University of Oxford

Robust hedging of digital double touch barrier options

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Oxford, June of 2009

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Acknowledgement

I express my heartfelt gratitude to my Supervisor Dr. Jan Obłój for his guidance, support and patience for my thesis. His valuable suggestions and encouragement during my work made it possible.

I would be grateful to the help of my classmates Yameng Wu and Sergey Shahverdyan. During writing my master thesis, we had quite a few fruitful discussion where I received much useful advice about C++ programming.

Last but not least, I would like to devote my deepest gratitude to my parents for their continuous love and moral support to me.

Abstract

In this dissertation, we present basic idea and key results for model-free pricing and hedging of digital double barrier options. Besides we extend this model to the market with non-zero interest rate by allowing some model-based trading. Moreover we apply this hedging strategies to Heston stochastic volatility model and compare its performances with that of delta hedging strategies in such setting. Finally we further interpret these numerical results to show the advantages and disadvantages of these two types of hedging strategies.

1 Introduction

It is standard in Mathematical Finance literature to price and hedge financial derivatives by postulating the existence of a filtered probability space $(\Omega, F, (F_t)_{t\geq 0}, \mathbb{Q})$ on which an underlying price is defined, calibrating it to the market data and then deriving prices and hedging strategies. However, in reality it is difficult to find a suitable mathematical model to describe the price process of the underlying very well, and it would lead to mispricing and bad performance of model-based hedging strategies. Thus model-free approach was introduced and developed to reduce such systematic hedging error due to model mispecification.

In this thesis our work is to derive the price bounds and replicate a digital double touch barrier option by model-free approach. More precisely, our problem which is discussed in this thesis is the following question. Given information about all the call prices on a fixed underlying with the same maturity, we want to deduce the price bounds for given exotic option written on the same underlying with the same maturity and how to hedge it based on the above market information without model specification. Besides we also investigate the preference of robust hedging strategy and dynamic hedging strategy in such settings. Here we only consider a digital double touch barrier option struck at $(\underline{b}, \overline{b})$ as an example. A digital double barrier option is a financial derivative which pays 1 if the stock price reaches both $\overline{b}$ and $\underline{b}$ before maturity T . For cases with zero interest rate, this problem has been solved in literature Cox and Obłój [1]. In our thesis we will extend this theory to cases with non-zero interest rate r. From numerical results we find out the bounds derived from this method might be too wide to be used for pricing from practical point of view; however the super/subhedging strategies are quite competitive compared with delta hedging especially in presence of market frictions.

1.1 SEP and robust hedging

The model-free approach, which we outline now, is based on the solution to the Skorokhod embedding problem (SEP). Solving SEP we fix a probability distribution and a stochastic process we try to design a random stopping time such that the process at this time has this specific distribution. It was first formulated and solved by A.V.Skorokhod in 1961 [2]. The detailed account of the problem and its history can be found in Obłój [7]. The main principle is to use the construction of Skorokhod embeddings to identify extremal process so as to find out the optimal super/sub-hedging strategies. Then a model based on the extremal solution to the SEP allows us to deduce that the price bounds implied by such super/sub-replicating strategies are tight. In other words there exists a specific probability measure under which these price bounds are attainable(please see details in [7]).

There are a growing research about SEP-driven methodology in robust hedging pathwise options, for example the work of Hobson [3](lookback option), Brown, Hobson and Rogers [5](single barrier options) and Cox & Obłój [8](digital double no touch barrier option). In [1] pathwise inequality for the payoff of digital double touch barrier option were constructed depending on the market input within the market with zero interest rate. Precise conditions to determine optimal robust hedging strategy were characterized as well. Based on [1] we devise a hybrid super/subhedge strategy to deal with the problem that interest rate is non-zero and compare it with delta hedge by carrying out a numerical example.
The structure of this thesis is as follows. In Chapter 2, we present main results about model-free appoaches to evaluate and replicate digital double touch barrier option for zero interest rate cases, which is cited from [1]. In Chapter 3 we discuss how to derive hybrid super/sub replicating strategy for non-zero interest rate cases in detail. In the next chapter we illustrate the numerical procedure to compare the performance of robust hedges and delta hedges in Black-Scholes model and Heston model. Numerical results for Heston Model are shown in Chapter 5.

1.2 Modeling setup

In our model, we assume that the barriers are constant and interest rate is constant as well. Our assumption on the stock price process $(S_t)_{t\geq 0}$ is merely that it has continuous path. We denote the running maximum and minimum of the price process as $\underline{S}_t = \inf_{u\geq t} S_u$ and $\overline{S}_t = \sup_{u\geq t} S_u$ respectively. It is convenient to introduce the notation of the first hitting times of levels $H_x = \inf\{t : S_t = x\}, x \geq 0$.

We assume that we have market data as follows. For a fixed maturity $T > 0$, assume that we observe the initial spot price S_0 and the market price of European calls $C(K)$ for all strikes $K \in \mathbb{K}$ where the set of available strike in this market is $\mathbb{K}$. We assume that $C(K)$ is twice differentiable and strictly convex on $(0, \infty)$.The market can be represented by the triple

$$(S_0, r, C(K) : K \in \mathbb{K}) \tag{1.1}$$

which is called the *market input*. The first element denotes which the initial price of the underlying we have in the market, the second element is interest rate and the third element is the set of call price in the market. When we discuss the theoretical part, we often assume that $\mathbb{K} = R^+$ (see [1]). However when we consider empirical example, $\mathbb{K}$ is usually a finite set.

Consider a market with interest rate r in which one we can trade all calls, forward, and constant cash, and we denoted this set as $\mathcal{X}$ and $Lin(\mathcal{X})$ is the space of their finite linear combinations. For convenience we introduce the pricing operator $\mathcal{P}$ which to a portfolio with payoff X at maturity T associates its initial price at time zero. We assume that this pricing operator is linear on $Lin(\mathcal{X})$. Then we introduce the definition of no arbitrage (or quasi-static arbitrage) [1].

Definition 1.2.1 (No arbitrage) *The market admits no arbitrage iff any portfolio of initially traded assets with a non-negative payoff has a non-negative price:*

$$\forall X \in Lin(\mathcal{X}) : \mathcal{X} \geq 0 : \mathcal{P}\mathcal{X} \geq 0. \tag{1.2}$$

Here we do not have any probability measure yet, by $X \geq 0, X \in \mathcal{X}$ we mean that the payoff is non-negative for any continuous non-negative stock price path $(S_t)_{t\leq T}$.

From now on we assume that the market admits no arbitrage and pricing operator $\mathcal{P}$ should satisfies the following conditions.

1. $\mathcal{P}(x - e^{-r(T-H_x)}S_T)1_{H_x \leq T} = 0$, that means that we can enter a forward transaction without cost.

2. $\mathcal{P}(K) = e^{-rT}K$ where K is a constant.

We have

$$(K - S_T)^+ = (S_T - e^{rT}S_0) + (e^{rT}S_0 - K) - (S_T - K)^+.$$

It shows that the European put can be seen as the linear combination of a forward, a European call and a constant payoff and thus $Lin(\mathcal{X})$ contains all the puts too. Then by linearity of the pricing operator $\mathcal{P}$ on $\mathcal{X}$ and no arbitrage assumption on the market, we can derive the put-call parity:

$$P(K) := \mathcal{P}(K - S_T)^+ = e^{-rT}K - S_0 + C(K).$$

where $P(K)$ is the price of European put with maturity T. This definition of linear operator $\mathcal{P}$ is a natural extension of that in [1].

Finally we introduce the definition of a *market model.* We say that there exits a market model, if there is a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ with a continuous $\mathbb{Q}$-martingale (S_t) which matches the market input (1.1) where $\mathbb{K} = R^+$. Consider that the model is under risk neutral measure and interest rate is r, so the pricing operator is the expectation $\mathcal{P} = e^{-rT}\mathbb{E}$. To make $(S_t)_{t\leq T}$ match the market input,

$$\begin{aligned} e^{-rT}\mathbb{E}^Q((S_T - K)^+) &= C(K), \forall K \in \mathbb{K} \\ e^{-rT}\mathbb{E}^Q(S_T) &= S_0 \end{aligned}$$

and

$$e^{-rT}E((S_T - K)^+) = C(K), \forall K \geq 0 \Leftrightarrow \mu(dK) = e^{rT}C''(K), \forall K \in \mathbb{K}$$

where μ is the law of the terminal price S_T and it is implied by the market price of calls.

2 Robust Hedging Strategies and Pricing

In this chapter, we summarize the key results of robust hedging strategies and pricing for double touch barrier option, which were established and developed in Cox & Obłój [1]. We do not give details of proof here. An interested reader is referred to [1].

2.1 Model-free Hedging Strategies

2.1.1 Superhedges

Following [1] we present four super-replicating strategies which will be the optimal model-free superhedges. There are different optimal superhedge strategies which are depend on market condition and positions of barriers. Intuition is as follows: when two barriers are sort of symmetrical, i.e both close to or both far away from the initial price of the underlying, it is very likely(unlikely) to hit both barriers. However if the barriers are very unsymmetrical, i.e. the initial price is very close to one barrier, but quite far away from the other, the digital double touch barrier option behaves quite similar to one barrier option. It is natural for us to consider the distances between two barriers and initial underlying price as an explicit criterion determining which strategy is best. In the following section we present formal criteria by using labels, e.g. '$\underline{b} \ll \overline{b}$', which provide an intuitive classification. The formal meaning of these labels can be found in [1].

$\overline{H}^1$: superhedge for $\underline{b} \ll S_0 < \overline{b}$.
The super-replication follows from the following inequality

$$\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} \leq \frac{(K - S_T)^+}{K - \underline{b}} + \frac{S_T - \underline{b}}{K - \underline{b}} \mathbf{1}_{\underline{S}_T \leq \underline{b}} =: \overline{H}^1(K) \tag{2.1}$$

where $K \in (\underline{b}, \infty)$. This equality is solved for any continuous path $S_t(\omega)$ for $t \in [0, T]$. We interpret this as a superhedge strategy: we buy $\frac{1}{K-\underline{b}}$ puts with strike K and when the stock price reaches $\underline{b}$ we buy $\frac{1}{K-\underline{b}}$ forward contracts to super-replicate one unit of digital double barrier option.

$\overline{H}^2$: superhedge for $\underline{b} < S_0 \ll \overline{b}$.
This is a mirror image of $\overline{H}^1$. The super-replication inequalities is given as follows.

$$\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} \leq \frac{(S_T - K)^+}{\overline{b} - K} + \frac{\overline{b} - S_T}{\overline{b} - K} \mathbf{1}_{\overline{S}_T \geq \overline{b}} =: \overline{H}^2(K) \tag{2.2}$$

where $K \in (0, \overline{b})$. It means that in order to superhedge digital double barrier option, it suffice to buy $\frac{1}{\overline{b}-K}$ calls with strike K and when the stock price reaches $\overline{b}$, and enter into $\frac{1}{\overline{b}-K}$ forward contracts.

