

TEICHMÜLLER HARMONIC MAP FLOW FROM CYLINDERS

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Abstract

We define a geometric flow that is designed to change surfaces of cylindrical type spanning two disjoint boundary curves into solutions of the Douglas-Plateau problem of finding minimal surfaces with given boundary curves. We prove that also in this new setting and for arbitrary initial data, solutions of the Teichmüller harmonic map flow exist for all times. Furthermore, for solutions for which a three-point-condition does not degenerate as $t \rightarrow \infty$, we show convergence along a sequence $t_i \rightarrow \infty$ to a critical point of the area given either by a minimal cylinder or by two minimal discs spanning the given boundary curves.

1 Introduction

Teichmüller harmonic map flow, introduced in the joint work [12] with Peter Topping for closed surfaces, is a geometric flow that is designed to change parametrised surfaces into critical points of the area. Indeed, for closed surfaces in a non-positively curved target manifold, the flow *always* succeeds in changing, or more generally decomposing, the initial surface into (a union of) branched minimal immersions through globally defined smooth solutions [15].

Here we take a first step to generalize this approach to the problem of flowing surfaces with boundaries to a solution of the Douglas-Plateau problem of finding a minimal surface spanning given boundary curves. Namely, given two disjoint, closed C^3 Jordan curves Γ_{\pm} in Euclidean space we investigate how to flow a surface of cylindrical type in order to find a minimal surface spanning the two boundary curves.

As in [12] this flow will be constructed as a gradient flow of the Dirichlet energy

$$E(u, g) = \frac{1}{2} \int_{C_0} |du|_g^2 dv_g$$

considered as a function of two variables: a map $u : C_0 \rightarrow \mathbb{R}^n$ parametrising the evolving surface over a fixed domain, here the cylinder $C_0 = [-1, 1] \times S^1$, and a Riemannian metric g on the domain.

We remark that if a pair (u, g) is a critical point of E then it is also a critical point of the area, to be more precise either a constant map or a (possibly branched) minimal immersion [7]. A key idea of Teichmüller harmonic map flow is to consider E on the set of equivalence classes of maps and metrics modulo the symmetries of E , compare [12]. Thus we identify

- $(u, g) \sim (u, \lambda \cdot g)$ for all functions $\lambda : C_0 \rightarrow \mathbb{R}^+$ due to the conformal invariance of E , and
- $(u, g) \sim (u \circ f, f^*g)$ for diffeomorphisms $f : C_0 \rightarrow C_0$ that are homotopic to the identity.

As in [12] we shall then define Teichmüller harmonic map flow from cylinders as an L^2 gradient flow on the resulting set of equivalence classes

$$\mathcal{A} := \{(u, g) : u : C_0 \rightarrow \mathbb{R}^n \text{ so that } u|_{\{\pm 1\} \times S^1} \text{ parametrises } \Gamma_{\pm}, g \text{ a metric on } C_0\} / \sim.$$

One important point to be understood in order to truly define such a flow is how to define an L^2 -metric on \mathcal{A} and the closely related question of how to best represent a curve in \mathcal{A} through pairs of maps and metrics.

In the definition of Teichmüller harmonic map flow on closed surfaces in the joint work [12] with P. Topping we chose a canonical representative by asking that g has constant curvature $K_g \equiv 1, 0, -1$ (depending on the genus) and that $\|\partial_t g\|_{L^2}$ is minimal. For the resulting L^2 -gradient flow this means that the symmetries are used to maximally simplify the evolution equation for the metric component in the sense that g only moves by the part of the gradient of E that is orthogonal to the action of the symmetries. At the same time, the map component evolves with the full gradient, i.e. the tension. As a result, cf. [13], for closed surfaces the evolution of the metric turns out to be well controlled as long as $\text{inj}(M, g(t)) \not\rightarrow 0$, while the map component shows a similar behaviour as the solutions of the corresponding flow for fixed metrics, i.e. the harmonic map heat flow of Eells-Sampson, which is well understood for closed domain surfaces and general target manifolds.

In the present setting of flowing surfaces with boundary the situation is somewhat different, mainly because our boundary condition is not of Dirichlet-type, but only of Plateau-type, i.e. prescribing the boundary values only up to reparametrisation. As such, even for a fixed metric g_0 , one cannot expect strong regularity results for the gradient flow of $u \mapsto E(g_0, u)$ unless one imposes a three-point-condition for $u|_{\partial C_0}$.

For maps u parametrised over the closed disc $(D_1(0), g_{\text{eucl}})$ such a gradient flow of maps was introduced and studied by Chang and Liu in [1, 2, 3] who considered both maps into Euclidean space and into Riemannian manifolds. The more general case of flowing to discs of prescribed mean curvature (and prescribed Plateau-boundary condition) has been considered more recently by Duzaar and Scheven [6]. They show that an isoperimetric condition on the prescribed mean curvature ensures the existence of global weak solutions and that these solutions subconverge to a disc with the prescribed mean curvature. In both cases, the flows are given as equations for only a map component u . This is consistent with our approach as the special structure of the disc makes it unnecessary to also evolve a metric on the domain; namely, the moduli space of the disc consists of only one point and one can furthermore pull-back any map by a suitable Möbius transform to obtain a map that obeys a three-point-condition. Since Möbiustransforms do not change the conformal structure this means that one can replace (u, g_{eucl}) by a representative of the same point of \mathcal{A} whose map component satisfies a three-point-condition without having to adapt the metric component at all.

These special features of maps and metrics on the disc are not present for any other surface with boundary, though in case of the cylinder the moduli space has a very simple structure as it is one dimensional. But even in this case, the group of conformal diffeomorphisms from (C_0, g) to itself is not sufficiently large to impose a three-point-condition for the map component without having to adjust the metric suitably. As some kind of

restriction on how $u|_{\partial C_0}$ parametrises the boundary curves Γ_{\pm} is needed to obtain a flow that admits global solutions, we shall thus use the symmetries in a slightly different way than in the case of closed surfaces. Namely, we use only most, but not all, symmetries to ensure that the evolution of the metric is regular, while also setting aside a number of degrees of freedom (3 per boundary curve) to prevent a formation of singularities of the map at the boundary by imposing a three-point-condition.

We remark that the Douglas-Plateau problem has been considered by many authors and we refer to the books [5], [4], [9], [17] and the references therein for an overview of existing results. It is in particular well known that while one can in general not prescribe the topological type of the desired minimal surface, one can always obtain a minimal surface that is parametrised either over the original domain, for us the cylinder, or over a surface of a simpler topological type, in the present situation two discs.

The paper is organised as follows. To begin with, we discuss how to best represent curves in the set of equivalence classes \mathcal{A} and consequently give the precise definition of the flow. We then state our main results which guarantee the existence of global weak solutions for arbitrary initial data, see Theorem 2.6, as well as subconvergence to either a minimal cylinder or to two minimal discs spanning the given boundary curves for solutions for which the three-point-condition does not degenerate, see Theorem 2.7. The rest of the paper is then dedicated to the proof of these results. In section 3 we prove short-time existence based on a time-discretisation scheme and derive a priori estimates on the map and metric component which are crucial for the proof of both existence and asymptotic convergence. This asymptotic analysis is carried out in section 5 but before that, in section 4, we establish that solutions exist for all times.

2 Definition of the flow

2.1 Representing a curve in \mathcal{A} : Admissible variations

As preparation for the definition of Teichmüller harmonic map flow on cylinders we discuss ways of representing curves in the set of equivalence classes \mathcal{A} through suitably chosen pairs of maps and metrics. We do not claim that our choice is canonical but rather that it is designed for the purpose of obtaining a gradient flow of energy that admits global regular solution.

To begin with, we need to identify a suitable representative of a conformal class \mathbf{c} of (smooth) metrics on C_0 . While one can always consider constant curvature, here flat, metrics with geodesic boundary curves, it turns out that this particular representative is in general not the natural one to flow surfaces with boundary towards minimal surfaces. In particular, one would like to avoid the possibility that a boundary curve of the domain (on which we after all impose our boundary condition) can shrink to a point and thus be lost.

For the cylinder we shall thus consider smooth metrics compatible with \mathbf{c} which are hyperbolic, i.e. have constant (Gauss-)curvature -1 , and for which the boundary curves have both the same constant geodesic curvature.

We first recall the following standard fact of complex analysis

Lemma 2.1. *To any smooth conformal structure \mathbf{c} on C_0 there exists a unique number $Y > 0$ such that (C_0, \mathbf{c}) is conformally equivalent to $([-Y, Y] \times S^1, ds^2 + d\theta^2)$.*

On such a cylinder $([-Y, Y] \times S^1, ds^2 + d\theta^2)$ we can then use the following hyperbolic metrics whose structure is well known from the Collar lemma [11]

Lemma 2.2. *On $([-Y, Y] \times S^1, ds^2 + d\theta^2)$ there is a one parameter family of collar metrics*

$$g_\ell = \rho_\ell(s)^2(ds^2 + d\theta^2)$$

where

$$\rho_\ell(s) = \frac{\ell}{2\pi \cos(\frac{\ell s}{2\pi})}, \quad \ell \in (0, L_0(Y)), \quad L_0(Y) := \frac{\pi^2}{Y},$$

which are all hyperbolic and whose boundary curves have the same constant geodesic curvature $\kappa \equiv \kappa_{\ell, Y}$.

As admissible metrics for our flow we shall thus consider

$$\begin{aligned} \mathcal{M}_{-1} &:= \{f^*g \text{ where } f : C_0 \rightarrow [-Y, Y] \times S^1 \text{ is a smooth diffeomorphism,} \\ &\quad g = g_\ell \text{ a collar metric as in Lemma 2.2, } \ell, Y \in (0, \infty)\}. \end{aligned}$$

While Lemma 2.2 does not yet give a canonical representative of a conformal class, such a choice can be made so that the following splitting of the tangent space is respected

Lemma 2.3. *For any $g \in \mathcal{M}_{-1}$ we have*

$$T_g \mathcal{M}_{-1} = \{L_X g : X \in \Gamma(TC_0)\} \oplus \text{Re}(\mathcal{H}(g)) \oplus \text{span}\{\psi_g \cdot g\}$$

and $\text{Re}(\mathcal{H}(g))$ is $L^2(C_0, g)$ -orthogonal to $\{L_X g\} \oplus \text{span}\{\psi_g \cdot g\}$.

Here $\mathcal{H}(g)$ is the real vector space of quadratic differentials that are holomorphic in the interior of C_0 , continuous up to the boundary and whose traces on ∂C_0 are real. Furthermore $\Gamma(TC_0)$ stands for the space of smooth vectorfields on C_0 which are tangential to ∂C_0 on ∂C_0 and $\psi_g : C_0 \rightarrow \mathbb{R}$ is characterised by

$$\psi_g \cdot g = f^*\left(\frac{d}{d\ell}(\rho_\ell^2)(ds^2 + d\theta^2)\right)|_{\ell=\ell_0}$$

for $g = f^*g_{\ell_0}$, g_ℓ the collar metrics of Lemma 2.2.

We recall that for the cylinder the space \mathcal{H} is simply made up by elements of the form cdz^2 , $c \in \mathbb{R}$, $z = s + i\theta$, for collar coordinates $(s, \theta) \in [-Y, Y] \times S^1$ as in Lemma 2.2.

We also remark that the orthogonality relation claimed in the lemma is a simple consequence of the fact that the real part of a holomorphic quadratic differential is trace and divergence free.

This lemma implies that the most *efficient* way (i.e. with least L^2 velocity) to lift a curve $[g(\cdot)]$ from Teichmüller space $\mathcal{M}_{-1}/\mathcal{D}_0$ to \mathcal{M}_{-1} is as a *horizontal* curve, moving only in the direction of $\text{Re}(\mathcal{H}(g))$. Here \mathcal{D}_0 denotes the space of smooth diffeomorphisms from C_0 to itself that are homotopic to the identity.

For cylinders we can describe such horizontal curves of metrics explicitly by the following lemma which is proved in the appendix

Lemma 2.4. *Let $\eta > 0$ be any fixed number. We define $Y = Y_\eta : (0, \infty) \rightarrow (0, \infty)$ by*

$$Y(\ell) = \frac{2\pi}{\ell} \left(\frac{\pi}{2} - \text{atan}(\eta \cdot \ell) \right)$$

and $f_\ell = f_\ell^\eta : C_0 \rightarrow [-Y(\ell), Y(\ell)] \times S^1$ by

$$f_\ell(x, \phi) = (s_\ell^\eta(x), \phi) = \left(\frac{2\pi}{\ell} \text{atan}\left(\frac{\ell_0}{\ell} \cdot \tan\left(\frac{\ell_0}{2\pi}x\right)\right), \phi \right)$$

where $\ell_0 = \ell_0^\eta$ is determined through the condition $Y(\ell_0) = 1$.

Then the family $G_\ell := (f_\ell)^*(\rho_\ell^2(s)(ds^2 + d\theta^2))$ is horizontal, i.e. for every ℓ

$$\frac{d}{d\ell}G_\ell \in \text{Re}(\mathcal{H}(C_0, G_\ell)).$$

Since $Y(\cdot)$ is a bijection, we can combine the above result with Lemmas 2.1 and 2.2 to conclude that any horizontal curve of metrics in \mathcal{M}_{-1} must be of the form $f^*(G_{\ell(t)}^\eta)$ for some fixed $\eta > 0$ and a fixed diffeomorphism $f : C_0 \rightarrow C_0$.

As such, we shall from now on consider $\eta > 0$ to be fixed and will in particular allow all constants to depend on this number as well as on the boundary curves Γ_\pm (and their parametrisations α_\pm) without further mentioning this.

To describe the space of admissible maps, we first recall that the prescribed boundary curves Γ_\pm are assumed to be disjoint, closed C^3 Jordan curves of which we shall fix proper C^3 parametrisations

$$\alpha_\pm : S^1 \rightarrow \Gamma_\pm.$$

We then consider maps in the space

$$H_\Gamma^1(C_0, g) := \{u \in H^1((C_0, g), \mathbb{R}^n) \text{ such that } u|_{\partial C_\pm} \rightarrow \Gamma_\pm \text{ is weakly monotone} \}$$

i.e. H^1 maps so that the traces $u|_{\partial C_\pm}$ can be written in the form

$$u|_{\partial C_\pm} = \alpha_\pm \circ \varphi_\pm$$

for some weakly monotone functions $\varphi_\pm : S^1 \rightarrow S^1$. Here and in the following we identify $\partial C_\pm := \{\pm 1\} \times S^1$ with S^1 when convenient.

It is well known that the space H_Γ^1 is not closed under weak H^1 convergence as one can find sequences of maps with bounded energy for which the boundary curves Γ_\pm are parametrised over smaller and smaller arcs of ∂C_\pm thus resulting in a weak limit that no longer spans Γ_\pm . The standard way to deal with this loss of completeness is to impose a three-point-condition. So we shall restrict the set of admissible maps for our flow to

$$H_{\Gamma,*}^1(C_0) := \{u \in H_\Gamma^1(C_0) : u|_{\partial C_\pm} = \alpha_\pm \circ \varphi_\pm \text{ for } \varphi_\pm \text{ satisfying} \\ \varphi_\pm(\theta_k) = \theta_k \text{ for } \theta_k = \frac{2\pi}{3}k, k = 0, 1, 2\}.$$

To compensate for the (in our case 6) lost degrees of freedom we need to allow the metric to move not only in horizontal direction but also through the pull-back by select diffeomorphisms.

For this purpose we will define (and discuss) a suitable family of diffeomorphisms $h_{b,\phi}$, $\phi = (\phi^+, \phi^-) \in \mathbb{R}^2$, $b = (b^+, b^-) \in \mathbb{C}^2$, with $|b^\pm| < 1$ later on in section 4.1.1. We remark that by using the one Killing field that is available for the cylinder we could reduce the number of degrees of freedom to 5 instead of 6 (e.g. by asking that $\phi^+ + \phi^- = 0$) though this would not lead to a significant simplification.

All in all we then say that a curve $(u, g)(t)$ is an admissible representative of a curve of equivalence classes $[(u, g)(t)] \in \mathcal{A}$ if

- $g(t) = h_{\phi(t), b(t)}^* \tilde{g}(t)$ for a horizontal curve of metrics $\tilde{g}(\cdot) \in \mathcal{M}_{-1}$ and continuous families of parameters $(b, \phi)(\cdot) \in \Omega_h$ where $\Omega_h := (D_1(0))^2 \times \mathbb{R}^2 \subset \mathbb{C}^2 \times \mathbb{R}^2$ is the domain of parameters for the diffeomorphisms $h_{b,\phi}$ that will be defined in section 4.1.1.

- $u(t) \in H_{\Gamma,*}^1(C_0, \mathbb{R}^n)$ for every t .

We remark that we can and will assume without loss of generality that the initial metric $\tilde{g}(0)$ of the horizontal curve (\tilde{g}) is given by one of the metrics G_ℓ described in Lemma 2.4 simply by pulling-back the whole problem (including the parametrisations α_\pm used in the three-point-condition) by a fixed diffeomorphism. As such we shall from now on consider metrics in the set

$$\widetilde{\mathcal{M}} := \{h_{b,\phi}^* G_\ell : (b, \phi) \in \Omega_h, \ell \in (0, \infty)\}.$$

2.2 Definition of the flow

As we consider a problem with a Plateau boundary condition, the space of admissible variations does not form a vectorspace. As such the flow that we shall define will not be governed by a system of PDEs with prescribed boundary values but rather, for the map component, by a partial differential inequality.

To motivate the definition of the flow given at the end of this section we first make some general formal computations and remarks on how the differential inequalities are obtained. Of course none of this is new in any way, but this part is rather included for the convenience of the reader. We remark in particular that the differential inequalities we derive correspond to the ones obtained in [1] and [6] in case of the domain being a disc.

Given a functional \mathcal{F} defined on some (Hilbert)manifold B we may want to define a gradient flow under the restriction that the velocity $\partial_t w$ at each time is constrained to some closed convex cone $X(w(t)) \subset T_{w(t)}B$, e.g. because we want to constrain the flow to some convex set A and thus the velocity to the corresponding solid tangent cone.

One can formally define such a gradient flow by asking that

$$\partial_t w = P^{X(w)} \left(-\nabla \mathcal{F}(w) \right)$$

where $P^{X(p)} : T_p B \rightarrow X(p)$ is the nearest point projection. We recall that $P^{X(p)}(v)$, $v \in T_p B$, can be equivalently characterised as the minimiser of $z \mapsto \frac{1}{2}\|z\|^2 - \langle z, v \rangle$ on $X(p)$.

As such $\partial_t w$ is characterised as the direction of an admissible variation w_ε which minimises

$$\frac{d}{d\varepsilon} \big|_{\varepsilon=0} \mathcal{F}(w_\varepsilon) + \frac{1}{2} \|\partial_\varepsilon w_\varepsilon\|^2 = \langle \nabla \mathcal{F}(w(t)), \partial_\varepsilon w \rangle + \frac{1}{2} \|\partial_\varepsilon w_\varepsilon\|^2, \quad (2.1)$$

and thus equivalently as solution of the variational inequality

$$\begin{aligned} 0 &\leq \frac{d}{ds} \big|_{s=0} \left(\frac{1}{2} \|(1-s)\partial_t w(t) + sv\|^2 + \langle (1-s)\partial_t w(t) + sv, \nabla \mathcal{F}(w(t)) \rangle \right) \\ &= \langle \partial_t w(t) + \nabla \mathcal{F}(w(t)), v - \partial_t w(t) \rangle \end{aligned}$$

for $v \in X(p)$. As $X(p)$ is a cone and thus in particular also $0, 2\partial_t w \in X(p)$ we note that the above inequality implies that $\langle \partial_t w(t) + \nabla \mathcal{F}(w(t)), \partial_t w(t) \rangle = 0$ and thus

$$\langle \partial_t w(t) + \nabla \mathcal{F}(w(t)), v \rangle \geq 0 \text{ for all } v \in X(p). \quad (2.2)$$

Going back to our problem of defining a gradient flow of the Dirichlet energy on the set \mathcal{A} we recall that the negative L^2 -gradient of the energy with respect to the metric

variable can be written in the form $\frac{1}{4}Re(\Phi(u, g))$, where $\Phi(u, g)$ is the Hopf-differential which is given in isothermal coordinates $(s, \theta) = (s_g, \theta_g)$ of (C_0, g) by $\Phi(u, g) = (|u_s|^2 - |u_\theta|^2 - 2i\langle u_s, u_\theta \rangle) \cdot dz^2$, $z = z_g = s + i\theta$.

