Additive
Higher Representation Theory

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Abstract

This thesis is devoted to the study of higher representation theory as introduced in [Rou4]. As this theory is in its early days, it is essential to seek out modules that can rightfully be named building blocks and allow one to express as much of the structure of arbitrary modules as possible in their terms. We contribute towards this undertaking in the case of additive higher representation theory. Inspiration is drawn from Soergel bimodules which categorify the Hecke algebra.

We introduce functorially cyclic modules as well as (strongly) universal cell modules. Examples include the minimal categorifications of [Rou4]. Properties of such modules are discussed and universal properties in terms of representable 2-functors are established. This leads to constructions and classifications in terms of split Frobenius objects, using a new variant of the Barr-Beck theorem for additive categories. Furthermore, we encounter a new class of modules so called coinvariant modules which arise from automorphism group actions. We also construct canonical cofiltrations and demonstrate why the Jordan-Hölder theory of [Rou4] does not readily generalise. Throughout, we comment on the succession [MaMi1]-[MaMi5] that tackles the same questions, however arrives at different conclusions.

As applications, we first show that the 2-category of singular Soergel bimodules of [Wi2] arises naturally within the additive higher representation theory of Soergel bimodules. Second, we establish (weak) equivalences between certain associated universal cell modules together with a categorification of cell module homomorphisms of the Hecke algebra. Third, we show that singular Soergel bimodules constructed with a faithful representation categorify the Schur algebroid, generalising the main result of [Li]. Fourth given a group and a subgroup, we recover the additive monoidal category of representations of the subgroup from the corresponding category for the group without invoking Tannakian formalism.

Contents

1 Introduction **1**

2 The Classical Case **13**

2.1 Rings and modules with positive structure coefficients 14
2.2 Structure of the category of representations 18
2.3 Cells and cyclicity 20
2.4 Annihilators 22
2.5 Universal cell modules 24
2.6 Hecke algebras 27
2.6.1 Basic definitions 28
2.6.2 W-graphs and $\mathcal{H} - \mathrm{Mod}^{(\mathbb{Z}[v,v^{-1}],\mathbb{N}_0[v,v^{-1}])}$ 31
2.6.3 W-graphs and cell modules in type B_2 34

3 Higher Representation Theory I - Basics **37**

3.1 2-categories 37
3.2 The centre of an object in a 2-category 43
3.3 Additive higher representations 45
3.3.1 Decategorification 46
3.3.2 Structure of the 2-category of higher representations 50
3.3.3 Annihilators and ideals 52
3.3.4 Representability and uniqueness 55

4 Higher Representation Theory II - New Developments **60**

4.1 Centres and base change 61
4.2 Cyclicity 64
4.2.1 1-cyclicity 64
4.2.2 Refinement for the action of a 2-category 68
4.2.3 2-cyclicity 69
4.2.4 Functorially cyclic modules 73

4.2.5 Choices for functorially cyclic modules . . . 77
4.3 Split Frobenius objects and the additive Barr-Beck theorem . . . 78
4.3.1 Split Frobenius objects . . . 79
4.3.2 Split Frobenius objects on additive categories . . . 84
4.3.3 Split Frobenius objects on 2-functors . . . 92
4.3.4 Split Frobenius objects on additive higher representations . . . 103
4.3.5 The split Frobenius 2-category . . . 109
4.3.6 Preorders on split Frobenius objects . . . 112
4.4 Restriction along quotients and its adjoints . . . 114
4.5 Functorially cyclic modules revisited . . . 120
4.5.1 Functorially cyclic modules and split Frobenius objects . . . 121
4.5.2 Free functorially cyclic modules . . . 125
4.5.3 Equivalences and weak equivalences . . . 127
4.5.4 Semi-free functorially cyclic modules . . . 129
4.5.5 Universal cell modules . . . 132
4.6 Coinvariant theory . . . 138
4.7 Compatibility with quotients . . . 155
4.8 Induced tensor products on functorially cyclic modules . . . 158
4.9 Towards a Jordan-Hölder theory . . . 171
4.9.1 Cofiltrations . . . 171
4.9.2 Covers and limitations . . . 176

5 Applications to Soergel Bimodules **180**
5.1 Singular Soergel bimodules . . . 180
5.2 Conjectures on Duflo involutions and split Frobenius objects . . . 185
5.3 Singular Soergel bimodules revisited . . . 191
5.4 Universal cell modules . . . 196
5.5 Coinvariant theory . . . 199
5.6 Control from 2-morphisms . . . 203
5.7 Reduction to non reflection faithful representations . . . 204

6 Applications to Group Theory **209**
6.1 Basic notation . . . 209
6.2 Functorially cyclic representations . . . 210
6.3 Subgroups . . . 211
6.4 Defect groups . . . 221

7 Future Directions **222**
7.1 Passage between monoidal and 2-categories 222
7.2 Twisted singular Soergel bimodules 223

Bibliography **224**

Chapter 1

Introduction

Background

Higher representations are ubiquitous in modern representation theory. Already in 1995, [Ber] started analysing general actions of monoidal categories. 13 years later, [Rou4] detailed the category theoretic framework for such actions. More precisely given 2-categories $\mathfrak{A}$ and $\mathfrak{D}$, [Rou4] defines the *2-category* $\mathfrak{Hom}(\mathfrak{A}, \mathfrak{D})$ *of 2-representations of* $\mathfrak{A}$ *in* $\mathfrak{D}$. Objects are 2-functors $\mathfrak{A} \to \mathfrak{D}$, 1-arrows are morphisms of 2-functors, and 2-arrows are morphisms of morphisms of 2-functors. In most examples relevant to representation theory, $\mathfrak{D}$ takes the form of $\mathfrak{A}dd$ (resp. $\mathfrak{A}dd^{\mathrm{idem}}$, $\mathfrak{L}in_{\mathrm{k}}$, $\mathfrak{A}b$ or $\mathfrak{T}ri$), which is the 2-category of additive (resp. idempotent closed additive, k-linear, Abelian or triangulated) categories with additive (resp. additive, k-linear, Abelian or triangulated) functors and transformations as 1- and 2-arrows.

The formal definition of higher representation theory was targeted towards the study of the *2-Kac-Moody algebras* $\mathfrak{A}_{\mathfrak{g}}$ associated to any Kac-Moody algebra $\mathfrak{g}$, see [Rou4] as well as [ChRo, La, KhLa]. The 2-Kac-Moody algebra $\mathfrak{A}_{\mathfrak{g}}$ categorifies Lusztig's idempotented version of the quantized enveloping algebra $U_q(\mathfrak{g})$, see [KhLa, We]. At the same time quiver Hecke algebras came into being, see [Rou4, KhLa]. It was shown that they categorify one-half of $U_q(\mathfrak{g})$, see [Rou5, VaVa], in the same time generalising the work of [Ar] on cyclotomic Hecke algebras. Another stepping stone was the proof of Broué's Abelian defect group conjecture for symmetric groups, see [ChRo]. Furthermore, Rouquier announced the construction of tensor products for dg-representations of $\mathfrak{A}_{\mathfrak{g}}$ which shall ultimately give rise to 4-dimensional TQFT's as envisaged by [CrFr]. For a more comprehensive summary, see [Br]. The Jordan-Hölder theorem for $\mathfrak{A}_{\mathfrak{g}}$ in [Rou4] motivated us to undertake this work.

Our focus is on additive higher representation theory of a general additive 2-category $\mathfrak{A}$. More precisely, we analyse building blocks of this 2-category. Such $\mathfrak{A}$-modules should be small, should satisfy good universal properties and should allow one to describe as much of the structure of arbitrary $\mathfrak{A}$-modules as possible in their terms. This is the central quest that drives this thesis and ultimately leads us to introduce *strongly universal cell modules.* Closely related research was undertaken in [MaMi1]-[MaMi5] yet under finiteness and duality assumptions which we do not impose. They arrive at a different and complimentary conclusion, namely cell modules that do not admit non-trivial quotients, which are called *simple transitive 2-representation.*

Let us clarify the term additive higher representation theory. First, we assume that $\mathfrak{A}$ is idempotent closed. This makes the exposition cleaner however most results can be expressed with additional notational burden in the general case. We denote by $\mathfrak{A} - \mathrm{Mod}$ the 1-2-full sub-2-category of $\mathfrak{Hom}(\mathfrak{A}, \mathfrak{A}dd)$ whose objects are additive 2-functors. The 0-, 1- and 2-arrows of $\mathfrak{A} - \mathrm{Mod}$ are referred to as $\mathfrak{A}$-modules, $\mathfrak{A}$-functors, and $\mathfrak{A}$-transformations respectively. The 1-2-full sub-2-category of $\mathfrak{A} - \mathrm{Mod}$ with $\mathfrak{A}$-functors that map objects to idempotent closed additive categories is denoted $\mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}}$. Note that in most relevant examples for $\mathfrak{D}$, we have forgetful 1-2-faithful 2-functors $\mathfrak{D} \to \mathfrak{A}dd$, which induce 1-2-faithful 2-functors $\mathfrak{Hom}(\mathfrak{A}, \mathfrak{D}) \to \mathfrak{A} - \mathrm{Mod}$, which further justifies our decision to focus on $\mathfrak{A} - \mathrm{Mod}$. For the remainder of this introduction, we assume that $\mathfrak{A}$ is strict.

Since we will base our constructions on *principal representations* and their quotients, we recall their definition here. This terminology originated in [MaMi1] within $\mathfrak{Hom}(\mathfrak{A}, \mathcal{A}b)$, however in essence these modules are introduced in [Rou4]. Crucially, the *minimal categorifications* $\mathcal{L}(\lambda) \in \mathfrak{A}_{\mathfrak{g}} - \mathrm{Mod}$ of the finite dimensional, simple representations $L(\lambda)$ of $\mathfrak{g}$ are examples. For an object $a \in \mathfrak{A}$, $\mathcal{P}_a := \mathfrak{A}(a, -) \in \mathfrak{A} - \mathrm{Mod}$ defines the principal representation of weight a. Part of the datum of any $R \in \mathfrak{A} - \mathrm{Mod}$ is an additive category $R(a)$ and further structure that allows one to extend the assignment $\Phi_a(R) := R(a)$ to a 2-functor $\Phi_a : \mathfrak{A} - \mathrm{Mod} \to \mathfrak{A}dd$, called the *$a$-weight space 2-functor.* The 2-Yoneda Lemma 3.3.49 implies that $\mathcal{P}_a$ represents Φ_a. Given any $M \in R(a)$, we have a left ideal $\mathcal{A}\mathrm{nn}(M) \trianglelefteq \mathcal{P}_a$ given by all 2-arrows that act trivially on M. On the other hand, to any left ideal $\mathcal{I} \trianglelefteq \mathcal{P}_a$ on can associate a sub-2-functor $\Phi_a^{\mathcal{I}}$ of Φ_a, by specifying $\Phi_a^{\mathcal{I}}(R)$ to be the full additive subcategory of $R(a)$ whose objects M satisfy $\mathcal{A}\mathrm{nn}(M) \subset \mathcal{I}$. Moreover, there exists a canonical quotient $\mathcal{P}_a \twoheadrightarrow \mathcal{P}_a^{\mathcal{I}}$ in $\mathfrak{A} - \mathrm{Mod}$ with associated ideal $\mathcal{I} \trianglelefteq \mathcal{P}_a$. One then proves that $\mathcal{P}_a^{\mathcal{I}}$ represents $\Phi_a^{\mathcal{I}}$.

Main results

A *split Frobenius object* $\hat{\Theta}$ in $\mathfrak{A}_a = \mathfrak{A}(a,a)$ is a triple $\hat{\Theta} = (\Theta, \Delta, \mu)$, where $\Theta \in \mathfrak{A}_a$, $\Delta \in \mathfrak{A}_a(\Theta, \Theta\Theta)$ and $\mu \in \mathfrak{A}_a(\Theta\Theta, \Theta)$ such that the Frobenius relations hold and such that Δ splits μ. In formulas, we require $(\Theta\mu) \circ (\Delta\Theta) = \Delta \circ \mu = (\mu\Theta) \circ (\Theta\Delta)$ and $\mu \circ \Delta = \Theta$, where Θ denotes the identity 2-arrow on Θ as usual. To our knowledge the following theorem is a new result, which we call the additive Barr-Beck theorem.

Theorem 1 (Theorem 4.3.26). *Let $\mathcal{M} \in \mathfrak{A}dd^{\mathrm{idem}}$ and let $\hat{\Theta}$ be a split Frobenius object in $\mathfrak{A}dd(\mathcal{M}, \mathcal{M})$. Then, there exists ${}_{\hat{\Theta}}\mathcal{M} \in \mathfrak{A}dd^{\mathrm{idem}}$, additive functors ${}_{\hat{\Theta}}\mathcal{M} \xrightarrow{\vartheta^{\hat{\Theta}}} \mathcal{M} \xrightarrow{\vartheta_{\hat{\Theta}}} {}_{\hat{\Theta}}\mathcal{M}$ and transformations $id_{{}_{\hat{\Theta}}\mathcal{M}} \xRightarrow{\alpha} \vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}} \xRightarrow{\beta} id_{{}_{\hat{\Theta}}\mathcal{M}}$ with $\beta \circ \alpha = id_{{}_{\hat{\Theta}}\mathcal{M}}$ and $\hat{\Theta} = (\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}}, \vartheta^{\hat{\Theta}}\alpha\vartheta_{\hat{\Theta}}, \vartheta^{\hat{\Theta}}\beta\vartheta_{\hat{\Theta}})$. On the other hand, given $\mathcal{D} \in \mathfrak{A}dd$, additive functors $\mathcal{D} \xrightarrow{G} \mathcal{M} \xrightarrow{F} \mathcal{D}$ and transformations $id_{\mathcal{D}} \xRightarrow{\alpha'} FG \xRightarrow{\beta'} id_{\mathcal{D}}$ with $\beta' \circ \alpha' = id_{\mathcal{D}}$, then $\hat{\Theta} := (GF, G\alpha'F, G\beta'F)$ is a split Frobenius object in $\mathfrak{A}dd(\mathcal{M}, \mathcal{M})$ and we have an equivalence ${}_{\hat{\Theta}}\mathcal{M} \cong \mathcal{D}^{\mathrm{idem}}$ of additive categories.*

In fact, we will provide an explicit construction of ${}_{\hat{\Theta}}\mathcal{M}$ that can be regarded as projective $\hat{\Theta}$-modules in $\mathcal{M}$. The construction is similar in spirit to the Eilenberg-Moore category associated to a monad, however it does not consider all possible actions of a monad on objects, but instead only direct summands of free objects. Note that an analogous theorem within $\mathfrak{A} - \mathrm{Mod}$ also holds.

Next, assume that $\mathcal{I} \trianglelefteq \mathcal{P}_a$ and let $\hat{\Theta}$ be a split Frobenius object in $\mathfrak{A}_a$ that stabilizes $\mathcal{I}$. Then, $\hat{\Theta}$ induces a split Frobenius object in the category of endofunctors of $\Phi_a^{\mathcal{I}}(R)$ for all $R \in \mathfrak{A} - \mathrm{Mod}$. Since $\Phi_a^{\mathcal{I}}$ restricts to a 2-functor $\mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$, Theorem 1 yields additive categories ${}_{\hat{\Theta}}\Phi_a^{\mathcal{I}}(R) := {}_{\hat{\Theta}}\left(\Phi_a^{\mathcal{I}}(R)\right)$ and we will prove that this assignment extends to a 2-functor ${}_{\hat{\Theta}}\Phi_a^{\mathcal{I}} : \mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$. Furthermore, we show that there exist morphisms of 2-functors and morphisms of morphisms of 2-functors that recover $\hat{\Theta}$ in $\mathcal{E}nd_{\mathfrak{A}}(\mathcal{P}_a^{\mathcal{I}})$ as in Theorem 1.

Theorem 2 (Theorem 4.5.6). *The 2-functor ${}_{\hat{\Theta}}\Phi_a^{\mathcal{I}}$ is represented by an explicitly defined $\mathfrak{A}$-module ${}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$. In other words, we have equivalences of idempotent closed additive categories $\mathcal{H}om_{\mathfrak{A}}({}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}, R) \cong {}_{\hat{\Theta}}\Phi_a^{\mathcal{I}}(R)$ which are 2-functorial in $R \in \mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}}$.*

The proof of Theorem 2 relies on keeping track of all the 2-categorical structure and two generalisations of Theorem 1: The first applies to split Frobenius objects on 2-functors with codomain $\mathfrak{A}dd^{\mathrm{idem}}$ and the second applies to split Frobenius objects on additive higher representations. For simplicity, we will carry out the explicit construction of ${}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$ only in the case of an additive monoidal category, rather than an additive 2-category $\mathfrak{A}$. Note that ${}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$ is equivalent to a not necessarily full, idempotent closed sub-$\mathfrak{A}$-module of $\mathcal{P}_a^{\mathcal{I}}$.

Let us give some flavour how Theorem 2 relates to the classical case. Denote by $A := [\mathfrak{A}]$ the split Grothendieck ring of $\mathfrak{A}$, and denote by $A' := A \otimes_{\mathbb{Z}} \mathbb{Q}$, see [Ma] for a detailed discussion of (pre-)categorifications. Given an idempotent $e' \in A'$, we have a projective module $A'e'$ and canonical isomorphisms $\mathrm{Hom}_{A'}(A'e', M') \cong e'M'$ for all $M' \in A' - \mathrm{Mod}$. This is very similar to Theorem 2. However, when working with A subtleties arise. We can find $e \in A$ with $e' = e \otimes \frac{1}{n}$ for some $n \in \mathbb{N}_{>1}$ and $e^2 = ne$. The element e may still be virtual, *i.e.* the difference of two elements that come from $\mathfrak{A}$. In this case, $\mathfrak{A}$ cannot tell us much about $A'e'$ *a priori* . Not even if e is categorifiable by a split Frobenius object, *i.e.* if there exists $\mathcal{M} \in \mathfrak{A} - \mathrm{Mod}$ with $[\mathcal{M}] = M$, and a split Frobenius object $\hat{\Theta}$ on $\mathcal{M}$ with $[\Theta] = e$, is an isomorphism $\mathrm{Hom}_A(Ae, M) \cong eM$ guaranteed in general. Only if $\hat{\Theta}$ satisfies certain intricate $\hat{\Theta}$-descent properties, one can deduce this isomorphism from Theorem 2. We will study this point in the Soergel bimodule application.

While the constructions presented thus far are *ad hoc*, the main body of this thesis provides a derivation from first principles. Starting with classical notions of cyclicity, we discuss the analogue of *1-cyclicity* in $\mathfrak{A} - \mathrm{Mod}$. Each 1-cyclic module comes with generators. If every non-trivial object of an $\mathfrak{A}$-module is a generator, we call it a *cell module*. Next, we introduce *2-cyclic* modules which are 1-cyclic and have 2-arrows that are controlled by $\mathfrak{A}$. *Functorially cyclic modules* are a further specialisation that have a readily testable definition. This leads naturally to the definition of *weak equivalences*. The definition of a functorially cyclic module $\mathcal{C}$ requires existence of $a \in \mathfrak{A}$, $\mathcal{I} \trianglelefteq \mathcal{P}_a$, structure very similar to the one present in Theorem 1, and an additional indecomposablility requirement. For example, $\mathcal{P}_a$ is functorially cyclic if the unit object $I_a \in \mathfrak{A}_a$ is *strongly indecomposable*, *i.e.* the only non-trivial idempotent of I_a is its identity 2-arrow. In the same way as in Theorem 1, we associate to $\mathcal{C}$ a split Frobenius object $\hat{\Theta}$ in $\mathcal{E}nd_{\mathfrak{A}}(\mathcal{P}_a^{\mathcal{I}})$. Theorem 4.5.4 summarizes several bijections between split Frobenius objects and certain higher representations, one of which implies $\mathcal{C}^{\mathrm{idem}} \cong {}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$. We will also see that non-trivial $\mathfrak{A}$-functors between cell modules only exist if the annihilators of the respective cell modules have the same maximal thick subideal. This observation will make it natural to define *universal cell modules*, which are $\mathfrak{A}$-modules that allow for non-trivial $\mathfrak{A}$-functors to any other cell module that has the same maximal thick ideal in its annihilator. A *strongly universal cell module* satisfies in addition that its endomorphism category consists of a single left cell, and is therefore unique up to weak equivalence. From our perspective, strongly universal cell modules are an appropriate substitute for simple modules.

For every additive 2-category $\mathfrak{A}$, let $W(\mathfrak{A})$ denote the set of isomorphism classes of indecomposable 1-arrows, and denote by B_x a representative for $x \in W(\mathfrak{A})$. Following [KaLu], $W(\mathfrak{A})$ carries a left order $\leq_L$ and a 2-sided order $\leq_{LR}$, where for example $x \leq_L y$ if there exists $z \in W(\mathfrak{A})$ and a split injection of B_x into $B_z B_y$. An equivalence class of $\leq_L$ (resp. $\leq_{LR}$) is called a left (resp. 2-sided) cell. For each 2-cell $\underline{c} \subset W(\mathfrak{A})$, we will associated a canonical *maximal* $\underline{c}$*-cosubquotient 2-functor* $\mathcal{Q}_{\underline{c}}$, which is an endo-2-functor of $\mathfrak{A} - \mathrm{Mod}$ that comes as close as possible to being a cosubquotient of a canonical cofiltration of $\mathfrak{A} - \mathrm{Mod}$. If $\mathcal{Q}_{\underline{c}}$ is a projector, we say that $\mathfrak{A}$ has $\underline{c}$-cell modules. Equivalently, $\mathfrak{A}$ has $\underline{c}$-cell modules if there exists $x,\, z \in \underline{c}$ with a split injection from B_x to $B_z B_x$ and where x lies in a minimal left cell with respect to $\leq_L$ in $\underline{c}$. We say that a cell module $\mathcal{C} \in \mathfrak{A} - \mathrm{Mod}$ belongs to $\underline{c}$ if $\mathcal{C} = \mathcal{Q}_{\underline{c}}(\mathcal{C})$. Equivalently, a cell module $\mathcal{C} \in \mathfrak{A} - \mathrm{Mod}$ belongs to $\underline{c}$ if the ideal $\mathfrak{J}_{\not\geq_{LR}} \trianglelefteq \mathfrak{A}$ generated by all B_x for $x \in W(\mathfrak{A})$ with $x \not\geq_{LR} \underline{c}$ is the maximal thick ideal in the annihilator of $\mathcal{C}$. Further, we call $\mathfrak{A}$ *weakly regular* if for all 2-sided cells $\underline{c}$ and all left cells $c \subset \underline{c}$ there exists only finitely many left cells $c' \subset \underline{c}$ that are comparable in the left order with c.

Theorem 3 (Theorem 4.5.34). *Let $\mathfrak{A}$ be a weakly regular additive 2-category and let $\underline{c}$ be a 2-sided cell of $\mathfrak{A}$. Then, there exists an idempotent closed, strongly universal cell module that belongs to $\underline{c}$ if and only if there exist $a \in \mathfrak{A}$ and a right indecomposable, split Frobenius object $\hat{\Theta}$ in $\mathfrak{A}/\mathfrak{J}_{\not\geq_{LR}\underline{c}}(a,a)$ whose underlying object lies in a left cell minimal with respect to $\leq_L$ and such that ${}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$ is a cell module, where $\mathcal{I} := \mathfrak{J}_{\not\geq_{LR}}(a,-)$.*

Let us sketch the proof: Let $\mathcal{C}$ be a strongly universal cell module that belongs to $\underline{c}$. It is not hard to associated $a \in \mathfrak{A}$ and a right indecomposable split Frobenius object $\hat{\Theta}$ in $\mathfrak{A}/\mathfrak{J}_{\not\geq_{LR}\underline{c}}(a,a)$ to $\mathcal{C}$, where the underlying object of $\hat{\Theta}$ is a sum of indecomposable objects that are associated to the lowest 2-sided cell of $\mathfrak{A}/\mathfrak{J}_{\not\geq_{LR}\underline{c}}(a,a)$. Here, we used that right indecomposable split Frobenius objects in $\mathfrak{A}/\mathfrak{J}_{\not\geq_{LR}\underline{c}}(a,a)$ correspond to indecomposable split Frobenius objects in $\mathcal{E}nd_{\mathfrak{A}}(\mathcal{P}_a^{\mathcal{I}})$, since $\mathcal{E}nd_{\mathfrak{A}}(\mathcal{P}_a^{\mathcal{I}}) \cong \Phi_a^{\mathcal{I}}(\mathcal{P}_a^{\mathcal{I}}) = \mathfrak{A}/\mathfrak{J}_{\not\geq_{LR}\underline{c}}(a,a)$ by the 2-Yoneda lemma. Essentially from Theorem 1, we obtain an $\mathfrak{A}$-functor $\mathcal{C} \to {}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$, which becomes an equivalence after taking the idempotent closure of $\mathcal{C}$. The subtlety is that we cannot guarantee that $\mathcal{C}^{\mathrm{idem}}$ remains a cell module (nor that it has the other two properties of a strongly universal cell module). However if $\mathcal{C} = \mathcal{C}^{\mathrm{idem}}$, we have $\mathcal{C} \cong {}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$ and can deduce the additional properties of $\hat{\Theta}$, as well. On the other hand, given $\hat{\Theta}$ as in Theorem 3, we will use simple properties of annihilators to show that ${}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$, is a universal cell module. Using Theorem 2, we have $\mathcal{E}nd_{\mathfrak{A}}({}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}) \cong {}_{\hat{\Theta}}\Phi_a^{\mathcal{I}}({}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}) \cong {}_{\hat{\Theta}}\left(\mathfrak{A}/\mathfrak{J}_{\not\geq_{LR}\mathcal{C}}(a,a)\right)_{\hat{\Theta}}$, where the right hand side is an additive monoidal category explicitly defined in Section 4.3.5. This equivalence allows us to deduce that ${}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$ is strongly universal.

Further findings

There are another four constructions provided in this thesis that deserve attention. Each of them is interesting in their own right. At the same time, they complement and support the development of our main results.

First, *base change over the centre*, see Section 4.1. This is the observation that an *ad hoc* construction of [Rou4], which extends a module by a multiplicity category, is actually a very generic and powerful tool to produce new higher representations. More precisely given $R \in \mathfrak{A}-\mathrm{Mod}$ and a category $\mathcal{D}$ that is linear over the centre $\mathcal{Z}_{\mathfrak{A}}(R)$ of R, we construct an $\mathfrak{A}$-module $R \otimes_{\mathcal{Z}_{\mathfrak{A}}(R)} \mathcal{D}$ whose objects are given by $\mathrm{Ob}(R(a)) \times \mathrm{Ob}(\mathcal{D})$ for all $a \in \mathfrak{A}$.

Second, *induction and coinduction 2-functors*, see Section 4.4. For every 2-functor $\pi : \mathfrak{A} \to \mathfrak{A}'$, there exists a natural *restriction 2-functor* $\mathrm{Res}_\pi : \mathfrak{A}'-\mathrm{Mod} \to \mathfrak{A}-\mathrm{Mod}$. We will discuss explicit constructions of left adjoint 2-functors Ind_π and of a right adjoint 2-functor Coind_π, provided that π is a *quotient functor*. Also, we construct Coind_π explicitly in general. It remains an open task to construct Ind_π and to identify the appropriate notion of adjoint 2-functors that applies in the latter two cases. Amongst other uses, these 2-functors will allow us to define cofiltrations for $\mathfrak{A}-\mathrm{Mod}$ and to define *isotypic $\mathfrak{A}$-modules* associated to 2-sided cells.

Third, *coinvariant modules*, see Section 4.6. A group action of group G on an $\mathfrak{A}$-module R is a monoidal functor $G \to \mathcal{H}om_{\mathfrak{A}}(R, R)$, where we regard G as a monoidal category in an obvious way. Given a group action, we construct explicitly a module $R_G \in \mathfrak{A}-\mathrm{Mod}$ together with a faithful $\mathfrak{A}$-functor $\iota_G : R \to R_G$ and various $\mathfrak{A}$-transformations and show that R_G satisfies a universal property relative to R. On the level of split Grothendieck groups, we recover the standard notion of coinvariant modules by proving $[R_G] \cong [R]_G$. Applied to a universal cell modules, we obtain under certain conditions another universal cell module which has an interesting Grothendieck group. Note that R_G is typically not idempotent closed. Curiously, we will see that the coinvariant module R_G is weakly equivalent to R.

Fourth, *tensor products on functorially cyclic modules*, see Section 4.8. Given an additive monoidal category $\mathcal{B}$, we will define *symmetrically central, split Frobenius objects* in $\mathcal{B}$, which can be viewed as (co)commutative split Frobenius objects in the Drinfeld centre of $\mathcal{B}$. In this case, we show that the associated functorially cyclic module embeds naturally into its endomorphism category, thereby inheriting a monoidal structure. This construction is the basis of Theorem 7 presented below.

Applications

Given a *Coxeter system* (W, S), the graded additive monoidal category $\mathcal{S}oe^\bullet = \mathcal{S}oe^\bullet(W, S)$ of *Soergel bimodules* is a purely algebraic analogue of certain categories of perverse sheaves that appear in geometric representation theory. The category $\mathcal{S}oe^\bullet$ originated in [Soe1] and it was shown in [Soe2] that $W \times \mathbb{Z} \cong W(\mathcal{S}oe^\bullet)$ and that the split Grothendieck ring $[\mathcal{S}oe^\bullet]$ is isomorphic the Hecke algebra $\mathcal{H} = \mathcal{H}(W, S)$. In characteristic zero, *Soergel's Conjecture* states that we have $[B_x] = \underline{H}_x$ for all $x \in W$, *i.e.* that the basis of $\mathcal{H}$ induced from the indecomposable objects of $\mathcal{S}oe^\bullet$ coincides with the Kazhdan-Lusztig basis. This conjecture was shown to hold for various Coxeter systems mostly by drawing on deep geometric theorems, see [Soe1, Hä, Soe5, Fi]. After a third of a century, and yet faster than many expected, the final breakthrough on Soergel's conjecture came in [ElWi2]. Furthermore following initial steps in [Li3, ElKh, Kl, El2], explicit generators and relations for $\mathcal{S}oe^\bullet$ were obtained in [ElWi1] modulo one inexplicit relation for each parabolic subgroup of type H_3. Beyond research intrinsic to $\mathcal{S}oe^\bullet$, one of the most important properties of $\mathcal{S}oe^\bullet$ is that it encodes homomorphism spaces between projective objects of the BGG-category $\mathcal{O}$, see [Soe1, Soe2, St]. Furthermore using the Rouquier complexes introduced in [Rou1], $\mathcal{S}oe^\bullet$ found crucial applications to braid groups and knot theory, see for example [Kh1, KhRo, Rou6, LiWi].

The first application of our theory is concerned with the graded additive 2-category $\mathfrak{S}oe^\bullet$ of *singular Soergel bimodules* which appeared in disguise in [St] for Weyl groups and was introduced for all Coxeter systems (W, S) in [Wi2]. Note that [MaStVa] provides a different generalisation of $\mathcal{S}oe^\bullet$ in type A. The objects of $\mathfrak{S}oe^\bullet$ are given by all *finitary* subsets of S, *i.e.* subsets that generate a finite subgroup. In [Wi1] it is shown that $\mathfrak{S}oe^\bullet$ categorifies the Schur algebroid, which is an idempotent completed generalisation to arbitrary Coxeter groups of the q-Schur algebra associated to symmetric groups. More precisely, Williamson introduces graded additive categories $\mathfrak{S}oe^\bullet(J, I) = {}^I\mathcal{S}oe^{\bullet J}$ that come with a faithful embedding into $\mathcal{S}oe^\bullet$. Further, he defines the Schur algebroid to have objects finitary subsets of S and specifies morphisms as certain subsets ${}^I\mathcal{H}^J$ of $\mathcal{H}$ for all finitary $I,\ J \subset S$. The composition rule is defined by a certain rescaling of the multiplication rule of $\mathcal{H}$. Then, it is shown that $[{}^I\mathcal{S}oe^{\bullet J}] \cong {}^I\mathcal{H}^J$ in a way compatible with the composition rules. Important for our work is that [Wi2] points out that it is more natural define the Schur algebroid using $\mathrm{Hom}_{\mathcal{H}}(\mathcal{H}^I, \mathcal{H}^J) \cong {}^I\mathcal{H}^J$, where $\mathcal{H}^I = {}^{\emptyset}\mathcal{H}^I$ are the $\mathcal{H}$-modules that are induced from so called trivial representations of $\mathcal{H}(W_I, I)$. Analogously, we denote $\mathcal{S}oe^{\bullet I} = {}^{\emptyset}\mathcal{S}oe^{\bullet I}$. We will show that this approach can be followed categorically:

Theorem 4 (Theorem 5.3.1)**.** *Let $I,\ J \subset S$ be finitary. Then, $\mathcal{S}oe^{\bullet I}$ is a free functorially cyclic $\mathcal{S}oe^{\bullet}$-module and we have an equivalence ${}^I\mathcal{S}oe^{\bullet J} \cong \mathcal{H}om_{\mathcal{S}oe^{\bullet}}(\mathcal{S}oe^{\bullet I}, \mathcal{S}oe^{\bullet J})$ of additive categories. Further, there exist split Frobenius objects $\hat{\Theta}_{w_0^I}$ and $\hat{\Theta}_{w_0^J}$ in $\mathcal{S}oe^{\bullet}$, such that $\mathcal{S}oe^{\bullet I} \cong \mathcal{S}oe^{\bullet}{}_{\hat{\Theta}_{w_0^I}}$ in $\mathcal{S}oe^{\bullet}-\mathrm{Mod}$ and such that ${}^I\mathcal{S}oe^{\bullet J} \cong {}_{\hat{\Theta}_{w_0^I}}\mathcal{S}oe^{\bullet}{}_{\hat{\Theta}_{w_0^J}}$ as additive categories.*

In fact, we construct $\hat{\Theta}_{w_0^I}$ explicitly, by equipping $B_{w_0^I}$ shifted upwards by the Bruhat length of w_0^I with a structure of a split Frobenius object. Thus, Theorem 4 provides a new, explicit construction of singular Soergel bimodules from Soergel bimodules. This might allow one to deduce generators and relations for singular Soergel bimodules from [ElWi1]. Moreover, we construct an enlargement $\mathfrak{F}rob(\mathcal{S}oe^{\bullet})$ of $\mathfrak{S}oe^{\bullet}$, and show that it is equivalent as a 2-category to the 2-category of non-trivial, idempotent closed, free functorially cyclic $\mathcal{S}oe^{\bullet}$-modules. Conjecturally, $\mathfrak{F}rob(\mathcal{S}oe^{\bullet})$ has (at least) one object per *Duflo involution* of W, while $\mathfrak{S}oe^{\bullet}$ has one object per parabolic longest element. Recall that for finite Weyl groups, *Duflo involutions* go back to [Jo], while the definition for general Coxeter groups is based on intricate degree properties of Kazhdan-Lusztig polynomials. We will not recall this original definition, we will however give a definition in terms of Soergel bimodules in the main body of the thesis. Also, note that parabolic longest elements are Duflo involutions. Another noteworthy point is that subject to the existence of an induction 2-functor $\mathrm{Ind}_\iota : \mathcal{S}oe^{\bullet}_I - \mathrm{Mod} \to \mathcal{S}oe^{\bullet} - \mathrm{Mod}$ associated to the faithful embedding $\iota : \mathcal{S}oe^{\bullet}_I = \mathcal{S}oe^{\bullet}(W_I, I) \to \mathcal{S}oe^{\bullet}$ which is discussed in [Rou6, Section 3.3], our theory implies that $\mathcal{S}oe^{\bullet I} \cong \mathrm{Ind}_\iota((\mathcal{S}oe^{\bullet}_I)^I)$ in $\mathcal{S}oe^{\bullet} - \mathrm{Mod}$. Since for finite W, $\mathcal{S}oe^{\bullet S}$ is a categorification of the so called trivial module of $\mathcal{H}$, this would make the analogy to the classical case complete. This should also be compared to [El1], where an *ad hoc* approach to induced trivial representations is discussed.

The second application concerns certain cyclic and cell modules of $\mathcal{S}oe^{\bullet}$. Let $\underline{c} \subset W$ be a 2-sided cell, and denote by $\overline{\mathcal{H}}$ the quotient of $\mathcal{H}$ by the ideal generated from those Kazhdan-Lusztig basis elements that are associated to 2-sided cells that are not greater or equal to $\underline{c}$ with respect to $\leq_{LR}$. Let $d \in \underline{c}$ be a Duflo involution. Note that every left cell contains a unique Duflo involution. We denote by $c(d) \subset \underline{c}$ the left cell of W that contains d and by $C_{c(d)} := \overline{\mathcal{H}}\underline{H}_d$ the associated left cell representation of $\mathcal{H}$, which is generated by the image in $\overline{\mathcal{H}}$ of the Kazhdan-Lusztig basis element $\underline{H}_d$. We now categorify this situation. Denote by $\mathcal{I} \trianglelefteq \mathcal{S}oe^\bullet$ the maximal thick ideal that does not contain the indecomposable objects associated to $\underline{c}$, and denote by $\overline{\mathcal{S}oe^\bullet} = \mathcal{S}oe^\bullet/\mathcal{I}$ the associated (monoidal) quotient. Further, denote by $\hat{B}_d$ the indecomposable Soergel bimodule associated to d shifted upwards by the value of Lusztig's **a**-function evaluated at d. Then, there exists a triple $\overline{\Theta}_d = (\hat{B}_d, \Delta_d, \mu_d)$ in $\overline{\mathcal{S}oe^\bullet}$, where both μ_d and Δ_d are of degree zero and for which $\mu_d \circ \Delta_d = id_{\hat{B}_d}$. We make several conjectures about these triples in Section 5.2 and show that these conjectures hold for parabolic longest elements. In particular, we conjecture that $\overline{\Theta}_d$ is a split Frobenius object. Thus, we obtain a strongly universal cell module $\mathcal{C}_d := {}_{\overline{\Theta}_d}\overline{\mathcal{S}oe^\bullet} = {}_{\overline{\Theta}_d}\mathcal{P}^{\mathcal{I}}$. Note that $\mathcal{S}oe^\bullet$ has only a single principal representation $\mathcal{P} = \mathcal{S}oe^\bullet$. Another conjecture of Section 5.2 implies that $[\mathcal{C}_d] \cong C_{c(d)}$.

Theorem 5 (Theorem 5.4.1 & Corollary 5.4.4)**.** *Let d_1, $d_2 \in \underline{c}$ be Duflo involutions, and assume that the conjectures of Section 5.2 hold for d_1 and d_2. Then, $\mathcal{C}_{d_1}$ and $\mathcal{C}_{d_2}$ are weakly equivalent, we have an equivalence $\mathcal{H}om_{\mathcal{S}oe^\bullet}(\mathcal{C}_{d_1}, \mathcal{C}_{d_2}) \cong {}_{\overline{\Theta}_{d_1}}\overline{\mathcal{S}oe^\bullet}_{\overline{\Theta}_{d_2}}$, and the associated split Grothendieck groups are isomorphic to $\underline{H}_{d_1}\overline{\mathcal{H}}\underline{H}_{d_2}$. Furthermore if $\underline{c}$ is strongly regular and contains a parabolic longest element, then we have an equivalence $\mathcal{C}_{d_1} \cong \mathcal{C}_{d_2}$ in $\mathcal{S}oe^\bullet - \mathrm{Mod}$.*

This theorem is closely related to results of [MaMi1] on equivalences of 2-cell representations in the regular case and of [MaMi3] on endomorphism categories of 2-cell representations. We conjecture that one can recover their categorification $\mathbf{C}_{c(d)}$ of $C_{c(d)}$ as a quotient of a base change over the centre of $\mathcal{C}_d$. Theorem 5.4.1 also implies that there exists a split surjection $\mathrm{Hom}_{\mathcal{H}}(C_{c(d_1)}, C_{c(d_2)}) \twoheadrightarrow \mathbb{Z}[v, v^{-1}][c(d_1)^{-1} \cap c(d_2)]$. For Weyl groups, we deduce in Corollary 5.4.6 that this surjection is an isomorphism using the classical result [Lu2, Proposition 12.15].

The third application generalises the main result of [Li] from $\mathcal{S}oe^\bullet$ to $\mathfrak{S}oe^\bullet$. Denote by $\overline{\mathfrak{S}oe^\bullet}$ the category of singular Soergel bimodules constructed using the geometric representation V' of a Coxeter group. It follows from [Soe6] that $V' \subset V$, where V is the reflection faithful representation that is normally used to construct Soergel bimodules. Note that $V = V'$ if W is finite. Now, denote by $R := \mathcal{O}(V)$ the regular functions on V and denote $\overline{Z} := R^W / \{r \in V^* | r(v) = 0 \ \forall v \in V'\}$.

Theorem 6 (Theorem 5.7.6)**.** *We have a quotient 2-functor* $\mathfrak{Soe}^\bullet \to \overline{\mathfrak{Soe}}^\bullet$*. Further, this quotient 2-functor induces a 2-equivalence* $\mathfrak{Soe}^\bullet \otimes_{R^W} \overline{Z} \cong \overline{\mathfrak{Soe}}^\bullet$*, as well as an isomorphism* $[\mathfrak{Soe}^\bullet] \cong \left[\overline{\mathfrak{Soe}}^\bullet\right]$ *of the associated graded split Grothendieck rings. Therefore,* $\overline{\mathfrak{Soe}}^\bullet$ *categorifies the Schur algebroid.*

The fourth application concerns additive monoidal categories of representations of groups. Given a group G, we denote by $\hat{G}$ its additive monoidal category of representations in $k-\mathrm{Mod}$ for some fix unital commutative ring k. Provided that $|G|$ is invertible in k, we will show that all functorially cyclic $\hat{G}$-modules are weakly equivalent. Further for all subgroups $H \leq G$, we will construct a *symmetrically central, split Frobenius object* $\hat{\Theta}_H$ in $\hat{G}$. We denote the associated functorially cyclic $\hat{G}$-module by $\mathcal{C}_H$. By the fourth further finding discussed above, $\mathcal{C}_H$ embeds into $\mathcal{E}nd_{\hat{G}}(\mathcal{C}_H)$ and thereby inherits a monoidal structure.

Theorem 7 (Theorem 6.3.6 & Theorem 6.3.11)**.** *Let* G *be a group and let* $H \leq G$ *be a subgroup. Then, we have an equivalence* $\mathcal{C}_H \cong \hat{H}$ *of additive monoidal categories. Furthermore given a normal subgroup* $N \trianglelefteq G$*, we have a monoidal equivalence* $\mathcal{E}nd_{\hat{G}}(\hat{N}) \cong \bigoplus_{|G/N|} \hat{N}$*, where* $\bigoplus_{|G/N|} \hat{N}$ *carries the monoidal structure inherited from* $\hat{N}$ *and the group multiplication of* G/N*.*

Outline of the thesis

Chapter 2 discusses the classical analogue of additive higher representation theory. To our knowledge this classical structure has not been studied in the way presented here, and is thus of interest in its own right. Section 2.1 introduces rings and modules with positive structure coefficients. Section 2.2 shows that the category of modules with positive structure coefficients is not Abelian and in fact not even additive. Section 2.3 relates cyclic and cell modules to the left order and left cells respectively, as well as quotients of the base ring to 2-sided cells. Section 2.4 introduces annihilators and discusses immediate connections with 2-sided cells. Section 2.5 introduces universal cell modules, defines when cell modules belong to a 2-sided cell, and introduce regularity notions. In the weakly regular case, necessary and sufficient conditions for the existence of cell modules that belong to 2-sided cells are given, as well as an explicit construction of a universal cell module. Furthermore in the weakly regular case with finitely many 2-sided cells, it is shown that the set of cell modules modulo linkage by morphisms is in bijection with the set of 2-sided cells. Section 2.6 introduces Coxeter groups, the Hecke algebra and the Schur algebroid, comments on W-graphs and foreshadows challenges faced in this thesis by discussing the cell modules of the Hecke algebra of type B_2.

Chapter 3 discusses the basics of higher representation theory. Section 3.1 follows [Rou4] and recalls the definitions of 2-categories, 2-functors, morphism of 2-functors, morphisms of morphisms of 2-functors, as well as the definition of a 2-category of higher representations of a 2-category. The graded case is also covered and monoidal categories are mentioned. Section 3.2 discusses centres of objects of 2-categories. Section 3.3 defines (graded) additive higher representations and discusses (graded) decategorification using the split Grothendieck construction. Further, kernels, cokernels, (thick) annihilators and (thick) ideals are introduced and related to the corresponding classical concepts. Finally, universal properties, the 2-Yoneda lemma, and (quotients of) principal representations are covered, which yields Theorem 2 for the identity split Frobenius object.

Chapter 4 is the technical centre piece of this work. Section 4.1 discusses centres and base change of $\mathfrak{A}$-modules. Section 4.2 introduces 1-cyclicity which is based on objects, 2-cyclicity which also takes morphisms into account, and functorially cyclic modules which are 2-cyclic, but posses a more readily testable definition. Section 4.3 introduces (indecomposable) split Frobenius objects and discusses the additive Barr-Beck theorem, first for additive categories (Theorem 1), then for 2-functors with codomain $\mathfrak{A}dd^{\mathrm{idem}}$, and finally for universal properties of idempotent closed additive higher representations. More general than Theorem 2, it is shown that representability of a universal properties is preserved under the additive Barr-Beck construction. Furthermore, the split Frobenius 2-category $\mathfrak{F}rob(\mathcal{B})$ is introduced, as well as certain preorders for split Frobenius objects. Section 4.4 is a technical intermezzo about restriction 2-functors and their adjoints. Section 4.5 revisits functorially cyclic modules. Using the additive Barr-Beck theorem, bijections between equivalence classes of certain modules (including functorially cyclic modules) and isomorphism classes of split Frobenius objects, a universal property of functorially cyclic modules, as well as an equivalence between the 2-category of free functorially cyclic modules and the split Frobenius 2-category are established. Further, $\mathfrak{F}rob(\mathcal{B})$ is identified with the 2-category of free functorially cyclic $\mathcal{B}$-modules and the preorders on split Frobenius are related to representation theory. Finally, (strongly) universal cell modules are introduced and Theorem 3 is established. Section 4.6 discusses coinvariant theory. Section 4.7 studies the compatibility of additive Barr-Beck construction with monoidal quotient functors, which is needed for Theorem 6. Section 4.8 shows that certain functorially cyclic modules can be equipped with a natural monoidal structure, which is needed for Theorem 7. Section 4.9 addresses Jordan-Hölder theory. Cofiltrations of the 2-categories of additive higher representations associated to refinements of the ordering of the set of 2-sided cells, modules isotypic with respect to a 2-sided cell, as well as isotypic covers are introduced, and various challenges that make it hard to establish results are discussed.

Chapter 5 is devoted to the higher representation theory of Soergel bimodules. Section 5.1 recalls definitions and the relevant literature. Section 5.2 makes conjectures about existence and properties of split Frobenius objects in $\mathcal{S}oe^\bullet$ associated to Duflo involutions, and presents examples mainly associated to parabolic longest elements. Section 5.3 establishes Theorem 4 showing that singular Soergel bimodules categorify the Schur algebroid in the representation theoretic sense. Furthermore, related centers are calculated. Section 5.4 discusses (universal) cell modules and their Grothendieck combinatorics by showing Theorem 5. Section 5.5 applies the coinvariant theory of Section 4.6 to certain cell modules of Soergel bimodules for several Coxeter systems. Section 5.6 observes that certain endomorphisms of the unit object of $\mathcal{S}oe$ are closely tied to the structure of $\mathcal{S}oe - \mathrm{Mod}$. Finally, Section 5.7 generalises the main result of [Li] to singular Soergel bimodules.

Chapter 6 discusses the higher representation theory of the additive monoidal category of representations of a group over a ring. Section 6.1 introduces notation. Section 6.2 discuses functorially cyclic modules and shows that there is only a single weak equivalence class of functorially cyclic modules provided an assumption that is satisfied for example if the order of the group is invertible in the base field. Independent of Section 6.2 and without invoking Tannakian formalism, Section 6.3 reconstructs the corresponding category of representations for all subgroups. Section 6.3 does not require us to work over a field, nor does it require any finiteness assumptions. Section 6.4 applies the additive Barr-Beck theorem to module categories that arise in the context of defect groups.

Finally, Chapter 7 puts forward two questions that we believe deserve the attention of future research, and provides pointers to other questions that are posed in earlier chapters.

Chapter 2

The Classical Case

In this chapter, we discuss the classical analogue of additive higher representation theory. To our knowledge this classical structure has not been studied in the way presented here. Thus, this chapter is of interest in its own right and also allows us to distinguish truly 2-categorical properties from classical observations. Section 2.1 introduces rings and modules with positive structure coefficients, which are the classical counterparts of additive higher representations. Section 2.2 shows that the category of modules with positive structure coefficients is neither additive nor that a canonical morphism, which is required to be an isomorphism in an Abelian category, is invertible in general. Section 2.3 relates cyclic and cell modules to the left order and left cells respectively. 2-sided cells and their relation to quotients are also discussed. Section 2.4 introduces annihilators associated to modules with positive structure coefficients and discusses immediate connections with 2-sided cells. Section 2.5 introduces universal cell modules, defines when cell modules belong to a 2-sided cell, and covers regularity notions. In the weakly regular case, necessary and sufficient conditions for the existence of cell modules that belong to 2-sided cells are given, as well as an explicit construction of a universal cell module. Furthermore in the weakly regular case with finitely many 2-sided cells, it is shown that the set of cell modules modulo linkage by morphisms is in bijection with the set of 2-sided cells. Section 2.6 introduces Coxeter groups, the Hecke algebra and the Schur algebroid, comments on W-graphs and foreshadows challenges faced in this thesis by discussing the cell modules of the Hecke algebra of type B_2.

2.1 Rings and modules with positive structure coefficients

In this section, we introduce rings and modules with positive structure coefficients, which are the classical analogues of $\mathfrak{A}$ and $\mathfrak{A} - \mathrm{Mod}$. We follow [EtKh, Os1], however we take the freedom to make slight adjustments. Further, we change the notation to better align with the standard notation used for the Hecke algebra, see Section 2.6.

Definition 2.1.1. *Let k be a commutative ring with unit, and let $O_k \subset k$ be a submonoid that is closed under multiplication, that contains the unit of k, that has no nonzero zero divisors, and that carries a preorder $\leq$. For all $n_1, n_2 \in O_k$, assume that $0 < 1 \leq n_1$, that $n_1 \leq n_1 + n_2$, and that $n_1 + n_2 \leq 1$ implies that $n_1 = 0$ or $n_2 = 0$. Then, we call the pair (k, O_k) a* ring with positivity structure.

Example 2.1.2. For us, the most relevant examples of rings with positivity structure are $(\mathbb{Z}, \mathbb{N}_0)$ and $(\mathbb{Z}[v, v^{-1}], \mathbb{N}_0[v, v^{-1}])$, where the order of $\mathbb{N}_0$ is the standard order and the order of $\mathbb{N}_0[v, v^{-1}]$ is induced from the order of $\mathbb{N}_0$ by evaluating $v \mapsto 1$.

Definition 2.1.3. *Let (k, O_k) be a ring with positivity structure.*

1. *An O_k-*basis *of a k-algebra A is a k-basis $\{b_w\}_{w\in W} \subset A$ with a distinguished subset $U \subset W$, such that $b_{w_1}b_{w_2} = \sum_{w_3\in W} h_{w_1,w_2,w_3} b_{w_3}$ where $h_{w_1,w_2,w_3} \in O_k$, such that $\{b_u\}_{u\in U}$ consists of orthogonal idempotents, and such that for all $a \in A$ there exists a finite set $U_a \subset U$ with $a = \sum_{u\in U_a} b_u a = \sum_{u\in U_a} ab_u$ and $0 = b_u a = ab_u$ for all $u \in U \setminus U_a$.*

2. *A (k, O_k)-*ring *A is a k-algebra A together with a fixed O_k-basis of A.*

3. *A (k, O_k)-*module over a *(k, O_k)-*ring *A is a k-free A-module that has a fixed basis $\{m_x\}_{x\in X}$ for some set X such that $b_w m_{x_1} = \sum_{x_2\in X} \mu_{w,x_1,x_2} m_{x_2}$ with $\mu_{w,x_1,x_2} \in O_k$, and such that for all $m \in M$ there exists a finite set $U_m \subset U$ with $m = \sum_{u\in U_m} b_u m$ and $0 = b_u m$ for all $u \in U \setminus U_m$.*

4. *A* morphism *$f : M_1 \to M_2$ of (k, O_k)-modules M_1 and M_2 over A with bases $\{m^1_x\}_{x\in X}$ and $\{m^2_y\}_{y\in Y}$ is a map $f : X \to O_k[Y]$ such that the k-linear map $\tilde{f} : M_1 \to M_2$ defined by $\tilde{f}(m^1_x) := \sum_{y\in Y} f_{x,y} m^2_y$ for all $x \in X$, where $f(x) = \sum_{y\in Y} f_{x,y} y$, is an A-module homomorphism.*

5. *An* isomorphism *is a morphism $f : M_1 \to M_2$ of (k, O_k)-modules over A that is invertible. A morphism f is an isomorphism if and only if f factors over Y and the induced map $X \to Y$ is a bijection. We say M_1 and M_2 are isomorphic and write $M_1 \cong M_2$.*

6. *The* direct sum *of (k, O_k)-modules M_1 and M_2 over A is given by the A-module $M_1 \oplus M_2$ together with the basis given by the union of the bases of M_1 and M_2.*

7. *An* indecomposable *(k, O_k)-module over A is a (k, O_k)-module that is not isomorphic to the direct sum of two nonzero (k, O_k)-modules.*

8. *A* sub-(k, O_k)-module *of a (k, O_k)-module M over A with basis $\{m_x\}_{x \in X}$ is an sub-A-module of M with basis $\{m_y\}_{y \in Y}$ for some $Y \subset X$.*

9. *A* cyclic (k, O_k)-module *is a (k, O_k)-module with basis $\{m_x\}_{x \in X}$ such that provided $X \neq \emptyset$ there exists $x_0 \in X$ such that M is the only sub-(k, O_k)-module that contains m_{x_0}.*

10. *A* cell (k, O_k)-module *M is a cyclic (k, O_k)-module that is non-trivial and generated by every basis element.*

Notation 2.1.4. Let M be a (k, O_k)-module, and let m_1, $m_2 \in M$ be elements of the O_k-linear span of the basis elements of M. We write $m_1 \leq m_2$ provided that this relation holds for each coefficient in the basis expansions. Whenever we write $m_1 \leq m_2$ for m_1, $m_2 \in M$, we take it as implicitly understood that m_1 and m_2 lie in the O_k-linear span of the basis elements of M.

Notation 2.1.5. Let A be a (k, O_k)-ring. Following definition 2.1.3, we denote by $A - \mathrm{Mod}^{(k, O_k)}$ the category of (k, O_k)-modules over A.

Remark 2.1.6. We would have preferred to call the rings and modules under discussion based rings and based modules. However, this terminology has already been claimed in [Lu4] and has been used in [Os1]. The definition of based rings requires the existence of a unit, a duality and a trace form which is compatible with the unit and the duality. Neither of these three ingredients exist in (some of) our applications.

The main example of interest is the Hecke algebra $\mathcal{H}$ together with the Kazhdan-Lusztig basis, see Section 2.6. As a little preview, we describe a special case (the Hecke algebra of type A_1) explicitly:

Example 2.1.7. Let us define a $(\mathbb{Z}[v,v^{-1}],\mathbb{N}_0[v,v^{-1}])$-ring $\mathcal{H}=\mathcal{H}(A_1)$ with basis $\{b_w\}_{w\in W}$ where $W=\{e,s\}$. Let $b_sb_s=(v+v^{-1})b_s$ and assume that b_e is a unit. All $(\mathbb{Z}[v,v^{-1}],\mathbb{N}_0[v,v^{-1}])$-modules M over $\mathcal{H}$ are thus completely determined by a diagram with vertices corresponding tthe basis of M and arrows labelled by the coefficients for the action of b_s. We omit the label if it is equal to 1 and draw no arrow if the label is 0. Similarly, we can describe morphisms by arrows of the form $\rightsquigarrow$. For example the morphism $\cdot b_s:\mathcal{H}\to\mathcal{H}$ given by right multiplication with b_s, is encoded as follows:

$$\begin{tikzcd}
e \arrow[d] \arrow[dr, rightsquigarrow] & e \arrow[d] \\
s \arrow[loop left, "v+v^{-1}"] \arrow[r, rightsquigarrow, "v+v^{-1}"] & s \arrow[loop right, "v+v^{-1}"]
\end{tikzcd}$$

Example 2.1.8. We give another example of a morphism in $\mathcal{H}-\mathrm{Mod}^{(\mathbb{Z}[v,v^{-1}],\mathbb{N}_0[v,v^{-1}])}$:

$$\begin{tikzcd}
e \arrow[d] \arrow[r, rightsquigarrow] & e \arrow[loop right, "v"] \arrow[d, shift right] \\
s \arrow[loop left, "v+v^{-1}"] \arrow[ur, rightsquigarrow, "v"] \arrow[r, rightsquigarrow] & s' \arrow[loop right, "v^{-1}"] \arrow[u, shift right]
\end{tikzcd}$$

The domain of this morphism is $\mathcal{H}$ itself and the codomain is $\mathcal{H}$ equipped with the basis $\{b_e,b_{s'}\}$ where $b_{s'}:=b_s-vb_e$. Note that this morphism induces the identity morphism on the Abelian group that underlies $\mathcal{H}$. However this morphism is not invertible in $\mathcal{H}-\mathrm{Mod}^{(\mathbb{Z}[v,v^{-1}],\mathbb{N}_0[v,v^{-1}])}$. The reader familiar with Hecke algebras will note that this morphism is given by the change of bases from the Kazhdan-Lusztig to the standard bases, which will be introduced in Section 2.6. Let us also note that the analogous base change map is a morphism in $\mathcal{H}-\mathrm{Mod}^{(\mathbb{Z}[v,v^{-1}],\mathbb{N}_0[v,v^{-1}])}$ for all Hecke algebras $\mathcal{H}$. This follows from the Positivity Conjecture 2.6.17 and the multiplication formula (2.1) stated in Section 2.6.2.

For $(k,O_k)=(\mathbb{Z},\mathbb{N}_0)$, the majority of our Definition 2.1.3 resembles definitions in [Os1], particularly [Os1, Definition 1], which in turn follow [EtKh, Definition 1.1]. We would like to comment on the differences:

Remark 2.1.9. In [EtKh] (resp. [Os1]) $\mathbb{Z}_+$-algebras and $\mathbb{Z}_+$-representations are discussed in a $\mathbb{C}$-linear (resp. $\mathbb{Z}$-linear) setting. Part (1.) is a relaxation of [EtKh, Definition 1.1] where the unit is required to be part of the basis. A similar relaxation was made in [Os1, Definition 2.a], however only the case of a finite set U is discussed. If U is finite, $\sum_{u\in U}b_u$ is a unit, while there exists no unit for infinite U. Part (3.) does not require from a module to be unital in general, however for finite U this is the case. This is in line with [EtKh] and [Os1], which consider unital modules only. Part (5.) is called an equivalence in both [EtKh] and [Os1] rather than an isomorphism. Part (9.) is a new addendum that will be discussed in Section 2.3. Part (10.) is equivalent to the requirement that M has two distinct submodules, namely 0 and M. This is in line with the standard definition of irreducible objects and has been used in [Os1, Definition 1 vii] (though without the non-triviality assumption). For consistency with say [MaSt] and much of the literature on categorical modules, we prefer the name cell module.

Remark 2.1.10. Let M be a (k, O_k)-module with basis $\{m_x\}_{x\in X}$ over a (k, O_k)-ring A with basis $\{b_w\}_{w\in W}$ and distinguished subset $U \subset W$. For all $u \in U$, we denote $X_u := \{x \in X | b_u m_x = m_x\}$. Since $\{b_u\}_{u\in U}$ are orthogonal idempotents that resemble a unit a A, we have $X = \coprod_{w\in U} X_w$. Thus, we obtain a decomposition $M = \bigoplus_{u\in U} M_u$, where the basis of M_u is given by X_u for all $u \in U$. For this argument, Definition 2.1.1 becomes important, since it implies that for all $x \in X$ there exists a unique $u \in U$ with $b_u m_x = m_x$.

For completeness, we state the following proposition. Its proof extends easily to the $(\mathbb{Z}[v, v^{-1}], \mathbb{N}_0[v, v^{-1}])$ case. We will show a comparable result in Corollary 2.5.8.

Proposition 2.1.11 ([Ga], [Os1, Proposition 1])**.** *Let A be a $(\mathbb{Z}, \mathbb{N}_0)$-ring of finite rank over $\mathbb{Z}$. Then, the set of isomorphism classes of cell modules in $A - \mathrm{Mod}^{(\mathbb{Z},\mathbb{N}_0)}$ is finite.*

It is important to note another difference between the category of based modules over based rings studied in [Os1] and the module category studied here:

Remark 2.1.12. After defining $\mathbb{Z}_+$-algebras and $\mathbb{Z}_+$-representations, [Os1] proceeds to study based rings and based modules over based rings. Based modules admit a scalar product, and it follows that indecomposable modules are cell modules, see [Os1, Lemma 1]. On the other contrary for many (k, O_k)-rings A, $A \in A - \mathrm{Mod}^{(k,O_k)}$ is an indecomposable module which is not a cell module. Example 2.1.7 provides a $(\mathbb{Z}[v, v^{-1}], \mathbb{N}_0[v, v^{-1}])$-ring with this property.

Remark 2.1.13. Multisemigroups, see [KuMa], encode all combinatorial information about the left-, right-, and 2-sided cells of a $(\mathbb{Z}, \mathbb{N}_0)$-ring. More precisely, such multisemigroups encode whether the structure coefficients of the $(\mathbb{Z}, \mathbb{N}_0)$-ring are positive or zero. In [MaMi2, Remark 8], it is noted that one could retain more information by using multimultisemigroups. We expect that multimultisemigroups would encode the same information as the $(\mathbb{Z}, \mathbb{N}_0)$-rings introduced here.

2.2 Structure of the category of representations

In this section, we show that a canonical morphism that is required to be an isomorphism in any Abelian category is not invertible in $A - \mathrm{Mod}^{(k,O_k)}$. In fact, we start by observing that $A - \mathrm{Mod}^{(k,O_k)}$ is not even additive. For convenience, we fix a (k, O_k)-ring A with basis $\{b_w\}_{w \in W}$.

Remark 2.2.1. Recall that $A - \mathrm{Mod}^{(k,O_k)}$ has direct sums. It is also fairly obvious that $A - \mathrm{Mod}^{(k,O_k)}$ has a zero object and that it is an O_k-linear category. Thus in order for $A - \mathrm{Mod}^{(k,O_k)}$ to be additive, it would be required that $\mathbb{Z} \subset O_k$. This contradicts Definition 2.1.1 of rings with positivity structure.

Proposition 2.2.2. *Let $f : M_1 \to M_2$ be a morphism in $A - \mathrm{Mod}^{(k,O_k)}$. Then, f admits a kernel $ker f$ and a cokernel $coker f$. Further, the canonical morphism* $\operatorname{coker}(\ker f \to M_1) \longrightarrow \ker(M_2 \to \operatorname{coker} f)$ *fails to be an isomorphism in general.*

Proof. Aside of examples, the remainder of this section is dedicated to this proof. □

For convenience, let us fix M, $N \in A - \mathrm{Mod}^{(k,O_k)}$ with bases $\{m_x\}_{x \in X}$, $\{n_y\}_{y \in Y}$ respectively. We will denote the structure coefficients of M (resp. N) using the symbol μ (resp. ν). Further, let us fix a morphism $f : M \to N$ in $A - \mathrm{Mod}^{(k,O_k)}$.

Lemma 2.2.3. *Specifying $X' := \{x \in X | f_{x,y} = 0 \ \forall y \in Y\}$ and $\mu'_{w,x_1,x_2} := \mu_{w,x_1,x_2}$ for all $w \in W$ and for all x_1, $x_2 \in X'$ defines a k-free A-module that satisfies the universal property required from* $\ker(f)$.

Proof. First, we show that $\mu_{w,x_1,x_2} = 0$ for all $w \in W$, $x' \in X'$ and $x \in X \setminus X'$. This implies that the definition of the structure coefficients μ' given in the statement define a (k, O_k)-module X' over A and that the obvious embedding $X' \hookrightarrow X \hookrightarrow O_k[X]$ is a morphism in $A - \mathrm{Mod}^{(k,O_k)}$. Let $w \in W$, $x' \in X'$, and $x \in X \setminus X'$. We have $\sum_{z \in X,\, y \in Y} \mu_{w,x',z} f_{z,y} n_y = \tilde{f}(\sum_{z \in X} \mu_{w,x',z} m_z) = \tilde{f}(b_w m_{x'}) = b_w \tilde{f}(m_{x'}) = 0$. Thus, $\sum_{z \in X} \mu_{w,x',z} f_{z,y} = 0$ for all $y \in Y$ and by assumption on O_k, we have $\mu_{w,x',z} f_{z,y} = 0$ for all $z \in X$, $y \in Y$. Since $x \in X \setminus X'$ there exist $y_0 \in Y$ with $f_{x,y_0} \neq 0$, hence $\mu_{w,x',x} = 0$ follows from $\mu_{w,x',x} f_{x,y_0} = 0$ and the properties of O_k, see Definition 2.1.1. Second, let M_1 be another (k, O_k)-module over A with basis X_1 and let $g : M_1 \to M$ be a morphism with $0 = f \circ g$. We have to show that $g_{x_1,x} \neq 0$ implies $x \in X'$

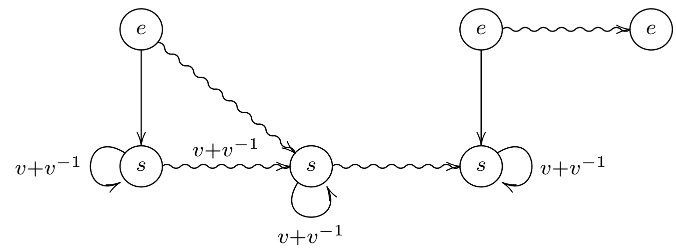
for all $x_1 \in X_1$, $x \in X$. We have $0 = f(g(m_{x_1})) = \sum_{x \in X,\, y \in Y} f_{x,y} g_{x_1,x} n_y$ and thus $0 = f_{x,y} g_{x_1,x}$ for all $y \in Y$. Since (k, O_k) is a ring with positivity structure, $g_{x_1,x} \neq 0$ implies that $0 = f_{x,y}$ for all $y \in Y$, and thus $x \in X'$. $\square$

Lemma 2.2.4. *Specifying $Y' := \{y \in Y | f_{x,y} = 0 \ \forall x \in X\}$ and $\nu'_{w,y_1,y_2} := \nu_{w,y_1,y_2}$ for all $w \in W$ and for all y_1, $y_2 \in Y'$ defines a k-free A-module that satisfies the universal property required from* coker(f).

