A laboratory model of post-Newtonian gravity with high power lasers and $4^{\rm th}$ generation light sources

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Abstract.

Using the post-Newtonian formalism of gravity, we attempt to calculate the x-ray Thomson scattering cross section of electrons that are accelerated in the field of a high intensity optical laser. We show that our results are consistent with previous calculations, suggesting that the combination of high power laser and $4^{\rm th}$ generation light sources may become a powerful platform to test models exploring high order corrections to the Newtonian gravity.

1. Introduction

A common misperception of theories that attempt to describe quantum mechanics in curved space-time is that they require accessing energies up to the Planck scale of 10^{19} GeV, which is unattainable from any conceivable particle collider. However, quantum electro-dynamics, as a highly successful quantum field theory, has demonstrated how renormalization involving high energy states can influence low energy physical observables. This is testified, for example, by the observable nature of the Casimir effect, which results from a renormalization of quantum vacuum through boundary conditions set by conducting plates at low energy. Similarly, the experiment of Colella, Overhauser and Werner [1, 2] showed a phase shift in the neutron's wave function caused by the lowest order gravity correction in the quantum mechanical Hamiltionian. Later on, the same result was obtained for neutrons in accelerated frames [3], confirming the validity of the weak equivalence principle [4, 5, 6] for quantum systems. To obtain a consistent description of experimental findings, it was shown that the Hamiltonian must consist of two parts, the usual kinetic term and a gravitational term. This forms the basis of a sub-class of equations known as Newton-Schrödinger Hamiltonians [7, 8]. While the Newton-Schrödinger formalism has an important role in macroscopic quantum gravity theories as it provides laboratory testable predictions, it should be also pointed out that the Newton-Schrödinger equations are not the semiclassical limit of the Einstein equations [9] and thus do not give a self-consistent description of gravity at scales where curvature, radiation (*i.e.,* gravitational waves) and particle production are important.

Indeed, the latter is the most prominent manifestation of gravity affecting vacuum fluctuations: black holes radiate energy at a universal temperature - the Hawking temperature [10, 11]. This is a quite general fact, not confined to black holes. As shown by Davies, Unruh and Fulling [12, 13, 14], an observer in a uniformly accelerated frame experiences the surrounding vacuum as filled with thermal radiation with temperature $T_{DU} = \hbar a/2\pi k_B c = 4.05 \times 10^{-23} a$ K, where a is the acceleration (in cm/s^2) and k_B is the Boltzmann constant. Thanks to the development of chirped pulse amplification of optical laser light [15], state-of-art laser systems can now achieve focused intensities $I \gtrsim 10^{22}$ W/cm^2, thus being able to accelerate the electron to unprecedented values: $a \gtrsim 10^{27}$ cm/s^2. And, for this reason, high power lasers have been advocated as an ideal tool to study gravity models beyond the Newton-Schrödinger approximation [16, 17, 18]. While several experimental proposals have been put forward, at present it remains unclear whether or not such effects are measurable in the laboratory frame [19].

In this paper, however, we take a step back and only focus on possible experimental tests of the Newton-Schrödinger approximation in the post-Newtonian formalism. We give a detailed derivation of dynamic structure factor of electrons scattering from a high power optical laser light in an accelerated frame. This work provides the theoretical background underpinning the ideas originally proposed by some of us in the paper of Crowley et al. [20]. While the experimental setup, the photometrics and the expected signal from scattering of linearly accelerated electrons are presented there, we feel that our original work needs a more in depth discussion of the underlying theory and its connection with other approaches. We indeed show that our results are consistent with the concept of mass shift [21, 22], and they can be interpreted as a frequency shift of radiation emitted by a particle experiencing a strong ponderomotive potential. By virtue of the equivalence principle, this work thus provide an important step towards an experimental platform to test higher order gravity corrections to the Newton-Schrödinger Hamiltonian.

2. Post-Newtonian gravity approximation

We now describe the non-relativistic Hamiltonian in the post-Newtonian gravity theory [4, 23]. By post-Newtonian gravity we refer to a classical metric model of weak gravity which has the standard Newtonian gravity as its limit for null curvature [6]. The gravity is generated by a system of mass M surrounded by a isotropic and homogeneous universe. In this approximation the metric is given by:

$$g_{\mu\nu} = \begin{bmatrix} -1+2\frac{U}{c^2}-2\beta\left(\frac{U}{c^2}\right)^2 & 0 & 0 & 0 \\ 0 & 1+2\gamma\frac{U}{c^2} & 0 & 0 \\ 0 & 0 & 1+2\gamma\frac{U}{c^2} & 0 \\ 0 & 0 & 0 & 1+2\gamma\frac{U}{c^2} \end{bmatrix}, \tag{1}$$

with the indices μ and ν running from 0 to 3 (0 is used for the time-like coordinate), and c is the speed of light. The Newtonian potential at a distance $r = (x^a x_a)^{1/2}$ from the mass is $U = GM/r$ (G is the gravitational constant, and latin indices are used for the spatial components of the 4-vectors). The parameters β and γ describe the specific gravitational model, with β measuring the amount of non-linearity and γ the degree of space-time curvature. Einstein's general relativity requires $\beta = \gamma = 1$. We also notice that $U = 0$ gives $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ is the Minkowski metric of flat space.

