OXFORD CENTRE FOR COLLABORATIVE APPLIED MATHEMATICS

Report Number 09/38

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Abstract

In this article, we discuss an alternate approach to a priori error estimates for the semidiscrete Galerkin approximation to a time dependent parabolic integro-differential equation with nonsmooth initial data. It is based on energy arguments and on a repeated use of time integration, but without using parabolic type duality technique. Optimal L^2-error estimate is derived for the semidiscrete approximation, when the initial data is in L^2.

Key words. Parabolic integro-differential equation, finite element method,semidiscrete solution, energy arguments, optimal error estimate, and nonsmooth initial data.

1 Introduction

In this paper, we discuss an alternate approach to *a priori* L^2-error estimate for a semidiscrete finite element Galerkin approximation to the following parabolic integro-differential equation:

$$
\begin{aligned}
u_t + \mathcal{A}(t)u &= \int_0^t B(t,s)u(s)\,ds \qquad && \text{in } \ \Omega \times J, \\
u &= 0 && \text{on } \ \partial\Omega \times J, \\
u(\cdot,0) &= u_0 && \text{in } \ \Omega.
\end{aligned} \tag{1.1}
$$

with $u_0 \in L^2(\Omega)$, where $\Omega \subset I\!R^d$, $d = 2, 3$ is a bounded convex polygon or polyhedron, $J = (0, T]$, $T < \infty$. Here, $u = u(x,t)$ is a real-valued function in $\Omega \times J$ and $u_t = \frac{\partial u}{\partial t}$. Further, $\mathcal{A}(t)$ is a second-order selfadjoint, uniformly positive definite elliptic operator of the form

$$\mathcal{A}(t) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\Big(a_{ij}(x,t)\frac{\partial}{\partial x_i}\Big) + a_0(x,t)I,$$

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and $B(t,s)$ is a general second-order elliptic differential operator

$$B(t,s) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\Big(b_{ij}(x;t,s)\frac{\partial}{\partial x_i}\Big) + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\Big(b_j(x;t,s)\frac{\partial}{\partial x_j}\Big) + b_0(x;t,s)I,$$

Equations of the type described above arise naturally in nonlocal flows in porous media (cf. Cushman and Glinn [3] and Dagan [4]) and heat conduction through materials with memory (cf. Renardy *et al.* [14]).

We use usual notations for the L^2, H_0^1 and H^2 spaces and their norms and semi-norms. Let $\mathcal{A}(t;\cdot,\cdot)$ and $\mathcal{B}(t,s\,;\cdot,\cdot)$ be bilinear forms on $H_0^1 \times H_0^1$ corresponding to operators $\mathcal{A}(t)$ and $B(t,s)$, respectively, i.e.

$$\mathcal{A}(t;\phi,\psi) = \int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij}(x,t)\frac{\partial\phi}{\partial x_i}\frac{\partial\psi}{\partial x_j} + a_0(x,t)\phi\psi\Big)\,dx,$$

and

$$\mathcal{B}(t,s;\phi(s),\psi) = \int_\Omega \Big(\sum_{i,j=1}^{d} b_{ij}(x;t,s)\frac{\partial\phi(s)}{\partial x_i}\frac{\partial\psi}{\partial x_j} + \sum_{j=1}^{d} b_j(x;t,s)\frac{\partial\phi(s)}{\partial x_j}\psi + b_0(x;t,s)\phi\psi\Big)\,dx.$$

The weak formulation for (1.1) may be stated as: find $u : J \to H_0^1$ such that

$$\begin{aligned} (u_t,\phi) + \mathcal{A}(t;u,\phi) &= \int_0^t \mathcal{B}(t,s;u(s),\phi)ds, \quad \forall \phi \in H_0^1, \quad t \in J \\ u(0) &= u_0. \end{aligned} \tag{1.2}$$

Now, we define a semidiscrete Galerkin approximation of u. Let h with $0 < h < 1$ be the discretizing parameter of a regular triangulation of Ω. Let us denote, as S_h, the corresponding finite dimensional subspace of H_0^1 such that for all $v \in H_0^1 \cap H^2$, $k \in \{1,2\}$,

$$\inf_{\phi_h \in S_h} \|v - \phi_h\|_j \le \rho_0 h^{k-j}\|v\|_k, \quad j \in \{0,1\}, \tag{1.3}$$

where ρ_0 is independent of h.
The semi-discrete Galerkin approximation to a solution u of (1.1) is to find $u_h \in S_h$ satisfying

$$(u_{ht},\phi_h) + \mathcal{A}(t;u_h,\phi_h) = \int_0^t B(t,s;u_h(s),\phi_h)ds, \quad \forall \phi_h \in S_h, \quad t > 0 \tag{1.4}$$

with $u_h(0) = P_h u_0$, where $P_h u_0$ is an L^2 projection of u_0 onto S_h.

Below, we present our main result on L^2 error estimate of $e = u - u_h$, when the initial data $u_0 \in L^2(\Omega)$.

Theorem 1.1 *Let u and u_h be the solution of the equations (1.2) and (1.4), respectively, with $u(0) = u_0$ and $u_h(0) = P_h u_0$. Then, there exists a positive constant C independent of h such that*

$$\|u - u_h\| \le Ch^2 t^{-1}\|u_0\|.$$

Earlier, Yanik and Fairweather [17] have derived optimal error estimates for smooth solutions for a nonlinear problem with first order partial differential operator B. Canon and Lin [1, 2], Lin *et al.* [6] and Lin and Zhang [7], Pani *et al.* [13] have proved *a priori* error estimates for parabolic integro-differential equations for smooth initial data using Ritz-Volterra projection as against elliptic projection which is normaly used for optimal error estimates of parabolic type equations. Thomée and Zhang [15] have obtained optimal L^2-error estimates for smooth and nonsmooth initial data using semi-group theoretic approach combined with an use of inverse of the associate elliptic operator, when $\mathcal{A}$ is independent of time. Subsequently, based on energy argument and parabolic type duality, Pani and Sinha [11] have proved optimal L^2-estimate for semi-discrete Galerkin approximation to a more general time dependent parabolic integro-differential equation with nonsmooth initial data. Pani and Peterson [9], Pani and Sinha [12] have discussed the effect of quadrature for non-smooth initial data using a combination of integration in time and a use of the inverse of the associated elliptic operator. For completely discerete scheme which is based on backward Euler method, optimal error estimates are derived in [16] and [10].

In this paper, an alternate approach to prove Theorem 1.1 is discussed for a semidiscerete Galerkin formulation (1.4), when $u_0 \in L^2$ using energy arguement, but without resorting to a use of parabolic type duality. Essentially, the proof technique is based on a combination of energy arguement and a repeated use of time integration instead of a use of the inverse of the associated discrete elliptic operator. Compared to Thomée and Zhang [15] and Pani and Sinha [11], the argument presented in this paper is mainly based on the energy technique which follows the standard pattern of error analysis related to parabolic integro-differential equations with smooth data.

Throughout this article, we denote by C, a generic positive constant which may vary from context to context. Further, we also use the following notation:

$$\hat{\phi}(t) = \int_0^t \phi(s)ds.$$

The plan of the paper is as follows. While Section 1 is introductory in nature, Section 2 deals with some *a priori* estimates and regularity results for the exact solution. In Section 3, Ritz Volterra projection is introduced and related estimates are carried out. Section 4 focuses on optimal L^2-error estimates, when nonsmooth initial data $u_0 \in L^2(\Omega)$.

2 A *priori* estimates

In this section, we derive some a priori bounds which are needed in our subsequent error analysis.

For our future use, we assume that the principal part of $\mathcal{A}(t)$ is uniformly elliptic and the coefficient $a_0 \geq 0$. Further, we assumae that all the coefficients of $\mathcal{A}(t)$ and $B(t,s)$ are smooth and their derivatives are also uniformly bounded in their domain of definition. Based on the assumptions on the coefficients, it is straightforward to show that the bilinear form $\mathcal{A}$ is coercive, i.e., there is positive constant ρ_1 independent of t, such that

$$\mathcal{A}(t;\phi,\phi) \geq \rho_1 \|\phi\|_1^2, \quad \phi \in H_0^1. \tag{2.1}$$

Also, the domain being convex polygonal or polyhedral, there is a positive constant ρ_2, independent of t, such that

$$\|\phi\|_2 \leq \rho_2 \|\mathcal{A}(t)\phi\|, \quad \phi \in H_0^1 \cap H^2. \tag{2.2}$$

Finally, there are positive constants ρ_3 and ρ_4, independent of t, such that

$$|\mathcal{A}(t;\phi,\psi)| \leq \rho_3 \|\phi\|_1 \|\psi\|_1, \quad \phi,\psi \in H_0^1. \tag{2.3}$$

$$|\mathcal{B}(t,s;\phi(s),\psi)| \leq \rho_4 \|\phi(s)\|_1 \|\psi\|_1, \quad \phi(s),\psi \in H_0^1. \tag{2.4}$$

We define the bilinear form $\mathcal{A}_t(t;\cdot,\cdot) : H_0^1 \times H_0^1 \to I\!R$ by

$$\mathcal{A}_t(t;\phi,\psi) = \int_\Omega \Big(\sum_{i,j=1}^{d} \frac{\partial a_{ij}(x,t)}{\partial t} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} + \frac{\partial a_0(x,t)}{\partial t} \phi\psi \Big)\, dx, \quad \phi,\psi \in H_0^1.$$

As the coefficients and their derivatives are bounded uniformly in time, we conclude that, there exists a positive constant ρ_5, independent of t, such that

$$|\mathcal{A}_t(t;\phi,\psi)| \leq \rho_5 \|\phi\|_1 \|\psi\|_1, \quad \phi,\psi \in H_0^1. \tag{2.5}$$

We present below a priori estimates and regularity results for the solution of (1.1), when $u_0 \in L^2(\Omega)$. For a proof, we refer to [11].

