# A Sieve Problem over the Gaussian Integers

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# Contents

This thesis is dedicated to

my parents

for their love, support and sacrifices

# Abstract

Our main result is that there are infinitely many primes of the form $a^2 + b^2$ such that $a^2 + 4b^2$ has at most 5 prime factors.

We prove this by first developing the theory of $L$-functions for Gaussian primes by using standard methods. We then give an exposition of the Siegel–Walfisz Theorem for Gaussian primes and a corresponding Prime Number Theorem for Gaussian Arithmetic Progressions.

Finally, we prove the main result by using the developed theory together with Sieve Theory and specifically a weighted linear sieve result to bound the number of prime factors of $a^2+4b^2$. For the application of the sieve, we need to derive a specific version of the Bombieri–Vinogradov Theorem for Gaussian primes which, in turn, requires a suitable version of the Large Sieve. We are also able to get the number of prime factors of $a^2+4b^2$ as low as 3 if we assume the Generalised Riemann Hypothesis.

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# Chapter 1

# Introduction

## 1.1 Historical Background

### 1.1.1 $L$-functions

Many people (see [Dav80]) think that Analytic Number Theory began with Dirichlet's 1837 memoir [Dir37] on the existence of primes in arithmetic progressions. In this memoir Dirichlet proved his theorem that there are infinitely many primes in arithmetic progressions where the first term is coprime to the difference of the terms. To prove this, he introduced $L$-functions in the form

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $\chi$ is a Dirichlet character, and exploited their analytic properties. All of these ideas have since become fundamental in Analytic Number Theory. Dirichlet's work was followed by work of Riemann, who realised better than anyone before him that the $\zeta$-function in the whole complex plane and in particular its zeroes hold the "secret of primes". Riemann published his results in his sole but outstanding paper on number theory [Rie59] in 1859. In particular, Riemann's methods were used by Hadamard [Had96] and de la Vallée Poussin [dlVP96] to independently prove the Prime Number Theorem in 1896. The Prime Number Theorem, originally formulated by Legendre in 1796, states that the number $\pi(x)$ of primes up to a given number $x$ is approximated by the function $\frac{x}{\ln(x)}$. In the two papers [Hec18] and [Hec20] in the early 1920s, Erich Hecke extended the notion of characters to an arbitrary number field and called these *Grössencharacters*, which also gave rise to new $L$-functions over these fields. Grössencharacters could be of infinite order, although we will only deal with finite characters in this thesis. In 1936, Chevalley [Che36] introduced ideles and the idele group of an algebraic number field and reinterpreted Hecke's Grössencharacters as characters of the idele class groups. In 1945 Artin and Whaples [AW45] defined the adele ring of an algebraic number field and finally in 1950 John Tate in his thesis [Tat67], and independently Iwasawa (though the latter probably not as far reaching), showed how to use harmonic analysis of adele groups to prove Hecke's theorems about $L$-functions. We shall not be concerned with the advanced means used for this, even though they can be regarded as a more natural approach. It is however our aim in the second chapter, to use straightforward standard arguments to prove the standard $L$-function theorems for the very specific Gaussian field.

### 1.1.2 Sieve Theory

Having developed the standard $L$-function results, we will try to apply an advanced sieve using these theorems to prove our result. Sieve Theory goes back to Erastosthenes, in about 300 BC, who seems to be the first person to develop an algorithm for tabulating primes. After that it took a long time, until Legendre, as mentioned above, took up the subject and formulated the Prime Number Theorem in 1796. This can be seen as the first use of the inclusion-exclusion principle. However, the first one to develop an effective sieve method, which was significantly more advanced than the sieve of Erastosthenes, was Viggo Brun [Bru15] in 1915, and Brun's sieve is still a very powerful method today. Despite this, Brun's sieve was first seen as complicated and even Brun himself seemed to discourage further study. Hardy, when using Brun's sieve in one of his collaborations with Littlewood, in which they derived an early version of the Brun–Titchmarsh inequality, said that the sieve did not seem "…sufficiently powerful or sufficiently profound to lead to a solution [of Goldbach's Problem]". Landau later acknowledged, when he finally read Brun's work, that it had not received the attention it deserved. In 1947 Selberg [Sel47] published an account of his upper bound sieve, which gave sieve theory a new impetus, and it seemed to consist of more elegant methods than Brun's original work. Selberg also indicated how his upper-bound sieve could be used to construct lower-bound sieves, however he never published the details. These were first supplied in the late 1950s by Vinogradov, Levin and Barban, but it was not until Ankeny and Onishi in 1964 [AO65], as well as Jurkat, Richert and Buchstab in 1965 that the power of this was fully exploited. We shall apply this advanced weighted sieve due to Richert, which was based on the results by Ankeny and Onishi. It turns out that this gives us a very strong result in Chapter 3 if we assume the Generalised Riemann Hypothesis. The final chapter will then be devoted to removing this assumption. A version of the Bombieri–Vinogradov Theorem will serve as the unconditional substitute for the Riemann Hypothesis to bound the error term in the Prime Number Theorem for Gaussian Arithmetic progressions.

### 1.1.3 The Bombieri–Vinogradov Theorem

The Bombieri–Vinogradov Theorem is a theorem of the form

$$\sum_{q\leqslant Q} \max_{(a,q)=1} \left| \pi(x;q,a) - \frac{\pi(x)}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}$$

for $A > 0$ and $Q = \frac{\sqrt{x}}{(\log x)^B}$, where $B$ only depends on $A$. Thus it gives a bound on the mean error in the Prime Number Theorem for arithmetic progressions. It is often used as an unconditional subsitute for the Generalised Riemann Hypothesis in many important arguments.

It was Barban in 1961 who proved the first result of this kind in [Bar61]. The standard version of the Bombieri–Vinogradov Theorem for rational integers given above is a refinement of Barban's method. The theorem is named after Enrico Bombieri [Bom65] and A. I. Vinogradov [Vin65], the latter of which published the result as part of a paper on the density hypothesis, in 1965, although some people believe that Barban's name is often unjustly forgotten. The result is a major application of the large sieve method, which was developed by Yuri Linnik [Lin41] in 1941.

The Bombieri–Vinogradov Theorem has many applications in Analytic Number Theory. It is often applied to sieve theory as an unconditional replacement for the Generalized Riemann Hypothesis. A famous applications is that every large even integers can be written in the form $p+P_3$. This result has since been superseded by

Chen's Theorem [Che73] which improved it to $p + P_2$ for which he used a related but different mean value theorem. Chen also used almost identical methods to prove (in the same paper) the best approximation to Goldbach's Problem to date, namely that every sufficiently large even integer is the sum of a prime and a product of at most two primes.

Early proofs of the Bombieri–Vinogradov Theorem needed the fact that very few $L$-functions have zeroes close to 1. Later the Siegel–Walfisz theorem was used to improve such results by limiting the region where such "Siegel zeroes" can occur. It should however be noted that results relying on the Siegel–Walfisz theorem are ineffective. In all of these proofs $Q$ can usually be little less than $\sqrt{x}$ and this barrier is of great importance. For our purpose, we need a theorem of this type that takes a slightly complicated form, in the sense that we need the theorem to apply to Gaussian integers, but we will only count rational integer moduli for our application and we have a larger factor inside the sum which we need to account for. These facts make it impossible to use earlier proofs of the Bombieri–Vinogradov Theorem for arbitrary number fields, as given in [Wil69], [Hin88] and in particular Huxley's version in [Hux71].

### 1.1.4 Hypothesis H

In 1958 Andrzej Schinzel and Wacław Sierpiński published a paper [SS58] which presented Schinzel's Hypothesis H which broadly says that for a set of irreducible polynomials in one variable with integer coefficients that do not have a common fixed prime divisor there exist infinitely many values for which all of the polynomials will represent prime numbers simultaneously. For example, this hypothesis covers the twin prime conjecture as a special case. Whilst Schinzel's hypothesis H is out of reach today, one can change or simplify the problem a bit to make it more accessible. The most famous such result is probably Chen's proof [Che73] where he shows that for any positive even integer $h$, there are infinitely many primes $p$ such that $p + h$ is either prime or a product of two primes.

Hypothesis H is what motivates our problem, which in a sense is the simplest possible question one can ask about polynomials in two variables. We show that there are infinitely many integers $a, b$ for which the quadratic form $a^2 + b^2$ is prime and $a^2 + 4b^2$ is almost prime taking $P_3$ or $P_5$ values infinitely often, conditional and unconditional on the Generalised Riemann Hypothesis respectively. By a $P_n$ value we mean a value which has at most $n$ prime factors.

Related work has been done by Gihan Marasingha [Mar06] in which he shows that the product of a pair of quadratic forms satisfying some reasonable conditions takes $P_5$-values infinitely often. The difference is that we are insisting on one of the quadratic forms to be prime, but we are only covering one special case of quadratic forms in this thesis, although we believe certain generalisations should be possible without much further work, though probably not to arbitrary quadratic forms as mentioned below. Marasingha's work uses a similar sieve approach, although his sieve has dimension 2 as opposed to our dimension 1 (linear) sieve, but relies on a level of distribution formula that he proves to estimate the error terms $\sum_d |R_d|$ instead of $L$-function theory over number fields and a custom Bombieri–Vinogradov theorem that we will prove and use in this thesis. Our approach also has a natural way of introducing the Generalised Riemann Hypothesis as a condition to get a stronger result.

### 1.1.5 Possible future work

We only had time to analyse this most basic case, though we believe that certain generalisations will be possible without too much extra work. For example one could replace the second form by $a^2 + B^2b^2$ for a constant integer $B$ and most likely get the same results without needing much modification. Replacing the second form by $a^2 + Bb^2$ for a non-square constant B should also be possible to some extent, though the set of primes one has to sift by will change, as $-B$ will need to be a quadratic residue modulo these primes. One can also think about modifying the first form, which we require to be prime to a more general form $a^2 + Db^2$ for a positive constant $D$, and we believe that generalisation to that form will not be too difficult if $\mathbb{Q}(\sqrt{-D})$ has class number 1. Other forms however would require to work over more complicated fields and in particular one would need to work with ideals instead of algebraic integers and it is not clear to us if our argumentation would still work, in particular whether we can still use a rational integer sieve result or whether one would need more sophisticated techniques over general number fields.

One could also think about improving the results given in this paper for example by improving the Bombieri–Vinogradov theorem that we prove. This could potentially be possible by summing non-trivially over the special character sums that we get. However this does not seem to be an easy task and at the moment we do not know how we would approach such a problem.

## 1.2 Overview

The thesis consists of four chapters: an introduction and the three chapters where all results are derived. The second chapter derives standard theorems about $L$-functions over the Gaussian field for finite Grössencharacters of the form $\chi(\mathrm{i}) = 1$. We do this by using standard methods. Chapter 3 then sets up the sieve and proves the following strong form of the result, which is conditional on the Generalised Riemann Hypothesis.

**Theorem 1.2.1** (Main Result 1)**.** *Assuming the Generalised Riemann Hypothesis, there are infinitely many primes of the form $a^2+b^2$ such that $a^2+4b^2$ has at most 3 prime factors.*

Finally, Chapter 4 removes this condition by proving an appropriate form of the Bombieri–Vinogradov Theorem, which, in turn, requires an appropriate form of the Large Sieve. We conclude that chapter with our main result:

**Theorem 1.2.2** (Main Result 2)**.** *Unconditionally, there exist infinitely many primes of the form $a^2 + b^2$ such that $a^2 + 4b^2$ has at most 5 prime factors.*

## 1.3 Notation

- We will write $s = \sigma + \mathrm{i}t$, with $\sigma, t \in \mathbb{R}$ whenever we analyse our functions at a point $s$.
- $\rho = \beta + \mathrm{i}\gamma$ will be the non-trivial zeroes of our function.
- $\pi \in \mathbb{Z}[\mathrm{i}]$ are meant to be Gaussian primes, whereas $p \in \mathbb{Z}$ are rational primes.
- $\mathrm{N}(\alpha)$ is the norm of an algebraic number being the product of all conjugates of $\alpha$.
- $\mathrm{Tr}(\alpha)$ is the trace of an algebraic number that is the sum of all conjugates of $\alpha$.
- $\chi_d$ is a character with modulus $d$.
- $\chi_0$ is the principal character.
- $\chi_1$ is a primitive character.
- Usually $L(s,\chi), \zeta(s), \psi(x), \Lambda(n), \ldots$ are meant to be the corresponding functions over the rationals, whereas $L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$, $\zeta_{\mathbb{Q}[\mathrm{i}]}(s)$, $\psi_{\mathbb{Q}[\mathrm{i}]}(x,\chi)$, $\Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha)$,... etc are over the Gaussian field.
- $\sum_{\alpha}^{\sim}, \prod_{\alpha}^{\sim}$ are the sum and product over all non-associate alpha.
- $\sum_{d}^{\#}$ is the sum over all squarefree $d$.
- $\sum_{d}^{\dagger}$ is the sum over all squarefree $d$ such that if $p \mid d$ then $p \equiv 1 \pmod 4$.
- $\sum_{d}^{\ddagger}$ is the sum over all squarefree $d$ such that if $p \mid d$ then $p \not\equiv 3 \pmod 4$.
- $\sum_{\chi}^{(0)}$ is the sum over all $\chi$ with $\chi(\mathrm{i}) = 1$.
- $\sum_{\chi}^{*}$ is the sum over all primitive $\chi$.

- $\sum_{\rho}'$ is the sum over all non-trivial zeroes $\rho$ which are not exceptional (ie not Siegel zeroes).
- $P_i$ is a rational integer with at most $i$ prime factors.
- $\Phi(d)$ is the Euler $\phi$-function for the Gaussian Integers (so that $\Phi(d)$ is the order of $(\mathbb{Z}[\mathrm{i}]/d\mathbb{Z}[\mathrm{i}])^{\times}$).
- $\phi(d)$ is the Euler $\phi$-function for rational integers.
- $\mathrm{d}(n)$ is the divisor function.
- $\nu(n)$ denotes the number of distinct rational prime factors of $n$.
- $\log(x)$ is the natural logarithm of $x$.
- $\log_n(x)$ is the logarithm of $x$ to the basis $n$.
- $\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$ is the logarithmic integral of $x$.
- $g(x) = \mathrm{O}(f(x))$ means that there exists a constant $C$ and a real number $x_0$ such that for all $x > x_0$ we have $|g(x)| \leqslant C|f(x)|$.
- $g(x) \ll f(x)$ is Vinogradov's notation and means $g(x) = \mathrm{O}(f(x))$.
- $g(x) \ll_n f(x)$ means that the implied constant depends on $n$.
- $\#\mathscr{A}$ is the number of elements in the set $\mathscr{A}$.
- $\mathrm{r}(n)$ is the number of representations of $n$ as a sum of two squares.

# Chapter 2

# Hecke Characters and $L$-functions

## 2.1 Introduction

In this chapter we investigate the Hecke $L$-functions over the Gaussian primes. We prove most standard results by using standard methods. Some things are similar to what happens in the rational field, others are different. Many differences arise because we need to estimate the number of integer points less than a given norm, which is easy for rational integers but is hard for the Gaussians. We avoid using estimates that have been proved for Gauss' Circle Problem. After establishing the Prime Number Theorem for Gaussian Primes, we proceed to proving an analog of the Siegel–Walfisz Theorem, which we will need for the results in the following chapters. Most of the facts in this chapter have been proven for general number fields by Erich Hecke in [Hec18] and [Hec20], as well as later generalised in John Tate's celebrated thesis [Tat67]. Further results about zero-free regions and the general Siegel–Walfisz Theorem for general number fields can be found in [Col90] and [Gol70]. Our aim, however, is to use elementary methods and concentrate on the Gaussian field.

## 2.2 The Dedekind $\zeta_{\mathbb{Q}[\mathrm{i}]}$-function

Let $s = \sigma + \mathrm{i}t$. We define the Dedekind Zeta Function over $\mathbb{Q}[\mathrm{i}]$ to be

$$\zeta_{\mathbb{Q}[\mathrm{i}]}(s) = \frac{1}{4} \sum_{\alpha \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \frac{1}{\mathrm{N}(\alpha)^s}.$$

First thing to note is that $\zeta_{\mathbb{Q}[i]}(s) \neq 0$ for $\sigma > 1$. As in the rational case, since $\mathbb{Z}[i]$ is a unique factorization domain, we have the Euler product (for $\sigma > 1$)

$$\zeta_{\mathbb{Q}[i]}(s) = \prod_{\pi \text{prime}}^{\sim} \frac{1}{1 - \frac{1}{\mathrm{N}(\pi)^s}},$$

where by $\prod^{\sim}$ we mean that the primes $\pi$ range only over non-associate primes (for example only those in the first quadrant of the complex plane).

From here, since we know what Gaussian primes are, we can easily deduce the following factorisation of $\zeta_{\mathbb{Q}[i]}(s)$:

$$\begin{aligned}
\zeta_{\mathbb{Q}[i]}(s) &= \prod_{\pi \text{ prime}}^{\sim} \frac{1}{1 - \frac{1}{\mathrm{N}(\pi)^s}} \\
&= \frac{1}{1 - \frac{1}{2^s}} \times \left( \prod_{p \equiv 1 \pmod 4} \frac{1}{1 - \frac{1}{p^s}} \right)^2 \times \prod_{p \equiv 3 \pmod 4} \frac{1}{1 - \frac{1}{p^{2s}}} \\
&= \frac{1}{1 - \frac{1}{2^s}} \times \left( \prod_{p \equiv 1 \pmod 4} \frac{1}{1 - \frac{1}{p^s}} \right)^2 \times \prod_{p \equiv 3 \pmod 4} \frac{1}{1 - \frac{1}{p^s}} \cdot \frac{1}{1 + \frac{1}{p^s}} \\
&= \zeta(s) \times \prod_{p \equiv 1 \pmod 4} \frac{1}{1 - \frac{1}{p^s}} \times \prod_{p \equiv 3 \pmod 4} \frac{1}{1 + \frac{1}{p^s}} \\
&= \zeta(s) \times \prod_{p \text{ prime}} \frac{1}{1 - \frac{\chi_4(p)}{p^s}} = \zeta(s) L(s, \chi_4),
\end{aligned}$$

where $\chi_4$ is the non-trivial Dirichlet character $\pmod 4$ over the ordinary integers.

In order to prove that the definition of our $\zeta_{\mathbb{Q}[i]}(s)$ is non-zero (in the real part region $\sigma > 1$), it is sufficient to show that its logarithm is finite:

$$|\log \zeta_{\mathbb{Q}[i]}(s)| = \left| \frac{1}{4} \sum_{\pi \text{ prime}} -\log\left(1 - \frac{1}{\mathrm{N}(\pi)^s}\right) \right|.$$

Now, $\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$, and hence

$$\begin{aligned}
|\log \zeta_{\mathbb{Q}[i]}(s)| &= \left| \frac{1}{4} \sum_{\pi \text{ prime}} \sum_{k=1}^{\infty} \frac{1}{k\mathrm{N}(\pi)^{sk}} \right| \leqslant \frac{1}{4} \sum_{\pi \text{ prime}} \sum_{k=1}^{\infty} \left| \frac{1}{\mathrm{N}(\pi)^{sk}} \right| \\
&= \frac{1}{4} \sum_{\pi \text{ prime}} \left| \frac{1}{1 - \frac{1}{|\mathrm{N}(\pi)^s|}} \cdot \frac{1}{\mathrm{N}(\pi)^s} \right| = \frac{1}{4} \sum_{\pi \text{ prime}} \left| \frac{1}{|\mathrm{N}(\pi)^s| - 1} \right| \\
&\leqslant \frac{1}{2} \sum_{n=2}^{\infty} \left| \frac{1}{|n^s| - 1} \right| = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n^\sigma - 1} < \infty.
\end{aligned}$$

The last inequalities hold because there are at most two non-associate primes of a given norm, and $\sigma > 1$.

## 2.3 Characters over $\mathbb{Q}$[i]

For the purpose of this thesis, we shall only be concerned with "finite" periodic characters over $\mathbb{Q}$[i], even though these are not all the characters one can define, and the infinite Hecke characters ("Grössencharaktere", as they were called by Hecke) give a bigger variety of $L$-functions (see [Neu99]). For our purpose the following definition is sufficient.

**Definition 2.3.1.** Let $d$ be a non-zero Gaussian integer in $\mathbb{Z}$[i]. Then a *character* $\chi$ over $\mathbb{Q}$[i] with *conductor* $d$ is a homomorphism

$$\chi : (\mathbb{Z}[\mathrm{i}]/d)^{\times} \longrightarrow \mathbb{C}^{\times},$$

such that $\chi(\alpha) = 0$ whenever $(\alpha, d) \neq 1$. In this thesis we shall be particularly interested in characters $\chi$ for which $\chi(\mathrm{i}) = 1$ (which then makes them characters on ideals).

**Definition 2.3.2.** A character $\chi_0 = \chi_{0,d}$ is called *principal* if $\chi_0(\alpha) = 1$ for all $\alpha$ coprime to $d$.

**Definition 2.3.3.** A character $\chi \pmod{d}$ is said to be *induced* by a character $\chi_1 \pmod{d_1}$ if $d_1 \mid d$ whenever $\chi(\alpha) = \chi_1(\alpha)$ whenever $(\alpha, d_1) = 1$.

**Definition 2.3.4.** A non-principal character with the conductor $d$ is called *primitive* if it is not induced by any other character. Otherwise it is called *imprimitive.*

## 2.4 Hecke $L_{\mathbb{Q}[\mathrm{i}]}$-functions

Correspondingly, for a character $\chi$ satisfying $\chi(\mathrm{i}) = 1$, we define a Hecke (or Dirichlet) $L$-function over $\mathbb{Q}$[i] to be

$$L_{\mathbb{Q}[\mathrm{i}]}(s, \chi) = \frac{1}{4} \sum_{\alpha \in \mathbb{Z}[\mathrm{i}]} \frac{\chi(\alpha)}{\mathrm{N}(\alpha)^s},$$

where $\chi$ is a character $\pmod{d}$ over $\mathbb{Q}[\mathrm{i}]$, as defined in 2.3.1 and $\mathrm{N}(d) \neq 1$. Of course, we get an Euler product representation

$$L_{\mathbb{Q}[\mathrm{i}]}(s,\chi) = \prod_{\pi \text{ prime}}^{\sim} \frac{1}{1 - \frac{\chi(\pi)}{\mathrm{N}(\pi)^s}}.$$

Just as before, we can prove that it is non-zero for $\sigma > 1$. The proof works exactly the same way as it does for the $\zeta$-function, and so we will avoid repetition.

## 2.5 Functional Equation

We will first make a short digression and present a proof of the functional equation for the Dedekind $\zeta$-function as we will need to refer to a specific form of it later.

### 2.5.1 $\zeta_{\mathbb{Q}[\mathrm{i}]}$-function

**Theorem 2.5.1** (Functional Equation)**.** *The function $\zeta_{\mathbb{Q}[\mathrm{i}]}$ satisfies the following functional equation:*

$$\pi^{-s}\Gamma(s)\zeta_{\mathbb{Q}[\mathrm{i}]}(s) \quad = \quad \pi^{-(1-s)}\Gamma(1-s)\zeta_{\mathbb{Q}[\mathrm{i}]}(1-s).$$

*Proof.* We start off with the usual definition of the $\Gamma$-function for $\sigma > 0$:

$$\Gamma(s) \quad = \quad \int_0^\infty \mathrm{e}^{-t} t^{s-1} \mathrm{d}t.$$

Put $t = \mathrm{N}(\alpha)\pi x$ and suppose that $\sigma > 1$:

$$\begin{aligned}
\Gamma(s) &= \int_0^\infty \mathrm{e}^{-\mathrm{N}(\alpha)\pi x} x^{s-1} \mathrm{N}(\alpha)^s \pi^s \mathrm{d}x; \\
\pi^{-s}\Gamma(s)\mathrm{N}(\alpha)^{-s} &= \int_0^\infty \mathrm{e}^{-\mathrm{N}(\alpha)\pi x} x^{s-1} \mathrm{d}x; \\
4\pi^{-s}\Gamma(s)\zeta_{\mathbb{Q}[\mathrm{i}]}(s) &= \int_0^\infty x^{s-1} \left( \sum_{\alpha \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \mathrm{e}^{-\mathrm{N}(\alpha)\pi x} \right) \mathrm{d}x.
\end{aligned}$$

Note that we now include associates in the sum. Let

$$\varpi(x) = \sum_{\alpha \in \mathbb{Z}[\mathrm{i}] \setminus \{0\}} \mathrm{e}^{-\mathrm{N}(\alpha)\pi x}. \tag{2.1}$$

Then

$$\begin{aligned}\varpi(x) &= \sum_{\substack{m,n\in\mathbb{Z}\\ m,n \text{ not both } 0}} \mathrm{e}^{-(m^2+n^2)\pi x}\\ &= 4\left(\sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi x}\right)^2 + 4\left(\sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi x}\right) = 4(\omega(x)^2+\omega(x)), \qquad (2.2)\end{aligned}$$

where

$$\omega(x) := \sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi x}.$$

**Lemma 2.5.2.** *We have*

$$\omega(x) \ll \max\left(1, \frac{1}{\sqrt{x}}\right)\mathrm{e}^{-\pi x}.$$

*Proof.* We first note that

$$\begin{aligned}\left(\int_1^{\infty}\mathrm{e}^{-y^2\pi x}\mathrm{d}y\right)^2 &= \int_1^{\infty}\mathrm{e}^{-y^2\pi x}\mathrm{d}y\int_1^{\infty}\mathrm{e}^{-z^2\pi z}\mathrm{d}z\\ &= \int_1^{\infty}\int_1^{\infty}\mathrm{e}^{-(y^2+z^2)\pi x}\mathrm{d}y\mathrm{d}z\\ &\leqslant \int_0^{\frac{\pi}{2}}\int_1^{\infty} r\mathrm{e}^{-r^2\pi x}\mathrm{d}r\mathrm{d}\theta\\ &= \frac{\pi}{2}\int_1^{\infty} r\mathrm{e}^{-r^2\pi x}\mathrm{d}r\\ &= \frac{1}{4x}\mathrm{e}^{-\pi x}.\end{aligned}$$

We now apply this as follows

$$\begin{aligned}\omega(x) &= \sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi x}\\ &\leqslant \mathrm{e}^{-\pi x} + \int_1^{\infty}\mathrm{e}^{-y^2\pi x}\mathrm{d}y\\ &= \mathrm{e}^{-\pi x} + \frac{1}{2\sqrt{x}}\mathrm{e}^{-\frac{\pi x}{2}}\\ &\ll \max\left(1, \frac{1}{\sqrt{x}}\right)\mathrm{e}^{-\pi x}.\end{aligned}$$

■

Now we split the integral:

$$\begin{aligned}\pi^{-s}\Gamma(s)\zeta_{\mathbb{Q}[i]}(s) &= \int_1^\infty x^{s-1}\varpi(x)\mathrm{d}x + \int_0^1 x^{s-1}\varpi(x)\mathrm{d}x \\ &= \int_1^\infty x^{s-1}\varpi(x)\mathrm{d}x + \int_1^\infty x^{-s-1}\varpi(x^{-1})\mathrm{d}x,\end{aligned}$$

which can be done since by Lemma 2.5.2 both integrals converge. We further know that

$$\omega(x^{-1}) = -\frac{1}{2} + \frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$$

(see [Dav80]), and hence

$$\omega(x^{-1})^2 + \omega(x^{-1}) = -\frac{1}{4} + \frac{1}{4}x + x(\omega(x)^2 + \omega(x)).$$

Thus

$$\varpi(x^{-1}) = -1 + x + x\varpi(x),$$

and so we get

$$4\pi^{-s}\Gamma(s)\zeta_{\mathbb{Q}[i]}(s) = -\frac{1}{s} + \frac{1}{s-1} + \left(\int_1^\infty (x^{s-1} + x^{-s})\varpi(x)\mathrm{d}x\right). \qquad (2.3)$$

We note that this integral is regular everywhere and so we have an analytic continuation of $\zeta_{\mathbb{Q}[i]}(s)$ for $s \neq 0, 1$. As the right hand side stays unchanged if we replace $s$ by $1-s$, we get the functional equation. ■

**Lemma 2.5.3.** *The function*

$$\xi_{\mathbb{Q}[i]}(s) := s(s-1)\pi^{-s}\Gamma(s)\zeta_{\mathbb{Q}[i]}(s)$$

*is regular everywhere and hence $\zeta_{\mathbb{Q}[i]}(s)$ has an analytic continuation to the whole complex plane.*

*Proof.* By (2.3) we have that

$$4\xi_{\mathbb{Q}[i]}(s) = 1 + s(s-1)\int_1^\infty (x^{s-1} + x^{-s})\varpi(x)\mathrm{d}x.$$

Recall that $\varpi(x) = 4(\omega(x)^2 + \omega(x))$ where $\omega(x) = \mathrm{O}(\mathrm{e}^{-\pi x})$ as $x \to \infty$. Hence $\varpi(x) = \mathrm{O}(\mathrm{e}^{-\pi x})$, and so the integral is bounded for every $s$ and thus $\xi_{\mathbb{Q}[i]}$ represents an everywhere regular function. ■

Since $s\Gamma(s)$ does not have any zeroes, this lemma also proves that $\zeta_{\mathbb{Q}[i]}(s)$ can only have a pole at 1. As the integral in (2.3) is regular everywhere, the pole must in fact be simple with residue $\frac{\pi}{4}$. Further, as $\zeta_{\mathbb{Q}[i]}(s)$ has no zeroes for $\sigma > 1$ , the only zeroes for $\sigma < 0$ are (trivial) simple zeroes at $-1, -2, -3, \ldots$, where $\Gamma(s)$ has simple poles.

### 2.5.2 $L_{\mathbb{Q}[i]}$-functions

**Definition 2.5.4.** (Gauss sum) We define the *Gauss sum* over $\mathbb{Q}[i]$ to be

$$\tau(\chi, y) \ := \sum_{x \pmod d} \chi(x) e^{2\pi i \mathrm{Tr}(\frac{xy}{2d})}, \tag{2.4}$$

where by the sum we mean that $x$ ranges over all coset representatives of $\mathbb{Z}[i]/d\mathbb{Z}[i]$. We also define

$$\tau(\chi) := \tau(\chi, 1).$$

We need to check that the sum in (2.4) is indeed well-defined on $(\mathbb{Z}[i]/d\mathbb{Z}[i])$, so suppose $x_1 \equiv x_2 \pmod d$. Then $x_1 = \alpha d + x_2$ for some Gaussian integer $\alpha$. Hence

$$\mathrm{Tr}\left(\frac{x_1 y}{2d}\right) = \mathrm{Tr}\left(\frac{x_2 y}{2d} + \frac{\alpha y}{2}\right) = \mathrm{Tr}\left(\frac{x_2 y}{2d}\right) + \mathrm{Tr}\left(\frac{\alpha y}{2}\right) \equiv \mathrm{Tr}\left(\frac{x_2 y}{2d}\right) \pmod{\mathbb{Z}}.$$

Thus the Gauss sum is well-defined. The same argument also proves that $\tau(\chi, y)$ is a function on $\mathbb{Z}[i]/d\mathbb{Z}[i]$.

**Theorem 2.5.5.** *If $\chi$ is a primitive character with conductor $d$ over $\mathbb{Q}[i]$, then*

$$\tau(\chi, y\alpha) = \overline{\chi}(y)\tau(\chi, \alpha).$$

*Proof.* First suppose that $(y, d) = 1$ (by which we mean that the corresponding ideals are coprime). Then

$$\overline{\chi}(y)\tau(\chi, \alpha) = \sum_{x \pmod d} \overline{\chi}(y)\chi(x) e^{2\pi i \mathrm{Tr}(\frac{x\alpha}{2d})}.$$

Now, as $(y, d) = 1$, for each $x \in (\mathbb{Z}[i]/d\mathbb{Z}[i])^{\times}$ we can find an $h \in (\mathbb{Z}[i]/d\mathbb{Z}[i])^{\times}$ such that $yh \equiv x \pmod d$. As $x$ ranges over all residue classes modulo $d$, we get that $h$ will also range over all residue classes. Hence

$$\overline{\chi}(y)\tau(\chi, \alpha) = \sum_{h \pmod d} \overline{\chi}(y)\chi(yh) e^{2\pi i \mathrm{Tr}(\frac{yh\alpha}{2d})} = \sum_{h \pmod d} \chi(h) e^{2\pi i \mathrm{Tr}(\frac{yh\alpha}{2d})} = \tau(\chi, y\alpha).$$

If, however, $(y,d) = d_1$ then, if $\chi$ is primitive, we can choose $\beta \in (\mathbb{Z}[\mathrm{i}]/d\mathbb{Z}[\mathrm{i}])^\times$ such that

$$\beta \equiv 1 \pmod{d_2}$$

and

$$\chi(\beta) \neq 1,$$

where $d = d_1 d_2$, since otherwise $\chi$ would be induced by a character with conductor $d_2$. Then $d_1 \mid y$ and $d_2 \mid \beta - 1$. Hence $y(\beta - 1) \equiv 0 \pmod{d}$ and $y\beta \equiv y \pmod{d}$. However, $\beta$ is coprime to $d$, and so we can use the first part of our proof to get

$$\overline{\chi}(\beta)\tau(\chi, \alpha y) = \tau(\chi, \beta\alpha y) = \tau(\chi, \alpha y).$$

The second equality holds because, as mentioned before, $\tau(\chi, y)$ is a function on $\mathbb{Z}[\mathrm{i}]/d\mathbb{Z}[\mathrm{i}]$. But as $\overline{\chi}(\beta) \neq 1$, we get that

$$\tau(\chi, \alpha y) = 0.$$

Also, clearly

$$\overline{\chi}(y)\tau(\chi, y) = 0,$$

as $(y,d) \neq 1$ implies $\overline{\chi}(y) = 0$. Thus the two expressions agree in this case as well. Hence the theorem holds for all primitive characters $\chi$. ■

**Lemma 2.5.6.**

$$S(x) = \sum_{y \pmod{d}} \mathrm{e}^{2\pi \mathrm{i} \mathrm{Tr}(\frac{xy}{2d})} = \begin{cases} \mathrm{N}(d) & \text{if } x \equiv 0 \pmod{d}; \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* First, note that

$$\mathrm{e}^{2\pi \mathrm{i} \mathrm{Tr}(\frac{x}{2d})} S(x) = \sum_{y \pmod{d}} \mathrm{e}^{2\pi \mathrm{i} \mathrm{Tr}(\frac{x(y+1)}{2d})} = S(x).$$

This is because $y+1$ ranges over the all residue classes whenever $y$ does. Hence $S(x) = 0$ or $\mathrm{e}^{2\pi \mathrm{i} \mathrm{Tr}(\frac{x}{2d})} = 1$. We get the latter case if and only if $\mathrm{Tr}(\frac{x}{2d}) \in \mathbb{Z}$, which happens if and only if $d \mid \mathrm{Re}(x)$. Similarly using

$$\mathrm{e}^{2\pi \mathrm{i} \mathrm{Tr}(\frac{\mathrm{i}x}{2d})} S(x)$$

we find that $S(x) = 0$ or $d \mid \mathrm{Im}(x)$. Thus $S(x)$ vanishes unless $d \mid x$ in which case $S(x)$ is clearly equal to $\mathrm{N}(d)$. ■

**Theorem 2.5.7.** *If $\chi$ is a primitive character modulo $d$ over $\mathbb{Q}[\mathrm{i}]$, then*

$$|\tau(\chi)| = \sqrt{|\mathrm{N}(d)|}.$$

*Proof.* Here a similar proof to the rational case works. Let $y \in (\mathbb{Z}[\mathrm{i}]/d\mathbb{Z}[\mathrm{i}])^\times$. By Theorem 2.5.5 we get

$$\begin{aligned} |\chi(y)|^2|\tau(\chi)|^2 &= \tau(\chi, y)\overline{\tau(\chi, y)} \\ &= \sum_{x_1 \pmod d} \chi(x_1)\mathrm{e}^{2\pi\mathrm{i}\mathrm{Tr}(\frac{x_1 y}{2d})} \times \overline{\sum_{x_2 \pmod d} \chi(x_2)\mathrm{e}^{2\pi\mathrm{i}\mathrm{Tr}(\frac{x_2 y}{2d})}} \\ &= \sum_{x_1 \pmod d}\sum_{x_2 \pmod d} \chi(x_1)\overline{\chi}(x_2)\mathrm{e}^{2\pi\mathrm{i}\mathrm{Tr}(\frac{(x_1-x_2)y}{2d})}. \end{aligned}$$

Now consider summing both sides over all $y \in (\mathbb{Z}[\mathrm{i}]/d\mathbb{Z}[\mathrm{i}])$. By Lemma 2.5.6, all the terms with $x_1 \not\equiv x_2 \pmod d$ cancel, and the terms with $x_1 \equiv x_2 \pmod d$ sum to $\mathrm{N}(d)$. So

$$\sum_{y \pmod d} |\chi(y)|^2|\tau(\chi)|^2 = \sum_{x \pmod d} |\chi(x)|^2\mathrm{N}(d),$$

and hence

$$|\tau(\chi)|^2 = \mathrm{N}(d).$$

■

**Theorem 2.5.8** (Functional equation)**.** *Let $\chi$ be a primitive character of $\mathbb{Q}[\mathrm{i}]$ with conductor $d \in \mathbb{Z}[\mathrm{i}]$ (not a unit). Then $L_{\mathbb{Q}[\mathrm{i}]}(s, \chi)$ satisfies the following functional equation:*

$$\left(\frac{|d|}{\pi}\right)^{1-s}\Gamma(1-s)L_{\mathbb{Q}[\mathrm{i}]}(\overline{\chi}, 1-s) = \frac{|d|}{\tau(\chi)}\left(\frac{|d|}{\pi}\right)^{s}\Gamma(s)L_{\mathbb{Q}[\mathrm{i}]}(s, \chi).$$

*Proof.* We have that

$$\Gamma(s) = \int_0^\infty \mathrm{e}^{-t}t^{s-1}\mathrm{d}t.$$

Put $t = \frac{\mathrm{N}(\alpha)\pi x}{|d|}$, then

$$\begin{aligned} \Gamma(s) &= \int_0^\infty \mathrm{e}^{-\frac{\mathrm{N}(\alpha)\pi x}{|d|}} x^{s-1}\mathrm{N}(\alpha)^s\pi^s|d|^{-s}\mathrm{d}x; \\ \left(\frac{|d|}{\pi}\right)^s\Gamma(s)\frac{\chi(\alpha)}{\mathrm{N}(\alpha)^s} &= \int_0^\infty \chi(\alpha)\mathrm{e}^{-\frac{\mathrm{N}(\alpha)\pi x}{|d|}} x^{s-1}\mathrm{d}x. \end{aligned}$$