The next step consists in describing the quantum mechanical behavior of an electron (excluding spin effects) in such metric. The approach we use is to generalize the standard Hilbert-Schrödinger formalism to curved space-time represented in terms of a metric that depends continuously on the coordinates. In other words, we do not concern ourselves with the quantized nature of the gravitational field itself. This semiclassical approach seems reasonable for weak gravity due to an external source that is not coupled to the motion of the particles under consideration. The generalization of the Klein Gordon equation to curved space-time is done by introducing the covariant derivatives in the Laplacian operator [24]:

$$\begin{aligned} &\left(g^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{m^2c^2}{\hbar^2}\right)\psi \\ &= \left(\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu - \frac{m^2c^2}{\hbar^2}\right)\psi = 0, \end{aligned} \tag{2}$$

where the covariant derivative of a 4-vector T^ν is defined as $\nabla_\mu T^\nu = \partial_\mu T^\nu + \Gamma^\nu_{\sigma\mu} T^\sigma$, with $\Gamma^\nu_{\sigma\mu}$ the Christoffel symbols and $\partial_\mu = \partial/\partial x^\mu$. In the Klein Gordon equation, $g = -\det(g_{\mu\nu})$, m is the electron mass, and ψ the electron wavefunction. Following closely the approach of Ref. [25, 23], we write the Klein Gordon wave function as

$$\psi(x) = \exp\left\{\frac{i}{\hbar}\left[c^2 K_0(x) + K_1(x) + c^{-2}K_2(x) + ...\right]\right\}, \tag{3}$$

where $x \equiv x^\mu$. By treating c^2 as our expansion parameter, and substituting the above ansatz in the Klein Gordon equation gives $K_0 = \pm mt$. The two solutions indicates that positive and negative energy states are well defined. Taking $K_0 = -mt$, we obtain to the lowest order $\psi = \exp(-imc^2 t/\hbar)$ which corresponds positive energy solutions. The procedure works only if we can neglect negative energies throughout, *i.e.,* particle creation is sufficiently small to be ignored. Particle production by the Hawking-Unruh effect is thus neglected in this approximation.

The next order approximation is obtained by setting $\psi = \exp(-iK_1/\hbar)$ and by keeping terms of order c^0. This gives the lowest order Newton-Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{p^2}{2m}\psi - mU\psi, \tag{4}$$

where $p_a = -i\hbar\partial_a$ is the usual quantum mechanical momentum operator, and thus $p^2 = p_a p^a$. This corresponds to the standard Schrödinger equation except for the addition of the gravitational potential. The experiments of Colella, Overhauser and Werner [1, 2] showed that the gravitational potential in the Newton-Schrödinger is needed in order to explain the interference of neutrons in the Earth's gravitational field. Further, keeping the terms in order c^{-2} leads to an Hamiltonian with additional relativistic correction terms [23]. We also require the Hamiltonian to be Hermitian with respect to a scalar product. The procedure can be greatly simplified if the wave functions and the Hamiltonian are redefined as [23]

$$\psi \to \left(1 + \frac{p^2}{m^2c^2}\right)^{1/4} \mathcal{J}^{1/4}\psi$$

$$\mathcal{H} \to \left(1 + \frac{p^2}{m^2c^2}\right)^{1/4} \mathcal{J}^{1/4}\mathcal{H} \tag{5}$$

$$\times \left(1 + \frac{p^2}{m^2c^2}\right)^{-1/4} \mathcal{J}^{-1/4} + i\hbar\partial_0\mathcal{J}^{-1/4}, \tag{6}$$

where $\mathcal{J} = \det(g_{ab})$ and thus the scalar product acquires the usual form of "flat" space:

$$\langle\phi|\psi\rangle = \int \phi^*\psi\, dV, \tag{7}$$

with $dV = dx^1dx^2dx^3$. The final result is thus [23]

$$i\hbar\frac{\partial\psi}{\partial t} = \mathcal{H}\psi, \tag{8}$$

where,

$$\mathcal{H} = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} - mU +$$

$$\frac{2\gamma+1}{2mc^2}\left(-Up^2+i\hbar\partial_a Up^a\right)-\left(\frac{1}{2}-\beta\right)\frac{mU^2}{c^2}+$$
$$3\gamma\frac{\hbar\partial_a\partial^a U}{4mc^2}. \tag{9}$$

Since this Hamiltonian is defined with respect to the usual non-relativistic scalar product (7), and thus all the terms in the equation maintain their normal meaning. Finally the effect of an external electromagnetic field $\mathbf{A}$ is introduced into the Hamiltonian (9) via a gauge transformation $\mathbf{p}\to\mathbf{p}-e\mathbf{A}$, where e is the electron charge.

3. The variable mass metric

Let's now consider the special case of $\beta = 0$ and $\gamma = -1$, which gives $g_{\mu\nu} = h^2(x)\eta_{\mu\nu}$, where $h^2 = 1-2U/c^2$. This metric corresponds to the Friedmann-Lemaître-Robertson-Walker metric of an isotropic expanding universe where the physical time has been replaced by the conformal time. For reasons that will become clear later on, we will refer to this particular metric as the variable mass metric. An important property of the variable mass metric is that the propagation of massless particles (*e.g.*, photons) is completely unaffected by this metric.