$\overline{H}^3$: superhedge for $\underline{b} \ll S_0 \ll \overline{b}$.
This superhedging strategy corresponds to an a.s. inequality

$$\begin{aligned}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T} \leq &+ \alpha_1(S_T - K_1)^+ + \alpha_2(S_T - K_2)^+ + \alpha_3(K_3 - S_T)^+ + \alpha_4(K_4 - S_T)^+ \\ &- \beta_1 \mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \wedge T} + \beta_2 \mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \wedge T} + \beta_3 \mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \wedge T} - \beta_4 \mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \wedge T} \\ &=: \overline{H}^3(K_1, K_2, K_3, K_4)\end{aligned} \tag{2.3}$$

where $0 < K_4 < \underline{b} < K_3 < K_2 < \overline{b} < K_1$ and

$$\alpha_3 = \frac{(K_1 - K_2)(\underline{b} - K_4)(\overline{b} - \underline{b}) - (K_1 - \overline{b})(\overline{b} - K_2)(\underline{b} - K_4)}{(K_1 - K_2)(K_3 - K_4)(\overline{b} - \underline{b})^2 - (K_3 - \underline{b})(K_1 - \overline{b})(\overline{b} - K_2)(\underline{b} - K_4)}$$

$$\begin{cases} \alpha_1 = (1 - \alpha_3 \frac{K_3 - K_4}{\underline{b} - K_4}(\overline{b} - \underline{b}))(K_1 - \overline{b})^{-1} \\ \alpha_2 = (1 - \alpha_3 \frac{K_3 - K_4}{\underline{b} - K_4}(\overline{b} - \underline{b}))(\overline{b} - K_2^{-1}) \\ \alpha_3 = \frac{K_3 - \underline{b}}{\underline{b} - K_4}\alpha_3 \end{cases} \quad \begin{cases} \beta_1 = \alpha_1 + \alpha_2 \\ \beta_2 = \alpha_3 + \alpha_4 \\ \beta_3 = \alpha_3 + \beta_1 \\ \beta_4 = \alpha_2 + \beta_2 \end{cases} \tag{2.4}$$

$\overline{H}^4$: superhedge for $\underline{b} < S_0 < \overline{b}$.

$$\begin{aligned}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T} \leq &+ \alpha_1(S_T - K_1)^+ + \alpha_2(S_T - K_2)^+ + \alpha_3(S_T - S_0) + \alpha_4 \\ &- \beta_1 \mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \wedge T} + \beta_2 \mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \wedge T} + \beta_3 \mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \wedge T} - \beta_4 \mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \wedge T} \\ &:= \overline{H}^4(K_1, K_2),\end{aligned} \tag{2.5}$$

where $0 < K_2 < \underline{b} < S_0 < \overline{b} < K_1$ and

$$\begin{cases} \alpha_1 = \frac{1}{K_1 - \underline{b}} \\ \alpha_2 = \frac{1}{\overline{b} - K_2} \\ \alpha_3 = \frac{(K_1 - \underline{b}) - (\overline{b} - K_2)}{(K_1 - \underline{b})(\overline{b} - K_2)} \\ \alpha_4 = \frac{\overline{b}\underline{b} - K_1 K_2}{(K_1 - \underline{b})(\overline{b} - K_2)} + \alpha_3 S_0 \end{cases} \quad \begin{cases} \beta_1 = \alpha_1 + \alpha_3 = 1/(\overline{b} - K_2) \\ \beta_2 = \alpha_2 - \alpha_3 = 1/(K_1 - \underline{b}) \\ \beta_3 = \alpha_1 = 1/(K_1 - \underline{b}) \\ \beta_4 = \alpha_2 = 1/(\overline{b} - K_2) \end{cases} \tag{2.6}$$

2.1.2 Subhedges

In this subsection, we present the subhedges from [1] which will turn out to be the best model-free subhedges depending on the relative distance of barriers to the spot. Actually except for cases $\underline{b} \ll S_0 \ll \bar{b}$ these three subhedges follows from the same form of an almost-sure inequality (2.7), but the parameters are different,.

$$\begin{aligned}\mathbf{1}_{\overline{S}_T \geq \bar{b}, \underline{S}_T} \geq & \alpha_0 + \alpha_1(S_T - S_0) - \alpha_2(S_T - K_2)^+ + \alpha_3(S_T - \underline{b})^+ - \alpha_3(S_T - K_3)^+ \\ & + \alpha_3(S_T - \bar{b})^+ - (\alpha_3 - \alpha_2)(S_T - K_1)^+ - \gamma_1 \mathbf{1}_{S_T > \underline{b}} + \gamma_2 \mathbf{1}_{S_T \geq \bar{b}} \\ & + (\alpha_2 - \alpha_1)(S_T - \underline{b})\mathbf{1}_{H_{\underline{b}} < H_{\bar{b}} \wedge T} - \alpha_2(S_T - \bar{b})\mathbf{1}_{H_{\underline{b}} < H_{\bar{b}} < T} \\ & - (\alpha_3 - \alpha_2 + \alpha_1)(S_T - \bar{b})\mathbf{1}_{H_{\bar{b}} < H_{\underline{b}} \wedge T} + (\alpha_3 - \alpha_2)(S_T - \underline{b})\mathbf{1}_{H_{\bar{b}} < H_{\underline{b}} < T}\end{aligned} \tag{2.7}$$

where $0 < K_1 < \underline{b} < S_0 < \bar{b} < K_1$.

$\underline{H}_1$: subhedge for $\underline{b} < S_0 < \bar{b}$.

$$\begin{cases} \alpha_0 = \frac{S_0(K_1+K_2-\bar{b}-\underline{b})+\bar{b}\underline{b}-K_1K_2}{(\bar{b}-K_2)(K_1-\underline{b})} \\ \alpha_1 = \frac{K_1+K_2-\bar{b}-\underline{b}}{(\bar{b}-K_2)(K_1-\underline{b})} \\ \alpha_2 = \frac{1}{\bar{b}-K_2} \\ \alpha_3 = \frac{\bar{b}-K_2+K_1-\underline{b}}{(\bar{b}-K_2)(K_1-\underline{b})} \end{cases} \qquad \begin{cases} K_3 = \frac{\bar{b}K_1-\underline{b}K_2}{\bar{b}-K_2-\underline{b}+K_1} \\ \gamma_1 = \frac{\bar{b}-\underline{b}}{\bar{b}-K_2} \\ \gamma_2 = \frac{\bar{b}-\underline{b}}{K_1-\underline{b}} \end{cases} \tag{2.8}$$

$\underline{H}_2$: subhedge for $\underline{b} < S_0 \ll \bar{b}$.

$$\begin{cases} \alpha_0 = \frac{(\bar{b}\underline{b}+S_0K_3)(K_1-K_2)-(\underline{b}K_1+S_0\bar{b})(K_3-K_2)-(\bar{b}K_2+S_0\underline{b})(K_1-K_3)}{(\bar{b}-\underline{b})(K_1-K_3)(\bar{b}-K_2)} \\ \alpha_1 = \frac{K_3(K_1-K_2)-\underline{b}(K_1-K_3)-\bar{b}(K_3-K_2)}{(\bar{b}-\underline{b})(K_1-K_3)(\bar{b}-K_2)} \\ \alpha_2 = \frac{1}{\bar{b}-K_2} \\ \alpha_3 = \frac{K_1-K_2}{(K_1-K_3)(\bar{b}-K_2)} \end{cases} \qquad \begin{cases} \gamma_1 = \frac{(K_3-\underline{b})(K_1-K_2)}{(\bar{b}-K_2)(K_1-K_3)} \\ \gamma_2 = \frac{(\bar{b}-K_3)(K_1-K_2)}{(\bar{b}-K_2)(K_1-K_3)} \end{cases} \tag{2.9}$$

where $K_3 \leq \bar{b}\frac{K_1-\underline{b}}{(K_1-\underline{b})+(\bar{b}-K_2)} + \underline{b}\frac{\bar{b}-K_2}{(K_1-\underline{b})+(\bar{b}-K_2)}$.

$\underline{H}_3$: subhedge for $\underline{b} \ll S_0 < \bar{b}$.

$$\begin{cases} \alpha_0 = \frac{(\bar{b}\underline{b}+S_0K_3)(K_1-K_2)-(\underline{b}K_1+S_0\bar{b})(K_3-K_2)-(\bar{b}K_2+S_0\underline{b})(K_1-K_3)}{(\bar{b}-\underline{b})(K_3-K_2)(K_1-\underline{b})} \\ \alpha_1 = \frac{K_3(K_1-K_2)-\underline{b}(K_1-K_3)-\bar{b}(K_3-K_2)}{(\bar{b}-\underline{b})(K_3-K_2)(K_1-\underline{b})} \\ \alpha_2 = \frac{K_1-K_3}{(K_3-K_2)(K_1-\underline{b})} \\ \alpha_3 = \frac{K_1-K_2}{(K_3-K_2)((K_3-K_2)(K_1-\underline{b})} \end{cases} \qquad \begin{cases} \gamma_1 = \frac{(K_3-\underline{b})(K_1-K_2)}{(K_1-\underline{b})(K_3-K_2)} \\ \gamma_2 = \frac{(\bar{b}-K_3)(K_1-K_2)}{(K_1-\underline{b})(K_3-K_2))} \end{cases} \tag{2.10}$$

where $K_3 \geq \bar{b}\frac{K_1-\underline{b}}{(K_1-\underline{b})+(\bar{b}-K_2)} + \underline{b}\frac{\bar{b}-K_2}{(K_1-\underline{b})+(\bar{b}-K_2)}$.
$\underline{H}_4$:Subhedge for $\underline{b} \ll S_0 \ll \bar{b}$.

In this case we use a fourth (trivial) subhedge, which has zero payment and is interpreted as an empty portfolio.

2.2 Model-free pricing of double touch options

Consider the double touch digital barrier option with payoff $\mathbf{1}_{\overline{S}\geq\bar{b},\underline{S}_T\leq\underline{b}}$, the upper and lower bound of its price is an immediate consequence of the superhedging and subhedging strategy described in the previous section. Moreover it can be proved that these bounds are tight, which is not obvious and the detailed proof can be found in [1].

Theorem 2.2.1 *Given the market input(1.1), no-arbitrage in the class of portfolios* $Lin(\mathcal{X}\bigcup\{\mathbf{1}_{\overline{S}\geq\bar{b},\underline{S}_T\leq\underline{b}}\})$ *implies that the following inequality holds*

$$\mathcal{P}\mathbf{1}_{\overline{S}\geq\bar{b},\underline{S}_T\leq\underline{b}} \leq \inf\{\mathcal{P}(\overline{H})^1(K),\mathcal{P}\overline{H}^2(K'),\mathcal{P}(\overline{H})^3(K_1,K_2,K_3,K_4),\mathcal{P}\overline{H}^4(K_1,K_4)\} \quad (2.11)$$

where the infimum is taken over $K>\underline{b}, K'<\bar{b}$ *and* $0<K_4<\underline{b}<K_3<K_2<\bar{b}<K_1$ *and where* $\overline{H}^1,\overline{H}^2,\overline{H}^3,\overline{H}^4$ *are given by (2.1), (2.2), (**??**)and (2.5) respectively. Moreover there exists a market model such that the infimum in (2.11) is the actual price of the double barrier option.*

Theorem 2.2.2 *Given the market input 1.1, no-arbitrage in the class of portfolio* $Lin(\mathcal{X}\bigcup\{\mathbf{1}_{\overline{S}\geq\bar{b},\underline{S}_T\leq\underline{b}}\},\mathbf{1}_{S_t>\bar{b}},\underline{S}_T\leq\underline{b})$ *implies the following inequality between the prices*

$$\mathcal{P}\mathbf{1}_{\overline{S}_T\geq\bar{b},\underline{S}_T\leq\underline{b}} \geq \sup\{\mathcal{P}\underline{H}_1(K_1,K_2),\mathcal{P}\underline{H}_2(K_1,K_2,K_3),\mathcal{P}\underline{H}_3(K_1,K_2,K_3),0\} \quad (2.12)$$

where the supremum is take over $0<K_2<\underline{b}<K_3<\bar{b}<K_1$ *and* $\underline{H}_1,\underline{H}_2,\underline{H}_3$ *are given by (2.7) and the solutions to the relevant set of (2.8), (2.9)and equation (2.10). Similarly it can be shown that there exists a market model such that the supremum in (2.12)is the actual price of the double barrier option.*

3 Model extension for non-zero interest rate

In our previous discussion, we assumed that the interest rate is zero; however in this chapter we will further consider the market with constant non-zero interest rate $r > 0$. For non-zero interest rate, the stock price process is not a martingale any more under risk neutral martingale measure and we can not apply our robust hedging strategy into this case simply. Our goal is to devise a hybrid approach of model-specific hedging and robust hedging to replicate the double touch barrier option. There are two possible way to do it. The first one is to regard this option with double barrier option of forward stock price with time-dependent barrier; the other one is to take advantage of change of measure technique. We will present those two methods in detail as follows.