If we now assume $\sigma > 1$ and sum both sides over $\alpha \in \mathbb{Z}[\mathrm{i}]\backslash\{0\}$ we get

$$4\left(\frac{|d|}{\pi}\right)^s \Gamma(s) L_{\mathbb{Q}[\mathrm{i}]}(s,\chi) = \int_0^\infty x^{s-1}\left(\sum_{\alpha\in\mathbb{Z}[\mathrm{i}]} \chi(\alpha) \mathrm{e}^{-\frac{\mathrm{N}(\alpha)\pi x}{|d|}}\right) \mathrm{d}x. \tag{2.5}$$

Now let

$$\psi(x,\chi) = \sum_{\alpha\in\mathbb{Z}[\mathrm{i}]} \chi(\alpha) \mathrm{e}^{-\frac{\mathrm{N}(\alpha)\pi x}{|d|}},$$

and consider

$$\begin{aligned}
\tau(\overline{\chi})\psi(x,\chi) &= \sum_{\alpha\in\mathbb{Z}[\mathrm{i}]} \tau(\overline{\chi},\alpha) \mathrm{e}^{-\frac{\mathrm{N}(\alpha)\pi x}{|d|}} \qquad \text{(by Theorem 2.5.5)}\\
&= \sum_{\alpha\in\mathbb{Z}[\mathrm{i}]} \left(\sum_{\beta \pmod d} \overline{\chi}(\beta) \mathrm{e}^{2\pi \mathrm{i}\mathrm{Tr}(\frac{\alpha\beta}{2d})}\right) \mathrm{e}^{-\frac{\mathrm{N}(\alpha)\pi x}{|d|}}\\
&= \sum_{\alpha\in\mathbb{Z}[\mathrm{i}]} \sum_{\beta \pmod d} \overline{\chi}(\beta) \mathrm{e}^{-\frac{\mathrm{N}(\alpha)\pi x}{|d|} + 2\pi\mathrm{i}\mathrm{Tr}(\frac{\alpha\beta}{2d})}.
\end{aligned}$$

Putting $\alpha = p\mathrm{i} + q$ and $\frac{\beta}{d} = u\mathrm{i} + v$ we get

$$\begin{aligned}
\tau(\overline{\chi})\psi(x,\chi) &= \sum_{\alpha\in\mathbb{Z}[\mathrm{i}]} \sum_{\beta \pmod d} \overline{\chi}(\beta) \mathrm{e}^{-\frac{(p^2+q^2)\pi x}{|d|} + 2\pi\mathrm{i}(vq-up)}\\
&= \sum_{\beta \pmod d} \overline{\chi}(\beta) \left(\sum_{p=-\infty}^{\infty} \mathrm{e}^{-\frac{p^2\pi x}{|d|} - 2\pi\mathrm{i}up}\right) \times \left(\sum_{q=-\infty}^{\infty} \mathrm{e}^{-\frac{q^2\pi x}{|d|} + 2\pi\mathrm{i}vq}\right),
\end{aligned}$$

and we can handle these sums separately, as we would do in the case of the ordinary Dirichlet $L$-function:

$$\begin{aligned}
\tau(\overline{\chi})\psi(x,\chi) &= \sum_{\beta \pmod d} \overline{\chi}(\beta) \left(\left(\frac{x}{|d|}\right)^{-\frac{1}{2}} \sum_{p=-\infty}^{\infty} \mathrm{e}^{\frac{-(p-u)^2\pi|d|}{x}}\right) \times\\
&\quad \left(\left(\frac{x}{|d|}\right)^{-\frac{1}{2}} \sum_{q=-\infty}^{\infty} \mathrm{e}^{\frac{-(q+v)^2\pi|d|}{x}}\right)\\
&= \frac{|d|}{x} \sum_{\beta \pmod d} \overline{\chi}(\beta) \sum_{\alpha\in\mathbb{Z}[\mathrm{i}]} \mathrm{e}^{-\left((p-u)^2+(q+v)^2\right)\frac{\pi|d|}{x}}.
\end{aligned}$$

Note that $\alpha + \overline{\left(\frac{\beta}{d}\right)} = (p-u)\mathrm{i} + (q+v)$. Thus

$$\begin{aligned}
\tau(\overline{\chi})\psi(x,\chi) &= \frac{|d|}{x} \sum_{\beta \,(\mathrm{mod}\ d)} \overline{\chi}(\beta) \sum_{\alpha \in \mathbb{Z}[\mathrm{i}]} \mathrm{e}^{-\mathrm{N}\left(\alpha + \overline{\left(\frac{\beta}{d}\right)}\right)\frac{\pi|d|}{x}} \\
&= \frac{|d|}{x} \sum_{\beta \,(\mathrm{mod}\ d)} \overline{\chi}(\beta) \sum_{\alpha \in \mathbb{Z}[\mathrm{i}]} \mathrm{e}^{-\mathrm{N}\left[d\left(\overline{\alpha} + \left(\frac{\beta}{d}\right)\right)\right]\frac{\pi}{|d|x}} \\
&= \frac{|d|}{x} \sum_{\beta \,(\mathrm{mod}\ d)} \overline{\chi}(\beta) \sum_{\alpha \in \mathbb{Z}[\mathrm{i}]} \mathrm{e}^{-\mathrm{N}[\overline{\alpha}d + \beta]\frac{\pi}{|d|x}}.
\end{aligned}$$

Put $\gamma = \overline{\alpha}d + \beta$. Then each pair $(\alpha, \beta)$ corresponds to exactly one $\gamma \in \mathbb{Z}[\mathrm{i}]$ such that $\gamma \equiv \beta \,(\mathrm{mod}\ d)$, hence we can regroup the terms as follows:

$$\tau(\overline{\chi})\psi(x,\chi) = \frac{|d|}{x} \sum_{\gamma \in \mathbb{Z}[\mathrm{i}]} \overline{\chi}(\gamma)\mathrm{e}^{-\frac{\mathrm{N}(\gamma)\pi}{|d|x}} = \frac{|d|}{x}\psi(x^{-1}, \overline{\chi}). \tag{2.6}$$

Now that the functional equation for $\psi(x,\chi)$ has been obtained, we can deduce the functional equation for $L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$:

$$\begin{aligned}
&4\left(\frac{|d|}{\pi}\right)^{s} \Gamma(s) L_{\mathbb{Q}[\mathrm{i}]}(s,\chi) \\
&= \int_0^\infty x^{s-1}\psi(x,\chi)\mathrm{d}x \\
&= \int_1^\infty x^{s-1}\psi(x,\chi)\mathrm{d}x + \int_1^\infty x^{-s-1}\psi(x^{-1},\chi)\mathrm{d}x \\
&= \int_1^\infty x^{s-1}\psi(x,\chi)\mathrm{d}x + \frac{\tau(\chi)}{|d|}\int_1^\infty x^{-s}\psi(x,\overline{\chi})\mathrm{d}x.
\end{aligned} \tag{2.7}$$

Theorem 2.5.7 implies that $|\tau(\chi)| = \sqrt{\mathrm{N}(d)} = |d|$. Furthermore, we note that for the $L_{\mathbb{Q}[\mathrm{i}]}$-function to be non-zero, we need $\chi(\mathrm{i}) = 1$, and hence $\chi(-1) = 1$ and thus we can deduce that $\tau(\overline{\chi}, x) = \overline{\tau(\chi, x)}$. Hence $\tau(\overline{\chi}, x)\tau(\chi, x) = \mathrm{N}(d) = |d|^2$.

The integral in (2.7) is everywhere regular and so gives, as in the rational case, an analytic continuation for $L_{\mathbb{Q}[\mathrm{i}]}(\chi, s)$. Also we note that if $s$ is replaced by $(1-s)$, and $\chi$ by $\overline{\chi}$, the integrals become

$$\begin{aligned}
& 4\left(\frac{|d|}{\pi}\right)^{1-s}\Gamma(1-s)L_{\mathbb{Q}[\mathrm{i}]}(1-s,\overline{\chi}) \\
= & \int_1^\infty x^{-s}\psi(x,\overline{\chi})\mathrm{d}x + \frac{\tau(\overline{\chi})}{|d|}\int_1^\infty x^{s-1}\psi(x,\chi)\mathrm{d}x \\
= & \frac{|d|}{\tau(\chi)}\left(\frac{\tau(\chi)}{|d|}\int_1^\infty x^{-s}\psi(x,\overline{\chi})\mathrm{d}x + \int_1^\infty x^{s-1}\psi(x,\chi)\mathrm{d}x\right) \\
= & 4\frac{|d|}{\tau(\chi)}\left(\frac{|d|}{\pi}\right)^{s}\Gamma(s)L_{\mathbb{Q}[\mathrm{i}]}(s,\chi),
\end{aligned}$$

and we get our functional equation. ■

A quick analysis of this reveals that, as $L_{\mathbb{Q}[\mathrm{i}]}(s)$ has no zeroes or poles for $\sigma > 1$, the only zeroes for $\sigma < 0$ are at the poles of $\Gamma(s)$, which are at $0, -1, -2, \ldots$. We also get that the non-trivial zeroes of $L_{\mathbb{Q}[\mathrm{i}]}(s)$ are symmetric with respect to the line $\sigma = \frac{1}{2}$, and if $\chi$ is a real character, then they are also symmetric with respect to the real line (this follows from the definition of $L_{\mathbb{Q}[\mathrm{i}]}(s)$).

## 2.6 Hadamard Product

**Lemma 2.6.1.** *If $\sigma > 2$ then $|\zeta_{\mathbb{Q}[\mathrm{i}]}(s)| = \mathrm{O}(1)$ as $|s| \to \infty$.*

*Proof.* We have

$$\begin{aligned}
|\zeta_{\mathbb{Q}[\mathrm{i}]}(s)| &= \left|\frac{1}{4}\sum_{\alpha\in\mathbb{Z}[\mathrm{i}]\backslash\{0\}}\frac{1}{\mathrm{N}(\alpha)^s}\right| = \left|\frac{1}{4}\sum_{\substack{p,q\in\mathbb{Z} \\ p,q \text{ not both } 0}}^{\infty}\frac{1}{(p^2+q^2)^s}\right| \\
&= \left|\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{1}{(p^2+q^2)^s} + \sum_{p=1}^{\infty}\frac{1}{p^{2s}}\right| = \left|\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{1}{(p^2+q^2)^s} + \zeta(2s)\right| \\
&\leqslant \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\left|\frac{1}{(2pq)^s}\right| + |\zeta(2s)| \leqslant \frac{1}{2}\left|\zeta^2(\sigma)\right| + |\zeta(2\sigma)| = \mathrm{O}(1).
\end{aligned}$$

■

For a proof that $|\zeta(s)| < C|s|$ see [Dav80, p. 79]. Unfortunately the idea of counting the number of integer points (which is used in [Dav80]) does not generalise easily to a general number field. In $\mathbb{Q}[\mathrm{i}]$ we would end up using Gauss Circle

Problem estimations to count the number of integer points in a circle, which we want to avoid. Instead we consider the interval $\frac{1}{2} \leqslant \sigma \leqslant 2$ separately using the analytic continuation of $\zeta_{\mathbb{Q}[i]}(s)$ to prove that $\xi_{\mathbb{Q}[i]}$ is of order 1.

**Theorem 2.6.2.** *$\xi_{\mathbb{Q}[i]}$ is an integral function of order 1.*

*Proof.* As with the ordinary $\zeta$-function, since we already know by Lemma 2.5.3 that $\xi_{\mathbb{Q}[i]}(s)$ is an integral function and since $\xi_{\mathbb{Q}[i]}(s) = \xi_{\mathbb{Q}[i]}(1-s)$, it is sufficient to show that

$$|\xi_{\mathbb{Q}[i]}(s)| < \mathrm{e}^{C|s|\log|s|}$$

for $\sigma \geqslant \frac{1}{2}$.

Clearly, $|s(s-1)\pi^{-s}| < \mathrm{e}^{C|s|}$. By Stirling's Formula, $|\Gamma(s)| < \mathrm{e}^{C|s|\log|s|}$, as for $\sigma \geqslant \frac{1}{2}$ we have $-\frac{1}{2}\pi < \arg(s) < \frac{1}{2}\pi$. These results, combined with Lemma 2.6.1, prove the theorem for $\sigma > 2$.

Now suppose $\frac{1}{2} \leqslant \sigma \leqslant 2$. Since the integral in (2.3) is absolutely convergent, we have

$$|\xi_{\mathbb{Q}[i]}(s)| \leqslant 1 + |s(s-1)| \int_1^\infty \left|(x^{s-1} + x^{-s})\varpi(x)\right| \mathrm{d}x.$$

Thus, for $\frac{1}{2} \leqslant \sigma \leqslant 2$,

$$|\xi_{\mathbb{Q}[i]}(s)| \leqslant C_0|s|^2 \leqslant \mathrm{e}^{C_1|s|}.$$

Consequently we can conclude that in the range $\sigma \geqslant \frac{1}{2}$ the function $\xi_{\mathbb{Q}[i]}$ is an integral function of order 1 and, by the functional equation, on the whole of the complex plane. ■

As $\sigma \to \infty$, $\zeta_{\mathbb{Q}[i]}(\sigma)$ is monotonically decreasing and also bounded below by 1 (in fact it is easy to see that it tends to 1). Thus, as $\log\Gamma(\sigma) \sim \sigma\log\sigma$, we cannot possibly have $|\xi_{\mathbb{Q}[i]}(s)| < \mathrm{e}^{C|s|}$. Hence we can deduce that $\xi_{\mathbb{Q}[i]}(s)$ has infinitely many zeroes $\rho$ with $\sum_\rho \frac{1}{|\rho|^{1+\varepsilon}} < \infty$ for any $\varepsilon > 0$ but $\sum_\rho \frac{1}{|\rho|}$ diverges. Also we get the representation of $\xi_{\mathbb{Q}[i]}(s)$ as a Hadamard product over its zeroes:

$$\xi_{\mathbb{Q}[i]}(s) = \mathrm{e}^{A+Bs} \prod_\rho \left(1 - \frac{s}{\rho}\right) \mathrm{e}^{\frac{s}{\rho}}. \tag{2.8}$$

As the only zeroes of $\xi_{\mathbb{Q}[i]}(s)$ are the non-trivial zeroes of $\zeta_{\mathbb{Q}[i]}(s)$, the Hadamard product runs over zeroes in the critical strip $0 \leqslant \sigma \leqslant 1$.

We now define a Hadamard product for the $L$-functions.

**Theorem 2.6.3.** *The function*

$$\xi_{\mathbb{Q}[i]}(s,\chi) := \left(\frac{|d|}{\pi}\right)^s \Gamma(s) L_{\mathbb{Q}[i]}(s,\chi) \tag{2.9}$$

*is an integral function of order 1 and hence there is a Hadamard product, which is a product over its zeroes of the form*

$$\xi_{\mathbb{Q}[i]}(s,\chi) = \mathrm{e}^{A(\chi)+B(\chi)s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \mathrm{e}^{\frac{s}{\rho}} \tag{2.10}$$

*for some constants $A(\chi), B(\chi)$ and zeroes $\rho$, which, again, are the non-trivial zeroes of $L(s,\chi)$.*

*Proof.* We will use what we already know for $\xi_{\mathbb{Q}[i]}(s)$ to bound $\xi_{\mathbb{Q}[i]}(s,\chi)$ for $\sigma \geqslant 2$:

$$\begin{aligned} |\xi_{\mathbb{Q}[i]}(s,\chi)| &\leqslant \left|\frac{d^s}{\pi^s}\right| |\Gamma(s)||L_{\mathbb{Q}[i]}(s,\chi)| \\ &\leqslant \frac{d^\sigma}{\pi^\sigma}\Gamma(\sigma)\zeta_{\mathbb{Q}[i]}(\sigma) \\ &\leqslant \frac{\xi_{\mathbb{Q}[i]}(\sigma)}{\sigma(\sigma-1)} d^\sigma < \mathrm{e}^{C\sigma\log\sigma}. \end{aligned}$$

To get $\frac{1}{2} \leqslant \sigma < 2$ we use exactly the same idea as we used for the $\zeta$-function, and apply the integral from the functional equation (2.7). Once again the stronger inequality $|\xi_{\mathbb{Q}[i]}(s,\chi)| < \mathrm{e}^{C|s|}$ does not hold so that we get the same results for the sums of reciprocals of zeroes as we did for $\zeta_{\mathbb{Q}[i]}(s)$. ■

**Theorem 2.6.4.**

$$\frac{L'_{\mathbb{Q}[i]}(s,\chi)}{L_{\mathbb{Q}[i]}(s,\chi)} = -\log\frac{|d|}{\pi} - \frac{\Gamma'(s)}{\Gamma(s)} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + B(\chi).$$

*Proof.* This follows by logarithmic differentiation of (2.9) and (2.10). ■

**Lemma 2.6.5.** *If $\chi$ is a primitive character* (mod $d$), *then*

$$\mathrm{Re}\,(B(\chi)) = -\sum_{\rho} \mathrm{Re}\left(\frac{1}{\rho}\right).$$

*Proof.* By logarithmic differentiation of (2.10) we get that

$$B(\chi) = \frac{\xi'_{\mathbb{Q}[\mathrm{i}]}(0,\chi)}{\xi_{\mathbb{Q}[\mathrm{i}]}(0,\chi)}.$$

The functional equation states that

$$\xi_{\mathbb{Q}[\mathrm{i}]}(s,\chi) = \frac{\tau(\chi)}{|d|}\xi_{\mathbb{Q}[\mathrm{i}]}(1-s,\overline{\chi}),$$

implying that

$$\frac{\xi'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{\xi_{\mathbb{Q}[\mathrm{i}]}(s,\chi)} = -\frac{\xi'_{\mathbb{Q}[\mathrm{i}]}(1-s,\overline{\chi})}{\xi_{\mathbb{Q}[\mathrm{i}]}(1-s,\overline{\chi})},$$

and hence

$$B(\chi) = \frac{\xi'_{\mathbb{Q}[\mathrm{i}]}(0,\chi)}{\xi_{\mathbb{Q}[\mathrm{i}]}(0,\chi)} = -\frac{\xi'_{\mathbb{Q}[\mathrm{i}]}(1,\overline{\chi})}{\xi_{\mathbb{Q}[\mathrm{i}]}(1,\overline{\chi})} = -B(\overline{\chi}) - \sum_{\overline{\rho}}\left(\frac{1}{1-\overline{\rho}} + \frac{1}{\overline{\rho}}\right).$$

Note that the last equality holds because $L(s,\overline{\chi}) = \overline{L(s,\chi)}$, and so $\rho$ is a zero of $L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$ if and only if $\overline{\rho}$ is a zero of $L_{\mathbb{Q}[\mathrm{i}]}(s,\overline{\chi})$. For the same reason it follows from (2.9) that

$$\frac{\xi'_{\mathbb{Q}[\mathrm{i}]}(0,\overline{\chi})}{\xi_{\mathbb{Q}[\mathrm{i}]}(0,\overline{\chi})} = \overline{\left(\frac{\xi'_{\mathbb{Q}[\mathrm{i}]}(0,\chi)}{\xi_{\mathbb{Q}[\mathrm{i}]}(0,\chi)}\right)},$$

and so $B(\overline{\chi}) = \overline{B(\chi)}$. The zeroes of $\xi_{\mathbb{Q}[\mathrm{i}]}(s)$ that is the non-trivial zeroes of $\zeta_{\mathbb{Q}[\mathrm{i}]}(s)$, are symmetric with respect to the critical line, and hence if $\rho$ is a zero then so is $1-\overline{\rho}$. Consequently, we can regroup the terms of the sum, as it is absolutely convergent, to get

$$\sum_{\overline{\rho}}\left(\frac{1}{1-\overline{\rho}} + \frac{1}{\overline{\rho}}\right) = \sum_{\rho}\left(\frac{1}{\rho} + \frac{1}{\overline{\rho}}\right).$$

Thus

$$\mathrm{Re}B(\chi) = -\sum_{\rho}\mathrm{Re}\frac{1}{\rho}.$$

∎

## 2.7 Zero-free Region

### 2.7.1 $\zeta_{\mathbb{Q}[\mathrm{i}]}$-function

As for the ordinary zeta function, our proof of the zero-free region revolves around the inequality

$$3 + 4\cos\theta + \cos 2\theta \geqslant 0. \tag{2.11}$$

By logarithmic differentiation of the Euler Product we get

$$-\frac{\zeta'_{\mathbb{Q}[i]}(s)}{\zeta_{\mathbb{Q}[i]}(s)} = \sum_{\alpha\in\mathbb{Z}[i]\backslash\{0\}}^{\sim} \frac{\Lambda_{\mathbb{Q}[i]}(\alpha)}{\mathrm{N}(\alpha)^s}, \tag{2.12}$$

where

$$\Lambda_{\mathbb{Q}[i]}(\alpha) = \begin{cases} \log \mathrm{N}(\pi) & \text{if } \alpha = \pi^m \text{ for some prime } \pi \text{ and some integer } m; \\ 0 & \text{otherwise.} \end{cases}$$

(recall that we count only one prime from each class of associates). Thus

$$-\mathrm{Re}\frac{\zeta'_{\mathbb{Q}[i]}(s)}{\zeta_{\mathbb{Q}[i]}(s)} = \sum_{\alpha\in\mathbb{Z}[i]\backslash\{0\}}^{\sim} \frac{\Lambda_{\mathbb{Q}[i]}(\alpha)}{\mathrm{N}(\alpha)^\sigma}\cos(t\log \mathrm{N}(\alpha)).$$

Hence by (2.11) we get

$$-3\frac{\zeta'_{\mathbb{Q}[i]}(\sigma)}{\zeta_{\mathbb{Q}[i]}(\sigma)} - 4\mathrm{Re}\frac{\zeta'_{\mathbb{Q}[i]}(\sigma+\mathrm{i}t)}{\zeta_{\mathbb{Q}[i]}(\sigma+\mathrm{i}t)} - \mathrm{Re}\frac{\zeta'_{\mathbb{Q}[i]}(\sigma+2\mathrm{i}t)}{\zeta_{\mathbb{Q}[i]}(\sigma+2\mathrm{i}t)} \geqslant 0. \tag{2.13}$$

We now want to estimate the three terms of this inequality, to get a condition on the zeroes close to the left of the line $\sigma = 1$.

As $\zeta_{\mathbb{Q}[i]}(s)$ has a simple pole at 1 with residue $\frac{\pi}{4}$, we get that $-\zeta'_{\mathbb{Q}[i]}(s)$ has a double pole, and $\frac{1}{\zeta_{\mathbb{Q}[i]}(s)}$ has a simple zero. So, if we look at the coefficients of the Laurent expansions, we find that $\frac{\zeta'_{\mathbb{Q}[i]}(s)}{\zeta_{\mathbb{Q}[i]}(s)}$ has a simple pole with residue -1. Hence we get

$$-\frac{\zeta'_{\mathbb{Q}[i]}(\sigma)}{\zeta_{\mathbb{Q}[i]}(\sigma)} < \frac{1}{\sigma-1} + \mathrm{O}(1) \tag{2.14}$$

in the range $1 < \sigma \leqslant 2$. We only need to consider this range, as we are only interested in the zeroes of $\zeta_{\mathbb{Q}[i]}(s)$ just to the left of 1.

Logarithmic differentiation of (2.8) gives that

$$\frac{\xi'_{\mathbb{Q}[i]}(s)}{\xi_{\mathbb{Q}[i]}(s)} = B + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Also from logarithmic differentiation of $\xi_{\mathbb{Q}[i]}(s)$ and the identity $s\Gamma(s) = \Gamma(s+1)$, we get

$$\frac{\xi'_{\mathbb{Q}[i]}(s)}{\xi_{\mathbb{Q}[i]}(s)} = -\log(\pi) + \frac{1}{s-1} + \frac{\Gamma'(s+1)}{\Gamma(s+1)} + \frac{\zeta'_{\mathbb{Q}[i]}(s)}{\zeta_{\mathbb{Q}[i]}(s)},$$

and so

$$\frac{\zeta'_{\mathbb{Q}[i]}(s)}{\zeta_{\mathbb{Q}[i]}(s)} = B + \log(\pi) - \frac{1}{s-1} - \frac{\Gamma'(s+1)}{\Gamma(s+1)} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{2.15}$$

We know that in the given region

$$\frac{\Gamma'(s+1)}{\Gamma(s+1)} = \log(s+1) + \mathrm{O}(|s+1|^{-1})$$

and thus

$$\mathrm{Re}\frac{\Gamma'(s+1)}{\Gamma(s+1)} = \log|s+1| + \mathrm{O}(|s+1|^{-1}) \leqslant A\log(t+2)$$

for $1 < \sigma \leqslant 2$ and $t \geqslant 2$. Also $B + \log(\pi) + \frac{1}{s-1} = \mathrm{O}(\log(t+2))$ in the same range of $s$, as $|\mathrm{Re}z| \leqslant |z|$ for all $z \in \mathbb{C}$. Thus

$$-\mathrm{Re}\frac{\zeta'_{\mathbb{Q}[i]}(s)}{\zeta_{\mathbb{Q}[i]}(s)} \leqslant A\log(t+2) - \sum_{\rho}\mathrm{Re}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

It is easy to see that the sum is, in fact, positive. This is because each term is positive (as $\sigma > 1 \geqslant \mathrm{Re}\rho$). Hence for a zero $\rho = \beta + \mathrm{i}\gamma$ with $\gamma \geqslant 0$ choose $s = \sigma + \mathrm{i}\gamma$ (that is, $t = \gamma$). Then, considering just the first term of the sum, we get a valid inequality

$$-\mathrm{Re}\frac{\zeta'_{\mathbb{Q}[i]}(\sigma+\mathrm{i}t)}{\zeta_{\mathbb{Q}[i]}(\sigma+\mathrm{i}t)} \leqslant A\log(t+2) - \frac{1}{\sigma-\beta}, \tag{2.16}$$

which gives an estimation for the second term of (2.13). Finally, for the last term, the sum may be omitted altogether.

$$-\mathrm{Re}\frac{\zeta'_{\mathbb{Q}[i]}(\sigma+2\mathrm{i}t)}{\zeta_{\mathbb{Q}[i]}(\sigma+2\mathrm{i}t)} \leqslant A\log(t+2). \tag{2.17}$$

Using these estimates in (2.13), we get

$$\frac{3}{\sigma-1} + \mathrm{O}(1) + 4A\log(t+2) - \frac{4}{\sigma-\beta} + A\log(t+2) > 0,$$

and hence

$$\begin{aligned}
\frac{3}{\sigma-1} + C\log(t+2) &> \frac{4}{\sigma-\beta};\\
\frac{\sigma-\beta}{4} &> \frac{\sigma-1}{3+C\log(t+2)(\sigma-1)};\\
\beta &< \sigma - \frac{4(\sigma-1)}{3+C\log(t+2)(\sigma-1)}.
\end{aligned}$$

Now let $\sigma = 1 + \frac{\delta}{\log(t+2)}$. Then

$$\beta < 1 + \left(\delta - \frac{4\delta}{3 + C\delta}\right)\frac{1}{\log(t+2)}.$$

Choose $\delta$ so that $D := \frac{4\delta}{3+C\delta} - \delta > 0$. Then

$$\beta < 1 - \frac{D}{\log(t+2)}.$$

So there cannot be a zero $s = \sigma + it$ such that $\sigma \geqslant 1 - \frac{D}{\log(t+2)}$ whenever $t \geqslant 2$. However, in view of the simple pole of $\zeta_{\mathbb{Q}[i]}(s)$ at $s = 1$, we can easily use (2.11) in exactly the same way as we do for the ordinary zeta function, to find that $\zeta_{\mathbb{Q}[i]}(s)$ does not have any zeroes on the line $\sigma = 1$. Hence, by continuity of $\zeta_{\mathbb{Q}[i]}(s)$ for $0 < t < 2$, there must be a positive constant $E$, such that $\zeta_{\mathbb{Q}[i]}(s)$ does not have any zeroes in the region $\sigma \geqslant 1 - \frac{E}{\log(t+2)}$. Thus the constant $F := \min(D, E)$ gives us a zero-free region $\sigma \geqslant 1 - \frac{F}{\log(|t|+2)}$ (for $t < 0$ we just use the fact that whenever $\rho$ is a zero, $\overline{\rho}$ is also a zero). This gives us an analogous zero-free region for $\zeta_{\mathbb{Q}[i]}(s)$ just as we have for the ordinary zeta function:

**Theorem 2.7.1.** *There exists a positive constant $F$ such that $\zeta_{\mathbb{Q}[i]}(s)$ does not have any zeroes $s = \sigma + \mathrm{i}t$ for*

$$\sigma \geqslant 1 - \frac{F}{\log(|t|+2)}.$$

### 2.7.2 $L_{\mathbb{Q}[i]}$-functions

For $s = \sigma + \mathrm{i}t$

$$-\frac{L'_{\mathbb{Q}[i]}(s,\chi)}{L_{\mathbb{Q}[i]}(s,\chi)} = \sum_{\alpha \in \mathbb{Z}[i]}^{\infty\ \sim} \Lambda_{\mathbb{Q}[i]}(\alpha)\mathrm{N}(\alpha)^{-\sigma}\chi(\alpha)\mathrm{e}^{-\mathrm{i}t\log \mathrm{N}(\alpha)}.$$

Now, if $(\alpha, d) = 1$, let $\theta$ be such that $\mathrm{Re}\left(\chi(\alpha)\mathrm{e}^{-\mathrm{i}t\log n}\right) = \cos\theta$. Then we get that $\mathrm{Re}\left(\chi^2(\alpha)\mathrm{e}^{-i(2t)\log \mathrm{N}(\alpha)}\right) = \cos 2\theta$. Obviously, $\mathrm{Re}\left(\chi_0(\alpha)\mathrm{e}^{-i(0t)\log \mathrm{N}(\alpha)}\right) = \mathrm{Re}(1) = 1$ and so we get the inequality

$$-\left[3\frac{L'_{\mathbb{Q}[i]}(\sigma,\chi_0)}{L_{\mathbb{Q}[i]}(\sigma,\chi_0)} + 4\mathrm{Re}\frac{L'_{\mathbb{Q}[i]}(\sigma+\mathrm{i}t,\chi)}{L_{\mathbb{Q}[i]}(\sigma+\mathrm{i}t,\chi)} + \mathrm{Re}\frac{L'_{\mathbb{Q}[i]}(\sigma+2\mathrm{i}t,\chi^2)}{L_{\mathbb{Q}[i]}(\sigma+2\mathrm{i}t,\chi^2)}\right] \geqslant 0. \qquad (2.18)$$

Let $\chi$ be a primitive character of conductor $d$. As in the rational case, $\zeta_{\mathbb{Q}[\mathrm{i}]}(s)$ has a simple pole at $s = 1$ and hence we get an easy estimation of the first term:

$$-\frac{L'_{\mathbb{Q}[\mathrm{i}]}(\sigma, \chi_0)}{L_{\mathbb{Q}[\mathrm{i}]}(\sigma, \chi_0)} \leqslant -\frac{\zeta'_{\mathbb{Q}[\mathrm{i}]}(\sigma)}{\zeta_{\mathbb{Q}[\mathrm{i}]}(\sigma)} < \frac{1}{\sigma - 1} + \mathrm{O}(1) \tag{2.19}$$

for $1 < \sigma \leqslant 2$.

For the second term we can use Theorem 2.6.4 to get

$$-\mathrm{Re}\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s, \chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s, \chi)} = \log\frac{|d|}{\pi} + \mathrm{Re}\frac{\Gamma'(s)}{\Gamma(s)} - \mathrm{Re}\sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \mathrm{Re}B(\chi).$$

By Lemma 2.6.5

$$-\mathrm{Re}\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s, \chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s, \chi)} = \log\frac{|d|}{\pi} + \mathrm{Re}\frac{\Gamma'(s)}{\Gamma(s)} - \mathrm{Re}\sum_{\rho}\left(\frac{1}{s-\rho}\right). \tag{2.20}$$

Now, for $t \geqslant 0$ and $\sigma \geqslant \frac{1}{2}$ we have $\frac{\Gamma'(s)}{\Gamma(s)} = \mathrm{O}\left(\log(t+2)\right)$, and so (for primitive characters $\chi$) we get

$$-\mathrm{Re}\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s, \chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s, \chi)} < A\mathscr{L} - \sum_{\rho}\mathrm{Re}\frac{1}{s-\rho}, \tag{2.21}$$

where

$$\mathscr{L} = \log|d| + \log(t+2).$$

Again, the series is positive (as, again, $s$ lies to the right of the line $\sigma = 1$ and any possible zero lies to the left of it). We can, as before, omit any of its parts and still get a valid inequality.

We now need to differentiate the cases when $\chi$ is real (quadratic) and when $\chi$ is complex, as $\chi^2$ is principal if and only if $\chi$ is real. So first suppose that $\chi$ is complex. Then $\chi^2$ is non-principal (and we do not run into difficulties with the pole at $s = 1$).

If $\rho = \beta + \mathrm{i}\gamma$, we again choose $t = \gamma$ and so, analogously to (2.16), we get an estimation for the second term

$$-\mathrm{Re}\frac{L'_{\mathbb{Q}[\mathrm{i}]}(\sigma + \mathrm{i}t, \chi)}{L_{\mathbb{Q}[\mathrm{i}]}(\sigma + \mathrm{i}t, \chi)} < A\mathscr{L} - \frac{1}{\sigma - \beta}. \tag{2.22}$$

Suppose first that $\chi^2$ is primitive. Then, by the estimate above,

$$-\mathrm{Re}\frac{L'_{\mathbb{Q}[i]}(s,\chi^2)}{L_{\mathbb{Q}[i]}(s,\chi^2)} < C\mathscr{L}. \tag{2.23}$$

If $\chi^2$ is imprimitive however, then let $\chi_1$ be the primitive character inducing $\chi^2$. We get

$$L(s,\chi) = \prod_{\pi\nmid d}^{\sim} \frac{1}{1-\frac{\chi(\pi)}{\mathrm{N}(\pi)^s}} = \prod_{\pi\nmid d}^{\sim} \frac{1}{1-\frac{\chi_1(\pi)}{\mathrm{N}(\pi)^s}} = L(s,\chi_1) \times \prod_{\pi|d}^{\sim}\left(1-\frac{\chi_1(\pi)}{\mathrm{N}(\pi)^s}\right).$$

Logarithmic differentiation gives

$$\begin{aligned}\frac{L'_{\mathbb{Q}[i]}(s,\chi)}{L_{\mathbb{Q}[i]}(s,\chi)} &= \frac{L'_{\mathbb{Q}[i]}(s,\chi_1)}{L_{\mathbb{Q}[i]}(s,\chi_1)} + \sum_{\pi|d}^{\sim} \frac{\chi_1(\pi)\log(\mathrm{N}(\pi))}{\mathrm{N}(\pi)^s(1-\frac{\chi_1(\pi)}{\mathrm{N}(\pi)^s})} \\ &= \frac{L'_{\mathbb{Q}[i]}(s,\chi_1)}{L_{\mathbb{Q}[i]}(s,\chi_1)} + \sum_{\pi|d}^{\sim} \frac{\chi_1(\pi)\log(\mathrm{N}(\pi))}{(\mathrm{N}(\pi)^s-\chi_1(\pi))},\end{aligned}$$

and so

$$\begin{aligned}\left|\frac{L'_{\mathbb{Q}[i]}(s,\chi)}{L_{\mathbb{Q}[i]}(s,\chi)} - \frac{L'_{\mathbb{Q}[i]}(s,\chi_1)}{L_{\mathbb{Q}[i]}(s,\chi_1)}\right| &\leqslant \sum_{\pi|d}^{\sim}\left(\left|\frac{\frac{\chi_1(\pi)}{\mathrm{N}(\pi)^s}}{1-\frac{\chi_1(\pi)}{\mathrm{N}(\pi)^s}}\right|\log(\mathrm{N}(\pi))\right) \\ &\leqslant \sum_{\pi|d}^{\sim}\left(\frac{\left|\frac{\chi_1(\pi)}{\mathrm{N}(\pi)^s}\right|}{\left|1-\left|\frac{\chi_1(\pi)}{\mathrm{N}(\pi)^s}\right|\right|}\log(\mathrm{N}(\pi))\right) \\ &= \sum_{\pi|d}^{\sim}\left(\frac{\left|\frac{\chi_1(\pi)}{\mathrm{N}(\pi)^s}\right|}{1-\left|\frac{\chi_1(\pi)}{\mathrm{N}(\pi)^s}\right|}\log(\mathrm{N}(\pi))\right).\end{aligned}$$

The last step holds because $0 \leqslant \left|\frac{\chi_1(\pi)}{\mathrm{N}(\pi)^s}\right| < 1$. Further, $\frac{x}{1-x}$ increases in the range $0 \leqslant x < 1$. Hence

$$\left|\frac{L'_{\mathbb{Q}[i]}(s,\chi)}{L_{\mathbb{Q}[i]}(s,\chi)} - \frac{L'_{\mathbb{Q}[i]}(s,\chi_1)}{L_{\mathbb{Q}[i]}(s,\chi_1)}\right| \leqslant \sum_{\pi|d}^{\sim}\left(\frac{\frac{1}{\mathrm{N}(\pi)^\sigma}}{1-\frac{1}{\mathrm{N}(\pi)^\sigma}}\log(\mathrm{N}(\pi))\right).$$

Also, $0 < \frac{1}{\mathrm{N}(\pi)^\sigma} < \frac{1}{2}$, and so $0 < \frac{\frac{1}{\mathrm{N}(\pi)^\sigma}}{1-\frac{1}{\mathrm{N}(\pi)^\sigma}} < 1$. Thus

$$\left|\frac{L'_{\mathbb{Q}[i]}(s,\chi)}{L_{\mathbb{Q}[i]}(s,\chi)} - \frac{L'_{\mathbb{Q}[i]}(s,\chi_1)}{L_{\mathbb{Q}[i]}(s,\chi_1)}\right| \leqslant \sum_{\pi|d}^{\sim}\log(\mathrm{N}(\pi)) \leqslant \log(\mathrm{N}(d)) = 2\log|d|. \tag{2.24}$$

Hence our bound (2.23) remains valid.

Combining the three bounds (2.19), (2.22) and (2.23) and plugging them into (2.18) again, we get

$$\frac{3}{\sigma-1}+D\mathscr{L}>\frac{4}{\sigma-\beta},$$

and hence, as in the previous section, we can find a positive constant $E$ such that any zero $\rho=\beta+\mathrm{i}\gamma$ of $L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$ satisfies

$$\beta<1-\frac{E}{\log|d|+\log(|\gamma|+2)}.$$

This gives us the zero-free region whenever $\chi$ is complex and primitive. The result can easily be extended to imprimitive complex $\chi$: if $\chi_1$ is the primitive character inducing $\chi$, then the only zeroes of $L(s,\chi)$ additional to those of $L(s,\chi_1)$ are the zeroes of the factors $1-\chi_1(\pi)\mathrm{N}(\pi)^{-s}$ for $\pi\mid d$. This means that we need $|\mathrm{N}(\pi)^s|=1$ and hence $\sigma=0$. So the extra zeroes cannot lie in our zero-free region and hence the result generalises to imprimitive characters.