The Lagrangian for a particle of a test particle of mass m in the gravitational field is

$$\mathcal{L} = \left(-\frac{v^\mu g_{\mu\nu}v^\nu}{c^2}\right)^{1/2} mc^2, \tag{10}$$

where $v^\mu = \partial x^\mu/\partial t = (c,\mathbf{v})$ is the particle velocity. In the case of the variable mass metric, the above Lagrangian specializes to

$$\mathcal{L} = \frac{h}{\gamma_v}mc^2, \tag{11}$$

where $\gamma_v = (1-v^2/c^2)^{-1/2}$. We can then determine the energy via a Legendre transformation:

$$\mathcal{E} = \mathbf{p}\cdot\mathbf{v}-\mathcal{L} = h\gamma_v mc^2, \tag{12}$$

where the canonical momentum is $\mathbf{p}\equiv p_a = \partial\mathcal{L}/\partial v^a$. This gives,

$$\mathcal{E} = \left(p^2c^2+h^2m^2c^4\right)^{1/2}. \tag{13}$$

Using Hamilton's equations, $\dot{p}_a = -\partial\mathcal{E}/\partial x^a$, the acceleration is then $\mathbf{a} = \dot{\mathbf{p}}/h\gamma_v m$, which gives

$$a_a = -\frac{c^2}{\gamma_v^2}\frac{\partial \ln h}{\partial x^a}. \tag{14}$$

If the gravitational potential is small, we can expand $h \sim 1-U/c^2$, and thus, assuming spherical symmetry,

$$\mathbf{a} = \frac{1}{\gamma_v^2}\frac{\partial U}{\partial\mathbf{r}}. \tag{15}$$

Let's consider now the case of a test electron in Minkowski space. We assume that this particle is accelerated by an external laser field, with acceleration given by $\mathbf{a} = -e\mathbf{E}/\gamma_v m$, where $\mathbf{E}$ is the laser electric field. Taking electron velocity to be $\mathbf{v} = \mathbf{p}/\gamma_v m$, with similar considerations as before, we arrive at

$$\mathcal{E} = \left(p^2c^2 + \gamma_v^2 m^2 c^4\right)^{1/2}. \tag{16}$$

Hence, the dynamics of an accelerated test electron in Minkowski space-time is the same as an electron in curved space-time described by the variable mass metric with $h = \gamma_v$. By taking the time derivative of this expression, we readily obtain

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{c^2}{\gamma_v^2}\frac{\partial \ln h}{\partial x^a}, \tag{17}$$

which is the same as Eq. (14) but with the reverse sign. Hence, the equivalence with the gravitational acceleration as defined previously, is obtained by setting $U = -\gamma_v^2 \mathbf{a}\cdot\mathbf{r} = \gamma_v e\mathbf{E}\cdot\mathbf{r}/m + U_0$, where U_0 is a constant which depends on the choice of the boundary conditions. Assuming $U = 0$ at $\mathbf{r} = 0$, we can then take $U_0 = 0$. One important point to consider is that, since the electron is accelerated in a space-time which was initially flat, the metric which we use to describe its motion needs to have null curvature. We note that for the case of the variable mass metric, the Ricci curvature is zero if $\nabla^2 h = 0$, as it is the case for uniform acceleration.

4. Thomson scattering of high frequency photons

We follow here closely the derivation outlined in Ref. [20]. Our goal is to design an experiment where it may be possible to test higher order correction in the Hamiltonian of Eq. (9) in the laboratory by emulating the variable mass metric through the acceleration of electrons in an intense optical laser field. The laser electric field is

$$|\mathbf{E}| = \left(\frac{2I_L}{\epsilon_0 c}\right)^{1/2} \tag{18}$$

where I_L is the laser intensity. Following Ref. [20], let's now consider the case of an electron oscillating in the field produced by the superposition of two linearly polarized laser beams with orthogonal polarizations, with vector potential $\mathbf{A} = \mathbf{A}_\mathrm{o} + \mathbf{A}_\mathrm{x}$, and $\mathbf{A}_\mathrm{o}\cdot\mathbf{A}_\mathrm{x} = 0$. We take $\mathbf{A}_\mathrm{o}$ to be a strong low-frequency optical laser field in which the motion of the electron can be treated classically. Thus $\mathbf{E} = -\dot{\mathbf{A}}_\mathrm{o}$ and $|e\dot{\mathbf{A}}_\mathrm{o}| \gg (\hbar m \Omega^3)^{1/2}$, with Ω the optical laser field frequency. Let the field component $\mathbf{A}_\mathrm{x}$ be a relatively weak perturbative high-frequency (x-ray) laser field, the effects of which on the electron are treated quantum-mechanically [32, 31]. Hence, $|e\dot{\mathbf{A}}_\mathrm{x}| \ll (\hbar m \omega_\mathrm{x}^3)^{1/2}$, with $\omega_\mathrm{x} \gg \Omega$. We can decompose the canonical momentum as $(\mathbf{p} - e\mathbf{A})^2 \sim (\mathbf{p}_\mathrm{x} - e\mathbf{A}_\mathrm{x})^2 + (\gamma_v m v)^2$, which yields that the energy of the electron as $\mathcal{E} = [(\mathbf{p}_\mathrm{x} - e\mathbf{A}_\mathrm{x})^2c^2 + \gamma_v^2 m^2 c^4]^{1/2}$. For the field $\mathbf{A}_\mathrm{x}$ let's take instead the one generated with a fourth generation source, or Free Electron Laser (FEL) [26]. Without loss of generality, for the rest of the paper we will replace $\mathbf{p}_\mathrm{x}$ by $\mathbf{p}$, and $\mathbf{A}_\mathrm{x}$ by $\mathbf{A}$.