Lemma 2.1 *Let u be a solution of parabolic integro-differential equation (1.1) and $u_0 \in L^2(\Omega)$. Then, the following estimates hod for $t \in J$:*

$$t\|u(t)\|_1^2 + \int_0^t s\|u_s(s)\|_1^2 ds \leq C\|u_0\|^2, \tag{2.6}$$

$$t^2\|u_t(t)\|^2 + \int_0^t s^2\|u_s(s)\|_1^2 ds \leq C\|u_0\|^2, \tag{2.7}$$

$$\|\hat{u}(t)\|_2 \leq C\|u_0\|, \tag{2.8}$$

$$t\|u(t)\|_2 \leq C\|u_0\|. \tag{2.9}$$

Next, we discuss the estimate for $\|u_t\|_1$ and $\|u_t\|_2$, again when $u_0 \in L^2(\Omega)$.

Lemma 2.2 *Let u be a solution of parabolic integro-differential equation (1.1) and $u_0 \in L^2(\Omega)$. Then, the following estimate holds for $k \in \{1,2\}$ and $t \in J$:*

$$\|u_t\|_k \leq Ct^{-(1+k/2)}\|u_0\|. \tag{2.10}$$

Proof. Differentiate (1.1) with respect to time and obtain

$$u_{tt} + \mathcal{A}(t)u_t + \mathcal{A}_t(t)u = B(t,t)u(t) + \int_0^t B_t(t,s)u(s)ds. \tag{2.11}$$

Multiply (2.11) with $t^3\mathcal{A}(t)u_t$, integrate over Ω and rewrite the resulting equation as

$$\begin{aligned}(u_{tt}, t^3\mathcal{A}(t)u_t) + (\mathcal{A}(t)u_t, t^3\mathcal{A}(t)u_t) &= -(\mathcal{A}_t(t)u, t^3\mathcal{A}(t)u_t) \\ &\quad + (B(t,t)u(t), t^3\mathcal{A}(t)u_t) + \int_0^t (B_t(t,s)u(s), t^3\mathcal{A}(t)u_t)ds.\end{aligned} \tag{2.12}$$

Observe that

$$\begin{aligned}(2.13)\qquad \frac{d}{dt}(u_t, t^3\mathcal{A}(t)u_t) &= \frac{d}{dt}t^3\mathcal{A}(t;u_t,u_t)\\ &= 3t^2\mathcal{A}(t;u_t,u_t)+t^3\mathcal{A}_t(t;u_t,u_t)+2t^3\mathcal{A}(t;u_{tt},u_t).\end{aligned}$$

Using integration by parts, we can rewrite the last term of (2.12) as

$$\begin{aligned}(2.14)\qquad \int_0^t (B_t(t,s)u(s), t^3\mathcal{A}(t)u_t)ds &= t^3(B_t(t,t)\hat{u}(t),\mathcal{A}(t)u_t)\\ &\quad -t^3\int_0^t (B_{ts}(t,s)\hat{u}(s),\mathcal{A}(t)u_t)ds.\end{aligned}$$

On substituting (2.13)-(2.14) in (2.12), we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{t^3\mathcal{A}(t;u_t,u_t)\} \;+\; t^3\|\mathcal{A}(t)u_t\|^2 &= \frac{3}{2}t^2\mathcal{A}(t;u_t,u_t)+\frac{1}{2}t^3\mathcal{A}_t(t;u_t,u_t)\\ &\quad -t^3(\mathcal{A}_t(t)u,\mathcal{A}(t)u_t)+t^3(B(t,t)u(t),\mathcal{A}(t)u_t)\\ &\quad +t^3(B_t(t,t)\hat{u}(t),\mathcal{A}(t)u_t)+t^3\int_0^t (B_{ts}(t,s)\hat{u}(s),\mathcal{A}(t)u_t)ds.\end{aligned}$$

Integrate the above equation with respect to time from 0 to t, use (2.1), (2.2), smoothness of coefficients of $\mathcal{A}(t)$ and $B(t,s)$ and Young's inequality to arrive at

$$\begin{aligned}t^3\|u_t\|_1^2+\int_0^t s^3\|u_s(s)\|_2^2 &\le C\int_0^t s^2(\|u_s(s)\|_1^2+\|u(s)\|_2^2)ds\\ &\quad +C\int_0^t s^3\|\hat{u}(s)\|_2^2ds.\end{aligned}$$

We obtain after applying Lemma 2.1

$$(2.15)\qquad t^3\|u_t\|_1^2+\int_0^t s^3\|u_s(s)\|_2^2\le C\|u_0\|^2.$$

Now multiply (2.11) by t^3u_{tt} and integrate over Ω to get

$$\begin{aligned}t^3\|u_{tt}\|^2+\mathcal{A}(t;u_t,t^3u_{tt})+\mathcal{A}_t(t;u,t^3u_{tt}) &= \mathcal{B}(t,t;u(t),t^3u_{tt})\\ &\quad +\int_0^t \mathcal{B}_t(t,s;u(s),t^3u_{tt})ds.\end{aligned}$$

Note that

$$(2.16)\qquad \frac{d}{dt}\{t^3\mathcal{A}(t;u_t,u_t)\}=3t^2\mathcal{A}(t;u_t,u_t)+t^3\mathcal{A}_t(t;u_t,u_t)+2t^3\mathcal{A}(t;u_t,u_{tt}),$$

and

$$(2.17)\qquad \int_0^t \mathcal{B}_t(t,s;u(s),t^3u_{tt})ds=t^3\mathcal{B}_t(t,t,\hat{u}(t),u_{tt})-t^3\int_0^t \mathcal{B}_{ts}(t,s;\hat{u}(s),u_{tt})\,ds.$$

Hence, we arrive at

$$\begin{aligned}(2.18)\; t^3\|u_{tt}\|^2+\frac{1}{2}\frac{d}{d}\{t^3\mathcal{A}(t;u_t,u_t)\} &= \frac{3}{2}t^2\mathcal{A}(t;u_t,u_t)+\frac{1}{2}t^3\mathcal{A}_t(t;u_t,u_t)-t^3\mathcal{A}_t(t;u,u_{tt})\\ &\quad +t^3\mathcal{B}(t,t;u,u_{tt})+t^3\mathcal{B}_t(t,t,\hat{u}(t),u_{tt})-t^3\int_0^t \mathcal{B}_{ts}(t,s;\hat{u}(s),u_{tt})ds\end{aligned}$$

Use Cauchy-Schwarz inequality alongwith Young's inequality and then integrate with respect to time from 0 to t to obtain

$$t^3\mathcal{A}(t;u_t,u_t)+\int_0^t s^3\|u_{ss}(s)\|^2ds \le C\int_0^t \Big\{s^2(\|u_s(s)\|_1^2+\|u(s)\|_2^2)+\|\hat{u}(s)\|_2^2\Big\}\,ds.$$

Lemma 2.1 yields with the help of (2.1)

$$t^3\|u_t\|_1^2+\int_0^t s^3\|u_{ss}(s)\|^2ds \le C\|u_0\|^2. \tag{2.19}$$

Now differentiate (2.11) with respect to time to obtain

$$u_{ttt}+\mathcal{A}(t)u_{tt}+2\mathcal{A}_t(t)u_t-\mathcal{A}_{tt}u = B(t,t)u_t+2B(t,t)u+\int_0^t B_{tt}(t,s)u(s)\,ds. \tag{2.20}$$

Now multiply t^4u_{tt} in the above equation and integrate over Ω and rewrite it as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{t^4\|u_{tt}\|^2\}+t^4\mathcal{A}(t;u_{tt},u_{tt}) &= 2t^3\|u_{tt}\|^2-2t^4\mathcal{A}_t(t;u_t,u_{tt})-t^4\mathcal{A}_{tt}(t;u,u_{tt})\\ &\quad +t^4\mathcal{B}(t,t;u,u_{tt})+\ 2t^4\mathcal{B}_t(t,t;u_t,u_{tt})+t^4\int_0^t\mathcal{B}_{tt}(t,s;u(s),u_{tt})ds\\ &\le 2t^3\|u_{tt}\|^2+\varepsilon\|u_{tt}\|_1^2+C(\varepsilon)t^4\Big(\|u_t\|_1^2+\|u\|_1^2+\int_0 4\|u(s)\|_1^2\,ds\Big).\end{aligned}$$

