

Limit theorems for multipower variation in the presence of jumps

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February 25, 2005

Abstract

In this paper we provide a systematic study of the robustness of probability limits and central limit theory for realised multipower variation when we add finite activity and infinite activity jump processes to an underlying Brownian semimartingale.

Keywords: Bipower variation; Infinite activity; Multipower variation; Power variation; Quadratic variation; Semimartingales; Stochastic volatility.

1 Introduction

Multipower variation is the probability limit of normalised partial sums of powers of lags of absolute high frequency increments of a semimartingale as the sampling frequency goes to infinity. It was introduced by Barndorff-Nielsen and Shephard in a series of papers motivated by some problems in financial econometrics. Realised multipower variation estimates this process and was shown, by Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005), to reveal integrated volatility powers in general Brownian semimartingales. They also derived the corresponding central limit theorem. Some detailed discussion of the econometric uses of these results are given in Barndorff-Nielsen, Graversen, Jacod, and Shephard (2005). Such continuous sample path processes are, of course, stimulating, however Barndorff-Nielsen and Shephard were also interested in realised multipower variation as they showed it has some features which are robust to finite activity jump processes (i.e. jump components with finite numbers of jumps in

finite time). In this paper we return to that issue, sharpening their results in the finite activity case and giving an analysis of the case where there are an infinite number of jumps.

Measuring the variation of price processes is a central topic in financial economics. A survey of this area is given by Andersen, Bollerslev, and Diebold (2005). The standard method is to use various quantities computed off the realised quadratic variation (QV) process. For a log-price process Y , which will be assumed to be a semimartingale, the realised QV process is

$$[Y_\delta]_t^{[2]} = \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2,$$

where $\delta > 0$ is some time gap, for example 10 minutes, and

$$y_j = Y_{j\delta} - Y_{(j-1)\delta},$$

are high frequency returns. Interest in this type of process is motivated by the advent of complete records of quotes or transaction prices for many financial assets. Although market microstructure effects (e.g. discreteness of prices, bid/ask bounce, irregular trading etc.) mean that there is a mismatch between asset pricing theory based on semimartingales and the data at very fine time intervals it does suggest the desirability of establishing an asymptotic distribution theory for estimators as we use more and more highly frequent observations. Papers which directly model the impact of market microstructure noise on these realised quantities include Bandi and Russell (2003), Hansen and Lunde (2003) and Zhang, Mykland, and Aït-Sahalia (2005). Related work in the probability literature on the impact of noise on discretely observed diffusions can be found in Gloter and Jacod (2001a) and Gloter and Jacod (2001b), while Delattre and Jacod (1997) report results on the impact of rounding on sums of functions of discretely observed diffusions. Papers which try to overcome the impact of noise include Zhang, Mykland, and Aït-Sahalia (2005), Barndorff-Nielsen, Hansen, Lunde, and Shephard (2004) and Zhang (2004). In this paper we ignore these effects.

Clearly, if the Y process is a semimartingale then as $\delta \downarrow 0$ so

$$[Y_\delta]_t^{[2]} \xrightarrow{p} [Y]_t^{[2]},$$

the QV process, where the convergence is also locally uniform in time. Recall the QV process is defined as

$$[Y]_t^{[2]} = \text{p-lim}_{M \rightarrow \infty} \sum_{j=1}^M (Y_{t_j} - Y_{t_{j-1}})^2, \quad (1)$$

for any sequence of partitions $t_0 = 0 < t_1 < \dots < t_M = t$ with $\sup_j \{t_{j+1} - t_j\} \rightarrow 0$ for $M \rightarrow \infty$. Here p-lim denotes the probability limit of the sum. Under the stronger condition that Y is a

Brownian semimartingale

$$Y_t = \int_0^t a_u du + \int_0^t \sigma_u dW_u,$$

where a is predictable, W is standard Brownian motion and σ is càdlàg, then

$$[Y]_t^{[2]} = \int_0^t \sigma_u^2 du,$$

the integrated variance. The convergence in probability result can be strengthened to a central limit theory under this Brownian semimartingale assumption. Results developed by Jacod (1994), Jacod and Protter (1998), Barndorff-Nielsen and Shephard (2002) and Barndorff-Nielsen and Shephard (2004a) imply that

$$\delta^{-1/2} \left([Y_\delta]_t^{[2]} - [Y]_t^{[2]} \right) \rightarrow \sqrt{2} \int_0^t \sigma_u^2 dB_u,$$

where convergence holds stably as a process and B is a new standard Brownian motion independent of W , a and σ .

When we extend this analysis to where we observe

$$X = Y + Z,$$

where Z is a jump process, then

$$[X]_t^{[2]} = \int_0^t \sigma_u^2 du + \sum_{u \leq t} (\Delta Z_u)^2.$$

Jacod and Protter (1998) have studied a central limit theorem for quantities related to $[X_\delta]_t^{[2]} - [X]_t^{[2]}$ under some strong conditions. These results are important, but they give us no device for learning about $\int_0^t \sigma_u^2 du$ in the presence of jumps. To tackle this problem Barndorff-Nielsen and Shephard (2004b) and Barndorff-Nielsen and Shephard (2005) introduced generalised measures which are now called realised multipower variation. In the simple case of order 1,1 realised bipower variation, which is the sole focus of Barndorff-Nielsen and Shephard (2005), they worked with quantities of the type

$$[X_\delta]_t^{[1,1]} = \sum_{j=2}^{\lfloor t/\delta \rfloor} |x_{j-1}| |x_j|,$$

and showed that

$$[X_\delta]_t^{[1,1]} \xrightarrow{p} [X]_t^{[1,1]} = \mu_1^2 \int_0^t \sigma_u^2 du = [Y]_t^{[1,1]},$$

when Z is a finite activity jump process and $\mu_1 = E|u|$, where $u \sim N(0,1)$. This implies that $\mu_1^{-2} [X_\delta]_t^{[1,1]}$ is a consistent estimator of integrated variance, while obviously $[X_\delta]_t^{[2]} - \mu_1^{-2} [X_\delta]_t^{[1,1]}$ estimates the quadratic variation of Z . Barndorff-Nielsen and Shephard (2005) have developed a central limit theorem for bipower variation when there are no jumps, which can be used to

construct tests for the hypothesis that there are no jumps. Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005) and Barndorff-Nielsen, Graversen, Jacod, and Shephard (2005) have deepened these results by giving rather general central limit theorems for realised multipower variation objects, under significantly weaker assumptions but again under the hypothesis that there are no jumps.

In this paper we ask two new questions: (i) do these kinds of robustness results also hold when the jump process has infinite activity, (ii) is it possible to construct central limit theorems for realised multipower variation processes when there are jumps in X ? For a closely related analysis see Woerner (2004). In Section 2 of the paper we establish notation and provide various definitions. This is followed in Section 3 with an analysis of multipower variation in the case where the processes are Brownian semimartingales plus jumps. In Section 4 we specialise the discussion to the case where the jumps are Lévy or OU processes. The results from a simulation experiment are reported in Section 5, while in Section 6 we draw our conclusions.

2 Some definitions

2.1 Brownian Semimartingales

Brownian semimartingales (denoted \mathcal{BSM}) are defined as the class of continuous semimartingales Y for which the (usual) decomposition $Y = A + M$ is such that

$$A_t = \int_0^t a_u du \quad (2)$$

$$M_t = \int_0^t \sigma_u dW_u \quad (3)$$

where a is predictable, W is standard Brownian motion and σ is càdlàg.

2.2 Multipower Variation (MPV)

Let X be an arbitrary stochastic process. Then the realised multipower variation (MPV) of X is based on high frequency returns, recorded every $\delta > 0$ time periods,

$$x_j = X_{j\delta} - X_{(j-1)\delta}, \quad j = 1, 2, \dots, \lfloor t/\delta \rfloor.$$

It can be defined via the unnormalised version

$$[X_\delta]_t^{[r]} = [X_\delta]_t^{[r_1, \dots, r_m]} = [X_\delta, \dots, X_\delta]^{[r_1, \dots, r_m]} = \sum_{j=m}^{\lfloor t/\delta \rfloor} |x_{j-m+1}|^{r_1} \cdots |x_j|^{r_m},$$

or more commonly through its normalised version

$$\{X_\delta\}_t^{[r]} = \{X_\delta\}_t^{[r_1, \dots, r_m]} = \delta^{1-r+2} [X_\delta]_t^{[r]}$$

where \mathbf{r} is short for r_1, \dots, r_m and

$$r_+ = \sum_{j=1}^m r_j.$$

It will be convenient to write

$$\max r = \max\{r_1, \dots, r_m\}.$$

Similarly, for arbitrary processes $X^{(1)}, \dots, X^{(m)}$ we let

$$[X_\delta^{(1)}, \dots, X_\delta^{(m)}]_t^{[\mathbf{r}]} = \sum_{j=m}^{\lfloor t/\delta \rfloor} |x_{j-m+1}^{(1)}|^{r_1} \dots |x_j^{(m)}|^{r_m},$$

while we always assume that $r_j \geq 0$ and $r_+ > 0$.

2.3 MPVCiP and MPVCLT for \mathcal{BSM}

We say that the Brownian semimartingale Y satisfies CiP (converges in probability) for MPV (denoted MPVCiP) provided that

$$\{Y_\delta\}_t^{[\mathbf{r}]} \xrightarrow{p} d_{\mathbf{r}} \sigma_t^{r_+*} = d_{\mathbf{r}} \int_0^t \sigma_u^{r_+} du,$$

where $d_{\mathbf{r}}$ is a known constant depending only on \mathbf{r} . Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005) have shown that this result holds if $Y \in \mathcal{BSM}$. Barndorff-Nielsen and Shephard (2004b) and Barndorff-Nielsen and Shephard (2005) show that this result continues to hold when we add finite activity jumps to a Brownian semimartingale. But what happens when the jumps are of infinite activity? We will provide a fairly detailed answer to this.