For the present, we assume that the scattering takes place on a sufficiently short timescale and is confined to a region of the FEL spot in which the acceleration field is relatively homogeneous. Accordingly, a can be considered to be constant. As it is customary, we regard the scattering part of the Hamiltonian to be determined solely by the $\mathbf{A}^\dagger \cdot \mathbf{A}$ term. Indeed, for an ensemble of weakly-coupled free electrons (*e.g.*, an ideal electron gas) the polarization contribution from second order terms involving $\mathbf{p} \cdot \mathbf{A}$ terms can be neglected [27, 28]. This is particularly true to the scattering of x-rays if the FEL photon energy is much higher than the plasma frequency of the electron gas. Thus the interaction Hamiltonian is

$$\mathcal{H}' = \frac{e^2}{2m} \mathrm{e}^{-\mathbf{q}\cdot\mathbf{r}} \mathbf{A}^\dagger(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}), \tag{19}$$

where

$$\mathrm{e}^{-\mathbf{q}\cdot\mathbf{r}} = \left(1 + \frac{U}{c^2}\right) = \left(1 - \frac{\gamma_v^2 \mathbf{a}\cdot\mathbf{r}}{c^2}\right), \tag{20}$$

with $\mathbf{q} \approx \gamma_v^2 \mathbf{a}/c^2$. We note that in absence of any gravitational effect, Eq. (19) corresponds to the standard perturbation Hamiltonian in scattering theory [27].

We use a second quantized representation for the electromagnetic field, confined to a volume V,

$$\mathbf{A}(\mathbf{r}) = -i\sqrt{\frac{\hbar}{V\epsilon_0}} \sum_{\mathbf{k},\epsilon} \frac{\epsilon \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{\omega}} \hat{\mathbf{a}}_{\mathbf{k},\epsilon}, \tag{21}$$

where $\hat{\mathbf{a}}_{\mathbf{k},\epsilon}$ is the annihilation operator for a photon in a state of wavenumber $\mathbf{k}$, frequency ω, polarization ϵ, and ϵ_0 is the vacuum permittivity. Such a representation follows directly from Eq. (2) with the variable mass metric when $m = 0$. Since the variable mass metric only affects particles with a finite rest mass, photons are unaffected by it. We note that in the weak gravity limit, when pair creation is negligible, the vacuum state remains unchanged, and the Bogolubov transformation is not required. Consequently the standard representation of the electromagnetic field in Minkowski space-time is applicable.

Assume a photon of momentum $\hbar\mathbf{k}_0$ and energy $\hbar\omega_0$ is scattered by the free electrons into the solid angle $d\Omega$. Let $\hbar\mathbf{k}_1$ and $\hbar\omega_1$ be the momentum and energy of the photon after the collision. Energy and momentum conservation imply

$$\hbar\mathbf{k} = \hbar\mathbf{k}_0 - \hbar\mathbf{k}_1 \tag{22}$$

$$\hbar\omega = \hbar\omega_1 - \hbar\omega_0 \equiv \hbar\omega_{01}, \tag{23}$$

where $\hbar\mathbf{k}$ and $\hbar\omega$ are the momentum and energy transferred. For low energy (compared to the electron rest mass) collisions, $|\mathbf{k}_0| \approx |\mathbf{k}_1|$, and $k = (4\pi/\lambda)\sin(\theta/2)$, where λ is the wavelength of the probe photons and θ the scattering angle. If we denote with $|0\rangle$ and $|1\rangle$ the initial and the final state of the electron, then the probability per unit time that the composite system of electron plus the photon makes a transition from $|0; \mathbf{k}_0, \epsilon^{(0)}\rangle$ to $|1; \mathbf{k}_1, \epsilon^{(1)}\rangle$, where $\epsilon^{(0)}$ and $\epsilon^{(1)}$ are the polarization unit vectors of the incident and scattered photons, is given by the Fermi's "golden rule" of first-order perturbation theory:

$$\Gamma(0;\mathbf{k}_0,\epsilon^{(0)}|1;\mathbf{k}_1,\epsilon^{(1)}) = \frac{2\pi}{\hbar}|\langle 1;\mathbf{k}_1,\epsilon^{(1)}|\mathcal{H}'_{int}|0;\mathbf{k}_0,\epsilon^{(0)}\rangle|^2\delta(\hbar\omega-\hbar\omega_{01}), \quad (24)$$

where $\mathcal{H}'_{int}$ is the perturbation part of the interaction Hamiltonian. For scattering of a single electron $\mathcal{H}_{int} \equiv \mathcal{H}'$.