Use (2.1) and then choose $\varepsilon=\rho_1/2$. Finally integrate and use (2.19) with Lemma 2.1 to conclude

$$t^4\|u_{tt}\|^2+\int_0^t s^4\|u_{ss}(s)\|_1^2ds \le C\|u_0\|^2. \tag{2.21}$$

Rewrite (2.11) as

$$\begin{aligned}\mathcal{A}(t)u_t &= u_{tt}-\mathcal{A}_t(t)u-B(t,t)u-\int_0^t B_t(t,s)u(s)ds\\ &= u_{tt}-\mathcal{A}_t(t)u-B(t,t)u-B_t(t,t)\hat{u}(t)+\int_0^t B_t(t,s)\hat{u}(s)ds.\end{aligned}$$

Using elliptic regularity (2.2), we arrive at

$$\|u_t\|_2^2 \ \le\ C\Big[\|u_{tt}\|^2+\|u\|_2^2+\|\hat{u}\|_2^2+\int_0^t\|\hat{u}(s)\|_2^2\,ds\Big]. \tag{2.22}$$

Multiply by t^4, use (2.21) and Lemma 2.1 to obtain

$$\|u_t\|_2 \ \le\ Ct^{-2}\|u_0\|, \tag{2.23}$$

and this completes the rest of the proof. □

3 Ritz-Volterra projection

In this section, we discuss Ritz-Volterra projection and the related error estimates which will prove useful for the proof of our main theorem.

Following Lin *et al.* [6] (see [1, 2]), define Ritz-Volterra projection $W_h : (0,T] \rightarrow S_h$ satisfying

$$\mathcal{A}(t;(u-W_hu)(t),\phi_h) = \int_0^t \mathcal{B}(t,s;(u-W_hu)(s),\phi_h)ds, \qquad \forall \phi_h \in S_h. \tag{3.1}$$

We refere to Cannon and Lin [2] and Lin *et al.* [6] to see that Ritz-Volterra projection is well defined. We also use the Ritz-projection $R_h = R_h(t) : H_0^1 \rightarrow S_h$ defined by

$$\mathcal{A}(t;u-R_hu,\phi_h) = 0, \quad \forall \phi_h \in S_h, \quad u \in H_0^1. \tag{3.2}$$

With $\theta = u - R_hu$, we discuss below some estimates for θ. For a proof, we refer to [8].

Lemma 3.1 *For θ as defined above and $u \in H_0^1 \cap H^2$ with $u_0 \in L^2$, there is a positive constant C independent of h such that for $k \in \{1,2\}$,and $j \in \{0,1\}$*

$$\begin{aligned} \|\theta\|_j &\le Ch^{k-j}\|u\|_k \le Ch^{k-j}t^{-k/2}\|u_0\|, \\ \|\theta_t\|_j &\le Ch^{k-j}\{\|u\|_k + \|u_t\|_k\} \le Ch^{k-j}t^{-(1+k/2)}\|u_0\|. \end{aligned}$$

Next, we present an estimate of $\hat{\theta}$.

Lemma 3.2 *For θ as defined above and $u \in H_0^1 \cap H^2$ with $u_0 \in L^2$, there exists a positive constant C such that for $k \in \{1,2\}$ and $j \in \{0,1\}$, the following estimate holds:*

$$\|\hat{\theta}\|_j \le Ch^{k-j}\|u_0\|.$$

Proof. We recall that for $\phi_h \in S_h$,

$$\mathcal{A}(t;\theta,\phi_h) = 0,$$

We integrate with respect to time to obtain

$$\mathcal{A}(t;\hat{\theta},\phi_h) - \int_0^t \mathcal{A}_s(s;\hat{\theta}(s),\phi_h)ds = 0. \tag{3.3}$$

Now we use the coercive property of $\mathcal{A}$ and (3.3) to find that

$$\begin{aligned} \rho_1\|\hat{\theta}\|_1^2 &\le \mathcal{A}(t;\hat{\theta},\hat{\theta}) = \mathcal{A}(t;\hat{\theta},\hat{u}-\chi) + \mathcal{A}(t,\hat{\theta},\chi - R_h\hat{u}), \quad \chi \in S_h, \\ &= \mathcal{A}(t;\hat{\theta},\hat{u}-\chi) + \int_0^t \mathcal{A}_s(s;\hat{\theta}(s),\chi - R_h\hat{u}), \\ &\le C\|\hat{\theta}\|_1\|\hat{u}-\chi\|_1 + \int_0^t \mathcal{A}_s(s;\hat{\theta}(s),(\chi-\hat{u})+\hat{\theta}), \\ &\le C\Big(\|\hat{\theta}\|_1 + \int_0^t \|\hat{\theta}\|_1\, ds\Big)\|\hat{u}-\chi\|_1 + C\Big(\int_0^t \|\hat{\theta}(s)\|_1\, ds\Big)\|\hat{\theta}\|_1. \end{aligned}$$

Using Young's inequality, we arrive at

$$\|\hat{\theta}\|_1^2 \;\le\; C\|\hat{u}-\chi\|_1^2 + C\int_0^t \|\hat{\theta}(s)\|_1^2 ds.$$

From (1.3) and (2.8), we obtain

$$\|\hat{\theta}\|_1^2 \;\le\; Ch^2\|u_0\|^2 + C\int_0^t \|\hat{\theta}(s)\|_1^2 ds.$$

Use Gronwall's lemma to conclude that

$$\|\hat{\theta}\|_1 \;\le\; Ch\|u_0\|. \tag{3.4}$$

For the L^2 estimate, we use Aubin-Nitche duality argument. Thus, we consider the following auxiliary problem. Find $\phi \in H_0^1(\Omega)$ such that

$$\mathcal{A}(t)\phi \;=\; \hat{\theta}, \quad \text{on } \Omega.$$

From the elliptic regularity result, we note that

$$\|\phi\|_2 \;\le\; C\|\hat{\theta}\|. \tag{3.5}$$

Now,

$$\begin{aligned}
\|\hat{\theta}\|^2 &= \mathcal{A}(t;\phi,\hat{\theta}),\\
&= \mathcal{A}(t;\phi-\chi,\hat{\theta}) + \mathcal{A}(t;\chi,\hat{\theta}) \qquad \text{for some } \chi\in S_h,\\
&= \mathcal{A}(t;\hat{\theta},\phi-\chi) + \int_0^t \mathcal{A}_s(s;\hat{\theta}(s),\chi)\,ds \qquad \text{from (3.3)},\\
&= \mathcal{A}(t;\hat{\theta},\phi-\chi) + \int_0^t \mathcal{A}_s(s;\hat{\theta}(s),\chi-\phi)\,ds + \int_0^t \mathcal{A}_s(s;\hat{\theta}(s),\phi)ds.\\
&\le C\Big(\|\hat{\theta}\|_1 + \int_0^t \|\hat{\theta}(s)\|_1\,ds\Big)\|\phi-\chi\|_1 + C\Big(\int_0^t \|\hat{\theta}(s)\|\,ds\Big)\|\phi\|_2.
\end{aligned}$$

Use approximation property (1.3) of S_h and then (3.4) and (3.5) to obtain

$$\|\hat{\theta}\| \;\le\; Ch^2\|u_0\| + \int_0^t \|\hat{\theta}(s)\|ds.$$

A use of Gronwall's lemma completes the rest of the proof. □

In the rest of this section, we will prove the estimates of $\eta = u - W_h u$. Using Ritz projection, we set $\eta = \theta - \rho$, where $\theta = u - R_h u$ and $\rho = W_h u - R_h u$. Hence, we now rewrite (3.1) using (3.2) as

$$\mathcal{A}(t;\rho,\phi_h) = \int_0^t \mathcal{B}(t,s;\rho(s),\phi_h)ds - \int_0^t \mathcal{B}(t,s;\theta(s),\phi_h)ds, \quad \forall\phi_h\in S_h. \tag{3.6}$$

Moreover, using integration by parts in time as we again rewrite (3.6) in the following form:

$$\begin{aligned}
\mathcal{A}(t;\rho,\phi_h) &= \mathcal{B}(t,t;\hat{\rho}(t),\phi_h) - \int_0^t \mathcal{B}_s(t,s;\hat{\rho}(s),\phi_h)ds\\
&- \mathcal{B}(t,t;\hat{\theta}(t),\phi_h) + \int_0^t \mathcal{B}_s(t,s;\hat{\theta}(s),\phi_h)ds, \quad \forall\phi_h\in S_h.
\end{aligned} \tag{3.7}$$

Now, we reperesent

Lemma 3.3 *For η as defined above and $u(t) \in H_0^1 \cap H^2$ with $u_0 \in L^2$, there exists a positive constant C independent of h such that for $k \in \{1,2\}$, $j \in \{0,1\}$, the following estimates hold:*

$$\|\eta\|_j \;\leq\; Ch^{k-j}t^{-k/2}\|u_0\|, \tag{3.8}$$

$$\|\hat{\eta}\|_j \;\leq\; Ch^{k-j}\|u_0\|. \tag{3.9}$$

Proof. Set $\phi_h = \rho$ in (3.7) to obtain

$$\mathcal{A}(t;\rho,\rho) = \mathcal{B}(t,t;\hat{\rho},\rho) - \int_0^t \mathcal{B}_s(t,s;\hat{\rho}(s),\rho)ds - \mathcal{B}(t,t;\hat{\theta},\rho) + \int_0^t \mathcal{B}_s(t,s;\hat{\theta}(s),\rho)ds. \tag{3.10}$$