We say that Y satisfies a central limit theorem (CLT) for MPV (denoted MPVCLT) provided that

$$\delta^{-1/2} \left(\{Y_\delta\}_t^{[\mathbf{r}]} - d_{\mathbf{r}} \sigma_t^{r_+*} \right) \xrightarrow{law} c_{\mathbf{r}} \int_0^t \sigma_u^{r_+} dB_u$$

where B is a Brownian motion, $Y \perp\!\!\!\perp B$ (i.e. Y is independent of B), and $c_{\mathbf{r}}$ is a known constant depending only on \mathbf{r} . Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005) have shown that under some mild additional assumptions on the σ process, such a CLT holds.

3 MPV for \mathcal{BSM} + jump process

We will now study what happens to the limiting distribution when we add jumps to the Brownian semimartingale Y . The only existing results we know of are due to Jacod and Protter (1998) who studied the case where $\mathbf{r} = 2$, Y is a Brownian semimartingale and the jumps came from a purely discontinuous Lévy process, and Woerner (2004) who derives closely related results to ours. Thus we shall discuss various extensions of MPVCiP and MPVCLT for \mathcal{BSM} to processes of the form

$$X = Y + Z$$

where $Y \in \mathcal{BSM}$ while Z is a process exhibiting jumps.

We assume that Y satisfies MPVCiP or MPVCLT and consider to which extent this limiting behaviour remains the same when Z is added to Y , i.e. whether the influence of Z is negligible (in this respect). Thus we ask whether:

- For the CiP case

$$\{X_\delta, \dots, X_\delta\}^{[r_1, \dots, r_m]} - \{Y_\delta, \dots, Y_\delta\}^{[r_1, \dots, r_m]} = o_p(1).$$

- For the CLT case

$$\{X_\delta, \dots, X_\delta\}^{[r_1, \dots, r_m]} - \{Y_\delta, \dots, Y_\delta\}^{[r_1, \dots, r_m]} = o_p(\delta^{1/2}).$$

We shall use the following fact

Proposition 1 *The Brownian semimartingale Y satisfies*

$$\delta^{-1/2}|Y_{j\delta} - Y_{(j-1)\delta}| = O_p(|\log \delta|^{1/2}) \quad (4)$$

uniformly in j .

Proof. First we split

$$|Y_{j\delta} - Y_{(j-1)\delta}| \leq \left| \int_{(j-1)\delta}^{j\delta} a_u du \right| + \left| \int_{(j-1)\delta}^{j\delta} \sigma_u dW_u \right|$$

and note that the first part is $O_p(\delta)$ whereas, by the Dubins-Schwarz theorem and stochastic integration,

$$M_t = \int_0^t \sigma_u dW_u = B_{\int_0^t \sigma_s^2 ds}$$

for a standard Brownian motion B . Lévy's theorem on the uniform modulus of continuity of Brownian motion states that

$$\mathbb{P} \left(\limsup_{\varepsilon \downarrow 0} \left(\sup_{0 \leq t_1 < t_2 \leq T: t_2 - t_1 \leq \varepsilon} \frac{|B_{t_2} - B_{t_1}|}{\sqrt{2\varepsilon |\log(\varepsilon)|}}} \right) = 1 \right) = 1.$$

Since

$$\int_{t_1}^{t_2} \sigma_s^2 ds \leq |t_2 - t_1| \sup_{0 \leq s \leq T} \sigma_s^2$$

and the latter supremum is a.s. finite, we deduce that

$$\mathbb{P} \left(\limsup_{\varepsilon \downarrow 0} \left(\sup_{0 \leq t_1 < t_2 \leq T: t_2 - t_1 \leq \varepsilon} \frac{|Y_{t_2} - Y_{t_1}|}{\sqrt{2\varepsilon |\log(\varepsilon)|}}} \right) < \infty \right) = 1$$

as required. ■

Without the sup over t_1 and t_2 , for fixed t , i.e. without uniformity, the result holds with log replaced by log log.

3.1 Finite activity case

Consider the m -th order MPV process

$$[X_\delta]^{[r]} = [X_\delta, \dots, X_\delta]^{[r_1, \dots, r_m]}.$$

When Z is a finite activity jump process then pathwise the number of jumps of Z is finite and, for sufficiently small δ , none of the additive terms in $[X_\delta, \dots, X_\delta]^{[r_1, \dots, r_m]}$ involves more than one jump. Each of the terms in $[X_\delta, \dots, X_\delta]^{[r_1, \dots, r_m]}$ that contains no jumps are of order $O_p((\delta |\log \delta|)^{r_+/2})$. Any of the terms that do include a jump is of order $O_p((\delta |\log \delta|)^{(r_+ - \max r)/2})$.

Hence

$$\begin{aligned} \delta^{1-r_+/2}([X_\delta]^{[r]} - [Y_\delta]^{[r]}) &= \delta^{1-r_+/2} O_p((\delta |\log \delta|)^{(r_+ - \max r)/2}) \\ &= O_p(\delta^{1-\max r/2} |\log \delta|^{(r_+ - \max r)/2}). \end{aligned}$$

So:

- CiP is not influenced by Z so long as $\max r < 2$, while CLT continues to hold so long as $\max r < 1$.

The bound $\max r < 2$ seems quite a tight condition for when $m = 1$ and $r = 2$

$$[X_\delta]^{[2]} \xrightarrow{p} [Y]^{[2]} + [Z]^{[2]},$$

where

$$[Z]_t^{[2]} = \sum_{s \leq t} |\Delta X_s|^2,$$

i.e. jumps do impact the limit.

The above CLT result is of some importance. It means that we can use multipower variation to make mixed Gaussian inference about $\int_0^t \sigma_u^2 du$, integrated variance, in the presence of finite activity jumps processes so long as $\max r < 1$ and $r_+ = 2$. An example of this is where $m = 3$ and we take $r_1 = r_2 = r_3 = 2/3$ (that is using Tripower Variation (TPV)).

3.2 Some inequalities

As a preliminary to treating the infinite activity case we now recall or derive several inequalities.

We shall refer to the following classical mathematical inequalities. Below a, b, c etc. denote arbitrary real numbers with $a = b + c$.

- (i) For $0 < r \leq 1$,

$$\left| \sum_{j=1}^n |a_j|^r - \sum_{j=1}^n |b_j|^r \right| \leq \sum_{j=1}^n |c_j|^r. \quad (5)$$

(ii) For $1 \leq p$

$$\left| \left(\sum_{j=1}^n |a_j|^p \right)^{1/p} - \left(\sum_{j=1}^n |b_j|^p \right)^{1/p} \right| \leq \left(\sum_{j=1}^n |c_j|^p \right)^{1/p}. \quad (6)$$

Formula (6) is a consequence of Minkovsky's inequality: If $p \geq 1$ then, for arbitrary random variables A, B and C with $A = B + C$,

$$E\{|A|^p\}^{1/p} \leq E\{|B|^p\}^{1/p} + E\{|C|^p\}^{1/p}.$$

Inequality (i) implies that for $0 < r, s \leq 1$,

$$\begin{aligned} \left| \sum_{j=2}^n |a_{j-1}|^r |a_j|^s - \sum_{j=2}^n |b_{j-1}|^r |b_j|^s \right| &\leq \sum_{j=2}^n |c_{j-1}|^r |c_j|^s + \sum_{j=2}^n |c_{j-1}|^r |b_j|^s + \sum_{j=2}^n |b_{j-1}|^r |c_j|^s \\ &= \sum_{j=2}^n |c_{j-1}|^r |c_j|^s + \sum_{j=2}^n |c_{j-1}|^r |b_j|^s \quad [2]. \end{aligned} \quad (7)$$

Similarly we find for $0 < r, s, u \leq 1$

$$\begin{aligned} &\left| \sum_{j=3}^n |a_{j-2}|^r |a_{j-1}|^s |a_j|^u - \sum_{j=3}^n |b_{j-2}|^r |b_{j-1}|^s |b_j|^u \right| \\ &\leq \sum_{j=3}^n |c_{j-2}|^r |c_{j-1}|^s |c_j|^u + \sum_{j=3}^n |c_{j-2}|^r |c_{j-1}|^s |b_j|^u \quad [3] + \sum_{j=3}^n |c_{j-2}|^r |b_{j-1}|^s |b_j|^u \quad [3], \end{aligned} \quad (8)$$

etc.

Applying this to MPV with $\max r \leq 1$ we find that

$$\begin{aligned} \left| [X_\delta, \dots, X_\delta]^{[r_1, \dots, r_m]} - [Y_\delta, \dots, Y_\delta]^{[r_1, \dots, r_m]} \right| &\leq [Z_\delta, \dots, Z_\delta]^{[r_1, \dots, r_m]} \\ &\quad + [Z_\delta, \dots, Z_\delta, Y_\delta]^{[r_1, \dots, r_m]} \left[\binom{m}{1} \right] \\ &\quad + [Z_\delta, \dots, Z_\delta, Y_\delta, Y_\delta]^{[r_1, \dots, r_m]} \left[\binom{m}{2} \right] \\ &\quad + \dots \\ &\quad + [Z_\delta, Y_\delta, Y_\delta, \dots, Y_\delta]^{[r_1, \dots, r_m]} \left[\binom{m}{m-1} \right] \end{aligned} \quad (9)$$

where the binomial coefficients indicate the relevant number of similar terms.