5. The differential cross section

We need now to generalize the previous formalism for an ensemble of N electrons, all subject to the same acceleration in the focal spot of the high intensity optical laser. While its full treatment of has already been given in our previous work [20, 27, 28], it lends to be rather involved. In the spirit of keeping a more straightforward derivation of the relevant equations in one singe place, we repeat them here with some additional simplifications.

We can assume for the perturbation Hamiltonian (in the interaction picture) a Fermi pseudo-potential approximation, *i.e.*, given by the sum of local terms

$$\mathcal{H}'_{int} = \int_V dV\,\mathcal{H}'(\mathbf{r})\sum_{j=1}^{N}\delta(\mathbf{r}-\mathbf{r}_j), \quad (25)$$

where $\mathbf{r}_j$ is the position vector of the jth electron. The matrix element in Eq. (24) now reads

$$\begin{aligned}&\langle 1;\mathbf{k}_1,\epsilon^{(1)}|\mathcal{H}'_{int}|0;\mathbf{k}_0,\epsilon^{(0)}\rangle = \\ &\frac{\hbar e^2}{2mV\epsilon_0}\frac{\epsilon^{(1)}\cdot\epsilon^{(0)}}{\sqrt{\omega_1\omega_0}}\int_V dV\sum_{j=1}^{N}\langle 1|\mathrm{e}^{-i(\mathbf{k}_1-\mathbf{k}_0-i\mathbf{q})\cdot\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}_j)|0\rangle = \\ &\frac{\hbar e^2\cos\beta}{2mV\epsilon_0\omega_0}\,\langle 1|\rho_{-\mathbf{k}-i\mathbf{q}}|0\rangle \end{aligned} \quad (26)$$

where in the last term we have used the fact that $\omega_1 \simeq \omega_0$ ($k_1 \simeq k_0$) in the low energy limit, and β is the angle between the two polarization vectors. The electron density operator is defined for any complex wavenumber $\tilde{\mathbf{k}}$ as

$$\rho_{\tilde{\mathbf{k}}} = \sum_{j=1}^{N}\mathrm{e}^{-i\tilde{\mathbf{k}}\cdot\mathbf{r}_j}, \quad (27)$$

and $\rho^{\dagger}_{\mathbf{k}+i\mathbf{q}} = \rho_{-\mathbf{k}+i\mathbf{q}}$, with both $\mathbf{k}$ and $\mathbf{q}$ real vectors. The transition probability (24) is then written as

$$\begin{aligned}&\Gamma(0;\mathbf{k}_0,\epsilon^{(0)}|1;\mathbf{k}_1,\epsilon^{(1)}) = \\ &\frac{2\pi}{\hbar^2}\left(\frac{e^2\hbar\cos\beta}{2m\epsilon_0 V\omega_0}\right)^2|\langle 1|\rho_{-\mathbf{k}-i\mathbf{q}}|0\rangle|^2\delta(\omega-\omega_{01}).\end{aligned} \quad (28)$$

In a scattering measurement, only $\mathbf{k}_1$ is measured, thus the transition probability needs to be summed over all possible initial and final states (see Ref. [27]). We denote with $p_{\{0\}}$ is the distribution of the initial states with $\sum p_{\{0\}} = 1$. Similarly, $p_{\{1\}}$ is the distribution of the final states. Usually, for a system in thermodynamic equilibrium, $p_{\{0\}}$ corresponds to the density matrix of the canonical ensemble, that is the Boltzmann factor divided by the sum of states (*e.g.,* the partition function). If we further introduce the integral representation of the delta function,

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t}, \tag{29}$$

we get

$$\begin{aligned}&|\langle 1|\rho_{-\mathbf{k}-i\mathbf{q}}|0\rangle|^2\delta(\omega-\omega_{01}) = \\ &\frac{1}{2\pi}\sum_{\{0\}}\sum_{\{1\}} p_{\{0\}}\int_{-\infty}^{\infty} |\langle 1|\rho_{-\mathbf{k}-i\mathbf{q}}|0\rangle|^2 \mathrm{e}^{i(\omega-\omega_{01})t} dt,\end{aligned} \tag{30}$$

where $\{0\}$ and $\{1\}$ refer to the ensemble of initial and final states, respectively. This can be simplified by noticing that [29]

$$\begin{aligned}&\mathrm{e}^{-i\omega_{01}t}|\langle 1|\rho_{-\mathbf{k}-i\mathbf{q}}|0\rangle|^2 = \\ &\mathrm{e}^{-iE_1t/\hbar}\mathrm{e}^{iE_0t/\hbar}\langle 0|\rho_{\mathbf{k}-i\mathbf{q}}|1\rangle\langle 1|\rho_{-\mathbf{k}-i\mathbf{q}}|0\rangle = \\ &\mathrm{e}^{-iE_1t/2\hbar}\mathrm{e}^{iE_0t/2\hbar}\langle 0|\rho_{\mathbf{k}-i\mathbf{q}}|1\rangle\mathrm{e}^{-iE_1t/2\hbar}\mathrm{e}^{iE_0t/2\hbar}\langle 1|\rho_{-\mathbf{k}-i\mathbf{q}}|0\rangle = \\ &\langle 0|\mathrm{e}^{i\mathcal{H}t/2\hbar}\rho_{\mathbf{k}-i\mathbf{q}}\mathrm{e}^{-i\mathcal{H}t/2\hbar}|1\rangle\langle 1|\mathrm{e}^{-i\mathcal{H}t/2\hbar}\rho_{-\mathbf{k}-i\mathbf{q}}\mathrm{e}^{i\mathcal{H}t/2\hbar}|0\rangle = \\ &\langle 0|\rho_{\mathbf{k}-i\mathbf{q}}(t/2)|1\rangle\langle 1|\rho_{-\mathbf{k}-i\mathbf{q}}(-t/2)|0\rangle,\end{aligned} \tag{31}$$