This almost works for real characters. There is a complication, however, since $\chi^2$ is the principal character, and so $L(s,\chi^2)$ has a pole at 1. Hence $\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi^2)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi^2)}$ needs to be compared to $\frac{\zeta'_{\mathbb{Q}[\mathrm{i}]}(s)}{\zeta_{\mathbb{Q}[\mathrm{i}]}(s)}$. Using the same argument as in (2.24) we deduce

$$\left|\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi^2)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi^2)}-\frac{\zeta'_{\mathbb{Q}[\mathrm{i}]}(s)}{\zeta_{\mathbb{Q}[\mathrm{i}]}(s)}\right|\leqslant 2\log|d|.$$

Using (2.15), we get that

$$-\mathrm{Re}\frac{\zeta'_{\mathbb{Q}[\mathrm{i}]}(s)}{\zeta_{\mathbb{Q}[\mathrm{i}]}(s)}<\mathrm{Re}\frac{1}{s-1}+C'\log(t+2),$$

and hence

$$-\mathrm{Re}\frac{L'_{\mathbb{Q}[\mathrm{i}]}(\sigma+2\mathrm{i}t,\chi^2)}{L_{\mathbb{Q}[\mathrm{i}]}(\sigma+2\mathrm{i}t,\chi^2)}<\mathrm{Re}\frac{1}{\sigma-1+2\mathrm{i}t}+C(\log|d|+\log(t+2)).$$

Hence, choosing $t=\gamma$, as before, and combining this with (2.19) and (2.22) in (2.18), gives us

$$\frac{4}{\sigma-\beta}<\frac{3}{\sigma-1}+\mathrm{Re}\frac{1}{\sigma-1+2\mathrm{i}\gamma}+D\mathscr{L}.$$

Now, take $\sigma=1+\frac{\delta}{\mathscr{L}}$. Then

$$\frac{4}{\sigma-\beta}<\frac{3\mathscr{L}}{\delta}+\mathrm{Re}\frac{1}{\frac{\delta}{\mathscr{L}}+2\mathrm{i}\gamma}+D\mathscr{L}=\frac{3\mathscr{L}}{\delta}+\frac{\frac{\delta}{\mathscr{L}}}{(\frac{\delta}{\mathscr{L}})^2+4\gamma^2}+D\mathscr{L}$$

for a positive constant $D$. If $\gamma \geqslant \frac{\delta}{\mathscr{L}}$ then

$$\frac{4}{\sigma - \beta} < \frac{3\mathscr{L}}{\delta} + \frac{\mathscr{L}}{5\delta} + D\mathscr{L},$$

and hence choosing $\delta$ sufficiently small in relation to $D$ (see [Dav80, p.91]) gives us the same zero-free region as before, but now subject to $\gamma \geqslant \frac{\delta}{\mathscr{L}}$.

As for the Dirichlet $L$-function over rational integers, it can be proven that there is at most one exceptional simple zero in that region if $\gamma < \frac{\delta}{\mathscr{L}}$ (and hence this zero must be real, as complex zeroes always come in conjugate pairs for a real character $\chi$). To prove this we use (2.21) at $s = \sigma > 1$ and consider only the terms corresponding to the two conjugate zeroes or to the two real zeroes (or one double zero). Then we use the relation $-\frac{L'_{\mathbb{Q}[i]}(\sigma,\chi)}{L_{\mathbb{Q}[i]}(\sigma,\chi)} \geqslant \frac{\zeta'_{\mathbb{Q}[i]}(\sigma)}{\zeta_{\mathbb{Q}[i]}(\sigma)}$ and apply the estimation (2.14) to get that $\beta < 1 - \frac{\delta}{\log|d|}$ for a sufficiently small $\delta$. Since it works exactly as in [Dav80, p.92], the proof is not given in detail.

Hence we have proven the following result:

**Theorem 2.7.2.** *There is a positive constant $\delta$ for a real character $\chi$ over $\mathbb{Q}[i]$, such that there is at most one simple real zero (called the Siegel zero) $\rho = i\gamma + \beta$ of $L_{\mathbb{Q}[i]}(s,\chi)$ satisfying*

$$|\gamma| < \frac{\delta}{\log|d|}, \qquad \beta > 1 - \frac{\delta}{\log|d|}.$$

Together with the result for complex characters we get

**Theorem 2.7.3.** *There is a positive constant $F$, such that if $\chi$ is a complex character with conductor $d$, then $L_{\mathbb{Q}[i]}(s,\chi)$ has no zero in the region defined by*

$$\sigma \geqslant \begin{cases} 1 - \frac{F}{\log|mt|} & \text{if } |t| \geqslant 1; \\ 1 - \frac{F}{\log|t|} & \text{if } |t| \leqslant 1. \end{cases}$$

*If $\chi$ is a real (but nonprincipal) character, then the only possible zero in this region is a simple real zero.*

## 2.8 Numbers of Zeroes

We now want to find estimations for the number $\mathrm{N}(T,\chi)$ of non-trivial zeroes of $L_{\mathbb{Q}[i]}(s,\chi)$ in the critical strip from height $-T$ up to the height $T$. (Note that the contour used in [Dav80] cannot be used, as it depends on $\zeta(s)$ not having any non-trivial real zeroes, which might not be true for $L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$). To do this we note that for any holomorphic function $f(s)$ Cauchy's Residue Theorem for any closed contour $\Gamma$ implies

$$\frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{f'(s)}{f(s)}\mathrm{d}s = \mathrm{N}_\Gamma(\mathrm{f}) - \mathrm{P}_\Gamma(\mathrm{f}),$$

where $N_\Gamma(f)$ is the number of zeroes inside $\Gamma$ and $P_\Gamma(f)$ the number of poles. Now, if $f(s)$ is holomorphic and non-zero inside an area $U$, then for any path $\gamma$ we have

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f'(s)}{f(s)}\mathrm{d}s &= \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{d}}{ds}\log f(s)\mathrm{d}s \\ &= \frac{1}{2\pi\mathrm{i}}\left(\int_\gamma \frac{\mathrm{d}}{ds}\log|f(s)|\mathrm{d}s + \int_\gamma \frac{\mathrm{d}}{ds}\mathrm{i}\arg f(s)\mathrm{d}s\right).\end{aligned}$$

The first integral is clearly zero and so

$$2\pi(N_\Gamma(f) - P_\Gamma(f)) = \int_\gamma \frac{\mathrm{d}}{ds}\arg f(s)\mathrm{d}s =: \Delta_\gamma \arg f(s).$$

The second integral is the change in argument of $f(s)$ as $s$ traverses $\gamma$, for which we use the above notation. We use this to estimate $\mathrm{N}(T,\chi)$ and for that we will employ the function $\xi_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$ which, as we have shown, is regular everywhere and only has zeroes that are non-trivial zeroes of $L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$. So in fact

$$2\pi\mathrm{N}(T,\chi) = \Delta_R(\arg\xi_{\mathbb{Q}[\mathrm{i}]}(s)),$$

where we choose $R$ to be the rectangle with vertices at $-\frac{3}{2}-\mathrm{i}T, \frac{5}{2}-\mathrm{i}T, \frac{5}{2}+\mathrm{i}T, -\frac{3}{2}+\mathrm{i}T$ as shown in Figure 2.1. For simplicity we will assume that $T \geqslant 2$ was chosen so that there is no zero of the form $\beta \pm \mathrm{i}T$. (We know that $\xi_{\mathbb{Q}[\mathrm{i}]}(s)$ is non-zero on the contours lines, so this does make sense.) Now, the functional equation tells us

$$\xi_{\mathbb{Q}[\mathrm{i}]}(1-s,\overline{\chi}) = \frac{\tau(\chi)}{|d|}\xi_{\mathbb{Q}[\mathrm{i}]}(s,\chi),$$

and since

$$\xi_{\mathbb{Q}[\mathrm{i}]}(\overline{s},\overline{\chi}) = \overline{\xi_{\mathbb{Q}[\mathrm{i}]}(s,\chi)},$$

we get that

$$\arg\xi_{\mathbb{Q}[\mathrm{i}]}(s,\chi) = \arg\frac{|d|}{\tau(\chi)} + \arg\xi_{\mathbb{Q}[\mathrm{i}]}(1-s,\overline{\chi}) = C + \arg\overline{\xi_{\mathbb{Q}[\mathrm{i}]}(1-\overline{s},\chi)}$$

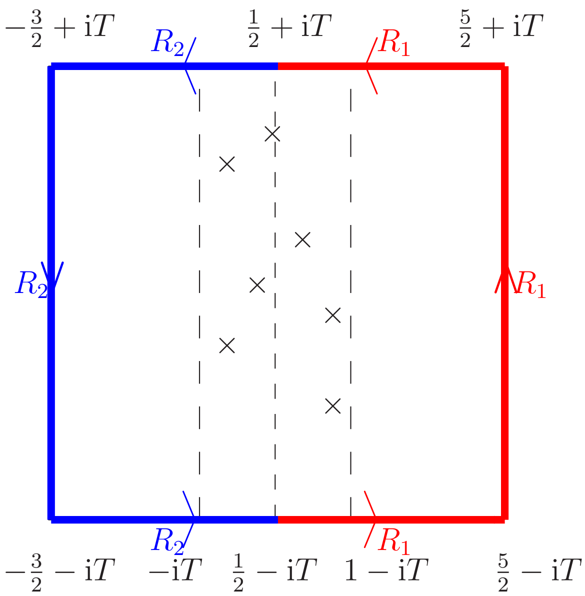


Figure 2.1: Our contour

for some constant $C$ independent of $s$. This means that the change in argument along the segment $R_1$ is equal to the change of argument along the segment $R_2$ (since we have $\arg \overline{z} = -\arg z$). So for now we will just concentrate on the segment $R_1$ and multiply by 2 later. Recall that $\xi_{\mathbb{Q}[\mathrm{i}]}(s,\chi) := \left(\frac{|d|}{\pi}\right)^s \Gamma(s) L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$. Thus

$$\Delta_{R_1} \arg \xi_{\mathbb{Q}[\mathrm{i}]}(s,\chi) = \Delta_{R_1} \arg \left(\frac{|d|}{\pi}\right)^s + \Delta_{R_1} \arg \Gamma(s) + \Delta_{R_1} \arg L_{\mathbb{Q}[\mathrm{i}]}(s,\chi).$$

Now, clearly

$$\Delta_{R_1} \arg \left(\frac{|d|}{\pi}\right)^s = \Delta_{R_1} t \log \left(\frac{|d|}{\pi}\right) = 2T \log \left(\frac{|d|}{\pi}\right). \tag{2.25}$$

We consider the vertical component of $R_1$ first (the horizontal components together turn out to be negligible).

$$\begin{aligned}
\Delta_{R_1} \arg \Gamma(s) &= \int_{\frac{5}{2}-\mathrm{i}T}^{\frac{5}{2}+\mathrm{i}T} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{Im} \log \Gamma(s) \mathrm{d}s \\
&= \mathrm{Im} \log \Gamma(\frac{5}{2}+\mathrm{i}T) - \mathrm{Im} \log \Gamma(\frac{5}{2}-\mathrm{i}T) \\
&= 2\mathrm{Im} \log \Gamma(\frac{5}{2}+\mathrm{i}T).
\end{aligned}$$

By Stirling's Formula we get that

$$\begin{aligned}
\Delta_{R_1} \arg \Gamma(s) &= 2\mathrm{Im} \left( (2+\mathrm{i}T) \log(\frac{5}{2}+\mathrm{i}T) - (\frac{5}{2}+\mathrm{i}T) + \mathrm{O}(1) \right) \\
&= 2 \left( 2 \arg \left( \frac{5}{2}+\mathrm{i}T \right) + T \log \left| \frac{5}{2}+\mathrm{i}T \right| - T + \mathrm{O}(1) \right).
\end{aligned}$$

But note that $0 \leqslant \arg\left(\frac{5}{2}+\mathrm{i}T\right) \leqslant \frac{\pi}{2}$ and also

$$\begin{aligned} T\log\left|\frac{5}{2}+\mathrm{i}T\right| &= \frac{1}{2}T\log\left(\frac{25}{2}+T^2\right) = \frac{1}{2}T\left(\log T^2+\log\left(\frac{25}{2T^2}+1\right)\right) \\ &= T\log T+\mathrm{O}(T^{-1}). \end{aligned}$$

Hence

$$\Delta_{R_1}\arg\Gamma(s) = 2T\log T - 2T + \mathrm{O}(1). \tag{2.26}$$

Precisely the same argument can be used to show that the horizontal contribution is $\mathrm{O}(T^{-1}) = \mathrm{O}(1)$. Thus it remains to find an estimate for the last term $\Delta_{R_1}\arg L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$. Since $L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$ does not have any zeroes or poles on the vertical component and is absolutely bounded on that path, the integral is also absolutely bounded on that path and is hence $\mathrm{O}(1)$. For the horizontal components, we apply the following lemma (which we will not prove here, as it follows from (2.21) very similarly to the example in [Dav80, p.99]).

**Lemma 2.8.1.** *If $\rho = \beta+\mathrm{i}\gamma$ runs through the nontrivial zeroes of $L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$, where $\chi$ is primitive, then for any real $t$ we get*

$$\sum_\rho \frac{1}{1+(t-\gamma)^2} = \mathrm{O}(\mathscr{L}),$$

*where $\mathscr{L} = \log|d|(|t|+2)$.*

Hence, following [Dav80], we can deduce that if $t$ does not coincide with the ordinate of a zero, and $-\frac{3}{2} \leqslant \sigma \leqslant \frac{5}{2}$, then

$$\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)} = \sum_\rho{}' \frac{1}{s-\rho} + \mathrm{O}(\mathscr{L}),$$

where now the sum runs over all zeroes such that $|t-\gamma| < 1$. So for the lower horizontal component we get

$$\Delta_{R_1}\arg L_{\mathbb{Q}[\mathrm{i}]}(s,\chi) = \int_{\frac{1}{2}-iT}^{\frac{5}{2}-\mathrm{i}T} \mathrm{Im}\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}\mathrm{d}s = \int_{\frac{1}{2}-iT}^{\frac{5}{2}-\mathrm{i}T} \mathrm{Im}\left(\sum_\rho{}'\frac{1}{s-\rho}+\mathrm{O}(\mathscr{L})\right)\mathrm{d}s.$$

The argument of $\frac{1}{s-\rho}$ has the value at most $\pi$, and we can easily see from Lemma 2.8.1 that there are at most $\mathrm{O}(\mathscr{L})$ zeroes such that $|\gamma - T| < 1$. The same estimation works for the upper horizontal component. Thus in fact

$$\Delta_{R_1}\arg L_{\mathbb{Q}[\mathrm{i}]}(s,\chi) = \mathrm{O}(\mathscr{L}). \tag{2.27}$$

Combining (2.25), (2.26) and (2.27) we get for $T \geqslant 2$

$$\mathrm{N}(T,\chi) = \frac{T}{\pi}\log\frac{|d|^2T}{\pi^2} - \frac{T}{\pi} + \mathrm{O}(\log T + \log|d|). \tag{2.28}$$

## 2.9 The Explicit Formula

In order to prove a version of the Prime Number Theorem for arithmetic progressions of Gaussian primes, we need to deduce a version of the explicit formula for $L$-functions and we also need to estimate the error term. This will not be done in great detail, because it is very similar to the rational case.

For a character $\chi$ such that $\chi(\mathrm{i}) = 1$, we define

$$\psi_{\mathbb{Q}[\mathrm{i}]}(x,\chi) = \sideset{}{^\sim}\sum_{\substack{\alpha\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}\\ \mathrm{N}(\alpha)\leqslant x}} \chi(\alpha)\Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha).$$

We also define a $\psi_0(x,\chi)$ which is $\psi_{\mathbb{Q}[\mathrm{i}]}(x,\chi)$ everywhere except for the points of discontinuity, where it is the average between the value from the left and the value from the right. Recall the previously used identity

$$-\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)} = \sideset{}{^\sim}\sum_{\alpha\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha)\frac{\chi(\alpha)}{\mathrm{N}(\alpha)^s}.$$

This implies that

$$\psi_0(x,\chi) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left[\frac{x^s}{s}\left(-\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}\right)\right]\mathrm{d}s.$$

We now use the following Lemma from [Dav80]

**Lemma 2.9.1.** *Let*

$$\delta(y) = \begin{cases} 0 & \text{if } 0 < y < 1;\\ \frac{1}{2} & \text{if } y = 1;\\ 1 & \text{if } y > 1; \end{cases}$$

*and*

$$I(y,T) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+\mathrm{i}T}\frac{y^s}{s}ds.$$

*Then for $y > 0, c > 0, T > 0$ we get*

$$|I(y,T) - \delta(y)| < \begin{cases} y^c\min(1, T^{-1}|\log y|^{-1}) & \text{if } y \neq 1,\\ cT^{-1} & \text{if } y = 1. \end{cases}$$

Now let $c > 1$ and

$$J(x,\chi,T) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} \left[ \frac{x^s}{s} \left( -\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)} \right) \right] \mathrm{d}s.$$

Applying Lemma 2.9.1 with $y = \frac{x}{\mathrm{N}(\alpha)}$, we get

$$\begin{aligned} &|\psi_0(x,\chi) - J(x,\chi,T)| \\ < &\sum_{\substack{\alpha\in\mathbb{Z}[\mathrm{i}]\setminus\{0\} \\ \mathrm{N}(\alpha)\neq x}} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha) \left(\frac{x}{\mathrm{N}(\alpha)}\right)^c \min\left(1, T^{-1}\left|\log\frac{x}{\mathrm{N}(\alpha)}\right|^{-1}\right) + cT^{-1}\Lambda(x). \end{aligned}$$

Note that the last term contains the ordinary $\Lambda$-function over $\mathbb{Q}$. This works, since if $\Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha) \neq 0$, then $\alpha$ is a power of a Gaussian prime, and so if $x = \mathrm{N}(\alpha)$, then $x$ is a power of $\mathrm{N}(\pi)$ which is a power of an ordinary prime, and so we have that $\Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha) \leqslant 2\Lambda(x) \leqslant 2\Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha)$ (as $\mathrm{N}(\pi) = p$ or $\mathrm{N}(\pi) = p^2$ for an ordinary prime $p$).

As before, we can express $\psi_0(x,\chi)$ as a principal value integral which is the sum of residues of $-\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}\frac{x^s}{s}$ provided that we can move the line of integration to infinity at the left. However we also want to estimate the error term if we just integrate in the rectangle $c - \mathrm{i}T, c + \mathrm{i}T, -U + \mathrm{i}T, -U - iT$ where $1 < c < 2$ and $U$ is $\frac{n}{2}$ for some positive odd integer $n$ (we have to see that the zeroes do not lie on our line of integration). There is a slight complication in calculating the residues because $L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$ has a zero at $s = 0$, and hence $-\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}\frac{x^s}{s}$ has a double pole at $s = 0$.

However, if we do a brief analysis of Laurent's expansions of $\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}$ and $\frac{x^s}{s}$ we find that

$$\begin{aligned} L_{\mathbb{Q}[\mathrm{i}]}(s,\chi) &= Cs + \mathrm{O}(s^2); \\ L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi) &= C + \mathrm{O}(s); \\ \frac{1}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)} &= \frac{(Cs)^{-1}}{1+\mathrm{O}(s)} = (Cs)^{-1}(1+\mathrm{O}(s)); \\ \frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)} &= (C+\mathrm{O}(s))(Cs)^{-1}(1+\mathrm{O}(s)) = \frac{1}{s} + b(\chi) + \mathrm{O}(s); \\ \frac{x^s}{s} &= \frac{1}{s}(1 + s\log x + \frac{(s\log x)^2}{2!} + \cdots), \end{aligned}$$

where $b(\chi)$ is a constant that can be expressed in terms of $B(\chi)$ by using our Theorem 2.6.4. Consequently

$$\begin{aligned} -\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}\frac{x^s}{s} &= -(\frac{1}{s}+b(\chi)+\mathrm{O}(s))(\frac{1}{s}+\log x+\mathrm{O}(s)) \\ &= -\frac{1}{s^2}-(b(\chi)+\log x)\frac{1}{s}+\mathrm{O}(1). \end{aligned}$$

Thus the residue at $s=0$ is $-(b(\chi)+\log x)$.

If we now follow [Dav80, p. 107], we find that everything Davenport mentions works in our case with slightly different constants as we can have more than one prime of a given norm, but since there are at most two, we just need to modify the constants, which will later be neglected anyway.

We first get the estimate (which is independent of $\chi$)

$$|\psi_0(x,\chi)-J(x,\chi,T)| \ll \frac{x(\log x)^2}{T}+(\log x)\min\left(1,\frac{x}{T\langle x\rangle}\right),$$

where $\langle x\rangle$ denotes the distance from $x$ to the norm of the nearest Gaussian prime. We now suppose that $x \geqslant 2$ and $T \geqslant 2$. By using Lemma 2.8.1, we can deduce that in the range $-1 \leqslant \sigma \leqslant 2$

$$\frac{L'_{\mathbb{Q}[\mathrm{i}]}(\sigma\pm \mathrm{i}T,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(\sigma\pm \mathrm{i}T,\chi)} = \mathrm{O}(\log^2|d|T).$$

So the contribution of the horizontal integrals in this range is $\ll \frac{x\log^2|d|T}{T\log x}$ and hence negligible. Now let us consider $-U \leqslant \sigma \leqslant -1$. For this range we use the functional equation in its unsymmetric form, that is

$$L_{\mathbb{Q}[\mathrm{i}]}(1-s,\chi) = \frac{|d|}{\tau(\chi)}\left(\frac{|d|}{\pi}\right)^{2s-1}\frac{\sin\pi s}{\pi}\Gamma(s)^2 L_{\mathbb{Q}[\mathrm{i}]}(s,\overline{\chi}),$$

and we then logarithmically differentiate it. It is easy to see that one of the terms to bound is $\frac{1}{\tan\pi s}$. In order to achieve that, we exclude circles of radius $\frac{1}{4}$ around the trivial zeroes of $L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$ that is around all the negative integers. If we do that, then the logarithmic derivative gives a bound for $\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}$ for $\sigma \leqslant -1$ in exactly the same way as it is done in [Dav80], namely

$$\left|\frac{L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}\right| \ll \log 2|s||d|.$$

Thus the remaining parts of the horizontal integrals contribute $\ll \frac{\log |d| T}{T x \log x}$, from which it follows that if $2 \leqslant T \leqslant x$ then

$$\psi_{\mathbb{Q}[i]}(x, \chi) = -\sum_{|\gamma|<T} \frac{x^\rho}{\rho} - b(\chi) + R_1(x, T),$$

with

$$|R_1(x, T)| \ll x T^{-1} \log^2 |d| x.$$

We then can proceed to estimate $b(\chi)$ and make the possible Siegel zero explicitly visible in our formula. We get that

$$b(\chi) = \mathrm{O}(\log |d|) - \sum_{|\gamma|<1} \frac{1}{\rho}.$$

Hence our estimate becomes

$$\psi_{\mathbb{Q}[i]}(x, \chi) = -\sum_{|\gamma|<T} \frac{x^\rho}{\rho} + \sum_{|\gamma|<1} \frac{1}{\rho} + R_2(x, T),$$

where the order of $R_2(x, T)$ is the same as the order of $R_1(x, T)$. Further, if $\chi$ is real and there is an exceptional Siegel zero, then we can extract the terms corresponding to the Siegel zeroes at $\sigma = \beta_1$ and $\sigma = 1 - \beta_1$ and use the estimations as in [Dav80] to get

$$\psi_{\mathbb{Q}[i]}(x, \chi) = -\frac{x^{\beta_1}}{\beta_1} - {\sum_{|\gamma|<T}}' \frac{x^\rho}{\rho} + R_3(x, T),$$

where the sum is meant to exclude the terms corresponding to $\rho = \beta_1$ and $\rho = 1 - \beta_1$, and where

$$|R_3(x, T)| \ll \frac{x}{T} \log^2(|d| x) + x^{\frac{1}{4}} \log x.$$

Of course the term $-\frac{x^{\beta_1}}{\beta_1}$ can only occur if $\chi$ is a quadratic character. The argument to extend this to imprimitive characters is exactly the same as in [Dav80, p.119]. It uses the fact that if $\chi$ is induced by $\chi_1$, then the difference between $\psi_{\mathbb{Q}[i]}(x, \chi)$ and $\psi_{\mathbb{Q}[i]}(x, \chi_1)$ is of order $\log^2 |d| x$ and the fact that all zeroes of $\chi_1$ are also zeroes of $\chi$ including the Siegel zero (though it might not be a Siegel zero for $\chi$). And so we get the following theorem

**Theorem 2.9.2.** *If $\chi$ is a nonprincipal character with conductor $d$ and $2 \leqslant T \leqslant x$, then*

$$\psi_{\mathbb{Q}[\mathrm{i}]}(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} - \sum_{|\gamma|<T}{}' \frac{x^\rho}{\rho} + R_3(x,T),$$

*where the sum is meant to exclude the terms corresponding to $\rho = \beta_1$ and $\rho = 1-\beta_1$ and where*

$$|R_3(x,T)| \ll \frac{x}{T}\log^2(|d|x) + x^{\frac{1}{4}}\log x,$$

*and the term $-\frac{x^{\beta_1}}{\beta_1}$ is to be omitted unless $\chi$ is a real character for which $L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)$ has a real zero $\beta_1$ such that $\beta_1 > 1 - \frac{F}{\log|d|}$ for some positive constant $F$.*

## 2.10 The Prime Number Theorem

Of course it does not make much sense to analyse $\pi_{\mathbb{Q}[\mathrm{i}]}(x)$, the number of Gaussian Primes of norm less than $x$ (not counting associates), as we know that

$$\pi_{\mathbb{Q}[\mathrm{i}]}(x) = 1 + 2\pi(x;4,1) + \pi(\sqrt{x};4,-1),$$

where $\pi(x;q,a)$ denotes the number of ordinary primes congruent to $a$ modulo $q$. Hence we can apply the Prime Number Theorem for arithmetic progressions (for fixed $q$) which states that if $(q,a)=1$ then $\pi(x;q,a) \sim \frac{\mathrm{Li}(x)}{\phi(q)}$. Thus

$$\lim_{x\to\infty}\frac{\pi_{\mathbb{Q}[\mathrm{i}]}(x)}{\mathrm{Li}(x)} = \lim_{x\to\infty}\frac{2\pi(x;4,1)}{\mathrm{Li}(x)} + \lim_{x\to\infty}\frac{\pi(\sqrt{x};4,-1)}{\mathrm{Li}(x)} = 1+0=1,$$

and hence the immediate result is

$$\pi_{\mathbb{Q}[\mathrm{i}]}(x) \sim \mathrm{Li}(x).$$

We also know that

$$\begin{aligned}
\psi_{\mathbb{Q}[i]}(x) &= \sum_{\substack{\alpha\in\mathbb{Z}[i]\\ \mathrm{N}(\alpha)\leqslant x}}^{\sim} \Lambda_{\mathbb{Q}[i]}(\alpha) = \sum_{\substack{\pi \text{ prime}\\ \mathrm{N}(\pi^m)\leqslant x}}^{\sim} \log \mathrm{N}(\pi^m)\\
&= \sum_{2^m\leqslant x} \log 2 + 2 \sum_{\substack{p\equiv 1 \pmod 4\\ p^m\leqslant x}} \log p + \sum_{\substack{p\equiv -1 \pmod 4\\ p^{2m}\leqslant x}} \log p\\
&= \sum_{2^m\leqslant x} \log 2 + 2 \sum_{\substack{p\equiv 1 \pmod 4\\ p^m\leqslant x}} \log p + \sum_{\substack{p\equiv -1 \pmod 4\\ p^{m}\leqslant \sqrt{x}}} \log p\\
&= \log 2 \sum_{m\leqslant \frac{\log x}{\log 2}} 1 + 2\psi(x;4,1) + \psi(\sqrt{x};4,-1)\\
&= \mathrm{O}(\log x) + \frac{1}{\phi(4)} \sum_{\chi} \left(2\overline{\chi}(1)\psi(x,\chi) + \overline{\chi}(-1)\psi(\sqrt{x},\chi)\right).
\end{aligned}$$

There are only 2 characters modulo 4: the principal one ($\chi_0$) and one for which $\chi(-1) = -1$, which we will call $\chi_1$. Thus

$$\begin{aligned}
\psi_{\mathbb{Q}[i]}(x) &= \mathrm{O}(\log x) + \psi(x,\chi_0) + \psi(x,\chi_1)\\
&\quad + \frac{1}{2}\left(\psi(\sqrt{x},\chi_0) - \psi(\sqrt{x},\chi_1)\right). \qquad (2.29)
\end{aligned}$$

We can thus get an estimate for $\psi_{\mathbb{Q}[i]}(x)$ using the well-known bound for $\psi(x,\chi)$ (for $d = 4$) and get

$$\begin{aligned}
\psi_{\mathbb{Q}[i]}(x) &= \mathrm{O}(\log x) + x + \mathrm{O}\left(x\mathrm{e}^{-C_1(\log x)^{\frac{1}{2}}}\right) + \mathrm{O}\left(x\mathrm{e}^{-C_2(\log x)^{\frac{1}{2}}}\right)\\
&\quad + \frac{1}{2}\sqrt{x} + \mathrm{O}\left(\sqrt{x}\mathrm{e}^{-C_3(\log x)^{\frac{1}{2}}}\right) + \mathrm{O}\left(\sqrt{x}\mathrm{e}^{-C_4(\log x)^{\frac{1}{2}}}\right).
\end{aligned}$$

Hence we can deduce an equivalent of the Prime Number Theorem in de Vallée Poussin's form.

**Theorem 2.10.1.** *We have*

$$\psi_{\mathbb{Q}[i]}(x) = x + \mathrm{O}\left(x\mathrm{e}^{-C(\log x)^{\frac{1}{2}}}\right),$$

*where $C$ is a positive constant.*

We are aiming to prove an equivalent theorem for Gaussian Arithmetic Progressions. We start by looking at the principal character. First we note

$$\begin{aligned}
|\psi_{\mathbb{Q}[\mathrm{i}]}(x,\chi_0) - \psi_{\mathbb{Q}[\mathrm{i}]}(x)| &= \left| \sideset{}{^\sim}\sum_{\substack{\alpha\in\mathbb{Z}[\mathrm{i}]\\ \mathrm{N}(\alpha)\leqslant x}} \chi_0(\alpha)\Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha) - \sideset{}{^\sim}\sum_{\substack{\alpha\in\mathbb{Z}[\mathrm{i}]\\ \mathrm{N}(\alpha)\leqslant x}} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha)\right| \\
&= \left| \sideset{}{^\sim}\sum_{\substack{(\alpha,d)\neq 1\\ \mathrm{N}(\alpha)\leqslant x}} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha)\right| \\
&= \left| \sideset{}{^\sim}\sum_{\substack{\pi \text{ prime}\\ \pi|d}} \sum_{\substack{v\\ \mathrm{N}(\pi^v)\leqslant x}} \log \mathrm{N}(\pi)\right|.
\end{aligned}$$

Since $\mathrm{N}(\pi^v) \leqslant x$, we have $\mathrm{N}(\pi)^v \leqslant x$ and thus $v \log \mathrm{N}(\pi) \leqslant \log x$. As $\mathrm{N}(\pi)$ is at least 2, there are at most $\frac{\log x}{\log \mathrm{N}(\pi)}$ possible $v$, and so

$$|\psi_{\mathbb{Q}[\mathrm{i}]}(x,\chi_0) - \psi_{\mathbb{Q}[\mathrm{i}]}(x)| \ll \log x \left| \sideset{}{^\sim}\sum_{\substack{\pi \text{ prime}\\ \pi|d}} \log \mathrm{N}(\pi)\right| \ll \log x \log \mathrm{N}(d). \quad (2.30)$$

We now define

**Definition 2.10.2.**

$$\psi_{\mathbb{Q}[\mathrm{i}]}(x;d,\alpha) = \sideset{}{^\sim}\sum_{\substack{\mathrm{N}(\beta)\leqslant x\\ \beta\equiv\tilde{\alpha} \ (\mathrm{mod}\ d)}} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta),$$

where $\tilde{\alpha}$ runs over the conjugates of $\alpha$.

It is a standard property of characters that for $(\beta, d) = 1$, then

$$\frac{1}{\Phi(d)} \sum_{\chi_d} \overline{\chi}(\alpha)\chi(\beta) = \begin{cases} 1 & \text{if } \beta \equiv \alpha \pmod{d}; \\ 0 & \text{otherwise.} \end{cases}$$

However, $\psi_{\mathbb{Q}[\mathrm{i}]}(x,\chi)$ is only defined for $\chi$ for which $\chi(\mathrm{i}) = 1$. We introduce a new notation $\sum_{\chi}^{(0)}$, by which we mean that the sum runs only over the characters for which $\chi(\mathrm{i}) = 1$.

**Lemma 2.10.3.** *If* $(\beta, d) = 1$*, then*

$$\frac{H_d}{\Phi(d)} \sum_{\chi_d}^{(0)} \overline{\chi}(\alpha)\chi(\beta) = \begin{cases} 1 & \text{if } \beta \equiv \tilde{\alpha} \pmod{d} \text{ with } \tilde{\alpha} \text{ some associate of } \alpha; \\ 0 & \text{otherwise,} \end{cases}$$

*where*

$$H_d = \begin{cases} 1 & \text{if } \mathrm{N}(d) = 1 \text{ or } \mathrm{N}(d) = 2; \\ 2 & \text{if } \mathrm{N}(d) = 4; \\ 4 & \text{otherwise.} \end{cases}$$

*Proof.* Since $\chi(\mathrm{i}) = 1$ we can view $\chi$ as a character of the quotient group given by $\frac{(\mathbb{Z}[\mathrm{i}]/d\mathbb{Z}[\mathrm{i}])^\times}{\{1,-1,\mathrm{i},-\mathrm{i}\}^\times}$, which has $\frac{\Phi(d)}{H_d}$ elements, with $\Phi(d)$ being the order of the multiplicative group $(\mathbb{Z}[\mathrm{i}]/d\mathbb{Z}[\mathrm{i}])^\times$. ∎

Note that the Gaussian $\Phi$-function behaves like the ordinary Euler $\phi$-function, in the sense that it is a multiplicative arithmetic function (see [Cro83]). We now note the important relationship between $\psi_{\mathbb{Q}[\mathrm{i}]}(x, \chi)$ and $\psi_{\mathbb{Q}[\mathrm{i}]}(x; d, \alpha)$:

$$\begin{aligned} \psi_{\mathbb{Q}[\mathrm{i}]}(x; d, \alpha) &= \sum_{\substack{\mathrm{N}(\beta)\leqslant x \\ \beta\equiv\alpha \pmod{d}}}^{\sim} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta) \\ &= \sum_{\mathrm{N}(\beta)\leqslant x}^{\sim} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta) \frac{H_d}{\Phi(d)} \sum_{\chi_d}^{(0)} \overline{\chi}(\alpha)\chi(\beta) \\ &= \frac{H_d}{\Phi(d)} \sum_{\chi_d}^{\sim} \overline{\chi}(\alpha) \sum_{\mathrm{N}(\beta)\leqslant x}^{(0)} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\chi(\beta) \\ &= \frac{H_d}{\Phi(d)} \sum_{\chi_d}^{(0)} \overline{\chi}(\alpha)\psi_{\mathbb{Q}[\mathrm{i}]}(x, \chi). \end{aligned} \tag{2.31}$$

Using this relationship, every estimation of $\psi_{\mathbb{Q}[\mathrm{i}]}(x, \chi)$ has an equivalent estimation of $\psi_{\mathbb{Q}[\mathrm{i}]}(x; d, \alpha)$, which, in turn, can be made into a version of the prime number theorem by partial summation in the same way we do this for rational primes. We will hence from here on only prove facts about $\psi_{\mathbb{Q}[\mathrm{i}]}(x, \chi)$. We use our bound for $\psi_{\mathbb{Q}[\mathrm{i}]}(x, \chi)$ from Theorem 2.9.2. First we estimate the sum over the non-exceptional zeroes

$$S := \sum_{|\gamma|<T}{}' \frac{x^\rho}{\rho}.$$

We recall that all (non-exceptional) zeroes $\rho = \beta + \mathrm{i}\gamma$ satisfy $\beta < 1 - \frac{F}{\log |d|T}$ for some constant $F$. Thus

$$|x^\rho| = x^\beta < x\mathrm{e}^{-F\frac{\log x}{\log |d|T}}.$$

It hence remains to estimate

$$S := \sum_{|\gamma|<T}{}' \frac{1}{|\rho|},$$

which we split up to get

$$S = \sum_{1<|\gamma|<T}{}' \frac{1}{|\rho|} + \sum_{|\gamma|\leqslant 1}{}' \frac{1}{|\rho|} = S_1 + S_2.$$

First suppose $|\gamma| > 1$. Then

$$S_1 \ll \sum_{1\leqslant|\gamma|<T}{}' \frac{1}{|\gamma|} = \int_1^T \frac{1}{t}\mathrm{dN}(t,\chi) = \frac{1}{T}\mathrm{N}(T,\chi) + \int_1^T \frac{\mathrm{N}(t,\chi)}{t^2}\mathrm{d}t.$$

We now use (2.28) which states

$$\begin{aligned}\mathrm{N}(T,\chi) &= \frac{T}{\pi}\log\frac{|d|^2T}{\pi^2} - \frac{T}{\pi} + \mathrm{O}(\log T + \log|d|)\\ &\ll T\log(|d|T),\end{aligned}$$

and so

$$\begin{aligned}S_1 &\ll \frac{1}{T}\mathrm{N}(T,\chi) + \int_1^T \frac{\mathrm{N}(t,\chi)}{t^2}\mathrm{d}t\\ &\ll \log(|d|T) + \int_1^T \frac{\log(|d|t)}{t}\mathrm{d}t\\ &\ll \log^2(|d|T) \ll \log^2(|d|x) \qquad (2.32)\end{aligned}$$

since $2 \leqslant T \leqslant x$. For $|\gamma| \leqslant 1$ we have

$$\frac{1}{|\rho|} = \frac{1}{\sqrt{\beta^2+\gamma^2}} \leqslant \frac{1}{|\beta|}.$$

Now $\beta < 1 - \frac{F}{\log|d|}$ for the zeroes in question, which are symmetric with respect to $\sigma = \frac{1}{2}$, so $\beta > \frac{F}{\log|d|}$. Thus

$$\frac{1}{|\rho|} = \mathrm{O}(\log|d|).$$

By (2.28) with $T = 2$ we also get that there are $\mathrm{O}(\log|d|)$ of these zeroes and hence

$$S_2 = \sum_{|\gamma|\leqslant 1}{}' \frac{1}{|\rho|} = \mathrm{O}(\log^2|d|). \qquad (2.33)$$

Using this in Theorem 2.9.2 we get

**Theorem 2.10.4.** *If $\chi$ is a nonprincipal character with conductor $d$ and suppose $2 \leqslant T \leqslant x$, then*

$$\psi_{\mathbb{Q}[i]}(x, \chi) = -\frac{x^{\beta_1}}{\beta_1} + R_4(x, T),$$

*where*

$$|R_4(x, T)| \ll x \log^2(|d|x) \mathrm{e}^{-F \frac{\log x}{\log |d| T}} + \frac{x}{T} \log^2(|d|x) + x^{\frac{1}{4}} \log x,$$

*and the term $-\frac{x^{\beta_1}}{\beta_1}$ is to be omitted unless $\chi$ is a real character for which $L_{\mathbb{Q}[i]}(s, \chi)$ has a real zero $\beta_1$ for which $\beta_1 > 1 - \frac{F}{\log |d|}$ for some positive constant $F$.*

Since this is very similar to the rational case, the rest is exactly as in [Dav80]. We impose a condition on $d$ that

$$|d| \leqslant \mathrm{e}^{C \log^{\frac{1}{2}} x},$$

where $C$ is an arbitrary positive constant, and we choose

$$T = \mathrm{e}^{C \log^{\frac{1}{2}} x}.$$

With these conditions we get that all terms of $R_4(x, T)$ are

$$\ll x\mathrm{e}^{-C' \log^{\frac{1}{2}} x},$$

and so

$$|R_4(x, T)| \ll x\mathrm{e}^{-C' \log^{\frac{1}{2}} x}$$

for some constant $C'$.