Thus in the $m = 1$ case we have that

$$\delta^{(1-r)/2} |[X_\delta]^{[r]} - [Y_\delta]^{[r]}| \leq \delta^{(1-r)/2} [Z_\delta]^{[r]}. \quad (10)$$

When $m = 2$

$$\begin{aligned}
\delta^{1-(r+s)/2} |[X_\delta, X_\delta]^{[r,s]} - [Y_\delta, Y_\delta]^{[r,s]}| &\leq \delta^{1-(r+s)/2} [Z_\delta, Z_\delta]^{[r,s]} \\
&\quad + \delta^{1-r/2} [Z_\delta, \delta^{-1/2} Y_\delta]^{[r,s]} \\
&\quad + \delta^{1-s/2} [\delta^{-1/2} Y_\delta, Z_\delta]^{[r,s]} \\
&= \delta^{1-(r+s)/2} [Z_\delta, Z_\delta]^{[r,s]} \\
&\quad + \delta^{1-r/2} [Z_\delta, \delta^{-1/2} Y_\delta]^{[r,s]} [2].
\end{aligned} \tag{11}$$

For $m = 3$

$$\begin{aligned}
&\delta^{1-(r+s+u)/2} |[X_\delta, X_\delta, X_\delta]^{[r,s,u]} - [Y_\delta, Y_\delta, Y_\delta]^{[r,s,u]}| \\
&\leq \delta^{1-(r+s+u)/2} [Z_\delta, Z_\delta, Z_\delta]^{[r,s,u]} \\
&\quad + \delta^{1-(r+s)/2} [Z_\delta, Z_\delta, \delta^{-1/2} Y_\delta]^{[r,s,u]} [3] \\
&\quad + \delta^{1-r/2} [Z_\delta, \delta^{-1/2} Y_\delta, \delta^{-1/2} Y_\delta]^{[r,s,u]} [3].
\end{aligned} \tag{12}$$

3.3 Infinite activity case

In discussing CiP and CLT for infinite activity we shall, for simplicity, mostly restrict consideration to the case $r_1 = \dots = r_m = r$.

Recall

$$X = Y + Z$$

where $Y \in \mathcal{BSM}$.

Sufficient conditions for MPVCiP, respectively MPVCLT, are (see the beginning of the present Section), that

$$\delta^{1-mr/2} ([X_\delta, \dots, X_\delta]^{[r, \dots, r]} - [Y_\delta, \dots, Y_\delta]^{[r, \dots, r]}) = o_p(1)$$

respectively

$$\delta^{(1-mr)/2} ([X_\delta, \dots, X_\delta]^{[r, \dots, r]} - [Y_\delta, \dots, Y_\delta]^{[r, \dots, r]}) = o_p(1).$$

3.3.1 Convergence in probability

We need to distinguish between the cases $0 < r \leq 1$ and $r > 1$.

When $0 < r \leq 1$ we have, by (9),

$$\begin{aligned}
& \delta^{1-mr/2} \left| [X_\delta, \dots, X_\delta]^{[r, \dots, r]} - [Y_\delta, \dots, Y_\delta]^{[r, \dots, r]} \right| \\
\leq & \delta^{1-mr/2} [Z_\delta, \dots, Z_\delta]^{[r, \dots, r]} \\
& + \delta^{1-(m-1)r/2} [Z_\delta, \dots, Z_\delta, \delta^{-1/2} Y_\delta]^{[r, \dots, r]} \left[\binom{m}{1} \right] \\
& + \delta^{1-(m-2)r/2} [Z_\delta, \dots, Z_\delta, \delta^{-1/2} Y_\delta, \delta^{-1/2} Y_\delta]^{[r, \dots, r]} \left[\binom{m}{2} \right] \\
& + \dots \\
& + \delta^{1-r/2} [Z_\delta, \delta^{-1/2} Y_\delta, \dots, \delta^{-1/2} Y_\delta]^{[r, \dots, r]} \left[\binom{m}{m-1} \right].
\end{aligned} \tag{13}$$

Thus for MPVCiP, when $0 < r \leq 1$ it suffices that the following conditions are met:

$$\delta^{1-mr/2} [Z_\delta, \dots, Z_\delta]^{[r, \dots, r]} = o_p(1) \tag{14}$$

$$\delta^{1-(m-1)r/2} |\log \delta|^{r/2} [Z_\delta, \dots, Z_\delta]^{[r, \dots, r]} \left[\binom{m}{1} \right] = o_p(1) \tag{15}$$

.....

$$\delta^{1-r/2} |\log \delta|^{(m-1)r/2} [Z_\delta]^{[r]} \left[\binom{m}{1} \right] = o_p(1). \tag{16}$$

For the power variation case, where $m = 1$, for the convergence in probability result (denoted PCiP) this reduces to

$$\delta^{1-r/2} [Z_\delta]^{[r]} = o_p(1).$$

For the bipower variation case, where $m = 2$, for the convergence in probability result (denoted BPVCiP) the conditions are

$$\delta^{1-r} [Z_\delta, Z_\delta]^{[r, r]} = o_p(1) \tag{17}$$

$$\delta^{1-r/2} [Z_\delta, \delta^{-1/2} Y_\delta]^{[r, r]} = o_p(1). \tag{18}$$

The latter relation is equivalent to

$$\delta^{1-r/2} |\log \delta|^{r/2} [Z_\delta]^{[r]} = o_p(1). \tag{19}$$

For $r > 1$ we have

$$\left| \left(\delta^{1-mr/2} [X_\delta, \dots, X_\delta]^{[r, \dots, r]} \right)^{1/r} - \left(\delta^{1-mr/2} [Y_\delta, \dots, Y_\delta]^{[r, \dots, r]} \right)^{1/r} \right| \leq \left(\delta^{1-mr/2} \mathbf{S} \right)^{1/r}$$

where, in a compact notation,

$$\mathbf{S} = \sum_{j=m}^{\lfloor t/\delta \rfloor} \left| \sum_{\omega} \prod y_k \prod z_l \right|^r$$

and

$$\sum_{\omega} \prod y_k \prod z_l = (y_{j-m+1} + z_{j-m+1}) \cdots (y_j + z_j) - y_{j-m+1} \cdots y_j,$$

where ω runs over all selections of one factor from each of the parentheses in the above equation, except the one leading to $y_{j-m+1} \cdots y_j$.

Now, if

$$\delta^{1-mr/2} \mathbf{S} = o_p(1) \quad (20)$$

then, on account of the previously established fact that $\delta^{1-mr/2}[Y_\delta, \dots, Y_\delta]^{[r, \dots, r]}$ converges in probability to a positive random variable, we can conclude from the Minkovski inequality that

$$\left(\delta^{1-mr/2}\right)^{1/r} \left(\left([X_\delta, \dots, X_\delta]^{[r, \dots, r]}\right)^{1/r} - \left([Y_\delta, \dots, Y_\delta]^{[r, \dots, r]}\right)^{1/r}\right) = o_p(1).$$

To determine a sufficient condition for (20), and hence for MPVCiP, we note that in view of the inequality

$$|b + c|^r \leq 2^{r-1}(|b|^r + |c|^r)$$

there exists a constant C such that

$$|\sum_{\omega} \prod y_k \prod z_l|^r \leq C \sum_{\omega} |\prod y_k \prod z_l|^r.$$

This yields

$$\mathbf{S} \leq C \sum_{j=m}^{\lfloor t/\delta \rfloor} \sum_{\omega} |\prod y_k \prod z_l|^r = C \sum_{\omega} \sum_{j=m}^{\lfloor t/\delta \rfloor} |\prod y_k \prod z_l|^r.$$

It follows that (20) will hold if, for all ω ,

$$\delta^{1-mr/2} \sum_{j=1}^{\lfloor t/\delta \rfloor} |\prod y_k \prod z_l|^r = o_p(1).$$

But this is equivalent to the set of conditions (14)-(16), which were previously established as sufficient for MPVCiP in the case $r \leq 1$.

So sufficient for CiP is:

$$\delta^{1-mr/2}[Z_\delta, \dots, Z_\delta]^{[r, \dots, r]} = o_p(1) \quad (21)$$

$$\delta^{1-(m-1)r/2} |\log \delta|^{r/2} [Z_\delta, \dots, Z_\delta]^{[r, \dots, r]} \left[\binom{m}{1}\right] = o_p(1) \quad (22)$$

.....

$$\delta^{1-r/2} |\log \delta|^{(m-1)r/2} [Z_\delta]^{[r]} \left[\binom{m}{1}\right] = o_p(1). \quad (23)$$

3.3.2 Central limit theorem

In the IA setting, for CLT we are assuming that $r \leq 1$. It will be seen, from the examples to be discussed in the next Section, that the restriction to $r \leq 1$ is essentially necessary. From (13) we find:

For MPVCLT it suffices that the following conditions are met for $r \leq 1$:

$$\delta^{(1-mr)/2} [Z_\delta, \dots, Z_\delta]^{[r, \dots, r]} = o_p(1) \quad (24)$$

$$\delta^{(1-(m-1)r)/2} |\log \delta|^{r/2} [Z_\delta, \dots, Z_\delta]^{[r, \dots, r]} \left[\binom{m}{1} \right] = o_p(1) \quad (25)$$

.....

$$\delta^{(1-r)/2} |\log \delta|^{(m-1)r/2} [Z_\delta]^{[r]} \left[\binom{m}{1} \right] = o_p(1). \quad (26)$$

For PCLT this reduces to

$$\delta^{(1-r)/2} [Z_\delta]^{[r]} = o_p(1)$$

which can only be satisfied for $r < 1$.

For BPCLT the conditions (in the general $[r, s]$ case) are

$$\delta^{(1-r-s)/2} [Z_\delta, Z_\delta]^{[r, s]} = o_p(1) \quad (27)$$

$$\delta^{(1-r)/2} [Z_\delta, \delta^{-1/2} Y_\delta]^{[r, s]} = o_p(1) \quad (28)$$

$$\delta^{(1-s)/2} [\delta^{-1/2} Y_\delta, Z_\delta]^{[r, s]} = o_p(1). \quad (29)$$

Due to assumption (4), sufficient for the relations (28) and (29) are

$$\delta^{(1-r)/2} |\log \delta|^{s/2} [Z_\delta]^{[r]} = o_p(1) \quad (30)$$

and

$$\delta^{(1-s)/2} |\log \delta|^{r/2} [Z_\delta]^{[s]} = o_p(1). \quad (31)$$

Sufficient for the first of these latter relations is $0 < r < 1$ and $\sup_\delta [Z_\delta]^{[r]} < \infty$. And similarly for the second.