where E_0 and E_1 are the energies of the initial and final states, respectively, and $\mathcal{H}$ is the unperturbed Hamiltonian of the system. Using completeness, $\sum|1\rangle\langle 1| = 1$, we then have

$$\begin{aligned}&|\langle 1|\rho_{-\mathbf{k}-i\mathbf{q}}|0\rangle|^2\delta(\omega-\omega_{01}) = \frac{1}{2\pi}\sum_{\{0\}} p_{\{0\}} \times \\ &\int_{-\infty}^{\infty}\langle 0|\rho_{\mathbf{k}-i\mathbf{q}}(t/2)\rho_{-\mathbf{k}-i\mathbf{q}}(-t/2)|0\rangle \mathrm{e}^{i\omega t}\, dt.\end{aligned} \tag{32}$$

It only remains to weight over all possible initial states, but this is equivalent to taking a thermal average [29]. Thus,

$$\begin{aligned}&|\langle 1|\rho_{-\mathbf{k}-i\mathbf{q}}|0\rangle|^2\delta(\omega-\omega_{01}) = \\ &\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t}\langle\rho_{\mathbf{k}-i\mathbf{q}}(t/2)\rho_{-\mathbf{k}-i\mathbf{q}}(-t/2)\rangle,\end{aligned} \tag{33}$$

where now $\langle ... \rangle$ refers to a thermal average.

The differential cross section for Thomson scattering is then defined as the probability of transition per unit time into $d\omega_1 d\Omega$ divided by the total number of electron scatterers (N) times the incident flux of photons (c/V). Since the number of final states $\mathbf{k}_1$ is given by $V/(2\pi)^3 d^3k_1$, we have

$$\frac{d^2\sigma}{d\omega_1 d\Omega} d\omega_1 d\Omega = \frac{\Gamma V/(2\pi)^3 k_1^2 d(\omega_1/c) d\Omega}{Nc/V}, \tag{34}$$

or, using (28), (33) and the condition $\omega_0 \simeq \omega_1$,

$$\frac{d^2\sigma}{d\omega_1 d\Omega} = (r_e\cos\beta)^2\, S(\mathbf{k}+i\mathbf{q},\omega), \tag{35}$$

where $r_e = e^2/4\pi\epsilon_0 mc^2$ is the classical electron radius, and the quantity

$$S(\tilde{\mathbf{k}},\omega) = \frac{1}{2\pi N}\int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t}\langle \rho^{\dagger}_{-\tilde{\mathbf{k}}}(t/2)\rho_{-\tilde{\mathbf{k}}}(-t/2)\rangle, \quad (36)$$

is the extension of the usual dynamic structure factor to a complex wavenumber $\tilde{\mathbf{k}}$, as obtained in, *e.g.*, Ref. [27]. We note that Eq. (36) is written in the less common time-symmetric form [20], which is more appropriate to non-equilibrium systems. This preserves the reality of the density-density correlation function for any value of $\tilde{\mathbf{k}}$ and ω, which is an essential requirement since the differential cross section represents an experimental observable (*i.e.*, the number of scattered photons).

6. Scattering in accelerated frames

All the static and dynamic details associated to the scattering are described by the structure factor $S(\mathbf{k}+i\mathbf{q},\omega)$. This differs from the usual structure factor in Minkowski space by the additional imaginary dependence on the wavenumber, a fact that was first discussed in Ref. [20]. In more physical terms, the appearance of an imaginary component in the structure factor is associated with the fact that the external gravitational force (or induced acceleration) imposes a collective motion that is added to the background (thermal) motion of the particles. In other words, macroscopic inhomogneneities are emerging out of the initial equilibrium state in absence of gravity.

In general, given the presence of a thermal background, the position vector of the ith electron in the system is $\mathbf{r}_i(t) = \mathbf{r}_i^0(t) + \mathbf{R}(t)$, where $\mathbf{R}(t)$ describes the motion of cold electrons induced by the superimposed high intensity laser, and $\mathbf{r}_i^0(t)$ describes the random movements of the electrons at a given temperature. Substituting this into Eq. (27) and using the formal definition of the structure factor given by (36), we obtain

$$S(\mathbf{k}+i\mathbf{q},\omega) = \int S^{\mathrm{eq}}(\mathbf{k},\omega')F(\mathbf{k}+i\mathbf{q},\omega-\omega')d\omega', \quad (37)$$

