Lambda \in \mathbf{H}_\Lambda$  hence  $\varphi'_\Lambda$  factors through  $\mathbf{H}_N$ :

$$\begin{array}{ccc}
 \mathbf{H}_N^{\text{eff}} & \xrightarrow{\varphi'_\Lambda} & \mathbf{H}_\Lambda \\
 \downarrow \iota & \nearrow \varphi_\Lambda & \\
 \mathbf{H}_N & & 
 \end{array} \tag{5.9}$$

## 6. BASIC LEMMA, AND APPLICATIONS

Now we would like to construct a morphism  $\varphi_N$  in the other direction. The construction relies on Nori's functor which associates to an affine variety a complex in  $\mathbf{HM}^{\text{eff}}$  computing its homology, and which in turn relies on the “Basic Lemma”. We recall them both in this section (basically following [13]), while we prove the existence of  $\varphi_N$  in the next section.

We first recall the Basic Lemma in the form Nori formulated it [26, Thm. 2.1]; see also [21, Lem. 4.4]. It was independently proven by Beilinson in a more general context [7, Lem. 3.3].

**Fact 6.1** (Basic Lemma). *Let  $X$  be an affine variety of dimension  $n$ , and  $W \subset X$  a closed subvariety of dimension  $\leq n-1$ . Then there exists a closed subvariety  $W \subset Z \subset X$  of dimension  $\leq n-1$  such that  $H_\bullet(X^{\text{an}}, Z^{\text{an}}; \mathbb{Z})$  is a free abelian group concentrated in degree  $n$ .*

We call a pair  $(X, Z, n)$  *very good* (cf. [20, Def. D.1]) if either  $X$  is affine of dimension  $n$ ,  $Z$  is of dimension  $\leq n-1$ ,  $X \setminus Z$  is smooth, and  $H_\bullet(X^{\text{an}}, Z^{\text{an}}; \mathbb{Z})$  is a free abelian group concentrated in degree  $n$ , or if  $X = Z$  is affine of dimension less than  $n$ . Thus the Basic Lemma implies that any pair  $(X, W, n)$  with  $\dim(X) = n > \dim(W)$  can be embedded into a very good pair  $(X, Z, n)$ .

Nori applied this result to construct “cellular decompositions” of affine varieties as follows. Let  $X$  be an affine variety. A *filtration* of  $X$  is an increasing sequence  $F_\bullet = (F_i X)_{i \in \mathbb{Z}}$  of closed subvarieties of  $X$  such that

- $\dim(F_i X) \leq i$  for all  $i$  (in particular,  $F_{-1} X = \emptyset$ ),
- $F_n X = X$  for some  $n \in \mathbb{Z}$ .



The minimal  $n \in \mathbb{Z}$  such that  $F_n X = X$  is called the *length* of  $F_\bullet$  (by convention, for  $X = \emptyset$  this length is defined to be  $-\infty$ ). Clearly the filtrations of  $X$  form a directed set. A filtration  $F_\bullet$  is called *very good* if  $(F_i X, F_{i-1} X, i)$  is a very good pair for each  $i$ . The following result says that very good filtrations form a cofinal set. It is also used in [26].

**Corollary 6.2.** *Let  $X$  be an affine variety, and  $F_\bullet$  a filtration of  $X$ . Then there exists a very good filtration  $G_\bullet \supset F_\bullet$  of  $X$ . In particular, every affine variety of dimension  $n$  admits a very good filtration of length  $n$ .*

*Proof.* We do induction on the length  $n$  of  $F_\bullet$ . Every filtration of length  $n = -\infty$  or  $n = 0$  is very good. Assume now  $n > 0$ . Set  $G_i X = X$  for all  $i \geq n$ . If  $\dim(X) < n$ , let  $G_{n-1} X = X$ . If  $\dim(X) = n$  then, applying the Basic Lemma to the pair  $(X, F_{n-1} X)$ , we obtain a closed subvariety  $F_{n-1} X \subset Z \subset X$  such that  $(X, Z, n)$  is very good, and we set  $G_{n-1} X = Z$  in this case. Now apply the induction hypothesis to the filtration  $\emptyset \subset F_0 X \subset \cdots \subset F_{n-2} X \subset G_{n-1} X$ .  $\square$

To any filtration  $F_\bullet$  of  $X$  we associate the complex  $H_\bullet(X, F_\bullet) = H_\bullet(X, F_\bullet; \Lambda)$ ,

$$H_n(X^{\text{an}}, F_{n-1} X^{\text{an}}) \rightarrow H_{n-1}(F_{n-1} X^{\text{an}}, F_{n-2} X^{\text{an}}) \rightarrow \cdots \rightarrow H_0(F_0 X^{\text{an}}, \emptyset),$$

concentrated in the range of degrees  $[0, n]$ , where the differentials are the boundary maps from the homology sequence of a triple. It follows that  $H_\bullet(X, F_\bullet)$  can be considered as an object of  $\mathbf{Cpl}(\mathbf{HM}^{\text{eff}})$ . For a very good filtration, this complex computes the singular homology of  $X^{\text{an}}$  for the same reason that cellular homology and singular homology agree. (For a more precise statement see Fact 6.3 below.) It then follows from the corollary that also

$$C(X) := \varinjlim_{F_\bullet} H_\bullet(X, F_\bullet) \in \mathbf{Cpl}(\text{Ind } \mathbf{HM}^{\text{eff}})$$

computes singular homology of  $X^{\text{an}}$ .

Given a morphism of affine varieties  $f : X \rightarrow Y$ , and a filtration  $F_\bullet$  on  $X$ , we obtain a filtration

$$\overline{f(X)} \supset \overline{f(F_{n-1} X)} \supset \cdots \supset \overline{f(F_0 X)} \supset \emptyset$$

of  $\overline{f(X)}$ . Let  $m = \dim(Y)$ , and define a filtration  $G_\bullet$  on  $Y$  by

$$G_i Y = \begin{cases} \overline{f(F_i X)} & : i < m \\ Y & : i \geq m. \end{cases}$$

This induces a morphism  $H_\bullet(X, F_\bullet) \rightarrow H_\bullet(Y, G_\bullet)$  in  $\mathbf{Cpl}(\mathbf{HM}^{\text{eff}})$ . It follows that  $C$  defines a functor  $\text{AffVar} \rightarrow \mathbf{Cpl}(\text{Ind } \mathbf{HM}^{\text{eff}})$  on affine varieties.

Now given filtrations  $F_\bullet$  and  $G_\bullet$  on affine varieties  $X$  and  $Y$ , respectively, we form the filtration  $(F \times G)_\bullet$  on  $X \times Y$ , setting  $(F \times G)_i(X \times Y)$  to be  $\cup_{p+q=i} F_p X \times G_q Y$ . There is a canonical morphism  $H_\bullet(X, F_\bullet) \otimes H_\bullet(Y, G_\bullet) \rightarrow H_\bullet(X \times Y, (F \times G)_\bullet)$  which induces a morphism  $C(X) \otimes C(Y) \rightarrow C(X \times Y)$ . One can check that this endows  $C$  with a lax monoidal structure.

To go further we have to make precise the relation between the functors  $C$  and  $\text{Sg} \circ \text{An}$ . For this, following [13], we consider the subcomplex  $P(X)$  of  $\text{Sg}(X^{\text{an}})$  which in degree  $p$  consists of singular  $p$ -chains in  $X^{\text{an}}$  whose image is contained in  $Z^{\text{an}}$  for some closed subvariety  $Z \subset X$  of dimension  $\leq p$ , and whose boundary lies in  $W^{\text{an}}$  for some closed subvariety  $W \subset X$  of dimension  $\leq p-1$ . Such a singular chain defines a homology class in  $H_p(Z^{\text{an}}, W^{\text{an}}; \Lambda)$  hence there is a canonical map

$P(X) \rightarrow oC(X)$  (here, as usual,  $o$  forgets the comodule structure). The following result follows from the Basic Lemma and some linear algebra.

**Fact 6.3** ([13, Lem. 4.14]). *Let  $X$  be an affine variety. Both maps of chain complexes of  $\Lambda$ -modules*

$$oC(X) \leftarrow P(X) \rightarrow \mathrm{Sg}(X^{\mathrm{an}})$$

*are quasi-isomorphisms.*

It is clear that  $P$  defines a functor  $\mathrm{AffVar} \rightarrow \mathbf{Cpl}(\Lambda)$  and that the two maps above are natural in  $X$ . Moreover  $P$  comes with a canonical lax monoidal structure induced from the one on  $\mathrm{Sg}$  (the Eilenberg-Zilber transformation), and which is compatible with the one on  $C$  defined before.

**Corollary 6.4.** *The maps of the previous lemma define monoidal transformations between lax monoidal functors*

$$oC \leftarrow P \rightarrow \mathrm{Sg} \circ \mathrm{An}$$

*from  $\mathrm{AffVar}$  to  $\mathbf{Cpl}(\Lambda)$ . If  $\Lambda$  is a principal ideal domain then after composing with the canonical (lax monoidal) functor  $\mathbf{Cpl}(\Lambda) \rightarrow \mathbf{D}(\Lambda)$  these become monoidal transformations of monoidal functors.*

*Proof.* Given affine varieties  $X$  and  $Y$ , we have a commutative diagram:

$$\begin{array}{ccccc} oC(X) \otimes oC(Y) & \longleftarrow & P(X) \otimes P(Y) & \longrightarrow & \mathrm{Sg}(X^{\mathrm{an}}) \otimes \mathrm{Sg}(Y^{\mathrm{an}}) \\ \downarrow & & \downarrow & & \downarrow \\ oC(X \times Y) & \longleftarrow & P(X \times Y) & \longrightarrow & \mathrm{Sg}(X^{\mathrm{an}} \times Y^{\mathrm{an}}) \end{array}$$

$\mathrm{Sg}$  takes values in (complexes of) free  $\Lambda$ -modules, and  $oC(X)$  is a direct limit of (complexes of) finitely generated free  $\Lambda$ -modules (by Corollary 6.2) hence is itself a complex of flat  $\Lambda$ -modules. If  $\Lambda$  is a principal ideal domain then  $P$  necessarily takes values in (complexes of) free  $\Lambda$ -modules as well. In conclusion, under the assumptions of the corollary, all tensor factors in the diagram above are flat.

It follows that all horizontal arrows in the diagram above are quasi-isomorphisms, and that in the upper row, all tensor products are equal to their derived versions. The corollary now follows from the fact that the right-most vertical arrow is a quasi-isomorphism.  $\square$

**Remark 6.5.** There are several ways to extend the functor  $C$  to all varieties, as explained in [26, p. 9]. However, for our purposes this will not be necessary as in the end we are interested only in the induced functor  $\mathbf{DA}^{\mathrm{eff}} \rightarrow \mathbf{D}(\mathrm{Ind} \mathbf{HM}^{\mathrm{eff}})$ , and here we can use the equivalence between  $\mathbf{DA}^{\mathrm{eff}}$  and  $\mathbf{DA}_{\mathrm{aff}}^{\mathrm{eff}}$  of Lemma 3.2.

## 7. MOTIVIC REALIZATION

We would now like to explain how the lax monoidal functor  $C$  constructed in the previous section induces a functor on categories of motives. The case of effective motives is treated in §7.1 and as an application we deduce a morphism of bialgebras  $\varphi'_N : \mathbf{H}_A^{\mathrm{eff}} \rightarrow \mathbf{H}_N^{\mathrm{eff}}$  in §7.2. In §7.3 and §7.4 we treat the case of effective motives with transfers and non-effective motives, respectively.

**7.1. Construction.** First we define the functor

$$C^* : \mathbf{USmAff} \rightarrow \mathbf{Cpl}(\mathrm{Ind} \mathbf{HM}^{\mathrm{eff}})$$

by the coend formula for a “left dg Kan extension” (cf. [10, Fact 2.1])

$$K \mapsto \int^{X \in \mathbf{SmAff}} K(X) \otimes C(X),$$

where the comodule structure is induced from the one on the right tensor factor. Recall that the coend appearing in the definition is nothing but the coequalizer of the diagram

$$\bigoplus_{X \rightarrow Y} K(Y) \otimes C(X) \rightrightarrows \bigoplus_X K(X) \otimes C(X),$$

where the two arrows are induced by the functoriality of  $K$  and  $C$ , respectively. We will prove below that  $C^*$  is left Quillen for the projective model structure on the domain (even induced from the injective model structure on  $\mathbf{Cpl}(\Lambda)$ ) and the injective model structure on the codomain (cf. Fact C.2).

**Proposition 7.1.** *Let  $\Lambda$  be a principal ideal domain.  $LC^*$  inherits a monoidal structure, and takes  $(\mathbb{A}^1, \tau)$ -local equivalences to quasi-isomorphisms. Moreover, it makes the following square commutative up to monoidal triangulated isomorphism.*

$$\begin{array}{ccc} \mathbf{DA}_{\mathrm{aff}}^{\mathrm{eff}} & \xrightarrow{\sim} & \mathbf{DA}^{\mathrm{eff}} \\ \downarrow LC^* & & \downarrow \mathrm{Bti}^{\mathrm{eff},*} \\ \mathbf{D}(\mathrm{Ind} \mathbf{HM}^{\mathrm{eff}}) & \xrightarrow[\mathbf{D}(o)]{} & \mathbf{D}(\Lambda) \end{array}$$

**Remark 7.2.** In [26, p. 9], Nori remarks that for an arbitrary variety  $X$  and an affine open cover  $\mathcal{U} = (U_1, \dots, U_q)$  of  $X$ , the complex

$$\mathrm{Tot}(\cdots \rightarrow \bigoplus_{1 \leq i_1 < \dots < i_p \leq q} C(U_{i_1} \cap \cdots \cap U_{i_p}) \rightarrow \cdots) \quad (7.3)$$

“computes the homology of  $X$ ”. This can also be explained using the proposition, at least if  $X$  is smooth. Namely, in that case, the complex

$$\cdots \rightarrow \bigoplus_{1 \leq i_1 < \dots < i_p \leq q} \Lambda(U_{i_1} \cap \cdots \cap U_{i_p}) \rightarrow \cdots$$

defines a cofibrant replacement of  $\Lambda(X)$  (as we will see in Lemma 8.1), and  $C^*$  applied to it is just (7.3), as follows from [10, Lemma 3.21]. Hence the proposition tells us that the underlying complex of  $\Lambda$ -modules in (7.3) is nothing but  $\mathrm{Bti}^{\mathrm{eff},*} \Lambda(X) \cong \mathrm{Sg}(X^{\mathrm{an}})$  (by Proposition 3.7).

We will come back to this explicit description of  $LC^*$  in section 8.1 where (7.3) is denoted by  $C^{\mathcal{U}}(X)$ .

*Proof of Proposition 7.1.* Just as  $C$  admits a left Kan extension, so do  $P$ ,  $\mathrm{Sg}$  and  $\mathrm{An}$ :

$$\begin{array}{ccc} \mathbf{SmAff} & \xrightarrow{\quad} & \mathbf{USmAff} \\ \downarrow C & \swarrow C^* & \downarrow \mathrm{An}^* \\ \mathbf{Cpl}(\mathrm{Ind} \mathbf{HM}^{\mathrm{eff}}) & & \\ \downarrow o & \swarrow P^* & \\ \mathbf{Cpl}(\Lambda) & \xleftarrow{\mathrm{Sg}^*} & \mathbf{UMan}_{\mathbb{C}} \end{array}$$

Endow  $\mathbf{Cpl}(\Lambda)$  and  $\mathbf{Cpl}(\mathrm{Ind} \mathbf{HM}^{\mathrm{eff}})$  with the injective model structures (cf. Fact C.2), and the presheaf categories with the projective model structures deduced from the injective model structure on  $\mathbf{Cpl}(\Lambda)$ . We then use [10, Lemma 2.5] to prove that all these Kan extensions are left Quillen functors. For  $\mathrm{Sg}$ ,  $P$ , and  $C$  this follows from the fact that they take values in complexes of flat objects (see the proof of Corollary 6.4) hence the tensor product with these complexes is a left Quillen functor for the injective model structure. For  $\mathrm{An}$ , this is because evaluation at a smooth affine scheme  $X$  clearly preserves (trivial) fibrations.

Also,  $C^*$  as well as  $P^*$  and  $\mathrm{Sg}^* \mathrm{An}^* \cong (\mathrm{Sg} \circ \mathrm{An})^*$  inherit canonically lax monoidal structures ([10, Lemma 2.2]). From Corollary 6.4 we deduce monoidal transformations

$$o \circ C^* \leftarrow P^* \rightarrow \mathrm{Sg}^* \circ \mathrm{An}^* \quad (7.4)$$

of lax monoidal functors defined on  $\mathbf{USmAff}$  taking values in  $\mathbf{Cpl}(\Lambda)$ . They give rise to monoidal triangulated transformations between the corresponding left derived functors (this uses Lemma C.4), and to prove that these transformations are invertible, it suffices to check it on objects of the form  $\Lambda(X)$ , where  $X \in \mathbf{SmAff}$  (by [10, Lemma 3.20] these compactly generate the derived category). These objects are cofibrant, and we conclude since the maps

$$o \circ C^* \Lambda(X) \leftarrow P^* \Lambda(X) \rightarrow \mathrm{Sg}^* \circ \mathrm{An}^* \Lambda(X)$$

are identified with the quasi-isomorphisms

$$o \circ C(X) \leftarrow P(X) \rightarrow \mathrm{Sg} \circ \mathrm{An}(X)$$

(see Fact 6.3).