Proof. We show that $Y \setminus Y'$ defines a sub-(k, O_k)-module over A of N. This implies that the quotient (k, O_k)-module of N by this sub-(k, O_k)-module has a basis parametrised by Y' and that the structure coefficients of this quotient (k, O_k)-module are indeed given by ν' as defined above. The second step of the proof goes along the lines of the second step of the proof of Lemma 2.2.4 up to relabelling. For completeness, let us mention that the morphism $N \to \mathrm{coker}(f)$ is given by the map $Y \to O_k[Y']$ which sends $y \mapsto y$ if $y \in Y'$ and $y \mapsto 0$ otherwise. Let $w \in W$ and $y_0 \in Y \setminus Y'$. We need to show that $b_w n_{y_0} = \sum_{y \in Y} \nu_{w,y_0,y} n_y$ is contained in the k-span of $Y \setminus Y'$. Thus for all $y \in Y'$, we need to show that $\nu_{w,y_0,y} = 0$. Since $y_0 \in Y \setminus Y'$, there exists $x_0 \in X$ such that $f_{x_0,y_0} \neq 0$. Writing out the equation $b_w \widetilde{f}(m_{x_0}) = \widetilde{f}(b_w m_{x_0})$, we obtain $\sum_{y_1,y_2 \in Y} f_{x_0,y_1} \nu_{w,y_1,y_2} n_{y_2} = \sum_{x \in X, y_2 \in Y} \mu_{w,x_0,x} f_{x,y_2} n_{y_2}$. In particular, $\sum_{y_1 \in Y} f_{x_0,y_1} \nu_{w,y_1,y} = \sum_{x \in X} \mu_{w,x_0,x} f_{x,y} = 0$, where we used that $f_{x,y} = 0$ for all $x \in X$. By the properties of O_k, see Definition 2.1.1, $f_{x_0,y_0} \nu_{w,y_0,y} = 0$ holds as well. Further, $f_{x_0,y_0} \neq 0$ follows and thus $\nu_{w,y_0,y} = 0$. $\square$

Remark 2.2.5. The basis of $\mathrm{im}(f) := \mathrm{coker}(\ker(f) \to M)$ is indexed by $X \setminus X'$ and the basis of $\mathrm{coim}(f) := \ker(N \to \mathrm{coker}(f))$ is indexed by $Y \setminus Y'$. Thus, there is *a priori* no reason for the canonical morphism $\mathrm{im}(f) \to \mathrm{coim}(f)$ to be an isomorphism. The following two examples show that this canonical morphism is indeed not guaranteed to be an isomorphism.

Example 2.2.6. The morphism $\cdot b_s : \mathcal{H} \to \mathcal{H}$ of Example 2.1.7 has trivial kernel. Thus, $\mathrm{im}(\cdot b_s) = \mathcal{H}$. The associated morphisms im $\to$ coim $\to \mathcal{H} \to$ coker are encoded as follows:



Example 2.2.7. Example 2.1.8 provides a family of morphism that have trivial kernels and cokernels, that induce the identity morphism on the underlying Abelian groups, and that nevertheless fail to be invertible.

2.3 Cells and cyclicity

In this section, we relate cyclic and cell modules with positive structure coefficients to the left order and left cells respectively, which are defined in analogy with [KaLu]. We also discuss 2-sided cells and their relation to quotients of a ring with positive structure coefficients. For convenience, we fix a (k, O_k)-ring A with basis $\{b_w\}_{w\in W}$.

Definition 2.3.1. *Let M be a (k, O_k)-module over A with basis $\{m_x\}_{x\in X}$. The* left order $\leq_L$ *of X is defined as $x \leq_L y$ for all x, $y \in X$ if and only if m_x is contained in all sub-(k, O_k)-modules of M that contain m_y. The* 2-sided order $\leq_{LR}$ *of W is defined as $x \leq_{LR} y$ for all x, $y \in W$ if and only if m_x is contained in all sub-(k, O_k)-bimodules of A that contain m_y. The equivalence relation induced by the order $\leq_L$ (resp. $\leq_{LR}$) is denoted by $\sim_L$ (resp. $\sim_{LR}$), and the associated equivalence classes are called the* left cells *(resp.* 2-sided cells*) of A.*

Remark 2.3.2. Let w, $w' \in W$. Then, we have $w \leq_L w'$ if and only if there exists $w_1 \in W$ such that $b_w \leq b_{w_1} b_{w'}$. This holds due to the properties of O_k, see Definition 2.1.3. Similarly, we have $w \leq_{LR} w'$ if and only if there exists $w_1, w_2 \in W$ such that $b_w \leq b_{w_1} b_{w'} b_{w_2}$.

Notation 2.3.3. Let $x \in W$. We denote by $c(x) := \{y \in W | y \sim_L x\}$ the left cell (resp. by $\underline{c}(x) := \{y \in W | y \sim_{LR} x\}$ the 2-sided cell) of W associated to x.

Definition 2.3.4. *Let $x \in W$. We call $cp(x) := \{y \in \underline{c}(x) | y \geq_L x \text{ or } x \geq_L y\}$ the* left companion set of x in W.

Remark 2.3.5. For all $x \in W$, we have $c(x) \subset cp(x) \subset \underline{c}(x)$. For the Hecke algebra associated to any Coxeter system (W, S), it is a well known fact that $c(x) = cp(x)$. On the other extreme, consider the one-variable Heisenberg algebra H, that is the unital $\mathbb{Z}$-algebra generated by two indeterminate symbols p and q that satisfy the relation $pq = qp + 1$. We can regard H as a $(\mathbb{Z}, \mathbb{N}_0)$-algebra with basis $\{q^n p^m\}_{(n,m)\in W}$ where $W := \mathbb{N}_0 \times \mathbb{N}_0$. It is easy to see that H has a single 2-sided cell and that the left cells are given by $\mathbb{N}_0 \times \{m\}$ for all $m \in \mathbb{N}_0$. Therefore, we have $cp(x) = \underline{c}(x) = W$ for all $x \in W$. It would be interesting to study the categorification of the infinitely many variable version of H, see [Kh2], in the context of this thesis.

Definition 2.3.6. *Let $M \in A - \mathrm{Mod}^{(k,O_k)}$ with basis labelled by X and let $Y \subset X$ be a subset. If $Y = \{z \in W | \exists y \in Y : z \geq_L y\} \cap \{z \in W | \exists y \in Y : z \leq_L y\}$, we say that Y* has no $\leq_L$-gaps*. If $Y = \{x \in X | \exists y \in Y : x \leq_L y\}$, we call Y* an ideal of X*. Further if $X = W$ and $Y = \{x \in X | \exists y \in Y : x \leq_{LR} y\}$, we call Y a* two-sided ideal.

Notation 2.3.7. Let $Y \subset X$. We denote by $I(Y) := \{x \in X | \exists y \in Y : x \leq_L y\}$ the minimal ideal of X that contains Y.

Remark 2.3.8. A subset $Y \subset X$ has no $\leq_L$-gaps if and only if $I(Y) \setminus Y$ is an ideal.

Remark 2.3.9. The ideals of X are in 1:1 correspondence with sub-(k, O_k)-modules of M and thus we will refer to an ideal $Y \subset X$ also as an *ideal of M*. Similarly, the two-sided ideals of W are also referred to as *two-sided ideals of A* and correspond to quotient (k, O_k)-rings of A.

We have the following characterisations of cyclic modules and cell modules:

Proposition 2.3.10. *Let $M \in A - \mathrm{Mod}^{(k,O_k)}$ with basis labelled by X. Then, M is cyclic if and only if $X = \emptyset$, if and only if X has a maximal $\leq_L$-equivalence class. Further, M is a cell module if and only if X consists of a single equivalence class for $\leq_L$, if and only if the set of ideals of M is given by $\{\emptyset, X\}$, if and only if every non-trivial morphism $A \to M$ has trivial cokernel.*

Proof. The statement follows directly from the definitions and the description of cokernels in Lemma 2.2.4. $\square$

Remark 2.3.11. In Section 2.6.3, we will provide concrete examples of cyclic (k, O_k)-modules M which have a generator m_x that cannot distinguish morphisms with domain M. It is a non-trivial property of a generator to distinguish all such morphisms, compare Theorem 5.4.1 and Remark 5.4.3. On the other hand, we do not know an example of a cyclic or cell module that does not have a generator with this property.

Notation 2.3.12. Let $X \subset W$. If X is an ideal in W, we denote by $M(X)$ the submodule of A with basis $\{b_w\}_{w \in X}$. If more generally X has no $\leq_L$-gaps, we denote by $M(X)$ the cokernel of the embedding $M(I(X) \setminus X) \subset M(I(X))$.

Notation 2.3.13. Let $X \subset W$. Then, $\{\ngeq_{LR} X\} := \{z \in W | z \ngeq_{LR} x \ \forall x \in X\}$ is a two-sided ideal of A and we denote by $A / \langle \ngeq_{LR} X \rangle$ the associated quotient (k, O_k)-ring of A.

Remark 2.3.14. Let $X \subset W$. Then, the basis of $M(X)$ is labelled by X. Furthermore, $M(X)$ is a sub-(k, O_k)-quotient of $A/\langle \not\geq_{LR} X\rangle$ regarded as a (k, O_k)-module over A. Note that $M(X)$ can also be viewed as the restriction to $A - \mathrm{Mod}^{(k,O_k)}$ of a module in $A/\langle \not\geq_{LR} X\rangle - \mathrm{Mod}^{(k,O_k)}$. Provided X is contained in a 2-sided cell $\underline{c}$ and $I(X) \cap \underline{c} = X$, we have a chain of sub-(k, O_k)-modules $M(X) \subset M(\underline{c}) \subset A/\langle \not\geq_{LR} X\rangle$ in $A - \mathrm{Mod}^{(k,O_k)}$.

Remark 2.3.15. Remark 2.3.14 equally applies to $A/\langle <_{LR} X\rangle$ instead of $A/\langle \not\geq_{LR} X\rangle$.

2.4 Annihilators

In this section, we introduce annihilators associated to modules with positive structure coefficients and discuss immediate connections with 2-sided cells. For convenience, we fix a (k, O_k)-ring A with basis $\{b_w\}_{w\in W}$.

Remark 2.4.1. Let $M \in A - \mathrm{Mod}$, where we regard A as a k-algebra. In the literature, the set $\{a \in A | a.m = 0 \ \forall m \in M\}$ is called the annihilator of M. In general, this set is not generated by a subset of the basis elements of A. For example, the 2-dimensional modules in Section 2.6.3 are acted upon in the same, non-trivial way by two different basis elements. Thus, we have to amend this standard definition.

Definition 2.4.2. *Let $M \in A - \mathrm{Mod}^{(k,O_k)}$. We define* the annihilator of M *in A to be the set* $\mathrm{Ann}_A(M) := \{w \in W | b_w m = 0 \ \ \forall m \in M\}$.

Proposition 2.4.3. *Let $M \in A - \mathrm{Mod}^{(k,O_k)}$. Then, $\mathrm{Ann}_A(M)$ is a two-sided ideal.*

Proof. Let $w \in \mathrm{Ann}_A(M)$ and let $w' \in W$ such that $w' \leq_{LR} w$. Then there exists $w_1, w_2 \in W$ such that $b_{w'} \leq b_{w_1} b_w b_{w_2}$. Since b_w acts by zero on M, so does $b_{w_1} b_w b_{w_2}$ and thus $b_{w'}$ as well. □

Corollary 2.4.4. *Let $M \in A - \mathrm{Mod}^{(k,O_k)}$, and let $\underline{c}$ be a 2-side cell of A such that $\underline{c} \cap \mathrm{Ann}_A(M) = \emptyset$. Then, $\{\geq_{LR} \underline{c}\} := \{y \in W | y \geq_{LR} \underline{c}\} \subset W \setminus \mathrm{Ann}_A(M)$.*

Proposition 2.4.5. *Let $Y \subset W$ be a two-sided ideal. Then, $\mathrm{Ann}_A(A/Y) = Y$.*

Proof. For $x \in \mathrm{Ann}_A(A/Y)$, we have $\overline{b_x} = b_x \overline{1} = 0$ in A/Y. Thus, $\mathrm{Ann}_A(A/Y) \subset Y$. On the other hand given $x \in Y$ and $y \in W$, $b_x b_y$ is a O_k-linear combination of basis elements b_z with $z \leq_{LR} x$ and thus $z \in Y$. Thus, $b_x \overline{b_y} = 0$ in A/Y and $x \in \mathrm{Ann}_A(A/Y)$ follows. □

Lemma 2.4.6. *Let $f : M \to N$ be a morphism in $A - \mathrm{Mod}^{(k,O_k)}$. If* $\mathrm{coker}(f) = 0$*, then* $\mathrm{Ann}_A(M) \subset \mathrm{Ann}_A(N)$*. If* $\ker(f) = 0$*, then* $\mathrm{Ann}_A(M) \supset \mathrm{Ann}_A(N)$*.*

Proof. Let $f : M \to N$ be a morphism in $A - \mathrm{Mod}^{(k,O_k)}$. Firstly, assume that $\mathrm{coker}(f) = 0$. Then, for all basis elements $n \in N$ there exists a basis element $m \in M$ such that $n \leq f(m)$. Now given $x \in \mathrm{Ann}_A(M)$, we have $b_x n \leq b_x f(m) = f(b_x m) = f(0) = 0$ and thus $x \in \mathrm{Ann}_A(N)$. Secondly, assume that $\ker(f) = 0$. Let $x \in \mathrm{Ann}_A(N)$ and let $m \in M$ be a basis element. Assume, there exists a basis element $m' \in M$ such that $m' \leq b_x m$. Then $f(m') \leq f(b_x m) = b_x f(m) = 0$, which contradicts $\ker(f) = 0$. Thus, $b_x m = 0$ and $x \in \mathrm{Ann}_A(M)$. $\square$

Next, we deduce that agreement of annihilators is a necessary condition for the existence of non-trivial morphisms between cell modules. If this were a sufficient condition, we could call this a linkage principle. Under mild assumptions, we will actually be able to establish sufficiency in Corollary 2.5.8.

Corollary 2.4.7. *Let $C \to C'$ be a non-trivial morphism between cell modules in $A - \mathrm{Mod}^{(k,O_k)}$. Then,* $\mathrm{Ann}_A(C) = \mathrm{Ann}_A(C')$*.*

Proof. Cell modules have no non-trivial proper submodules. Thus, the kernel and the cokernel of any non-trivial morphism between cell modules has to be trivial. $\square$

Further, Lemma 2.4.6 also implies that subquotients of a module have larger annihilators than the module. Since $M(cp(x)) \subset A/\langle \ngeq_{LR} \underline{c}(x)\rangle$ and since $M(c(x))$ is a subquotient of $M(cp(x))$ for all $x \in W$, we obtain the following corollary.

Corollary 2.4.8. *For all $x \in W$,* $\mathrm{Ann}_A(M(c(x))) \supset \mathrm{Ann}_A(M(cp(x))) \supset \{\ngeq_{LR} x\}$*.*

In fact, we can determine the annihilator of certain cell modules:

Proposition 2.4.9. *Let $C \in A - \mathrm{Mod}^{(k,O_k)}$ be a cell module and let $\underline{c}$ be a 2-sided cell of A. Assume that $\{<_{LR} \underline{c}\} \subset \mathrm{Ann}_A(C)$ and that $\underline{c} \cap \mathrm{Ann}_A(C) = \emptyset$. Then,* $\mathrm{Ann}_A(C) = \{\ngeq_{LR} \underline{c}\}$*. Furthermore, $\underline{c}$ is unique with this property.*

Proof. Let $C \in A - \mathrm{Mod}^{(k,O_k)}$ be a cell module, let $\underline{c}$ be a 2-sided cell of A and denote $Z := \{\ngeq_{LR} \underline{c}\}$. Taking the assumption $\underline{c} \cap \mathrm{Ann}_A(C) = \emptyset$ together with the fact that $\mathrm{Ann}_A(C)$ is a two-sided ideal, we obtain $\mathrm{Ann}_A(C) \subset Z$ and have to show equality. Assume there exists $z \in Z \setminus \mathrm{Ann}_A(C)$. Then, there exists a basis element m_x of C such that $b_z m_x \neq 0$. By Definition 2.1.1 of a ring with positivity structure, the module axioms of C, and since $\underline{c} \cap \mathrm{Ann}_A(C) = \emptyset$, there exists $w \in \underline{c}$ such that $b_w b_z m_x \neq 0$. Again by the assumptions on a positivity structure, there exists $u \in W$

such that $b_u \leq b_w b_z$ and $b_u m_x \neq 0$. Thus, u satisfies three properties, namely $u \leq_L z$, $u \leq_R w$ and $u \notin \mathrm{Ann}_A(C)$. The latter two properties together with the assumption $\{<_{LR} \underline{c}\} \subset \mathrm{Ann}_A(C)$ imply that $u \in \underline{c}$. Now, the first property implies that $z \geq_{LR} \underline{c}$ which contradicts $z \in Z$. Thus, $\mathrm{Ann}_A(C) \subset Z$ holds. Given another 2-sided cell $\underline{c}'$ with the same properties as $\underline{c}$, we deduce that $Z = \{\ngeq_{LR} \underline{c}'\}$. Taking the complement, it follows that $\underline{c} \leq_{LR} \underline{c}' \leq_{LR} \underline{c}$ and thus $\underline{c} = \underline{c}'$. □

Corollary 2.4.10. *Let $C \in A - \mathrm{Mod}^{(k,O_k)}$ be a cell module and assume that A has finitely many 2-sided cells. Then, there exists a unique 2-sided cell $\underline{c}$ of A such that $\mathrm{Ann}_A(C) = \{\ngeq_{LR} \underline{c}\}$.*

Proof. By Remark 2.1.10, $W \setminus \mathrm{Ann}_A(C) \neq \emptyset$. Since there exist only finitely many 2-sided cells by assumption, we can find a 2-sided cell with the properties required in Proposition 2.4.9, and the corollary follows. □

Proposition 2.4.9 also implies a criterion for equality in Corollary 2.4.8.

Corollary 2.4.11. *Let $\underline{c}$ be a 2-sided cell of A and let x, y, $z \in \underline{c}$. Further assume that $b_z \leq b_y b_x$ and $x \sim_L z$. Then, $\mathrm{Ann}_A(M(c(x))) = \mathrm{Ann}_A(M(cp(x))) = \{\ngeq_{LR} \underline{c}\}$.*

2.5 Universal cell modules

In this section, we introduce universal cell modules, define when cell modules belong to a 2-sided cell, and introduce regularity notions. Then, Theorem 2.5.6 provides necessary and sufficient conditions for the existence of cell modules that belong to 2-sided cells in the weakly regular case. Under these assumptions, Corollary 2.5.7 gives a simple explicit construction of a universal cell module. Furthermore in the weakly regular case with finitely many 2-sided cells, Corollary 2.5.8 establishes a linkage principle, which implies that the set of cell modules modulo linkage by morphisms is in bijection with the set of 2-sided cells.

Corollary 2.4.7 makes the following definition natural:

Definition 2.5.1. *Let A be a (k, O_k)-ring and let $C \in A-\mathrm{Mod}^{(k,O_k)}$ be a cell module. Then, we call C a* universal cell module *if there exists a non-trivial morphism $C \to C'$ in $A - \mathrm{Mod}^{(k,O_k)}$ for all cell modules $C' \in A - \mathrm{Mod}^{(k,O_k)}$ with $\mathrm{Ann}_A(C) = \mathrm{Ann}_A(C')$.*

Proposition 2.4.9 motivates the following definition:

Definition 2.5.2. *Let A be a (k, O_k)-ring and let $\underline{c}$ be a 2-sided cell of A. We say that A* has $\underline{c}$-cell modules*, if there exists a cell module $C \in A - \mathrm{Mod}^{(k,O_k)}$ such that $\mathrm{Ann}_A(C) = \{\not\geq_{LR} \underline{c}\}$. In this situation, we also say that C* belongs to $\underline{c}$.

Remark 2.5.3. We focus on cell modules that belong to 2-sided cells. Essentially by the argument of Corollary 2.4.10, this class comprises all cell modules typically studied in the literature, see [MaMi5, Section 7.2] for a related discussion. In [Rou4] the main object of study are integrable representations, see [Rou4, Definition 5.1]. Any cyclic integrable representation of a 2-Kac-Moody algebra can be regarded as the restriction of a representation of a quotient 2-category of this 2-Kac-Moody algebra that has finitely many 2-sided cells. Similarly, the finitary 2-categories of [MaMi1] have finitely many 2-sided cells by definition.

Example 2.5.4. Consider $\mathbb{Z}[X]$ as a $(\mathbb{Z}, \mathbb{N}_0)$-algebra with basis $\{X^n\}_{n \in \mathbb{N}_0}$. Each 2-sided cell consists of a single basis element. For all $n \in \mathbb{N}_0$, we have a cell module $\mathbb{Z}[X]/\langle X - n \rangle$. The annihilator of $\mathbb{Z}[X]/\langle X \rangle$ is given by $\mathbb{N}$ and this cell module thus belongs to the maximal 2-sided cell. On the other hand for $n \neq 0$, the annihilator of $\mathbb{Z}[X]/\langle X - n \rangle$ is trivial and the module does therefore not belong to any 2-sided cell. Also, note that attempting the same definition for $n < 0$ would not result in a module with positive structure coefficients.

We now recall the regularity notions of [MaMi1], and add a weaker variant.

Definition 2.5.5. *Let A be a (k, O_k)-ring and let $\underline{c}$ be a 2-sided cell of A.*

- *We call $\underline{c}$* weakly regular *if $cp(x)$ consists of finitely many left cells for all $x \in \underline{c}$.*
- *We call $\underline{c}$* regular *if $cp(x) = c(x)$ for all $x \in \underline{c}$,*
- *We call $\underline{c}$* strongly regular *if $\underline{c}$ is regular and if $|c \cap \widetilde{c}| = 1$ for all left cells c and right cells $\widetilde{c}$ that are contained in $\underline{c}$.*
- *We call A* (weakly/strongly) regular*, if all 2-sided cells of A are (weakly/strongly) regular.*

The following theorem is the main result of this section:

Theorem 2.5.6. *Let A be a (k, O_k)-ring and let $\underline{c}$ be a 2-sided cell of A. Assume that $\underline{c}$ is weakly regular. Then, the following are equivalent:*

1. *There exist a cell module of A that belongs to $\underline{c}$.*

2. *There exist a universal cell module of A that belongs to $\underline{c}$.*

3. *For all $z \in \underline{c}$ there exist $x, y \in cp(z)$ such that $b_x \leq b_y b_x$ and such that $c(x)$ is minimal with respect to $\leq_L$ in $\underline{c}$.*

4. *There exist $x, y \in \underline{c}$ such that $b_x \leq b_y b_x$ and such that $c(x)$ is minimal with respect to $\leq_L$ in $\underline{c}$.*

In fact, the proof of Theorem 2.5.6 implies the following:

Corollary 2.5.7. *Let A be a (k, O_k)-ring, let $\underline{c}$ be a weakly regular 2-sided cell of A that has cell modules, and let $c \subset \underline{c}$ be a left cell that is minimal with respect to $\leq_L$ inside $\underline{c}$. Then, $M(c)$ is a universal cell module.*

Proof of Theorem 2.5.6. Let A be a (k, O_k)-ring and let $\underline{c}$ be a weakly regular 2-sided cell of A. The implications (3) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) are obvious. Thus, it remains to show the three implications (1) $\Rightarrow$ (3), (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2). To show the first implication, let us assume (1), *i.e.* that A has $\underline{c}$-cell modules. Let $C \in A - \mathrm{Mod}^{(k,O_k)}$ be a cell module that belongs to $\underline{c}$, and let $z \in \underline{c}$. Since $\underline{c}$ is weakly regular, we can choose $\widetilde{x} \in cp(z)$ such that $c(\widetilde{x})$ is the minimal left cell in $cp(z)$ with respect to $\leq_L$. Since $\mathrm{Ann}_A(C) \cap \underline{c} = \emptyset$ by assumption, there exists a basis element m of C such that $b_{\widetilde{x}} m \neq 0$. Consider the morphism $A \to C$ given by acting on m. Since $\mathrm{Ann}_A(C) = \{\ngeq_{LR}\}$, we obtain a factorisation $A/\langle \ngeq_{LR} \rangle \to C$ which induces a non-trivial morphism $M(c(\widetilde{x})) \to C$. By Corollary 2.4.7, we have $\mathrm{Ann}_A(M(c(\widetilde{x}))) = \{\ngeq_{LR} \underline{c}\}$. Therefore there exists $x \in c$ such that $b_z b_x \neq 0$. Thus, we have $x' \in c$ such that $b_{x'} \leq b_z b_x$. Since $x' \sim_L x$, there exists $w \in W$ such that $b_x \leq b_w b_{x'}$, and we deduce $b_x \leq b_w b_z b_x$. By the assumptions on the positivity structure, there exists $y \in W$ such that $b_y \leq b_w b_z$ and $b_x \leq b_y b_x$. The former property implies that $y \leq_L z$ and the latter implies that $y \notin \mathrm{Ann}_A(C)$. Since $z \in \underline{c}$, we obtain $y \in \underline{c}$ and more precisely $y \in cp(z)$. Thus, we have established (3). To show the second and third implication, let us assume (4), *i.e.* that that there exist $x, y \in \underline{c}$ such that $b_x \leq b_y b_x$, and such that $c(x)$ is minimal with respect to $\leq_L$ in $\underline{c}$. By Corollary 2.4.11, it follows that $M(c(x))$ is a cell module that belongs to $\underline{c}$, *i.e.* (1). Now, the proof of the implication (1) $\Rightarrow$ (3) shows that for every cell module C that belongs to $\underline{c}$, there exists a non-trivial morphism $M(c(x)) \to C$. Thus, $M(c(x))$ is a universal cell modules and we have established the last outstanding implication. $\square$

Combining Theorem 2.5.6 with Corollaries 2.4.10 and 2.4.7, we can deduce the linkage principle that we had announced. This should be compared with Proposition 2.1.11 due to [Ga].

Corollary 2.5.8. *Let A be a weakly regular (k, O_k)-ring that has finitely many 2-sided cells. Then, there exists finitely many linkage classes of cell modules.*

Let us give a simple example for linkage of cell modules which is inspired by [MaMi5, Section 3.4]. In particular, it provides an example of a cell module that does not come from a left cell. An example of a cell module that is not a universal cell module can be found in the type $\widetilde{A}_2$ example in Section 5.5.

Example 2.5.9. Let $A = \mathbb{Z}[X]/\langle X^2 - 2X\rangle$ be the the $(\mathbb{Z}, \mathbb{N}_0)$ ring with basis $\{1, X\}$. Clearly, A is strongly regular and 1 and X are generalised idempotents. Specialisation yields two cell modules $C_1 := A|_{X=0}$ and $C_X := A|_{X=2}$ that belong to the 2-sided cells $\{1\}$ and $\{X\}$ respectively. Interestingly, there exists another cell module C_{tw} associated to $\{X\}$. The underlying Abelian group of C_{tw} is given by $\mathbb{Z} \times \mathbb{Z}$ and the action of X is given by $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. It is easy to write down a morphism from A to C_X (resp. C_{tw}) as well as a morphism in the other direction. In both cases one composition is equal to $X id_A$. The other compositions are equal to $2 id_{C_X}$ and $id_{C_{tw}} + S$ respectively, where $S((z_1, z_2)) := (z_2, z_1)$ for all $(z_1, z_2) \in C_{tw}$. For a categorification of A and the modules discussed here, see Example 4.5.7.

Before we can state another corollary, we need to introduce the following terminology:

Definition 2.5.10. *Let A be a (k, O_k)-ring and let $\underline{c}$ be a 2-sided cell of A. We say that $\underline{c}$ has* generalised idempotents*, if there exists $x \in \underline{c}$ such that $b_x \leq b_x b_x$.*

Remark 2.5.11. Note that we do not require that $b_x b_x$ is equal to a O_k-multiple of b_x. Similarly, this will not be required from split Frobenius objects, see Section 4.3.1.

Corollary 2.5.12. *Let A be a (k, O_k)-ring and let $\underline{c}$ be a strongly regular 2-sided cell of A. Then, A has an universal cell module that belongs to $\underline{c}$, if and only if A has $\underline{c}$-cell modules, if and only if each left cell of A inside $\underline{c}$ contains a generalised idempotent.*

Proof. By Theorem 2.5.6, the first two properties are both equivalent to the existence of elements x, y in each companion set in $\underline{c}$ such that $b_x \leq b_y b_x$. Since we assume that A is regular, each companion set is a left cell. Further $b_x \leq b_y b_x$ implies that y is contained in the right cell associated to x. Since we assume that A is strongly regular, we deduce that $x = y$. □

Remark 2.5.13. The notion of generalised idempotents is new. In the special case of strongly regular $(\mathbb{Z}, \mathbb{N}_0)$-rings categorifiable by finitary 2-categories, existence of generalised idempotents follows from [MaMi1, Proposition 34].

2.6 Hecke algebras

This section, we introduce the Hecke algebra, which is a $(\mathbb{Z}[v,v^{-1}], \mathbb{N}_0[v,v^{-1}])$-ring. Section 2.6.1 recalls basics about Coxeter groups, the Hecke algebra and the Schur algebroid. Section 2.6.2 recalls the W-graphs of [KaLu] and observes that not all W-graphs fit into the positive structure coefficient framework discussed here. Section 2.6.3 discuses the cell modules of the Hecke algebra of type B_2, which foreshadows most of the challenges that we are faced with in this thesis.

2.6.1 Basic definitions

In this section, we recall basics about Coxeter groups, which can be found for example in [Hum]. Then, we introduce the Hecke algebra and the Schur algebroid. Here, we use the notation of [Soe3], since it is most convenient when relating the Hecke algebra to Soergel bimodules later on.

Definition 2.6.1. *A* Coxeter system (W, S) *is a group* W*, called a* Coxeter group*, together with a subset* $S \subset W$*, such that* $W = \{s \in S | s^2 = 1, (st)^{m_{s,t}} = 1\}$ *for a symmetric function* $m : S \times S \to \mathbb{N} \amalg \{\infty\}$ *that satisfies* $m_{s,t} = 1$ *if and only if* $s = t$*.*

Definition 2.6.2. *Let* (W, S) *be a Coxeter system, and let* $I \subset S$*. The* standard parabolic subgroup associated to I *is defined as* $W_I := \langle s \in I \rangle \leq W$*. The set* I *is called* finitary *provided that* $|W_I| < \infty$*.*

Remark 2.6.3. Given a Coxeter system (W, S) and a subset $I \subset S$, it is a non-trivial fact that (W_I, I) is a Coxeter system.

Definition 2.6.4. *Let* (W, S) *be a Coxeter system and let* $w \in W$*. An expression* $w = s_1..s_k$*, where* $s_i \in S$ *for all* $i \in \{1, ..., n\}$*, is called a* reduced expression *if* $k \in \mathbb{N}_0$ *is as small as possible. The length* k *of a reduced expressions of* $w \in W$ *is called the* length of w *and is denoted by* $l(w)$*.*

Remark 2.6.5. By the definition of Coxeter systems, reduced expressions exist for every element of the Coxeter group. It is a non-trivial fact that two reduced expression can be obtained from each other via the *braid relations* $\underbrace{s\,t\,...}_{m_{s,t}-elements} = \underbrace{t\,s\,...}_{m_{s,t}-elements}$ which hold for all s, $t \in S$.

Definition 2.6.6. *Let* (W, S) *be a Coxeter system and let* $x, w \in W$*. We write* $x \leq w$*, if there exists a reduced expression of* w *such that* x *is equal to a subexpression. The thus defined order* $\leq$ *of* (W, S) *is called the* Bruhat order*.*

Remark 2.6.7. Given a Coxeter system (W, S) and $I \subset S$, the Bruhat order of (W_I, I) agrees with the order on W_I induced from the Bruhat order of (W, S). Further if I is finitary, there exists a unique longest element of W_I with respect to both the Bruhat order and the length function of Definition 2.6.4.

Definition 2.6.8. *Let (W, S) be a Coxeter system and let $I \subset S$ be finitary. Then, we denote by w_0^I the longest element of W_I and refer to w_0^I as a* parabolic longest element*. If W is finite, we denote by $w_0 := w_0^S$, which is called the* longest element*.*

Definition 2.6.9. *Let (W, S) be a Coxeter system and let $w \in W$. The* left descent set *of w is defined as $L(w) := \{s \in S | l(sw) < l(w)\}$. Analogously, the* right descent set *of w is defined as $R(w) := \{s \in S | l(ws) < l(w)\}$.*

We finally come to the definition of the Hecke algebra.

Definition 2.6.10. *Let (W, S) be a Coxeter system. We denote by $\mathcal{H}(W, S)$ the $\mathbb{Z}[v, v^{-1}]$-algebra with generators $\{H_s\}_{s \in S}$, such that for all $s, t \in S$ we have braid relations*

$$\underbrace{H_s\, H_t \ldots}_{m_{s,t}-elements} = \underbrace{H_t\, H_s \ldots}_{m_{s,t}-elements} \qquad and$$
$$(H_s + v)(H_s - v^{-1}) = 0.$$

The algebra $\mathcal{H}(W, S)$ is called the Iwahori-Hecke algebra with equal parameters of the Coxeter system (W, S)*, and will be referred to as the* Hecke algebra*.*

Notation 2.6.11. Whenever statements are independent of the specific Coxeter system (W, S) or a Coxeter system (W, S) is fixed, we abbreviate $\mathcal{H} = \mathcal{H}(W, S)$. Furthermore for $I \subset S$, we identify $\mathcal{H}(W_I, I)$ with the $\mathbb{Z}[v, v^{-1}]$-subalgebra $\mathcal{H}_I$ of $\mathcal{H}$ generated by $\{H_s\}_{s \in I}$.

Remark 2.6.12. The Hecke algebra $\mathcal{H}$ arises naturally in various representation theoretic contexts. Also, note that $\mathcal{H}$ it is deformation of the integral group algebra of W. More precisely, we have $\mathbb{Z}[W] \cong \mathcal{H}(W, S)|_{v=1}$.

Remark 2.6.13. The Hecke algebra has the following important bases:

- The *standard basis* $\{H_w\}_{w \in w}$ is defined by $H_w := H_{s_1} \ldots H_{s_{l(w)}}$, where $w = s_1 .. s_{l(w)}$ is a reduced expression. It is well defined due to Remark 2.6.5.
- The famous *Kazhdan-Lusztig basis* $\{\underline{H}_w\}_{w \in w}$, see Theorem 2.6.14 below.

- *Soergel's basis* is defined in terms of a categorification $\mathcal{S}oe^\bullet$ of $\mathcal{H}$, see Theorem 5.1.13, which states that the split Grothendieck ring of $\mathcal{S}oe^\bullet$ is isomorphic to the Hecke algebra. Via this isomorphism, Soergel's basis is induced from the indecomposable objects of $\mathcal{S}oe^\bullet$. Conjecturally, this basis depends only on the characteristic of the field used to define $\mathcal{S}oe^\bullet$. According to Soergel's Conjecture 5.1.14, Soergel's basis in characteristic zero agrees with the Kazhdan-Lusztig basis. Over $\mathbb{R}$ this has recently been established in [ElWi2]. In characteristic p, the induced basis of the Hecke algebra is called the *p-canonical basis*.

For all $s \in S$, we have $H_s^{-1} = H_s + (v - v^{-1})H_e$. Thus, all elements of the standard basis are invertible. This allows us to define the $\mathbb{Z}[v, v^{-1}]$-anti-linear *bar-involution* of $\mathcal{H}$ by setting $\overline{H_w} := H_{w^{-1}}^{-1}$ and $\overline{v} := v^{-1}$.

Theorem 2.6.14 ([KaLu]). *There exists a unique $\mathbb{Z}[v, v^{-1}]$-basis $\{\underline{H}_w\}_{w \in W}$ of the Hecke algebra $\mathcal{H}(W, S)$, called the* Kazhdan-Lusztig basis*, such that for all $w \in W$*

1. *$\underline{H}_w$ is self dual, i.e. $\underline{H}_w = \overline{\underline{H}_w}$, and*
2. *$\underline{H}_w = H_w + \sum_{x<w} h_{x,w} H_x$ with $h_{x,w} \in v\mathbb{Z}[v]$.*

Remark 2.6.15. The Kazhdan-Lusztig basis becomes the Kazhdan-Lusztig basis $\{C'_w\}$ originally defined in [KaLu] when replacing v by $-q^{\frac{1}{2}}$ and H_x by $\widetilde{T}_x = q^{-l(x)} T_x$. In this way, the polynomials $h_{x,w}$ are identified with the so called *Kazhdan-Lusztig polynomials*.

Example 2.6.16. An element $w \in W$ is called smooth, if $\underline{H}_w = \sum_{x \leq w} v^{l(w)-l(x)} H_w$. All parabolic longest elements are smooth. Furthermore, every element in a dihedral group is smooth.

For completeness, we would like to mention the *Positivity Conjecture*. As explained in [Soe6], it follows from Soergel's Conjecture 5.1.14 which identifies the polynomials $h_{x,w}$ with certain graded dimensions, and is thus a theorem.

Theorem 2.6.17 (Positivity Conjecture, [Soe6], [ElWi2]). *For all $x, w \in W$ such that $x < w$, we have $h_{x,w} \in v\mathbb{N}_0[v]$ and $h_{w,w} = 1$.*

Next, we introduce the Schur algebroid following [Wi2]. For finitary $I \subset S$, it is well known that $\underline{H}_s \underline{H}_{w_0^I} = (v + v^{-1}) \underline{H}_{w_0^I} = \underline{H}_{w_0^I} \underline{H}_s$ for all $s \in I$. Thus for finitary $I \subset S$, we can introduce the notation $\mathcal{H}^I := \{H \in \mathcal{H} | H \underline{H}_s = (v + v^{-1})H \;\; \forall s \in I\}$ and ${}^I\mathcal{H} := \{H \in \mathcal{H} | \underline{H}_s H = (v + v^{-1})H \;\; \forall s \in I\}$. While we will not use the following fact, it provides some intuition for this notation:

Remark 2.6.18. For finitary $I \subset S$, we have $\mathcal{H}^I \cong \mathrm{Ind}_{\mathcal{H}_I}^{\mathcal{H}} \mathrm{tr} = \mathcal{H} \otimes_{\mathcal{H}_I} \mathrm{tr}$ where tr is the *trivial* $\mathcal{H}_I$-module. More precisely, tr is given by $\mathbb{Z}[v, v^{-1}]$ on which H_s acts by v^{-1} for all $s \in S$. Thus, $\underline{H}_s$ acts by $v + v^{-1}$. This module is called trivial since under the isomorphism $\mathbb{Z}[W] \cong \mathcal{H}|_{v=1}$ the specialisation $\mathrm{tr}|_{v=1}$ corresponds to the trivial module of $\mathbb{Z}[W]$.

Definition 2.6.19. *Let* (W, S) *be a Coxeter system. The* Schur algebroid *of* (W, S) *is the* $\mathbb{Z}[v, v^{-1}]$*-linear category with objects finitary subsets* $I \subset S$*, with morphisms between objects* I *and* J *given by* $\mathrm{Hom}_{\mathcal{H}}(\mathcal{H}^I, \mathcal{H}^J)$*, and with the obvious composition law induced from the composition of functions.*

In [Wi1], the Schur algebroid was defined in a more *ad hoc*manned. However in the introduction of the published version [Wi2] (aside from working with right rather than left modules), it was pointed out that Definition 2.6.19 is an equivalent and more natural way to define this algebroid. For the readers convenience, let us discuss this alternative definition and explain the algebroid isomorphism between the two version:

Remark 2.6.20. For all finitary I, $J \subset S$, denote ${}^I\mathcal{H}^J := {}^I\mathcal{H} \cap \mathcal{H}^J$. Further, define an operation $*_J : {}^I\mathcal{H}^J \times {}^J\mathcal{H}^K \to {}^I\mathcal{H}^K$ for all finitary I, J, $K \subset S$ by dividing the multiplication of $\mathcal{H}$ by the *Poincaré polynomial* $\pi(I) \in \mathbb{Z}[v, v^{-1}]$ of W_I. This polynomial is given by $\pi(I) := \sum_{w \in W_I} v^{2l(w) - l(w_0^I)}$, and is uniquely determined by the equation $\underline{H}_{w_0^I} \underline{H}_{w_0^I} = \pi(I) \underline{H}_{w_0^I}$. Thus for $I = J$ or $J = K$, $\underline{H}_{w_0^J}$ acts as an identity. Therefore, we have defined an algebroid with objects finitary subsets of S. Now for all finitary I, $J \subset S$, the operation $*_I$ induces a morphism ${}^I\mathcal{H}^J \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}^I, \mathcal{H}^J)$. In fact, this morphism is an isomorphism and defines an algebroid isomorphism to the Schur algebroid of Definition 2.6.19.

2.6.2 W-graphs and $\mathcal{H} - \mathrm{Mod}^{(\mathbb{Z}[v,v^{-1}], \mathbb{N}_0[v,v^{-1}])}$

In this section, we recall the definition of W-graphs which are a combinatorial analogue of certain $\mathcal{H}$-modules. The definition was originally formulated in [KaLu] in terms of the standard basis of $\mathcal{H}$, whereas we give an equivalent definition formulated in terms of the Kazhdan-Lusztig basis. We will also observe that not all W-graphs give rise to an object of $\mathcal{H} - \mathrm{Mod}^{(\mathbb{Z}[v,v^{-1}], \mathbb{N}_0[v,v^{-1}])}$. In this thesis, W-graphs do not play a role. However, W-graphs play an important role in the literature on Hecke algebras, and thus we highlight commonalities and differences.

Example 2.6.21. The codomain of the morphism discussed in Example 2.1.8 is an object of $\mathcal{H} - \mathrm{Mod}^{(\mathbb{Z}[v,v^{-1}], \mathbb{N}_0[v,v^{-1}])}$ that cannot be afforded by a W-graph.

The basic idea behind W-graphs is that $\mathcal{H}$ is generated as a $\mathbb{Z}[v, v^{-1}]$-algebra by $\{\underline{H}_s\}_{s\in S}$, and thus one can encode any $\mathcal{H}$-module which is free over $\mathbb{Z}[v, v^{-1}]$ in terms of the structure coefficients associated to these generators. As a further motivation, recall the following multiplication formula, which determines the structure of $\mathcal{H}$ as a $\mathcal{H}$-module, see [Wi1, Formula (2.1.3)] which in turn is based on the proof of [Soe3, Behauptung 2.3]:

$$\underline{H}_s\underline{H}_w = \begin{cases} (v+v^{-1})\underline{H}_w & \text{if } s \in L(w) \\ \sum_{x\in W; s\in L(x)} \mu(x,w)\underline{H}_x & \text{if } s \notin L(w) \end{cases} \tag{2.1}$$

Here, $\mu : W \times W \to \mathbb{Z}$ is the symmetric function, which is 0 on pairs (x, w) that are not compatible in the Bruhat order, and is given by the coefficient of v in the Kazhdan-Lusztig polynomial $h_{x,w}$ for all pairs $(x, w) \in W \times W$ with $x \le w$ in the Bruhat order. The analogous formula for right multiplication holds as well and thus one might expect that $\mathcal{H}$ together with the Kazhdan-Lusztig basis is a $(\mathbb{Z}[v, v^{-1}], \mathbb{N}_0[v, v^{-1}])$-ring. To prove this directly is a highly non-trivial task. However, the following proposition is known to hold due to Theorem 5.1.13 and Soergel's Conjecture 5.1.14.

Proposition 2.6.22. *The Hecke algebra $\mathcal{H}$ together with the Kazhdan-Lusztig basis is a $(\mathbb{Z}[v, v^{-1}], \mathbb{N}_0[v, v^{-1}])$-ring. The regular representation $\mathcal{H}$ and all its subquotients lie in $\mathcal{H} - \mathrm{Mod}^{(\mathbb{Z}[v,v^{-1}],\mathbb{N}_0[v,v^{-1}])}$.*

By formula (2.1), the regular representation of $\mathcal{H}$ and its subquotients can be encoded in a (left) W-graph, a terminology that we define now:

Definition 2.6.23. *A W-graph is an triple (X, I, μ), where X is the set of vertices, $I : X \to \mathcal{P}(S)$ is a labelling of the vertices and $\mu : X \times X \setminus \{(x, x)|x \in X\} \to \mathbb{Z}$ is a labelling of the edges. This data is subject to the requirement that for all $s \in S$ and $x \in X$*

$$\underline{H}_s.x := \begin{cases} (v+v^{-1})x & \text{if } s \in I(x) \\ \sum_{x'\in X: s\in I(x')} \mu(x',x)x' & \text{if } s \notin I(x) \end{cases}$$

defines the structure of a $\mathcal{H}$-module on the free $\mathbb{Z}[v, v^{-1}]$-module with basis X. We say that the W-graph (X, I, μ) has positive coefficients if and only if the action of all Kazhdan-Lusztig basis elements has coefficients in $\mathbb{N}_0[v, v^{-1}]$.

Remark 2.6.24. The edge labelling function μ is not required to be symmetric, while it happens to be symmetric in the case of the regular representation of $\mathcal{H}$.

The following definition will typically be applied to a field k on which $\mathbb{Z}[v, v^{-1}]$ acts by evaluating v at $1 \in k$.

Definition 2.6.25. *A W-graph* affords a W-representation *over a $\mathbb{Z}[v,v^{-1}]$-algebra k, if the $k[W]$-module associated to the W-representation is isomorphic to the base change to the group algebra $k[W] = \mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} k$ of the $\mathcal{H}$-module associated to the W-graph.*

Remark 2.6.26. In [Gy] it is proven that all irreducible representation of finite Coxeter groups over a splitting field can be afforded by a W-graph over the ring of integers of this splitting field. Hence for a Weyl group W, W-graphs over $\mathbb{Z}$ are sufficient to afford all irreducible representations of $\mathbb{Q}[W] \cong \mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}$. For affine Weyl groups this statement no longer holds, see [Gy, Lu5]. Furthermore, there exist irreducible representations that are afforded by W-graphs, which do not have positive coefficients. In particular, such W-graphs do not give rise to objects of $\mathcal{H} - \mathrm{Mod}^{(\mathbb{Z}[v,v^{-1}],\mathbb{N}_0[v,v^{-1}])}$. More concretely, we will see in Section 2.6.3 that there exist W-graphs in type B_2 that have a positive μ-function, but do not have positive structure coefficients.

Definition 2.6.27. *A morphism of W-graphs $f : (X, I_X, \mu_X) \to (Y, I_Y, \mu_Y)$ over $\mathbb{Z}$ is a map $f : X \to \mathbb{Z}[v,v^{-1}]\langle Y\rangle$ sending $x \mapsto \sum_{y\in Y} f_{x,y}y$, such that for all $x \in X$ and $y \in Y$*

$$\begin{array}{lll} s \in I_X(x) & \Rightarrow & f(x) \in \mathbb{Z}[v,v^{-1}]\left\langle I_Y^{-1}(s)\right\rangle \\ s \notin I_X(x) & \Rightarrow & \sum_{x'\in I_X^{-1}(s)} f_{x',y}\mu_X(x',x) = \begin{cases} \sum_{y'\in Y\setminus\{y\}} f_{x,y'}\mu_Y(y,y') & \text{if } s \in I_Y(y) \\ 0 & \text{if } s \notin I_Y(y) \end{cases} . \end{array}$$

We say that f is a morphism with positive coefficients if and only if both (X, I_X, μ_X) and (Y, I_Y, μ_Y) have coefficients in $\mathbb{N}_0[v,v^{-1}]$ and $f_{x,y} \in \mathbb{N}_0[v,v^{-1}]$ for all $x \in X$ and $y \in Y$.

This definition amounts to saying that f induces a homomorphism between the associated $\mathcal{H}$-modules. This is the case if and only if $f(\underline{H}_s.x) = \underline{H}_s.f(x)$ for all $s \in S$ and $x \in X$, which is exactly the condition stated above.

The following proposition provides some insight in the structure of W-graphs. It can be deduced from the more precise lemma stated afterwards:

Proposition 2.6.28. *Let (W, S) be a connected Coxeter system. Let (X, I, μ) be a W-graph with positive coefficients, and assume that the associated $\mathcal{H}$-module is a cell module. Then, we have either*

$$S = \bigcup_{x\in X} I(x)$$

or $X = \{x\}$ and $I(x) = \emptyset$.

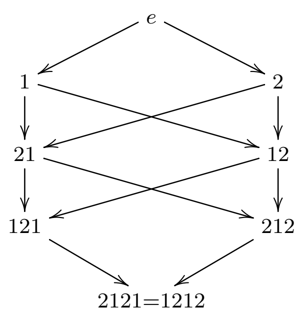
Lemma 2.6.29. *Let (X, I, μ) be a W-graph with positive coefficients, $x \in X$ and $s \in I(x)$. Let further $t \in S$ be such that $m_{s,t} \geq 3$. Then, either $t \in I(x)$ or there exists $x' \in X$ such that $t \in I(x')$, $s \notin I(x')$ and $\mu(x, x') \neq 0 \neq \mu(x', x)$. In the latter case, $\underline{H}_{sts}.x = 0$ if and only if x' is the only such element and $\mu(x, x')\mu(x', x) = 1$.*

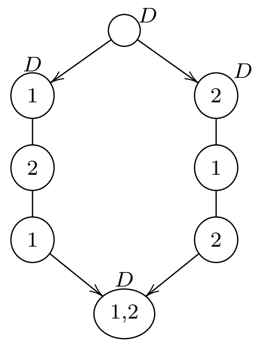
Proof. Assume $t \notin I(x)$. We have $\underline{H}_s\underline{H}_t\underline{H}_s = \underline{H}_{sts} + \underline{H}_s$. Since $\underline{H}_{sts}$ acts with positive coefficients on x and $\underline{H}_s$ acts by $v + v^{-1}$ on x, $\underline{H}_t$ cannot act trivially. Furthermore due to the above equation, there exists x' in the image of $\underline{H}_t$ such that x occurs with non-trivial coefficient in $\underline{H}_s.x$. Thus, $s \notin I(x')$ and $\mu(x, x') \neq 0$ as well. The remaining claim follows from positivity of the coefficients. $\square$

2.6.3 W-graphs and cell modules in type B_2

In this section, we provide examples that served as testing ground throughout the process of writing this thesis. We will refer to these examples on multiple occasions, most notably in Section 5.5 where we will categorify the structure presented in this section.

The Weyl group of type B_2 has the presentation $W = \langle s_1, s_2 | s_1^2 = s_2^2 = (s_1s_2)^4 = 1\rangle$. We abbreviate in the following $s_1 = 1$ and $s_2 = 2$. The associated Bruhat graph and the W-graph of the regular representation $\mathcal{H} = \mathcal{H}(W)$ can be visualised as follows:

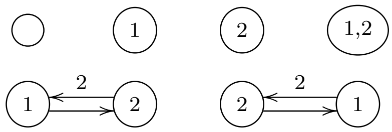




Note that in that the μ function of this W-graph takes the values 0 and 1 only, and can thus be visualized using an arrow $x \to x'$ if $\mu(x, x') = 1$ and $\mu(x', x) = 0$ or by a line $x - x'$ if $\mu(x, x') = \mu(x', x) = 1$. For the readers convenience, we have also highlighted the Duflo involutions (see Definition 5.2.1) using the symbol D.

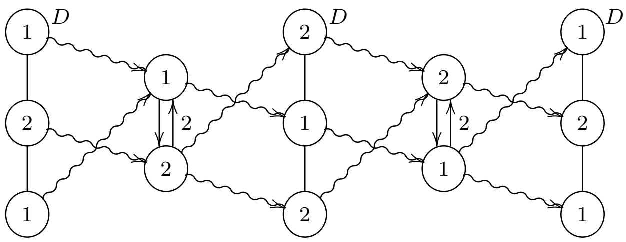
From the above W-graph, one can infer that the Weyl group B_2 has four left cells, namely $\{e\}$, $\{1212\}$, $\{1, 21, 121\}$ and $\{2, 12, 212\}$, see Definition 2.3.1. Also, it is not hard to see that there are three 2-sided cells, namely $\{e\}$, $\{1212\}$ and $\{1, 2, 12, 21, 121, 212\}$.

There are 5 non-isomorphic irreducible W-representations over $\mathbb{Q}$ which are afforded by the following W-graphs:

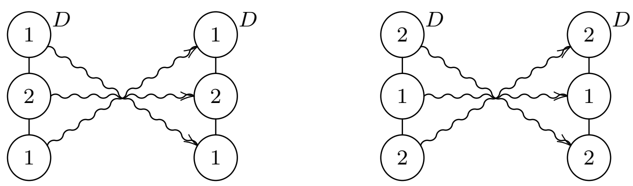


The two rank 2 W-graphs afford isomorphic W-representations over $\mathbb{Q}$ (and more generally over any ring k with $2 \in k^*$). As $\mathbb{Q}[W]$-modules, the cell module of the left cell $\{1, 21, 121\}$ becomes isomorphic to a direct sum of the two dimensional representation and the one dimensional representation associated to the second W-graph displayed above. Even though the μ function of all these W-graphs has values only in $\mathbb{N}_0$, the second and third W-graph do not have positive coefficients by the following argument: Since $\underline{H}_2$ acts by zero on the second W-graph, we have that $0 = (\underline{H}_1\underline{H}_2\underline{H}_1). = \underline{H}_{121}. + \underline{H}_1.$ and thus $\underline{H}_{121}$ acts by $-(v + v^{-1})$. On the other hand, it is not hard to see that the remaining four W-graphs displayed above lie in $\mathcal{H} - \mathrm{Mod}^{\mathbb{Z}[v,v^{-1}],\mathbb{N}_0[v,v^{-1}])}$. Furthermore, the two cell modules with bases $\{1, 21, 121\}$ and $\{2, 12, 212\}$ lie in this category as well.

Next, we describe all morphisms between the 6 W-graphs with positive structure coefficients discussed above. The only non-trivial morphism are given by $\mathbb{N}_0[v, v^{-1}]$-linear combinations of suitable compositions of the morphisms π_1, ι_2, π_2 and ι_1 given by



as well as the morphisms s_1 and s_2



and identity morphisms. Note that the composition $\iota_1 \circ \pi_2 \circ \iota_2 \circ \pi_1$ (resp. $\iota_2 \circ \pi_1 \circ \iota_1 \circ \pi_2$) agrees with the sum of s_1 (resp. s_2) and the respective identity morphism. Provided that 2 is invertible, we can divide both these compositions by 2 and obtain idempotents. These idempotents project onto the 2-dimensional modules. Note that the complementary 1-dimensional summand are associated to an idempotent, say $\frac{1}{2}(id - s_1)$, which is a morphism that has negative coefficients and thus is only a morphism of W-representations over $\mathbb{Q}$, but is not a morphism in $\mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q} - \mathrm{Mod}^{(\mathbb{Q},\mathbb{N}_0)}$.

Remark 2.6.30. The four morphisms π_1, ι_2, π_2 and ι_1 are build by one of the following principles: Either two basis elements are mapped to the same target or one basis element is mapped to two basis elements. The former is the case for π_1 and π_2, the latter applies to ι_1 and ι_2. In Section 5.5, we will understand this process in categorical term as in the former case adding an isomorphism (compare Lemma 4.6.10) or in the latter case taking the idempotent closure.

Chapter 3

Higher Representation Theory I - Basics

In this chapter, we discuss the basics of higher representation theory. Section 3.1 follows [Rou4] and recalls the definitions of 2-categories, 2-functors, morphism of 2-functors, morphisms of morphisms of 2-functors, as well as the definition of a 2-category of higher representations of a 2-category. The graded case is also covered and monoidal categories are mentioned. Section 3.2 discusses centres of objects of 2-categories. Section 3.3 defines (graded) additive higher representations and discusses (graded) decategorification using the split Grothendieck construction. Further, kernels, cokernels, (thick) annihilators and (thick) ideals are introduced and related to the classical case. Finally, universal properties, the 2-Yoneda lemma, and (quotients of) principal representations are covered.

3.1 2-categories

In this section, we recall the definitions of 2-categories, 2-functors, morphism of 2-functors and morphisms of morphisms of 2-functors. We follow closely the notation in [Rou4], reserving $\mathfrak{A}$, $\mathfrak{D}$,... to denote 2-categories. We also recall the definition of the 2-category of higher representations $\mathfrak{Hom}(\mathfrak{A}, \mathfrak{D})$ of $\mathfrak{A}$ in $\mathfrak{D}$, as well as their additive and graded analogues. Finally, we comment on the close relation between 2-categories and monoidal categories, which we typically denote by $\mathcal{B}$. Once and for all, we fix a sufficiently large universe so that all categorical structures considered in this thesis are (essentially) small and we will not mention this technical requirement again.

We denote by $\mathcal{C}at$ the category of categories and functors. A direct way to climb the categorical ladder is to consider categories enriched in $\mathcal{C}at$ (in order to make sense of this statement, one has to equip $\mathcal{C}at$ with the structure of a monoidal category using direct products), see [Ke] for the notion of enriched categories. Categories enriched in $\mathcal{C}at$ are the same as strict 2-categories. An example of a strict 2-category is $\mathfrak{C}at$, the 2-category of categories, functors and natural transformations.

We will need a more general version of 2-categories, which are often referred to as weak 2-categories or bicategories:

Definition 3.1.1. *A* 2-category $\mathfrak{A}$ *consists of*

- *a set of objects* $\mathfrak{A}_0$
- *categories* $\mathfrak{A}(a_1, a_2)$ *for* $a_1, a_2 \in \mathfrak{A}_0$
- *functors* $\mathfrak{A}(a_1, a_2) \times \mathfrak{A}(a_2, a_3) \to \mathfrak{A}(a_1, a_3)$, $(b_1, b_2) \mapsto b_2 b_1$ *for* $a_1, a_2, a_3 \in \mathfrak{A}_0$
- *an identity object* $I_a \in \mathfrak{A}_a := \mathfrak{A}(a, a)$ *for* $a \in \mathfrak{A}_0$
- *isomorphisms* $\mathrm{can}_{b_3,b_2,b_1} : (b_3 b_2) b_1 \overset{\sim}{\Longrightarrow} b_3(b_2 b_1)$ *in* $\mathfrak{A}(a_1, a_4)$ *natural in* $b_i \in \mathfrak{A}(a_i, a_{i+1})$ *for* $a_1, .., a_4 \in \mathfrak{A}_0$
- *isomorphisms* $\mathrm{can}^r_b : bI_{a_1} \overset{\sim}{\Longrightarrow} b$ *and* $\mathrm{can}^l_b : I_{a_2} b \overset{\sim}{\Longrightarrow} b$ *in* $\mathfrak{A}(a_1, a_2)$ *natural in* $b \in \mathfrak{A}(a_1, a_2)$ *for* $a_1, a_2 \in \mathfrak{A}_0$

such that the pentagon axiom and the unit axiom are satisfied, that is the diagrams

$$\begin{tikzcd}
& \big((b_4b_3)b_2\big)b_1 \arrow[rr, Rightarrow, "\mathrm{can}_{b_4,b_3,b_2}\ b_1"] \arrow[dl, Rightarrow, "\mathrm{can}_{b_4b_3,b_2,b_1}"'] & & \big(b_4(b_3b_2)\big)b_1 \arrow[dr, Rightarrow, "\mathrm{can}_{b_4,b_3b_2,b_1}"] & \\
(b_4b_3)(b_2b_1) \arrow[drr, Rightarrow, "\mathrm{can}_{b_4,b_3,b_2b_1}"'] & & & & b_4\big((b_3b_2)b_1\big) \arrow[dll, Rightarrow, "b_4\ \mathrm{can}_{b_3,b_2,b_1}"] \\
& & b_4\big(b_3(b_2b_1)\big) & &
\end{tikzcd}$$

$$\begin{tikzcd}
(b_2I_a)b_1 \arrow[rr, Rightarrow, "\mathrm{can}_{b_2,I_a,b_1}"] \arrow[dr, Rightarrow, "\mathrm{can}^r_{b_2}\ b_1"'] & & b_2(I_ab_1) \arrow[dl, Rightarrow, "b_2\ \mathrm{can}^l_{b_1}"] \\
& b_2b_1 &
\end{tikzcd}$$

commute.

We call $\mathfrak{A}$ a *strict 2-category* if the above isomorphisms are identity morphisms. The functors in the definition are called *horizontal composition* (functors) and are also known as juxtaposition. The composition within the categories $\mathfrak{A}(a_1, a_2)$ is called *vertical composition*. We denote the former composition without a symbol and the latter using $\circ$.

Notation 3.1.2. For objects a of $\mathfrak{A}$, we will abuse notation and write $a \in \mathfrak{A}$ instead of $a \in \mathfrak{A}_0$. However, whenever we refer to the entire set $\mathfrak{A}_0$ we will keep the index. The objects (resp. morphisms) of the categories $\mathfrak{A}(a_1, a_2)$ are called *1-arrows* (resp. *2-arrows*). We write $b : a_1 \to a_2$ for a 1-arrow between $a_1, a_2 \in \mathfrak{A}$ and we write $c : b_1 \Rightarrow b_2$ for a 2-arrow between 1-arrows $b_1, b_2 : a_1 \to a_2$. Further, we abbreviate $\mathfrak{A}_{a_1 \to a_2}(b_1, b_2) := (\mathfrak{A}(a_1, a_2))\,(b_1, b_2)$.

Remark 3.1.3. Given suitably composable 2-arrows α, β, γ and δ, the axioms of a 2-category imply that $(\delta \circ \gamma)(\beta \circ \alpha) = (\delta\beta) \circ (\gamma\alpha)$. This fact is commonly referred to as *the interchange law*. In particular, the following diagram is well defined and motivates the terms horizontal and vertical composition:

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] \arrow[l, bend right=60] \arrow[l, bend left=60] \arrow[l, phantom, shift right=3, "\Downarrow\gamma"] \arrow[l, phantom, shift left=3, "\Downarrow\delta"] & \bullet \arrow[l] \arrow[l, bend right=60] \arrow[l, bend left=60] \arrow[l, phantom, shift right=3, "\Downarrow\alpha"] \arrow[l, phantom, shift left=3, "\Downarrow\beta"]
\end{tikzcd}$$

Definition 3.1.4. *Given a 2-category $\mathfrak{A}$, we define the* reverse 2*-category* $\mathfrak{A}^{\mathrm{rev}}$ *by setting $\mathfrak{A}_0^{\mathrm{rev}} := \mathfrak{A}_0$ and $\mathfrak{A}^{\mathrm{rev}}(a, a') := \mathfrak{A}(a', a)$, and by reversing the horizontal composition of $\mathfrak{A}$, i.e. $\mathfrak{A}^{\mathrm{rev}}(a_1, a_2) \times \mathfrak{A}^{\mathrm{rev}}(a_2, a_3) \to \mathfrak{A}^{\mathrm{rev}}(a_1, a_3)$ is given on objects by $(b_1, b_2) \mapsto b_2 b_1$. The rest of the structure is inherited from $\mathfrak{A}$.*

Definition 3.1.5. *A* (graded) additive category *is a category, whose morphism spaces are (graded) Abelian groups, composition is (graded) bilinear and finite direct sums exist. In addition, we require that a graded additive category $\mathcal{A}^\bullet$ has an auto-equivalence* $[1] : \mathcal{A} \to \mathcal{A}$ *and natural isomorphisms* $\mathcal{A}^n(a_1, a_2) \cong \mathcal{A}^{n+1}(a_1[1], a_2) \cong \mathcal{A}^{n-1}(a_1, a_2[1])$ *for all $n \in \mathbb{Z}$.*

A (graded) additive functor *is a functor between (graded) additive categories, which is given on morphisms spaces by homomorphism of (graded) Abelian groups. In the graded case it has to commute up to isotransformation with the auto-equivalences* [1] *and this isotransformation has to be compatible with the natural isomorphisms above.*

A (graded) additive category $\mathcal{C}$ is idempotent closed *if for all idempotents $e \in \mathcal{C}(C, C)$ (of degree zero), that is $e^2 = e$, there exists $C' \in \mathrm{Ob}(\mathcal{C})$ and (degree zero) morphisms $C' \xrightarrow{\iota} C \xrightarrow{\pi} C'$ such that $e = \iota \circ \pi$ and $id_{C'} = \pi \circ \iota$.*

Remark 3.1.6. By graded we always mean $\mathbb{Z}$-graded, whereas in [Rou4] graded merely means existence of an auto-equivalence. Using the latter terminology, there is an equivalence between graded additive categories and categories that are enriched in graded Abelian groups and have finite direct sums. [Rou4] provides the argument for the $\mathbb{Z}$-graded case. The $\mathbb{Z}/n\mathbb{Z}$-graded case is similar. From this perspective, it would be more consistent to call our graded categories bi-graded categories with compatible gradings.