The variational inequality (2.2) thus translates to the condition that

$$\left[\int \langle dv, du \rangle_g + v \cdot \partial_t u dv_g \right] + \int \left\langle -\frac{1}{4}Re(\Phi(u, g)) + \partial_t g, h \right\rangle dv_g \geq 0$$

for all $(v, h) = \partial_\varepsilon(u_\varepsilon, g_\varepsilon)$ where $(u_\varepsilon, g_\varepsilon)$ are admissible variations as described above.

On the one hand, the resulting differential inequality for g can be simply recast as a differential equation

$$\partial_t g = \frac{1}{4}P_g^\mathcal{V}(Re(\Phi(u, g))) \quad (2.3)$$

to be solved on $\widetilde{\mathcal{M}}$. Here $P_g^\mathcal{V}$ is the L^2 -orthogonal projection onto the tangent space $\mathcal{V}(g)$ of $\widetilde{\mathcal{M}} = \{h_{b,\phi}^* G_\ell : (b, \phi) \in \Omega_h, \ell \in (0, \infty)\}$ which is given by

$$\mathcal{V}(g) := Re(\mathcal{H}(g)) \oplus \{L_{h_{b,\phi}^* X} g : X \in \mathcal{X}(b, \phi)\},$$

for $g = h_{b,\phi}^* G_\ell$ where $\mathcal{X}(b, \phi)$ is the 6 dimensional space of vectorfields generating the diffeomorphisms $h_{b,\phi}$, compare section 4.1.1.

On the other hand, admissible variations of the map can be described as follows. Given $u \in H_{\Gamma,*}^1(C_0)$ we let φ_\pm be such that $u|_{\partial C_\pm} = \alpha_\pm \circ \varphi_\pm$. Then functions of the form $\alpha_\pm \circ (\varphi_\pm + \varepsilon \cdot \beta_\pm + O(\varepsilon^2))$ are again weakly monotone parametrisations of Γ_\pm , for ε in a small one-sided interval $[0, \varepsilon_0)$, if β_\pm can be written in the form $\beta_\pm = \lambda_\pm \cdot (\psi_\pm - \varphi_\pm)$ for some numbers $\lambda_\pm > 0$ and weakly monotone functions ψ_\pm .

As variations $\frac{d}{d\varepsilon}|_{\varepsilon=0}u$ of the map component we thus consider elements of

$$T_u^+ H_{\Gamma,*}^1(C_0) := \{v \in H^1(C_0, \mathbb{R}^n) : v|_{\partial C_\pm} = \lambda_\pm \alpha'_\pm(\varphi) \cdot (\psi_\pm - \varphi_\pm) \text{ for } \lambda_\pm > 0 \text{ and } \psi_\pm \in C^0(S^1, S^1) \text{ weakly monotone with } \psi_\pm(\theta_k) = \theta_k\}.$$

We remark that it is proven in [6, Lemma 2.1] that to any $v \in T_u^+ H_{\Gamma,*}^1(C_0)$ there is a one-sided variation $(u_\varepsilon) \subset H_{\Gamma,*}^1(C_0)$, $\varepsilon \in [0, \varepsilon_0)$ with $\frac{d}{d\varepsilon}|_{\varepsilon=0}u_\varepsilon = v$.

As $T_u^+ H_{\Gamma,*}^1$ is in general not a vector space, but only a convex cone, we cannot reduce the resulting partial differential inequality

$$\langle du, dw \rangle_{L^2(C_0, g)} + \int w \cdot \partial_t u dv_g \geq 0 \text{ for all } w \in T_u^+ H_{\Gamma,*}^1(C_0) \quad (2.4)$$

to a PDE with a standard boundary condition though of course u satisfies the heat equation $\partial_t u = \Delta_g u$ in the interior.

We furthermore remark that, as pointed out in [6], the additional condition

$$\int \langle du, dw \rangle + \Delta_g u \cdot w dv_g \geq 0 \text{ for all } w \in T_u^+ H_{\Gamma,*}^1,$$

can be seen as a weak Neumann-type boundary condition, though this will not be used in the present paper.

Given that (2.3) and (2.4) were motivated by the idea that $\partial_t(u, g)$ should minimise the functional (2.1), the so called stationarity condition, asking that

$$\frac{1}{4} \int Re(\Phi(u, g)) L_X g dv_g + \int Du(X) \cdot \Delta_g u dv_g = 0 \text{ for every } X \in \Gamma(TC_0)_*, \quad (2.5)$$

where

$$\Gamma(TC_0)_* := \{Y \in \Gamma(TC_0) : Y(\pm 1, \theta_k) = 0\}, \quad (2.6)$$

results if one considers variations of the form $(u(t + \varepsilon) \circ f_\varepsilon, g)$. Similarly, from (2.1), one expects the energy to be non-increasing along the flow, compare (2.9) below.

All in all, we define

Definition 2.5. A weak solution of Teichmüller harmonic map flow on the cylinder C_0 is represented by a curve of maps

$$u \in L^\infty([0, T], H_{\Gamma,*}^1(C_0, \mathbb{R}^n)) \cap H^1([0, T] \times C_0)$$

and a curve of metrics $g \in C^{0,1}([0, T], \widetilde{\mathcal{M}})$ which satisfy

$$\int_{[0,T] \times C_0} \langle du, dw \rangle_g + \partial_t u \cdot w \, dv_{g(t)} \, dt \geq 0 \text{ for all } w \in L^2([0, T], T_u^+ H_{\Gamma,*}^1(C_0)) \quad (2.7)$$

and

$$\partial_t g = \frac{1}{4} P_g^\vee (Re(\Phi(u, g))) \quad \text{for a.e. } t \in [0, T]. \quad (2.8)$$

Such a weak solution is called stationary if it satisfies (2.5) for almost every t , and we say (u, g) satisfies the energy inequality if for almost every $t_1 < t_2$

$$E(u, g)(t_1) - E(u, g)(t_2) \geq \frac{1}{2} \int_{t_1}^{t_2} \int_{C_0} |\partial_t u|^2 \, dv_g \, dt + \int_{t_1}^{t_2} \|\partial_t g\|_{L^2(C_0, g)}^2 \, dt. \quad (2.9)$$

2.3 Main results

For the flow we just defined we will prove the following two main results

Theorem 2.6 (Existence of global solutions). *Let Γ_\pm be two disjoint closed C^3 Jordan curves. Then to any initial data $(u_0, g_0) \in H_{\Gamma,*}^1(C_0) \times \widetilde{\mathcal{M}}$ there exists a stationary weak solution (u, g) of Teichmüller harmonic map flow which is defined for all times and satisfies the energy inequality.*

The above solution flows to a minimal surface in the sense that

Theorem 2.7 (Asymptotics). *Assume that $(u, g)(t)$, $t \in [0, \infty)$, is a stationary weak solution of Teichmüller harmonic map flow that satisfies the energy inequality and for which the three-point-condition does not degenerate in the sense that $\limsup_{t \rightarrow \infty} 1 - \max_\pm |b^\pm(t)| > 0$. Then there is a sequence of times $t_i \rightarrow \infty$ such that the equivalence classes $[(u, g)(t_i)]$ converge to a critical point of the area in one of the following ways:*

I (Non-degenerate case) If $\text{inj}(C_0, g(t_i)) \not\rightarrow 0$ for $i \rightarrow \infty$ then $f_i^(u(t_i), g(t_i))$ converges to a limit (u_∞, g_∞) where $g_\infty \in \widetilde{\mathcal{M}}$ and where $u_\infty \in H_{\Gamma,*}^1(C_0, \mathbb{R}^n) \cap C^0(C_0)$ is a (possibly branched) minimal immersion.*

Here

$$f_i := h_{0, 2\pi n_i}, \quad n_i^\pm = \lfloor \frac{\phi^\pm(t_i)}{2\pi} \rfloor \quad (2.10)$$

and the convergence for the metric component is smooth convergence on all of C_0 while the maps converge uniformly on the whole cylinder C_0 as well as strongly in $H^1(C_0)$ and weakly in $H_{loc}^2(C_0 \setminus \bigcup_{j,\pm} P_j^\pm)$ away from the points $P_j^\pm = (\pm 1, \theta_j)$, $j = 0, 1, 2$ at which the three-point-condition is imposed.

II (Degenerate case) If $\text{inj}(C_0, g(t_i)) \rightarrow 0$ then $f_i^*(u(t_i), g(t_i))$ converges locally on $C_0 \setminus (\{0\} \times S^1)$ to a limit (u_∞, g_∞) which is such that

- each of the cylinders (C_\pm, g_∞) , $C_\pm := \{0 < \pm s \leq 1\} \times S^1$ is isometric to the hyperbolic cusp

$$([0, \infty) \times S^1, \rho_0(s)^2(ds^2 + d\theta^2)), \quad \rho_0(s) = \frac{1}{2\pi\eta + s},$$

- the two maps $u_\infty|_{C_\pm}$ can be extended across the punctures to give two (possibly branched) minimal immersion $\bar{u}_\infty^\pm \in H_{\Gamma_\pm, *}^1(\bar{D}) \cap C^0(\bar{D})$, parametrised over closed discs \bar{D} in \mathbb{R}^2 , each of which spans the corresponding boundary curve Γ^\pm .

Here the convergence is smooth local convergence for the metrics and weak H_{loc}^2 convergence on $(C_- \cup C_+) \setminus \bigcup_{j, \pm} P_j^\pm$ as well as locally uniform and strong H_{loc}^1 convergence on $C_- \cup C_+$ for the maps and the diffeomorphisms are again given by (2.10).

We remark that while we only obtain convergence in $H^1 \cap C^0(C_0)$, respectively in H^2 away from P_j^\pm , the limit u_∞ is indeed far more regular than that. Namely, classical regularity theory for solutions of the Plateau-Problem, see e.g. [17] or [5], yields that u_∞ is of class $C^{2, \alpha}$, $\alpha < 1$, upto the boundary.

Remark 2.8. We do not claim that a degeneration of the three-point-condition at infinite time cannot happen. However the asymptotic analysis in this case requires a somewhat different approach as the degeneration of the three-point-condition will result in the formation of a 'boundary bubble' so this case will be addressed in future work.

3 Short-time existence of solutions

We shall prove short-time existence of solutions to arbitrary initial data based on a time-discretisation scheme. We remark that this well known method has been carried out successfully to obtain solutions of several other geometric flows, e.g. by Haga et. al. [8] for harmonic map flow and by Moser [10] for biharmonic map flow, and that also the solutions for the evolution to minimal discs by Chang-Liu [1] respectively to discs of prescribed mean curvature of Duzaar-Scheven [6] were obtained this way.

A key part of this section consists in proving suitable a priori estimates for the approximate solutions resulting from such a time-discretisation. For some of these estimates we will be able to appeal to work of Duzaar and Scheven [6] whose delicate estimates allowed them to prove H^2 bounds upto the boundary but away from the points P_j^\pm despite their equation being non-linear.

In the present paper the challenges are somewhat different as we do not have to deal with a non-linear equation for the map component but instead have to understand the interplay of the map and the metric component of the flow. What makes this particular aspect of the flow quite delicate is that this relation involves a non-local projection operator. This forces us to prove estimates that are valid not just near most boundary points but rather in neighbourhood of every boundary point, including the points P_j^\pm at which we impose the three-point-condition.

3.1 The time-discretisation scheme

To begin with we outline the time-discretisation scheme and show that it is well defined.

Given an initial pair $(u_0, g_0) \in H_{\Gamma,*}^1(C_0) \times \widetilde{\mathcal{M}}$ and a (small) number $h > 0$ we construct an approximate solution of Teichmüller harmonic map flow using the following time-discretisation:

For $j = 0, 1, 2, \dots$ we let $t_j = t_j^h = j \cdot h$, set $u^h(t) = u_0$ for $t \in [t_0, t_1]$ and then construct iteratively the approximate solution (u^h, g^h) on the interval $[t_j^h, t_{j+1}^h]$ as follows:

First determine $g^h(\cdot)$ on $(t_j^h, t_{j+1}^h]$ as the solution of

$$\partial_t g(t) := \frac{1}{4} P_{g(t)}^\nu(\operatorname{Re}(\Phi(u^h(t_j), g(t)))) \text{ with } g(t_j) = g^h(t_j). \quad (3.1)$$

Then select $u^h(t_{j+1})$ as a minimiser of the functional $\mathcal{F}_{g^h(t_{j+1}), u^h(t_j)}^h$ where

$$\mathcal{F}_{g,v}^h(w) = E(w, g) + \frac{1}{2h} \|w - v\|_{L^2(C_0, g)}^2. \quad (3.2)$$

The existence of a minimiser of this functional is assured by the direct method of calculus of variation thanks to the H^1 -weak-lower semicontinuity of $u \mapsto E(u, g)$ as well as the Courant Lebesgue Lemma and the resulting equicontinuity of the traces $u|_{\partial C_0}$, c.f. Appendix A.1.

To be more precise, we have

Lemma 3.1. *For any $g \in \mathcal{M}_{-1}$, any map $\bar{u} \in H_{\Gamma,*}^1(C_0)$ and any $h > 0$ there exists a minimiser $w \in H_{\Gamma,*}^1(C_0)$ of*

$$\mathcal{F}(w) = E(w, g) + \frac{1}{2h} \|w - \bar{u}\|_{L^2(C_0, g)}^2$$

and w satisfies

$$\int_{C_0} \langle dw, dv \rangle_g + \frac{1}{h} \cdot (w - \bar{u}) \cdot v \, dv_g \geq 0 \text{ for all } v \in T_w^+ H_{\Gamma,*}^1(C_0), \quad (3.3)$$

in particular $\Delta_g w = \frac{1}{h} \cdot (w - \bar{u})$ in the interior of C_0 .

Furthermore w satisfies the stationarity equation

$$\frac{1}{4} \int \operatorname{Re}(\Phi(w, g)) \cdot L_X g \, dv_g + \int dw(X) \cdot \Delta_g w \, dv_g = 0 \text{ for all } X \in \Gamma(TC_0)_* \quad (3.4)$$

and the energy inequality

$$E(w, g) + \frac{1}{2h} \cdot \|w - \bar{u}\|_{L^2(C_0, g)}^2 \leq E(\bar{u}, g). \quad (3.5)$$

Remark 3.2. We note furthermore that the minimiser w obtained in the above lemma is bounded by

$$\|w\|_{L^\infty(C_0)} \leq \|\bar{u}\|_{L^\infty(C_0)}.$$

Indeed, if the above estimate would not be satisfied, we could compose w with the nearest point projection to the ball with radius $\|\bar{u}\|_{L^\infty}$ to obtain a function with smaller energy \mathcal{F} . As a consequence, the L^∞ norm of the map component is non-increasing in time along the discretised flow.

In the present setting of metrics on a cylinder, short-time existence of a solution to the differential equation (3.1) on $\widetilde{\mathcal{M}}$ is a simple consequence of the fact that \mathcal{H} is one-dimensional since this means that the evolution of the metric could be expressed as a system of (in total 7) ordinary differential equations. As such it is easy to check that the projection $P^\mathcal{V}$ satisfies the following Lipschitz-estimates

Lemma 3.3. *Let K be a compact subset of the set of admissible metrics $\widetilde{\mathcal{M}} = \{g = h_{b,\phi}^* G_\ell, \ell \in (0, \infty), (b, \phi) \in \Omega_h\}$ and let $P_g^\mathcal{V}$ be the L^2 -orthogonal projection onto $\mathcal{V}(g) := T_g \widetilde{\mathcal{M}}$. Then*

$$\begin{aligned} \|P_{g_1}^\mathcal{V}(Re(\Psi_1)) - P_{g_2}^\mathcal{V}(Re(\Psi_2))\|_{C^k} &\leq C \|g_1 - g_2\|_{C^k} \cdot \|\Psi_1\|_{L^1(C_0, g_1)} \\ &\quad + C \|Re(\Psi_1) - Re(\Psi_2)\|_{L^1(C_0, g_1)} \end{aligned}$$

for all $g_i \in K$, all quadratic differentials Ψ_i on (C_0, g_i) , $i = 1, 2$, and every $k \in \mathbb{N}$, where C depends only on k and K . Here and in the following the C^k norms are computed with respect to the fixed coordinates on $C_0 = [-1, 1] \times S^1$.

We remark that the real part of the Hopf-differential can be written equivalently as

$$Re(\Phi(u, g)) = 2u^* g_{\mathbb{R}^n} - |du|_g^2 \cdot g \quad (3.6)$$

so that we can bound the differences of Hopf-differentials by

$$\|Re(\Phi(u, g)) - Re(\Phi(\tilde{u}, \tilde{g}))\|_{L^1} \leq C \|g - \tilde{g}\|_{C^0} + C \|u - \tilde{u}\|_{H^1} \quad (3.7)$$

for a constant C that depends only on an upper bound on the energies $E(u, g)$, $E(\tilde{u}, \tilde{g})$.

Combined, Lemmas 3.1 and 3.3 thus imply that solutions of the time-discretisation scheme exist for as long as the injectivity radius $\text{inj}(C_0, g) = 2\ell$ of the domain (C_0, g) is bounded away from zero and infinity and the parameters (b, ϕ) remain in a compact subset of Ω_h .

We furthermore remark that the energy of such an approximate solution is non-increasing, namely on the open interval (t_k, t_{k+1}) it decreases by

$$\begin{aligned} E((u^h, g^h)(t_k)) - E((u^h(t_k), g^h(t_{k+1}))) &= \int_{[t_k, t_{k+1}]} \|\partial_t g^h\|_{L^2(C_0, g^h)}^2 dt \\ &= \frac{1}{16} \int_{[t_k, t_{k+1}]} \|P^\mathcal{V}(Re(\Phi(u^h(t_k), g^h(t))))\|_{L^2(C_0, g^h)}^2 dt \end{aligned}$$

while at t_{k+1} there is a further loss of energy of no less than

$$E(u^h(t_k), g^h(t_{k+1})) - E((u^h, g^h)(t_{k+1})) \geq \frac{1}{2h} \|u^h(t_k) - u^h(t_{k+1})\|_{L^2(C_0, g^h(t_{k+1}^h))}^2,$$

compare (3.5).

All in all we can thus estimate

$$E((u^h, g^h)(t_k)) \leq E((u^h, g^h)(t_{\tilde{k}})) - \int_{t_{\tilde{k}}}^{t_k} \frac{1}{2} \|D_t^h u^h\|_{L^2(C_0, \tilde{g}^h)}^2 + \|\partial_t g^h\|_{L^2(C_0, g^h)}^2 dt \quad (3.8)$$

for $\tilde{k} < k$ where compute the norm of the difference quotient

$$D_t^h u^h(x, t) = \frac{1}{h} \cdot (u^h(x, t+h) - u^h(x, t))$$

with respect to the piecewise constant curve of metrics $\tilde{g}^h|_{[t_k, t_{k+1}]} = g^h(t_{k+1})$.

We obtain in particular that the length of the curve of metrics is bounded uniformly by

$$L_{L^2}(g^h|_{[0,t]}) \leq E_0^{1/2} t^{1/2} \text{ for every } h > 0,$$

E_0 an upper bound on the initial energy $E(u(0), g(0))$.

Consequently, to any given (u_0, g_0) there exist numbers $\bar{\varepsilon} > 0$, $T > 0$ and $\bar{C} < \infty$ such that for every parameter $h > 0$ the solution of the time-discretisation scheme exists at least on the interval $[0, T]$ and so that the metric component $g^h = h_{b,\phi}^* G_\ell$ satisfies estimates of the form

$$\text{inj}(C_0, g^h) = 2\ell \geq \bar{\varepsilon} \text{ and } |b_j^\pm| \leq 1 - \bar{\varepsilon}, \quad |\phi^\pm| \leq \bar{C} \quad (3.9)$$

on this interval.

Similarly we could obtain an upper bound for ℓ on $[0, T]$, but we remark that ℓ is indeed bounded from above uniformly in time in terms of only the initial energy. Namely, let $\delta_\Gamma := \text{dist}(\Gamma_+, \Gamma_-) > 0$ be the distance between our prescribed disjoint boundary curves. Then the energy of any map $w \in H_\Gamma^1(C_0)$ with respect to a metric g of the form $g = h_{b,\phi}^* G_\ell$ is bounded from below by

$$\begin{aligned} E(w, h_{b,\phi}^* G_\ell) &= E(\tilde{w}, G_\ell) = \frac{1}{2} \int_0^{2\pi} \int_{-Y(\ell)}^{Y(\ell)} |\tilde{w}_s|^2 + |\tilde{w}_\theta|^2 ds d\theta \\ &\geq \frac{c}{Y(\ell)} \left(\int_0^{2\pi} \int_{-Y(\ell)}^{Y(\ell)} |\tilde{w}_s| ds d\theta \right)^2 \geq \frac{\tilde{c} \delta_\Gamma^2}{Y(\ell)}, \end{aligned}$$

for some $\tilde{c} > 0$ and $\tilde{w} = w \circ h_{b,\phi}^{-1}$. The resulting lower bound on $Y(\ell)$ results in an upper bound for ℓ and thus for the injectivity radius of the form

$$\ell \leq \bar{L}(E_0), \quad (3.10)$$

where $\bar{L}(E_0)$ depends only on an upper bound E_0 on the initial energy $E(u_0, g_0)$ (and as usual the geometric setting and the fixed number η).