3.1 Method 1

Since under the risk neutral measure the discounted price is a martingale, to take advantage of model free approach in the previous chapter, it is natural to consider the new underlying price process $S_t^* = S_t e^{-rt}$. We have:

$$\begin{aligned}
\mathbf{1}_{(\overline{S}_T \geq \bar{b}, \underline{S}_T \leq \underline{b})} &= \mathbf{1}_{(\exists t \leq T, u \leq T: S_t \geq \bar{b}, \leq T, S_u \leq \underline{b})} \\
&= \mathbf{1}_{(\exists t \leq T: S_t \geq \bar{b})} \mathbf{1}_{(\exists u \leq T: S_u \leq \underline{b})} \\
&= \mathbf{1}_{(\exists t \leq T: S_t e^{-rt} \geq \bar{b} e^{-rt})} \mathbf{1}_{(\exists u \leq T: S_u e^{-ru} \leq \underline{b} e^{-ru})} \\
&= \mathbf{1}_{(\exists t \leq T: S_t^* \geq \bar{b} e^{-rt})} \mathbf{1}_{(\exists u \leq T: S_u^* \leq \underline{b} e^{-ru})} \\
&= \mathbf{1}_{(\exists t \leq T, u \leq T: S_t^* \geq \bar{b} e^{-rt}, S_u^* \leq \underline{b} e^{-ru})}
\end{aligned}$$

However, when we regard S_t^* as a new underlying, this digital double touch barrier option has time-dependent barriers and we still can not apply techniques in the previous chapter to this case simply. To deal with non-zero interest rate, we propose to decompose the payoff of digital double touch barrier option into two parts:

$$\begin{aligned}
& \mathbf{1}_{\{\exists t \leq T, u \leq T: S_t^* \geq \bar{b} e^{-rt}), S_u^* \leq \underline{b} e^{-ru}\}} \\
= \; & \mathbf{1}_{\{\exists t \leq T, u \leq T: S_t^* \geq \bar{b}, S_u^* \leq \underline{b}\}} + \left(\mathbf{1}_{\{\exists t \leq T, u \leq T: S_t^* \geq \bar{b} e^{-rt}, S_u^* \leq \underline{b} e^{-ru}\}} - \mathbf{1}_{\{\exists t \leq T, u \leq T: S_t^* \geq \bar{b}, S_u^* \leq \underline{b}\}}\right) \\
:= \; & X^1 + X^2 \qquad (3.1)
\end{aligned}$$

For the first part we can use the robust hedging in the previous chapter applied to this fictional market with underlying S_t^* and do the delta hedging for the second part. The intuitive behind such procedure is that X^2 should be small and hence has smaller delta-hedging error, which might result in better performance of hybrid robust hedges compared with that of delta-hedges in some circumstances.

To illustrate this approach more clearly, we introduce the definition of hypothetical market to the model with non-zero interest rate. The true market can be represented by the triple $(S_t, r, C(K) : K \in \mathbb{K})$, where $\mathbb{K} = \{K_1, K_2, ..., K_n\}$ is the set of call prices for given finite available strikes in true market. Our hypothetical market is constructed by changing the underlying S_t^* and the interest rate 0. We want to derive the available strike set for hypothetical market corresponding to the true market $(S_t, r, \mathbb{K})$. Considering that

$$(S_T - K)^+ = (S_T^* e^{rT} - K)^+ = e^{rT}(S_T^* - e^{-rT}K)^+ \tag{3.2}$$

It suggests that in the hypothetical market there are available strike sets $\mathbb{K}^*$, where $\mathbb{K}^* = \{e^{-rT}K_1, e^{-rT}K_2, ..., e^{-rT}K_n\}$. Following (3.2), we obtain:

$$\begin{aligned} C^*(Ke^{-rT}) &= \mathcal{P}(S_T^* - Ke^{-rT})^+ = \mathbb{E}^Q(S_T^* - Ke^{-rT})^+ = e^{-rT}\mathbb{E}^Q(S_T^* e^{rT} - K)^+ \\ &= e^{-rT}\mathbb{E}^Q(S_T - K)^+ = \mathcal{P}(S_T - K)^+ = C(K) \end{aligned} \tag{3.3}$$

where$K \in \mathbb{K}$.
Thus the hypothetical market input can be represented by the triple

$$(S_0^*, 0, C^*(K^*) : K^* \in \mathbb{K}^*) = (S_0, 0, C(e^{rT}K^*) : K^* \in \mathbb{K}^*) = (S_0, 0, C(K) : K \in \mathbb{K}) \tag{3.4}$$

It means that the price of one unit of European Call on the underlying S_t^* with strike $K \in \mathbb{K}^*$ in hypothetical market equals to the price of one unit of European Call on the underlying S_t with strike $e^{rT}K$ in the true market. This provides a way to determine the call prices in hypothetical market based on information in true market, which is useful for us to derive robust hedging strategy in hypothetical market.
Considering subhedges we need to take digital call in those two markets into account. Similarly

$$\mathbf{1}_{\{S_T^* > b\}} = \mathbf{1}_{\{S_T > be^{rT}\}} \tag{3.5}$$

$$\mathcal{P}^*\mathbf{1}_{\{S_T* > b\}} = \mathbb{E}^Q\mathbf{1}_{\{S_T > be^{rT}\}} = e^{rT}\mathcal{P}\mathbf{1}_{\{S_T > be^{rT}\}} \tag{3.6}$$

Since the payoff one unit of digital call with strike b in hypothetical market equals to that of one unit of digital call with strike be^{rT} in real market, the price of one unit of one digital call with strike b in hypothetical market equals to that of e^{rT} unit with strike be^{rT} in true market. It is necessary to assume that in true market digital call with strike $\bar{b}e^{rT}$ and $\underline{b}e^{rT}$ are available to trade. Then we can apply typical robust hedging strategy which is covered in Chapter 3 for zero interest rate case to replicate $\mathbf{1}_{\{\overline{S^*}_t \geq \bar{b}, \underline{S}_u^* \leq \underline{b}\}}$ in hypothetic market.

For delta hedging X^2, we can regard it as delta hedge of long position of $\mathbf{1}_{\{\overline{S}_T \geq \bar{b}, \underline{S}_T \leq \underline{b}\}}$ in true market and delta hedge of short position of $\mathbf{1}_{\{\overline{S}^*_T \geq \bar{b}, \underline{S}^*_T \leq \underline{b}\}}$. Following delta hedging procedure, we can derive these two delta hedging strategies separately. Discussion about this will be left in Section 4.2.3.

Finally we should translate the hedging strategy in hypothetical market to the corresponding strategy in true market. The rules are listed as follows:

1. To trade one unit of put (call) with strike $K \in \mathbb{K}^*$ in hypothetic market at time 0 means to trade e^{-rT} unit of put (call) with strike Ke^{rT} at time 0 in true market;
2. To save x amount of cash at time t in the hypothetic market means to save $xe^{-r(T-t)}$ amount of cash at time t in true market;
3. To trade one unit of digital call with strike $b \in \{\underline{b}, \bar{b}\}$ in the hypothetical market at time 0 means to trade one unit of digital call with strike be^{rT} in the true market at time 0;
4. To trade one unit of underlying S^*_t in hypothetical market at time t means to trade e^{-rt} unit of the underlying S_t in true market at time t.

3.2 Method 2

Considering that

$$\begin{aligned} E^{\mathbb{Q}}[\mathbf{1}_{(\overline{S}_T \geq \bar{b}, \underline{S}_T \leq \underline{b})}] &= E^{\Theta}[\frac{d\mathbb{Q}}{d\Theta}\mathbf{1}_{(\overline{S}_T \geq \bar{b}, \underline{S}_T \leq \underline{b})}] \\ &= E^{\Theta}[\mathbf{1}_{(\overline{S}_T \geq \bar{b}, \underline{S}_T \leq \underline{b})}] + E^{\Theta}[(\frac{d\mathbb{Q}}{d\Theta} - 1)\mathbf{1}_{(\overline{S}_T \geq \bar{b}, \underline{S}_T \leq \underline{b})}] \end{aligned} \tag{3.7}$$

where $\mathbb{Q}$ is risk neutral measure, and Θ is the measure under which S_t is a martingale. To do so we should specify which risk measure $\mathbb{Q}$ we have, and then based on that we can calculate Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\Theta}$. Therefore we can robust hedge for the first part and delta-hedge the second part. But this method is model-specific and we focus on method 1 in this thesis.

4 Construction of Numerical Procedure

4.1 Practical Consideration

In practice, the upper and lower bounds derived by the techniques given by robust hedging is unlike to be successful; however the superhedging and subhedging strategy might be helpful. Consider a trader who has sold a double barrier option and who wishes to hedge the resulting risk. In Black-Scholes (BS) Model the trader can remove the risk from his position by delta-hedging the short position, but there are a number of practical considerations that would lead to an unsatisfactory performance of this approach.

1. **Discrete Hedging:** The adjustment of delta hedging on a periodic basis will result in an inexact hedge of the position. Besides, there is a significant organizational cost (and risk) to set up such a hedge operation.

2. **Transaction Cost:** To perfectly delta-hedge the option, the trader should trade the stocks frequently, which makes the transaction cost very large. To minimize the total transaction cost, the trader would like to trade as infrequently as possible. That means that the trader should trade off discretisation errors with expensive transaction costs.

3. **Model risk:** In real market, it is very difficult to find a mathematical model which describes the stock prices process perfectly. The incorrect model or the error of the estimated parameters will lead to systematic hedging error, an possibly large losses.

Robust hedging which we investigate in this thesis is model-free and it does not need frequent trading, which will significantly decrease the transaction costs compared with delta-hedging. We went to investigate numerically if in practice it might outperform standard delta-replicating strategy.

4.2 Structure of Numerical Examples

In the numerical examples, we divide the whole procedure into three major stages, that is, the stock prices process is in Black-Scholes Model, Heston Model with zero interest rate and Heston Model with non-zero interest rate. In each model, as to compare robust hedging and delta hedging, the main tasks are listed as follows.

1. We simulate stock price process in each model and compute the time-0 call prices under this model at a range of strike prices, and the time-0 price of a double barrier option;

2. We compute the optimal super- and sub-hedges for digital double touch barrier option based on the calculated call prices.

3. We carry out the dynamic part of robust hedges and compare with delta-hedging. We compare the average utility and cumulative distribution of hedging errors from delta-hedging and robust hedging.

We choose C++ to do the programming to achieve all the numerical results we need.

4.2.1 Stage 1: Black-Scholes-Merton Model

To replicate digital double barrier option via delta hedging, we should compute its price so that based on it we can compute its first derivative with respect to the stock price S_t to determine the delta hedging strategy. We can apply Brownian Bridge technique to improve the performance of Monte-Carlo Method in pricing double barrier option. However if we use it to compute the delta hedging strategy, it is still quite time-consuming because at each time step we should price this option twice and Monte-Carlo Method requires us to simulate a large number of paths to determine this option's price. Therefore an alternative way to price this option is necessary, and the better choice is Finite Difference Method. We use standard finite different θ-scheme by applying boundary condition

$$V(S,T) = 0, \text{ for } S \ \in [\overline{b}, \underline{b}]$$

and knocking in condition

$$\begin{cases} V(\overline{b},t) = e^{-r(T-t)}\mathbb{E}^Q[\mathbf{1}_{\exists u\in[t,T],\underline{S}_u<\underline{b}}|S_t = \overline{b}] \\ V(\underline{b},t) = e^{-r(T-t)}\mathbb{E}^Q[\mathbf{1}_{\exists u\in[t,T],\overline{S}_u<\overline{b}}|S_t = \underline{b}] \end{cases}$$

We note that a unit of digital double touch barrier option at time t with spot price $S_t = \overline{b}$ is equivalent to a unit of digital down-and-in barrier option with maturity $T-t$, initial stock price $\overline{b}$ and barrier $\underline{b}$. Similarly the digital double touch barrier option at time t with spot price $S_t = \underline{b}$ is equivalent to digital up-and-in barrier option with maturity $T-t$, initial stock price $\underline{b}$ and barrier $\overline{b}$. Recall the analytic form of the price of digital one touch barrier options:

$$\begin{aligned} e^{-rT}\mathbb{E}^Q[\mathbf{1}_{\exists t\in[0,T],\overline{S}_t>b}] &= \begin{cases} e^{-rT}(p_1 + \kappa^{-\nu/\sigma}(1-p_2)), if \ S_0 < b \\ e^{-rT}, \ otherwise \end{cases} \\ e^{-rT}\mathbb{E}^Q[\mathbf{1}_{\exists t\in[0,T],\underline{S}_t<b}] &= \begin{cases} e^{-rT}(1-(q_1 - \kappa^{-\nu/\sigma}q_2), if \ S_0 > b \\ e^{-rT}, \ otherwise \end{cases} \end{aligned}$$

where $N(.)$ is cumulative distribution of standard normal distribution function and

$$\begin{cases} \kappa = (\frac{S_0}{b})^2 \\ \nu = (2r - \sigma^2)/(2\sigma) \\ p_1 = N((\log(S_0/b) + \frac{(r-\sigma^2/2)T}{\sigma\sqrt{T}}) \\ p_2 = N((\log(S_0/b/\kappa) + \frac{(r-\sigma^2/2)T}{\sigma\sqrt{T}}) \\ q_1 = N(\frac{\log(\kappa)}{2\sigma\sqrt{T}} + \nu\sqrt{T}) \\ q_2 = N(-\frac{\log(\kappa)}{2\sigma\sqrt{T}} + \nu\sqrt{T}) \end{cases}$$

The above equations can be proved by reflection principle and found in Shreve [11]. Using these formulas, we can obtain the explicit formula of knocking out condition for digital double touch barrier option, which would be used in finite difference scheme.