We have now constructed a diagram of lax monoidal triangulated functors

$$\begin{array}{ccc} \mathbf{D}(\mathrm{Ind} \mathbf{HM}^{\mathrm{eff}}) & \xleftarrow{LC^*} & \mathbf{D}(\mathbf{USmAff}) \\ \mathbf{D}(o) \downarrow & \swarrow LP^* & \downarrow \mathrm{An}^* \\ \mathbf{D}(\Lambda) & \xleftarrow{LSg^*} & \mathbf{D}(\mathbf{UMan}_{\mathbb{C}}) \end{array}$$

which commutes up to monoidal triangulated isomorphism. Using the identification  $\mathbf{DA}_{\mathrm{aff}}^{\mathrm{eff}} \xrightarrow{\sim} \mathbf{DA}^{\mathrm{eff}}$  (Lemma 3.2) the result therefore follows from Proposition 3.7.  $\square$

**7.2. Bialgebra morphism.** Since  $\mathbf{H}_N^{\mathrm{eff}}$  is a flat  $\Lambda$ -module,  $\mathbf{Cpl}(\mathrm{Ind} \mathbf{HM}^{\mathrm{eff}})$  is canonically equivalent to the category of  $\mathbf{H}_N^{\mathrm{eff}}$ -comodules in  $\mathbf{Cpl}(\Lambda)$  (Fact C.1), and we can consider the following composition of monoidal functors:

$$\mathcal{R}_N : \mathbf{DA}^{\mathrm{eff}} \simeq \mathbf{DA}_{\mathrm{aff}}^{\mathrm{eff}} \xrightarrow{LC^*} \mathbf{D}(\mathrm{Ind} \mathbf{HM}^{\mathrm{eff}}) \rightarrow \mathbf{coMod}(\mathbf{H}_N^{\mathrm{eff}})^{\mathbf{D}(\Lambda)}, \quad (7.5)$$

where the last term denotes the category of  $\mathbf{H}_N^{\mathrm{eff}}$ -comodules in  $\mathbf{D}(\Lambda)$ . The upshot of the discussion so far is that we obtain a diagram

$$\begin{array}{ccc} \mathbf{DA}^{\mathrm{eff}} & \xrightarrow{\mathcal{R}_N} & \mathbf{coMod}(\mathbf{H}_N^{\mathrm{eff}})^{\mathbf{D}(\Lambda)} \\ & \searrow \mathrm{Bti}^{\mathrm{eff},*} & \downarrow o \\ & & \mathbf{D}(\Lambda) \end{array}$$

of monoidal functors which commutes up to monoidal isomorphism. To invoke the universal property of  $\mathcal{H}_A^{\mathrm{eff}}$  (Fact 4.3) we still need to verify the following Lemma.

**Lemma 7.6.** *Let  $K \in \mathbf{D}(\Lambda)$ ,  $\Lambda$  a principal ideal domain. Then the coaction of  $\mathbf{H}_N^{\text{eff}}$  on  $\mathcal{R}_N(K_{\text{cst}})$  is trivial.*

*Proof.* We may assume that  $K$  is projective cofibrant consisting of free  $\Lambda$ -modules in each degree (for example by [10, Proposition 4.4]).  $K_{\text{cst}}$  is projective cofibrant, and in each degree consists of a direct sum of  $\Lambda(\text{Spec}(k))$ , the presheaf represented by  $\text{Spec}(k)$ . Hence  $\text{LC}^*(K_{\text{cst}}) = C^*(K_{\text{cst}})$  is a complex which in each degree consists of a direct sum of  $C(\text{Spec}(k))$  on which  $\mathbf{H}_N^{\text{eff}}$  coacts trivially. Hence  $\mathbf{H}_N^{\text{eff}}$  also coacts trivially on  $\mathcal{R}_N(K_{\text{cst}})$ .  $\square$

**Corollary 7.7.** *Assume that  $\Lambda$  is a principal ideal domain. There is a morphism of bialgebras  $\mathcal{H}_A^{\text{eff}} \rightarrow \mathbf{H}_N^{\text{eff}}$  inducing  $\varphi'_N : \mathbf{H}_A^{\text{eff}} \rightarrow \mathbf{H}_N^{\text{eff}}$  and rendering the following diagrams commutative up to monoidal isomorphism:*

$$\begin{array}{ccc} \mathbf{DA}^{\text{eff}} & \xrightarrow{\mathcal{R}_N} & \mathbf{coMod}(\mathbf{H}_N^{\text{eff}})^{\mathbf{D}(\Lambda)} \\ \widetilde{\text{Bti}}^{\text{eff},*} \downarrow & \nearrow & \downarrow o \\ \mathbf{coMod}(\mathcal{H}_A^{\text{eff}}) & \xrightarrow{o} & \mathbf{D}(\Lambda) \end{array} \quad \begin{array}{ccc} \mathbf{DA}^{\text{eff}} & \xrightarrow{H_0 \mathcal{R}_N} & \mathbf{coMod}(\mathbf{H}_N^{\text{eff}}) \\ H_0 \widetilde{\text{Bti}}^{\text{eff},*} \downarrow & \nearrow \varphi'_N & \downarrow o \\ \mathbf{coMod}(\mathbf{H}_A^{\text{eff}}) & \xrightarrow{o} & \mathbf{Mod}(\Lambda) \end{array}$$

There are two ways to obtain similar statements in the stable setting. The easier one is to check that  $\mathcal{H}_A^{\text{eff}} \rightarrow \mathbf{H}_N^{\text{eff}}$  passes to the localizations  $\mathcal{H}_A \rightarrow \mathbf{H}_N$  and consider the composition

$$\mathbf{DA} \xrightarrow{\widetilde{\text{Bti}}^*} \mathbf{coMod}(\mathcal{H}_A) \rightarrow \mathbf{coMod}(\mathbf{H}_N)^{\mathbf{D}(\Lambda)}.$$

This will be sufficient for our main theorem, and we will pursue it in §8.1. However, it might seem more natural and lead to stronger results to extend the construction of  $C^*$  to the level of spectra and derive the resulting functor. This will be done in §7.4.

**7.3. Transfers.** The remainder of §7 will not be strictly necessary for our main theorem but the results obtained here are of independent interest. In §7.3, our goal is Proposition 7.9 where we prove that  $\text{LC}^*$  extends to effective motives with transfers.

Recall ([26, §3.1]) that to a finite correspondence  $X \rightarrow S^d Y$  of degree  $d$  between affine schemes, Nori associates a morphism  $C(X) \rightarrow C(Y)$ , defined as the composition

$$C(X) \rightarrow C(S^d Y) \xleftarrow{\sim} C(Y^d)_{\Sigma_d} \xrightarrow{\sum_{i=1}^d C(p_i)} C(Y),$$

where  $(\bullet)_{\Sigma_d}$  denotes the  $\Sigma_d$ -coinvariants, and where the  $p_i : Y^d \rightarrow Y$  are the canonical projections. As proved in [16, Thm. 4.8.1], this induces a functor  $C_{\text{tr}} : \mathbf{SmAffCor} \rightarrow \mathbf{Cpl}(\text{Ind } \mathbf{HM}^{\text{eff}})$  on smooth affine correspondences, and we thus obtain a commutative triangle

$$\begin{array}{ccc} \mathbf{SmAff} & \xrightarrow{C} & \mathbf{Cpl}(\mathbf{HM}^{\text{eff}}) \\ \downarrow & \nearrow C_{\text{tr}} & \\ \mathbf{SmAffCor} & & \end{array} \quad (7.8)$$

where the vertical arrow is the canonical inclusion. The same procedure as above yields a left Quillen functor  $C_{\text{tr}}^* : \mathbf{USmAffCor} \rightarrow \mathbf{Cpl}(\text{Ind } \mathbf{HM}^{\text{eff}})$  for the projective model structure on the domain and the injective model structure on the codomain.

**Proposition 7.9.** *Let  $\Lambda$  be a principal ideal domain.  $LC_{\text{tr}}^*$  inherits a monoidal structure, and takes  $(\mathbb{A}^1, \tau)$ -local equivalences to quasi-isomorphisms. Moreover, (7.8) induces the following diagram, commutative up to monoidal triangulated isomorphism.*

$$\begin{array}{ccccc}
 \mathbf{DA}^{\text{eff}} & \xleftarrow{\sim} & \mathbf{DA}_{\text{aff}}^{\text{eff}} & \xrightarrow{LC^*} & \mathbf{D}(\text{Ind } \mathbf{HM}^{\text{eff}}) \\
 \downarrow & & \downarrow & \nearrow & \\
 \mathbf{DM}^{\text{eff}} & \xleftarrow{\sim} & \mathbf{DM}_{\text{aff}}^{\text{eff}} & \xrightarrow{LC_{\text{tr}}^*} & 
 \end{array}$$

*Proof.* It is proved in [16, Thm. 4.9.6] that the lax monoidal structure on  $C$  is natural with respect to finite correspondences so that (7.8) becomes a commutative triangle of lax monoidal functors. It follows that  $C_{\text{tr}}^*$  and  $LC_{\text{tr}}^*$  inherit lax monoidal structures, compatible with those of  $C^*$  and  $LC^*$ . Fix a smooth affine scheme  $X$  and consider the natural transformation (in  $F$ )

$$C_{\text{tr}}(X) \otimes^L LC_{\text{tr}}^*(F) \rightarrow LC_{\text{tr}}^*(\Lambda(X) \otimes F) \quad (7.10)$$

of functors  $\mathbf{D}(\mathbf{U}(\text{SmAffCor})) \rightarrow \mathbf{D}(\text{Ind } \mathbf{HM}^{\text{eff}})$ . Since the representable presheaves compactly generate the triangulated category  $\mathbf{D}(\mathbf{U}(\text{SmAffCor}))$  (see [10, Lemma 3.20]), (7.10) will be an isomorphism for all  $F$  if it is so for  $F = \Lambda(Y)$  a smooth affine scheme. But in this case, (7.10) can be identified with

$$C_{\text{tr}}(X) \otimes C_{\text{tr}}(Y) \rightarrow C_{\text{tr}}(X \times Y),$$

hence by (7.8) with

$$C(X) \otimes C(Y) \rightarrow C(X \times Y),$$

which we know to be a quasi-isomorphism. Similarly, fix an object  $F \in \mathbf{D}(\mathbf{U}(\text{SmAffCor}))$  and consider now the natural transformation (in  $G$ )

$$LC_{\text{tr}}^*(G) \otimes^L LC_{\text{tr}}^*(F) \rightarrow LC_{\text{tr}}^*(G \otimes^L F)$$

of functors  $\mathbf{D}(\mathbf{U}(\text{SmAffCor})) \rightarrow \mathbf{D}(\text{Ind } \mathbf{HM}^{\text{eff}})$ . Again, it will be an isomorphism for all  $G$  if it is so on representables  $G = \Lambda(X)$ . But this we just proved. We conclude that  $LC_{\text{tr}}^*$  is monoidal.

We now claim that  $LC_{\text{tr}}^*$  takes  $(\mathbb{A}^1, \tau)$ -local equivalences to quasi-isomorphisms. Notice that by the theory of Bousfield localizations, this is equivalent to the claim that the right adjoint  $C_{\text{tr},*}$  takes fibrant objects  $K$  to  $(\mathbb{A}^1, \tau)$ -fibrant presheaves of complexes. In other words ([10, Thm. 5.7]), we need to check that

- $C_{\text{tr},*}K$  satisfies descent with respect to  $\tau$ -hypercovers; and
- $C_{\text{tr},*}K(X) \rightarrow C_{\text{tr},*}K(\mathbb{A}_X^1)$  is a quasi-isomorphism for every smooth affine scheme  $X$ .

Both these properties can be checked on the site  $\text{SmAff}$  (instead of  $\text{SmAffCor}$ ) but restricted to this site  $C_{\text{tr},*}K$  coincides with  $C_*K$ . Thus we conclude with Proposition 7.1.

Commutativity of the left square in the statement is obvious. The fact that the top horizontal arrow is an equivalence is Lemma 3.2. Similarly, the bottom horizontal arrow is an equivalence ([10, Cor. 5.16]).  $\square$

**7.4. Stabilization.** In this subsection we will develop the stable motivic realizations for motives with and without transfers in parallel. Statements containing the symbol (tr) thus have two obvious interpretations.

For any flat complex of comodules  $K$ , there is an injective stable model structure on the category of symmetric  $K$ -spectra, by Proposition C.3. Denote by  $T$ , as in section 3, a cofibrant replacement of  $\Lambda(\mathbb{A}^1)/\Lambda(\mathbb{G}_m)$  and set  $T_{\text{tr}} = a_{\text{tr}}T$ . Notice that, canonically,  $C_{\text{tr}}^*T_{\text{tr}} \cong C^*T$ .

**Lemma 7.11.**

(1) *The canonical morphism of bialgebras  $\iota : \mathbf{H}_N^{\text{eff}} \rightarrow \mathbf{H}_N$  induces a functor*

$$\mathbf{Spt}_{C^*T}^{\Sigma} \mathbf{Cpl}(\text{Ind } \mathbf{HM}^{\text{eff}}) \xrightarrow{\bar{\iota}} \mathbf{Spt}_{iC^*T}^{\Sigma} \mathbf{Cpl}(\text{Ind } \mathbf{HM})$$

*which preserves stable weak equivalences.*

(2) *There is a canonical Quillen equivalence*

$$(\text{Sus}_{iC^*T}^0, \text{Ev}_0) : \mathbf{Cpl}(\text{Ind } \mathbf{HM}) \rightarrow \mathbf{Spt}_{iC^*T}^{\Sigma} \mathbf{Cpl}(\text{Ind } \mathbf{HM}).$$

*Proof.* The functor is obtained by applying  $\bar{\iota}$  levelwise (cf. the following proof for the details, or [1, Déf. 4.3.16]). The first part is then obvious, and the second part follows from [18, Thm. 9.1] since, as proved in the following section, tensoring with  $C^*T[2]$  (and hence with  $C^*T$ ) is a Quillen equivalence.  $\square$

We will prove in the following proposition that  $C_{(\text{tr})}^*$  induces a left Quillen functor  $\mathcal{R}_{(\text{tr}),s}^*$  on the level of spectra. Thus we may define the compositions

$$\mathcal{R}_{N,s} : \mathbf{DA} \simeq \mathbf{DA}_{\text{aff}} \xrightarrow{\text{LC}_s^*} \mathbf{Hot}(\mathbf{Spt}_{C^*T}^{\Sigma} \mathbf{Cpl}(\text{Ind } \mathbf{HM}^{\text{eff}})) \xrightarrow{\text{REv}_0 \circ \bar{\iota}} \mathbf{D}(\text{Ind } \mathbf{HM}),$$

$$\mathcal{R}_{N,\text{tr},s} : \mathbf{DM} \simeq \mathbf{DM}_{\text{aff}} \xrightarrow{\text{LC}_{\text{tr},s}^*} \mathbf{Hot}(\mathbf{Spt}_{C^*T}^{\Sigma} \mathbf{Cpl}(\text{Ind } \mathbf{HM}^{\text{eff}})) \xrightarrow{\text{REv}_0 \circ \bar{\iota}} \mathbf{D}(\text{Ind } \mathbf{HM}).$$

These are triangulated functors, and we will prove that they are in addition monoidal, at least if  $\Lambda$  is a field.

**Proposition 7.12.**

(1) *The functors  $C^*$  and  $C_{\text{tr}}^*$  induce canonically lax monoidal left Quillen functors*

$$C_s^* : \mathbf{Spt}_T^{\Sigma} \mathbf{U}(\text{SmAff})/(\mathbb{A}^1, \tau) \rightarrow \mathbf{Spt}_{C^*T}^{\Sigma} \mathbf{Cpl}(\text{Ind } \mathbf{HM}^{\text{eff}}),$$

$$C_{\text{tr},s}^* : \mathbf{Spt}_{T_{\text{tr}}}^{\Sigma} \mathbf{U}(\text{SmAffCor})/(\mathbb{A}^1, \tau) \rightarrow \mathbf{Spt}_{C^*T}^{\Sigma} \mathbf{Cpl}(\text{Ind } \mathbf{HM}^{\text{eff}}).$$

(2) *The following triangles commute up to triangulated isomorphisms*

$$\begin{array}{ccc} \mathbf{DA} & \xrightarrow{\mathcal{R}_{N,s}} & \mathbf{D}(\text{Ind } \mathbf{HM}) \\ & \searrow \text{Bti}^* & \downarrow \mathbf{D}(o) \\ & & \mathbf{D}(\Lambda) \end{array} \quad \begin{array}{ccc} \mathbf{DA} & \xrightarrow{\mathcal{R}_{N,s}} & \mathbf{D}(\text{Ind } \mathbf{HM}) \\ \text{La}_{\text{tr}} \downarrow & \nearrow \mathcal{R}_{N,\text{tr},s} & \\ \mathbf{DM} & & \end{array} \quad (7.13)$$

(3) *If  $\Lambda$  is a field then  $\mathcal{R}_{N,(\text{tr}),s}$  is monoidal, and the triangles in (7.13) commute up to monoidal isomorphisms.*

(4) *The Nori realization functors restrict to functors*

$$\mathcal{R}_{N,s} : \mathbf{DA}_{\text{ct}} \rightarrow \mathbf{D}^b(\mathbf{HM}), \quad \mathcal{R}_{N,\text{tr},s} : \mathbf{DM}_{\text{ct}} \rightarrow \mathbf{D}^b(\mathbf{HM})$$

*on the categories of constructible motives.*



Recall that the category of constructible motives is the thick subcategory generated by smooth schemes.

*Proof.* We will prove the first part for  $C^*$  but the case with transfers is literally the same.  $C^*$  together with the natural transformation  $\theta : C^*T \otimes C^*(\bullet) \rightarrow C^*(T \otimes \bullet)$  induces a functor

$$C_s^* : \mathbf{Spt}_T^\Sigma \mathbf{USmAff} \rightarrow \mathbf{Spt}_{C^*T}^\Sigma \mathbf{Cpl}(\mathrm{Ind} \mathbf{HM}^{\mathrm{eff}})$$

(cf. [1, Déf. 4.3.16]). Explicitly, it takes a symmetric  $T$ -spectrum  $\mathbf{E}$  to the symmetric  $C^*T$ -spectrum which in level  $n$  is given by  $C^*(\mathbf{E}_n)$  and whose bonding maps are given by

$$C^*T \otimes C^*(\mathbf{E}_n) \xrightarrow{\theta} C^*(T \otimes \mathbf{E}_n) \rightarrow C^*(\mathbf{E}_{n+1}),$$

the second arrow being induced by the bonding map of  $\mathbf{E}$ . The lax monoidal structure on  $C^*$  induces canonically a lax monoidal structure on  $C_s^*$ .