This uniform upper bound on ℓ and the resulting control on the metrics G_ℓ near the boundary ∂C_0 will allow us to derive a priori bounds for the map component near ∂C_0 that are independent of $\text{inj}(C_0, g)$. This will be crucial for the asymptotic analysis (in the degenerate case) carried out later on. Conversely, all estimates near the boundary will depend on b as the case $|b_\pm| \rightarrow 1$ corresponds to the degeneration (in collar coordinates) of the three-point-condition.

To prove that the above time-discretisation scheme converges to a solution of the flow, we need to derive a priori estimates for the map component where we will distinguish between

- the interior of the cylinder where standard estimates for the heat equation apply
- the boundary region away from the points $P_k^\pm = (\pm 1, \theta_k)$, on which we shall be able to appeal to results of Duzaar-Scheven [6]
- the region near the points P_k^\pm at which the three-point-condition is imposed.

3.2 A priori estimates in the interior and near general boundary points

Let \bar{u} be any fixed map, $h > 0$, let $g = h_{b,\phi}^* G_\ell$ be a metric as described in Lemma 2.4 and let u be a minimiser of the functional $\mathcal{F}_{g,\bar{u}}^h = E(u, g) + \frac{1}{2h} \|u - \bar{u}\|_{L^2(C_0, g)}^2$.

Then, setting $f = \frac{1}{h}(u - \bar{u})$ we know that

$$\int_{C_0} \langle du, dv \rangle dv_g + \int_{C_0} v \cdot f dv_g \geq 0 \text{ for } v \in T_u^+ H_{\Gamma,*}^1(C_0). \quad (3.11)$$

In particular $\Delta_g u = f$ in the interior, so standard elliptic estimates combined with the upper bound (3.10) on ℓ yield

Lemma 3.4. *Given any $\ell_0 > 0$, $E_0 < \infty$ and any $\delta > 0$ there exists a constant $C < \infty$ such that the following holds true. Let $u \in H_{\Gamma,*}^1(C_0)$ be any map of energy $E(u, g) \leq E_0$ which satisfies (3.11) for some $f \in L^2(C_0)$ and $g \in \widetilde{\mathcal{M}}$ with $2 \operatorname{inj}(C_0, g) > \ell_0$. Then*

$$\int_{[-1+\delta, 1-\delta] \times S^1} |\nabla_g^2 u|^2 + |du|_g^4 dv_g \leq C \cdot (E(u, g) + \|f\|_{L^2(C_0, g)}^2). \quad (3.12)$$

We remark that the region in which G_ℓ degenerates as $\ell \rightarrow 0$ is contained in what corresponds to arbitrarily small cylinders $[-\delta, \delta] \times S^1$ with respect to the fixed coordinates $(x, \theta) \in C_0 = [-1, 1] \times S^1$ since we use hyperbolic rather than flat metrics to represent a conformal class, cf. Remark A.5 in the appendix. Therefore

Remark 3.5. The analogue of (3.12) is valid with a constant independent of ℓ_0 on every compact region of $(-1, 1) \times S^1 \setminus (\{0\} \times S^1)$ and for every metric $g \in \widetilde{\mathcal{M}}$ as well as for the metrics $h_{b,\phi} G_{\ell=0}$ given in (A.9) which describe hyperbolic cusps.

Near the boundary but away from the points P_j^\pm we can use the results of Duzaar-Scheven [6] which apply to more general (in particular non-linear) equations. Namely, as explained in Appendix A.1, we can easily derive the following a priori estimates from Theorem 8.3 of [6]

Proposition 3.6. *For any $b_0 < 1$, $\ell_0 > 0$ and $E_0 < \infty$ there exist constants $\varepsilon_1, r_0 > 0$ and $C < \infty$ such that the following holds true. Let $g = h_{b,\phi}^* G_\ell$ for some $\ell \geq \ell_0$ and $|b^\pm| \leq b_0$. Suppose furthermore that $f \in L^2(C_0, g)$ and that $u \in H_{\Gamma,*}^1(C_0)$ has energy $E(u, g) \leq E_0$. Then, if u is so that (3.11) is satisfied for all variations $v \in T^+ H_{\Gamma,*}^1(C_0)$ with support in a ball $B_r^g(p)$, where $p \in C_0$ and $r \in (0, r_0)$ are such that*

$$B_r^g(p) \cap \{P_j^\pm, j = 0, 1, 2\} = \emptyset$$

and if the energy on this ball is small in the sense that

$$E(u, B_r^g(p)) := \frac{1}{2} \int_{B_r^g(p)} |du|_g^2 dv_g \leq \varepsilon_1$$

then $u \in H^2(B_{r/2}^g(p), g)$ with

$$\int_{B_{r/2}^g(p)} |\nabla_g^2 u|^2 + |du|_g^4 dv_g \leq \frac{C}{r^2} E(u, B_r^g(p)) + C \int_{B_r^g(p)} |f|^2 dv_g. \quad (3.13)$$

Here and in the following we denote geodesic balls in (C_0, g) by $B_r^g(p) := \{\tilde{p} \in C_0 : d_g(\tilde{p}, p) < r\}$ and compute the energy on a geodesic ball with respect to the corresponding metric unless indicated otherwise.

As we shall use this and the subsequent lemmas to control the map near the boundary, it is important to remark

Remark 3.7. The above result is valid also without imposing a lower bound on ℓ , and in particular also for the metric G_0 defined in (A.9), as long as one considers only points p contained in a compact subset $K \subset C_0 \setminus (\{0\} \times S^1)$ and allows the constants to depend also on this set K . Similarly, on compact sets $K \subset C_0 \setminus (\{0\} \times S^1)$ the a priori estimates derived in the subsequent Lemma 3.8 and in Corollary 3.13 are valid for all metrics $h_{b,\phi}^* G_\ell$ with $\ell \geq 0$ and $|b_\pm| \leq b_0 < 1$, again with a constant that also depends on K .

As our target is Euclidean space which 'supports no bubbles', i.e. to which there are no non-trivial harmonic maps from S^2 , we can furthermore exclude a concentration of the energy near general points of the boundary

Lemma 3.8. *To any numbers $\Lambda, E_0 < \infty$, $b_0 < 1$, $d, \ell_0 > 0$ and any $\varepsilon > 0$ there exists a radius $r > 0$ such that the following holds true.*

Let $u \in H_{\Gamma,}^1(C_0)$ be a map of energy $E(u, g) \leq E_0$ which satisfies (3.11) for a function $f \in L^2(C_0, g)$ with $\|f\|_{L^2} \leq \Lambda$, a metric $g = h_{b,\phi}^* G_\ell$, with $\ell \geq \ell_0$ and $|b^\pm| \leq b_0$ and variations $v \in T_u^+ H_{\Gamma,*}^1(C_0)$ with $\text{supp}(v) \subset C^* := C_0 \setminus \bigcup_{j,\pm} P_j^\pm$.*

Then the energy is small

$$E(u, B_r^g(p)) \leq \varepsilon$$

on balls around points

$$p \in C_g^*(d) := C_0 \setminus \bigcup_{j,\pm} B_d^g(P_j^\pm).$$

In particular, the estimate

$$\|u\|_{H^2(C_g^*(d),g)} \leq C \cdot E(u, g) + C \|f\|_{L^2}^2$$

holds true with a constant C that depends only on $\Lambda, E_0, \ell_0, b_0$ and d .

Proof. In order to prove the first part of the lemma we argue by contradiction. So assume that for some numbers $\varepsilon, d > 0$, $b_0 < 1$ and $E_0, \Lambda < \infty$ there is a sequence of (u_i, g_i, f_i) as in the lemma and a sequence of radii $\tilde{r}_i \rightarrow 0$ for which $\sup_{x \in C_{g_i}^*(d)} E(u, B_{\tilde{r}_i}^{g_i}(x)) > \varepsilon$. Here we can of course assume that $\varepsilon \leq \varepsilon_1$, the number of Proposition 3.6.

We first prove

Claim: There exist radii $r_i \rightarrow 0$, points $p_i \in C_{g_i}^*(d/2)$ and numbers $\lambda_i \rightarrow \infty$ so that

$$E(u, B_{r_i}^{g_i}(p_i)) = \varepsilon = \max_{p \in B_{\lambda_i r_i}^{g_i}(p_i)} E(u, B_{r_i}^{g_i}(p)).$$

To prove this claim let us first choose points y_i and radii $r_i \rightarrow 0$ so that $E(u, B_{r_i}^{g_i}(y_i)) = \varepsilon = \max_{p \in C_{g_i}^*(d)} E(u, B_{r_i}^{g_i}(p))$. Then the claim is trivially true for $p_i = y_i$ unless the points y_i converge to the boundary (relative to C_0) of the set $C_{g_i}^*(d)$ defined in the lemma.

So assume that, after passing to a subsequence, $\text{dist}_{g_i}(y_i, P_j^\pm) \rightarrow d$ for one of the point P_j^\pm , say for P_0^+ .

We then consider concentric annuli

$$A_i^k := B_{d-2kr_i}^{g_i} \setminus B_{d-2(k+1)r_i}^{g_i}(P_0^+)$$

constructed so that two balls of radius r_i one having its centre in A_k the other in A_{k+2} are always disjoint. Thus the number of such annuli that contain a point p for which $E(u, B_{r_i}^{g_i}(p)) > \varepsilon$ can be no more than $K_\varepsilon = 2 \lfloor \frac{E_0}{\varepsilon} \rfloor$.

We now choose $N_i \rightarrow \infty$ so that for i large $(K_\varepsilon + 1) \cdot N_i r_i \leq d/2$ and observe that $B_d^{g_i} \setminus B_{d/2}^{g_i}(x_0)$ must contain an annulus $\bigcup_{k=k_i}^{k_i+N_i-1} A_i^k$ of thickness $N_i r_i$ which does not contain any point p with $E(u, B_{r_i}^{g_i}(p)) > \varepsilon$. Possibly reducing r_i so that the maximum of $p \mapsto E(u, B_{r_i}^{g_i}(p))$ on $B_{2d}^{g_i} \setminus B_{d-k_i r_i}^{g_i}(x_0)$ is equal to ε and selecting p_i to be a point at which this maximum is achieved then implies the claim.

Based on this claim we now derive a contradiction using a standard blow-up argument where we distinguish between

Case 1: $\text{dist}_{g_i}(p_i, \partial C_0) r_i^{-1} \rightarrow \infty$ for some subsequence

Case 2: $\text{dist}(p_i, \partial C_0) r_i^{-1} \leq C$ for some constant $C < \infty$.

In the first case, we rescale to maps $v_i(x) = u(\exp_{p_i}^{g_i}(r_i x))$ that are defined on larger and larger balls in \mathbb{R}^2 . We also observe that as always in such a bubbling argument the resulting metrics can be written as $(\exp_{p_i}^{g_i}(r_i \cdot))^* g_i = r_i^2 \tilde{g}_i$ for metrics \tilde{g}_i that converge locally to the euclidean metric g_{eucl} .

In particular, Proposition 3.6 implies that the H^2 -norm of the maps v_i on compact subsets of $(\mathbb{R}^2, g_{\text{eucl}})$ are bounded uniformly and that $\Delta v_i \rightarrow 0$ locally in L^2 since we have assumed $\|\Delta_{g_i} u_i\|_{L^2} = \|f_i\|_{L^2}$ to be bounded.

Thus, after passing to a subsequence, we conclude that v_i converges strongly in H_{loc}^1 and weakly in H_{loc}^2 to a limit $v_\infty : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ which is harmonic and has bounded energy and is thus constant. At the same time $E(v_\infty, D_1(0)) = \lim_{i \rightarrow \infty} E(v_i, D_1(0), \tilde{g}_i) = \lim_{i \rightarrow \infty} E(u_i, B_{r_i}^{g_i}(p_i)) > \varepsilon$ resulting in the desired contradiction.

In the second case we rescale not around the points p_i themselves, but rather around their nearest point projection \tilde{p}_i to the boundary of C_0 using local isothermal coordinates. The resulting maps, defined on larger and larger subsets of the halfplane $\mathbb{H} = \{y \in \mathbb{R}^2, y_2 \geq 0\}$, satisfy uniform H^2 bounds on compact sets of \mathbb{H} and have energy at least ε in the ball $B_{C+1}(0) \cap \mathbb{H}$.

We again obtain a harmonic limit with bounded energy, now defined only on the half-plane, but furthermore constant on the axis $\partial \mathbb{H}$, since the maps $u_i|_{\partial C_0}$ are equicontinuous, compare Corollary A.2 in the appendix. Thus also this limit must be constant leading again to a contradiction. This completes the proof of the first part of the lemma.

The second part now immediately follows from Proposition 3.6 and the first part if we choose $\varepsilon = \varepsilon_1$ to be the constant of that proposition. \square

A further consequence of Proposition 3.6 is

Corollary 3.9. *Let $u_i \in H_{\Gamma,*}^1(C_0)$ be any sequence of maps with uniformly bounded energy for which (3.11) is satisfied for functions f_i with $\sup_i \|f_i\|_{L^2(M, g_i)} < \infty$ and metrics $g_i = h_{b_i, \phi_i}^* G_{\ell_i}$ with $\sup_i |b_i^\pm| < 1$ and $\inf_i \ell_i > 0$ and for variations $v \in T_{u_i}^+ H_{\Gamma,*}^1(C_0)$ with $\text{supp}(v) \subset C^* = C_0 \setminus \bigcup_{j,\pm} P_j^\pm$. If no energy concentrates at the points P_j^\pm in the sense that*

$$\lim_{r \rightarrow 0} \sup_i E(u_i, B_r^{g_i}(P_j^\pm)) = 0, \quad j = 0, 1, 2 \quad (3.14)$$

then the maps u_i are equicontinuous on all of C_0 .

We note that the above lemma implies in particular that any map $u \in H_{\Gamma,*}^1(C_0)$ satisfying (3.11) for variations as described above is continuous in the points P_j^\pm where the three-point-condition is imposed.

Proof. Let u_i be a sequence of maps as described in the lemma. As Lemma 3.8 yields uniform H^2 -bounds and thus equicontinuity for the u_i 's on any compact subset of C^* it is enough to prove that to any number $\varepsilon > 0$ there exists a radius $r_0 > 0$ so that

$$\sup_{B_{r_0}^{g_i}(P_j^\pm)} u_i < \varepsilon, \quad j = 0, 1, 2, \quad i \in \mathbb{N}.$$

To begin with, we recall from Corollary A.2 that the traces $u_i|_{\partial C_0}$ are equicontinuous so that for $r_0 > 0$ sufficiently small

$$\sup_{B_{r_0}^{g_i}} u_i \leq \sup_{r \in (0, r_0]} \sup_{\partial B_r^{g_i}} u_i + \sup_{r \in (0, r_0]} \sup_{\partial B_r^{g_i}} u_i \leq \sup_{r \in (0, r_0]} \sup_{\partial B_r^{g_i}} u_i + \frac{1}{2} \varepsilon$$

where $\partial B_r \subset \check{C}_0$ is the boundary relative to C_0 and where we consider balls with centre P_j^\pm unless indicated otherwise.

We bound the oscillation over $\partial B_r^{g_i}$ by deriving suitable H^2 -estimates on annuli $A_r^i := B_{\frac{5}{4}r}^{g_i} \setminus B_{\frac{3}{4}r}^{g_i}$. First of all, by (3.14) we can assume that $r_0 > 0$ is small enough so that $E(u_i, B_{2r_0}^{g_i}) < \varepsilon_1$, the number of Proposition 3.6. Then, given any $r \in (0, r_0]$ we cover the above annulus A_r^i by balls $B_{r/4}^{g_i}(x_k)$ so that the corresponding balls with double the radius are contained in $B_{2r}^{g_i} \setminus \{P_j^\pm\}$ and so that no point is contained in more than K of these larger balls $B_{r/2}^{g_i}(x_k)$, K independent of r and i . Since (3.3) is satisfied for (u_i, g_i, f_i) , at least for variations supported on C^* , we can apply Proposition 3.6 to bound

$$\int_{B_{r/4}^{g_i}(x_k)} |\nabla_{g_i}^2 u_i|^2 dv_{g_i} \leq Cr^{-2} E(u_i, B_{r/2}^{g_i}(x_k)) + C \|f_i\|_{L^2(B_{r/2}^{g_i}(x_k))}^2$$

and thus also

$$\int_{A_r^i} |\nabla_{g_i}^2 u_i|^2 dv_{g_i} \leq Cr^{-2} E(u_i, B_{2r}^{g_i}) + C \|f_i\|_{L^2(B_{2r}^{g_i})}^2$$

for a constant C that is independent of i .

Observe that while the oscillation is invariant under the rescaling $\tilde{u}_i(x) = u_i(\exp_{P_j^\pm}(rx))$, the left-hand-side of the above estimate transforms as

$$\begin{aligned} \int_{A_1(0)} |\nabla_{g_{euc}}^2 \tilde{u}_i|^2 dx &\leq Cr^2 \int_{A_r} |\nabla_{g_i}^2 u_i|^2 dv_{g_i} + CE(u_i, B_{2r}^{g_i}) \\ &\leq CE(u_i, B_{2r_0}^{g_i}) + Cr^2 \|f_i\|_{L^2(B_{2r_0}^{g_i})}^2. \end{aligned}$$

Applying the Sobolev embedding theorem on a suitable subset of the fixed half-annulus $A_1(0) \subset \mathbb{R}^2$ thus allows us to conclude that

$$\left(\sup_{\partial B_{r_0}^{g_i}} u_i \right)^2 \leq C \cdot \int_{A_1(0)} |\nabla_{g_{euc}}^2 \tilde{u}_i|^2 + |d\tilde{u}_i|^2 dx \leq CE(u_i, B_{2r_0}^{g_i}) + Cr^2 \|f_i\|_{L^2(B_{2r_0}^{g_i})}^2$$

for every $r \in (0, r_0]$ again with constants that are independent of i .

Since the L^2 -norms of f_i are uniformly bounded and since we assumed that there is no concentration of energy at the points P_j^\pm we can thus choose r_0 small enough so that the above expression is less than $\varepsilon/2$ which gives the desired estimate. \square

Remark 3.10. We observe that the claim of the above corollary remains true on arbitrary compact regions of $C_0 \setminus (\{0\} \times S^1)$ also without the assumption of a lower bound on ℓ . Similarly, as all arguments are carried out locally, knowing that u satisfies (3.3) for variations supported in an open set $U \setminus \bigcup_{j,\pm} P_j^\pm$ is sufficient to conclude that u is continuous on every compact subset K of U with modulus of continuity depending only on K , U and the bounds on $|b^\pm|$, $E(u, g)$ as well as the local L^2 -norm of f .

3.3 No concentration of energy at points P_j^\pm .

As the flow of metrics is determined in terms of the (non-local) projection of the Hopf-differential onto \mathcal{V} , we need to exclude the possibility that a non-trivial amount of energy (and thus possibly of L^1 norm of Φ) is concentrating near one of the points P_j^\pm . Such a concentration of energy would be lost in a limiting process meaning that we could not expect the evolution of the limiting metric to be described by the projection of the limiting Hopf-differential.

To this end we prove the following key lemma

Lemma 3.11. *To any given numbers $\Lambda, M, E_0 < \infty$, $b_0 < 1$ and $\varepsilon, d_0 > 0$ there exists a radius $r > 0$ such that the following holds true. Let $u \in H_{\Gamma,*}^1(C_0)$ be a map with energy $E(u, g) \leq E_0$ that is bounded by $\|u\|_{L^\infty} \leq M$ and that weakly solves the differential inequality (3.11) for a metric $g = h_{b,\phi}^* G_\ell$ for which $|b^\pm| \leq b_0$ and for a function f with $\|f\|_{L^2(C_0,g)} \leq \Lambda$. Then the estimate*

$$E(u, B_r^g(x_0)) < \varepsilon$$

holds true for every point $x_0 = (x, \theta) \in C_0$ with $|x| \geq d_0$, in particular for $x_0 = P_j^\pm$.