Next we compare prices computed via Finite Difference Method and Monte-Carlo (Brownian Bridge) Method (see Paul [9]) while varying the upper barrier and lower barrier and give the tables to show these results. Here we choose $S_0 = 100$, $\sigma = 0.5$ and $T = 1$. Following lecture notes of Reisinger [10], the θ-scheme is conditional stable for $\theta \in [0,\ 0.5)$ and unconditional stable for $\theta \in [0.5, 1]$. We use that the number of time step of generating stock process by Monte Carlo Method $nT = 5000$ and the number of simulated paths $N = 20000$. It is know that Monte-Carlo Method has the convergence rate of $\mathcal{O}(\frac{1}{\sqrt{N}})$. We realized θ-scheme for finite difference method and test a number of pairs (nS, nT_{FD}), where nS and nT_{FD} are denoted as the number of grids and the number of time step for finite difference method respectively. Based on the test results we find out that for parameters given by us explicit scheme has better convergence rate than implicit method. Thus we decide to use explicit scheme in this numerical example by keeping that the length of grid $\Delta S = 0.5$ and the number of time step in Finite Difference Method $nT_{FD} = 300000$. We can not choose large nS, because in order to make explicit finite difference work properly, the number of time step nT_{FD} should increase dramatically corresponding to the change of nS. Now we compute the initial price of digital double touch barrier option for case with both zero interest rate and non-zero interest rate case respectively. The results are given in the following Tables 4.1. From this table, the absolute value of difference of digital double barrier options prices computed via two methods are less than 0.02, which can show that our finite difference scheme performs well in pricing digital double barrier option and it is much more efficient than Monte-Carlo Method in respect of computation time.

Then we will investigate how the delta hedging and robust hedging perform in this case. To do this, we should specify what the hedging error means for each type of hedging strategy.
The delta-hedge error for replication of digital double barrier option is defined as

$$\varepsilon^{\Delta} = H_T^{\Delta} - \mathbf{1}_{\overline{S}_T \geq \bar{b}, \underline{b}_T \leq \underline{b}}$$

Table 4.1: Comparison of price of digital double barrier option computed by Finite Difference Method and Monte Carlo Method

r=0

$(\underline{b}, \overline{b})$	(70,130)	(80,120)	(83,117)	(85,115)	(90,110)
Finite Difference	0.130901	0.374808	0.464528	0.525366	0.679811
Monte Carlo	0.12745	0.36305	0.4542	0.5142	0.6717
$(\underline{b}, \overline{b})$	(95,105)	(97,103)	(95,120)	(80,105)	(75,110)
Finite Difference	0.837815	0.902145	0.587891	0.624054	0.447863
Monte Carlo	0.8221	0.88765	0.5721	0.6132	0.44155

r=0.05

$(\underline{b}, \overline{b})$	(70,130)	(80,120)	(83,117)	(85,115)	(90,110)
Finite Difference	0.122519	0.356304	0.442501	0.500901	0.648725
Monte Carlo	0.112578	0.342633	0.4435187	0.491025	0.62843
$(\underline{b}, \overline{b})$	(95,105)	(97,103)	(95,120)	(80,105)	(75,110)
Finite Difference	0.798859	0.859538	0.577521	0.576986	0.413691
Monte Carlo	0.785287	0.845358	0.571499	0.56465	0.408743

where H_T^{Δ} is the payoff of delta hedging strategy.
The superhedge error is given by

$$\overline{\varepsilon} = \overline{H}^* - \mathbf{1}_{\overline{S}_T \geq \bar{b}, \underline{S}_T \leq \underline{b}} - \mathcal{P}\overline{H}^* + \mathcal{P}\mathbf{1}_{\overline{S}_T \geq \bar{b}, \underline{S}_T \leq \underline{b}} \tag{4.1}$$

where $\overline{H}^*$ is the outcome of the optimal superhedging strategy. Similarly the subhedge error is denoted as $\underline{\varepsilon}$, which satisfies

$$\underline{\varepsilon} = \mathbf{1}_{\overline{S}_T \geq \bar{b}, \underline{S}_T \leq \underline{b}} - \underline{H}^* + \mathcal{P}\underline{H}^* - \mathcal{P}\mathbf{1}_{\overline{S}_T \geq \bar{b}, \underline{b}_T \leq \underline{b}}$$

where $\underline{H}^*$ is the outcome of optimal subhedge strategy.

We consider cases with and without transaction costs respectively. Since in Black-Scholes Model the underlying is geometric Brownian motion, theoretically the delta hedge should perform almost perfectly if the rebalancing interval is sufficiently small and transaction cost are not taken into account. As our hedging strategies are self-financing and hedging errors are defined as above, the mean of hedging errors should be zero if the market does not have transaction costs. We plot the empirical cumulative distribution function of hedging error from both delta hedges and robust hedges and calculate their corresponding average exponential utility. Here assume that we can trade at any sampled time and we adopt parameters $\sigma = 0.5$, $S_0 = 100$,$r = 0$,$\underline{b} = 90$ and $\bar{b} = 110$. We present our results in Figure 4.1 and Figure 4.2.

In these two cases we simulate 10000 paths and compute the average utility. Results are

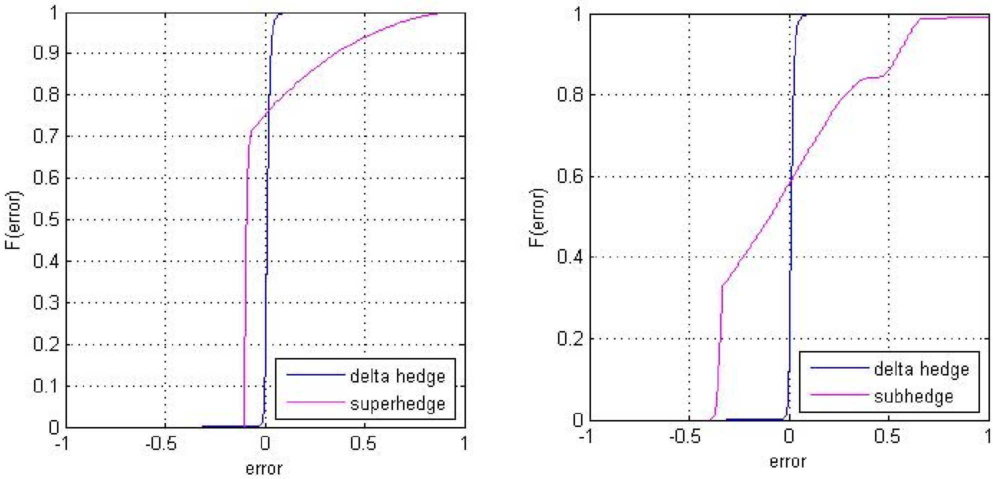

Figure 4.1: Cumulative distribution of hedging errors under different scenarios of replicating digital double barrier option with barriers at 90 and 110 under Black-Scholes Model. We assume that there is no transaction costs in the market.

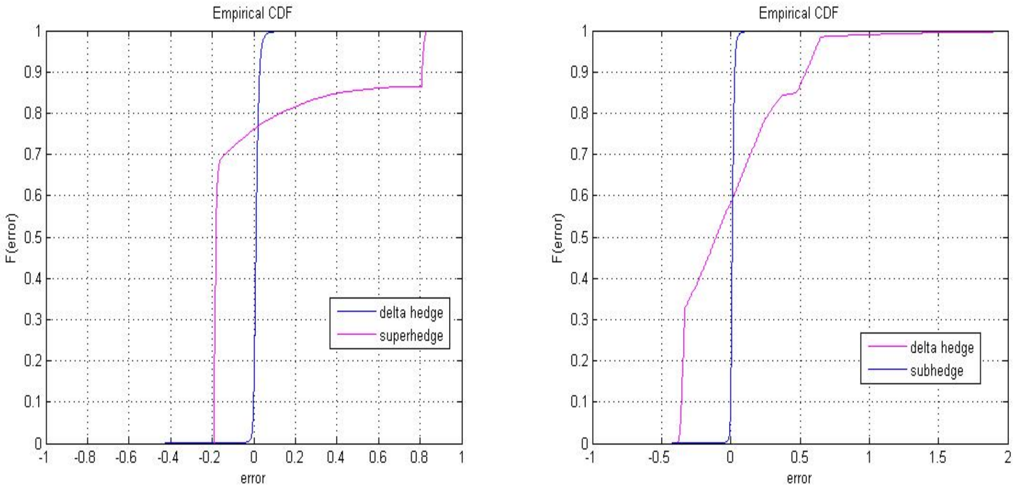

Figure 4.2: Cumulative distribution of hedging errors under different scenarios of replicating digital double barrier option with barriers at 90 and 110 under Black-Scholes Model. We assume that there are transaction cost rate of stock= 0.005 and transaction cost rate of option = 0.01 in the market.

summarized in Table 4.2.1. From the empirical density of hedging error we can conclude that delta hedging performs better than robust hedging in this case, for it has smaller absolute average hedging error and higher average utility compared to robust hedging. In the end of this subsection, we compare the average error and absolute value of er-

	case without transaction cost	case with transaction cost
Delta hedge	**0.0128324**	**0.012581**
Superhedge	-0.037228	-0.0509704
Subhedge	-0.0795055	-0.0742017

Table 4.2: Comparison of exponential utilities of hedging errors in digital double option with barriers at 90 and 110 in Black-Scholes Model. In each case, preferred hedge is highlight.

ror for all the hedging strategies. We consider non-transaction cost cases while varying barrier pairs. The results are summarized in Table 4.2.1 and Table 4.2.1. Table 4.2.1 presents that the average hedging errors from robust hedging and delta hedging are all close to zero. And Table 4.2.1 shows that average absolute value of delta hedging error is smaller compared with corresponding robust hedging errors and it becomes smaller as rebalancing becomes more frequently.

4.2.2 Stage 2: Heston Model with zero interest rate

In the rest of this chapter we assume that the underlying process follows Heston stochastic volatility model (see Heston's paper [12]). First of all we briefly introduce Heston Model, which is one of the most well-known and popular of all stochastic volatility models. More precisely, in Heston model we assume that the underlying stock satisfies the following stochastic differential equation.

$$\begin{cases} dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_t^1, S_0 = S_0, \nu_0 = \sigma_0 \\ d\nu_t = \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t} dW_t^2, d\langle W^1, W^2 \rangle_t = \rho dt \end{cases} \tag{4.2}$$

In this subsection we keep interest rate $r = 0$.

Step 1: *Simulation of stock price process under Heston Model*
From the discussion of several ways to simulate Heston Process in Gatheral [4], such as Euler discretization, Milstein Discretization or an implicit scheme, we choose Milstein Discretization, because compare with Euler scheme, it is always to be preferred for it substantially reduces the frequency of negative variance. Besides it is much simpler than than implicit scheme.