It is clear that  $C_s^*$  is cocontinuous hence admits a right adjoint, by the adjoint functor theorem for locally presentable categories. Let  $f$  be a projective cofibration in  $\mathbf{Spt}_T^\Sigma \mathbf{USmAff}$ . Then  $f$  is in particular levelwise a cofibration ([1, Cor. 4.3.23]) and by the discussion in the previous section,  $C^*$  takes these to monomorphisms. Thus  $C_s^*(f)$  is a monomorphism. The same argument shows that  $C_s^*$  takes projective cofibrations which are levelwise  $(\mathbb{A}^1, \tau)$ -local equivalences to monomorphisms which are levelwise quasi-isomorphisms. In other words,  $C_s^*$  is a left Quillen functor for the *unstable* model structures. To prove the first part of the proposition, it remains to prove that  $C_s^*$  takes the morphism

$$\zeta_n^D : \mathrm{Sus}_T^{n+1}(T \otimes D) \rightarrow \mathrm{Sus}_T^n D$$

to a stable equivalence for every cofibrant object  $D$  and every  $n \geq 0$  (cf. [18, Def. 8.7]). But in the unstable homotopy category we can factor the image of  $\zeta_n^D$  as follows:

$$\begin{aligned} C_s^* \mathrm{Sus}_T^{n+1}(T \otimes D) &\leftarrow \mathrm{Sus}_{C^*T}^{n+1} C^*(T \otimes D) \\ &\leftarrow \mathrm{Sus}_{C^*T}^{n+1} (C^*T \otimes C^*D) \\ &\rightarrow \mathrm{Sus}_{C^*T}^n C^*D \\ &\rightarrow C_s^* \mathrm{Sus}_T^n D. \end{aligned}$$

The first, second and fourth arrows are all levelwise quasi-isomorphisms because  $LC^*$  is monoidal on the level of derived categories. Moreover, the third arrow is a stable equivalence by definition.

We now come to the second part of the proposition. Commutativity of the triangle on the right follows from (the proof of) Proposition 7.9. For the triangle on the left, recall that  $\mathrm{Sg}^* : \mathbf{U}(\mathrm{Man}_{\mathbb{C}})/(\mathbb{D}^1, \mathrm{usu}) \rightarrow \mathbf{Cpl}(\Lambda)$  is a lax monoidal left Quillen functor. As for  $C^*$  above this implies that there is an induced lax monoidal left Quillen functor  $\mathrm{Sg}_s^*$  on the level of spectra (for the projective, respectively injective stable model structures). The Betti realization can then also be described as the following composition:

$$\mathbf{DA} \xrightarrow{\mathrm{An}^*} \mathbf{AnDA} \xrightarrow[\sim]{\mathrm{LSg}_s^*} \mathbf{Hot}(\mathbf{Spt}_{\mathrm{Sg}^* \mathbf{An}^* T}^\Sigma \mathbf{Cpl}(\Lambda)) \xrightarrow[\sim]{\mathrm{REv}_0} \mathbf{D}(\Lambda).$$

Analogously,  $\mathbf{D}(o)\mathcal{R}_{N,s}$  can be described as the composition

$$\mathbf{DA}_{\mathrm{aff}} \xrightarrow{\mathrm{LC}_s^*} \mathbf{Hot}(\mathbf{Spt}_{C^*T}^\Sigma \mathbf{Cpl}(\mathrm{coMod}(\mathbf{H}_N^{\mathrm{eff}}))) \xrightarrow{\mathbf{D}(o)} \mathbf{Hot}(\mathbf{Spt}_{oC^*T}^\Sigma \mathbf{Cpl}(\Lambda)) \xrightarrow[\sim]{\mathrm{REv}_0} \mathbf{D}(\Lambda).$$

One is then essentially reduced to compare  $\mathbf{D}(o)LC_s^*$  and  $\mathbf{LSg}_s^* \mathbf{An}^*$  which is done, as in the effective case, by means of the intermediate functor  $P$ .

We come to the third part, and assume now that  $\Lambda$  is a field. Using Lemma C.5 together with [18, Thm. 8.11] we see that the categories occurring in the definition of  $\mathcal{R}_{N,(\mathrm{tr}),s}$  all carry induced monoidal structures. By the previous lemma,  $\mathrm{REv}_0 \circ \bar{\iota}$  is lax monoidal, as is  $LC_{(\mathrm{tr}),s}^*$  by the first part of the proposition. It follows that  $\mathcal{R}_{N,(\mathrm{tr}),s}$  is a lax monoidal functor, and the comparisons in part 2 are compatible with these lax monoidal structures.

Monoidality of  $\mathcal{R}_{N,s}$  now follows from monoidality of  $\mathrm{Bti}^*$  and the fact that the derived forgetful functor is conservative. Monoidality of  $\mathcal{R}_{N,\mathrm{tr},s}$  in the étale case follows from this since  $\mathrm{La}_{\mathrm{tr}}$  is an equivalence of categories (cf. [4, Cor. B.14]). Finally, the Nisnevich realization factors through the étale realization via a monoidal functor.

The last part of the proposition holds because  $\mathcal{R}_{N,(\mathrm{tr}),s}$  takes a smooth affine scheme into  $\mathbf{D}^b(\mathbf{HM})$ . (For this we use that  $\mathbf{D}^b(\mathbf{HM})$  is a full subcategory of  $\mathbf{D}^b(\mathrm{Ind} \mathbf{HM})$ ; see [23, Pro. 8.6.11 and Thm. 15.3.1.(i)].)  $\square$

**Remark 7.14.** During the preparation of the present article, Ivorra in [21] independently defined such a motivic realization for étale motives without transfers. While his construction is more general in that it applies also to a relative case (involving his generalization of Nori motives to “perverse Nori motives” over a base), he does not consider monoidality of the functor nor its behaviour with respect to transfers.

Denote by  $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{pol}}$  the category of polarizable mixed  $\mathbb{Q}$ -Hodge structures, and by  $\mathrm{Ind} \mathbf{MHS}_{\mathbb{Q}}^{\mathrm{pol}}$  its Ind-category. There is a monoidal exact mixed Hodge realization for Nori motives whose composition with the forgetful functor yields the forgetful functor on Nori motives. Composing its derived counterpart with  $\mathcal{R}_{N,(\mathrm{tr}),s}$  from the previous proposition yields the following immediate corollary.

**Corollary 7.15.** *There are mixed Hodge realization functors*

$$\mathcal{R}_{\mathrm{H}} : \mathbf{DA}_{\mathbb{Q}} \longrightarrow \mathbf{D}(\mathrm{Ind} \mathbf{MHS}_{\mathbb{Q}}^{\mathrm{pol}}), \quad \mathcal{R}_{\mathrm{H},\mathrm{tr}} : \mathbf{DM}_{\mathbb{Q}} \longrightarrow \mathbf{D}(\mathrm{Ind} \mathbf{MHS}_{\mathbb{Q}}^{\mathrm{pol}})$$

*satisfying the following properties:*

- (1) *They are triangulated monoidal.*
- (2) *They make the following triangles commute up to monoidal triangulated isomorphisms.*

$$\begin{array}{ccc} \mathbf{DA}_{\mathbb{Q}} & \xrightarrow{\mathcal{R}_{\mathrm{H}}} & \mathbf{D}(\mathrm{Ind} \mathbf{MHS}_{\mathbb{Q}}^{\mathrm{pol}}) \\ & \searrow \mathrm{Bti}^* & \downarrow \mathbf{D}(o) \\ & & \mathbf{D}(\mathbb{Q}) \end{array} \quad \begin{array}{ccc} \mathbf{DA}_{\mathbb{Q}} & \xrightarrow{\mathcal{R}_{\mathrm{H}}} & \mathbf{D}(\mathrm{Ind} \mathbf{MHS}_{\mathbb{Q}}^{\mathrm{pol}}) \\ \mathrm{La}_{\mathrm{tr}} \downarrow & \nearrow \mathcal{R}_{\mathrm{H},\mathrm{tr}} & \\ \mathbf{DM}_{\mathbb{Q}} & & \end{array}$$

- (3) *They restrict to triangulated monoidal functors*

$$\mathcal{R}_{\mathrm{H}} : \mathbf{DA}_{\mathbb{Q},\mathrm{ct}} \longrightarrow \mathbf{D}^b(\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{pol}}), \quad \mathcal{R}_{\mathrm{H},\mathrm{tr}} : \mathbf{DM}_{\mathbb{Q},\mathrm{ct}} \longrightarrow \mathbf{D}^b(\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{pol}})$$

*on the categories of constructible motives.*

## 8. ALMOST SMOOTH PAIRS

In the sequel we will want to manipulate the Morel-Voevodsky motives of pairs of varieties  $(X, Z)$ , and describe their images under certain functors explicitly. This is easy if both  $X$  and  $Z$  are smooth but turns out to be rather difficult in general. What we need is a class of pairs which on the one hand are close enough to smooth ones so that explicit computations are feasible, and on the other hand flexible enough so that we are able to reduce our arguments from general pairs to this smaller class. This is provided by the class of *almost smooth* pairs, i.e. pairs of varieties  $(X, Z)$  where  $X$  is smooth and  $Z$  a simple normal crossings divisor. By resolution of singularities and excision, every good pair receives a morphism from an almost smooth one which induces isomorphisms in Betti homology. In this section, we will give rather explicit motivic models for almost smooth pairs, both on the effective and the stable level, and compute their images under various functors. One immediate consequence of our discussion here is that the morphism of bialgebras  $\varphi'_N$  passes to the stable level.

**8.1. Effective level.**  $(X, Z)$  will now be our running notation for an almost smooth pair. We always denote the irreducible components of  $Z$  by  $Z_1, \dots, Z_p$  and endow them with the reduced structure. The (smooth) intersection of  $Z_i$  and  $Z_j$  is denoted by  $Z_{ij}$ , and similarly for intersections of more than two components. The presheaf  $\Lambda(X, Z)$  is defined to be the cokernel of the morphism  $\bigoplus_{i=1}^p \Lambda(Z_i) \rightarrow \Lambda(X)$ .

In addition, let  $\mathcal{Y} = (Y_1, \dots, Y_q)$  be an open affine cover of  $X$ . For any functor  $F : \mathbf{SmAff} \rightarrow \mathbf{Cpl}(\mathcal{C})$  into the category of complexes on an abelian category  $\mathcal{C}$ , we define  $F^{\mathcal{Y}}(X, Z_{\bullet}) \in \mathbf{Cpl}(\mathcal{C})$  to be the (sum) total complex of the tricomplex whose  $(i, j, k)$ -th term is

$$\bigoplus_{a_0 < \dots < a_i, b_1 < \dots < b_j} F_k(Y_{a_0 \dots a_i} \cap Z_{b_1 \dots b_j}),$$

where by convention the empty intersection of the  $Z_i$ 's is  $X$ . This can also be understood as the mapping cone of the morphism

$$F^{\mathcal{Y} \cap Z_{\bullet}}(Z_{\bullet}) \rightarrow F^{\mathcal{Y}}(X)$$

with an obvious interpretation of the first term. If  $F$  is defined on all smooth schemes, we set  $F(X, Z_{\bullet})$  to be  $F^{(X)}(X, Z_{\bullet})$ , and if  $F$  is defined on all affine varieties, we can similarly define  $F^{\mathcal{Y}}(X, Z)$ .

For example, we can consider the presheaf of complexes  $\Lambda^{\mathcal{Y}}(X, Z_{\bullet})$  with the canonical map to  $\Lambda(X, Z)$ . This defines a cofibrant replacement as we now prove.

**Lemma 8.1.** *The canonical morphism  $\Lambda^{\mathcal{Y}}(X, Z_{\bullet}) \rightarrow \Lambda(X, Z)$  is a cofibrant replacement for the  $\tau$ -local model structure, and the complexes*

$$C^* \Lambda^{\mathcal{Y}}(X, Z_{\bullet}) \xrightarrow{\sim} C^{\mathcal{Y}}(X, Z_{\bullet}) \xrightarrow{\sim} C^{\mathcal{Y}}(X, Z)$$

*all provide models for  $LC^* \Lambda(X, Z)$ .*

*Proof.* To prove the first statement consider the following morphism of distinguished triangles in the derived category of presheaves on smooth schemes:

$$\begin{array}{ccccccc} \Lambda^{\mathcal{Y}}(Z_{\bullet}) & \longrightarrow & \Lambda^{\mathcal{Y}}(X) & \longrightarrow & \Lambda^{\mathcal{Y}}(X, Z_{\bullet}) & \longrightarrow & \Lambda^{\mathcal{Y}}(Z_{\bullet})[-1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Lambda(Z_{\bullet}) & \longrightarrow & \Lambda(X) & \longrightarrow & \Lambda(X, Z) & \longrightarrow & \Lambda(Z_{\bullet})[-1] \end{array}$$

(It should be clear what the first term denotes although we haven't formally defined it above. That the second row arises from a short exact sequence of complexes of presheaves (and hence is indeed a distinguished triangle) is [33, 2.1.4]; see [30, Lem. 1.4] for a proof.) The second vertical arrow is a  $\tau$ -local equivalence as is the left vertical arrow by induction on the number of irreducible components of  $Z$ . It follows that the third vertical arrow is a  $\tau$ -local equivalence as well. Since  $\Lambda^{\mathcal{Y}}(X, Z_{\bullet})$  is a bounded below complex of representables, it is projective cofibrant ([10, Fact 3.10]).

We now come to the second statement of the lemma. It is clear that the first arrow is invertible. For the second arrow consider the following diagram:

$$\begin{array}{ccccc} oC^{\mathcal{Y}}(X, Z_{\bullet}) & \longleftarrow & P^{\mathcal{Y}}(X, Z_{\bullet}) & \longrightarrow & (\mathrm{Sg})^{\mathcal{Y}}(X, Z_{\bullet}) \\ \downarrow & & \downarrow & & \downarrow \\ oC^{\mathcal{Y}}(X, Z) & \longleftarrow & P^{\mathcal{Y}}(X, Z) & \longrightarrow & (\mathrm{Sg})^{\mathcal{Y}}(X, Z) \end{array}$$

By the discussion in section 6, we know that the horizontal arrows are all quasi-isomorphisms. Since the right-most vertical arrow is a quasi-isomorphism so is the left-most.  $\square$

Recall that we equipped the Betti realization of any effective Morel-Voevodsky motive  $M$  with a coaction of  $\mathbf{H}_N^{\mathrm{eff}}$ , and we denoted the resulting comodule by  $\mathcal{R}_N(M)$  (see (7.5)). Of course, for this to be a sensible construction, the comodule structure should better be compatible with the canonical one on  $\Lambda$ -modules of the form  $H_n(X, Z)$ . The following lemma states that this is the case for  $(X, Z)$  almost smooth.

Define the following zig-zag of morphisms of complexes of  $\Lambda$ -modules:

$$oC^{\mathcal{Y}}(X, Z) \leftarrow P^{\mathcal{Y}}(X, Z) \rightarrow (\mathrm{Sg})^{\mathcal{Y}}(X, Z) \rightarrow \mathrm{Sg}(X, Z). \quad (8.2)$$

**Lemma 8.3.** *Assume that  $(X, Z)$  is an almost smooth pair, and that  $\Lambda$  is a principal ideal domain. Then (8.2) induces an isomorphism of  $\mathbf{H}_N^{\mathrm{eff}}$ -comodules*

$$H_n \mathcal{R}_N \Lambda(X, Z) \xrightarrow{\sim} H_n(X, Z)$$

for all  $n \in \mathbb{Z}$ .

*Proof.* By Jouanolou's trick there exists a smooth affine variety  $X'$  and a Zariski locally trivial morphism  $p : X' \rightarrow X$  whose fibers are isomorphic to affine space. Setting  $Z'_i = Z_i \times_X X'$  we obtain an almost smooth pair  $(X', Z')$  with  $X'$  affine, and a morphism  $p : (X', Z') \rightarrow (X, Z)$  which induces an isomorphism in singular homology.

Let  $\mathcal{Y}'$  be the pullback of the affine cover to  $X'$  and consider the following commutative diagram, where all the arrows are the canonical ones:

$$\begin{array}{ccccc}
oC^{\mathcal{Y}}(X, Z) & \longleftarrow & P^{\mathcal{Y}}(X, Z) & \longrightarrow & (\mathrm{Sg})^{\mathcal{Y}}(X, Z) \\
\uparrow & & \uparrow & & \uparrow \\
oC^{\mathcal{Y}'}(X', Z') & \longleftarrow & P^{\mathcal{Y}'}(X', Z') & \longrightarrow & (\mathrm{Sg})^{\mathcal{Y}'}(X', Z') \\
\downarrow & & \downarrow & & \downarrow \\
oC(X', Z') & \longleftarrow & P(X', Z') & \longrightarrow & \mathrm{Sg}(X', Z') \\
& & \downarrow & & \downarrow \\
& & P(X')/P(Z') & \longrightarrow & \mathrm{Sg}(X')/\mathrm{Sg}(Z')
\end{array}$$

By the discussion in section 6, we know that the top horizontal arrows are both quasi-isomorphisms. All vertical arrows are quasi-isomorphisms. We thus reduce to prove that the zig-zag of morphisms  $oC(X', Z') \leftarrow P(X', Z') \rightarrow \mathrm{Sg}(X')/\mathrm{Sg}(Z')$  induces an  $\mathbf{H}_N^{\mathrm{eff}}$ -comodule (iso)morphism in the  $n$ -th homology. Writing  $(X, Z)$  for  $(X', Z')$ , this is expressed by commutativity of the following diagram, where the vertical arrows are the coaction of  $\mathbf{H}_N^{\mathrm{eff}}$  on the objects in question:

$$\begin{array}{ccccc}
H_n oC(X, Z) & \xleftarrow{\sim} & H_n P(X, Z) & \xrightarrow{\sim} & H_n(X, Z) \\
\mathrm{ca} \downarrow & & & & \downarrow \mathrm{ca} \\
H_n oC(X, Z) \otimes \mathbf{H}_N^{\mathrm{eff}} & \xleftarrow{\sim} & H_n P(X, Z) \otimes \mathbf{H}_N^{\mathrm{eff}} & \xrightarrow{\sim} & H_n(X, Z) \otimes \mathbf{H}_N^{\mathrm{eff}}
\end{array}$$