**Theorem 2.10.5.** *If $\chi$ is a nonprincipal character with conductor $d$ and*

$$|d| \leqslant \mathrm{e}^{C \log^{\frac{1}{2}} x},$$

*then*

$$\psi_{\mathbb{Q}[i]}(x, \chi) = -\frac{x^{\beta_1}}{\beta_1} + \mathrm{O}(x\mathrm{e}^{-C' \log^{\frac{1}{2}} x}).$$

*and the term $-\frac{x^{\beta_1}}{\beta_1}$ is to be omitted unless $\chi$ is a real character for which $L_{\mathbb{Q}[i]}(s, \chi)$ has a real zero $\beta_1$ for which $\beta_1 > 1 - \frac{F}{\log |d|}$ for some positive constant $F$.*

We could now prove a weak form of the Prime Number Theorem as done in [Dav80], by using a weak upper bound on $\beta_1$, but it is not required for the purpose of this thesis, so we will skip this step and instead proceed directly to Siegel's Theorem, which gives us a strong bound for $\beta_1$ which we can use to deduce the required version of the Prime Number Theorem. However, before we do that, we will finish off this section by looking at what happens if we assume the Generalised Riemann Hypothesis for Gaussian Integers, which states that the Dedekind $\zeta$-function as well as all $L$-functions have their non-trivial zeroes on the critical line $\sigma = \frac{1}{2}$.

**Theorem 2.10.6.** *If we assume the Generalised Riemann Hypothesis, then*

$$\psi_{\mathbb{Q}[i]}(x, \chi_0) = x + \mathrm{O}(x^{\frac{1}{2}} \log^2 x).$$

*Proof.* By (2.29) we have

$$\psi_{\mathbb{Q}[i]}(x) = \mathrm{O}(\log x) + \psi(x, \chi_0) + \psi(x, \chi_1) + \frac{1}{2}\left(\psi(\sqrt{x}, \chi_0) - \psi(\sqrt{x}, \chi_1)\right),$$

and we know from the theory of the usual $\zeta$-function ([Dav80, pp.113,125])

$$\begin{aligned} \psi(x, \chi_0) &= x + \mathrm{O}(x^{\frac{1}{2}} \log^2 x) \\ \psi(x, \chi_1) &\ll x^{\frac{1}{2}} \log^2 x, \end{aligned}$$

and so

$$\psi_{\mathbb{Q}[i]}(x) = x + \mathrm{O}(x^{\frac{1}{2}} \log^2 x).$$

By (2.30) we have

$$|\psi_{\mathbb{Q}[i]}(x, \chi_0) - \psi_{\mathbb{Q}[i]}(x)| \ll \log x \log \mathrm{N}(4).$$

Thus

$$\psi_{\mathbb{Q}[i]}(x, \chi_0) = x + \mathrm{O}(x^{\frac{1}{2}} \log^2 x).$$

■

**Theorem 2.10.7.** *If we assume the Generalised Riemann Hypothesis and if* $\chi \neq \chi_0$ *and* $|d| \leqslant x$*, then*

$$\left|\psi_{\mathbb{Q}[\mathrm{i}]}(x,\chi)\right| \ll x^{\frac{1}{2}} \log^2 x.$$

*Proof.* By Theorem 2.9.2 we have for $2 \leqslant T \leqslant x$

$$\psi_{\mathbb{Q}[\mathrm{i}]}(x,\chi) \ll \frac{x^{\beta_1}}{\beta_1} + \left|{\sum_{|\gamma|<T}}' \frac{x^\rho}{\rho}\right| + \frac{x}{T}\log^2(|d|x) + x^{\frac{1}{4}} \log x.$$

The Generalised Riemann Hypothesis tells us that all non-trivial zeroes lie on the line $\sigma = \frac{1}{2}$, and thus

$$\psi_{\mathbb{Q}[\mathrm{i}]}(x,\chi) \ll x^{\frac{1}{2}} + x^{\frac{1}{2}} {\sum_{|\gamma|<T}}' \left|\frac{1}{\rho}\right| + \frac{x}{T}\log^2(|d|x) + x^{\frac{1}{4}} \log x.$$

By (2.32) and (2.33) we know

$${\sum_{|\gamma|<T}}' \frac{1}{|\rho|} \ll \log^2(|d|x),$$

and we take $T = x^{\frac{1}{2}}$ to get

$$\psi_{\mathbb{Q}[\mathrm{i}]}(x,\chi) \ll x^{\frac{1}{2}} \log^2(|d|x).$$

■

## 2.11 The Siegel–Walfisz Theorem

We will now give a proof for Siegel's Theorem which will essentially follow the proof first given by Estermann [Est48] using standard methods. The reader might also be interested in seeing the general proof for arbitrary number fields given by Goldstein in [Gol70]. As can be seen, the generalised proof is not much more complicated, especially if one is only interested in real characters and in fields of class number 1.

We start with a lemma:

**Lemma 2.11.1.** *Let* $\chi$ *be a non-principal character. Then in the circle* $|s-2| = \frac{3}{2}$ *we have*

$$|L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)| \ll |d|^{\frac{3}{2}}.$$

*Proof.* Recall the formula for the $L$-function (2.7), which is valid on the whole of the complex plane:

$$4\left(\frac{|d|}{\pi}\right)^s \Gamma(s) L_{\mathbb{Q}[\mathrm{i}]}(s,\chi) = \int_1^\infty x^{s-1}\psi(x,\chi)\mathrm{d}x + \frac{\tau(\chi)}{|d|}\int_1^\infty x^{-s}\psi(x,\overline{\chi})\mathrm{d}x,$$

that is

$$L_{\mathbb{Q}[\mathrm{i}]}(s,\chi) = \frac{\pi^s}{|d|^s 4\Gamma(s)}\left(\int_1^\infty x^{s-1}\psi(x,\chi)\mathrm{d}x + \frac{\tau(\chi)}{|d|}\int_1^\infty x^{-s}\psi(x,\overline{\chi})\mathrm{d}x\right), \quad (2.34)$$

where now $\psi(x,\chi)$ is the theta function

$$\psi(x,\chi) = \sum_{\alpha\in\mathbb{Z}[\mathrm{i}]} \chi(\alpha)\mathrm{e}^{-\frac{\mathrm{N}(\alpha)\pi x}{|d|}}.$$

For $x \geqslant 1$ we know that

$$\begin{aligned} |\psi(x,\chi)| &= \left|\sum_{\alpha\in\mathbb{Z}[\mathrm{i}]} \chi(\alpha)\mathrm{e}^{-\frac{\mathrm{N}(\alpha)\pi x}{|d|}}\right| \\ &\ll \sum_{\substack{m,n\in\mathbb{Z} \\ m,n \text{ not both } 0}} \mathrm{e}^{-\frac{(m^2+n^2)\pi x}{|d|}} \\ &= \varpi\left(\frac{x}{|d|}\right), \end{aligned}$$

where $\varpi(x)$ is as in (2.1). We hence know from (2.2) that

$$|\psi(x,\chi)| = \varpi\left(\frac{x}{|d|}\right) = 4\left(\omega\left(\frac{x}{|d|}\right)^2 + \omega\left(\frac{x}{|d|}\right)\right).$$

We now use Lemma 2.5.2 to get

$$\begin{aligned} |\psi(x,\chi)| &= 4\left(\omega\left(\frac{x}{|d|}\right)^2 + \omega\left(\frac{x}{|d|}\right)\right) \\ &\ll \max\left(1, \frac{|d|^2}{x^2}\right)\mathrm{e}^{-\frac{\pi x}{|d|}}. \end{aligned} \quad (2.35)$$

In the circle given by $|s-2| = \frac{3}{2}$ we have that $\Gamma(s)$ is non-zero, and hence

$$\frac{1}{\Gamma(s)} = \mathrm{O}(1).$$

Further in that region we get

$$\left(\frac{\pi}{|d|}\right)^s \ll |d|^{-\frac{1}{2}}.$$

By Theorem 2.5.7 we also know that $|\tau(\chi)| = |d|$. Using all of these estimations in (2.34), we get that in the above region

$$\begin{aligned} |L_{\mathbb{Q}[i]}(s,\chi)| &\ll |d|^{-\frac{1}{2}} \left| \int_1^\infty x^{s-1}\psi(x,\chi)\mathrm{d}x + \frac{\tau(\chi)}{|d|} \int_1^\infty x^{-s}\psi(x,\overline{\chi})\mathrm{d}x \right| \\ &\ll |d|^{-\frac{1}{2}} \left(|d|^2 + |d|^2\right) \ll |d|^{\frac{3}{2}}. \end{aligned}$$

■

We first note that if $\chi_1$, $\chi_2$ are both real primitive (and hence nonprincipal) characters to the distinct moduli $d_1$, $d_2$ respectively, then $\chi_1\chi_2$ is a character to the modulus $d_1 d_2$, which is nonprincipal. The former is a standard fact from character theory and the latter follows from the fact that if it were principal, then, since $\chi_1$ and $\chi_2$ are real, it would mean that $\chi_1$ and $\chi_2$ would agree on values coprime to $d_1 d_2$, which would in turn mean that they both induce the same character to the modulus $d_1 d_2$. This is impossible since the conductor of a character is a well-defined property.

We now define

$$F(s) = \zeta_{\mathbb{Q}[i]}(s) L_{\mathbb{Q}[i]}(s,\chi_1) L_{\mathbb{Q}[i]}(s,\chi_2) L_{\mathbb{Q}[i]}(s,\chi_1\chi_2)$$

and note that since all characters involved are non-principal, $F(s)$ is regular everywhere except for a simple pole at $s = 1$, where the residue is

$$\lambda = L_{\mathbb{Q}[i]}(1,\chi_1) L_{\mathbb{Q}[i]}(1,\chi_2) L_{\mathbb{Q}[i]}(1,\chi_1\chi_2).$$

**Lemma 2.11.2.** *If $s$ is real with $\frac{7}{8} \leqslant s < 1$ then we have*

$$F(s) > \frac{1}{2} - \frac{E\lambda}{1-s} |d_1 d_2|^{12(1-s)}$$

*for some positive constant $E$.*

*Proof.* Since all of the functions involved have Euler products, we know that $F(s)$ can be written as

$$F(s) = \frac{1}{4} \sum_{\alpha \in \mathbb{Z}[i]} \frac{a_\alpha}{\mathrm{N}(\alpha)^s}$$

whenever $\sigma > 1$ with $a_1 = 1$. We further know that

$$\begin{aligned}\log F(s) &= \log \zeta_{\mathbb{Q}[\mathrm{i}]}(s) + \log L_{\mathbb{Q}[\mathrm{i}]}(s,\chi_1) + \log L_{\mathbb{Q}[\mathrm{i}]}(s,\chi_2) + \log L_{\mathbb{Q}[\mathrm{i}]}(s,\chi_1\chi_2)\\ &= \sum_{\pi}\sum_{k=1}^{\infty} \frac{\left(1+\chi_1(\pi^k)\right)\left(1+\chi_2(\pi^k)\right)}{k\mathrm{N}(\pi)^{ks}}.\end{aligned}$$

Thus it follows from the general theory of Dirichlet series that, since the coefficients in that sum are non-negative, so are the coefficients $a_\alpha$. We can now use de Vallée Poussin's argument and expand $F(s)$ as a Taylor series around 2. We know that the radius of convergence of that series is 1. We get

$$F(s) = \sum_{k=0}^{\infty} \frac{F^{(k)}(2)(s-2)^k}{k!}.$$

We can calculate $F^{(k)}(2)$ from the Dirichlet series and get

$$F^{(k)}(2) = \frac{(-1)^k}{4} \sum_{\alpha\in\mathbb{Z}[\mathrm{i}]} \frac{a_\alpha \log^k \mathrm{N}(\alpha)}{\mathrm{N}(\alpha)^2} = (-1)^k b_k,$$

where $b_k \geqslant 0$, and so

$$F(s) = \sum_{k=0}^{\infty} c_k (2-s)^k,$$

where $c_k = \frac{b_k}{k!} \geqslant 0$. We now remove the singularity that $F(s)$ has at 1 to get that

$$F(s) - \frac{\lambda}{s-1}$$

is a holomorphic function in the disc around 2 with radius of at least 2. The second term has the Taylor expansion

$$\frac{\lambda}{s-1} = \frac{\lambda}{1-(2-s)} = \sum_{k=0}^{\infty} \lambda(2-s)^k,$$

which is valid for $|s-2| < 1$. Thus in that region we have

$$F(s) - \frac{\lambda}{s-1} = \sum_{k=0}^{\infty} (c_k - \lambda)(2-s)^k. \tag{2.36}$$

By our observation above the left hand side is regular for $|s-2| < 2$ and uniqueness of Laurent expansions then implies that the right side must also be valid in this region.

We now use Lemma 2.11.1 to get that for $|s-2| = \frac{3}{2}$ we have

$$\begin{aligned} |L(s,\chi_1)| &\ll |d_1|^{\frac{3}{2}} \\ |L(s,\chi_2)| &\ll |d_2|^{\frac{3}{2}} \\ |L(s,\chi_1\chi_2)| &\ll |d_1 d_2|^{\frac{3}{2}}. \end{aligned}$$

We also note that on that circumference $\zeta_{\mathbb{Q}[\mathrm{i}]}(s)$ is bounded. Thus

$$|F(s)| \ll |d_1 d_2|^3,$$

and we also have

$$\left|\frac{\lambda}{s-1}\right| \ll \frac{L_{\mathbb{Q}[\mathrm{i}]}(1,\chi_1) L_{\mathbb{Q}[\mathrm{i}]}(1,\chi_2) L_{\mathbb{Q}[\mathrm{i}]}(1,\chi_1\chi_2)}{|s-1|} \ll |d_1 d_2|^3.$$

We can hence apply Cauchy's inequalities for the coefficients of a power series to (2.36) and get

$$|c_k - \lambda| \ll |d_1 d_2|^3 \left(\frac{2}{3}\right)^k.$$

Suppose now $\frac{7}{8} \leqslant s < 1$. We have

$$\begin{aligned} \sum_{k=M}^{\infty} |c_k - \lambda|(2-s)^k &\ll \sum_{k=M}^{\infty} |d_1 d_2|^3 \left|\frac{2}{3}(2-s)\right|^k \\ &\ll |d_1 d_2|^3 \sum_{k=M}^{\infty} \left|\frac{2}{3} \times \frac{9}{8}\right|^k \\ &\ll |d_1 d_2|^3 \sum_{k=M}^{\infty} \left(\frac{3}{4}\right)^k \\ &\ll |d_1 d_2|^3 \left(\frac{3}{4}\right)^M \ll |d_1 d_2|^3 \mathrm{e}^{-\frac{M}{4}}. \end{aligned}$$

We apply that $c_0 \geqslant 1$ and $c_k \geqslant 0$ and get for $\frac{7}{8} \leqslant s < 1$

$$\begin{aligned} F(s) - \frac{\lambda}{s-1} &= \sum_{k=0}^{\infty} (c_k - \lambda)(2-s)^k \\ &\geqslant 1 - \sum_{k=0}^{M-1} \lambda (2-s)^k - C|d_1 d_2|^3 \mathrm{e}^{-\frac{M}{4}} \\ &\geqslant 1 - \lambda \frac{(2-s)^M}{1-s} - C|d_1 d_2|^3 \mathrm{e}^{-\frac{M}{4}}. \end{aligned}$$

We now choose $M$ such that

$$\frac{1}{2}\mathrm{e}^{-\frac{1}{4}} \leqslant C|d_1 d_2|^3 \mathrm{e}^{-\frac{M}{4}} < \frac{1}{2}$$

and get

$$F(s) > \frac{1}{2} - \frac{\lambda}{1-s}(2-s)^M,$$

where

$$-\log 2 - \frac{1}{4} \leqslant -\frac{M}{4} + 3\log|d_1 d_2| + D.$$

Consequently

$$M \leqslant 12\log|d_1 d_2| + D'.$$

We can thus conclude that

$$(2-s)^M = \mathrm{e}^{M\log(2-s)} = \mathrm{e}^{M\log(1+1-s)} < \mathrm{e}^{M(1-s)} \ll |d_1 d_2|^{12(1-s)}.$$

We have hence proven that for $\frac{7}{8} \leqslant s < 1$

$$F(s) > \frac{1}{2} - \frac{E\lambda}{1-s}|d_1 d_2|^{12(1-s)}$$

for some constant $E$. ■

We now want to prove that $L(\sigma, \chi) \ll \log|d|$ for $1 - \log|d| < \sigma \leqslant 1$. To do that we have to refer to a version of the Phragmen–Lindelöf principle, the proof of which is well-known (see [PL08]) and hence omitted.

**Theorem 2.11.3.** *(Lindelöf's Theorem) Suppose $f(\sigma + \mathrm{i}t)$ is holomorphic in the strip $a < \sigma < b$ and continuous on $a \leqslant \sigma \leqslant b$, and suppose that $|f(\sigma+\mathrm{i}t)|$ is bounded by $M$ on the edges $\sigma = a$ and $\sigma = b$ and satisfies $f = \mathrm{O}(\mathrm{e}^{|t|^C})$ as $|t| \to \infty$ for some constant $C$. Then $|f(\sigma + \mathrm{i}t)| \leqslant M$ for $a \leqslant \sigma \leqslant b$.*

**Lemma 2.11.4.** *If $1 - \frac{1}{\log|d|} \leqslant \sigma \leqslant 1$, then*

$$L_{\mathbb{Q}[\mathrm{i}]}(1, \chi) \ll \log|d|.$$

*Proof.* Define

$$G(s) := \frac{L_{\mathbb{Q}[\mathrm{i}]}(s,\chi)}{(s+2)^3|d|^{-3(s-1)}},$$

and let $a=-1$, $b = 1+\frac{1}{\log|d|}$. For $\sigma>1$ we know that

$$|L_{\mathbb{Q}[\mathrm{i}]}(\sigma+\mathrm{i}t,\chi)| \leqslant \zeta_{\mathbb{Q}[\mathrm{i}]}(\sigma) = \zeta(\sigma)L(\sigma,\chi_4) \ll \sum_{n=1}^{\infty}\frac{1}{n^\sigma} \ll \frac{1}{\sigma-1},$$

where $\chi_4$ is the non-trivial character $\pmod 4$. For $\sigma = b$ we get

$$\left|L_{\mathbb{Q}[\mathrm{i}]}\left(1+\frac{1}{\log|d|}+\mathrm{i}t,\chi\right)\right| \ll \log|d|.$$

To get a bound for the boundary $\sigma = a$, we use the functional equation. By the above fact we have

$$|L_{\mathbb{Q}[\mathrm{i}]}(2+\mathrm{i}t,\chi)| \ll 1.$$

By the functional equation we know

$$\frac{\pi}{|d|}\left|\Gamma(-1+\mathrm{i}t)L_{\mathbb{Q}[\mathrm{i}]}(\chi,-1+\mathrm{i}t)\right| = \left(\frac{|d|}{\pi}\right)^2\left|\Gamma(2+\mathrm{i}t)L_{\mathbb{Q}[\mathrm{i}]}(2+\mathrm{i}t,\overline{\chi})\right|.$$

It is well known that $\Gamma(s+1) = s\Gamma(s)$, so

$$\Gamma(2+\mathrm{i}t) = (1+\mathrm{i}t)\mathrm{i}t(-1+\mathrm{i}t)\Gamma(-1+\mathrm{i}t),$$

and thus

$$\left|L_{\mathbb{Q}[\mathrm{i}]}(-1+\mathrm{i}t,\chi)\right| \leqslant \left(\frac{|d|}{\pi}\right)^3(t^2+1)^{\frac{3}{2}}\left|L_{\mathbb{Q}[\mathrm{i}]}(2+\mathrm{i}t,\overline{\chi})\right| \ll |d|^3(t^2+1)^{\frac{3}{2}}.$$

We now return to $G(s)$. If $\sigma=-1$, we have

$$G(-1+\mathrm{i}t) = \frac{L_{\mathbb{Q}[\mathrm{i}]}(-1+\mathrm{i}t,\chi)}{(1+\mathrm{i}t)^3|d|^{-3(-2+\mathrm{i}t)}} \ll \frac{|d|^3(t^2+1)^{\frac{3}{2}}}{|d|^6(t^2+1)^{\frac{3}{2}}} \ll 1,$$

and for $\sigma = 1+\frac{1}{\log|d|}$ we have

$$G\left(1+\frac{1}{\log|d|}+\mathrm{i}t\right) = \frac{L_{\mathbb{Q}[\mathrm{i}]}(1+\frac{1}{\log|d|+\mathrm{i}t},\chi)}{\left(3+\frac{1}{\log|d|}+\mathrm{i}t\right)^3|d|^{-3\left(\frac{1}{\log|d|}+\mathrm{i}t\right)}} \ll \frac{\log|d|}{(t^2+16)^{\frac{3}{2}}\mathrm{e}^{-3}} \ll \log|d|.$$

Hence, there exists a constant $D$ with

$$|G(s)| \leqslant D\log|d|$$

on the edges $a$ and $b$. We know that $G$ is holomorphic and continuous in the strip, thus for Theorem 2.11.3 to be applicable we still need to show that $G(s) = \mathrm{O}(\mathrm{e}^{|t|^C})$ in this strip as $|t| \to \infty$. To do that we follow the same strategy we used in Lemma 2.11.1. We know from (2.34) that

$$L_{\mathbb{Q}[\mathrm{i}]}(s,\chi) = \frac{\pi^s}{|d|^s 4\Gamma(s)} \left( \int_1^\infty x^{s-1}\psi(x,\chi)\mathrm{d}x + \frac{\tau(\chi)}{|d|} \int_1^\infty x^{-s}\psi(x,\overline{\chi})\mathrm{d}x \right). \quad (2.37)$$

By (2.35) we also know that

$$|\psi(x,\chi)| \ll \max\left(1, \frac{|d|^2}{x^2}\right) \mathrm{e}^{-\frac{\pi x}{|d|}}. \quad (2.38)$$

Hence if $a \leqslant \sigma \leqslant b$

$$\begin{aligned} L_{\mathbb{Q}[\mathrm{i}]}(s,\chi) &\ll \frac{1}{|d|^{-1}\Gamma(s)} \left( \int_1^\infty x^{\sigma-1} |\psi(x,\chi)|\, \mathrm{d}x + \int_1^\infty x^{-\sigma} |\psi(x,\overline{\chi})|\, \mathrm{d}x \right) \\ &\ll \frac{|d|^3}{\Gamma(s)}. \end{aligned}$$

It is a known fact that from Stirling's formula for $\log\Gamma(s)$ that

$$\Gamma(s) \gg \mathrm{e}^{-|s|\log|s|}.$$

Hence

$$L_{\mathbb{Q}[\mathrm{i}]}(s,\chi) \ll |d|^3 \mathrm{e}^{|s|\log|s|} = |d|^3 \mathrm{e}^{(2+t^2)^{\frac{1}{2}} \log\left((2+t^2)^{\frac{1}{2}}\right)}$$

in our strip. And hence, from the definition of $G(s)$, we deduce that $G(s) = \mathrm{O}(\mathrm{e}^{|t|^C})$ in this strip as $|t| \to \infty$. We can therefore apply Lindelöf's Theorem to $G(s)$ to deduce that

$$|G(\sigma + \mathrm{i}t)| \leqslant D \log|d|$$

for $-1 \leqslant \sigma \leqslant 1 + \frac{1}{\log|d|}$. This means that

$$L_{\mathbb{Q}[\mathrm{i}]}(\sigma + \mathrm{i}t, \chi) \ll (|t|+2)^3 |d|^{-3(\sigma-1)} \log|d| \quad (2.39)$$

for the same range. However, if we reduce the range to $1 - \frac{1}{\log|d|} \leqslant \sigma \leqslant 1 + \frac{1}{\log|d|}$, then

$$L_{\mathbb{Q}[\mathrm{i}]}(\sigma,\chi) \ll |d|^{-3(\sigma-1)} \log|d| \ll \log|d|.$$

■

**Theorem 2.11.5** (Siegel's Theorem I). *For any $\varepsilon > 0$ there exists a positive number $C_\varepsilon$, such that if $\chi$ is any real, non-principal character with modulus $d$, then*

$$L_{\mathbb{Q}[i]}(1,\chi) > C_\varepsilon |d|^{-\varepsilon}.$$

*Proof.* Following Estermann we will use the inequality we have just established in Lemma 2.11.2 to prove Siegel's Theorem in our case.

- Suppose first that there is a character $\chi$ such that $L_{\mathbb{Q}[i]}(s,\chi)$ has a real zero between $1 - \frac{1}{24}\varepsilon$ and 1. We choose $\chi_1$ to be that character and $\beta_1$ to be that zero, and we get $F(\beta_1) = 0$ for any $\chi_2$.

- If such a zero does not exist, then we let $\chi_1$ be any real primitive character and $\beta_1$ be any number between $1 - \frac{1}{24}\varepsilon$ and 1. We note that $F(\sigma) > 0$ for $\sigma > 1$ (since the Euler products of all factors are positive) and $F(s)$ has a simple pole at $s = 1$, ie after removing the singularity $(s-1)F(s)$ does not have a zero at $s = 1$. Thus $(\sigma - 1)F(\sigma) > 0$ for $1 < \sigma$ and $(s-1)F(s) \neq 0$ for $1 - \frac{1}{24}\varepsilon < \sigma$ (after removing the singularity). Thus we must have $(\sigma-1)F(\sigma) > 0$ for $1 - \frac{1}{24}\varepsilon < \sigma \leqslant 1$, which, in turn, means that $F(\sigma) < 0$ for $1 - \frac{1}{24}\varepsilon < \sigma < 1$.

In both cases we thus have $F(\beta_1) \leqslant 0$. We now use Lemma 2.11.2 to deduce

$$0 \geqslant F(\beta_1) > \frac{1}{2} - \frac{E\lambda}{1-\beta_1}|d_1 d_2|^{12(1-\beta_1)},$$

which implies

$$\begin{aligned} \frac{E\lambda}{1-\beta_1}|d_1 d_2|^{12(1-\beta_1)} &> \frac{1}{2}; \\ E\lambda &> \frac{1}{2}(1-\beta_1)|d_1 d_2|^{-12(1-\beta_1)}. \end{aligned}$$

We now choose $\chi_2$ to be any real primitive character to a modulus $d_2$ such that $|d_2| > |d_1|$. By Lemma 2.11.4 we know

$$\lambda < D \log|d_1| L_{\mathbb{Q}[i]}(1,\chi_2) D \log|d_1 d_2|,$$

which means

$$\begin{aligned} L_{\mathbb{Q}[\mathrm{i}]}(1,\chi_2) &> \frac{\lambda}{D^2 \log|d_1| \log|d_1 d_2|} \\ &> \frac{\frac{1}{2}(1-\beta_1)|d_1 d_2|^{-12(1-\beta_1)}}{ED^2 \log|d_1| \log|d_1 d_2|} \\ &> C_\varepsilon \frac{|d_2|^{-12(1-\beta_1)}}{\log|d_2|}, \end{aligned}$$

where $C_\varepsilon$ only depends on $\varepsilon$ since this is how we chose all parameters in the procedure. As $12(1-\beta_1) < \frac{1}{2}\varepsilon$, for sufficiently large $|d_2|$ (which we may assume) we get

$$L_{\mathbb{Q}[\mathrm{i}]}(1,\chi_2) > C_\varepsilon |d_2|^{-\varepsilon}.$$

■

**Lemma 2.11.6.** *If* $1 - \frac{1}{\log|d|} \leqslant \sigma \leqslant 1$, *then*

$$L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi) \ll \log^2|d|.$$

*Proof.* By Cauchy's Differentiation Formula we have

$$L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi) = \frac{1}{2\pi\mathrm{i}} \oint \frac{L_{\mathbb{Q}[\mathrm{i}]}(s+x,\chi)}{x^2} \mathrm{d}x,$$

and the integral is around a circle path of radius $\frac{1}{\log|d|}$ around 0. From (2.39) we know

$$L_{\mathbb{Q}[\mathrm{i}]}(\sigma+\mathrm{i}t,\chi) \ll (|t|+2)^3 d^{-3(\sigma-1)} \log|d|$$

for $-1 \leqslant \sigma \leqslant 1 + \frac{1}{\log|d|}$. This means that if $1 - \frac{2}{\log|d|} \leqslant \sigma \leqslant 1 + \frac{1}{\log|d|}$ and $|t| \leqslant 1$, then

$$L_{\mathbb{Q}[\mathrm{i}]}(\sigma+\mathrm{i}t,\chi) \ll \log|d|.$$

We note that the circle path above lies within this region if $s = \sigma + \mathrm{i}t$ and for $1 - \frac{1}{\log|d|} \leqslant \sigma \leqslant 1$. We hence have

$$\begin{aligned} \left|L'_{\mathbb{Q}[\mathrm{i}]}(s,\chi)\right| &= \left|\frac{1}{2\pi\mathrm{i}} \oint \frac{L_{\mathbb{Q}[\mathrm{i}]}(s+x,\chi)}{x^2} \mathrm{d}x\right| \\ &\ll \frac{1}{2\pi} \left( \frac{2\pi}{\log|d|} \times \frac{\log|d|}{\left(\frac{1}{\log|d|}\right)^2} \right) \ll \log^2|d|. \end{aligned}$$

■

**Theorem 2.11.7.** *(Siegel's Theorem II) For any $\varepsilon > 0$ there exists a positive number $C_\varepsilon$, such that if $\chi$ is any real, non-principal character with modulus $d$, then $L_{\mathbb{Q}[i]}(\sigma, \chi) \neq 0$ for*

$$\sigma > 1 - C_\varepsilon |d|^{-\varepsilon}.$$

*Proof.* We deduce this form of Siegel's Theorem from the original form (Theorem 2.11.5) and our Lemma 2.11.6. Suppose for a contradiction that there exists such a real zero $\beta$. We may suppose that $|d|$ is sufficiently large in which case $\beta$ lies in the range $1 - \frac{1}{\log |d|} \leqslant \beta \leqslant 1$. By the Mean Value Theorem, there exists $\sigma_0$ with $\beta < \sigma_0 < 1$ such that

$$L'_{\mathbb{Q}[i]}(\sigma_0, \chi) = \frac{L_{\mathbb{Q}[i]}(1, \chi) - L_{\mathbb{Q}[i]}(\beta, \chi)}{1 - \beta} = \frac{L_{\mathbb{Q}[i]}(1, \chi)}{1 - \beta},$$

which means

$$L_{\mathbb{Q}[i]}(1, \chi) = (1 - \beta) L'_{\mathbb{Q}[i]}(\sigma_0, \chi) \ll C_\varepsilon |d|^{-\varepsilon} \log^2 |d|.$$

Now if $\chi$ is primitive then this contradicts Theorem 2.11.5 if we take the $\varepsilon$ in that theorem to be slightly smaller, such as $\frac{1}{2}\varepsilon$, say.

If $\chi$ is not primitive, then let $\chi_1$ be the inducing character with conductor $d_1 \mid d$ (in particular $|d_1| < |d|$). We know that $L_{\mathbb{Q}[i]}(\sigma, \chi)$ can only be zero if $L_{\mathbb{Q}[i]}(\sigma, \chi_1)$ is zero, and we also have

$$\sigma > 1 - C_\varepsilon |d_1|^{-\varepsilon} > 1 - C_\varepsilon |d|^{-\varepsilon}.$$

Thus we can take $C_\varepsilon$ to be the same as for the primitive characters. ∎

**Theorem 2.11.8** (Siegel–Walfisz Theorem)**.** *Suppose that*

$$|d| \leqslant (\log x)^N$$

*for some positive constant $N$. Then for any non-principal $\chi$ of conductor $d$ we have*

$$\left|\psi_{\mathbb{Q}[i]}(x, \chi)\right| \ll x \mathrm{e}^{-C_N \log^{\frac{1}{2}} x}.$$

*Proof.* Theorem 2.10.5 tells us that if $\chi$ is a nonprincipal character and

$$|d| \leqslant \mathrm{e}^{C \log^{\frac{1}{2}} x},$$

then

$$\psi(x, \chi) = -\frac{x^{\beta_1}}{\beta_1} + \mathrm{O}(x\mathrm{e}^{-C' \log^{\frac{1}{2}} x}),$$

where $\beta_1$ is the exceptional real zero. Now Siegel's Theorem II (Theorem 2.11.7) tells us that

$$\beta_1 \leqslant 1 - C_\varepsilon |d|^{-\varepsilon},$$

and hence

$$x^{\beta_1} \leqslant x\mathrm{e}^{-C_\varepsilon |d|^{-\varepsilon} \log x}.$$

We now suppose that $|d| \leqslant (\log x)^N$ and take $\varepsilon = \frac{1}{2N}$. Then

$$|d|^\varepsilon \leqslant (\log x)^{\frac{1}{2}},$$

and so

$$x^{\beta_1} \leqslant x\mathrm{e}^{-C_N (\log x)^{\frac{1}{2}}}.$$

Thus

$$\left|\psi_{\mathbb{Q}[\mathrm{i}]}(x, \chi)\right| \ll x\mathrm{e}^{-C_N \log^{\frac{1}{2}} x},$$

as required. ∎

# Chapter 3

# The Conditional Result

## 3.1 Introduction

In this chapter we shall setup and use a weighted linear sieve due to Richert, based on results by Ankeny and Onishi [AO65], to show that there are infinitely many almost primes with 3 prime factors in the set

$$\mathscr{A} = \left\{a^2 + 4b^2 : a, b \in \mathbb{Z}, 2 \mid b, a^2 + b^2 = p, a^2 + b^2 < x\right\}.$$

We will show that the sieve is applicable to our problem, by checking all of the conditions, and we will use the conditional results of the previous chapter to derive this strong result, which is conditional on the Generalised Riemann Hypothesis. Most of this chapter, however, will be devoted to getting the sieve to work with our problem and getting the error terms in a form that is adequate for the next chapter. In particular it will be seen that the major savings come from only requiring rational integer moduli in the sieve and only special characters in the error term.

## 3.2 The Sieve

**Theorem 3.2.1.** *Let*

$$\mathscr{A} := \left\{a^2 + 4b^2 : a, b \in \mathbb{Z}, 2 \mid b, a^2 + b^2 = p, a^2 + b^2 < x\right\}$$

*and*

$$\mathscr{A}_d := \{m \in \mathscr{A} : d \mid m\}.$$

*Then for squarefree $d$ we have*

$$\#\mathscr{A}_d = \frac{\omega(d)}{d} X + R_d,$$

*with*

$$X = 2\mathrm{Li}(x),$$

$$\omega(d) = \begin{cases} \frac{d2^{\nu(d)}}{\phi(d)} & \text{if } p \mid d \Rightarrow p \equiv 1 \pmod 4; \\ 0 & \text{otherwise,} \end{cases}$$

*and*

$$R_d = \begin{cases} \mathrm{O}\left(x\mathrm{e}^{-C(\log x)^{\frac{1}{2}}}\right) & \text{if } d = 1; \\ \mathrm{O}\left(\sum_{\beta \in S_2} P_K\right) + \mathrm{O}(E_1(x; 2d)) & \text{if } d \neq 1 \text{ and} \\ & p \mid d \Rightarrow p \equiv 1 \pmod 4; \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

*where*

$$P_K = \left(x^{\frac{1}{K}} + K^{2\nu(d)} \frac{x^{\frac{1}{2}}}{d^2}\right) \log x.$$

*for a fixed integer $K > 2$. $E_1(x; 2d)$ is given by* (3.7), (3.9) *and* (3.10)*. Finally, $S_2$ is a set of residue classes $\beta \pmod{2d}$, to be defined in the proof.*

*Proof.* Let $d \in \mathbb{N}$ be odd. We want to check which residue classes $\pmod d$ exist in $\mathscr{A}_d$. First suppose $q$ is a rational prime. Let $\alpha = a + \mathrm{i}b$ and let $q \mid a^2 + 4b^2$, then $a^2 \equiv -(2b)^2 \pmod q$. If $q \equiv 1 \pmod 4$, we know that $\left(\frac{-1}{q}\right) = 1$ and so there does exist an $x \in (\mathbb{Z}/q\mathbb{Z})$ such that $a \equiv \pm 2xb \pmod q$. Hence for each $b \in (\mathbb{Z}/q\mathbb{Z})$ there are two possible $a$. Thus for each integer prime $q \equiv 1 \pmod 4$ there are $2(q-1)$ residue classes $\alpha \pmod q$ for which $q \mid a^2 + 4b^2$. Obviously if $q \equiv -1 \pmod 4$, then there are no such residue classes $\alpha$. Also we note that $q \equiv 2 \pmod 4$ is impossible since we require $b$ to be even and $a^2 + b^2$ is prime, so $a$ is odd, and thus $a^2 + 4b^2$ is odd.

Now suppose $d$ is squarefree and $d = q_1 q_2 q_3 \dots q_n$ then we can apply the Chinese Remainder Theorem to the following set of equations:

$$\begin{aligned} a^2 &\equiv -(2b)^2 \pmod{q_1}; \\ a^2 &\equiv -(2b)^2 \pmod{q_2}; \\ \vdots & \;\vdots\; \vdots \\ a^2 &\equiv -(2b)^2 \pmod{q_n}; \end{aligned}$$

to get the number of solutions to

$$a^2 \equiv -(2b)^2 \pmod{d}.$$

As we know, there will be

$$2(q_1 - 1)2(q_2 - 1)\dots 2(q_n - 1) = 2^n \phi(d) = 2^{\nu(d)}\phi(d)$$

residue classes $\alpha = a + \mathrm{i}b \pmod{d}$ if $d$ is such that $d \mid a^2 + 4b^2$ if all $q_i \equiv 1 \pmod 4$ and 0 otherwise.