A use of coercivity of $\mathcal{A}$ with Cauchy-Schwarz inequality yields

$$\|\rho\|_1 \;\leq\; C\Big(\|\hat{\rho}\|_1 + \int_0^t \|\hat{\rho}(s)\|_1 ds + \|\hat{\theta}\|_1 + \int_0^t \|\hat{\theta}(s)\|_1 ds\Big). \tag{3.11}$$

For finding $\|\hat{\rho}\|_1$, we integrate (3.7) and obtain

$$\begin{aligned}\mathcal{A}(t;\hat{\rho},\phi_h) - \int_0^t \mathcal{A}_s(s;\hat{\rho}(s),\phi_h)ds &= \int_0^t \mathcal{B}(s,s;\hat{\rho}(s),\phi_h)ds - \int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat{\rho}(\tau),\phi_h)d\tau ds \\ &\quad - \int_0^t \mathcal{B}(s,s;\hat{\theta}(s),\phi_h)ds + \int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat{\theta}(\tau),\phi_h)d\tau ds.\end{aligned}$$

Choose $\phi_h = \hat{\rho}$, apply Cauch-Schwarz inequality and coercivity of $\mathcal{A}$ to obtain

$$\|\hat{\rho}\|_1 \;\leq\; C\Big(\int_0^t \|\hat{\theta}(s)\|_1 ds + \int_0^t \|\hat{\rho}(s)\|_1 ds\Big). \tag{3.12}$$

Now an application of Lemma 3.2 yields

$$\|\hat{\rho}\|_1 \;\leq\; Ch^{k-1}\|u_0\| + C\int_0^t \|\hat{\rho}(s)\|_1 ds. \tag{3.13}$$

Apply Gronwall's Lemma to arrive at

$$\|\hat{\rho}\|_1 \;\leq\; Ch^{k-1}\|u_0\|. \tag{3.14}$$

Using Lemma 3.2 and triangle inequality, we obtain

$$\|\hat{\eta}\|_1 \;\leq\; \|\hat{\theta}\|_1 + \|\hat{\rho}\|_1 \leq Ch^{k-1}\|u_0\|. \tag{3.15}$$

Now substitute estimate of $\|\hat{\rho}\|_1$ from (3.14) in (3.11) and use Lemma 3.2 to arrive at

$$\|\rho\|_1 \;\leq\; Ch^{k-1}\|u_0\|. \tag{3.16}$$

Again use triangle inequality and Lemma 3.2 to obtain

$$\|\eta\|_1 \;\leq\; \|\theta\|_1 + \|\rho\|_1 \leq Ch^{k-1}t^{-k/2}\|u_0\|. \tag{3.17}$$

Now consider the following auxiliary problem:

$$\mathcal{A}(t)\zeta \;=\; \hat{\eta} \quad \text{in } \Omega, \tag{3.18}$$

$$\zeta \;=\; 0 \quad \text{on } \partial\Omega, \tag{3.19}$$

where ζ satisfies the regularity result

$$\|\zeta\|_2 \leq C\|\hat{\eta}\|. \tag{3.20}$$

Now

$$\begin{aligned} \|\hat{\eta}\|^2 &= \mathcal{A}(t;\zeta,\hat{\eta}) \\ &= \mathcal{A}(t;\zeta-\chi,\hat{\eta}) + \mathcal{A}(t;\chi,\hat{\eta}) \quad \text{for some } \chi \in S_h. \end{aligned} \tag{3.21}$$

On integration (3.1) and using the fact that $\frac{d}{dt}\hat{\phi}(t) = \phi(t)$, we obtain

$$\begin{aligned} \mathcal{A}(t;\hat{\eta},\phi_h) - \int_0^t \mathcal{A}_s(s;\hat{\eta}(s),\phi_h)ds &= \int_0^t \mathcal{B}(s,s;\hat{\eta}(s),\phi_h)ds \\ &- \int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat{\eta}(\tau),\phi_h)d\tau ds \end{aligned} \tag{3.22}$$

We obtain on substituting (3.22) with $\phi_h = \chi$ in (3.21)

$$\begin{aligned} \|\hat{\eta}\|^2 &= \mathcal{A}(t;\zeta-\chi,\hat{\eta}) + \int_0^t \mathcal{A}_s(s;\hat{\eta}(s),\chi)ds + \int_0^t \mathcal{B}(s,s;\hat{\eta}(s),\chi)\,ds \\ &- \int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat{\eta}(\tau),\chi)d\tau ds \\ &= \mathcal{A}(t;\zeta-\chi,\hat{\eta}) - \int_0^t \mathcal{A}_s(s;\hat{\eta}(s),\zeta-\chi)ds - \int_0^t \mathcal{B}(s,s;\hat{\eta}(s),\zeta-\chi)\,ds \\ &+ \int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat{\eta}(\tau),\zeta-\chi)\,d\tau ds + \int_0^t (\hat{\eta}(s),\mathcal{A}_s^*(s)\zeta)\,ds \\ &+ \int_0^t (\hat{\eta}(s),\mathcal{B}^*(s,s)\zeta)\,ds - \int_0^t\int_0^s (\hat{\eta}(\tau),\mathcal{B}_\tau^*(s,\tau)\zeta)\,d\tau ds \\ &\leq C\Big(\|\hat{\eta}\|_1 + \int_0^t \|\hat{\eta}(s)\|_1\,ds\Big)\|\zeta-\chi\|_1 + C\Big(\int_0^t \|\hat{\eta}(s)\|\,ds\Big)\|\zeta\|_2. \end{aligned} \tag{3.23}$$

Here, $\mathcal{A}_t^*$, $\mathcal{B}^*(t,t)$, and $\mathcal{B}_s^*(t,s)$ are formal adjoint of $\mathcal{A}_t$, $\mathcal{B}(t,t)$, and $\mathcal{B}_s(t,s)$, respectively. Using the approximation property (1.3) for S_h, we obtain from (3.17) and (3.15)

$$\|\hat{\eta}\| \leq Ch^2\|u_0\| + \int_0^t \|\hat{\eta}(s)\|\,ds. \tag{3.24}$$

Use Gronwall's lemma to obtain the desired estimate for $\|\hat{\eta}\|$.

For finding the estimate of $\|\eta\|$, we again consider the auxiliary problem (3.18)-(3.19) by replacing the right hand side function $\hat{\eta}$ by η. Then, we proceed similarly as in the estimate of $\|\hat{\eta}\|$ and obtain

$$\|\eta\| \leq Ch\Big(\|\eta\|_1 + \|\hat{\eta}\|_1 + \int_0^t \|\hat{\eta}(s)\|_1\,ds\Big) + C\Big(\|\hat{\eta}\| + + \int_0^t \|\hat{\eta}(s)\|\,ds\Big). \tag{3.25}$$

A use of (3.9)with (3.17) yields the desired estimate of $\|\eta\|$ and this completes the rest of the proof. □

Below, we discuss the estimate of $\|\eta_t\|$.

Lemma 3.4 *For η as defined by (3.1) and $u_0 \in L^2$, let both $u(t)$ and $u_t(t)$ be in $H_0^1 \cap H^2$, for $t \in J$. Then there is a positive constantC independent of h such that*

$$\|\eta_t\| \;\leq\; Ch^2 t^{-2}\|u_0\|.$$

Proof. To find the estimate of $\|\eta_t\|$, first we obtain an estimate for $\|\rho_t\|_1$. Now differentiate (3.6) with respect to time to arrive at

$$\begin{aligned}\mathcal{A}(t;\rho_t,\phi_h)+\mathcal{A}_t(t;\rho,\phi_h) &= \mathcal{B}(t,t;\rho,\phi_h)+\int_0^t \mathcal{B}_t(t,s;\rho(s),\phi_h)\,ds\\ &- \mathcal{B}(t,t;\theta,\phi_h)-\int_0^t \mathcal{B}_t(t,s;\theta(s),\phi_h)\,ds.\end{aligned} \tag{3.26}$$

Using integration by parts in time, we rewrite (3.26) as

$$\begin{aligned}\mathcal{A}(t;\rho_t,\phi_h)+\mathcal{A}_t(t;\rho,\phi_h) &= \mathcal{B}(t,t;\rho,\phi_h)+\mathcal{B}_t(t,t;\hat{\rho},\phi_h)-\int_0^t \mathcal{B}_{ts}(t,s;\hat{\rho}(s),\phi_h)\,ds\\ &- \mathcal{B}(t,t;\theta,\phi_h)-\mathcal{B}_t(t,t;\hat{\theta}(s),\phi_h)+\int_0^t \mathcal{B}_{ts}(t,s;\hat{\theta}(s),\phi_h)\,ds.\end{aligned} \tag{3.27}$$