Start with any  $[(f, g)] \in H_n P(X, Z)$ . Thus there exist  $X_n \subset X$ ,  $Z_{n-1} \subset Z$  closed subvarieties of dimension at most  $n$  and  $n-1$ , respectively, such that  $f \in \mathrm{Sg}_n(X_n)$ ,  $g = \pm \partial f \in \mathrm{Sg}_{n-1}(Z_{n-1})$  (depending on the sign conventions for the mapping cone). It is then clear from the definition of the natural transformations  $oC \leftarrow P \rightarrow \mathrm{Sg}$  that we reduce to prove commutativity of the following diagram

$$\begin{array}{ccccc}
H_n oC(X, Z) & \xleftarrow{\sim} & H_n(X_n, Z_{n-1}) & \xrightarrow{\sim} & H_n(X, Z) \\
\mathrm{ca} \downarrow & & \mathrm{ca} \downarrow & & \downarrow \mathrm{ca} \\
H_n oC(X, Z) \otimes \mathbf{H}_N^{\mathrm{eff}} & \xleftarrow{\sim} & H_n(X_n, Z_{n-1}) \otimes \mathbf{H}_N^{\mathrm{eff}} & \xrightarrow{\sim} & H_n(X, Z) \otimes \mathbf{H}_N^{\mathrm{eff}}
\end{array}$$

which is obvious.  $\square$

This lemma will be important later on as well but one immediate application is that it allows us to extend the morphism of bialgebras  $\varphi'_N : \mathbf{H}_A^{\mathrm{eff}} \rightarrow \mathbf{H}_N^{\mathrm{eff}}$  constructed in section 7 to a morphism  $\varphi_N : \mathbf{H}_A \rightarrow \mathbf{H}_N$ . Indeed, we see that there is the following isomorphism of  $\mathbf{H}_N^{\mathrm{eff}}$ -comodules:

$$\begin{aligned}
H_0 \overline{\varphi_N} \widetilde{\mathrm{Bti}}^{\mathrm{eff},*}(T[2]) &\xrightarrow{\sim} H_0 \mathcal{R}_N(T[2]) && \text{by Cor. 7.7} \\
&\xrightarrow{\sim} H_0 \mathcal{R}_N \Lambda(\mathbb{G}_m, \{1\})[1] \\
&\xrightarrow{\sim} H_1(\mathbb{G}_m, \{1\}),
\end{aligned}$$

the last isomorphism by the previous lemma. One deduces easily that  $\varphi'_N(s_A) = s_N \in \mathbf{H}_N^{\mathrm{eff}}$  and hence the morphism  $\mathcal{H}_A^{\mathrm{eff}} \rightarrow \mathbf{H}_N^{\mathrm{eff}}$  from Corollary 7.7 passes to the

localization and induces the following commutative squares:

$$\begin{array}{ccc} \mathcal{H}_A^{\text{eff}} & \longrightarrow & \mathbf{H}_N^{\text{eff}} \\ \downarrow \iota & & \downarrow \iota \\ \mathcal{H}_A & \longrightarrow & \mathbf{H}_N \end{array} \quad \begin{array}{ccc} \mathbf{H}_A^{\text{eff}} & \xrightarrow{\varphi'_N} & \mathbf{H}_N^{\text{eff}} \\ \downarrow \iota & & \downarrow \iota \\ \mathbf{H}_A & \xrightarrow{\varphi_N} & \mathbf{H}_N \end{array}$$

**Remark 8.4.** In particular, we can now define a stable version of the functor  $\mathcal{R}_N$  constructed in the effective case in section 7 (still assuming that  $\Lambda$  is a principal ideal domain). Indeed, we set it to be the composition

$$\mathcal{R}_N : \mathbf{DA} \xrightarrow{\widetilde{\text{Bti}}^*} \mathbf{coMod}(\mathcal{H}_A) \rightarrow \mathbf{coMod}(\mathbf{H}_N)^{\mathbf{D}(\Lambda)}.$$

As the composition of two monoidal functors,  $\mathcal{R}_N$  is again monoidal. It follows also that the diagrams analogous to the ones in Corollary 7.7 commute

$$\begin{array}{ccc} \mathbf{DA} & \xrightarrow{\mathcal{R}_N} & \mathbf{coMod}(\mathbf{H}_N)^{\mathbf{D}(\Lambda)} \\ \widetilde{\text{Bti}}^* \downarrow & \nearrow & \downarrow o \\ \mathbf{coMod}(\mathcal{H}_A) & \xrightarrow{o} & \mathbf{D}(\Lambda) \end{array} \quad \begin{array}{ccc} \mathbf{DA} & \xrightarrow{H_0 \mathcal{R}_N} & \mathbf{coMod}(\mathbf{H}_N) \\ H_0 \widetilde{\text{Bti}}^* \downarrow & \nearrow \overline{\varphi_N} & \downarrow o \\ \mathbf{coMod}(\mathbf{H}_A) & \xrightarrow{o} & \mathbf{Mod}(\Lambda) \end{array}$$

as does the following square:

$$\begin{array}{ccc} \mathbf{DA}^{\text{eff}} & \xrightarrow{\mathcal{R}_N} & \mathbf{coMod}(\mathbf{H}_N^{\text{eff}})^{\mathbf{D}(\Lambda)} \\ \text{LSus}_T^0 \downarrow & & \downarrow \bar{\iota} \\ \mathbf{DA} & \xrightarrow{\mathcal{R}_N} & \mathbf{coMod}(\mathbf{H}_N)^{\mathbf{D}(\Lambda)} \end{array}$$

**8.2. Stable level.** We continue our study of almost smooth pairs but now we work in the context of non-effective motives. For such a pair, we will provide a rather explicit model for both its homological as well as cohomological Morel-Voevodsky motive in Theorem 8.5, and then similarly for its analytification in Theorem 8.9. Subsequently we prove that the Betti realization is in some sense compatible with these models (Lemmas 8.10 and 8.12).

Let  $(X, Z)$  be an almost smooth pair. The inclusion of the complement  $U = X \setminus Z \rightarrow X$  is denoted by  $j$ . Recall (from [4, §2.2.4]) the following constructions. Given a presheaf  $K$  of complexes on smooth schemes, one defines  $K(X, Z)$  to be the kernel of the map  $K(X) \rightarrow \prod_{i=1}^p K(Z_i)$ . The endofunctor  $\underline{\text{hom}}((X, Z), \bullet)$  is defined as the right adjoint to tensoring with  $\Lambda(X, Z)$ . Explicitly,

$$\underline{\text{hom}}((X, Z), K)(Y) = K(Y \times X, Y \times Z)$$

for any presheaf of complexes  $K$  and for any smooth scheme  $Y$ .  $\underline{\text{hom}}((X, Z), \bullet)$  canonically extends to an endofunctor on symmetric  $T$ -spectra of presheaves of complexes.

In general, we denote the internal hom in symmetric  $T$ -spectra by  $\underline{\text{Hom}}$ . We note that for a complex of presheaves  $K$  and a symmetric  $T$ -spectrum  $\mathbf{E}$ , the object  $\underline{\text{Hom}}(\text{Sus}_T^0 K, \mathbf{E})$  admits the following simple description. In level  $n$ , it is given by  $\underline{\text{Hom}}(K, \mathbf{E}_n)$ , the action of  $\Sigma_n$  is on  $\mathbf{E}_n$ , and the bonding maps are given by the composition

$$T \otimes \underline{\text{Hom}}(K, \mathbf{E}_n) \rightarrow \underline{\text{Hom}}(K, T \otimes \mathbf{E}_n) \rightarrow \underline{\text{Hom}}(K, \mathbf{E}_{n+1}),$$

where the first arises from the adjunction  $(\otimes, \underline{\text{Hom}})$ , and the second uses the bonding maps from  $\mathbf{E}$ . To emphasize this description we write simply  $\underline{\text{Hom}}(K, \mathbf{E})$  for this symmetric  $T$ -spectrum.

Using the notation from §8.1,  $\Lambda(X, Z_\bullet)$  denotes the augmented complex

$$\cdots \rightarrow \bigoplus_{i_1 < \cdots < i_l} \Lambda(Z_{i_1 \dots i_l}) \rightarrow \cdots \rightarrow \bigoplus_i \Lambda(Z_i) \rightarrow \Lambda(X),$$

the last term being in homological degree 0.

For the next result, recall that on presheaves of complexes on smooth schemes there is also an *injective*  $(\mathbb{A}^1, \tau)$ -local model structure, obtained by  $(\mathbb{A}^1, \tau)$ -localization from the “injective model structure” ([1, Déf. 4.5.12]). The cofibrations and weak equivalences of the latter are defined objectwise. One deduces then the existence of an “injective stable  $(\mathbb{A}^1, \tau)$ -local model structure” on symmetric  $T$ -spectra as described in §3 (cf. [1, Déf. 4.5.21]).

**Theorem 8.5.** *Let  $(X, Z)$  be almost smooth.*

- (1) *Let  $\mathbf{E}$  be a projective stable  $(\mathbb{A}^1, \tau)$ -fibrant symmetric  $T$ -spectrum of presheaves of complexes on  $\text{Sm}$ . Then  $\underline{\text{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E})$  provides a model for  $R\pi_* j_! \mathbf{E}|_U$  in  $\mathbf{DA}$ . Moreover, this identification is functorial in  $\mathbf{E}$ .*
- (2) *If  $\mathbf{E}$  is injective stable  $(\mathbb{A}^1, \tau)$ -fibrant, then one can replace  $\underline{\text{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E})$  by  $\underline{\text{hom}}((X, Z), \mathbf{E})$  in the statement above.*
- (3) *For  $\mathbf{E}$  an injective stable  $(\mathbb{A}^1, \tau)$ -fibrant replacement of the unit spectrum,  $\underline{\text{hom}}((X, Z), \mathbf{E})$  provides a model for  $\mathcal{R}_A(X, Z, 0)^\vee$  in  $\mathbf{DA}$ .*
- (4) *In  $\mathbf{DA}$ ,  $\text{LSus}_T^0 \Lambda(X, Z) \cong \mathcal{R}_A(X, Z, 0)$  canonically.*

*Proof.* (1) Let  $K_\bullet(\mathbf{E})$  be the object

$$\mathbf{E}|_X \rightarrow \oplus_i i_{i*} \mathbf{E}|_{Z_i} \rightarrow \oplus_{i < j} i_{ij*} \mathbf{E}|_{Z_{ij}} \rightarrow \cdots$$

with differentials given by the canonical “restriction” morphism (the unit of an adjunction) with a suitable sign:

$$i_{j_1 \dots j_m \dots j_l}^* \mathbf{E}|_{Z_{j_1 \dots j_m \dots j_l}} \xrightarrow{(-1)^m} i_{j_1 \dots j_l}^* \mathbf{E}|_{Z_{j_1 \dots j_l}}.$$

There is a canonical morphism

$$j_! \mathbf{E}|_U \rightarrow \text{Tot}(K_\bullet(\mathbf{E})) =: K(\mathbf{E}) \quad (8.6)$$

(the totalization functor is applied levelwise; up to canonical isomorphism it doesn’t matter whether  $\text{Tot}^\oplus$  or  $\text{Tot}^\Pi$  is used), and we claim that this is a projective stable  $(\mathbb{A}^1, \tau)$ -fibrant resolution. In other words, we claim that

- (a) (8.6) is a stable  $(\mathbb{A}^1, \tau)$ -local equivalence,
- (b)  $K(\mathbf{E})$  is levelwise projective  $(\mathbb{A}^1, \tau)$ -fibrant, and
- (c)  $K(\mathbf{E})$  is an  $\Omega$ -spectrum.

*Proof of (a).* One can use conservativity of the couple  $(j^*, Li^*)$ ,  $i : Z \hookrightarrow X$  being the closed immersion. It is obvious that  $j^*$  applied to (8.6) is an equivalence while in the case of  $Li^*$  it is an easy induction argument on the number of irreducible components of  $Z$ .  $\square$

*Proof of (b).* Fix a level  $n$  and set  $E = \mathbf{E}_n$ . We know that for each  $l$ ,  $K_l(E)$  is  $\tau$ -fibrant hence so is  $K(E)$  by [10, Lemma 5.20]. The same argument shows that  $K(E)$  is  $\mathbb{A}^1$ -local.  $\square$



*Proof of (c).* Since  $K_l(\mathbf{E}_n) \rightarrow \underline{\mathrm{hom}}(T, K_l(\mathbf{E}_{n+1}))$  is an  $(\mathbb{A}^1, \tau)$ -local equivalence for each  $l$  so is the totalization

$$\mathrm{Tot}(K_\bullet(\mathbf{E}_n)) \rightarrow \mathrm{Tot}(\underline{\mathrm{hom}}(T, K_\bullet(\mathbf{E}_{n+1}))) = \underline{\mathrm{hom}}(T, \mathrm{Tot}(K_\bullet(\mathbf{E}_{n+1}))).$$

□

It follows that in  $\mathbf{DA}$ ,

$$\begin{aligned} \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E}) &= \mathrm{Tot}(\underline{\mathrm{Hom}}(\Lambda(X), \mathbf{E}) \rightarrow \oplus_i \underline{\mathrm{Hom}}(\Lambda(Z_i), \mathbf{E}) \rightarrow \cdots) \\ &= \mathrm{Tot}(\pi_* \pi^* \mathbf{E} \rightarrow \oplus_i \pi_{Z_i} \pi_{Z_i}^* \mathbf{E} \rightarrow \cdots) \\ &= \pi_* \mathrm{Tot}(K_\bullet(\mathbf{E})) \\ &\xleftarrow{\sim} R\pi_* j_! \mathbf{E}|_U. \end{aligned} \tag{8.7}$$

Finally, functoriality in  $\mathbf{E}$  is clear.

(2) We find an  $(\mathbb{A}^1, \tau)$ -local equivalence

$$\Lambda(X, Z_\bullet) \rightarrow \Lambda(X, Z) \tag{8.8}$$

between injective cofibrant objects hence  $\underline{\mathrm{Hom}}(\bullet, \mathbf{E})$  will transform (8.8) into an (un)stable  $(\mathbb{A}^1, \tau)$ -local equivalence. Since  $\mathbf{E}$  is in particular projective stable  $(\mathbb{A}^1, \tau)$ -fibrant it follows from the first part of the theorem just proved that

$$\begin{aligned} R\pi_* j_! \mathbf{E}|_U &\xrightarrow{\sim} \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E}) \\ &\xrightarrow[\text{(8.8)}]{\sim} \underline{\mathrm{Hom}}(\Lambda(X, Z), \mathbf{E}) = \underline{\mathrm{hom}}((X, Z), \mathbf{E}). \end{aligned}$$

(3) This is an immediate consequence of the second part and the identification  $\mathcal{R}_\Lambda(X, Z, 0)^\vee = R\pi_* j_! \mathbb{1}$ .

(4) This follows from the third part by duality.

□

We will need a similar result in the analytic setting. Thus let  $X$  be a complex manifold, and  $Z$  a closed subset which is the union of finitely many complex submanifolds. We call this an *almost smooth analytic pair*, and as before, we denote by  $Z_1, \dots, Z_p$  the “components” of  $Z$ , namely the connected components of the normalization of  $Z$ . We can then define, analogously,  $\Lambda(X, Z_\bullet)$ ,  $\Lambda(X, Z)$  and  $\underline{\mathrm{hom}}((X, Z), \bullet)$  (cf. [4, §2.2.1]).

**Theorem 8.9.** *Let  $\mathbf{E}$  be a projective stable  $(\mathbb{D}^1, \mathrm{usu})$ -fibrant presheaf of complexes on  $\mathrm{Man}_\mathbb{C}$ . Then  $\underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E})$  provides a model for  $R\pi_* j_! \mathbf{E}|_U$  in  $\mathbf{AnDA}$ . Moreover, this identification is functorial in  $\mathbf{E}$ .*

*Proof.* The proof is very similar to the one of the last theorem and we omit the details. (Also, the other parts of the previous theorem are equally true in the analytic setting, with almost identical proofs.) □

Later on, we will use the following relation between the two descriptions of motives we just gave. Choose a projective stable  $(\mathbb{A}^1, \tau)$ -fibrant replacement  $\mathbf{E}$  of the unit spectrum  $\mathbb{1}$ . Also choose a projective stable  $(\mathbb{D}^1, \mathrm{usu})$ -fibrant replacement  $\mathbf{E}'$  of  $\mathbb{1}$ .  $\mathrm{An}^* \mathbb{1} \cong \mathbb{1} \rightarrow \mathbf{E}'$  induces, by adjunction,  $\mathbb{1} \rightarrow \mathrm{An}_* \mathbf{E}'$  and the latter is

projective stable  $(\mathbb{A}^1, \tau)$ -fibrant. It follows that there is a morphism  $\mathbf{E} \rightarrow \mathrm{An}_* \mathbf{E}'$  which induces  $\mathrm{An}^* \mathbf{E} \rightarrow \mathbf{E}'$  rendering the triangle

$$\begin{array}{ccc} \mathbb{1} & \longrightarrow & \mathrm{An}^* \mathbf{E} \\ & \searrow & \downarrow \\ & & \mathbf{E}' \end{array}$$

commutative. Notice that the morphism  $\mathrm{An}^* \mathbf{E} \rightarrow \mathbf{E}'$  is necessarily a stable  $(\mathbb{D}^1, \mathrm{usu})$ -local equivalence.