Proof. Thanks to Lemma 3.8 it is sufficient to establish the claim for the points P_j^\pm , say for $x_0 = P_0^+$.

So let us assume that for some fixed numbers $\Lambda, M, E_0 < \infty$ and $b_0 < 1$ there exists a number $\varepsilon_2 > 0$ such that there are triples $(u_i, g_i = h_{b_i, \phi_i}^* G_{\ell_i}, f_i)$ for which all the assumptions of the lemma are satisfied, but for which energy concentrates at x_0 in the sense that

$$E(u_i, B_{r_i}^{g_i}(x_0)) \geq \varepsilon_2$$

for a sequence of radii $r_i \rightarrow 0$.

We remark that the diffeomorphisms $h_{b,\phi}$ defined later on in section 4.1.1 are such that $h_{b,\phi} \equiv h_{b,\tilde{\phi}}$ in a neighbourhood of ∂C_0 if the parameters ϕ^\pm and $\tilde{\phi}^\pm$ agree modulo 2π . Thus, after passing to a subsequence, the metrics converge smoothly to some limiting metric $g = h_{b,\phi}^* G_\ell$, $\ell \geq 0$, at least in a neighbourhood U of ∂C_0 , compare Appendix A.2.

Away from the points P_j^\pm we can apply Lemma 3.8 to conclude that, after passing to a further subsequence, the maps converge on $U^* := U \setminus \bigcup_{j,\pm} P_j^\pm$ in the sense that

$$u_i \rightarrow u_\infty \text{ weakly in } H_{loc}^2(U^*) \text{ and strongly in } W_{loc}^{1,p}(U^*) \text{ for every } p < \infty.$$

Furthermore, the uniform bounds on the energy imply that the maps u_i converge to u_∞ weakly in H^1 on all of U while the uniform L^2 bounds on f_i yield, after passing to a further subsequence, weak L^2 convergence to a limit f_∞ on U . Here and in the following there is no need to specify with respect to which metric g_i the norms are computed as they are uniformly equivalent.

Furthermore the traces $u_i|_{\partial C_0}$ converge uniformly to $u_\infty|_{\partial C_0}$ thanks to the equicontinuity obtained from the Courant-Lebesgue Lemma, so u_∞ can be extended to an element of $H_{\Gamma,*}^1(C_0)$.

We finally remark that the convergence of $(u_i, g_i, f_i) \rightarrow (u_\infty, g, f_\infty)$ implies that the differential inequality (3.11) is again satisfied for (u_∞, g, f_∞) at least for variations supported in U^* , see also Appendix A.1.

The basic idea of the proof, working without modification only if the image of $u_i|_{\partial C_0 \cap B_{r_0}^{g_i}(x_0)}$ happens to be the subset of a straight line, is now the following.

Since u_∞ is an element of H^1 , we can choose $r_0 > 0$ so that $E(u_\infty, B_{2r_0}^g(x_0))$ is far smaller than ε_2 and thus in particular far smaller than the energy of the maps u_i on this ball. We would thus like to consider variations u_ε of u_i which have the form $u_i + \varepsilon \lambda \cdot (u_\infty - u_i)$, λ a cut-off function supported on $B_{2r_0}^{g_i}(x_0)$. Then, if we could insert $v = \frac{d}{d\varepsilon}|_{\varepsilon=0} u_\varepsilon$ as test-function into (3.3) we would get a contradiction since the first term would give a negative contribution of roughly $-\varepsilon_2$ which could not be compensated by the second possibly positive term.

Of course, having Plateau- rather than Dirichlet-boundary conditions, the maps $u_i + \varepsilon \lambda \cdot (u_\infty - u_i)$ are in general not in H_Γ^1 . To obtain an admissible variation we shall thus use the following lemma which is proved later on.

Lemma 3.12. *Let Γ be a regular closed Jordan curve in \mathbb{R}^n of class at least C^3 . Then there exist constants $\hat{r} = \hat{r}(\Gamma) > 0$ and $C = C(\Gamma)$ such that for any point $p \in \Gamma$ and any $r \in (0, \hat{r}(\Gamma))$ there exists a C^2 -diffeomorphism $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $\Phi = id$ outside of $B_{2r}(p)$ and $\Phi(p) = p$ which satisfies*

$$r^{-2} \|\Phi - id\|_{L^\infty} + r^{-1} \|d\Phi - id\|_{L^\infty} + \|d^2\Phi\|_{L^\infty} \leq C \quad (3.15)$$

and which straightens out the curve Γ in a neighbourhood of p in the sense that

$$\Phi : \Gamma \cap B_r(p) \rightarrow \{p\} + T_p\Gamma.$$

Returning to the proof of Lemma 3.11 we let $\bar{r} \in (0, \hat{r}(\Gamma_+))$ be a fixed number that we determine later on. Then, as the traces $u_i|_{\partial C_0}$ are equicontinuous we can choose $r_0 > 0$ small enough so that

$$u_i(B_{2r_0}^{g_i}(x_0) \cap \partial C_0) \subset B_{\bar{r}}(p_0)$$

where $p_0 := u_i(x_0) = \alpha^+(\theta_0) \in \mathbb{R}^n$ is prescribed by the three-point-condition.

Let now Φ be the map given by Lemma 3.12 which straightens out the boundary curve on the ball $B_{\bar{r}}(p_0) \subset \mathbb{R}^n$ and let $\lambda_i \in C_0^\infty(B_{2r_0}^{g_i}(x_0))$ be radial cut-off functions which are identically 1 on $B_{r_0}^{g_i}(x_0) \subset C_0$, non-increasing in radial direction and whose derivatives satisfy $|\nabla_{g_i}^k \lambda_i| \leq C r_0^{-k}$, $k = 0, 1, 2$, C independent of i .

We then define

$$u_i^\varepsilon := \Phi^{-1} \circ [\Phi \circ u_i + \varepsilon \cdot \lambda_i \cdot (\Phi \circ u_\infty - \Phi \circ u_i)]$$

and claim that, for $\varepsilon > 0$ sufficiently small and after possibly rescaling u_∞ , this is an admissible variation of u_i , i.e. that

$$u_i^\varepsilon \in H_{\Gamma,*}^1(C_0).$$

As u_i^ε is clearly of class H^1 , and as $u_i^\varepsilon \equiv u_i$ away from $\text{supp}(\lambda_i)$ it is enough to show that $u_i^\varepsilon(x_0) = p_0$ and that $u_i^\varepsilon|_{\partial C_0 \cap B_{2r_0}^{g_i}(x_0)}$ is a weakly monotone parametrisation of a subarc of Γ (namely of $u_i(\partial C_0 \cap B_{2r_0}^{g_i}(x_0))$). The first claim is trivial since the three-point-condition is satisfied for all u_i 's and thus also for u_∞ . The choice of r_0 and the properties of Φ imply furthermore that the restriction of $\Phi \circ u_i$ to $\partial C_0 \cap B_{2r_0}^{g_i}(x_0)$ gives a weakly monotone parametrisation of a segment in the tangent $t_{p_0, \Gamma} = \{p_0\} + T_{p_0}\Gamma_+$. The same holds true also for $\Phi \circ u_\infty$ and thus any interpolation of these two maps gives a parametrisation of such a segment.

If this parametrisation is monotone, then $w_i := \frac{d}{d\varepsilon}|_{\varepsilon=0} u_i^\varepsilon \in T_{u_i}^+ H_{\Gamma,*}^1$ is an admissible test function for the differential inequality

$$\int_{C_0} \langle du_i, dw_i \rangle dv_{g_i} + \int_{C_0} f_i \cdot w_i dv_{g_i} \geq 0 \quad (3.16)$$

which will lead to the desired contradiction for i sufficiently large provided the above construction is carried out on sufficiently small balls $B_{r_0}^{g_i}$ and $B_{\bar{r}}(p_0)$. To this end, we first remark that

$$w_i = \frac{d}{d\varepsilon}|_{\varepsilon=0} u_i^\varepsilon = \lambda_i \cdot (d\Phi^{-1})(\Phi(u_i)) \cdot (\Phi(u_\infty) - \Phi(u_i))$$

is supported in the small ball $B_{2r_0}^{g_i}(x_0)$ and bounded $\|w_i\|_{L^\infty} \leq C_2$ by a constant depending only on Γ^+ and the bound M imposed on the L^∞ norms of the u_i .

We can thus bound the second term in (3.16) by

$$|\int_{C_0} f_i \cdot w_i dv_{g_i}| \leq C_2 \cdot \|f_i\|_{L^1(B_{2r_0}^{g_i}(x_0))} \leq Cr_0 \|f_i\|_{L^2(C_0)} \leq C\Lambda r_0 < \frac{\varepsilon_2}{4}$$

provided r_0 is chosen sufficiently small.

The first term of (3.16) on the other hand can be bounded from above by

$$\begin{aligned} \int \langle du_i, dw_i \rangle dv &\leq \int \lambda_i \cdot \left\langle \left[(d\Phi^{-1})(\Phi(u_i)) \cdot d\Phi(u_\infty) \cdot du_\infty - du_i \right], du_i \right\rangle dv \\ &\quad + C \sup_{x \in B_{2r_0}^{g_i}} (|(d^2\Phi)(u_i(x))| \cdot |\Phi(u_\infty(x)) - \Phi(u_i(x))|) \cdot \int \lambda_i |du_i|^2 dv \\ &\quad + \frac{C}{r_0} \int_{\text{supp}(d\lambda_i)} |\Phi(u_\infty) - \Phi(u_i)| \cdot |du_i| dv \\ &\leq -\left[1 - \frac{1}{4} - C \cdot \omega_i(r_0)\right] \cdot \int \lambda_i \cdot |du_i|^2 dv \\ &\quad + C \cdot E(u_\infty, B_{2r_0}^{g_i}) + C \cdot \|u_i - u_\infty\|_{L^\infty(B_{2r_0}^{g_i} \setminus B_{r_0}^{g_i})}^2 + CE(u_i, B_{2r_0}^{g_i} \setminus B_{r_0}^{g_i}) \end{aligned} \quad (3.17)$$

where all balls are to be taken with centre x_0 , all integrals and norms are computed with respect to g_i and where we set

$$\omega_i(r_0) := \sup_{x \in B_{2r_0}^{g_i}} (|(d^2\Phi)(u_i(x))| \cdot |u_\infty(x) - u_i(x)|).$$

We recall that $u_i \rightarrow u_\infty$ in $W_{loc}^{1,p}(U^*)$ for every $p < \infty$ and that the metrics converge. Thus the penultimate term in (3.17) tends to zero as $i \rightarrow \infty$, and is in particular $\leq \frac{1}{4}\varepsilon_2$ for i large. Furthermore, for i large, the last term in (3.17) is bounded by $CE(u_\infty, B_{3r_0}^g(x_0)) + \frac{1}{4}\varepsilon_2 \leq \frac{1}{2}\varepsilon_2$, where the last inequality holds provided r_0 is chosen sufficiently small.

Given that $\frac{1}{2} \int \lambda_i \cdot |du_i|^2 dv_{g_i} \geq E(u_i, B_{r_0}^{g_i}) \geq \varepsilon_2$, we can thus estimate (for i large)

$$\int_{C_0} \langle du_i, dw_i \rangle dv_{g_i} + \int_{C_0} f_i \cdot w_i dv_{g_i} \leq -\left(\frac{1}{4} - C\omega_i(r_0)\right) \cdot \int \lambda_i \cdot |du_i|^2 dv_{g_i}, \quad (3.18)$$

which leads to the desired contradiction to (3.16) provided we show that $r_0 > 0$ can be chosen so that

$$C\omega_i(r_0) < \frac{1}{4} \text{ for } i \text{ large.} \quad (3.19)$$

To prove this last claim we recall that $d^2\Phi$ vanishes identically outside the ball $B_{2\bar{r}}(p)$. This means that ω_i is obtained as supremum over a set on which the oscillation of the

function u_i is a priori no more than $4\bar{r}$, for a number $\bar{r} > 0$ that we can still reduce if needed. This aspect of the construction is crucial as we have no control on the behaviour of u_i near x_0 , so could in particular not hope for the oscillation of u_i over the full ball $B_{2r_0}^{g_i}$ to be uniformly small.

For $\bar{r} > 0$ sufficiently small and i large, we can in particular estimate

$$\begin{aligned} C\omega_i(r_0) &\leq C \cdot \sup_{B_{2r_0}^{g_i}(x_0) \cap \text{supp}(d^2\Phi \circ u_i)} |u_i - p_0| + C \cdot \sup_{B_{2r_0}^{g_i}(x_0)} |u_\infty - p_0| \\ &\leq C\bar{r} + C \cdot \text{osc}_{B_{4r_0}^g(x_0)} u_\infty \leq \frac{1}{8} + C \cdot \text{osc}_{B_{4r_0}^g(x_0)} u_\infty. \end{aligned}$$

We finally recall that u_∞ satisfies (3.3) for the function $f_\infty \in L^2$, the limiting metric g and for variations supported in U^* . As such Corollary 3.9 and Remark 3.10 imply that u_∞ is continuous at least in a neighbourhood of ∂C_0 and thus in particular in the points P_j^\pm . Carrying out the above argument for a small enough radius r_0 , which might depend on u_∞ but is independent of i , we thus find that (3.19) indeed holds.

While the parametrisation of $u_i^\varepsilon|_{\partial C_0}$ might not be monotone, we can always replace u_∞ by $\tilde{u}_\infty = u_\infty \circ \rho$ where ρ is a dilation around x_0 with a small factor which, thanks to the uniform convergence of the traces, can be chosen so that for all i large enough $\tilde{u}_\infty(\partial C_0 \cap B_{2r_0}^{g_i}) \subset u_i(\partial C_0 \cap B_{r_0}^{g_i})$. The resulting u_i^ε are then admissible and the above argument still applies, as energy and oscillation of \tilde{u}_∞ are bounded by $CE(u_\infty, B_{2r_0}^{g_i})$ and $\text{osc}_{B_{2r_0}^{g_i}} u_\infty$ while $\|u_i - \tilde{u}_\infty\|_{L^\infty(B_{2r_0}^{g_i} \setminus B_{r_0}^{g_i})} \leq \|u_i - u_\infty\|_{L^\infty(B_{2r_0}^{g_i} \setminus B_{r_0}^{g_i})} + \text{osc}_{B_{2r_0}^{g_i}} u_\infty$.

□

It remains to give the

Proof of Lemma 3.12. Let Γ be a C^3 closed Jordan curve, let $p_0 \in \Gamma \subset \mathbb{R}^n$, let $t_{p_0, \Gamma} = p_0 + T_{p_0} \Gamma$ be the tangent to Γ at p_0 and let $\pi : \mathbb{R}^n \rightarrow t_{p_0, \Gamma}$ be the nearest point projection onto $t_{p_0, \Gamma}$.

Observe that for $\hat{r} > 0$ chosen sufficiently small, in particular so that $\Gamma \cap B_{3\hat{r}}(p)$ is connected, this projection induces a C^2 bijection from $\Gamma \cap B_{3\hat{r}}(p)$ to a segment in $t_{p_0, \Gamma}$. Furthermore, after possibly reducing $\hat{r} > 0$, we have that $t_{p_0, \Gamma} \cap B_{2r}(p) \subset \pi(\Gamma \cap B_{3r}(p))$ for all $r \in (0, \hat{r})$.

We now consider $\psi : t_{p_0, \Gamma} \cap B_{2\hat{r}}(p) \rightarrow \mathbb{R}^n$ defined by $\psi = id|_{t_{p_0, \Gamma}} - (\pi|_{\Gamma \cap B_{3\hat{r}}(p)})^{-1}$ and claim that given any number $r \in (0, \hat{r})$

$$\Psi = \Psi_r := id + \lambda_r \cdot \psi \circ \pi = id + \lambda_r \cdot [\pi - (\pi|_{\Gamma \cap B_{3\hat{r}}(p)})^{-1} \circ \pi]$$

gives the desired diffeomorphism. Here $\lambda_r \in C_0^\infty(B_{2r}(p), \mathbb{R})$ is given by a cut-off function which is identically 1 on $B_r(p)$ and which satisfies the usual estimates of $|D^k \lambda| \leq Cr^{-k}$, $k = 0, 1, 2$. Since $\pi(\Gamma \cap B_{2r}(p)) \subset B_{2r}(p) \cap t_{p_0, \Gamma} \subset \pi(\Gamma \cap B_{3\hat{r}}(p))$ the map Ψ is well defined for any radius $r \in (0, \hat{r})$.

To prove that Ψ has the properties we asked for in Lemma 3.12 we first remark that $\pi(p) = p$ and thus $\psi(p) = 0$, i.e. $\Psi(p) = p$. More generally, given any point $x \in \Gamma \cap B_r(p)$ we obtain that

$$\Psi(x) = x + \lambda_r \cdot [\pi(x) - x] = \pi(x)$$

so Ψ straightens the curve $\Gamma \cap B_r(p)$ to a line as described in the lemma. Since λ_r and thus also $\Psi - id$ is supported in $B_{2r}(p)$, it remains to show that the estimate (3.15) claimed in the lemma holds true with a constant independent of r .

Since Γ is of class C^3 and since we project onto the tangent to Γ , an estimate of the form $|\pi(x) - x| \leq Cr^2$ is valid for all $x \in \Gamma \cap B_{3r}(p)$ (recall that $\Gamma \cap B_{3r}(p)$ is connected).

We then use that we can write any $y \in B_{2r}(p) \cap t_{p_0, \Gamma}$ as $y = \pi(x)$ for an $x \in \Gamma \cap B_{3r}(p)$ to conclude that $|\psi(y)| = |\pi(x) - x| \leq Cr^2$. In particular

$$\|\Psi - id\|_{L^\infty} \leq C \sup_{B_{2r}(p)} |\psi \circ \pi| \leq C \sup_{B_{2r}(p) \cap t_{p_0, \Gamma}} |\psi| \leq Cr^2$$

holds true with a constant C depending only on Γ . We furthermore remark that since the derivative of the function ψ (which is defined only on a line) vanishes in the point p there exists a constant C (again depending only on Γ) so that $\|d\psi\|_{L^\infty(t_{p, \Gamma} \cap B_{2r}(p))} \leq Cr$. We can thus bound

$$\begin{aligned} \|d\Psi - id\|_{L^\infty} &\leq \|d\lambda\|_{L^\infty} \cdot \|\psi \circ \pi\|_{L^\infty} + \|d\psi\|_{L^\infty} \cdot \|d\pi\|_{L^\infty} \\ &\leq Cr^{-1} \cdot r^2 + C\|d\psi\|_{L^\infty} \leq Cr \end{aligned}$$

as well as

$$\|d^2\Psi\|_{L^\infty} \leq \|d^2\lambda\|_{L^\infty} \cdot \|\psi \circ \pi\|_{L^\infty} + \|d\lambda\|_{L^\infty} \cdot \|d\psi\|_{L^\infty} + C\|d^2(\psi \circ \phi)\|_{L^\infty} \leq C.$$

□

An important consequence we can derive from our key Lemma 3.11 is

Corollary 3.13. *Let $\Lambda, M < \infty$ and let K be a compact subset of $\widetilde{\mathcal{M}}$. Then to every $\varepsilon > 0$ there exists a constant $\delta > 0$ such that the following holds true:*

Let u_1 and u_2 be such that (3.11) is satisfied for functions f_i with $\|f_i\|_{L^2} \leq \Lambda$ and metrics $g_i \in K$. Suppose furthermore that $E(u_i, g_i) \leq E_0$, that $\|u_i\|_{L^\infty} \leq M, i = 1, 2$, and that

$$\|u_1 - u_2\|_{L^2(C_0, g)} \leq \delta$$

for some $g \in K$. Then also

$$\|u_1 - u_2\|_{H^1(C_0, g)} < \varepsilon.$$

Proof. To begin with, we remark that all metrics in K are uniformly equivalent since K is compact. Thus it is sufficient to show the claim for norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^1}$ that are computed with respect to some fixed $g \in K$.

We then argue by contradiction. Assume there is a number $\varepsilon_1 > 0$ and triples $(u_1^k, g_1^k, f_1^k)_{k \in \mathbb{N}}$ as well $(u_2^k, g_2^k, f_2^k)_{k \in \mathbb{N}}$ so that all the assumptions of the lemma are satisfied (for each k) but for which

$$\|u_1^k - u_2^k\|_{L^2(C_0)} \rightarrow 0 \text{ while } \|u_1^k - u_2^k\|_{H^1(C_0)} \geq \varepsilon. \quad (3.20)$$

Then, after passing to a subsequence and using Proposition 3.6 and Lemmas 3.4 and 3.8, we find that locally on C^* the maps $u_{1,2}^k$ converge to limits $u_{1,2}$ strongly in H^1 , where by construction these two limits must agree.