(Milstein Scheme) To achieve a higher order in the Ito-Taylor expansion of $\nu(t_i + \Delta t)$,

$(\underline{b}, \overline{b})$	suphedge	subhedge	delta hedge
$(70, 130)$	0.00635962	-0.00918839	0.0104111
$(80, 120)$	0.0197268	-0.0180604	0.0188008
$(83, 117)$	0.0195771	-0.0164211	0.0197582
$(85, 115)$	0.0156873	-0.0235922	0.019984
$(90, 110)$	0.0182838	-0.0214928	0.020291
$(95, 105)$	0.0237703	-0.017436	0.0207605
$(97, 103)$	0.0227167	-0.0145936	0.0207409
$(95, 120)$	0.0196382	-0.0238206	0.0188548
$(80, 105)$	0.0254215	-0.0308586	0.0215463
$(75, 110)$	0.0222513	-0.0202368	0.0199563

Table 4.3: Comparison of average hedging errors from superhedge, subhedge and delta hedge in digital double touch option with barriers at 90 and 110 in Black-Scholes Model. In this case, it is allowed for the trader to rebalance his positions every time step.

$(\underline{b}, \overline{b})$	suphedge	subhedge	delta hedge
$(70, 130)$	0.0115229	-0.0147047	0.0110038
$(80, 120)$	0.0207064	-0.0137855	0.0198412
$(83, 117)$	0.0214493	-0.0203589	0.020177
$(85, 115)$	0.0176973	-0.0229617	0.0199101
$(90, 110)$	0.020877	-0.0245558	0.019745
$(95, 105)$	0.0219335	-0.0202344	0.0218116
$(97, 103)$	0.0203785	-0.0230547	0.021049
$(95, 120)$	0.017592	-0.0117527	0.0188076
$(80, 105)$	0.0244856	-0.0216979	0.021435
$(75, 110)$	0.0204217	-0.0210513	0.0200696

Table 4.4: Comparison of average hedging errors from superhedge, subhedge and delta hedge in digital double touch option with barriers at 90 and 110 in Black-Scholes Model. In this case, it is allowed for the trader to rebalance his positions daily.

$(\underline{b}, \overline{b})$	suphedge	subhedge	delta hedge
$(70, 130)$	0.1534	0.184669	0.0144067
$(80, 120)$	0.155651	0.274888	0.0229946
$(83, 117)$	0.181675	0.284368	0.0231966
$(85, 115)$	0.172226	0.282111	0.0231353
$(90, 110)$	0.147965	0.296894	0.0225998
$(95, 105)$	0.104202	0.35713	0.0218847
$(97, 103)$	0.073605	0.394785	0.0215078
$(95, 120)$	0.142142	0.301915	0.0214434
$(80, 105)$	0.165908	0.321028	0.0240563
$(75, 110)$	0.165986	0.291067	0.0239518

Table 4.5: Comparison of average absolute value of hedging errors from superhedge, subhedge and delta hedge in digital double touch option with barriers at 90 and 110 in Black-Scholes Model. In this case, it is allowed for the trader to rebalance his positions every time step.

$(\underline{b}, \overline{b})$	suphedge	subhedge	delta hedge
$(70, 130)$	0.156422	0.181635	0.034251
$(80, 120)$	0.159693	0.273917	0.0514526
$(83, 117)$	0.184735	0.287427	0.0506898
$(85, 115)$	0.174474	0.287831	0.0481351
$(90, 110)$	0.15135	0.299482	0.0469388
$(95, 105)$	0.106855	0.356444	0.0438798
$(97, 103)$	0.0794097	0.381941	0.0434094
$(95, 120)$	0.142435	0.311027	0.044751
$(80, 105)$	0.166988	0.32678	0.0502126
$(75, 110)$	0.169942	0.29429	0.052665

Table 4.6: Comparison of average absolute value of hedging errors from superhedge, subhedge and delta hedge in digital double touch option with barriers at 90 and 110 in Black-Scholes Model. In this case, it is allowed for the trader to rebalance his positions daily.

we arrive at the following discretization of the variance process

$$\nu_{i+1} = \nu_i - \kappa(\theta - \nu_i)dt + \xi\sqrt{\nu_i \Delta t}Z + \frac{\xi^2}{4}\Delta t(Z^2 - 1) \quad (4.3)$$

where $\nu_i = \nu(t_i)$ and Z is a standard normal random variable.This method is called Milstein Scheme.
Using Euler Scheme for the first SDE of Heston Model, we will have

$$S_{i+1} = S_i + S_i r\Delta t + S_i\sqrt{\nu_i \Delta t}(\rho Z + \sqrt{1-\rho}W) \quad (4.4)$$

where W is a standard normal random variable which is independent with Z.
To test our simulations of stock price processes driven by Heston Model, we generate a large number of stock price processes and compute the European call prices with varied strikes via Monte Carlo method. From Gatheral [4] call price in Heston model has analytical form of solutions. We compare those call prices computed by two methods and plot call prices vs strikes.[1] Results are shown in Figure 4.3. From Figure 4.3, the two curves are very close and it can convince us that our simulation of stock price process is successful.

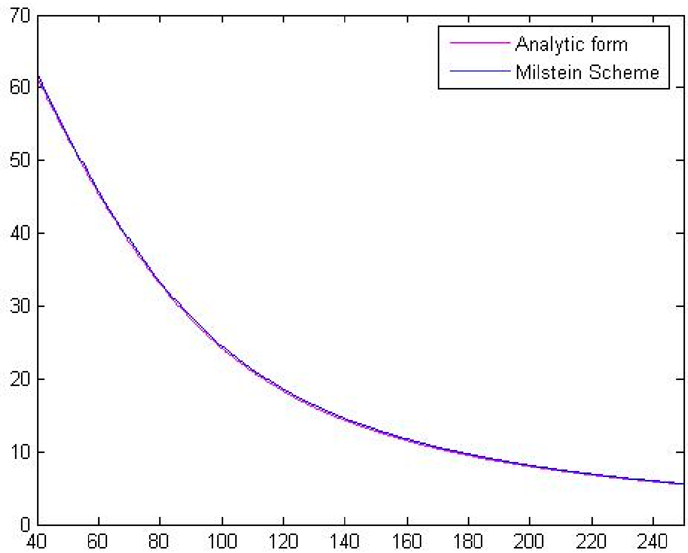

Figure 4.3: Comparison of prices of European Call in Heston Model computed with Milstein Scheme simulations and analytic form. The parameters for Milstein Scheme are given as follows: the number of time step $nT = 2000$ and the number of simulations of paths are 100000. Other parameters are chosen as follows:$S_0 = 100, \sigma_0 = 0.5, \kappa = 0.6, \theta = 1, \xi = 1.3, \rho = 0.15$.

[1]The codes for analytic form of call prices were given to us and were computed by A.M.G.Cox.

Step 2: *Delta-Hedge of Digital Double Barrier Option in Heston Model*
In reality the trader is uncertain about what model the market follows and maybe it is reasonable for him to believe that the market is still in classical Black Scholes model. He does not want to update the implied volatility as he wants to hedge within Black-Scholes model. What we have to do now is to find out what the implied volatility of this market is. There are various ways to do this. Here we use implied volatility of a standard European call at the money. More clearly, we compute the call price with initial value S_0 and the strike S_0 and calculate the implied volatility such that the Black-Scholes price with this this implied volatility matches that call price. Then the trader continue to delta-hedge this digital double-touch barrier option using this implied volatility $\hat{\sigma}$ until maturity T.
Step 3: *Robust Hedges of Digital Double Touch Barrier Option in Heston Model*
Since the robust hedging strategy is model-free, for zero interest rate cases we use the same techniques of robust hedging in GBM model to replicate this digital double barrier option. The only difference is that the underlying stock price process is simulated by Milstein Scheme. Due to discrete rebalancing time, the robust hedging strategy might not trade forward at the exact barriers, which leads to the correction term in the errors. Thus we introduce the corrected error term, which assume that we trade the forward at the exact barriers.The correction term is actually the difference of the theoretical robust hedging strategy and the real one. Details are provided and discussed in Chapter 5.

4.2.3 Stage 3: Heston Model with non-zero interest rate

Recall method 1 in Chapter 3 to replicate digital double touch barrier option in cases with non-zero interest rate.

$$\mathbf{1}_{\{\overline{S}_T \geq \bar{b}, \underline{S}_T \leq \underline{b}\}} = \mathbf{1}_{\{\exists t \leq T, u \leq T: S_t^* \geq \bar{b}, S_u^* \leq \underline{b}\}} + (\mathbf{1}_{\{\overline{S}_T \geq \bar{b}, \underline{S}_T \leq \underline{b}\}} - \mathbf{1}_{\{\exists t \leq T, u \leq T: S_t^* \geq \bar{b}, S_u^* \leq \underline{b}\}}) := X^1 + X^2$$

where $S_t^* = S_t e^{-rt}$. We delta hedge X^2 and robust hedge X^1 and then we explain it in detail. In true market using techniques of delta hedging digital double barrier option we covered before, we can find out the implied volatility $\hat{\sigma}$ for true market and delta-hedge this double barrier option following the trader's belief that the underlying S_t is within Black-Sholes world with interest rate r and the volatility $\hat{\sigma}$. Our implied volatility $\hat{\sigma}$ is extracted from the initial call price with strike S_0.

$$BS(S_0, 0; \hat{\sigma}, S_0, r) = C(S_0) = e^{-rT}\mathbb{E}^Q(S_T - S0)^+.$$

where $BS(S, t; \sigma, K, r)$ is Black-Scholes formula for the European call with strike K settled at maturity t with constant volatility σ and interest rate r.
Similarly the hypothetic market can be regarded within Black Scholes world with the underlying S_t^*, zero interest rate and volatility σ^*, which satisfies

$$BS(S_0, 0; \hat{\sigma}^*, S_0^*, 0) = C^*(S_0) = C(e^{rT} S_0).$$

The detailed description about robust hedging digital double barrier option in hypothetical market is covered in Chapter 3 and we do not repeat it here.

5 Numerical Results and analysis in Heston Model

In this section, we present the numerical results for performance of robust hedging and delta hedging in Heston Model. We plot cumulative distributions of hedging errors and the expected mean of exponential utility function of hedging errors for judging preference of those replicating strategies.

5.1 Case with zero interest rate

We choose parameters

$$S_0 = 100, \sigma_0 = 0.5, \kappa = 0.6, \theta = 1, \xi = 1.3, \rho = 0.15 \tag{5.1}$$

Transaction in S_t carry a 0.5% transaction cost and buying or selling call/put options carries a 1% transaction cost. These are used in [1] and for comparison purposes we adopt them here.

Firstly, we want to find out which robust hedging strategy is optimal to use to replicate digital option with different barriers. We choose two barrier pairs $(\underline{b}, \overline{b}) = (90, 115)$ and $(\underline{b}, \overline{b}) = (70, 130)$. In the first case with barriers (90, 115), the optimal superhdge strategy is method 4, its corresponding strikes are $K_1 = 278.01$ and $K_2 = 48.01$, and the value of this superhedge portfolio is 0.789738. The optimal subhedge strategy is method 1, its corresponding strikes $K_1 = 182.51$, $K_2 = 53.51$ and $K_3 = 105.51$ and the value of this subhedge strategy is 0.311265. The price of this digital double barrier option is 0.59571, which is actually within the interval [0.311265, 0.789738]. In the second case with barriers (70, 130), the optimal superhdge strategy is method 3, its corresponding strikes are $K_1 = 150.51$, $K_2 = 114.01$, $K_3 = 83.51$ and $K_4 = 56.01$, and the value of this superhedge portfolio is 0.39235. The optimal subhedge strategy is method 2, its corresponding strikes are $K_1 = 499.51$, $K_2 = 19.01$ and $K_3 = 106.01$ and the value of this subhedge strategy is 0.0291333. The price of this digital double barrier option is 0.22442, which is actually within the interval [0.0291333, 0.39235]. It shows that the robust hedging strategy indeed provides upper and lower bounds for digital double barrier option, although these bounds might be too wide to use in practice.