We now note that if $\alpha$ is an admissible residue class, then $-\alpha$ is an admissible residue class. Generally $\alpha$ and $-\alpha$ will be distinct $\pmod{d}$, unless $d \mid 2\alpha$ so that $d \mid 2$, and since $d$ is odd, that means $d = 1$. We hence assume for now that $d \neq 1$ and will deal with the case $d = 1$ later. Hence suppose $d \neq 1$ and $d$ is squarefree and such that $p \mid d \Rightarrow p \equiv 1 \pmod 4$. Then if we count only one residue class from each admissible pair $\{\alpha, -\alpha\}$, we have $2^{\nu(d)-1}\phi(d)$ possible residue classes in the set $S_1$, say, of permissible residue classes. However, our aim is to find the possible residue classes of the primes used in $\mathscr{A}_d$ and this has the further condition $2 \mid b$. To count the numbers of residue classes of elements of $\mathscr{A}_d$ we hence need to look at the congruence classes $\pmod{2d}$ now. The possible congruence classes $\beta \pmod{2d}$ with $\beta \equiv \alpha \pmod{d}$ for $\alpha \in S_1$ are $\alpha$, $\alpha + d$, $\alpha + \mathrm{i}d$ and $\alpha + d + \mathrm{i}d$. As $d$ is odd, of these only 2 can have even imaginary part and of those only one will have odd real part. If both real and imaginary part were even then the norm could not be prime, as it would be divisible by 4. We hence know that for each $\alpha \in S_1$ there is exactly one $\beta$, such that $\beta \equiv \alpha \pmod{d}$ and $\beta$ has even imaginary part and odd real part. Let $S_2$, say, be the set formed by such $\beta$. We have hence shown that there is a bijective correspondence between the elments $\alpha \in S_1$ and the elements $\beta \in S_2$. Thus both sets have exactly $2^{\nu(d)-1}\phi(d)$ elements. We also note that if $\beta \in S_2$, then none of the associates $-\beta$, $\mathrm{i}\beta$, $-\mathrm{i}\beta$ are in $S_2$.

We now define $\tilde{\pi}(x; 2d, \beta)$ to be the number of associate classes of primes which are congruent to $\beta \,(\mathrm{mod}\ 2d)$. Let $\pi$ be a Gaussian prime coprime to 2 such that $\pi \equiv \pm\alpha \,(\mathrm{mod}\ d)$ for $\alpha \in S_1$ and $2 \mid \mathrm{Im}(\pi)$. Since $2 \nmid \pi$, we have that $\mathrm{Re}(\pi)$ is odd. By the analysis above it means that there is a corresponding $\beta \in S_2$ such that $\pi \equiv \beta \,(\mathrm{mod}\ 2d)$. However for $k \in \{1, 2, 3\}$ we have $\mathrm{i}^k\pi \not\equiv \beta'$ for any $\beta' \in S_2$. As such there is a bijective correspondence between the pairs of primes $\pm\pi = a + \mathrm{i}b$ with $a^2 + 4b^2 \in \mathscr{A}_d$ and the classes of associates $\{\pi, \mathrm{i}\pi, -\pi, -\mathrm{i}\pi\}$ for which exactly one member is congruent to $\beta \in S_2$.

Hence we have for

$$\#\mathscr{A}_d = \begin{cases} 2\sum\limits_{\beta \in S_2} \tilde{\pi}(x; 2d, \beta) & \text{for } d \neq 1 \text{ such that } p \mid d \Rightarrow p \equiv 1 \,(\mathrm{mod}\ 4); \\ 0 & \text{otherwise, if } d \neq 1. \end{cases}$$

We know that

$$\tilde{\pi}(x; d, \alpha) = \sideset{}{^\sim}\sum_{\substack{\beta \in \mathbb{Z}[\mathrm{i}]\setminus\{0\} \\ \beta \equiv \alpha \ (\mathrm{mod}\ d) \\ \mathrm{N}(\beta) \leqslant x}} \frac{\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)}{\log \mathrm{N}(\beta)} - \sideset{}{^\sim}\sum_{\substack{k \geqslant 2 \\ \pi \text{ prime} \\ \pi^k \equiv \alpha \ (\mathrm{mod}\ d) \\ \mathrm{N}(\pi^k) \leqslant x}} \frac{\log \mathrm{N}(\pi)}{\log \mathrm{N}(\pi^k)}$$

For the second sum we know

$$\begin{aligned} \sideset{}{^\sim}\sum_{\substack{k \geqslant 2 \\ \pi \text{ prime} \\ \pi^k \equiv \alpha \ (\mathrm{mod}\ d) \\ \mathrm{N}(\pi^k) \leqslant x}} \frac{\log \mathrm{N}(\pi)}{\log \mathrm{N}(\pi^k)} &= \sideset{}{^\sim}\sum_{2 \leqslant k \leqslant \log_2 x} \frac{1}{k} \sum_{\substack{k \geqslant 2 \\ \pi \text{ prime} \\ \pi^k \equiv \alpha \ (\mathrm{mod}\ d) \\ \mathrm{N}(\pi^k) \leqslant x}} 1 \\ &\ll \log x \sideset{}{^\sim}\sum_{\substack{k \geqslant 2 \\ \pi \text{ prime} \\ \pi^k \equiv \alpha \ (\mathrm{mod}\ d) \\ \mathrm{N}(\pi^k) \leqslant x}} 1. \end{aligned} \tag{3.2}$$

We need to estimate the last sum.

**Lemma 3.2.2.** *Let $K > 2$ be fixed. Then*

$$\sideset{}{^\sim}\sum_{\substack{k \geqslant 2 \\ \pi \text{ prime} \\ \pi^k \equiv \alpha \ (\mathrm{mod}\ d) \\ \mathrm{N}(\pi^k) \leqslant x}} 1 \ll x^{\frac{1}{K}} + K^{2\nu(d)} \frac{x^{\frac{1}{2}}}{d^2}.$$

*Proof.* We have

$$\sideset{}{^\sim}\sum_{\substack{k\geqslant 2\\ \pi \text{ prime}\\ \pi^k\equiv\alpha \ (\mathrm{mod}\ d)\\ \mathrm{N}(\pi^k)\leqslant x}} 1 = \left|\{\pi : \pi^k \equiv \alpha \,(\mathrm{mod}\ d), \mathrm{N}(\pi^k) \leqslant x, k \geqslant 2\}\right|.$$

First suppose $k > K$, then

$$\begin{aligned}\left|\{\pi : \pi^k \equiv \alpha \,(\mathrm{mod}\ d), \mathrm{N}(\pi^k) \leqslant x\}\right| &\leqslant \left|\{\pi : \mathrm{N}(\pi^k) \leqslant x\}\right| \\ &= \left|\{\pi : \mathrm{N}(\pi) \leqslant x^{\frac{1}{K}}\}\right| \ll x^{\frac{1}{K}}. \qquad (3.3)\end{aligned}$$

Now let $2 \leqslant k \leqslant K$, then

$$\begin{aligned}&\left|\{\pi : \pi^k \equiv \alpha \,(\mathrm{mod}\ d), \mathrm{N}(\pi^k) \leqslant x\}\right| \\ &= \sum_{k=2}^{K} \sum_{\substack{\beta \ (\mathrm{mod}\ d)\\ \beta^k\equiv\alpha \ (\mathrm{mod}\ d)}} \left|\{\pi : \pi \equiv \beta \,(\mathrm{mod}\ d), \mathrm{N}(\pi^k) \leqslant x\}\right|.\end{aligned}$$

Estimating the number of possible $\pi$ trivially gives

$$\left|\{\pi : \pi^k \equiv \alpha \,(\mathrm{mod}\ d), \mathrm{N}(\pi^k) \leqslant x\}\right| \ll \sum_{k=2}^{K} \sum_{\substack{\beta \ (\mathrm{mod}\ d)\\ \beta^k\equiv\alpha \ (\mathrm{mod}\ d)}} \left(\frac{x^{\frac{1}{2k}}}{d} + 1\right)^2.$$

Modulo each Gaussian prime $\pi \mid d$ there are $k$ residue classes $\beta$, say, such that $\beta^k \equiv \alpha \,(\mathrm{mod}\ \pi)$. There are at most $2\nu(d)$ primes dividing $d$, and so by the Chinese Remainder Theorem there are at most $k^{2\nu(d)}$ residue classes $\beta$ with $\beta^k \equiv \alpha \,(\mathrm{mod}\ d)$. Thus

$$\begin{aligned}\left|\{\pi : \pi^k \equiv \alpha \,(\mathrm{mod}\ d), \mathrm{N}(\pi^k) \leqslant x\}\right| &\ll \sum_{k=2}^{K} k^{2\nu(d)} \left(\frac{x^{\frac{1}{2k}}}{d} + 1\right)^2 \\ &\ll_K K^{2\nu(d)} \sum_{k=2}^{K} \left(\frac{x^{\frac{1}{k}}}{d^2} + 1\right) \\ &\ll_K K^{2\nu(d)} \left(\frac{x^{\frac{1}{2}}}{d^2} + 1\right). \qquad (3.4)\end{aligned}$$

The combination of (3.3) and (3.4) yields the required result. ■

Using this Lemma in (3.2) gives

$$\sum^{\sim}_{\substack{k\geqslant 2\\ \pi \text{ prime}\\ \pi^k\equiv\alpha \pmod d\\ \mathrm{N}(\pi^k)\leqslant x}} \frac{\log \mathrm{N}(\pi)}{\log \mathrm{N}(\pi^k)} \ll \left(x^{\frac{1}{K}} + K^{2\nu(d)}\left(\frac{x^{\frac{1}{2}}}{d^2}+1\right)\right)\log x =: P_K \qquad (3.5)$$

Consequently

$$\tilde{\pi}(x;d,\alpha) = \sum^{\sim}_{\substack{\beta\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}\\ \beta\equiv\alpha \pmod d\\ \mathrm{N}(\beta)\leqslant x}} \frac{\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)}{\log \mathrm{N}(\beta)} + \mathrm{O}\left(P_K\right). \qquad (3.6)$$

In order to apply partial summation to the main term, we can express it as $\sum_{n\leqslant x} f(n)F(n)$ with

$$f(n) = \sum^{\sim}_{\substack{\beta\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}\\ \beta\equiv\alpha \pmod d\\ \mathrm{N}(\beta)=n}} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta);$$

$$F(n) = \frac{1}{n}.$$

Partial summation then gives

$$\sum^{\sim}_{\substack{\beta\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}\\ \beta\equiv\alpha \pmod d\\ \mathrm{N}(\beta)\leqslant x}} \frac{\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)}{\log \mathrm{N}(\beta)} = \sum_{n=1}^{x} f(n)F(n) = S(x)F(x) - \int_1^x S(t)F'(t)\mathrm{d}t,$$

where

$$S(t) = \sum_{n=1}^{t}{}^{\sim} f(n) = \sum_{\substack{\beta\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}\\ \beta\equiv\alpha \pmod d\\ \mathrm{N}(\beta)\leqslant t}} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta) = \psi_{\mathbb{Q}[\mathrm{i}]}(t;d,\alpha).$$

Hence we have

$$\sum^{\sim}_{\substack{\beta\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}\\ \beta\equiv\alpha \pmod d\\ \mathrm{N}(\beta)\leqslant x}} \frac{\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)}{\log \mathrm{N}(\beta)} = \frac{\psi_{\mathbb{Q}[\mathrm{i}]}(x;d,\alpha)}{\log x} + \int_2^x \frac{\psi_{\mathbb{Q}[\mathrm{i}]}(t;d,\alpha)}{t\log^2 t}\mathrm{d}t.$$

We shall define

$$E(x;d,\alpha) = \psi_{\mathbb{Q}[\mathrm{i}]}(x;d,\alpha) - \frac{H_d x}{\Phi(d)}. \qquad (3.7)$$

Then

$$\sum_{\substack{\beta\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}\\ \beta\equiv\alpha \pmod{d}\\ \mathrm{N}(\beta)\leqslant x}}^{\sim} \frac{\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)}{\log \mathrm{N}(\beta)} = \frac{E(x;d,\alpha)}{\log x} + \int_2^x \frac{E(t;d,\alpha)}{t\log^2 t}\mathrm{d}t + \frac{H_d}{\Phi(d)}\left(\frac{x}{\log x} + \int_2^x \frac{t}{\log^2 t}\mathrm{d}t\right). \tag{3.8}$$

For the last term we have

$$\begin{aligned}\frac{x}{\log x} + \int_2^x \frac{t}{\log^2 t}\mathrm{d}t &= \frac{x}{\log x} + \int_2^x \frac{t}{\log^2 t}\mathrm{d}t\\ &= \frac{x}{\log x} + \frac{2}{\log 2} - \frac{x}{\log x} + \int_2^x \frac{1}{\log t}\mathrm{d}t = \mathrm{Li}(x) + \frac{2}{\log 2}.\end{aligned}$$

Thus using (3.8) in (3.6) gives

$$\tilde{\pi}(x;d,\alpha) = \frac{H_d}{\Phi(d)}\mathrm{Li}(x) + \frac{E(x;d,\alpha)}{\log x} + \int_2^x \frac{E(t;d,\alpha)}{t\log^2 t}\mathrm{d}t + \mathrm{O}(P_K).$$

If we write

$$E_0(x;d) := \sum_{\beta\in S_2} E(x;d,\beta) \tag{3.9}$$

then for $d \neq 1$ with $p \mid d \Rightarrow p \equiv 1 \pmod 4$ we have

$$\begin{aligned}\#\mathscr{A}_d &= 2\sum_{\beta\in S_2}\tilde{\pi}(x;2d,\beta)\\ &= 2\sum_{\beta\in S_2}\left(\frac{H_d}{\Phi(d)}\mathrm{Li}(x) + \mathrm{O}(P_K)\right) + 2\frac{E_0(x;2d)}{\log x} + \int_2^x \frac{E_0(t;2d)}{t\log^2 t}\mathrm{d}t.\end{aligned}$$

We further define

$$E_1(x;d) := \max_{y\leqslant x}|E_0(y;d)| \tag{3.10}$$

and can then deduce

$$\#\mathscr{A}_d = 2\sum_{\beta\in S_2}\left(\frac{H_d}{\Phi(d)}\mathrm{Li}(x) + \mathrm{O}(P_K)\right) + \mathrm{O}\left(E_1(x;2d)\right).$$

We now also notice that $\Phi(d) = \phi^2(d)$ if $d$ only has prime factors $p \equiv 1 \pmod 4$ and $\Phi(2) = 2$. Proofs of this can be found in [Cro83]. It follows that

$$\#\mathscr{A}_d = 2\sum_{\beta\in S_2}\tilde{\pi}(x;2d,\beta) = \sum_{\beta\in S_2}\left(\frac{4}{\phi^2(d)}\mathrm{Li}(x) + \mathrm{O}(P_K)\right) + \mathrm{O}(E_1(x;2d)).$$

So if we take for $d \neq 1$

$$\omega(d) = \begin{cases} \frac{d2^{\nu(d)}}{\phi(d)} & \text{if } p \mid d \Rightarrow p \equiv 1 \pmod 4; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$X = 2\mathrm{Li}(x),$$

then

$$\sum_{\beta \in S_2} \frac{4\mathrm{Li}(x)}{\Phi(d)} = \#S_2 \frac{4\mathrm{Li}(x)}{\Phi(d)} = 2^{\nu(d)-1}\phi(d)\frac{4\mathrm{Li}(x)}{\Phi(d)} = \frac{\omega(d)}{d}X.$$

Consequently, for $d \neq 1$ we have

$$\#\mathscr{A}_d = \frac{\omega(d)}{d}X + \mathrm{O}\left(\sum_{\beta \in S_2} P_K\right) + \mathrm{O}(E_1(x; 2d)).$$

If, however, $d = 1$ then $\mathscr{A}_d = \mathscr{A}$, which contains as many elements as there are Gaussian primes with an even imaginary part whose norm is a rational prime. We know that for any real prime $p \equiv 1 \pmod 4$ there are 2 conjugate non-associate Gaussian primes and vice-versa, 8 primes if we do count associates. Exactly half of those will have an even imaginary part, so for each ordinary prime there are 4 Gaussian primes which produce elements in $\mathscr{A}$. By the Prime Number Theorem for real primes there are

$$\frac{\mathrm{Li}(x)}{\phi(4)} + \mathrm{O}(x\mathrm{e}^{-C(\log x)^{\frac{1}{2}}}) = \frac{\mathrm{Li}(x)}{2} + \mathrm{O}(x\mathrm{e}^{-C(\log x)^{\frac{1}{2}}})$$

rational primes such that $p \equiv 1 \pmod 4$. So in fact

$$\mathscr{A}_1 = 2\mathrm{Li}(x) + \mathrm{O}(x\mathrm{e}^{-C(\log x)^{\frac{1}{2}}}) = \frac{\omega(d)}{d}X + \mathrm{O}(x\mathrm{e}^{-C(\log x)^{\frac{1}{2}}}).$$

Combining these results, we have for all $d$

$$\#\mathscr{A}_d = \frac{\omega(d)}{d}X + R_d,$$

where

$$R_d = \begin{cases} \mathrm{O}\left(x\mathrm{e}^{-C(\log x)^{\frac{1}{2}}}\right) & \text{if } d = 1; \\ \mathrm{O}\left(\sum_{\beta \in S_2} P_K\right) + \mathrm{O}(E_1(x; 2d)) & \text{if } d \neq 1 \text{ and} \\ & p \mid d \Rightarrow p \equiv 1 \pmod 4; \\ 0 & \text{otherwise,} \end{cases}$$

■

This sets up our sieve. In particular this gives $\omega(p) = \frac{2p}{p-1}$ for all primes $p \equiv 1 \pmod 4$ and $\omega(p) = 0$ otherwise. So intuitively $\omega(p)$ is about 2 for every second prime and 0 for all the others, so on average it is about 1. Thus we are expecting the sieve weight $\kappa$ to be 1, and hence that we will use a linear sieve. In fact, we will apply a linear weighted sieve to get a lower bound for the number of almost primes in $\mathscr{A}$.

## 3.3 Weighted Linear Sieve

We will use the weighted sieve developed by Richert (see [HR74]) based on weighting results by Kuhn, which were extended by Ankeny and Onishi in [AO65]. We will use the sieve as stated in [HR74, p. 253]. We define the numbers

$$\Lambda_r = r + 1 - \frac{\log 4}{(1+3^{-r})\log 3}.$$

**Theorem 3.3.1.** *Let $\mathscr{A}$ and $\mathscr{A}_d$ be as defined above, and assume the following conditions hold:*

*(1)*

$$0 \leqslant \frac{\omega(p)}{p} \leqslant 1 - \frac{1}{A}.$$

*(2) If $2 \leqslant v \leqslant w$, then*

$$-B \log\log 3X \leqslant \sum_{v \leqslant p < w} \frac{\omega(p)}{p} \log p - \log \frac{w}{v} \leqslant B.$$

*(3) If $2 \leqslant z \leqslant y$, then*

$$\sum_{z \leqslant p < y} |A_{p^2}| \leqslant C \left( \frac{X \log X}{z} + y \right).$$

*(4) If $X \geqslant 2$, then there exists $0 < \alpha < 1$ such that*

$$\sum_{\substack{d < \frac{X^\alpha}{(\log X)^D} \\ (d, \overline{\mathfrak{P}})=1}} \mu^2(d) 3^{\nu(d)} |R_d| \leqslant E \frac{X}{\log^2 X}.$$

*Then if*

*(a) We have $(a, \overline{\mathfrak{P}}) = 1$ for all $a \in \mathscr{A}$;*

*(b) There is* $\delta \in \mathbb{R}$ *such that* $0 < \delta \leqslant \frac{2}{3}$ *and* $r \geqslant 2$ *is so large that*

$$|a| \leqslant X^{\alpha(\Lambda_r - \delta)}$$

*for all* $a \in \mathscr{A}$*;*

*we have*

$$|\{P_r : P_r \in \mathscr{A}\}| \geqslant \frac{\delta}{\alpha} \prod_p \frac{1 - \omega(p)/p}{1 - 1/p} \frac{X}{\log X} \text{ for } X \geqslant X_0.$$

*Thus there are* $\gg \frac{X}{\log X}$ *almost primes* $P_r$ *with at most* $r$ *prime factors in the set* $\mathscr{A}$*.*

Note that $\overline{\mathfrak{P}}$ is the complement of $\mathfrak{P}$ (the set of primes we are sifting by) in the set of all primes. But as we want to sift by all integer primes $p \equiv 1 \pmod 4$, $\overline{\mathfrak{P}}$ consists of all $p \equiv 2, -1 \pmod 4$. We have shown above that if such $p$ divides $d$, then $R_d = 0$ and hence the condition $(d, \overline{\mathfrak{P}}) = 1$ is actually redundant.

## 3.4 Application of the Sieve

We will now apply this theorem to the sieve as set up in Theorem 3.2.1. Our aim is to prove the main Theorem 1.2.1. We need to check all of the conditions of Theorem 3.3.1, of which the error term estimation (4) will be the most important one and will determine the number of prime factors of $a^2 + 4b^2$.

(1)

$$0 \leqslant \frac{\omega(p)}{p} \leqslant 1 - \frac{1}{A}.$$

*Proof.* If $p \not\equiv 1 \pmod 4$ then $\omega(p) = 0$, otherwise $\omega(p) = \frac{2p}{p-1}$. Hence for $p \geqslant 5$ we have

$$0 \leqslant \frac{\omega(p)}{p} = \frac{2}{p-1} \leqslant \frac{2}{4} = \frac{1}{2}.$$

We can hence take $A = 2$, say. ■

(2) If $2 \leqslant v \leqslant w$, then

$$-B \log\log 3X \leqslant \sum_{v \leqslant p < w} \frac{\omega(p)}{p} \log p - \log \frac{w}{v} \leqslant B.$$

In fact, in our case we will show that

$$-B \leqslant \sum_{v \leqslant p < w} \frac{\omega(p)}{p} \log p - \log \frac{w}{v} \leqslant B.$$

.

*Proof.* Note that

$$\begin{aligned} S &:= \sum_{v \leqslant p < w} \frac{\omega(p)}{p} \log p = \sum_{\substack{v \leqslant p < w \\ p \equiv 1 \pmod 4}} \frac{2 \log p}{p-1} \\ &= 2 \sum_{\substack{v \leqslant p < w \\ p \equiv 1 \pmod 4}} \frac{\log p}{p} + 2 \sum_{\substack{v \leqslant p < w \\ p \equiv 1 \pmod 4}} \frac{\log p}{p(p-1)}. \end{aligned}$$

Note that the second sum converges by comparison with $\sum_n \frac{1}{n^{2-\varepsilon}}$. We will now use two results which are proved in [Dav80, pp.57–58]. The first is

$$\sum_{p<x} \frac{\log p}{p} = \log x + \mathrm{O}(1).$$

and the second is

$$\sum_{n<x} \frac{\chi(n)\Lambda(n)}{n} = \mathrm{O}(1).$$

Choosing $\chi$ to be the non-trivial character $\pmod 4$ and adding the two expressions above, we get

$$2 \sum_{\substack{p<x \\ p \equiv 1 \pmod 4}} \frac{\log(p)}{p} + \sum_{\substack{p^k<x \\ k \geqslant 2}} \frac{\chi(p^k) \log p}{p^k} = \log x + \mathrm{O}(1).$$

Clearly the second sum above converges absolutely by comparison with the sum $\sum_n \frac{1}{n^{2-\varepsilon}}$, again. Thus

$$2 \sum_{\substack{p<x \\ p \equiv 1 \pmod 4}} \frac{\log(p)}{p} = \log x + \mathrm{O}(1).$$

Hence

$$2\sum_{\substack{v\leqslant p<w\\ p\equiv 1\ (\mathrm{mod}\ 4)}}\frac{\log p}{p} = 2\sum_{\substack{p<w\\ p\equiv 1\ (\mathrm{mod}\ 4)}}\frac{\log p}{p}-2\sum_{\substack{p<v\\ p\equiv 1\ (\mathrm{mod}\ 4)}}\frac{\log p}{p}$$
$$= \log w-\log v+\mathrm{O}(1)=\log\frac{w}{v}+\mathrm{O}(1).$$

That deals with the first sum component of $S$, and the second sum clearly converges. Thus

$$S=\log\frac{w}{v}+\mathrm{O}(1),$$

and hence

$$-B\leqslant S-\log\frac{w}{v}\leqslant B$$

for some constant $B$, as required. ■

(3) If $2\leqslant z\leqslant y$, then

$$\sum_{z\leqslant p<y}|\mathscr{A}_{p^2}|\leqslant C\left(\frac{X\log X}{z}+y\right).$$

*Proof.* This condition is slightly out of line, as until now we have only considered squarefree $d$ (and thus our estimates so far are not applicable) and in fact we will only need squarefree $d$ after this. Luckily we do not need a strong estimate here, in fact a very weak one turns out to be sufficient.

Let $r(n)$ be the number of ways an integer $n$ can be written as a sum of two squares. We first prove a helpful lemma.

**Lemma 3.4.1.**

$$r(p^2m)\leqslant 3r(m).$$

*Proof.* We know that there are $r(m)$ different Gaussian integers $\alpha_i$, such that $\mathrm{N}(\alpha_i)=m$. Now suppose that $\beta$ is a Gaussian integer, such that $\mathrm{N}(\beta)=p^2m$. Let the unique prime factorisation be $\beta=\pi_1\pi_2\dots\pi_n$ (up to units and order with multiplicities), where each $\pi_i$ is a Gaussian prime. Taking norms on both sides gives

$$\mathrm{N}(\beta)=\mathrm{N}(\pi_1)\mathrm{N}(\pi_2)\dots\mathrm{N}(\pi_n)=p_1^{e_1}p_2^{e_2}\dots p_n^{e_n},$$

where each $e_i$ is either 1 or 2 (depending on whether $p_i$ splits or not). From the unique factorisation of ordinary integers, we can deduce that if $p$ splits, then $p_i = p$ and $p_j = p$ for some $i \neq j$, and if $p$ is inert or is ramified then $p_i = p$ for some $i$. In any case it means that

$$\beta = \varpi\alpha,$$

such that $\mathrm{N}(\varpi) = p^2$ and $\mathrm{N}(\alpha) = m$. By the above hypothesis, this means that $\alpha = \alpha_i$ for some $i$. There are at most 3 different possible $\varpi$ (up to units), as there are at most 3 different Gaussian integers of norm $p^2$: If $p$ does not split in $\mathbb{Z}[\mathrm{i}]$, there is just one, if $p = \pi_1\pi_2$ does split then there are 3 possibilities $\varpi_1 = \pi_1\pi_1$, $\varpi_2 = \pi_1\pi_2 = p$ and $\varpi_3 = \pi_2\pi_2$. This now means that for each $\beta$ of norm $p^2m$, there exists a $\varpi_i$ and an $\alpha_j$ such that $\beta = u\varpi_i\alpha_j$ for some unit $u$. Hence for each representation of $m$ as a sum of two squares, there are at most 3 representations of $p^2m$ as a sum of two squares, which yields the required result. ■

We now continue with the original proof and apply Lemma 3.4.1. It is clear that

$$|\mathscr{A}_{p^2}| \leqslant \sum_{\substack{m \leqslant x \\ p^2 | m}} r(m) = \sum_{n=1}^{\left[\frac{x}{p^2}\right]} r(p^2 n) \leqslant 3 \sum_{n=1}^{\left[\frac{x}{p^2}\right]} r(n).$$

We will use a very crude estimate to Gauss' Circle Problem. Clearly the sum on the right is the number of integer lattice points within a circle of radius $\left[\frac{x}{p}\right]$. We will estimate that by the number of lattice points in the square of side length $\left[\frac{x}{p}\right] + 1$, which is $\left(\left[\frac{x}{p}\right] + 1\right)^2$. So

$$|\mathscr{A}_{p^2}| \leqslant 3\left(\left[\frac{x}{p}\right] + 1\right)^2 \leqslant 6\left(\left[\frac{\sqrt{x}}{p}\right]^2 + 1\right) \leqslant 6 + \frac{6x}{p^2}.$$

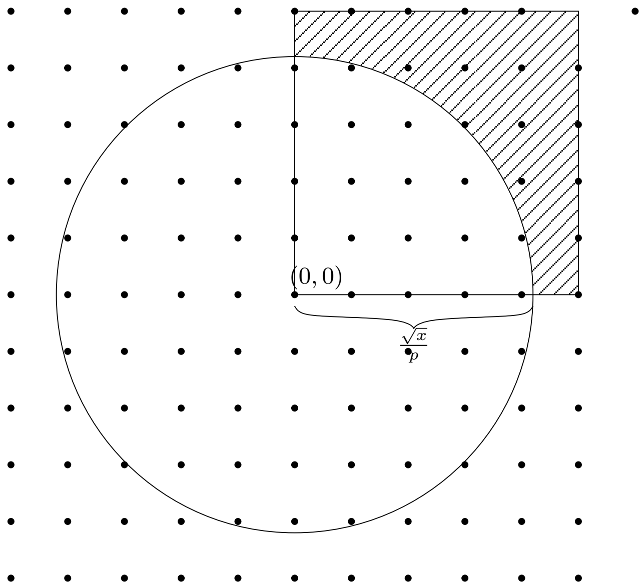


Figure 3.1: Gauss' Circle Problem

Thus for $2 \leqslant z \leqslant y$ we get

$$\begin{aligned}
\sum_{z \leqslant p < y} |\mathscr{A}_{p^2}| &\leqslant \sum_{z \leqslant p < y} \left(6 + \frac{6x}{p^2}\right) \\
&\leqslant 6y + \sum_{z \leqslant p < y} \frac{6x}{p^2} \\
&\leqslant 6y + 6x \sum_{z \leqslant p < y} \frac{1}{p^2} \\
&\leqslant 6y + 6x \int_{z-1}^{y} \frac{dt}{t^2} \\
&\leqslant 6y + 6x \left((z-1)^{-1} - y^{-1}\right) \\
&\leqslant 6y + \frac{6x}{z-1} \\
&\leqslant 6y + \frac{12x}{z}.
\end{aligned}$$

Also note that $X = 2\mathrm{Li}(x)$ and we know that

$$\mathrm{Li}(x) \sim \frac{x}{\log x},$$

and hence

$$\mathrm{Li}(x) = \mathrm{O}\left(\frac{x}{\log x}\right).$$

Also

$$\log x = \mathrm{O}\left(\log\left(\frac{x}{\log x}\right)\right) = \mathrm{O}\left(\log\left(\mathrm{Li}(x)\right)\right), \tag{3.11}$$

and so

$$x = \mathrm{O}\left(\mathrm{Li}(x)\log x\right) = \mathrm{O}\left(\mathrm{Li}(x)\log\left(\mathrm{Li}(x)\right)\right) = \mathrm{O}(X\log X). \tag{3.12}$$

It follows that

$$\sum_{z\leqslant p<y} |\mathscr{A}_{p^2}| \leqslant C\left(\frac{X\log X}{z} + y\right),$$

as required. ■

(4) If $X \geqslant 2$ then we need to find an $\alpha$ such that

$$\sum_{\substack{d<\frac{X^\alpha}{(\log X)^D}\\(d,\overline{\mathfrak{P}})=1}} \mu^2(d)3^{\nu(d)}\,|R_d| \leqslant E\frac{X}{\log^2 X}.$$

This condition is the most important one and the rest of the thesis will be dedicated to finding an admissible $\alpha$, which will determine the number of prime factors of $a^2+4b^2$ in the main theorem.

## 3.5 Error Term Estimation

Initially we shall only be concerned with $d>1$. We recall that we only have non-zero error terms if $d$ has only prime factors that are equal to $1 \pmod 4$. We also note that the sum we need to estimate only counts squarefree values of $d$. We shall hence introduce a new notation $\sum^{\#}$ which shall mean that the sum ranges only over squarefree values of the index.

**Lemma 3.5.1.** *For any fixed integer $n$ and any $Q \geqslant 1$ we have*

$$\sum_{k<Q}^{\#}\left(\frac{n^{\nu(k)}}{\phi(k)}\right) \ll_n (\log Q+1)^n.$$

*Proof.* Note that $n^{\nu(d)}$ is exactly the number of factorisations $d_1d_2\ldots d_n$ of $d$ such that $(d_1,d_2,\ldots,d_n)=1$ because as the sum ranges only over squarefree $d$, each prime factor occurs once and each prime factor can be part of exactly one $d_i$. There are $\nu(d)$ prime factors, each of which can be part of exactly one of the $n$ divisors $d_i$, giving $n^{\nu(d)}$ combinations. Thus

$$\begin{aligned}\sum_{k<Q}^{\#}\left(\frac{n^{\nu(k)}}{\phi(k)}\right) &\ll \sum_{\substack{d_1d_2\dots d_n<Q\\(d_1,d_2,\dots,d_n)=1}}\frac{1}{\phi(d_1)\phi(d_2)\dots\phi(d_n)}\\ &\ll \left(\sum_{d<Q}\frac{1}{\phi(d)}\right)^n \ll_n (\log Q+1)^n.\end{aligned}$$

■

Note by (2.31) that for relevant $d \neq 1$ we have

$$\psi_{\mathbb{Q}[i]}(x;2d,\alpha) = \frac{4}{\Phi(2d)}\sum_{\chi_{2d}}^{(0)}\overline{\chi}(\alpha)\psi_{\mathbb{Q}[i]}(x,\chi).$$

We now define

$$\psi'_{\mathbb{Q}[i]}(x,\chi) = \begin{cases}\psi_{\mathbb{Q}[i]}(x,\chi) & \text{if } \chi\neq\chi_0;\\ \psi_{\mathbb{Q}[i]}(x,\chi)-\frac{4x}{\Phi(2d)} & \text{if } \chi=\chi_0.\end{cases}$$

Then

$$E(x;2d,\beta) = \frac{4}{\Phi(2d)}\sum_{\chi_{2d}}^{(0)}\overline{\chi}(\beta)\psi'_{\mathbb{Q}[i]}(x,\chi).$$

Now consider

$$\begin{aligned}\sum_{\beta\in S_2}E(x;2d,\beta) &= \sum_{\beta\in S_2}^{(0)}\frac{4}{\Phi(2d)}\sum_{\chi_{2d}}\overline{\chi}(\beta)\psi'_{\mathbb{Q}[i]}(x,\chi) \\ &= \frac{4}{\Phi(2d)}\sum_{\chi_{2d}}^{(0)}\sum_{\beta\in S_2}\overline{\chi}(\beta)\psi'_{\mathbb{Q}[i]}(x,\chi).\end{aligned} \tag{3.13}$$

**Lemma 3.5.2.**

$$\left|\sum_{\beta\in S_2}\chi(\beta)\right| \ll \begin{cases}2^{\nu(d)}\phi(d) & \text{if } \chi\Big|_{(\mathbb{Z}/2d\mathbb{Z})^\times}=1;\\ 0 & \text{otherwise.}\end{cases}$$

*so the sum is zero, unless $\chi$ is principal on the rational integers.*

*Proof.* From the beginning of the proof of Theorem 3.2.1 where we construct the set $S_1$, it is clear that if $\alpha \in S_1$ then $\pm n\alpha \in S_1$ for every $n \in (\mathbb{Z}/d\mathbb{Z})^\times$. Thus we can deduce that if $\beta \in S_2$, then $\pm n\beta \in S_2$ for all $n \in (\mathbb{Z}/2d\mathbb{Z})^\times$, as it will not change the parities of the real and imaginary parts. Hence

$$\sum_{\beta\in S_2}\chi(\beta) = \frac{1}{2}\sum_{\beta\in S_3}\sum_{n\in(\mathbb{Z}/2d\mathbb{Z})^\times}\chi(n\beta),$$

where $S_3$ consists of coset representations of $S_2$ over $(\mathbb{Z}/2d\mathbb{Z})^\times$. We hence get

$$\begin{aligned} 2\sum_{\beta\in S_2}\chi(\beta) &= \sum_{\beta\in S_3}\sum_{n\in(\mathbb{Z}/2d\mathbb{Z})^\times}\chi(n\beta) \\ &= \sum_{\beta\in S_3}\sum_{n\in(\mathbb{Z}/2d\mathbb{Z})^\times}\chi(n)\chi(\beta) = \sum_{\beta\in S_3}\chi(\beta)\sum_{n\in\mathbb{Z}/2d\mathbb{Z}}\chi(n). \end{aligned} \tag{3.14}$$

As there are $2^{\nu(d)-1}\phi(d)$ elements in $S_1$, and there are $\frac{1}{2}\phi(2d)=\frac{1}{2}\phi(d)$ elements in $\left((\mathbb{Z}/2d\mathbb{Z})^\times\right)/\{\pm1\}$, so we have $2^{\nu(d)}$ elements in $S_3$. Since the restriction of $\chi$ to $\mathbb{Z}/2d\mathbb{Z}$ is also a character, we know that

$$\sum_{n\in\mathbb{Z}/2d\mathbb{Z}}\chi(n) = \begin{cases} \phi(d) & \text{if } \chi\Big|_{(\mathbb{Z}/2d\mathbb{Z})^\times} = 1; \\ 0 & \text{if } \chi\Big|_{(\mathbb{Z}/2d\mathbb{Z})^\times} \neq 1. \end{cases}$$

And thus

$$\begin{aligned} \left|\sum_{\beta\in S_2}\chi(\beta)\right| &= \frac{1}{2}\left|\sum_{\beta\in S_3}\chi(\beta)\right|\left|\sum_{n\in\mathbb{Z}/2d\mathbb{Z}}\chi(n)\right| \leqslant \frac{1}{2}\sum_{\beta\in S_3}1\left|\sum_{n\in\mathbb{Z}/2d\mathbb{Z}}\chi(n)\right| \\ &= \begin{cases} 2^{\nu(d)-1}\phi(d) & \text{if } \chi\Big|_{(\mathbb{Z}/2d\mathbb{Z})^\times} = 1; \\ 0 & \text{if } \chi\Big|_{(\mathbb{Z}/2d\mathbb{Z})^\times} \neq 1. \end{cases} \end{aligned}$$

■

Applying this lemma to (3.13) gives

$$\sum_{\beta\in S_2}E(x;2d,\beta) \ll \frac{2^{\nu(d)}\phi(d)}{\Phi(2d)}\sum_{\substack{\chi_{2d} \\ \chi|_{(\mathbb{Z}/2d\mathbb{Z})^\times}=1}}^{(0)}\left|\psi'_{\mathbb{Q}[i]}(x,\chi)\right|.$$

However for the relevant $d$ this can be simplified to

$$\sum_{\beta\in S_2}E(x;2d,\beta) \ll \frac{2^{\nu(d)}}{\phi(d)}\sum_{\substack{\chi_{2d} \\ \chi|_{(\mathbb{Z}/2d\mathbb{Z})^\times}=1}}^{(0)}\left|\psi'_{\mathbb{Q}[i]}(x,\chi)\right|. \tag{3.15}$$

We now note that if $\chi$ is induced by a primitive character $\chi_1$, then

$$\psi'_{\mathbb{Q}[i]}(y,\chi_1)-\psi'_{\mathbb{Q}[i]}(y,\chi) = \sum_{\substack{\mathrm{N}(\pi^k)\leqslant y \\ \pi|d}}\chi_1(\pi^k)\log\mathrm{N}(\pi).$$

It is clear that $\mathrm{N}(\pi^k) = \mathrm{N}(\pi)^k$, and hence there are at most $\frac{\log y}{\log \mathrm{N}(\pi)}$ integers $k$ such that $\mathrm{N}(\pi^k) \leqslant y$. Thus

$$\begin{aligned}\psi'_{\mathbb{Q}[i]}(y,\chi_1) - \psi'_{\mathbb{Q}[i]}(y,\chi) &\ll \sum_{\pi \mid d} \frac{\log y}{\log \mathrm{N}(\pi)} \log \mathrm{N}(\pi) \leqslant \log y \sum_{\pi \mid d} \log \mathrm{N}(\pi) \\ &\ll \log(y)\log(\mathrm{N}(d)) \ll \log(yd)^2,\end{aligned}$$

and we hence get

$$\sum_{\beta \in S_2} E(y; 2d, \beta) \ll \frac{d2^{\nu(d)}}{\phi(d)} \log(yd)^2 + \frac{2^{\nu(d)}}{\phi(d)} \sideset{}{^{(0)}}\sum_{\substack{\chi_{2d} \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times} = 1}} \left|\psi'_{\mathbb{Q}[i]}(y,\chi_1)\right|, \tag{3.16}$$

where $\chi_1$ is the primitive character inducing $\chi$.