**Lemma 8.10.** *Under the identifications of the two previous theorems, the following square commutes ( $(X, Z)$  is an almost smooth pair):*

$$\begin{array}{ccc} \mathrm{An}^* \mathrm{R}\pi_* j_! \mathbb{1} & \xlongequal{\quad} & \mathrm{An}^* \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E}) \\ \downarrow & & \downarrow \\ \mathrm{R}\pi_*^{\mathrm{an}} j_!^{\mathrm{an}} \mathbb{1} & \xlongequal{\quad} & \underline{\mathrm{Hom}}(\Lambda(X^{\mathrm{an}}, Z_\bullet^{\mathrm{an}}), \mathbf{E}') \end{array}$$

*Proof.* By adjunction, this is equivalent to the commutativity of the following outer diagram:

$$\begin{array}{ccc} \mathrm{R}\pi_* j_! \mathbf{E}|_U & \xlongequal{\quad} & \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E}) \\ \mathrm{adj} \downarrow & & \downarrow \mathrm{adj} \\ \mathrm{R}\pi_* j_! (\mathrm{An}_* \mathrm{An}^* \mathbf{E})|_U & & \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathrm{An}_* \mathrm{An}^* \mathbf{E}) \\ \downarrow & & \downarrow \\ \mathrm{R}\pi_* j_! (\mathrm{An}_* \mathbf{E}')|_U & \xlongequal{\quad} & \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathrm{An}_* \mathbf{E}') \\ \sim \downarrow & & \downarrow \sim \\ \mathrm{RAn}_* \mathrm{R}\pi_*^{\mathrm{an}} j_!^{\mathrm{an}} \mathbf{E}'|_{U^{\mathrm{an}}} & \xlongequal{\quad} & \mathrm{An}_* \underline{\mathrm{Hom}}(\Lambda(X^{\mathrm{an}}, Z_\bullet^{\mathrm{an}}), \mathbf{E}') \end{array}$$

Commutativity of the upper part follows from the functoriality statement in Theorem 8.5. For the lower square, one checks that  $\mathrm{An}_*$  commutes with the relevant equalities in (8.7). The main point is that  $\mathrm{An}_* K(\mathbf{E}') = K(\mathrm{An}_* \mathbf{E}')$ .  $\square$

This lemma states that the analytification functor is compatible with our choices of models for the relative motives. We now want to prove the analogous statement for the Betti realization functor. Factoring the latter as  $\Gamma \mathrm{Ev}_0 \mathrm{An}^*$  (where  $\Gamma$  is the global sections functor, cf. section 3), we reduce to showing this compatibility for the composed functor  $\Gamma \mathrm{Ev}_0$ . Thus let  $(X, Z)$  be an almost smooth analytic pair. By what we saw in section 5 (or rather appendix B, specifically Fact B.3), the object  $\Gamma \mathrm{Ev}_0 \mathrm{R}\pi_* j_! \mathbb{1}$  is modeled by the complex of relative singular cochains on  $(X, Z)$ . Now suppose that in the situation of the previous lemma, we choose  $\mathbf{E}'$  to be  $\mathbf{Sg}^\vee$  of Remark 3.9. Then we find a canonical quasi-isomorphism

$$\begin{aligned} \mathrm{Sg}(X, Z)^\vee &\xrightarrow{\sim} \mathrm{Tot}(\mathrm{Sg}(X)^\vee \rightarrow \oplus_i \mathrm{Sg}(Z_i)^\vee \rightarrow \cdots) \\ &= \Gamma \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{Sg}^\vee) \\ &= \Gamma \mathrm{Ev}_0 \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{Sg}^\vee). \end{aligned} \tag{8.11}$$

**Lemma 8.12.** *The following square commutes:*

$$\begin{array}{ccc} \Gamma \mathrm{Ev}_0 \mathrm{R}\pi_* j_! \mathbb{1} & \xlongequal{\quad} & \Gamma \mathrm{Ev}_0 \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{Sg}^\vee) \\ \sim \downarrow & & \uparrow \sim \\ \mathrm{R}\tilde{\pi}_* \tilde{j}_! \Lambda_{\mathrm{cst}} & \xlongequal{\quad} & \mathrm{Sg}(X, Z)^\vee \end{array} \quad (8.11)$$

Here, we temporarily decorate the functors operating on sheaves on locally compact topological spaces with a tilde to distinguish them from their counterparts in the complex analytic world.

*Proof.* Since the identification of Theorem 8.9 inducing the top horizontal arrow is levelwise, we may prove the lemma staying completely on the effective level, thus decomposing the square as follows:

$$\begin{array}{ccccccc} \mathrm{R}\Gamma \mathrm{R}\pi_* j_! \Lambda & \xrightarrow{\sim} & \Gamma \pi_* K(\mathrm{Sg}^\vee) & \xlongequal{\quad} & \mathrm{Sg}^\vee(X, Z_\bullet) & & \\ \sim \downarrow & & \parallel & & \nearrow \sim & & \\ \mathrm{R}\tilde{\pi}_* \mathrm{R}\iota_{X*} j_! \Lambda & \xrightarrow{\sim} & \tilde{\pi}_* \iota_{X*} K(\mathrm{Sg}^\vee) & \xrightarrow{\sim} & \tilde{\pi}_* \mathrm{Tot}(\mathcal{S}_X \rightarrow \oplus_i i_{i*} \mathcal{S}_{Z_i} \rightarrow \cdots) & & \\ \sim \uparrow & & \sim \uparrow & & \sim \uparrow & & \\ \mathrm{R}\tilde{\pi}_* \tilde{j}_! \Lambda_{\mathrm{cst}} & \xrightarrow{\sim} & \tilde{\pi}_*(\mathcal{S}_X \otimes \Lambda_U) & \xrightarrow{\sim \alpha} & \tilde{\pi}_* \mathcal{K}_{(X, Z)} & \xleftarrow{\sim \beta} & \mathrm{Sg}(X, Z)^\vee \end{array}$$

Recall (from appendix B) that  $\mathcal{S}_X$  is the sheaf of singular cochains on the topological space  $X$ ,  $U = X \setminus Z$ , and  $\mathcal{K}_{(X, Z)}$  is the kernel of the canonical morphism  $\mathcal{S}_X \rightarrow i_* \mathcal{S}_Z$ .  $K(\mathrm{Sg}^\vee)$  is defined as in the proof of Theorem 8.5, the maps  $\alpha$  and  $\beta$  are also defined in appendix B.

Everything except possibly the lower left inner diagram clearly commutes. Commutativity of this remaining diagram can be proved before applying  $\mathrm{R}\tilde{\pi}_*$ . We replace the constant presheaf  $\Lambda$  by  $\mathrm{Sg}^\vee$ , and the constant sheaf  $\Lambda_{\mathrm{cst}}$  by  $\mathcal{S}_U$ . Then the lemma follows from commutativity of the following diagram, which is obvious.

$$\begin{array}{ccccc} \mathrm{R}\iota_{X*} j_! \mathrm{Sg}^\vee|_U & \rightarrow & \iota_{X*} \mathrm{Tot}(\mathrm{Sg}^\vee|_X \rightarrow \oplus_i i_{i*} \mathrm{Sg}^\vee|_{Z_i} \rightarrow \cdots) & \rightarrow & \mathrm{Tot}(\mathcal{S}_X \rightarrow \oplus_i i_{i*} \mathcal{S}_{Z_i} \rightarrow \cdots) \\ \uparrow & \nearrow & & \nearrow & \uparrow \\ \tilde{j}_! \iota_{U*} \mathrm{Sg}^\vee|_U & \xrightarrow{\quad} & \tilde{j}_! \mathcal{S}_U & \xlongequal{\quad} & \mathcal{S}_X \otimes \Lambda_U \end{array}$$

□

We end this section with the following result expressing a duality between relative Morel-Voevodsky motives associated to complements of two different divisors in a smooth projective scheme. We will make essential use of it in the following section.

**Lemma 8.13** (cf. [26, p. 13], [20, Lem. 1.13], [24, Lem. I.IV.2.3.5]). *Let  $W$  be a smooth projective scheme of dimension  $d$ ,  $W_0 \cup W_\infty$  a simple normal crossings divisor. Then there is a canonical isomorphism*

$$\mathcal{R}_A(W - W_\infty, W_0 - W_\infty, n)^\vee \cong \mathcal{R}_A(W - W_0, W_\infty - W_0, 2d - n)(-d)$$

in **DA**.

*Proof.* Fix the notation as in the following diagram:

$$\begin{array}{ccc} W - W_\infty & \xrightarrow{j_\infty} & W \\ j'_0 \uparrow & & \uparrow j_0 \\ W - (W_\infty \cup W_0) & \xrightarrow{j'_\infty} & W - W_0 \end{array}$$

The left hand side of the equality to establish is

$$\pi_{W*} j_{\infty*} j'_0! j_0^! j_\infty^* \pi_W^* \mathbb{1}[-n] \cong \pi_{W!} j_{\infty*} j'_0! j_\infty^* j_0^! \pi_W^* \mathbb{1}(-d)[2d-n]$$

by relative purity, hence to prove the lemma it suffices to provide a canonical isomorphism  $j_0! j_{\infty*} \mathbb{1} \cong j_{\infty*} j'_0! \mathbb{1}$ . The candidate morphism is obtained by adjunction from the composition

$$j'_{\infty*} \xrightarrow[\sim]{\text{adj}} j'_{\infty*} j'_0! j_0^! \cong j_0^! j_{\infty*} j'_0!.$$

It is clear that the candidate morphism is invertible on  $W - W_0$ , hence by localization, it remains to prove the same on  $W_0$ . Denote by  $i_\bullet$  the closed immersion complementary to  $j_\bullet$ . We add a second subscript 0 (resp.  $\infty$ ) to denote the pullback of a morphism along  $i_0$  (resp.  $i_\infty$ ).

Note that  $i_0^* j_0! = 0$  hence it suffices to prove  $i_0^* j_{\infty*} j'_0! \mathbb{1} = 0$ . By one of the localization triangles for the couple  $(W - W_\infty, W - (W_\infty \cup W_0))$  we can equivalently prove that the morphism

$$\text{adj} : i_0^* j_{\infty*} \mathbb{1} \rightarrow i_0^* j_{\infty*} i'_0! \mathbb{1} \quad (8.14)$$

is invertible. The codomain of this morphism is isomorphic to

$$i_0^* j_{\infty*} i'_0! \mathbb{1} \cong i_0^* i_{0*} j_{\infty 0*} \mathbb{1} \cong j_{\infty 0*} \mathbb{1},$$

and under this identification, (8.14) corresponds to the morphism

$$i_0^* j_{\infty*} \mathbb{1} \xrightarrow{\text{bc}} j_{\infty 0*} i'_0! \mathbb{1} \cong j_{\infty 0*} \mathbb{1}. \quad (8.15)$$

Here, as in the rest of the proof, bc denotes the canonical base change morphism of the functors involved. Consider now the following diagram:

$$\begin{array}{ccccccc} i_0^* i_{\infty!} i_\infty^! \mathbb{1} & \longrightarrow & i_0^* \mathbb{1} & \longrightarrow & i_0^* j_{\infty*} \mathbb{1} & \longrightarrow & i_0^* i_{\infty!} i_\infty^! \mathbb{1}[-1] \\ \alpha \downarrow & & \sim \downarrow & & (8.15) \downarrow & & \downarrow \alpha \\ i_{\infty 0!} i_{\infty 0}^! \mathbb{1} & \longrightarrow & \mathbb{1} & \longrightarrow & j_{\infty 0*} \mathbb{1} & \longrightarrow & i_{\infty 0!} i_{\infty 0}^! \mathbb{1}[-1] \end{array}$$

The bottom row is a localization triangle, the top row arises from such by application of  $i_0^*$ . It is clear that the middle square commutes.  $\alpha$  is defined as the composition

$$i_0^* i_{\infty!} i_\infty^! \mathbb{1} \xrightarrow[\sim]{\text{bc}} i_{\infty 0!} i_{\infty 0}^* i_\infty^! \mathbb{1} \xrightarrow{\text{bc}} i_{\infty 0!} i_{\infty 0}^! i_0^* \mathbb{1} \cong i_{\infty 0!} i_{\infty 0}^! \mathbb{1},$$

and it is again easy to see that the left square commutes. Since there is only the zero morphism from  $i_0^* i_{\infty!} i_\infty^! \mathbb{1}[-1]$  to  $j_{\infty 0*} \mathbb{1}$ , this implies commutativity of the whole diagram. Now we have a morphism of distinguished triangles, and to prove invertibility of (8.15) (and therefore (8.14)) it suffices to prove invertibility of  $\alpha$ . Only the middle arrow in its definition needs to be considered, and for this we note that  $\text{bc} : i_0^* i_{\infty!} i_\infty^! \mathbb{1} \rightarrow i_{\infty 0!} i_{\infty 0}^* i_\infty^! \mathbb{1}$  is invertible by purity.  $\square$

## 9. MAIN RESULT

The goal of this section is to prove the following theorem. The two main inputs are Proposition 9.2 and Theorem 9.3 which we prove subsequently.

**Theorem 9.1.** *Assume that  $\Lambda$  is a principal ideal domain. Then  $\varphi_A$  and  $\varphi_N$  are isomorphisms of Hopf algebras  $\mathbf{H}_A \cong \mathbf{H}_N$ , inverse to each other. In particular, there is an isomorphism of affine pro-group schemes over  $\text{Spec}(\Lambda)$ :*

$$\mathcal{G}_A \cong \mathcal{G}_N.$$

*Proof.* We are going to prove

- $\varphi_N \varphi_A : \mathbf{H}_N \rightarrow \mathbf{H}_N$  is the identity;
- $\varphi_A$  is surjective.

It will then follow that  $\varphi_A$  and  $\varphi_N$  are bialgebra isomorphisms, inverse to each other. And since an antipode of a Hopf algebra is unique, they are automatically isomorphisms of Hopf algebras.

*Proof of  $\varphi_N \varphi_A = \text{id}$ .* Consider the following triangle:

$$\begin{array}{ccc} \mathcal{D}_N^g & \xrightarrow{\overline{\varphi_N} \text{H}_0 \widetilde{\text{Bti}}^* \mathcal{R}_A} & \mathbf{coMod}^f(\mathbf{H}_N) \\ \tilde{\text{H}}_\bullet \downarrow & \nearrow \varepsilon & \\ \mathbf{coMod}^f(\mathbf{H}_N^{\text{eff}}) & & \end{array}$$

By construction, the triangle commutes (up to u. g. m. isomorphism) for  $\varepsilon = \overline{\varphi_N \varphi'_A}$ . By Proposition 9.2 below it also commutes (up to u. g. m. isomorphism) for  $\varepsilon = \bar{\iota}$ , where  $\iota : \mathbf{H}_N^{\text{eff}} \rightarrow \mathbf{H}_N$  is the canonical localization map. By universality of Nori's category (Theorem 2.3), we must therefore have a monoidal isomorphism of functors  $\overline{\varphi_N \varphi'_A} \cong \bar{\iota}$ . By [29, II, 3.3.1], we must then have  $\varphi_N \varphi'_A = \iota$ . Hence  $\varphi_N \varphi_A : \mathbf{H}_N \rightarrow \mathbf{H}_N$  is the identity.  $\square$

*Proof of surjectivity of  $\varphi_A$ .* We use Theorem 9.3 below to deduce that the comultiplication  $\text{cm} : \mathbf{H}_A \rightarrow \mathbf{H}_A \otimes \mathbf{H}_A$  is equal to

$$\mathbf{H}_A \xrightarrow{\text{ca}} \mathbf{H}_N \otimes \mathbf{H}_A \xrightarrow{\varphi_A \otimes \text{id}} \mathbf{H}_A \otimes \mathbf{H}_A$$

for some coaction  $\text{ca}$  of  $\mathbf{H}_N$  on  $\mathbf{H}_A$ . Composing  $\text{cm}$  with the counit  $\text{id} \otimes \text{cu} : \mathbf{H}_A \otimes \mathbf{H}_A \rightarrow \mathbf{H}_A \otimes \Lambda \cong \mathbf{H}_A$  yields the identity, therefore also

$$\text{id} = (\text{id} \otimes \text{cu}) \circ (\varphi_A \otimes \text{id}) \circ \text{ca} = \varphi_A \circ (\text{id} \otimes \text{cu}) \circ \text{ca}.$$

It follows that  $\varphi_A$  is surjective.  $\square$

$\square$

**Proposition 9.2.** *Assume that  $\Lambda$  is a principal ideal domain. The following square commutes up to an isomorphism of u. g. m. representations:*

$$\begin{array}{ccc} \mathcal{D}_N^g & \xrightarrow{\mathcal{R}_A} & \mathbf{DA} \\ \tilde{\text{H}}_\bullet \downarrow & & \downarrow \text{H}_0 \circ \widetilde{\text{Bti}}^* \\ \mathbf{coMod}(\mathbf{H}_N) & \xleftarrow{\overline{\varphi_N}} & \mathbf{coMod}(\mathbf{H}_A) \end{array}$$

*Proof.* This is a rather long and tedious verification. We will proceed in several steps.

**Step 1.** By Proposition 5.7, we already know that after composition with the forgetful functor  $\mathbf{coMod}(\mathbf{H}_N) \rightarrow \mathbf{Mod}(\Lambda)$ , the two u.g.m. representations are naturally isomorphic. Call the isomorphism  $\eta$ . It therefore suffices to prove that the components of  $\eta$  are compatible with the  $\mathbf{H}_N$ -comodule structure.

**Step 2.** Let  $v = (X, Z, n)$  be an arbitrary vertex in  $\mathcal{D}_N^g$ . We find by resolution of singularities a vertex  $v' = (X', Z', n)$  and an edge  $p : v' \rightarrow v$  such that  $(X', Z')$  is almost smooth and  $H_\bullet(p)$  is an isomorphism. Consider the following commutative square in  $\mathbf{Mod}(\Lambda)$ :

$$\begin{array}{ccc} H_\bullet(v') & \xrightarrow{\eta_{v'}} & \overline{\varphi}_N H_0 \widetilde{\text{Bti}}^* \mathcal{R}_A(v') \\ H_\bullet(p) \downarrow & & \downarrow \overline{\varphi}_N H_0 \widetilde{\text{Bti}}^* \mathcal{R}_A(p) \\ H_\bullet(v) & \xrightarrow{\eta_v} & \overline{\varphi}_N H_0 \widetilde{\text{Bti}}^* \mathcal{R}_A(v) \end{array}$$

All arrows are invertible and both vertical arrows are  $\mathbf{coMod}(\mathbf{H}_N)$ -morphisms. We may therefore assume that  $(X, Z)$  is almost smooth.

**Step 3.** Now consider the following diagram in  $\mathbf{Mod}(\Lambda)$ :

$$\begin{array}{ccc} H_0 \text{Bti}^* \mathcal{R}_A(X, Z, n) & \xrightarrow[\text{Thm. 8.5}]{\sim} & H_0 \text{Bti}^{\text{eff},*} \Lambda(X, Z)[n] \\ \sim \downarrow \eta & & \text{Cor. 7.7} \downarrow \sim \\ H_n(X, Z) & \xleftarrow[(8.2)]{\sim} & H_0 \mathcal{R}_N \Lambda(X, Z)[n] \end{array}$$

By Lemma 8.3, the bottom horizontal arrow is compatible with the  $\mathbf{H}_N^{\text{eff}}$ -coaction; the same is clearly true for the top horizontal and the right vertical one. We are thus reduced to show commutativity of this square, and it suffices to do so before applying  $H_n$ .