where

$$S^{\mathrm{eq}}(\mathbf{k},\omega) = \frac{1}{2\pi N}\int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t}\langle \rho^{0\dagger}_{-\mathbf{k}}(t/2)\rho^{0}_{-\mathbf{k}}(-t/2)\rangle$$
$$= \frac{1}{2\pi N}\int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t}\langle \rho^{0}_{\mathbf{k}}(t)\rho^{0}_{-\mathbf{k}}(0)\rangle, \quad (38)$$

is the usual structure factor for an equilibrium plasma (where time averages are equivalent to ensemble averages) with thermal density fluctuations

$$\rho^0_{\mathbf{k}} = \sum_{j=1}^{N} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}_j^0(t)}, \quad (39)$$

and

$$F(\mathbf{k}+i\mathbf{q},\omega) = \frac{1}{2\pi}\int \exp\{i\omega t' - i\mathbf{k}\cdot[\mathbf{R}(t+t'/2)-\mathbf{R}(t-t'/2)]$$
$$-\mathbf{q}\cdot[\mathbf{R}(t+t'/2)-\mathbf{R}(t-t'/2)]\}\, dt'. \quad (40)$$

If we consider a linearly polarized high intensity optical laser, and we probe the electrons for a short time, $t \ll c/a$, at the peak of the laser acceleration, then, taking the acceleration to be uniform, $\mathbf{R}(t) \approx \mathbf{a}t^2/2$, where we have assumed the initial electron velocity is small and the motion remains non-relativistic, $\gamma_v \sim 1$. This was the regime considered in Ref. [20]. Under this condition equation (40) becomes

$$F(\mathbf{k}+i\mathbf{q},\omega) = \frac{c\,\mathrm{e}^{-2\mathbf{q}\cdot\mathbf{R}(t)}}{\sqrt{2\pi a^2}} \exp\left[-\frac{c^2}{2a^2}(\omega-\mathbf{k}\cdot\mathbf{v})^2\right]$$
$$\sim \frac{c}{\sqrt{2\pi a^2}} \exp\left[-\frac{c^2}{2a^2}(\omega-\mathbf{k}\cdot\mathbf{v})^2\right], \tag{41}$$

where $\mathbf{q}\cdot\mathbf{R}(t) \sim a^2t^2/2c^2 \ll 1$.

We also note that in an homogeneous plasma, the approximate dynamic structure factor of free electrons is given by the dielectric superposition principle [30]:

$$S^{\mathrm{eq}}(\mathbf{k},\omega) = \frac{S^0(\mathbf{k},\omega)}{|\epsilon(\mathbf{k},\omega)|^2}, \tag{42}$$

where $\epsilon(\mathbf{k},\omega)$ is the dielectric response function, $S^0(\mathbf{k},\omega)$ is the structure factor for an ideal (non-interacting) gas, which is used as the reference system:

$$S^0(\mathbf{k},\omega) = \int d\mathbf{v} f(\mathbf{v})\delta(\omega-\mathbf{k}\cdot\mathbf{v}), \tag{43}$$

and $f(\mathbf{v})$ is the equilibrium distribution function at the temperature T. For a classical gas, this is a Maxwell-Boltzmann distribution:

$$f(\mathbf{v}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m|\mathbf{v}|^2}{2k_B T}\right). \tag{44}$$

At sufficiently high temperatures, electron motions are uncorrelated and $\epsilon(\mathbf{k},\omega) \approx 1$.

Putting together Equations (41) and (43), we obtain the following expression for the structure factor of a uniformly accelerated thermal ensemble of electrons [20],

$$S(\mathbf{k}+i\mathbf{q},\omega) = \sqrt{\frac{m}{2\pi k^2 k_B T^*}} \exp\left[-\frac{m(\omega-\mathbf{k}\cdot\mathbf{v})^2}{2k^2 k_B T^*}\right], \tag{45}$$

where,

$$T^* = T + \frac{ma^2}{k^2 k_B c^2}. \tag{46}$$

This results can be interpreted as implying an additional uncertainty, expressed as a standard deviation, of $\hbar\Delta\omega$ in the energy of the outgoing photon as given by

$$\hbar\Delta\omega = \hbar k\sqrt{\frac{k_B(T^*-T)}{m}} = \frac{\hbar a}{c} = 2\pi k_B T_{DU}, \tag{47}$$

which, apart from a numerical factor, is just the Davies-Unruh temperature. This is also consistent with the energy uncertainty, due to the imposition of non-relativistic motion, predicted at the start.

Alternatively, this uncertainty in the outgoing photon momentum can be interpreted as a change in the rest mass of the electron [21]. To see this, consider the case when $\Delta\omega$ is small, and of the same order as the frequency Ω of the optical photons. This means that the electrons experience an energy change resonant with the bath of optical photons which surrounds them. Because of this change of energy, the work done by the optical laser photons in accelerating the electrons is slightly different from the one done in decelerating them. This difference in energy is thus attributed to a difference in the rest mass of the electron. Hence, from Eq. (47) and by setting $\Delta\omega = \Omega$,

$$\Delta m^2 = \frac{e^2 E^2}{\Omega^2 c^2}, \tag{48}$$

which is the usual expression for the mass shift of an electron in an intense laser beam [21, 31, 32].