We define

$$\begin{aligned}R'_d &:= E_1(x;2d) = \max_{y \leqslant x} \left| \sum_{\beta \in S_2} E(y;2d,\beta) \right|; \\ R''_d &:= \sum_{\beta \in S_2} P_K,\end{aligned}$$

so that by (3.1), if $d \neq 1$ and $d$ is such that it only has prime factors which are congruent to $1 \pmod 4$ then

$$R_d = R'_d + R''_d.$$

We apply (3.16) to get

$$R'_d \ll \frac{d2^{\nu(d)}}{\phi(d)} \log(xd)^2 + \frac{2^{\nu(d)}}{\phi(d)} \sideset{}{^{(0)}}\sum_{\substack{\chi_{2d} \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times} = 1}} \max_{y \leqslant x} \left|\psi'_{\mathbb{Q}[i]}(y,\chi_1)\right|. \tag{3.17}$$

We now recall the condition (4) of Theorem 3.3.1 which says that we need to show that for $X \geqslant 2$ there exists $0 < \alpha < 1$ with

$$\sum_{d < \frac{X^\alpha}{(\log X)^D}} \mu^2(d) 3^{\nu(d)} |R_d| \leqslant E \frac{X}{\log^2 X}.$$

We define

$$R := \sum_{d<Q} \mu^2(d) 3^{\nu(d)} |R_d| \tag{3.18}$$

and

$$R' \quad := \sum_{1<d<Q} \mu^2(d) 3^{\nu(d)} \left| R'_d \right| \tag{3.19}$$

and finally

$$R'' \quad := \sum_{1<d<Q} \mu^2(d) 3^{\nu(d)} \left| R''_d \right|. \tag{3.20}$$

This gives

$$R \ll |R_1| + R' + R''.$$

We will first estimate $R''$ and then $R'$.

**Lemma 3.5.3.** *If $Q < x$, we have*

$$R'' \ll \left( x^{\frac{1}{K}} Q^2 + x^{\frac{1}{2}} \right) (\log x)^{6K^2+1}.$$

*Proof.* Recall

$$\begin{aligned}
R'' \quad &:= \sum_{1<d<Q} \mu^2(d) 3^{\nu(d)} \left| \sum_{\beta \in S_2} \left( x^{\frac{1}{K}} + K^{2\nu(d)} \left( \frac{x^{\frac{1}{2}}}{d^2} + 1 \right) \right) \log x \right| \\
&\ll \sideset{}{^\#}\sum_{1<d<Q} 3^{\nu(d)} \left| 2^{\nu(d)} \phi(d) \left( x^{\frac{1}{K}} + K^{2\nu(d)} \left( \frac{x^{\frac{1}{2}}}{d^2} + 1 \right) \right) \log x \right| \\
&= \sideset{}{^\#}\sum_{1<d<Q} 6^{\nu(d)} \phi(d) \left( x^{\frac{1}{K}} + K^{2\nu(d)} \left( \frac{x^{\frac{1}{2}}}{d^2} + 1 \right) \right) \log x \\
&= \log x \sideset{}{^\#}\sum_{1<d<Q} 6^{\nu(d)} \phi(d) x^{\frac{1}{K}} + \log x \sideset{}{^\#}\sum_{1<d<Q} 6^{\nu(d)} \phi(d) K^{2\nu(d)} \frac{x^{\frac{1}{2}}}{d^2} \\
&\quad + \log x \sideset{}{^\#}\sum_{1<d<Q} 6^{\nu(d)} \phi(d) K^{2\nu(d)} \\
&=: S_1 + S_2 + S_3.
\end{aligned}$$

We will estimate $S_1, S_2$ and $S_3$ individually. For $S_1$ we have that

$$\phi(d) \leqslant d \leqslant \frac{Q^2}{d} \leqslant \frac{Q^2}{\phi(d)}, \tag{3.21}$$

since $d < Q$. Thus

$$S_1 \leqslant x^{\frac{1}{K}} Q^2 \log x \sideset{}{^\#}\sum_{1<d<Q} \frac{6^{\nu(d)}}{\phi(d)}.$$

We now apply Lemma 3.5.1 to get

$$S_1 \ll x^{\frac{1}{K}} Q^2 \log x (\log Q)^6.$$

For $S_2$ we have

$$S_2 \ll x^{\frac{1}{2}} \log x \sum_{1<d<Q}^{\#} \frac{(6K^2)^{\nu(d)}}{d} \ll x^{\frac{1}{2}} \log x (\log Q)^{6K^2}.$$

And finally for $S_3$ we have

$$S_3 \ll \log x \sum_{1<d<Q}^{\#} 6^{\nu(d)} \phi(d) K^{2\nu(d)}.$$

By (3.21) we, again, get

$$S_3 \ll Q^2 \log x \sum_{1<d<Q}^{\#} \frac{(6K^2)^{\nu(d)}}{d} \ll Q^2 \log x (\log Q)^{6K^2}.$$

Combining the results for $S_1, S_2$ and $S_3$ gives

$$\begin{aligned} R'' &\ll x^{\frac{1}{K}} Q^2 \log x (\log Q)^6 + x^{\frac{1}{2}} \log x (\log Q)^{6K^2} + Q^2 \log x (\log Q)^{6K^2} \\ &\ll x^{\frac{1}{K}} Q^2 \log x (\log Q)^{6K^2} + x^{\frac{1}{2}} \log x (\log Q)^{6K^2}. \end{aligned}$$

The result now follows provided $Q < x$. ■

We now turn to $R'$.

**Lemma 3.5.4.** *For $Q \leqslant \sqrt{x}$, we have*

$$R' \ll (\log x)^{19} \sqrt{x} (R_0)^{\frac{1}{2}},$$

*where*

$$R_0 := \sum_{1<d<Q} \mu^2(d) 2^{-\nu(d)} |R'_d|.$$

*Proof.* We note

$$R' = \sum_{1<d<Q} \left( \frac{\mu^2(d) 6^{\nu(d)}}{\sqrt{d}} \times \mu^2(d) 2^{-\nu(d)} \sqrt{d} |R'_d| \right). \tag{3.22}$$

We now state a well-known inequality, that we will not prove here.

**Lemma 3.5.5** (Cauchy–Schwarz Inequality)**.**

$$\left|\sum_i a_i b_i\right|^2 \leqslant \sum_i |a_i|^2 \sum_i |b_i|^2.$$

We apply Cauchy–Schwarz to get

$$R' \leqslant \left(\sum_{1<d<Q} \frac{\mu^2(d) 36^{\nu(d)}}{d}\right)^{\frac{1}{2}} \left(\sum_{d<Q} \mu^2(d) 4^{-\nu(d)} d |R'_d|^2\right)^{\frac{1}{2}}. \tag{3.23}$$

The first factor can be estimated by Lemma 3.5.1 as

$$\sum_{1<d<Q} \frac{\mu^2(d) 36^{\nu(d)}}{d} \ll (\log Q)^{36}. \tag{3.24}$$

For the second factor, one of the $|R'_d|$ can be estimated trivially by using (3.1) and the fact that there are $2^{\nu(d)-1}\phi(d)$ residue classes in the sum to get

$$\begin{aligned} R'_d &\ll \sum_{\beta \in S_2} \left(\max_{y \leqslant x} \left|\psi_{\mathbb{Q}[\mathrm{i}]}(y; 2d, \beta) - \frac{4y}{\Phi(2d)}\right|\right) \\ &\ll 2^{\nu(d)} \phi(d) \left(\left(\frac{\sqrt{x}}{2d} + 1\right)^2 \log x + \frac{x}{\Phi(2d)}\right). \end{aligned}$$

It is a well known theorem (see [HW79, p.267]) that

$$\liminf_{n \to \infty} \frac{\phi(n) \log\log n}{n} = \mathrm{e}^{-\gamma},$$

and we hence deduce that

$$\frac{1}{\phi(d)} \ll \frac{\log\log d}{d}.$$

For relevant $d$ we thus know that

$$\frac{1}{\Phi(2d)} = \frac{1}{2\phi(d)^2} \ll \frac{(\log\log d)^2}{d^2}. \tag{3.25}$$

Consequently the trivial estimation gives

$$R'_d \ll 2^{\nu(d)} \frac{x}{d} \log x.$$

We use this together with (3.24) in (3.23) to get

$$R' \ll (\log Q)^{18} \sqrt{x \log x} \left(\sum_{1<d<Q} \mu^2(d) 2^{-\nu(d)} |R'_d|\right)^{\frac{1}{2}}.$$

Thus if $Q \leqslant \sqrt{x}$ we have

$$R' \ll (\log x)^{19}\sqrt{x}\left(\sum_{1<d<Q} \mu^2(d)2^{-\nu(d)}|R'_d|\right)^{\frac{1}{2}} = (\log x)^{19}\sqrt{x}(R_0)^{\frac{1}{2}}, \tag{3.26}$$

where

$$R_0 := \sum_{1<d<Q} \mu^2(d)2^{-\nu(d)}|R'_d|.$$

■

From now on, our main task will be to estimate $R_0$. Since $d < Q$, it follows from (3.17) that

$$\begin{aligned} R_0 \ \ll\ & \sum_{1<d<Q}^{\dagger} \frac{d}{\phi(d)} \log(xd)^2 \\ & + \sum_{1<d<Q}^{\dagger} \frac{1}{\phi(d)} \sum_{\substack{\chi_{2d} \\ \chi\big|_{(\mathbb{Z}/2d\mathbb{Z})^\times}=1}}^{(0)} \max_{y \leqslant x} \left|\psi'_{\mathbb{Q}[\mathrm{i}]}(y,\chi_1)\right|, \end{aligned} \tag{3.27}$$

where we need to find the maximal $\alpha$ such that $Q = \frac{X^\alpha}{(\log X)^D}$. By $\sum^{\dagger}$ we mean that the sum is only over the relevant $d$ (squarefree and having only prime factors congruent to 1 (mod $d$)). We will deal with each of the two terms separately. As $d < Q$, it follows that

$$\begin{aligned} \sum_{1<d<Q}^{\dagger} \frac{d}{\phi(d)} \log(xd)^2 \ &\ll\ Q\log(xQ)^2 \sum_{1<d<Q}^{\dagger} \frac{1}{\phi(d)} \\ &\ll\ Q\log(xQ)^2 \sum_{1<d<Q}^{\#} \frac{1}{\phi(d)}. \end{aligned}$$

Applying Lemma 3.5.1 we get

$$\sum_{1<d<Q}^{\dagger} \frac{d}{\phi(d)} \log(xd)^2 \ \ll\ Q\log(xQ)^3,$$

and so

$$R_0 \ \ll\ Q\log(xQ)^3 + \sum_{1<d<Q}^{\dagger} \frac{1}{\phi(d)} \sum_{\substack{\chi_{2d} \\ \chi\big|_{(\mathbb{Z}/2d\mathbb{Z})^\times}=1}}^{(0)} \max_{y \leqslant x} \left|\psi'_{\mathbb{Q}[\mathrm{i}]}(y,\chi_1)\right|. \tag{3.28}$$

**Lemma 3.5.6.** *If $\chi$ is a non-principal Gaussian character to a rational prime modulus $p$ such that*

$$\chi\Big|_{(\mathbb{Z}/p\mathbb{Z})^\times} = 1,$$

*then $\chi$ is primitive (with conductor $p$).*

*Proof.* For a contradiction, suppose that $\chi$ is induced by $\chi_\pi$ with $\pi \mid p$. Suppose $\pi = a\mathrm{i} + b$ with $a \neq 0$. Then

$$\chi_\pi(n) = 1$$

for any $n \in \mathbb{Z}/p\mathbb{Z}^\times$. Since $\chi$ is non-principal, there exists a non-rational

$$\alpha = c\mathrm{i} + d \in (\mathbb{Z}[\mathrm{i}]/\pi\mathbb{Z}[\mathrm{i}])^\times$$

such that $c \not\equiv 0 \pmod{\pi}$ and

$$\chi_\pi(\alpha) \neq 1.$$

Now consider $\beta = a\alpha - c\pi = ad - bc$. Suppose $\pi \mid \beta$. Then since $\pi \nmid a$, we must have that $\pi \mid \alpha$, which is a contradiction, to the assumption $\alpha \in (\mathbb{Z}[\mathrm{i}]/\pi\mathbb{Z}[\mathrm{i}])^\times$. Thus $\pi \nmid \beta$ meaning that $p \nmid \beta$ and $\beta \in (\mathbb{Z}/p\mathbb{Z})^\times$. Consequently

$$\chi_\pi(\beta) = 1.$$

However, $\beta \equiv a\alpha \pmod{\pi}$ and also $p \nmid a$. Thus

$$1 = \chi_\pi(\beta) = \chi_\pi(a\alpha) = \chi_\pi(a)\chi_\pi(\alpha) = \chi_\pi(\alpha) \neq 1,$$

which is the required contradiction. ■

**Lemma 3.5.7.** *If $\chi$ is a non-principal Gaussian character modulo a squarefree rational integer $d$, such that*

$$\chi\Big|_{(\mathbb{Z}/d\mathbb{Z})^\times} = 1.$$

*Then the conductor of $\chi$ is also a rational integer.*

*Proof.* We will prove this by induction on the number of (real) prime factors of $d$. The base case is then if $d$ is prime, which is hence dealt with by Lemma 3.5.6. We now suppose that the Lemma is true for all numbers with fewer prime factors than $d$. Now suppose that $d = uv$, where $u$ and $v$ are rational integers. Then $(u, v) = 1$, since $d$ is squarefree. As $\mathbb{Z}$ is a Euclidean Domain, we can choose numbers $\overline{u}, \overline{v} \in \mathbb{Z}$ such that $u\overline{u} + v\overline{v} = 1$. We now construct two characters $\chi_u$ and $\chi_v$ with moduli $u$,$v$ respectively by defining

$$\chi_u(\alpha) := \chi(\alpha v\overline{v} + u\overline{u});$$
$$\chi_v(\alpha) := \chi(v\overline{v} + \alpha u\overline{u}).$$

We need to show that these in fact are well-defined characters. Without loss of generality it is enough to do this for $\chi_u$. For that we need to show that it is a well-defined multiplicative function on $\mathbb{Z}[\mathrm{i}]/u\mathbb{Z}[\mathrm{i}]$, satisfying $\chi_u(\alpha) = 0$ for $(\alpha, u) \neq 1$ and $\chi_u(\alpha) \neq 0$ for $(\alpha, u) = 1$.

(i) Clearly if $\alpha \equiv \beta \pmod u$ then

$$\chi_u(\alpha) = \chi(\alpha v\overline{v} + u\overline{u}) = \chi((\beta + \gamma u)v\overline{v} + u\overline{u}) = \chi(\beta v\overline{v} + \gamma d\overline{v} + u\overline{u}),$$

but $\chi$ is a character modulo $d$, hence

$$\chi_u(\alpha) = \chi(\beta v\overline{v} + u\overline{u}) = \chi_u(\beta).$$

(ii)

$$\begin{aligned} \chi_u(\alpha)\chi_u(\beta) &= \chi((\alpha v\overline{v} + u\overline{u})(\beta v\overline{v} + u\overline{u})) \\ &= \chi(\alpha\beta v^2\overline{v}^2 + \beta uv\overline{uv} + \alpha uv\overline{uv} + u^2\overline{u}^2) \\ &= \chi(\alpha\beta v^2\overline{v}^2 + \beta d\overline{uv} + \alpha d\overline{uv} + u^2\overline{u}^2) \\ &= \chi(\alpha\beta v^2\overline{v}^2 + u^2\overline{u}^2). \end{aligned}$$

We know $v\overline{v} = -u\overline{u} + 1$. Thus

$$v^2\overline{v}^2 = -u\overline{u}v\overline{v} + v\overline{v} = -d\overline{uv} + v\overline{v} \equiv v\overline{v} \pmod d,$$

and symmetrically we get $u^2\overline{u}^2 \equiv u\overline{u} \pmod d$. Hence

$$\chi_u(\alpha)\chi_u(\beta) = \chi(\alpha\beta v\overline{v} + u\overline{u}) = \chi_u(\alpha\beta).$$

(iii) Suppose $(\alpha, u) = \beta \neq 1$ then $\beta \mid \alpha v\overline{v} + u\overline{u}$ and $\beta \mid uv = d$. Then we have that $(d, \alpha v\overline{v} + u\overline{u}) \neq 1$ and hence

$$\chi_u(\alpha) = 0.$$

(iv) Suppose $(\alpha, u) = 1$ and $\chi_u(\alpha) = 0$. Let $\beta = (d, \alpha v\overline{v} + u\overline{u}) \neq 1$, whence there exists a Gaussian prime $\pi$ such that $\pi \mid \beta$. As $d = uv$ we have $\pi \mid u$ or $\pi \mid v$. Suppose $\pi \mid u$, then since $\pi \mid \alpha v\overline{v} + u\overline{u} = \alpha(-u\overline{u} + 1) + u\overline{u}$ we have $\pi \mid \alpha$, which is a contradiction to $(\alpha, u) = 1$. Suppose on the other hand that $\pi \mid v$, then $\pi \mid \alpha v\overline{v} + u\overline{u} = \alpha v\overline{v} - v\overline{v} + 1$, whence $\pi \mid 1$, which is impossible. Hence we have $\chi_u(\alpha) \neq 0$.

We have hence shown that $\chi_u$ and $\chi_v$ are Gaussian characters. We also note that

$$\begin{aligned} \chi_u\Big|_{(\mathbb{Z}/u\mathbb{Z})^\times} &= 1; \\ \chi_v\Big|_{(\mathbb{Z}/v\mathbb{Z})^\times} &= 1, \end{aligned}$$

because if $a$ is a rational integer, then $av\overline{v} + u\overline{u}$ is also a rational integer (as $a, u, \overline{u}, v, \overline{v} \in \mathbb{Z}$). Furthermore,

$$\begin{aligned} \chi_u(\alpha)\chi_v(\alpha) &= \chi(\alpha v\overline{v} + u\overline{u})\chi(v\overline{v} + \alpha u\overline{u}) \\ &= \chi(\alpha v^2\overline{v}^2 + \alpha u^2\overline{u}^2) \\ &= \chi(\alpha). \end{aligned}$$

Consequently,

$$\chi = \chi_u\chi_v.$$

$\chi_u$ and $\chi_v$ cannot both be principal, since otherwise $\chi$ would be principal as well. We can apply the inductive hypothesis to the non-principal characters and get that

$$\chi = \chi_{u_1}\chi_{v_1}\chi_{u0}\chi_{v0},$$

where $\chi_{u_1}, \chi_{v_1}$ are primitive with rational integer moduli $u_1 \mid u$ and $v_1 \mid v$. Also $\chi_{u0}$ and $\chi_{v0}$ are principal with moduli $u, v$. (Note that if $\chi_u$ or $\chi_v$ is principal, the corresponding $\chi_{u_1}$ or $\chi_{v_1}$ would not exist). From the theory of Dirichlet characters [Dav80] we know that $\chi_{u0}\chi_{v0} = \chi_0$ where $\chi_0$ is the principal character modulo $d$. We also know that $\chi_{u_1}\chi_{v_1} = \chi_1$ is a primitive character with conductor $u_1v_1 \in \mathbb{Z}$. Thus $\chi = \chi_1\chi_0$, and so $\chi$ has an integer conductor. ■

By Lemma 3.5.7, all the primitive characters $\chi_1$ in (3.16) have integer conductors which are divisors of $2d$ and hence the estimate can be rewritten as

$$R_0 \ll Q\log(xQ)^3 + \sideset{}{^\ddagger}\sum_{1<d\leqslant 2Q} \sideset{}{^\ddagger}\sum_{k\leqslant \frac{2Q}{d}} \left(\frac{1}{\phi(kd)}\right) \sideset{}{^{(0)*}}\sum_{\substack{\chi_d \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \max_{y\leqslant x}\left|\psi'_{\mathbb{Q}[i]}(y,\chi)\right|, \tag{3.29}$$

where by $\sum^\ddagger$ we mean that the sum is over all squarefree $d$ such that if $p \mid d$ then $p \not\equiv 3 \pmod 4$. Note that from now on $d$ can be even, but it still is squarefree and it still cannot have any prime factors $\equiv 3 \pmod 4$. Applying the well-known fact that $\phi(kd) \geqslant \phi(k)\phi(d)$ to $R_0$ gives

$$R_0 \ll Q\log(xQ)^3 + \sideset{}{^\ddagger}\sum_{1<d\leqslant 2Q} \frac{1}{\phi(d)} \sideset{}{^\ddagger}\sum_{k\leqslant \frac{2Q}{d}} \left(\frac{1}{\phi(k)}\right) \sideset{}{^{(0)*}}\sum_{\substack{\chi_d \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \max_{y\leqslant x}\left|\psi'_{\mathbb{Q}[i]}(y,\chi)\right|. \tag{3.30}$$

We apply Lemma 3.5.1 to the sum over $k$ to get

**Lemma 3.5.8.**

$$R_0 \ll Q\log(xQ)^3 + \log Q \sideset{}{^\ddagger}\sum_{1<d\leqslant 2Q} \frac{1}{\phi(d)} \sideset{}{^{(0)*}}\sum_{\substack{\chi_d \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \max_{y\leqslant x}\left|\psi'_{\mathbb{Q}[i]}(y,\chi)\right|.$$

If we now use our Theorems 2.10.6 and 2.10.7, which assume the Generalised Riemann Hypothesis, then we get for $\chi \neq \chi_0$ and $\mathrm{N}(d) \leqslant x$ that

$$\left|\psi'_{\mathbb{Q}[i]}(y,\chi)\right| \ll y^{\frac{1}{2}}\log^2 y,$$

and also

$$\psi_{\mathbb{Q}[i]}(x,\chi_0) = x + O\left(x^{\frac{1}{2}}(\log x)^2\right).$$

Thus we immediately get

$$R_0 \ll Q\log(xQ)^3 + \log Q \sideset{}{^\ddagger}\sum_{1<d\leqslant 2Q} \frac{1}{\phi(d)} \sideset{}{^{(0)*}}\sum_{\substack{\chi_d \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \left|x^{\frac{1}{2}}(\log x)^2\right|.$$

**Lemma 3.5.9.** *If $d$ has no prime factor $p \equiv 3 \pmod 4$ then there are at most $2\phi(d)$ characters $\chi$ on $\mathbb{Z}[\mathrm{i}]/d\mathbb{Z}[\mathrm{i}]$ with*

$$\chi\Big|_{(\mathbb{Z}/d\mathbb{Z})^\times} = 1.$$

*Proof.* Characters on $\mathbb{Z}[\mathrm{i}]/d\mathbb{Z}[\mathrm{i}]$ that are principal on $\mathbb{Z}/d\mathbb{Z}$ can be viewed as characters on $(\mathbb{Z}[\mathrm{i}]/d\mathbb{Z}[\mathrm{i}])^\times/(\mathbb{Z}/d\mathbb{Z})^\times$, which is a group of order $\frac{\Phi(d)}{\phi(d)} \leqslant 2\phi(d)$, and so there are at most $2\phi(d)$ of them. ■

Note that we have a further restriction $\chi(\mathrm{i}) = 1$ in this sum, but since we are not interested in constant multiples anymore, this is irrelevant. Thus

$$R_0 \ll Q\log(xQ)^3 + (\log Q)x^{\frac{1}{2}}\log^2 xQ \sum_{1<d\leqslant Q}^{\dagger} \frac{1}{\phi(d)},$$

and by Lemma 3.5.1 we have

$$R_0 \ll Q\log(xQ)^3 + (\log Q)^2(\log x)^2 x^{\frac{1}{2}}Q.$$

Hence for $Q = x^\gamma$ with $0 < \gamma \leqslant \frac{1}{2}$ we have

$$R_0 \ll x^{\frac{1}{2}+\gamma}(\log x)^4.$$

By (3.26) this means

$$R' \ll (\log x)^{19}\sqrt{x}(R_0)^{\frac{1}{2}} \ll x^{\frac{3}{4}+\frac{\gamma}{2}}(\log x)^{21}.$$

Since $\frac{3}{4} + \frac{\gamma}{2} \leqslant 1$ and by (3.11) and (3.12) we get

$$R' \ll X^{\frac{3}{4}+\frac{\gamma}{2}}(\log X)^{22}.$$

We recall that by Lemma 3.5.3 that

$$\begin{aligned} R'' &\ll \left(x^{\frac{1}{K}}Q^2 + x^{\frac{1}{2}}\right)(\log x)^{6K^2+1} \\ &\ll \left(x^{\frac{1}{K}+2\gamma} + x^{\frac{1}{2}}\right)(\log x)^{6K^2+1}. \end{aligned}$$

If $\gamma \geqslant \frac{1}{4}$, then

$$R'' \ll x^{\frac{1}{K}+2\gamma}(\log x)^{6K^2+1} \ll X^{\frac{1}{K}+2\gamma}(\log X)^{6K^2+2}.$$

Combining $R'$ and $R''$ we get

$$\begin{aligned} R &= \sum_{d<Q} \mu^2(d) 3^{\nu(d)} |R_d| \ll R_1 + R' + R'' \\ &\ll \frac{X}{\log^2 X} + X^{\frac{3}{4}+\frac{\gamma}{2}} (\log X)^{10} + X^{\frac{1}{K}+2\gamma} (\log X)^{6K^2+2}. \end{aligned}$$

Thus for any $\gamma < \frac{1}{2}$, we can pick $K = \left[\frac{1}{1-2\gamma}\right] + 1$ to get

$$\sum_{d<Q} \mu^2(d) 3^{\nu(d)} |R_d| \ll \frac{X}{(\log X)^2}.$$

Now, $\Lambda_3 = 4 - \frac{\log 4}{(1+3^{-3}) \log 3} \approx 2.78$. Thus taking any $\alpha$ in the range $\frac{1}{2.7} < \alpha < \frac{1}{2}$ would satisfy the conditions of Theorem 3.3.1, which means that there are $\gg \frac{X}{\log X}$ $P_3$ almost primes in $\mathscr{A}$. This completes the proof of

**Theorem 3.5.10** (Main Result 1). *Assuming the Generalised Riemann Hypothesis, there are infinitely many primes of the form $a^2 + b^2$ such that $a^2 + 4b^2$ has at most 3 prime factors.*

It is of course our aim to avoid the Riemann Hypothesis, and the next Chapter will be dedicated to that by proving a Bombieri–Vinogradov type theorem.

# Chapter 4

# The Unconditional Result

## 4.1 Introduction

The previous result, whilst interesting in its own right, is conditional on the Riemann Hypothesis, an assumption which we would like to avoid. We will hence try to establish an unconditional result in this chapter. The initial setup is almost the same. In fact all of the conditions in Theorem 3.3.1 are unchanged apart from the very last one which estimates the error terms.

Thus the purpose of this final chapter is to establish condition (4), so if $X \geqslant 2$, then there exists $0 < \alpha < 1$ such that

$$\sum_{d < \frac{X^\alpha}{(\log X)^D}} \mu^2(d) 3^{\nu(d)} |R_d| \leqslant E \frac{X}{\log^2 X}. \tag{4.1}$$

Previous work on this is not sufficient for our purpose. The existing versions of the Bombieri–Vinogradov result for arbitrary number fields (eg [Wil69], [Hux71] for sums over ideals or [Hin88] for sums over prime elements) count all moduli $d \in \mathbb{Z}[\mathrm{i}]$ whereas we want to only count rational integer moduli. Furthermore our theorem will take the form

$$\sum_{d \leqslant Q} \frac{d^2}{\Phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times} = 1}} \max_{y \leqslant x} \left|\psi_{\mathbb{Q}[\mathrm{i}]}(y, \chi)\right| \ll \frac{x}{(\log x)^C}$$

whereas standard results and in particular the general result from [Hux71] require the factor $\frac{1}{\Phi(d)}$ instead of the larger $\frac{d^2}{\Phi(d)}$ in the sum. We are also able to get some saving by using the fact that we are only interested in some of the characters (mod $d$).

To estimate this sum unconditionally, we will derive a version of the Bombieri–Vinogradov Theorem for the Gaussian field. Since $R_d$ is zero if $d$ has a prime factor not congruent to $1 \pmod 4$, we shall only be concerned with squarefree $d$ that have only prime factors that are $1 \pmod 4$, and we will call these the 'relevant' $d$.

## 4.2 The Large Sieve

In order to prove a suitable version of the Large Sieve for our application, we are going to generalise a result of Heath-Brown that is proven in [HB01, pp.79]. In fact, we are only going to prove it for 2 dimensions (instead of 3 given by Heath-Brown), but it looks like the theorem can be extended to arbitrary dimensions.

**Theorem 4.2.1.** *Let* $\mathbf{a} = (a_1, a_2)^T$ *and* $\mathbf{b} = (b_1, b_2)$. *Also let* $c(\mathbf{b})$ *be arbitrary constants,*

$$\mathscr{S}_{\mathbf{z},l} = \left\{ \mathbf{b} \in \mathbb{Z}^2 : z_i \leqslant b_i \leqslant z_i + l \right\}$$

*and*

$$S(\mathbf{a}, \mathscr{S}_{\mathbf{z},l}) = \sum_{\mathbf{b} \in \mathscr{S}_{\mathbf{z},l}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{a}\cdot\mathbf{b}} c(\mathbf{b}).$$

*Then we have*

$$\sideset{}{'}\sum_{Q<d\leqslant 2Q} \sum_{\mathbf{x} \pmod d} \left| S(d^{-1}\mathbf{x}, \mathscr{S}_{\mathbf{z},l}) \right|^2 \ll (l^2 + Q^2 l + Q^3) \sum_{\mathbf{b} \in \mathscr{S}_{\mathbf{z},l}} |c(\mathbf{b})|^2 .$$

*where* $\sum'$ *is over* $d$ *for which* $(x_1, x_2, d) = 1$.

*Proof.* We first note that

$$S(\mathbf{a}, \mathscr{S}_{\mathbf{z},l}) = \mathrm{e}^{-2\pi \mathrm{i} \mathbf{a}\cdot\mathbf{y}} \sum_{\mathbf{b} \in \mathscr{S}_{\mathbf{z}+\mathbf{y},l}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{a}\cdot\mathbf{b}} c(\mathbf{b} - \mathbf{y}),$$

so that for any translation of the square $\mathscr{S}$ by an integral vector $\mathbf{y}$ there exist constants $c'(\mathbf{b})$ such that $|S(\mathbf{a}, \mathscr{S})|$ remains the same. Thus it is sufficient to prove the theorem for one fixed square, such as $S_{\mathbf{0},l}$, say. We define

$$S(\mathbf{a}) = S(\mathbf{a}, \mathscr{S}_{\mathbf{0},l}) = \sum_{\mathbf{b} \in \mathscr{S}_{\mathbf{0},l}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{a}\cdot\mathbf{b}} c(\mathbf{b}) = \sum_{b_1=0}^{l} \sum_{b_2=0}^{l} \mathrm{e}^{2\pi \mathrm{i}(a_1 b_1 + a_2 b_2)} c(\mathbf{b}).$$

We start with the Sobolev–Gallagher inequality ([Gal67], [Mon71, Lemma 1.1]) which states

**Lemma 4.2.2** (Sobolev–Gallagher)**.** *Let $a < b$ be real numbers, and let $f$ be a continuous complex-valued function on $[a, b]$ with continuous first derivative on $(a, b)$. Then*

$$|f(0)| \leqslant \frac{1}{2\delta}\int_{-\delta}^{\delta} |f(t)|\,\mathrm{d}t + \frac{1}{2}\int_{-\delta}^{\delta} |f'(t)|\,\mathrm{d}t.$$

We apply this three times to functions of two variables to get

$$\begin{aligned}
|f(0,0)| &\leqslant \frac{1}{2\delta}\int_{-\delta}^{\delta} |f(t_1,0)|\,\mathrm{d}t_1 + \frac{1}{2}\int_{-\delta}^{\delta} |f_{t_1}(t_1,0)|\,\mathrm{d}t_1;\\
|f(t_1,0)| &\leqslant \frac{1}{2\delta}\int_{-\delta}^{\delta} |f(t_1,t_2)|\,\mathrm{d}t_2 + \frac{1}{2}\int_{-\delta}^{\delta} |f_{t_2}(t_1,t_2)|\,\mathrm{d}t_2;\\
|f_{t_1}(t_1,0)| &\leqslant \frac{1}{2\delta}\int_{-\delta}^{\delta} |f_{t_1}(t_1,t_2)|\,\mathrm{d}t_2 + \frac{1}{2}\int_{-\delta}^{\delta} |f_{t_1t_2}(t_1,t_2)|\,\mathrm{d}t_2.
\end{aligned}$$

Substituting the second and third inequalities in the first one gives

$$\begin{aligned}
|f(0,0)| \leqslant\ & \frac{1}{4\delta^2}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f(t_1,t_2)|\,\mathrm{d}t_1\mathrm{d}t_2 + \frac{1}{4\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f_{t_2}(t_1,t_2)|\,\mathrm{d}t_1\mathrm{d}t_2\\
&+\frac{1}{4\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f_{t_1}(t_1,t_2)|\,\mathrm{d}t_1\mathrm{d}t_2 + \frac{1}{4}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f_{t_1t_2}(t_1,t_2)|\,\mathrm{d}t_1\mathrm{d}t_2.
\end{aligned}$$

We now apply Cauchy's inequality to get

$$\begin{aligned}
|f(0,0)|^2 \leqslant\ & \left(\frac{1}{2\delta^2}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f(t_1,t_2)|\,\mathrm{d}t_1\mathrm{d}t_2\right)^2\\
&+\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f_{t_2}(t_1,t_2)|\,\mathrm{d}t_1\mathrm{d}t_2\right)^2\\
&+\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f_{t_1}(t_1,t_2)|\,\mathrm{d}t_1\mathrm{d}t_2\right)^2\\
&+\left(\frac{1}{2}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f_{t_1t_2}(t_1,t_2)|\,\mathrm{d}t_1\mathrm{d}t_2\right)^2.
\end{aligned}$$

We apply Cauchy–Schwarz again to each of the terms:

$$\begin{aligned}
|f(0,0)|^2 \leqslant\ & \frac{1}{\delta^2}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f(t_1,t_2)|^2\,\mathrm{d}t_1\mathrm{d}t_2 + \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f_{t_2}(t_1,t_2)|^2\,\mathrm{d}t_1\mathrm{d}t_2\\
&+\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f_{t_1}(t_1,t_2)|^2\,\mathrm{d}t_1\mathrm{d}t_2 + \delta^2\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |f_{t_1t_2}(t_1,t_2)|^2\,\mathrm{d}t_1\mathrm{d}t_2.
\end{aligned}$$

Let us now leverage this with

$$\begin{aligned} f(t,s) &= f(\mathbf{t}) = S(d^{-1}\mathbf{x}+\mathbf{t}) = S(d^{-1}\mathbf{x}+\mathbf{t}, S_{\mathbf{0},l}); \\ \delta &= \frac{1}{l}. \end{aligned}$$

We get

$$\begin{aligned} &\left|S(d^{-1}\mathbf{x})\right| \\ \leqslant\ & l^2 \int_{-1/l}^{1/l}\int_{-1/l}^{1/l} \left|S(d^{-1}\mathbf{x}+\mathbf{t})\right|^2 \mathrm{d}t_1\mathrm{d}t_2 + \int_{-1/l}^{1/l}\int_{-1/l}^{1/l} \left|S_{t_1}(d^{-1}\mathbf{x}+\mathbf{t})\right|^2 \mathrm{d}t_1\mathrm{d}t_2 \\ &+ \int_{-1/l}^{1/l}\int_{-1/l}^{1/l} \left|S_{t_2}(d^{-1}\mathbf{x}+\mathbf{t})\right|^2 \mathrm{d}t_1\mathrm{d}t_2 + l^{-2}\int_{-1/l}^{1/l}\int_{-1/l}^{1/l} \left|S_{t_1t_2}(d^{-1}\mathbf{x}+\mathbf{t})\right|^2 \mathrm{d}t_1\mathrm{d}t_2. \end{aligned}$$

It is obvious from the definition that $S(\mathbf{t}) = S(\mathbf{t}')$ if $\mathbf{t} = \mathbf{t}+\mathbf{n}$ for some integer vector $\mathbf{n}$. Thus we can reduce $\mathbf{x}$ modulo $d$ above to get $\mathbf{x}'$, with $0 \leqslant x_i' < d$, and do a substitution $\mathbf{t}' = d^{-1}\mathbf{x}' + \mathbf{t}$.

$$\begin{aligned} &\left|S(d^{-1}\mathbf{x})\right| \\ \leqslant\ & l^2 \int_{-\frac{1}{l}+\frac{x_1'}{d}}^{\frac{1}{l}+\frac{x_1'}{d}}\int_{-\frac{1}{l}+\frac{x_2'}{d}}^{\frac{1}{l}+\frac{x_2'}{d}} |S(\mathbf{t})|^2 \mathrm{d}t_1\mathrm{d}t_2 + \int_{-\frac{1}{l}+\frac{x_1'}{d}}^{\frac{1}{l}+\frac{x_1'}{d}}\int_{-\frac{1}{l}+\frac{x_2'}{d}}^{\frac{1}{l}+\frac{x_2'}{d}} |S_{t_1}(\mathbf{t})|^2 \mathrm{d}t_1\mathrm{d}t_2 \\ &+ \int_{-\frac{1}{l}+\frac{x_1'}{d}}^{\frac{1}{l}+\frac{x_1'}{d}}\int_{-\frac{1}{l}+\frac{x_2'}{d}}^{\frac{1}{l}+\frac{x_2'}{d}} |S_{t_2}(\mathbf{t})|^2 \mathrm{d}t_1\mathrm{d}t_2 + l^{-2}\int_{-\frac{1}{l}+\frac{x_1'}{d}}^{\frac{1}{l}+\frac{x_1'}{d}}\int_{-\frac{1}{l}+\frac{x_2'}{d}}^{\frac{1}{l}+\frac{x_2'}{d}} |S_{t_1t_2}(\mathbf{t})|^2 \mathrm{d}t_1\mathrm{d}t_2. \end{aligned}$$

We replace the integration limits by an indicator function

$$\iota_{\mathbf{x},d}(\mathbf{t}) = \begin{cases} 1 & \text{if } |d^{-1}x_i' - t_i| \leqslant \frac{1}{l} \text{ for } i = 1,2; \\ 0 & \text{otherwise.} \end{cases}$$

and we also note that since $\frac{x_i'}{d} \leqslant 1$ we get $-1 \leqslant -\frac{1}{l} + \frac{x_i'}{d}$ and $\frac{1}{l} + \frac{x_i'}{d} \leqslant 2$. Hence we can state

$$\begin{aligned} \left|S(d^{-1}\mathbf{x})\right| \leqslant\ & l^2 \int_{-1}^{2}\int_{-1}^{2} \iota_{\mathbf{x},d}(\mathbf{t})\,|S(\mathbf{t})|^2 \mathrm{d}t_1\mathrm{d}t_2 + \int_{-1}^{2}\int_{-1}^{2} \iota_{\mathbf{x},d}(\mathbf{t})\,|S_{t_1}(\mathbf{t})|^2 \mathrm{d}t_1\mathrm{d}t_2 \\ +\ & \int_{-1}^{2}\int_{-1}^{2} \iota_{\mathbf{x},d}(\mathbf{t})\,|S_{t_2}(\mathbf{t})|^2 \mathrm{d}t_1\mathrm{d}t_2 + l^{-2}\int_{-1}^{2}\int_{-1}^{2} \iota_{\mathbf{x},d}(\mathbf{t})\,|S_{t_1t_2}(\mathbf{t})|^2 \mathrm{d}t_1\mathrm{d}t_2. \end{aligned}$$

We generalise this to the sum

$$
\begin{aligned}
\sum_{Q<d\leqslant 2Q}{}' \sum_{\mathbf{x} \pmod d} \left|S(d^{-1}\mathbf{x})\right|^2 \leqslant\ & l^2 \int_{-1}^{2}\int_{-1}^{2} \#v_{Q,d}(\mathbf{t})\, |S(\mathbf{t})|^2 \,\mathrm{d}t_1\mathrm{d}t_2 \\
&+ \int_{-1}^{2}\int_{-1}^{2} \#v_{Q,d}(\mathbf{t})\, |S_{t_1}(\mathbf{t})|^2 \,\mathrm{d}t_1\mathrm{d}t_2 \\
&+ \int_{-1}^{2}\int_{-1}^{2} \#v_{Q,d}(\mathbf{t})\, |S_{t_2}(\mathbf{t})|^2 \,\mathrm{d}t_1\mathrm{d}t_2 \\
&+ l^{-2} \int_{-1}^{2}\int_{-1}^{2} \#v_{Q,d}(\mathbf{t})\, |S_{t_1t_2}(\mathbf{t})|^2 \,\mathrm{d}t_1\mathrm{d}t_2, \quad (4.2)
\end{aligned}
$$

where

$$
\begin{aligned}
v_{Q,d}(\mathbf{t}) &= \sum_{Q<d\leqslant 2Q}{}' \sum_{\mathbf{x} \pmod d} \iota_{\mathbf{x},d}(\mathbf{t}) \\
&= \left|\left\{(d,\mathbf{x}) : |d^{-1}x_i' - t_i| \leqslant l \text{ for } Q<d\leqslant 2Q, (d,x_1,x_2)=1\right\}\right|.
\end{aligned}
$$

We let $\overline{v}_{Q,d} = \max_{\mathbf{t}} v_{Q,d}(\mathbf{t})$. Such a maximum clearly exists, as there are only finitely many elements $(d,\mathbf{x})$. We get

$$
\begin{aligned}
\sum_{Q<d\leqslant 2Q}{}' \sum_{\mathbf{x} \pmod d} \left|S(d^{-1}\mathbf{x})\right|^2 \ll\ & l^2\overline{v}_{Q,d} \int_{-1}^{2}\int_{-1}^{2} |S(\mathbf{t})|^2 \,\mathrm{d}t_1\mathrm{d}t_2 \\
&+ \overline{v}_{Q,d} \int_{-1}^{2}\int_{-1}^{2} |S_{t_1}(\mathbf{t})|^2 \,\mathrm{d}t_1\mathrm{d}t_2 \\
&+ \overline{v}_{Q,d} \int_{-1}^{2}\int_{-1}^{2} |S_{t_2}(\mathbf{t})|^2 \,\mathrm{d}t_1\mathrm{d}t_2 \\
&+ l^{-2}\overline{v}_{Q,d} \int_{-1}^{2}\int_{-1}^{2} |S_{t_1t_2}(\mathbf{t})|^2 \,\mathrm{d}t_1\mathrm{d}t_2. \quad (4.3)
\end{aligned}
$$

We now consider Parseval's identity applied to the space $L^2[(0,1)^2]$ and the orthonormal basis $\{\mathrm{e}^{2\pi\mathrm{i}(n_1t_1+n_2t_2)}\}$.