**Step 4.** Modulo the identification of  $\text{Bti}^{\text{eff},*}$  with  $\text{LSg}^* \circ \text{An}^*$  of Proposition 3.7, the composition of the right vertical and the bottom horizontal arrow can be equivalently described as the composition

$$\text{LSg}^* \text{An}^* \Lambda(X, Z) \rightarrow \text{LSg}^* \Lambda(X^{\text{an}}, Z^{\text{an}}) \rightarrow \text{Sg}(X^{\text{an}}, Z^{\text{an}}).$$

Thus the square above will commute if the following diagram does:

$$\begin{array}{ccccc} \text{Bti}^* L\pi_! Rj_* j^* \pi^! \mathbb{1} & \xrightarrow{\sim} & \text{Bti}^* \text{LSus}_T^0 \Lambda(X, Z_\bullet) & \xrightarrow{\sim} & \text{LSg}^* \text{An}^* \Lambda(X, Z_\bullet) \\ \sim \downarrow & & \sim \downarrow & & \downarrow \sim \\ (\text{Bti}^* R\pi_* j_! \mathbb{1})^\vee & \xlongequal{\quad} & (\text{Bti}^* \underline{\text{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E}))^\vee & & \text{LSg}^* \Lambda(X^{\text{an}}, Z_\bullet^{\text{an}}) \\ \sim \downarrow & & \sim \downarrow & & \downarrow \sim \\ (R\tilde{\pi}_* \tilde{j}_!^{\text{an}} \Lambda_{\text{cst}})^\vee & \xlongequal{\quad} & \text{Sg}(X^{\text{an}}, Z^{\text{an}})^{\vee\vee} & \xleftarrow{\sim} & \text{Sg}(X^{\text{an}}, Z^{\text{an}}) \end{array}$$

The arrows in the top left square are induced by the identifications in Theorem 8.5 ( $\mathbf{E}$  is a projective stable  $(\mathbb{A}^1, \tau)$ -fibrant replacement of the unit spectrum) and duality which makes the square clearly commutative. Commutativity of the lower left square is Lemma 8.10 and 8.12. We are reduced to prove commutativity of the right half.

**Step 5.** Consider the following diagram (all “arrows” are isomorphisms, either canonical or introduced before):

$$\begin{array}{ccccccc}
 (\text{Bti}^* \text{LSus}^0 \Lambda(X, Z_\bullet))^\vee & \xrightarrow{\quad} & (\text{Bti}^{\text{eff},*} \Lambda(X, Z_\bullet))^\vee & \xrightarrow{\quad} & (\text{LSg}^* \text{An}^* \Lambda(X, Z_\bullet))^\vee & \xrightarrow{\quad} & (\text{LSg}^* \Lambda(X^{\text{an}}, Z_\bullet^{\text{an}}))^\vee \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \text{R}\Gamma\text{REv}_0(\text{An}^* \text{LSus}^0 \Lambda(X, Z_\bullet))^\vee & \xrightarrow{\quad} & \text{R}\Gamma\text{REv}_0(\text{LSus}^0 \Lambda(X^{\text{an}}, Z_\bullet^{\text{an}}))^\vee & \xrightarrow{\quad} & \text{R}\Gamma\text{REv}_0 \text{LSus}^0 \Lambda(X^{\text{an}}, Z_\bullet^{\text{an}})^\vee & \xrightarrow{\quad} & \text{LSg}^* \Lambda(X^{\text{an}}, Z_\bullet^{\text{an}})^\vee \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \text{Bti}^* \underline{\text{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E}) & \xrightarrow{\quad} & \text{R}\Gamma\text{REv}_0 \underline{\text{Hom}}(\Lambda(X^{\text{an}}, Z_\bullet^{\text{an}}), \mathbf{Sg}^\vee) & \xrightarrow{\quad} & \text{R}\Gamma \underline{\text{Hom}}(\Lambda(X^{\text{an}}, Z_\bullet^{\text{an}}), \mathbf{Sg}^\vee) & \xrightarrow{\quad} & \text{LSg}^* \underline{\text{Hom}}(\Lambda(X^{\text{an}}, Z_\bullet^{\text{an}}), \mathbf{Sg}^\vee)
 \end{array}$$

The upper part clearly commutes as does the lower right square. For the lower left square we need to prove commutative

$$\begin{array}{ccc}
 \text{An}^* \underline{\text{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E}) & \xlongequal{\quad} & \text{An}^* \text{R} \underline{\text{Hom}}(\text{LSus}^0 \Lambda(X, Z_\bullet), \mathbb{1}) \\
 \downarrow & & \downarrow \\
 \underline{\text{Hom}}(\Lambda(X^{\text{an}}, Z_\bullet^{\text{an}}), \mathbf{Sg}^\vee) & \xlongequal{\quad} & \text{R} \underline{\text{Hom}}(\text{LSus}^0 \Lambda(X^{\text{an}}, Z_\bullet^{\text{an}}), \mathbb{1})
 \end{array}$$

and this is done as in Lemma 8.10. The lower middle square is easily seen to commute hence, using duality, it only remains to prove that the composition of the dotted arrows is equal to

$$\text{R}\Gamma \underline{\text{Hom}}(\Lambda(X^{\text{an}}, Z_\bullet^{\text{an}}), \mathbf{Sg}^\vee) \leftarrow \mathbf{Sg}^\vee(X^{\text{an}}, Z^{\text{an}}) \rightarrow (\text{LSg}^* \Lambda(X^{\text{an}}, Z_\bullet^{\text{an}}))^\vee.$$

**Step 6.** Notice that  $\mathbf{Sg}^\vee = \mathbf{Sg}_* \Lambda$ . Then, writing  $B$  for  $\Lambda(X^{\text{an}}, Z_\bullet^{\text{an}})$ , we reduce to prove commutative:

$$\begin{array}{ccccc}
 \Gamma \underline{\text{Hom}}(B, \mathbf{Sg}_* \Lambda) & \xrightarrow{\quad} & \text{LSg}^* \underline{\text{Hom}}(B, \mathbf{Sg}_* \Lambda) & \xlongequal{\quad} & \text{LSg}^* \text{R} \underline{\text{Hom}}(B, \Lambda) \\
 \downarrow & & \downarrow & & \downarrow \\
 \Gamma \mathbf{Sg}_* \underline{\text{Hom}}(\mathbf{Sg}^* B, \Lambda) & \xrightarrow{\quad} & \text{LSg}^* \mathbf{Sg}_* \underline{\text{Hom}}(\mathbf{Sg}^* B, \Lambda) & & \text{R} \underline{\text{Hom}}(\text{LSg}^* B, \Lambda) \\
 & \searrow & \downarrow & \swarrow & \\
 & & \underline{\text{Hom}}(\mathbf{Sg}^* B, \Lambda) & & 
 \end{array}$$

Here we used that  $\mathbf{Sg} \circ \iota_{\text{pt}} : \text{pt} \rightarrow \mathbf{Cpl}(\Lambda)$  takes the value  $\mathbf{Sg}(\text{pt}) \simeq \Lambda$  hence  $\Gamma \circ \mathbf{Sg}_*$  is canonically quasi-isomorphic to the identity. Since the undecorated functors  $\mathbf{Sg}^*$  and  $\mathbf{Sg}_*$  appearing are only applied to cofibrant, respectively fibrant, objects they can be identified with their derived counterparts and the diagram is easily seen to commute.

□

**Theorem 9.3.** *Assume that  $\Lambda$  is a principal ideal domain. The bialgebra  $\mathbf{H}_\Lambda$  considered as a comodule over itself lies in the essential image of*

$$\mathbf{coMod}(\mathbf{H}_\Lambda) \xrightarrow{\overline{\varphi_\Lambda}} \mathbf{coMod}(\mathbf{H}_\Lambda).$$

*Proof.* We may prove this statement for the Nisnevich topology. Ayoub gives in [4, Thm. 2.67] an explicit model for the symmetric  $T$ -spectrum  $\mathrm{Bti}_*^{\mathrm{Nis}} \Lambda$  which we are now going to describe at a level of detail appropriate for our proof.

Recall the category  $\mathcal{V}_{\mathrm{ét}}(\overline{\mathbb{D}}^n/\mathbb{A}^n)$  ( $n \geq 0$ ) whose objects are étale neighborhoods of the closed polydisk  $\overline{\mathbb{D}}^n$  inside affine space  $\mathbb{A}^n$  (for the precise definition see [4, §2.2.4]). It is a cofiltered category. Forgetting the presentation as a scheme over  $\mathbb{A}^n$  defines a canonical functor  $\overline{\mathbb{D}}_{\mathrm{ét}}^n : \mathcal{V}_{\mathrm{ét}}(\overline{\mathbb{D}}^n/\mathbb{A}^n) \rightarrow \mathrm{Sm}$ . In other words we obtain a pro-smooth scheme. We write  $\overline{\mathbb{D}}_{\mathrm{ét}}$  for the associated cocubical object in pro-smooth schemes where the faces  $d_{i,\varepsilon}$  are induced from the faces in  $\mathbb{A}^n$  (the “coordinate hyperplanes” through 0 and 1). For  $n \in \mathbb{N}$  and  $\varepsilon = 0, 1$  write  $\partial_\varepsilon \overline{\mathbb{D}}_{\mathrm{ét}}^n$  for the union of the faces  $d_{i,\varepsilon}(\overline{\mathbb{D}}_{\mathrm{ét}}^{n-1})$ , where  $i$  runs through  $1, \dots, n$ . Also write  $\partial \overline{\mathbb{D}}_{\mathrm{ét}}^n$  for the union  $\partial_0 \overline{\mathbb{D}}_{\mathrm{ét}}^n \cup \partial_1 \overline{\mathbb{D}}_{\mathrm{ét}}^n$ , and  $\partial_{1,1} \overline{\mathbb{D}}_{\mathrm{ét}}^n$  for the union of all  $d_{i,\varepsilon}(\overline{\mathbb{D}}_{\mathrm{ét}}^{n-1})$  except  $(i, \varepsilon) = (1, 1)$ .

We obtain the bicomplex  $N(\underline{\mathrm{hom}}(\overline{\mathbb{D}}_{\mathrm{ét}}, K))$  which in degree  $n$  (in the direction of the cocubical dimension) is given by  $\underline{\mathrm{hom}}((\overline{\mathbb{D}}_{\mathrm{ét}}^n, \partial_{1,1} \overline{\mathbb{D}}_{\mathrm{ét}}^n), K)$ , and whose differential in degree  $n > 0$  is  $d_{1,1}$ .<sup>9</sup> In particular, the cycles in degree  $n > 0$  are given by  $\underline{\mathrm{hom}}((\overline{\mathbb{D}}_{\mathrm{ét}}^n, \partial \overline{\mathbb{D}}_{\mathrm{ét}}^n), K)$ . Thus we obtain a canonical morphism of bicomplexes

$$\underline{\mathrm{hom}}((\overline{\mathbb{D}}_{\mathrm{ét}}^n, \partial \overline{\mathbb{D}}_{\mathrm{ét}}^n), K)[-n] \rightarrow N^{\leq n}(\underline{\mathrm{hom}}(\overline{\mathbb{D}}_{\mathrm{ét}}, K))$$

where the right hand side denotes the bicomplex truncated at degree  $n$  from above. One can check that this induces a quasi-isomorphism on the associated total complexes whenever  $K$  is injective fibrant.

Taking the total complex of the bicomplex  $N(\underline{\mathrm{hom}}(\overline{\mathbb{D}}_{\mathrm{ét}}, K))$  (resp.  $N^{\leq n}(\underline{\mathrm{hom}}(\overline{\mathbb{D}}_{\mathrm{ét}}, K))$ ) we obtain an endofunctor  ${}^n \underline{\mathrm{Sg}}_{\mathrm{ét}}^{\mathbb{D}}$  (resp.  ${}^n \underline{\mathrm{Sg}}_{\mathrm{ét}}^{\mathbb{D}^{\leq n}}$ ) of presheaves of complexes on smooth schemes. It extends canonically to an endofunctor on symmetric  $T$ -spectra. Let  $\mathbf{E}$  be an injective stable  $(\mathbb{A}^1, \mathrm{Nis})$ -fibrant replacement of the unit spectrum  $\mathbb{1}$ . [4, Thm. 2.67] states that  $\mathrm{Bti}_*^{\mathrm{Nis}} \mathbb{1}$  is given explicitly by the symmetric  $T$ -spectrum

$$\mathrm{Sing}_{\mathrm{ét}}^{\mathbb{D}, \infty}(\mathbf{E}) := \varinjlim_r s_-^r {}^n \underline{\mathrm{Sg}}_{\mathrm{ét}}^{\mathbb{D}}(\mathbf{E})[2r],$$

where  $s_-$  denotes the “shift down” functor (so that  $s_-(\mathbf{E})_m = \mathbf{E}_{m+1}$ ; see [1, Déf. 4.3.13]). As we will not need a description of the transition morphisms in the sequential colimit above, we content ourselves with referring to [4, Déf. 2.65]. Let  $Q : \mathbf{Spt}_T^{\Sigma} \mathrm{USm} \rightarrow \mathbf{DA}^{\mathrm{Nis}}$  denote the canonical localization functor, and consider the following canonical morphisms:

$$\begin{aligned} \varinjlim_{r,n} H_0 \widetilde{\mathrm{Bti}}^{\mathrm{Nis},*} Q(s_-^r \underline{\mathrm{hom}}((\overline{\mathbb{D}}_{\mathrm{ét}}^n, \partial \overline{\mathbb{D}}_{\mathrm{ét}}^n), \mathbf{E})[2r-n]) &\rightarrow \varinjlim_{r,n} H_0 \widetilde{\mathrm{Bti}}^{\mathrm{Nis},*} Q(s_-^r {}^n \underline{\mathrm{Sg}}_{\mathrm{ét}}^{\mathbb{D}^{\leq n}}(\mathbf{E})[2r]) \\ &\rightarrow H_0 \widetilde{\mathrm{Bti}}^{\mathrm{Nis},*} Q(\mathrm{Sing}_{\mathrm{ét}}^{\mathbb{D}, \infty}(\mathbf{E})) \end{aligned} \quad (9.4)$$

in  $\mathbf{coMod}(\mathbf{H}_A)$ . The last term is the bialgebra  $\mathbf{H}_A$  considered as a comodule over itself. We are going to show

- (1) that the composition in (9.4) is invertible, and

<sup>9</sup>Given a pro-object  $(X_i, Z_i)_{i \in I}$  of almost smooth pairs,  $\underline{\mathrm{hom}}((X_i, Z_i)_i, K)$  takes a smooth scheme  $Y$  to

$$\varinjlim_{i \in I} K(Y \times X_i, Y \times Z_i).$$



- (2) that the comodules in the filtered system on the left hand side are in the essential image of  $\overline{\varphi}_A$ .

This is enough since, as seen in the proof of Theorem 9.1, the previous proposition implies that  $\overline{\varphi}_N \overline{\varphi}_A \cong \text{id}$  hence the essential image of  $\overline{\varphi}_A$  is a full subcategory of  $\mathbf{coMod}(\mathbf{H}_A)$  (since both  $\overline{\varphi}_A$  and  $\overline{\varphi}_N$  are faithful) closed under small colimits (by Fact C.1).

*Proof of (1).* It suffices to prove this after forgetting the comodule structure. Just as in the case of the étale singular complex there is an endofunctor  $\mathbf{Sing}^{\mathbb{D}, \infty}$  on symmetric  $\mathbf{An}^*(T)$ -spectra defined using  $\mathbf{Sg}^{\mathbb{D}}$  instead of  ${}^n\mathbf{Sg}_{\text{ét}}^{\mathbb{D}}$  (cf. [4, Déf. 2.45]). Denote by  $F : \mathbf{Spt}_T^{\Sigma} \mathbf{USm} \rightarrow \mathbf{Mod}(\Lambda)$  the composition of functors  $\mathbf{H}_0 \Gamma \text{Ev}_0 \mathbf{Sing}^{\mathbb{D}, \infty} \mathbf{An}^*$  and notice that

- (a)  $F$  commutes with filtered colimits, by construction;
- (b)  $F$  takes levelwise quasi-isomorphisms of symmetric  $T$ -spectra to isomorphisms of modules, as follows essentially from [4, Lem. 2.55];
- (c)  $F$  applied to a projective stable  $(\mathbb{A}^1, \text{Nis})$ -fibrant spectrum  $\mathbf{K}$  is a model for  $\mathbf{H}_0 \text{Bti}^{\text{Nis}, *}\mathbf{K}$ , by [4, Lem. 2.72 and Thm. 2.48].