In the tripower variation (TPV) case of $m = 3$, with $r, s, u \leq 1$ we have

$$\begin{aligned} |[X]^{[r, s, u]} - [Y]^{[r, s, u]}| &\leq [Z_\delta, Z_\delta, Z_\delta]^{[r, s, u]} \\ &\quad + [Z_\delta, Z_\delta, Y_\delta]^{[r, s, u]} [3] \\ &\quad + [Z_\delta, Y_\delta, Y_\delta]^{[r, s, u]} [3] \end{aligned}$$

and sufficient for TPVCLT is that

$$\delta^{(1-r-s-u)/2} [Z_\delta, Z_\delta, Z_\delta]^{[r, s, u]} = o_p(1)$$

$$\delta^{(1-r-s)/2} |\log \delta|^{u/2} [Z_\delta, Z_\delta]^{[r, s]} = o_p(1) \quad [3]$$

$$\delta^{(1-r)/2} |\log \delta|^{(s+u)/2} [Z_\delta]^{[r]} = o_p(1) \quad [3].$$

4 Lévy processes with no continuous component

4.1 Preliminaries on Lévy processes and their small-time behaviour

Lévy processes, i.e. processes with stationary independent increments, are a versatile class of jump processes on which we can apply the deterministic criteria derived in the previous section. Whether MPVCiP or MPVCLT hold, depends on the characteristics of the Lévy process. Notably the number of small jumps is important. We have seen that finite activity restricts $\max r < 2$ and $\max r < 1$, respectively for MPVCiP and MPVCLT. We will get further restrictions, in general, when we have infinite activity.

Standard references on Lévy processes are Bertoin (1996) and Sato (1999). Let us recall here some important facts. The most general (real-valued) Lévy process can be decomposed into a continuous component $\kappa B_t + bt$, a Brownian motion with drift (hence a Brownian semimartingale), and a part Z_t “with no continuous component” that incorporates jumps $(\Delta Z_t)_{t \geq 0}$, which form a homogeneous Poisson point process, whose intensity measure, called the Lévy measure, we will write as Π . Π is a Radon measure on $\mathbb{R}^* = \mathbb{R} - \{0\}$ with

$$\int_{\mathbb{R}^*} (|x|^2 \wedge 1) \Pi(dx) < \infty. \quad (32)$$

If the stronger condition $\int_{\mathbb{R}^*} (|x| \wedge 1) \Pi(dx) < \infty$ holds, then we choose the drift b so that

$$Z_t = \sum_{s \leq t} \Delta Z_s \quad \text{and} \quad \mathbb{E}(\exp\{i\lambda Z_t\}) = \exp\{-t\Psi(\lambda)\}, \quad \text{where } \Psi(\lambda) = \int_{\mathbb{R}^*} (1 - e^{i\lambda x}) \Pi(dx),$$

and Z has paths of locally bounded variation. If $\int_{\mathbb{R}^*} (|x| \wedge 1) \Pi(dx) = \infty$, there is no canonical choice of drift $b \in \mathbb{R}$, in general. We create some redundancy and allow an additional drift parameter $a \in \mathbb{R}$ so that

$$\mathbb{E}(\exp\{i\lambda Z_t\}) = \exp\{-t\Psi(\lambda)\}, \quad \text{where } \Psi(\lambda) = -i\lambda a + \int_{\mathbb{R}^*} (1 - e^{i\lambda x} + i\lambda x 1_{\{|x| \leq 1\}}) \Pi(dx),$$

and in this case Z has paths of locally unbounded variation. The function Ψ is the so-called characteristic exponent of Z .

We define an index

$$\alpha = \inf \left\{ \gamma \geq 0 : \int_{[-1,1]} |x|^\gamma \Pi(dx) < \infty \right\} \in [0, 2].$$

The number α measures how heavily infinite Π is at zero, i.e. how many small jumps Z has. If $\alpha = 0$, then Π is finite, or only just infinite.

Example 1 For a Variance Gamma process Z , we have

$$\Pi(dx) = \nu |x|^{-1} \exp\{-\lambda |x|\} dx$$

and so an infinite Lévy measure with $\alpha = 0$.

If $0 < \alpha < 2$, then Z is locally comparable to a stable process of index α . If $\alpha = 2$, then Π only just satisfies the integrability condition (32). An example is the Lévy measure

$$\Pi(dx) = |x|^{-3} |\log |x/2||^{-1-\beta} 1_{[-1,1]}(x) dx,$$

for $\beta > 0$. This would not be a Lévy measure without the log-term.

Clearly, if Z has bounded variation, then $0 \leq \alpha \leq 1$. If Z has unbounded variation, then $1 \leq \alpha \leq 2$. The boundary $\alpha = 1$ is attained for both bounded and unbounded variation processes. $\Pi(dx) = |x|^{-2} |\log |x/2||^{-1-\beta} 1_{[-1,1]}(x) dx$ is an example for a bounded variation process with $\alpha = 1$.

Example 2 For a Normal Inverse Gaussian process Z , we have

$$\Pi(dx) = \frac{1}{\pi} \delta \sqrt{\gamma^2 + \beta^2} |x|^{-1} K_1 \left(\sqrt{\gamma^2 + \beta^2} |x| \right) e^{\beta x} dx,$$

and the well-known asymptotic property of the Bessel function $K_1(x) \sim |x|^{-1}$, as $x \downarrow 0$, shows that Z has unbounded variation, and $\alpha = 1$.

The index α can be seen to be greater than or equal (usually equal) to the Blumenthal and Gettoor (1961) upper index

$$\alpha^* = \inf \{ \gamma \geq 0 : \limsup_{\lambda \rightarrow \infty} |\Psi(\lambda)| / \lambda^\gamma = 0 \} \in [0, 2].$$

Let Z be a Lévy process with no continuous component. Then without loss of generality we can decompose Z into

$$Z_t = Z_t^{(1)} + Z_t^{(2)},$$

where $Z^{(1)}$ and $Z^{(2)}$ are independent processes and $Z^{(2)}$ is defined as

$$Z_t^{(2)} = \sum_{s \leq t} \Delta Z_s I(|\Delta Z_s| > 1).$$

Clearly $Z^{(2)}$ is a compound Poisson process, and hence of finite activity. The effect of $Z^{(2)}$ on MPVCiP and MPVCLT was studied in the previous Section and so from now on in this Section we can, without loss of generality, set $Z^{(2)}$ to zero, i.e. assume Π is concentrated on $[-1, 1]$.

Lemma 1 *Let Z be a Lévy process with no continuous component and index α . Then*

$$\sup_{\delta > 0} \frac{\mathbb{E}|Z_\delta|^\gamma}{\delta} < \infty,$$

for all $\alpha < \gamma \leq 1$ if Z has finite mean and bounded variation, and for all $1 \leq \alpha < \gamma \leq 2$ if Z is a zero-mean Lévy process with finite variance.

Proof. Let $\alpha < 1$. From (5) and the compensation formula for Poisson point processes we get for all $\alpha < \gamma \leq 1$

$$\mathbb{E} |Z_\delta|^\gamma = \mathbb{E} \left| \sum_{0 \leq s \leq \delta} \Delta Z_s \right|^\gamma \leq \mathbb{E} \sum_{0 \leq s \leq \delta} |\Delta Z_s|^\gamma = \delta \int_{\mathbb{R}^*} |z|^\gamma \Pi_Z(dz) < \infty.$$

If $1 \leq \alpha < 2$, we use Monroe embedding $Z_t = B_{T_t}$ into a Brownian motion B , for a subordinator T_t of stopping times for B , with $\mathbb{E}(T_t) = \mathbb{E}(Z_t^2) < \infty$. Using the explicit embedding of Winkel (2005), we have as Lévy measure of T

$$\Pi_T = \int_{\mathbb{R}^*} \rho_{|x|} \Pi(dx) + \int_{\mathbb{R}^*} \int_0^{|x|} \frac{|x|}{y^2} \rho_{|x|} * \rho_{|x|} dy \Pi(dx),$$

where ρ_x is the distribution of the first passage time at x of a three-dimensional Bessel process starting from zero. In particular, $R_x \sim \rho_x$ has first moment $\mathbb{E}(R_x) = x^2/3$, so that for all $2 \geq \gamma > \alpha \geq 1$, by Jensen's inequality,

$$\int_{\mathbb{R}^*} \mathbb{E}(R_{|x|}^{\gamma/2}) \Pi(dx) \leq \int_{\mathbb{R}^*} (\mathbb{E}(R_{|x|}))^{\gamma/2} \Pi(dx) = \left(\frac{1}{3}\right)^{\gamma/2} \int_{\mathbb{R}^*} |x|^\gamma \Pi(dx) < \infty,$$

and similarly

$$\begin{aligned} \int_{\mathbb{R}^*} \int_0^{|x|} \frac{|x|}{y^2} \mathbb{E}((R_{|y|} + \tilde{R}_{|y|})^{\gamma/2}) dy \Pi(dx) &\leq \left(\frac{2}{3}\right)^{\gamma/2} \int_{\mathbb{R}^*} \int_0^{|x|} \frac{|x|}{y^2} y^\gamma dy \Pi(dx) \\ &= \left(\frac{2}{3}\right)^{\gamma/2} \frac{1}{\gamma-1} \int_{\mathbb{R}^*} |x|^\gamma \Pi(dx) < \infty. \end{aligned}$$

The sum of the left hand sides is $\int_{(0,\infty)} |z|^{\gamma/2} \Pi_T(dz)$, so that the index of T is (at most) $\alpha/2$.