Let us recall the standard notion of indecomposability and introduced a somewhat more refined version that will frequently be used in this thesis:

Definition 3.1.7. *An object in an additive category is called* indecomposable *if it is non-trivial and does not allow for a direct sum decomposition into two non-trivial objects. We call an object* strongly indecomposable *if it is indecomposable in the idempotent closure or equivalently if it has exactly two distinct idempotents.*

Notation 3.1.8. We denote by $\mathcal{A}dd$ the category of additive categories and additive functors. Similarly, $\mathcal{A}dd^{\mathrm{idem}}$ denotes the 1-2-full subcategory of $\mathcal{A}dd$ with objects idempotent closed additive categories. Similarly, $\mathcal{A}dd^{\bullet}$ and $\mathcal{A}dd^{\bullet\mathrm{idem}}$ denote the graded counterparts. $\mathfrak{A}dd$ denotes the strict 2-category of additive categories, additive functors and natural transformations. Note that we do not impose additional requirements on the natural transformations. Similarly, $\mathfrak{A}dd^{\mathrm{idem}}$ denotes the 1-2-full sub-2-category of $\mathfrak{A}dd$ with objects idempotent closed additive categories. Again, $\mathfrak{A}dd^{\bullet}$ and $\mathfrak{A}dd^{\bullet\mathrm{idem}}$ denote the graded counterparts.

Definition 3.1.9. *A* (graded) additive 2-category *is a 2-category $\mathfrak{A}$ where the categories $\mathfrak{A}(a_1, a_2)$ are (graded) additive categories and the composition functors are (graded) additive functors. A (graded) additive 2-category is* idempotent closed *if all (graded) additive categories $\mathfrak{A}(a_1, a_2)$ are idempotent closed.*

Example 3.1.10. The 2-category $\mathfrak{A}dd$ (resp. $\mathfrak{A}dd^{\bullet}$) is a (graded) additive 2-category. The 2-category $\mathfrak{A}dd^{\mathrm{idem}}$ (resp. $\mathfrak{A}dd^{\bullet\mathrm{idem}}$) is an idempotent closed (graded) additive 2-category.

Definition 3.1.11. *A* 2-functor $R : \mathfrak{A} \to \mathfrak{D}$ *between* 2*-categories is the data of*

- *a map* $R : \mathfrak{A}_0 \to \mathfrak{D}_0$
- *functors* $R : \mathfrak{A}(a_1, a_2) \to \mathfrak{D}(R(a_1), R(a_2))$ *for all* $a_1, a_2 \in \mathfrak{A}$
- *invertible* 2*-arrows* $\mathrm{can}^R_{b_2,b_1} : R(b_2)R(b_1) \overset{\sim}{\Longrightarrow} R(b_2 b_1)$ *in* $\mathfrak{D}$ *which are natural in* $b_1 \in \mathfrak{A}(a_1, a_2)$ *and* $b_2 \in \mathfrak{A}(a_2, a_3)$
- *invertible* 2*-arrows* $\mathrm{can}^R_a : I_{R(a)} \overset{\sim}{\Longrightarrow} R(I_a)$ *in* $\mathfrak{D}$ *for all* $a \in \mathfrak{A}$

such that the following diagrams commute:

$$\begin{tikzcd}
\big(R(b_3)R(b_2)\big)R(b_1) \arrow[r, Rightarrow, "\mathrm{can}^R_{b_3,b_2}\ R(b_1)"] \arrow[d, Rightarrow, "\mathrm{can}_{R(b_3),R(b_2),R(b_1)}"'] & R(b_3b_2)R(b_1) \arrow[r, Rightarrow, "\mathrm{can}^R_{b_3b_2,b_1}"] & R\big((b_3b_2)b_1\big) \arrow[d, Rightarrow, "R(\mathrm{can}_{b_3,b_2,b_1})"] \\
R(b_3)\big(R(b_2)R(b_1)\big) \arrow[r, Rightarrow, "R(b_3)\ \mathrm{can}^R_{b_2,b_1}"'] & R(b_3)R(b_2b_1) \arrow[r, Rightarrow, "\mathrm{can}^R_{b_3,b_2b_1}"'] & R\big(b_3(b_2b_1)\big)
\end{tikzcd}$$

$$\begin{tikzcd}
R(b)I_{R(a)} \arrow[r, Rightarrow, "R(b)\ \mathrm{can}^R_a"] \arrow[d, Rightarrow, "\mathrm{can}^r_{R(b)}"'] & R(b)R(I_a) \arrow[d, Rightarrow, "\mathrm{can}^R_{b,I_a}"] \\
R(b) & R(bI_a) \arrow[l, Rightarrow, "R(\mathrm{can}^r_b)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
I_{R(a')}R(b) \arrow[r, Rightarrow, "\mathrm{can}^R_{a'}\ R(b)"] \arrow[d, Rightarrow, "\mathrm{can}^l_{R(b)}"'] & R(I_{a'})R(b) \arrow[d, Rightarrow, "\mathrm{can}^R_{I_{a'},b}"] \\
R(b) & R(I_{a'}b) \arrow[l, Rightarrow, "R(\mathrm{can}^l_b)"]
\end{tikzcd}$$

We call a 2-functor *strict*, if all 2-arrows can^R_a and $\mathrm{can}^R_{b_1,b_2}$ are identity 2-arrows. Note that the requirement for $\mathrm{can}^R_{b_1,b_2}$ to be an identity 2-arrow was missing in [Rou4]. Strict 2-functors are called strict pseudo-functors in [Gr].

Definition 3.1.12. *A* (graded) additive 2-functor $R : \mathfrak{A} \to \mathfrak{D}$ *between (graded) additive 2-categories* $\mathfrak{A}$ *and* $\mathfrak{D}$ *is a 2-functor, where the functors* $R : \mathfrak{A}(a_1, a_2) \to \mathfrak{D}(R(a_1), R(a_2))$ *are (graded) additive functors. In the graded case, we require the associated invertible 2-arrows to be of degree zero.*

Notation 3.1.13. We denote by $2-\mathcal{C}at$ the category of 2-categories and 2-functors. Similarly, we denote by $2-\mathcal{A}dd$ (resp. $2-\mathcal{A}dd^\bullet$) the additive category of (graded) additive 2-categories and additive 2-functors.

Remark 3.1.14. The 2-category $2-\mathcal{A}dd^\bullet$ is not a graded additive category, but only comes with a non-trivial auto-equivalence which stabilises the objects.

Definition 3.1.15. *Let* $R, R' : \mathfrak{A} \to \mathfrak{D}$ *be 2-functors. A* morphism of 2-functors $\sigma : R \to R'$ *is the data of*

- 1*-arrows* $\sigma(a) : R(a) \to R'(a)$ *in* $\mathfrak{D}$ *for* $a \in \mathfrak{A}$
- *invertible* 2*-arrows* $\mathrm{can}^\sigma_b : R'(b)\sigma(a_1) \overset{\sim}{\Longrightarrow} \sigma(a_2)R(b)$ *in* $\mathfrak{D}$ *natural in* $b \in \mathfrak{A}(a_1, a_2)$

such that the following diagrams commute:

$$\begin{tikzcd}
\big(R'(b_2)R'(b_1)\big)\sigma(a_1) \arrow[r, Rightarrow, "\mathrm{can}_{R'(b_2),R'(b_1),\sigma(a_1)}"] \arrow[d, Rightarrow, "\mathrm{can}^R_{b_2,b_1}\ \sigma(a_1)"] & R'(b_2)\big(R'(b_1)\sigma(a_1)\big) \arrow[r, Rightarrow, "R'(b_2)\ \mathrm{can}^\sigma_{b_1}"] & R'(b_2)\big(\sigma(a_2)R(b_1)\big) \arrow[d, Rightarrow, "\mathrm{can}^{-1}_{R'(b_2),\sigma(a_2),R(b_1)}"'] \\
R'(b_2b_1)\sigma(a_1) \arrow[d, Rightarrow, "\mathrm{can}^\sigma_{b_2b_1}"] & & \big(R'(b_2)\sigma(a_2)\big)R(b_1) \arrow[d, Rightarrow, "\mathrm{can}^\sigma_{b_2}\ R(b_1)"'] \\
\sigma(a_3)R(b_2b_1) & \sigma(a_3)\big(R(b_2)R(b_1)\big) \arrow[l, Rightarrow, "\sigma(a_3)\ \mathrm{can}^R_{b_2,b_1}"] & \big(\sigma(a_3)R(b_2)\big)R(b_1) \arrow[l, Rightarrow, "\mathrm{can}_{\sigma(a_3),R(b_2),R(b_1)}"]
\end{tikzcd}$$

$$\begin{tikzcd}
I_{R'(a)}\sigma(a) \arrow[r, Rightarrow, "\mathrm{can}^l_{R'(a)}"] \arrow[d, Rightarrow, "\mathrm{can}^{R'}_a\ \sigma(a)"'] & \sigma(a) \arrow[r, Rightarrow, "\left(\mathrm{can}^r_{R(a)}\right)^{-1}"] & \sigma(a)I_{R(a)} \arrow[d, Rightarrow, "\sigma(a)\ \mathrm{can}^R_a"] \\
R'(I_a)\sigma(a) \arrow[rr, Rightarrow, "\mathrm{can}^\sigma_{I_a}"'] & & \sigma(a)R(I_a)
\end{tikzcd}$$

These are called quasi-natural transformations with invertible 2-arrows in [Gr].

Definition 3.1.16. *A* (graded) morphism of (graded) additive 2-functors *is a morphism of the underlying 2-functors. In the graded case, we require the invertible 2-arrows to be of degree zero.*

Remark 3.1.17. In general, we cannot extend the category $2-\mathcal{C}at$ to a 2-category by defining the 2-arrows to be morphisms of 2-functors. The reason is that the composition of morphisms of 2-functors is typically associative and unital only up to an invertible morphism of morphisms as defined below. The correct notion to deal with this detail are 3-categories, however we have no need to go there. The reader might want to note that 2-functors $\mathfrak{A} \to \mathfrak{D}$ and morphism between such 2-functors do form a category, provided that $\mathfrak{D}$ is strict.

Definition 3.1.18. *Let R, $R' : \mathfrak{A} \to \mathfrak{D}$ be 2-functors and σ, $\tilde{\sigma} : R \to R'$ be morphisms of* 2*-functors. A* morphism of morphisms of 2-functors *$\gamma : \sigma \to \tilde{\sigma}$ is the data of* 2*-arrows $\gamma(a) : \sigma(a) \Rightarrow \tilde{\sigma}(a)$ in $\mathfrak{D}$ for all $a \in \mathfrak{A}$ such that the following diagrams commute:*

$$\begin{tikzcd}
R'(b)\sigma(a_1) \arrow[r, Rightarrow, "R'(b)\gamma(a_1)"] \arrow[d, Rightarrow, "\mathrm{can}^\sigma_b"'] & R'(b)\tilde{\sigma}(a_1) \arrow[d, Rightarrow, "\mathrm{can}^{\tilde{\sigma}}_b"] \\
\sigma(a_2)R(b) \arrow[r, Rightarrow, "\gamma(a_2)R'(b)"'] & \tilde{\sigma}(a_2)R(b)
\end{tikzcd}$$

Morphisms of morphisms of 2-functors are called modifications in [Gr].

Definition 3.1.19. *A* (graded) morphism of (graded) morphisms of (graded) additive 2-functors *is simply a morphism of the underlying morphisms of 2-functors.*

Definition 3.1.20. *Let $\mathfrak{A}$ and $\mathfrak{D}$ be 2-categories. We denote by $\mathfrak{Hom}(\mathfrak{A}, \mathfrak{D})$ the* 2*-category of* 2*-functors $\mathfrak{A} \to \mathfrak{D}$, morphisms between these 2-functors and morphisms between these morphisms. Provided that $\mathfrak{A}$ and $\mathfrak{D}$ are (graded) additive 2-categories, we abuse notation and denote by $\mathfrak{Hom}(\mathfrak{A}, \mathfrak{D})$ the 2-category of (graded) additive 2-functors, (graded) morphism of (graded) additive 2-functors and (graded) morphisms of (graded) morphisms of (graded) additive 2-functors.*

If $\mathfrak{D}$ is a strict 2-category, then $\mathfrak{Hom}(\mathfrak{A},\mathfrak{D})$ is strict as well. It is worth reiterating that for additive 2-categories $\mathfrak{A}$ and $\mathfrak{D}$ the 2-category $\mathfrak{Hom}(\mathfrak{A},\mathfrak{D})$ is only a 1-2-full sub-2-category of $\mathfrak{Hom}(\mathrm{For}(\mathfrak{A}),\mathrm{For}(\mathfrak{D}))$ where $\mathrm{For} : 2-\mathcal{A}dd \to 2-\mathcal{C}at$ is the functor that forgets the additive structure. The analogous statement in the graded case yields only a 2-full sub-2-category.

We have intentionally omitted the usage of the symbol $\mathfrak{B}$. The reason is that we will frequently deal with monoidal categories, as defined for example in [Ka]. To make the distinction to 2-categories easier, we reserve the calligraphic letter $\mathcal{B}$ for (additive) monoidal categories. For the horizontal composition and the unit object of $\mathcal{B}$, we will use the standard notation $\otimes$ and I.

Example 3.1.21. Let $\mathfrak{A}$ be a 2-category, and let $a \in \mathfrak{A}$. Then, $\mathfrak{A}_a$ is a monoidal category. Similarly, the analogous restriction of any 2-functors yields a monoidal functor. Conversely, given a monoidal category (resp. a monoidal functor) one obtains an obvious one object 2-category (resp. a 2-functor between one object 2-categories).

Remark 3.1.22. Noteworthy, monoidal transformations do not correspond to morphisms of 2-functors in general. Invertible monoidal transformations correspond to morphisms of 2-functors between one object 2-categories, however there exist morphisms of 2-functors between one object 2-categories that are not invertible monoidal transformations. This distinction is explained in detail in [Lei2, Example 1.5.11] and in the exposition leading up to it.

Remark 3.1.23. We will make use of monoidal structures frequently in this thesis, predominantly in the context of additive higher representations of additive monoidal categories, see Remark 3.3.4, and sometimes when discussing equivalences of monoidal categories. Thus, all monoidal transformations considered will be invertible.

3.2 The centre of an object in a 2-category

In this section, we discuss the centre construction. Usually, centres are introduced as the endotransformations of identity functors. We will provide a more general definition that applies to objects in 2-categories. We will employ this construction in order to construct new higher representations by base change, see Section 4.1.

The centre of a monoid (or ring) A is given by $Z(A) := \{a \in A | ab = ba \ \forall b \in A\}$. If we consider A as a category with one object, we can describe $Z(A)$ as the set of natural transformations of the identity functor. This is a categorical description within $\mathfrak{C}at$. More generally, we make the following definition:

Definition 3.2.1. *Let $\mathfrak{A} \in 2-\mathcal{C}at$ and let $a \in \mathfrak{A}$.* The centre $\mathcal{Z}_{\mathfrak{A}}(a)$ of *a is defined as $\mathcal{Z}_{\mathfrak{A}}(a) := \mathfrak{A}_a(I_a, I_a)$.*

Remark 3.2.2. By the Eckmann-Hilton argument $\mathcal{Z}_{\mathfrak{A}}(a)$ is commutative monoid. If the category $\mathfrak{A}_a$ is additive, then $\mathcal{Z}_{\mathfrak{A}}(a)$ is a ring.

Definition 3.2.3. *Let $\mathcal{C} \in \mathcal{C}at$. The* centre $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ *is defined as $\mathcal{Z}(\mathcal{C}) := \mathcal{Z}_{\mathfrak{C}at}(\mathcal{C})$.*

The following observation is well known, see [Mi, Section 11], and in some sense inverse to Remark 3.2.2:

Proposition 3.2.4. *Let $\mathcal{C} \in \mathcal{A}dd$. Then, $\mathcal{C}$ is a category enriched in $\mathcal{Z}(\mathcal{C}) - \mathrm{Mod}$.*

The following definition is in the spirit of [Mi, Section 11]:

Definition 3.2.5. *Let R be a commutative ring. A R-*linear category *is an additive category $\mathcal{M}$ with a ring homomorphism $R \to \mathcal{Z}(\mathcal{M})$. Given commutative rings R_1 and R_2, a R_1-R_2-*linear category *is a $R_1 \otimes_{\mathbb{Z}} R_2$-linear category.*

Remark 3.2.6. Equivalently, one can define an R-linear category as a category that is enriched in $R - \mathrm{Mod}$ and that has finite biproducts, see [Ke] for the definition of enriched categories. Since will not discuss enriched categories, we prefer the definition given above. Also, note that the terminology R_1-R_2-linear category is in line with the standard naming convention for bimodules.

Proposition 3.2.7. *Let $\mathfrak{A} \in 2-\mathcal{A}dd$ and let $a_1, a_2 \in \mathfrak{A}$. Then, $\mathcal{Z}\left(\mathfrak{A}(a_1, a_2)\right)$ is a $\mathcal{Z}_{\mathfrak{A}}(a_1)$-$\mathcal{Z}_{\mathfrak{A}}(a_2)$-algebra. Thus, $\mathfrak{A}(a_1, a_2)$ is a $\mathcal{Z}_{\mathfrak{A}}(a_1)$-$\mathcal{Z}_{\mathfrak{A}}(a_2)$-linear category.*

Proof. Note that the identity objects I_{a_1} and I_{a_2} induce identity functors on $\mathfrak{A}(a_1, a_2)$. Similarly, the structure of $\mathfrak{A}$ induces ring homomorphisms between the respective endomorphisms rings, i.e. $\mathcal{Z}_{\mathfrak{A}}(a_i) \to \mathcal{Z}(\mathfrak{A}(a_1, a_2))$ for $i \in \{1, 2\}$. Note that $\mathcal{Z}_{\mathfrak{A}}(a_2)$ is commutative and thus agrees with its opposite ring. It remains to show that any element in the images of these ring homomorphisms commute with each other, which follows easily from the interchange law. $\square$

3.3 Additive higher representations

In this section, we define the 2-category of (graded) additive higher representations of a (graded) additive 2-category. For the reader's convenience, we provide the special case of additive higher representations of an additive monoidal category, in which the definition simplifies greatly. Section 3.3.1 discusses decategorification using the split Grothendieck construction in the graded and non-graded case. Section 3.3.2 shows that the 2-category of higher representations admits kernels and cokernels, and shows that cokernels can differ from what might be expected classically. Section 3.3.3 introduces (thick) annihilators and (thick) ideals, makes standard observations, and discusses the connection to the classical case. Section 3.3.4 discusses universal properties of higher representations, the 2-Yoneda lemma, and (quotients of) principal representations as examples.

For the remainder of this section, let $\mathfrak{A}$ (resp. $\mathfrak{A}^\bullet$) be an additive (resp. graded additive) 2-category. Following [Rou4], the 2-category of higher representations of $\mathfrak{A}$ in some 2-category $\mathfrak{D}$ is defined as $\mathfrak{A} - \mathrm{Mod}(\mathfrak{D}) := \mathfrak{H}om(\mathfrak{A}, \mathfrak{D})$, see Definition 3.1.20. For the purpose of this work, we are only interested in (graded) additive higher representations:

Definition 3.3.1. *We denote by* $\mathfrak{A} - \mathrm{Mod} := \mathfrak{H}om(\mathfrak{A}, \mathfrak{A}dd)$ *the strict 2-category of* additive higher representations of $\mathfrak{A}$. *Further* $\mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}} := \mathfrak{H}om(\mathfrak{A}, \mathfrak{A}dd^{\mathrm{idem}})$ *denotes the 2-category of* idempotent closed additive higher representations of $\mathfrak{A}$. *Similarly,* $\mathfrak{A}^\bullet - \mathrm{Mod} := \mathfrak{H}om(\mathfrak{A}^\bullet, \mathfrak{A}dd^\bullet)$ *(resp.* $\mathfrak{A}^\bullet - \mathrm{Mod}^{\mathrm{idem}} := \mathfrak{H}om(\mathfrak{A}^\bullet, \mathfrak{A}dd^{\bullet\mathrm{idem}})$*) denotes the (idempotent closed) graded additive 2-category of* (idempotent closed) graded additive higher representations *of* $\mathfrak{A}^\bullet$.

We regard $\mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}}$ (resp. $\mathfrak{A}^\bullet - \mathrm{Mod}^{\mathrm{idem}}$) as a 1-2-full sub-2-category of $\mathfrak{A} - \mathrm{Mod}$ (resp. $\mathfrak{A}^\bullet - \mathrm{Mod}$) via the inclusion 2-functor $\mathfrak{A}dd^{\mathrm{idem}} \to \mathfrak{A}dd$ (resp. $\mathfrak{A}dd^{\bullet\mathrm{idem}} \to \mathfrak{A}dd^\bullet$).

Notation 3.3.2. We call $R \in \mathfrak{A} - \mathrm{Mod}$ an $\mathfrak{A}$-*module*. For R, $R' \in \mathfrak{A} - \mathrm{Mod}$, we abbreviate $\mathcal{H}om_{\mathfrak{A}}(R, R') := \mathfrak{A} - \mathrm{Mod}(R, R')$ and call its objects $\mathfrak{A}$-*functors* and its morphisms $\mathfrak{A}$-*transformations*. For two $\mathfrak{A}$-functors σ, $\sigma' : R \to R'$, we denote by $\mathcal{H}om_{R\to R'}(\sigma, \sigma') = \mathcal{H}om^{\mathfrak{A}}_{R\to R'}(\sigma, \sigma')$ the Abelian group of $\mathfrak{A}$-transformations $\sigma \Rightarrow \sigma'$.

Let us give a first example of higher representations, that has a fairly nice universal property, a notion to be defined in Definition 3.3.51:

Definition 3.3.3. *For all $a \in \mathfrak{A}$,* the principal representations *are defined as* $\mathcal{P}_a^{\mathfrak{A}} := \mathfrak{A}(a,-) \in \mathfrak{A}-\mathrm{Mod}(\mathfrak{D}) = \mathfrak{Hom}(\mathfrak{A}, \mathfrak{D})$. *The action functor* $\mathfrak{A}(a_1, a_2) \to \mathfrak{D}(\mathcal{P}_a^{\mathfrak{A}}(a_1), \mathcal{P}_a^{\mathfrak{A}}(a_2))$ *is induced from the functor* $\mathfrak{A}(a, a_1) \times \mathfrak{A}(a_1, a_2) \to \mathfrak{A}(a, a_2)$ *which is part of the datum of* $\mathfrak{A}$*.*

When clear from the context, we abuse notation and write $\mathcal{P}_a = \mathcal{P}_a^{\mathfrak{A}}$.

For an additive monoidal category $\mathcal{B}$, the definition of $\mathcal{B} - \mathrm{Mod}$ can be expressed in a simpler fashion. We present this simplified version below, and introduce some notational abuses at the same time which will be used throughout.

Remark 3.3.4. A *$\mathcal{B}$-module* is a pair $\mathcal{M} = (\mathcal{M}, \rho)$, where $\mathcal{M}$ is an additive category and $\rho : \mathcal{B} \to \mathfrak{A}dd(\mathcal{M}, \mathcal{M})$ is an additive monoidal functor. We will refer to ρ as the *structure functor*. At times, we avoid making ρ explicit, and write $B. = \rho(B)$ for $B \in \mathcal{B}$ and $b. = \rho(b)$ for $b \in \mathcal{B}(B_1, B_2)$.

A *$\mathcal{B}$-functor* is a pair $F = (F, \{\mathrm{can}_{F,B}\}_{B\in\mathcal{B}}) : \mathcal{M} \to \mathcal{N}$, where $F : \mathcal{M} \to \mathcal{N}$ is an additive functor and $\mathrm{can}_{F,B} : (B.)F \overset{\sim}{\Longrightarrow} F(B.)$ is an isotransformation for all $B \in \mathcal{B}$, such that $\{\mathrm{can}_{F,B}\}_{B\in\mathcal{B}}$ is natural in B and is compatible the structure maps of the structure functors.

A *$\mathcal{B}$-transformation* $\tau : F \Rightarrow G$ is a transformation $\tau : F \Rightarrow G$ of the underlying additive functors, such that $\mathrm{can}_{G,B} \circ ((B.)\tau) = ((B.)\tau) \circ \mathrm{can}_{F,B}$ for all $B \in \mathcal{B}$.

Example 3.3.5. For the principal representation $\mathcal{B} = (\mathcal{B}, \rho)$, we have $\rho(B_1)(B_2) = B_1B_2$ for all $B_1, B_2 \in \mathcal{B}$, where we consider B_1 as an object in the acting monoidal category and B_2 as an object of the principal representation.

3.3.1 Decategorification

In this section, we discuss the passage to the classical case using the split Grothendieck constructions for the acting (graded) additive 2-category, the (graded) additive higher representations, as well as for their morphisms. This process is commonly referred to as decategorification. Since this is fairly standard, we omit all proofs.

Let $\mathfrak{A}$ be an additive 2-category. Recall that we always assume that $\mathfrak{A}$ is essentially small. Thus, there exists a set $\langle\mathfrak{A}\rangle$ of 1-arrows in $\mathfrak{A}$ such that every 1-arrow in $\mathfrak{A}$ is isomorphic to a unique element of $\langle\mathfrak{A}\rangle$. This necessary technical assumption will be suppressed in the remainder of this thesis. For a 1-arrow $b : a_1 \to a_2$ in $\mathfrak{A}$, we denote $\langle b\rangle$ the corresponding element in $\langle\mathfrak{A}\rangle$.

Notation 3.3.6. We denote by $W(\mathfrak{A})$ the set of indecomposable 1-arrows of $\mathfrak{A}$ up to isomorphism. We often fix a choice of an embedding $W(\mathfrak{A}) \hookrightarrow \langle\mathfrak{A}\rangle$ and denote the indecomposable 1-arrow associated to $x \in W(\mathfrak{A})$ by B_x.

Definition 2.3.1 can now be generalised. For simplicity, we provide this generalisation for the base ring only.

Definition 3.3.7. *The* left order $\leq_L$ *of* $W(\mathfrak{A})$ *is defined as* $x \leq_L y$ *for all* x, $y \in W(\mathfrak{A})$ *if and only if there exists a 1-arrow* b *in* $\mathfrak{A}$ *and a split injection from* B_x *to* bB_y*. The* 2-sided order $\leq_{LR}$ *of* $W(\mathfrak{A})$ *is defined as* $x \leq_{LR} y$ *for all* x, $y \in W(\mathfrak{A})$ *if and only there exist 1-arrows* b_1, b_2 *in* $\mathfrak{A}$ *such that there exists a split injection from* B_x *to* $b_1B_yb_2$*. The equivalence relation induced from* $\leq_L$ *(resp.* $\leq_{LR}$*) is denoted by* $\sim_L$ *(resp.* $\sim_{LR}$*), and the associated equivalence classes are called the* left cells *(resp.* 2-sided cells*) of* $\mathfrak{A}$ *in* $W(\mathfrak{A})$.

Remark 3.3.8. Note that we do not require that $\mathfrak{A}$ is idempotent closed. Thus for a given $x \in W(\mathfrak{A})$, B_x might not be strongly indecomposable. Therefore, we decided not to define $x \leq_L y$ by requiring that there exists a 1-arrow b in $\mathfrak{A}$ such that B_x is a direct summand of bB_y.

Definition/Proposition 3.3.9. *The* split Grothendieck group $[\mathfrak{A}]$ *of* $\mathfrak{A}$ *is the quotient of the free Abelian group* $\mathbb{Z}\langle\mathfrak{A}\rangle$ *by the relation* $\langle b_1\rangle + \langle b_2\rangle = \langle b_1 \oplus b_2\rangle$ *for all 1-arrows* $b_1, b_2 : a_1 \to a_2$*. We denote by* $[b]$ *the image of* $\langle b\rangle$ *in* $[\mathfrak{A}]$*. The Abelian group* $[\mathfrak{A}]$ *becomes a ring, the* split Grothendieck ring*, by setting*

$$[b_2][b_1] := \begin{cases} [b_2b_1] & \text{if } a_2 = a_3 \\ 0 & \text{otherwise} \end{cases}$$

for all 1-arrows $b_1 : a_1 \to a_2$ *and* $b_2 : a_3 \to a_4$ *in* $\mathfrak{A}$.

Remark 3.3.10. We have an obvious ring homomorphism $\mathbb{Z}[W(\mathfrak{A})] \to [\mathfrak{A}]$.

Similarly, let $R \in \mathfrak{A}-\mathrm{Mod}$ and let $\langle R\rangle$ be the set of representatives of isomorphism classes of objects in $R(a)$ for all $a \in \mathfrak{A}$.

Definition/Proposition 3.3.11. *The* split Grothendieck group $[R]$ *of* $R \in \mathfrak{A}-\mathrm{Mod}$ *is the quotient of the free Abelian group* $\mathbb{Z}\langle R\rangle$ *by the relation* $\langle r_1\rangle + \langle r_2\rangle = \langle r_1 \oplus r_2\rangle$ *for all* $a \in \mathfrak{A}$ *and all objects* $r_1, r_2 \in R(a)$*. We denote by* $[r]$ *the image of* $\langle r\rangle$ *in* $[R]$*. The Abelian group* $[R]$ *becomes an* $[\mathfrak{A}]$*-module, the* split Grothendieck module*, by setting*

$$[b].[r] := \begin{cases} [R(b)(r)] & \text{if } a_1 = a_3 \\ 0 & \text{otherwise} \end{cases}$$

for all 1-arrows $b : a_1 \to a_2$ *in* $\mathfrak{A}$ *and all objects* $r \in R(a_3)$.

Example 3.3.12. For an additive 2-category $\mathfrak{A}$, $[\mathfrak{A}]$ considered as a left $[\mathfrak{A}]$-module is isomorphic to $[\bigoplus_{a\in\mathfrak{A}}\mathcal{P}_a]$, see Definition 3.3.3 and Remark 3.3.58.

Next, we discuss the analogue notions for a graded additive 2-category $\mathfrak{A}^\bullet$. We have a set $\langle\mathfrak{A}^\bullet\rangle$ of representatives of degree zero isomorphism classes of 1-arrows in $\mathfrak{A}$, that is every 1-arrow in $\mathfrak{A}$ is isomorphic via two degree zero 2-morphisms to a unique element in $\langle\mathfrak{A}^\bullet\rangle$. For a 1-arrow $b : a_1 \to a_2$ in $\mathfrak{A}$, we denote $\langle b\rangle$ the corresponding element in $\langle\mathfrak{A}^\bullet\rangle$.

Definition/Proposition 3.3.13. *The* graded split Grothendieck group $[\mathfrak{A}^\bullet]$ *of* $\mathfrak{A}^\bullet$ *is the quotient of the* $\mathbb{Z}[v, v^{-1}]$*-linear free Abelian group* $\mathbb{Z}[v, v^{-1}]\langle\mathfrak{A}^\bullet\rangle$ *by the relations* $\langle b_1\rangle + \langle b_2\rangle = \langle b_1 \oplus b_2\rangle$ *and* $\langle b_1[1]\rangle = v\langle b_1\rangle$ *for all 1-arrows* $b_1, b_2 : a_1 \to a_2$*. We denote by* $[b]$ *the image of* $\langle b\rangle \in \langle\mathfrak{A}^\bullet\rangle$ *in* $[\mathfrak{A}^\bullet]$*. The* $\mathbb{Z}[v, v^{-1}]$*-linear Abelian group* $[\mathfrak{A}^\bullet]$ *becomes a* $\mathbb{Z}[v, v^{-1}]$*-algebra, the* graded split Grothendieck ring*, by setting*

$$[b_2][b_1] := \begin{cases} [b_2b_1] & \text{if } a_2 = a_3 \\ 0 & \text{otherwise} \end{cases}$$

for all 1-arrows $b_1 : a_1 \to a_2$ *and* $b_2 : a_3 \to a_4$ *in* $\mathfrak{A}^\bullet$*.*

Remark 3.3.14. We have an obvious ring homomorphism $\mathbb{Z}[v, v^{-1}][W(\mathfrak{A}^\bullet)] \to [\mathfrak{A}^\bullet]$.

Similarly, let $R^\bullet \in \mathfrak{A}^\bullet - \mathrm{Mod}$ and let $\langle R^\bullet\rangle$ be the set of representatives of degree zero isomorphism classes of objects in $R^\bullet(a)$ for all $a \in \mathfrak{A}$.

Definition/Proposition 3.3.15. *The* split Grothendieck group $[R^\bullet]$ *of* $R^\bullet \in \mathfrak{A}^\bullet -$ Mod *is the quotient of* $\mathbb{Z}[v, v^{-1}]$*-linear free Abelian group* $\mathbb{Z}\langle R\rangle$ *by the relation* $\langle r_1\rangle + \langle r_2\rangle = \langle r_1 \oplus r_2\rangle$ *and* $\langle r_1[1]\rangle = v\langle r_1\rangle$ *for all* $a \in \mathfrak{A}$ *and* $r_1, r_2 \in R(a)$*. We denote by* $[r]$ *the image of* $\langle r\rangle$ *in* $[R^\bullet]$*. The* $\mathbb{Z}[v, v^{-1}]$*-linear Abelian group* $[R^\bullet]$ *becomes an* $[\mathfrak{A}^\bullet]$*-module, the* split Grothendieck module*, by setting*

$$[b].[r] := \begin{cases} [R^\bullet(b)(r)] & \text{if } a_1 = a_3 \\ 0 & \text{otherwise} \end{cases}$$

for all 1-arrows $b : a_1 \to a_2$ *in* $\mathfrak{A}^\bullet$ *and all objects* $r \in R^\bullet(a_3)$*.*

In order to to provide the link to the classical case of Section 2.1, we need the following definition:

Definition 3.3.16. *A (graded) additive category has the* (graded) Krull-Schmidt property*, if every object has an up to (degree zero) isomorphism unique decomposition into indecomposable objects. In the graded case for all objects* a*, we require* *in addition that there exists no degree zero isomorphism between a and $a[n]$ for all $n \in \mathbb{Z} \setminus \{0\}$.*

A (graded) additive 2-category $\mathfrak{A}$ has the (graded) Krull-Schmidt property if $\mathfrak{A}(a_1, a_2)$ has the (graded) Krull-Schmidt property for all a_1, $a_2 \in \mathfrak{A}$, if the unit object I_a is indecomposable for all $a \in \mathfrak{A}$. An $\mathfrak{A}$-module R has the (graded) Krull-Schmidt property if $R(a)$ has the (graded) Krull-Schmidt property for all $a \in \mathfrak{A}$.

Remark 3.3.17. The (graded) Krull-Schmidt property implies that the ring homomorphism of Remark 3.3.10 (resp. Remark 3.3.14) is an isomorphism.

Remark 3.3.18. Krull-Schmidt categories are categories whose objects can be decomposed into finite direct sums of indecomposable objects that have local endomorphism rings. The properties of Krull-Schmidt categories imply the Krull-Remak-Schmidt theorem and in particular the uniqueness of decompositions into indecomposable objects up to a permuting isomorphism, see [Kr, Theorem 4.2]. Motivated by the example of Soergel bimodules in which indecomposable objects do not have local endomorphism rings (the endomorphism ring of the unit object for example is a polynomial ring), we do not discuss Krull-Schmidt categories. Note however that Soergel bimodules satisfy the Krull-Schmidt property.

Notation 3.3.19. Given a (graded) additive 2-category $\mathfrak{A}$ that satisfies the Krull-Schmidt property, we denote by $\mathfrak{A}-\mathrm{Mod}_{KS}$ the 1-2-full sub-2-category of $\mathfrak{A}-\mathrm{Mod}$ with objects 2-representations $R : \mathfrak{A} \to \mathfrak{A}dd$ (resp. $R : \mathfrak{A} \to \mathfrak{A}dd^{\bullet}$) that have the (graded) Krull-Schmidt property. The analogous 1-2-full sub-2-category of $\mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}}$ is denoted by $\mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}}_{KS}$.

We fix a choice of isomorphism classes of indecomposable objects and assume that the unit objects I_a for all $a \in \mathfrak{A}$ are included in this choice. In the graded case, we speak of isomorphism classes of indecomposable objects up to shifts. The following statement is now obvious:

Proposition 3.3.20. *Let $\mathfrak{A}$ be an additive 2-category that satisfies the Krull-Schmidt property. Then, $[\mathfrak{A}]$ is a $(\mathbb{Z}, \mathbb{N}_0)$-ring with basis given by the isomorphism classes of indecomposable objects. Provided that $\mathfrak{A}_0$ is finite, the unit is given by $\sum_{a\in\mathfrak{A}}[I_a]$. Given $R \in \mathfrak{A}-\mathrm{Mod}_{KS}$, then $[R] \in [\mathfrak{A}]-\mathrm{Mod}^{(\mathbb{Z},\mathbb{N}_0)}$. Given R, $R' \in \mathfrak{A}-\mathrm{Mod}_{KS}$ and an $\mathfrak{A}$-functor $F : R \to R'$, then we have a morphism $[F] : [R] \to [R']$ in $[\mathfrak{A}] - \mathrm{Mod}^{(\mathbb{Z},\mathbb{N}_0)}$.*

Analogously, let $\mathfrak{A}^{\bullet}$ be a graded additive 2-category that satisfies the Krull-Schmidt property. Then, $[\mathfrak{A}^{\bullet}]$ is a $(\mathbb{Z}[v, v^{-1}], \mathbb{N}_0[v, v^{-1}])$-ring with a basis given by the isomorphism classes of indecomposable objects up to shift. If $\mathfrak{A}^{\bullet}_0$ is finite, the unit is *given by* $\sum_{a \in \mathfrak{A}^\bullet}[I_a]$. *Furthermore,* $F^\bullet \in \mathfrak{A}^\bullet - \mathrm{Mod}_{KS}(R^\bullet, R^{\bullet\prime})$ *yields a morphism* $[F^\bullet] : [R^\bullet] \to [R^{\bullet\prime}]$ *in* $[\mathfrak{A}^\bullet] - \mathrm{Mod}^{(\mathbb{Z}[v,v^{-1}], \mathbb{N}_0[v,v^{-1}])}$.

3.3.2 Structure of the 2-category of higher representations

In this section, we show that the 2-category of higher representations admits kernels and cokernels in the same fashion as in the classical case. For non-idempotent closed higher representations, we will show that cokernels can differ from what might be expected classically. For simplicity, we work with an additive monoidal category $\mathcal{B}$.

We begin by recalling a definition given in [Rou4]:

Definition 3.3.21. *Let* $b : a \to a'$ *be a 1-arrow in an additive 2-category* $\mathfrak{A}$. *A* kernel of b *is an object* $\ker b \in \mathfrak{A}$ *together with a 1-arrow* $\ker b \to a$ *such that for all* $a'' \in \mathfrak{A}$ *the induced functor* $\mathfrak{A}(a'', \ker b) \to \mathfrak{A}(a'', a)$ *is fully faithful with essential image given by those 1-arrows* $b' : a'' \to a$ *for which* $bb' = 0$. *Similarly, a* cokernel of b *is an kernel of* b *considered as a 1-arrow* $a' \to a$ *in* $\mathfrak{A}^{\mathrm{rev}}$.

Remark 3.3.22. As stated in [Rou4] if kernels and cokernels exist, they are unique up to equivalence unique up to unique isomorphism.

Definition 3.3.23. *Let* $F : M_1 \to M_2$ *be a 1-arrow in a 2-category that has kernels and cokernels. The* image of F *is defined as* $\operatorname{im} F := \operatorname{coker}(\ker F \to M_1)$. *The* coimage of F *is defined as* $\operatorname{coim} F := \ker(M_2 \to \operatorname{coker} F)$.

Remark 3.3.24. We have a canonical 1-arrow $\operatorname{im} F \to \operatorname{coim} F$. To see this apply first the properties of the cokernel and then the properties of the kernel.

Proposition 3.3.25. *The additive 2-category* $\mathcal{B} - \mathrm{Mod}$ *admits kernels and cokernels. Given a* $\mathcal{B}$*-functor* F*, the canonical* $\mathcal{B}$*-functor* $\operatorname{im} F \to \operatorname{coim} F$ *is not an equivalence in general.*

Lemma 3.3.26. *Let* $F \in \mathcal{H}om_{\mathcal{B}}(\mathcal{M}, \mathcal{N})$. *Then,* $\ker F$ *exists and the underlying additive category is given by the full subcategory of* $\mathcal{M}$ *with objects all* $M \in \mathcal{M}$ *such that* $F(M) = 0$. *Similarly,* $\operatorname{coker} F$ *exist and the underlying additive category is given by the additive quotient of* $\mathcal{N}$ *by the family* $F(m)$ *indexed by all morphisms* m *of* $\mathcal{M}$.

Proof. Since F is a $\mathcal{B}$-functor, the set of objects M in $\mathcal{M}$ with $F(M) = 0$ is stable under the $\mathcal{B}$-action. Thus, $\ker F$ as defined in the statement indeed defines a sub-$\mathcal{B}$-module of $\mathcal{M}$. Similarly, the set of morphisms in $\mathcal{N}$ of the form $F(m)$ for some morphisms m of $\mathcal{M}$ is also stable under the $\mathcal{B}$-action. Thus, the structure functor of $\mathcal{N}$ descents along the quotient functor $\mathcal{N} \to \operatorname{coker} F$, equipping $\operatorname{coker} F$ with the structure of a $\mathcal{B}$-module such that the quotient functors is a $\mathcal{B}$-functor. Now, it is obvious that the properties of kernels and cokernels are satisfied. In fact, the functors that are required to be an equivalence in Definition 3.3.21 are isomorphisms of categories. $\square$

Given a $\mathfrak{A}$-functor $\sigma : R \to R'$ in $\mathfrak{A} - \mathrm{Mod}_{KS}$, it follows from the universal properties that we have morphisms $[\ker \sigma] \to \ker[\sigma]$ and $\operatorname{coker}[\sigma] \to [\operatorname{coker} \sigma]$.

Proposition 3.3.27. *We have an isomorphisms* $[\ker \sigma] \xrightarrow{\sim} \ker[\sigma]$ *and provided that* R' *is idempotent closed* $\operatorname{coker}[\sigma] \to [\operatorname{coker} \sigma]$ *is an isomorphism, as well.*

Proof. The statement for kernels follows directly from the explicit description of kernels for both higher representations and the classical analogue. For cokernels, one has to note that an object becomes zero in an additive quotient if and only if its identity morphism factors through one of the morphisms by which the quotient is taken. In our case, an object that becomes zero in the quotient is a summand of an object of the form $\sigma(a)(r)$ for all $a \in \mathfrak{A}$ and $r \in R(a)$. $\square$

In general, neither $\operatorname{coker}[\sigma] \to [\operatorname{coker} \sigma]$ nor $\operatorname{im}(\sigma) \to \operatorname{coim}(\sigma)$ is an isomorphism. To give a concrete example, we use Soergel bimodules of type A_1, see Section 5.1 for background on Soergel bimodules. Note that this example applies only in the non-graded setting (as well as the $\mathbb{Z}/2\mathbb{Z}$-graded setting).

Example 3.3.28. Let $\mathcal{M} \in \mathcal{S}oe - \mathrm{Mod}$ denote either $\mathcal{S}oe \otimes_R R/\langle 1 - 2\alpha_s \rangle$ or $\mathcal{S}oe \otimes_R R[\alpha_s^{-1}]$. We have obvious $\mathcal{S}oe$-functors $\iota : \mathcal{S}oe \to \mathcal{M}$ as well as $K : \mathcal{M} \to \mathcal{M}^{\mathrm{idem}}$. It is easy to see that $[\mathcal{S}oe] = [\mathcal{M}]$ and thus that $[\iota] = id$. On the other hand $[K]$ is the morphism provided in example 2.1.8 (after specialising v^2 to 1). Thus $\ker[K] = \operatorname{coker}[K] = 0$. Further let $F := - \otimes B_s : \mathcal{S}oe \to \mathcal{S}oe$. Clearly, $[F]$ is the morphism given in Example 2.2.6, which also implies that the canonical $\mathcal{S}oe$-functor $\operatorname{im} F \to \operatorname{coim} F$ is not an equivalence. We have $[\iota F] = [\iota] \circ [F] = [F]$. Therefore, $\operatorname{coim}[\iota F]$ is a proper submodule of $[\mathcal{S}oe]$ and $\operatorname{coker}[\iota F]$ is non-trivial, see Example 2.2.6. However, $\operatorname{coker}(\iota F) = \operatorname{coker}(K\iota F) = 0$ and thus $[\operatorname{coim}(\iota F)] = [\mathcal{S}oe] \neq \operatorname{im}[\iota F]$ as well as $0 = [\operatorname{coker}(\iota F)] \neq \operatorname{coker}[\iota F]$.

3.3.3 Annihilators and ideals

In this section, we introduce annihilators and ideals for additive higher representations, and make standard observations. Further for thick annihilators and thick ideals, we discuss the connection to the classical case.

Definition 3.3.29. *Let $\mathfrak{A}$ be an additive 2-category. An ideal $\mathfrak{J} \trianglelefteq \mathfrak{A}$ consist of a family of Abelian subgroups $\mathfrak{J}(b_1, b_2) \leq \mathfrak{A}_{a_1 \to a_2}(b_1, b_2)$ indexed by 1-arrows $b_1, b_2 \in \mathfrak{A}(a_1, a_2)$ where $a_1, a_2 \in \mathfrak{A}$ varies. Furthermore, $\mathfrak{J}$ shall be closed under horizontal and vertical composition with arbitrary 2-arrows in $\mathfrak{A}$. An ideal $\mathfrak{J} \trianglelefteq \mathfrak{A}$ is called* thick*, if it is generated by identity 2-arrows.*

Remark 3.3.30. One might call an ideal a two-sided ideal, though we will not use this terminology. An ideal is thick if and only if every 2-arrow contained in the ideal factors over an identity 2-arrow that is also contained in this ideal. In formulas, $\mathfrak{J}$ is thick if and only if for all $c \in \mathfrak{J}(b_1, b_2)$ there exists a 1-arrow b_3 and 2-arrows $c_1 : b_1 \Rightarrow b_3$, $c_2 : b_3 \Rightarrow b_2$ such that $id_{b_3} \in \mathfrak{J}(b_3, b_3)$ and $c = c_2 \circ c_1$. Note as well that if $b_4 \leq b_3$ and $id_{b_3} \in \mathfrak{J}(b_3, b_3)$, than $id_{b_4} \in \mathfrak{J}(b_4, b_4)$.

The following definition is natural and will be used at various occasions:

Definition 3.3.31. *We call a functor between categories a* quotient functor *if it is full and essentially surjective. We call a 2-functor $F : \mathfrak{A} \to \mathfrak{D}$ a* quotient functor*, if F is locally a quotient functor, and we call $\mathfrak{D}$* a quotient of *$\mathfrak{A}$. Further, we call an $\mathfrak{A}$-functor $\sigma : R \to R'$ a* quotient $\mathfrak{A}$-functor *if $\sigma(a) : R(a) \to R'(a)$ is a quotient functor for all $a \in \mathfrak{A}$, and we call R' an* $\mathfrak{A}$-quotient of R.

We have the following standard observation:

Proposition 3.3.32. *Given an ideal $\mathfrak{J} \trianglelefteq \mathfrak{A}$, one can form the quotient 2-category $\mathfrak{A}/\mathfrak{J}$ and we have a quotient 2-functor $\pi_{\mathfrak{J}} : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$. On the other hand given an additive 2-functor $F : \mathfrak{A} \to \mathfrak{D}$, the family $\mathfrak{J}(F)$ of all 2-arrows of $\mathfrak{A}$ that are send to zero under F is an ideal $\mathfrak{J}(F) \trianglelefteq \mathfrak{A}$ and F factors over $\pi_{\mathfrak{J}(F)}$. Further if F is a quotient 2-functor, then the factorisation over $\pi_{\mathfrak{J}(F)}$ is an equivalence. Furthermore given an ideal $\mathfrak{J} \trianglelefteq \mathfrak{A}$, we have $\mathfrak{J}(\pi_{\mathfrak{J}}) = \mathfrak{J}$.*

Definition 3.3.33. *The* annihilator of a 2-representation $R : \mathfrak{A} \to \mathfrak{A}dd$ *is defined as* $\mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R) := \mathfrak{J}(R) \trianglelefteq \mathfrak{A}$.

Remark 3.3.34. Note that any 2-representation $R : \mathfrak{A} \to \mathfrak{A}dd$ of a 2-category $\mathfrak{A}$ factors over the quotient 2-functor $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R)$. Thus, one can say that R arises as the restriction of a 2-representation of $\mathfrak{A}/\mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R)$. We will make this precise in Section 4.4 which will enable us to write that $R = \mathrm{Res}_\pi(\mathrm{Ind}_\pi(R))$. Further for all $R' \in \mathfrak{A} - \mathrm{Mod}$, we will be able to deduce that $R'' := \mathrm{Res}_\pi(\mathrm{Ind}_\pi(R'))$ is an $\mathfrak{A}$-quotient of R', that $R'' = \mathrm{Res}_\pi(\mathrm{Ind}_\pi(R''))$ and that $\mathfrak{J}(\pi)$ is a subideal of $\mathfrak{A}\mathrm{nn}(R'')$. The cofiltration of $\mathfrak{A} - \mathrm{Mod}$ that will be introduced in Section 4.9.1 can be regarded as a further refinement to these observations.

We have the following generalisation of Lemma 2.4.6. Note that unlike in the classical case where we required a trivial kernel, we have to be more precise now and require a faithfulness condition:

Lemma 3.3.35. *Let $\sigma : R \to R'$ be an $\mathfrak{A}$-functor. If* $\mathrm{coker}(\sigma) = 0$*, then* $\mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R) \subset \mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R')$*. If σ is locally faithful, then* $\mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R) \supset \mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R')$*.*

Proof. For each $c \in \mathfrak{A}_{a_1 \to a_2}(b_1, b_2)$, the canonical 2-arrow $R'(c)\sigma(a_1) \overset{\sim}{\Longrightarrow} \sigma(a_2)R(c)$ is an isotransformation. Thus if $c \in \mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R)$, then $R(c) = 0$ and thus $R'(c)\sigma(a_1)$, *i.e.* $R'(c) = 0$ on all objects in the image of σ on thus also on the idempotent closure. Hence, $\mathrm{coker}(\sigma) = 0$ implies $\mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R) \subset \mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R')$. On the other hand if $c \in \mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R')$, $R'(c) = 0$ and thus $\sigma(a_2)R(c) = 0$, *i.e.* $R(c)$ is made up of morphisms that are send to zero under $\sigma(a_2)$. Hence if σ is locally faithful, it follows that $R(c)$ is made up of morphisms that are zero, *i.e.* $R(c) = 0$, and this implies $\mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R) \supset \mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(R')$. □

Remark 3.3.36. In [MaMi1], coideals are introduced as the subset of 1-arrows in $\mathfrak{A}$ which are closed under the left and right action of $\mathfrak{A}$ as well as under idempotents. These subsets correspond to our thick annihilators.

Definition 3.3.37. *An* ideal $\mathcal{I}$ in a 2-representation R*, denoted $\mathcal{I} \trianglelefteq R$, consists of a family of subobjects $\mathcal{I}(a)(r_1, r_2)$ of $R(a)(r_1, r_2)$ for all $a \in \mathfrak{A}$ and $r_1, r_2 \in R(a)$. Furthermore, $\mathcal{I}$ shall be closed under composition and application of the functors $R(b) : R(a) \to R(a')$ for all 1-arrows $b : a \to a'$ in $\mathfrak{A}$. An ideal $\mathcal{I} \trianglelefteq R$ is called* thick*, if it is generated by identity 2-arrows.*

Let us now give an example in $\mathcal{V}ect_{\mathbb{C}} - \mathrm{Mod}$, which induces an identity morphism on the level of the split Grothendieck groups, and yet provides insights on the non-triviality of the categorical structures we have just defined:

Example 3.3.38. Consider two categories, constructed as additive closures from $\mathbb{C}$-linear quivers as follows:

$$\mathcal{L}in_{\mathbb{C}}\left(\cdot_1 \longrightarrow \cdot_2 \right) \text{ and } \mathcal{L}in_{\mathbb{C}}\left(\cdot_3 \qquad \cdot_4 \right).$$

The objects in both categories can be described as pairs $(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0$ and can be thought of as $\mathbb{C}^n \oplus \mathbb{C}^m$. The morphism spaces from (n, m) to (k, l) are of complex dimension $n * k + n * l + m * l$ in the first and $n * k + m * l$ in the second case. Both categories become $\mathcal{V}ect_{\mathbb{C}}$-modules in an obvious way using the tensor product $\otimes_{\mathbb{C}}$ of $\mathcal{V}ect_{\mathbb{C}}$. The second module thus constructed is equivalent as $\mathcal{V}ect_{\mathbb{C}}$-module to the direct sum of the principal $\mathcal{V}ect_{\mathbb{C}}$-module $\mathcal{V}ect_{\mathbb{C}}$, see Definition 3.3.3, with itself. Let F be the $\mathcal{V}ect_{\mathbb{C}}$-functor from the first to the second $\mathcal{V}ect_{\mathbb{C}}$-module determined by $F(\cdot_1) = \cdot_3$ and $F(\cdot_2) = \cdot_4$. Clearly, F has to send all morphisms $\cdot_1 \to \cdot_2$ to zero. We have $\ker F = 0$ and $\operatorname{coker} F = 0$, and thus $\operatorname{im} F \neq \operatorname{coim} F$. The ideal $\mathcal{I}(F)$ contained in the first category, see Proposition 3.3.39, consists of all morphisms $\cdot_1 \to \cdot_2$, the zero morphisms of $\cdot_1$ and $\cdot_2$, the zero morphism $\cdot_2$ to $\cdot_1$, as well as all direct sums of these morphisms. On the other hand, the annihilators of both modules, see Definition 3.3.33, (as well as the annihilators of object to be defined in Definition 3.3.44) are trivial. We will reformulate this example in an equivalent fashion in Example 4.1.8.

Proposition 3.3.39. *Let $\mathcal{I} \trianglelefteq R$ be an ideal. Then, there exists an $\mathfrak{A}$-quotient $R/\mathcal{I}$ of R and a quotient $\mathfrak{A}$-functor $\pi_{\mathcal{I}} : R \to R/\mathcal{I}$. On the other hand let $\sigma : R \to R'$ be an $\mathfrak{A}$-functor. Then, the family of morphisms in $R(a)$ that are send to zero under $\sigma(b)$ for all a, $a' \in \mathfrak{A}$ and all 1-arrows $b : a \to a'$ in $\mathfrak{A}$ forms an ideal which we denote by $\mathcal{I}(\sigma) \trianglelefteq R$ and σ factors over $\pi_{\mathcal{I}(\sigma)}$. Further, let $\sigma' : R' \to R''$ be another $\mathfrak{A}$-functor. Then, we have $\mathcal{I}(\sigma) \subset \mathcal{I}(\sigma' \, \sigma)$. If σ is a quotient $\mathfrak{A}$-functor, then the factorisation of $\pi_{\mathcal{I}}$ is an equivalence in $\mathfrak{A} - \mathrm{Mod}$. Furthermore given $\mathcal{I} \trianglelefteq R$, we have $\mathcal{I}(\pi_{\mathcal{I}}) = \mathcal{I}$.*

Proof. This is standard. □

Remark 3.3.40. Given $\sigma \in \mathcal{H}om_{\mathfrak{A}}(R, R')$, $\ker(\sigma)$ is the maximal full sub-$\mathfrak{A}$-module of R whose morphism are contained in $\mathcal{I}(\sigma)$. Note that the minimal ideal in R that contains all morphisms of $\ker(\sigma)$ is thick and can be described as $\mathcal{I}\left(R \to \operatorname{im}(\sigma)\right)$. Moreover in $\mathfrak{A} - \mathrm{Mod}$, we have the following commutative diagram:

$$\begin{tikzcd}
\ker(\sigma) \arrow[r, hook] \arrow[dr, "0"'] & R \arrow[rr, "\sigma"] \arrow[d] \arrow[dr, "\pi_{\mathcal{I}(\sigma)}"] & & R' \arrow[r, two heads] & \operatorname{coker}(\sigma) \\
& \operatorname{im}(\sigma) \arrow[r, two heads] & R/\mathcal{I}(\sigma) \arrow[ur, hook] \arrow[r, hook] & \operatorname{coim}(\sigma) \arrow[u, hook] \arrow[ur, "0"'] &
\end{tikzcd}$$

Here $\hookrightarrow$ (resp. $\twoheadrightarrow$) indicates that the $\mathfrak{A}$-functor is locally faithful (resp. locally full).

Remark 3.3.41. Cokernels of locally full $\mathfrak{A}$-functors in $\mathfrak{Hom}(\mathfrak{A}, \mathfrak{D})$ were already explained in [Rou4], under the assumption that the 2-category $\mathfrak{D}$ has cokernels.

Remark 3.3.42. In [MaMi2] given a 2-functor $F : \mathfrak{A} \to \mathfrak{D}$, the associated ideal in $\mathfrak{J}(F) \trianglelefteq \mathfrak{A}$ is denoted by $\operatorname{Ker}(F)$. We propose the notation $\mathfrak{J}(F)$, since kernels do not exist within the category of 2-categories, in the same way as they do not exist in the category of unital rings. In [MaMi2] given a 2-representation $R : \mathfrak{A} \to \mathfrak{A}dd$, $\operatorname{Ker}(R)$ is used to denote $\mathfrak{Ann}_{\mathfrak{A}}(R)$. We propose the notation $\mathfrak{Ann}_{\mathfrak{A}}(R)$ (instead of $\mathfrak{J}(R)$) to highlight that $R \in \mathfrak{A} - \mathrm{Mod}$.

Remark 3.3.43. Given an ideal $\mathfrak{J} \trianglelefteq \mathfrak{A}$, one can consider $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{J}$ as $\mathfrak{A}$-modules and there is a quotient $\mathfrak{A}$-functor $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$. Morally, this should lead to a bijection $\mathfrak{Ann}_{\mathfrak{A}}(\mathfrak{A}/\mathfrak{J}) \cong \mathcal{I}(\pi)$. However only if $\mathfrak{A}$ has a single object, this is the case. We will make a precise statement in Remark 3.3.58.

Finally, we introduce annihilators of objects within 2-representations. The reader will note that the annihilator of a higher representation agrees (in an appropriate sense) with the intersection of all annihilators of its objects.

Definition 3.3.44. *Let $a \in \mathfrak{A}$, $R \in \mathfrak{A} - \mathrm{Mod}$ and $r \in R(a)$. The* annihilator of r, *denoted by $\mathcal{A}\mathrm{nn}_a(r) \trianglelefteq \mathcal{P}_a$, consists of all $c \in \mathfrak{A}_{a \to a'}(b_1, b_2)$ where b_1, b_2 and a' vary such that $0 = R(c)_r \in R(a')\left(R(b_1)(c), R(b_2)(c)\right)$.*

Remark 3.3.45. Definition 3.3.44 is taken from [MaMi2]. More conceptually, one can use the equivalence $\mathcal{H}\mathrm{om}_{\mathfrak{A}}(\mathcal{P}_a, R) \cong R(a)$ to obtain an evaluation $\mathfrak{A}$-functor $\mathrm{ev}_r : \mathcal{P}_a \to R$ determined by $I_a \mapsto r$. Then, $\mathcal{A}\mathrm{nn}_a(r) = \mathcal{I}(\mathrm{ev}_r) \trianglelefteq \mathcal{P}_a$. Also, note that the maximal thick subideal of $\mathcal{A}\mathrm{nn}_a(r)$ can be described as $\mathcal{I}\left(\mathcal{P}_a \to \mathrm{im}(\mathrm{ev}_r)\right)$, and that the indecomposable objects of $\ker(\mathrm{ev}_r)$ are in one to one correspondence with the annihilator of r as defined in [MaMi1].

3.3.4 Representability and uniqueness

In this section, we discuss representable 2-functors. Applied to 2-functors with domain $\mathfrak{A} - \mathrm{Mod}$, this is the appropriate generalisation of a universal property. We make the standard observation, based on the 2-Yoneda lemma, that higher representations that satisfy a univgersal property are unique up to equivalence. Constructions of such universal properties and their representing objects forms an integral part of this thesis. As examples, we discuss (sub-2-functors of) weight space 2-functors, and (quotients of) principal representations.

For the remainder of this section, we fix an additive 2-categories $\mathfrak{A}$ and a 2-category $\mathfrak{D}$ such that the 2-functor $\mathfrak{A}(a,-) : \mathfrak{A} \to \mathfrak{D}$ is well defined for all $a \in \mathfrak{A}$. For our purposes it is enough to think of the cases $\mathfrak{D} = \mathfrak{A}dd$, $\mathfrak{A}dd^{\text{idem}}$, $\mathfrak{A}dd^{\bullet}$ or $\mathfrak{A}dd^{\bullet\text{idem}}$.

Definition 3.3.46. *A 2-functor $R : \mathfrak{A} \to \mathfrak{D}$ is called* representable*, if there exists $a \in \mathfrak{A}$ and an equivalence of 2-functors $\mathfrak{A}(a,-) \cong R$ in $\mathfrak{Hom}(\mathfrak{A},\mathfrak{D})$.*

Example 3.3.47. For all $a \in \mathfrak{A}$, the principal representations $\mathcal{P}_a : \mathfrak{A} \to \mathfrak{A}dd$ is representable by definition.

Objects that represent a given 2-functor are essentially unique:

Proposition 3.3.48. *Let a_1, $a_2 \in \mathfrak{A}$. Then, there exists an equivalence $a_1 \cong a_2$ in $\mathfrak{A}$ if and only if there exists an equivalence $\mathfrak{A}(a_1,-) \cong \mathfrak{A}(a_2,-)$ in $\mathfrak{Hom}(\mathfrak{A},\mathfrak{D})$.*

Proof. The first implication holds due to the interchange law. In order to show the converse implication, let us denote $R_i := \mathfrak{A}(a_i,-)$. By assumption, we have morphisms of 2-functors $\sigma_1 : R_1 \to R_2$ and $\sigma_2 : R_2 \to R_1$, and morphisms of morphisms $\gamma_1 : id_{R_1} \xRightarrow{\sim} \sigma_2\sigma_1$ and $\gamma_2 : id_{R_2} \xRightarrow{\sim} \sigma_1\sigma_2$. Setting $b_i := \sigma_i(a_i)(I_{a_i})$, we define 1-arrows $a_2 \xrightarrow{b_1} a_1$ and $a_1 \xrightarrow{b_2} a_2$ in $\mathfrak{A}$. We have an isotransformation $\gamma_1(a_1) : id_{\mathfrak{A}(a_1,a_1)} \xrightarrow{\sim} \sigma_2(a_1)\sigma_1(a_1)$ of endofunctors of $\mathfrak{A}(a_1,a_1)$ and thus isomorphisms $\gamma_1(a_1)_{I_{a_1}} : I_{a_1} \xrightarrow{\sim} \sigma_2(a_1)(b_1)$. Further, $\sigma_2(a_1)(b_1) \cong \sigma_2(a_1)(b_1 I_{a_2}) \cong b_1\sigma_2(a_2)(I_{a_2}) = b_1b_2$, where the isomorphism comes from evaluating the invertible 2-arrow $\sigma_2(a_1)R_2(b_1) \xRightarrow{\sim} R_1(b_1)\sigma_2(a_2)$ associated to the morphisms of 2-functors σ_2 at I_{a_2}. It follows that $I_{a_1} \xrightarrow{\sim} b_1b_2$ and $I_{a_2} \xrightarrow{\sim} b_2b_1$ is shown analogously. $\square$

Representability of functors is a useful concept due to the *Yoneda lemma*, which has a vast rage of applications. It states that for any functor $F : \mathcal{C} \to \mathcal{S}et$, where $\mathcal{S}et$ denotes the category of sets, we have a bijection $\mathfrak{C}at_{\mathcal{C}\to\mathcal{S}et}(\mathcal{C}(c,-),F) \cong F(c)$. In particular, an isotransformation $\mathcal{C}(c_1,-) \xRightarrow{\sim} \mathcal{C}(c_2,-)$ is the same as an isomorphism $c_2 \xrightarrow{\sim} c_1$ in $\mathcal{C}$. In the 2-categorical setting, we have to replace isomorphisms by equivalences and choose a good replacement for $\mathcal{S}et$. For the remainder of this section, we assume that the categories $\mathfrak{Hom}(\mathfrak{A},\mathfrak{D})(R,R')$ are objects in $\mathfrak{D}$ for all 2-functors R, $R' : \mathfrak{A} \to \mathfrak{D}$. For notational clarity, we write $\mathfrak{Hom}_{\mathfrak{A}\to\mathfrak{D}} = \mathfrak{Hom}(\mathfrak{A},\mathfrak{D})$. The following statement is called the *2-Yoneda lemma*, see [Rou4, Section 2.2.4] for the first statement in Lemma 3.3.49 in the $\mathfrak{D} = \mathfrak{C}at$ case.