We claim that the morphism of  $\Lambda$ -modules underlying (9.4) can be identified with the composition

$$\begin{aligned} \varinjlim_{r, n} F(s_-^r \underline{\text{hom}}((\overline{\mathbb{D}}_{\text{ét}}^n, \partial \overline{\mathbb{D}}_{\text{ét}}^n), \mathbf{E})[2r - n]) &\rightarrow \varinjlim_{r, n} F(s_-^r {}^n\mathbf{Sg}_{\text{ét}}^{\mathbb{D} \leq n}(\mathbf{E})[2r]) \\ &\rightarrow F(\mathbf{Sing}_{\text{ét}}^{\mathbb{D}, \infty}(\mathbf{E})). \end{aligned} \quad (9.5)$$

This follows from (c) because both  $\mathbf{Sing}_{\text{ét}}^{\mathbb{D}, \infty}(\mathbf{E})$  and  $\underline{\text{hom}}((\overline{\mathbb{D}}_{\text{ét}}^n, \partial \overline{\mathbb{D}}_{\text{ét}}^n), \mathbf{E})$  are projective stable  $(\mathbb{A}^1, \text{Nis})$ -fibrant, as follows from [4, Thm. 2.67] for the first, and from our proof of Theorem 8.5 together with [4, Lem. 2.69] for the second. But the first arrow in (9.5) is invertible by (b), and the second one by (a) so we conclude that (9.4) is invertible.  $\square$

*Proof of (2).* Next we fix  $(r, n) \in \mathbb{N}^2$  and consider the canonical morphism

$$\begin{aligned} \varinjlim_{(X, x) \in \mathcal{V}_{\text{ét}}(\overline{\mathbb{D}}^n / \mathbb{A}^n)} \mathbf{H}_0 \widetilde{\text{Bti}}^{\text{Nis}, *} Q(s_-^r \underline{\text{hom}}((X, \partial X), \mathbf{E})[2r - n]) &\rightarrow \\ \mathbf{H}_0 \widetilde{\text{Bti}}^{\text{Nis}, *} Q(s_-^r \underline{\text{hom}}((\overline{\mathbb{D}}_{\text{ét}}^n, \partial \overline{\mathbb{D}}_{\text{ét}}^n), \mathbf{E})[2r - n]). \end{aligned}$$

The same argument as above establishes invertibility of this arrow and reduces us to show that the comodules in the filtered system on the left hand side lie in the essential image of  $\overline{\varphi}_A$ . Hence fix  $(X, x) \in \mathcal{V}_{\text{ét}}(\overline{\mathbb{D}}^n / \mathbb{A}^n)$ . By resolution of singularities there is a smooth projective scheme  $W$  and a simple normal crossings divisor  $W_0 \cup W_{\infty}$  on  $W$  together with a projective surjective morphism  $p : W - W_{\infty} \rightarrow X$  such that  $p^{-1}(\partial X) = W_0 - W_{\infty}$  and  $p|_{W - p^{-1}(\partial X)} : W - p^{-1}(\partial X) \rightarrow X - \partial X$  is an isomorphism. Therefore, canonically,  $\mathcal{R}_A(X, \partial X, 0) \cong \mathcal{R}_A(W - W_{\infty}, W_0 - W_{\infty}, 0)$ ,

and we obtain in  $\mathbf{DA}^{\text{Nis}}$ :

$$\begin{aligned}
Q(s^r \underline{\text{hom}}((X, \partial X), \mathbf{E})[2r - n]) &\cong \underline{\text{hom}}((X, \partial X), \mathbf{E})[-n](r) \\
&\cong \mathcal{R}_A(X, \partial X, 0)^\vee[-n](r) \\
&\cong \mathcal{R}_A(W - W_\infty, W_0 - W_\infty, 0)^\vee[-n](r) \\
&\cong \mathcal{R}_A(W - W_0, W_\infty - W_0, n)(r - n) \\
&\cong \mathcal{R}_A(W - W_0, W_\infty - W_0, n) \otimes^L \mathcal{R}_A(\mathbb{G}_m, \{1\}, 1)^{\otimes^L(r-n)},
\end{aligned}$$

where we used [1, Thm. 4.3.38] for the first, Theorem 8.5 for the second, and Lemma 8.13 for the penultimate isomorphism. Applying  $\widetilde{\text{H}_0\text{Bti}}^*$  to these isomorphisms, and using (5.8) as well as (5.9) we obtain the following sequence of isomorphisms

$$\begin{aligned}
&\widetilde{\text{H}_0\text{Bti}}^{\text{Nis},*} Q(s^r \underline{\text{hom}}((X, \partial X), \mathbf{E})[2r - n]) \\
&\cong \widetilde{\text{H}_0\text{Bti}}^{\text{Nis},*} \left( \mathcal{R}_A(W - W_0, W_\infty - W_0, n) \otimes^L \mathcal{R}_A(\mathbb{G}_m, \{1\}, 1)^{\otimes^L(r-n)} \right) \\
&\cong \overline{\varphi}_A \left( \tilde{\mathbf{H}}_\bullet(W - W_0, W_\infty - W_0, n) \otimes \tilde{\mathbf{H}}_\bullet(\mathbb{G}_m, \{1\}, 1)^{\otimes(r-n)} \right),
\end{aligned}$$

which concludes the proof.  $\square$

$\square$

#### APPENDIX A. NORI'S TANNAKIAN FORMALISM IN THE MONOIDAL SETTING

In this section we indicate briefly which modifications to [20, App. B] have to be made in order to justify our arguments in the main body of the text regarding Nori's Tannakian formalism. Most importantly we seek to obtain a universality statement for Nori's construction in the monoidal setting. Something similar was undertaken by Bruguières in [9], and for the main proof below we follow his ideas. However the results there on monoidal representations do not seem to apply directly to Nori's construction since there is no obvious monoidal structure (in the sense of [9]) on Nori's diagrams.<sup>10</sup>

A *graded diagram* and a *commutative product structure* on such a graded diagram are defined as in [20, Def. B.14]. From now on, fix such a graded diagram  $\mathcal{D}$  with a commutative product structure. Let  $(\mathcal{C}, \otimes)$  be an additive (symmetric, unitary) monoidal category. A *graded multiplicative representation*  $T : \mathcal{D} \rightarrow \mathcal{C}$  is a representation of  $\mathcal{D}$  in  $\mathcal{C}$  together with a choice of isomorphisms

$$\tau_{(f,g)} : T(f \times g) \rightarrow T(f) \otimes T(g)$$

for any vertices  $f$  and  $g$  of  $\mathcal{D}$ , satisfying (1)-(5) of [20, Def. B.14]. *Unital graded multiplicative (u. g. m.) representations* are then defined as in [20, Def. B.14]. A *u. g. m. transformation*  $\eta : T \rightarrow U$  between two unital graded multiplicative representations  $T, U : \mathcal{D} \rightarrow \mathcal{C}$  is a family of morphisms in  $\mathcal{C}$ :

$$\eta_f : T(f) \rightarrow U(f),$$

compatible with edges in  $\mathcal{D}$  and the choices of isomorphisms  $\tau$ , and such that  $\eta_{\text{id}} = \text{id}$ .  $\eta$  is a *u. g. m. isomorphism* if all its components are invertible.

From now on, fix also a u. g. m. representation  $T : \mathcal{D} \rightarrow \mathbf{Mod}^f(\Lambda)$  taking values in projective modules (we assume  $\Lambda$  to be of global dimension at most 2; see [9,

<sup>10</sup>This is related to the problem discussed in [20, Rem. B.13].

§5.3]). By Nori's theorem ([26, Thm. 1.6], [20, Pro. B.8]), there is a universal abelian  $\Lambda$ -linear category  $\mathcal{C}(T)$  with a representation  $\tilde{T} : \mathcal{D} \rightarrow \mathcal{C}(T)$ , through which  $T$  factors via a faithful exact  $\Lambda$ -linear functor  $o_T : \mathcal{C}(T) \rightarrow \mathbf{Mod}^f(\Lambda)$ . Nori also showed (see [20, Pro. B.16]) that in this case  $\mathcal{C}(T)$  carries naturally a (right exact) monoidal structure such that  $o_T$  is a monoidal functor. It is obvious from the construction of this monoidal structure that  $\tilde{T}$  is a u.g.m. representation. The following theorem states that these data are *universal*.

**Theorem A.1.** *Given a right exact monoidal abelian  $\Lambda$ -linear category  $\mathcal{C}$  and a factorization of  $T$  into*

$$\mathcal{D} \xrightarrow{S} \mathcal{C} \xrightarrow{o_S} \mathbf{Mod}^f(\Lambda)$$

*with  $S$  u.g.m. and  $o_S$  a faithful exact  $\Lambda$ -linear monoidal functor, there exists a monoidal functor (unique up to unique monoidal isomorphism)  $F : \mathcal{C}(T) \rightarrow \mathcal{C}$  making the following diagram commutative (up to monoidal isomorphism).*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{S} & \mathcal{C} \\ \tilde{T} \downarrow & \nearrow F & \downarrow o_S \\ \mathcal{C}(T) & \xrightarrow{o_T} & \mathbf{Mod}^f(\Lambda) \end{array}$$

Moreover,  $F$  is faithful exact  $\Lambda$ -linear.

Explicitly, there exists a monoidal functor  $F : \mathcal{C}(T) \rightarrow \mathcal{C}$ , a u.g.m. isomorphism  $\alpha : S\tilde{T} \xrightarrow{\sim} F$ , and a monoidal isomorphism  $\beta : o_T \xrightarrow{\sim} o_SF$  such that  $o_S\alpha = \beta\tilde{T}$ . Moreover given another triple  $(F', \alpha', \beta')$  satisfying these conditions, there exists a unique monoidal isomorphism  $\gamma : F' \xrightarrow{\sim} F$  transforming  $\alpha$  to  $\alpha'$  and  $\beta$  to  $\beta'$ .

*Proof.* Given [20, Pro. B.8 and B.16], the only thing left to prove is that the functor and transformations whose existence is asserted are monoidal. This could be proven by going through the construction of these and checking monoidality directly. Alternatively, one can deduce the monoidal structure from the existence of the functor and transformations alone without referring to their construction. We sketch the latter proof which is due to Bruguières. For the details we refer the reader to [9].

Let  $\Psi_{X,Y}$  be the composition

$$o_S(FX \otimes FY) = o_SFX \otimes o_SFY \cong o_SF(X \otimes Y),$$

for any  $X, Y \in \mathcal{C}(T)$ . This defines a natural isomorphism of functors. Since  $o_S$  is faithful, it suffices to construct morphisms  $\Phi_{X,Y} : FX \otimes FY \rightarrow F(X \otimes Y)$  which realize  $\Psi_{X,Y}$ . Thus consider the class

$$L = \{(X, Y) \in \mathcal{C}(T) \times \mathcal{C}(T) \mid \exists \Phi_{X,Y} : o_S\Phi_{X,Y} = \Psi_{X,Y}\}.$$

Notice that  $L$  contains all pairs in the image of  $\tilde{T}$ :

$$F\tilde{T}f \otimes F\tilde{T}g \cong Sf \otimes Sg \rightarrow S(f \times g) \cong F\tilde{T}(f \times g) \rightarrow F(\tilde{T}f \otimes \tilde{T}g)$$

can (and has to) be taken as  $\Phi_{\tilde{T}f, \tilde{T}g}$ . Now for fixed  $f$  the functors  $F\tilde{T}f \otimes F(\bullet)$  and  $F(\tilde{T}f \otimes \bullet)$  are exact hence one can define  $\Phi_{\tilde{T}f, \bullet}$  on the subcategory of  $\mathcal{C}(T)$  containing the image of  $\tilde{T}$  and closed under kernels, cokernels and direct sums. But this is all of  $\mathcal{C}(T)$ . By symmetry one sees that  $L$  contains all pairs  $(X, Y)$  where one of  $X$  or  $Y$  is contained in the image of  $\tilde{T}$ . Now a similar argument shows that  $L$  also contains all pairs  $(X, Y)$  where one of  $o_TX$  or  $o_TY$  is projective (since then

the functors considered above are still exact). Finally, one uses that every object in  $\mathcal{C}(T)$  is a quotient of an object with underlying projective  $\Lambda$ -module to conclude that  $L$  consists of all pairs of objects in  $\mathcal{C}(T)$ .

It is obvious from the definition of  $\Phi_{\tilde{T}f, \tilde{T}g}$  that  $\alpha$  is monoidal, and from the definition of  $\Psi_{X,Y}$  that  $\beta$  is as well. It is an easy exercise to prove that  $\gamma$  is monoidal as well.  $\square$

## APPENDIX B. RELATIVE COHOMOLOGY

It is well-known that singular and sheaf cohomology agree on locally contractible topological spaces. The same is true for pairs of such spaces. However, we have not been able to find in the literature the statements in the form we need them in the main body of the chapter (in particular in section 5) although the book of Bredon [8] comes close. We will freely use the results of [8, §III.1].  $\Lambda$  is a fixed principal ideal domain. All topological spaces are assumed locally contractible and paracompact.

**B.1. Model.** For a topological space  $X$ , denote by  $\mathcal{S}_X$  the complex of sheaves of singular cochains on  $X$  with values in  $\Lambda$ . This is a flabby resolution of the constant sheaf  $\Lambda$ . Moreover, the canonical map  $\text{Sg}(X)^\vee \rightarrow \mathcal{S}_X(X)$  is a quasi-isomorphism.

Now let  $i : Z \hookrightarrow X$  a closed subset with open complement  $j : U \hookrightarrow X$ . We denote by  $\Lambda_U$  (respectively  $\Lambda_Z$ ) the constant sheaf  $\Lambda$  supported at  $U$  (respectively  $Z$ ), i.e.  $\Lambda_U = j_! j^* \Lambda_X$  (resp.  $\Lambda_Z = i_* i^* \Lambda_X$ ). The canonical morphism  $\mathcal{S}_X \otimes \Lambda_Z \rightarrow i_* \mathcal{S}_Z$  induces the diagram of solid arrows in the category of complexes of sheaves on  $X$  with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_{(X,Z)} & \longrightarrow & \mathcal{S}_X & \longrightarrow & i_* \mathcal{S}_Z \\ & & \uparrow \alpha & & \parallel & & \uparrow \\ 0 & \longrightarrow & \mathcal{S}_X \otimes \Lambda_U & \longrightarrow & \mathcal{S}_X & \longrightarrow & \mathcal{S}_X \otimes \Lambda_Z \end{array} \quad (\text{B.1})$$

We obtain a unique morphism  $\alpha$  rendering the diagram commutative. It induces a quasi-isomorphism after taking global sections.

Similarly,  $\beta$  is the unique morphism of complexes making the following diagram commute:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_{(X,Z)}(X) & \longrightarrow & \mathcal{S}_X(X) & \longrightarrow & i_* \mathcal{S}_Z(X) \\ & & \uparrow \beta & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \text{Sg}(X, Z)^\vee & \longrightarrow & \text{Sg}(X)^\vee & \longrightarrow & \text{Sg}(Z)^\vee \end{array} \quad (\text{B.2})$$

Again, it is a quasi-isomorphism.

Now,  $\mathcal{S}_X \otimes \Lambda_U$  is a resolution of  $\Lambda_U$  which computes derived global sections hence we deduce the following result.

**Fact B.3.** *The zigzag of  $\alpha$  and  $\beta$  exhibits  $\text{Sg}(X, Z)^\vee$  as a model for  $\text{R}\Gamma(X, \Lambda_U) = \text{R}\pi_* j_! \Lambda$  in  $\mathbf{D}(\Lambda)$ , where  $\pi : X \rightarrow \text{pt}$ .*

**B.2. Functoriality.** We now turn to functoriality of these constructions. Suppose we are given a morphism of pairs of topological spaces  $f : (X, Z) \rightarrow (X', Z')$ . We keep the notation from above, decorating the symbols with a prime when the objects are associated to the second pair.

**Lemma B.4.** *The following diagram commutes in  $\mathbf{D}(\Lambda)$ :*

$$\begin{array}{ccc} R\pi'_* j'_! \Lambda & \longrightarrow & R\pi_* j_! \Lambda \\ \beta^{-1} \alpha \downarrow \sim & & \beta^{-1} \alpha \downarrow \sim \\ \mathrm{Sg}(X', Z')^\vee & \xrightarrow{\mathrm{Sg}(f)^\vee} & \mathrm{Sg}(X, Z)^\vee \end{array}$$

Here the top horizontal arrow is defined as  $R\pi'_*$  applied to

$$j'_! \Lambda \xrightarrow{\mathrm{adj}} Rf_* f^* j'_! \Lambda \rightarrow Rf_* j_! f^* \Lambda \xrightarrow{\sim} Rf_* j_! \Lambda.$$

*Proof.* We will construct the two middle horizontal arrows below, and then prove that they make each square in the following diagram commute:

$$\begin{array}{ccc} R\pi'_* j'_! \Lambda & \longrightarrow & R\pi_* j_! \Lambda \\ \parallel & & \parallel \\ \mathcal{S}_{X'} \otimes \Lambda_{U'}(X') & \xrightarrow{\text{(B.5)}} & \mathcal{S}_X \otimes \Lambda_U(X) \\ \alpha \downarrow \sim & & \alpha \downarrow \sim \\ \mathcal{K}_{(X', Z')}(X') & \xrightarrow{\mathcal{K}_f} & \mathcal{K}_{(X, Z)}(X) \\ \beta \uparrow \sim & & \beta \uparrow \sim \\ \mathrm{Sg}(X', Z')^\vee & \xrightarrow{\mathrm{Sg}(f)^\vee} & \mathrm{Sg}(X, Z)^\vee \end{array}$$

From the inclusion  $f^{-1}(U') \subset U$  we obtain a canonical morphism of sheaves on  $X$ :

$$f^* \Lambda_{U'} \xrightarrow{\sim} \Lambda_{f^{-1}(U')} \rightarrow \Lambda_U.$$

Composition with  $f$  induces a morphism  $\mathcal{S}_f : \mathcal{S}_{X'} \rightarrow f_* \mathcal{S}_X$  and thus by adjunction also  $f^* \mathcal{S}_{X'} \rightarrow \mathcal{S}_X$ . Together we obtain a morphism

$$f^*(\mathcal{S}_{X'} \otimes \Lambda_{U'}) \xrightarrow{\sim} f^* \mathcal{S}_{X'} \otimes f^* \Lambda_{U'} \rightarrow \mathcal{S}_X \otimes \Lambda_U. \quad (\text{B.5})$$

Similarly, we define morphisms

$$f^*(\mathcal{S}_{X'} \otimes \Lambda_{Z'}) \xrightarrow{\sim} f^* \mathcal{S}_{X'} \otimes f^* \Lambda_{Z'} \rightarrow \mathcal{S}_X \otimes \Lambda_Z$$

and

$$f^* i'_* \mathcal{S}_{Z'} \rightarrow i_* f^* \mathcal{S}_{Z'} \rightarrow i_* \mathcal{S}_Z.$$

It is then clear that the following diagram commutes

$$\begin{array}{ccc}
f^*(\mathcal{S}_{X'} \otimes \Lambda_{U'}) & \longrightarrow & \mathcal{S}_X \otimes \Lambda_U \\
\downarrow & & \downarrow \\
f^*\mathcal{S}_{X'} & \longrightarrow & \mathcal{S}_X \\
\downarrow & & \downarrow \\
f^*(\mathcal{S}_{X'} \otimes \Lambda_{Z'}) & \longrightarrow & \mathcal{S}_X \otimes \Lambda_Z \\
\downarrow & & \downarrow \\
f^*i'_*\mathcal{S}_{Z'} & \longrightarrow & i_*\mathcal{S}_Z
\end{array}$$

so that, in particular, we deduce the existence of a morphism  $f^*\mathcal{K}_{(X',Z')} \rightarrow \mathcal{K}_{(X,Z)}$  rendering the following two squares commutative:

$$\begin{array}{ccc}
f^*\mathcal{K}_{(X',Z')} & \longrightarrow & \mathcal{K}_{(X,Z)} \\
\downarrow & & \downarrow \\
f^*\mathcal{S}_{X'} & \longrightarrow & \mathcal{S}_X
\end{array}
\quad
\begin{array}{ccc}
f^*\mathcal{K}_{(X',Z')} & \longrightarrow & \mathcal{K}_{(X,Z)} \\
\uparrow \alpha & & \uparrow \alpha \\
f^*(\mathcal{S}_{X'} \otimes \Lambda_{U'}) & \longrightarrow & \mathcal{S}_X \otimes \Lambda_U
\end{array}$$

Denote by  $\mathcal{K}_f : \mathcal{K}_{(X',Z')} \rightarrow f_*\mathcal{K}_{(X,Z)}$  the morphism obtained by adjunction. We now claim that also the following square of complexes commutes:

$$\begin{array}{ccc}
\mathcal{K}_{(X',Z')}(X') & \xrightarrow{\mathcal{K}_f} & \mathcal{K}_{(X,Z)}(X) \\
\beta \uparrow & & \uparrow \beta \\
\mathrm{Sg}(X', Z')^\vee & \xrightarrow[\mathrm{Sg}(f)^\vee]{} & \mathrm{Sg}(X, Z)^\vee
\end{array}$$

Indeed, using the injection  $\mathcal{K}_{(X,Z)}(X) \hookrightarrow \mathcal{S}_X(X)$  one reduces to prove commutativity of

$$\begin{array}{ccc}
\mathcal{S}_{X'}(X') & \xrightarrow{\mathcal{S}_f} & \mathcal{S}_X(X) \\
\uparrow & & \uparrow \\
\mathrm{Sg}(X')^\vee & \xrightarrow[\mathrm{Sg}(f)^\vee]{} & \mathrm{Sg}(X)^\vee
\end{array}$$

which is clear.