**Lemma 4.2.3** (Parseval's identity)**.** *Let*

$$
f(\mathbf{t}) = \sum_{\mathbf{b}\in\mathbb{Z}^2} \mathrm{e}^{2\pi\mathrm{i}\mathbf{t}\cdot\mathbf{b}} v(\mathbf{b}).
$$

*Then*

$$
\int_0^1\int_0^1 |f(\mathbf{t})|^2 \mathrm{d}t_1\mathrm{d}t_2 = \sum_{\mathbf{b}\in\mathbb{Z}^2} |v(\mathbf{b})|^2.
$$

**Definition 4.2.4.** For a set $I = (i_1, i_2) \in \{0,1\} \times \{0,1\}$ let

$$f^{(I)}(\mathbf{t})$$

denote partial derivative of $f(\mathbf{t})$ of order $\#I = i_1 + i_2$, derived $i_k$ times with respect to $t_k$.

Note that

$$S^{(I)}(\mathbf{t}) \;=\; \sum_{b_1=0}^{l}\sum_{b_2=0}^{l}(2\pi \mathrm{i} b_1)^{i_1}(2\pi \mathrm{i} b_2)^{i_2}\mathrm{e}^{2\pi \mathrm{i}(t_1 b_1 + t_2 b_2)}c(\mathbf{b}).$$

We apply Lemma 4.2.3 to $S(\mathbf{t})^{(I)}$ to get

$$\begin{aligned}\int_0^1\int_0^1 |S(\mathbf{t})^{(I)}|^2 \mathrm{d}t_1\mathrm{d}t_2 \;&=\; \sum_{b_1=0}^{l}\sum_{b_2=0}^{l}\left|(2\pi \mathrm{i} b_1)^{i_1}(2\pi \mathrm{i} b_2)^{i_2}c(\mathbf{b})\right|^2 \\ &\leqslant\; (2\pi l)^{2\#I}\sum_{b_1=0}^{l}\sum_{b_2=0}^{l}|c(\mathbf{b})|^2 .\end{aligned}$$

As we noted above, $S(\mathbf{t})$ is periodic modulo 1 in each variable, and hence we can modify the limits to get

$$\int_{-1}^2\int_{-1}^2 |S(\mathbf{t})^{(I)}|^2 \mathrm{d}t_1\mathrm{d}t_2 \;\leqslant\; 9(2\pi l)^{2\#I}\sum_{n_1=1}^{l}\sum_{n_2=1}^{l}|c(\mathbf{n})|^2 .$$

We now use this in (4.3) to get

$$\sum_{Q<d\leqslant 2Q}{}' \sum_{\mathbf{x} \pmod d}\left|S(d^{-1}\mathbf{x})\right|^2 \;\ll\; l^2\overline{v}_{Q,d}\sum_{n_1=1}^{l}\sum_{n_2=1}^{l}|c(\mathbf{n})|^2 . \tag{4.4}$$

It hence remains to prove the following lemma

**Lemma 4.2.5.**

$$v_{Q,d}(\mathbf{t}) \ll 1 + \frac{Q^2}{l} + \frac{Q^3}{l^2}.$$

*Proof.* Suppose that $v_{Q,d}(\mathbf{t})$ is not empty. We can choose $(r,\mathbf{y}) \in v_{Q,d}(\mathbf{t})$ and classify the elements of $v_{Q,d}(\mathbf{t})\backslash\{(r,\mathbf{y})\}$ into the sets $v_1$ and $v_2$ such that $(d,\mathbf{x}) \in v_j$ if and only if $\frac{x_j}{d} \neq \frac{y_j}{r}$. Since we are only considering $(d,\mathbf{x})$ with $(d,x_1,x_2) = 1$, the vectors $d^{-1}\mathbf{x}$ will be necessarily distinct. Thus each element of $v_{Q,d}(\mathbf{t})$ apart from $(r, \mathbf{y})$ will lie in at least one of the two classes $v_j$. We will try to estimate the sizes of the two classes $v_1, v_2$. Let $(d, \mathbf{x}) \in v_j$. Then

$$0 \neq \left| \frac{y_j}{r} - \frac{x_j}{d} \right| \leqslant \frac{2}{l}, \tag{4.5}$$

which follows from the triangle inequality, as the distance from each of the two numbers to $t_j$ is $\leqslant \frac{1}{l}$. We define

$$s_j = dy_j - rx_j. \tag{4.6}$$

Since $Q \leqslant r, d \leqslant 2Q$, we use (4.5) and get

$$0 \neq |s_j| \leqslant \frac{8Q^2}{l}.$$

Note also that if $u_j = (y_j, r)$, then $u_j \mid s_j$. As $s_j$ is an integer, there are at most $\frac{16Q^2}{lu_j}$ possible values for $s_j$. Once $s_j$ is defined, then we have

$$d\frac{y_j}{u_j} \equiv \frac{s_j}{u_j} \pmod{\frac{r}{u_j}}.$$

Furthermore, since $\frac{y_j}{u_j}$ and $\frac{r}{u_j}$ are coprime, $d$ is defined modulo $\frac{r}{u_j}$, and so $d = n\frac{r}{u_j} + m$, where the only free parameter now is $n$. But again, since $Q \leqslant r, d \leqslant 2Q$, we have $n\frac{Q}{u_j} \leqslant n\frac{r}{u_j} \leqslant 2Q$ and thus there are at most $2u_j$ choices for $n$. This way $y_j, r$ and $d$ are defined uniquely and hence, by (4.6), $x_j$ is also determined. Thus we have at most $\frac{16Q^2}{lu_j} \times 2u_j = \mathrm{O}\left(\frac{Q^2}{l}\right)$ possible choices for the pair $(d, x_j)$. Finally consider $x_k$ for $k \neq j$. We now only have the upper bound

$$\left| \frac{y_k}{r} - \frac{x_k}{d} \right| \leqslant \frac{2}{l},$$

and hence

$$\left| x_k - d\frac{y_k}{r} \right| \leqslant \frac{2d}{l} \leqslant \frac{4Q}{l}$$

elements in $v_j$. In an interval of length $\frac{4Q}{l}$ there are at most $1 + \frac{4Q}{l}$ integers, and hence we have $\mathrm{O}(1 + \frac{Q}{l})$ choices for $x_k$. Overall it means that there are (together with $(r, \mathbf{y})$) $\mathrm{O}\left(1 + \frac{Q^2}{l} + \frac{Q^3}{l^2}\right)$ elements in $v_j$ for each $j = 1, 2$. □

The proof of our theorem is thus complete. ■

For $\alpha = a_1 + a_2\mathrm{i} \in \mathbb{Q}[\mathrm{i}]$ and $\beta = b_1 + b_2\mathrm{i}$, we define

$$S(\alpha) \quad = \sum_{\substack{\beta\in\mathbb{Z}[\mathrm{i}]\\ \mathrm{N}(\beta)\leqslant N}} \mathrm{e}^{2\pi\mathrm{i}(a_1b_1-a_2b_2)}c(\beta).$$

**Lemma 4.2.6.** *Let*

$$T(\chi) = \sum_{\mathrm{N}(\beta)\leqslant N} c(\beta)\chi(\beta).$$

*Then*

$$\frac{d^2}{\Phi(d)}\sum_{\chi \pmod d}^{*} |T(\chi)|^2 \leqslant \sum_{\substack{\alpha \pmod d\\ (\alpha,d)=1}} \left|S\left(\frac{\alpha}{d}\right)\right|^2.$$

*Proof.* We start off with the known expression, deduced from Theorem 2.5.5

$$\chi(\beta) = \frac{\tau(\overline{\chi},\beta)}{\tau(\overline{\chi},1)} = \frac{1}{\tau(\overline{\chi})}\sum_{\alpha \pmod d} \overline{\chi}(\alpha)\mathrm{e}^{2\pi\mathrm{i}\mathrm{Tr}\left(\frac{\alpha\beta}{2d}\right)}.$$

Multiply both sides by $c(\beta)$ and sum over $\mathrm{N}(\beta) \leqslant N$ to get

$$\begin{aligned} T(\chi) &= \frac{1}{\tau(\overline{\chi})}\sum_{\alpha \pmod d}\overline{\chi}(\alpha)\sum_{\mathrm{N}(\beta)\leqslant N}\mathrm{e}^{2\pi\mathrm{i}\mathrm{Tr}\left(\frac{\alpha\beta}{2d}\right)}c(\beta)\\ &= \frac{1}{\tau(\overline{\chi})}\sum_{\alpha \pmod d}\overline{\chi}(\alpha)S\left(\frac{\alpha}{d}\right). \end{aligned}$$

We now use the fact that $|\tau(\overline{\chi})| = d$ for primitive $\chi$ to get that

$$P := \sum_{\chi \pmod d}^{*} |T(\chi)|^2 \quad = \frac{1}{d^2}\sum_{\chi \pmod d}^{*}\left|\sum_{\alpha \pmod d}\overline{\chi}(\alpha)S\left(\frac{\alpha}{d}\right)\right|^2. \tag{4.7}$$

Finally

$$\begin{aligned} P &\leqslant \frac{1}{d^2}\sum_{\chi \pmod d}\left|\sum_{\alpha \pmod d}\overline{\chi}(\alpha)S\left(\frac{\alpha}{d}\right)\right|^2\\ &= \frac{1}{d^2}\sum_{\chi \pmod d}\left(\sum_{\alpha \pmod d}\overline{\chi}(\alpha)S\left(\frac{\alpha}{d}\right)\times\overline{\sum_{\alpha \pmod d}\overline{\chi}(\alpha)S\left(\frac{\alpha}{d}\right)}\right)\\ &= \frac{1}{d^2}\sum_{\chi \pmod d}\sum_{\alpha \pmod d}\sum_{\gamma \pmod d}\overline{\chi}(\alpha)S\left(\frac{\alpha}{d}\right)\times\overline{\overline{\chi}(\gamma)S\left(\frac{\gamma}{d}\right)}\\ &= \frac{1}{d^2}\sum_{\alpha \pmod d}\sum_{\gamma \pmod d}S\left(\frac{\alpha}{d}\right)\times\overline{S\left(\frac{\gamma}{d}\right)}\sum_{\chi \pmod d}\overline{\chi}(\alpha)\chi(\gamma). \end{aligned}$$

If $\alpha$ or $\gamma$ is not coprime to $d$, the character sum will be zero. If both $\alpha$ and $\gamma$ are coprime to $d$, we can write $\gamma \equiv \delta\alpha \,(\mathrm{mod}\ d)$. Further, as $\gamma$ varies over all residue classes modulo $d$, so does $\delta$. Thus

$$\begin{aligned} P &\leqslant \frac{1}{d^2} \sum_{\substack{\alpha \,(\mathrm{mod}\ d) \\ (\alpha,d)=1}} \sum_{\substack{\delta \,(\mathrm{mod}\ d) \\ (\delta,d)=1}} S\left(\frac{\alpha}{d}\right) \times \overline{S\left(\frac{\delta\alpha}{d}\right)} \sum_{\chi \,(\mathrm{mod}\ d)} \overline{\chi}(\alpha)\chi(\delta\alpha) \\ &= \frac{1}{d^2} \sum_{\substack{\alpha \,(\mathrm{mod}\ d) \\ (\alpha,d)=1}} \sum_{\substack{\delta \,(\mathrm{mod}\ d) \\ (\delta,d)=1}} S\left(\frac{\alpha}{d}\right) \times \overline{S\left(\frac{\delta\alpha}{d}\right)} \sum_{\chi \,(\mathrm{mod}\ d)} \chi(\delta). \end{aligned}$$

Now, the character sum is zero apart from when $\delta = 1$, in which case it is $\Phi(d)$, and hence

$$\begin{aligned} P &\leqslant \frac{1}{d^2} \sum_{\substack{\alpha \,(\mathrm{mod}\ d) \\ (\alpha,d)=1}} \sum_{\substack{\delta \,(\mathrm{mod}\ d) \\ (\delta,d)=1}} S\left(\frac{\alpha}{d}\right) \times \overline{S\left(\frac{\delta\alpha}{d}\right)} \sum_{\chi \,(\mathrm{mod}\ d)} \chi(\delta) \\ &= \frac{\Phi(d)}{d^2} \sum_{\substack{\alpha \,(\mathrm{mod}\ d) \\ (\alpha,d)=1}} S\left(\frac{\alpha}{d}\right) \times \overline{S\left(\frac{\alpha}{d}\right)} \\ &= \frac{\Phi(d)}{d^2} \sum_{\substack{\alpha \,(\mathrm{mod}\ d) \\ (\alpha,d)=1}} \left|S\left(\frac{\alpha}{d}\right)\right|^2. \end{aligned}$$

■

We can deduce an immediate corollary for our character sums.

**Corollary 4.2.7.**

$$\sum_{d \leqslant Q} \frac{d^2}{\Phi(d)} \sum_{\chi}{}^{*} |T(\chi)|^2 \ll (N + Q^2\sqrt{N} + Q^3) \sum_{\mathrm{N}(\beta) \leqslant N} |c(\beta)|^2.$$

*Proof.* If we take $M = 1 + [\sqrt{N}]$ and $l = 2M$, then

$$\begin{aligned} S(\alpha) &= \sum_{\substack{\beta \in \mathbb{Z}[\mathrm{i}] \\ \mathrm{N}(\beta) \leqslant N}} \mathrm{e}^{2\pi\mathrm{i}(a_1b_1 - a_2b_2)} c(\beta) = \sum_{b_1=-M}^{M} \sum_{b_2=-M}^{M} \mathrm{e}^{2\pi\mathrm{i}(a_1b_1 - a_2b_2)} c(\mathbf{b}) \\ &= S(\mathbf{a}, \mathscr{S}_{\mathbf{z}+\mathbf{m},l}), \end{aligned}$$

where $\mathbf{a} = (a_1, -a_2)$, $\mathbf{m} = (-M, -M)$ and

$$c(\mathbf{b}) = \begin{cases} c(b_1 + \mathrm{i}b_2) & \text{if } b_1^2 + b_2^2 \leqslant N \\ 0 & \text{otherwise} \end{cases}$$

The result now follows immediately from Lemma 4.2.6 and Theorem 4.2.1, since if $(\alpha, d) = 1$ then $(a_1, a_2, d) = 1$. ■

## 4.3 An Application of the Large Sieve

We want to apply the Large Sieve to our sieve error term. To do this we first need the following Lemma which we will use several times.

**Lemma 4.3.1.**

$$\begin{aligned} S \quad &:= \quad \sum_{d\leqslant Q}\frac{d^2}{\Phi(d)}\sum_{\chi}{}^{*}\max_{u}\left|\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\substack{\mathrm{N}(\beta)\leqslant N\\ \mathrm{N}(\alpha\beta)\leqslant u}}a_\alpha b_\beta\chi(\alpha\beta)\right| \\ &\ll \quad (N^2+Q^2N+Q^3)^{\frac{1}{2}}(M^2+Q^2M+Q^3)^{\frac{1}{2}} \\ &\quad\quad \times\left(\sum_{\mathrm{N}(\alpha)\leqslant M}|a_\alpha|^2\right)^{\frac{1}{2}}\left(\sum_{\mathrm{N}(\beta)\leqslant N}|b_\beta|^2\right)^{\frac{1}{2}}\log 2MN. \end{aligned}$$

*Proof.* By the large sieve inequality and more specifically Collorary 4.2.7, we know that

$$\sum_{d\leqslant Q}\frac{d^2}{\Phi(d)}\sum_{\chi}{}^{*}\left|\sum_{\mathrm{N}(\alpha)\leqslant N}a_\alpha\chi(\alpha)\right|^2 \ll (N+Q^2\sqrt{N}+Q^3)\sum|a_\alpha|^2. \tag{4.8}$$

As in [Dav80], we use the equality

$$\int_{-T}^{T}\mathrm{e}^{\mathrm{i}t\alpha}\frac{\sin t\beta}{t}\mathrm{d}t \quad = \quad \begin{cases} \pi+\mathrm{O}(T^{-1}(\beta-|\alpha|)^{-1}) & \text{if } |\alpha|<\beta; \\ \mathrm{O}(T^{-1}(|\alpha|-\beta)^{-1}) & \text{if } |\alpha|>\beta. \end{cases} \tag{4.9}$$

Let

$$\begin{aligned} A(t,\chi) \quad &= \quad \sum_{\mathrm{N}(\alpha)\leqslant M}\frac{a_\alpha\chi(\alpha)}{\mathrm{N}(\alpha)^{\mathrm{i}t}}; \\ B(t,\chi) \quad &= \quad \sum_{\mathrm{N}(\beta)\leqslant N}\frac{b_\beta\chi(\beta)}{\mathrm{N}(\beta)^{\mathrm{i}t}}. \end{aligned}$$

Then if $\beta = \log u$, then by (4.9) we get

$$\begin{aligned}
&\int_{-T}^{T} A(t,\chi)B(t,\chi)\frac{\sin(t\log u)}{\pi t}\mathrm{d}t \\
= &\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\mathrm{N}(\beta)\leqslant N} a_\alpha b_\beta \chi(\alpha\beta)\int_{-T}^{T}\frac{\sin(t\log u)}{\pi t\mathrm{N}(\alpha\beta)^{\mathrm{i}t}}\mathrm{d}t \\
= &\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\mathrm{N}(\beta)\leqslant N} a_\alpha b_\beta \chi(\alpha\beta)\int_{-T}^{T}\mathrm{e}^{-\log(\mathrm{N}(\alpha\beta))\mathrm{i}t}\frac{\sin(t\log u)}{\pi t}\mathrm{d}t \\
= &\ \pi\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\substack{\mathrm{N}(\beta)\leqslant N\\ \mathrm{N}(\alpha\beta)\leqslant u}} a_\alpha b_\beta\chi(\alpha\beta) \\
&+\mathrm{O}\left(\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\mathrm{N}(\beta)\leqslant N}|a_\alpha b_\beta|\left|T\log\frac{\mathrm{N}(\alpha\beta)}{u}\right|^{-1}\right). \qquad (4.10)
\end{aligned}$$

Now, suppose that $u = k + \frac{1}{2}$ for an integer $k$, such that $0 \leqslant k \leqslant MN$. Then

$$\left|\log\frac{\mathrm{N}(\alpha\beta)}{u}\right| \gg \frac{1}{u} \gg \frac{1}{MN}$$

and

$$\sin(t\log u) \ll \min(1, |t|\log 2MN).$$

Thus, it follows from (4.10) that

$$\begin{aligned}
&\left|\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\substack{\mathrm{N}(\beta)\leqslant N\\ \mathrm{N}(\alpha\beta)\leqslant u}} a_\alpha b_\beta\chi(\alpha\beta)\right| \\
\ll &\int_{-T}^{T}|A(t,\chi)B(t,\chi)|\min(\frac{1}{|t|},\log 2MN)\mathrm{d}t + \frac{MN}{T}\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\mathrm{N}(\beta)\leqslant N}|a_\alpha b_\beta| \\
= &\int_{-T}^{T}\left|\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\mathrm{N}(\beta)\leqslant N} a_\alpha b_\beta\chi(\alpha\beta)\right|\min(\frac{1}{|t|},\log 2MN)\mathrm{d}t \\
&+\frac{MN}{T}\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\mathrm{N}(\beta)\leqslant N}|a_\alpha b_\beta|. \qquad (4.11)
\end{aligned}$$

We apply this to $S$ as originally defined in Lemma 4.3.1 to get

$$\begin{aligned}
S \ll &\sum_{d\leqslant Q}\frac{d^2}{\Phi(d)}\sum_{\chi}{}^{*}\int_{-T}^{T}\left|\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\mathrm{N}(\beta)\leqslant N} a_\alpha b_\beta\chi(\alpha\beta)\right|\min(\frac{1}{|t|},\log 2MN)\mathrm{d}t \\
&+\sum_{d\leqslant Q}\frac{d^2}{\Phi(d)}\sum_{\chi}{}^{*}\frac{MN}{T}\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\mathrm{N}(\beta)\leqslant N}|a_\alpha b_\beta|. \qquad (4.12)
\end{aligned}$$

By Cauchy–Schwarz (Lemma 3.5.5) and (4.8) we know that

$$\begin{aligned}
&\sum_{d\leqslant Q}\frac{d^2}{\Phi(d)}\sum_{\chi}{}^{*}\left|\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\mathrm{N}(\beta)\leqslant N} a_\alpha b_\beta \chi(\alpha\beta)\right| \\
\ll\ & (N+Q^2\sqrt{N}+Q^3)^{\frac{1}{2}}(M+Q^2\sqrt{M}+Q^3)^{\frac{1}{2}} \\
&\times\left(\sum_{\mathrm{N}(\alpha)\leqslant M}|a_\alpha|^2\right)^{\frac{1}{2}}\left(\sum_{\mathrm{N}(\beta)\leqslant M}|b_\beta|^2\right)^{\frac{1}{2}}. \qquad (4.13)
\end{aligned}$$

By (double) Cauchy–Schwarz we also know that

$$\begin{aligned}
\sum_{\mathrm{N}(\alpha)\leqslant M}\sum_{\mathrm{N}(\beta)\leqslant N}|a_\alpha b_\beta| &\leqslant \sum_{\mathrm{N}(\alpha)\leqslant M}|a_\alpha|\, N\left(\sum_{\mathrm{N}(\beta)\leqslant N}|b_\beta|^2\right)^{\frac{1}{2}} \\
&\leqslant MN\left(\sum_{\mathrm{N}(\alpha)\leqslant M}|a_\alpha|^2\right)^{\frac{1}{2}}\left(\sum_{\mathrm{N}(\beta)\leqslant N}|b_\beta|^2\right)^{\frac{1}{2}}. \qquad (4.14)
\end{aligned}$$

If we now apply (4.13) to the first term of (4.12) and (4.14) to the second. then we get

$$\begin{aligned}
S \ll\ & (N+Q^2\sqrt{N}+Q^3)^{\frac{1}{2}}(M+Q^2\sqrt{M}+Q^3)^{\frac{1}{2}} \\
&\times\left(\sum_{\mathrm{N}(\alpha)\leqslant M}|a_\alpha|^2\right)^{\frac{1}{2}}\left(\sum_{\mathrm{N}(\beta)\leqslant M}|b_\beta|^2\right)^{\frac{1}{2}} \\
&\times\int_{-T}^{T}\min\left(\frac{1}{|t|},\log 2MN\right)\mathrm{d}t \\
&+\sum_{d\leqslant Q}\frac{d^2}{\Phi(d)}\sum_{\chi}{}^{*}\frac{M^2N^2}{T}\left(\sum_{\mathrm{N}(\alpha)\leqslant M}|a_\alpha|^2\right)^{\frac{1}{2}}\left(\sum_{\mathrm{N}(\beta)\leqslant N}|b_\beta|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Note in the second term that the outer sum has $Q$ terms and that there are $\Phi(d)$

characters modulo $d$. Hence we can estimate it by

$$\begin{aligned} S &\ll (N + Q^2\sqrt{N} + Q^3)^{\frac{1}{2}}(M + Q^2\sqrt{M} + Q^3)^{\frac{1}{2}} \\ &\times \left(\sum_{\mathrm{N}(\alpha)\leqslant M} |a_\alpha|^2\right)^{\frac{1}{2}} \left(\sum_{\mathrm{N}(\beta)\leqslant M} |b_\beta|^2\right)^{\frac{1}{2}} \\ &\times \int_{-T}^{T} \min\left(\frac{1}{|t|}, \log 2MN\right) \mathrm{d}t \\ &+ \frac{Q^3M^2N^2}{T} \left(\sum_{\mathrm{N}(\alpha)\leqslant M} |a_\alpha|^2\right)^{\frac{1}{2}} \left(\sum_{\mathrm{N}(\beta)\leqslant N} |b_\beta|^2\right)^{\frac{1}{2}}. \end{aligned}$$

We now note that

$$\int_{-T}^{T} \min(\frac{1}{|t|}, \log 2MN)\mathrm{d}t \ll \log 2MN + \int_{1}^{T} \frac{1}{t}\mathrm{d}t = \log 2MN + \log T.$$

We apply that to our estimation and we let $T = M^2N^2$ to get our final result

$$\begin{aligned} S &\ll (N + Q^2\sqrt{N} + Q^3)^{\frac{1}{2}}(M + Q^2\sqrt{M} + Q^3)^{\frac{1}{2}} \\ &\times \left(\sum_{\mathrm{N}(\alpha)\leqslant M} |a_\alpha|^2\right)^{\frac{1}{2}} \left(\sum_{\mathrm{N}(\beta)\leqslant N} |b_\beta|^2\right)^{\frac{1}{2}} \log 2MN. \end{aligned}$$

■

## 4.4 Vaughan's Method

We will now generalise Vaughan's Method, which is presented in [Dav80], to the Gaussian field. The following sums and products only include one associate per class. Let

$$\begin{aligned} F(s) &= \sum_{\mathrm{N}(\alpha)\leqslant U}^{\sim} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha)\mathrm{N}(\alpha)^{-s}; \\ G(s) &= \sum_{\mathrm{N}(\alpha)\leqslant V}^{\sim} \mu_{\mathbb{Q}[\mathrm{i}]}(\alpha)\mathrm{N}(\alpha)^{-s}. \end{aligned}$$

We use the identity

$$\begin{aligned} -\frac{\zeta'_{\mathbb{Q}[\mathrm{i}]}(s)}{\zeta_{\mathbb{Q}[\mathrm{i}]}(s)} &= F(s) - \zeta_{\mathbb{Q}[\mathrm{i}]}(s)F(s)G(s) - \zeta'_{\mathbb{Q}[\mathrm{i}]}(s)G(s) \\ &+ \left(-\frac{\zeta'_{\mathbb{Q}[\mathrm{i}]}(s)}{\zeta_{\mathbb{Q}[\mathrm{i}]}(s)} - F(s)\right)\left(1 - \zeta_{\mathbb{Q}[\mathrm{i}]}(s)G(s)\right) \qquad (4.15) \end{aligned}$$

and (2.12) which is

$$-\frac{\zeta'_{\mathbb{Q}[i]}(s)}{\zeta_{\mathbb{Q}[i]}(s)} = \sum_{\alpha\in\mathbb{Z}[i]}^{\sim} \frac{\Lambda_{\mathbb{Q}[i]}(\alpha)}{\mathrm{N}(\alpha)}.$$

By equating the coefficients in this Dirichlet series, we get that

$$\Lambda_{\mathbb{Q}[i]}(\alpha) = a_1(\alpha) + a_2(\alpha) + a_3(\alpha) + a_4(\alpha), \tag{4.16}$$

with the $a_i$ being the Dirichlet coefficients in (4.15), namely

$$\begin{aligned}
a_1(\alpha) &= \begin{cases} \Lambda_{\mathbb{Q}[i]}(\alpha) & \text{if } \mathrm{N}(\alpha) \leqslant U; \\ 0 & \text{otherwise}; \end{cases} \\
a_2(\alpha) &= -\sum_{\substack{\beta\gamma\delta=\alpha \\ \mathrm{N}(\beta)\leqslant U \\ \mathrm{N}(\gamma)\leqslant V}} \Lambda_{\mathbb{Q}[i]}(\beta)\mu_{\mathbb{Q}[i]}(\gamma); \\
a_3(\alpha) &= \sum_{\substack{\varepsilon\gamma=\alpha \\ \mathrm{N}(\gamma)\leqslant V}} \mu_{\mathbb{Q}[i]}(\gamma)\log \mathrm{N}(\varepsilon); \\
a_4(\alpha) &= -\sum_{\substack{\beta\eta=\alpha \\ \mathrm{N}(\beta)>U \\ \mathrm{N}(\eta)>1}} \Lambda_{\mathbb{Q}[i]}(\beta) \left( \sum_{\substack{\gamma\nu=\eta \\ \mathrm{N}(\gamma)\leqslant V}} \mu_{\mathbb{Q}[i]}(\gamma) \right).
\end{aligned}$$

Our aim is to use Vaughan's Method to estimate the sum

$$\sum_{d\leqslant Q} \frac{d^2}{\Phi(d)} \sum_{\substack{\chi \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}}^{*} \max_{y\leqslant x} |\psi(y,\chi)|.$$

We have

$$\psi(x,\chi) = \sum_{\mathrm{N}(\alpha)\leqslant x} \chi(\alpha)\Lambda_{\mathbb{Q}[i]}(\alpha). \tag{4.17}$$

Combining this with (4.16), gives

$$\psi(y,\chi) = S_1 + S_2 + S_3 + S_4,$$

where $S_i = \sum_{\mathrm{N}(\alpha)\leqslant y} a_i(\alpha)\chi(\alpha)$. Thus we are left with estimating the following sums

$$S_1 = \sum_{\mathrm{N}(\alpha)\leqslant U} \Lambda_{\mathbb{Q}[i]}(\alpha)\chi(\alpha) \ll U; \tag{4.18}$$

$$\begin{aligned} S_2 &= -\sum_{\mathrm{N}(\alpha)\leqslant y}\left(\sum_{\substack{\beta\gamma\delta=\alpha\\ \mathrm{N}(\beta)\leqslant U\\ \mathrm{N}(\gamma)\leqslant V}}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\right)\chi(\alpha)\\ &= -\sum_{\substack{\mathrm{N}(\varepsilon\delta)\leqslant y\\ \mathrm{N}(\varepsilon)\leqslant UV}}\left(\sum_{\substack{\beta\gamma=\varepsilon\\ \mathrm{N}(\beta)\leqslant U\\ \mathrm{N}(\gamma)\leqslant V}}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\right)\chi(\varepsilon\delta)\\ &= -\sum_{\mathrm{N}(\varepsilon)\leqslant UV}\left(\sum_{\substack{\beta\gamma=\varepsilon\\ \mathrm{N}(\beta)\leqslant U\\ \mathrm{N}(\gamma)\leqslant V}}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\right)\sum_{\mathrm{N}(\delta)\leqslant\frac{y}{\mathrm{N}(\varepsilon)}}\chi(\varepsilon\delta); \end{aligned}$$

$$\begin{aligned} S_3 &= \sum_{\mathrm{N}(\alpha)\leqslant y}\left(\sum_{\substack{\varepsilon\gamma=\alpha\\ \mathrm{N}(\gamma)\leqslant V}}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\log\mathrm{N}(\varepsilon)\right)\chi(\alpha)\\ &= \sum_{\mathrm{N}(\gamma)\leqslant V}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\sum_{\mathrm{N}(\varepsilon)\leqslant\frac{y}{\mathrm{N}(\gamma)}}\chi(\gamma\varepsilon)\log\mathrm{N}(\varepsilon)\\ &= \sum_{\mathrm{N}(\gamma)\leqslant V}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\sum_{\mathrm{N}(\varepsilon)\leqslant\frac{y}{\mathrm{N}(\gamma)}}\chi(\gamma\varepsilon)\int_1^{\mathrm{N}(\varepsilon)}\frac{1}{t}\mathrm{d}t\\ &= \int_1^y\sum_{\mathrm{N}(\gamma)\leqslant V}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\sum_{t\leqslant\mathrm{N}(\varepsilon)\leqslant\frac{y}{\mathrm{N}(\gamma)}}\chi(\gamma\varepsilon)\frac{1}{t}\mathrm{d}t\\ &\ll (\log y)\sum_{\mathrm{N}(\gamma)\leqslant V}\max_t\left|\sum_{t\leqslant\mathrm{N}(\varepsilon)\leqslant\frac{y}{\mathrm{N}(\gamma)}}\chi(\gamma\varepsilon)\right|; \end{aligned}$$

and finally

$$\begin{aligned} S_4 &= -\sum_{\mathrm{N}(\alpha)\leqslant y}\left(\sum_{\substack{\beta\eta=\alpha\\ \mathrm{N}(\beta)>U\\ \mathrm{N}(\eta)>1}}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\left(\sum_{\substack{\gamma\nu=\eta\\ \mathrm{N}(\gamma)\leqslant V}}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\right)\right)\chi(\alpha)\\ &= -\sum_{\mathrm{N}(\beta\eta)\leqslant y}\left(\sum_{\substack{\mathrm{N}(\beta)>U\\ \mathrm{N}(\eta)>1}}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\left(\sum_{\substack{\gamma\mid\eta\\ \mathrm{N}(\gamma)\leqslant V}}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\right)\right)\chi(\beta\eta). \end{aligned}$$

However, since whenever $1 < \mathrm{N}(\nu) \leqslant V$ we have

$$\sum_{\substack{\gamma\mid\nu\\ \mathrm{N}(\gamma)\leqslant V}}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma) = 0,$$

the last sum can be rewritten as

$$\begin{aligned} S_4 &= -\sum_{\mathrm{N}(\beta\eta)\leqslant y}\left(\sum_{\substack{\mathrm{N}(\beta)>U\\ \mathrm{N}(\eta)>V}}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\left(\sum_{\substack{\gamma\mid\eta\\ \mathrm{N}(\gamma)\leqslant V}}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\right)\right)\chi(\beta\eta)\\ &= -\sum_{U<\mathrm{N}(\beta)\leqslant\frac{y}{V}}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\sum_{V<\mathrm{N}(\eta)\leqslant\frac{y}{\mathrm{N}(\beta)}}\left(\sum_{\substack{\gamma\mid\eta\\ \mathrm{N}(\gamma)\leqslant V}}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\right)\chi(\beta\eta). \end{aligned}$$

