The Aizerman and Kalman Conjectures using Symmetry ⋆

Ross Drummond [a], Stephen Duncan [a],

[a] *Department of Engineering Science, University of Oxford, 17 Parks Road, OX1 3PJ Oxford, United Kingdom*

Abstract

Using a decomposition of a Lurie system in terms of symmetric and skew-symmetric matrices, this paper presents reformulations of the classical conjectures of Aizerman and Kalman which give valid conditions for absolutely stability. Under this decomposition, it is shown that a restatement of the Aizerman conjecture implies stability while the re-stated Kalman conjecture implies contraction.

Key words: Kalman Conjecture, Symmetry, Absolute Stability Theory.

1 Introduction

The classical conjectures of Aizerman [1] and Kalman [10] propose Routh-Hurwitz type stability criteria for a class of nonlinear systems known as Lurie systems [11]. These conjectures have been shown to be false by counter-example [5, 2, 8], but in this paper, restatements of these conjectures are given such that they represent valid stability tests. The key result underpinning the restatements is the decomposition of the square matrices of the "closed-form" Lurie system into symmetric and skew-symmetric components. It is noted that the presented results are applicable to generic SISO Lurie systems and do not rely upon the system admitting a symmetric realisation.

2 Notation

The set of real matrices of dimension $n \times m$ are denoted by $\mathbb{R}^{n\times m}$, the set of symmetric matrices of dimension n as $\mathbb{S}^n$, the set of skew-symmetric matrices of dimension n as $\mathbb{A}^n$. The square matrix of dimension $n \times n$ containing 0s is denoted $\mathbf{0}^{n\times n}$. We adopt the notation of contraction theory from [6]. $\mathcal{M}$ is used to define a manifold of dimension n. $T_x\mathcal{M}$ denotes the tangent vector spaces at $x \in \mathcal{M}$ with $\delta x \in T_x\mathcal{M}$ a vector in the tangent space at x. The tangent bundle $T\mathcal{M}$ is the disjoint union of these tangent vector spaces over $\mathcal{M}$.

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Email addresses: `ross.drummond@eng.ox.ac.uk` (Ross Drummond), `stephen.duncan@eng.ox.ac.uk` (Stephen Duncan).

3 Lurie Systems

The class of nonlinear systems considered by the Aizerman and Kalman conjectures are of the Lurie type.

Definition 1 (Lurie System) *The Lurie system is defined by the vector field*

$$\begin{cases} \dot{x} &= Ax + B\phi(y) \\ y &= Cx \end{cases} \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ and $x \in \mathcal{M}$. We restrict our attention to the case of $\phi(y) : \mathcal{Y} \subseteq \mathbb{R} \to \mathbb{R}$ being a single nonlinear term that is assumed to be a C^2 function, Lipschitz on $\mathcal{Y}$ satisfying $\phi(0) = 0$, globally sector bounded

$$\frac{\phi(y)}{y} \in \Delta := [\underline{\mu}, \overline{\mu}] \quad \forall y \in \mathcal{Y} \subseteq \mathbb{R} \tag{2a}$$

and slope restricted

$$\frac{d\phi(y)}{dy} \in \Pi := [\underline{\pi}, \overline{\pi}] \quad \forall y \in \mathcal{Y} \subseteq \mathbb{R}. \tag{2b}$$

The following (false) conjectures of Aizerman [1] and Kalman [10] were proposed for verifying the stability of the Lurie system (1) in terms of Routh-Hurwitz type criteria. The form of the conjectures used here is in terms of the state-space matrices of the Lurie system.

[Aizerman Conjecture] If the matrix inequalities

$$A + kBC < 0 \quad \forall k \in \Delta \tag{3}$$

hold, then the nonlinear Lurie system is stable.

[Kalman Conjecture] Similarly, if the matrix inequalities

$$A + kBC < 0 \quad \forall k \in \Pi \tag{4}$$

hold, then the Lurie system (1) is stable.