Choose $\phi_h=\rho_t$ in (3.27) to obtain

$$\begin{aligned}\mathcal{A}(t;\rho_t,\rho_t) &= -\mathcal{A}_t(t;\rho,\rho_t)+\mathcal{B}(t,t;\rho,\rho_t)+\mathcal{B}_t(t,t;\hat{\rho},\rho_t)-\int_0^t \mathcal{B}_{ts}(t,s;\hat{\rho}(s),\rho_t)\,ds\\ &- \mathcal{B}(t,t;\theta,\rho_t)-\mathcal{B}_t(t,t;\hat{\theta},\rho_t)+\int_0^t \mathcal{B}_{ts}(t,s;\hat{\theta}(s),\rho_t)\,ds.\end{aligned}$$

Using (2.1), smoothness of coefficients of $\mathcal{A}(t)$ and $B(t,s)$, we obtain

$$\|\rho_t\|_1 \;\leq\; C\Big(\|\rho\|_1+\|\hat{\rho}\|_1+\|\theta\|_1+\|\hat{\theta}\|_1+\int_0^t(\|\hat{\rho}(s)\|_1+\|\hat{\theta}(s)\|_1)ds\Big). \tag{3.28}$$

Use of (2.1), Lemma 3.1, Lemma 3.2 and (3.14) help us to arrive at

$$\|\rho_t\|_1 \;\leq\; Cht^{-1/2}\|u_0\|. \tag{3.29}$$

Hence use of triangle inequality and Lemma 3.1 yield

$$\|\eta_t\|_1 \;\leq\; \|\theta_t\|_1+\|\rho_t\|_1 \leq Cht^{-2}\|u_0\|. \tag{3.30}$$

For L^2-estimate, we again consider the following auxiliary problem:

$$\mathcal{A}(t)\phi \;=\; \eta_t \qquad \text{in } \;\Omega, \tag{3.31}$$

$$\phi \;=\; 0 \qquad \text{on } \;\partial\Omega, \tag{3.32}$$

with ϕ satisfying the regularity result

$$\|\phi\|_2 \;\leq\; C\|\eta_t\|. \tag{3.33}$$

Now,

$$\begin{aligned}\|\eta_t\|^2 &= \mathcal{A}(t;\phi,\eta_t)\\ &= \mathcal{A}(t;\eta_t,\phi-\chi)+\mathcal{A}(t;\eta_t,\chi) \quad \text{for} \quad \chi\in S_h\\ &= \mathcal{A}(t;\eta_t,\phi-\chi)-\mathcal{A}_t(t;\eta,\chi)+\mathcal{B}(t,t,\eta,\chi)+\int_0^t \mathcal{B}_t(t,s;\eta(s),\chi)\,ds.\end{aligned}$$

Note that we have differentiated (3.1) with respect to time and then substituted the value of $\mathcal{A}(t;\eta_t,\chi)$. Note that

$$\begin{aligned}
\|\eta_t\|^2 &= \mathcal{A}(t;\eta_t,\phi-\chi)+\mathcal{A}_t(t;\eta_t,\phi-\chi)-\mathcal{A}_t(t;\eta,\phi)-\mathcal{B}(t,t,\eta,\phi-\chi)\\
&- \mathcal{B}(t,t,\eta,\phi)-\int_0^t \mathcal{B}_t(t,s;\eta(s),\phi-\chi)\,ds+\int_0^t \mathcal{B}_t(t,s;\eta(s),\phi)\,ds\\
&= \mathcal{A}(t;\eta_t,\phi-\chi)+\mathcal{A}_t(t;\eta_t,\phi-\chi)-\mathcal{A}_t(t;\eta_t,\phi)-\mathcal{B}(t,t,\eta,\phi-\chi)\\
&- \mathcal{B}(t,t,\eta,\phi)-\mathcal{B}_t(t,t;\hat{\eta}(s),\phi-\chi)+\int_0^t \mathcal{B}_{ts}(t,s;\hat{\eta}(s),\phi-\chi)ds\\
&+ \mathcal{B}_t(t,s;\hat{\eta}(t),\phi)-\int_0^t \mathcal{B}_{ts}(t,s;\hat{\eta}(s),\phi)\,ds.
\end{aligned}$$

Using the smoothness of coefficients of $\mathcal{A}(t)$ and $B(t,s)$, we obtain

$$\begin{aligned}
\|\eta_t\|^2 &\leq C\Big(\|\eta_t\|_1+\|\eta\|_1+\|\hat{\eta}\|_1+\int_0^t \|\hat{\eta}(s)\|_1\,ds\Big)\|\phi-\chi\|_1\\
&+ C\Big(\|\eta\|+\|\hat{\eta}\|+\int_0^t \|\hat{\eta}(s)\|_1\,ds\Big)\|\phi\|_2.
\end{aligned} \tag{3.34}$$

Using approximation property (1.3), Lemma 3.3 and (3.30), it follows that

$$\|\eta_t\| \leq Ch^2t^{-2}\|u_0\|. \tag{3.35}$$

This completes the rest of the proof. □

4 Semidiscrete Error Estimates for Non-smooth Data

In this section, we discuss the proof of our main theorem that is Theorem 1.1.

Observe that $e=u-u_h$ satisfies the following equation

$$(e_t,\phi_h)+\mathcal{A}(t;e,\phi_h)=\int_0^t B(t,s;e(s),\phi_h)ds, \quad \forall\ \phi_h\in S_h,\quad t>0. \tag{4.1}$$

Using Ritz-Volterra projection W_hu of u, we rewrite

$$e=u-u_h=(u-W_hu)-(u_h-W_hu)=:\eta-\xi. \tag{4.2}$$

Using (3.1), the equation (4.1) can be written as

$$(\xi_t,\phi_h)+\mathcal{A}(t;\xi,\phi_h) = (\eta_t,\phi_h)+\int_0^t \mathcal{B}(t,s;\xi(s),\phi_h)ds, \quad \forall\ \phi_h\in S_h. \tag{4.3}$$

With $u_h(0)=P_hu_0$, we now integrate (4.3) once and twice to obtain

$$\begin{aligned}
(\xi,\phi_h)+\mathcal{A}(t;\hat{\xi},\phi_h)-\int_0^t \mathcal{A}_s(s;\hat{\xi}(s),\phi_h)ds &= (\eta,\phi_h)+\int_0^t \mathcal{B}(s,s;\hat{\xi}(s),\phi_h)ds\\
&-\int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat{\xi}(\tau),\phi_h)d\tau\,ds, \quad \forall\ \phi_h\in S_h,
\end{aligned} \tag{4.4}$$

and

$$(4.5)\qquad (\hat{\xi},\phi_h) + \mathcal{A}(t;\hat{\hat{\xi}},\phi_h) - 2\int_0^t \mathcal{A}_s(s;\hat{\hat{\xi}}(s),\phi_h)ds + \int_0^t\int_0^s \mathcal{A}_{\tau\tau}(\tau;\hat{\hat{\xi}}(\tau),\phi_h)\,d\tau\,ds$$
$$= (\hat{\eta},\phi_h) + \int_0^t \mathcal{B}(s,s;\hat{\hat{\xi}}(s),\phi_h)\,ds - 2\int_0^t\int_0^s \mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\phi_h)\,d\tau ds$$
$$+ \int_0^t\int_0^s\int_0^\tau \mathcal{B}_{\tau\tau'}(\tau,\tau';\hat{\hat{\xi}}(\tau'),\phi_h)\,d\tau' d\tau\,ds,\quad \forall\,\phi_h\in S_h,$$

respectively. Below, we prove a series of Lemmas involving estimates of ξ.

Lemma 4.1 *Let $\hat{\hat{\xi}}$ satisfy (4.5). Then, there exists a positive constant C independent of h such that*

$$(4.6)\qquad \|\hat{\hat{\xi}}\|^2 + \int_0^t \|\hat{\hat{\xi}}(s)\|_1^2 ds \;\le\; Cth^4\|u_0\|^2,$$

and

$$(4.7)\qquad \|\hat{\hat{\xi}}\|_1^2 + \int_0^t \|\hat{\xi}(s)\|^2\,ds \;\le\; Cth^4\|u_0\|^2.$$

Proof. Choose $\phi_h = \hat{\hat{\xi}}(t)$ in (4.5) to obtain

$$(4.8)\quad \frac{1}{2}\frac{d}{dt}\|\hat{\hat{\xi}}\|^2 + \mathcal{A}(t;\hat{\hat{\xi}},\hat{\hat{\xi}}) = 2\int_0^t \mathcal{A}_s(s;\hat{\hat{\xi}}(s),\hat{\hat{\xi}})ds - \int_0^t\int_0^s \mathcal{A}_{\tau\tau}(\tau;\hat{\hat{\xi}}(\tau),\hat{\hat{\xi}})d\tau\,ds$$
$$+ (\hat{\eta},\hat{\hat{\xi}}) + \int_0^t \mathcal{B}(s,s;\hat{\hat{\xi}}(s),\hat{\hat{\xi}})ds - 2\int_0^t\int_0^s \mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\hat{\hat{\xi}})d\tau ds$$
$$+ \int_0^t\int_0^s\int_0^\tau \mathcal{B}_{\tau\tau'}(\tau,\tau';\hat{\hat{\xi}}(\tau'),\hat{\hat{\xi}})\,d\tau' d\tau\,ds,$$

and hence, using Cauchy-Schwarz inequality alongwith Yong's inequality, we integrate the resulting inequality to arrive at

$$\|\hat{\hat{\xi}}\|^2 + \int_0^t \|\hat{\hat{\xi}}(s)\|_1^2\,ds \;\le\; C\int_0^t \|\hat{\eta}(s)\|^2\,ds + C\int_0^t \left(\|\hat{\hat{\xi}}(s)\|^2 + \int_0^s \|\hat{\hat{\xi}}(\tau)\|_1^2\,d\tau\right)\,ds.$$

Use Lemma 3.3 and then apply Gronwall's lemma to conclude the estimate (4.6).