In particular, for any fixed number $r > 0$ we have

$$\|u_1^k - u_2^k\|_{H^1(C_0 \setminus \bigcup_{\pm, j} B_r^g(P_j^\pm))} \rightarrow 0 \quad (3.21)$$

as $k \rightarrow \infty$.

We can then choose $r > 0$ so small that Lemma 3.11, combined with the equivalence of the metrics in K , implies that

$$\|u_i^k\|_{H^1(B_r^g(x_0))} \leq C \|u_i^k\|_{H^1(B_{Cr}^{g_{i_0}^k}(x_0))} \leq \frac{\varepsilon}{24} \text{ for all } k \in \mathbb{N}, i = 1, 2, \text{ and } x_0 \in C_0.$$

Applying this estimates for $x_0 = P_j^\pm$ and combining this with (3.21) yields a contradiction to (3.20). □

3.4 Convergence of the time-discretisation scheme

Given any initial data $(u_0, g_0) \in H_{\Gamma,*}^1(C_0) \times \widetilde{\mathcal{M}}$ we consider the approximate solutions of Teichmüller harmonic map flow $(u_j, g_j) := (u^{h_j}, g^{h_j})$, $h_j = 2^{-j}$, obtained by the time discretisation scheme described in section 3.1. We can analyse the maps u_j using the results of the previous section since $u_j(t)$ can be seen as a stationary solution of

$$\int \langle du_j(t), dw \rangle dv_{\tilde{g}_j(t)} + \int D_t^{h_j} u_j(t) \cdot w dv_{\tilde{g}_j(t)} \geq 0 \text{ for } w \in T_{u_j(t)}^+ H_{\Gamma,*}^1 \text{ and } t \in [0, T],$$

where \tilde{g}_j is piecewise constant so that $\tilde{g}_j(t) = g_j(t_k^{h_j})$, $t \in [t_k^{h_j}, t_{k+1}^{h_j})$.

Based on the results of the previous sections we can pass to a subsequence, still denoted by (u_j, g_j) , of approximate solutions which converge to a limiting curve of maps and metrics (u, g) as described below.

To begin with, we claim that u_j converges uniformly in time with respect to L^2 in space, i.e. that

$$\sup_{t \in [0, T]} \|u_j(t) - u(t)\|_{L^2} \rightarrow 0, \quad \text{for } j \rightarrow \infty \quad (3.22)$$

and that $u \in C^0([0, T], L^2(C_0))$. Here and in the following there is no need to specify with respect to which of the metrics $g_j(t)$ the above convergence is to be understood as these metrics are all uniformly equivalent on the interval $[0, T]$, compare (3.9).

To prove this claim we let $t \mapsto \tilde{u}_j(t)$ be piecewise linear with $\tilde{u}_j(t_k^{h_j}) = u_j(t_k^{h_j})$ for every k . Then (3.8) gives uniform $C_t^{0, \frac{1}{2}} L_x^2$ estimates for $\tilde{u}_j(t)$, namely

$$\begin{aligned} \|\tilde{u}_j(t_2) - \tilde{u}_j(t_1)\|_{L^2} &\leq \int_{t_1}^{t_2} \|D_t^{h_j} u_j\|_{L^2} dt \leq (t_2 - t_1)^{1/2} \left(\int_{t_1}^{t_2} \|D_t^{h_j} u_j\|_{L^2}^2 dt \right)^{1/2} \\ &\leq (2E_0)^{1/2} (t_2 - t_1)^{1/2}. \end{aligned}$$

Thus, after passing to a subsequence, \tilde{u}_j converges in $C_t^0 L_x^2$ to a limit u . Combined with the estimate

$$\sup_t \|u_j(t) - \tilde{u}_j(t)\|_{L^2} \leq \sup_k \int_{t_k^{h_j}}^{t_{k+1}^{h_j}} \|D_t^{h_j} u_j\|_{L^2} dt \leq h_j^{1/2} (2E_0)^{1/2} \rightarrow 0 \text{ for } j \rightarrow \infty$$

that follows from (3.8) this yields (3.22).

We remark that the uniform bounds on the energy furthermore imply that the limiting map is in $L^\infty([0, T], H^1(M, g_0))$ and that the spatial derivatives converge

$$du_j \rightharpoonup du \text{ weakly in } L^2([0, T] \times C_0).$$

We recall that the traces of u_j on ∂C_0 are equicontinuous, so we also have that

$$u_j(t)|_{\partial C_0} \rightarrow u(t)|_{\partial C_0} \text{ uniformly on } \partial C_0 \text{ for all } t \in [0, T],$$

where we stress that we do not claim that the rate of this convergence is uniform in time.

Additionally, the energy inequality (3.8) gives uniform $L^2(C_0 \times [0, T - h_j])$ estimates for the difference quotients $D_t^{h_j} u_j$. Consequently, u is weakly differentiable in time on $C_0 \times [0, T]$ with $D_t^{h_j} u_j \rightharpoonup \partial_t u$ in $L^2_{loc}(C_0 \times [0, T])$.

For the metric component g_j we can apply Lemma 3.3 to get uniform $C^{0,1}$ estimates in time with respect to any C^k metric in space since the L^1 norm of the Hopf-differential is bounded in terms of the (non-increasing) energy. We can thus get convergence of $g_j \rightarrow g$ in $C^{0,\alpha}([0, T], C^k(C_0))$, $\alpha < 1$, with the limiting curve being again a Lipschitz-continuous map from $[0, T]$ to the finite dimensional manifold $\widetilde{\mathcal{M}}$ and thus in particular differentiable in time for almost every $t \in [0, T]$.

We shall now prove that the limit (u, g) obtained in this way gives the desired solution of Teichmüller harmonic map flow, namely

Proposition 3.14. *Let (u^{h_j}, g^{h_j}) be a sequence of approximate solutions to a fixed initial data (u_0, g_0) converging as described above to some limiting curve (u, g) as $h_j \rightarrow 0$. Then the limit (u, g) is a stationary weak solution of Teichmüller harmonic map flow which also satisfies the energy-inequality (for a.e. $t_1 < t_2$).*

We remark that while g is clearly again an admissible curve, we need to prove that its derivative is actually given by the projection of the Hopf-differential of the limit. As the projection operator is non-local, for this part of the proof the key Lemma 3.11 and its Corollary 3.13 are crucial to get strong H^1 convergence for the map u and thus strong L^1 convergence for the Hopf-differential on all of C_0 , in particular also near the points P_j^\pm where Proposition 3.6 does not apply.

Conversely, the analysis of the map component can be carried out very similarly to the work of Duzaar and Scheven [6] and is indeed less involved than the corresponding arguments since the metric is well controlled and since our equation for the map is linear.

Proof of Proposition 3.14. We first infer from the energy inequality (3.8) that for any $\Lambda < \infty$ the set of times

$$A_j^\Lambda := \{t \in [0, T] \text{ so that } \|D_t^{h_j} u_j(t)\|_{L^2} \leq \Lambda\}$$

has measure $\mathcal{L}^1(A_j^\Lambda) \geq T - \frac{CE_0}{\Lambda^2}$, so in particular $\mathcal{L}^1(A^\Lambda) \geq T - \frac{CE_0}{\Lambda^2}$ also for

$$A^\Lambda = \limsup A_j^\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j^\Lambda.$$

We recall that the maps $u_j(t)$ satisfy (3.11) for $f = D_t^{h_j} u_j(t)$ so it is precisely bounds of the form $\|D_t^{h_j} u_j(t)\|_{L^2} \leq \Lambda$ that are required in order to be able to apply the results derived in the previous section.

We begin by analysing the metric component. To prove that g indeed solves (2.3) we show that it agrees with the solution $\hat{g}(t) \in C_t^{0,1} \widetilde{\mathcal{M}}$ of the initial value problem

$$\partial_t \hat{g} = P_{\hat{g}}^\mathcal{V}(Re((\Phi(u, \hat{g}))), \quad \hat{g}(0) = g_0.$$

We will prove this claim based on Corollary 3.13. So let K be the compact set of metrics $h_{b,\phi}^* G_\ell$ satisfying (3.9) as well as (3.10) and let M be a bound on the L^∞ norm of the initial map which therefore also serves as bound for $\|u_j\|_{L^\infty}$, compare Remark 3.2.

Then given any numbers $\varepsilon > 0$ and $\Lambda < \infty$ we let $\delta > 0$ be the number given by Corollary 3.13 and select j_0 so that

$$\sup_{t \in [0, T]} \|u_j(t) - u_k(t)\|_{L^2} \leq \delta \text{ for } j, k \geq j_0.$$

This implies that

$$\|u_k(t) - u_j(t)\|_{H^1(C_0)} \leq \varepsilon \text{ for all } t \in A_j^\Lambda \cap A_k^\Lambda \text{ and } j, k \geq j_0$$

which in turn yields the same bound for $\|u(t) - u_j(t)\|_{H^1(C_0)}$ for $t \in A_j^\Lambda \cap A^\Lambda$.

For times in $A_j^\Lambda \cap A^\Lambda$ the difference of the Hopf-differentials is thus controlled by

$$\begin{aligned} \|Re_{\hat{g}}(\Phi(u, \hat{g})(t)) - Re_{g_j}(\Phi(u_j, g_j)(t))\|_{L^1} &\leq C \cdot \|(g_j - \hat{g})(t)\|_{C^0} \cdot E_0 \\ &\quad + C \cdot E_0^{\frac{1}{2}} \cdot \|(u_j - u)(t)\|_{H^1(C_0)} \\ &\leq C \cdot \|(g_j - \hat{g})(t)\|_{C^0} + C \cdot \varepsilon, \end{aligned}$$

c.f. (3.6). Here and in the following constants C may depend on E_0 , K and T but not on ε or j unless indicated otherwise.

Based on Lemma 3.3 we can thus conclude that

$$\|\partial_t(\hat{g} - g_j)(t)\|_{C^k} \leq C \cdot \|(\hat{g} - g_j)(t)\|_{C^k} + C\varepsilon$$

for any $t \in A_j^\Lambda \cap A^\Lambda$ and for $j \geq j_0(\Lambda, \varepsilon)$.

On the other hand, we can always bound the C^k norms of both $\partial_t g_j$ and $\partial_t \hat{g}$ by $C \cdot E_0$, compare Lemma 3.3. We shall use these trivial bounds on the set $[0, T] \setminus (A_{h_j}^\Lambda \cap A^\Lambda)$ on which we cannot apply any of the results of the previous section since we lack the necessary control on the inhomogeneity of (3.11).

Combining these two cases, we obtain that for almost every t

$$\|\partial_t(\hat{g} - g_j)(t)\|_{C^k} \leq C \cdot \|(\hat{g} - g_j)(t)\|_{C^k} + h_j^\Lambda(t)$$

where

$$h_j^\Lambda = C \cdot \varepsilon + CE_0 \chi_{[0, T] \setminus (A_j^\Lambda \cap A^\Lambda)}.$$

Based on Gronvall's Lemma, we can thus conclude that for any $t \in [0, T]$ and $j \geq j_0(\varepsilon, \Lambda)$

$$\|(\hat{g} - g_j)(t)\|_{C^k} \leq e^{CT} \cdot \int_0^t |h_j^\Lambda| dt \leq C\varepsilon + \frac{C}{\Lambda^2}.$$

Choosing $\Lambda \rightarrow \infty$ and $\varepsilon \rightarrow 0$ and corresponding values of $j_0(\varepsilon, \Lambda) \rightarrow \infty$ yields the claim that $g_j \rightarrow \hat{g}$ uniformly and thus that $g = \hat{g}$ is indeed the solution of (2.3).

We now turn to the analysis of the map component where we follow largely the arguments of [6]. To begin with, we observe that for almost every time $t \in [0, T]$ there exists a number $\Lambda < \infty$ such that $t \in A^\Lambda$. Choosing a subsequence along which $\|D_t^{h_j} u_j(t)\|_{L^2(C_0)} \rightarrow \liminf_{j \rightarrow \infty} \|D_t^{h_j} u_j(t)\|_{L^2(C_0)} \leq \Lambda$ we conclude that

$$u_j(t) \rightarrow u(t)$$

converges not only strongly in L^2 , but thanks to Corollary 3.13 indeed strongly in H^1 on all of C_0 and, thanks to Lemma 3.8, also weakly in $H_{loc}^2(C^*)$ where $C^* := C_0 \setminus \bigcup \{P_j^\pm\}$. We stress that the choice of this subsequence is allowed to depend on the time t we are considering.

We furthermore remark that combining the uniform H^2 -estimates for u_j valid on subsets $\Omega \subset\subset C^*$ with the uniform convergence of the metrics $g_j \rightarrow g$ yields that also $\Delta_{g_j(t)} u_j(t) \rightharpoonup \Delta_{g(t)} u(t)$ weakly in $L^2(\Omega)$ for each such Ω . Using the L^2 bound on $\Delta_{g_j(t)} u_j(t) = D_t^{h_j} u_j$ valid for $t \in A_j^\Lambda$ we can thus conclude that

$$\|\Delta_{g(t)} u(t)\|_{L^2(\Omega)} \leq \lim_{j \rightarrow \infty} \|\Delta_{g_j(t)} u_j(t)\|_{L^2(\Omega)} \leq \Lambda \text{ for each } \Omega \subset\subset C^*.$$

Passing to the limit in both the Euler-Lagrange-equation and the stationarity condition, cf. Appendix A.1, thus yields that $u(t)$ is a stationary solution of (3.11) for a function $f(t) = \Delta_{g(t)} u(t)$ whose $L^2(C_0, g)$ -norm is again bounded by Λ , or indeed more precisely by $\liminf_{j \rightarrow \infty} \|D_t^{h_j} u_j(t)\|_{L^2(C_0, g_j)}$.

Repeating this argument for a.e. $t \in [0, T]$ we thus obtain a function $f : [0, T] \times C_0$ which must have bounded $L^2(C_0 \times [0, T])$ -norm since

$$\|f\|_{L^2(C_0 \times [0, T])}^2 \leq \int_0^T \liminf_{j \rightarrow \infty} \|D_t^{h_j} u_j(t)\|_{L^2(C_0)}^2 dt \leq \liminf_{j \rightarrow \infty} \|D_t^{h_j} u_j\|_{L^2(C_0 \times [0, T])}^2 \leq 2E_0$$

where the last inequality follows from (3.8).

We now wish to show that f agrees with the time derivative of u . To this end, proceeding as in [6], we set

$$\tilde{u}_j^\Lambda(t) = \begin{cases} u(t) & \text{if } t \in B_j^\Lambda := [0, T] \setminus A_j^\Lambda \\ u_j(t) & \text{if } t \in A_j^\Lambda \end{cases} \text{ and } \tilde{f}_j^\Lambda(t) = \begin{cases} f(t) & \text{if } t \in B_j^\Lambda \\ D_t^{h_j} u_j(t) & \text{if } t \in A_j^\Lambda \end{cases}$$

in order to obtain a new sequence of pairs satisfying (3.11) for $\tilde{g}_j(t)$ but for which the estimate $\|\tilde{f}_j^\Lambda(t)\|_{L^2(C_0)} \leq \Lambda$ is now satisfied for every j and every $t \in [0, T]$.

We first claim that for any sequence $\Lambda_j \rightarrow \infty$

$$\tilde{f}_j^{\Lambda_j} \rightharpoonup \partial_t u \text{ weakly in } L^2(C_0 \times [0, T]) \text{ for } j \rightarrow \infty,$$

or, as we know that $D_t^{h_j} u_j \rightharpoonup \partial_t u$, equivalently $\tilde{f}_j^{\Lambda_j} - D_t^{h_j} u_j \rightharpoonup 0$. Indeed, given any function $\phi \in L^2(C_0 \times [0, T])$ we have that

$$\begin{aligned} \int_{[0, T] \times C_0} \phi \cdot (D_t^{h_j} u_j - \tilde{f}_j^{\Lambda_j}) dv_g dt &\leq \|\phi\|_{L^2(B_j^{\Lambda_j} \times C_0)} \cdot (\|\tilde{f}_j^{\Lambda_j}\|_{L^2([0, T] \times C_0)} + \|D_t^{h_j} u_j\|_{L^2([0, T] \times C_0)}) \\ &\leq CE_0^{\frac{1}{2}} \|\phi\|_{L^2(B_j^{\Lambda_j} \times C_0)} \end{aligned}$$

which tends to zero as $\Lambda_j \rightarrow \infty$ since the measure $\mathcal{L}(B_j^{\Lambda_j}) \rightarrow 0$.

At the same time we claim that $d\tilde{u}_j^{\Lambda_j}$ converges not just weakly, which would be evident from an argument just as carried out above, but indeed strongly in $L^2([0, T] \times C_0)$ to du . Indeed, let us first consider $du - d\tilde{u}_j^\Lambda$ for a fixed number Λ . Since this difference vanishes on B_j^Λ and since we can apply Corollary 3.13 to $u(t)$ and $\tilde{u}_j^\Lambda(t)$ for $t \in A_j^\Lambda$ (and j large enough so that the maps are L^2 close) we obtain that

$$\|du - d\tilde{u}_j^\Lambda\|_{L^2(C_0 \times [0, T])} \rightarrow 0 \text{ for every fixed } \Lambda. \quad (3.23)$$

But the set of times $A_j^{\Lambda_j} \Delta A_j^\Lambda \subset B_j^{\Lambda_j} \cup B_j^\Lambda$, on which $d\tilde{u}_j^\Lambda$ and $d\tilde{u}_j^{\Lambda_j}$ do not agree, has measure no more than $C(\Lambda^{-2} + \Lambda_j^{-2})$. Combined with the uniform bound on the energy this means that (3.23) suffices to conclude that indeed $\|du - d\tilde{u}_j^{\Lambda_j}\|_{L^2(C_0 \times [0, T])} \rightarrow 0$.

Thanks to the strong H^1 convergence we can furthermore approximate each test function $w \in L^2([0, T], T_u^+ H_{\Gamma, *}^1(C_0))$ by elements $w_i \in L^2([0, T], T_{u_i}^+ H_{\Gamma, *}^1(C_0))$ in the sense that $\|dw - dw_i\|_{L^2(C_0 \times [0, T])} \rightarrow 0$, compare Appendix A.1. All in all we thus conclude that

$$\int_0^T \int_{C_0} du \cdot dw + \partial_t u \cdot w dv_g dt \geq 0$$

for all such w . As we have already shown that g satisfies (2.3), we thus obtain that (u, g) is indeed a weak solution of Teichmüller harmonic map flow.

It remains to prove that the stationarity condition (2.5) and the energy identity (2.9) are satisfied for almost every time. To this end we recall that $\bigcup_{\Lambda > 0} A^\Lambda \subset [0, T]$ has full measure and that for every t_1 in this set we can choose a sequence $h_j \rightarrow 0$ such that $D_t^{h_j} u_j(t_1)$ is bounded in L^2 and thus, after passing to a further subsequence using Corollary 3.13, so that $u_j(t_1) \rightarrow u(t_1)$ converges strongly in $H^1(C_0)$.

On the one hand this implies that the Hopf-differentials $\Phi(u_j, g_j)(t_1) \rightarrow \Phi(u, g)(t_1)$ converge strongly in L^1 and that, after passing to a further subsequence, $\Delta_{g_j} u_j(t_1) \rightharpoonup \Delta_g u(t_1)$ weakly in L^2 . We can thus pass to the limit in the stationarity condition (3.4) which is satisfied for $u_j(t_1)$ to conclude that (2.5) holds true for any such time t_1 and thus for a.e. $t \in [0, T]$.

On the other hand, strong $H^1(C_0)$ convergence of $u_j(t_1)$ implies also convergence of the energies $E((u_j, g_j)(t_1)) \rightarrow E((u, g)(t_1))$. Thus as u_j satisfies the energy inequality

$$E(u_j, g_j)(t_1) - E(u_j, g_j)(t) \geq \int_{t_1}^t \|\partial_t g_j\|_{L^2(C_0, g_j)}^2 dt + \frac{1}{2} \int_{t_1}^t \|\partial_t u_j\|_{L^2(M, \tilde{g}_j(t))}^2 dt$$

for all times $t \in [t_1, T]$ and as $u_j(t)$ converges at least weakly in H^1 to $u(t)$ for every T (a further strongly convergent subsequence could be found for almost every t but is not needed) we obtain that

$$E(u, g)(t_1) - E(u, g)(t) \geq \int_{t_1}^t \|\partial_t g\|_{L^2(C_0, g)}^2 dt + \frac{1}{2} \int_{t_1}^t \|\partial_t u\|_{L^2(C_0, g)}^2 dt$$

for every $t \geq t_1$. This completes the proof of the theorem. \square

4 Long time existence

4.1 A priori estimates for the metric component

Before we can analyse admissible curves of metrics in more detail we finally need to decide how to select the family of diffeomorphisms $h_{b, \phi}$ which we use to compensate for the lost degrees of freedom resulting from imposing the three-point-condition.