Now we invoke Revuz and Yor (1999, Exercise V.(1.23)):

$$\mathbb{E} |B_\tau|^{2p} \leq C_p \mathbb{E}(\tau^p), \quad (33)$$

for all (bounded, but then all) stopping times τ with $\mathbb{E}(\tau^p) < \infty$, all $p > 0$, and universal constants C_p ; see also Revuz and Yor (1999, Theorem IV.(4.10)).

This implies

$$\mathbb{E} |Z_\delta|^\gamma = \mathbb{E} |B_{T_\delta}|^\gamma \leq C_p \mathbb{E} T_\delta^{\gamma/2}$$

and an application of the bounded variation case to the subordinator T completes the proof. ■

Inspecting the proof of the lemma for weaker assumptions, one obtains the following corollary.

We recall that a function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be subadditive if, for all $x, y \in [0, \infty)$,

$$f(x+y) \leq f(x) + f(y);$$

f is called regular varying at 0, with index β if

$$\frac{f(\lambda x)}{f(x)} \xrightarrow{x \downarrow 0} \lambda^\beta$$

for all $\lambda > 0$ (cf. Bingham, Goldie, and Teugels (1989)).

Corollary 1 (i) Let Z be a finite mean bounded variation Lévy process with no continuous component and f an even nonnegative measurable function with $f(0) = 0$, subadditive on $[0, \infty)$. Then

$$\int_{\mathbb{R}^*} f(x) \Pi(dx) < \infty \quad \Rightarrow \quad \sup_{\delta > 0} \frac{\mathbb{E}f(Z_\delta)}{\delta} < \infty.$$

(ii) Let Z be a zero-mean finite-variance Lévy process with no continuous component, and f an even nonnegative continuous function with $f(0) = 0$, increasing concave on $[0, \infty)$, regularly varying at 0 with index $\beta \in (1/2, 1]$. Then

$$\int_{\mathbb{R}^*} f(x^2) \Pi(dx) < \infty \quad \Rightarrow \quad \sup_{\delta > 0} \frac{\mathbb{E}f(Z_\delta^2)}{\delta} < \infty.$$

Proof. (i) is clear. For (ii) note that concavity with $f(0) = 0$ implies $f(|x + y|) \leq f(|x|) + f(|y|)$ so that we will be able to apply (i). We do not repeat the whole argument here, but point out that the assumption of regular variation is used to ensure that as $x \downarrow 0$ so

$$\int_0^{|x|} \frac{|x|}{y^2} f\left(\frac{2}{3}y^2\right) dy \asymp f\left(\frac{2}{3}x^2\right) \asymp f(x^2),$$

cf. Bingham, Goldie, and Teugels (1989). Here $f \asymp g$ means that $0 < \liminf (f/g) \leq \limsup (f/g) < \infty$. For the appropriate generalisation of (33) to more general functions f , we refer to Revuz and Yor (1999, Theorem IV.(4.10)). ■

Examples of functions other than $f(x) = |x|^\gamma$ to which the results apply can be built from $\tilde{f}_{1,\kappa}(x) = |x|(\log(1/|x|))^\kappa$ and $\tilde{f}_{2,\kappa}(x) = |x|(\log \log(1/|x|))^\kappa$ for $\kappa \geq 0$, which only fail to be increasing and concave on all of $[0, \infty)$. Since they have these properties in a neighbourhood of 0, we can take a linear continuation $f_{j,\kappa}$ of $\tilde{f}_{j,\kappa}$ outside its monotonicity/concavity domain. To prove monotonicity and concavity for small x , calculate for $x > 0$

$$\begin{aligned} \tilde{f}'_{1,\kappa}(x) &= \left(\log\left(\frac{1}{x}\right)\right)^{\kappa-1} \left(\log\left(\frac{1}{x}\right) - \kappa\right) \\ \tilde{f}''_{1,\kappa}(x) &= \kappa \frac{1}{x} \left(\log\left(\frac{1}{x}\right)\right)^{\kappa-2} \left((\kappa-1) - \log\left(\frac{1}{x}\right)\right) \end{aligned}$$

and

$$\begin{aligned} \tilde{f}'_{2,\kappa}(x) &= \left(\log \log\left(\frac{1}{x}\right)\right)^{\kappa-1} \left(\log\left(\frac{1}{x}\right)\right)^{-1} \left(\log \log\left(\frac{1}{x}\right) \log\left(\frac{1}{x}\right) - \kappa\right) \\ \tilde{f}''_{2,\kappa}(x) &= \kappa \frac{1}{x} \left(\log \log\left(\frac{1}{x}\right)\right)^{\kappa-2} \left(\log\left(\frac{1}{x}\right)\right)^{-2} \left((\kappa-1) - \log \log\left(\frac{1}{x}\right) - \log \log\left(\frac{1}{x}\right) \log\left(\frac{1}{x}\right)\right) \end{aligned}$$

4.2 General results on multipower variation for \mathcal{BSM} plus Lévy

We recall that we are working with

$$X = Y + Z,$$

where Y is a Brownian semimartingale. No assumptions are made regarding dependence between Y and Z .

We can now show the following general result

Theorem 1 *Let Z be a Lévy process with no continuous component and with index $\alpha \in [0, 2]$. Then*

- $0 < r < 2 \Rightarrow \text{PCiP is valid,}$
- $\alpha < 2 \text{ and } 0 < \max r < 2 \Rightarrow \text{MPVCiP is valid,}$
- $\alpha < 1 \text{ and } \alpha/(2 - \alpha) < r < 1 \Rightarrow \text{PCLT is valid,}$
- $\alpha < 1 \text{ and } \alpha/(2 - \alpha) < \min r \leq \max r < 1 \Rightarrow \text{MPVCLT is valid.}$

Apart from a finer distinction on the boundaries such as $\alpha = 2$ or $r = \alpha/(2 - \alpha)$ in terms of powers of logs or integral criteria, we believe that the ranges for α and r cannot be extended. Some evidence for this is given in form of examples in the next subsection.

Proof. For the PCiP, note that $\Psi(\lambda)/\lambda^2 \rightarrow 0$ as $\lambda \rightarrow \infty$ since we have no Gaussian coefficient (cf. Bertoin (1996, Proposition I.2)). Therefore

$$\mathbb{E} \left(\exp \left\{ i\lambda \frac{Z_\delta}{\delta^{1/2}} \right\} \right) = \exp \left\{ -\delta \Psi \left(\frac{\lambda}{\delta^{1/2}} \right) \right\} \rightarrow 1$$

i.e. $Z_\delta/\delta^{1/2} \rightarrow 0$ in probability as $\delta \downarrow 0$. Since also $E(Z_\delta^2) = c\delta$, we have that $(Z_\delta/\delta^{1/2})_{\delta>0}$ is bounded in L^2 , i.e. convergent in L^r , $1 \leq r < 2$, and it is easily seen that this extends to $0 < r < 2$ (e.g. by raising $Z_\delta/\delta^{1/2}$ to a small power and applying the argument again). Therefore

$$\mathbb{E} \left(\delta^{1-r/2} [Z_\delta]_t^{[r]} \right) = \delta \lfloor t/\delta \rfloor \frac{\mathbb{E}|Z_\delta|^r}{\delta^{r/2}} \rightarrow 0.$$

By (21), this shows PCiP. For the MPVCiP, the argument works for (21) since, by independence of increments,

$$\mathbb{E} \left(\delta^{1-r_+/2} [Z_\delta, \dots, Z_\delta]_t^{[r]} \right) = \delta \lfloor 1 - m + t/\delta \rfloor \prod_{j=1}^m \frac{\mathbb{E}|Z_\delta|^{r_j}}{\delta^{r_j/2}} \rightarrow 0,$$

but fails for (22-23) because of the log-terms e.g. in (23). However, if $\alpha < 2$, we can adapt the argument as follows. By Lemma 1, we have

$$\sup_{\delta>0} \frac{\mathbb{E}|Z_\delta|^\gamma}{\delta} < \infty$$

for all $\alpha^* \leq \alpha < \gamma \leq 2$. As above, we have $Z_\delta/\delta^{1/\gamma} \rightarrow 0$ in probability, and hence in L^r for $r < \gamma$. This allows us to check (23) for $0 < r_j < \gamma$:

$$\mathbb{E} \left(\delta^{1-r_j/2} \left(\log \left(\frac{1}{\delta} \right) \right)^{r_+-r_j} [Z_\delta]_t^{[r_j]} \right) = \delta \lfloor t/\delta \rfloor \frac{\mathbb{E}|Z_\delta|^{r_j}}{\delta^{r_j/2} \left(\log \left(\frac{1}{\delta} \right) \right)^{r_j-r_+}} \leq \delta \lfloor t/\delta \rfloor \frac{\mathbb{E}|Z_\delta|^{r_j}}{\delta^{r_j/\gamma}} \rightarrow 0,$$

and similarly all (21)-(23).