To estimate (4.7), set $\phi_h = \hat{\xi}(t)$ in (4.5) to obtain

$$(4.9)\|\hat{\xi}\|^2 + \mathcal{A}(t;\hat{\hat{\xi}},\hat{\xi}) = 2\int_0^t \mathcal{A}_s(s;\hat{\hat{\xi}}(s),\hat{\xi})ds - \int_0^t\int_0^s \mathcal{A}_{\tau\tau}(\tau;\hat{\hat{\xi}}(\tau),\hat{\xi})\,d\tau\,ds$$
$$+ (\hat{\eta},\hat{\xi}) - \int_0^t \mathcal{B}(s,s;\hat{\hat{\xi}}(s),\hat{\xi})\,ds + 2\int_0^t\int_0^s \mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi})\,d\tau\,ds$$
$$+ \int_0^t\int_0^s\int_0^\tau \mathcal{B}_{\tau\tau'}(\tau,\tau';\hat{\hat{\xi}}(\tau'),\hat{\xi})\,d\tau'\,d\tau\,ds.$$

Since

$$\frac{d}{dt}\mathcal{A}(t;\hat{\hat{\xi}},\hat{\hat{\xi}}) = \mathcal{A}_t(t;\hat{\hat{\xi}},\hat{\hat{\xi}}) + 2\mathcal{A}(t;\hat{\hat{\xi}},\hat{\xi}),$$

we rewrite (4.9) as

$$\begin{aligned}
\|\hat{\xi}\|^2 &+ \frac{1}{2}\frac{d}{dt}\mathcal{A}(t;\hat{\hat{\xi}},\hat{\hat{\xi}}) = \frac{1}{2}\mathcal{A}_t(t;\hat{\hat{\xi}},\hat{\hat{\xi}}) + (\hat{\eta},\hat{\xi}) - 2\mathcal{A}(t;\hat{\hat{\xi}},\hat{\hat{\xi}}) + 2\frac{d}{dt}\left(\int_0^t \mathcal{A}(s;\hat{\hat{\xi}}(s),\hat{\hat{\xi}})ds\right) \\
(4.10) \quad &+ \mathcal{B}(t,t;\hat{\hat{\xi}},\hat{\hat{\xi}}) - \frac{d}{dt}\left(\int_0^t \mathcal{B}(s,s;\hat{\hat{\xi}}(s),\hat{\hat{\xi}})ds\right) + \int_0^t \mathcal{A}_{ss}(s;\hat{\hat{\xi}}(s),\hat{\hat{\xi}})ds \\
&- 2\int_0^t \mathcal{B}_s(s,s;\hat{\hat{\xi}}(s),\hat{\hat{\xi}})ds - \frac{d}{d}\left(\int_0^t\int_0^s \left(\mathcal{A}_{\tau\tau}(\tau;\hat{\hat{\xi}}(\tau),\hat{\hat{\xi}}) - 2\mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\hat{\hat{\xi}})\right)d\tau\, ds\right) \\
&+ \frac{d}{dt}\left(\int_0^t\int_0^s\int_0^\tau \mathcal{B}_{\tau\tau'}(\tau,\tau';\hat{\hat{\xi}}(\tau'),\hat{\hat{\xi}})d\tau' d\tau\, ds\right) - \int_0^t\int_0^s \mathcal{B}_{s\tau}(s,\tau;\hat{\hat{\xi}}(\tau),\hat{\hat{\xi}})d\tau ds
\end{aligned}$$

Integrate (4.10) with respect to time, use coecivity property (2.1) with smoothness of coefficients of $\mathcal{A}(t)$ and $B(t,s)$. Then, an application of Cauchy-Schwarz inequality with Young's inequality yields

$$\int_0^t \|\hat{\xi}(s)\|^2\, ds + \|\hat{\hat{\xi}}(t)\|_1^2 \;\leq\; C\int_0^t \|\hat{\eta}\|^2 ds + C\int_0^t \|\hat{\hat{\xi}}(s)\|_1^2 ds.$$

Using (4.6) and Lemma 3.3, we obtain

$$\|\hat{\hat{\xi}}\|_1^2 + \int_0^t \|\hat{\xi}(s)\|^2 ds \;\leq\; Cth^4\|u_0\|^2, \tag{4.11}$$

and this completes the rest of the proof. □

Lemma 4.2 *Let $\hat{\xi}$ be such that it satisfies the equation (4.4). Then, there exists a positive constant C independent of h such that*

$$t\|\hat{\xi}\|^2 + \int_0^t s\|\hat{\xi}(s)\|_1^2\, ds \;\leq\; Cth^4\|u_0\|^2, \tag{4.12}$$

and

$$t^2\|\hat{\xi}\|_1^2 + \int_0^t s^2\|\xi\|^2 ds \;\leq\; Cth^4\|u_0\|^2. \tag{4.13}$$

Proof. Choose $\phi_h = t\hat{\xi}(t)$ in (4.4) to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\{t\|\hat{\xi}\|^2\} + t\mathcal{A}(t;\hat{\xi},\hat{\xi}) &= \frac{1}{2}\|\hat{\xi}\|^2 + t\int_0^t \mathcal{A}_s(s;\hat{\xi}(s),\hat{\xi})\, ds + t(\eta,\hat{\xi}) + t\int_0^t \mathcal{B}(s,s;\hat{\xi}(s),\hat{\xi})\, ds \\
&\quad -t\int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat{\xi}(\tau),\hat{\xi})\, d\tau\, ds,
\end{aligned}$$

and an application of integration by parts with respect to time t yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\{t\|\hat{\xi}\|^2\} + t\mathcal{A}(t;\hat{\xi},\hat{\xi}) &= \frac{1}{2}\|\hat{\xi}\|^2 + t\mathcal{A}_t(t;\hat{\hat{\xi}}(t),\hat{\xi}) - t\int_0^t \mathcal{A}_{ss}(s;\hat{\hat{\xi}}(s),\hat{\xi})\, ds + t(\eta,\hat{\xi}) \\
(4.14) \quad &\quad +t\mathcal{B}(t,t;\hat{\hat{\xi}}(t),\hat{\xi}) - 2t\int_0^t \mathcal{B}_s(s,s;\hat{\hat{\xi}}(s),\hat{\xi})ds \\
&\quad +t\int_0^t\int_0^s \mathcal{B}_{\tau\tau}(s,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi})d\tau ds.
\end{aligned}$$

Now integrate (4.14) with respect to time to obtain

$$\begin{aligned}
\frac{1}{2}t\|\hat{\xi}\|^2 &+ \int_0^t s\mathcal{A}(s;\hat{\xi}(s),\hat{\xi}(s))\,ds = \frac{1}{2}\int_0^t \|\hat{\xi}(s)\|^2 ds + \int_0^t s\mathcal{A}_s(s;\hat{\hat{\xi}}(s),\hat{\xi}(s))\,ds \\
&- \int_0^t \int_0^s s\mathcal{A}_{\tau\tau}(\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s))\,d\tau\,ds + \int_0^t s(\eta,\hat{\xi})\,ds + \int_0^t s\mathcal{B}(s,s;\hat{\hat{\xi}}(s),\hat{\xi}(s))\,ds \\
&- 2\int_0^t \int_0^s s\mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s))\,ds + \int_0^t \int_0^s \int_0^\tau s\mathcal{B}_{\tau'\tau'}(\tau,\tau';\hat{\hat{\xi}}(\tau'),\hat{\xi}(s))\,d\tau'\,d\tau\,ds.
\end{aligned}$$

Therefore, using coercive property (2.1), Cauchy-Schwarz inequality and Young's inequality, it follows that

$$t\|\hat{\xi}\|^2 + \int_0^t s\|\hat{\xi}(s)\|_1^2\,ds \;\le\; C\int_0^t (\|\hat{\xi}(s)\|^2 + s^2\|\eta(s)\|^2 + \|\hat{\hat{\xi}}(s)\|_1^2)\,ds.$$

Now, use Lemma 3.3 and (4.7) to arrive at (4.12).