7. Conclusion

In this paper we have developed a model of x-ray Thomson scattering of electrons that are accelerated in the field of a high intensity optical laser. Due to the strong low-frequency optical laser field, the motion of the electrons is treated classically, meanwhile, due to its relatively weak perturbative high-frequency field, the effects of the 4^{th} generation free electron laser probe on the electrons are treated quantum-mechanically. We assume that the scattering takes place on a sufficiently short timescale and is confined to a region of the FEL spot in which the acceleration field is relatively homogeneous. For this situation we derive a general form for the dynamic structure factor of the accelerated electrons. We show that this differs from the usual structure factor in Minkowski space-time by the additional imaginary dependence on the wavenumber. The appearance of this imaginary component is associated with the fact that the external gravitational force, *i.e.*, the induced acceleration, imposes a collective motion that is added to the background motion of the particles.

We show that our results are consistent with the concept of mass shift, that is, an uncertainty in the outgoing photon energy due to the acceleration. The theoretical results discussed here provide the underlying framework that supports the conclusions of Ref. [20], and rely on the validity of the equivalence principle in describing the acceleration of the electron in terms of a variable mass metric. These results are thus important for the development of future experimental tests of post-Newtonian gravity using high power lasers and 4^{th} generation light sources [20].

References

[1] A. W. Overhauser and R. Collela, Phys. Rev. Lett. **33**, 1237–1239 (1974).
[2] R. Colella, A.W. Overhauser, and S.A. Werner, Phys. Rev. Lett. **34**, 1472–1474 (1975).
[3] U. Bonseand and T. Wroblewski, Phys. Rev. Lett. **51**,1401–1404 (1983).
[4] C. Kiefer, *Quantum Gravity* (Oxford, 2007).
[5] C. Kiefer and C. Weber, Ann. Phys. **14**, 253–278 (2005).
[6] C. M. Will, *Theory and Experiments in Gravitational Physics* (Cambridge, 1993).
[7] R. Penrose, Phil. Trans. R. Soc. **356**, 1927–1939 (1998).
[8] S. Carlip, Class. Quantum Grav. **25**, 154010 (2008).
[9] C. Anastopoulos and B. L. Hu, New J. Phys. **16**, 085007 (2014).

[10] S. W. Hawking, Nature **248**, 30–31 (1974).
[11] S.W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[12] P. C. W. Davies, J. Phys. A **8**, 609–616 (1975).
[13] W. G. Unruh, Phys. Rev. D **14**, 870–892 (1976).
[14] P. C. W Davies and S. A. Fulling, Proc. R. Soc. Lond. A **356**, 237–257 (1977).
[15] D. Strickland and G. Mourou, Opt. Commun. **55**, 447–449 (1985).
[16] P. Chen and T. Tajima, Phys. Rev. Lett. **83**, 256–259 (1999).
[17] G. A. Mourou, T. Tajima and S. V. Bulanov, Rev. Mod. Phys. **78**, 309–371 (2006).
[18] G. Brodin, M. Marklund, R. Bingham, J. Collier and R. Evans, Class. Quantum Grav. **25**, 145005 (2008).
[19] G. W. Ford and R. F. O'Conell, Phys. Lett. A **350**, 17 (2006).
[20] B. J. B. Crowley, R. Bingham, R. G. Evans, D. O. Gericke, O. L. Landen, C. D. Murphy, P. A. Norreys, S. J. Rose, Th Tschentscher, C. H.-T Wang, J. S. Wark and G. Gregori, Sci. Rep. **2**, 491 (2012).
[21] T. W. B. Kibble, Phys. Rev. **138**, B740 (1965).
[22] H. R. Reiss, Phys. Rev. A **89**, 022116 (2014).
[23] C. Lämmerzahl, Phys. Lett. A **203**, 12–7 (1995).
[24] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge, 1994).
[25] C. Kiefer and T. P. Singh, Phys. Rev. D **44** 1067–1076 (1991).
[26] P. Emma, R. Akre, J. Arthur, R. Bionta, C. Bostedt, J. Bozek, A. Brachmann, P. Bucksbaum, R. Coffee, F.-J. Decker, Y . Ding, D. Dowell, S. Edstrom, A. Fisher, J. Frisch, S. Gilevich, J. Hastings, G. Hays, Ph. Hering, Z. Huang, R. Iverson, H. Loos, M. Messerschmidt, A. Miahnahri, S. Moeller, H.-D. Nuhn, G. Pile, D. Ratner, J. Rzepiela, D. Schultz, T. Smith, P. Stefan, H. Tompkins, J. Turner, J. Welch, W. White, J. Wu, G. Yocky and J. Galayda, Nature Photonics **4**, 641–647 (2010).
[27] B. J. B. Crowley and G. Gregori, New J. Phys. **15** 015014 (2013).
[28] B. J. B. Crowley and G. Gregori, High Energy Dens. Physics **13**, 55-083 (2014).
[29] J.-P. Hansen and I.R. McDonald, *Theory of Simple Liquids* (Academic, London, 2000).
[30] S. Ichimaru, Rev. Mod. Phys. **54**, 1017 (1982).
[31] C. Harvey, T. Heinzl, A. Ilderton and M. Marklund, Phys. Rev. Lett. **109**, 100402 (2012).
[32] C. Harvey, T. Heinzl and A. Ilderton, Phys. Rev. A **79**, 6 (2009).