For the MPVCLT note that $\alpha < 1$ implies that Z has bounded variation. Furthermore, we can assume that Z has no drift, as this can be placed in the Y process. Now, Lemma 1 gives the basis for the above MPVCiP argument to apply here, for $\alpha < \gamma < 1$, and we can check (26):

$$\mathbb{E} \left(\delta^{1/2-r_j/2} \left(\log \left(\frac{1}{\delta} \right) \right)^{r_+-r_j} [Z_\delta]_t^{[r_j]} \right) = \delta \lfloor t/\delta \rfloor \frac{\mathbb{E}|Z_\delta|^{r_j}}{\delta^{r_j/2+1/2} \left(\log \left(\frac{1}{\delta} \right) \right)^{r_j-r_+}} \leq \delta \lfloor t/\delta \rfloor \frac{\mathbb{E}|Z_\delta|^{r_j}}{\delta^{r_j/\gamma}} \rightarrow 0$$

if and only if $r_j/2 + 1/2 < r_j/\gamma$, i.e. $r_j > \gamma/(2-\gamma) \downarrow \alpha/(2-\alpha)$ as $\gamma \downarrow \alpha$. It is now easy to repeat the argument and check that then also (24-26) hold. ■

Remark 1 A more elementary and instructive (partial) proof is as follows. For $\alpha < 1$, note that for all $0 < r \leq 1$,

$$[Z]_t^{[r]} = \lim_{\delta \downarrow 0} [Z_\delta]_t^{[r]} = \sum_{0 \leq s \leq t} |\Delta Z_s|^r. \quad (34)$$

Specifically, we use (5) to see that

$$[Z_\delta]_t^{[r]} = \sum_{j=1}^{\lfloor t/\delta \rfloor} \left| \sum_{(j-1)\delta < s \leq j\delta} \Delta Z_s \right|^r \leq \sum_{0 \leq s \leq t} |\Delta Z_s|^r.$$

For the opposite inequality we choose $\varepsilon > 0$ and denote $T_0 = 0$ and jump times of jumps of size $> \varepsilon$ by $T_n = \inf\{t > T_{n-1} : |\Delta Z_t| > \varepsilon\}$. Then, for all $\delta < \min\{T_n - T_{n-1} : T_n \leq t\}$, we have

$$[Z_\delta]_t^{[r]} \geq \sum_{n: T_n \leq t} |Z_{\delta\lceil T_n/\delta \rceil} - Z_{\delta\lfloor T_n/\delta \rfloor}|^r \rightarrow \sum_{n: T_n \leq t} |\Delta Z_{T_n}|^r,$$

as $\delta \downarrow 0$, by the càdlàg property of sample paths. Since this holds for all $\varepsilon > 0$, this establishes (34).

Furthermore, by the exponential formula for the Poisson process $(\Delta Z_t)_{t \geq 0}$ of jumps of Z with intensity measure Π ,

$$\sum_{0 \leq s \leq t} |\Delta Z_s|^r < \infty \iff \int_{\mathbb{R}^*} (|x|^r \wedge 1) \Pi(dx) < \infty,$$

which holds if (and essentially only if) $r > \alpha$. Now, if we scale $[X_\delta]_t^{[r]}$ by $\delta^{(1-r)/2} \rightarrow 0$, we deduce that PCLT holds for $\alpha < r < 1$, but this does not allow to make a statement about

$$\frac{\alpha}{2-\alpha} < r \leq \alpha.$$

To improve the lower bound to $\alpha/(2-\alpha)$, we had to take into account the interplay between $\delta^{(1-r)/2} \rightarrow 0$ and $[Z_\delta]_t^{[r]} \rightarrow \infty$.

4.3 Examples

In the examples we shall discuss Z is a Lévy jump process and $r_1 = \dots = r_m = r$. However, as will be noted at the end of this Section, quite similar results hold for Z being a process of OU type.

Example 3 Suppose Z is the $\Gamma(\nu, \lambda)$ subordinator, i.e. Z is the Lévy process for which the law of Z_1 is the gamma distribution with probability density function

$$\frac{\lambda^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\lambda x}.$$

This has infinite activity. In fact, $\alpha = 0$ is its index.

Then, for $t \downarrow 0$,

$$\mathbb{E}\{|Z_t|^p\} = \lambda^{-p} \frac{\Gamma(t\nu + p)}{\Gamma(t\nu)} \sim O(t)$$

whatever the value of $p > 0$. (Here we have used that $t\Gamma(t) \rightarrow 1$ as $t \rightarrow 0$.) Thus $[Z_\delta]^{[r]} = O_p(1)$, $[Z_\delta, Z_\delta]^{[r,r]} = O_p(\delta)$, $[Z_\delta, Z_\delta, Z_\delta]^{[r,r,r]} = O_p(\delta^2)$, etc.

Consequently:

- MPVCiP is valid for all $m = 1, 2, \dots$ and $0 < r < 2$.
- MPVCLT is valid for all $m = 1, 2, \dots$ and $0 < r < 1$.

On the other hand we have, for example, that BPVCLT does not hold if $r = 1$ and $Y \perp\!\!\!\perp Z$.

Example 4 Let Z be the $IG(\phi, \gamma)$ subordinator, i.e. Z is the Lévy process for which the law of Z_1 is the inverse Gaussian distribution with density function

$$\frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma x - 3/2} e^{-\frac{1}{2}(\phi^2 x^{-1} + \gamma^2 x)}.$$

Then, as $t \downarrow 0$, (proof below)

$$\mathbb{E}\{|Z_t|^p\} \sim \begin{cases} O(t) & \text{if } p > \frac{1}{2} \\ O(t|\log t|) & \text{if } p = \frac{1}{2} \\ O(t^{2p}) & \text{if } 0 < p < \frac{1}{2}, \end{cases} \quad (35)$$

so that, for $\frac{1}{2} < r < 1$ we have $[Z_\delta]^{[r,r]} = O_p(\delta)$ and $[Z_\delta]^{[r]} = O_p(1)$.

Consequently:

- MPVCiP is valid for all $m = 1, 2, \dots$ and $0 < r < 2$.
- MPVCLT is valid for all m if $\frac{1}{2} < r < 1$.

In particular, MPVCLT holds for tripower variation with $r = \frac{2}{3}$.

When $r = \frac{1}{2}$, $[Z_\delta, Z_\delta]^{[\frac{1}{2}, \frac{1}{2}]} = O_p(\delta^{1-\varepsilon})$ for ε arbitrarily close to 0 and $[Z_\delta]^{[\frac{1}{2}]} = O_p(|\log \delta|)$ and BPCLT holds. For $0 < r < \frac{1}{2}$ we find $[Z_\delta, Z_\delta]^{[r, r]} = O_p(\delta^{4r-1})$ and $[Z_\delta]^{[r]} = O_p(\delta^{2r-1})$. Hence BPCLT holds if $\frac{1}{3} < r < \frac{1}{2}$. In fact $\alpha = \frac{1}{2}$ is the index of Z , and $\frac{1}{3} = \frac{\alpha}{2-\alpha}$ is the lower bound established in Theorem 1.

Proof of (35): Recall that the density of the generalised inverse Gaussian distribution, denoted $GIG(\nu, \phi, \gamma)$, is

$$p(x) = \frac{(\gamma/\phi)^\nu}{2K_\nu(\phi\gamma)} x^{\nu-1} e^{-\frac{1}{2}(\phi^2 x^{-1} + \gamma^2 x)}, \quad x \in R_+.$$

where $K_\nu(\cdot)$ is a modified Bessel function of the third kind. We note that

$$K_{-\frac{1}{2}}(x) = K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}} x^{-\frac{1}{2}} e^{-x}, \quad x \in R_+.$$

Hence, for p positive real

$$E\{|Z_t|^p\} = \sqrt{\frac{2}{\pi}} e^{\phi\gamma t} \gamma^{\frac{1}{2}-p} \phi^{\frac{1}{2}+p} K_{|p-\frac{1}{2}|}(\phi\gamma t) t^{\frac{1}{2}+p}.$$

For $x \downarrow 0$ and $\nu > 0$ we have

$$K_\nu(x) \sim \Gamma(\nu) 2^{\nu-1} x^{-\nu}$$

whereas

$$K_0(x) \sim |\log x|.$$

Thus, as $t \downarrow 0$,

$$E\{|Z_t|^p\} \sim \begin{cases} O(t) & \text{if } p > \frac{1}{2} \\ O(t|\log t|) & \text{if } p = \frac{1}{2} \\ O(t^{2p}) & \text{if } 0 < p < \frac{1}{2}. \end{cases}$$

Example 5 If Z is the Variance Gamma Lévy process (also known as a normal Gamma process, written $NT(\nu, \lambda)$) then, for $t \downarrow 0$,

$$E\{|Z_t|^q\} \sim O(t) \tag{36}$$

whatever the value of $q > 0$. Consequently:

- MPVCiP is valid for all $m = 1, 2, \dots$ and $0 < r < 2$.
- MPVCLT is valid for all m and $0 < r < 1$.

Proof of (36): One can use the fact that if Z is the Variance Gamma Lévy process with parameters ν and λ , then it can be written as

$$Z_t = B_{T_t},$$

where B is Brownian motion and T is a $\Gamma(\nu, \lambda)$ subordinator, while $B \perp\!\!\!\perp T$. This means that

$$E\{|Z_t|^q\} = E\{|T_t|^{q/2}\}.$$

Example 6 Let Z be the $NIG(\gamma, 0, 0, \phi)$ Lévy process. This is representable as the subordination of a Brownian motion B by the $IG(\phi, \gamma)$ subordinator. Hence, $E\{|Z_t|^q\}$ behaves asymptotically as in (35) with $p = q/2$. Consequently:

- MPVCiP is valid for all $m = 1, 2, \dots$ and $0 < r < 2$.
- MPVCLT does not hold for any value of r .

Remark 2 From a modelling perspective it will often be more natural to have Z as an OU process V with a background driving Lévy process (BDLP) L . Letting

$$V_t^* = \int_0^t V_s ds$$

we have, since V is by definition the solution of $dV_t = -\lambda V_t + dL_{\lambda t}$, that

$$V_t = V_0 - \lambda V_t^* + L_{\lambda t}.$$

Hence, letting $Y' = Y + V_0 - \lambda V^*$ we see that Y' satisfies the condition (4). Therefore the asymptotics are the same whether $Z = V$ or $Z = L$. In the latter case we are back in the setting of the above examples.

5 Simulation experiments

5.1 Simulation design

In the first design we will repeatedly simulate, over the unit interval, standard Brownian motion B plus four different types of jump process Z . The B and Z processes are drawn independently of one another. The jump process Z_t will have a zero mean, be symmetrically distributed and have a unit unconditional variance. Further, the four jump processes will be setup to share identical first four moments. The specifics of the jump processes are as follows.