Rather than just writing down a possible family, we shall first describe which properties we require in the present context of flowing to minimal surfaces. We will then later give an example of such a family, see (4.2), but do not claim this choice to be in any way unique.

To begin with, in order to obtain solutions of the flow that exist for all times we need the following L^2 -completeness property:

Lemma 4.1. *Let $(h_{b,\phi})$ be the family of diffeomorphisms defined in (4.2). Assume that $g(t) = h_{b(t),\phi(t)}^* G_{\ell(t)}$, $t \in [0, T)$, is a Lipschitz continuous curve of metrics such that the diffeomorphisms $h_{b,\phi}$ become singular as $t \rightarrow T$, in the sense that (at least) one of the values $|b^\pm| \rightarrow 1$ or $|\phi^\pm| \rightarrow \infty$ as $t \rightarrow T$. Then*

$$\int_0^T \|\partial_t g\|_{L^2(C_{0,g})} dt = \infty.$$

A further requirement we want to impose in preparation for the asymptotic analysis carried out later in section 5 is

Lemma 4.2. *Let $(h_{b,\phi})$ be the family of diffeomorphisms defined in (4.2) and let $\chi(b, \phi)$ the space of generating vectorfields of $h_{b,\phi}$. Then*

$$\Gamma(TC_0) = \Gamma(TC_0)_* \oplus h_{b,\phi}^* \chi(b, \phi)$$

for all $(b, \phi) \in \Omega_h := D_1(0)^2 \times \mathbb{R}^2$.

Here and in the following $\Gamma(TC_0)_*$ is defined as in (2.6) while $\chi(b, \phi) \subset \Gamma(TC_0)$ denotes the 6 dimensional vectorspace which is spanned by the vectorfields generating the diffeomorphisms $h_{b,\phi}$, i.e. by $Y_{\phi^\pm}(b, \phi)$ characterised by

$$\frac{d}{d\phi^\pm} h_{b,\phi}(s, \theta) = Y_{\phi^\pm}(b, \phi)(h_{b,\phi}(s, \theta)), \quad (4.1)$$

together with the vectorfields $Y_{Re(b^\pm)}(b, \phi)$ and $Y_{Im(b^\pm)}(b, \phi)$ defined by the analogue of (4.1) or, if $b^\pm \neq 0$, equivalently together with the vectorfields $Y_{|b^\pm|}(b, \phi)$, $Y_{Arg(b^\pm)}(b, \phi)$ corresponding to variations of the absolute value respectively the argument of b^\pm .

The final property we shall ask of the diffeomorphisms $h_{b,\phi}$ is that they agree with the identity in a neighbourhood of the central geodesic $\{0\} \times S^1$. This will have the advantage that the modification by these diffeomorphisms does not interfere with the analysis of a possible collapse of the central geodesic, cf. Lemma 4.4. It will furthermore prove to be useful to choose the $h_{b,\phi}$ so that the support of the induced variations of the metrics with respect to the parameters ϕ^\pm on the one hand and b^\pm on the other hand are disjoint.

4.1.1 Choice of diffeomorphisms

A simple way of assuring that our diffeomorphisms satisfy Lemma 4.2 is to choose them as restrictions of Möbiustransforms on the boundary of C_0 .

Given numbers $\phi^\pm \in \mathbb{R}$ and $b^\pm \in \mathbb{C}$ with $|b^\pm| < 1$ we consider the functions $f_{b^\pm, \phi^\pm} : \mathbb{R} \rightarrow \mathbb{R}$ which are induced by the Möbiustransforms M_{b^\pm, ϕ^\pm} , i.e. chosen so that $f_{0,0}(\theta) = \theta$ and $e^{if_{b^\pm, \phi^\pm}(\theta)} = M_{b^\pm, \phi^\pm}(e^{i\theta})$, where

$$M_{b,\phi}(z) := e^{i\phi} \frac{z + b}{1 + \bar{b} \cdot z}, \text{ for } b \in D_1(0) \subset \mathbb{C}, \phi \in \mathbb{R}, z \in \overline{D_1(0)} \subset \mathbb{C}.$$

To extend the maps induced by f_{b^\pm, ϕ^\pm} on ∂C_0 to a suitable diffeomorphism $h_{b,\phi} : C_0 \rightarrow C_0$ we choose smooth cut-off functions $\lambda_{1,2}$ such that $\lambda_1 \equiv 0$ on $[-1, \frac{3}{4}]$ with $\lambda_1 \equiv 1$ on $[\frac{7}{8}, 1]$ while $\lambda_2 \equiv 0$ on $[-1, \frac{1}{2}]$ with $\lambda_2 \equiv 1$ on $[\frac{5}{8}, 1]$. We then define $h_{b,\phi} : C_0 \rightarrow C_0$, $b = (b^-, b^+)$, $\phi = (\phi^-, \phi^+)$ through

$$h_{b,\phi}(s, \theta) = (s, \lambda_1(s) \cdot f_{b^+, \phi^+}(\theta) + (1 - \lambda_1(s)) \cdot (\theta + \lambda_2(s) \cdot \phi^+)) \quad (4.2)$$

if $s \geq 0$ respectively by the analogue formula, replacing (b^+, ϕ^+) with (b^-, ϕ^-) and s by $-s$, if $s \leq 0$.

Since $f_{b,\phi}(\theta) = f_{b,0}(\theta) + \phi$ and $\lambda_2 \equiv 1$ on $\text{supp}(\lambda_1)$, this formula reduces to $h_{b,\phi}(s, \theta) = (s, \theta + \lambda_1 \cdot (f_b(\theta) - \theta) + \lambda_2 \cdot \phi_+)$ where we abbreviate $f_b = f_{b,\phi=0}$.

In order to show that this family of diffeomorphisms satisfies Lemma 4.1 we observe that a change of one of the parameters, say of $|b^+|$, induces a change of the metric of

$$\frac{d}{d|b^+|}(h_{b,\phi}^* G) = h_{b,\phi}^* L_{Y_{|b^+|}} G = L_{h_{b,\phi}^* Y_{|b^+|}} g$$

for $g = h_{b,\phi}^* G$ and that the resulting Lie derivatives of the collar metrics G satisfy the following estimates

Lemma 4.3. *Let $(h_{b,\phi})$ be the family of diffeomorphisms defined in (4.2), let $Y_{|b^\pm|}$, $Y_{Arg(b^\pm)}$ and Y_{ϕ^\pm} be its generating vectorfields and let (G_ℓ) be the family of metrics defined in Lemma 2.4 for some fixed number $\eta > 0$. Then to any number $L_0 < \infty$ there exist constants $C_{1,2,3,4} \in (0, \infty)$ (depending only on L_0 and η) such that the following estimates hold true for any metric $G = G_\ell$ with $\ell < L_0$ and any (b, ϕ)*

$$\|L_{Y_{|b^+|}} G\|_{L^2(C_0, G)} \geq \frac{C_1}{1 - |b^+|} - C_2 \quad \text{and} \quad C_3 \leq \|L_{Y_{\phi^+}} G\|_{L^2(C_0, G)} \leq C_4. \quad (4.3)$$

Furthermore $L_{Y_{\phi^+}} G$ is $L^2(C_0, G)$ -orthogonal to both $L_{Y_{Re(b^+)}} G$ and $L_{Y_{Im(b^+)}} G$ and for $b^+ \neq 0$ also $L_{Y_{|b^+|}} G$ and $L_{Y_{Arg(b^+)}} G$ are $L^2(C_0, G)$ -orthogonal to each other.

The claims made above for variations with respect to ϕ^+ and b^+ are of course valid also for variations with respect to b^- and ϕ^- and from the construction it is evident that variations with respect to (ϕ^+, b^+) on the one hand and (ϕ^-, b^-) on the other hand have disjoint support so result in Lie-derivatives that are trivially orthogonal.

With regards to the proof of this lemma, we observe that the orthogonality of $L_{Y_{\phi^+}} G$ to the variations with respect to b^+ follows since Y_{ϕ^+} is given by the Killing field $\frac{\partial}{\partial \theta}$ on the support of $Y_{Re(b^+)}$ and $Y_{Im(b^+)}$. The orthogonality of $L_{Y_{|b^+|}} G$ and $L_{Y_{Arg(b^+)}} G$ on the other hand will follow from the different symmetry properties of these two tensors. The proof of this last part and of the estimates claimed in the lemma is not difficult, though a bit technical, so we include it in Appendix A.3.

As a consequence of Lemma 4.3 we can now prove Lemma 4.1 for this particular choice of diffeomorphism

Proof of Lemma 4.1. Let $g(t) = h_{b(t), \phi(t)}^* G_{\ell(t)}$, $t \in [0, T]$, be an admissible curve of metrics which is Lipschitz, and thus in particular differentiable a.e., for which

$$L_{L^2}(g) := \int_0^T \|\partial_t g\|_{L^2(C_0, g)} dt < \infty.$$

We first recall that $Re(\mathcal{H}(g))$ is orthogonal to $\{L_X g\}$ so that both $\|\partial_t g\|_{L^2(C_0, G)} \leq \|\partial_t g\|_{L^2(C_0, g)}$ and $\|\frac{d}{d\varepsilon}|_{\varepsilon=0} h_{b(t+\varepsilon), \phi(t+\varepsilon)} G(t)\|_{L^2(C_0, g)} \leq \|\partial_t g(t)\|_{L^2(C_0, g(t))}$ must have finite integral over $[0, T]$.

On the one hand, this implies that $\ell(t)$ is bounded from above by a constant \bar{L} depending only on the initial metric and $L_{L^2}(g)$, compare (A.7).

Using the orthogonality of $L_{Y_{\phi^+}} G$ to the variations generated by a change of any of the other parameters, as well as estimate (4.3), we know furthermore that

$$\|\partial_t g\|_{L^2(C_0, g)} \geq \left| \frac{d}{dt} \phi^+ \right| \cdot \left\| \frac{d}{d\phi^+} h_{b, \phi}^* G \right\|_{L^2(C_0, h_{b, \phi}^* G)} = \left| \frac{d}{dt} \phi^+ \right| \cdot \|L_{Y_{\phi^+}} G\|_{L^2(C_0, G)} \geq C_3 \cdot \left| \frac{d}{dt} \phi^+ \right|$$

where $C_3 > 0$ depends only on the upper bound on ℓ obtained above. This implies that ϕ^+ , and by the same argument also ϕ^- , remains bounded.

It remains to consider the behaviour of b^\pm , or by symmetry just of b^+ . The orthogonality relations of Lemma 4.3 combined with (4.3) imply

$$\|\partial_t g\|_{L^2(C_0, g)} \geq \left\| \frac{d}{dt} |b^+| \cdot L_{Y_{|b^+|}} G \right\|_{L^2(C_0, G)} \geq \left[\frac{C_1}{1 - |b^+|} - C_2 \right] \left| \frac{d}{dt} |b^+| \right|.$$

In particular, for $|b^+|$ sufficiently close to 1, an estimate of the form

$$\left| \frac{d}{dt} \log(1 - |b^+|) \right| \leq C \|\partial_t g\|_{L^2(M, g)}$$

holds true which prevents b^+ from reaching $\partial D_1(0)$ if the curve g has finite L^2 length. \square

We remark that the Teichmüller space of the cylinder equipped with the L^2 -metric that results from representing conformal structures by hyperbolic metrics $f^* G_\ell^\eta$ as described in Lemmas 2.1, 2.2 and 2.3 is not complete. Indeed, as explained in Appendix A.2, for general curves in \mathcal{M} and ℓ small we can only bound

$$\left| \frac{d\ell}{dt} \right| \leq C \cdot \|\partial_t g\|_{L^2} \cdot \ell^{1/2}$$

so that ℓ can tend to zero along curves of finite length.

Nonetheless, for Teichmüller harmonic map flow a degeneration of the metric in finite time is excluded since we can prove

Lemma 4.4. *To any numbers $\ell_1 > 0$ and $M, T, E_0 < \infty$ there exist constant $C < \infty$ and $\varepsilon_0 > 0$ such that the following holds true. Let $(u_0, g_0) \in H_{\Gamma, *}^1(C_0) \times \mathcal{M}$ be any initial data so that $E(u_0, g_0) \leq E_0$, $\|u_0\|_{L^\infty} \leq M$ and $\text{inj}(C_0, g_0) \geq 2\ell_1$ and let (u, g) be the corresponding stationary weak solution of Teichmüller harmonic map flow whose existence on some interval $[0, T_1)$ is assured by Proposition 3.14.*

Then the following weighted energy is bounded uniformly on $[0, \min(T, T_1))$

$$I(t) := \int_{C_0} e(u(t), g(t)) \rho^{-2}(t) dv_{g(t)} \leq C, \quad (4.4)$$

and so is the injectivity radius

$$\text{inj}(C_0, g) \geq \varepsilon_0, \text{ for all } t \in [0, \min(T, T_1)).$$

Here $e(u, g) = \frac{1}{2} |du|_g^2$ denotes the energy density while $\rho(t)(x, \theta) = \rho_{\ell(t)}(s_{\ell(t)}(x))$ is the conformal factor of the hyperbolic collar.

This result is essentially a special case of results proven in the joint work [15] with P. Topping for Teichmüller harmonic map flow from closed surfaces into non-positively curved targets. The reason why the argument of [15, section 5] carry over with only minor modifications to the flow from the cylinder is that the action of the diffeomorphisms $h_{b, \phi}$ does not affect the region near the central geodesic. We also recall that while the metric component g is in general not smooth, it is Lipschitz continuous in time with respect to any metric in space. So u is in $C_t^{1, \alpha} C_x^k$ in the interior of C_0 and away from time $t = 0$ which is enough to apply the arguments of [15].

Proof of Lemma 4.4. We first explain why a bound on the weighted energy I results in a bound on the injectivity radius. We recall that the evolution of $g(t) = h_{(b,\phi)(t)}^* G(t)$, $G(t) = G_{\ell(t)}$, splits L^2 -orthogonally into the projection of the Hopf-differential onto the subspace $\{L_{h_{b,\phi}^*} Xg, X \in \chi(b, \phi)\}$ and into the projection onto $Re(\mathcal{H}(g))$ and thus that $\partial_t G = \frac{1}{4} P^{Re(\mathcal{H}(G))}(Re(\Phi))$. As the space $\mathcal{H}(G)$ consists only of tensors that can be written in the form $a_0 \cdot dz^2$, $a_0 \in \mathbb{R}$, with respect to collar coordinates $z = s + i\theta$, $(s, \theta) \in [-Y(\ell), Y(\ell)] \times S^1$, we have

$$\begin{aligned} \partial_t G &= \frac{1}{4 \|Re(dz^2)\|_{L^2}^2} \cdot \langle Re(\Phi), Re(dz^2) \rangle Re(dz^2) \\ &= \frac{1}{2 \|dz^2\|_{L^2}^2} \int (|u_s|^2 - |u_\theta|^2) 2\rho^{-2} ds d\theta \cdot Re(dz^2), \end{aligned} \quad (4.5)$$

compare (A.2), where $\|dz^2\|_{L^2}^2$ is given by (A.4) for small values of ℓ .

As the length of the shortest closed geodesic evolves by $\frac{d\ell}{dt} = -\frac{2\pi^2}{\ell} a_0$ if $\partial_t G = a_0 Re(dz^2)$ we conclude that

$$\left| \frac{d}{dt} \log \ell + \frac{1}{16\pi^3} \cdot \ell \int_{C(\ell)} (|u_s|^2 - |u_\theta|^2) \rho^{-2} ds d\theta \right| \leq C\ell \|\Phi(u, g)\|_{L^1} \leq CE_0,$$

compare also [15, section 5].

In particular $\left| \frac{d}{dt} \log(\ell) \right| \leq C\ell + \ell I(t)$ is bounded if $I(t)$ is bounded, resulting in the desired lower bound on $\ell = 2 \text{inj}(C_0, G_\ell)$.

For the proof of (4.4) we follow the arguments of sections 3 and 5 of [15]. Namely, the results of [15, section 3], in particular Proposition 3.6, give angular energy estimates for maps from hyperbolic collars into compact non-positively curved targets. Since we know that $\|u\|_{L^\infty} \leq M$ these results apply without change also to the present situation. As in [15, section 5] we consider a cut-off version of the weighted energy given by

$$\mathcal{I}(t) := \int_{C(\ell(t))} e(u(t), g(t)) \rho^{-2}(t) \varphi(\rho(t)) dv_{g(t)}, \quad (4.6)$$

where $\varphi \in C_0^\infty([0, 2\delta], [0, 1])$ is a cut-off function with $\varphi \equiv 1$ on $[0, \delta]$, and where $\delta > 0$ can be chosen to be any fixed number. In the present situation we choose $\delta > 0$ sufficiently small so that the diffeomorphisms $h_{b,\phi}$ agree with the identity on the support of $\varphi \circ \rho$. The reasons that make this possible are that $h_{b,\phi} \equiv \text{id}$ in a neighbourhood of $\{0\} \times S^1$ and that on compact subsets of $C_0 \setminus \{0\} \times S^1$ the injectivity radius, and thus also $\rho \circ s_\ell$, is bounded away from zero uniformly in ℓ , cf. Remark A.5. The metric thus evolves only by $\partial_t g = Re(c(t) dz^2)$ on the support of $\varphi \circ \rho$ which implies in particular that the evolution equation for the conformal factor described in Lemma 5.4 of [15] applies without change in this region.

This allows us to argue precisely as in the proof of in [15, Lemma 5.1] to obtain that

$$\left| \frac{d}{dt} \log(1 + \mathcal{I}) \right| \leq C \left(1 + \|\Delta_g u\|_{L^2(C_0, g)}^2 \right) \leq C \left(1 + \|\partial_t u\|_{L^2(C_0, g)}^2 \right)$$

where C depends only on M , the initial energy and the choice of δ . Thus \mathcal{I} and consequently also $I \leq \mathcal{I} + C_\delta E_0$ is bounded uniformly on every compact time interval as claimed in the lemma. \square

From Lemma 4.4 we thus conclude that for arbitrary initial data $(u_0, g_0) \in H_{\Gamma, *}^1 \times \mathcal{M}_{-1}$ solutions to Teichmüller harmonic map flow from the cylinder indeed exist for all times as claimed in Theorem 2.6.

5 Asymptotics of global solutions

We now turn to the proof of the second main result of the paper, the asymptotic convergence for the global weak solutions whose existence we have just proven. As noted in Remark 2.8, in the present paper we only analyse solutions for which the three-point-condition does not degenerate as $t \rightarrow \infty$ in the sense that $\limsup_{t \rightarrow \infty} 1 - \max_{\pm} |b^{\pm}| > 0$.

Given a global stationary weak solution (u, g) of Teichmüller harmonic map flow which satisfies this property as well as the energy inequality, we first remark that we can choose times $t_i \rightarrow \infty$ with $\liminf_{i \rightarrow \infty} 1 - |b^{\pm}(t_i)| > 0$ for which the stationarity condition is satisfied and for which

$$\|\Delta_{g(t_i)} u(t_i)\|_{L^2(C_0, g(t_i))} \rightarrow 0, \quad (5.1)$$

$$\|P_g^{\mathcal{V}}(Re(\Phi(u, g)(t_i)))\|_{L^2(C_0, g(t_i))} \rightarrow 0. \quad (5.2)$$

After passing to a subsequence we can thus assume that

$$b_i^{\pm} = b^{\pm}(t_i) \rightarrow b_{\infty}^{\pm} \in D_1(0) \subset \mathbb{C} \text{ and } \tilde{\phi}_i^{\pm} := \phi_i^{\pm} - n_i^{\pm} \cdot 2\pi \rightarrow \phi_{\infty}^{\pm},$$

as $i \rightarrow \infty$ where $n_i = \lfloor \frac{\phi(t_i)^{\pm}}{2\pi} \rfloor$. We pull-back the maps and metrics by the diffeomorphisms $f_i := h_{0, 2\pi n_i} = h_{b_i, \phi_i}^{-1} \circ h_{b_i, \tilde{\phi}_i}$ and consider the resulting metrics $g_i := f_i^* g(t_i) = h_{b_i, \tilde{\phi}_i}^* G_{\ell(t_i)}$ and maps $u_i = u(t_i) \circ f_i$. We note that since f_i agrees with the identity in a neighbourhood of the boundary, the maps u_i still satisfy the three-point-condition, i.e. are again elements of $H_{\Gamma, *}^1(C_0)$.