Lemma 3.3.49. *Let $a \in \mathfrak{A}$. Then, we have $\mathfrak{Hom}_{\mathfrak{A}\to\mathfrak{D}}(\mathfrak{A}(a,-),R) \cong R(a)$ in $\mathfrak{D}$. Further, we have $\mathfrak{Hom}_{\mathfrak{A}\to\mathfrak{D}}(\mathfrak{A}(a,-),\mathfrak{A}(a,-)) \cong \mathfrak{A}^{\text{rev}}(a,a)$ as monoidal categories.*

Proof. We provide a sketch only. First, define a 1-arrow $\mathfrak{Hom}_{\mathfrak{A}\to\mathfrak{D}}(\mathfrak{A}(a,-),R) \to R(a)$ in $\mathfrak{D}$ as follows: A morphism of 2-functors $\sigma : \mathfrak{A}(a,-) \to R$ is send to $\sigma(a)(I_a) \in R(a)$ and a morphism of morphisms $\gamma : \sigma \to \tilde{\sigma}$ is send to $\gamma(a)_{I_a}$. Second, the 1-arrow $R(a) \to \mathfrak{Hom}_{\mathfrak{A}\to\mathfrak{D}}(\mathfrak{A}(a,-),R)$ in $\mathfrak{D}$ is defined as follows: An object $r \in R(a)$ is send to the morphism of 2-functors $\mathfrak{A}(a,-) \to R$ which for all $a' \in \mathfrak{A}$ sends $b \in \mathfrak{A}(a,a')$ to $R(b)(r)$ and for all b_1, $b_2 \in \mathfrak{A}(a,a')$ sends a 2-arrow $c : b_1 \Rightarrow b_2$ to $R(c)_r$. The natural isomorphisms necessary to complete the definition of the 2-functor $\mathfrak{A}(a,-) \to R$ are induced by the natural isomorphisms that are part of the datum of the 2-functor R. We leave it to the reader to check that the respective compositions of these 1-arrows are isomorphic to the respective unit 1-arrows, which establishes the first statement. For the second statement, we take two morphisms of 2-functors σ_1, $\sigma_2 : \mathfrak{A}(a,-) \to \mathfrak{A}(a,-)$. The composition of their images $\sigma_1(a)(I_a)$ and $\sigma_2(a)(I_a)$ is $\sigma_1(a)(I_a)\sigma_2(a)(I_a)$ by the definition of the reversed category. This agrees up to isomorphism with the image $(\sigma_2\sigma_1)(a)(I_a)$ of their composition $\sigma_2\sigma_1$. Since $(\sigma_2\sigma_1)(a)(I_a) = \sigma_2(a)(\sigma_1(a)(I_a)) \cong \sigma_2(a)((\sigma_1(a)(I_a)).I_a) \cong (\sigma_1(a)(I_a))\sigma_2(a)(I_a)$, where we used both the left unit in $\mathfrak{A}(a,a)$ as well as the structure map of the 2-functor σ_2, it is straightforward to turn the equivalence into a monoidal equivalence. □

Remark 3.3.50. The 2-Yoneda lemma (for $\mathfrak{D} = \mathfrak{C}at$) has a long history. See [Lei1] for a an independent treatment of this statement and the most important references. The crucial application to the Yoneda embedding for bicategories, as well as the corollary that every 2-category is biequivalent to a strict 2-category can also be found there. It is worth noting that according to their own introduction, [Mo] provides the first complete proof of this statement in 1997.

Definition 3.3.51. *Let $a \in \mathfrak{A}$. We call the obvious 2-functor $\Phi_a : \mathfrak{A} - \mathrm{Mod}(\mathfrak{D}) \to \mathfrak{D}$ that is given by $\Phi_a(R) = R(a)$ for all $\mathfrak{D}$-modules the a-*weight space 2-functor.

Provided our assumptions on $\mathfrak{A}$ and $\mathfrak{D}$, one deduces the following statement. This statement has already been used implicitly in [Rou4, Section 5.1.3] for $\mathfrak{D} = \mathfrak{C}at$ and has been stated explicitly for (k-linear) principal representations of a strict 2-category $\mathfrak{A}$ in [MaMi2, Lemma 9].

Proposition 3.3.52. *Let $a \in \mathfrak{A}$. Then, $\mathcal{P}_a$ represents Φ_a.*

Proof. By the 2-Yoneda lemma applied to the pair $(\mathfrak{A}-\mathrm{Mod}(\mathfrak{D}),\mathfrak{D})$, we have an equivalence $\mathfrak{Hom}_{\mathfrak{A}-\mathrm{Mod}(\mathfrak{D})\to\mathfrak{D}}(\mathcal{H}om_{\mathfrak{A}}(\mathcal{P}_a,-),\Phi_a) \cong \Phi_a(\mathcal{P}_a) = \mathcal{P}_a(a) = \mathfrak{A}(a,a)$ in $\mathfrak{D}$. Thus, $I_a \in \mathfrak{A}(a,a)$ gives rise to a morphism of 2-functor $\mathcal{H}om_{\mathfrak{A}}(\mathcal{P}_a,-) \to \Phi_a$. Evaluated at $R \in \mathfrak{A} - \mathrm{Mod}(\mathfrak{D})$, this morphism agrees with the functor $\mathcal{H}om_{\mathfrak{A}}(\mathcal{P}_a,R) \to R(a) =$

$\Phi_a(R)$ of the 2-Yoneda lemma, which is an equivalence. Thus, $\mathcal{H}om_{\mathfrak{A}}(\mathcal{P}_a, -) \to \Phi_a$ is locally an equivalence, and it is not hard to check that these equivalences actually define an equivalence of 2-functors. □

We will now see that $\mathfrak{A}$ is fully encoded in its category of higher representations. For this purpose, denote by $\mathcal{P}\mathrm{rin}(\mathfrak{A})$ the 1-2-full subcategory of $\mathfrak{A} - \mathrm{Mod}$ that has all principal representations as objects. In the same way as in the 2-Yoneda Lemma 3.3.49, the reversed category comes into play:

Corollary 3.3.53. *The assignment $a \mapsto \mathcal{P}_a$ extends to an equivalence $\mathfrak{A}^{\mathrm{rev}} \cong \mathcal{P}\mathrm{rin}(\mathfrak{A})$. In particular, $\mathcal{B}^{\mathrm{rev}} \cong \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{B}, \mathcal{B})$ as monoidal categories.*

Proof. We leave it to the reader to assert that the equivalences provided in Proposition 3.3.52 actually defines 2-functors and morphisms of 2-functors that together yield this equivalence. Let us however provide an explanation for the appearance of the reversed 2-category $\mathfrak{A}^{\mathrm{rev}}$ through the following up to isotransforamtions commutative diagram:

$$\begin{tikzcd}
\mathfrak{A}^{\mathrm{rev}}(a_1, a_2) \times \mathfrak{A}^{\mathrm{rev}}(a_2, a_3) \arrow[r] \arrow[d, equal] & \mathfrak{A}^{\mathrm{rev}}(a_1, a_3) \arrow[d, equal] \\
\mathfrak{A}(a_2, a_1) \times \mathfrak{A}(a_3, a_2) \arrow[r] \arrow[d, "\sim"] & \mathfrak{A}(a_3, a_1) \arrow[d, "\sim"] \\
\mathcal{H}\mathrm{om}_{\mathfrak{A}}(\mathcal{P}_{a_1}, \mathcal{P}_{a_2}) \times \mathcal{H}\mathrm{om}_{\mathfrak{A}}(\mathcal{P}_{a_2}, \mathcal{P}_{a_3}) \arrow[r] & \mathcal{H}\mathrm{om}_{\mathfrak{A}}(\mathcal{P}_{a_1}, \mathcal{P}_{a_3})
\end{tikzcd}$$

□

Remark 3.3.54. We will prove an enhanced version of the second statement of Corollary 3.3.53 in Theorem 4.5.10.

Remark 3.3.55. Proposition 3.3.48 clearly follows from Corollary 3.3.53. We decided to present the statements in this order, since we expect that most readers are more familiar with Proposition 3.3.48 than with Corollary 3.3.53.

Now let $a \in \mathfrak{A}$, and let $\mathcal{I} \trianglelefteq \mathcal{P}_a$ be an ideal of the principal representation $\mathcal{P}_a$ of $\mathfrak{A}$. We denote by $\mathcal{P}_a^{\mathcal{I}} := \mathcal{P}_a/\mathcal{I}$ the corresponding $\mathfrak{A}$-quotient. There exists an obvious 2-functor $\Phi_a^{\mathcal{I}} : \mathfrak{A} - \mathrm{Mod} \to \mathfrak{A}dd$ that sends an $\mathfrak{A}$-module R to the full subcategory of $R(a)$ with objects M that satisfy $\mathcal{I} \subset \mathcal{A}\mathrm{nn}_a(M)$. Clearly, this 2-functor comes with a locally fully faithful morphism $\Phi_a^{\mathcal{I}} \to \Phi_a$ of 2-functors. Since, $\mathcal{P}_a \to \mathcal{P}_a^{\mathcal{I}}$ is a quotient $\mathfrak{A}$-functor, it induces a fully faithful functor $\mathcal{H}om_{\mathfrak{A}}(\mathcal{P}_a^{\mathcal{I}}, R) \to \mathcal{H}om_{\mathfrak{A}}(\mathcal{P}_a, R)$. The equivalence $\mathcal{H}om_{\mathfrak{A}}(\mathcal{P}_a, R) \cong \Phi_a(R)$ now clearly induces equivalences $\mathcal{H}om_{\mathfrak{A}}(\mathcal{P}_a^{\mathcal{I}}, R) \cong \Phi_a^{\mathcal{I}}(R)$ which are natural in R. We have obtained the following result, a special case of which has been used in [Rou4, Section 5.1.3]:

Proposition 3.3.56. *Let $a \in \mathfrak{A}$ and let $\mathcal{I} \trianglelefteq \mathcal{P}_a$. Then, $\Phi_a^{\mathcal{I}}$ is represented by $\mathcal{P}_a^{\mathcal{I}}$.*

Example 3.3.57.

1. Let $\mathfrak{A}_{\mathfrak{g}}$ be a 2-Kac-Moody algebra and $\lambda \in \mathfrak{A}_{\mathfrak{g}}$. In [Rou4], the λ-weight space 2-functor $\mathfrak{A} - \mathrm{Mod}(\mathfrak{Lin}_k) \to \mathfrak{C}at$ that sends $R \mapsto R(\lambda)$ is discussed. It agrees with the composition of Φ_λ with the forgetful 2-functor $\mathfrak{Lin}_k \to \mathfrak{C}at$. Further, it is stated that a 2-representation denoted $\mathcal{H}om(\lambda, -) \in \mathfrak{A}_{\mathfrak{g}} - \mathrm{Mod}(\mathfrak{Lin}_k)$ represents this composition. In our notation, $\mathcal{H}om(\lambda, -) = \mathcal{P}_\lambda$.

2. In [MaMi1], the corresponding 2-Yoneda lemma for Abelian higher representations is proven for the Abelian principal representations $\mathbf{P}_a$. In [MaMi2], an abelianization 2-functor is constructed and it is shown that $\mathcal{P}_a \mapsto \mathbf{P}_a$. They also provide an adjoint (though do not state that it is one), which one may use to deduce the universal property of $\mathbf{P}_a$ from $\mathcal{P}_a$. Working out the details should also lead to a connection between $\mathcal{P}_a^{\mathcal{I}}$ and the modules $\mathbf{P}_a^{\mathbf{I}}$ defined in [MaMi1].

Having introduced principal representations and their $\mathfrak{A}$-quotients, we can make Remark 3.3.43 precise:

Remark 3.3.58. Let $\mathfrak{J} \trianglelefteq \mathfrak{A}$ be an ideal. For all $a \in \mathfrak{A}$, we obtain an ideal $\mathcal{I}_a \trianglelefteq \mathcal{P}_a$ by setting $\mathcal{I}_a(a') := \mathfrak{J}(a, a')$ for all $a' \in \mathfrak{A}$, and we denote the associated quotient $\mathfrak{A}$-functor by $\pi_a : \mathcal{P}_a \to \mathcal{P}_a^{\mathcal{I}_a}$. Denoting $\hat{\pi} := \oplus_{a \in \mathfrak{A}} \pi_a$, we have $\mathcal{I}(\hat{\pi})(a') = \bigoplus_{a \in \mathfrak{A}} \mathfrak{J}(a, a')$ for all $a' \in \mathfrak{A}$. Now, one may abuse notation in two different ways by writing $\mathfrak{A} = \bigoplus_{a \in \mathfrak{A}} \mathcal{P}_a$ and $\mathfrak{A}/\mathfrak{J} = \bigoplus_{a \in \mathfrak{A}} \mathcal{P}_a^{\mathcal{I}_a}$ in $\mathfrak{A} - \mathrm{Mod}$. Since $\mathrm{Res}_{\pi_{\mathfrak{J}}}(\mathcal{P}_a^{\mathfrak{A}/\mathfrak{J}}) = (\mathcal{P}_a^{\mathfrak{A}})^{\mathcal{I}_a}$, these notational abuses lead to the statement $\mathfrak{A}/\mathfrak{J} = \mathrm{Res}_\pi(\mathfrak{A}/\mathfrak{J})$ in $\mathfrak{A} - \mathrm{Mod}$. In this sense, we have $\pi_{\mathfrak{J}} = \hat{\pi}$ as $\mathfrak{A}$-functors. However, one has to be careful with such identifications: The ideal $\mathcal{I}(\hat{\pi})$ in the $\mathfrak{A}$-module $\mathfrak{A}$ contains the morphisms of the ideal $\mathfrak{Ann}_{\mathfrak{A}}(\mathfrak{A}/\mathfrak{J}) = \mathfrak{J}(\pi_{\mathfrak{J}}) = \mathfrak{J}$ in the 2-category $\mathfrak{A}$ in multiple form, as the above formula shows. Only if $\mathfrak{A}$ has precisely one object, $\mathcal{I}(\hat{\pi})$ and $\mathfrak{Ann}_{\mathfrak{A}}(\mathfrak{A}/\mathfrak{J})$ are identical.

Remark 3.3.59. Representability is usually formulated for $\mathfrak{D} = \mathfrak{C}at$, which is sufficient to prove uniqueness up to equivalence of any representing object. For the relation to ideals however, we used the existence of zero 2-arrows. Further, it will be crucial later on to consider 2-functors for example with $\mathfrak{D} = \mathfrak{A}dd^{\mathrm{idem}}$. This was the motivation for framing this section with flexible $\mathfrak{D}$.

Chapter 4

Higher Representation Theory II - New Developments

This chapter is the technical centre piece of this work. All statements apply to arbitrary additive 2-categories $\mathfrak{A}$. For simplicity, we often focus on an additive monoidal category $\mathcal{B}$ instead of $\mathfrak{A}$. Section 4.1 discusses centres and base change of $\mathfrak{A}$-modules. Section 4.2 introduces three generalisations of the classical notion of cyclicity: First, a direct generalisation called 1-cyclicity, second 2-cyclicity which takes morphisms into account and implies desirable properties for the functor categories with 2-cyclic domain, and third functorially cyclic modules which are 2-cyclic modules and posses a more readily testable definition. Section 4.3 introduces (indecomposable) split Frobenius objects and discusses the additive Barr-Beck theorem, first for additive categories, then for 2-functors with codomain $\mathfrak{A}dd^{\mathrm{idem}}$ and finally for universal properties of additive higher representations. It is shown that representability of universal properties is preserved under the additive Barr-Beck construction. Furthermore, the split Frobenius 2-category is introduce as well as preorders for split Frobenius objects. Section 4.4 is a technical intermezzo about restriction 2-functors and their adjoints, which is kept as brief as possible with our applications in mind. Section 4.5 revisits functorially cyclic modules. Using the additive Barr-Beck theorem, bijections between equivalence classes of certain modules (including functorially cyclic modules) and isomorphism classes of split Frobenius objects, a universal property of functorially cyclic modules, as well as an equivalence between the 2-category of free functorially cyclic modules and the split Frobenius 2-category are established. Further, the preorders on split Frobenius are related to representation theory. Finally, (strongly) universal cell modules are introduced and for weakly regular additive 2-categories it is shown that existence of strongly universal cell modules is equivalent to the existence of split Frobenius objects that satisfy certain technical conditions.

Section 4.6 discusses coinvariant theory. Section 4.7 studies the compatibility of additive Barr-Beck construction with monoidal quotient functors, and shows that the induced functors remain full and, provided a technical condition, also essential surjective. Section 4.8 shows that functorially cyclic modules can be equipped with a natural monoidal structure, provided that the associated split Frobenius object is symmetrically central. Finally, Section 4.9 provides an abstract framework in which Jordan-Hölder theory can be addressed. First, cofiltrations of the 2-categories of additive higher representations associated to refinements of the ordering of the set of 2-sided cells, as well as modules isotypic with respect to a 2-sided cell are introduced. Second, a framework for Jordan-Hölder theory in terms of isotypic covers is lined out and various challenges that make it hard to establish results are discussed.

4.1 Centres and base change

In this section, we discus centres and base change of $\mathfrak{A}$-modules. The latter notion had already been used in [Rou4], however the role of centres had not been made transparent. For notational simplicity, we mostly work in $\mathcal{B} - \mathrm{Mod}$, with obvious generalisations to $\mathfrak{A} - \mathrm{Mod}$. Recall from Remark 3.2.2 that the centres we discussed in Section 3.2 are commutative.

Definition 4.1.1. *Let $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$. We define* the centre of $\mathcal{M}$ as a $\mathcal{B}$-module *by $\mathcal{Z}_{\mathcal{B}}(\mathcal{M}) := \mathcal{Z}_{\mathcal{B}-\mathrm{Mod}}(\mathcal{M})$.*

Remark 4.1.2. The centre $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ of a $\mathcal{B}$-module $\mathcal{M}$ can be viewed as a subring of the centre $\mathcal{Z}(\mathcal{M})$ of the underlying additive category $\mathcal{M}$. This subring is given by all those transformations compatible with the $\mathcal{B}$-action. In formulas, we have $\mathcal{Z}_{\mathcal{B}}(\mathcal{M}) = \{(z_M)_{M\in\mathcal{M}} \in \mathcal{Z}(\mathcal{M}) | B.z_M = z_{B.M} \ \ \forall B \in \mathcal{B}\}$.

Example 4.1.3. One can verify explicitly that $\mathcal{Z}_{\mathcal{B}}(\mathcal{B}) \cong \mathcal{B}(I, I)$. Provided certain cyclicity assumptions, we will show the more general Corollary 4.2.21 that provides a method to calculate the centre of higher representations in terms of a subring of the endomorphisms of a cyclic generator. A less general but more explicit version follows in Corollary 4.5.28.

Let $\mathcal{M}$ be a Z-linear category, that is $\mathcal{M}$ has finite biproducts and we have a ring homomorphism $Z \to \mathcal{Z}(\mathcal{M})$. Let A be a Z-algebra. We can form a new category $\mathcal{M} \otimes_Z A$ with the same objects as $\mathcal{M}$ and morphism spaces defined by $(\mathcal{M}\otimes_Z A)(M_1, M_2) := \mathcal{M}(M_1, M_2)\otimes_Z A$. More generally, given a Z-linear category $\mathcal{A}$, we can form a category $\mathcal{M} \otimes_Z \mathcal{A}$ with objects pairs (M, A) where $M \in \mathcal{M}$ and $A \in \mathcal{A}$. The morphisms spaces of $\mathcal{M} \otimes_Z \mathcal{A}$ are defined as $\mathcal{M} \otimes_Z \mathcal{A}((M_1, A_1), (M_2, A_2)) := \mathcal{M}(M_1, M_2) \otimes_Z \mathcal{A}(A_1, A_2)$. If $\mathcal{M}$ and $\mathcal{A}$ are additive, we abuse notation and denote by $\mathcal{M} \otimes_Z \mathcal{A}$ the additive closure of the category defined above. Now, let $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ and $\mathcal{A}$ be a $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$-linear category. Since the underlying additive category $\mathcal{M}$ is $\mathcal{Z}(\mathcal{M})$-linear, and thus $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$-linear as well, we can form the additive category $\mathcal{M} \otimes_{\mathcal{Z}_{\mathcal{B}}(\mathcal{M})} \mathcal{A}$. In fact, we have the following statements:

Proposition 4.1.4. *Let $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ and let $\mathcal{A}$ be a $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$-linear category. Then, $\mathcal{M} \otimes_{\mathcal{Z}_{\mathcal{B}}(\mathcal{M})} \mathcal{A} \in \mathcal{B} - \mathrm{Mod}$.*

The following proposition may serve as a motivation to compute $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ and $\mathcal{H}om_{\mathcal{B}}(\mathcal{M}, \mathcal{N})$. We will be able to do so provided $\mathcal{M}$ is 2-cyclic.

Proposition 4.1.5. *Let $\mathcal{M}, \mathcal{N} \in \mathcal{B} - \mathrm{Mod}$. Then, evaluation defines a $\mathcal{B}$-functor $\mathcal{M} \otimes_{\mathcal{Z}_{\mathcal{B}}(\mathcal{M})} \mathcal{H}om_{\mathcal{B}}(\mathcal{M}, \mathcal{N}) \to \mathcal{N}$.*

Remark 4.1.6. Note that $\mathcal{H}om_{\mathcal{B}}(\mathcal{M}, \mathcal{N})$ is a $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$-linear category, and therefore that the domain of the evaluation $\mathcal{B}$-functor is well defined. The remaining details of the proof pose no significant challenges. Since neither Proposition 4.1.4 nor Proposition 4.1.5 will be used in any proofs this thesis, we do not prove these propositions here.

Remark 4.1.7. Proposition 4.1.5 is used to state the Jordan-Hölder theorem of [Rou4] for a 2-Kac Moody algebra $\mathfrak{A}_{\mathfrak{g}}$. In particular, [Rou4, Lemma 5.4] uses the application of this proposition to the minimal categorifications $\mathcal{L}(\lambda) \in \mathfrak{A}_{\mathfrak{g}} - \mathrm{Mod}$. The tensor product was taken over a quotient of $(\mathfrak{A}_{\mathfrak{g}})_{\lambda \to \lambda}(I_\lambda, I_\lambda)$ which is indeed $\mathcal{Z}_{\mathfrak{A}_{\mathfrak{g}}}(\mathcal{L}(\lambda))$ but also agrees with $\mathcal{Z}(\mathcal{L}(\lambda))$. The new observation here is that $\mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ is the maximal subring $Z' \leq \mathcal{Z}(\mathcal{M})$ that allows one to construct a canonical $\mathcal{B}$-action on the additive category $\mathcal{M} \otimes_{Z'} \mathcal{A}$ for all Z'-linear categories $\mathcal{A}$.

Example 4.1.8. The $\mathbb{C}$-linear quivers of Example 3.3.38 yield $\mathbb{C}$-algebras A and A'. The algebra A (resp. A') has complex dimension 3 (resp. 2) and has a basis given by the idempotents $\cdot_1$, $\cdot_2$ and the arrow $\cdot_1 \to \cdot_2$ (resp. a basis given by the idempotents $\cdot_3$ and $\cdot_4$). Furthermore, we have a $\mathbb{C}$-algebra morphism $A \to A'$ which is defined analogously to the $\mathcal{V}ect_{\mathbb{C}}$-functor F of Example 3.3.38. Since $\mathcal{Z} := \mathcal{Z}_{\mathcal{V}ect_{\mathbb{C}}}(\mathcal{V}ect_{\mathbb{C}}) = \mathbb{C}$, we obtain $\mathcal{V}ect_{\mathbb{C}}$-modules $\mathcal{V}ect_{\mathbb{C}} \otimes_{\mathcal{Z}} A$ and $\mathcal{V}ect_{\mathbb{C}} \otimes_{\mathcal{Z}} A'$, whose idempotent closures are equivalent to the modules of Example 3.3.38. Further, the $\mathcal{Z}$-algebra morphism $A \to A'$ induces a $\mathcal{V}ect_{\mathbb{C}}$-functor $\mathcal{V}ect_{\mathbb{C}} \otimes_{\mathcal{Z}} A \to \mathcal{V}ect_{\mathbb{C}} \otimes_{\mathcal{Z}} A'$ which up to idempotent closure and equivalence corresponds to F.

There is another centre that is heavily studied in the literature. Clearly this center is not a commutative ring, but rather a monoidal category.

Definition 4.1.9. *The* Drinfeld centre *of an additive 2-category $\mathfrak{A}$ can be defined as $\mathcal{Z}^{\mathrm{DR}}(\mathfrak{A}) = \mathfrak{Hom}_{\mathfrak{A}\to\mathfrak{A}}(id_{\mathfrak{A}}, id_{\mathfrak{A}})$.*

It is well known that $\mathcal{Z}^{\mathrm{DR}}(\mathfrak{A})$ is an additive braided monoidal category, see for example [Ka] for the standard definition describing objects and morphisms explicitly. The unit object is given by the 2-functor $id_{\mathfrak{A}}$ and the necessary additional structure comes from the left and right unit transformations.

Remark 4.1.10. Let $R \in \mathfrak{A} - \mathrm{Mod}$. Then, we have $\mathcal{E}nd_{\mathfrak{A}}(R) \in \mathcal{Z}^{\mathrm{DR}}(\mathfrak{A}) - \mathrm{Mod}$.

We will not discuss the Drinfeld centre in this thesis, we will however use the endomorphism ring of its identity object. This endomorphism ring will for example play a role when we discuss central quotients in Section 4.4.

Definition 4.1.11. *We denote by $\mathcal{Z}^{\otimes}(\mathfrak{A})$ the ring of endomorphisms of the identity morphism of the identity 2-functor of $\mathfrak{A}$. In formulas, $\mathcal{Z}^{\otimes}(\mathfrak{A}) = \mathcal{Z}^{\mathrm{Dr}}(\mathfrak{A})(id_{\mathfrak{A}}, id_{\mathfrak{A}})$.*

Proposition 4.1.12. *Let R, $R' \in \mathfrak{A} - \mathrm{Mod}$. Then, we have a canonical ring homomorphism $\mathcal{Z}^{\otimes}(\mathfrak{A}) \to \mathcal{Z}_{\mathfrak{A}}(R)$, that is further compatible with the analogous morphism $\mathcal{Z}^{\otimes}(\mathfrak{A}) \to \mathcal{Z}_{\mathfrak{A}}(R')$ in the sense that the $\mathcal{Z}_{\mathfrak{A}}(R)$-$\mathcal{Z}_{\mathfrak{A}}(R')$-linear category $\mathcal{H}\mathrm{om}_{\mathfrak{A}}(R, R')$ can be regarded as a $\mathcal{Z}_{\mathfrak{A}}(R) \otimes_{\mathcal{Z}^{\otimes}(\mathfrak{A})} \mathcal{Z}_{\mathfrak{A}}(R')$-linear category. Furthermore, we have an injection $\mathcal{Z}^{\otimes}(\mathfrak{A}) \hookrightarrow \mathcal{Z}_{\mathfrak{A}}(\bigoplus_{a\in\mathfrak{A}} \mathcal{P}_a)$.*

Proof. An element $z \in \mathcal{Z}^{\otimes}(\mathfrak{A})$ is nothing but a family index by $a \in \mathfrak{A}$ of 2-arrows $z_a : I_a \Rightarrow I_a$ in $\mathfrak{A}(a, a)$ that satisfies certain compatibility properties. Similarly, an element $\omega \in \mathcal{Z}_{\mathfrak{A}}(R)$ is given by a family of morphisms $(\omega_a)_b : b \to b$ indexed by $a \in \mathfrak{A}$ and $b \in R(a)$. Defining $(\omega_a)_b$ as the composition $b = id_{R(a)}(b) \cong R(I_a)(b) \xrightarrow{R(z_a)} R(I_a)(b) \cong b$, yields indeed an element $\omega \in \mathcal{Z}_{\mathfrak{A}}(R)$. We leave it to the reader to check the details, as well as that the thus defined map $\mathcal{Z}^{\otimes}(\mathfrak{A}) \to \mathcal{Z}_{\mathfrak{A}}(R)$ is a ring homomorphism. The reader may want to note that the arguments presented here are a special case of some of the arguments needed to prove Remark 4.1.10. Further, the abstract nonsense in Proposition 3.2.7 implies that $\mathcal{H}\mathrm{om}_{\mathfrak{A}}(R, R')$ is a $Z_{\mathfrak{A}}(R)$-$\mathcal{Z}_{\mathfrak{A}}(R')$-linear category. We leave it as an exercise to check that the actions of $\mathcal{Z}^{\otimes}(\mathfrak{A})$ on the morphism spaces of $\mathcal{H}\mathrm{om}_{\mathfrak{A}}(R, R')$ induced via $Z_{\mathfrak{A}}(R)$ and $\mathcal{Z}_{\mathfrak{A}}(R')$ agree. Establishing this implies that $\mathcal{H}\mathrm{om}_{\mathfrak{A}}(R, R')$ is a $\mathcal{Z}_{\mathfrak{A}}(R) \otimes_{\mathcal{Z}^{\otimes}(\mathfrak{A})} \mathcal{Z}_{\mathfrak{A}}(R')$-linear category. Furthermore for $R = \bigoplus_{a'\in\mathfrak{A}} \mathcal{P}_{a'}$, the image of $z = (z_a)_{a\in\mathfrak{A}}$ is given by the transformation which is given for all $a \in \mathfrak{A}$ and $b \in \bigoplus_{a'\in\mathfrak{A}} \mathcal{P}_{a'}(a)$ by the composition $b \cong bI_a \xrightarrow{bz_a} bI_a \cong b$. Choosing $b = I_a \in \mathfrak{A}(a, a) \subset^{\oplus} \bigoplus_{a'\in\mathfrak{A}} \mathcal{P}_{a'}(a)$ implies injectivity. $\square$

4.2 Cyclicity

In this section, we study generalisations of the classical notion of cyclicity. Section 4.2.1 introduces a direct generalisation called 1-cyclic modules and provides furhter justifications for this notion. Section 4.2.2 extends the discussion of 1-cyclic modules to an acting 2-category and discusses the refined structure that arises in this case. Section 4.2.3 introduces 2-cyclic modules, which are 1-cyclic and whose morphisms are controlled by the acting 2-category in particular way. A fairly explicit description of the functor categories with domain given by a 2-cyclic modules is provided. Section 4.2.4 introduces functorially cyclic modules and shows that they form a subclass of 2-cyclic modules. Section 4.2.5 discusses the non-uniqueness of the structure functors and transformations of a functorially cyclic module.

4.2.1 1-cyclicity

In this section, we introduce 1-cyclic modules which are the direct counterpart of the classical notion of cyclicity. We also provide several justifications for this notion.

The discussion of the classical counterpart and especially Proposition 2.3.10 suggest the following definition. Recall the notion of strong indecomposability introduced in Definition 3.1.7.

Definition 4.2.1. *Let $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$. Then, $\mathcal{C}$ is called* 1-cyclic *if $\mathcal{C}$ is trivial or if there exists an strongly indecomposable $\mathcal{B}$-functor $\mathcal{B} \to \mathcal{C}$ with trivial cokernel.*

Remark 4.2.2. Classical notions of cyclicity can be unified as follows: An object is cyclic if and only if there exists an epimorphism from a free object of rank 1. For example for modules over a ring A this free object of rank one would be the A-module A, and for (Hausdorff topological) groups it would be $\mathbb{Z}$ (equipped with the discrete topology). This motivates an alternative definition of 1-cyclicity, namely that $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$ is 1-cyclic if and only if there exists a $\mathcal{B}$-functor $\mathcal{B} \to \mathcal{M}$ which is an "epimorphism", that is it induces a "monomorphism" $\mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M}) \to \mathcal{H}om_{\mathcal{B}}(\mathcal{B}, \mathcal{M})$ for all $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$. Unfortunately, there is no good notion of "monomorphisms" between additive categories.

Nevertheless, there is a criterion for 1-cyclicity formulated in terms of the categories of $\mathcal{B}$-functors and $\mathcal{B}$-transformations as the following proposition shows.

Proposition 4.2.3. *Let $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$ be non-trivial. The following are equivalent:*

1. *$\mathcal{C}$ is 1-cyclic.*

2. *There exists a strongly indecomposable $\mathcal{B}$-functor $\mathcal{B} \to \mathcal{C}$ such that for all $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ the induced functor $\mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M}) \to \mathcal{H}om_{\mathcal{B}}(\mathcal{B}, \mathcal{M})$ is faithful.*

3. *There exists a strongly indecomposable object $C \in \mathcal{C}$ such that for all $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ and for all $F, G \in \mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M})$ the morphism $\mathcal{H}om_{\mathcal{C}\to\mathcal{M}}(F, G) \to \mathcal{M}(F(C), G(C))$ which sends $\alpha \mapsto \alpha_C$ is injective.*

4. *There exists a* cyclic generator, i.e. *a strongly indecomposable object $C \in \mathcal{C}$ such that for all $C' \in \mathcal{C}$ there exists $B \in \mathcal{B}$ with $C' \subset^{\oplus} B.C$.*

Proof. Recall that $\mathcal{H}om_{\mathcal{B}}(\mathcal{B}, \mathcal{M}) \cong \mathcal{M}$ for all $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ by Proposition 3.3.52. Thus, a $\mathcal{B}$-functor $F : \mathcal{B} \to \mathcal{M}$ is strongly indecomposable if and only if one and all objects $M \in \mathcal{M}$ with $M \cong F(I)$ are strongly indecomposable. This implies that the strong indecomposability statements in (1)-(4) are all equivalent. We can thus show that (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) without having to mention strong indecomposability again.

(1) $\Rightarrow$ (4): Let $\pi : \mathcal{B} \to \mathcal{C}$ be a $\mathcal{B}$-functor with $\mathrm{coker}(\pi) = 0$. Define $C := \pi(I)$. Recall that $\mathrm{coker}(\pi)$ can be constructed as the additive quotient of $\mathcal{C}$ by all objects of the form $\pi(B)$ for $B \in \mathcal{B}$. Since $\mathrm{coker}(\pi) = 0$, all objects of $\mathcal{C}$ have to be a direct summand of $\pi(B)$ for some $B \in \mathcal{B}$. Using the right unit axiom of $\mathcal{B}$ and the fact that π is a $\mathcal{B}$-functor, $\pi(B) \cong \pi(B.I) \cong B.C$ and the claim follows.

(4) $\Rightarrow$ (3): Let $C \in \mathcal{C}$ such that for all $C' \in \mathcal{C}$ there exists $B \in \mathcal{B}$ with $C' \subset^{\oplus} B.C$. Choose $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ and $F, G \in \mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M})$ arbitrarily. It is enough to show that any $\alpha \in \mathcal{H}om_{\mathcal{C}\to\mathcal{M}}(F, G)$ is uniquely determined by α_C. By definition of $\mathcal{B}$-transformations, α is a family of morphisms $\alpha_{C'}$ for $C' \in \mathcal{C}$ with additional properties. It is enough to show that $\alpha_{C'}$ is determined by α_C for a fixed $C' \in \mathcal{C}$. Since α is a $\mathcal{B}$-transformation, this follows from the commutative diagram

$$\begin{tikzcd}
B_{C'}.F(C) \arrow[r, "can^F_{B'_C, C}"] \arrow[d, "B_{C'}.\alpha_C"] & F(B_{C'}.C) \arrow[d, "\alpha_{B_{C'}.C}"] & F(C') \arrow[l, hook', "F(\iota)"'] \arrow[d, "\alpha_{C'}"] \\
B_{C'}.G(C) \arrow[r, "can^G_{B'_C, C}"] & G(B_{C'}.C) \arrow[r, two heads, "G(\pi)"] & G(C')
\end{tikzcd} .$$

(3) $\Rightarrow$ (2): Let $C \in \mathcal{C}$ be an object such that evaluating $\mathcal{B}$-transformations between $\mathcal{B}$-functors with domain $\mathcal{C}$ is a faithful map. By the equivalence $\mathcal{H}om_{\mathcal{B}}(\mathcal{B}, \mathcal{C}) \cong \mathcal{C}$, we have a $\mathcal{B}$-functor $\pi : \mathcal{B} \to \mathcal{C}$ with $\pi(I) \cong C$. Let $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ be arbitrary. We need to show that $\mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M}) \xrightarrow{-\pi} \mathcal{H}om_{\mathcal{B}}(\mathcal{B}, \mathcal{M})$ is faithful. For F, $G \in \mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M})$,

we thus have to show that $\mathcal{H}om_{\mathcal{C}\to\mathcal{M}}(F,G) \to \mathcal{H}om_{\mathcal{B}\to\mathcal{M}}(F\pi, G\pi)$ is injective. Using again the equivalence $\mathcal{H}om_{\mathcal{B}}(\mathcal{B},\mathcal{M}) \cong \mathcal{M}$, we obtain the first of the following equivalences $\mathcal{H}om_{\mathcal{B}\to\mathcal{M}}(F\pi, G\pi) \cong \mathcal{M}(F\pi(I), G\pi(I)) \cong \mathcal{M}(F(C), G(C))$, where the latter follows from $\pi(I) \cong C$. We leave it to the reader to check that the thus constructed map coincides with the injective map $\alpha \mapsto \alpha_C$ of property (3). (2) $\Rightarrow$ (1): Let $\pi : \mathcal{B} \to \mathcal{C}$ be a $\mathcal{B}$-functor such that $\mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{C},\mathcal{M}) \xhookrightarrow{-\pi} \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{B},\mathcal{M})$ is faithful for all $\mathcal{M} \in \mathcal{B}-\mathrm{Mod}$. Denote by $\mathrm{can} : \mathcal{C} \twoheadrightarrow \mathrm{coker}(\pi)$ the canonical quotient $\mathcal{B}$-functor for which $\mathrm{can}\,\pi = 0$, see Lemma 3.3.26. Using $\mathcal{M} = \mathrm{coker}(\pi)$, we obtain a sequence $\mathcal{H}om_{\mathcal{B}}(\mathrm{coker}(\pi), \mathrm{coker}(\pi)) \xrightarrow{-\mathrm{can}} \mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathrm{coker}(\pi)) \xhookrightarrow{-\pi} \mathcal{H}om_{\mathcal{B}}(\mathcal{B}, \mathrm{coker}(\pi))$ where the second arrow is faithful and the composition is zero. By this sequence, we have $id_{\mathrm{coker}(\pi)} \mapsto id_{\mathrm{can}} \mapsto id_{\mathrm{can}\,\pi} = 0$. Since, the second arrow in the sequence is faithful, $id_{\mathrm{can}} = 0$ follows and thus $\mathrm{can} = 0$. Since can is a quotient functor $\mathrm{coker}(\pi) = 0$. $\square$

Following Proposition 2.3.10, we ought to make the following definition as well:

Definition 4.2.4. *Let $\mathcal{C} \in \mathcal{B}-\mathrm{Mod}$. We call $\mathcal{C}$ a* cell module *if there exists strongly indecomposable $\mathcal{B}$-functor $\mathcal{B} \to \mathcal{C}$ and if* $\mathrm{coker}(\pi) = 0$ *holds for all nonzero $\mathcal{B}$-functors $\pi : \mathcal{B} \to \mathcal{C}$.*

Remark 4.2.5. It is interesting to note that the classical counterpart $[\mathcal{C}]$ in $[\mathcal{B}]-\mathrm{Mod}^{(\mathbb{Z},\mathbb{N}_0)}$ of a cell module $\mathcal{C} \in \mathcal{B}-\mathrm{Mod}$, might not be a cell module. This can occur for cell modules that are not idempotent closed. Example 3.3.28 provides two such modules $\mathcal{M} \in \mathcal{B}-\mathrm{Mod}$ that arise from $\mathcal{B}$ (which is not a cell module) through manipulation of morphisms. These manipulations do not change the Grothendieck group of the modules and thus the classical counterparts are not cell modules. However in type A_1 (and similarly for appropriate generalisations in other types), the thus constructed modules are cell modules.

Example 4.2.6. If the unit object of $\mathcal{B}$ is strongly indecomposable, it follows that the principal representation $\mathcal{B} \in \mathcal{B}-\mathrm{Mod}$ is 1-cyclic. In this case, $\mathcal{B}$ may or may not be a cell module. Soergel bimodules provide an example in which this is not the case, whereas $\hat{G} \in \hat{G}-\mathrm{Mod}$ as introduced in Section 6.2 is a cell module provided certain assumptions on the ground field are satisfied.

The following examples gives a first motivation why we required the cyclic generator to be strongly indecomposable. We will elaborate further upon the indecomposability requirement in Example 4.2.29, as well as upon strong indecomposability in Example 4.2.28.

Example 4.2.7. Let $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$ be a 1-cyclic (resp. cell) module and let A be a unital $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$. Then, $\mathcal{C} \otimes_{\mathcal{Z}_{\mathcal{B}}(\mathcal{C})} A$ may no longer be 1-cyclic (resp. no longer be a cell module). For example, this is the case if A that has two (or more) orthogonal idempotents, since any cyclic generator of $\mathcal{C}$ will no longer be strongly indecomposable in $\mathcal{C} \otimes_{\mathcal{Z}_{\mathcal{B}}(\mathcal{C})} A$.

Remark 4.2.8. In [MaMi1], a different notion of cyclicity was proposed. Therein, an Abelian higher representation $\mathbf{M} \in \mathcal{B} - \mathrm{Mod}(\mathfrak{A}b^{\mathrm{idem}})$ is cyclic if and only if there exists $M \in \mathbf{M}$ such that for all $M' \in \mathbf{M}$ there exists $B \in \mathcal{B}$ such that $M' \cong B.M$ and such that $\hat{\mathcal{B}}(B, B') \to \mathbf{M}(B.M, B'.M)$ is surjective. Note that $\hat{\mathcal{B}}$ denotes the *Abelian envelope* of $\mathcal{B}$ which may be characterised by its universal property that it represents the forgetful functor $\mathfrak{A} - \mathrm{Mod}(\mathfrak{A}b^{\mathrm{idem}}) \to \mathfrak{A}b^{\mathrm{idem}}$. In other words, an Abelian $\mathcal{B}$-module is cyclic in the terminology of [MaMi1] if and only if it is a $\mathcal{B}$-quotient of $\hat{\mathcal{B}}$. While this is a reasonable assumption in the Abelian (which leads to the important results in [MaMi1]-[MaMi5]), it is quite a strong assumption from our additive viewpoint. Our 1-cyclic modules certainly do not satisfy such an assumptions. This will become clearer through property (2) in Definition 4.2.17, where we will introduce a subclass of 1-cyclic modules.

Let us make another comment related to the lack of a good notion of "monomorphisms" between additive categories and whether there is a better requirement than faithfulness:

Remark 4.2.9. The reader may wonder whether there is a better requirement that could be imposed on the functor $\mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M}) \to \mathcal{H}om_{\mathcal{B}}(\mathcal{B}, \mathcal{M})$. Here, better may refer to a definition that is easier to test or to a definition that implies new useful properties. A first idea that might come to mind are faithful functors that are injective on objects. A second idea would be faithful functors that reflect isomorphisms. Neither idea is useful as discussed next. See Remark 4.2.20 for another idea.
The first idea has the clear drawback that it leads to a notion of cyclicity that is not stable under equivalences. The second idea takes care of this issue. By Remark 2.3.11, the second idea can however at best hold for certain choices of strongly indecomposable $\mathcal{B}$-functors $\mathcal{B} \to \mathcal{C}$. Even if we are able to make such a good choice, we can prove this requirement only if $\mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M})$ satisfies the Krull-Schmidt property (and also use a stronger notion of cyclicity introduced in Section 4.2.4). Furthermore, the Krull-Schmidt property of the functor category $\mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M})$ neither obvious in examples nor do we expect it to hold in general.

4.2.2 Refinement for the action of a 2-category

For future reference, we extend our definition of 1-cyclicity to $\mathfrak{A}-\mathrm{Mod}$ for an arbitrary additive 2-category $\mathfrak{A}$. Similar to the classical Remark 2.1.10, one can refine the definition according to objects of $\mathfrak{A}$. Proofs are given only where new concepts are involved as compared to the $\mathcal{B}-\mathrm{Mod}$ case.

To motivate the following definition, note that the only 2-functor $\mathfrak{A}-\mathrm{Mod} \to \mathfrak{A}dd$ that may reasonably be call the forgetful functor sends $R \mapsto \prod_{a\in\mathfrak{A}} R(a)$. This functor is represented by $\bigoplus_{a\in\mathfrak{A}} \mathcal{P}_a$, the direct sum over all principal representations.

Definition 4.2.10. *Let $R \in \mathfrak{A}-\mathrm{Mod}$. We say R is* 1-cyclic *if R is trivial or if there exists a strongly indecomposable $\mathfrak{A}$-functor $\bigoplus_{a\in\mathfrak{A}} \mathcal{P}_a \to R$ with trivial cokernel. For all $a \in \mathfrak{A}$, we say that R is a-*cyclic *if there exists a strongly indecomposable $\mathfrak{A}$-functor $\mathcal{P}_a \to R$ with trivial cokernel.*

The following characterisation is the obvious generalization of Proposition 4.2.3 in the $\mathcal{B}-\mathrm{Mod}$ case:

Proposition 4.2.11. *Let $R \in \mathfrak{A}-\mathrm{Mod}$ be non-trivial. The following are equivalent:*

1. *R is 1-cyclic.*
2. *There exits $a \in \mathfrak{A}$ and a strongly indecomposable $\mathfrak{A}$-functor $\mathcal{P}_a \to R$ such that for all $R' \in \mathfrak{A}-\mathrm{Mod}$ the induced functor $\mathcal{H}om_{\mathfrak{A}}(R, R') \to \mathcal{H}om_{\mathfrak{A}}(\mathcal{P}_a, R')$ is faithful.*
3. *There exists $a \in \mathfrak{A}$ and a strongly indecomposable object $b \in R(a)$ such that for all $R' \in \mathfrak{A}-\mathrm{Mod}$ and $\sigma, \tilde{\sigma} \in \mathcal{H}om_{\mathfrak{A}}(R, R')$ the morphism $\mathcal{H}om_{R\to R'}(\sigma, \tilde{\sigma}) \to R'(a)(F(b), G(b))$ which sends $\gamma \mapsto \gamma(a)_b$ is injective.*
4. *There exists $a \in \mathfrak{A}$ and a cyclic generator $b \in R(a)$,* i.e. *a strongly indecomposable object b such that for all $a' \in \mathfrak{A}$ and $b' \in R(a')$ there exists $b \in \mathfrak{A}(a, a')$ with $b' \subset^{\oplus} R(b)b$.*
5. *There exists $a \in \mathfrak{A}$ such that R is a-cyclic.*

The objects of $\mathfrak{A}$ carry a partial order which was of course trivial in the case of a monoidal category (*i.e.* 1-object 2-category) $\mathcal{B}$:

Definition 4.2.12. *Let $a, a' \in \mathfrak{A}$. We write $a \leq a'$ if and only if there exists $b_1 \in \mathfrak{A}(a, a')$ and $b_2 \in \mathfrak{A}(a', a)$ such that $I_a \subset^{\oplus} b_2 b_1$. An equivalence class of the partial order $\leq$ is called a* cell *in $\mathfrak{A}$.*

Example 4.2.13. Let $\mathfrak{A}_{\mathfrak{g}}$ be a 2-Kac-Moody algebra. The objects of $\mathfrak{A}_{\mathfrak{g}}$ are given by the set of weights X of the underlying Kac-Moody algebra $\mathfrak{g}$. It follows from $\mathfrak{A}_{\mathfrak{sl}_2}$-theory that $\lambda \leq \lambda'$ if and only if the λ'-weight space of $\mathcal{L}(\tilde{\lambda})$ is nonzero, where $\{\tilde{\lambda}\} = W\lambda \cap X^+$. Thus, the maximal $\leq$-equivalence classes are given by the pairwise distinct orbits of 0 and all fundamental weights ω_i for $i = 1, .., \mathrm{rk}_{\mathbb{Z}}(X)$.

Proposition 4.2.14. *We have $a \leq a'$ if and only if $\mathcal{P}_a$ is a'-cyclic.*

Proof. By definition, $a \leq a'$ if and only if there exists $b_1 \in \mathfrak{A}(a, a')$ and $b_2 \in \mathfrak{A}(a', a)$ such that $I_a \subset^{\oplus} b_2 b_1$. This is equivalent to the statement that for all $a'' \in \mathfrak{A}$ and $b' \in \mathcal{P}_a(a'')$ we have $b' \cong b' \circ I_a \subset^{\oplus} b'(b_2 b_1) \cong (b' b_2) b_1$. This is precisely property (4) of Proposition 4.2.11 with a' being the object that was required to exist. Property (5) relates this to a'-cyclicity. $\square$

Corollary 4.2.15. *Let $a \leq a'$ and $R \in \mathfrak{A} - \mathrm{Mod}$ be a-cyclic. Then, R is a'-cyclic.*

Proof. This follows for example using the second criterion in Proposition 4.2.11 and the fact that the composition of two injective maps is injective. $\square$

Remark 4.2.16. In particular, if $\{a_i\}_{i \in \{1,\dots,n\}}$ are representatives of the maximal equivalence classes, then for every cyclic module R there exists $i \in \{1, \dots, n\}$ such that R is a_i-cyclic. We haven't studied under which conditions i is unique.

4.2.3 2-cyclicity

In this section, we introduce 2-cyclic $\mathcal{B}$-modules. In addition to the requirements of 1-cyclicity, the morphisms of 2-cyclic modules are controlled by the action of $\mathcal{B}$ in particular way. This somewhat surprising requirement will be justified by Theorem 4.2.19 that controls $\mathcal{B}$-transformations of $\mathcal{B}$-functors with 2-cyclic domain. For simplicity, we do not discuss 2-cyclicity in $\mathfrak{A} - \mathrm{Mod}$.

Definition 4.2.17. *Let $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$. We say $\mathcal{C}$ is* 2-cyclic *if $\mathcal{C}$ is trivial or if $\mathcal{C}$ satisfies the following properties:*

1. *There exists a* cyclic generator*, that is a strongly indecomposable object $C \in \mathcal{C}$ such that for all $C' \in \mathcal{C}$ there exists $B_{C'} \in \mathcal{B}$, $\iota_{C'} \in \mathcal{C}(C', B_{C'}.C)$ and $\pi_{C'} \in \mathcal{C}(B_{C'}.C, C')$ with $\pi_{C'} \circ \iota_{C'} = id_{C'}$.*

2. *For all $c \in \mathcal{C}(C', C'')$ there exists $b_c \in \mathcal{B}(B_{C'}, B_{C''})$ such that*

$$\begin{tikzcd} C' \arrow[r, "i_{C'}"] \arrow[d, "c"] & B_{C'}.C \arrow[r, "\pi_{C'}"] \arrow[d, "b_c.id_C"] & C' \arrow[d, "c"] \\ C'' \arrow[r, "i_{C''}"] & B_{C''}.C \arrow[r, "\pi_{C''}"] & C'' \end{tikzcd} \quad \textit{commutes.}$$

3. *For all $B \in \mathcal{B}$ and $C' \in \mathcal{C}$ there exists $\mathrm{can}_{B,C'} \in \mathcal{B}(B \otimes B_{C'}, B_{B.C'})$ such that*

$$\begin{tikzcd} B.C' \arrow[r, "B.i_{C'}"] \arrow[d, "i_{B.C'}"] & B.(B_{C'}.C) \arrow[r, "B.\pi_{C'}"] \arrow[d, "\sim"', "\mathrm{can}"] & B.C' \\ B_{B.C'}.C & (B \otimes B_{C'}).C \arrow[l, "\mathrm{can}_{B,C'}.id_C"'] \arrow[r, "can_{B,C'}.id_C"] & B_{B.C'}.C \arrow[u, "\pi_{B.C'}"'] \end{tikzcd} \quad \textit{commutes,}$$

where $\mathrm{can} : B.(B_{C'}.C) \xrightarrow{\sim} (B \otimes B_{C'}).C$ *comes from the monoidal structure functor of $\mathcal{C}$.*

Let us comment on the properties required from a non-trivial 2-cyclic module $\mathcal{C}$:

Remark 4.2.18. 1. This property corresponds to property (4) of Proposition 4.2.3. Thus, a 2-cyclic module is 1-cyclic.

2. This property states that $\mathcal{C}$ is very close to being a quotient of $\mathcal{B}$. Note however that not every morphism of $\mathcal{C}$ lies in the image of the action of $\mathcal{B}$. In examples, this does apply to the idempotent $e_{C'} := \iota_{C'} \circ \pi_{C'}$ associated to certain $C' \in \mathcal{C}$. This allows for a wider range of Grothendieck combinatorics across all 2-cyclic modules as a definition made in analogy to the one discussed in Remark 4.2.8 would.

3. We do not require that $\mathrm{can}_{B,C'}$ is an isomorphism. However if $\mathrm{can}_{B,C'}$ is an isomorphism, the idempotents $B.e_{C'}$ and $e_{B.C'}$ are conjugate. In almost all examples considered in this thesis $\mathrm{can}_{B,C'}$ is an isomorphism, compare however Remark 4.2.31.

Apart of (strong) indecomposability, the following theorem uses precisely the requirements we made in the definition of 2-cyclic modules and will allow us to calculate centres in $\mathcal{B} - \mathrm{Mod}$ in Corollary 4.2.21.

Theorem 4.2.19. *Let $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$ be 2-cyclic with cyclic generator $C \in \mathcal{C}$, let $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ and let $F, G \in \mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M})$. Then, the canonical morphism*

$$\mathcal{H}om_{\mathcal{C} \to \mathcal{M}}(F, G) \to \mathcal{M}(F(C), G(C))$$

of $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \otimes_{\mathcal{Z}^{\otimes}(\mathcal{B})} \mathcal{Z}_{\mathcal{B}}(\mathcal{M})$-modules which sends $\alpha = (\alpha_{C'})_{C' \in \mathcal{C}} \mapsto \alpha_C$ splits. An explicit splitting is given in the proof.

Proof. Since $\mathcal{C}$ is 2-cyclic, it is 1-cyclic and thus $\mathcal{H}om_{\mathcal{C} \to \mathcal{M}}(F, G) \to \mathcal{M}(F(C), G(C))$ is an injection by property (3) of Proposition 4.2.3. We repeat the proof here in order to motivate the construction of the splitting. Fix a pair $(i_{C'}, \pi_{C'})$ for each $C' \in \mathcal{C}$ as in property (1) of Definition 4.2.17. Contemplating the commutative diagram

$$\begin{tikzcd}
F(C') \arrow[rr, hook, "F(i_{C'})"] \arrow[dr, equal] \arrow[dd, "\alpha_{C'}"] & & F(B_{C'}.C) \arrow[dl, two heads, "F(\pi_{C'})"'] \arrow[dd, "\alpha_{B_{C'}.C}"] & B.F(C) \arrow[l, "\sim"', "\text{can}"] \arrow[dd, "B_{C'}.\alpha_C"] \\
& F(C') \arrow[dd, "\alpha_{C'}"] & & \\
G(C') \arrow[rr, hook, "G(i_{C'})"] \arrow[dr, equal] & & G(B_{C'}.C) \arrow[dl, two heads, "G(\pi_{C'})"'] & B_{C'}.G(C) \arrow[l, "\sim"', "\text{can}"] \\
& G(C') & &
\end{tikzcd}$$

for any $\alpha \in \mathcal{H}om(F, G)$, we observe that α_C determines $\alpha_{C'}$ for all $C' \in \mathcal{C}$, which implies injectivity. Note that so far the proof is independent of the choice of the pairs $(i_{C'}, \pi_{C'})$, however going forward the particular compatibilities required in Definition 4.2.17 will become important.

The above diagram suggests a definition for a splitting: For every $\widetilde{\alpha} \in \mathcal{M}(F(C), G(C))$, we define a $\mathcal{B}$-transformation $\alpha = (\alpha_{C'})_{C' \in \mathcal{C}}$ by specifying $\alpha_{C'}$ for every $C' \in \mathcal{C}$ as the composition $F(C') \xhookrightarrow{F(i_{C'})} F(B_{C'}.C) \xleftarrow[\text{can}]{\sim} B_{C'}.F(C) \xrightarrow{B.\widetilde{\alpha}} B_{C'}.G(C) \xrightarrow[\text{can}]{\sim} G(B_{C'}.C) \xrightarrow{G(\pi_{C'})} G(C')$. It remains to check that this splitting is well defined, *i.e.* that $\alpha \in \mathcal{H}om_{\mathcal{C} \to \mathcal{M}}(F, G)$. First, we need to check that it is indeed a transformation. Second, we need to check compatibility with the $\mathcal{B}$-action.

For the first task, we fix for every $c \in \mathcal{C}(C', C'')$ a morphism $b_c \in \mathcal{B}(B_{C'}, B_{C''})$ as in property (2) of Definition 4.2.17. The diagram

$$\begin{tikzcd}
F(C') \arrow[r, hook, "F(i_{C'})"] \arrow[d, "F(c)"] & F(B_{C'}.C) \arrow[d, "F(b_c.id_C)"] & B_{C'}.F(C) \arrow[l, "\sim"', "\text{can}"] \arrow[r, "B_{C'}.\widetilde{\alpha}"] \arrow[d, "b_c.F(id_C)"] & B_{C'}.G(C) \arrow[r, "\sim", "\text{can}"'] \arrow[d, "b_c.G(id_C)"] & G(B_{C'}.C) \arrow[r, two heads, "G(\pi_{C'})"] \arrow[d, "G(b_c.id_C)"] & G(C') \arrow[d, "G(c)"] \\
F(C'') \arrow[r, hook, "F(i_{C''})"] & F(B_{C''}.C) & B_{C''}.F(C) \arrow[l, "\sim"', "\text{can}"] \arrow[r, "B_{C''}.\widetilde{\alpha}"] & B_{C''}.G(C) \arrow[r, "\sim", "\text{can}"'] & G(B_{C''}.C) \arrow[r, two heads, "G(\pi_{C''})"] & G(C'')
\end{tikzcd}$$

commutes for all $c \in \mathcal{C}(C', C'')$. The first and last square commutes by property (2) and functoriality of F and G. The second and fourth square commute since F, G are $\mathcal{B}$-functors and the square in the middle commutes due to functoriality of the $\mathcal{B}$-action.

For the second task, we fix for every pair $(B, C') \in \mathcal{B} \times \mathcal{C}$ a morphism $\text{can}_{B,C'} \in$

$\mathcal{B}(B \otimes B_{C'}, B_{B.C'})$ using property (3) of Definition 4.2.17. Now, we can deduce that α is a $\mathcal{B}$-transformation from the following commutative diagram:

$$
\begin{tikzcd}
B.F(C') \arrow[rr, "\sim", "\mathrm{can}"'] \arrow[d, hook, "B.F(i_{C'})"] & & F(B.C') \arrow[dl, "F(B.i_{C'})"'] \arrow[d, hook, "F(i_{B.C'})"] \\
B.F(B_{C'}.C) \arrow[r, "\sim", "\mathrm{can}"'] & F(B.B_{C'}.C) \arrow[r, "\sim", "F(can_{B,C'}.id_C)"'] & F(B_{B.C'}.C) \\
B.B_{C'}.F(C) \arrow[u, "\sim", "B.\,\mathrm{can}"'] \arrow[ur, "\sim", "\mathrm{can}"'] \arrow[rr, "\sim", "\mathrm{can}_{B,C'}.id_{F(C)}"'] \arrow[d, "B.B_{C'}.\widetilde{\alpha}"] & & B_{B.C'}.F(C) \arrow[u, "\sim"', "\mathrm{can}"] \arrow[d, "B_{B.C'}.\widetilde{\alpha}"] \\
B.B_{C'}.G(C) \arrow[rr, "\sim", "\mathrm{can}_{B,C'}.id_{G(C)}"'] \arrow[d, "\sim"', "B.\,\mathrm{can}"] \arrow[dr, "\sim", "\mathrm{can}"'] & & B_{B.C'}.G(C) \arrow[d, "\sim"', "\mathrm{can}"] \\
B.G(B_{C'}.C) \arrow[r, "\sim", "\mathrm{can}"'] \arrow[d, "B.G(\pi_{C'})"] & G(B.B_{C'}.C) \arrow[r, "\sim", "G(\mathrm{can}_{B,C'}.id_C)"'] \arrow[dr, "G(B.\pi_{C'})"'] & G(B_{B.C'}.C) \arrow[d, "G(\pi_{B.C'})"] \\
B.G(C') \arrow[rr, "\sim", "\mathrm{can}"'] & & G(B.C')
\end{tikzcd}
.
$$

The two triangles in the left half of the diagram and the central square commute since F and G are $\mathcal{B}$-functors, *i.e.* compatible with the associativity constraint of $\mathcal{B}$, and since F and G are compatible with the structure functor $\mathcal{B} \to \mathrm{End}(\mathcal{M})$ of $\mathcal{M}$. The two triangles in the right half of the diagram commute due to property (3) of Definition 4.2.17 and functoriality of F and G. The remaining squares commute due to further properties required in the definition of $\mathcal{B}$-functors.

That the thus constructed splitting is a $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \otimes_{\mathcal{Z}^{\otimes}(\mathcal{B})} \mathcal{Z}_{\mathcal{B}}(\mathcal{M})$-module homomorphism, can be checked easily and we leave this to the reader. $\square$

Remark 4.2.20. Similar to Proposition 4.2.3, it may be possible to characterise 2-cyclic modules in terms of a $\mathcal{B}$-functor $\mathcal{B} \to \mathcal{C}$ such that for all $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ the induced functor $\mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M}) \to \mathcal{H}om_{\mathcal{B}}(\mathcal{B}, \mathcal{M})$ satisfies certain properties. Certainly, these functors would need to be faithful and would also need to have some property related to splitness. We have not attempted to work this out.

Recall that $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ denotes the centre of $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$, see Definition 4.1.1. Putting $F = G = id_{\mathcal{C}}$, we obtain the following corollary. Note that the splitting in the statement is not a splitting of rings.

Corollary 4.2.21. *We have a split injection $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \hookrightarrow \mathcal{C}(C, C)$. A splitting is given by the assignment $c \mapsto \alpha(c) = (\alpha(c)_{C'})_{C' \in \mathcal{C}}$ where $\alpha(c)_{C'} := \pi_{C'} \circ (B_{C'}.c) \circ \iota_{C'}$.*

Remark 4.2.22. In the next section, we introduce a special class of 2-cyclic modules and show in Section 4.3.4 that they represent certain 2-functors. We encourage the reader to check that Theorem 4.3.53 implies Theorem 4.2.19 for this special class. The reader should also note that 4.3.53 provides additional information, since it controls $\mathcal{B}$-functors that have this special class as the domain in addition to $\mathcal{B}$-transformations between such $\mathcal{B}$-functors.

4.2.4 Functorially cyclic modules

In this section, we introduce functorially cyclic modules and show that they form a subclass of 2-cyclic modules. One clear benefit of this definition is that it can be verified easily in many examples.

Definition 4.2.23. *Let $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$. We call $\mathcal{C}$* functorially cyclic *if there exist a quotient $\mathcal{B}$-functor $\pi : \mathcal{B} \longrightarrow \overline{\mathcal{B}}$, $\mathcal{B}$-functors $\mathcal{C} \xrightarrow{G} \overline{\mathcal{B}} \xrightarrow{F} \mathcal{C}$ with F being strongly indecomposable or trivial, and $\mathcal{B}$-transformations $id_{\mathcal{C}} \overset{\alpha}{\Longrightarrow} FG \overset{\beta}{\Longrightarrow} id_{\mathcal{C}}$ such that $\beta \circ \alpha = id_{\mathcal{C}}$. A functorially cyclic module is called* semi-free *if $\overline{\mathcal{B}}$ is a quotient of $\mathcal{B}$ by a thick ideal. A functorially cyclic module is called* free *if $\pi = id_{\mathcal{B}}$.*

Notation 4.2.24. To simplify the notation, we call a functorially cyclic module $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$ interchangeably (F, G)-*cyclic*, (F, G, α, β)-*cyclic*, $(F, G, (\overline{\mathcal{B}}, \pi))$-*cyclic* or $(F, G, \alpha, \beta, (\overline{\mathcal{B}}, \pi))$-*cyclic* depending on the level of detail that is required.

Remark 4.2.25. Functorially cyclic modules in $\mathfrak{A} - \mathrm{Mod}$ are defined analogously, where we have to use $\bigoplus_{a \in \mathfrak{A}} \mathcal{P}_a$ as the domain of the quotient $\mathfrak{A}$-functor π. Note that since the structure functor F is assumed to be strongly indecomposable, so is $F\pi$, which thus has to factor over $\mathcal{P}_a$ for a unique $a \in \mathfrak{A}$. Furthermore the proof of Proposition 4.2.14 implies that we can change a (and the other structure maps) so that a is minimal with respect to the order $\leq$ of Definition 4.2.12.

Example 4.2.26. Let $\mathcal{I} \trianglelefteq \mathcal{P}_a \in \mathfrak{A} - \mathrm{Mod}$. A sufficient condition for $\mathcal{P}_a^{\mathcal{I}}$ to be functorially cyclic is that the image of $I_a \in \mathcal{P}_a(a) = \mathfrak{A}(a, a)$ in $\mathcal{P}_a^{\mathcal{I}}(a) = \mathfrak{A}(a, a)/\mathcal{I}(a)$ under the canonical quotient $\mathfrak{A}$-functor is strongly indecomposable or trivial.

Remark 4.2.27. It is a challenging task to classify all functorially cyclic $\mathfrak{A}$-modules. How challenging it is depends heavily on the complexity of $\mathfrak{A}$ and what type of classification one is aiming for. One challenge is posed by the non-uniqueness of the quintuple $(F, G, \alpha, \beta, (\overline{\mathcal{B}}, \pi))$ which we will discuss in Section 4.2.5. For Soergel bimodules of type A_1, we classified all $\mathcal{B}$-quotients $\overline{\mathcal{B}}$ in terms of generators of the associated ideal while keeping track of the conditions that the generators of the different morphism spaces impose on each other. To obtain such a detailed description for Soergel bimodules of arbitrary type seems rather difficult. In Section 4.5, we provide several results towards this goal. To achieve this goal, we will first have to introduce the notion of split Frobenius objects in Section 4.3 and study them in great detail.

The following example of a $\mathcal{B}$-module that we certainly would not want to call cyclic highlights the importance of the indecomposability requirement on F in Definition 4.2.23, as well as the analogous requirements in other cyclicity notions introduced earlier.