The Aizerman criterion was posed as a question in [1] while the Kalman conjecture was formulated as a criterion in [10]. Counter-examples have been found that disprove both conjectures in general [3], with a comprehensive discussion of these conjectures given in [12]. For first and second order continuous time systems, the Aizerman conjecture is known to be true [14], but counter-examples exist for third order systems [5]. Similarly, third order counter-examples have been found for the Kalman conjecture [2] and a second order counter example was found for the discrete time case in [8]. The Aizerman conjecture was shown to be true when the linear system is described by a "strictly negative imaginary" transfer function [4] or when it has a positive impulse response [7]. Even though the conjectures were shown to be false in general, valid techniques have been developed to verify stability of Lurie type systems, with this problem known as the absolute stability problem [11].

4 Symmetrical Decompositions of Lurie Systems

This section introduces the symmetrical decomposition of the Lurie system matrices that is used in the restatements of the Aizerman and Kalman conjectures. The essential result is that the closed-form vector field of the Lurie system can be decomposed into symmetrical and skew-symmetrical components. The proof of this result relies on the following basic property of square matrices.

Lemma 1 *Each square matrix $M \in \mathbb{R}^{n\times n}$ can be uniquely decomposed into the sum of a symmetrical matrix $M_s = \frac{1}{2}(M + M^T) \in \mathbb{S}^{n\times n}$ and a skew-symmetrical matrix $M_a = \frac{1}{2}(M - M^T) \in \mathbb{A}^{n\times n}$ such that $M = M_s + M_a$.*

Proof. The result is well known from linear algebra. See [9] for instance. ■

Define the *closed-form* Lurie system as

$$\dot{x} = Ax + \frac{\phi(y)}{y}BCx \tag{5}$$

where both A and BC are square matrices in $\mathbb{R}^{n\times n}$. This form is obtained by noting that $B\phi(y) = B\frac{\phi(y)}{y}y = \frac{\phi(y)}{y}BCx$ for the SISO case considered. Lemma 1 can then be used to uniquely decompose this vector field into symmetric and skew-symmetric components

$$\dot{x} = A_s x + A_a x + \frac{\phi(y)}{y}(BC)_s x + \frac{\phi(y)}{y}(BC)_a x \tag{6}$$

where $\{A_s, (BC)_s\} \in \mathbb{S}^{n\times n}$ and $\{A_a, (BC)_a\} \in \mathbb{A}^{n\times n}$ with the subscripts s and a respectively denoting the symmetric and skew-symmetric components of the square matrices.

For the re-statement of the stability conjectures, the following pair of matrices characterising the closed loop Lurie system dynamics of (5) are defined.

Definition 2 (Σ, Σ_s, Σ_a) *Define the matrix pairs $\Sigma = \{A, BC\}$, $\Sigma_s = \{A_s, (BC)_s\}$, $\Sigma_a = \{A_a, (BC)_a\}$.*

5 Stability and the Aizerman Conjecture

A re-statement of the Aizerman conjecture is now presented that provides a valid test for stability.

Theorem 1 *If the matrix Hurwitz condition of the Aizerman conjecture*

$$A + kBC < 0 \quad \forall k \in \Delta \tag{7}$$

holds when the matrix pair $\Sigma = \{A,\ BC\}$ are replaced by $\Sigma_s = \{A_s,\ (BC)_s\}$, then the origin is an asymptotically equilibrium point of the Lurie system.

Proof. The result is based upon the candidate Lyapunov function $V_A(x) = x^T x$ being the energy of the states of the Lurie system. The time domain derivative of $V_A(x)$ along the trajectories of (1) is $\dot{V}_A(x) = x^T\dot{x} + \dot{x}^T x$ where

$$\dot{V}_A = x^T\left(A + \frac{\phi(y)}{y}BC + \left(A + \frac{\phi(y)}{y}BC\right)^T\right)x \tag{8a}$$

$$= x^T\left(A_s + A_a + \frac{\phi(y)}{y}(BC)_s + \frac{\phi(y)}{y}(BC)_a\right. \tag{8b}$$

$$\left. + \left(A_s + A_a + \frac{\phi(y)}{y}(BC)_s + \frac{\phi(y)}{y}(BC)_a\right)^T\right)x \tag{8c}$$

$$= 2x^T\left(A_s + \frac{\phi(y)}{y}(BC)_s\right)x \tag{8d}$$

since $M_s + M_s^T = 2M_s$ and $M_a + M_a^T = \mathbf{0}^{n\times n}$ for $M_s \in \mathbb{S}^{n\times n}$ and $M_a \in \mathbb{A}^{n\times n}$. Then, if (3) holds for the matrix pair Σ_s, $\dot{V}_A < 0\ \forall \phi(y)/y \in \Delta$ and the level sets of $V_A(x)$ are positively invariant. ■