We note that

$$\left|\sum_{\substack{\gamma\mid\eta\\ \mathrm{N}(\gamma)\leqslant V}}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\right| \leqslant \sum_{\substack{\gamma\mid\eta\\ \mathrm{N}(\gamma)\leqslant V}}\left|\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\right| \leqslant d_{\mathbb{Q}[\mathrm{i}]}(\eta). \tag{4.19}$$

If we combine this with Lemma 4.3.1 using

$$a_\alpha = \Lambda_{\mathbb{Q}[\mathrm{i}]}(\alpha)$$

and

$$b_\beta = \sum_{\substack{\gamma\mid\beta\\ \mathrm{N}(\gamma)\leqslant V}}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma),$$

we may split the interval for $\mathrm{N}(\beta)$ into dyadic ranges to get

$$\sum_{d\leqslant Q}\frac{d^2}{\Phi(d)}\sum_{\chi}{}^{*}\max_{y\leqslant x}\left|\sum_{\substack{U<\mathrm{N}(\beta)\leqslant\frac{y}{V}\\ M<\mathrm{N}(\beta)\leqslant 2M}}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\sum_{V<\mathrm{N}(\eta)\leqslant\frac{y}{\mathrm{N}(\beta)}}\left(\sum_{\substack{\gamma|\eta\\ \mathrm{N}(\gamma)\leqslant V}}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\right)\chi(\beta\eta)\right|$$

$$\ll (M+Q^2\sqrt{M}+Q^3)^{\frac{1}{2}}\left(\frac{x}{M}+Q^2\sqrt{\frac{x}{M}}+Q^3\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{M<\mathrm{N}(\beta)\leqslant 2M}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)^2\right)^{\frac{1}{2}}\left(\sum_{\mathrm{N}(\eta)\leqslant\frac{x}{M}}d_{\mathbb{Q}[\mathrm{i}]}(\eta)^2\right)^{\frac{1}{2}}\log x. \tag{4.20}$$

Now we note that

$$\sum_{M\leqslant\mathrm{N}(\beta)\leqslant 2M}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)^2\leqslant(\log\mathrm{N}(2M))\sum_{M\leqslant\mathrm{N}(\beta)\leqslant 2M}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\ll M\log M, \tag{4.21}$$

and also

$$\sum_{\mathrm{N}(\eta)\leqslant\frac{x}{M}}d_{\mathbb{Q}[\mathrm{i}]}(\eta)^2 = \sum_{\mathrm{N}(\eta)\leqslant\frac{x}{M}}\sum_{\gamma|\eta}d_{\mathbb{Q}[\mathrm{i}]}(\gamma^2)\ll\sum_{\mathrm{N}(\gamma)\leqslant\frac{x}{M}}d_{\mathbb{Q}[\mathrm{i}]}(\gamma^2)\frac{x}{M\mathrm{N}(\gamma)}$$

since

$$\left|\left\{\eta:\mathrm{N}(\eta)\leqslant\frac{x}{M},\gamma\mid\eta\right\}\right|=\left|\left\{\rho:\mathrm{N}(\rho)\leqslant\frac{x}{M\mathrm{N}(\gamma)}\right\}\right|\ll\frac{x}{M\mathrm{N}(\gamma)}$$

by the trivial Gauss Circle Problem estimation. Hence

$$\begin{aligned}\sum_{\mathrm{N}(\eta)\leqslant\frac{x}{M}}d_{\mathbb{Q}[\mathrm{i}]}(\eta)^2 &\ll \left(\frac{x}{M}\right)\sum_{\mathrm{N}(\gamma)\leqslant\frac{x}{M}}\frac{d_{\mathbb{Q}[\mathrm{i}]}(\gamma^2)}{\mathrm{N}(\gamma)}\\ &\ll \left(\frac{x}{M}\right)\prod_{\mathrm{N}(\pi)\leqslant\frac{x}{M}}\left(1+\frac{d_{\mathbb{Q}[\mathrm{i}]}(\pi^2)}{\mathrm{N}(\pi)}+\frac{d_{\mathbb{Q}[\mathrm{i}]}(\pi^4)}{\mathrm{N}(\pi^2)}+\frac{d_{\mathbb{Q}[\mathrm{i}]}(\pi^6)}{\mathrm{N}(\pi^3)}+\cdots\right)\\ &\ll \left(\frac{x}{M}\right)\prod_{\mathrm{N}(\pi)\leqslant\frac{x}{M}}\left(1+\frac{3}{\mathrm{N}(\pi)}+\frac{5}{\mathrm{N}(\pi^2)}+\frac{7}{\mathrm{N}(\pi^3)}+\cdots\right)\\ &\ll \left(\frac{x}{M}\right)\prod_{2<\mathrm{N}(\pi)\leqslant\frac{x}{M}}\left(1-\frac{3}{\mathrm{N}(\pi)}\right)^{-1}\\ &\ll \left(\frac{x}{M}\right)\prod_{2<\mathrm{N}(\pi)\leqslant\frac{x}{M}}\left(1-\frac{1}{\mathrm{N}(\pi)}\right)^{-3}.\end{aligned}$$

There are at most two Gaussian primes $\pi$ of a given (prime-power) norm and their norm will be $p$ or $p^2$ depending on the residue class modulo 4. Thus $\mathrm{N}(\pi) \geqslant p$ for the corresponding norm $p$. Consequently

$$\begin{aligned}\sum_{\mathrm{N}(\eta)\leqslant \frac{x}{M}} d_{\mathbb{Q}[\mathrm{i}]}(\eta)^2 &\ll \left(\frac{x}{M}\right)\prod_{p\leqslant\frac{x}{M}}\left(1-\frac{1}{p}\right)^{-6} \\ &\ll \left(\frac{x}{M}\right)\left(\log\frac{2x}{M}\right)^6 \ll \left(\frac{x}{M}\right)(\log x)^6. \qquad (4.22)\end{aligned}$$

Substituting (4.21) and (4.22) in (4.20)

$$\begin{aligned}&\sum_{d\leqslant Q}\frac{d^2}{\Phi(d)}\sum_{\chi}{}^{*}\max_{y\leqslant x}\left|\sum_{\substack{U<\mathrm{N}(\beta)\leqslant\frac{y}{V}\\ M<\mathrm{N}(\beta)\leqslant 2M}}\Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\sum_{V<\mathrm{N}(\eta)\leqslant\frac{y}{\mathrm{N}(\beta)}}\left(\sum_{\substack{\gamma\mid\eta\\ \mathrm{N}(\gamma)\leqslant V}}\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)\right)\chi(\beta\nu)\right| \\ \ll\ & (M+Q^2\sqrt{M}+Q^3)^{\frac12}\left(\frac{x}{M}+Q^2\sqrt{\frac{x}{M}}+Q^3\right)^{\frac12}x^{\frac12}(\log x)^{\frac92} \\ \ll\ & \Big(x+x^{\frac34}QM^{\frac14}+x^{\frac12}Q^{\frac32}M^{\frac12}+xQM^{-\frac14}+x^{\frac34}Q^2+x^{\frac12}Q^{\frac52}M^{\frac14} \\ & +xQ^{\frac32}M^{-\frac12}+x^{\frac34}Q^{\frac52}M^{-\frac14}+x^{\frac12}Q^3\Big)(\log x)^{\frac92}.\end{aligned}$$

Now we apply this to $M=2^k$ for each $k$ such that $\frac{U}{2}<2^k\leqslant\frac{x}{V}$, and, since there are only $\mathrm{O}(\log x)$ such $M$, we get

$$\begin{aligned}&\sum_{d\leqslant Q}\frac{d^2}{\Phi(d)}\sum_{\chi}{}^{*}\max_{y\leqslant x}|S_4| \\ \ll\ & \Big(x+xQV^{-\frac14}+xQ^{\frac32}V^{-\frac12}+xQU^{-\frac14}+x^{\frac34}Q^2 \\ & +x^{\frac34}Q^{\frac52}V^{-\frac14}+xQ^{\frac32}U^{-\frac12}+x^{\frac34}Q^{\frac52}U^{-\frac14}+x^{\frac12}Q^3\Big)(\log x)^{\frac{11}{2}}. \qquad (4.23)\end{aligned}$$

We will treat $S_2$ just as Davenport does in [Dav80], with the exception that we need a suitable Pólya–Vinogradov type inequality which we get from a result by Landau that can be found in [Lan18]. We start off by separating $S_2$ into two ranges of $\varepsilon$:

$$S_2=\sum_{\mathrm{N}(\varepsilon)\leqslant UV}=\sum_{\mathrm{N}(\varepsilon)\leqslant U}+\sum_{U<\mathrm{N}(\varepsilon)\leqslant UV}=S_2'+S_2''.$$

We first deal with $S_2''$ which is very similar to $S_4$. The contribution to (4.17) from those terms of $S_2$ with $M < \mathrm{N}(\varepsilon) \leqslant 2M$ is

$$\overline{S_2''} = \sum_{d\leqslant Q} \frac{d^2}{\Phi(d)} \sum_{\chi}{}^{*} \max_{y\leqslant x} \left| \sum_{\substack{U<\mathrm{N}(\varepsilon)\leqslant UV \\ M<\mathrm{N}(\varepsilon)\leqslant 2M}} \left( \sum_{\substack{\beta\gamma=\varepsilon \\ \mathrm{N}(\beta)\leqslant U \\ \mathrm{N}(\gamma)\leqslant V}} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma) \right) \sum_{\mathrm{N}(\delta)\leqslant \frac{y}{\mathrm{N}(\varepsilon)}} \chi(\varepsilon\delta) \right|.$$

We now write

$$a_\varepsilon = \sum_{\substack{\beta\gamma=\varepsilon \\ \mathrm{N}(\beta)\leqslant U \\ \mathrm{N}(\gamma)\leqslant V}} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma)$$

and note that

$$|a_\varepsilon| \leqslant \sum_{\substack{\beta|\varepsilon \\ \mathrm{N}(\beta)\leqslant U}} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta) \leqslant \log \mathrm{N}(\varepsilon).$$

Thus

$$\overline{S_2''} \ll \sum_{d\leqslant Q} \frac{d^2}{\Phi(d)} \sum_{\chi}{}^{*} \max_{y\leqslant x} \left| \sum_{\substack{U<\mathrm{N}(\varepsilon)\leqslant UV \\ M<\mathrm{N}(\varepsilon)\leqslant 2M}} a_\varepsilon \sum_{\mathrm{N}(\delta)\leqslant \frac{y}{\mathrm{N}(\varepsilon)}} \chi(\varepsilon\delta) \right|.$$

We now again apply Lemma 4.3.1 to this and get

$$\begin{aligned}
\overline{S_2''} &\ll (M + Q^2\sqrt{M} + Q^3)^{\frac{1}{2}} \left( \frac{x}{M} + Q^2\sqrt{\frac{x}{M}} + Q^3 \right)^{\frac{1}{2}} \\
&\quad \times \left(\frac{x}{M}\right)^{\frac{1}{2}} \left( \sum_{M<\mathrm{N}(\varepsilon)\leqslant 2M} (\log \mathrm{N}(\varepsilon))^2 \right)^{\frac{1}{2}} \log x \\
&\ll (M + Q^2\sqrt{M} + Q^3)^{\frac{1}{2}} \left( \frac{x}{M} + Q^2\sqrt{\frac{x}{M}} + Q^3 \right)^{\frac{1}{2}} \left(\frac{x}{M}\right)^{\frac{1}{2}} \left(M(\log M)^2\right)^{\frac{1}{2}} \log x \\
&\ll (M + Q^2\sqrt{M} + Q^3)^{\frac{1}{2}} \left( \frac{x}{M} + Q^2\sqrt{\frac{x}{M}} + Q^3 \right)^{\frac{1}{2}} x^{\frac{1}{2}} (\log x)^2 \\
&\ll \Big( x + x^{\frac{3}{4}} Q M^{\frac{1}{4}} + x^{\frac{1}{2}} Q^{\frac{3}{2}} M^{\frac{1}{2}} + xQM^{-\frac{1}{4}} + x^{\frac{3}{4}} Q^2 + x^{\frac{1}{2}} Q^{\frac{5}{2}} M^{\frac{1}{4}} \\
&\quad + xQ^{\frac{3}{2}} M^{-\frac{1}{2}} + x^{\frac{3}{4}} Q^{\frac{5}{2}} M^{-\frac{1}{4}} + x^{\frac{1}{2}} Q^3 \Big) (\log x)^2.
\end{aligned}$$

Again, if we sum over $M = 2^k$ for $\frac{1}{2}U < 2^k \leqslant UV$ we see that

$$\begin{aligned}
&\sum_{d\leqslant Q} \frac{d^2}{\Phi(d)} \sum_{\chi}{}^{*} \max_{y\leqslant x} |S_2''| \\
\ll\ & \left(x + x^{\frac{3}{4}}QU^{\frac{1}{4}}V^{\frac{1}{4}} + x^{\frac{1}{2}}Q^{\frac{3}{2}}U^{\frac{1}{2}}V^{\frac{1}{2}} + xQU^{-\frac{1}{4}} + x^{\frac{3}{4}}Q^2 \right.\\
&\left. + x^{\frac{1}{2}}Q^{\frac{5}{2}}U^{\frac{1}{4}}V^{\frac{1}{4}} + xQ^{\frac{3}{2}}U^{-\frac{1}{2}} + x^{\frac{3}{4}}Q^{\frac{5}{2}}U^{-\frac{1}{4}} + x^{\frac{1}{2}}Q^3\right)(\log x)^3. \qquad (4.24)
\end{aligned}$$

To treat $S_2'$ note

$$\begin{aligned}
S_2' &= -\sum_{\mathrm{N}(\varepsilon)\leqslant U} \left( \sum_{\substack{\beta\gamma=\varepsilon \\ \mathrm{N}(\beta)\leqslant U \\ \mathrm{N}(\gamma)\leqslant V}} \Lambda_{\mathbb{Q}[\mathrm{i}]}(\beta)\mu_{\mathbb{Q}[\mathrm{i}]}(\gamma) \right) \sum_{\mathrm{N}(\delta)\leqslant \frac{y}{\mathrm{N}(\varepsilon)}} \chi(\varepsilon\delta) \\
&\ll \log U \sum_{\mathrm{N}(\varepsilon)\leqslant U} \left| \sum_{\mathrm{N}(\delta)\leqslant \frac{y}{\mathrm{N}(\varepsilon)}} \chi(\delta) \right|. \qquad (4.25)
\end{aligned}$$

We now use a special case for $\mathbb{Q}[\mathrm{i}]$ of Landau's general result from [Lan18] which states (after translating from ideals to numbers) that if $\chi$ is a primitive, non-principal character modulo $d$, then

$$\sum_{\mathrm{N}(\alpha)\leqslant x} \chi(\alpha) \ll \mathrm{N}(d)^{\frac{1}{3}}(\log \mathrm{N}(d))^2 x^{\frac{1}{3}}. \qquad (4.26)$$

We note that in contrast to the Pólya–Vinogradov inequality in the rational case, there is a dependence on $x$ here. We combine this with (4.25) to get that if $d > 1$, then

$$S_2' \ll \log U \sum_{\mathrm{N}(\varepsilon)\leqslant U} \left| d^{\frac{2}{3}}(\log d)^2 \left(\frac{y}{\mathrm{N}(\varepsilon)}\right)^{\frac{1}{3}} \right| \ll y^{\frac{1}{3}} d^{\frac{2}{3}} (\log Ud)^3 \sum_{\mathrm{N}(\varepsilon)\leqslant U} \mathrm{N}(\varepsilon)^{-\frac{1}{3}}.$$

Now note

$$\sum_{\mathrm{N}(\varepsilon)\leqslant U} \mathrm{N}(\varepsilon)^{-\frac{1}{3}} = \sum_{n=1}^{U} \mathrm{r}(n) n^{-\frac{1}{3}}.$$

By the trivial Gauss Circle Problem estimation we know

$$F(x) := \sum_{n\leqslant x} \mathrm{r}(n) = \mathrm{O}(x).$$

Thus we can use partial summation to get

$$\begin{aligned}
\sum_{\mathrm{N}(\varepsilon)\leqslant U} \mathrm{N}(\varepsilon)^{-\frac{1}{3}} &= F(U)U^{-\frac{1}{3}} - \int_1^U F(t)\frac{\mathrm{d}}{\mathrm{d}t}t^{-\frac{1}{3}}\mathrm{d}t \\
&= F(U)U^{-\frac{1}{3}} + \frac{1}{3}\int_1^U F(t)t^{-\frac{4}{3}}\mathrm{d}t \\
&= \mathrm{O}\left(U^{\frac{2}{3}}\right) + \mathrm{O}\left(\int_1^U t^{-\frac{1}{3}}\mathrm{d}t\right) \\
&= \mathrm{O}\left(U^{\frac{2}{3}}\right) + \mathrm{O}\left(U^{\frac{2}{3}}\right) = \mathrm{O}\left(U^{\frac{2}{3}}\right).
\end{aligned}$$

Thus

$$S_2' \ll U^{\frac{2}{3}} y^{\frac{1}{3}} d^{\frac{2}{3}} (\log Ud)^3.$$

For $d = 1$ we use the trivial bound

$$\begin{aligned}
S_2' &\ll \log U \sum_{\mathrm{N}(\varepsilon)\leqslant U} \frac{y}{\mathrm{N}(\varepsilon)} \ll y \log U \sum_{\mathrm{N}(\varepsilon)\leqslant U} \frac{1}{\mathrm{N}(\varepsilon)} \\
&\ll y \log U \left(\sum_{n\leqslant U} \frac{\mathrm{r}(n)}{n}\right) \ll y(\log U)^2,
\end{aligned}$$

since $\sum_{n\leqslant U} \frac{\mathrm{r}(n)}{n} = \mathrm{O}(\log U)$ by the same partial summation argument as above. Overall, if we only consider the relevant characters, we get

$$\sum_{d\leqslant Q} \frac{d^2}{\Phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \max_{y\leqslant x} |S_2'| \ll (Q^{\frac{8}{3}} x^{\frac{1}{3}} U^{\frac{2}{3}} + x)(\log xU)^3 \tag{4.27}$$

for all $d$. This is because there are only $\mathrm{O}(Q)$ of the relevant characters, as shown in Lemma 3.5.9. Finally we deal with $S_3$, which is basically the same as $S_2'$:

$$\begin{aligned}
&\sum_{d\leqslant Q} \frac{d^2}{\Phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \max_{y\leqslant x} |S_3| \\
\ll\ &\sum_{d\leqslant Q} \frac{d^2}{\Phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \max_{y\leqslant x} \left((\log y) \sum_{\mathrm{N}(\gamma)\leqslant V} \max_t \left| \sum_{t\leqslant \mathrm{N}(\varepsilon)\leqslant \frac{y}{\mathrm{N}(\gamma)}} \chi(\gamma\varepsilon)\right|\right) \\
\ll\ &(Q^{\frac{8}{3}} x^{\frac{1}{3}} V^{\frac{2}{3}} + x)(\log xV)^3.
\end{aligned} \tag{4.28}$$

Combine the estimates (4.18), (4.27), (4.24), (4.28) and (4.23), then we get (assuming $Q^2 < x$)

$$\begin{aligned}
&\sum_{d\leqslant Q}\frac{d^2}{\Phi(d)}\sideset{}{^*}\sum_{\substack{\chi\\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}}\max_{y\leqslant x}|\psi(y,\chi)|\\
\ll\;& \Big(Q^2U + x + xQV^{-\frac14} + xQ^{\frac32}V^{-\frac12} + xQU^{-\frac14} + x^{\frac34}Q^2 + x^{\frac34}Q^{\frac52}V^{-\frac14}\\
&+xQ^{\frac32}U^{-\frac12} + x^{\frac34}Q^{\frac52}U^{-\frac14} + x^{\frac12}Q^3 + x^{\frac12}Q^{\frac52}U^{\frac14}V^{\frac14} + x^{\frac34}QU^{\frac14}V^{\frac14}\\
&+x^{\frac12}Q^{\frac32}U^{\frac12}V^{\frac12} + x^{\frac13}Q^{\frac83}U^{\frac23} + x^{\frac13}Q^{\frac83}V^{\frac23}\Big)(\log xUV)^{\frac{11}{2}}.
\end{aligned}\tag{4.29}$$

We note that omitting the first term, this expression, $F(U,V)$ say, is symmetric in $U$ and $V$. We now wish to choose the parameters $U$ and $V$ to our best advantage, by looking for values that minimise $F(U,V)$.

**Lemma 4.4.1.** *We have $F(U,V) \geqslant F(\sqrt{UV},\sqrt{UV})$, and hence $F$ can be minimised with a choice in which $U = V$.*

*Proof.* It is sufficient to prove the lemma for the simplest general symmetric form $F(U,V) = U^aV^b + U^bV^a$, as any other symmetric form will be a sum of these. Suppose $F(U,V)$ is minimized with $U = U_0$, $V = V_0$. We have

$$\begin{aligned}
(x+1)^2 &\geqslant 0;\\
x^2+1 &\geqslant 2x;\\
x+\frac{1}{x} &\geqslant 2.
\end{aligned}$$

We now choose $x = \left(\frac{U}{V}\right)^{\frac{a-b}{2}}$ and get

$$\begin{aligned}
\left(\frac{U_0}{V_0}\right)^{\frac{a-b}{2}} + \left(\frac{V_0}{U_0}\right)^{\frac{a-b}{2}} &\geqslant 2\\
U_0^{a-b} + V_0^{a-b} &\geqslant 2\sqrt{U_0V_0}^{\,a-b}\\
U_0^aV_0^b + V_0^aU_0^b &\geqslant 2\sqrt{U_0V_0}^{\,a+b}\\
F(U_0,V_0) &\geqslant F(\sqrt{U_0V_0},\sqrt{U_0V_0}).
\end{aligned}$$

This means that for any $U_0$, $V_0$ there exist $U_1 = V_1 = \sqrt{U_0V_0}$ such that we have $F(U_0,V_0) \geqslant F(U_1,V_1)$. Thus there always exist $U = V$ such that $F(U,V)$ is minimised. ■

Since the expression (4.29) is almost symmetric, we will assume $U = V$ in order to minimise it. It thus simplifies to

$$\sum_{d\leqslant Q}\frac{d^2}{\Phi(d)}\sideset{}{^*}\sum_{\substack{\chi\\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}}\max_{y\leqslant x}|\psi(y,\chi)| \ll \Big(Q^2U + x + xQ^{\frac{3}{2}}U^{-\frac{1}{2}} + xQU^{-\frac{1}{4}} + x^{\frac{3}{4}}Q^2$$
$$+x^{\frac{3}{4}}Q^{\frac{5}{2}}U^{-\frac{1}{4}} + x^{\frac{1}{2}}Q^3 + x^{\frac{1}{2}}Q^{\frac{5}{2}}U^{\frac{1}{2}} + x^{\frac{3}{4}}QU^{\frac{1}{2}} + x^{\frac{1}{2}}Q^{\frac{3}{2}}U + x^{\frac{1}{3}}Q^{\frac{8}{3}}U^{\frac{2}{3}}\Big)$$
$$\times(\log xU)^{\frac{11}{2}}. \tag{4.30}$$

We shall denote

$$\begin{aligned} S &:= Q^2U + x + xQ^{\frac{3}{2}}U^{-\frac{1}{2}} + xQU^{-\frac{1}{4}} + x^{\frac{3}{4}}Q^2 + x^{\frac{3}{4}}Q^{\frac{5}{2}}U^{-\frac{1}{4}} + \\ &\quad x^{\frac{1}{2}}Q^3 + x^{\frac{1}{2}}Q^{\frac{5}{2}}U^{\frac{1}{2}} + x^{\frac{3}{4}}QU^{\frac{1}{2}} + x^{\frac{1}{2}}Q^{\frac{3}{2}}U + x^{\frac{1}{3}}Q^{\frac{8}{3}}U^{\frac{2}{3}}. \end{aligned}$$

We now want to find an optimal $U$ to minimise the above expression. We will later use this estimation for dyadic sums of the form

$$\sum_{k=0}^{\lceil\log_2 Q\rceil}\frac{1}{2^k}\sideset{}{^\ddagger}\sum_{2^k\leqslant d\leqslant 2^{k+1}}\frac{d^2}{\Phi(d)}\sideset{}{^*}\sum_{\substack{\chi\\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}}\max_{y\leqslant x}\left|\psi'_{\mathbb{Q}[\mathrm{i}]}(y,\chi)\right|,$$

which will need to be $\mathrm{O}(\frac{x}{\log^A x})$ to be applicable in our Main Sieve Theorem 3.3.1. This however requires $S \ll \frac{Qx}{\log^{A+1} x}$. We see from above that $x^{\frac{3}{4}}Q^2 \leqslant S$ and thus the best general result we can hope for requires $Q \leqslant x^{\frac{1}{4}}$ since for $Q > x^{\frac{1}{4}}$ we have $Qx < x^{\frac{3}{4}}Q^2 \leqslant S$. We thus assume $Q \leqslant x^{\frac{1}{4}}$. In that case we have

$$Q^2U \ll x^{\frac{1}{2}}U \ll x^{\frac{1}{2}}Q^{\frac{3}{2}}U.$$

Furthermore

$$x^{\frac{1}{2}}Q^3 \ll x^{\frac{3}{4}}Q^2.$$

Thus we have

$$\begin{aligned} S &\ll x + xQ^{\frac{3}{2}}U^{-\frac{1}{2}} + xQU^{-\frac{1}{4}} + x^{\frac{3}{4}}Q^2 + x^{\frac{3}{4}}Q^{\frac{5}{2}}U^{-\frac{1}{4}} \\ &\quad + x^{\frac{1}{2}}Q^{\frac{5}{2}}U^{\frac{1}{2}} + x^{\frac{3}{4}}QU^{\frac{1}{2}} + x^{\frac{1}{2}}Q^{\frac{3}{2}}U + x^{\frac{1}{3}}Q^{\frac{8}{3}}U^{\frac{2}{3}}. \end{aligned}$$

It turns out that $U = x^{\frac{1}{3}}$ balances two pairs of terms and is a suitable choice for our purpose. We hence choose

$$U = x^{\frac{1}{3}}.$$

With that choice of $U$ for $Q \leqslant x^{\frac{1}{4}}$ we get

$$\begin{aligned} S &\ll x + x^{\frac{5}{6}}Q^{\frac{3}{2}} + x^{\frac{11}{12}}Q + x^{\frac{3}{4}}Q^2 + x^{\frac{2}{3}}Q^{\frac{5}{2}} + \\ &\quad x^{\frac{2}{3}}Q^{\frac{5}{2}} + x^{\frac{11}{12}}Q + x^{\frac{5}{6}}Q^{\frac{3}{2}} + x^{\frac{5}{9}}Q^{\frac{8}{3}} \\ &\ll x + x^{\frac{5}{6}}Q^{\frac{3}{2}} + x^{\frac{11}{12}}Q + x^{\frac{3}{4}}Q^2 + x^{\frac{2}{3}}Q^{\frac{5}{2}} + x^{\frac{5}{9}}Q^{\frac{8}{3}}. \end{aligned}$$

We also note that

$$x^{\frac{5}{9}}Q^{\frac{8}{3}} = x^{\frac{5}{9}}Q^{\frac{5}{2}}Q^{\frac{1}{6}} \ll x^{\frac{5}{9}}x^{\frac{1}{24}}Q^{\frac{5}{2}} = x^{\frac{43}{72}}Q^{\frac{5}{2}} \ll x^{\frac{2}{3}}Q^{\frac{5}{2}},$$

and hence

$$S \ll x + x^{\frac{5}{6}}Q^{\frac{3}{2}} + x^{\frac{11}{12}}Q + x^{\frac{3}{4}}Q^2 + x^{\frac{2}{3}}Q^{\frac{5}{2}}.$$

First suppose $x^{\frac{1}{6}} \leqslant Q$. Then

$$\begin{aligned} x &\leqslant x^{\frac{2}{3}}Q^{\frac{5}{2}}; \\ x^{\frac{5}{6}}Q^{\frac{3}{2}} &\leqslant x^{\frac{2}{3}}Q^{\frac{5}{2}}; \\ x^{\frac{11}{12}}Q &\leqslant x^{\frac{2}{3}}Q^{\frac{5}{2}}; \\ x^{\frac{3}{4}}Q^2 &\leqslant x^{\frac{2}{3}}Q^{\frac{5}{2}}; \end{aligned}$$

and hence

$$S \ll x^{\frac{2}{3}}Q^{\frac{5}{2}}.$$

If otherwise $Q \leqslant x^{\frac{1}{6}}$, then

$$\begin{aligned} x^{\frac{5}{6}}Q^{\frac{3}{2}} &\leqslant x^{\frac{11}{12}}Q; \\ x^{\frac{3}{4}}Q^2 &\leqslant x^{\frac{11}{12}}Q; \\ x^{\frac{2}{3}}Q^{\frac{5}{2}} &\leqslant x^{\frac{11}{12}}Q; \end{aligned}$$

and hence

$$S \ll x + x^{\frac{11}{12}}Q$$

in this range. Thus for any $Q \leqslant x^{\frac{1}{4}}$ we have

$$S \ll x + x^{\frac{11}{12}}Q + x^{\frac{2}{3}}Q^{\frac{5}{2}}.$$

Using this in (4.30) we have hence proved an analogue of a Bombieri–Vinogradov Result:

**Theorem 4.4.2.** *We have*

$$\sum_{d\leqslant Q} \frac{d^2}{\Phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \max_{y\leqslant x} \left|\psi_{\mathbb{Q}[i]}(y,\chi)\right| \ll \left(x + x^{\frac{11}{12}}Q + x^{\frac{2}{3}}Q^{\frac{5}{2}}\right)(\log x)^{\frac{11}{2}}.$$

## 4.5 Error Term Estimation

As in the previous chapter, we again estimate $R_0$ to get an estimation for $R$. To do this, we will use the estimation from Lemma 3.5.8 which was

$$R_0 \ll Q\log(xQ)^3 + \log Q \sideset{}{^\ddagger}\sum_{1<d\leqslant 2Q} \frac{1}{\phi(d)} \sideset{}{^{(0)*}}\sum_{\substack{\chi_{2d} \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \max_{y\leqslant x} \left|\psi'_{\mathbb{Q}[i]}(y,\chi)\right|.$$

To estimate the sum in the second term without the Riemann Hypothesis, we use Theorem 4.4.2 above. First, however, we need to get the expression into a form the result is applicable to. We first split the sum into ranges of $d$ as follows

$$\begin{aligned} T &:= \sideset{}{^\ddagger}\sum_{d\leqslant 2Q} \frac{1}{\phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \left|\psi'_{\mathbb{Q}[i]}(y,\chi)\right| \\ &\ll \sum_{k=0}^{\lceil \log_2(2Q)\rceil} \sideset{}{^\ddagger}\sum_{2^k<d\leqslant 2^{k+1}} \frac{1}{\phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \left|\psi'_{\mathbb{Q}[i]}(y,\chi)\right|. \end{aligned}$$

Now, in the inner sum $\frac{d}{2^k} \geqslant 1$ and $\frac{d}{\phi(d)} \geqslant 1$, thus

$$T \ll \sum_{k=0}^{\lceil \log_2(2Q)\rceil} \sideset{}{^\ddagger}\sum_{2^k<d\leqslant 2^{k+1}} \frac{d}{2^k}\frac{d}{\phi(d)}\frac{1}{\phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}} \left|\psi'_{\mathbb{Q}[i]}(y,\chi)\right|.$$

For the $d$ we are summing over we have $\Phi(d) = \phi(d)^2$, and hence

$$T \ll \sum_{k=0}^{\lceil \log_2(2Q) \rceil} \frac{1}{2^k} \sideset{}{^\ddagger}\sum_{2^k < d \leqslant 2^{k+1}} \frac{d^2}{\Phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi\big|_{(\mathbb{Z}/d\mathbb{Z})^\times} = 1}} \left|\psi'_{\mathbb{Q}[\mathrm{i}]}(y,\chi)\right| .$$

We will now split this sum into small and large $d$

$$\begin{aligned} T \ll & \sideset{}{^\ddagger}\sum_{d \leqslant Q_1} \frac{1}{\phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi\big|_{(\mathbb{Z}/d\mathbb{Z})^\times} = 1}} \left|\psi'_{\mathbb{Q}[\mathrm{i}]}(y,\chi)\right| + \\ & \sum_{k=\lceil \log_2 Q_1 \rceil}^{\lceil \log_2(2Q) \rceil} \frac{1}{2^k} \sideset{}{^\ddagger}\sum_{2^k < d \leqslant 2^{k+1}} \frac{d^2}{\Phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi\big|_{(\mathbb{Z}/d\mathbb{Z})^\times} = 1}} \left|\psi'_{\mathbb{Q}[\mathrm{i}]}(y,\chi)\right| =: T_1 + T_2 \qquad (4.31) \end{aligned}$$

and consider each sum, separately starting with $T_2$. We want to apply Theorem 4.4.2 to this. We can apply this with $Q = 2^{k+1}$ to $T_2$ and get

$$\begin{aligned} T_2 &:= \sum_{k=\lceil \log_2 Q_1 \rceil}^{\lceil \log_2(2Q) \rceil} \frac{1}{2^k} \sideset{}{^\ddagger}\sum_{2^k < d \leqslant 2^{k+1}} \frac{d^2}{\Phi(d)} \sideset{}{^*}\sum_{\substack{\chi \\ \chi\big|_{(\mathbb{Z}/d\mathbb{Z})^\times} = 1}} \left|\psi'_{\mathbb{Q}[\mathrm{i}]}(y,\chi_1)\right| \\ &\ll (\log x)^6 \sum_{k=\lceil \log_2 Q_1 \rceil}^{\lceil \log_2(2Q) \rceil} \frac{1}{2^k} \left( x + x^{\frac{11}{12}} 2^{k+1} + x^{\frac{2}{3}} (2^{k+1})^{\frac{5}{2}} \right) \\ &\ll (\log x)^6 \sum_{k=\lceil \log_2 Q_1 \rceil}^{\lceil \log_2(2Q) \rceil} \left( \frac{x}{2^k} + x^{\frac{11}{12}} + x^{\frac{2}{3}} (2^k)^{\frac{3}{2}} \right) \\ &\ll (\log x)^6 \sum_{k=\lceil \log_2 Q_1 \rceil}^{\lceil \log_2(2Q) \rceil} \left( \frac{x}{Q_1} + x^{\frac{11}{12}} + x^{\frac{2}{3}} Q^{\frac{3}{2}} \right). \end{aligned}$$

There are $\mathrm{O}(\log Q)$ terms in this sum so

$$T_2 \ll (\log x)^6 \log Q \left( \frac{x}{Q_1} + x^{\frac{11}{12}} + x^{\frac{2}{3}} Q^{\frac{3}{2}} \right).$$

Since we need this to be applicable to our sieve we choose $Q_1 = (\log x)^A$ to get

$$T_2 \ll (\log x)^6 \log Q \left( \frac{x}{(\log x)^A} + x^{\frac{2}{3}} Q^{\frac{3}{2}} \right).$$

As for $T_1$, we will apply the Siegel–Walfisz Theorem proven earlier (Theorem 2.11.8). We have

$$\begin{aligned} T_1 &:= \sum_{d\leqslant(\log x)^A}^{\ddagger} \frac{1}{\phi(d)} \sum_{\substack{\chi \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}}^{*} \left|\psi'_{\mathbb{Q}[\mathrm{i}]}(y,\chi_1)\right| \\ &\ll \sum_{d\leqslant(\log x)^A}^{\ddagger} \frac{1}{\phi(d)} \sum_{\substack{\chi \\ \chi|_{(\mathbb{Z}/d\mathbb{Z})^\times}=1}}^{*} \frac{x}{(\log x)^{2A}}. \end{aligned}$$

We use Lemma 3.5.9 to estimate the number of characters that are counted by the inner sum to get

$$T_1 \ll \sum_{d\leqslant(\log x)^A}^{\ddagger} \frac{\phi(d)}{\phi(d)} \frac{x}{(\log x)^{2A}} \ll \frac{x}{(\log x)^A}.$$

We combine $T_1$ and $T_2$

$$T \ll (\log x)^6 \log Q \left(\frac{x}{(\log x)^A} + x^{\frac{2}{3}} Q^{\frac{3}{2}}\right).$$

We use this in Lemma 3.5.8 to get

$$R_0 \ll Q\log(xQ)^3 + (\log xQ)^8 \left(\frac{x}{(\log x)^A} + x^{\frac{2}{3}} Q^{\frac{3}{2}}\right).$$

Note that while the second term in Lemma 3.5.8 contains only characters for which $\chi(\mathrm{i}) = 1$, it does not matter in the estimation. Let $Q = x^\gamma$ where $\gamma < \frac{2}{9}$. Then

$$R_0 \ll (\log x)^8 \left(\frac{x}{(\log x)^A} + x^{\frac{2}{3}+\frac{3\gamma}{2}}\right) \ll x(\log x)^{8-A}.$$

By (3.26) this means

$$R' \ll (\log x)^{19}\sqrt{x}(R_0)^{\frac{1}{2}} \ll x(\log x)^{23-\frac{A}{2}}.$$

We now use (3.11) and (3.12)

$$R' = \sum_{d<Q} \mu^2(d) 3^{\nu(d)} \left|R'_d\right| \ll X(\log X)^{24-\frac{A}{2}}.$$

We recall that by Lemma 3.5.3

$$\begin{aligned} R'' &\ll \left(x^{\frac{1}{K}} Q^2 + x^{\frac{1}{2}}\right) (\log x)^{6K^2+1} \\ &\ll \left(x^{\frac{1}{K}+2\gamma} + x^{\frac{1}{2}}\right) (\log x)^{6K^2+1}. \end{aligned}$$

As $\gamma < \frac{2}{9}$ this means that choosing $K = 18$ gives

$$R'' \ll x^{\frac{1}{2}}(\log x)^{6K^2+1}.$$

We use (3.11) and (3.12) again

$$R'' \ll X^{\frac{1}{2}}(\log x)^{6K^2+2},$$

and hence

$$R \ll |R_1| + R' + R'' \ll \frac{X}{\log^2 X} + X(\log X)^{24-\frac{A}{2}}.$$

We choose $A = 52$:

$$R = \sum_{d<Q} \mu^2(d) 3^{\nu(d)} |R_d| \ll \frac{X}{(\log X)^2}.$$

So any $Q = x^\gamma$ such that $\gamma < \frac{2}{9}$ would yield

$$\sum_{d<Q} \mu^2(d) 3^{\nu(d)} |R_d| \ll \frac{X}{(\log X)^2}.$$

Now $\Lambda_5 = 6 - \frac{\log 4}{(1+3^{-5})\log 3} \approx 4.74$. Thus taking any $\alpha$ in the range $\frac{1}{4.7} < \alpha < \frac{1}{4.5} = \frac{2}{9}$ would satisfy the conditions of Theorem 3.3.1. We have thus proven Theorem 1.2.2:

**Theorem 4.5.1** (Main Result 2)**.** *Unconditionally, there exist infinitely many primes of the form $a^2 + b^2$ such that $a^2 + 4b^2$ has at most 5 prime factors.*

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