Example 4.2.28. We have $\mathcal{B}$-functors $\mathcal{B}\times\mathcal{B} \xrightarrow{\oplus} \mathcal{B} \xrightarrow{\Delta} \mathcal{B}\times\mathcal{B}$, given by $(B_1, B_2) \mapsto B_1 \oplus B_2$ and $B \mapsto (B, B)$. The composition $\Delta\oplus$ sends (B_1, B_2) to $(B_1 \oplus B_2, B_1 \oplus B_2)$. As $\mathcal{B}$-functors, $id_{\mathcal{B}\times\mathcal{B}}$ is a direct summand of $\Delta\oplus$ due to the diagram

$$\begin{tikzcd}
(B_1,B_2) \arrow[r, "(i_1,i_2)"] \arrow[d, "(b_1,b_2)"] & (B_1\oplus B_2,B_1\oplus B_2) \arrow[r, "(\pi_1,\pi_2)"] \arrow[d, "(b_1\oplus b_2,b_1\oplus b_2)"] & (B_1,B_2) \arrow[d, "(b_1,b_2)"] \\
(B_1',B_2') \arrow[r, "(i_1,i_2)"] & (B_1'\oplus B_2',B_1'\oplus B_2') \arrow[r, "(\pi_1,\pi_2)"] & (B_1',B_2')
\end{tikzcd}$$

that commutes for all $(b_1, b_2) \in \mathcal{B}\times\mathcal{B}\,((B_1, B_2), (B_1', B_2'))$ and where we denoted by i_j the canonical inclusion $\mathcal{B}$-functor into (resp. by π_j the canonical projection $\mathcal{B}$-functor from) the j-th factor for $j \in \{1, 2\}$. However, $\Delta = \iota_1 \oplus \iota_2$ where ι_1 (resp. ι_2) is the $\mathcal{B}$-functor given by $B \mapsto (B, 0)$ (resp. $B \mapsto (0, B)$), *i.e.* Δ is not indecomposable.

Let us refine this example a little further to also highlight the importance of strong indecomposability:

Example 4.2.29. Consider the diagonal subcategory $\mathcal{D}$ of $\mathcal{B}\times\mathcal{B}$, *i.e.* the full subcategory with objects given by $\{(B, B)|B \in \mathcal{B}\}$. As before, we have $\mathcal{B}$-functors $\mathcal{D} \xrightarrow{\oplus} \mathcal{B} \xrightarrow{\Delta} \mathcal{D}$ and $id_{\mathcal{D}} \subset^{\oplus} \Delta\oplus$. What is different is that Δ is now indecomposable, however Δ is not strongly indecomposable. Thus, $\mathcal{D}$ is not functorially cyclic either. We would like to highlight the importance of this requirement even further by pointing out that this example extends to n-fold products.

Proposition 4.2.30. *Functorially cyclic modules are 2-cyclic.*

Proof. Let $\mathcal{C}$ be $(F, G, \alpha, \beta, (\overline{\mathcal{B}}, \pi))$-cyclic. We have a diagram

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{B}}(\mathcal{C},\mathcal{M}) \arrow[r, hook, shift left, "-F"] & \mathcal{H}om_{\mathcal{B}}(\overline{\mathcal{B}},\mathcal{M}) \arrow[l, shift left, "-G"] \arrow[r, hook, "-\pi"] & \mathcal{H}om_{\mathcal{B}}(\mathcal{B},\mathcal{M})
\end{tikzcd}$$

The functor $-F$ is faithful, since $-id_{\mathcal{C}} = id_{\mathcal{H}om_{\mathcal{B}}(\mathcal{C},\mathcal{M})}$ is a direct summand of the composition $(-G)(-F) = -(FG)$. The functor $-\pi$ is fully faithful since π is full and essentially surjective. As a composition of faithful functors, $-\pi F$ is faithful as well. Furthermore, as F is strongly indecomposable, the same holds for $F\pi$. Thus, $\mathcal{C}$ is 1-cyclic by property (2) of Proposition 4.2.3. The same proposition also shows that property (1) of Definition 4.2.17 is satisfied.

Let us give another proof of property (1) of Definition 4.2.17 which will guide us towards establishing properties (2) and (3) of the same Definition and thus proving this proposition. Define $C := F\pi(I)$ and for all $C' \in \mathcal{C}$ set $B_{C'} := G(C')$, where we identified $\mathrm{Ob}(\overline{\mathcal{B}}) = \mathrm{Ob}(\mathcal{B})$ w.l.o.g in the latter definition. First note that C is strongly indecomposable, since $\mathcal{C}(C,C) \cong \mathcal{H}om_{\mathcal{B}\to\mathcal{C}}(F\pi, F\pi)$ by the 2-Yoneda Lemma 3.3.49 together with our assumption that F is strongly indecomposable. Next for all $C' \in \mathcal{C}$, define a morphism $i_{C'} : C' \to B_{C'}.C$ as the composition

$$C' \xrightarrow{\alpha_{C'}} FG(C') \xrightarrow{F\left(\pi(u^r_{G(C)})^{-1}\right)} F(G(C') \otimes I) = F(G(C').I) \xrightarrow{(can^F_{G(C'),I})^{-1}} G(C').F(I) = B_{C'}.C\ ,$$

where $u^r_B : B \otimes I \xrightarrow{\sim} B$ denotes the right unit transformation of the monoidal category $\mathcal{B}$ evaluated at B, and $can^F_{B,\overline{B}} : F(B.\overline{B}) \xrightarrow{\sim} B.F(\overline{B})$ denotes the canonical morphism associated with the $\mathcal{B}$-functor F and any $B \in \mathcal{B}$ and $\overline{B} \in \overline{\mathcal{B}}$. We define $\pi_{C'} := \beta_{C'} \circ can^F_{G(C'),I} \circ F(\pi(u^r_{G(C)}))$ analogously. Obviously, $\pi_{C'} \circ \iota_{C'} = \beta_{C'} \circ \alpha_{C'} = id_{C'}$ holds. Thus, $F(I)$ is a cyclic generator of $\mathcal{C}$ and we have established property (1) a second time.

Property (2) required in Definition 4.2.17 can now be derived as follows: For all morphism c in $\mathcal{C}$, we choose a preimage b_c under π of $G(c)$. The diagrams required in property (2) commute since $\iota_{C'}$ and $\pi_{C'}$ are constructed using the transformations α and β and natural isomorphisms.

Property (3) of Definition 4.2.17 can now be verified as follows. Define the morphism $\mathrm{can}_{B,C'} := \mathrm{can}^G_{B,C'} : B \otimes B_{C'} = B \otimes G(C') \xrightarrow{\sim} G(B.C') = B_{B.C'}$. Now, contemplate the commutative diagram

$$\begin{tikzcd}
B.C' \arrow[r, "B.\alpha_{C'}"] \arrow[d, "\alpha_{B.C'}"] \arrow[rrr, bend left, "B.\iota_{C'}"] \arrow[ddd, bend right, "\iota_{B.C'}"'] & B.F(G(C')) \arrow[d, "can"] \arrow[dl, "can"] & B.F(G(C')\otimes I) \arrow[l, "B.F(u^r_{G(C')})"'] \arrow[d, "can"] & B.(G(C').F(R)) \arrow[l, "B.can"'] \arrow[ddd, "can"] \\
F(G(B.C')) & F(B\otimes G(C')) \arrow[l, "F(can)"'] & F(B\otimes(G(C')\otimes I)) \arrow[l, "F(B.u^r_{G(C')})"'] & \\
F(G(B.C')\otimes I) \arrow[u, "F(u^r_{G(B.C')})"] & & F((B\otimes G(C'))\otimes I) \arrow[ll, "F(can\otimes id_I)"'] \arrow[u, "F(a_{(B,G(C'),R)})"'] \arrow[ul, "F(u^r_{B\otimes G(C')})"] & \\
G(B.C').F(R) \arrow[u, "can"] & & & (B\otimes G(C')).F(R) \arrow[lll, "can.id_{F(R)}"'] \arrow[ul, "can"']
\end{tikzcd}$$

to deduce that the left hand side of the diagram in property (3) of Definition 4.2.17 commutes. The right hand side of the same diagram commutes by an analogous argument, where one uses the above diagram with all arrows that involve α (resp. ι) reversed and α (resp. ι) replaced by β (resp. π). $\square$

Remark 4.2.31. Note that $\mathrm{can}_{B,C'}$ as defined in the previous proof is an isomorphism which we had not required in Definition 4.2.17. Motivated by this observation, one may define a larger class of *lax-functorially cyclic modules* by requiring G to be a lax-$\mathcal{B}$-functor, *i.e.* a $\mathcal{B}$-functor for which the associated 2-arrows $\mathrm{can}^G_{B,C'}$ are not necessarily invertible. Clearly, such lax-functorially cyclic modules are 2-cyclic. In the only example of a lax-functorially cyclic module with truly lax $\mathcal{B}$-functor G that we are aware of, one can replace G by a different $\mathcal{B}$-functor, see Remark 6.3.8. Thus, we do not provide any details. We would like to note however that one can extend most results of the following sections including Section 4.3.4 to this lax setting using an appropriate notion of lax-Frobenius objects.

Let us restate Theorem 4.2.19 in the functorially cyclic setting.

Corollary 4.2.32. *Let $\mathcal{C} \in \mathcal{B}-\mathrm{Mod}$ be $(F, G, \alpha, \beta, (\overline{\mathcal{B}}, \pi))$-cyclic. Denote $C := F\pi(I)$ and $B_{C'} := G(C') \in \mathcal{B}$ for all $C' \in \mathcal{C}$. Given H_1, $H_2 \in \mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M})$, we have a split injection*

$$\mathcal{H}om_{\mathcal{C}\to\mathcal{M}}(H_1, H_2) \hookrightarrow \mathcal{M}(H_1(C), H_2(C)).$$

which sends $\alpha \mapsto \alpha_C$. A splitting $m \mapsto \alpha(m) = (\alpha(m)_{C'})_{C'\in\mathcal{C}}$ can be defined by setting $\alpha(m)_{C'} = H_2(\beta_{C'}) \circ H_2F\left(\overline{u^r_{B_{C'},I}}\right) \circ \mathrm{can}^{H_2F}_{B_{C'},I} \circ (B_{C'}.m) \circ \left(\mathrm{can}^{H_1F}_{B_{C'},I}\right)^{-1} \circ H_1F\left(\overline{(u^r_{B_{C'},I})^{-1}}\right) \circ H_1(\alpha_{C'})$ for all $C' \in \mathcal{C}$.

Remark 4.2.33. Let $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$ be (F, G)-cyclic, let H_1, $H_2 \in \mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M})$ and assume that $FH_1 \cong FH_2$. Then, we have $H_1 \subset^{\oplus} GFH_1 \cong GFH_2 \supset^{\oplus} H_2$, however this does not imply $H_1 \cong H_2$ in general. Compare Remark 4.2.37 for some intuition on this point. If $\mathcal{H}om_{\mathcal{B}}(\mathcal{C}, \mathcal{M})$ satisfies the Krull-Schmidt property and $GF \cong id_{\mathcal{C}} \oplus ... \oplus id_{\mathcal{C}}$, one can conclude that $H_1 \cong H_2$.

4.2.5 Choices for functorially cyclic modules

In this section, we discuss the non-uniqueness of the quintuple $(F, G, \alpha, \beta, (\overline{\mathcal{B}}, \pi))$ that exist for every functorially cyclic $\mathcal{B}$-module.

Remark 4.2.34. In order to show that the trivial module is functorially cyclic module, we can choose an arbitrary $\mathcal{B}$-quotient $\overline{\mathcal{B}}$ of $\mathcal{B}$. The remaining structure maps are then unique. For the remainder of the section treat the case of non-trivial functorially cyclic modules.

Remark 4.2.35. For all 1- or 2-cyclic modules $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$ with cyclic generator $C_1 \in \mathcal{C}$, the following statement can be deduced directly from the definitions: Let $B \in \mathcal{B}$ and let $C_2 \in \mathcal{C}$ be a strongly indecomposable object such that $C_1 \subset^{\oplus} B.C_2$. Then, C_2 is a cyclic generator of $\mathcal{C}$, as well.

The analogous statement for functorially cyclic modules requires some additional assumptions:

Proposition 4.2.36. *Let $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$ be $(F, G, (\overline{\mathcal{B}}, \pi))$-cyclic and let $C \in \mathcal{C}$ be an strongly indecomposable object such that $F(I) \subset^{\oplus} B.C$ for some $B \in \mathcal{B}$. Assume that $\otimes B : \mathcal{B} \to \mathcal{B}$ induces a functor $\overline{\otimes B} : \overline{\mathcal{B}} \to \overline{\mathcal{B}}$ and that $\mathrm{ev}_C : \mathcal{B} \to \mathcal{C}$ induces a functor $\overline{\mathrm{ev}_C} : \overline{\mathcal{B}} \to \mathcal{C}$. Then $\mathcal{C}$ is $(\overline{\mathrm{ev}_C}, \overline{\otimes B} \circ G, (\overline{\mathcal{B}}, \pi))$-cyclic.*

Proof. Without loss of generality, let us identify $\mathrm{Ob}(\mathcal{B}) = \mathrm{Ob}(\overline{\mathcal{B}})$. Then $F \circ \pi \cong \mathrm{ev}_{F(I)} \subset^{\oplus} \mathrm{ev}_{B.C} \cong \mathrm{ev}_{(I \otimes B).C} = (\overline{\mathrm{ev}_C} \circ \overline{\otimes B}) \circ \pi$ as $\mathcal{B}$-functors. Since π is a quotient $\mathcal{B}$-functor this implies that $F \subset^{\oplus} \overline{\mathrm{ev}_C} \circ \overline{\otimes B}$ as $\mathcal{B}$-functors. This allows us to conclude that $id_{\mathcal{C}} \subset^{\oplus} F \circ G \subset^{\oplus} \overline{\mathrm{ev}_C} \circ (\overline{\otimes B} \circ G)$ as $\mathcal{B}$-functors. It remains to show that $\overline{\mathrm{ev}_C}$ is indecomposable. Since π is a quotient $\mathcal{B}$-functor this is equivalent to $\mathrm{ev}_C = \overline{\mathrm{ev}_C} \circ \pi$ being strongly indecomposable, which in turn is equivalent to C being strongly indecomposable. The claim follows. $\square$

The properties of a particular choice of the pair (F, G) and the associated generator can differ. The following situation arises in examples, see Section 5.5 and the related Remark 2.3.11.

Remark 4.2.37. Let $\mathcal{C}$ be a $\mathcal{B}$-module that is (F_1, G_1)-cyclic and (F_2, G_2)-cyclic. Assume $F_1G_1 \cong id_{\mathcal{C}} \oplus id_{\mathcal{C}}$ and $F_2G_2 \cong id_{\mathcal{C}} \oplus S$ for some $S \in \mathcal{E}nd_{\mathcal{B}}(\mathcal{C})$. Assume further S is not isomorphic to $id_{\mathcal{C}}$ and that $SS \cong id_{\mathcal{C}}$. Since $F_2 id_{\mathcal{C}} \cong F_2 S$ while $F_1 id_{\mathcal{C}}$ and $F_1 S$ are not isomorphic, F_1 can distinguish between $id_{\mathcal{C}}$ and S, whereas F_2 cannot.

There may also exist many choices of α and β:

Remark 4.2.38. The composition FG might contain several copies of the identity functor as direct summands. In this case, there are many choices for α and β. Section 5.3 provides a concrete example. Note that this example is a true example only after forgetting the grading, however can be adopted to a true example by shifting the F and G.

The following proposition shows that $\overline{\mathcal{B}}$ may be replaced by a canonical quotient. See Remark 4.5.40 for a more concrete example.

Proposition 4.2.39. *Let $\mathcal{C}$ be $(F, G, (\overline{\mathcal{B}}, \pi))$-cyclic. Denote by $\tilde{F}$ the canonical $\mathcal{B}$-morphism* $\operatorname{im} F \to \mathcal{C}$*, by $\tilde{G}$ the composition* $\mathcal{C} \xrightarrow{G} \overline{\mathcal{B}} \to \operatorname{im} F$ *and by $\tilde{\pi}$ the composition* $\mathcal{B} \xrightarrow{\pi} \overline{\mathcal{B}} \to \operatorname{im} F$*. Then, $\mathcal{C}$ is $(\tilde{G}, \tilde{F}, (\operatorname{im} F, \tilde{\pi}))$-cyclic. In particular, $\overline{\mathcal{B}}$ can always be replaced by* $\operatorname{im} F$.

Proof. Since both π and the canonical morphism $\operatorname{can} : \overline{\mathcal{B}} \to \operatorname{im} F$ are full and essentially surjective, so is $\widetilde{\pi}$. Further, $id_{\mathcal{C}} \subset^{\oplus} FG \cong \widetilde{F} \operatorname{can} G = \widetilde{F}\widetilde{G}$. Since can is a quotient $\mathcal{B}$-functor, it follows that F is strongly indecomposable if and only if $\widetilde{F}$ is strongly indecomposable. □

4.3 Split Frobenius objects and the additive Barr-Beck theorem

In this section, we introduce split Frobenius objects and discuss the related additive Barr-Beck theorem in various levels of complexity. Section 4.3.1 introduces (right) indecomposable split Frobenius objects and provides several examples, including how to associate one such object to a non-trivial functorially cyclic module. Section 4.3.2 studies split Frobenius objects on additive categories, establishes the additive Barr-Beck theorem, and comments on the related literature. Section 4.3.3 generalises to a compatible family of Frobenius objects on idempotent closed additive categories or in other words to a split Frobenius object on a 2-functor with codomain $\mathfrak{A}dd^{\mathrm{idem}}$. Section 4.3.4 specialises to additive universal properties of idempotent closed additive higher representations, and shows that representability is preserved after applying the Frobenius construction. Section 4.3.5 assigns to an additive monoidal category its so called split Frobenius 2-category. Section 4.3.6 introduces the Frobenius preorder and strong equivalences of Frobenius objects.

4.3.1 Split Frobenius objects

In this section, we introduce (split) Frobenius objects, as well as several notions of indecomposability, which will become important later on. Further in Proposition 4.3.20, we associate an indecomposable, split Frobenius object to any non-trivial functorially cyclic module. Comments on related notions in the literature follow at the end of the section.

Definition 4.3.1. *Let $\mathcal{D}$ be a monoidal category, let $\mathfrak{D}$ be a 2-category and let $d \in \mathfrak{D}$. A* Frobenius object in $\mathcal{D}$ *is a triple $\hat{\Theta} = (\Theta, \Delta, \mu)$, where $\Theta \in \mathcal{D}$, $\Delta \in \mathcal{D}(\Theta, \Theta\Theta)$ and $\mu \in \mathcal{D}(\Theta\Theta, \Theta)$ such that $(\Theta\mu) \circ \mathrm{can} \circ (\Delta\Theta) = \Delta \circ \mu = (\mu\Theta) \circ \mathrm{can}^{-1} \circ (\Theta\Delta)$, where $\mathrm{can} : (\Theta\Theta)\Theta \xrightarrow{\sim} \Theta(\Theta\Theta)$ comes from the associator of $\mathcal{D}$. We say that a Frobenius object is* split *if $\mu \circ \Delta = \Theta$. Further, we define a* (split) Frobenius object on d *to be a (split) Frobenius object in the monoidal category $\mathfrak{D}(d, d)$.*

Definition 4.3.2. *Let $\hat{\Theta}_1$ and $\hat{\Theta}_2$ be Frobenius objects either in a monoidal category or on an object of a 2-category. We say that $\hat{\Theta}_1$* and *$\hat{\Theta}_2$* are isomorphic*, if there exists an isomorphism $\tau : \Theta_1 \to \Theta_2$ such that $\Delta_2 \circ \tau = (\tau\tau) \circ \Delta_1$ and $\tau \circ \mu_1 = \mu_2 \circ (\tau\tau)$.*

Example 4.3.3. The unit object I of any additive monoidal category can be equipped with the structure a split Frobenius object by defining μ as the left (or equivalently right) unit transformation evaluated at I and by defining $\Delta := \mu^{-1}$.

Let us give a non-example:

Example 4.3.4. Given a triple (Θ, Δ, μ) with $\mu \circ \Delta = \Theta$, the Frobenius relations are far from trivial. To give a simple example in which they do not hold, consider the tensor algebra $T(V)$ of a k-vector space V, where k is a commutative ring with $2 \in k^*$. Then, we have linear maps $\Delta : T(V) \to T(V) \otimes_k T(V)$ given by $v \mapsto v \otimes 1 + 1 \otimes v$ and $\mu : T(V) \otimes_k T(V) \to T(V)$ given by $v \otimes u \mapsto \frac{1}{2}uv$.

The following will be revisited several times to highlight subtleties about split Frobenius objects:

Example 4.3.5. Let k be a field, and let $\mathcal{B}_2$ denote the additive monoidal category $\mathcal{V}ect_k$ equipped with $\oplus$ as the tensor product. Fix $V \in \mathcal{B}_2$. For n, $m \in \mathbb{N}$, denote by $\iota_n : V \to V^{\oplus n}$ the inclusion into the n-th factor, by $\iota_n : V \to V^{\oplus n}$ the projection onto the n-th factor, by $\iota_{n,m} := \iota_m \circ \pi_n$ and by $\pi_{n,m} := \pi_m \circ \iota_n$. Then, $\hat{V}_1 := (V, \iota_1, \pi_1)$, $\hat{V}_2 := (V, \iota_2, \pi_2)$ and $\hat{V}_{S_2} = (V \oplus V, \iota_{1,1} \oplus \iota_{2,4}, \pi_{1,1} \oplus \pi_{4,2})$ are split Frobenius objects. It will become clear later that $\hat{V}_{S_2}$ arises from an action of the symmetric group S_2. Further, note that $\hat{k}_1$ and $\hat{k}_2$ are the only split Frobenius structures with which we can equip k.

Remark 4.3.6. One might be tempted to define a category of (split) Frobenius objects whose invertible morphisms correspond to Definition 4.3.2. For our purposes it is however more appropriate to regard (split) Frobenius objects as objects of a 2-category, see Remark 4.3.22, and we will recover Definition 4.3.2 naturally in this setting, see Remark 4.3.46. Note however that the non-isomorphic split Frobenius objects $\hat{k}_1$ and $\hat{k}_2$ of Example 4.3.5 are not equivalent, though one of them is equivalent to $\hat{k}_{\Sigma_2}$.

The following property will be useful throughout.

Lemma 4.3.7. *Let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a split Frobenius object. Then, μ is associative and Δ is coassociative.*

Proof. Let $\hat{\Theta}$ be a split Frobenius object (either in a monoidal category or on an object of a 2-category). W.l.o.g, we assume that the monoidal category is strict. We have

$$\begin{aligned}
\mu \circ (\mu\Theta) &= \mu \circ \Delta \circ \mu \circ (\mu\Theta) && (\hat{\Theta} \text{ is split})\\
&= \mu \circ (\Theta\mu) \circ (\Delta\Theta) \circ (\mu\Theta) && (\text{Frobenius relation})\\
&= \mu \circ (\Theta\mu) \circ ((\Delta \circ \mu)\Theta) && (\text{Remark 3.1.3})\\
&= \mu \circ (\Theta\mu) \circ (((\mu\Theta) \circ (\Theta\Delta))\Theta) && (\text{Frobenius relation})\\
&= \mu \circ (\Theta\mu) \circ (\mu\Theta\Theta) \circ (\Theta\Delta\Theta) && (\text{Remark 3.1.3})\\
&= \mu \circ (\mu\mu) \circ (\Theta\Delta\Theta) && (\text{Remark 3.1.3})
\end{aligned}$$

and repeating analogous steps in the opposite order we arrive at $\mu \circ (\mu\Theta) = \mu \circ (\Theta\mu)$. The same argument shows $(\Delta\Theta) \circ \Delta = (\Theta\Delta) \circ \Delta$. $\square$

Notation 4.3.8. Let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a (split) Frobenius object in $\mathcal{D}_1$ and let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a monoidal functor. We denote the induced (split) Frobenius object $\left(F(\Theta), \operatorname{can}^F_{\Theta,\Theta} \circ F(\Delta), F(\mu) \circ \left(\operatorname{can}^F_{\Theta,\Theta}\right)^{-1}\right)$ in $\mathcal{D}_2$ by $F(\hat{\Theta})$.

The following example is presented for strict $\mathcal{B}$, however the obvious generalisation to the non-strict case is equally valid.

Example 4.3.9. Let $\hat{\Theta}_1 = (\Theta_1, \Delta_1, \mu_1)$ and $\hat{\Theta}_2 = (\Theta_2, \Delta_2, \mu_2)$ be Frobenius objects in $\mathcal{B}$. Then, $\hat{\Theta}_1- = (\Theta_1-, \Delta_1-, \mu_1-)$ and $-\hat{\Theta}_2 = (-\Theta_2, -\Delta_2, -\mu_2)$ are Frobenius objects on $\mathcal{B}$. The notation $\hat{\Theta}_1-$ shall suggest the use of the monoidal left multiplication functor $\mathcal{B} \to \mathcal{E}nd(\mathcal{B})$ given on objects by $B \mapsto B- := (B' \mapsto BB')$. Similarly, $-\hat{\Theta}_2$ uses the right multiplication functor $\mathcal{B}^{\text{rev}} \to \mathcal{E}nd(\mathcal{B})$. Since the objects underlying objects of $\hat{\Theta}_1-$ and $-\hat{\Theta}_2$ commute, we obtain a new Frobenius object $\widetilde{\Theta_1\Theta_2} := (\Theta_1 - \Theta_2, \Delta_1 - \Delta_2, \mu_1 - \mu_2)$. Note that the composition of the underlying object of $\widetilde{\Theta_1\Theta_2}$ with itself is given by $\Theta_1\Theta_1 - \Theta_2\Theta_2$.

Definition 4.3.10. *Let $\mathcal{M} \in \mathfrak{A}dd$ and let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a Frobenius object on $\mathcal{M}$. A morphism $f \in \mathcal{M}(\Theta M_1, \Theta M_2)$ satisfies* $\hat{\Theta}$-descent *if $f = \mu_{M_2} \circ (\Theta f) \circ \Delta_{M_1}$.*

Notation 4.3.11. Whenever we speak of a morphism that satisfies $\hat{\Theta}$-decent, we understand implicitly that the morphism is a morphism between objects in the image of the underlying object of $\hat{\Theta}$. If we want to emphasise that a morphism is a morphism between objects in the image of the underlying object of $\hat{\Theta}$, we say that the morphism is *eligible for $\hat{\Theta}$-descent*.

Definition 4.3.12. *Let $(\mathcal{M}, \rho) \in \mathcal{B} - Mod$ and let $\hat{\Theta}$ be a Frobenius object in $\mathcal{B}$. A morphism in $\mathcal{M}$ satisfies* $\hat{\Theta}$-descent *if it satisfies $\rho(\hat{\Theta})$-descent.*

For the sake of readability, we state the following definition for strict $\mathcal{B}$ and leave it to the reader to fill in the obvious associativity isomorphisms in the non-strict case.

Definition 4.3.13. *Let $\mathcal{B}$ be an additive monoidal category and let $\hat{\Theta}$ be a Frobenius object in $\mathcal{B}$. A morphism in $\mathcal{B}$ satisfies* $\hat{\Theta}$-descent *(resp.* right $\hat{\Theta}$-descent*) if it satisfies $(\hat{\Theta}-)$-decent (resp. $(-\hat{\Theta})$-decent). A morphism in $\mathcal{B}$ satisfies* $\hat{\Theta}_1$-$\hat{\Theta}_2$-descent*, if it satisfies $\hat{\Theta}_1$-descent and right $\hat{\Theta}_2$-descent. A morphism in $\mathcal{B}$ satisfies* two-sided $\hat{\Theta}$-descent*, if it satisfies $\hat{\Theta}$-$\hat{\Theta}$-descent.*

Remark 4.3.14. Following Example 4.3.9, $\hat{\Theta}_1$-$\hat{\Theta}_2$-descent can equivalently be defined as $\widetilde{\Theta_1\Theta_2}$-descent.

Definition 4.3.15. *Let $\mathcal{B}$ be a monoidal category and let $\hat{\Theta}$ be a Frobenius object in $\mathcal{B}$. We say $\mathcal{B}$ satisfies* $\hat{\Theta}$-descent*, if all idempotents of $\mathcal{B}$ which are eligible for $\hat{\Theta}$-descent satisfy $\hat{\Theta}$-descent. For $\mathcal{B}$ to satisfy* right $\hat{\Theta}$-descent, $\hat{\Theta}_1$-$\hat{\Theta}_2$-descent or two-sided $\hat{\Theta}$-descent *is defined analogously.*

Definition 4.3.16. *A Frobenius object (Θ, Δ, μ) is called* (right) indecomposable*, if $\Theta \neq 0$ and if $e = id_\Theta$ for all non-trivial idempotents e of Θ that satisfy (right) $\hat{\Theta}$-descent.*

Remark 4.3.17. Let $\mathcal{D}$ be an additive monoidal category that satisfies the Krull-Schmidt property, and let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a split Frobenius object in $\mathcal{D}$. Then, $[\Theta] \leq [\Theta][\Theta]$. If Θ is indecomposable, then $[\Theta]$ is a generalised idempotent, see Definition 2.5.10, whilst it is not sufficient that $\hat{\Theta}$ is (right) indecomposable.

Example 4.3.18. Let us continue with Example 4.3.5. For all $0 \neq V \in \mathcal{B}_2$, it is easy to see that $\hat{V_1}$ is indecomposable, but only right indecomposable if $\dim_k(V) = 1$. Similarly, $\hat{V_2}$ is right indecomposable, but only indecomposable if $\dim_k(V) = 1$. More interestingly, $\hat{k}_{S_2}$ is both indecomposable and right indecomposable. Let us show that $\hat{k}_{S_2}$ is indecomposable. Assume that $E = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$ is an idempotent of the underlying object $k \oplus k$. Then $\hat{k}_{S_2}$-descent implies that $e_{11} = 1$. Since $E = EE$, the top left entry of this matrix equation implies $e_{12}e_{21} = 0$. W.l.o.g assume that $e_{12} = 0$. Thus, the bottom right entry implies that e_{22} is an idempotent and thus $e_{22} = 1$. Now the bottom left entry yields $e_{21} = 2e_{21}$ and thus $e_{21} = 0$. The claim is established.

Remark 4.3.19. If $\hat{\Theta}$ is a right indecomposable Frobenius object in $\mathcal{B}$, then it is an indecomposable Frobenius object in $\mathcal{B}^{\text{rev}}$ and $Y(\hat{\Theta})$ is an indecomposable Frobenius object on $\mathcal{B}$, where $Y : \mathcal{B}^{\text{rev}} \xrightarrow{\sim} \mathcal{H}\text{om}_{\mathcal{B}}(\mathcal{B}, \mathcal{B})$ denotes the monoidal functor provided in Corollary 3.3.53. Also, note that in order to be compliant with Definition 4.3.10 in the case of non-strict monoidal categories, we should have used an idempotent e of FI (resp. IF) where I is the appropriate unit object.

By Remark 4.2.25 the following proposition applies to all functorially cyclic modules. We will introduce certain equivalence relations later on that will allow us to show in Theorem 4.5.4, that the following assignment induces a bijection.

Proposition 4.3.20. *Let $\mathcal{C} \in \mathfrak{A} - \text{Mod}$ be $(F, G, \alpha, \beta, (\mathcal{P}_a^{\mathcal{I}}, \pi))$-cyclic, where $a \in \mathfrak{A}$ and $\mathcal{I} \trianglelefteq \mathcal{P}_a$ is an ideal. Then, $\hat{\Theta} := (GF, G\alpha F, G\beta F)$ is a split Frobenius object on $\mathcal{P}_a^{\mathcal{I}}$. If $\mathcal{C}$ is non-trivial, then $\hat{\Theta}$ is indecomposable.*

Proof. The triple $\hat{\Theta}$ is a split Frobenius object, since $(G\beta FGF) \circ (GFG\alpha F) = G\beta\alpha F = (G\alpha F) \circ (G\beta F) = (GFG\beta F) \circ (G\alpha FGF)$ and since $(G\beta F) \circ (G\alpha F) = G(\beta \circ \alpha)F = GF$. Next, let us assume that $\mathcal{C}$ is non-trivial. Let e be a non-trivial idempotent of GF that satisfies $\hat{\Theta}$-descent. Define $\widetilde{e} := (\beta F) \circ (Fe) \circ (\alpha F)$. Then, $G\widetilde{e} = \mu \circ (GFe) \circ \Delta = e$ due to $\hat{\Theta}$-descent. In particular, $\widetilde{e}$ is non-trivial. Further, $\widetilde{e}$ is an idempotent since $\widetilde{e} \circ \widetilde{e} = (\beta \circ \alpha)(\widetilde{e} \circ \widetilde{e}) = (\beta F) \circ (FG\widetilde{e}) \circ (FG\widetilde{e}) \circ (\alpha F) = (\beta F) \circ (Fe) \circ (Fe) \circ (\alpha F) = (\beta F) \circ (Fe) \circ (\alpha F) = \widetilde{e}$ where we made multiple use of the interchange law discussed in Remark 3.1.3. Since F is strongly indecomposable by the definition of functorial cyclicity, it follows that $\widetilde{e} = id_F$ and thus $e = id_{GF}$. $\square$

Remark 4.3.21. By Proposition 3.3.56, we have $\mathcal{E}nd_{\mathfrak{A}}(\mathcal{P}_a^{\mathcal{I}}) \cong \Phi_a^{\mathcal{I}}(\mathcal{P}_a^{\mathcal{I}})$ where the right hand side is given by the full subcategory of $\mathcal{P}_a^{\mathcal{I}}(a)$ whose objects have the property that their identity morphism is annihilated by $\mathcal{I}$. If $\mathcal{I}$ is stable under the right action of $\mathfrak{A}(a,a)$, we thus have $\mathcal{E}nd_{\mathfrak{A}}(\mathcal{P}_a^{\mathcal{I}}) \cong \mathcal{P}_a^{\mathcal{I}}(a) = \mathfrak{A}^{\mathrm{rev}}(a,a)/\mathcal{I}(a)$, see Corollary 3.3.53 for the need to use the reversed category. In this case (and assuming that $\mathcal{I}$ is a proper ideal), we can interpret the indecomposable split Frobenius object associated to $\mathcal{C}$ in Proposition 4.3.20 as a right indecomposable split Frobenius object in $\mathfrak{A}(a,a)/\mathcal{I}(a)$.

There is the following alternative to Definition 4.3.1:

Remark 4.3.22. One may define a universal Frobenius object 2-category $\mathfrak{F}r$ as the 2-category with a single object whose endomorphism category is the strict monoidal category generated by one object Θ, and by morphisms $\Delta : \Theta \to \Theta\Theta$ and $\mu : \Theta\Theta \to \Theta$ that satisfy the relations $(\Theta\mu)\circ(\Delta\Theta) = \Delta\circ\mu = (\mu\Theta)\circ(\Theta\Delta)$. Similarly, $\mathfrak{F}r^{sp}$ is defined imposing the additional relation $\mu \circ \Delta = \Theta$. Then, the objects of $\mathfrak{H}om(\mathfrak{F}r, \mathfrak{C})$ are in 1:1 correspondence to Frobenius objects on objects of $\mathfrak{C}$ and similarly the objects of $\mathfrak{H}om(\mathfrak{F}r^{\mathrm{sp}}, \mathfrak{C})$ are in 1:1 correspondence with split Frobenius objects on objects in $\mathfrak{C}$. It will become apparent that this is conceptually the correct way to study (split) Frobenius objects. We have chosen to give the more concrete Definition 4.3.1 and to follow up with Definition 4.3.43 since it allows us to simplify the exposition, see also Remark 4.3.47.

Remark 4.3.23. *A priori* it is unclear whether there exist split Frobenius objects in a given monoidal category beyond Example 4.3.3 (and the trivial split Frobenius object in the additive case). We do not have a general recipe that allows us to construct or identify them. However, we will provide a classification up to a certain weak equivalence relation in Section 4.5.3 using Lusztig's 2-sided cells. With regards to this classification, one should note that even if a 2-sided cell contains an (indecomposable) object that carries the structure of a split Frobenius object, there may exist objects in this 2-sided cell that cannot be equipped with the structure of a split Frobenius object (see Section 5.5 for an example in type B_2), or there may exist objects that can carry different non-isomorphic structures as Frobenius objects (see Example 5.3.9).

Remark 4.3.24. The terminology Frobenius object is chosen in analogy to Frobenius algebras. These are very important objects in many branches of mathematics, notably topological quantum field theory. In this context μ and Δ represent a classical multiplication and comultiplication. Only after completing this thesis, we noticed that a similar notion of special Frobenius algebras has been used in [FS]. The obvious difference between our and their definition is that we do not require unit and counit morphisms. Further, associativity of μ and coassociativity of Δ is a requirement in there definition, while we deduce it from splitness. More subtle is the difference with respect to the requirement for a Frobenius algebra to be special. The example that motivated us to make our definition, see Example 5.2.9, cannot be equipped with a unit morphism. However, we can change the multiplication μ to a multiplication $\widetilde{\mu}$ that allows for a unit. In fact, we obtain in this way a Frobenius algebra. Yet the theory develop in [FS] is not applicable since $\widetilde{\mu} \circ \Delta = 0$, *i.e.* the Frobenius algebra is not a special Frobenius algebra.

Remark 4.3.25. Frobenius algebras also have a link to (co)monads and the Barr-Beck theorem. We will get back to this point in Remark 4.3.28.

4.3.2 Split Frobenius objects on additive categories

In this section, we study split Frobenius objects on additive categories and establish the additive Barr-Beck Theorem 4.3.26. Remark 4.3.28 comments on the similarity to the well-known Barr-Beck theorem and results in the literature that are comparable to ours.

Theorem 4.3.26. *Let $\hat{\Theta}$ be a split Frobenius object on $\mathcal{M} \in \mathfrak{Add}^{\mathrm{idem}}$. Then, there exists ${}_{\hat{\Theta}}\mathcal{M} \in \mathfrak{Add}^{\mathrm{idem}}$, additive functors ${}_{\hat{\Theta}}\mathcal{M} \xrightarrow{\vartheta^{\hat{\Theta}}} \mathcal{M} \xrightarrow{\vartheta_{\hat{\Theta}}} {}_{\hat{\Theta}}\mathcal{M}$ and transformations $id_{{}_{\hat{\Theta}}\mathcal{M}} \overset{\alpha}{\Longrightarrow} \vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}} \overset{\beta}{\Longrightarrow} id_{{}_{\hat{\Theta}}\mathcal{M}}$ with $\beta \circ \alpha = id_{{}_{\hat{\Theta}}\mathcal{M}}$ and $\hat{\Theta} = (\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}}, \vartheta^{\hat{\Theta}}\alpha\vartheta_{\hat{\Theta}}, \vartheta^{\hat{\Theta}}\beta\vartheta_{\hat{\Theta}})$. On the other hand, given $\mathcal{D} \in \mathfrak{Add}$, additive functors $\mathcal{D} \xrightarrow{G} \mathcal{M} \xrightarrow{F} \mathcal{D}$ and transformations $id_{\mathcal{D}} \overset{\alpha'}{\Longrightarrow} FG \overset{\beta'}{\Longrightarrow} id_{\mathcal{D}}$ with $\beta' \circ \alpha' = id_{\mathcal{D}}$, setting $\hat{\Theta} := (GF, G\alpha'F, G\beta'F)$ defines a split Frobenius object on $\mathcal{M}$ and we have an equivalence ${}_{\hat{\Theta}}\mathcal{M} \cong \mathcal{D}^{\mathrm{idem}}$ of additive categories.*

Proof. First, note that F, G, α' and β' extend to $\mathcal{D}^{\mathrm{idem}}$ and that $\hat{\Theta}$ defined using these extensions agrees with $\hat{\Theta}$ as defined in the statement. In Proposition 4.3.33, we calculate that $\hat{\Theta}$ as associated above to $\mathcal{D}$, F, G, α' and β' is indeed a split Frobenius object on $\mathcal{M}$. Further, we show that $\mathcal{D}^{\mathrm{idem}}$ represents the 2-functor ${}_{\hat{\Theta}}\Upsilon$ which we will associate to $\hat{\Theta}$ in Definition 4.3.29. The 2-Yoneda Lemma 3.3.49 thus implies equivalences between any idempotent closed additive category that admits F, G, α' and β' with the properties in the statement. In particular, the equivalence ${}_{\hat{\Theta}}\mathcal{M} \cong \mathcal{D}^{\mathrm{idem}}$ follows provided we can show existence of ${}_{\hat{\Theta}}\mathcal{M}$. This is achieved through the explicit construction in Definition 4.3.35 which can by Remark 4.3.36 be interpreted as the category of projective $\hat{\Theta}$-modules. $\square$

Remark 4.3.27. One enhance the above theorem as follows: Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be additive categories together with the structure as above such that the associated split Frobenius objects $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are isomorphic. Then, we have an equivalence of additive categories $\mathcal{D}_1^{\mathrm{idem}} \cong \mathcal{D}_2^{\mathrm{idem}}$. To prove this, one may write down an explicit isomorphism between ${}_{\hat{\Theta}_1}\Upsilon$ and ${}_{\hat{\Theta}_2}\Upsilon$. We will prove a more general statement in Proposition 4.3.48 after generalising Theorem 4.3.26 in Theorem 4.3.42.

Theorem 4.3.26 should be viewed as a additive alternative to the Barr-Beck theorem, see Remark 4.3.28. The split Frobenius object $\hat{\Theta}$ plays the role of the (co)monad and we will see in Remark 4.3.36that the category ${}_{\hat{\Theta}}\mathcal{M}$ can be understood as projective $\hat{\Theta}$-modules in $\mathcal{M}$. Also, note that we do not require any technical assumptions on morphisms nor do we use units or counits.

Remark 4.3.28. Monads and comonads are important notions in category theory. Any adjoint pair (L, R), where $L : \mathcal{C} \to \mathcal{D}$ is left adjoint to R, yields a monad T on $\mathcal{C}$ and a comonad on $\mathcal{D}$. Using the monad one can construct the Kleisli category $\mathcal{C}_T$ and the category $\mathcal{C}^T$ of T-algebras or how we prefer to call it T-modules. There are comparison functors $\mathcal{C}_T \to \mathcal{D} \to \mathcal{C}^T$. If L is essentially surjective, then $\mathcal{C}_T$ is equivalent to $\mathcal{D}$, see [ML, VI.5 Exercise 1]. On the other hand, the Barr-Beck theorem, see [ML, VI.7 and exercise 7], identifies the precise conditions on morphisms and coequalisers and their behaviour under R that are equivalent to the statement that $\mathcal{D} \to \mathcal{C}^T$ is an equivalence. By reversing arrows, analogues statements hold for $\mathcal{C}$ and comodules in $\mathcal{D}$ over the comonad.
The Barr-Beck theorem has been applied in Abelian (and triangulated) higher representation theory, see [Rou2, Theorem 8.3]. In a similar spirit yet without referring to Barr-Beck, [Os1, Section 3.3] shows that semisimple indecomposable objects of $\mathcal{B} - \mathrm{Mod}(\mathfrak{A}b)$ where $\mathcal{B}$ is a semisimple rigid monoidal category are equivalent to certain categories of modules for an algebra (*i.e.* a monad) in $\mathcal{B}$ associated to the Abelian $\mathcal{B}$-module. Even earlier for a strict Abelian tensor category $\mathcal{B}$ for which the tensor product is bilinear over the endomorphism ring of the identity object, [FS] studied modules in $\mathcal{B}$ over special Frobenius algebras in $\mathcal{B}$, see Remark 4.3.24, and it was shown in [FS, Proposition 5.1] that such module categories are Abelian. The papers [Rou2, Os1, FS] have in common that the module categories studied are Abelian (or triangulated), which distinguishes them from our additive setting, and that (co)units are used, which are not required to exist here.

For the remainder of the section, we fix an idempotent closed additive category $\mathcal{M} \in \mathfrak{A}dd^{\mathrm{idem}}$. Next, we define the 2-functor ${}_{\hat{\Theta}}\Upsilon$.

Definition 4.3.29. *Let $\hat{\Theta}$ be a split Frobenius object on $\mathcal{M}$. We denote by ${}_{\hat{\Theta}}\Upsilon$ the endo-2-functor of $\mathfrak{A}dd$ that is defined on any $\mathcal{N} \in \mathfrak{A}dd$, $H \in \mathfrak{A}dd(\mathcal{N}_1, \mathcal{N}_2)$ and $\beta \in \mathfrak{A}dd_{\mathcal{N}_1 \to \mathcal{N}_2}(H_1, H_2)$ by setting*

- $${}_{\hat{\Theta}}\Upsilon(\mathcal{N}) := \left\{ \begin{array}{ll} Obj: & (F, \iota, \pi) \text{ where } F \in \mathfrak{A}dd(\mathcal{M}, \mathcal{N}), \\ & \iota \in \mathfrak{A}dd(F, F\Theta) \text{ and } \pi \in \mathfrak{A}dd(F\Theta, F): \\ & (\iota\Theta) \circ \iota = (F\Delta) \circ \iota,\ \pi \circ (\pi\Theta) = \pi \circ (F\mu), \\ & \pi \circ \iota = F,\ \iota \circ \pi = (F\mu) \circ (\iota\Theta) = (\pi\Theta) \circ (F\Delta) \\ Mor: & {}_{\hat{\Theta}}\Upsilon(\mathcal{N})((F_1, \iota_1, \pi_1), (F_2, \iota_2, \pi_2)) := \\ & \{\alpha \in \mathfrak{A}dd(F_1, F_2) | (\alpha\Theta) \circ \iota_1 = \iota_2 \circ \alpha,\ \pi_2 \circ (\alpha\Theta) = \alpha \circ \pi_1\} \end{array} \right\}$$
- $${}_{\Theta}\Upsilon(H) := \left\{ \begin{array}{ll} Obj: & {}_{\hat{\Theta}}\Upsilon(H)((F, \iota, \pi)) := (HF, H\iota, H\pi) \\ Mor: & {}_{\hat{\Theta}}\Upsilon(H)(\alpha) := H\alpha \end{array} \right\}$$
- $${}_{\hat{\Theta}}\Upsilon(\beta) := \Big(({}_{\hat{\Theta}}\Upsilon(\beta))_{(F,\iota,\pi)}\Big)_{(F,\iota,\pi) \in {}_{\hat{\Theta}}\Upsilon(\mathcal{N}_1)} := (\beta F)_{(F,\iota,\pi) \in {}_{\hat{\Theta}}\Upsilon(\mathcal{N}_1)}$$

and by specifying all required isotransformations to be identity transformations.

Proof. It is an easy exercise to check that ${}_{\hat{\Theta}}\Upsilon(\mathcal{N})$ defines an additive category using the composition inherited from $\mathfrak{A}dd(\mathcal{M}, \mathcal{N})$. Similarly, one checks that ${}_{\hat{\Theta}}\Upsilon(H)$ defines an additive functor ${}_{\hat{\Theta}}\Upsilon(\mathcal{N}_1) \to {}_{\hat{\Theta}}\Upsilon(\mathcal{N}_2)$ using the interchange law, see Remark 3.1.3. That ${}_{\hat{\Theta}}\Upsilon(\mathcal{N})(\beta)$ defines a transformation ${}_{\hat{\Theta}}\Upsilon(H_1) \Rightarrow {}_{\hat{\Theta}}\Upsilon(H_2)$ follows again from the interchange law, as does compatibility with composition and identity transformations. Since $\mathfrak{A}dd$ is a strict 2-category, we obtain ${}_{\hat{\Theta}}\Upsilon(K)\,{}_{\hat{\Theta}}\Upsilon(H) = {}_{\hat{\Theta}}\Upsilon(KH)$ for $K \in \mathfrak{A}dd(\mathcal{N}_2, \mathcal{N}_3)$ and ${}_{\hat{\Theta}}\Upsilon(id_{\mathcal{N}}) = id_{{}_{\hat{\Theta}}\Upsilon(\mathcal{N})}$. Thus, all required isotransformations can be defined using identity transformations and all compatibilities required in Definition 3.1.11 are satisfied trivially. $\square$

Next, we state the for our purposes most important properties of ${}_{\hat{\Theta}}\Upsilon$. Subsequently, we will show that these properties determine ${}_{\hat{\Theta}}\Upsilon$ up to equivalence (after restricting the domain to idempotent closed categories).

Proposition 4.3.30. *There exist morphisms of 2-functors ${}_{\hat{\Theta}}\Upsilon \to \mathfrak{A}dd(\mathcal{M}, -) \to {}_{\hat{\Theta}}\Upsilon$ whose composition has the identity transformation as a summand. More precisely, these morphisms of 2-functors can be defined as follows: For all $\mathcal{N} \in \mathfrak{A}dd$, the functor ${}_{\hat{\Theta}}\Upsilon(\mathcal{N}) \to \mathfrak{A}dd(\mathcal{M}, \mathcal{N})$ sends $(F, \iota, \pi) \mapsto F$ and $\alpha \mapsto \alpha$, and the functor $\mathfrak{A}dd(\mathcal{M}, \mathcal{N}) \to {}_{\hat{\Theta}}\Upsilon(\mathcal{N})$ sends $F \mapsto (F\Theta, F\Delta, F\mu)$ and $\alpha \mapsto \alpha\Theta$.*

Proof. The functor ${}_{\hat{\Theta}}\Upsilon(\mathcal{N}) \to \mathfrak{A}dd(\mathcal{M}, \mathcal{N})$ as specified in the statement is well defined since ${}_{\hat{\Theta}}\Upsilon(\mathcal{N})$ inherits its composition rule from $\mathfrak{A}dd(\mathcal{M}, \mathcal{N})$. To see that $\mathfrak{A}dd(\mathcal{M}, \mathcal{N}) \to {}_{\hat{\Theta}}\Upsilon(\mathcal{N})$ as specified in the statement is well defined, we compute

$((F\Delta)\Theta) \circ (F\Delta) = F((\Delta\Theta) \circ \Delta) \overset{\text{Lem 4.3.7}}{=} F((\Theta\Delta) \circ \Delta) = ((F\Theta)\Delta) \circ (F\Delta)$. Similarly, we obtain $(F\mu) \circ ((F\mu)\Theta) = (F\mu) \circ ((F\Theta)\mu)$. Further, $(F\mu) \circ (F\Delta) = F(\mu \circ \Delta) = F\Theta$, $(F\Delta) \circ (F\mu) = F(\Delta \circ \mu) = F((\Theta\mu) \circ (\Delta\Theta)) = ((F\Theta)\mu) \circ (F\Delta)\Theta))$ and analogously $(F\Delta) \circ (F\mu) = F(\Delta \circ \mu) = ((F\mu)\Theta) \circ (F\Theta)\Delta))$. Therefore, we have $(F\Theta, F\Delta, F\mu) \in {}_{\hat{\Theta}}\Upsilon(\mathcal{N})$. Similarly for all $\alpha \in \mathfrak{A}dd_{\mathcal{M}\to\mathcal{N}}(F_1, F_2)$, we have $\alpha\Theta \in {}_{\hat{\Theta}}\Upsilon(\mathcal{N})((F_1\Theta, F_1\Delta, F_1\mu), (F_2\Theta, F_2\Delta, F_2\mu))$, since $((\alpha\Theta)\Theta) \circ (F_1\Delta) = \alpha\Delta = (F_2\Delta) \circ (\alpha\Theta)$ and similarly $(F_2\mu) \circ ((\alpha\Theta)\Theta) = (\alpha\Theta) \circ (F_1\mu)$. The thus well defined collection of functors ${}_{\hat{\Theta}}\Upsilon(\mathcal{N}) \to \mathfrak{A}dd(\mathcal{M}, \mathcal{N}) \to {}_{\hat{\Theta}}\Upsilon(\mathcal{N})$ can now be turned into 2-functors ${}_{\hat{\Theta}}\Upsilon \to \mathfrak{A}dd(\mathcal{M}, -) \to {}_{\hat{\Theta}}\Upsilon$ by specifying all required isotransformations to be identity transformations, making all compatibility requirements hold trivially.

It remains to define two morphisms of morphisms of 2-functors between the identity morphism on ${}_{\hat{\Theta}}\Upsilon$ to the composition of the above defined morphisms and to note that their composition is the identity morphism on the identity morphism of ${}_{\hat{\Theta}}\Upsilon$. The first of these morphisms is defined on an object $(F, \iota, \pi) \in {}_{\hat{\Theta}}\Upsilon(\mathcal{N})$ by the morphism $\iota : (F, \iota, \pi) \to (F\Theta, F\Delta, F\mu)$ in ${}_{\hat{\Theta}}\Upsilon(\mathcal{N})$. Note that ι is well defined since $(\iota\Theta) \circ \iota = (F\Delta) \circ \iota$ and $\iota \circ \pi = (F\mu) \circ (\iota\Theta)$, which hold by the definition of ${}_{\hat{\Theta}}\Upsilon$. Further, this collection of morphisms defines a transformation holds since $\iota \circ \alpha = (\alpha\Theta) \circ \iota$, which is required from any morphism α in ${}_{\hat{\Theta}}\Upsilon(\mathcal{N})$. The second of these morphisms is defined on an object $(F\Theta, F\Delta, F\mu) \in {}_{\hat{\Theta}}\Upsilon(\mathcal{N})$ by the morphism $\pi : (F\Theta, F\Delta, F\mu) \to (F, \iota, \pi)$. Note that π is well defined, by the same arguments as for ι using the complimentary requirements in the definition of ${}_{\hat{\Theta}}\Upsilon$. The only property of the 2-functor ${}_{\hat{\Theta}}\Upsilon$ that we have not used yet, is that $\pi \circ \iota = F$ for all $(F, \iota, \pi) \in {}_{\hat{\Theta}}\Upsilon(\mathcal{N})$. This final ingredient implies that the composition of the two morphisms of morphisms of 2-functors, which we have just defined, is the identity morphism on the identity morphism of ${}_{\hat{\Theta}}\Upsilon$. □

Theorem 4.3.31. *The 2-functor ${}_{\hat{\Theta}}\Upsilon$ restricts to a 2-functor $\mathfrak{A}dd^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$ and this restriction is representable.*

Proof. We split the proof in three steps. First, Lemma 4.3.32 shows that ${}_{\hat{\Theta}}\Upsilon$ preserves idempotent closed additive categories. Second, Proposition 4.3.33 shows that a category ${}_{\hat{\Theta}}\mathcal{M} \in \mathfrak{A}dd$ represents ${}_{\hat{\Theta}}\Upsilon$, given that it comes together with certain additive functors and transformations which give rise to the Frobenius object $\hat{\Theta}$. Third, Proposition 4.3.38 provides an explicit construction of ${}_{\hat{\Theta}}\mathcal{M}$ together with all the additive functors, transformations and relations that were required in Proposition 4.3.33. □

Lemma 4.3.32. *The following diagram commutes:*

$$\begin{tikzcd}
\mathfrak{A}dd \arrow[r, "{}_{\hat{\Theta}}\Upsilon"] & \mathfrak{A}dd \\
\mathfrak{A}dd^{\mathrm{idem}} \arrow[u, hook] \arrow[r, "{}_{\hat{\Theta}}\Upsilon|_{\mathfrak{A}dd^{\mathrm{idem}}}"] & \mathfrak{A}dd^{\mathrm{idem}} \arrow[u, hook]
\end{tikzcd}$$

Proof. Let $\mathcal{N} \in \mathfrak{A}dd^{\mathrm{idem}}$. We need to show that ${}_{\hat{\Theta}}\Upsilon(\mathcal{N})$ is idempotent closed. Let $(F, \iota, \pi) \in {}_{\hat{\Theta}}\Upsilon(\mathcal{N})$ and let e be an idempotent of (F, ι, π), that is a transformation $e : F \Rightarrow F$ such that $e^2 = e$, $(e\Theta) \circ \iota = \iota \circ e$ and $\pi \circ (e\Theta) = e \circ \pi$. Since $\mathcal{N}$ is idempotent closed, there exists an additive functor $F_e : \mathcal{M} \to \mathcal{N}$ and transformations $F_e \overset{\alpha}{\Longrightarrow} F \overset{\beta}{\Longrightarrow} F_e$ such that $\alpha \circ \beta = e$ and $\beta \circ \alpha = id_{F_e}$. Define $\iota_e := (\beta\Theta) \circ \iota \circ \alpha$ and $\pi_e := \beta \circ \pi \circ (\alpha\Theta)$. We leave it to the reader to check that $(F_e, \iota_e, \pi_e) \in {}_{\Theta}\Upsilon(\mathcal{N})$. Now, the morphism $\alpha : (F_e, \iota_e, \pi_e) \to (F, \iota, \pi)$ in ${}_{\Theta}\Upsilon(\mathcal{N})$ is well defined since $(\alpha\Theta) \circ \iota_e = (\alpha\Theta) \circ (\beta\Theta) \circ \alpha = ((\alpha \circ \beta)\Theta) \circ \alpha = \alpha$ and $\pi \circ (\alpha\Theta) = \alpha \circ \beta \circ \pi \circ (\alpha\Theta) = \alpha \circ \pi_e$. Similarly, the morphism $\beta : (F, \iota, \pi) \to (F_e, \iota_e, \pi_e)$ is well defined. Thus, we have constructed a summand of (F, ι, π) with associated idempotent $e = \alpha \circ \beta$. $\square$

Proposition 4.3.33. *Let ${}_{\hat{\Theta}}\mathcal{M} \in \mathfrak{A}dd^{\mathrm{idem}}$, let $\mathcal{M} \xrightarrow{\vartheta^{\hat{\Theta}}} {}_{\hat{\Theta}}\mathcal{M} \xrightarrow{\vartheta_{\hat{\Theta}}} \mathcal{M}$ be additive functors and let $id_{{}_{\hat{\Theta}}\mathcal{M}} \overset{\alpha}{\Longrightarrow} \vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}} \overset{\beta}{\Longrightarrow} id_{{}_{\hat{\Theta}}\mathcal{M}}$ be transformations such that $\beta \circ \alpha = id_{{}_{\hat{\Theta}}\mathcal{M}}$. Then, $\hat{\Theta} := (\Theta, \Delta, \mu) := (\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}})$ is a split Frobenius object on $\mathcal{M}$. Furthermore, we have an equivalence of 2-functors $\mathfrak{A}dd^{\mathrm{idem}}({}_{\hat{\Theta}}\mathcal{M}, -) \cong {}_{\hat{\Theta}}\Upsilon|_{\mathfrak{A}dd^{\mathrm{idem}}}$. Thus, $\hat{\Theta}$ determines ${}_{\hat{\Theta}}\mathcal{M}$ is uniquely up to equivalence.*

Proof. For convenience, we abuse notation and write ${}_{\hat{\Theta}}\Upsilon$ instead of ${}_{\hat{\Theta}}\Upsilon|_{\mathfrak{A}dd^{\mathrm{idem}}}$. First, $\hat{\Theta}$ is a Frobenius object, since $(\Theta\mu) \circ (\Delta\Theta) = (\Theta\vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}}) \circ (\vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}\Theta) = (\vartheta_{\hat{\Theta}}\alpha\beta\vartheta^{\hat{\Theta}}) = (\vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}) \circ (\vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}}) = \Delta \circ \mu$ and analogously $\Delta \circ \mu = (\mu\Theta) \circ (\Theta\Delta)$. Furthermore, $\hat{\Theta}$ is split since $\mu \circ \Delta = \vartheta_{\hat{\Theta}}(\beta \circ \alpha)\vartheta^{\hat{\Theta}} = \vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}} = \Theta$.

Next, we define a morphisms of 2-functors $\varphi_1 : \mathfrak{A}dd^{\mathrm{idem}}({}_{\hat{\Theta}}\mathcal{M}, -) \to {}_{\hat{\Theta}}\Upsilon$. We define morphisms $\iota := \alpha\vartheta^{\hat{\Theta}}$ and $\pi := \beta\vartheta^{\hat{\Theta}}$, and claim that $(\vartheta^{\hat{\Theta}}, \iota, \pi) \in {}_{\hat{\Theta}}\Upsilon({}_{\hat{\Theta}}\mathcal{M})$. Note that $(\iota\Theta) \circ \iota = (\alpha\vartheta^{\hat{\Theta}}\Theta) \circ (\alpha\vartheta^{\hat{\Theta}}) = \alpha\alpha\vartheta^{\hat{\Theta}} = (\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}) \circ (\alpha\vartheta^{\hat{\Theta}}) = (\vartheta^{\hat{\Theta}}\Delta) \circ \iota$. Similarly, $\pi \circ (\pi\Theta) = \pi \circ (\vartheta^{\hat{\Theta}}\mu)$ holds and $\pi \circ \iota = \vartheta^{\hat{\Theta}}$ holds obviously. Further, we have $\iota \circ \pi = (\alpha\vartheta^{\hat{\Theta}}) \circ (\beta\vartheta^{\hat{\Theta}}) = \alpha\beta\vartheta^{\hat{\Theta}} = (\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}}) \circ (\alpha\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}) = (\vartheta^{\hat{\Theta}}\mu) \circ (\iota\vartheta^{\hat{\Theta}})$ and similarly $\iota \circ \pi = \beta\alpha\vartheta^{\hat{\Theta}} = (\pi\Theta) \circ (\vartheta^{\hat{\Theta}}\Delta)$. Thus, $(\vartheta^{\hat{\Theta}}, \iota, \pi) \in {}_{\hat{\Theta}}\Upsilon({}_{\hat{\Theta}}\mathcal{M})$ holds and we can define φ_1 as the image of $(\vartheta^{\hat{\Theta}}, \iota, \pi)$ under the equivalence

$${}_{\hat{\Theta}}\Upsilon({}_{\hat{\Theta}}\mathcal{M}) \cong \mathcal{H}om_{\mathfrak{A}dd^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}}(\mathfrak{A}dd^{\mathrm{idem}}({}_{\hat{\Theta}}\mathcal{M}, -), {}_{\hat{\Theta}}\Upsilon),$$

which is specified by the 2-Yoneda Lemma 3.3.49.

Next, we define a morphisms of 2-functors $\varphi_2 : {}_{\hat{\Theta}}\Upsilon \to \mathfrak{A}dd^{\text{idem}}({}_{\hat{\Theta}}\mathcal{M}, -)$. To that end, we need to define a functor ${}_{\hat{\Theta}}\Upsilon(\mathcal{N}) \to \mathfrak{A}dd^{\text{idem}}({}_{\hat{\Theta}}\mathcal{M}, \mathcal{N})$ for each $\mathcal{N} \in \mathfrak{A}dd^{\text{idem}}$. We declare this functor to send any object $(F, \iota, \pi) \in {}_{\hat{\Theta}}\Upsilon(\mathcal{N})$ to $(F \circ \vartheta_{\hat{\Theta}}, e_F)$ where $e_F := (F\vartheta_{\hat{\Theta}}\beta) \circ (\iota\vartheta_{\hat{\Theta}})$. Further, any morphism $\alpha : (F_1, \iota_1, \pi_1) \to (F_2, \iota_2, \pi_2)$ in ${}_{\hat{\Theta}}\Upsilon(\mathcal{N})$ shall be send to $e_{F_2} \circ (\alpha\vartheta_{\hat{\Theta}}) \circ e_{F_1} : (F_1\vartheta_{\hat{\Theta}}, e_{F_1}) \to (F_2\vartheta_{\hat{\Theta}}, e_{F_2})$. To show that this functor is well defined, we need to show that e_F is an idempotent and that the assignment for morphisms is compatible with composition. The latter follows from the former since $e_{F_2} \circ (\alpha\vartheta_{\hat{\Theta}}) = (\alpha\vartheta_{\hat{\Theta}}) \circ e_{F_1}$, which in turn can be deduced easily from $(\alpha\Theta) \circ \iota_1 = \iota_2 \circ \alpha$. To show the former, we calculate that $e_F = (\pi\vartheta_{\hat{\Theta}}\beta) \circ (\iota\Theta\vartheta_{\hat{\Theta}}) \circ (\iota\vartheta_{\hat{\Theta}}) = (\pi\vartheta_{\hat{\Theta}}\beta) \circ (F\Delta\vartheta_{\hat{\Theta}}) \circ (\iota\vartheta_{\hat{\Theta}}) = (\pi\vartheta_{\hat{\Theta}}) \circ (F\vartheta_{\hat{\Theta}}\alpha) \circ (F\vartheta_{\hat{\Theta}}\beta) \circ (\iota\vartheta_{\hat{\Theta}}) = (\pi\vartheta_{\hat{\Theta}}) \circ (F\vartheta_{\hat{\Theta}}\alpha) \circ e_F$ and note that by the same argument $(\pi\vartheta_{\hat{\Theta}}) \circ (F\vartheta_{\hat{\Theta}}\alpha) \circ e_F = (\pi\vartheta_{\hat{\Theta}}) \circ (F\vartheta_{\hat{\Theta}}\alpha)$ holds. Hence, $e_F = (\pi\vartheta_{\hat{\Theta}}) \circ (F\vartheta_{\hat{\Theta}}\alpha)$ and moreover $e_F = (\pi\vartheta_{\hat{\Theta}}) \circ (F\vartheta_{\hat{\Theta}}\alpha) \circ e_F = e_F \circ e_F$, showing that e_F is an idempotent. To complete the definition of φ_2, note that the natural isomorphisms $\mathfrak{A}dd^{\text{idem}}({}_{\hat{\Theta}}\mathcal{M}, H) \circ \varphi_2(\mathcal{N}_1) \xrightarrow{\sim} \varphi_2(\mathcal{N}_2) \circ {}_{\hat{\Theta}}\Upsilon(H)$ for all $H \in \mathfrak{A}dd^{\text{idem}}(\mathcal{N}_1, \mathcal{N}_2)$, which are required in Definition 3.1.11, can be chosen as identity morphism. All necessary compatibilities thus hold trivially.