Then let us investigate in detail the hedging error from robust hedging. We plot the stock price at the maturity vs hedging error and corrected hedging error and present the

results in Figure 5.1, Figure 5.2 and Figure5.3.
In case with barrier pair $(90, 115)$, as we mentioned before, the optimal superhedge is method 3. Recall the payoff of superhedge $\overline{H}^4(K_1, K_2)$ which is given in (2.5) and (2.6) and the definition of the corrected superhedge error (4.1). In general when the terminal price of the underlying is known, whether the underlying price hit both barriers is not determined. That is why the payoff of the superhedge strategy is not one-to-one correspondence with the terminal underlying price, nor is the corrected error. We consider that $S_T \in (0, K_2]$, $(K_2,\ \underline{b}]$, $(\underline{b},\ S_0)$, $(S_0,\ \overline{b})$, $[\overline{b},\ K_1)$, $[K_1,\ \infty)$ respectively. We take $S_T \in (\underline{b}, S_0]$ as an example. The corrected error for this superhedge strategy should be

$$
\begin{aligned}
\overline{\varepsilon} &= \overline{H}^* - \mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} - p \\
&= \begin{cases} \alpha_3(S_T - S_0) + \alpha_4 - \beta_1(S_T - \overline{b}), \text{ if } H_{\overline{b}} < H_{\underline{b}} \wedge T \text{ and } H_{\underline{b}} > T; \\ \alpha_3(S_T - S_0) + \alpha_4 - \beta_1(S_T - \overline{b}) + \beta_3(S_T - \underline{b}) - 1 - p, \text{ if } H_{\overline{b}} < H_{\underline{b}} \wedge T \text{ and } H_{\underline{b}} \leq T; \\ \alpha_3(S_T - S_0) + \alpha_4 + \beta_2(S_T - \underline{b}) \text{ if } H_{\underline{b}} < H_{\overline{b}} \wedge T \text{ and } H_{\overline{b}} > T; \\ \alpha_3(S_T - S_0) + \alpha_4 + \beta_2(S_T - \underline{b}) - \beta_4(S_T - \overline{b}) - 1 - p, \text{ if } H_{\underline{b}} < H_{\overline{b}} \wedge T \text{ and } H_{\overline{b}} \leq T. \end{cases} \\
&= \begin{cases} -\alpha_1 S_T - \alpha_3 S_0 + \alpha_4 + \beta_1 \overline{b}, \text{ if } H_{\overline{b}} < H_{\underline{b}} \wedge T \text{ and } H_{\underline{b}} > T; \\ \alpha_2 S_T - \alpha_3 S_0 + \alpha_4 - \beta_2 \underline{b}) \text{ if } H_{\underline{b}} < H_{\overline{b}} \wedge T \text{ and } H_{\overline{b}} > T; \\ -\alpha_3 S_0 + \alpha_4 + \beta_1 \overline{b} - \beta_3 \underline{b} - p, \text{ otherwise}; \end{cases}
\end{aligned}
$$

where $p = \mathcal{P}\overline{H}^* - \mathcal{P}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}}$ is a constant.
Following this way we can derive analytic relationship between S_T and corrected error for both superhedge and subhedge. In Figure 5.3, Figure 5.1 we observe that the relationship between S_T and $\overline{\varepsilon}$ when $S_T \in (\underline{b},\ S_0)$ is exactly the same as we expected. Theoretically speaking this relationship can not be affected by rebalancing interval. It is consistent with what we have seen in the right graphs of those two figures as well. Besides from analytic formula the corrected error term is bounded below, which results in a sharp curve of cumulative distribution function of corrected error. This feature can be captured in left graphs of Figure 5.5, Figure 5.4and Figure 5.6. However as rebalancing interval increases, the trader might not be able to buy or sell the forward at exactly barrier prices, which leads to that robust hedging error clusters around the corresponding corrected hedging error. Moreover it adds a left tail to the cumulative distribution of robust hedging error compared with that of corrected error. When the trader rebalance his position and monitor barriers every time step, the difference between errors and corrected errors is very small, which comes from discrete approximation of continuous sample paths, and thus graphs drawn for both cases are almost the same. It is shown very clearly in Figure 5.5. Based on this point, it is not necessary for us to draw graphs for cases with other barrier pairs, e.g. (70, 130) if rebalancing frequency is every time step, because this information can be found in graphs about corrected error for cases with the same barrier pair and rebalancing interval is daily.

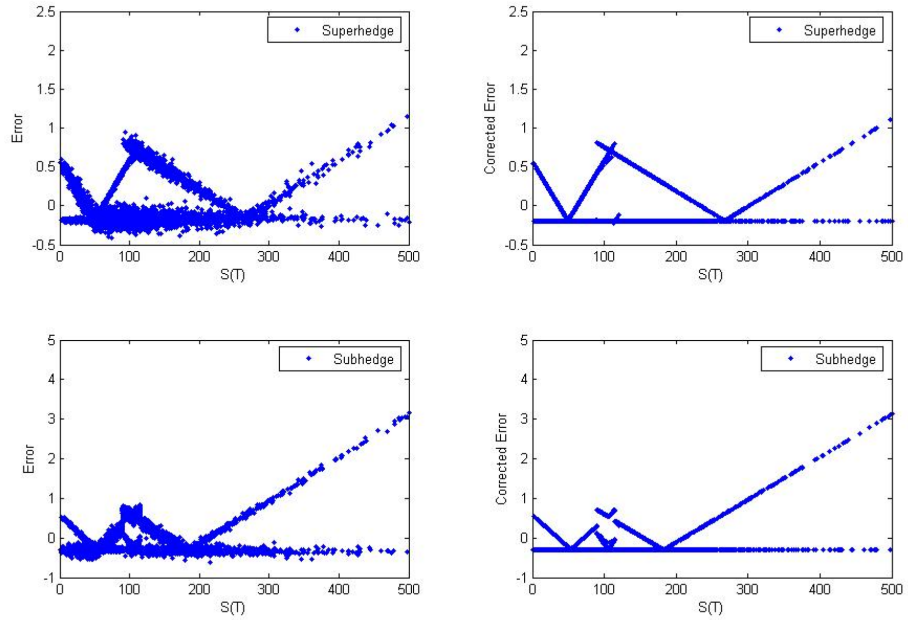

Figure 5.1: Terminal stock price and (corrected) hedging errors for super/sub-hedge in double touch option with barriers at 115 and 90 under Heston Model with parameters given in (5.1) and interest rate $r = 0$. It was allowed for the trader to rebalance position on daily basis.

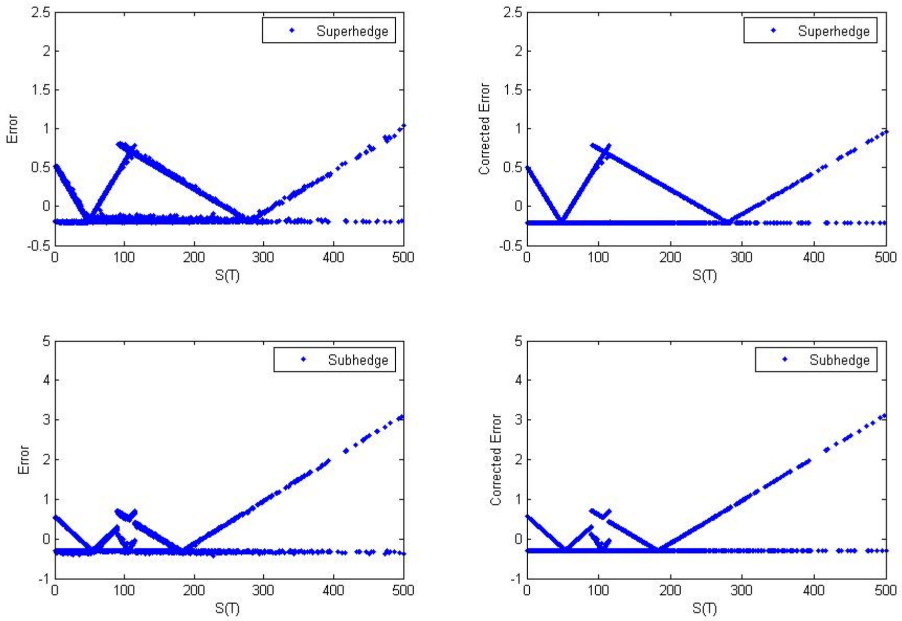

Figure 5.2: Terminal stock price and (corrected) hedging errors for super/sub-hedge in double touch option with barriers at 115 and 90 under Heston Model with parameters given in (5.1) and interest rate $r = 0$. It was allowed for the trader to monitor barrier crossing and rebalance his positions every time step.

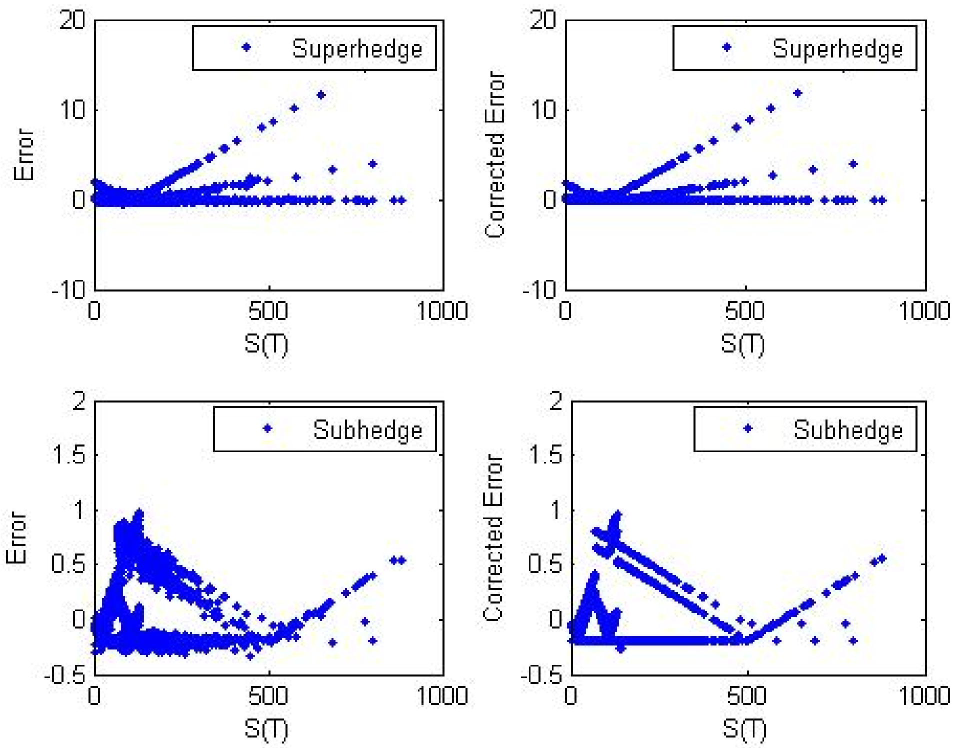

Figure 5.3: Terminal stock price and (corrected) hedging errors for super/sub-hedge in double touch option with barriers at 130 and 70 under Heston Model with parameters given in (5.1)and interest rate $r = 0$. It was allowed for the trader to rebalance position on daily basis.

After that we want to explore further the comparison of delta hedging and robust hedging. For this purpose, we still choose barrier pairs $(\underline{b}, \overline{b}) = (90, 115)$ and $(\underline{b}, \overline{b}) = (70, 130)$ and compute the empirical cumulative distribution function of errors from delta hedge and super/sub hedge. The results are shown in Figure 5.4, Figure 5.5 and Figure 5.6. Figure 5.4 and Figure 5.5 present the cumulative distribution of (corrected) hedging errors for barriers 115 and 90 for different rebalancing intervals. It is interesting to note that the robust hedges eliminates some extremely rare larger losses at the cost of frequent small losses.