We also remark that (5.1) and (5.2) are satisfied also for (u_i, g_i) and both the differential inequality

$$\int \langle du_i, dw \rangle dv_{g_i} + \int \Delta_{g_i} u_i \cdot w dv_{g_i} \geq 0 \text{ for all } w \in T_{u_i}^+ H_{\Gamma, *}^1 \quad (5.3)$$

and the stationarity equation

$$\int Re(\Phi(u_i, g_i)) \cdot L_X g_i + \Delta_{g_i} u_i \cdot du_i(X) dv_g = 0 \text{ for all } X \in \Gamma(TC_0)_* \quad (5.4)$$

hold true.

To prove convergence of (u_i, g_i) to a critical point of area as described in Theorem 2.7 we now distinguish between the non-degenerate case, $\ell(t_i) \nrightarrow 0$, in which we will obtain a (branched) minimal immersion parametrised over a cylinder, and the degenerate case $\ell(t_i) \rightarrow 0$ in which the surface splits into two minimal discs.

We begin with

Proof of Theorem 2.7 part (i): The non-degenerate case. We can assume, after passing to a further subsequence, that $\ell_i = \ell(t_i) \rightarrow \ell_{\infty} > 0$ which implies that the metrics converge $g_i \rightarrow g_{\infty} = h_{b_{\infty}, \phi_{\infty}}^* G_{\ell_{\infty}}$ smoothly on C_0 .

Furthermore, as u_i is a solution of (5.3) for which $\|\Delta_{g_i} u_i\|_{L^2}$ is bounded, we can apply the H^2 -estimates of Lemma 3.8 away from P_j^{\pm} as well as the H^1 estimates of Lemma 3.11 and Corollary 3.13 on the whole of C_0 . We conclude that a subsequence of the u_i converges to a limit $u_{\infty} \in H_{loc}^2(C^*) \cap H^1(C_0)$ where the obtained convergence is weak H_{loc}^2 and strong $W_{loc}^{1,p}$ convergence on $C^* := C_0 \setminus \bigcup P_j^{\pm}$ as well as strong H^1 convergence on all of C_0 . Furthermore, Corollary 3.9 implies that the maps u_i are equicontinuous near the boundary, so that the u_i converge uniformly on C_0 . In particular, $u_{\infty} \in C^0(C_0)$.

The above convergence implies not only that

$$\Delta_{g_\infty} u_\infty \equiv 0 \text{ on } C_0$$

and consequently that the Hopf-differential of the limit is holomorphic, but furthermore that the Hopf-differentials $\Phi(u_i, g_i) \rightarrow \Phi(u_\infty, g_\infty)$ converge strongly in L^1 on the whole cylinder C_0 .

From Lemma 3.3 we thus obtain that

$$P_{g_\infty}^\mathcal{V}(\Phi(u_\infty, g_\infty)) = \lim_{i \rightarrow \infty} P_{g_i}^\mathcal{V}(\Phi(u_i, g_i)) = 0. \quad (5.5)$$

On the one hand, this implies that

$$\int_{C_0} \operatorname{Re}(\Phi(u_\infty, g_\infty)) \cdot L_Y g_\infty dv_{g_\infty} = 0 \quad (5.6)$$

holds true for the vectorfields $Y \in h_{b_\infty, \tilde{\phi}_\infty}^* \chi(b_\infty, \tilde{\phi}_\infty)$ generating the diffeomorphisms $h_{b, \phi}$.

On the other hand, the convergence of the Hopf-differentials allows us to pass to the limit in the stationarity condition (5.4) to conclude that (5.6) holds true also for all vectorfields $X \in \Gamma(TC_0)_*$. Thus, by Lemma 4.2, we find that (5.6) is indeed true for any smooth vectorfield on C_0 which is tangential to ∂C_0 on ∂C_0 .

We now show that this forces $\Phi_\infty = \Phi(u_\infty, g_\infty)$ to be of the form cdz^2 for some $c \in \mathbb{R}$. We note that if Φ_∞ were smooth (or even just $W^{1,1}$) upto the boundary, we could directly combine (5.6) with Stokes theorem to conclude that Φ_∞ is real on the boundary and, as Φ_∞ is holomorphic, thus to conclude that $\Phi_\infty = cdz^2, c \in \mathbb{R}$.

However, while Φ_∞ is holomorphic and thus smooth in the interior as well as in $W^{1,p}$, $p < 2$ in a neighbourhood of general boundary points we know a priori only that Φ_∞ is in $L^1(C_0)$ due to the weaker control on the behaviour of the maps near the points P_j^\pm . Thus Φ_∞ could have a pole at such a point and we need to proceed with more care.

Given any fixed $X \in \Gamma(TC_0)$ we use that (5.6) implies that

$$\left| \int_{[-1+\varepsilon, 1-\varepsilon] \times S^1} L_X g_\infty \cdot \operatorname{Re}(\Phi_\infty) dv_{g_\infty} \right| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \quad (5.7)$$

and we initially work on such subcylinders where Φ_∞ is smooth.

We recall that $L_X g$ can be identified with $-\delta_g^* X$, where δ_g^* is the L^2 -adjoint of the divergence operator and that the real part of a holomorphic quadratic differential is divergence free. Thus switching to collar coordinates $(s, \theta) \in [-Y_\infty, Y_\infty] \times S^1$, $Y_\infty = Y(\ell_\infty)$ and applying Stokes theorem to (5.7) yields

$$\left| \int_{\{Y_\infty - \tilde{\varepsilon}\} \times S^1} \operatorname{Re}(\Phi_\infty) \left(\frac{\partial}{\partial s}, X \right) \rho^{-2} d\theta - \int_{\{-Y_\infty + \tilde{\varepsilon}\} \times S^1} \operatorname{Re}(\Phi_\infty) \left(\frac{\partial}{\partial s}, X \right) \rho^{-2} d\theta \right| \rightarrow 0 \text{ as } \tilde{\varepsilon} \rightarrow 0 \quad (5.8)$$

where $\rho = \rho_{\ell_\infty}(s)$.

Away from the boundary of $[-Y, Y] \times S^1$ we now represent Φ by its Fourier expansion $\Phi_\infty = \sum_{n \in \mathbb{Z}} (a_n + ib_n) e^{ns} e^{ni\theta}$, $a_n, b_n \in \mathbb{R}$ and apply (5.8) for vectorfields of the form $X = \lambda^\pm(s) \cos(m\theta) \cdot \frac{\partial}{\partial \theta}$ and $X = \lambda^\pm(s) \sin(m\theta) \cdot \frac{\partial}{\partial \theta}$, $m \in \mathbb{N}$, where λ^\pm are cut-off functions that are identically one in a neighbourhood of ± 1 and that vanish on $\{\pm s \leq \frac{1}{2}\}$.

Passing to the limit $\tilde{\varepsilon} \rightarrow 0$ in (5.8) yields

$$b_m e^{mY_\infty} = b_{-m} e^{-mY_\infty} \text{ and } b_m e^{-mY_\infty} = b_{-m} e^{mY_\infty}$$

as well as

$$a_m e^{mY_\infty} = -a_{-m} e^{-mY_\infty} \text{ and } a_m e^{-mY_\infty} = -a_{-m} e^{mY_\infty}.$$

so that all Fourier coefficients except for $c_0 = a_0 + ib_0$ need to be zero. Of course, testing with $X = \lambda^\pm \cdot \frac{\partial}{\partial \theta}$ furthermore gives that $b_0 = 0$ and thus that $\Phi_\infty = a_0 dz^2$ is indeed an element of $\mathcal{H}(g_\infty)$.

But (5.5) also implies that the projection of Φ_∞ onto $\mathcal{H}(g_\infty) = \{cdz^2, c \in \mathbb{R}\}$ vanishes so $\Phi_\infty \equiv 0$ and u_∞ is (weakly) conformal. Thus u_∞ is a weakly conformal and harmonic map which spans Γ and can thus in particular not be constant so must be a (possibly branched) minimal immersion [7]. \square

Proof of Theorem 2.7 part (ii): The degenerate case: Let (u_i, g_i) be as above and assume now that $\ell_i \rightarrow 0$. We let $C^+ = (0, 1] \times S^1$ and $C^- = [-1, 0) \times S^1$ and observe that the subcylinders (C^\pm, g_i) are isometric to

$$([0, Y_i] \times S^1, \rho_\ell^2(Y_i - s) \cdot (ds^2 + d\theta^2)), \quad Y_i = Y(\ell_i)$$

with an isometry given by $\tilde{f}_{\ell_i}^\pm : (x, \theta) \mapsto (Y_i \mp s_{\ell_i}(x), \theta)$. We remark that $\rho_\ell(Y(\ell) - s) \rightarrow \frac{1}{2\pi\eta + s}$ locally smoothly on $[0, \infty) \times S^1$ as $\ell \rightarrow 0$. At the same time \tilde{f}_ℓ^\pm converges locally to the diffeomorphism $\tilde{f}_\infty^\pm : C^\pm \rightarrow [0, \infty) \times S^1$ given by $f_\infty^\pm(x, \theta) = (\frac{2\pi}{\ell_0} \cdot \tan(\frac{\pi}{2} \mp \frac{\ell_0 x}{2\pi}) - 2\pi\eta, \theta)$.

Thus the metrics g_i converge smoothly locally on C^\pm to a metric g_∞ that is isometric to the hyperbolic cusp

$$([0, \infty) \times S^1, \rho_0^2(s) \cdot (ds^2 + d\theta^2))$$

described in the theorem.

At the same time, we get subconvergence for the maps $u_i = u(t_i) \circ h_{0, 2\pi n_i} \rightarrow u_\infty$ as described in the theorem since the bounds on $|b_i^\pm|$ allow us to apply the H^2 -estimates of Lemma 3.4 and Lemma 3.8 as well as the H^1 estimate of Lemma 3.11 and the equicontinuity result of Corollary 3.9 on every compact subset of C^\pm , see also Remarks 3.7 and 3.10.

The above convergence of maps and metrics implies in particular that $\Delta_{g_\infty} u_\infty = 0$ and thus that $\Phi_\infty = \Phi(u_\infty, g_\infty)$ is holomorphic on C^\pm . It furthermore allows us to pass to the limit in the stationarity condition (5.4) to conclude that

$$\int_{C^\pm} L_X g_\infty \cdot \text{Re}(\Phi_\infty) dv_{g_\infty} = 0 \tag{5.9}$$

provided we only consider vectorfields $X \in \Gamma(TC_0)_*$ whose support is contained in one of the subcylinders C^\pm .

We recall that also the support of each of the vectorfields Y_{ϕ^\pm} , $Y_{Re(b^\pm)}$ and $Y_{Im(b^\pm)}$ generating the diffeomorphisms $h_{b, \phi}$ is contained in C^\pm and that the projection of Φ onto the corresponding Lie-derivatives tends to zero, compare (5.2). Thus local strong convergence of u_i in H^1 and consequently of Φ_i in L^1 implies that (5.9) holds true also for these particular vectorfields, and thus, by Lemma 4.2, indeed for all vectorfields $X \in \Gamma(TC_0)$ whose support is contained in either C^+ or C^- .

As the Fourier expansion of $\Phi(u_\infty, g_\infty)$ on $(C^\pm, g_\infty) \simeq [0, \infty) \times S^1$ cannot have any exponentially growing terms (since $\|\Phi_\infty\|_{L^1} < CE_0 < \infty$) we can then argue as in the previous proof to conclude that in collar coordinates $(s, \theta) \in [0, \infty) \times S^1$

$$\Phi_\infty = c^\pm(ds + id\theta)^2 \quad \text{for some } c^\pm \in \mathbb{R}.$$

Pulling Φ_∞ back to the punctured disc $D^* = \overline{D_1(0)} \setminus \{0\}$ through conformal diffeomorphisms $f^\pm : (D^*, g_{eucl}) \rightarrow (C^\pm, g_\infty)$ we thus find that the Hopf-differential of u_∞ is represented by $c^\pm z^{-2} dz^2$, $c^\pm \in \mathbb{R}$, $z \in D^*$.

But as the limiting map u_∞ is a harmonic map from the punctured disc (D^*, g_{eucl}) whose energy is finite (since the energy is conformally invariant) we can extend u_∞ smoothly across the puncture using the Sacks-Uhlenbeck removable singularity theorem [16]. Thus Φ_∞ must be smooth on all of $D_1(0)$ and must thus vanish identically.

This proves that the maps $u_\infty^\pm = u_\infty|_{C^\pm}$ extend to weakly conformal harmonic maps from the disc and thus that in this case the flow changes the initial map into two (possibly branched) minimal immersions with each of them spanning one of the boundary curves Γ^\pm .

□

A Appendix

A.1 Courant-Lebesgue Lemma and properties of $H_{\Gamma,*}^1(C_0)$

Throughout the paper we made use of the Courant-Lebesgue Lemma of which we use the following version, see e.g. [9, Lemma 3.1.1] or [17, Lemma 4.4]

Lemma A.1. *Let $D_r(0)^+ = \{x \in \mathbb{R}^2 : |x| \leq r, x_1 \geq 0\}$ and let $u \in H^1(D_r(0)^+, \mathbb{R}^n)$ be any map that has energy $E(u, g_{eucl}) \leq E_0$, E_0 any fixed number. Then for any $\delta \in (0, \min(r, \frac{1}{2}))$ there exists $\rho \in (\delta, \sqrt{\delta})$ so that $u|_{\partial D_\rho(0)^+}$ is absolutely continuous and so that the estimate*

$$|u(x) - u(y)| \leq C \cdot |\log(\delta)|^{-1/2}, \quad \text{for all } x, y \in \partial D_\rho^+ := \{y : |y| = \rho, \ y_1 \geq 0\}$$

holds true with a constant C that depends only on E_0 .

We use in particular the following consequence for maps satisfying the three-point-condition

Corollary A.2. *Let $u_i \in H_{\Gamma,*}^1(C_0)$ be a sequence of maps that have uniformly bounded energy $E(u_i, g_i) \leq E_0 < \infty$ with respect to metrics $g_i = h_{b_i, \phi_i}^* G_{\ell_i}$ for which $\sup |b_i^\pm| < 1$. Then the traces $u_i|_{\partial C_0}$ are equicontinuous.*

Proof of Corollary A.2. Pulling back the maps and metrics by the diffeomorphism h_{b_i, ϕ_i}^{-1} one can reduce this corollary to the corresponding claim for the metrics G_{ℓ_i} and for maps \tilde{u}_i so that the functions φ_i^\pm that describe the traces $\tilde{u}_i|_{\partial C_0^\pm} = \alpha^\pm \circ \varphi_i^\pm$ are such that there are points $\tilde{\theta}_{i,k}$ with

$$\varphi_i(\tilde{\theta}_{i,k}) = \frac{2\pi}{3}k \text{ and } |\tilde{\theta}_{i,k+1} - \tilde{\theta}_{i,k}| \geq c, \quad k = 0, 1, 2 \quad (\text{A.1})$$

where $c > 0$ depends only on $1 - \sup |b^\pm|$ and where $\tilde{\theta}_{i,3} := \tilde{\theta}_{i,0} + 2\pi$.

We then remark that the upper bound on ℓ given by (3.10) implies that the induced metrics on the boundary of (C_0, G_ℓ) are all equivalent and that the numbers $Y(\ell)$ are bounded away from zero. Given any point $p = (\pm 1, \bar{\theta})$ we can thus apply Lemma A.1 on the set described by $\{(\pm 1, \bar{\theta})\} \mp D_r^+(0)$ in collar coordinates, with $r > 0$ depending only on the upper bounds on ℓ and $|b^\pm|$, namely chosen so that $r < \min(c/2, Y(\ell))$. The proof then follows by a standard argument: Given that the parametrisations are weakly monotone and that (A.1) does not permit that more than one of the three points $\alpha_\pm(\theta_k)$ is contained in the image of the small arc $(s, \theta) \in \{\pm 1\} \times [\bar{\theta} - r, \bar{\theta} + r]$, we obtain the desired bound on the modulus of continuity from the Courant-Lebesgue Lemma. \square

The above lemma implies in particular that any map u that is obtained as weak H^1 limit of a sequence of maps $u_i \in H_{\Gamma,*}^1(C_0)$ is again an element of $H_{\Gamma,*}^1(C_0)$.

In order to pass to the limit in the differential inequality (3.3) we use at several points in the paper that the tangent cones $T_u^+ H_{\Gamma,*}^1(C_0)$ depend continuously on u namely that

Lemma A.3. *Let $u \in H_{\Gamma,*}^1(C_0)$ and let $u^i \in H_{\Gamma,*}^1(C_0)$ be any sequence that converges strongly in $H^1(C_0)$ to u . Then any element $v \in T_u^+ H_{\Gamma,*}^1(C_0)$ can be approximated by elements in $v^i \in T_{u^i}^+ H_{\Gamma,*}^1(C_0)$ in the sense that*

$$v^i \rightarrow v \text{ strongly in } H^1(C_0).$$

Indeed, writing $u|_{\partial C_\pm} = \alpha_\pm \circ \varphi_\pm$ respectively $u^i|_{\partial C_\pm} = \alpha_\pm \circ \varphi_\pm^i$ and $v|_{\partial C_\pm} = \lambda_\pm \cdot \alpha'_\pm(\varphi_\pm) \cdot (\psi_\pm - \varphi_\pm)$ we can use that the traces converge both strongly in $H^{1/2}$ and uniformly and thus that also $v^i|_{\partial C_\pm} := \lambda_\pm \cdot \alpha'_\pm(\varphi_\pm^i) \cdot (\psi_\pm - \varphi_\pm^i)$ converge to $v|_{\partial C_\pm}$ in $H^{1/2}$. The desired elements of $T_{u^i}^+ H_{\Gamma,*}^1(C_0)$ are then obtained as harmonic extensions of these traces similarly to the proof of Lemma 2.1 in [6].

As a consequence we obtain

Corollary A.4. *Let $(u_i, g_i, f_i) \in H_{\Gamma,*}^1(C_0) \times \widetilde{\mathcal{M}} \times L^2(C_0)$ be such that (3.3) is satisfied and assume that $g_i \rightarrow g$, $u_i \rightarrow u_\infty$ strongly in $H^1(C_0)$ and $f_i \rightharpoonup f$ weakly in L^2 . Then (3.3) is satisfied also for the limit (u, g, f) .*

We finally outline how Proposition 3.6 can be derived from the corresponding estimates for maps from the disc proven by Duzaar and Scheven in [6, Theorem 8.3]

Sketch of the proof of Proposition 3.6. Because of the interior estimates of Lemma 3.4 it is sufficient to consider points p that are contained in $V^\pm = \{s : \pm s \in [\frac{1}{2}, 1]\} \times S^1$. We pull-back the maps and metrics first by $h_{b,\phi}^{-1}$, which we note maps V^\pm onto itself, then by a fixed conformal diffeomorphism that maps a neighbourhood of $\partial D_1 \subset (\overline{D_1}(0), g_{eucl})$ onto $V^\pm \subset (C_0, G_\ell) = (C_0, (h_{b,\phi}^{-1})^* g)$ and finally, to restore the three-point-condition, by the Möbius transform $M_{b^\pm, \phi^\pm} : \overline{D_1} \rightarrow \overline{D_1}$. The resulting triple $(\tilde{u}, \tilde{g}, \tilde{f}) = \psi^*(u, g, f)$ then satisfies equation (3.3) on a neighbourhood U of ∂D_1 . By construction the metric \tilde{g} is conformal to the Euclidian metric and we note that the corresponding conformal factor in $\tilde{g} = \lambda g_{eucl}$ is bounded uniformly both from above and away from zero since the bound on $1 - |b^\pm|$ allows us to control M_{b^\pm, ϕ^\pm} and since ρ_ℓ is controlled uniformly on V^\pm even if $\ell \rightarrow 0$. Given that (3.3) holds true also for $(\tilde{u}, g_{eucl}, \tilde{f}\lambda^2)$ we can then apply Theorem 8.3 of [6] to obtain the claimed estimates on corresponding balls $D_r(p)$ in the Euclidean disc.

We can then pull-back these estimates to give the claim of Proposition 3.6 since the bounds on $1 - |b^\pm|$ and on ρ (on V^\pm) give sufficient control on the involved metrics and diffeomorphisms. \square

A.2 Properties of hyperbolic collars and the horizontal family of metrics G_ℓ

In this part of the appendix we collect some well known properties of hyperbolic collars, where we refer to the appendix of [14] and the references therein for more information, as well as properties of the hyperbolic cylinders (C_0, G_ℓ) that are used throughout the paper. We furthermore give the proof that the family of metrics described in Lemma 2.4 is horizontal, i.e. that $\frac{d}{d\ell}G_\ell \in \text{Re}(\mathcal{H}(G_\ell))$.