It remains to show that $\varphi_1 \circ \varphi_2$ and $\varphi_2 \circ \varphi_1$ are isomorphic to the respective identity morphism of 2-functors. By the 2-Yoneda Lemma 3.3.49, $\varphi_2 \circ \varphi_1$ (as well as the identity morphism on the 2-functor $\mathfrak{A}dd^{\text{idem}}({}_{\hat{\Theta}}\mathcal{M}, -)$) is determined up to isomorphism by its evaluation at $id_{{}_{\hat{\Theta}}\mathcal{M}}$, which is given by $(\varphi_2 \circ \varphi_1)(id_{{}_{\hat{\Theta}}\mathcal{M}}) = \varphi_2\left((\vartheta^{\hat{\Theta}}, \alpha\vartheta^{\hat{\Theta}}, \beta\vartheta^{\hat{\Theta}})\right) = (\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}}, (\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}}\beta) \circ (\alpha\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}})) = (\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}}, \alpha \circ \beta) \cong id_{{}_{\hat{\Theta}}\mathcal{M}}$. The last isomorphism holds due to the assumption that $\beta \circ \alpha = id_{{}_{\hat{\Theta}}\mathcal{M}}$. Hence $\varphi_2 \circ \varphi_1 \cong id_{\mathfrak{A}dd^{\text{idem}}({}_{\hat{\Theta}}\mathcal{M}, -)}$.
Last, we establish the equivalence $\varphi_1 \circ \varphi_2 \cong id_{{}_{\hat{\Theta}}\Upsilon}$. For all $(F, \iota, \pi) \in {}_{\hat{\Theta}}\Upsilon(\mathcal{N})$, we have $(\varphi_1 \circ \varphi_2)\left((F, \iota, \pi)\right) = \varphi_1\left((F\vartheta_{\hat{\Theta}}, e_F)\right) = \left(((F\vartheta_{\hat{\Theta}})\vartheta^{\hat{\Theta}}, (F\vartheta_{\hat{\Theta}})\alpha\vartheta^{\hat{\Theta}}, (F\vartheta_{\hat{\Theta}})\beta\vartheta^{\hat{\Theta}}), e_F\vartheta^{\hat{\Theta}}\right) = ((F\Theta, F\Delta, F\mu), \iota \circ \pi)$, where $e_F\vartheta^{\hat{\Theta}} = (F\vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}}) \circ (\iota\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}) = (F\mu) \circ (\iota\Theta) = \iota \circ \pi$ holds by the definition of ${}_{\hat{\Theta}}\Upsilon$. For all morphisms $\alpha : (F_1, \iota_1, \pi_1) \to (F_2, \iota_2, \pi_2)$ in ${}_{\hat{\Theta}}\Upsilon(\mathcal{N})$, we have $(\varphi_1 \circ \varphi_2)(\alpha) = \varphi_1(e_{F_2} \circ (\alpha\vartheta_{\hat{\Theta}}) \circ e_{F_1}) = (e_{F_2} \circ (\alpha\vartheta_{\hat{\Theta}}) \circ e_{F_1})\vartheta^{\hat{\Theta}} = (e_{F_2}\vartheta^{\hat{\Theta}}) \circ (\alpha\Theta) \circ (e_{F_1}\vartheta^{\hat{\Theta}}) = \iota_2 \circ \pi_2 \circ (\alpha\Theta) \circ \iota_1 \circ \pi_1$. Now, recall from Proposition 4.3.30 that the composition ${}_{\hat{\Theta}}\Upsilon \to \mathfrak{A}dd(\mathcal{M}, -) \to {}_{\hat{\Theta}}\Upsilon$ specified there has the identity functor as a summand. The explicit formulas in the proof yield an idempotent on the composition ${}_{\hat{\Theta}}\Upsilon \to \mathfrak{A}dd(\mathcal{M}, -) \to {}_{\hat{\Theta}}\Upsilon$ and a comparison with the calculation above shows that the summand associated to this idempotent is precisely $\varphi_1 \circ \varphi_2$. $\square$

Remark 4.3.34. One may deduce from arguments in the previous proof together with Proposition 4.3.30 that there is a diagram

$$\begin{tikzcd}
 & & {}_{\hat\Theta}\Upsilon|_{\mathfrak{A}dd^{\mathrm{idem}}} \arrow[dd, leftrightarrow] \arrow[drr] & & \\
\mathfrak{A}dd^{\mathrm{idem}}(\mathcal{M},-) \arrow[urr] \arrow[drr] \arrow[r, phantom, "\overset{\sim}{\Longrightarrow}"] & {} & & {} \arrow[r, phantom, "\overset{\sim}{\Longrightarrow}"] & \mathfrak{A}dd^{\mathrm{idem}}(\mathcal{M},-) \\
 & -\vartheta^{\hat\Theta} & \mathfrak{A}dd^{\mathrm{idem}}({}_{\hat\Theta}\mathcal{M},-) \arrow[urr] & -\vartheta_{\hat\Theta} &
\end{tikzcd}.$$

where the vertical arrow represents the equivalence of Proposition 4.3.33.

We complete the proof of Theorem 4.3.31 by defining a category ${}_{\hat\Theta}\mathcal{M}$ and by showing in Proposition 4.3.38 that ${}_{\hat\Theta}\mathcal{M}$ admits all the structure that is required for Proposition 4.3.33 to apply.

Definition 4.3.35. *Let $\hat\Theta = (\Theta, \Delta, \mu)$ be a Frobenius object on $\mathcal{M} \in \mathfrak{A}dd^{\mathrm{idem}}$. We define ${}_{\hat\Theta}\mathcal{M} \in \mathfrak{A}dd^{\mathrm{idem}}$ as the idempotent closure of ${}_{\hat\Theta}\widetilde{\mathcal{M}} \in \mathfrak{A}dd$, which in turn is defined as the additive category with objects $\mathrm{Ob}({}_{\hat\Theta}\widetilde{\mathcal{M}}) := \mathrm{Ob}(\mathcal{M})$ and morphisms*

$$\widetilde{{}_{\hat\Theta}\mathcal{M}}(M_1, M_2) := \left\{ f \in \mathcal{M}(\Theta M_1, \Theta M_2) \;\middle|\; \begin{array}{rcl} (\Theta f) \circ \Delta_{M_1} & = & \Delta_{M_2} \circ f \ \textit{and} \\ \mu_{M_2} \circ (\Theta f) & = & f \circ \mu_{M_1} \end{array} \right\}.$$

for all M_1, $M_2 \in \mathrm{Ob}({}_{\hat\Theta}\widetilde{\mathcal{M}})$.

It is an easy exercise to check that ${}_{\hat\Theta}\mathcal{M}$ is well defined. The Frobenius relation of $\hat\Theta$ is not necessary for this purpose nor will it be needed to establish the first part of Proposition 4.3.38. We allow ourselves this notational convenience, since we will study ${}_{\hat\Theta}\mathcal{M}$ only for Frobenius objects $\hat\Theta$. In fact, $\hat\Theta$ will be split in all our applications. The following remark shows that in order to interpret ${}_{\hat\Theta}\mathcal{M}$ as the category of projective $\hat\Theta$-modules, $\hat\Theta$ has to be associative and coassociative.

Remark 4.3.36. Analogous to (co)modules for (co)monads, one can define a $\hat\Theta$-module to be a triple (M, δ, ρ) where $M \in \mathcal{M}$, $\delta \in \mathcal{M}(M, \Theta M)$ is the comodule map and $\rho \in \mathcal{M}(\Theta M, M)$ is the module map, which satisfy the comodule axiom $(\Delta M) \circ \delta = (\Theta\delta) \circ \delta$, the module axiom $\rho \circ (\mu M) = \rho \circ (\Theta\rho)$ and the compatibility axiom $\delta \circ \rho = (\Theta\rho) \circ (\Delta M) = (\mu M) \circ (\Theta\delta)$. Since $\hat\Theta$ does not have a (co)unit, we have no choice but to consider non-unital modules. Further if $\hat\Theta$ is associative and coassociative, we have a free $\hat\Theta$-module $\hat\Theta M := (\Theta M, \Delta M, \mu M)$ for all $M \in \mathcal{M}$. With the natural definition of morphisms of $\hat\Theta$-modules, it follows that ${}_{\hat\Theta}\widetilde{\mathcal{M}}$ is equivalent to the category of free $\hat\Theta$-modules $\mathcal{M}$. Therefore, one may refer to ${}_{\hat\Theta}\mathcal{M}$ as the category of projective $\hat{\Theta}$-modules. Now assume that $\hat{\Theta}$ is split. Then, $\delta \circ \rho$ is an idempotent of (M, δ, ρ). While not necessarily true in general, this idempotent is the identity morphism in case of a free $\hat{\Theta}$-modules and one can deduce from Lemma 4.3.37 that this is also true in the case of projective $\hat{\Theta}$-modules.

The following lemma will simplify proofs going forward (including the proof of Proposition 4.3.38), since it allows us to check one instead of two conditions. We will restate this lemma in Lemma 4.3.40 for the idempotent closure ${}_{\hat{\Theta}}\mathcal{M}$ using the tools to be developed in Proposition 4.3.38.

Lemma 4.3.37. *Let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a split Frobenius object on $\mathcal{M}$. Then, we have*

$$\widetilde{{}_{\hat{\Theta}}\mathcal{M}}(M_1, M_2) = \{f \in \mathcal{M}(\Theta M_1, \Theta M_2) | f \textit{ satisfies } \hat{\Theta}\textit{-descent}\}.$$

Proof. Let us fix $f \in \mathcal{M}(\Theta M_1, \Theta M_2)$. First, we assume that $f \in \widetilde{{}_{\hat{\Theta}}\mathcal{M}}(M_1, M_2)$. Then, we have $\mu_{M_2} \circ (\Theta f) \circ \Delta_{M_1} = f \circ \mu_{M_1} \circ \Delta_{M_1} = f$, where we used that $\hat{\Theta}$ is split. Conversely, we assume that $f = \mu_{M_2} \circ (\Theta f) \circ \Delta_{M_1}$. Then, we calculate

$$\begin{aligned}
(\Theta f) \circ \Delta_{M_1} &= (\Theta \{\mu_{M_2} \circ (\Theta f) \circ \Delta_{M_1}\}) \circ \Delta_{M_1} \\
&= (\Theta\mu_{M_2}) \circ (\Theta\Theta f) \circ (\Theta\Delta_{M_1}) \circ \Delta_{M_1} && \text{(Remark 3.1.3)} \\
&= (\Theta\mu_{M_2}) \circ (\Theta\Theta f) \circ (\Delta_{M_1}\Theta) \circ \Delta_{M_1} && \text{(Lemma 4.3.7)} \\
&= (\Theta\mu_{M_2}) \circ (\Delta_{M_2}\Theta) \circ (\Theta f) \circ \Delta_{M_1} && \text{(Remark 3.1.3)} \\
&= \Delta_{M_2} \circ \mu_{M_2} \circ (\Theta f) \circ \Delta_{M_1} && \text{(Frobenius relation)} \\
&= \Delta_{M_2} \circ f.
\end{aligned}$$

An analogous argument shows $f \circ \mu_{M_1} = \mu_{M_2} \circ (\Theta f)$. Thus, $f \in \widetilde{{}_{\hat{\Theta}}\mathcal{M}}(M_1, M_2)$. $\square$

Proposition 4.3.38. *Let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a Frobenius object on $\mathcal{M} \in \mathfrak{A}dd^{\text{idem}}$. Then, there exist additive functors $\mathcal{M} \xrightarrow{\vartheta^{\hat{\Theta}}} {}_{\hat{\Theta}}\mathcal{M} \xrightarrow{\vartheta_{\hat{\Theta}}} \mathcal{M}$ such that $\Theta = \vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}$. Further if $\hat{\Theta}$ is split, there exist transformations $id_{{}_{\hat{\Theta}}\mathcal{M}} \overset{\alpha}{\Rightarrow} \vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}} \overset{\beta}{\Rightarrow} id_{{}_{\hat{\Theta}}\mathcal{M}}$ with $\beta \circ \alpha = id_{{}_{\hat{\Theta}}\mathcal{M}}$ and we recover $\hat{\Theta} = (\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}})$.*

Proof. Let $\vartheta^{\hat{\Theta}} : \mathcal{M} \to {}_{\hat{\Theta}}\mathcal{M}$ be the additive functor defined by $\vartheta^{\hat{\Theta}}M := M$ for all $M \in \mathcal{M}$ and by $\vartheta^{\hat{\Theta}}f := \Theta f$ and $f \in \mathcal{M}(M_1, M_2)$. Since Δ and μ are transformations, $\vartheta^{\hat{\Theta}}f$ is well defined. Further Θ is additive, $\vartheta^{\hat{\Theta}}$ is additive, as well.

Next, we define $\vartheta_{\hat{\Theta}} : {}_{\hat{\Theta}}\mathcal{M} \to \mathcal{M}$. Since $\mathcal{M}$ is idempotent closed, it suffices to define the restriction of $\vartheta_{\hat{\Theta}}$ to $\widetilde{{}_{\hat{\Theta}}\mathcal{M}}$, where we set $\vartheta_{\hat{\Theta}}(M) := \Theta(M)$ for all $M \in \widetilde{{}_{\hat{\Theta}}\mathcal{M}}$ and $\vartheta_{\hat{\Theta}}(f) := f$ for all $f \in \widetilde{{}_{\hat{\Theta}}\mathcal{M}}(M_1, M_2)$. Doubtless, this assignment defines an additive functor. Clearly, $\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}} = \Theta$ holds.

To define α and β, it is sufficient to define their restrictions to $\widetilde{{}_{\hat{\Theta}}\mathcal{M}}$. We set $\alpha_M := \Delta_M$

and $\beta_M := \mu_M$ for all $M \in \mathrm{Ob}(\widetilde{{}_{\hat{\Theta}}\mathcal{M}})$. Note that indeed $\alpha_M \in \widetilde{{}_{\hat{\Theta}}\mathcal{M}}(M, \Theta M)$, since $\alpha_M \overset{\hat{\Theta}\text{ spl}}{=} \mu_{\Theta M} \circ \Delta_{\Theta M} \circ \Delta_M \overset{\text{Lem 4.3.7}}{=} \mu_{\Theta M} \circ (\Theta\Delta_M) \circ \Delta_M \overset{\hat{\Theta}\text{ spl}}{=} \mu_{\Theta M} \circ (\Theta\alpha_M) \circ \Delta_M$. Similarly, $\beta_M \in \widetilde{{}_{\hat{\Theta}}\mathcal{M}}(\Theta M, M)$ follows. That α and β are indeed transformations follows from the definition of the morphism spaces in $\widetilde{{}_{\hat{\Theta}}\mathcal{M}}$. Now, $\mu \circ \Delta = \Theta$ implies that $\beta \circ \alpha = {}_{\hat{\Theta}}\mathcal{M}$. Since, $\vartheta^{\hat{\Theta}}$ is the identity on objects and $\vartheta_{\hat{\Theta}}$ the identity on objects, we have $(\vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}})_M = \vartheta_{\hat{\Theta}}\alpha_{\vartheta^{\hat{\Theta}}M} = \alpha_M$ and thus $\vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}} = \Delta$. By an analogous argument, $\vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}} = \mu$ holds as well, and we have recovered the Frobenius object. $\square$

Remark 4.3.39. The notation $\vartheta^{\hat{\Theta}}$ is chosen to indicate that $\vartheta^{\hat{\Theta}}$ acts on morphism, which in the spirit of this thesis are the higher structure, whereas $\vartheta_{\hat{\Theta}}$ acts on objects.

It should be obvious that $\vartheta_{\hat{\Theta}}$ is faithful. In fact, the following concise generalisation of Lemma 4.3.37 describes the image explicitly:

Lemma 4.3.40. *Let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a split Frobenius object on $\mathcal{M} \in \mathfrak{Add}^{\mathrm{idem}}$. Then, we have an isomorphism*

$$ {}_{\hat{\Theta}}\mathcal{M}(M_1, M_2) \xrightarrow{\vartheta_{\hat{\Theta}}} \{f \in \mathcal{M}(\vartheta_{\hat{\Theta}}M_1, \vartheta_{\hat{\Theta}}M_2) | f \text{ satisfies } \hat{\Theta}\text{-descent}\}. $$

Remark 4.3.41. The functor $\vartheta_{\hat{\Theta}}$ induces an equivalence with an idempotent closed subcategory of $\mathcal{M}$ if and only if all idempotents in $\mathcal{M}$, which are eligible $\hat{\Theta}$-descent, satisfy $\hat{\Theta}$-descent.

4.3.3 Split Frobenius objects on 2-functors

In this section, we generalise the constructions and universal properties of Section 4.3.2 to a compatible family of Frobenius objects on idempotent closed additive categories.

Theorem 4.3.42. *Let $\Phi : \mathfrak{C} \to \mathfrak{Add}^{\mathrm{idem}}$ be a 2-functor and let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a split Frobenius object on Φ. Then, there exists a 2-functor ${}_{\hat{\Theta}}\Phi : \mathfrak{C} \to \mathfrak{Add}^{\mathrm{idem}}$ that sends $c \mapsto {}_{\hat{\Theta}(c)}\Phi(c)$ and that is uniquely determined up to an equivalence of 2-functors by the following properties:*

- *There exist morphisms of 2-functors $\Phi \xrightarrow{\vartheta^{\hat{\Theta}}} {}_{\hat{\Theta}}\Phi \xrightarrow{\vartheta_{\hat{\Theta}}} \Phi$.*
- *There exist morphisms of morphisms of 2-functors $id_{{}_{\hat{\Theta}}\Phi} \overset{\alpha}{\Longrightarrow} \vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}} \overset{\beta}{\Longrightarrow} id_{{}_{\hat{\Theta}}\Phi}$ with $\beta \circ \alpha = id_{{}_{\hat{\Theta}}\Phi}$.*
- *$\hat{\Theta} = (\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}})$.*

Proof. We split the proof in several parts which show more detailed results. Assuming that there is a well defined assignment $\Phi \mapsto {}_{\hat{\Theta}}\Phi$ with the properties as stated above, Proposition 4.3.48 shows (to the extend that is needed later on) that this assignment is 2-functorial with respect to morphisms of 2-functors and morphisms of morphisms of 2-functors that are compatible with the Frobenius objects. This allows us to deduce equivalences between different realisations of ${}_{\hat{\Theta}}\Phi$ under certain assumptions in Corollary 4.3.50. Thus, it remains to provide an explicit construction of ${}_{\hat{\Theta}}\Phi$ which is carried out in Proposition 4.3.52. $\square$

Definition 4.3.43. *Let $\mathfrak{C}$ be a strict 2-category and for $i \in \{1,2\}$ let $\hat{\Theta}_i = (\Theta_i, \Delta_i, \mu_i)$ be Frobenius objects on $c_i \in \mathfrak{C}$. We say that a 1-arrow $F : c_1 \to c_2$ in $\mathfrak{C}$ is* compatible with the Frobenius objects via can_F, *if there exists a 2-arrow $\mathrm{can}_F : F\Theta_1 \overset{\sim}{\Longrightarrow} \Theta_2 F$ in $\mathfrak{C}$ such that $(\Delta_2 F) \circ \mathrm{can}_F = (\Theta_2 \,\mathrm{can}_F) \circ (\mathrm{can}_F \Theta_1) \circ (F\Delta_1)$ and such that $\mathrm{can}_F \circ (F\mu_1) = (\mu_2 F) \circ (\Theta_2 \,\mathrm{can}_F) \circ (\mathrm{can}_F \Theta_1)$. Furthermore given another 1-arrow $G : c_1 \to c_2$ in $\mathfrak{C}$ that is compatible with the Frobenius object via can_G, we say that a 2-arrow $\tau : F \Rightarrow G$ in $\mathfrak{C}$ is* compatible with the Frobenius objects, *if $(\Theta_2\tau) \circ \mathrm{can}_F = \mathrm{can}_G \circ (\tau\Theta_1)$.*

Notation 4.3.44. Let $c_1 \xrightarrow{F} c_2 \xrightarrow{K} c_3$ be 1-arrows between objects and let a Frobenius object on each object be given. Assume that F (resp. K) is compatible with the Frobenius objects on its domain and codomain via can_F (resp. can_K). Then, $KF : c_1 \to c_3$ is compatible with the Frobenius objects via $\mathrm{can}_{KF} := (\mathrm{can}_K F) \circ (K \,\mathrm{can}_F)$.

Remark 4.3.45. Let $\hat{\Theta}$ be a Frobenius object on $c \in \mathfrak{C}$. Then, I_c is compatible with $\hat{\Theta}$ via Θ. Furthermore, $\alpha \in \mathfrak{C}_c(I_c, I_c)$ is compatible with the Frobenius object via Θ if and only if $\alpha\Theta = \Theta\alpha$.

Remark 4.3.46. Let $\hat{\Theta}_1$ and $\hat{\Theta}_2$ be Frobenius objects on $c \in \mathfrak{C}$ and recall Definition 4.3.2. Then, $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are isomorphic if and only if the unit object $I_c \in \mathfrak{C}(c,c)$ is compatible with the Frobenius objects $\hat{\Theta}_1$ and $\hat{\Theta}_2$.

Remark 4.3.47. Recall the 2-category $\mathfrak{Hom}(\mathfrak{Fr}, \mathfrak{C})$ which we had introduced in Remark 4.3.22. The structure described in Definition 4.3.43 corresponds precisely to the 1-arrows and 2-arrows between strict 2-functors in $\mathfrak{Hom}(\mathfrak{Fr}, \mathfrak{C})$. For the applications in this thesis, Definition 4.3.43 is sufficient and its concrete form simplifies the exposition, since we do not need to mention the additional (trivial) structure as well as the (trivial) strictness properties that this additional structure satisfies.

Proposition 4.3.48. *Let $\mathfrak{C}$ be a 2-category. For $i \in \{1,2\}$, let Φ_i and ${}_{\hat{\Theta}_i}\Phi_i$ be 2-functors from $\mathfrak{C}$ to $\mathfrak{Add}^{\mathrm{idem}}$, let $\Phi_i \xrightarrow{\vartheta^{\hat{\Theta}_i}} {}_{\hat{\Theta}_i}\Phi_i \xrightarrow{\vartheta_{\hat{\Theta}_i}} \Phi_i$ be morphisms of 2-functors,* *let $id_{{}_{\hat\Theta_i}\Phi_i} \overset{\alpha_i}{\Longrightarrow} \vartheta^{\hat\Theta_i}\vartheta_{\hat\Theta_i} \overset{\beta_i}{\Longrightarrow} id_{{}_{\hat\Theta_i}\Phi_i}$ be morphisms of morphisms of 2-functors such that $\beta_i \circ \alpha_i = id_{{}_{\hat\Theta_i}\Phi_i}$, and let $\hat\Theta_i := (\Theta_i, \Delta_i, \mu_i) := (\vartheta_{\hat\Theta_i}\vartheta^{\hat\Theta_i}, \vartheta_{\hat\Theta_i}\alpha_i\vartheta^{\hat\Theta_i}, \vartheta_{\hat\Theta_i}\beta_i\vartheta^{\hat\Theta_i})$ be the associated split Frobenius object on Φ_i. For all morphisms of 2-functors $F : \Phi_1 \to \Phi_2$ which are compatible with the Frobenius objects via can_F, one can assign a morphism of 2-functors $\widetilde{F} : {}_{\hat\Theta_1}\Phi_1 \to {}_{\hat\Theta_2}\Phi_2$ that fits into the diagram*

$$\begin{tikzcd}
\Phi_1 \arrow[r, "\vartheta^{\hat\Theta_1}"] \arrow[d, "F"'] & {}_{\hat\Theta_1}\Phi_1 \arrow[r, "\vartheta_{\hat\Theta_1}"] \arrow[d, "\widetilde{F}"] & \Phi_1 \arrow[d, "F"] \\
\Phi_2 \arrow[r, "\vartheta^{\hat\Theta_2}"'] \arrow[ur, Rightarrow, "\sim"] & {}_{\hat\Theta_2}\Phi_2 \arrow[r, "\vartheta_{\hat\Theta_2}"'] \arrow[ur, Rightarrow, "\sim"] & \Phi_2
\end{tikzcd} .$$

Further, for all morphism of morphisms of 2-functors $\tau : F \Rightarrow G$ which are compatible with the Frobenius objects, one can assign a morphism of morphisms of 2-functors $\widetilde{\tau} : \widetilde{F} \to \widetilde{G}$ in a functorial way. Furthermore, the assignment $F \mapsto \widetilde{F}$ is compatible with composition of morphisms of 2-functors and identity morphisms of 2-functors up to isomorphism and respects associativity.

Proof. For $i \in \{1, .., 4\}$ let Φ_i, ${}_{\hat\Theta_i}\Phi_i, ...$ be given. First, recall the argument in Proposition 4.3.33 which shows that $\hat\Theta_i = (\Theta_i, \Delta_i, \mu_i)$ is a split Frobenius object.

Next, let $F : \Phi_1 \to \Phi_2$ be a morphism of 2-functors which is compatible with the Frobenius objects via $\mathrm{can}_F : F\Theta_1 \overset{\sim}{\Longrightarrow} \Theta_2 F$. Now, we define a morphism of 2-functors $\widetilde{F} : {}_{\hat\Theta_1}\Phi_1 \to {}_{\hat\Theta_2}\Phi_2$ as follows. Set $\hat F := \vartheta^{\hat\Theta_2} F \vartheta_{\hat\Theta_1}$ and consider the endomorphism e_F of $\hat F$ defined by $e_F := (\beta_2 \hat F) \circ (\vartheta^{\hat\Theta_2}\,\mathrm{can}_F\,\vartheta_{\hat\Theta_1}) \circ (\hat F \alpha_1)$. Indeed, e_F is an idempotent since

$$\begin{aligned}
e_F \circ e_F &= (\beta_2 \hat F) \circ (\vartheta^{\hat\Theta_2}\vartheta_{\hat\Theta_2}\beta_2\hat F) \circ (\vartheta^{\hat\Theta_2}\vartheta_{\hat\Theta_2}\vartheta^{\hat\Theta_2}\,\mathrm{can}_F\,\vartheta_{\hat\Theta_1}) \\
&\qquad \circ (\vartheta^{\hat\Theta_2}\,\mathrm{can}_F\,\vartheta_{\hat\Theta_1}\vartheta^{\hat\Theta_1}\vartheta_{\hat\Theta_1}) \circ (\hat F \alpha_1 \vartheta^{\hat\Theta_1}\vartheta_{\hat\Theta_1}) \circ (\hat F \alpha_1) && \text{(Remark 3.1.3)} \\
&= (\beta_2 \hat F) \circ \left\{ \vartheta^{\hat\Theta_2} \left((\mu_2 F) \circ (\Theta_2\,\mathrm{can}_F) \circ (\mathrm{can}_F\,\Theta_1) \circ (F\Delta_1) \right) \vartheta_{\hat\Theta_1} \right\} \\
&\qquad \circ (\hat F \alpha_1) && \text{(Remark 3.1.3)} \\
&= e_F.
\end{aligned}$$

In the first equation, we used the interchange law to pull the term $\beta_2 \hat F$ of the second e_F along the left side all the way to the first term, and simultaneously the term $\hat F \alpha_1$ of the first e_F along the right to the very end. In the last equation, we used that F is compatible with the Frobenius object, see Definition 4.3.43, which implies that $(\mu_2 F) \circ (\Theta_2\,\mathrm{can}_F) \circ (\mathrm{can}_F\,\Theta_1) \circ (F\Delta_1) = \mathrm{can}_F \circ (F\mu_1) \circ (F\Delta_1) = \mathrm{can}_F$. Since $\hat F$ is a morphism of 2-functors with target $\mathfrak{A}dd^{\mathrm{idem}}$, we can now define $\widetilde{F}$ to be the morphism of 2-functors $\widetilde{F} := (\hat F, e_F) : {}_{\hat\Theta_1}\Phi_1 \to {}_{\hat\Theta_2}\Phi_2$.

Next, we show that $F\vartheta_{\hat\Theta_1} \cong \vartheta_{\hat\Theta_2}\widetilde{F} = (\Theta_2 F \vartheta_{\hat\Theta_1}, \vartheta_{\hat\Theta_2} e_F)$. We need to define two morphisms of morphisms of 2-functors $f_1 : F\vartheta_{\hat\Theta_1} \to \Theta_2 F \vartheta_{\hat\Theta_1}$ and $f_2 : \Theta_2 F \vartheta_{\hat\Theta_1} \to F\vartheta_{\hat\Theta_1}$

and show that $f_2 \circ f_1 = F\vartheta_{\hat{\Theta}_1}$ and $f_1 \circ f_2 = \vartheta_{\hat{\Theta}_2} e_F$. We set $f_1 := (\mathrm{can}_F\, \vartheta_{\hat{\Theta}_1}) \circ (F\vartheta_{\hat{\Theta}_1}\alpha_1)$ and $f_2 := (F\vartheta_{\hat{\Theta}_1}\beta_1) \circ (\mathrm{can}_F^{-1}\, \vartheta_{\hat{\Theta}_1})$. Now, $f_2 \circ f_1 = F\vartheta_{\hat{\Theta}_1}$ is obvious and we compute

$$
\begin{aligned}
f_1 \circ f_2 &= (\mathrm{can}_F\, \vartheta_{\hat{\Theta}_1}) \circ (F\mu_1\vartheta_{\hat{\Theta}_1}) \circ (\mathrm{can}_F^{-1}\, \Theta_1\vartheta_{\hat{\Theta}_1}) \circ (\Theta_2 F\vartheta_{\hat{\Theta}_1}\alpha_1) && \text{(Remark 3.1.3)}\\
&= (\mu_2 F\vartheta_{\hat{\Theta}_1}) \circ (\Theta_2\, \mathrm{can}_F\, \vartheta_{\hat{\Theta}_1}) \circ (\mathrm{can}_F\, \Theta_1\vartheta_{\hat{\Theta}_1})\\
&\qquad \circ (F\Theta_1\vartheta_{\hat{\Theta}_1}\alpha_1) \circ (\mathrm{can}_F^{-1}\, \vartheta_{\hat{\Theta}_1}) && \text{(Definition 4.3.43)}\\
&= (\mu_2 F\vartheta_{\hat{\Theta}_1}) \circ (\Theta_2\, \mathrm{can}_F\, \vartheta_{\hat{\Theta}_1}) \circ (\Theta_2 F\vartheta_{\hat{\Theta}_1}\alpha_1) && \text{(Remark 3.1.3)}\\
&= \vartheta_{\hat{\Theta}_2} e_F. && \text{(Remark 3.1.3)}
\end{aligned}
$$

Next, let us define $e'_F := (\hat{F}\beta_1) \circ (\vartheta^{\hat{\Theta}_2}\, \mathrm{can}_F^{-1}\, \vartheta_{\hat{\Theta}_1}) \circ (\alpha_2 \hat{F})$. We have

$$
\begin{aligned}
f_1 \circ f_2 &= (\Theta_2 F\vartheta_{\hat{\Theta}_1}\beta_1) \circ (\mathrm{can}_F\, \Theta_1\vartheta_{\hat{\Theta}_1}) \circ (F\Delta_1\vartheta_{\hat{\Theta}_1}) \circ (\mathrm{can}_F^{-1}\, \vartheta_{\hat{\Theta}_1}) && \text{(Remark 3.1.3)}\\
&= (\mathrm{can}_F\, \vartheta_{\hat{\Theta}_1}) \circ (F\Theta_1\vartheta_{\hat{\Theta}_1}\beta_1) \circ (\mathrm{can}_F^{-1}\, \Theta_1\vartheta_{\hat{\Theta}_1})\\
&\qquad \circ (\Theta_2\, \mathrm{can}_F^{-1}\, \vartheta_{\hat{\Theta}_1}) \circ (\Delta_2 F\vartheta_{\hat{\Theta}_1}) && \text{(Definition 4.3.43)}\\
&= (\Theta_2 F\vartheta_{\hat{\Theta}_1}\beta_1) \circ (\Theta_2\, \mathrm{can}_F^{-1}\, \vartheta_{\hat{\Theta}_1}) \circ (\Delta_2 F\vartheta_{\hat{\Theta}_1}) && \text{(Remark 3.1.3)}\\
&= \vartheta_{\hat{\Theta}_2} e'_F && \text{(Remark 3.1.3)}
\end{aligned}
$$

and thus $\vartheta_{\hat{\Theta}_2} e_F = \vartheta_{\hat{\Theta}_2} e'_F$. An application of $(\beta_2\hat{F}) \circ (\vartheta^{\hat{\Theta}_2} -) \circ (\alpha_2\hat{F})$ yields $e_F = e'_F$.

Next, we show that $\vartheta^{\hat{\Theta}_2} F \cong \widetilde{F}\vartheta^{\hat{\Theta}_1} = (\vartheta^{\hat{\Theta}_2} F\Theta_1, e_F\vartheta^{\hat{\Theta}_1})$. As for the argument given above, we need morphisms of morphisms of 2-functors $g_1 : \vartheta^{\hat{\Theta}_2} F \to \vartheta^{\hat{\Theta}_2} F\Theta_1$ and $g_2 : \vartheta^{\hat{\Theta}_2} F\Theta_1 \to \vartheta^{\hat{\Theta}_2} F$ such that $g_2 \circ g_1 = \vartheta^{\hat{\Theta}_2} F$ and $g_1 \circ g_2 = e_F\vartheta^{\hat{\Theta}_1}$. By a calculation analogous to the one given above, $g_1 := (\vartheta^{\hat{\Theta}_2}\, \mathrm{can}_F^{-1}) \circ (\alpha_2\vartheta^{\hat{\Theta}_2} F)$ and $g_2 := (\beta_2\vartheta^{\hat{\Theta}_2} F) \circ (\vartheta^{\hat{\Theta}_2}\, \mathrm{can}_F)$ satisfy these properties.

Next, let $G : \Phi_1 \to \Phi_2$ be another morphism of 2-functors which is compatible with the Frobenius objects via $\mathrm{can}_G : G\Theta_1 \xRightarrow{\sim} \Theta_2 G$ and let $\tau : F \Rightarrow G$ be a morphism of morphisms of 2-functors that is compatible with the Frobenius objects. We want to construct a morphism of morphism of 2-functors $\widetilde{\tau} : \widetilde{F} \to \widetilde{G}$ in a functorial way. Clearly, $\hat{\tau} := \vartheta^{\hat{\Theta}_2}\tau\vartheta_{\hat{\Theta}_1} : \hat{F} \to \hat{G}$ is a functorial assignment. Now, we set $\widetilde{\tau} := e_G \circ \hat{\tau} \circ e_F : \widetilde{F} \Rightarrow \widetilde{G}$. Note that, $\widetilde{id_F} = e_F = id_{\widetilde{F}}$. In order to show that the assignment $\tau \mapsto \widetilde{\tau}$ is compatible with composition, it suffices to show that $\widetilde{\tau} = e_G \circ \hat{\tau} = \hat{\tau} \circ e_F$. Since e_F and e_G are idempotents, the first equation follows from the latter. The latter holds since

$$
\begin{aligned}
\hat{\tau} \circ e_F &= (\beta_2\hat{G}) \circ (\vartheta^{\hat{\Theta}_2}\vartheta_{\hat{\Theta}_2}\hat{\tau}) \circ (\vartheta^{\hat{\Theta}_2}\, \mathrm{can}_F\, \vartheta_{\hat{\Theta}_1}) \circ (\hat{F}\alpha_1) && \text{(Remark 3.1.3)}\\
&= (\beta_2\hat{G}) \circ \{\vartheta^{\hat{\Theta}_2}((\Theta_2\tau) \circ \mathrm{can}_F)\vartheta_{\hat{\Theta}_1}\} \circ (\hat{F}\alpha_1) && \text{(Remark 3.1.3)}\\
&= (\beta_2\hat{G}) \circ \{\vartheta^{\hat{\Theta}_2}(\mathrm{can}_G \circ (\tau\Theta_1))\vartheta_{\hat{\Theta}_1}\} \circ (\hat{F}\alpha_1) && \text{(Definition 4.3.43)}\\
&= (\beta_2\hat{G}) \circ (\vartheta^{\hat{\Theta}_2}\, \mathrm{can}_G\, \vartheta_{\hat{\Theta}_1}) \circ (\hat{\tau}\vartheta^{\hat{\Theta}_1}\vartheta_{\hat{\Theta}_1}) \circ (\hat{F}\alpha_1) && \text{(Remark 3.1.3)}\\
&= e_G \circ \hat{\tau}. && \text{(Remark 3.1.3)}
\end{aligned}
$$

Next, we show that $id_{\hat{\Theta}_1\Phi_1} \cong \widetilde{id_{\Phi_1}}$. Recall from Remark 4.3.45 that id_{Φ_1} is compatible with the Frobenius object $\hat{\Theta}_1$ via Θ. Furthermore, we have the equations $\widetilde{id_{\Phi_1}} = \Big(\vartheta_{\hat{\Theta}_1}\vartheta^{\hat{\Theta}_1}, (\beta_1\vartheta_{\hat{\Theta}_1}\vartheta^{\hat{\Theta}_1}) \circ (\vartheta_{\hat{\Theta}_1}\Theta_1\vartheta^{\hat{\Theta}_1}) \circ (\vartheta_{\hat{\Theta}_1}\vartheta^{\hat{\Theta}_1}\alpha_1)\Big) \overset{\text{Rem 3.1.3}}{=} \Big(\vartheta_{\hat{\Theta}_1}\vartheta^{\hat{\Theta}_1}, \alpha_1 \circ \beta_1\Big)$. Since $\alpha_1 : id_{\Theta_1\Phi_1} \to \vartheta_{\hat{\Theta}_1}\vartheta^{\hat{\Theta}_1}$ and $\beta_1 : \vartheta_{\hat{\Theta}_1}\vartheta^{\hat{\Theta}_1} \to id_{\hat{\Theta}_1\Phi_1}$ satisfy $\beta_1 \circ \alpha_1 = id_{\hat{\Theta}_1\Phi_1}$, we obtain the isomorphism $id_{\hat{\Theta}_1\Phi_1} \cong \widetilde{id_{\Phi_1}}$.

Next, let $K : \Phi_2 \to \Phi_3$ be another morphism of 2-functors that is compatible with the Frobenius objects via $\mathrm{can}_K : K\Theta_2 \overset{\sim}{\Longrightarrow} \Theta_2 K$. We have to show that $\widetilde{KF} \cong \widetilde{K}\widetilde{F}$. For that purpose, we will define morphisms of morphisms of 2-functors $h_1 : \hat{K}\hat{F} \Rightarrow \widehat{KF}$ and $h_2 : \widehat{KF} \Rightarrow \hat{K}\hat{F}$. We will then show that $e_{KF} \circ h_1 = h_1 = h_1 \circ (e_K e_F)$ and $(e_K e_F) \circ h_2 = h_2 = h_2 \circ e_{KF}$, and thus obtain an induced morphism of morphisms between $\widetilde{KF}$ and $\widetilde{K}\widetilde{F}$. Furthermore, we show that $h_1 \circ h_2 = e_{KF}$ and $h_2 \circ h_1 = e_K e_F$, and therefore that $\widetilde{KF} \cong \widetilde{K}\widetilde{F}$.

Now, we define $h_1 := (\beta_3\widehat{KF}\beta_1) \circ (\vartheta^{\hat{\Theta}_3}\,\mathrm{can}_K\,\mathrm{can}_F^{-1}\,\vartheta_{\hat{\Theta}_1}) \circ (\hat{K}\alpha_2\hat{F})$ and set $h_2 := (\hat{K}\beta_2\hat{F}) \circ (\vartheta^{\hat{\Theta}_3}\,\mathrm{can}_K^{-1}\,\mathrm{can}_F\,\vartheta_{\hat{\Theta}_1}) \circ (\alpha_3\widehat{KF}\alpha_1)$. By the interchange law of Remark 3.1.3, we have $h_1 = (\beta_3\widehat{KF}) \circ (\vartheta^{\Theta_3}\,\mathrm{can}_K\,F\vartheta_{\Theta_1}) \circ (\hat{K}e'_F) = (\widehat{KF}\beta_1) \circ (\vartheta^{\hat{\Theta}_3}K\,\mathrm{can}_F^{-1}\,\vartheta_{\hat{\Theta}_1}) \circ (e_K\hat{F})$ and $h_2 = (\hat{K}e_F) \circ (\vartheta^{\hat{\Theta}_3}\,\mathrm{can}_K^{-1}\,F\vartheta_{\hat{\Theta}_1}) \circ (\alpha_3\widehat{KF}) = (e'_K\hat{F}) \circ (\vartheta^{\hat{\Theta}_3}K\,\mathrm{can}_F\,\vartheta_{\hat{\Theta}_1}) \circ (\widehat{KF}\alpha_1)$. Using that $e'_F = e_F = e_F \circ e_F$ and the analogously shown equation $e'_K = e_K = e_K \circ e_K$, we obtain that $h_1 \circ (e_K e_F) = h_1 \circ (\hat{K}e_F) \circ (e_K\hat{F}) = h_1 \circ (e_K\hat{F}) = h_1$ and similarly $(e_K e_F) \circ h_2 = h_2$. Once we show that $h_1 \circ h_2 = e_{KF}$ and $h_2 \circ h_1 = e_K e_F$, it follows that $h_1 = h_1 \circ (e_K e_F) = h_1 \circ h_2 \circ h_1 = e_{KF} \circ h_1$ and analogously that $h_2 = h_2 \circ e_{KF}$. We compute

$$
\begin{aligned}
h_1 \circ h_2 &= \{(\widehat{KF}\beta_1) \circ (\vartheta^{\hat{\Theta}_3}K\,\mathrm{can}_F^{-1}\,\vartheta_{\hat{\Theta}_1}) \circ (e_K\hat{F})\} \\
&\qquad \circ \{(e'_K\hat{F}) \circ (\vartheta^{\hat{\Theta}_3}K\,\mathrm{can}_F\,\vartheta_{\hat{\Theta}_1}) \circ (\widehat{KF}\alpha_1)\} \\
&= \{(\beta_3\widehat{KF}) \circ (\vartheta^{\hat{\Theta}_3}\,\mathrm{can}_K\,F\vartheta_{\hat{\Theta}_1}) \circ (\hat{K}e'_F)\} \\
&\qquad \circ \{(\vartheta^{\hat{\Theta}_3}K\,\mathrm{can}_F\,\vartheta_{\hat{\Theta}_1}) \circ (\widehat{KF}\alpha_1)\} \\
&= \{(\beta_3\widehat{KF}) \circ (\vartheta^{\hat{\Theta}_3}\,\mathrm{can}_K\,F\vartheta_{\hat{\Theta}_1})\} \circ \{\vartheta^{\hat{\Theta}_3}K\,\{(\vartheta_{\hat{\Theta}_2}e_F) \circ f_1\}\} \\
&= \{(\beta_3\widehat{KF}) \circ (\vartheta^{\hat{\Theta}_3}\,\mathrm{can}_K\,F\vartheta_{\hat{\Theta}_1})\} \circ \{\vartheta^{\hat{\Theta}_3}K\,\{f_1 \circ f_2 \circ f_1\}\} \\
&= \{(\beta_3\widehat{KF}) \circ (\vartheta^{\hat{\Theta}_3}\,\mathrm{can}_K\,F\vartheta_{\hat{\Theta}_1})\} \circ \{\vartheta^{\hat{\Theta}_3}Kf_1\} \\
&= \{(\beta_3\widehat{KF}) \circ (\vartheta^{\hat{\Theta}_3}\,\mathrm{can}_K\,F\vartheta_{\hat{\Theta}_1})\} \circ \{(\vartheta^{\hat{\Theta}_3}K\,\mathrm{can}_F\,\vartheta_{\hat{\Theta}_1}) \circ (\widehat{KF}\alpha_1)\} \\
&= e_{KF}.
\end{aligned}
$$

To show $h_2 \circ h_1 = e_K e_F$, we use the notations $g_1^K := (\vartheta^{\hat{\Theta}_3}\,\mathrm{can}_K^{-1}) \circ (\alpha_3\vartheta^{\hat{\Theta}_3}K)$ and $g_2^K := (\beta_3\vartheta^{\hat{\Theta}_3}K) \circ (\vartheta^{\hat{\Theta}_3}\,\mathrm{can}_K)$, which are analogous to the morphisms g_1 and g_2 introduced above. We have $g_2^K \circ g_1^K = \vartheta^{\Theta_3}K$ and $g_1^K \circ g_2^K = e_K\vartheta^{\hat{\Theta}_2}$. Thus, we obtain $h_2 \circ h_1 = (\hat{K}\beta_2\hat{F}) \circ \{(g_1^K \circ g_2^K)(f_1 \circ f_2)\} \circ (\hat{K}\alpha_2\hat{F}) = (\hat{K}\beta_2\hat{F}) \circ (e_K\vartheta^{\hat{\Theta}_2}\vartheta_{\hat{\Theta}_2}e_F) \circ (\hat{K}\alpha_2\hat{F}) =$

$e_K(\beta_2 \circ \alpha_2)e_F = e_K e_F$.

Last, let $L : \Phi_3 \to \Phi_4$ be another morphism that is compatible with the Frobenius objects. We show that the composition $\widetilde{L}\widetilde{K}\widetilde{F} \cong \widetilde{LK}\widetilde{F} \cong \widetilde{LKF}$ agrees with the composition $\widetilde{L}\widetilde{K}\widetilde{F} \cong \widetilde{L}\widetilde{KF} \cong \widetilde{LKF}$. We denote $h_{K,F} := h_1$ and $h^{-1}_{K,F} := h_2$. Recall that we have $h_{K,F} = (g_2^K F\vartheta_{\hat{\Theta}_1}) \circ (\hat{K}e_F) = (\vartheta^{\hat{\Theta}_3} K f_2^F) \circ (e_K \hat{F})$ and $h^{-1}_{K,F} = (\hat{K}e_F) \circ (g_1^K F \vartheta_{\hat{\Theta}_1}) = (e_K\hat{F}) \circ (\vartheta^{\hat{\Theta}_3} K f_1^F)$. Also, recall the relations of the form $(\vartheta_{\hat{\Theta}_2} e_f) \circ f_1^F = f_1^F$. Using the obvious notation, we have to show that $id_{\widetilde{LKF}} = h_{L,KF} \circ (\widetilde{L}h_{K,F}) \circ (h^{-1}_{L,K}\widetilde{F}) \circ h^{-1}_{LK,F}$.
We compute $h_{L,KF} \circ (\widetilde{L}h_{K,F}) \circ (h^{-1}_{L,K}\widetilde{F}) \circ h^{-1}_{LK,F}$
$= (g_2^L KF\vartheta_{\hat{\Theta}_1}) \circ (\hat{L}e_{KF}) \circ (e_L h_{K,F}) \circ (h^{-1}_{L,K} e_F) \circ (e_{LK}\hat{F}) \circ (\vartheta^{\hat{\Theta}_4} LK f_1^F)$
$= (g_2^L KF\vartheta_{\hat{\Theta}_1}) \circ (\hat{L}h_{K,F}) \circ (h^{-1}_{L,K}\hat{F}) \circ (\vartheta^{\hat{\Theta}_4} LK f_1^F)$
$= (g_2^L KF\vartheta_{\hat{\Theta}_1}) \circ (\hat{L}\vartheta^{\hat{\Theta}_3} K f_2^F) \circ (\hat{L}e_K\hat{F}) \circ (\hat{L}e_K\hat{F}) \circ (g_1^L K\vartheta_{\hat{\Theta}_2}\hat{F}) \circ (\vartheta^{\hat{\Theta}_4} LK f_1^F)$
$= (g_2^L KF\vartheta_{\hat{\Theta}_1}) \circ (\hat{L}g_2^K F\vartheta_{\hat{\Theta}_1}) \circ (\hat{L}\hat{K}e_F) \circ (g_1^L K f_1^F)$
$= (g_2^L KF\vartheta_{\hat{\Theta}_1}) \circ (\hat{L}g_2^K F\vartheta_{\hat{\Theta}_1}) \circ (g_1^L K f_1^F)$
$= (\beta_4\widehat{LKF}) \circ (\vartheta^{\hat{\Theta}_4} \operatorname{can}_L KF\vartheta_{\hat{\Theta}_1}) \circ (\vartheta^{\hat{\Theta}_4} L\mu_3 KF\vartheta_{\hat{\Theta}_1}) \circ (\hat{L}\vartheta^{\hat{\Theta}_3} \operatorname{can}_K F\vartheta_{\hat{\Theta}_1}) \circ (g_1^L K f_1^F)$
$= (\beta_4\widehat{LKF}) \circ (\vartheta^{\hat{\Theta}_4}\mu_4 LKF\vartheta_{\hat{\Theta}_1}) \circ (\vartheta^{\hat{\Theta}_4}\Theta_4 \operatorname{can}_L KF\vartheta_{\hat{\Theta}_1}) \circ (\vartheta^{\hat{\Theta}_4} \operatorname{can}_L \operatorname{can}_K F\vartheta_{\hat{\Theta}_1}) \circ (g_1^L K f_1^F)$
$= (\beta_4\widehat{LKF}) \circ (\vartheta^{\hat{\Theta}_4} \operatorname{can}_L KF\vartheta_{\hat{\Theta}_1}) \circ (g_2^L \operatorname{can}_K F\vartheta_{\hat{\Theta}_1}) \circ (g_1^L K f_1^F)$
$= (\beta_4\widehat{LKF}) \circ (\vartheta^{\hat{\Theta}_4} \operatorname{can}_L KF\vartheta_{\hat{\Theta}_1}) \circ (\vartheta^{\hat{\Theta}_4} L \operatorname{can}_K F\vartheta_{\hat{\Theta}_1}) \circ (\vartheta^{\hat{\Theta}_4} LK \operatorname{can}_F \vartheta_{\hat{\Theta}_1}) \circ (L\hat{K}F\alpha_1)$
$= e_{LKF}$
$= id_{\widetilde{LKF}}$
and are finally done. □

Remark 4.3.49. More compatibilities than stated in Proposition 4.3.48 are satisfied. For example, the composition $\widetilde{F} \overset{\widetilde{F}\alpha_1}{\Longrightarrow} \widetilde{F}\vartheta^{\hat{\Theta}_1}\vartheta_{\hat{\Theta}_1} \overset{\sim}{\Longrightarrow} \vartheta^{\hat{\Theta}_2} F\vartheta_{\hat{\Theta}_1} \overset{\sim}{\Longrightarrow} \vartheta^{\hat{\Theta}_2}\vartheta_{\hat{\Theta}_2}\widetilde{F}$ agrees with $\alpha_1\widetilde{F}$. Moreover, one may arrange and extend the constructions given into a 2-functor $\mathfrak{Hom}\left(\mathfrak{Fr}^{\mathrm{sp}}, \mathfrak{Hom}(\mathfrak{A} - \mathrm{Mod}, \mathfrak{Add}^{\mathrm{idem}})\right) \longrightarrow \mathfrak{Hom}(\mathfrak{A} - \mathrm{Mod}, \mathfrak{Add}^{\mathrm{idem}})$. Some of the steps required to establish this statement can be found in the proof of Proposition 4.3.52, however additional steps are necessary which have no counterpart in this proof.

Corollary 4.3.50. *For $i \in \{1, 2\}$, let $\Phi_i : \mathfrak{C} \to \mathfrak{Add}^{\mathrm{idem}}$ be a 2-functor and let $\hat{\Theta}_i$ be a split Frobenius objects on Φ_i. Assume that there exists an equivalence of Φ_1 and Φ_2 that is given by morphisms of 2-functors and morphisms of morphisms of 2-functors that are compatible with the Frobenius objects. Then, ${}_{\hat{\Theta}_1}\Phi_1$ and ${}_{\hat{\Theta}_2}\Phi_2$ are equivalent.*

Proof. By assumption, we have morphisms of 2-functors $F : \Phi_1 \to \Phi_2$ and $G : \Phi_2 \to \Phi_1$ that are compatible with the Frobenius objects. Furthermore, we have isomorphisms of morphisms of 2-functors $FG \cong id_{\Phi_2}$ and $GF \cong id_{\Phi_1}$ that are compatible with the Frobenius objects. We conclude $id_{{}_{\hat{\Theta}_1}\Phi_1} \cong \widetilde{id_{\Phi_1}} \cong \widetilde{GF} \cong \widetilde{G}\widetilde{F}$ and analogously $id_{{}_{\hat{\Theta}_2}\Phi_2} \cong \widetilde{F}\widetilde{G}$. □

Corollary 4.3.51. *Let $\Phi : \mathfrak{C} \to \mathfrak{Add}^{\mathrm{idem}}$ be a 2-functor and let $\hat{\Theta}_1$ and $\hat{\Theta}_2$ be isomorphic split Frobenius objects on Φ. Then, ${}_{\hat{\Theta}_1}\Phi$ and ${}_{\hat{\Theta}_2}\Phi$ are equivalent.*

Proof. By Remark 4.3.46, an isomorphism of Frobenius objects on a 2-functor is the same as the identity 2-morphism of this 2-functor being compatible with the Frobenius objects. Since the identity morphism on the identity morphism of Φ are trivially compatible with the Frobenius objects as well, we can apply Corollary 4.3.50. □

Having shown that a 2-functors ${}_{\hat{\Theta}}\Phi$ is uniquely determined up to equivalence by the properties stated in Proposition 4.3.48, we will now generalise Proposition 4.3.38 and show existence of ${}_{\hat{\Theta}}\Phi$. This is the last step of the proof of Theorem 4.3.42.

Proposition 4.3.52. *Let $\Phi : \mathfrak{C} \to \mathfrak{Add}^{\mathrm{idem}}$ be a 2-functor and let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a split Frobenius object on Φ. Then, there exists a 2-functor ${}_{\hat{\Theta}}\Phi : \mathfrak{C} \to \mathfrak{Add}^{\mathrm{idem}}$, morphisms of 2-functors $\Phi \xrightarrow{\vartheta^{\hat{\Theta}}} {}_{\hat{\Theta}}\Phi \xrightarrow{\vartheta_{\hat{\Theta}}} \Phi$ and morphisms of morphisms of 2-functors $id_{{}_{\hat{\Theta}}\Phi} \xRightarrow{\alpha} \vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}} \xRightarrow{\beta} id_{{}_{\hat{\Theta}}\Phi}$ such that $\beta \circ \alpha = id_{{}_{\hat{\Theta}}\Phi}$ and $\hat{\Theta} = (\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}})$. Furthermore, one can construct ${}_{\hat{\Theta}}\Phi$ explicitly such that ${}_{\hat{\Theta}}\Phi(c) = {}_{\hat{\Theta}(c)}\Phi(c)$ for all $c \in \mathfrak{C}$.*

Proof. Let $\Phi : \mathfrak{C} \to \mathfrak{Add}^{\mathrm{idem}}$ be a 2-functor and let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a split Frobenius object on Φ. In particular for all $c \in \mathfrak{C}$, $\hat{\Theta}(c) := (\Theta(c), \Delta(c), \mu(c))$ is a split Frobenius object on $\Phi(c) \in \mathfrak{Add}^{\mathrm{idem}}$. Thus, we can apply Proposition 4.3.38 and obtain categories ${}_{\hat{\Theta}(c)}\Phi(c) \in \mathfrak{Add}^{\mathrm{idem}}$, additive functors $\Phi(c) \xrightarrow{\vartheta^{\hat{\Theta}(c)}} {}_{\hat{\Theta}(c)}\Phi(c) \xrightarrow{\vartheta_{\hat{\Theta}(c)}} \Phi(c)$ such that $\vartheta_{\hat{\Theta}(c)}\vartheta^{\hat{\Theta}(c)} = \Theta(c)$, and transformations ${}_{\hat{\Theta}(c)}\Phi(c) \xRightarrow{\alpha_c} \vartheta_{\hat{\Theta}(c)}\vartheta^{\hat{\Theta}(c)} \xRightarrow{\beta_c} {}_{\hat{\Theta}(c)}\Phi(c)$ such that $\beta_c \circ \alpha_c = {}_{\hat{\Theta}(c)}\Phi(c)$, $\vartheta_{\hat{\Theta}(c)}\alpha_c\vartheta^{\hat{\Theta}(c)} = \Delta(c)$ and $\vartheta_{\hat{\Theta}(c)}\beta_c\vartheta^{\hat{\Theta}(c)} = \mu(c)$.

Next, we define the 2-functor ${}_{\hat{\Theta}}\Phi : \mathfrak{C} \to \mathfrak{Add}^{\mathrm{idem}}$, and establish the properties required in Definition 3.1.11. For $c \in \mathfrak{C}$, we define ${}_{\hat{\Theta}}\Phi(c) := {}_{\hat{\Theta}(c)}\Phi(c)$. For a 1-arrow $b : c \to c'$ in $\mathfrak{C}$, we set $\hat{\Phi}(b) := \vartheta^{\hat{\Theta}(c')}\Phi(b)\vartheta_{\hat{\Theta}(c)}$. Further, we denote by $e_b := (\beta_{c'}\vartheta^{\hat{\Theta}(c')}\Phi(b)\vartheta_{\hat{\Theta}(c)}) \circ (\vartheta^{\hat{\Theta}(c')} \operatorname{can}_b \vartheta_{\hat{\Theta}(c)}) \circ (\vartheta^{\hat{\Theta}(c')}\Phi(b)\vartheta_{\hat{\Theta}(c)}\alpha_c)$ an endotransformation of $\hat{\Phi}(b)$, where $\operatorname{can}_b : \Phi(b)\Theta(c) \xRightarrow{\sim} \Theta(c')\Phi(b)$ is the natural isomorphism associated to b and the endomorphism Θ of Φ. In fact e_b is an idempotent, which follows as for e_F in the proof of Proposition 4.3.48. The important ingredient being that $(\mu(c')\Phi(b)) \circ (\Theta(c') \operatorname{can}_b) \circ (\operatorname{can}_b \Theta(c)) = \operatorname{can}_b \circ (\Phi(b)\mu(c))$, which holds because μ is a morphism of morphisms 2-functors with target in the strict(!) 2-category $\mathfrak{Add}^{\mathrm{idem}}$. Now, we can define ${}_{\hat{\Theta}}\Phi(b) := (\hat{\Phi}(b), e_b)$. Further, for a 2-arrow $\gamma : b \Rightarrow b'$ in $\mathfrak{C}$ we set $\hat{\Phi}(\gamma) := \vartheta^{\hat{\Theta}(c')}\Phi(\gamma)\vartheta_{\hat{\Theta}}(c)$ and ${}_{\hat{\Theta}}\Phi(\gamma) := e_{b'} \circ \hat{\Phi}(\gamma) \circ e_b$. Clearly, ${}_{\hat{\Theta}(c)}\Phi(id_b) = e_b = id_{{}_{\hat{\Theta}}\Phi(b)}$. To show compatibility with composition, it suffices to show $\hat{\Phi}(\gamma) \circ e_b = e_{b'} \circ \hat{\Phi}(\gamma)$. This is proven analogously to the equation $e_G \circ \hat{\tau} = \hat{\tau} \circ e_F$ in the proof of Proposition

4.3.48, the main ingredient being here that $(\Theta(c')\Phi(\gamma)) \circ \mathrm{can}_b = \mathrm{can}_{b'} \circ (\Phi(\gamma)\Theta(c))$, which in turn holds since the family $\{\mathrm{can}_b\}_{b\in\mathfrak{C}(c,c')}$ is natural in b.

Furthermore, we have to provide several natural isomorphisms and prove certain compatibilities. We start by defining $h_{b_2,b_1} : {}_{\hat\Theta}\Phi(b_2)\,{}_{\hat\Theta}\Phi(b_1) \overset{\sim}{\Longrightarrow} {}_{\hat\Theta}\Phi(b_2b_1)$ for all 1-arrows $b_1 \in \mathfrak{C}(c_1, c_2)$ and $b_2 \in \mathfrak{C}(c_2, c_3)$. Denote by $\mathrm{can}_{b_2,b_1} : \Phi(b_2)\Phi(b_1) \to \Phi(b_2b_1)$ the natural isomorphisms that are part of the datum Φ. Now, we can introduce $h_{b_2,b_1} := (\beta_{c_3}\vartheta^{\Theta(c_3)}\,\mathrm{can}_{b_2,b_1}\,\vartheta_{\Theta(c_1)}\beta_{c_1}) \circ (\vartheta^{\Theta(c_3)}\,\mathrm{can}_{b_2}\,\mathrm{can}_{b_1}^{-1}\,\vartheta_{\Theta(c_1)}) \circ (\hat\Phi(b_2)\alpha_{c_2}\hat\Phi(b_1))$. *A priori* , we have defined a 2-arrow $h_{b_2,b_1} : \hat\Phi(b_2)\hat\Phi(b_1) \Rightarrow \hat\Phi(b_2b_1)$, and thus have to show that $h_{b_2,b_1} \circ (e_{b_2}e_{b_1}) = h_{b_2,b_1} = e_{b_2b_1} \circ h_{b_2,b_1}$. We claim that $h^{-1}_{b_2,b_1} = h'_{b_2,b_1} := (\hat\Phi(b_2)\beta_{c_2}\hat\Phi(b_1)) \circ (\vartheta^{\Theta(c_3)}\,\mathrm{can}^{-1}_{b_2}\,\mathrm{can}_{b_1}\,\vartheta_{\Theta(c_1)}) \circ (\alpha_{c_3}\vartheta^{\Theta(c_3)}\,\mathrm{can}^{-1}_{b_2,b_1}\,\vartheta_{\Theta(c_1)}\alpha_{c_1})$. Similar to arguments made before, we need to show that $h'_{b_2,b_1} \circ e_{b_2b_1} = h'_{b_2,b_1} = (e_{b_2}e_{b_1}) \circ h'_{b_2,b_1}$ in order to see that $h'_{b_2,b_1} : {}_{\hat\Theta}\Phi(b_2b_1) \Rightarrow {}_{\hat\Theta}\Phi(b_2)\,{}_{\hat\Theta}\Phi(b_1)$. Moreover, we need that $h_{b_2,b_1} \circ h'_{b_2,b_1} = e_{b_2b_1}$ and $h'_{b_2,b_1} \circ h_{b_2,b_1} = e_{b_2}e_{b_1}$, to conclude $h^{-1}_{b_2,b_1} = h'_{b_2,b_1}$. This can be shown by analogous arguments to those used to establish $\widetilde{KF} \cong \widetilde{K}\widetilde{F}$ in Proposition 4.3.38. However, we have to take into account that no analogue of can_{b_2,b_1} was present there. The proof carries over nevertheless, since instead of $(\mathrm{can}_K F) \circ (K\,\mathrm{can}_F) = \mathrm{can}_{KF}$ we now have $(\Theta(c_3)\,\mathrm{can}_{b_1,b_2}) \circ (\mathrm{can}_{b_2}\,\Phi(b_1)) \circ (\Phi(b_2)\,\mathrm{can}_{b_1}) \circ (\mathrm{can}^{-1}_{b_2,b_1}\,\Theta(c_1)) = \mathrm{can}_{b_1b_2}$ since Θ is an endomorphism of Φ.

Furthermore, we need to show that h_{b_2,b_1} is natural in b_1 and b_2. Let $\gamma_1 : b_1 \Rightarrow b'_1$ and $\gamma_2 : b_2 \Rightarrow b'_2$ be 2-arrows in $\mathfrak{C}$. Indeed, we can calculate

$$\begin{aligned}
h_{b'_2,b'_1} \circ ({}_{\hat\Theta}\Phi(\gamma_2)\,{}_{\hat\Theta}\Phi(\gamma_1)) &= h_{b'_2,b'_1} \circ \{(e_{b'_2} \circ \hat\Phi(\gamma_2))(e_{b'_1} \circ \hat\Phi(\gamma_1))\} \\
&= h_{b'_2,b'_1} \circ (e_{b'_2}e_{b'_1}) \circ (\hat\Phi(\gamma_1)\hat\Phi(\gamma_2)) \\
&= e_{b'_2b'_1} \circ h_{b'_2,b'_1} \circ (\hat\Phi(\gamma_1)\hat\Phi(\gamma_2)) \\
&= e_{b'_2b'_1} \circ \hat\Phi(\gamma_1\gamma_2) \circ h_{b_2,b_1} \\
&= {}_{\hat\Theta}\Phi(\gamma_1\gamma_2) \circ h_{b_2,b_1},
\end{aligned}$$

where we used that $h_{b'_2,b'_1} \circ (\hat\Phi(\gamma_1)\hat\Phi(\gamma_2)) = (\hat\Phi(\gamma_1\gamma_2)) \circ h_{b_2,b_1}$, which in turn holds since h_{b_2,b_1} is defined using morphisms that are natural.

Further, ${}_{\hat\Theta}\Phi((b_3b_2)b_1) = {}_{\hat\Theta}\Phi(\mathrm{can}_{b_3,b_2,b_1}) \circ h_{b_3,b_2b_1} \circ ({}_{\hat\Theta}\Phi(b_3)h_{b_2,b_1}) \circ (h^{-1}_{b_3,b_2}\,{}_{\hat\Theta}\Phi(b_1)) \circ h^{-1}_{b_3b_2,b_1}$ holds by a calculation analogous to one provided in the proof of Proposition 4.3.38.