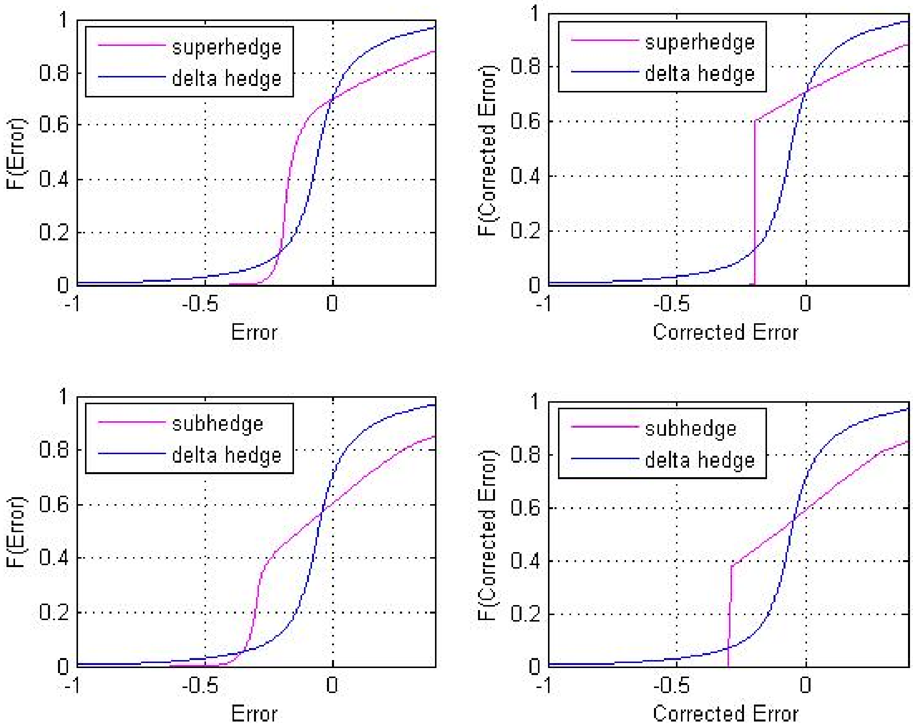

Figure 5.4: Cumulative distribution of (corrected) hedging errors via delta hedges and robust hedges in digital double touch option with barriers at 115 and 90 under Heston Model with parameters given in (5.1) and interest rate $r = 0$. It was allowed for the trader to rebalance his position on daily basis.

Finally we compute the expected utility of hedges and here we choose exponential utility function; that is, the utility function is $u(x) = 1 - exp(-x)$. The idea is following: for risk neutral trader, delta hedging generally is more attractive, because it has lower variance compared with other hedging strategies. However if the trader is risk averse, he might be very afraid of large loss and think that our robust hedging is more appealing as the corrected error of our hedging strategies is bounded below. To measure the preference of delta hedges and robust hedges for the trader of risk-aversion type, we use exponential utility function as judging criteria. We give the results in the following Table

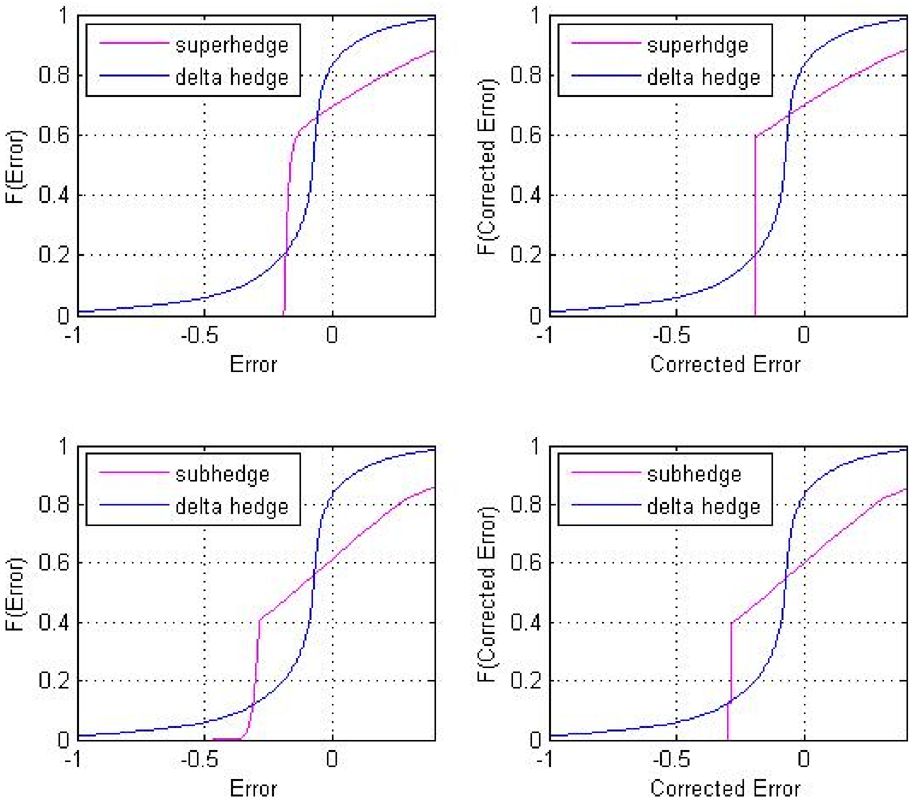

Figure 5.5: Cumulative distribution of (corrected) hedging errors via delta hedges and robust hedges in digital double touch option with barriers at 115 and 90 under Heston Model with parameters given in (5.1)and interest rate $r = 0$. It was allowed for the trader to monitor barrier crossing and rebalance his positions every time step.

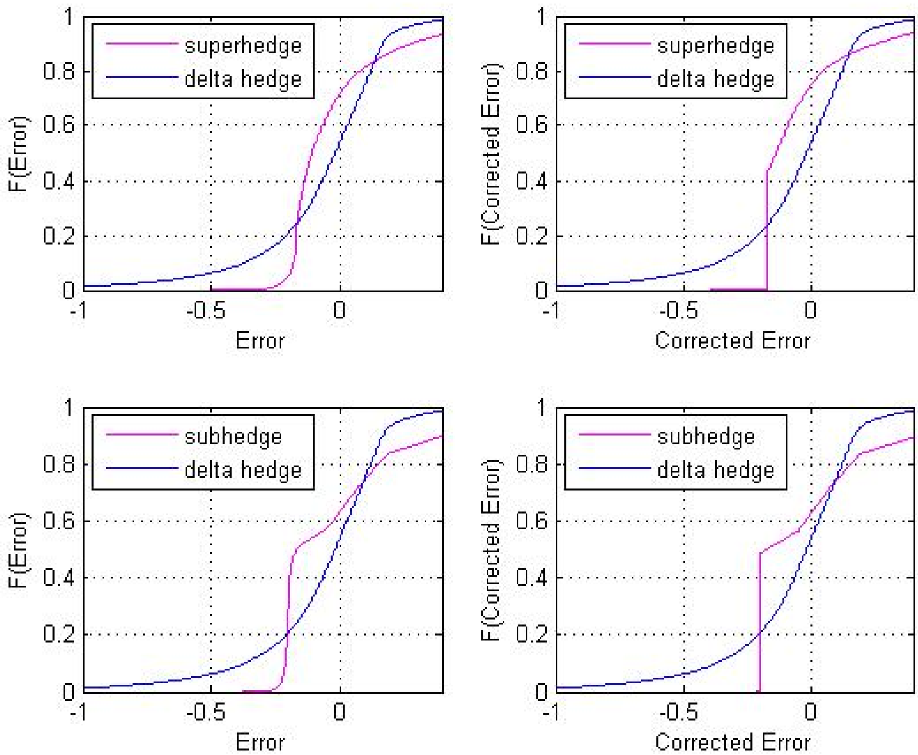

Figure 5.6: Cumulative distribution of (corrected) hedging errors via delta hedges and robust hedges in digital double touch option with barriers at 130 and 70 under Heston Model with parameters given in (5.1) and interest rate $r = 0$. It was allowed for the trader to rebalance his position on daily basis.

5.1 and Table 5.2. From those tables, we conclude that robust hedges outperform delta hedge in most of cases whether the rebalancing time interval is every time step or daily. Compared the expected utility of delta hedging errors for different rebalancing intervals, highly frequent rebalancing for delta hedge results in a significant amount of transaction costs. The trader should trade off between hedging accuracy and high transaction costs from frequent rebalancing. In our numerical example, daily delta hedge strategy is better than doing it every time step. But even the trader does delta hedges daily, robust hedging strategies are still superior than delta hedging, except for barriers are very close to each other. Generally speaking, our superhedge performs better than subhedge especially for cases in which two barriers are both very close to the initial underlying price. Our results are similar to numerical results in [1](using the same parameters for Heston Model), and both of them show that robust hedging is a better choice for trader with exponential utility. But our delta hedging does not perform as well as delta hedging in [1]. It might be because that in [1] vega hedging is combined to use which might improve the performance of delta hedging. We can draw conclusion that our static hedging strategies are competitive with delta hedging where market conditions are quite different from Black-Scholes world. If market mispecification deepens, we believe that our robust hedging strategy might be much more promising compared with standard delta hedging.

$r = 0$

$(\underline{b}, \overline{b})$	superhedge	subhedge	delta hedge
$(70, 130)$	-0.0235025	-0.0381601	-0.260627
$(80, 120)$	-0.0756644	-0.0533421	-0.242286
$(83, 117)$	-0.053446	-0.0591509	-0.219323
$(85, 115)$	-0.0468362	-0.0644652	-0.199717
$(90, 110)$	-0.0495323	-0.084778	-0.140946
$(95, 105)$	-0.0374856	-0.105639	-0.0876634
$(97, 103)$	-0.0265466	-0.120855	-0.0677742
$(95, 120)$	-0.03524	-0.0781962	-0.147172
$(80, 105)$	-0.0411524	-0.0783977	-0.224251
$(75, 110)$	-0.0668755	-0.0585624	-0.281872

Table 5.1: Comparison of exponential utilities of hedging errors in a digital double touch barrier option under Heston Model with 5.1 resulting from delta-hedge and our model-free super/sub-hedge. In this case it was allowed for the trader to monitor barriers every time step.

5.2 Case with non-zero interest rate

We choose the same parameters in Heston model as before, except for the interest rate. In this section we fix interest rate $r = 0.05$. Similarly in the previous section,

$r = 0$

$(\underline{b}, \overline{b})$	superhedge	subhedge	delta hedge
$(70, 130)$	-0.0448702	-0.0322499	-0.125559
$(80, 120)$	-0.0514704	-0.0544851	-0.14098
$(83, 117)$	-0.051174	-0.0577676	-0.113254
$(85, 115)$	-0.0592804	-0.0593399	-0.107374
$(90, 110)$	-0.0417653	-0.0826144	-0.103364
$(95, 105)$	-0.0358415	-0.110279	-0.0495612
$(97, 103)$	-0.0322341	-0.130506	-0.0393983
$(95, 120)$	-0.034395	-0.0780222	-0.0783498
$(80, 105)$	-0.0739575	-0.0761485	-0.113524
$(75, 110)$	-0.0600953	-0.0600324	-0.137082

Table 5.2: Comparison of exponential utilities of hedging errors in a double touch options under Heston Model with 5.1 resulting from delta hedging and our model-free super/sub-hedge. In this case it was allowed for the trader to rebalance his positions on daily basis.

firstly we investigate the relationship between the terminal stock price and the hedging errors resulted from hybrid sup/sub-hedging strategies. We still use barrier pairs $(\underline{b}, \overline{b}) = (90, 115)$ and $(\underline{b}, \overline{b}) = (70, 130)$. We plot robust hedging error against terminal stock price in Figure 5.7 and Figure 5.8. We notice that these two figures are quite different from the corresponding figures in the previous section. The left graphs and the right graphs in Figure 5.7 and Figure 5.8 are quite similar even for rebalancing interval is every day. It still makes sense, for the delta hedging X^2 produces hedging errors which blur pieceswise linearity of the corrected hedging error and S_T; however there is some pattern of the previous graphs left here. It also can explain that we observe that smooth cumulative distribution of corrected hedging errors seen in Figure 5.9.

Finally we compute the expected utility of hedging errors while varying upper and lower barriers and show numerical results in Table 5.3 and Table 5.4. Following the same reason as the previous section, we still choose exponential utility here. We present the results for only four typical barrier pairs. For delta hedges it is quite similar as zero interest case. The trader prefers to do delta hedges daily rather than every time step since high transaction costs. Besides It is presented that generally hybrid hedges consistently outperforms delta hedge for both different rebalancing intervals and delta hedges become more attractive when two barriers are very close. But even on such condition our hybrid hedges are still competitive with delta hedges especially subhedges. It is similar to the findings of zero interest rate case. Therefore we are convinced that at least in some circumstances our hybrid hedges are applicable and more attractive to replicate digital double touch barrier option for risk averse traders.