6 Contraction and the Kalman Conjecture

This section extends the asymptotic stability results of Section 5 by considering the convergence of the trajectories of the Lurie system towards each other with the conditions of the Kalman conjecture linked to contraction theory [13]. A result of contraction theory says that if the symmetric part of the Jacobian of the systems dynamics is Hurwitz on some region, then the system trajectories are contracting and converge towards each other. This definition was applied in [16] to Lurie systems.

The main concepts of contraction theory as given in [6] are now described.

Definition 3 (Contraction Metric [6]) *A contraction metric is defined as the positive definite function $F(x, \delta x) : \mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}_{\geq 0}$ mapping the tangent bundle to the line of non-negative real numbers for every $(x, \delta x) \in \mathcal{TM}$.*

Definition 4 (Forward Invariant Set [6]) *The flow of the Lurie system from the initial condition $x_0 \in \mathcal{M}$ at time t_0 is denoted $\psi_{t_0}(\cdot, x_0)$. If $\psi_{t_0}(t, x_0) \in \mathcal{C}$ for each $t \geq t_0 \geq 0$, then the set $\mathcal{C} \subseteq \mathcal{M}$ is called forward invariant and connected. It is assumed that every two points in $\mathcal{C}$ can be connected by a smooth curve $\gamma : I \times \mathcal{C}$ with $I \triangleq \{s \in \mathbb{R} \mid 0 \leq s \leq 1\}$ that satisfies $\gamma(0) = x_1$, and $\gamma(1) = x_2$.*

Definition 5 (Findler distance [6]) *Consider the contraction metric of Definition 3 defined on the manifold $\mathcal{M}$. For any two points $(x_1, x_2) \in \mathcal{C} \subseteq \mathcal{M}$, let $\Gamma(x_1, x_2)$ be the collection of piecewise C^1 curves γ. The distance $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0}$ induced by the contraction metric F satisfies*

$$d(x_1, x_2) \triangleq \inf_{\Gamma(x_1,x_2)} \int_I F(\gamma(s), \dot{\gamma}(s))ds \tag{9}$$

Semi-algebraic conditions exist for checking incremental stability of systems such as the Lurie system.

Theorem 2 ([6]) *Consider a system that evolves along a smooth manifold $\mathcal{M}$ and assume there exists a connected forward invariant set $\mathcal{C} \subseteq \mathcal{M}$. Let $F(x, \delta x)$ be a candidate contraction metric that satisfies $\dot{F}(x, \delta x) < -\alpha(F(x, \delta x))$ for each $t \in \mathbb{R}$, $x \in \mathcal{C} \subseteq \mathcal{M}$ and $\delta x \in T_x\mathcal{M}$ for some $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Then the system is said to be incrementally stable satisfying $d(\psi_{t_0}(t, x_1), \psi_{t_0}(t, x_2)) < 0$. If the system is incrementally stable, then it is said to be contracting via the metric $F(x, \delta x)$ on a contracting region $\mathcal{C} \subseteq \mathcal{M}$.*

This notion of a contracting system is linked to the Kalman conjecture by the following theorem.

Theorem 3 *If the matrix Hurwitz conditions of the Kalman conjecture*

$$A + kBC < 0 \quad \forall k \in \Pi \tag{10}$$

are satisfied when the matrix pair $\Sigma = \{A, BC\}$ are replaced by $\Sigma_s = \{A_s, (BC)_s\}$, then the Lurie system is a contracting system.