Next to estimate (4.13), set $\phi_h = t^2\xi(t)$ in (4.4) to obtain

$$t^2\|\xi\|^2 + t^2\mathcal{A}(t;\hat{\xi},\xi) - t^2\int_0^t \mathcal{A}_s(s;\hat{\xi}(s),\xi)\,ds = t^2(\eta,\xi) + \int_0^t \mathcal{B}(s,s;\hat{\xi}(s),t^2\xi)\,ds.$$

Note that

$$\frac{1}{2}\frac{d}{dt}\{t^2\mathcal{A}(t;\hat{\xi},\hat{\xi})\} = t\mathcal{A}(t;\hat{\xi},\hat{\xi}) + \frac{t^2}{2}\mathcal{A}_t(t;\hat{\xi},\hat{\xi}) + t^2\mathcal{A}(t;\hat{\xi},\xi),$$

and hence,

$$\begin{aligned}
t^2\|\xi\|^2 + \frac{1}{2}\frac{d}{dt}\{t^2\mathcal{A}(\hat{\xi},\hat{\xi})\} &= t\mathcal{A}(t;\hat{\xi},\hat{\xi}) + \frac{t^2}{2}\mathcal{A}_t(t;\hat{\xi},\hat{\xi}) + t^2\int_0^t \mathcal{A}_s(s;\hat{\xi}(s),\xi)ds + t^2(\eta,\xi) \\
&+ t^2\int_0^t \mathcal{B}(s,s;\hat{\xi}(s),\xi)\,ds - t^2\int_0^t \int_0^s \mathcal{B}_\tau(s,\tau;\hat{\xi}(\tau),\xi)\,d\tau\,ds.
\end{aligned}$$

Integrate the above equation with respect to time, from 0 to t and then rewrite the resulting equation as

$$\begin{aligned}
(4.15)\int_0^t s^2\|\xi(s)\|^2 ds &+ \frac{1}{2}t^2\mathcal{A}(t;\hat{\xi},\hat{\xi}) = \int_0^t \Big(s\mathcal{A}(s;\hat{\xi}(s),\hat{\xi}(s)) + \frac{s^2}{2}\mathcal{A}_s(s;\hat{\xi}(s),\hat{\xi}(s)) \\
&+ s^2(\eta(s),\xi(s)) - s^2\mathcal{A}_s(s;\hat{\xi}(s),\hat{\xi}(s)) - s^2\mathcal{B}(s,s;\hat{\xi}(s),\hat{\xi}(s))\Big)\,ds \\
&+ t^2\int_0^t \Big(\mathcal{A}_s(s;\hat{\xi}(s),\hat{\xi}(t)) + \mathcal{B}(s,s;\hat{\xi}(s),\hat{\xi}(t)\Big)\,ds \\
&- 2\int_0^t \int_0^s s\,\Big(\mathcal{A}_\tau(\tau;\hat{\xi}(\tau),\hat{\xi}(s)) + \mathcal{B}(\tau,\tau;\hat{\xi}(\tau),\hat{\xi}(s))\Big)\,d\tau\,ds \\
&- t^2\int_0^t \int_0^s \mathcal{B}_\tau(s,\tau;\hat{\xi}(\tau),\hat{\xi}(t))\,d\tau\,ds \\
&+ \int_0^t \int_0^s s^2\mathcal{B}_\tau(s,\tau;\hat{\xi}(\tau),\hat{\xi}(s))\,d\tau\,ds \\
&+ 2\int_0^t \int_0^s \int_0^\tau s\mathcal{B}_{\tau'}(\tau,\tau';\hat{\xi}(\tau'),\hat{\xi}(s))\,d\tau'\,d\tau\,ds \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

To estimate I_1 on the right hand side of (4.15), we obtain

$$|I_1| \le \frac{1}{2}\int_0^t s^2\|\xi(s)\|^2\,ds + \frac{1}{2}\int_0^t s^2\|\eta(s)\|^2\,ds + C\int_0^t \|\hat{\xi}(s)\|_1^2\,ds.$$

For I_2, use integration by parts to rewrite it as

$$\begin{aligned} I_2 &= t^2\int_0^t \Big(\mathcal{A}_s(s;\hat{\xi}(s),\hat{\xi}(t)) + \mathcal{B}(s,s;\hat{\xi}(s),\hat{\xi}(t)\Big)ds \\ &= t^2\Big(\mathcal{A}_t(t;\hat{\hat{\xi}}(t),\hat{\xi}(t)) + \mathcal{B}(t,t;\hat{\hat{\xi}}(t),\hat{\xi}(t)\Big) \\ &- t^2\int_0^t \Big(\mathcal{A}_{ss}(s;\hat{\hat{\xi}}(s),\hat{\xi}(t)) + \mathcal{B}_s(s,s;\hat{\hat{\xi}}(s),\hat{\xi}(t)\Big)\,ds, \end{aligned}$$

and hence,

$$|I_2| \le \frac{\rho_1}{4}t^2\|\hat{\xi}\|_1^2 + C\Big(\|\hat{\hat{\xi}}(t)\|_1^2 + \int_0^t \|\hat{\hat{\xi}}(s)\|_1^2\,ds\Big).$$

For I_4, we again use integration by parts in time to obtain

$$\begin{aligned} I_4 &= t^2\int_0^t\int_0^s \mathcal{B}_\tau(s,\tau;\hat{\xi}(\tau),\hat{\xi}(t))d\tau ds \\ &= t^2\int_0^t \mathcal{B}_s(s,s;\hat{\hat{\xi}}(s),\hat{\xi}(t))ds - t^2\int_0^t\int_0^s \mathcal{B}_{\tau\tau}(s,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(t))\,d\tau\,ds, \end{aligned}$$

and hence, estimate it as

$$|I_4| \le \frac{\rho_1}{4}t^2\|\hat{\xi}\|_1^2 + C\int_0^t \|\hat{\hat{\xi}}(s)\|_1^2\,ds.$$

Similarly, we can rewrite I_3, I_5 and I_6 respectively as:

$$\begin{aligned} I_3 &= 2\int_0^t\int_0^s s\Big(\mathcal{A}_\tau(\tau;\hat{\xi}(\tau),\hat{\xi}(s)) + \mathcal{B}(\tau,\tau;\hat{\xi}(\tau),\hat{\xi}(s))\Big)\,d\tau ds \\ &= 2\int_0^t s\Big(\mathcal{A}_s(s;\hat{\hat{\xi}}(s),\hat{\xi}(s)) + \mathcal{B}(s,s;\hat{\hat{\xi}}(s),\hat{\xi}(s))\Big)\,ds \\ &- 2\int_0^t\int_0^s s\Big(\mathcal{A}_{\tau\tau}(\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s)) + \mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s))\Big)\,d\tau\,ds, \end{aligned}$$

$$\begin{aligned} I_5 &= \int_0^t\int_0^s s^2\mathcal{B}_\tau(s,\tau;\hat{\xi}(\tau),\hat{\xi}(s))\,d\tau\,ds \\ &= \int_0^t s^2\mathcal{B}_s(s,s;\hat{\hat{\xi}}(s),\hat{\xi}(s))\,ds - \int_0^t\int_0^s s^2\mathcal{B}_{\tau\tau}(s,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s))\,d\tau\,ds, \end{aligned}$$

and

$$\begin{aligned} I_6 &= 2\int_0^t\int_0^s\int_0^\tau s\mathcal{B}_{\tau'}(\tau,\tau';\hat{\xi}(\tau'),\hat{\xi}(s))\,d\tau'\,d\tau\,ds \\ &= 2\int_0^t\int_0^s s\mathcal{B}_\tau(\tau,\tau;\hat{\hat{\xi}}(\tau),\hat{\xi}(s))d\tau ds - 2\int_0^t\int_0^s\int_0^\tau s\mathcal{B}_{\tau'\tau'}(\tau,\tau';\hat{\hat{\xi}}(\tau'),\hat{\xi}(s))\,d\tau'\,d\tau\,ds. \end{aligned}$$

Thus, we obtain

$$|I_3| + |I_5| + |I_6| \leq C \int_0^t s\|\hat{\xi}(s)\|_1^2\, ds + C \int_0^t \|\hat{\hat{\xi}}(s)\|_1^2\, ds.$$

Substituting the estimates of I_1 to I_6 in (4.15), a use of coercive property (2.1) yields

$$\begin{aligned} \int_0^t s^2\|\xi(s)\|^2 ds + t^2\|\hat{\xi}\|_1^2 \ &\leq \ C \int_0^t \Big(s\|\hat{\xi}(s)\|_1^2 + s^2\|\eta(s)\|^2 + \|\hat{\hat{\xi}}(s)\|_1^2 \Big)\, ds \\ &+ \ C\|\hat{\hat{\xi}}(t)\|_1^2. \end{aligned} \tag{4.16}$$

From (4.6), (4.7) and (4.12), we easily obtain estimate (4.13) and this completes the rest of the proof. □

Proof of the Theorem 1.1.