Furthermore, we have to provide the natural isomorphisms $u_c : id_{{}_{\hat\Theta}\Phi(c)} \overset{\sim}{\Longrightarrow} {}_{\hat\Theta}\Phi(I_c)$ for all $c \in \mathfrak{C}$. We denote by $\mathrm{can}_c : id_{\Phi(c)} \overset{\sim}{\Longrightarrow} \Phi(I_c)$ the natural isomorphisms associated to Φ. Set $u_c := (\vartheta^{\hat\Theta(c)}\,\mathrm{can}_c\,\vartheta_{\hat\Theta(c)}) \circ \alpha_c$ which is *a priori* a morphism $id_{{}_{\hat\Theta}\Phi(c)} \overset{\sim}{\Longrightarrow} \hat\Phi(I_c)$. However, $e_{I_c} \circ u_c = (\beta_c\hat\Phi(I_c)) \circ (\vartheta^{\hat\Theta(c)}\,\mathrm{can}_{I_c}\,\vartheta_{\hat\Theta(c)}) \circ (\vartheta^{\hat\Theta(c)}\,\mathrm{can}_c\,\Theta(c)\vartheta_{\hat\Theta(c)}) \circ (\alpha_c\alpha_c) = (\beta_c\hat\Phi(I_c)) \circ (\vartheta^{\hat\Theta(c)}\Theta(c)\,\mathrm{can}_c\,\vartheta_{\hat\Theta(c)}) \circ (\alpha_c\alpha_c) = u_c$, where we used that the natural isomorphisms can_c, which come with Φ, are compatible with the natural isomorphisms

can_{I_c}, which come with the endomorphisms Θ of Φ. Thus, $u_c : id_{{}_{\hat\Theta}\Phi(c)} \Rightarrow {}_{\hat\Theta}\Phi(I_c)$. We claim that the inverse of u_c is given by $\beta_c \circ (\vartheta^{\hat\Theta(c)} \mathrm{can}_c^{-1} \vartheta_{\hat\Theta(c)})$, which is a morphism ${}_{\hat\Theta}\Phi(I_c) \Rightarrow id_{{}_{\hat\Theta}\Phi(c)}$ by the same arguments as above. The other composition being obvious, we compute

$$\begin{aligned}
&u_c \circ \beta_c \circ (\vartheta^{\hat\Theta(c)} \mathrm{can}_c^{-1} \vartheta_{\hat\Theta(c)})\\
&\qquad = (\beta_c\hat\Phi(c)) \circ (\vartheta^{\hat\Theta(c)}\Theta(c)\,\mathrm{can}_c \vartheta_{\hat\Theta(c)}) \circ (\vartheta^{\hat\Theta(c)} \mathrm{can}_c^{-1} \Theta(c)\vartheta_{\hat\Theta(c)}) \circ (\hat\Phi(c)\alpha_c)\\
&\qquad = (\beta_c\hat\Phi(c)) \circ (\vartheta^{\hat\Theta(c)} \mathrm{can}_{I_c} \vartheta_{\hat\Theta(c)}) \circ (\vartheta^{\hat\Theta(c)} \mathrm{can}_c \Theta(c)\vartheta_{\hat\Theta(c)})\\
&\qquad\qquad \circ (\vartheta^{\hat\Theta(c)} \mathrm{can}_c^{-1} \Theta(c)\vartheta_{\hat\Theta(c)}) \circ (\hat\Phi(c)\alpha_c)\\
&\qquad = e_{I_c}.
\end{aligned}$$

It remains to show that the two squares in Definition 3.1.11 commute. For this purpose let $b : c \to c'$ be a 1-arrow in $\mathfrak{C}$ and write $\mathrm{can}(b) : bI_c \overset{\sim}{\Longrightarrow} b$ for the right unit in $\mathfrak{C}$. Recall that $\mathrm{can}_{I_c} \circ (\mathrm{can}_c \Theta(c)) = \Theta(c)\,\mathrm{can}_c$ by Definition 3.1.15 for Θ and that $\Phi(\mathrm{can}(b)) \circ \mathrm{can}_{b,I_c} \circ (\Phi(b)\,\mathrm{can}_c) = \Phi(b)$ by the left square in Definition 3.1.11 for Φ. Now, the left square in Definition 3.1.11 for ${}_{\hat\Theta}\Phi$ commutes since

$$\begin{aligned}
&{}_{\hat\Theta}\Phi(\mathrm{can}(b)) \circ h_{b,I_c} \circ ({}_{\hat\Theta}\Phi(b)u_c)\\
&\quad = \hat\Phi(\mathrm{can}(b)) \circ e_{bI_c} \circ h_{b,I_c} \circ (e_b u_c)\\
&\quad = \hat\Phi(\mathrm{can}(b)) \circ h_{b,I_c} \circ (\hat\Phi(b)u_c)\\
&\quad = \hat\Phi(\mathrm{can}(b)) \circ (\beta_{c'}\vartheta^{\hat\Theta(c')} \mathrm{can}_{b,I_c} \vartheta_{\hat\Theta(c)}\beta_c) \circ (\vartheta^{\hat\Theta(c')} \mathrm{can}_b \mathrm{can}_{I_c}^{-1} \vartheta_{\hat\Theta(c)})\\
&\qquad\quad \circ (\hat\Phi(b)\alpha_c\hat\Phi(I_c)) \circ (\hat\Phi(b)u_c)\\
&\quad = (\beta_{c'}\vartheta^{\hat\Theta(c')} \mathrm{can}_c^{-1} \vartheta_{\hat\Theta(c)}\beta_c) \circ (\vartheta^{\hat\Theta(c')} \mathrm{can}_b \mathrm{can}_{I_c}^{-1} \vartheta_{\hat\Theta(c)}) \circ (\hat\Phi(b)\alpha_c\hat\Phi(I_c)) \circ (\hat\Phi(b)u_c)\\
&\quad = (\beta_{c'}\hat\Phi(b)\beta_c) \circ (\vartheta^{\hat\Theta(c')} \mathrm{can}_b \Theta(c)\,\mathrm{can}_c^{-1} \vartheta_{\hat\Theta(c)}) \circ (\hat\Phi(b)\alpha_c\hat\Phi(I_c))\\
&\qquad\quad \circ (\hat\Phi(b)\vartheta^{\hat\Theta(c)} \mathrm{can}_c \vartheta_{\hat\Theta(c)}) \circ (\hat\Phi(b)\alpha_c)\\
&\quad = (\beta_{c'}\hat\Phi(b)\beta_c) \circ (\vartheta^{\hat\Theta(c')} \mathrm{can}_b \Theta(c)\vartheta_{\hat\Theta(c)}) \circ (\hat\Phi(b)\alpha_c\vartheta^{\hat\Theta(c)}\vartheta_{\hat\Theta(c)}) \circ (\hat\Phi(b)\alpha_c)\\
&\quad = e_b\\
&\quad = {}_{\hat\Theta}\Phi(b).
\end{aligned}$$

Commutativity of the other square follows analogously and the definition of the 2-functor ${}_{\hat\Theta}\Phi$ is finally complete.

Next, we define the morphisms $\Phi \xrightarrow{\vartheta^{\hat\Theta}} {}_{\hat\Theta}\Phi \xrightarrow{\vartheta_{\hat\Theta}} \Phi$. We start with $\vartheta_{\hat\Theta}$ and give the definition of $\vartheta^{\hat\Theta}$ thereafter without repeating the proof of the properties required by Definition 3.1.15. We set $\vartheta_{\hat\Theta}(c) := \vartheta_{\hat\Theta(c)}$ which is a functor ${}_{\hat\Theta}\Phi(c) = {}_{\hat\Theta(c)}\Phi(c) \to \Phi(c)$. Further for all 1-arrows $b : c \to c'$ in $\mathfrak{C}$, we need to specify a natural isomorphism $t_b : \Phi(b)\vartheta_{\hat\Theta(c)} \overset{\sim}{\Longrightarrow} \vartheta_{\hat\Theta(c')}\,{}_{\hat\Theta}\Phi(b)$. We set $t_b := (\mathrm{can}_b \vartheta_{\hat\Theta(c)}) \circ (\Phi(b)\vartheta_{\hat\Theta(c)}\alpha_c)$. *A priori* $t_b : \Phi(b)\vartheta_{\hat\Theta(c)} \Rightarrow \vartheta_{\hat\Theta(c')}\hat\Phi(b)$, however

$$
\begin{aligned}
(\vartheta_{\hat{\Theta}(c')}e_b) \circ t_b &= (\mu_{c'}\Phi(b)\vartheta_{\hat{\Theta}(c)}) \circ (\hat{\Theta}(c')\,\mathrm{can}_b\,\vartheta_{\hat{\Theta}(c)}) \circ (\mathrm{can}_b\,\Theta(c)\vartheta_{\hat{\Theta}(c)}) \circ (\Phi(b)\vartheta_{\hat{\Theta}(c)}\alpha_c\alpha_c) \\
&= (\mathrm{can}_b\,\vartheta_{\hat{\Theta}(c)}) \circ (\mu_{c'}\vartheta_{\hat{\Theta}(c)}) \circ (\Phi(b)\vartheta_{\hat{\Theta}(c)}\alpha_c\alpha_c) \\
&= t_b.
\end{aligned}
$$

That t_b is natural in b holds since can_b and $\Phi(b)$ are natural in b. We claim that $t_b^{-1} := (\Phi(b)\vartheta_{\hat{\Theta}(c)}\beta_c) \circ (\mathrm{can}_b^{-1}\,\vartheta_{\hat{\Theta}(c)})$ is the inverse of t_b. Analogously as for t_b, one shows that $t_b^{-1} : \vartheta_{\hat{\Theta}(c')}\,{}_{\hat{\Theta}}\Phi(b) \Rightarrow \Phi(b)\vartheta_{\hat{\Theta}(c)}$. Since $t_b^{-1} \circ t_b = id_{\Phi(b)\vartheta_{\hat{\Theta}(c)}}$ clearly holds, we only have to compute $t_b \circ t_b^{-1} = (\mathrm{can}_b\,\vartheta_{\hat{\Theta}(c)}) \circ (\Phi(b)\mu_c\vartheta_{\hat{\Theta}(c)}) \circ (\mathrm{can}_b^{-1}\,\vartheta_{\hat{\Theta}(c)}\alpha_c) = (\mu_{c'}\Phi(b)\vartheta_{\hat{\Theta}(c)}) \circ (\Theta(c')\,\mathrm{can}_b\,\vartheta_{\hat{\Theta}(c)}) \circ (\Theta(c')\Phi(b)\vartheta_{\hat{\Theta}(c)}\alpha_c) = \vartheta_{\hat{\Theta}(c')}e_b = \vartheta_{\hat{\Theta}(c')}id_{{}_{\hat{\Theta}}\Phi(b)}$. It remains to check the two compatibilities which are required in Definition 3.1.15. First, we need to show $t_{b_2b_1} \circ (\mathrm{can}_{b_2,b_1}\,\vartheta_{\hat{\Theta}(c_1)}) = (\vartheta_{\hat{\Theta}(c_3)}h_{b_2,b_1}) \circ (t_{b_2}\,{}_{\hat{\Theta}}\Phi(b_1)) \circ (\Phi(b_2)t_{b_1})$ for all 1-arrows $b_1 \in \mathfrak{C}(c_1, c_2)$ and $b_2 \in \mathfrak{C}(c_2, c_3)$. Second for all $c \in \mathfrak{C}$, we need to show that $t_{I_c} \circ (\mathrm{can}_c\,\vartheta_{\hat{\Theta}(c)}) = \vartheta_{\hat{\Theta}(c)}u_c$. The second compatibility is easier: $t_{I_c} \circ (\mathrm{can}_c\,\vartheta_{\hat{\Theta}(c)}) = (\mathrm{can}_{I_c}\,\vartheta_{\hat{\Theta}(c)}) \circ (\mathrm{can}_c\,\Theta(c)\vartheta_{\hat{\Theta}(c)}) \circ (\vartheta_{\hat{\Theta}(c)}\alpha_c) = (\Theta(c)\,\mathrm{can}_c\,\vartheta_{\hat{\Theta}(c)}) \circ (\vartheta_{\hat{\Theta}(c)}\alpha_c) = \vartheta_{\hat{\Theta}(c)}u_c$. To establish the first compatibility, we compute

$(\vartheta_{\hat{\Theta}(c_3)}h_{b_2,b_1}) \circ (t_{b_2}\,{}_{\hat{\Theta}}\Phi(b_1)) \circ (\Phi(b_2)t_{b_1})$

$$
\begin{aligned}
&= (\vartheta_{\hat{\Theta}(c_3)} h_{b_2,b_1}) \circ (t_{b_2} e_{b_1}) \circ (\Phi(b_2) t_{b_1}) \\
&= (\vartheta_{\hat{\Theta}(c_3)} h_{b_2,b_1}) \circ (t_{b_2} \hat{\Phi}(b_1)) \circ (\Phi(b_2) t_{b_1}) \\
&= (\vartheta_{\hat{\Theta}(c_3)} h_{b_2,b_1}) \circ (t_{b_2} \hat{\Phi}(b_1)) \circ (\Phi(b_2) t_{b_1}) \\
&= (\mu_3 \operatorname{can}_{b_2,b_1} \vartheta_{\hat{\Theta}(c_1)} \beta_{c_1}) \circ (\Theta(c_3) \operatorname{can}_{b_2} \operatorname{can}_{b_1}^{-1} \vartheta_{\hat{\Theta}(c_1)}) \circ (\Theta(c_3)\Phi(b_2)\Delta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \\
&\quad \circ (\operatorname{can}_{b_2} \Theta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2)\Delta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2) t_{b_1}) \\
&= (\Theta(c_3) \operatorname{can}_{b_2,b_1} \vartheta_{\hat{\Theta}(c_1)} \beta_{c_1}) \circ (\mu(c_3)\Phi(b_2) \operatorname{can}_{b_1}^{-1} \vartheta_{\hat{\Theta}(c_1)}) \\
&\quad \circ (\Theta(c_3) \operatorname{can}_{b_2} \Theta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \circ (\operatorname{can}_{b_2} \Theta(c_2)\Theta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \\
&\quad \circ (\Phi(b_2)\Theta(c_2)\Delta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2)\Delta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2) t_{b_1}) \\
&= (\Theta(c_3) \operatorname{can}_{b_2,b_1} \vartheta_{\hat{\Theta}(c_1)} \beta_{c_1}) \circ (\Theta(c_3)\Phi(b_2) \operatorname{can}_{b_1}^{-1} \vartheta_{\hat{\Theta}(c_1)}) \circ (\operatorname{can}_{b_2} \Theta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \\
&\quad \circ (\Phi(b_2)\mu(c_2)\Theta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2)\Delta(c_2)\Theta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \\
&\quad \circ (\Phi(b_2)\Delta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2) t_{b_1}) \\
&= (\Theta(c_3) \operatorname{can}_{b_2,b_1} \vartheta_{\hat{\Theta}(c_1)} \beta_{c_1}) \circ (\operatorname{can}_{b_2} \operatorname{can}_{b_1}^{-1} \vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2)\Delta(c_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)}) \\
&\quad \circ (\Phi(b_2) \operatorname{can}_{b_1} \vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)} \alpha_{c_1}) \\
&= (\Theta(c_3) \operatorname{can}_{b_2,b_1} \vartheta_{\hat{\Theta}(c_1)} \beta_{c_1}) \circ (\operatorname{can}_{b_2} \Phi(b_1)\Theta(c_1)\vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2) \operatorname{can}_{b_1} \Theta(c_1)\vartheta_{\hat{\Theta}(c_1)}) \\
&\quad \circ (\Phi(b_2)\Phi(b_1)\Delta(c_1)\vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)} \alpha_{c_1}) \\
&= (\operatorname{can}_{b_2 b_1} \vartheta_{\hat{\Theta}(c_1)} \beta_{c_1}) \circ (\operatorname{can}_{b_2,b_1} \Phi(b_1)\Theta(c_1)\Theta(c_1)\vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2)\Phi(b_1)\Delta(c_1)\vartheta_{\hat{\Theta}(c_1)}) \\
&\quad \circ (\Phi(b_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)} \alpha_{c_1}) \\
&= (\operatorname{can}_{b_2 b_1} \vartheta_{\hat{\Theta}(c_1)}) \circ (\operatorname{can}_{b_2,b_1} \Theta(c_1)\vartheta_{\hat{\Theta}(c_1)}) \circ (\Phi(b_2)\Phi(b_1)\vartheta_{\hat{\Theta}(c_1)} \alpha_{c_1}) \\
&= t_{b_2 b_1} \circ (\operatorname{can}_{b_2,b_1} \vartheta_{\hat{\Theta}(c_1)}).
\end{aligned}
$$

As previously announced, we now state the definition of $\vartheta^{\hat{\Theta}}$ without checking the details. We set $\vartheta^{\hat{\Theta}}(c) := \vartheta^{\hat{\Theta}(c)} : \Phi(c) \to {}_{\hat{\Theta}(c)}\Phi(c) = {}_{\hat{\Theta}}\Phi(c)$ for all $c \in \mathfrak{C}$ and define $k_b := (\beta_{c'}\vartheta^{\hat{\Theta}(c')}\Phi(b)) \circ (\vartheta^{\hat{\Theta}(c')} \operatorname{can}_b) : {}_{\hat{\Theta}}\Phi(b)\vartheta^{\hat{\Theta}(c)} \xRightarrow{\sim} \vartheta^{\hat{\Theta}(c')}\Phi(b)$ for all 1-arrows $b \in \mathfrak{C}(c, c')$. As above, we have $k_b \circ e_b = k_b$.

Next, we show that $\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}} = \Theta$. By construction, we have $(\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}})(c) = \vartheta_{\hat{\Theta}(c)}\vartheta^{\hat{\Theta}(c)} = \Theta(c)$ and it remains to check that the natural isomorphisms that come with $\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}$ and Θ agree. Indeed for all 1-arrows $b \in \mathfrak{C}(c, c')$ in $\mathfrak{C}$, we have $(\vartheta_{\hat{\Theta}(c')}k_b) \circ (t_b\vartheta^{\hat{\Theta}(c)}) = (\mu(c')\Phi(b)) \circ (\Theta(c') \operatorname{can}_b) \circ (\operatorname{can}_b \Theta(c)) \circ (\Phi(b)\Delta(c)) = \operatorname{can}_b \circ (\Phi(b)\mu(c)) \circ (\Phi(b)\Delta(c)) = \operatorname{can}_b$.

It remains to define the morphisms of morphisms of 2-functors ${}_{\hat{\Theta}}\Phi \xRightarrow{\alpha} \vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}} \xRightarrow{\beta} {}_{\hat{\Theta}}\Phi$, and to show that $\beta \circ \alpha = id_{{}_{\hat{\Theta}}\Phi}$ and that $\hat{\Theta} = (\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}})$. We define α by setting $\alpha(c) := \alpha_c : {}_{\hat{\Theta}(c)}\Phi(c) \Rightarrow \vartheta^{\hat{\Theta}(c)}\vartheta_{\hat{\Theta}(c)} = (\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}})(c)$. Since α has domain an identity morphisms of a 2-functor, the requirement of Definition 3.1.18 for all 1-

arrows $b \in \mathfrak{C}(c,c')$ specialises to $(\vartheta^{\hat{\Theta}(c')} t_b) \circ (k_b \vartheta_{\hat{\Theta}(c)}) \circ ({}_{\hat{\Theta}}\Phi(b)\alpha_c) = \alpha_c\, {}_{\hat{\Theta}}\Phi(b)$, where $(\vartheta^{\hat{\Theta}(c')} t_b) \circ (k_b \vartheta_{\hat{\Theta}(c)})$ is the canonical map associated to the composition $\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}}$. Now, $(\vartheta^{\hat{\Theta}(c')} t_b) \circ (k_b \vartheta_{\hat{\Theta}(c)}) \circ (e_b \alpha_c)$

$$
\begin{aligned}
&= (\vartheta^{\hat{\Theta}(c')} t_b) \circ (k_b \vartheta_{\hat{\Theta}(c)}) \circ (\hat{\Phi}(b)\alpha_c) \\
&= (\vartheta^{\hat{\Theta}(c')} \operatorname{can}_b \vartheta_{\hat{\Theta}(c)}) \circ (\hat{\Phi}(b)\alpha_c) \circ (\beta_{c'}\hat{\Phi}(b)) \circ (\vartheta^{\hat{\Theta}(c')} \operatorname{can}_b \vartheta_{\hat{\Theta}(c)}) \circ (\hat{\Phi}(b)\alpha_c) \\
&= (\vartheta^{\hat{\Theta}(c')} \operatorname{can}_b \vartheta_{\hat{\Theta}(c)}) \circ (\beta_{c'}\vartheta^{\hat{\Theta}(c')}\Phi(b)\Theta(c)\vartheta_{\hat{\Theta}(c)}) \circ (\vartheta^{\hat{\Theta}(c')} \operatorname{can}_b \Theta(c)\vartheta_{\hat{\Theta}(c)}) \\
&\qquad \circ (\vartheta^{\hat{\Theta}(c')}\Phi(b)\Delta(c)\vartheta_{\hat{\Theta}(c)}) \circ (\hat{\Phi}(b)\alpha_c) \\
&= (\beta_{c'}\vartheta^{\hat{\Theta}(c')}\Theta(c')\Phi(b)\vartheta_{\hat{\Theta}(c)}) \circ (\vartheta^{\hat{\Theta}(c')}\Delta(c')\Phi(b)\vartheta_{\hat{\Theta}(c)}) \circ (\vartheta^{\hat{\Theta}(c')} \operatorname{can}_b \vartheta_{\hat{\Theta}(c)}) \circ (\hat{\Phi}(b)\alpha_c) \\
&= (\alpha_{c'}\hat{\Phi}(b)) \circ (\beta_{c'}\hat{\Phi}(b)) \circ (\vartheta^{\hat{\Theta}(c')} \operatorname{can}_b \vartheta_{\hat{\Theta}(c)}) \circ (\hat{\Phi}(b)\alpha_c) \\
&= (\alpha_{c'}\hat{\Phi}(b)) \circ e_b \\
&= \alpha_{c'} e_b \\
&= \alpha_{c'} id_{{}_{\hat{\Theta}}\Phi(b)}.
\end{aligned}
$$

Analogously, we define $\beta : \vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}} \Rightarrow {}_{\hat{\Theta}}\Phi$ by setting $\beta(c) := \beta_c$. The requirements in Definition 3.1.18 are checked in the same way as for α. We have $\beta \circ \alpha = id_{{}_{\hat{\Theta}}\Phi}$, since $\beta_c \circ \alpha_c = id_{{}_{\hat{\Theta}(c)}\Phi(c)}$ holds by construction in Proposition 4.3.38 for all $c \in \mathfrak{C}$. Similarly $(\vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}})(c) = \vartheta_{\hat{\Theta}(c)}\alpha_c\vartheta^{\hat{\Theta}(c)} = \Delta_c = \Delta(c)$ holds for all $c \in \mathfrak{C}$, and thus $\vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}} = \Delta$. Analogously, $\vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}} = \mu$ holds. Thus, we have recovered the Frobenius object and the proof is finally complete. □

4.3.4 Split Frobenius objects on additive higher representations

In this section, we specialise the theory developed in Section 4.3.3 to the case $\mathfrak{C} = \mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}}$ where $\mathfrak{A}$ is an additive 2-category. The goal of this section is to proof the following theorem:

Theorem 4.3.53. *Let $\Phi : \mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$ be a representable 2-functor and let $\hat{\Theta}$ be a split Frobenius object on Φ. Then, ${}_{\hat{\Theta}}\Phi$ is representable.*

Proof. We split the proof in several parts: In Proposition 4.3.57, we show under which assumptions a higher representation $\mathcal{M}_{\hat{\Theta}} \in \mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}}$ represents ${}_{\hat{\Theta}}\Phi$. Thereafter, we construct such a representation explicitly in Proposition 4.3.58. This last step is where we take a short cut and only perform the construction in $\mathcal{B} - \mathrm{Mod}$. □

Remark 4.3.54. We do not claim that ${}_{\hat{\Theta}_1}\Phi_1 \cong {}_{\hat{\Theta}_2}\Phi_2$ implies $\Phi_1 \cong \Phi_2$. A counterexample can be constructed from Proposition 4.5.37 provided existence of certain split Frobenius objects.

Already in Notation 4.3.8, we explained how one can transport Frobenius objects along monoidal functors. In the following, we need a version of this statement that applies to 2-functors. In order to establish Theorem 4.3.53, we moreover have to ensure that we can transport this structure in a compatible manner. Lemma 4.3.56 will allow us to achieve this and indeed applies in out situation due to the following well known fact:

Remark 4.3.55. Let (F, G, η, ϵ') be a quadruple, where $F : c_1 \to c_2$ and $G : c_2 \to c_1$ are 1-arrows and $\eta : I_{c_1} \xRightarrow{\sim} GF$ and $\epsilon' : FG \xRightarrow{\sim} I_{c_2}$ are invertible 2-arrows. Setting $\epsilon := \epsilon' \circ (F\eta^{-1}G) \circ (FG(\epsilon')^{-1})$, we obtain adjoint equivalences (F, G, η, ϵ) and $(F, G, \epsilon^{-1}, \eta^{-1})$. Furthermore, if (F, G, η, ϵ') was an adjoint equivalence then $\epsilon' = \epsilon$.

Lemma 4.3.56. *Let $\mathfrak{C}$ be a strict 2-category, let c_1, $c_2 \in \mathfrak{C}$, let (F, G, η, ϵ) be an equivalence between c_1 and c_2, and let (Θ, Δ, μ) be a Frobenius object on c_2. Then, setting $\widetilde{\Theta} := G\Theta F$, $\widetilde{\Delta} := (G\Theta\epsilon^{-1}\Theta F) \circ (G\Delta F)$ and $\widetilde{\mu} := (G\mu F) \circ (G\Theta\epsilon\Theta F)$ defines a Frobenius object $(\widetilde{\Theta}, \widetilde{\Delta}, \widetilde{\mu})$ on c_1. Further, F (resp. G) is compatible with the Frobenius objects via $\mathrm{can}_F := \epsilon\Theta F$ (resp. $\mathrm{can}_G := G\Theta\epsilon^{-1}$). Furthermore, η is compatible with (Θ, Δ, μ) via $\mathrm{can}_{FG} := (\mathrm{can}_F G) \circ (F \mathrm{can}_G)$, and provided that (F, G, η, ϵ) is an adjoint equivalence, ϵ is compatible with $(\widetilde{\Theta}, \widetilde{\Delta}, \widetilde{\mu})$ via $\mathrm{can}_{GF} = (\mathrm{can}_G F) \circ (G \mathrm{can}_F)$.*

Proof. We start be checking the Frobenius property. We calculate

$$\begin{aligned}
\widetilde{\Delta} \circ \widetilde{\mu} &= (G\Theta\epsilon^{-1}\Theta F) \circ (G\Theta\mu F) \circ (G\Delta\Theta F) \circ (G\Theta\epsilon\Theta F) \\
&= (G\Theta FG\mu F) \circ (G\Theta\epsilon^{-1}\Theta\Theta F) \circ (G\Theta\Theta\epsilon\Theta F) \circ (G\Delta FG\Theta F) \\
&= (\widetilde{\Theta}G\mu F) \circ (G\Theta FG\Theta\epsilon\Theta F) \circ (G\Theta\epsilon^{-1}\Theta FG\Theta F) \circ (G\Delta F\widetilde{\Theta}) \\
&= (\widetilde{\Theta}\widetilde{\mu}) \circ (\widetilde{\Delta}\widetilde{\Theta})
\end{aligned}$$

and can show analogously that $\widetilde{\Delta} \circ \widetilde{\mu} = (\widetilde{\mu}\widetilde{\Theta}) \circ (\widetilde{\Theta}\widetilde{\Delta})$ holds.

Next, F is compatible with the Frobenius objects via can_F, see Definition 4.3.43, since

$$\begin{aligned}
\mathrm{can}_F \circ (F\widetilde{\mu}) &= (\epsilon\Theta F) \circ (FG\mu F) \circ (FG\Theta\epsilon\Theta F) \\
&= (\mu F) \circ (\epsilon\Theta\epsilon\Theta F) \\
&= (\mu F) \circ (\Theta\epsilon\Theta F) \circ (\epsilon\Theta FG\Theta F) \\
&= (\mu F) \circ (\Theta \mathrm{can}_F) \circ (\mathrm{can}_F \widetilde{\Theta})
\end{aligned}$$

and similarly

$$\begin{aligned}
(\Delta F) \circ \mathrm{can}_F &= (\epsilon\Theta\Theta F) \circ (FG\Delta F) \\
&= (\Theta\epsilon\Theta F) \circ (\epsilon\Theta\epsilon^{-1}\Theta F) \circ (FG\Delta F) \\
&= (\Theta\epsilon\Theta F) \circ (\epsilon\Theta FG\Theta F) \circ (FG\Theta\epsilon^{-1}\Theta F) \circ (FG\Delta F) \\
&= (\Theta \mathrm{can}_F) \circ (\mathrm{can}_F \widetilde{\Theta}) \circ (F\widetilde{\Delta})
\end{aligned}$$

holds.

Analogously, we can show that G is compatible with the Frobenius objects via can_G, *i.e.* the equations $(\widetilde{\Delta}G)\circ\operatorname{can}_G = (\widetilde{\Theta}\operatorname{can}_G)\circ(\operatorname{can}_G\Theta)\circ(G\Delta)$ as well as $\operatorname{can}_G\circ(G\mu) = (\widetilde{\mu}G)\circ(\widetilde{\Theta}\operatorname{can}_G)\circ(\operatorname{can}_G\Theta)$.

Next, we show that ϵ is compatible with (Θ,Δ,μ) via can_{FG}. We have $(\Theta\epsilon)\circ\operatorname{can}_{FG} = (\Theta\epsilon)\circ(\operatorname{can}_F G)\circ(F\operatorname{can}_G) = (\Theta\epsilon)\circ(\epsilon\Theta FG)\circ(FG\epsilon^{-1}) = (\epsilon\Theta)\circ(FG\Theta\epsilon)\circ(FG\Theta\epsilon^{-1}) = \epsilon\Theta$.

Finally, we show that η is compatible with $(\widetilde{\Theta},\widetilde{\Delta},\widetilde{\mu})$ assuming that (F,G,ϵ,η) is an adjoint equivalence. We thus assume that $(G\epsilon)\circ(\eta G) = G$ and $(\epsilon F)\circ(F\eta) = F$. In fact, we only need $\epsilon^{-1}F = F\eta$ which is equivalent to the latter equation. We have $\operatorname{can}_{GF}\circ(\eta\widetilde{\Theta}) = (\operatorname{can}_G F)\circ(G\operatorname{can}_F)\circ(\eta\widetilde{\Theta}) = (G\Theta\epsilon^{-1}F)\circ(G\epsilon\Theta F)\circ(\eta G\Theta F) = G\Theta\epsilon^{-1}F = G\Theta F\eta = \widetilde{\Theta}\eta$. $\square$

The next two propositions are the main steps in the proof of Theorem 4.3.53.

Proposition 4.3.57. *Let $\Phi : \mathfrak{A}-\mathrm{Mod}^{\mathrm{idem}} \longrightarrow \mathcal{A}dd^{\mathrm{idem}}$ be a 2-functor, let $\mathcal{M}$ be an idempotent closed $\mathfrak{A}$-module and let (F,G,η,ϵ) be an adjoint equivalence between $\mathcal{H}om_{\mathfrak{A}}(\mathcal{M},-)$ and Φ. Further, let $\hat{\Theta}_1 = (\Theta_1,\Delta_1,\mu_1)$ be a split Frobenius object on Φ. Furthermore, let $\mathcal{M}_{\hat{\Theta}_3}\in\mathfrak{A}-\mathrm{Mod}^{\mathrm{idem}}$, let $\mathcal{M}\xrightarrow{\vartheta^{\hat{\Theta}_3}}\mathcal{M}_{\hat{\Theta}_3}\xrightarrow{\vartheta_{\hat{\Theta}_3}}\mathcal{M}$ be $\mathfrak{A}$-functors and let $id_{\mathcal{M}_{\hat{\Theta}_3}}\xRightarrow{\alpha_3}\vartheta^{\hat{\Theta}_3}\vartheta_{\hat{\Theta}_3}\xRightarrow{\beta_3} id_{\mathcal{M}_{\hat{\Theta}_3}}$ be $\mathfrak{A}$-transformations with $\beta_3\circ\alpha_3 = \mathcal{M}_{\hat{\Theta}_3}$. Now, denote by ${}_{\hat{\Theta}_1}\Phi : \mathfrak{A}-\mathrm{Mod}^{\mathrm{idem}}\longrightarrow\mathcal{A}dd^{\mathrm{idem}}$ the 2-functor associated to $\hat{\Theta}_1$ according to Proposition 4.3.52. Further, denote by $\hat{\Theta}_2 = (\Theta_2,\Delta_2,\mu_2)$ the split Frobenius object on $\mathcal{H}om_{\mathfrak{A}}(\mathcal{M},-)$ constructed from $\hat{\Theta}_1$ and (F,G,η,ϵ) according to Lemma 4.3.56. Furthermore, denote $\hat{\Theta}_3 := (\Theta_3,\Delta_3,\mu_3) := (\vartheta_{\hat{\Theta}_3}\vartheta^{\hat{\Theta}_3}, \vartheta_{\hat{\Theta}_3}\alpha_3\vartheta^{\hat{\Theta}_3}, \vartheta_{\hat{\Theta}_3}\beta_3\vartheta^{\hat{\Theta}_3})$.
If $\hat{\Theta}_2$ and $-\hat{\Theta}_3$ are isomorphic, then $\mathcal{M}_{\hat{\Theta}_3}$ represents ${}_{\hat{\Theta}_1}\Phi$ and the diagram*

$$\begin{tikzcd}
\mathcal{H}om_{\mathfrak{A}}(\mathcal{M},-) \arrow[r, "F"] \arrow[d, "-\vartheta_{\hat{\Theta}_3}"] & \Phi \arrow[r, "G"] \arrow[d, "\vartheta^{\hat{\Theta}_1}"] & \mathcal{H}om_{\mathfrak{A}}(\mathcal{M},-) \arrow[d, "-\vartheta_{\hat{\Theta}_3}"] \\
\mathcal{H}om_{\mathfrak{A}}(\mathcal{M}_{\hat{\Theta}_3},-) \arrow[r] \arrow[d, "-\vartheta^{\hat{\Theta}_3}"] & {}_{\hat{\Theta}_1}\Phi \arrow[r] \arrow[d, "\vartheta_{\hat{\Theta}_1}"] & \mathcal{H}om_{\mathfrak{A}}(\mathcal{M}_{\hat{\Theta}_3},-) \arrow[d, "-\vartheta^{\hat{\Theta}_3}"] \\
\mathcal{H}om_{\mathfrak{A}}(\mathcal{M},-) \arrow[r, "F"] & \Phi \arrow[r, "G"] & \mathcal{H}om_{\mathfrak{A}}(\mathcal{M},-)
\end{tikzcd}$$

commutes up to isomorphisms.

Proof. As in Proposition 4.3.33, we see that $\hat{\Theta}_3$ is a split Frobenius object on $\mathcal{M}$. Since $\mathfrak{A}-\mathrm{Mod}$ is a strict 2-category $-\hat{\Theta}_3 = (-\Theta_3,-\Delta_3,-\mu_3)$ is a strict Frobenius object on $\mathcal{H}om_{\mathfrak{A}}(\mathcal{M},-)$. Since $\hat{\Theta}_2$ and $-\hat{\Theta}_3$ are isomorphic, Remark 4.3.46 implies that the identity morphism of the 2-functor $\mathcal{H}om_{\mathfrak{A}}(\mathcal{M},-)$ is compatible with the Frobenius objects $\hat{\Theta}_2$ and $-\hat{\Theta}_3$. Thus, Proposition 4.3.48 implies $\mathcal{H}om_{\mathfrak{A}}(\mathcal{M}_{\hat{\Theta}_3},-) \cong {}_{\hat{\Theta}_2}(\mathcal{H}om_{\mathfrak{A}}(\mathcal{M},-))$. Furthermore, the assumptions of Corollary 4.3.50 by construction of $\hat{\Theta}_2$. Hence, we have an equivalence ${}_{\hat{\Theta}_1}\Phi \cong {}_{\hat{\Theta}_2}(\mathcal{H}om_{\mathfrak{A}}(\mathcal{M},-))$. Thus, $\mathcal{M}_{\hat{\Theta}_3}$ represents ${}_{\hat{\Theta}_1}\Phi$. Existence of the morphisms $\mathcal{H}om_{\mathfrak{A}}(\mathcal{M}_{\hat{\Theta}},-) \to {}_{\hat{\Theta}_1}\Phi \to \mathcal{H}om_{\mathfrak{A}}(\mathcal{M}_{\hat{\Theta}},-)$ follows from Proposition 4.3.48. Compatibility with horizontal composition shown in the same proposition provides the isomorphisms up to which the above diagram commutes. □

Proposition 4.3.58. *Let $\mathcal{M} \in \mathcal{B}-\mathrm{Mod}^{\mathrm{idem}}$ and let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a split Frobenius object on $\mathcal{M}$. There exist $\mathcal{M}_{\hat{\Theta}} \in \mathcal{B}-\mathrm{Mod}^{\mathrm{idem}}$, $\mathcal{B}$-functors $\mathcal{M} \xrightarrow{\vartheta^{\hat{\Theta}}} \mathcal{M}_{\hat{\Theta}} \xrightarrow{\vartheta_{\hat{\Theta}}} \mathcal{M}$ and $\mathcal{B}$-transformations $\mathcal{M}_{\hat{\Theta}} \xRightarrow{\alpha} \vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}} \xRightarrow{\beta} \mathcal{M}_{\hat{\Theta}}$ such that $\beta \circ \alpha = id_{\mathcal{M}_{\hat{\Theta}}}$. Furthermore, this structure recovers the Frobenius object, that is $\hat{\Theta} = (\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}})$.*

Proof. Recall that Proposition 4.3.38 already provides us with an idempotent closed additive category ${}_{\hat{\Theta}}\mathcal{M}$, additive functors $\vartheta_{\hat{\Theta}}$ and $\vartheta^{\hat{\Theta}}$ and transformation α and β with properties as claimed in the proposition. Note that in the statement we used the notation $\mathcal{M}_{\hat{\Theta}}$ instead of ${}_{\hat{\Theta}}\mathcal{M}$. Since we will construct a left $\mathcal{B}$-action on this category, we consider this notation more appropriate.

It remains to equip ${}_{\hat{\Theta}}\mathcal{M}$ with a $\mathcal{B}$-action, as well as $\vartheta_{\hat{\Theta}}$ and $\vartheta^{\hat{\Theta}}$ with the structure of $\mathcal{B}$-functors. Then, we have to show that α and β are $\mathcal{B}$-transformations, as well as that the $\mathcal{B}$-structure of the Frobenius object is recovered. For the latter, we only have to check that $\hat{\Theta} = \vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}$ as $\mathcal{B}$-functors, since $\mathcal{B}$-transformation do not carry any additional structure as compared to the underlying additive transformations.

Recall that ${}_{\hat{\Theta}}\mathcal{M}$ is the idempotent closure of ${}_{\hat{\Theta}}\widetilde{\mathcal{M}}$ where $\mathrm{Ob}({}_{\hat{\Theta}}\widetilde{\mathcal{M}}) = \mathrm{Ob}(\mathcal{M})$ and ${}_{\hat{\Theta}}\widetilde{\mathcal{M}}(M_1, M_2) = \{m \in \mathcal{M}(\Theta M_1, \Theta M_2) | (\Theta m) \circ \hat{\Delta}_{M_1} = \hat{\Delta}_{M_2} \circ m,\ m \circ \hat{\mu}_{M_1} = \hat{\mu}_{M_2} \circ (\hat{\Theta} m)\}$.
It is sufficient to define a $\mathcal{B}$-action ψ on ${}_{\hat{\Theta}}\widetilde{\mathcal{M}}$, as it will extend to ${}_{\hat{\Theta}}\mathcal{M}$ by the universal property of the idempotent closure.

Let us define the structure-functor $\psi : \mathcal{B} \to \mathfrak{A}dd({}_{\hat{\Theta}}\widetilde{\mathcal{M}}, {}_{\hat{\Theta}}\widetilde{\mathcal{M}})$. For $B \in \mathcal{B}$, we define a functor $\psi(B) : {}_{\hat{\Theta}}\widetilde{\mathcal{M}} \to {}_{\hat{\Theta}}\widetilde{\mathcal{M}}$. For $M \in \mathrm{Ob}({}_{\hat{\Theta}}\widetilde{\mathcal{M}}) = \mathrm{Ob}(\mathcal{M})$, we set $\psi(B)(M) := B.M$ where the right hand side uses the $\mathcal{B}$-action on $\mathcal{M}$. Further for $m \in {}_{\hat{\Theta}}\widetilde{\mathcal{M}}(M_1, M_2)$, we define $\psi(B)(m) \in \mathcal{M}(\hat{\Theta}(B.M_1), \hat{\Theta}(B.M_2))$ by commutativity of the diagram

$$\begin{tikzcd} \Theta(B.M_1) \arrow[r, "\sim"] \arrow[d, "\psi(B)(m)"] & B.\Theta M_1 \arrow[d, "B.m"] \\ \Theta(B.M_2) \arrow[r, "\sim"] & B.\Theta M_2 \end{tikzcd}.$$

That $\psi(B)(m) \in {}_{\hat{\Theta}}\widetilde{\mathcal{M}}(B.M_1, B.M_2)$ follows from the commutative diagram

$$\begin{tikzcd}
\Theta(B.M_1) \arrow[r, "\sim"] \arrow[d, "\psi(B)(m)"] \arrow[rrrr, bend left, "\Delta_{B.M_1}"] & B.\Theta M_1 \arrow[r, "B.\Delta_{M_1}"] \arrow[d, "B.m"] & B.\Theta^2 M_1 \arrow[r, "\sim"] \arrow[d, "B.\Theta(m)"] & \Theta(B.\Theta M_1) \arrow[r, "\sim"] \arrow[d, "\Theta(B.m)"] & \Theta^2(B.M_1) \arrow[d, "\Theta(\psi(B)(m))"] \\
\Theta(B.M_2) \arrow[r, "\sim"] \arrow[rrrr, bend right, "\Delta_{B.M_2}"'] & B.\Theta M_2 \arrow[r, "B.\Delta_{M_1}"] & B.\Theta^2 M_2 \arrow[r, "\sim"] & \Theta(B.\Theta M_2) \arrow[r, "\sim"] & \Theta^2(B.M_2)
\end{tikzcd}$$

and $\hat{\mu}_{M_2} \circ \Theta(\psi(B)(m)) = \psi(B)(m) \circ \hat{\mu}_{M_1}$, which is shown analogously.
Since $B.$ is a functor, so is $\psi(B)$. Since Θ and $B.$ are additive, so is $\psi(B)$.
Given $b \in \mathcal{B}(B_1, B_2)$, we have to define a transformation $\psi(b) : \psi(B_1) \Rightarrow \psi(B_2)$. For $M \in {}_{\hat{\Theta}}\widetilde{\mathcal{M}}$, we set $\psi(b)_M := \Theta(b.id_M) = \vartheta_{\hat{\Theta}}(b.id_M) \in {}_{\hat{\Theta}}\widetilde{\mathcal{M}}(B_1.M, B_2.M)$. Since $\psi(B)(m)$ is defined using isomorphisms that are natural in B, it we can conclude that $\psi(B_2)(m) \circ \Theta(b.id_{M_1}) = \Theta(b.id_{M_2}) \circ \psi(B_1)(m)$ for all morphisms $b \in \mathcal{B}(B_1, B_2)$ and $m \in {}_{\hat{\Theta}}\widetilde{\mathcal{M}}(M_1, M_2)$. Hence, $\psi(b)$ is a transformation. Furthermore by additive functoriality of the action on $\mathcal{M}$ and of Θ, ψ is indeed an additive functor.
Next, we show that ψ is monoidal, which requires two steps. First, we define an isotransformation $\psi(B_1)\psi(B_2) \overset{\sim}{\Longrightarrow} \psi(B_1 \otimes B_2)$ for all B_1, $B_2 \in \mathcal{B}$. At all $M \in {}_{\hat{\Theta}}\widetilde{\mathcal{M}}$, we define this isotransformation to be given by the image under $\vartheta^{\hat{\Theta}}$ of the natural isomorphisms $B_1.(B_2.M) \overset{\sim}{\longrightarrow} (B_1 \otimes B_2).M$ coming from the action of $\mathcal{B}$ on $\mathcal{M}$. That this defines indeed a transformation follows from the commutative diagram

$$\begin{tikzcd}
\Theta(B_1.(B_2.M_1)) \arrow[r, "\sim"] \arrow[ddd, "\sim"] \arrow[rrr, bend left, "\psi(B_1)(\psi(B_2)(m))"] & B_1.\Theta(B_2.M_1) \arrow[d, "\sim"] & B_1.(B_2.\Theta(M_2)) \arrow[r, "\sim"] & \Theta(B_1.(B_2.M_2)) \arrow[ddd, "\sim"] \\
 & B_1.(B_2.\Theta(M_1)) \arrow[r, "B_1.(B_2.m)"] \arrow[d, "\sim"] & B_1.(B_2.\Theta(M_2)) \arrow[u, "\sim"] \arrow[d, "\sim"] & \\
 & (B_1 \otimes B_2).\Theta(M_1)) \arrow[r, "(B_1 \otimes B_2).m"] & (B_1 \otimes B_2).\Theta(M_2)) \arrow[dr, "\sim"] & \\
\Theta((B_1 \otimes B_2).M_1)) \arrow[ur, "\sim"] \arrow[rrr, "\psi(B_1 \otimes B_2)(m)"] & & & \Theta((B_1 \otimes B_2).M_2))
\end{tikzcd}$$

The compositions $\psi(B_1)\psi(B_2)\psi(B_3) \overset{\sim}{\Longrightarrow} \psi(B_1 \otimes B_2)\psi(B_3) \overset{\sim}{\Longrightarrow} \psi((B_1 \otimes B_2) \otimes B_3)$ and $\psi(B_1)\psi(B_2)\psi(B_3) \overset{\sim}{\Longrightarrow} \psi(B_1)\psi(B_2 \otimes B_3) \overset{\sim}{\Longrightarrow} \psi(B_1 \otimes (B_2 \otimes B_3)) \overset{\sim}{\Longrightarrow} \psi((B_1 \otimes B_2) \otimes B_3)$ agree, since we can apply Θ to the corresponding requirement on the action of $\mathcal{B}$ on $\mathcal{M}$ and use $\psi(B)(\Theta(m)) = \Theta(B.m)$.
Second, we need an isotransformation $\psi(I) \overset{\sim}{\Longrightarrow} {}_{\hat{\Theta}}\widetilde{\mathcal{M}}$: For all $M \in {}_{\hat{\Theta}}\widetilde{\mathcal{M}}$, we apply $\vartheta_{\hat{\Theta}}$ to the natural isomorphism $I.M \overset{\sim}{\longrightarrow} M$ that come from the action of $\mathcal{B}$

on $\mathcal{M}$. By the requirements on $\hat{\Theta}$, this isomorphism agrees with the composition $\Theta(I.M) \cong I.\Theta(M) \cong \Theta(M)$. From naturality of the second isomorphism isomorphism it follows that we have indeed defined a transformation $\psi(I) \Rightarrow {}_{\hat{\Theta}}\widetilde{\mathcal{M}}$. The composition $\psi(I)\psi(B) \xRightarrow{\sim} \psi(I \otimes B) \xRightarrow{\sim} \psi(B)$ agrees with the transformation induced from $\psi(I) \Rightarrow {}_{\hat{\Theta}}\widetilde{\mathcal{M}}$, by an application of $\hat{\Theta}$ to the corresponding requirement for the action of $\mathcal{B}$ on $\mathcal{M}$. The analogous requirement for the composition $\psi(B)\psi(I) \xRightarrow{\sim} \psi(B \otimes I) \xRightarrow{\sim} \psi(B)$ holds by the same argument.

Next, we equip $\vartheta_{\hat{\Theta}} : {}_{\hat{\Theta}}\mathcal{M} \to \mathcal{M}$ with the structure of a $\mathcal{B}$-functor. For all objects $B \in \mathcal{B}$, we need to specify a natural isotransformation $\mathrm{can}_{\vartheta_{\hat{\Theta}},B} : (B.)\vartheta_{\hat{\Theta}} \xRightarrow{\sim} \vartheta_{\hat{\Theta}}\psi(B)$. For all $M \in {}_{\hat{\Theta}}\widetilde{\mathcal{M}}$, we define $\left(\mathrm{can}_{\vartheta_{\hat{\Theta}},B}\right)_M := \mathrm{can}_{\Theta,B}$ since $(B.)\vartheta_{\hat{\Theta}}(M) = B.\Theta(M)$ and $\vartheta_{\hat{\Theta}}\psi(B)(M) = \vartheta_{\hat{\Theta}}(B.M) = \Theta(B.M)$. Indeed this defines a transformation $\mathrm{can}_{\vartheta_{\hat{\Theta}},B}$ since $\mathrm{can}_{\Theta,B}$ is one such. Naturality in B holds as well, since the morphisms $(\vartheta_{\hat{\Theta}}\psi(b))_M = \vartheta_{\hat{\Theta}}(\psi(b)_M) = \vartheta_{\hat{\Theta}}(\Theta(b.M)) = \Theta(b.M)$ and $((b.)\vartheta_{\hat{\Theta}})_M = b.\Theta(M)$ are intertwined by $\mathrm{can}_{\Theta,B_1}$ and $\mathrm{can}_{\Theta,B_2}$ for all $b \in \mathcal{B}(B_1, B_2)$. The requirements that the compositions $(B_1.)(B_2.)\vartheta_{\hat{\Theta}} \cong ((B_1 \otimes B_2).)\vartheta_{\hat{\Theta}} \cong \vartheta_{\hat{\Theta}}\psi(B_1 \otimes B_2) \cong \vartheta_{\hat{\Theta}}\psi(B_1)\psi(B_2) \cong (B_1.)\vartheta_{\hat{\Theta}}\psi(B_2) \cong (B_1.)(B_2.)\vartheta_{\hat{\Theta}}$ and $\vartheta_{\hat{\Theta}} \cong \psi(I)\vartheta_{\hat{\Theta}} \cong \vartheta_{\hat{\Theta}}(I.) \cong \vartheta_{\hat{\Theta}}$ agree with the respective identity morphisms are satisfied, since they are exactly the same as for Θ.

Similarly, we equip $\vartheta^{\hat{\Theta}}$ with the structure of a $\mathcal{B}$-functor. For all $B \in \mathcal{B}$, we need to specify a natural isotransformation $\mathrm{can}_{\vartheta^{\hat{\Theta}},B} : \psi(B)\vartheta^{\hat{\Theta}} \xRightarrow{\sim} \vartheta^{\hat{\Theta}}(B.)$. For all $M \in \mathcal{M}$, we define $\left(\mathrm{can}_{\vartheta^{\hat{\Theta}},B}\right)_M := id_{\Theta(B.M)}$ since $\psi(B)\vartheta^{\hat{\Theta}}(M) = \psi(B)(M) = B.M$ and $\vartheta^{\hat{\Theta}}(B.M) = B.M$. Note that $id_{\Theta(B.M)} = \Theta(id_{B.M})$ is indeed a morphism in ${}_{\hat{\Theta}}\widetilde{\mathcal{M}}$. The functors $\psi(B)\vartheta^{\hat{\Theta}}$ and $\vartheta^{\hat{\Theta}}(B.)$ also act in the same way on morphisms in $\mathcal{B}$. Thus, $\mathrm{can}_{\vartheta^{\hat{\Theta}},B}$ is a transformation as it is defined via identity morphism. The same argument shows naturality in B. The compatibility requirements that $\vartheta^{\hat{\Theta}}\psi(B_1)\psi(B_2) \cong \vartheta^{\hat{\Theta}}\psi(B_1 \otimes B_2) \cong ((B_1 \otimes B_2).)\vartheta^{\hat{\Theta}} \cong (B_1.)(B_2.)\vartheta^{\hat{\Theta}} \cong (B_1.)\vartheta^{\hat{\Theta}}\psi(B_2) \cong \vartheta^{\hat{\Theta}}\psi(B_1)\psi(B_2)$ and $\vartheta^{\hat{\Theta}} \cong (I.)\vartheta^{\hat{\Theta}} \cong \vartheta^{\hat{\Theta}}\psi(I) \cong \vartheta^{\hat{\Theta}}$ agree with the respective identity morphisms also holds almost trivially.

Next, we show $\Theta = \vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}$ as $\mathcal{B}$-functors. As we know this equality already for the underlying functors, we only need to check $\mathrm{can}_{\Theta,B} = (\vartheta_{\hat{\Theta}}\, \mathrm{can}_{\vartheta^{\hat{\Theta}},B}) \circ (\mathrm{can}_{\vartheta_{\hat{\Theta}},B}\, \vartheta^{\hat{\Theta}})$. This holds by the definitions of $\mathrm{can}_{\vartheta_{\hat{\Theta}},B}$ and $\mathrm{can}_{\vartheta^{\hat{\Theta}},B}$, since they involve identity morphism and $\mathrm{can}_{\Theta,B}$ respectively.

The last step is to show that α and β are $\mathcal{B}$-transformation. For all $B \in \mathcal{B}$ and all $M \in {}_{\hat{\Theta}}\widetilde{\mathcal{M}}$, we have $\psi(B)(\alpha_M) = \psi(B)(\Delta_M) = (\mathrm{can}_{\Theta,B})_{\Theta(M)} \circ (B.\Delta_M) \circ (\mathrm{can}_{\Theta,B})_M^{-1}$ $= \Theta((\mathrm{can}_{\Theta,B})_M^{-1}) \circ (\mathrm{can}_{\Theta\Theta,B})_M \circ (B.\Delta_M) \circ (\mathrm{can}_{\Theta,B})_M^{-1} = \Theta((\mathrm{can}_{\Theta,B})_M^{-1}) \circ \Delta_{B.M}$ $= \Theta((\mathrm{can}_{\Theta,B})_M^{-1}) \circ \alpha_{\psi(B)(M)}$, where we used that Θ is a $\mathcal{B}$-functor and that Δ is a

$\mathcal{B}$-transformation. Since $\mathrm{can}_{id_{\hat{\Theta}}\mathcal{M},B} = id_{\psi(B)}$, the above equations show that α is a $\mathcal{B}$-transformation. The analogous argument (using that μ is a $\mathcal{B}$-transformation) shows that β is a $\mathcal{B}$-transformation. $\square$

The following lemma will become important later on:

Lemma 4.3.59. *Let $\hat{\Theta}$ and $\vartheta^{\hat{\Theta}}$ be as in Proposition 4.3.58. Then, $\hat{\Theta}$ is indecomposable if and only if $\vartheta^{\hat{\Theta}}$ is strongly indecomposable.*

Proof. First, let $\hat{\Theta}$ be indecomposable and let e be an idempotent $\mathcal{B}$-transformation of $\vartheta^{\hat{\Theta}}$. Then, $\vartheta_{\hat{\Theta}}e$ is an idempotent of $\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}} = \Theta$. Furthermore, $\vartheta_{\hat{\Theta}}e$ satisfies $\hat{\Theta}$-descent, since $\mu \circ (\Theta\vartheta_{\hat{\Theta}}e) \circ \Delta = (\vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}}) \circ (\Theta\vartheta_{\hat{\Theta}}e) \circ (\vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}) = \vartheta_{\hat{\Theta}}(\beta \circ \alpha)e = \vartheta_{\hat{\Theta}}e$. By indecomposability of $\hat{\Theta}$, the $\mathcal{B}$-transformation $\vartheta_{\hat{\Theta}}e$ has to be trivial or the identity. Since $e = (\beta \circ \alpha)e = (\beta\vartheta^{\hat{\Theta}}) \circ (\vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}}e) \circ (\alpha\vartheta^{\hat{\Theta}})$, we deduce that e has to be trivial or the identity, as well. Thus, $\vartheta^{\hat{\Theta}}$ is strongly indecomposable.
Second, let $\vartheta^{\hat{\Theta}}$ be strongly indecomposable. Then, $\hat{\Theta}$ is indecomposable by the same argument which we used in the proof of Proposition 4.3.20. Note that strong indecomposability of $\vartheta^{\hat{\Theta}}$ implies that it is non-trivial, and thus that $\hat{\Theta}$ is non-trivial as well. $\square$

4.3.5 The split Frobenius 2-category

In this section, we assign to an additive monoidal category $\mathcal{B}$ its split Frobenius 2-category $\mathfrak{Frob}(\mathcal{B})$. For simplicity, we assume that $\mathcal{B}$ is strict. In Section 4.5.2, we will use this 2-category to describe free functorially cyclic modules.

Before we can introduce $\mathfrak{Frob}(\mathcal{B})$ in Definition 4.3.63, we need a preliminary definition which resembles Definition 4.3.35, and explain some additional structure.

Definition 4.3.60. *Let $\hat{\Theta}_1 = (\Theta_1, \Delta_1, \mu_1)$ and $\hat{\Theta}_2 = (\Theta_2, \Delta_2, \mu_2)$ be Frobenius objects in a strict additive monoidal category $\mathcal{B}$. We define ${}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2} \in \mathcal{A}dd^{\mathrm{idem}}$ as the idempotent closure of $\widetilde{{}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2}} \in \mathcal{A}dd$, where* $\mathrm{Ob}\left(\widetilde{{}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2}}\right) := \mathrm{Ob}(\mathcal{B})$ *and*

$$\widetilde{{}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2}}(B_1, B_2) := \left\{ f \in \mathcal{B}(\Theta_1 B_1 \Theta_2, \Theta_1 B_2 \Theta_2) \,\middle|\, f \textit{ satisfies } \hat{\Theta}_1\text{-}\hat{\Theta}_2\textit{-descent} \right\}.$$

for all B_1, $B_2 \in$ $\mathrm{Ob}\left(\widetilde{{}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2}}\right)$.

It is easy to see that ${}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2}$ is well defined. Note that for convenience, we have used $\hat{\Theta}_1$-$\hat{\Theta}_2$-descent in this definition rather than showing a statement analogous to Lemma 4.3.37. To define $\mathfrak{Frob}(\mathcal{B})$, we need the following additional structure:

Definition 4.3.61. *Let $\hat{\Theta}_1$, $\hat{\Theta}_2$ and $\hat{\Theta}_3$ be split Frobenius objects in an strict additive monoidal category $\mathcal{B}$. Then, we define the additive functor $\otimes_{\hat{\Theta}_2}$ by the following assignment:*

$$\begin{array}{llll} \otimes_{\hat{\Theta}_2}: & {}_{\hat{\Theta}_2}\mathcal{B}_{\hat{\Theta}_1} \times {}_{\hat{\Theta}_3}\mathcal{B}_{\hat{\Theta}_2} & \to & {}_{\hat{\Theta}_3}\mathcal{B}_{\hat{\Theta}_1} \\ & ((B_1, e_1), (B_2, e_2)) & \mapsto & (B_2\Theta_2 B_1, (e_2 B_1 \Theta_1) \circ (\Theta_3 B_2 e_1)) \\ & (f_1, f_2) & \mapsto & (f_2 B_1^2 \Theta_1) \circ (\Theta_3 B_2^1 f_1) \end{array}$$

where $f_1 \in {}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2}((B_1^1, e_1^1), (B_1^2, e_1^2))$ and $f_2 \in {}_{\hat{\Theta}_2}\mathcal{B}_{\hat{\Theta}_3}((B_2^1, e_2^1), (B_2^2, e_2^2))$.

The additive functor $\otimes_{\hat{\Theta}_2}$ is well defined by the following lemma, which in particular implies that $(e_2 B_1 \Theta_1) \circ (\Theta_3 B_2 e_1)$ is an idempotent:

Lemma 4.3.62. *Let $\hat{\Theta}_1$, $\hat{\Theta}_2$ and $\hat{\Theta}_3$ be split Frobenius objects in an strict additive monoidal category $\mathcal{B}$. Then, we have $(f_2 B_1^2 \Theta_1) \circ (\Theta_3 B_2^1 f_1) = (\Theta_3 B_2^2 f_1) \circ (f_2 B_1^1 \Theta_1)$ for all $f_1 \in {}_{\hat{\Theta}_2}\mathcal{B}_{\hat{\Theta}_1}((B_1^1, e_1^1), (B_1^2, e_1^2))$ and $f_2 \in {}_{\hat{\Theta}_3}\mathcal{B}_{\hat{\Theta}_2}((B_2^1, e_2^1), (B_2^2, e_2^2))$.*

Proof. This statement requires a simple calculation:

$$\begin{aligned} &(f_2 B_1^2 \Theta_1) \circ (\Theta_3 B_2^1 f_1) \\ &\quad = (f_2 B_1^2 \Theta_1) \circ (\Theta_3 B_2^1 \mu_2 B_1^2 \Theta_1) \circ (\Theta_3 B_2^1 \Delta_2 B_1^2 \Theta_1) \circ (\Theta_3 B_2^1 f_1) && (\hat{\Theta}_2 \text{ is split}) \\ &\quad = (\Theta_3 B_2^2 \mu_2 B_2^2 \Theta_1) \circ (f_2 B_1^2 \Theta_1) \circ (\Theta_3 B_2^1 f_1) \circ (\Theta_3 B_2^1 \Delta_2 B_1^1 \Theta_1) && (\text{Lemma 4.3.40}) \\ &\quad = (\Theta_3 B_2^2 \mu_2 B_2^2 \Theta_1) \circ (\Theta_3 B_2^2 f_1) \circ (f_2 B_1^1 \Theta_1) \circ (\Theta_3 B_2^1 \Delta_2 B_1^1 \Theta_1) && (\text{Remark 3.1.3}) \\ &\quad = (\Theta_3 B_2^2 f_1) \circ (f_2 B_1^1 \Theta_1) \end{aligned}$$

In the last equality, we have used the arguments of the previous three equalities in reverse order in a single step. □

Furthermore for every split Frobenius object $\hat{\Theta}$ in $\mathcal{B}$, we associate the idempotent $e_{\hat{\Theta}} := \Delta \circ \mu$. It is easy to see that $e_{\hat{\Theta}}$ satisfies two-sided $\hat{\Theta}$-decent, and thus we have $I_{\hat{\Theta}} := (I, e_{\hat{\Theta}}) \in {}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}}$, where I denotes the unit object of $\mathcal{B}$.

Now, we are prepared to state central definition of this section:

Definition 4.3.63. *Let $\mathcal{B}$ be a strict additive monoidal category. Then, we define $\mathfrak{F}rob(\mathcal{B})$ as the additive 2-category with the following datum:*

- $\mathfrak{F}rob(\mathcal{B})_0 := \{\hat{\Theta}$ *right indecomposable split Frobenius objects in* $\mathcal{B}\}$
- $\mathfrak{F}rob(\mathcal{B})(\hat{\Theta}_1, \hat{\Theta}_2) := {}_{\hat{\Theta}_2}\mathcal{B}_{\hat{\Theta}_1}$ *for all* $\hat{\Theta}_1, \hat{\Theta}_2 \in \mathfrak{F}rob(\mathcal{B})_0$
- $\mathfrak{F}rob(\mathcal{B})(\hat{\Theta}_1, \hat{\Theta}_2) \times \mathfrak{F}rob(\mathcal{B})(\hat{\Theta}_2, \hat{\Theta}_3) \to \mathfrak{F}rob(\mathcal{B})(\hat{\Theta}_1, \hat{\Theta}_3)$ *is given by* $\otimes_{\hat{\Theta}_2}$ *for all* $\hat{\Theta}_1, \hat{\Theta}_2, \hat{\Theta}_3 \in \mathfrak{F}rob(\mathcal{B})_0$

- $I_{\hat{\Theta}} \in \mathfrak{F}rob(\mathcal{B})_{\hat{\Theta}}$ *is the identity object for each* $\hat{\Theta} \in \mathfrak{F}rob(\mathcal{B})_0$
- *The associator is given by identity morphisms*
- *For all* $\hat{\Theta}_1, \hat{\Theta}_2 \in \mathfrak{F}rob(\mathcal{B})_0$*, the right (resp. left) unit on* $(B, e) \in \mathfrak{F}rob(\mathcal{B})(\hat{\Theta}_1, \hat{\Theta}_2)$ *is defined using* $e\mu_1 \in \mathfrak{F}rob(\mathcal{B})_{\hat{\Theta}_1 \to \hat{\Theta}_2}((B, e)I_{\hat{\Theta}_1}, (B, e))$ *and its inverse* $e\Delta_1$ *(resp.* $\mu_2 e \in \mathfrak{F}rob(\mathcal{B})_{\hat{\Theta}_1 \to \hat{\Theta}_2}(I_{\hat{\Theta}_2}(B, e), (B, e))$ *and its inverse* $\Delta_2 e$*)*

Proof. It remains to show that the unit transformations and their inverses are well-defined and invertible and further that the axioms of a monoidal category are satisfied. We leave this as an exercise. □

Remark 4.3.64. Even though we had assumed that $\mathcal{B}$ is strict, $\mathfrak{F}rob(\mathcal{B})$ will be strict only if $\mathfrak{F}rob(\mathcal{B})$ has no further objects apart from (I, id_I, id_I).

Remark 4.3.65. The definition of a 2-category Frob($\mathfrak{A}$) for an additive 2-categories $\mathfrak{A}$ is straight forward. Section 7.1 proposes to study this and more in the future.

Remark 4.3.66. In analogy to Remark 4.3.36, we can interpret ${}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2}$ as a category of projective $\hat{\Theta}_1$-$\hat{\Theta}_2$-bimodules. Furthermore, instead of considering $\hat{\Theta}_1$ and $\hat{\Theta}_2$ as split Frobenius objects in $\mathcal{B}$, we can consider the Frobenius object $\widetilde{\Theta_1\Theta_2}$ on $\mathcal{B}$ as constructed in Example 4.3.9, for which we have to consider $\mathcal{B}$ as an additive category only. Note that $\widetilde{\Theta_1\Theta_2}$ is also split in this situation. Using this notation, we have ${}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2} = \mathcal{B}_{\widetilde{\Theta_1\Theta_2}}$ where we used Definition 4.3.35. Remark 4.3.68 below provides tools that can also be used to construct $\widetilde{\Theta_1\Theta_2}$.

The following remark will be useful for Section 4.8.