Finally, notice that (B.5) is compatible with the coaugmentations  $\Lambda \rightarrow \mathcal{S}_{X'}$  and  $\Lambda \rightarrow \mathcal{S}_X$ , thus the lemma.  $\square$

**Lemma B.6.** *The following defines a morphism of distinguished triangles in  $\mathbf{D}(\Lambda)$ :*

$$\begin{array}{ccccccc}
\mathrm{R}\pi_*j_!\Lambda & \longrightarrow & \mathrm{R}\pi_*\Lambda & \longrightarrow & \mathrm{R}\pi_*i_*\Lambda & \longrightarrow & \mathrm{R}\pi_*j_!\Lambda[-1] \\
\sim \uparrow \alpha^{-1}\beta & & \uparrow \sim & & \sim \uparrow & & \sim \uparrow \alpha^{-1}\beta \\
\mathrm{Sg}(X, Z)^\vee & \longrightarrow & \mathrm{Sg}(X)^\vee & \longrightarrow & \mathrm{Sg}(Z)^\vee & \longrightarrow & \mathrm{Sg}(X, Z)^\vee[-1]
\end{array} \tag{B.7}$$

Here, the first row is induced by the localization triangle while the second row is the distinguished triangle associated to the short exact sequence consisting of the first three terms.

*Proof.* It is clear that the first two squares commute. We only need to prove this for the third one.

Extend the first square in (B.1) to a morphism of triangles in  $\mathbf{Cpl}(\mathbf{Sh}(X))$

$$\begin{array}{ccccccc} \mathcal{K}_{(X,Z)} & \xrightarrow{a} & \mathcal{S}_X & \longrightarrow & \text{cone}(a) & \longrightarrow & \mathcal{K}_{(X,Z)}[-1] \\ \uparrow \alpha & & \parallel & & \uparrow & & \uparrow \alpha \\ \mathcal{S}_X \otimes \Lambda_U & \xrightarrow{j} & \mathcal{S}_X & \longrightarrow & \text{cone}(j) & \longrightarrow & \mathcal{S}_X \otimes \Lambda_U[-1] \end{array}$$

using the mapping cones. Since  $\Gamma(X, \text{cone}(a)) = \text{cone}(a_X)$  the first square in (B.2) extends to a morphism of triangles in  $\mathbf{Cpl}(\Lambda)$ :

$$\begin{array}{ccccccc} \mathcal{K}_{(X,Z)}(X) & \xrightarrow{a_X} & \mathcal{S}_X(X) & \longrightarrow & \text{cone}(a)(X) & \longrightarrow & \mathcal{K}_{(X,Z)}(X)[-1] \\ \uparrow \beta & & \uparrow & & \uparrow & & \uparrow \beta \\ \text{Sg}(X, Z)^\vee & \xrightarrow{b} & \text{Sg}(X)^\vee & \longrightarrow & \text{cone}(b) & \longrightarrow & \text{Sg}(X, Z)^\vee[-1] \end{array}$$

Notice that under the canonical identification  $\text{cone}(b) \xrightarrow{\sim} \text{Sg}(Z)^\vee$ , the bottom row is precisely the bottom row of (B.7), while modulo the canonical identification  $\text{cone}(a)(X) \xrightarrow{\sim} \mathcal{S}_Z(Z)$ , the top row of the first diagram induces the top row of (B.7) (taking global sections). Indeed, the latter contention follows from the fact that in  $\mathbf{D}(\mathbf{Sh}(X))$  there is a *unique* morphism  $\delta$  making the following candidate triangle distinguished:

$$j_! \Lambda \longrightarrow \Lambda \longrightarrow i_* \Lambda \xrightarrow{\delta} j_! \Lambda[-1].$$

The lemma now follows from the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{S}_X \otimes \Lambda_Z(X) & \longleftarrow & \text{cone}(j)(X) \\ \downarrow & & \downarrow \\ i_* \mathcal{S}_Z(X) & \longleftarrow & \text{cone}(a)(X) \\ \uparrow & & \uparrow \\ \text{Sg}(Z)^\vee & \longleftarrow & \text{cone}(b) \end{array}$$

The first square commutes since the second square in (B.2) does, while the second square does since the second square in (B.1) does.  $\square$

**B.3. Monoidality.** We come to the last compatibility of the model, namely with the cup product. For this we fix a topological space  $X$  and two closed subspaces  $Z_1$  and  $Z_2$  of  $X$ . We write  $Z = Z_1 \cup Z_2$ , and we assume that there exist open neighborhoods  $V_i$  of  $Z_i$  in  $X$  such that  $V_i$  deformation retracts onto  $Z_i$  and  $V_1 \cap V_2$  deformation retracts onto  $Z_1 \cap Z_2$ . This is satisfied e. g. if  $X$  is a CW-complex and the  $Z_i$  are subcomplexes. The cup product in cohomology is denoted by  $\smile$ , and  $\text{Sg}(X, Z_1 + Z_2)$  is the free  $\Lambda$ -module on simplices in  $X$  which are neither contained in  $Z_1$  nor in  $Z_2$ .

**Lemma B.8.** *The following diagram commutes in  $\mathbf{D}(\Lambda)$ :*

$$\begin{array}{ccc} R\pi_* j_{1!} \Lambda \otimes^L R\pi_* j_{2!} \Lambda & \xrightarrow{\quad \sim \quad} & R\pi_* j_! \Lambda \\ \beta^{-1} \alpha \downarrow \sim & & \beta^{-1} \alpha \downarrow \sim \\ \mathrm{Sg}(X, Z_1)^\vee \otimes \mathrm{Sg}(X, Z_2)^\vee & \xrightarrow{\quad \sim \quad} \mathrm{Sg}(X, Z_1 + Z_2)^\vee & \xleftarrow{\quad \sim \quad} \mathrm{Sg}(X, Z)^\vee \end{array}$$

Here the top horizontal arrow is defined as the composition

$$R\pi_* j_{1!} \Lambda \otimes^L R\pi_* j_{2!} \Lambda \rightarrow R\pi_*(j_{1!} \Lambda \otimes^L j_{2!} \Lambda) \xrightarrow{\sim} R\pi_* j_! \Lambda. \quad (\text{B.9})$$

*Proof.* Notice that the composition

$$\mathrm{Sg}(X, V_i)^\vee \rightarrow \mathrm{Sg}(X, Z_i)^\vee \xrightarrow{\beta} \mathcal{K}_{(X, Z_i)}(X)$$

factors through  $\alpha : \mathcal{S}_X \otimes \Lambda_{U_i}(X) \rightarrow \mathcal{K}_{(X, Z_i)}(X)$  because  $V_i$  is open in  $X$  and  $\mathcal{S}_X \otimes \Lambda_{U_i}(X)$  consists of sections of  $\mathcal{S}_X$  whose support is contained in  $U_i$ . It follows that the left vertical arrow in the lemma is equal to the composition of the left vertical arrows in the following diagram.

$$\begin{array}{ccccc} (\mathcal{S}_X \otimes \Lambda_{U_1})(X) \otimes^L (\mathcal{S}_X \otimes \Lambda_{U_2})(X) & \xrightarrow{\quad \sim \quad} & (\mathcal{S}_X \otimes \Lambda_U)(X) \\ \sim \uparrow & & \sim \downarrow \alpha \\ \mathrm{Sg}(X, V_1)^\vee \otimes \mathrm{Sg}(X, V_2)^\vee & \xrightarrow{\quad \sim \quad} \mathrm{Sg}(X, V_1 + V_2)^\vee & \xrightarrow{\quad \sim \quad} \mathcal{K}_{(X, Z)}(X) \\ \sim \downarrow & & \sim \downarrow \\ \mathrm{Sg}(X, Z_1)^\vee \otimes \mathrm{Sg}(X, Z_2)^\vee & \xrightarrow{\quad \sim \quad} \mathrm{Sg}(X, Z_1 + Z_2)^\vee & \xleftarrow{\quad \sim \quad} \mathrm{Sg}(X, Z)^\vee \\ & & \sim \uparrow \beta \end{array} \quad (\text{B.10})$$

Recall that the sheaf  $\mathcal{S}_Z$  is the quotient of the presheaf  $V \mapsto \mathrm{Sg}(V)^\vee$  where a section  $f \in \mathrm{Sg}(V)^\vee$  becomes 0 in  $\mathcal{S}_Z(V)$  if there exists an open cover  $(W_i)_i$  of  $V$  such that  $f|_{W_i} = 0$  for all  $i$ . Now, start with  $f \in \mathrm{Sg}(X)^\vee$  vanishing on both  $V_1$  and  $V_2$ , i. e. an element of  $\mathrm{Sg}(X, V_1 + V_2)^\vee$ . These two open subsets of  $X$  cover  $Z$ , and by the description of  $\mathcal{S}_Z$  just given, we see that  $f$  defines the zero class in  $i_* \mathcal{S}_Z(X)$  hence lands in  $\mathcal{K}_{(X, Z)}(X)$ . This yields the right horizontal arrow in the middle row. It follows that the upper half of the diagram commutes. Evidently the lower left square does as well. For the lower right square denote by  $V$  the union of  $V_1$  and  $V_2$ . Then we may decompose this square as follows:

$$\begin{array}{ccc} & \sim & \\ \mathrm{Sg}(X, V_1 + V_2)^\vee & \xleftarrow{\quad \sim \quad} \mathrm{Sg}(X, V)^\vee & \xrightarrow{\quad \sim \quad} \mathcal{K}_{(X, Z)}(X) \\ \sim \downarrow & \downarrow & \nearrow \beta \\ \mathrm{Sg}(X, Z_1 + Z_2)^\vee & \xleftarrow{\quad \sim \quad} \mathrm{Sg}(X, Z)^\vee & \end{array}$$

Commutativity is now clear.

It remains to prove that the top horizontal arrow in (B.10) is a model for (B.9). This follows from the fact that the resolution  $\Lambda \xrightarrow{\sim} \mathcal{S}_X$  of the constant sheaf on  $X$  is *multiplicative*. Namely, this makes the right square of the following diagram



commutative; the left one clearly commutes.

$$\begin{array}{ccccc}
 R\pi_*\Lambda_{U_1} \otimes^L R\pi_*\Lambda_{U_2} & \longrightarrow & R\pi_*(\Lambda_{U_1} \otimes \Lambda_{U_2}) & \xrightarrow{\sim} & R\pi_*\Lambda_U \\
 \parallel & & \parallel & & \parallel \\
 \pi_*(\mathcal{S}_X \otimes \Lambda_{U_1}) \otimes^L \pi_*(\mathcal{S}_X \otimes \Lambda_{U_2}) & \longrightarrow & \pi_*(\mathcal{S}_X \otimes \Lambda_{U_1} \otimes \mathcal{S}_X \otimes \Lambda_{U_2}) & \xrightarrow{\sim} & \pi_*(\mathcal{S}_X \otimes \Lambda_U)
 \end{array}$$

□

### APPENDIX C. CATEGORIES OF COMODULES

In this section we recall some facts about categories of (complexes of) comodules used in the main body of the text. Throughout we fix a ring  $\Lambda$  and a *flat*  $\Lambda$ -coalgebra  $C$ . By a  $C$ -comodule we mean a counitary left  $C$ -comodule.  $\mathbf{coMod}(C)$  (respectively,  $\mathbf{coMod}^f(C)$ ) denotes the category of  $C$ -comodules (respectively,  $C$ -comodules finitely generated as  $\Lambda$ -modules).

The starting point is really the following result.

**Fact C.1.**

- (1)  $\mathbf{coMod}^f(C)$  and  $\mathbf{coMod}(C)$  are abelian  $\Lambda$ -linear categories, and there is a canonical equivalence of abelian  $\Lambda$ -linear categories  $\mathrm{Ind} \mathbf{coMod}^f(C) \simeq \mathbf{coMod}(C)$ .
- (2) The forgetful functor  $o : \mathbf{coMod}(C) \rightarrow \mathbf{Mod}(\Lambda)$  is exact  $\Lambda$ -linear and creates colimits and finite limits.
- (3)  $\mathbf{coMod}(C)$  is a Grothendieck category, copowered over  $\mathbf{Mod}(\Lambda)$ . In particular, it is bicomplete.

*Proof.* The first statement follows from [29, II, 2.0.6 and 2.2.3]. The rest is proved in [34], see [34, Cor. 3 and 9, Pro. 38, Cor. 26]. Explicitly, the copower of a  $\Lambda$ -module  $m$  and a  $C$ -comodule  $c$  is given by the tensor product (as  $\Lambda$ -modules)  $m \otimes c$  with the comodule coaction on  $c$ . □

Next, we are interested in different models for the derived category of  $\mathbf{coMod}(C)$ . The following result is true more generally for any Grothendieck category.

**Fact C.2** ([11, Thm. 1.2]).  $\mathbf{Cpl}(\mathbf{coMod}(C))$  is a proper cellular model category with quasi-isomorphisms as weak equivalences and monomorphisms as cofibrations.

The model structure in the statement is called the *injective model structure*.

From now on assume that  $C$  is a (commutative) bialgebra.  $\mathbf{coMod}(C)$  then becomes a monoidal  $\Lambda$ -linear category with  $C$  coacting on the tensor product (as  $\Lambda$ -modules)  $c \otimes d$  by

$$c \otimes d \xrightarrow{\mathrm{ca} \otimes \mathrm{ca}} (c \otimes C) \otimes (d \otimes C) \xrightarrow{\sim} (c \otimes d) \otimes (C \otimes C) \rightarrow (c \otimes d) \otimes C,$$

the last arrow being induced by the multiplication of  $C$ . In particular, the forgetful functor  $o : \mathbf{coMod}(C) \rightarrow \mathbf{Mod}(\Lambda)$  is monoidal. The category  $\mathbf{Cpl}(\mathbf{coMod}(C))$  inherits a monoidal structure in the usual way.

**Proposition C.3.** *Let  $T$  be a flat object in  $\mathbf{Cpl}(\mathbf{coMod}(C))$ . Then there is a proper cellular model structure on  $\mathbf{Spt}_T^\Sigma \mathbf{Cpl}(\mathbf{coMod}(C))$  with stable equivalences as weak equivalences and monomorphisms as cofibrations.*

*Proof.* The stable equivalences are described in [18, Def. 8.7], and the proof in [11, Pro. 6.31] applies.  $\square$

The model structure of the proposition is called the *injective stable model structure*.

Unfortunately, the monoidal structure does not, in general, interact well with the injective model structures. In the cases of interest in the main body of the text (namely, when  $\Lambda$  is a principal ideal domain) we have the following result, essentially due to Serre.

**Lemma C.4.** *Let  $\Lambda$  be a Dedekind domain. In  $\mathbf{Cpl}(\mathbf{coMod}(C))$  there exist functorial flat resolutions. In particular,  $\mathbf{D}(\mathbf{coMod}(C))$  admits naturally a monoidal structure.*

*Proof.* We follow [31, Pro. 3]. Let  $E$  be a comodule and consider the morphism of comodules  $\mathrm{ca}_E : E \rightarrow C \otimes E$  given by the coaction of  $C$  on  $E$ , where the target has a comodule structure induced by the comultiplication on  $C$  (sometimes called the “extended comodule associated to  $E$ ”). In fact, the coaction  $\mathrm{ca}_\bullet$  defines a natural transformation from the identity functor on  $\mathbf{coMod}(C)$  to the “extended comodule”-functor (this is the unit of an adjunction whose left adjoint is the forgetful functor  $o : \mathbf{coMod}(C) \rightarrow \mathbf{Mod}(\Lambda)$ ; cf. [4, Lem. 1.53]). Since  $E$  is counitary, this natural transformation is objectwise injective. Let  $F : \mathbf{Mod}(\Lambda) \rightarrow \mathbf{Mod}(\Lambda)$  be the functor which associates to a  $\Lambda$ -module  $M$  the free  $\Lambda$ -module  $\oplus_{m \in M} \Lambda$ . It comes with a natural transformation  $\eta : F \rightarrow \mathrm{Id}$  which is objectwise an epimorphism. We obtain a diagram

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{ca}_E} & C \otimes E \\ & & \uparrow 1 \otimes \eta_E \\ & & C \otimes F(E) \end{array}$$

in the category of  $C$ -comodules (the module in the bottom row is again an extended comodule). Since the forgetful functor from  $\mathbf{coMod}(C)$  to  $\mathbf{Mod}(\Lambda)$  commutes with finite limits, we see that the pullback of this diagram is a comodule  $E'$  which both maps surjectively onto  $E$ , and embeds into  $C \otimes F(E)$ . By assumption,  $C \otimes F(E)$  is torsion-free thus so is  $E'$ . It is clear from the construction that the association  $E \mapsto E'$  defines a functor together with a natural transformation  $\eta'$  from it to the identity functor.

Using that  $\mathbf{coMod}(C)$  is a Grothendieck category, the usual procedure leads to functorial flat resolutions.  $\square$

If  $\Lambda$  is a field then we can do better.

**Lemma C.5.** *Let  $\Lambda$  be a field. The injective model structure on  $\mathbf{Cpl}(\mathbf{coMod}(C))$  is a monoidal model structure.*

*Proof.* Indeed, since the forgetful functor is monoidal, exact and creates colimits, the conditions for the injective model structure to be monoidal can be checked in  $\mathbf{Cpl}(\Lambda)$ .  $\square$

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