Choose $\phi_h = t^3\xi(t)$ in (4.3) and find that

$$\frac{1}{2}\frac{d}{dt}(t^3\|\xi\|^2) + \mathcal{A}(t; \xi, t^3\xi) = \Big(\frac{3}{2}t^2\|\xi\|^2 + t^3(\eta_t, \xi)\Big) + t^3 \int_0^t \mathcal{B}(t, s; \xi(s), \xi)\, ds.$$

Integrate with respect to time to arrive at

$$\begin{aligned} t^3\|\xi\|^2 \ &+ \ 2\int_0^t s^3\mathcal{A}(s; \xi(s), \xi(s))\, ds = \int_0^t \Big(3s^2\|\xi(s)\|^2 + 2s^3(\eta_s(s), \xi(s))\Big) ds \\ &+ \ 2\int_0^t \int_0^s s^3\mathcal{B}(s, \tau; \xi(\tau), \xi(s))\, d\tau\, ds = I_1 + I_2. \end{aligned} \tag{4.17}$$

Now, we can write I_1 term as

$$I_1 \ \leq \ C \int_0^t s^2\|\xi(s)\|^2\, ds + C \int_0^t s^4\|\eta_s(s)\|^2\, ds.$$

From Lemma 3.4, we obtain

$$\|\eta_t\|^2 \ \leq \ Ch^4t^{-4}\|u_0\|^2,$$

and hence,

$$\int_0^t s^4\|\eta_s(s)\|^2 ds \ \leq \ Ch^4t\|u_0\|^2. \tag{4.18}$$

On substituting (4.18) and (4.13) in the estimate of $|I_1|$, we obtain

$$I_1 \ \leq \ Ch^4t\|u_0\|^2.$$

For I_2, with the help of integration by parts, we rewrite it as

$$\begin{aligned} I \ &= \ 2\int_0^t \int_0^s s^3\mathcal{B}(s, \tau; \xi(\tau), \xi(s))\, d\tau\, ds \\ &= \ 2\int_0^t s^3\mathcal{B}(s, s; \hat{\xi}(s), \xi(s))\, ds - 2\int_0^t \int_0^s s^3\mathcal{B}_\tau(s, \tau; \hat{\xi}(\tau), \xi(s))\, d\tau\, ds \\ &= \ 2\int_0^t s^3\mathcal{B}(s, s; \hat{\xi}(s), \xi(s))\, ds - 2\int_0^t s^3\mathcal{B}_s(s, s; \hat{\hat{\xi}}(s), \xi(s))\, ds \\ &+ \ 2\int_0^t \int_0^s s^3\mathcal{B}_{\tau\tau}(s, \tau; \hat{\hat{\xi}}(\tau), \xi(s))\, d\tau\, ds, \end{aligned} \tag{4.19}$$

and hence, using (4.6) and (4.12), we obtain

$$\begin{aligned} |I_2| &\le \rho_1 \int_0^t s^3 \|\xi(s)\|_1^2 \, ds + C \int_0^t \Big(s\|\hat{\xi}(s)\|_1^2 + \|\hat{\hat{\xi}}(s)\|_1^2 \Big) \, ds \\ &\le \rho_1 \int_0^t s^3 \|\xi(s)\|_1^2 \, ds + Ch^4 t \|u_0\|^2. \end{aligned}$$

Substituting the estimates of I_1 and I_2 in (4.17) and using the coecivity property (2.1), we arrive at

$$t^3\|\xi\|^2 + \int_0^t s^3 \|\xi(s)\|_1^2 \le Cth^4 \|u_0\|^2, \tag{4.20}$$

and hence,

$$\|\xi\| \le Ch^2 t^{-1} \|u_0\|. \tag{4.21}$$

Now, from Lemma 3.3 with $k = 2$ and $j = 0$ and (4.21), we conclude that

$$\|e\| \le Ch^2 t^{-1} \|u_0\|,$$

and this completes the proof of the theorem. □

Remarks:1.For completely discrete scheme based on backward Euler method, we obtain from Lemma 3.10 of Pani and Sinha [10] that at each time level t_n

$$\|U^n - u_h(t_n)\| \le Ckt_n^{-1} \Big(1 + \log \frac{1}{k}\Big) \|u_0\|, \tag{4.22}$$

where U^n denotes the backward Euler approximation at t_n. Note that at each time level $t = t_n$, we find from Theorem 1.1

$$\|u(t_n) - u_h(t_n)\| \le Ch^2 t_n^{-1} \|u_0\|. \tag{4.23}$$

Combining (4.22) and (4.23), we, therefore, arrive at the following final completely discerete error estimate:

$$\|u(t_n) - U^n\| \le Ct_n^{-1} \Big(h^2 + k\Big(1 + \log \frac{1}{k}\Big)\Big) \|u_0\|.$$

2. We can even obtain superconvergence result for ξ in H^1-norm. Now setting $\phi_h = t^4 \xi_t$ in (4.3), we obtain

$$t^4 \|\xi_t\|^2 + t^4 \mathcal{A}(t; \xi, \xi_t) = t^4 (\eta_t, \xi_t) + \int_0^t \mathcal{B}(t, s; \xi(s), t^4 \xi_t) ds. \tag{4.24}$$

Observe that

$$\frac{d}{dt}\{t^4 \mathcal{A}(t; \xi, \xi)\} = 4t^3 \mathcal{A}(t; \xi, \xi) + 2t^4 \mathcal{A}(t; \xi, \xi_t) + t^4 \mathcal{A}_t(t; \xi, \xi),$$

and

$$\frac{d}{dt}\{t^4 \mathcal{B}(t, s; \xi(s), \xi)\} = 4t^3 \mathcal{B}(t, s; \xi(s), \xi) + t^4 \mathcal{B}(t, s; \xi, \xi_t) + t^4 \mathcal{B}_t(t, s; \xi(s), \xi).$$

Therefore,

$$
\begin{aligned}
t^4\|\xi_t\|^2 + \frac{1}{2}\frac{d}{dt}\{t^4\mathcal{A}(t;\xi,\xi)\} \;&=\; t^4(\eta_t,\xi_t) + 2t^3\mathcal{A}(t;\xi,\xi) + \frac{1}{2}t^4\mathcal{A}_t(t;\xi,\xi)\\
&+\; \frac{d}{dt}\left(\int_0^t t^4\mathcal{B}(t,s;\xi(s),\xi)\,ds\right) - t^4\mathcal{B}(t,t;\xi,\xi)\\
&-\; \int_0^t \Big(4t^3\mathcal{B}(t,s;\xi(s),\xi) - t^4\mathcal{B}_t(t,s;\xi(s),\xi)\Big)\,ds.
\end{aligned}
$$

Integrate the above equation with respect to time and then use smoothness of coefficients of $\mathcal{A}(t)$ and $B(t,s)$ with Cauchy-Schwarz and Young's inequalities to obtain

$$
\begin{aligned}
(4.25)\quad t^4\mathcal{A}(t;\xi,\xi) + \int_0^t s^4\|\xi_s\|^2 ds \;&\le\; 2\int_0^t s^4\|\eta_s\|^2\,ds + C\int_0^t s^3(1+s)\|\xi(s)\|_1^2\,ds\\
&+\; \frac{\rho_1}{2}\|\xi\|_1^2.
\end{aligned}
$$

Using coercive property (2.1), we easily find that

$$
(4.26)\qquad t^4\|\xi\|_1^2 + \int_0^t s^4\|\xi_s\|^2 ds \;\le\; C\int_0^t s^4\|\eta_s\|^2 ds + C\int_0^t s^3\|\xi\|_1^2 ds.
$$

We conclude, using Lemma 3.4 and (4.20), that

$$
t^4\|\xi\|_1^2 + \int_0^t s^4\|\xi_s\|^2 ds \;\le\; Ch^4 t\|u_0\|^2,
$$

and hence,

$$
(4.27)\qquad \|\xi\|_1 \;\le\; Ch^2 t^{-3/2}\|u_0\|.
$$

3. Assuming that the triangulation is quasiuniform and $d=2$, we obtain, from the subspace-Sobolev inequality, that

$$
\|\chi\|_\infty \;\le\; C|\log h|^{1/2}\|\chi\|_1, \quad \forall\,\chi\in S_h,
$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm. Now, from (4.27), we arrive at

$$
(4.28)\qquad \|\xi\|_\infty \;\le\; Ch^2|\log h|^{1/2}t^{-3/2}\|u_0\|.
$$

Also, we know from [5] that for $u \in W^2_\infty \cap H^2 \cap H^1_0$,

$$
\|\eta\|_\infty = \|u - W_h u\|_\infty \;\le\; Ch^2|\log h|^{1/2}t^{-1}\|u_0\|_\infty.
$$

With above two estimates we conclude that

$$
(4.29)\qquad \|e\|_\infty \le \|\eta\|_\infty + \|\xi\|_\infty \le Ch^2|\log h|^{1/2}\Big(t^{-3/2}\|u_0\| + t^{-1}\|u_0\|_\infty\Big).
$$

Thus, it is possible to discuss maximum norm estimate, provided the triangulation is quasi-uniform and $d=2$.

Acknowledgment. The first author would like to thank CSIR, Government of India, for the financial support.The second author gratefully acknowledges the research support of the Department of Science and Technology, Government of India under DST-CNPq Indo-Brazil Project-DST/INT/Brazil/RPO-05/2007 (Grant No. 490795/2007-2). This publication is also based on work supported in part by Award No. KUK-C1-013-04, made by King Abdullah University of Science and Technology (KAUST).

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