Proof. Define $\mathcal{C}$ as the set of x for which $d\phi(y)/dy \in [\underline{\pi}, \overline{\pi}]$ and consider the contraction metric $F(x, \delta x) = \delta x^T \delta x$. It follows that $\dot{F}(x, \delta x) = \delta x^T \dot{\delta x} + \dot{\delta x}^T \delta x$ where δx is defined by the dynamics of the variational system

$$\begin{cases} \dot{\delta x} &= A\delta x + B\delta\phi(y) \\ \delta y &= C\delta x \end{cases} \tag{11}$$

which, by linearising the Lurie system, has closed-form

$$\dot{\delta x} = \left(A + \frac{d\phi(y)}{dy} BC\right) \delta x. \tag{12}$$

As the variational system (12) is defined by the same matrices as the Lurie system (5), it can also be decomposed into symmetrical and skew-symmetrical parts, and in an analogous manner to the proof of Theorem 1, the condition $\dot{F}(x, \delta x) < 0$ can be expressed as

$$\dot{F}(x, \delta x) = 2\delta x^T \left(A_s + \frac{d\phi(y)}{dy}(BC)_s\right) \delta x < 0. \tag{13}$$

If then the Hurwitz condition (4) of the Kalman conjecture holds for the matrix pair Σ_s, then $\dot{F} < 0 \; \forall x \in \mathcal{C}$, with the conditions of Theorem 2 then being satisfied. ∎

Remark 1 Note that the set of slopes from (2b) contains the set of sectors from (2a) with $\Pi \supseteq \Delta$. As such the Hurwitz condition of the Aizerman conjecture is less restrictive than that of the Kalman conjecture. ⋆

7 Numerical Examples

7.1 Sufficient Stability Test

Consider the linear system from [15]

$$G(s) = \frac{s^2 - 0.2s - 0.1}{s^3 + 2s^2 + s + 1}. \tag{14}$$

This system has a balanced realisation

$$A = \begin{bmatrix} 0.0394 & -0.8268 & 0.2528 \\ -0.8268 & -0.2358 & 0.4811 \\ 0.2528 & 0.4811 & -1.7249 \end{bmatrix},\ B = \begin{bmatrix} -0.3022 \\ 0.6585 \\ -0.8110 \end{bmatrix},$$
$$C = \begin{bmatrix} -0.3022 & 0.6585 & -0.8110 \end{bmatrix}$$

which is symmetric. The maximum value of k for which the matrix Hurwitz conditions of Theorem 1 hold is $k = 1.1$. This is within the stability range [15] and as such the theorem can provide a sufficient condition for stability. However, this maximum value for k is even more conservative than the circle criterion [15].

7.2 The Counter-Example of Barabanov

A counter example to the Kalman conjecture was developed in [2] where a system that satisfied the Kalman conjecture was shown to have a non-trivial periodic solution. This system was of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -2 \\ 1 & 0 & 1 & -1 \end{bmatrix}}_{A_{\mathrm{B}}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} \phi(y), \tag{16a}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1, & x_2, & x_3, & x_4 \end{bmatrix}^T. \tag{16b}$$

The eigenvalues of the matrix $\frac{1}{2}(A_{\mathrm{B}} + A_{\mathrm{B}})$ are $\{-1.7247,\ 0,\ 0,\ \underline{0.7247}\}$ and hence, due to the positive eigenvalue, the restated stability (contraction) conjectures cannot verify stability for slope restricted nonlinearities in $\frac{d\phi(y)}{dy} \in [0, \overline{\pi}]$. This system is therefore not a counter-example of Theorem 3.

8 Conclusion

This paper has presented restatements of the conjectures of Aizerman and Kalman such that they provide valid stability conditions for Lurie systems. These conditions exploit a decomposition of the system matrices into symmetrical and skew-symmetrical components. More specifically, the restated Aizerman conjecture is shown to imply convergence of the trajectories of the Lurie system towards the origin while the Kalman conjecture implies contraction. We conclude by commenting that compared to the many results on absolute stability that can be found in the literature, the results presented here seem trivial. However, the results of this paper imply that the conjectures of Aizerman and Kalman would have been correct if they had been framed in terms of a state-space realisation that considered the symmetry of the system.

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