- (i) a normal inverse Gaussian Lévy process such that $Z_1 \stackrel{L}{=} \varepsilon \sigma$ where $\varepsilon \perp\!\!\!\perp \sigma$, $\varepsilon \sim N(0, 1)$ and $\sigma \sim IG(c2, 2)$. This process has index of 1,

- (ii) a normal gamma Lévy process, which has index of 0,
- (iii) a stratified normal inverse Gaussian compound Poisson process (CPP) with a single jump per unit of time (which means the process will distribute an arrival randomly in the unit interval and the jumps are distributed as a normal inverse Gaussian random variable), which has index of 0,
- (iv) a stratified normal inverse Gaussian CPP process with 10 jumps per unit of time. Obviously the jumps in this process will have to have a smaller variance than in (iii) to compensate for the fact that there are more jumps. This process has index of 0.

The parameters of the normal gamma and normal inverse Gaussian distributions in (ii), (iii) and (iv) were selected to match the distributions in (i). When $c = 1$ the variance, per unit of time, of B and Z are equal. In empirical work based on 10 minute returns for the Dollar against the DM from 1986 to 1996, Barndorff-Nielsen and Shephard (2005) suggest that jumps account for around 10% of the variation of the price process. Hence in our Monte Carlo results we will study the cases where $c = 1/10$ and 1.

5.2 Convergence in probability

In this subsection we see how accurate the CiP predictions are in finite samples. Our simulations of the jumps processes will cover both finite activity and infinite activity processes. To assess the results we will compute the root mean square error of the following error terms

$$\begin{aligned} S_1 &= \left(\mu_1^{-2} \{Y_\delta\}_1^{[1,1]} - 1 \right), \\ S_2 &= \left(\mu_{2/3}^{-3} \{Y_\delta\}_1^{[2/3,2/3,2/3]} - 1 \right), \\ S_3 &= \{Y_\delta\}_1^{[2]} - \mu_1^{-2} \{Y_\delta\}_1^{[1,1]} - [Z_\delta]_1^{[2]}, \end{aligned}$$

and

$$S_4 = \{Y_\delta\}_1^{[2]} - \mu_{2/3}^{-3} \{Y_\delta\}_1^{[2/3,2/3,2/3]} - [Z_\delta]_1^{[2]}.$$

The first two terms are the errors of the scaled realised bipower and tripower estimators of the quadratic variation of the continuous component of the process, which is 1 in this case. The third and fourth terms are the corresponding errors in estimating the quadratic variation of the jump component.

The results in Table 1 are given only in the $c = 1$ case, as this is the most challenging. They suggest that the tripower variation based statistics provide better estimators than when there are jumps in the process. The two CPP cases are interesting. They suggest that the infrequent

n	NIG case				NGamma case			
	S_1	S_2	S_3	S_4	S_1	S_2	S_3	S_4
10	1.162	0.987	1.027	0.903	1.208	1.021	1.027	0.895
30	0.883	0.766	0.833	0.728	0.835	0.709	0.786	0.671
100	0.621	0.527	0.603	0.513	0.577	0.459	0.567	0.455
300	0.454	0.357	0.450	0.355	0.363	0.263	0.364	0.268
1,000	0.299	0.221	0.298	0.222	0.217	0.140	0.216	0.143
3,000	0.198	0.139	0.198	0.140	0.132	0.077	0.134	0.081
10,000	0.126	0.083	0.127	0.084	0.073	0.038	0.073	0.040
n	CPP 1 jump				CPP 10 jumps			
	S_1	S_2	S_3	S_4	S_1	S_2	S_3	S_4
10	0.851	0.684	0.802	0.747	1.175	1.002	1.022	0.897
30	0.550	0.432	0.560	0.480	0.846	0.727	0.785	0.678
100	0.314	0.227	0.329	0.257	0.585	0.477	0.575	0.475
300	0.180	0.126	0.191	0.147	0.398	0.299	0.396	0.301
1,000	0.102	0.066	0.108	0.079	0.245	0.167	0.244	0.168
3,000	0.060	0.036	0.063	0.043	0.150	0.092	0.151	0.094
10,000	0.031	0.019	0.035	0.024	0.084	0.046	0.086	0.048
n	Brownian motion							
	S_1	S_2	S_3	S_4				
10	0.957	1.014	0.540	0.749				
30	0.585	0.626	0.295	0.397				
100	0.320	0.345	0.158	0.208				
300	0.187	0.202	0.089	0.118				
1,000	0.101	0.109	0.049	0.065				
3,000	0.059	0.063	0.028	0.037				
10,000	0.032	0.035	0.015	0.020				

Table 1: Root mean square error of the estimators of various measures of variation. The four jump processes have the same first four moments. First two cases have infinite activity, next two finite activity. Code: winkel.ox.

jumps case is easier to deal with than the case with many, smaller jumps. The processes are set up so that if the number of jumps went off to infinity so the CPP case would converge to the NIG Lévy process. These are the expected results from the theory and detailed examples developed in the previous Section.

Table 1 also gives results when there are no jumps in the process. Here we compute the same quantities as above but now simulate from $\sqrt{2}$ times standard Brownian motion. The results are the ones expected from the theory given in Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005) and Barndorff-Nielsen, Graversen, Jacod, and Shephard (2005) which is that the bipower based quantities are slightly more accurate than the corresponding tripower versions.

5.3 Central limit theorem

The theory we have developed implies we should expect that when the jumps are of a finite activity case the CLT holds for realised tripower variation based estimators of integrated variance. However, it fails for realised power and realised bipower variation versions. In this subsection we use the above simulation design to assess the accuracy of these predictions in finite samples.

In the case where the continuous component is standard Brownian motion, if the CLT works then the bipower CLT takes on the form of

$$\frac{\sqrt{M} \left(\log \left(\mu_1^{-2} \{Y_\delta\}_1^{[1,1]} \right) - \log(1) \right)}{\sqrt{2.6090}} \xrightarrow{d} N(0, 1).$$

The corresponding tripower CLT is

$$\frac{\sqrt{M} \left(\log \left(\mu_{2/3}^{-3} \{Y_\delta\}_1^{[2/3, 2/3, 2/3]} \right) - \log(1) \right)}{\sqrt{3.0613}} \xrightarrow{d} N(0, 1).$$

We compute these t-statistics and record the mean and standard error, which should be roughly 0 and 1 if the CLT exactly holds. Also recorded is the percentage that the absolute value of the statistics are less than 1.96. This should be around 95% if the finite sample distribution is accurate.

5.3.1 Tripower variation

The results on realised tripower are given in Table 2 for the $c = 0.1$ case. They are in line with the above theory. It suggests that practically speaking the CLT does indeed seem to work for a moderately large sample size in the normal gamma, compound Poisson and Brownian motion cases and it shows signs of failure in the NIG case.

The theoretical results seem to continue to have some explanatory power in the extreme $c = 1$ case — which is less realistic in financial economics. The results are given in Table 3 in the NIG and normal gamma cases. Having said that, the convergence of the CLT for the normal gamma process is very slow indeed and the predictions from the theory would seem to be unreliable for any sensible finite samples. The Table shows the dramatic failure of the CLT to hold in the NIG case, as predicted.

The corresponding results for compound Poisson processes when $c = 1$ are also given in Table 3. In the single jump case the CLT seems to perform in a somewhat useful way, while when there are many jumps the CLT takes extreme large samples to produce reasonably accurate predictions. In the case of pure Brownian motion the theory gives accurate predictions even for quite small samples.

	NIG case			Ngamma case					
M	Bias	S.E.	Cove	Bias	S.E.	Cove			
10	-0.61	1.13	88.1	-0.61	1.12	87.7			
30	-0.21	1.05	93.3	-0.22	1.04	93.2			
100	0.04	1.03	94.0	-0.01	1.05	93.6			
300	0.17	1.02	93.9	0.08	1.03	93.7			
1,000	0.29	1.03	93.4	0.14	1.03	93.9			
3,000	0.38	1.02	92.8	0.17	1.02	94.0			
10,000	0.42	1.02	92.4	0.14	1.02	94.4			
	CPP 1 jump			CPP 40 jumps			Brownian motion		
M	Bias	S.E.	Cove	Bias	S.E.	Cove	Bias	S.E.	Cove
10	-0.59	1.12	88.1	-0.56	1.11	88.7	-0.70	1.12	86.5
30	-0.23	1.05	92.8	-0.24	1.05	93.0	-0.36	1.03	92.4
100	-0.03	1.03	94.1	0.04	1.04	93.9	-0.21	1.00	94.5
300	0.05	1.01	94.5	0.17	1.04	93.7	-0.12	0.99	94.8
1,000	0.09	1.00	94.7	0.26	1.02	93.7	-0.05	1.00	95.0
3,000	0.08	1.01	94.4	0.33	1.02	92.8	-0.03	1.01	94.4
10,000	0.11	1.01	94.7	0.39	1.01	92.8	-0.01	1.00	94.9

Table 2: $c = 0.1$ case. Bias and standard error of the realised tripower variation errors using the log-based asymptotics. Cove denotes estimated finite sample coverage using the asymptotic theory setting the nominal level at 95.0. Based on 5,000 replications. File: `simple.ox`.

5.3.2 Bipower variation

The corresponding results for realised bipower variation in the $c = 1$ case are given in Table 4. They confirm the theoretical predictions that the CLT fails when there are any form of jumps.

6 Conclusion

In this paper we have studied how the behaviour of realised multipower variation changes when we add jumps to a Brownian semimartingale. Previously Barndorff-Nielsen and Shephard have shown that the probability limit of these measures of variation are robust to finite activity jumps, whatever their relationship to the Brownian semimartingale. Here we show that this conclusion generalises to infinite activity Lévy processes provided the activity is not too high. Similar results hold for jump processes in wide generality, see the closely related work of Woerner (2004). Thus, in particular, we expect that realised multipower variation can be used to split up quadratic variation into that due to the continuous component of prices and that due to the jumps.