We first recall that the δ -thin part of a hyperbolic cylinder is described in collar coordinates $(s, \theta) \in (-Y(\ell), Y(\ell))$ by

$$(-\min(X_\delta(\ell), Y(\ell)), \min(X_\delta(\ell), Y(\ell))) \times S^1,$$

where

$$X_\delta(\ell) = \frac{2\pi}{\ell} \left(\frac{\pi}{2} - \arcsin \left(\frac{\sinh(\frac{\ell}{2})}{\sinh \delta} \right) \right)$$

for $\delta \geq \ell/2$, respectively zero for smaller values of δ .

Remark A.5. For the metrics $G_\ell = f_\ell^*(\rho_\ell^2(ds^2 + d\theta^2))$ this means that for each $\delta > 0$ there exists a number $c_0(\delta) > 0$ with $c_0(\delta) \rightarrow 0$ for $\delta \rightarrow 0$ so that δ -thin(C_0, G_ℓ) is contained in $(-c_0(\delta), c_0(\delta)) \times S^1$ (with respect to the fixed coordinates (x, θ) of C_0) for every $\ell > 0$; or, said differently, for every $c_1 > 0$ there exists a number $\delta(c_1) > 0$ so that

$$\text{inj}_{G_\ell}(x, \theta) \geq \delta(c_1) \text{ for all } |x| \geq c_1 \text{ and all } \ell > 0.$$

As $\text{inj}_G(x) \leq \pi\rho(s_\ell(x))$ this also yields a uniform lower bound on the conformal factor on this set.

We also note that

$$|dz^2|_g = 2\rho^{-2} = \sqrt{2}|\text{Re}(dz^2)| \quad (\text{A.2})$$

and hence that the norms of dz^2 on $([-Y(\ell), Y(\ell)] \times S^1, \rho_\ell^2(ds^2 + d\theta^2))$ can be computed as

$$\|dz^2\|_{L^\infty} = \frac{8\pi^2}{\ell^2} \quad (\text{A.3})$$

and

$$\|dz^2\|_{L^2}^2 = \frac{64\pi^4}{\ell^3} \cdot [\sin(\text{atan}(\eta\ell)) \cdot \cos(\text{atan}(\eta\ell)) + (\frac{\pi}{2} - \text{atan}(\eta\ell))].$$

For ℓ small we thus have that

$$\|dz^2\|_{L^2}^2 = \frac{32\pi^5}{\ell^3} + O(1), \quad (\text{A.4})$$

while for ℓ large

$$\|dz^2\|_{L^2}^2 = \frac{128\pi^4}{\eta\ell^4} + O(\ell^{-5}). \quad (\text{A.5})$$

We also recall the well known fact that if a metric g evolves by $\partial_t g = \text{Re}(\Psi)$ for a holomorphic quadratic differential Ψ then the length of the central geodesic changes by

$$\frac{d\ell}{dt} = -\frac{2\pi^2}{\ell} \text{Re}(c_0), \quad (\text{A.6})$$

where $c_0 dz^2$ is the principal part in the Fourier expansion of Ψ , or in our case simply the coefficient in $\Psi = a_0 dz^2$, $a_0 \in \mathbb{R}$.

For large values of ℓ we can thus bound the evolution of ℓ along a horizontal curve by

$$\left| \frac{d\ell}{dt} \right| \leq \frac{2\pi^2}{\ell} \frac{\|\partial_t g\|_{L^2}}{\|dz^2\|_{L^2}} \leq C \cdot \ell \|\partial_t g\|_{L^2} \quad (\text{A.7})$$

while for small values of ℓ we only obtain that

$$\left| \frac{d\ell}{dt} \right| \leq C \cdot \ell^{1/2} \|\partial_t g\|_{L^2} \quad (\text{A.8})$$

which allows for a degeneration of the metric along a curve of finite length.

Finally we explain how the formula for the horizontal families of metrics in \mathcal{M}_{-1} claimed in Lemma 2.4 can be derived.

Proof of Lemma 2.4. Let $t \mapsto g(t)$ be a curve of metrics in \mathcal{M}_{-1} which moves in horizontal direction, $\partial_t g(t) \in \text{Re}(\mathcal{H}(C_0, g(t)))$, and so that g is given as pull-back of a collar $([-Y(\ell), Y(\ell)] \times S^1, \rho_\ell(s)^2(ds^2 + d\theta^2))$ by a suitable diffeomorphism $f_\ell : C_0 \rightarrow [-Y(\ell), Y(\ell)] \times S^1$ where both $Y(\ell)$ and f_ℓ need to be determined.

To begin with, we derive a differential equation for $Y(\ell)$ by computing the evolution of the *width*

$$w(\ell(t)) := \text{dist}_{g(t)}(\{-1\} \times S^1, \{1\} \times S^1)$$

of the cylinder $(C_0, g(t))$.

Let t be any fixed time and let $(s, \theta) \in [-Y(\ell(t)), Y(\ell(t))] \times S^1$ be the corresponding collar coordinates. Then in these fixed coordinates, the evolution of g at time t is given by $a_0(ds^2 - d\theta^2)$ where a_0 is related to the evolution of the length ℓ of the central geodesic by (A.6).

Thus the width of the collar, which at time t is simply given by the length of the geodesics $s \mapsto (s, \theta_0)$, evolves according to

$$\begin{aligned} \frac{d}{dt} w(\ell(t)) &= \frac{d}{dt} \int_{-Y(\ell)}^{Y(\ell)} (g_{ss}(t))^{1/2} ds = \int_0^{Y(\ell)} (g_{ss}(t))^{-1/2} \cdot \partial_t g_{ss}(t) ds \\ &= a_0 \int_0^{Y(\ell)} \rho_\ell^{-1}(s) ds = a_0 \frac{2\pi}{\ell} \int_0^{Y(\ell)} \cos\left(\frac{\ell s}{2\pi}\right) ds \\ &= \left(\frac{2\pi}{\ell}\right)^2 a_0 \sin\left(\frac{\ell}{2\pi} Y(\ell)\right) = -\frac{2}{\ell} \sin\left(\frac{\ell}{2\pi} Y(\ell)\right) \frac{d\ell}{dt} \end{aligned}$$

where (A.6) is used in the last step. For $V(\ell)$ chosen so that $Y(\ell) = \frac{2\pi}{\ell}(\frac{\pi}{2} - V(\ell))$, this formula reduces to

$$\frac{dw}{d\ell} = -\frac{2}{\ell} \cos(V(\ell)).$$

On the other hand, we can directly compute $w(\ell(t))$ by working in collar coordinates of $g(t)$ as

$$\begin{aligned} w(\ell) &= 2 \int_0^{Y(\ell)} \rho_\ell(s) ds = 2 \int_0^{Y(\ell)} \frac{\ell}{2\pi \cos(\frac{\ell}{2\pi} s)} ds \\ &= 2h\left(\frac{\ell}{2\pi} Y(\ell)\right) = 2h\left(\frac{\pi}{2} - V(\ell)\right) \end{aligned}$$

where $h(x) := \log(\tan(\frac{x}{2} + \frac{\pi}{4}))$ is so that $h'(x) = \frac{1}{\cos(x)}$.

Thus

$$\frac{dw}{d\ell} = -2 \frac{1}{\sin(V(\ell))} \cdot \frac{d}{d\ell}(V(\ell))$$

meaning that V satisfies

$$\frac{1}{\ell} \cos(V(\ell)) = \frac{1}{\sin(V(\ell))} \cdot \frac{d}{d\ell}(V(\ell))$$

or equivalently

$$\frac{2V'}{\sin(2V)} = \ell^{-1}.$$

Thus $V(\ell) = \text{atan}(c_0 \cdot \ell)$ for some constant $c_0 > 0$ and therefore

$$Y(\ell) = \frac{2\pi}{\ell} \left(\frac{\pi}{2} - \text{atan}(c_0 \ell) \right).$$

We can argue similarly to derive the formula for the diffeomorphism $f_\ell(x, \theta) = (s_\ell(x), \theta)$. Namely, we use that $\partial_t g = f_\ell^*(a_0(ds^2 - d\theta^2))$ needs to agree with

$$\partial_t g = \frac{d\ell}{dt} \cdot \frac{d}{d\ell}(f_\ell^*(\rho_\ell^2 \cdot (ds^2 + d\theta^2))) = -\frac{2\pi^2}{\ell} a_0 \frac{d}{d\ell} [\rho_\ell^2(s_\ell(x)) \cdot ((\frac{\partial s_\ell}{\partial x})^2 \cdot dx^2 + d\theta^2)].$$

Comparing the two expressions for $(\partial_t g)_{\theta\theta}$ immediately yields the condition that

$$-1 = -\frac{2\pi^2}{\ell} \frac{d}{d\ell}(\rho_\ell^2 \circ s_\ell)$$

and thus that $(\frac{\ell}{2\pi})^2 - \rho_\ell^2 \circ s_\ell(x)$ is independent of ℓ , or equivalently that there exists a constant $c(x)$ with

$$\tan(\frac{\ell}{2\pi} s_\ell(x)) = \frac{c(x)}{\ell}.$$

Finally one can determine $c(x)$ so that $s_{\ell_0}(x) = x$ for the number $\ell_0 > 0$ for which $Y(\ell_0) = 1$ and check that for the resulting map f_ℓ also the two expressions for $(\partial_t g)_{xx}$ agree. \square

We furthermore remark that the metrics G_ℓ converge locally smoothly away from the central geodesic $\{0\} \times S^1$ as $\ell \rightarrow 0$ with the limiting metric G_0 given by

$$G_0|_{C_\pm} = f_\pm^*(\rho_0(s)^2(ds^2 + d\theta^2)), \quad (\text{A.9})$$

for $f_\pm : C_\pm \rightarrow [0, \infty) \times S^1$ defined by

$$f_\pm(x, \theta) = (\lim_{\ell \rightarrow 0} Y(\ell) \mp s_\ell(x), \theta) = (\frac{2\pi}{\ell_0} \cdot \tan(\frac{\pi}{2} \mp \frac{\ell_0 x}{2\pi}) - 2\pi\eta, \theta)$$

and $([0, \infty), \rho_0^2(ds^2 + d\theta))$ the hyperbolic cusp described in Theorem 2.7, case II.

A.3 Properties of the diffeomorphisms $h_{b,\phi}$

Here we provide (a sketch of) the proof of the properties of the diffeomorphisms $h_{b,\phi}$ introduced in (4.2).

Proof of Lemma 4.3. We recall that for $x > 0$ we can write $h_{b,\phi}(x, \theta) = (x, \lambda_1(x)f_{b^+}(\theta) + (1 - \lambda_1(x))\theta + \lambda_2(x)\phi^+)$. Thus $Y_{\phi^+} = \lambda_2(x) \cdot \frac{\partial}{\partial \theta}$ is a Killing field on $\text{supp}(\lambda_1) \subset \{\lambda_2 \equiv 1\}$ so in particular

$$L_{Y_{\text{Re}(b^+)}} G \perp L_{Y_{\phi^+}} G \text{ as well as } L_{Y_{\text{Im}(b^+)}} G \perp L_{Y_{\phi^+}} G.$$

To prove the other orthogonality relation we recall that a different choice of ϕ only results in a constant rotation on $\text{supp}(Y_{\text{Arg}(b)}) = \text{supp}(Y_{|b|})$. It is thus enough to consider the case $\phi = 0$ and we abbreviate in the following $h_b = h_{b,0}$. We also recall that all generating vectorfields have the form $Y = Y^\theta \cdot \frac{\partial}{\partial \theta}$ and we claim that for $x \geq 0$ and $\psi = \text{Arg}(b^+)$

$$Y_{\text{Arg}(b^+)}^\theta(h_b(x, \psi + \theta)) = Y_{\text{Arg}(b^+)}^\theta(h_b(x, \psi - \theta))$$

while

$$Y_{|b^+|}^\theta(h_b(x, \psi + \theta)) = -Y_{|b^+|}^\theta(h_b(x, \psi - \theta)).$$

Given that the conformal factor of the collar metric is independent of θ this immediately results in the claimed orthogonality of $L_{Y_{\text{Arg}(b^+)}} G$ and $L_{Y_{|b^+|}} G$.

To prove the symmetry relation for $Y_{\text{Arg}(b^+)}$ or equivalently for $\frac{d}{d\text{Arg}(b^+)} f_{b^+}$ we first observe that for $b^+ = a \cdot e^{i\psi}$, $a \in \mathbb{R}$, we have $M_{b^+}(e^{i(\psi+\theta)}) = e^{i\psi} M_a(e^{i\theta})$ and thus

$$f_{b^+}(\theta) = \psi + f_a(\theta - \psi). \quad (\text{A.10})$$

In particular

$$\frac{d}{d\text{Arg}(b^+)} f_{b^+}(\theta) = 1 - \partial_\theta f_a(\theta - \psi).$$

Differentiating the relation $M_a(e^{i\theta}) = e^{if_a(\theta)}$ we get

$$\begin{aligned} \partial_\theta f_a(\theta) &= (iM_a(e^{i\theta}))^{-1} \cdot \left(\frac{d}{dz} M_a(e^{i\theta}) \right) \cdot i \cdot e^{i\theta} \\ &= \frac{1 - a^2}{(1 + a \cdot \cos \theta)^2 + a^2 \sin^2(\theta)}. \end{aligned}$$

Thus indeed

$$\frac{d}{d\text{Arg}(b^+)} f_{b^+}(\psi + \theta) = 1 - \partial_\theta f_a(\theta) = 1 - \partial_\theta f_a(-\theta) = \frac{d}{d\text{Arg}(b^+)} f_{b^+}(\psi - \theta)$$

which implies the claimed symmetry of $Y_{\text{Arg}(b)}$.

On the other hand, again for $a \in \mathbb{R}$

$$\frac{d}{da} f_a(\theta) = (iM_a(e^{i\theta}))^{-1} \cdot \left(\frac{d}{da} M_a(e^{i\theta}) \right) = -\frac{2 \cdot \sin(\theta)}{(1 + a \cos \theta)^2 + a^2 \sin^2(\theta)}$$

so that for $b^+ = ae^{i\psi}$

$$\frac{d}{d|b^+|} f_{b^+}(\psi + \theta) = \frac{d}{da} f_a(\theta) = -\frac{d}{da} f_a(-\theta) = -\frac{d}{d|b^+|} f_{b^+}(\psi - \theta)$$

as claimed.

It remains to prove the estimates for $L_{Y_\phi^+} G$ and $L_{Y_{|b^+|}} G$. As $Y_{\phi^+} = \lambda_2(x) \frac{\partial}{\partial \theta}$, we have

$$L_{Y_{\phi^+}} G_\ell = \lambda_2'(x) \rho^2(s_\ell(x)) \cdot (dx \otimes d\theta + d\theta \otimes dx).$$

The claimed estimate thus follows from the uniform upper and lower bounds on $\rho_\ell \circ s_\ell$ on fixed cylinders $[\delta, 1] \times S^1 \subset C_0$, $\delta > 0$, that are valid for all $\ell \in (0, L_0]$, compare Remark A.5.

To analyse $L_{Y_{|b^+|}} G$, we first remark that

$$h_{b,\phi}^* G_\ell = \rho_\ell^2(s_\ell(x)) \cdot \left[\left[\left(\frac{\partial s_\ell}{\partial x} \right)^2 + \left(\frac{\partial h_{b,\phi}^\theta}{\partial x} \right)^2 \right] dx^2 + \left(\frac{\partial h_{b,\phi}^\theta}{\partial x} \right) \cdot \left(\frac{\partial h_{b,\phi}^\theta}{\partial \theta} \right) \cdot (dx \otimes d\theta + d\theta \otimes dx) + \left(\frac{\partial h_{b,\phi}^\theta}{\partial \theta} \right)^2 d\theta^2 \right]$$

and that in view of (A.10) we only need to consider the case that $b^+ = a \in \mathbb{R}$.

As we only wish to prove a lower bound on $\|L_{Y_{|b^+|}} G\|_{L^2}$ it is enough to consider the subcylinder $[\frac{7}{8}, 1] \times S^1$ on which the above expression reduces to

$$h_a^* G_\ell = \rho^2(s_\ell(x)) \cdot \left[\left(\frac{\partial s_\ell}{\partial x} \right)^2 dx^2 + \left(\frac{\partial f_a}{\partial \theta} \right)^2 d\theta^2 \right]$$

so that

$$\frac{d}{da} (h_a^* G) = 2\rho^2(s_\ell(x)) \cdot \frac{\partial f_a}{\partial \theta} \cdot \frac{d}{da} \frac{\partial f_a}{\partial \theta} d\theta^2.$$

On this part of the cylinder we furthermore have that $g^{\theta\theta} = \rho^{-2} \circ s_\ell \cdot \left(\frac{\partial f_a}{\partial \theta} \right)^{-2}$ and that ρ is bounded away from zero uniformly in ℓ . Thus

$$\begin{aligned} \|L_{Y_{|b|}} G\|_{L^2(C_0, G)}^2 &= \left\| \frac{d}{da} (h_a^* G) \right\|_{L^2(C_0, h_a^* G)}^2 \geq 4 \int_{[\frac{7}{8}, 1] \times S^1} \left(\frac{\partial f_a}{\partial \theta} \right)^{-2} \left| \frac{d}{da} \frac{\partial f_a}{\partial \theta} \right|^2 dv_g \\ &\geq c \int_{S^1} \left(\frac{\partial f_a}{\partial \theta} \right)^{-1} \left| \frac{d}{da} \frac{\partial f_a}{\partial \theta} \right|^2 d\theta \end{aligned} \quad (\text{A.11})$$

for some fixed constant $c > 0$. As $\partial_\theta f_a = \frac{1-a^2}{(1+a \cos \theta)^2 + a^2 \sin^2 \theta}$, we compute

$$\begin{aligned} \partial_a \partial_\theta f_a &= -|ae^{i\theta} + 1|^{-4} \cdot \left[2a(1 + a \cos \theta)^2 + 2a^3 \sin^2 \theta \right. \\ &\quad \left. + (1 - a^2)[2 \cos(\theta)(1 + a \cos \theta) + 2a \sin^2 \theta] \right] \\ &= -|ae^{i\theta} + 1|^{-4} \cdot [2a \sin^2 \theta + 2(1 + a \cos \theta) \cdot (a + \cos \theta)]. \end{aligned}$$

We set $\varepsilon = 1 - a$ and remark that for ε small and for θ given by $\theta = \pi + \lambda \cdot \varepsilon$

$$[2a \sin^2 \theta + 2(1 + a \cos \theta) \cdot (a + \cos \theta)] \geq 2 \cdot [\lambda^2 \varepsilon^2 - \varepsilon^2 + O(\varepsilon^3)]$$

so that this expression is bounded away from 0 by $2\varepsilon^2$ for angles $2\varepsilon \leq |\theta + \pi| \leq 3\varepsilon$.

Combined with (A.11) we thus find that

$$\|L_{Y_{|b|}} G\|_{L^2(C_0, G)}^2 \geq c \cdot \varepsilon^4 \int_{\pi+2\varepsilon}^{\pi+3\varepsilon} \left(\frac{1-a^2}{|ae^{i\theta} + 1|^2} \right)^{-1} \cdot |ae^{i\theta} + 1|^{-8} d\theta \geq c\varepsilon^{-2} = \frac{c}{(1-a)^2}$$

for $1 - a$ sufficiently small. This implies the claim of Lemma 4.3. \square

Sketch of Proof of Lemma 4.2. The property asked for in Lemma 4.2 is essentially a consequence of us choosing the diffeomorphisms as restrictions of Möbius transforms onto S^1 and the fact that given any two triples (w_1, w_2, w_3) and (z_1, z_2, z_3) of points on S^1 there is a unique Möbiustransform mapping z_i to w_i . To be more precise, using the group property of the Möbius transforms one can reduce the claim of Lemma 4.2 to proving that for any distinct $\vartheta_{1,2,3} \in [0, 2\pi)$ and any $a_0 \in [0, 1)$ the derivative of the map $(b, \psi) \mapsto (f_{b,\psi}(\vartheta_1), f_{b,\psi}(\vartheta_2), f_{b,\psi}(\vartheta_3))$ has full rank in the point $(b, \psi) = (a_0, 0) \in \mathbb{C} \times \mathbb{R}$. A short calculation then verifies this claim. \square

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