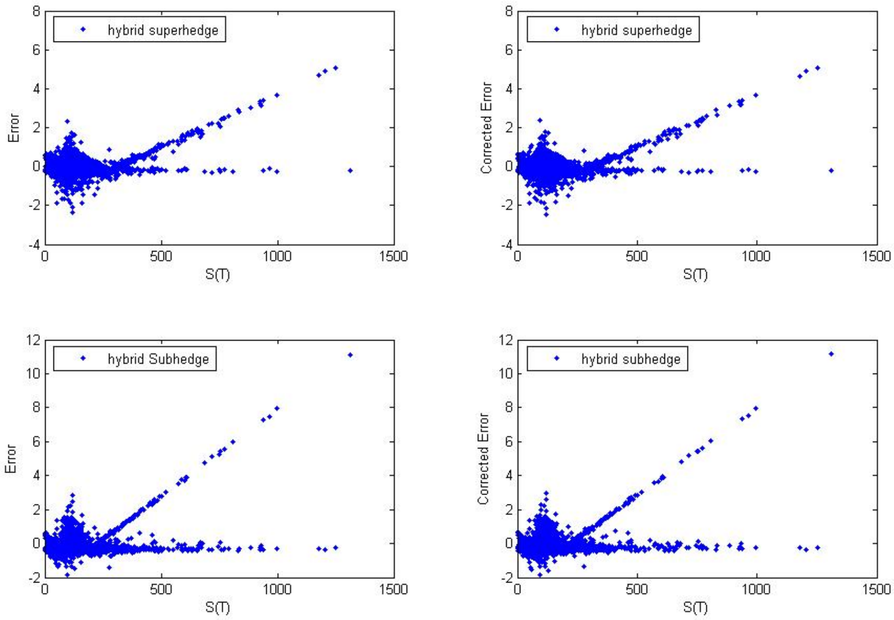

Figure 5.7: Terminal Stock price and (corrected) hedging errors for super/sub-hedge in double touch option with barriers at 115 and 90 under Heston Model with parameters given in (5.1)and interest rate $r = 0.05$. It was allowed for the trader to rebalance position on daily basis.

$r = 0.05$

$(\underline{b}, \overline{b})$	superhedge	subhedge	delta hedge
$(70, 130)$	-0.050444	-0.0605462	-0.132169
$(90, 110)$	-0.111615	-0.0769362	-0.148706
$(95, 115)$	-0.112024	-0.111147	-0.123283
$(85, 105)$	-0.118144	-0.0459819	-0.172881

Table 5.3: Comparison of exponential utilities of hedging errors in a double touch options under Heston Model with parameters given in (5.1) resulting from delta hedging and our model-free super/sub-hedge. In this case it was allowed for the trader to monitor barrier crossings and do their delta hedging every time step.

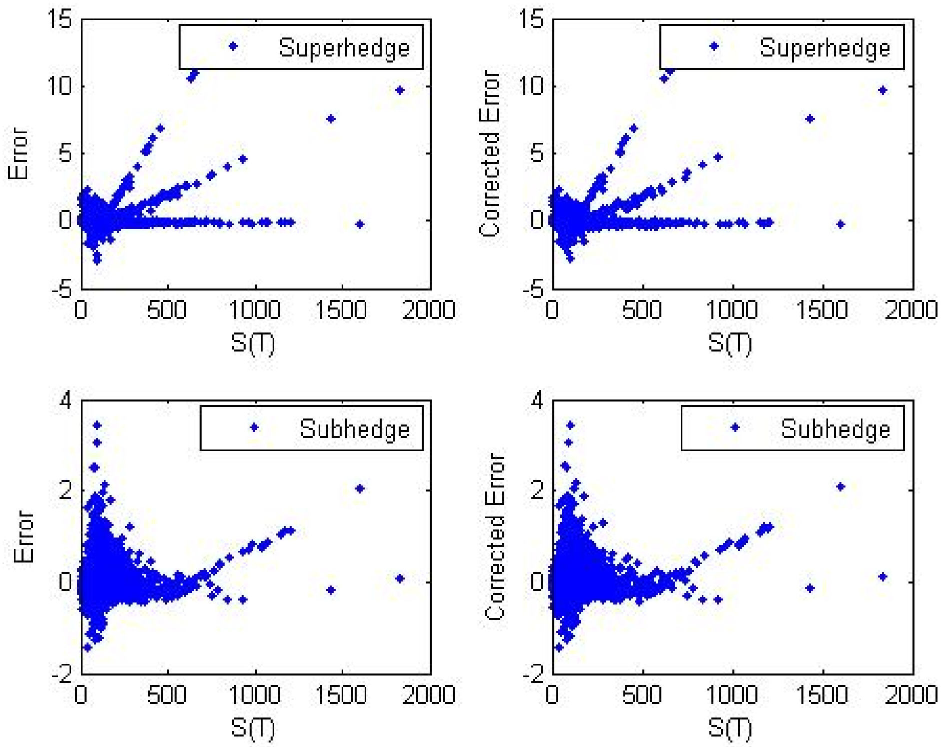

Figure 5.8: Terminal Stock price and (corrected) hedging errors for super/sub-hedge in double touch option with barriers at 130 and 70 under Heston Model with parameters given in (5.1) and interest rate $r = 0.05$. It was allowed for the trader to rebalance position on daily basis.

$r = 0.05$

$(\underline{b}, \overline{b})$	superhedge	subhedge	delta hedge
$(70, 130)$	-0.050444	-0.0605462	-0.132169
$(90, 110)$	-0.0948161	-0.072407	-0.0729002
$(95, 115)$	-0.099834	-0.108678	-0.0639093
$(85, 105)$	-0.110623	-0.0427418	-0.0894208

Table 5.4: Comparison of exponential utilities of hedging errors in a double touch options under Heston Model with (5.1) resulting from delta hedging and our model-free super/sub-hedge. In this case it was allowed for the trader to rebalance his positions on daily basis.

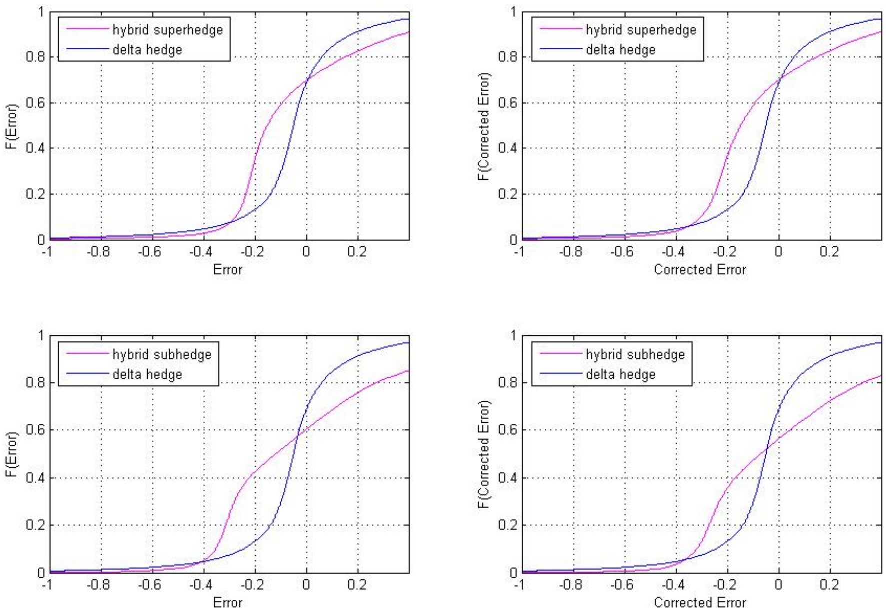

Figure 5.9: Cumulative distribution of (corrected) hedging errors via delta hedges and robust hedges in digital double touch option with barriers at 115 and 90 under Heston Model with parameters given in (5.1) and interest rate $r = 0.05$. It was allowed for the trader to rebalance his position on daily basis.

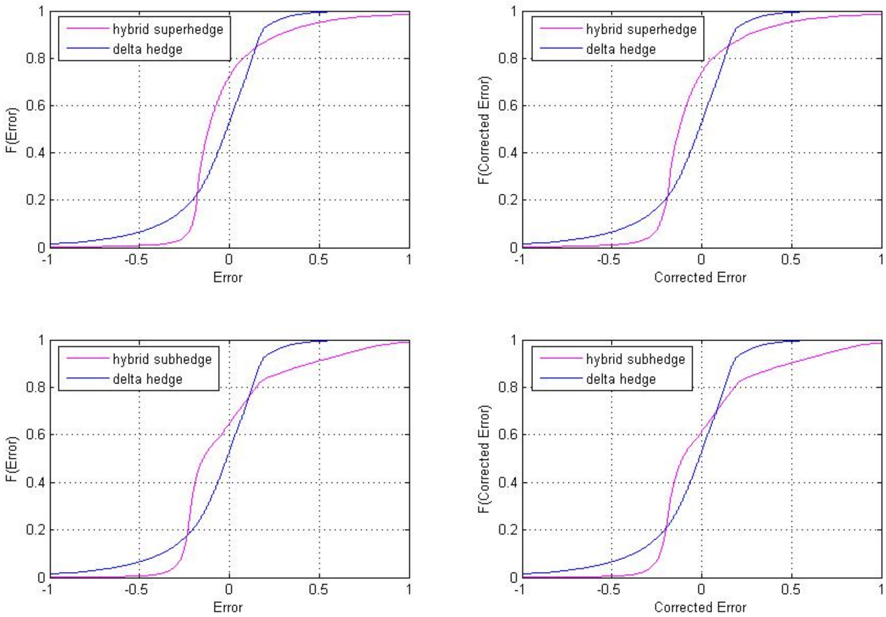

Figure 5.10: Cumulative distribution of (corrected) hedging errors via delta hedges and robust hedges in digital double touch option with barriers at 130 and 70 under Heston Model with parameters given in (5.1) and interest rate $r = 0.05$. It was allowed for the trader to rebalance position every timestep.

6 Conclusions and Final Remarks

In this thesis we study robust replicating strategy for digital double touch barrier option. We extend this approach for the case with zero interest rate (Cox & Obłój [1]) to the case with non-zero interest rate by combining model independent and model specific strategies to devise a hybrid type of hedge. We realize both delta hedge and model-free hedge by C++ and carefully investigate numerical performance of (hybrid) super/subhedge and delta hedge under Heston model. For Heston Model with zero or non-zeror interest rates, our (hybrid) super/sub-hedging is indeed an improvement on the classical hedges. Generally speaking, for high frequency of rebalancing, our hedging strategies outperform delta hedge in terms of expected utility. Besides the gap between two barriers becomes smaller, delta hedge performs better and better. For zero interest rate case we observe that superhedge seems to be more stable than subhedges, but for non-zero interest rate case subhedge turns out to work better to replicate digital barrier option. It needs to be further investigated.

Besides it might be interesting to discuss which strategy is preferred for other types of traders with different risk aversion profile. It is a natural extension for our numerical discussion. When designing a numerical experiment for comparing robust hedging and delta hedging, we might consider combining our model risk with jumps in the underlying price process and have a careful investigation of their performance.

Furthermore there are a large number of literatures (e.g. [6]) about improvement over standard delta hedging strategy which we have implemented, for example by doing vega hedging, choosing random moments of rebalancing (depending on gamma) and so on. These techniques also might contribute to enhance the performance of both delta-hedging and our hybrid strategies for non-zero interest rate cases. It is left for future research due to time constraints.

APPENDIX

The C++ codes for Black-Scholes Model and Heston Model with both zero interest rate or non-zero interest rate are not provided in this appendix, because it is too long; however it is recorded in CD-R. I submitted this thesis with this CD-R. If the reader is interested in our programming, they can check it or run it to do their own numerical test by their own. Codes for Heston Model with non-zero interest rate also can be applied into model with zero interest rate. But considering the efficiency of programming, we also give simplified codes specially for zero interest rate model.

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