The other contribution of the paper is to provide the first analysis of the asymptotic distribution of realised multipower variation when there are jumps. We showed that if our interest is in estimating integrated variance, in the presence of arbitrary finite activity jumps, then realised

	NIG case			Ngamma case					
M	Bias	S.E.	Cove	Bias	S.E.	Cove			
10	0.33	1.17	89.0	0.20	1.20	89.3			
30	1.12	1.13	76.8	0.98	1.20	79.6			
100	1.96	1.20	50.2	1.54	1.25	63.6			
300	2.64	1.21	28.6	1.80	1.31	55.9			
1,000	3.33	1.25	13.2	1.96	1.27	50.0			
3,000	3.80	1.24	5.6	1.92	1.25	51.3			
10,000	4.38	1.17	1.6	1.76	1.15	55.6			
30,000	4.74	1.15	0.7	1.57	1.11	64.2			
100,000	5.05	1.10	0.3	1.38	1.10	68.4			
300,000	5.36	1.09	0.0	1.11	1.04	79.9			
1,000,000	5.59	1.05	0.0	0.84	1.04	85.2			
	CPP 1 jump			CPP 40 jumps			Brownian motion		
M	Bias	S.E.	Cove	Bias	S.E.	Cove	Bias	S.E.	Cove
10	-0.24	1.20	89.1	0.26	1.16	90.9	-0.72	1.15	85.2
30	0.16	1.07	92.7	1.04	1.18	77.5	-0.39	1.02	92.9
100	0.36	1.13	90.0	1.73	1.18	59.4	-0.17	1.01	95.0
300	0.43	1.09	90.2	2.20	1.19	43.4	-0.15	1.01	94.9
1,000	0.40	1.08	90.4	2.43	1.22	35.7	-0.04	0.99	96.2
3,000	0.37	1.03	91.9	2.49	1.14	32.0	-0.02	0.98	95.6
10,000	0.26	1.02	93.4	2.36	1.15	37.0	-0.04	0.97	95.6
30,000	0.32	1.04	92.4	2.17	1.08	43.2	-0.07	0.99	94.6
100,000	0.21	1.01	93.8	1.86	1.08	54.7	0.00	0.99	94.8
300,000	0.20	1.00	95.0	1.60	1.03	63.7	-0.02	0.99	94.9
1,000,000	0.15	1.00	94.8	1.35	1.02	72.1	0.01	0.99	95.3

Table 3: $c = 1.0$ case. *Bias and standard error of the realised tripower variation errors using the log-based asymptotics. Cove denotes estimated finite sample coverage using the asymptotic theory setting the nominal level at 95.0. Based on 2,000 replications. File: simple.ox.*

tripower variation can do this and the corresponding standard non-jump central limit theorem continues to hold under jumps. This result does not hold in the case of bipower variation. When the jumps are of infinite activity the results are more complicated, as discussed. Simulation results suggest that the CLT does have some predictive power in finite samples when the share of the variation in the process due to jumps is moderate — which is realistic in financial economics. However, when jumps make up a large share of the movement in the process the predictions from the theory are often quite inaccurate.

7 Acknowledgments

We are grateful to Jeannette Woerner for useful discussions on the topic of this paper and for providing us with an early version of her notes on CiP for bipower variation. Ole E. Barndorff-Nielsen’s work is supported by CAF (www.caf.dk), which is funded by the Danish Social Science

	NIG case			Ngamma case					
M	Bias	S.E.	Cove	Bias	S.E.	Cove			
10	0.66	1.16	84.6	0.60	1.20	86.1			
30	1.48	1.19	66.5	1.33	1.28	67.5			
100	2.50	1.36	35.3	2.05	1.39	49.2			
300	3.48	1.47	15.0	2.73	1.65	32.6			
1,000	4.65	1.67	3.2	3.25	1.75	21.8			
3,000	5.76	2.03	0.8	3.61	1.85	17.5			
10,000	6.99	1.95	0.0	3.77	2.05	16.8			
30,000	8.15	2.17	0.0	4.00	2.07	13.3			
100,000	9.33	1.98	0.0	4.08	1.87	10.8			
300,000	10.5	1.96	0.0	3.62	1.70	16.4			
1,000,000	11.7	2.03	0.0	2.89	1.48	28.2			
	CPP 1 jump			CPP 40 jumps			Brownian motion		
M	Bias	S.E.	Cove	Bias	S.E.	Cove	Bias	S.E.	Cove
10	0.19	1.19	90.2	0.60	1.16	87.1	-0.50	1.13	89.1
30	0.52	1.28	84.7	1.46	1.17	66.5	-0.28	1.02	93.6
100	0.81	1.31	80.8	2.27	1.35	42.4	-0.11	1.00	95.2
300	1.00	1.45	76.3	3.02	1.50	23.4	-0.10	1.00	95.0
1,000	1.03	1.51	75.8	3.78	1.65	12.0	-0.02	1.00	95.8
3,000	1.12	1.57	73.6	4.34	1.72	5.9	0.00	0.98	95.5
10,000	1.17	1.49	73.2	4.76	1.89	3.0	-0.03	0.97	95.9
30,000	1.14	1.65	73.9	4.86	1.76	2.4	-0.07	0.99	95.1
100,000	1.20	1.52	72.2	5.07	1.84	2.2	0.01	0.99	95.6
300,000	1.19	1.64	74.5	5.13	1.74	1.6	-0.01	0.98	95.4
1,000,000	1.20	1.60	72.8	5.22	1.87	1.4	0.01	0.99	95.0

Table 4: $c = .01$ case. *Bias and standard error of the realised bipower variation errors using the log-based asymptotics. Cove denotes estimated finite sample coverage using the asymptotic theory setting the nominal level at 95.0. Based on 2,000 replications. File: simple.ox.*

Research Council. Neil Shephard’s research is supported by the UK’s ESRC through the grant “High frequency financial econometrics based upon power variation.”

References

- Andersen, T. G., T. Bollerslev, and F. X. Diebold (2005). Parametric and nonparametric measurement of volatility. In Y. Aït-Sahalia and L. P. Hansen (Eds.), *Handbook of Financial Econometrics*. Amsterdam: North Holland. Forthcoming.
- Bandi, F. M. and J. R. Russell (2003). Microstructure noise, realized volatility, and optimal sampling. Unpublished paper, Graduate School of Business, University of Chicago.
- Barndorff-Nielsen, O. E., S. E. Graversen, J. Jacod, M. Podolskij, and N. Shephard (2005). A central limit theorem for realised power and bipower variations of continuous semimartingales. In Y. Kabanov and R. Lipster (Eds.), *From Stochastic Analysis to Mathematical*

- Finance, Festschrift for Albert Shiryaev*. Springer. Forthcoming. Also Economics working paper 2004-W29, Nuffield College, Oxford.
- Barndorff-Nielsen, O. E., S. E. Graversen, J. Jacod, and N. Shephard (2005). Limit theorems for realised bipower variation in econometrics. Unpublished paper: Nuffield College, Oxford.
- Barndorff-Nielsen, O. E., P. R. Hansen, A. Lunde, and N. Shephard (2004). Regular and modified kernel-based estimators of integrated variance: the case with independent noise. Unpublished paper: Nuffield College, Oxford.
- Barndorff-Nielsen, O. E. and N. Shephard (2002). Econometric analysis of realised volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society, Series B* 64, 253–280.
- Barndorff-Nielsen, O. E. and N. Shephard (2004a). Econometric analysis of realised covariation: high frequency covariance, regression and correlation in financial economics. *Econometrica* 72, 885–925.
- Barndorff-Nielsen, O. E. and N. Shephard (2004b). Power and bipower variation with stochastic volatility and jumps (with discussion). *Journal of Financial Econometrics* 2, 1–48.
- Barndorff-Nielsen, O. E. and N. Shephard (2005). Econometrics of testing for jumps in financial economics using bipower variation. *Journal of Financial Econometrics*. Forthcoming.
- Bertoin, J. (1996). *Lévy Processes*. Cambridge: Cambridge University Press.
- Bingham, N. H., C. M. Goldie, and J. L. Teugels (1989). *Regular Variation*. Cambridge: Cambridge University Press.
- Blumenthal, R. M. and R. K. Gettoor (1961). Sample functions of stochastic processes with independent increments. *Journal of Math Mech* 10, 493–516.
- Delattre, S. and J. Jacod (1997). A central limit theorem for normalized functions of the increments of a diffusion process in the presence of round off errors. *Bernoulli* 3, 1–28.
- Gloter, A. and J. Jacod (2001a). Diffusions with measurement errors. I — local asymptotic normality. *ESAIM: Probability and Statistics* 5, 225–242.
- Gloter, A. and J. Jacod (2001b). Diffusions with measurement errors. II — measurement errors. *ESAIM: Probability and Statistics* 5, 243–260.
- Hansen, P. R. and A. Lunde (2003). An optimal and unbiased measure of realized variance based on intermittent high-frequency data. Unpublished paper, Department of Economics, Stanford University.

- Jacod, J. (1994). Limit of random measures associated with the increments of a Brownian semimartingale. Preprint number 120, Laboratoire de Probabilités, Université Pierre et Marie Curie, Paris.
- Jacod, J. and P. Protter (1998). Asymptotic error distributions for the Euler method for stochastic differential equations. *Annals of Probability* 26, 267–307.
- Revuz, D. and M. Yor (1999). *Continuous Martingales and Brownian motion* (3 ed.). Heidelberg: Springer-Verlag.
- Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.
- Winkel, M. (2005). Explicit constructions of Monroe’s embedding and a converse for Lévy processes. Unpublished paper, Department of Statistics, University of Oxford.
- Woerner, J. (2004). Power and multipower variation: inference for high frequency data. Unpublished paper.
- Zhang, L. (2004). Efficient estimation of stochastic volatility using noisy observations: a multi-scale approach. Unpublished paper: Department of Statistics, Carnegie Mellon University.
- Zhang, L., P. Mykland, and Y. Aït-Sahalia (2005). A tale of two time scales: determining integrated volatility with noisy high-frequency data. *Journal of the American Statistical Association*. Forthcoming.