Remark 4.3.67. Using Remark 4.3.19, we obtain an indecomposable split Frobenius object on $\mathcal{B} \in \mathcal{B}-\mathrm{Mod}$ by transporting $\hat{\Theta}_2$ along the equivalence $\mathcal{B}^{\mathrm{rev}} \to \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{B}, \mathcal{B})$ induced from taking tensor products on the right. By abuse of notation, we denote this indecomposable split Frobenius object on $\mathcal{B}$ also by $\hat{\Theta}_2$. Similarly by taking tensor products on the left, we obtain an indecomposable split Frobenius object $\hat{\Theta}_1$ on $\mathcal{B}$. Proposition 4.3.58, thus yields a category $\mathcal{B}_{\hat{\Theta}_2}$ together with $\mathcal{B}$-functor $\vartheta_{\hat{\Theta}_2}$ and $\vartheta^{\hat{\Theta}_2}$. Using the same argument as in Proposition 4.3.48, $\hat{\Theta}_1$ descents to a split Frobenius object $\hat{\hat{\Theta}}_1$ on $\mathcal{B}_{\hat{\Theta}_2}$. This action is no longer given by $\mathcal{B}$-functors, however Proposition 4.3.38 applies and we obtain an additive category ${}_{\hat{\hat{\Theta}}_1}\left(\mathcal{B}_{\hat{\Theta}_2}\right)$ together with additive functors ${}_{\hat{\hat{\Theta}}_1}\vartheta$ and ${}^{\hat{\hat{\Theta}}_2}\vartheta$, where we have put the indices on the other side to align notation below. Repeating all the arguments involved, one can apply the same constructions first for the left action (resulting in a right $\mathcal{B}$-module ${}_{\hat{\Theta}_1}\mathcal{B}$) and then for the right action (resulting in an additive category $\mathcal{B}$-module $\left({}_{\hat{\Theta}_1}\mathcal{B}\right)_{\hat{\hat{\Theta}}_2}$). All these constructions fit together into the following diagrams

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, "\vartheta^{\hat{\Theta}_2}"] \arrow[d, "{}^{\hat{\Theta}_1}\vartheta"'] \arrow[drr, "{}^{\hat{\Theta}_1}\vartheta^{\hat{\Theta}_2}"] & \mathcal{B}_{\hat{\Theta}_2} \arrow[r, "{}^{\hat{\hat{\Theta}}_1}\vartheta"] & {}_{\hat{\hat{\Theta}}_1}\left(\mathcal{B}_{\hat{\Theta}_2}\right) \arrow[d, "\sim"] \\
{}_{\hat{\Theta}_1}\mathcal{B} \arrow[r, "\vartheta^{\hat{\hat{\Theta}}_2}"] & \left({}_{\hat{\Theta}_1}\mathcal{B}\right)_{\hat{\hat{\Theta}}_2} \arrow[r, "\sim"] & {}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B} & \mathcal{B}_{\hat{\Theta}_2} \arrow[l, "\vartheta_{\hat{\Theta}_2}"'] & {}_{\hat{\hat{\Theta}}_1}\left(\mathcal{B}_{\hat{\Theta}_2}\right) \arrow[l, "{}_{\hat{\hat{\Theta}}_1}\vartheta"'] \\
{}_{\hat{\Theta}_1}\mathcal{B} \arrow[u, "{}_{\hat{\Theta}_1}\vartheta"] & \left({}_{\hat{\Theta}_1}\mathcal{B}\right)_{\hat{\hat{\Theta}}_2} \arrow[l, "\vartheta_{\hat{\hat{\Theta}}_2}"'] & {}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2} \arrow[l, "\sim"'] \arrow[u, "\sim"'] \arrow[ull, "{}_{\hat{\Theta}_1}\vartheta_{\hat{\Theta}_2}"]
\end{tikzcd},
$$

which commute up to isomorphisms. The two equivalences in each diagram will be explained in the following Remark 4.3.68, which provides an abstract argument that also implies commutativity of the diagrams above. Note that Lemma 4.5.12 will make the equivalences explicit. Going forward, we will abuse notation and denote the composition $\mathcal{B}_{\hat{\Theta}_2} \xrightarrow{{}^{\hat{\hat{\Theta}}_1}\vartheta} {}_{\hat{\hat{\Theta}}_1}\left(\mathcal{B}_{\hat{\Theta}_2}\right) \xrightarrow{\sim} {}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2}$ by ${}^{\hat{\Theta}_1}\vartheta$ and do the same for the other three analogous compositions involving isomorphisms. In fact, these functors can be written down explicitly in the same fashion as the original functors ${}^{\hat{\Theta}_1}\vartheta$, *etc.*

Remark 4.3.68. The applications in Remark 4.3.67 of Proposition 4.3.58 and Proposition 4.3.38, also yield additional transformations. These transformations allow us to equip $\vartheta_{\hat{\Theta}_2}\,{}_{\hat{\hat{\Theta}}_1}\vartheta\;{}^{\hat{\hat{\Theta}}_1}\vartheta\,\vartheta^{\hat{\Theta}_2}$ and ${}_{\hat{\Theta}_1}\vartheta\,\vartheta_{\hat{\hat{\Theta}}_2}\;\vartheta^{\hat{\hat{\Theta}}_2}\,{}^{\hat{\Theta}_1}\vartheta$. One can calculate explicitly that both induced split Frobenius are isomorphic to the split Frobenius object associated to ${}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2}$ with underlying object ${}_{\hat{\Theta}_2}\vartheta_{\hat{\hat{\Theta}}_1}\;{}^{\hat{\hat{\Theta}}_1}\vartheta^{\hat{\Theta}_2}$. The universal properties of split Frobenius structure thus implies equivalences of additive categories $\left({}_{\hat{\Theta}_1}\mathcal{B}\right)_{\hat{\Theta}_2} \cong {}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2} \cong {}_{\hat{\Theta}_1}\left(\mathcal{B}_{\hat{\Theta}_2}\right)$ as well as isotransformations between the structure functors.

Remark 4.3.69. The composition ${}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2} \cong \mathcal{H}om_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}_1}, \mathcal{B}_{\hat{\Theta}_2}) \xrightarrow{\mathrm{ev}} \mathcal{B}_{\hat{\Theta}_2}$, where ev is given by evaluation at $\vartheta_{\hat{\Theta}_1}I$, agrees with ${}_{\hat{\Theta}_1}\vartheta : {}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2} \to \mathcal{B}_{\hat{\Theta}_2}$ up to isomorphism.

4.3.6 Preorders on split Frobenius objects

In this section, we introduce the Frobenius preorder and strong equivalences of Frobenius objects. We will establish a connection between these notions and higher representation theory in Theorem 4.5.14.

Definition 4.3.70. *Let $\hat{\Theta}_1 = (\Theta_1, \Delta_1, \mu_1)$ and $\hat{\Theta}_2 = (\Theta_2, \Delta_2, \mu_2)$ be Frobenius objects in a strict additive monoidal category $\mathcal{B}$. The* Frobenius preorder $\leq$ *is defined by $\hat{\Theta}_1 \leq \hat{\Theta}_2$ if and only if $\Theta_1 \leq_{LR} \Theta_2$. We say that $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are* weakly equivalent *and write $\hat{\Theta}_1 \sim \hat{\Theta}_2$, if they are equivalent with respect to $\leq$.*

Showing that $\leq$ indeed defines a preorder is a simple exercise.

Remark 4.3.71. Let $\hat{\Theta}_1$ and $\hat{\Theta}_2$ be Frobenius objects in $\mathcal{B}$ with $\hat{\Theta}_1 \leq \hat{\Theta}_2$. Then, one can construct a Frobenius object $\hat{\Theta}_1 \oplus \hat{\Theta}_2$ with underlying object $\Theta_1 \oplus \Theta_2$. We will not use $\hat{\Theta}_1 \oplus \hat{\Theta}_2$ and thus we do not provide the details. However, let us note that if $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are right indecomposable split Frobenius objects, we will construct associated free functorially cyclic modules $\mathcal{B}_{\hat{\Theta}_1}$ and $\mathcal{B}_{\hat{\Theta}_2}$, see Corollary 4.5.9, and a generalisation of Example 4.2.28 applied to $\mathcal{B}_{\hat{\Theta}_1} \oplus \mathcal{B}_{\hat{\Theta}_2}$ can be used to obtain $\hat{\Theta}_1 \oplus \hat{\Theta}_2$ (which is no longer right indecomposable). The point we want to make here is that $\hat{\Theta}_1 \leq \hat{\Theta}_1 \oplus \hat{\Theta}_2 \sim \hat{\Theta}_2$.

Definition 4.3.72. *Let $\hat{\Theta}_1 = (\Theta_1, \Delta_1, \mu_1)$ and $\hat{\Theta}_2 = (\Theta_2, \Delta_2, \mu_2)$ be split Frobenius objects in a strict additive monoidal category $\mathcal{B}$. We say that $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are* strongly equivalent *and write $\hat{\Theta}_1 \simeq \hat{\Theta}_2$, if there exist objects $(A, e_A) \in {}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2}$ and $(B, e_B) \in {}_{\hat{\Theta}_2}\mathcal{B}_{\hat{\Theta}_1}$, and morphisms $\iota_1 \in {}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_1}(I, A\Theta_2 B)$, $\pi_1 \in {}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_1}(A\Theta_2 B, I)$, $\iota_2 \in {}_{\hat{\Theta}_2}\mathcal{B}_{\hat{\Theta}_2}(I, B\Theta_1 A)$ and $\pi_2 \in {}_{\hat{\Theta}_2}\mathcal{B}_{\hat{\Theta}_2}(B\Theta_1 A, I)$, such that $\pi_1 \circ \iota_1 = \Delta_1 \circ \mu_1$, $\iota_1 \circ \pi_1 = (e_A B\Theta_1) \circ (\Theta_1 A e_B)$, $\pi_2 \circ \iota_2 = \Delta_2 \circ \mu_2$ and $\iota_2 \circ \pi_2 = (e_B A\Theta_2) \circ (\Theta_2 B e_A)$.*

Remark 4.3.73. One can show that strong equivalence defines a preorder.

Remark 4.3.74. If $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are strongly equivalent, the morphisms $\iota_1 \circ \Delta_1$ and $\mu_1 \circ \pi_1$ imply that $\Theta_1 \subset^{\oplus} \Theta_1 A\Theta_2 B\Theta_1$. Thus, $\Theta_1 \leq_{LR} \Theta_2$ and therefore $\hat{\Theta}_1 \leq \hat{\Theta}_2$ holds. Analogously, $\hat{\Theta}_2 \leq \hat{\Theta}_1$ follows. Thus, modules that are strongly equivalent are also weakly equivalent.

Remark 4.3.75. The natural definition of equivalence of Frobenius objects, which is induced from their interpretation as objects of the 2-category $\mathbf{\mathfrak{Hom}}(\mathbf{\mathfrak{Fr}}, \mathbf{\mathfrak{C}})$, see Remark 4.3.22, where $\mathbf{\mathfrak{C}}$ is the one object 2-category associated to $\mathcal{B}$, does not agree with the definition of strong or weak equivalence given above. For $\hat{\Theta}_1$ and $\hat{\Theta}_2$ to be equivalent in this sense, objects A, $B \in \mathcal{B}$ are required such that $A\Theta_1 \cong B\Theta_2$, $BA\Theta_1 \cong \Theta_1$ and $AB\Theta_2 \cong \Theta_2$.

We will provide concrete examples of strong equivalences in Example 5.3.8 which are non-trivial. For now, we establish the following useful compatibility between the different notions:

Proposition 4.3.76. *Isomorphic split Frobenius objects are strongly equivalent.*

Proof. Let $\hat{\Theta}_1 = (\Theta_1, \Delta_1, \mu_1)$ and $\hat{\Theta}_2 = (\Theta_2, \Delta_2, \mu_2)$ be split Frobenius objects, and let $\tau : \Theta_1 \to \Theta_2$ be an isomorphism of $\hat{\Theta}_1$ and $\hat{\Theta}_2$. Define $A := B := I$ as the unit objects of $\mathcal{B}$. Further, set $e_A := (\Theta_1\tau) \circ e_{\hat{\Theta}_1} \circ (\Theta_1\tau^{-1})$ and $e_B := (\Theta_2\tau^{-1}) \circ e_{\hat{\Theta}_2} \circ (\Theta_2\tau)$. Clearly,

e_A and e_B are idempotents and, since τ is a isomorphism of Frobenius objects, they satisfy the necessary Frobenius descent properties. Next, we define the morphisms $\iota_1 := (\Theta_1\tau\Theta_1) \circ (\Delta_1\Theta_1)$, $\pi_1 := (\mu_1\Theta_1) \circ (\Theta_1\tau^{-1}\Theta_1)$, $\iota_2 := (\Theta_2\tau^{-1}\Theta_2) \circ (\Delta_2\Theta_2)$ and $\pi_2 := (\Theta_2\mu_2) \circ (\Theta_2\tau\Theta_2)$. Again since τ is a morphism of Frobenius objects, these morphisms satisfy the necessary Frobenius descent properties. Finally, we calculate $\pi_1 \circ \iota_1 = (\mu_1\Theta_1) \circ (\Delta_1\Theta_1) = \Delta_1 \circ \mu_1$, where we used the Frobenius relation of $\hat{\Theta}_1$,

$$
\begin{aligned}
\iota_1 \circ \pi_1 &= (\Theta_1\tau\Theta_1) \circ (\Delta_1\Theta_1) \circ (\mu_1\Theta_1) \circ (\Theta_1\tau^{-1}\Theta_1) \\
&= (\Theta_1\tau\Theta_1) \circ (e_{\hat{\Theta}_1}\Theta_1) \circ (\Theta_1\Delta_1) \circ (\Theta_1\mu_1) \circ (\Theta_1\tau^{-1}\Theta_1) \\
&= (\Theta_1\tau\Theta_1) \circ (e_{\hat{\Theta}_1}\Theta_1) \circ (\Theta_1\Delta_1) \circ (\Theta_1\tau^{-1}) \circ (\Theta_1\mu_2) \circ (\Theta_1\tau^{-1}\Theta_1) \\
&= (\Theta_1\tau\Theta_1) \circ (e_{\hat{\Theta}_1}\Theta_1) \circ (\Theta_1\tau^{-1}\tau^{-1}) \circ (\Theta_1\Delta_2) \circ (\Theta_1\mu_2) \circ (\Theta_1\Theta_2\tau) \\
&= (e_AB\Theta_1) \circ (\Theta_1Ae_B),
\end{aligned}
$$

where we used that $\hat{\Theta}_1$ is split as well as Remark 3.1.3. Similarly, $\pi_2 \circ \iota_2 = \Delta_2 \circ \mu_2$ and $\iota_2 \circ \pi_2 = (e_BA\Theta_2) \circ (\Theta_2Be_A)$ follow and the proof is complete. $\square$

Remark 4.3.77. In contrast to weak equivalence, strong equivalence between split Frobenius objects is not determined by the underlying object. In Remark 5.3.9, we will see that there exists indecomposable objects that can carry different structures as (right indecomposable) split Frobenius objects which are not strongly equivalent. Further, we will see in Example 4.5.13 that strongly equivalent indecomposable split Frobenius objects can have very different underlying objects.

4.4 Restriction along quotients and its adjoints

In this technical section, we discuss the restriction 2-functor between two 2-categories of additive higher representations associated to any additive monoidal functor between the acting additive monoidal categories. We will focus our attention on the quotient 2-functor case relevant for this thesis. For simplicity, we actually only discus monoidal quotient functors. In this case, we construct a left adjoint 2-functor called induction and a right adjoint 2-functor called coinduction. Further, we define central quotient functors and give a different construction of the induction 2-functor using base change that is available in this case.

We start by introducing a 2-functor $\mathrm{Res}_\phi : \mathcal{B}_2 - \mathrm{Mod} \to \mathcal{B}_1 - \mathrm{Mod}$ associated to a given additive monoidal functor $\phi : \mathcal{B}_1 \to \mathcal{B}_2$. Since $\mathcal{B} - \mathrm{Mod} = \mathfrak{Hom}(\mathcal{B}, \mathfrak{A}dd)$, it is fairly obvious that right composition with ϕ on each categorical level defines the 2-functor Res_ϕ. Let us nevertheless provide some details: Recall that any $\mathcal{M} \in$

$\mathcal{B}_2-\mathrm{Mod}$ is actually given by a pair $(\mathcal{M}, \rho_{\mathcal{M}})$ where $\mathcal{M}$ is an additive category and $\rho_{\mathcal{M}} : \mathcal{B}_2 \to \mathfrak{A}dd(\mathcal{M}, \mathcal{M})$ is an additive monoidal functor. Thus, $\mathrm{Res}_\phi(\mathcal{M}) := (\mathcal{M}, \rho_{\mathrm{Res}_\phi(\mathcal{M})}) := (\mathcal{M}, \rho_{\mathcal{M}} \circ \phi) \in \mathcal{B}_1-\mathrm{Mod}$. Now, let $F : \mathcal{M}_1 \to \mathcal{M}_2$ be a $\mathcal{B}_2$-functor. Recall that this is given by an additive functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ together with additive isotransformations $\mathrm{can}_{F,B} : F \circ \rho_{\mathcal{M}_1}(B) \overset{\sim}{\Longrightarrow} \rho_{\mathcal{M}_2}(B) \circ F$ for all $B \in \mathcal{B}_2$. Clearly, F is also an additive functor between the underlying additive categories of $\mathrm{Res}_\phi(\mathcal{M}_1)$ and $\mathrm{Res}_\phi(\mathcal{M}_2)$. Setting $\mathrm{can}_{\mathrm{Res}_\phi(F),B} := \mathrm{can}_{F,\phi(B)}$ for all $B \in \mathcal{B}_1$, which of course still satisfies the necessary relations, turns F into a $\mathcal{B}_1$-functor which we denote $\mathrm{Res}_\phi(F)$. Similarly for all $\mathcal{B}_2$-transformations $\alpha : F \Rightarrow G$, we set $\mathrm{Res}_\phi(\alpha) := \alpha$ which satisfies all the necessary relations since α does. All required compatibility isomorphisms that turn Res_ϕ into a 2-functor can be chosen as (being induced from) identity morphisms. Thus, we have fully specified a strict 2-functor Res_ϕ.

Definition 4.4.1. *We call* $\mathrm{Res}_\phi : \mathcal{B}_2-\mathrm{Mod} \to \mathcal{B}_1-\mathrm{Mod}$ restriction along ϕ*. Provided that there exists a 2-functor* $\mathrm{Ind}_\phi : \mathcal{B}_1-\mathrm{Mod} \to \mathcal{B}_2-\mathrm{Mod}$ *(resp.* $\mathrm{Coind}_\phi : \mathcal{B}_1-\mathrm{Mod} \to \mathcal{B}_2-\mathrm{Mod}$*) that is left (resp. right) 2-adjoint of* Res_ϕ*, it is called* induction (resp. restriction) along ϕ*.*

Remark 4.4.2. Definition 4.4.1 is not a precise definition and in fact it is not sufficient to apply the standard notion of an adjoint pair in a 2-category. This is not surprising in light of Remark 3.1.17, where we noted that there exists no appropriate 2-category of 2-categories, but rather a 3-category. Already in [Gr, Chapter I.7], various appropriate generalisations are discussed and we believe that the context discussed here fits in this framework. However, since we do not need this level of sophistication, we limit ourselves to discussing the properties essential for us. We leave it to future research to develop the appropriate notion in this context.

Whenever it is unambiguous along which ϕ we restrict, induce or coinduce, we will omit stating it explicitly.

Remark 4.4.3. Below, we will provide a general definition for Coind_ϕ and observe that the zig-zag equation does only hold up to an invertible morphism of morphisms of 2-functors. On the other hand if $\phi = \pi$ is a monoidal quotient functor, we will provide constructions of Ind_π and Coind_π together with unit and counit morphisms of 2-functors that satisfy the same zig-zag relations as in the case of the standard notion of adjoints in 2-categories.

In the remainder of the section, we discuss the 2-adjoints of Res_ϕ. As in the classical setting, coinduction can be described in terms of homomorphisms:

Definition/Proposition 4.4.4. $\mathrm{Coind}_\phi := \mathcal{H}om_{\mathcal{B}_1}(\mathcal{B}_2, -)$ *defines a strict 2-functor which is right 2-adjoint to* Res_ϕ.

Proof. Since we will not use Coind_ϕ in the remained of the work, we will only outline the main steps. One should note that in the last step of this outline, we will see that the standard notion of adjoints is not sufficient in this case.
First, note that we have $\mathcal{B}_s = (\mathcal{B}_2, \rho) \in \mathcal{B}_2 - \mathrm{Mod}$. Thus $(\mathcal{B}_2, \rho\phi) \in \mathcal{B}_1 - \mathrm{Mod}$. Second, while ρ was given by taking tensor products on the left, we can also take tensor products on the right and obtain $\mathcal{B}_2 \in (\mathcal{B}_2)^{\mathrm{rev}} - \mathrm{Mod}$. Since the two actions are compatible, it follows that $\mathcal{B}_2 \in \mathcal{B}_1 \times (\mathcal{B}_2)^{\mathrm{rev}} - \mathrm{Mod}$. For all $\mathcal{M} \in \mathcal{B}_1 - \mathrm{Mod}$, we can thus regard $\mathcal{H}om_{\mathcal{B}_1}(\mathcal{B}_2, \mathcal{M})$ as a $\mathcal{B}_2$-module. Third, one constructs the 2-functor Coind_ϕ by using (strict) functoriality of $\mathcal{H}om_{\mathcal{B}_2}(-,-)$ in the second argument. Fourth, one has to provide the unit and counit 2-arrows and show the zig-zag relations. As in the classical case, the counit transformation $\mathrm{Res}_\phi \mathrm{Coind}_\phi \Rightarrow id_{\mathcal{B}_1 - \mathrm{Mod}}$ is given by evaluating at the unit object of $\mathcal{B}_2$. The unit transformation $id_{\mathcal{B}_2 - \mathrm{Mod}} \Rightarrow \mathrm{Coind}_\phi \mathrm{Res}_\phi$ is given on any $\mathcal{M} \in \mathcal{B}_2 - \mathrm{Mod}$ by a $\mathcal{B}_2$-functor $\mathcal{M} \to \mathcal{H}om_{\mathcal{B}_1}(\mathcal{B}_2, \mathrm{Res}_\phi(\mathcal{M}))$ which in turn maps any $M \in \mathcal{M}$ to the $\mathcal{B}_1$-functor given by $B \mapsto B.M$. Now, one can compute the induced transformation $\mathrm{Res}_\phi \Rightarrow \mathrm{Res}_\phi \mathrm{Coind}_\phi \mathrm{Res}_\phi \Rightarrow \mathrm{Res}_\phi$ (resp. $\mathrm{Coind}_\phi \Rightarrow \mathrm{Coind}_\phi \mathrm{Res}_\phi \mathrm{Coind}_\phi \Rightarrow \mathrm{Coind}_\phi$). In the simplistic definition of an adjunction these composition should be given by the identity transformations. The functor obtained by evaluating the transformation at $\mathcal{M} \in \mathcal{B}_2 - \mathrm{Mod}$ (resp. $N \in \mathcal{N} \in \mathcal{B}_1 - \mathrm{Mod}$), sends an object $M \in \mathrm{Res}_\phi(\mathcal{M})$ (resp. $F \in \mathcal{H}om_{\mathcal{B}_1}(\mathcal{B}_2, \mathcal{N})$) to $I_2.M$ (resp. $(B \mapsto F(B.I_2))$, where I_2 denotes the unit object in $\mathcal{B}_2$. Neither composition yields the identity transformation, however in both cases the resulting transformation is isomorphic to the identity transformation via a morphism of morphisms of 2-functors that is straight forward to construct. □

From now on, we assume that $\phi = \pi : \mathcal{B}_1 \twoheadrightarrow \mathcal{B}_2$ is a monoidal quotient functor. To simplify matters further, we assume that ϕ induces the identity map on objects rather than merely being essentially surjective. The general case of essential surjectivity can be handled by choosing per object in $\mathcal{B}_2$ an isomorphic object in the image of π together with a fixed isomorphism.

Definition/Proposition 4.4.5. *For* $\mathcal{M} = (\mathcal{M}, \rho) \in \mathcal{B}_1 - \mathrm{Mod}$, *denote by* $\underline{\mathcal{M}}_\pi$ *the full subcategory of* $\mathcal{M}$ *with objects* $\mathrm{Ob}(\underline{\mathcal{M}}_\pi) := \{M \in \mathrm{Ob}(\mathcal{M}) | \alpha.id_M = 0 \ \forall \alpha \in \mathfrak{J}(\pi)\}$, *denote by* $i_{\mathcal{M}} : \underline{\mathcal{M}}_\pi \hookrightarrow \mathcal{M}$ *the inclusion functor and denote by* $\underline{\rho}_{\mathcal{M}}$ *the unique monoidal functor with* $i_{\mathcal{M}}\underline{\rho}_{\mathcal{M}} = \rho$. *Thus,* $(\underline{\mathcal{M}}_\pi, \underline{\rho}_\pi) \in \mathcal{B}_1 - \mathrm{Mod}$ *is a sub-*$\mathcal{B}_1$*-module of* $\mathcal{M}$. *Further, denote by* $\check{\rho}_{\mathcal{M}}$ *the unique additive monoidal* $\mathcal{B}_2$*-structure functor on* $\underline{\mathcal{M}}_\pi$

such that $\pi\check{\rho} = \underline{\rho}_{\mathcal{M}}$. *We define* $\mathrm{Coind}_\pi(\mathcal{M}) := (\underline{\mathcal{M}}_\pi, \check{\rho}) \in \mathcal{B}_2 - \mathrm{Mod}$. *For all* $\mathcal{B}_1$*-functors* $F : \mathcal{M}_1 \to \mathcal{M}_2$*, we define* $\mathrm{Coind}_\pi(F) : \mathrm{Coind}_\pi(\mathcal{M}_1) \to \mathrm{Coind}_\pi(\mathcal{M}_2)$ *as the unique* $\mathcal{B}_2$*-functor such that the following diagram of additive categories commutes:*

$$\begin{tikzcd} \mathcal{M}_1 \arrow[r, "F"] & \mathcal{M}_2 \\ \underline{\mathcal{M}_1}_\pi \arrow[u, hook] \arrow[r, dashed, "\mathrm{Coind}_\pi(F)"] & \underline{\mathcal{M}_2}_\pi \arrow[u, hook] \end{tikzcd}$$

For all $\mathcal{B}_1$*-transformations* $\alpha : F \Rightarrow G$*, we define* $\mathrm{Coind}_\pi(\alpha)$ *by setting* $\mathrm{Coind}_\pi(\alpha)_M := \alpha_M$ *for all* $M \in \mathcal{M}_1$*. The thus defined strict 2-functor* $\mathrm{Coind}_\pi : \mathcal{B}_1 - \mathrm{Mod} \to \mathcal{B}_2 - \mathrm{Mod}$ *is called* coinduction along π*.*

Proof. By definition, the ideal $\mathfrak{J} \trianglelefteq \mathcal{B}_1$ consists of all morphisms in $\mathcal{B}_1$ that vanish under π. By definition $\underline{\mathcal{M}}_\pi$ is a full additive subcategory of $\mathcal{M}$. Since $id_B.(\alpha.id_M)$ and $id_B.(\alpha.id_M)$ agree up to canonical isomorphisms (which come from the structure of $\mathcal{M}$ as a $\mathcal{B}_1$-module), we have indeed defined a sub-$\mathcal{B}_1$-module $(\underline{\mathcal{M}}_\pi, \rho|_{\underline{\mathcal{M}}_\pi}) \in \mathcal{B}_1 - \mathrm{Mod}$. Furthermore, any element of $\mathcal{I}$ acts trivially on this sub-$\mathcal{B}_1$-module. The existence of $\check{\rho}$ follows.
Since any $\mathcal{B}_1$-functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ commutes up to canonical isomorphisms with the $\mathcal{B}_1$ action, the image of $\underline{\mathcal{M}_1}_\pi$ lies indeed in $\underline{\mathcal{M}_2}_\pi$. Since the latter is a full subcategory of $\mathcal{M}_2$, all necessary morphisms to turn this factorisation into a $\mathcal{B}_2$ functor are available and we obtain the unique functor $\mathrm{Coind}_\pi(F)$ as claimed.
Finally given a $\mathcal{B}_1$-transformation $\alpha : F_1 \Rightarrow F_2$ where $F_1, F_2 : \mathcal{M}_1 \to \mathcal{M}_2$ are $\mathcal{B}_1$-functors, we define $\mathrm{Coind}_\pi(\alpha)$ by setting $\mathrm{Coind}_\pi(\alpha)_M := \alpha_M$ for all $M \in \mathcal{M}_1$.
All required compatibility isomorphisms that turn Coind_π into a 2-functor can be chosen as (being induced from) identity morphisms. Thus, we have defined fully specified the strict 2-functor Coind_π. ☐

Remark 4.4.6. For $\phi = \pi$ there is an equivalence of 2-functors $\mathrm{Coind}_\phi \cong \mathrm{Coind}_\pi$. In particular, we have equivalences $\mathcal{H}om_{\mathcal{B}_1}(\mathcal{B}_2, \mathcal{M}) \cong \mathrm{Coind}_\pi(\mathcal{M})$ in $\mathcal{B}_2 - \mathrm{Mod}$ which are natural in $\mathcal{M} \in \mathcal{B}_1 - \mathrm{Mod}$. Arbitrary additive monoidal functors ϕ induce $\mathcal{B}_1$-functors $\mathcal{B}_1 \to \mathrm{Res}_\pi(\mathcal{B}_2)$. This leads to the following diagram of functors and equivalences of additive categories.

$$\begin{tikzcd} \mathcal{H}om_{\mathcal{B}_1}(\mathcal{B}_1, \mathcal{M}) \arrow[rr, "\sim"] & & \mathcal{M} & \\ \mathcal{H}om_{\mathcal{B}_1}(\mathrm{Res}_\pi(\mathcal{B}_2), \mathcal{M}) \arrow[u, "- \circ \pi"] \arrow[r, "\sim"] & \mathcal{H}om_{\mathcal{B}_2}(\mathcal{B}_2, \mathrm{Coind}_\phi(\mathcal{M})) \arrow[rr, "\sim"] & & \mathrm{Coind}_\phi(\mathcal{M}) \arrow[ul, dashed] \end{tikzcd}$$

Thus up to equivalence, we had no choice when defining Coind_ϕ. In the case of $\phi = \pi$, the arrow $- \circ \pi$ is faithful, which is the basis of our definition of Coind_π.

Proposition 4.4.7. Coind_π *is right adjoint to* Res_π.

Proof. By construction, we have $\mathrm{Coind}_\pi \mathrm{Res}_\pi = id_{\mathcal{B}_2-\mathrm{Mod}}$ and this gives us unit $\mathcal{B}_2$-transformation $\eta : id_{\mathcal{B}_2-\mathrm{Mod}} \Rightarrow \mathrm{Coind}_\pi \mathrm{Res}_\pi$. Clearly, we have $\eta\, \mathrm{Coind}_\pi = id_{\mathrm{Coind}_\pi}$ and $\mathrm{Res}_\pi \eta = id_{\mathrm{Res}_\pi}$. The counit $\mathcal{B}_1$-transformation $\epsilon : \mathrm{Res}_\pi \mathrm{Coind}_\pi \Rightarrow id_{mcB_1-\mathrm{Mod}}$ evaluated at $\mathcal{M} \in \mathcal{B}_1-\mathrm{Mod}$ is given by the inclusion of the sub-$\mathcal{B}_1$-module $(\underline{\mathcal{M}}_\pi, \rho|_{\underline{\mathcal{M}}_\pi})$ into $\mathcal{M}$. If $\mathcal{M} = \mathrm{Res}_\pi(\mathcal{N})$ for $\mathcal{N} \in \mathcal{B}_2 - \mathrm{Mod}$, we have $\underline{\mathcal{M}}_\pi = \mathcal{M}$ and thus $\epsilon\, \mathrm{Res}_\pi = id_{Res_\pi}$. Similarly, $\mathrm{Coind}_\pi \epsilon = id_{\mathrm{Coind}_\pi}$ since Coind_π essentially picks a submodule on which then the action factors over $\mathcal{B}_2$. In the case of $\mathcal{M}$ this submodule is $\underline{\mathcal{M}}_\pi$. To conclude, we note that we have shown the zig-zag relations $(\epsilon\, \mathrm{Res}_\pi) \circ (\mathrm{Res}_\pi \eta) = \mathrm{Res}_\pi$ and $(\mathrm{Coind}_\pi \epsilon) \circ (\eta\, \mathrm{Coind}_\pi) = \mathrm{Coind}_\pi$ hold. □

Definition/Proposition 4.4.8. *For all* $\mathcal{M} = (\mathcal{M}, \rho) \in \mathcal{B}_1$*, we denote by* $\overline{\mathcal{M}}^\pi$ *the additive quotient of* $\mathcal{M}$ *by all morphisms of the form* $\rho(b)(id_M)$ *where* b *varies over all morphism in* $\mathcal{B}_1$ *with* $\pi(b) = 0$ *and* M *varies over* $\mathrm{Ob}(\mathcal{M})$*. Further, denote by* $\hat{\rho}(B)$ *the unique additive monoidal* $\mathcal{B}_2$*-structure functor on* $\overline{\mathcal{M}}^\pi$ *such that* $\hat{\rho}\pi = \rho$*. We define* $\mathrm{Ind}_\pi(\mathcal{M}) := (\overline{\mathcal{M}}^\pi, \hat{\rho}) \in \mathcal{B}_2 - \mathrm{Mod}$*. For all* $\mathcal{B}_1$*-functors* $F : \mathcal{M}_1 \to \mathcal{M}_2$*, we define* $\mathrm{Ind}_\pi(F) : \mathrm{Ind}_\pi(\mathcal{M}_1) \to \mathrm{Ind}_\pi(\mathcal{M}_2)$ *as the unique* $\mathcal{B}_2$*-functor such that the following diagram of additive categories commutes:*

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r, "F"] \arrow[d, "\pi_1"] & \mathcal{M}_2 \arrow[d, "\pi_2"] \\
\overline{\mathcal{M}_1}^\pi \arrow[r, dashed, "\exists!\, \mathrm{Ind}_\pi(F)"] & \overline{\mathcal{M}_2}^\pi
\end{tikzcd}$$

For all $\mathcal{B}_1$*-transformations* $\alpha : F \Rightarrow G$*, we define* $\mathrm{Ind}_\pi(\alpha) : \mathrm{Ind}_\pi(F) \to \mathrm{Ind}_\pi(G)$ *by setting* $\mathrm{Ind}_\pi(\alpha)_M := \pi_2(\alpha_M)$ *for all* $M \in \mathcal{M}_2$*. We call the thus defined strict additive 2-functor* $\mathrm{Ind}_\pi : \mathcal{B}_1 - \mathrm{Mod} \to \mathcal{B}_2 - \mathrm{Mod}$ induction along π.

Proof. Since the set of morphisms of the form $\rho(b)(id_M)$ is stable under the $\mathcal{B}_1$ action, we obtain a quotient $\mathcal{B}_1$-module. Since all morphisms in $\mathrm{Ker}(\pi)$ vanish by definition on this quotient $\mathcal{B}_1$-module, existence and uniqueness of $\hat{\rho}$ with $\hat{\rho}\pi = \rho$ follows. For all $\mathcal{B}_1$-functors $F : \mathcal{M}_1 \to \mathcal{M}_2$ and morphisms of the form $\rho_{\mathcal{M}_1}(b)(id_M)$, $F(\rho_{\mathcal{M}_1}(b)(id_M))$ agrees up to canonical isomorphisms with $\rho_{\mathcal{M}_2}(b)(id_{F(M)})$. Thus, we obtain a unique $\mathcal{B}_1$-functor between the quotient modules which can be regarded as a $\mathcal{B}_2$-functor. Thus, we have obtained $\mathrm{Ind}_\pi(F)$. Checking that $\mathrm{Ind}_\pi(\alpha)$ is well defined is straight forward. All required compatibility isomorphisms that turn Ind_π into a 2-functor can be chosen as (being induced from) identity morphisms. Thus, we have defined fully specified the strict 2-functor Ind_π. □

Proposition 4.4.9. *We have* $id_{\mathcal{B}_2-\mathrm{Mod}} \cong \mathrm{Ind}_\pi \mathrm{Res}_\pi$*.*

Proof. The statement is obvious. The only reason that we cannot write $id_{\mathcal{B}_2-\mathrm{Mod}} = \mathrm{Ind}_\pi \mathrm{Res}_\pi$ is because by the above definition we have to take a (trivial) quotient by all zero morphisms. $\square$

We denote the associated isomorphism of 2-functors by $\epsilon : \mathrm{Ind}_\pi \mathrm{Res}_\pi \Rightarrow id_{\mathcal{B}_2-\mathrm{Mod}}$.

Proposition 4.4.10. *The 2-functor* Ind_π *is left adjoint to* Res_π*.*

Proof. It remains to provide a transformation $\eta : id_{\mathcal{B}_1-\mathrm{Mod}} \Rightarrow \mathrm{Res}_\pi \mathrm{Ind}_\pi$ and to check the zig-zag equation. In fact, we have already constructed the necessary morphisms. For each $\mathcal{M} \in \mathcal{B}_1 - \mathrm{Mod}$, we set $\eta_\mathcal{M} : \mathcal{M} \to \overline{\mathcal{M}}^\pi$ the previously constructed quotient functor. In fact together with the trivial structure maps it becomes a $\mathcal{B}_1$-functor $\eta_\mathcal{M} : \mathcal{M} \to \mathrm{Res}_\pi \mathrm{Ind}_\pi(\mathcal{M})$. It is easy to see that this defines a transformation of 2-functors $\eta : id_{\mathcal{B}_1-\mathrm{Mod}} \Rightarrow \mathrm{Res}_\pi \mathrm{Ind}_\pi$.
By the discussion preceding this proposition, the induced transformation $\eta \mathrm{Res}_\pi$ is also essentially given by the identity transformation. To be more precise, we have the first zig-zag relation $(\mathrm{Res}_\pi \epsilon) \circ (\eta \mathrm{Res}_\pi) = id_{\mathrm{Res}_\pi}$. The second zig-zag relation, *i.e.* $(\epsilon \mathrm{Ind}_\pi) \circ (\mathrm{Ind}_\pi \eta) = id_{\mathrm{Ind}_\pi}$ holds as well, since this time $\mathrm{Ind}_\pi \eta$ is essentially the identity transformation. $\square$

For the remained of this section, we will deal with a further specialisation, namely the case of a central quotient functor π. In this case, we can provide a somewhat cleaner description of Ind_π which will be useful for applications, see Section 5.7. Recall Definition 4.1.11.

Definition 4.4.11. *An additive monoidal quotient functor* $\pi : \mathcal{B}_1 \to \mathcal{B}_2$ *is called* central *if there exists a quotient ring* $\overline{Z}$ *of* $Z := \mathcal{Z}^\otimes(\mathcal{B})$ *such that* $\mathcal{B}_2 \cong \mathcal{B}_1 \otimes_Z \overline{Z}$*.*

Remark 4.4.12. A monoidal quotient functor $\pi : \mathcal{B}_1 \to \mathcal{B}_2$ is central if and only every morphism in $\mathcal{I}(\pi)$ can be written as a sum of morphisms each of which is a composition of an arbitrary morphisms and a morphism α_B for some $B \in \mathcal{B}_1$ and some $\alpha \in \mathcal{Z}^\otimes(\mathcal{B})$.

For the rest of the section, we fix a central quotient functor $\pi : \mathcal{B} \to \overline{\mathcal{B}}$. Denoting $Z := \mathcal{Z}^\otimes(\mathcal{B})$, we can assume without loss of generality that there exists a quotient ring $\overline{Z}$ of Z, such that $\overline{\mathcal{B}} = \mathcal{B} \otimes_Z \overline{Z}$ and that π is the canonical monoidal quotient functor $\mathcal{B} \to \mathcal{B} \otimes_Z \overline{Z}$. In particular we have $\mathrm{Ob}(\mathcal{B}) = \mathrm{Ob}(\overline{\mathcal{B}})$, which puts us in the same situation as when constructing Ind_π. For all $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$, we have a natural ring homomorphism $Z \to Z_{\mathcal{M}} := \mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ into the centre of $\mathcal{M}$ as a $\mathcal{B}$-module. Denote by $\overline{Z}_{\mathcal{M}}$ the quotient of $Z_{\mathcal{M}}$ by the image of the kernel of $Z \twoheadrightarrow \overline{Z}$ under $Z \to Z_{\mathcal{M}}$. Clearly, we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, two heads] & Z_{\mathcal{M}} \arrow[d, two heads] \\ \overline{Z} \arrow[r] & \overline{Z}_{\mathcal{M}} \end{tikzcd} .$$

Now, recall from Section 4.1 the construction of the $\mathcal{B}$-module $\mathcal{M} \otimes_{Z_{\mathcal{M}}} \overline{Z}_{\mathcal{M}}$, denote its underlying additive category by $\overline{\mathcal{M}}$ and denote its structure functor by $\rho_{\mathcal{M} \otimes_{Z_{\mathcal{M}}} \overline{Z}_{\mathcal{M}}} : \mathcal{B} \to \mathfrak{A}dd(\overline{\mathcal{M}}, \overline{\mathcal{M}})$. Note that $\overline{\mathcal{M}}$ is isomorphic to $\overline{\mathcal{M}}^{\pi}$ as defined in Definition/Proposition 4.4.8. This isomorphism is induced by the standard isomorphisms $M \otimes_R R/I \cong M/I$ that hold for all right R-modules M and all left ideals $I \trianglelefteq R$. The remaining necessary arguments are exactly the same as in the proof of Definition/Proposition 4.4.8 where we constructed Ind_{π}. Since Ind_{π} is left adjoint to Res_{π}, see Proposition 4.4.10, and it is clear that the newly constructed 2-functor is isomorphic to Ind_{π}, we obtain the following corollary:

Corollary 4.4.13. *The obvious 2-functor $\mathcal{B} - \mathrm{Mod} \to \mathcal{B} \otimes_Z \overline{Z} - \mathrm{Mod}$, that is given on objects by $\mathcal{M} \mapsto \mathcal{M} \otimes_{Z_{\mathcal{M}}} \overline{Z}_{\mathcal{M}}$, is left adjoint to Res_{π}.*

4.5 Functorially cyclic modules revisited

In this section, we revisit functorially cyclic modules. Section 4.5.1 analyses the implications of the additive Barr-Beck theorem. Bijections between equivalence classes of certain modules (including functorially cyclic modules) and isomorphism classes of split Frobenius objects are established and the universal property of functorially cyclic modules is established. Section 4.5.2 makes connection between right indecomposable split Frobenius objects in a monoidal category and free functorially cyclic modules more explicit. Further, the equivalence between the 2-category of free functorially cyclic modules and the split Frobenius 2-category of Section 4.3.5. Section 4.5.3 discusses the representation theoretic meaning of the preorders on split Frobenius objects of Section 4.3.6, and shows that free functorially cyclic modules are determined up to weak equivalence by equivalence classes of split Frobenius objects under the Frobenius preorder. Furthermore, equivalence classes of free functorially cyclic modules correspond to strong equivalence classes of Frobenius objects. Section 4.5.4 discusses semi-free functorially cyclic modules which are restrictions of free functorially cyclic modules along quotient functors, studies relationships that arise when comparing different quotients, and calculates certain centres. Section 4.5.5 introduces (strongly) universal cell modules and proves for weakly regular additive 2-categories that existence of strongly universal cell modules is equivalent to the existence of split Frobenius objects that satisfy certain technical conditions.

4.5.1 Functorially cyclic modules and split Frobenius objects

In this section, we start to analyse the implications of the additive Barr-Beck theorem for additive higher representation theory. Theorem 4.5.4 provides several bijections between equivalence classes of certain modules and isomorphism classes of split Frobenius objects. Furthermore, we deduce Theorem 4.5.6 which states that the universal property associated to a split Frobenius object satisfied by the associated functorially cyclic module.

Recall from Proposition 4.3.20 that any functorially cyclic $\mathfrak{A}$-module gives rise to an indecomposable split Frobenius object in the endomorphism category of an $\mathfrak{A}$-quotient of a principal representation. In this section, we reverse this process by showing how to construct idempotent closed functorially cyclic modules from indecomposable split Frobenius objects on $\mathfrak{A}$-quotients of principal representations.

Definition 4.5.1. *Let R, R' be $\mathfrak{A}$-modules. We say that R* weakly dominates *R' if there exist $\mathfrak{A}$-functors $R' \xrightarrow{G} R \xrightarrow{F} R'$ such that $\mathrm{id}_{R'} \subset^{\oplus} FG$. Further, we say that R* dominates *R' if there exists F and G as above where F is indecomposable.*

Definition 4.5.2. *Let R, R_1 and R_2 be $\mathfrak{A}$-modules, let $R_i \xrightarrow{G_i} R \xrightarrow{F_i} R_i$ for $i \in \{1,2\}$ be $\mathfrak{A}$-functors such that $\mathrm{id}_{R_i} \subset^{\oplus} F_iG_i$ and let $R_1 \xrightarrow{K} R_2 \xrightarrow{L} R_1$ be $\mathfrak{A}$-functors. We say (K, L) is a dominated equivalence, if (K, L) is an equivalence and if we have iso-$\mathfrak{A}$-transformations $KF_1 \overset{\sim}{\Longrightarrow} F_2$ and $G_1 \overset{\sim}{\Longrightarrow} G_2K$ such that the following diagrams commute*

$$\begin{tikzcd}
G_1F_1 \arrow[r, Rightarrow, "\sim"] \arrow[d, Rightarrow, "G_1\alpha_1F_1"] & G_2KF_1 \arrow[r, Rightarrow, "\sim"] & G_2F_2 \arrow[d, Rightarrow, "G_2\alpha_2F_2"'] \\
G_1F_1G_1F_1 \arrow[r, Rightarrow, "\sim"] & G_2KF_1G_2KF_1 \arrow[r, Rightarrow, "\sim"] & G_2F_2G_2F_2
\end{tikzcd}
\qquad
\begin{tikzcd}
G_1F_1 \arrow[r, Rightarrow, "\sim"] & G_2KF_1 \arrow[r, Rightarrow, "\sim"] & G_2F_2 \\
G_1F_1G_1F_1 \arrow[r, Rightarrow, "\sim"] \arrow[u, Rightarrow, "G_1\beta_1F_1"'] & G_2KF_1G_2KF_1 \arrow[r, Rightarrow, "\sim"] & G_2F_2G_2F_2 \arrow[u, Rightarrow, "G_2\beta_2F_2"]
\end{tikzcd}$$

Remark 4.5.3. For simplicity, we have stated Definition 4.5.2 in a non-symmetric fashion. We leave it to the reader to check that the Definition also implies the existence of iso-$\mathfrak{A}$-transformations $F_2L \overset{\sim}{\Longrightarrow} L$ and $G_1L \overset{\sim}{\Longrightarrow} G_2$ together with the analogous diagrams as above and additional compatibilities between the new and the old transformations.

With these definitions, we can state the following theorem:

Theorem 4.5.4. *Let $R \in \mathfrak{A} - \mathrm{Mod}$, let $a \in \mathfrak{A}$, let $\mathcal{I}, \mathcal{I}' \trianglelefteq \mathcal{P}_a$ be ideals and assume that $\mathcal{I}'$ is stable under the right $\mathfrak{A}(a,a)$-action. Then, we have the following bijections*

$$\left\{\begin{array}{c}\textit{idempotent closed } \mathfrak{A}\textit{-modules}\\ \textit{weakly dominated by } R\\ \textit{up to dominated equivalence}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\textit{split Frobenius objects}\\ \textit{on } R\\ \textit{up to isomorphism}\end{array}\right\}$$

$$\left\{\begin{array}{c}\textit{idempotent closed } \mathfrak{A}\textit{-modules}\\ \textit{dominated by } R\\ \textit{up to dominated equivalence}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\textit{indecomposable, split}\\ \textit{Frobenius objects on } R\\ \textit{up to isomorphism}\end{array}\right\}$$

$$\left\{\begin{array}{c}\textit{idempotent closed, functorially}\\ \textit{cyclic } \mathfrak{A}\textit{-modules}\\ \textit{dominated by } \mathcal{P}_a^{\mathcal{I}}\\ \textit{up to dominated equivalence}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\textit{indecomposable,}\\ \textit{split Frobenius objects}\\ \textit{on } \mathcal{P}_a^{\mathcal{I}}\\ \textit{up to isomorphism}\end{array}\right\}$$

$$\left\{\begin{array}{c}\textit{idempotent closed, functorially}\\ \textit{cyclic } \mathfrak{A}\textit{-modules}\\ \textit{dominated by } \mathcal{P}_a^{\mathcal{I}'}\\ \textit{up to dominated equivalence}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\textit{right indecomposable,}\\ \textit{split Frobenius objects}\\ \textit{in } \mathfrak{A}(a,a)/\mathcal{I}'(a)\\ \textit{up to isomorphism}\end{array}\right\}$$

Proof. Most of the work has already been done, so let us however summarize the ingredients. The maps from the left to the right are given by the following principle: Given 1-arrows $R' \xrightarrow{G} R \xrightarrow{F} R'$ and 2-arrows $id_{R'} \overset{\alpha}{\Longrightarrow} FG \overset{\beta}{\Longrightarrow} id_{R'}$ with $\beta \circ \alpha = id_{R'}$, then $(GF, G\alpha F, G\beta F)$ is a split Frobenius object on R, see Proposition 4.3.20. On the other hand, the map from the right to the left is given by the construction of Proposition 4.3.58 (which was presented only within $\mathcal{B} - \mathrm{Mod}$, but readily generalises to $\mathfrak{A} - \mathrm{Mod}$). Lemma 4.3.59 stated that dominance on the left corresponds to indecomposability on the right. For the last arrow above, we have to transport the indecomposable split Frobenius object along the equivalence $\mathcal{E}nd_{\mathfrak{A}}(\mathcal{P}_a^{\mathcal{I}}) \cong \mathfrak{A}^{\mathrm{rev}}(a,a)/\mathcal{I}'(a)$ which was discussed in Remark 4.3.21. Thus, we obtain a right indecomposable split Frobenius object in $\mathfrak{A}(a,a)/\mathcal{I}'$.

It remains to show that we have constructed bijections after taking equivalence classes by dominated equivalence and by isomorphism respectively.

First, assume that (K, L) is a dominated equivalence as in Definition 4.5.2. Denote by τ_F (resp. τ_G) the iso-$\mathfrak{A}$-transformation $KF_1 \overset{\sim}{\Longrightarrow} F_2$ (resp. $G_1 \overset{\sim}{\Longrightarrow} G_2K$). We claim that $\tau := (G_2\tau_F) \circ (\tau_G F_1)$ defines an isomorphism of the associated Frobenius objects $(G_1F_1, G_1\alpha_1F_1, G_1\beta_1F_1)$ and $(G_2F_2, G_2\alpha_2F_2, G_2\beta_2F_2)$. This claim holds, since

$$\begin{aligned}(\tau\tau) \circ (G_1\alpha_1F_1) &= (G_2\tau_F G_2\tau_F) \circ (\tau_G F_1 \tau_G F_1) \circ (G_1\alpha_1F_1) && \text{(Remark 3.1.3)}\\ &= (G_2\tau_F\tau) \circ (G_2K\alpha_1F_1) \circ (\tau_G F_1) && \text{(Definition 4.5.2)}\\ &= (G_2\alpha_2F_2) \circ \tau\end{aligned}$$

and analogously $\tau \circ (G_1\beta_1F_1) = (G_2\beta_2F_2) \circ (\tau\tau)$.
Second, assume conversely that we are given $\mathfrak{A}$-modules R, R_1 and R_2, and $\mathfrak{A}$-functors $R_i \xrightarrow{G_i} R \xrightarrow{F_i} R_i$ for $i \in \{1,2\}$ such that $id_{R_i} \subset^{\oplus} F_iG_i$, and assume that there exists an isomorphism $\tau : (G_1F_1, G_1\alpha_1F_1, G_1\beta_1F_1) \to (G_2F_2, G_2\alpha_2F_2, G_2\beta_2F_2)$. In other words, the $\mathfrak{A}$-functor id_R is compatible with the Frobenius objects via τ and so is the identity $\mathfrak{A}$-transformation of id_R. Thus, we can apply Proposition 4.3.48 to these two identity arrows (using the stated functoriality properties) and obtain an equivalence (K, L) between R_1 and R_2. Of course, this requires an application of the 2-Yoneda Lemma 3.3.49. Furthermore, Remark 4.3.49 states that the thus obtained equivalence is actually a dominated equivalence. $\square$

Remark 4.5.5. This classification of functorially cyclic modules is somewhat unsatisfactory. The reason is that the notion of dominated equivalence depends on the choice of the structure functors, compare the discussion in Section 4.2.5. Note also the related Example 5.3.8, which discusses an explicit module that can be equipped with two different sets of structure functors which do not allow for a dominated equivalence. We address this issue for free functorially cyclic modules in Section 4.5.3.

Let us align the notation of Theorem 4.5.4 further with the notation used in the introduction. Let $a \in \mathfrak{A}$, let $\mathcal{I} \trianglelefteq \mathcal{P}_a$ be an ideal and let $\hat{\Theta}$ be a split Frobenius object on the $\mathfrak{A}$-quotient $\mathcal{P}_a^{\mathcal{I}} = \mathcal{P}_a/\mathcal{I}$ of the principal representation $\mathcal{P}_a$. By Proposition 3.3.56, this is the same as an object in $\mathfrak{A}(a,a)$ that stabilises $\mathcal{I}$ (when multiplied on the right!) together with structure maps $\Delta \in \mathfrak{A}(a, aa)$ and $\mu \in \mathfrak{A}(aa, a)$ (that also stabilise $\mathcal{I}$) such that the relations required from a split Frobenius object are satisfied in $\mathcal{P}_a^{\mathcal{I}}(a)$. Note that $\mathcal{P}_a^{\mathcal{I}}(a)$ is in general not a monoidal category, however its full subcategory with objects that come from $\mathcal{P}_a(a)$ and that stabilise $\mathcal{I}$ when multiplied on the right is monoidal, and thus we can speak of the Frobenius relation. Since $\mathcal{P}_a^{\mathcal{I}}$ represents the 2-functor $\Phi_a^{\mathcal{I}} : \mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$, this induces a Frobenius object $\hat{\Theta}$ on $\Phi_a^{\mathcal{I}}$. We obtain a 2-functor ${}_{\hat{\Theta}}\Phi_a^{\mathcal{I}} : \mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$ from Theorem 4.3.42. Subject to an explicit construction of the $\mathfrak{A}$-module ${}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$, which can be obtained through a straightforward generalisation of Proposition 4.3.58, Theorem 4.3.53 now implies Theorem 2 from the introduction:

Theorem 4.5.6. *The 2-functor ${}_{\hat{\Theta}}\Phi_a^{\mathcal{I}} : \mathfrak{A} - \mathrm{Mod}^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$ is represented by ${}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$.*

The following example categorifies Example 2.5.9 and is inspired by [MaMi5, Section 3.4], and continues Examples 4.3.5 and 4.3.18.

Example 4.5.7. Let $\mathcal{A} := k-\mathrm{Mod}$, and let $\mathcal{B}_2$ be the smallest k-linear additive subcategory of $\mathfrak{A}dd(\mathcal{A},\mathcal{A})$ with objects $I := \begin{pmatrix} id_{\mathcal{A}} & 0 \\ 0 & id_{\mathcal{A}} \end{pmatrix}$ and $F := \begin{pmatrix} id_{\mathcal{A}} & id_{\mathcal{A}} \\ id_{\mathcal{A}} & id_{\mathcal{A}} \end{pmatrix}$. Note that $\mathcal{B}_2$ is very similar to the 2-category defined in [MaMi5, Section 3.4] by a generator F and the relation $FF \cong F \oplus F$. Note however that our definition implies that $\mathcal{B}_2$ is symmetric monoidal and furthermore that F is a split Frobenius object. In an obvious way, we have $\mathcal{A} \oplus \mathcal{A} \in \mathcal{B}_2-\mathrm{Mod}$. Denote by $\overline{\mathcal{B}}_2$ the essential full image of $\mathcal{B}_2$ in $\mathfrak{A}dd(\mathcal{A}\oplus\mathcal{A},\mathcal{A}\oplus\mathcal{A})$, and denote by $\pi : \mathcal{B}_2 \to \overline{\mathcal{B}}_2$ the inclusion functor. Note that π is a quotient-$\mathcal{B}_2$-functor. Evaluation at $(k,0)$ defines a $\mathcal{B}_2$-functor $\overline{\mathcal{B}}_2 \to \mathcal{A}\oplus\mathcal{A}$, which is clearly indecomposable. Further, mapping both $(V,0)$ and $(0,V)$ to $\begin{pmatrix} -\otimes_k V & -\otimes_k V \\ -\otimes_k V & -\otimes_k V \end{pmatrix}$ defines a $\mathcal{B}_2$-functor $\mathcal{A}\oplus\mathcal{A} \to \overline{\mathcal{B}}_2$. The composition $\mathcal{A}\oplus\mathcal{A} \to \overline{\mathcal{B}}_2 \to \mathcal{A}\oplus\mathcal{A}$ is isomorphic to $id \oplus S$ as a $\mathcal{B}_2$-functor, where $S := \begin{pmatrix} 0 & id_{\mathcal{A}} \\ id_{\mathcal{A}} & 0 \end{pmatrix}$. Note that the idempotents associated to this decomposition do not lie in $\mathcal{B}_2$. Thus, $\mathcal{A}\oplus\mathcal{A}$ is functorially cyclic. The underlying object of the associated right indecomposable split Frobenius in $\overline{\mathcal{B}}_2$ is isomorphic to F.
Aside of $\mathcal{B}_2$ and $\mathcal{B}_2/F$, $\mathcal{B}_2$ has another functorially cyclic module, namely the sub-$\mathcal{B}_2$-module of $\mathcal{B}_2$ generated by F. Let us denote this sub-$\mathcal{B}_2$-module by $\mathcal{C}_F$. Evaluation at F gives an indecomposable $\mathcal{B}_2$-functor $\mathcal{B}_2 \to \mathcal{C}_F$, and we have an inclusion $\mathcal{B}_2$-functor $\mathcal{C}_F \to \mathcal{B}_2$. Since $FF \cong F \oplus F$, the composition $\mathcal{C}_F \to \mathcal{B}_2 \to \mathcal{C}_F$ is isomorphic to $id_{\mathcal{C}_F} \oplus id_{\mathcal{C}_F}$. Thus, $\mathcal{C}_F$ is functorially cyclic. The object underlying the associated split Frobenius object is again given by F.
Since $\mathcal{A}\oplus\mathcal{A}$ and $\mathcal{C}_F$ are clearly not equivalent, the two constructed split Frobenius objects cannot be equivalent. Considering the top left matrix entry only gives rise to $\hat{k}_1$ and $\hat{k}_2$ of Example 4.3.5 and we have thus shown that they are not equivalent as claimed in Remark 4.3.6.
Further, we can equip $\mathcal{A}$ with a $\mathcal{B}_2$ action by letting F act as $id_{\mathcal{A}} \oplus id_{\mathcal{A}}$. Note that $\mathcal{A}$ is equivalent to $\mathcal{C}_F$. One can show that the direct sum (resp. the diagonal map) defines a $\mathcal{B}_2$-functor $\mathcal{A}\oplus\mathcal{A} \to \mathcal{A}$ (resp. $\mathcal{A} \to \mathcal{A}\oplus\mathcal{A}$), which can be used to show that $\mathcal{A}$ and $\mathcal{A}\oplus\mathcal{A}$ are weakly equivalent. We will discuss an abstract reason for this in Example 4.6.7. It is important to note that the diagonal functor $\mathcal{A} \to \mathcal{A}\oplus\mathcal{A}$ is not decomposable as a $\mathcal{B}_2$-functor. Thus, we can equip $\mathcal{A}\oplus\mathcal{A}$ with another structure as a functorially cyclic module using the composition $\mathcal{B}_2 \to \mathcal{C}_F \to \mathcal{A}\oplus\mathcal{A}$ as the indecomposable structure $\mathcal{B}_2$-functor. The object underlying the associated split Frobenius object is given by FF, and by considering the top left matrix entry only one obtains $\hat{k}_{S_2}$ of Example 4.3.5. At the same time, we have shown that $\hat{k}_{S_2}$ has to

be equivalent to either $\hat{k}_2$ or $\hat{k}_2$, whichever arises from the structure of $\mathcal{A} \oplus \mathcal{A}$ as a functorially cyclic module discussed above.

4.5.2 Free functorially cyclic modules

In this section, we make connection between right indecomposable split Frobenius objects in $\mathcal{B}$ and free functorially cyclic modules more explicit, see Corollary 4.5.9. This is largely based on Theorem 4.3.53, however this time we keep track of the additional indecomposability assumptions. Second, we show in Theorem 4.5.10 that the 1-2-full sub-2-category of $\mathcal{B} - \mathrm{Mod}$ with objects free functorially cyclic modules is equivalent to $\mathfrak{Frob}(\mathcal{B})$.

In order to state Corollary 4.5.9, we associate certain objects to a right indecomposable split Frobenius object $\hat{\Theta}$ in $\mathcal{B}$. First, we construct a split Frobenius object (θ, δ, M) on the forgetful 2-functor $\mathrm{For} : \mathcal{B} - \mathrm{Mod}^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$. We define θ by setting $\theta\left((\mathcal{M}, \rho)\right) := \rho(\Theta)$ for all $\mathcal{M} = (\mathcal{M}, \rho) \in \mathcal{B} - \mathrm{Mod}^{\mathrm{idem}}$ and specifying $\mathrm{can}_{\theta,F} : F\theta \to \theta F$ for all $F = (F, \{\mathrm{can}_{F,B}\}_{B\in\mathcal{B}}) \in \mathcal{B} - \mathrm{Mod}^{\mathrm{idem}}\left((\mathcal{M}_1, \rho_1), (\mathcal{M}_2, \rho_2)\right)$ as $\mathrm{can}_{F,\Theta}^{-1}$. Now, θ is a well defined endomorphism of For due to the properties of the morphism of 2-functors $(F, \{\mathrm{can}_{F,B}\}_{B\in\mathcal{B}})$. Further, we define $\delta : \theta\theta \to \theta$ by setting $\delta\left((\mathcal{M}\rho)\right) := \left(\mathrm{can}^{\rho}_{\Theta,\Theta}\right)^{-1} \circ \rho(\Delta)$ where $\mathrm{can}^{\rho}_{B_1,B_2} : \rho(B_2)\rho(B_1) \Rightarrow \rho(B_1B_2)$ is the canonical transformation that comes with the monoidal functor ρ for all $B_1, B_2 \in \mathcal{B}$. Now, δ is well defined due to naturality of $\{\mathrm{can}_{F,B}\}_{B\in\mathcal{B}}$ and the properties of the morphism of 2-functors F. Similarly, we define $M : \theta\theta \to \theta$ using μ instead of Δ. It is straight forward to check that (θ, δ, M) is a split Frobenius object on For. Second, Theorem 4.3.42 yields a 2-functor ${}_{(\theta,\delta,M)}\mathrm{For} : \mathcal{B} - \mathrm{Mod}^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$. To lighten the notational burden, we will abuse notation by using the notation ${}_{\hat{\Theta}}(-) := {}_{(\theta,\delta,M)}\mathrm{For}$.

Remark 4.5.8. More conceptually, one obtains (θ, δ, M) from $\hat{\Theta}$ by applying Remark 4.3.55 and Lemma 4.3.56 to the equivalences $\mathfrak{Hom}_{\mathcal{B}-\mathrm{Mod}^{\mathrm{idem}}\to\mathfrak{A}dd^{\mathrm{idem}}}(\mathrm{For}, \mathrm{For}) \cong \mathfrak{Hom}_{\mathcal{B}-\mathrm{Mod}^{\mathrm{idem}}\to\mathfrak{A}dd^{\mathrm{idem}}}(\mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{B}, -), \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{B}, -)) \cong \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{B}, \mathcal{B}) \cong \mathcal{B}^{\mathrm{rev}}$, all of which are consequences of the 2-Yoneda Lemma 3.3.49.

Corollary 4.5.9. *Let $\hat{\Theta}$ be a right indecomposable split Frobenius object in $\mathcal{B}$. Then, the associated 2-functor ${}_{\hat{\Theta}}(-) : \mathcal{B} - \mathrm{Mod}^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$ is represented by an explicitly defined $\mathcal{B}$-module $\mathcal{B}_{\hat{\Theta}} \in \mathcal{B} - \mathrm{Mod}^{\mathrm{idem}}$ which comes together with $\mathcal{B}$-functors $\mathcal{B} \xrightarrow{\vartheta^{\hat{\Theta}}} \mathcal{B}_{\hat{\Theta}} \xrightarrow{\vartheta_{\hat{\Theta}}} \mathcal{B}$ and $\mathcal{B}$-transformations $id_{\mathcal{B}_{\hat{\Theta}}} \xRightarrow{\alpha} \vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}} \xRightarrow{\beta} id_{\mathcal{B}_{\hat{\Theta}}}$ such that $\beta \circ \alpha = id_{\mathcal{B}_{\hat{\Theta}}}$. Further, we have $-\hat{\Theta} = (\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}})$. If $\hat{\Theta}$ is right indecomposable, then $\mathcal{B}_{\hat{\Theta}}$ is a free functorially cyclic module. Furthermore, let $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}^{\mathrm{idem}}$ be a free* *functorially cyclic module with associated right indecomposable split Frobenius object $\hat{\Theta}'$ in $\mathcal{B}$ and assume that $\hat{\Theta}'$ is isomorphic to $\hat{\Theta}$. Then, we have $\mathcal{B}_{\hat{\Theta}} \cong \mathcal{M}$.*

Proof. Using Remark 4.3.19, we obtain an indecomposable split Frobenius object $-\hat{\Theta}$ on $\mathcal{B} \in \mathcal{B}-\mathrm{Mod}$ by transporting the right indecomposable split Frobenius object $\hat{\Theta}$ along the equivalence $\mathcal{B}^{\mathrm{rev}} \to \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{B},\mathcal{B})$ induced from taking tensor products on the right. Proposition 4.3.58 provides us a $\mathcal{B}$-module $\mathcal{B}_{-\hat{\Theta}}$ and all $\mathcal{B}$-functors, $\mathcal{B}$-transformation and their relations as claimed. By abuse of notation, we denote this module by $\mathcal{B}_{\hat{\Theta}} = \mathcal{B}_{-\hat{\Theta}}$ and proceed similarly for associated $\mathcal{B}$-functors. Note that $\vartheta^{\hat{\Theta}}$ is strongly indecomposable since $-\hat{\Theta}$ is indecomposable, see Lemma 4.3.59. Thus, $\mathcal{B}_{\hat{\Theta}}$ is a free functorially cyclic module that comes with all the structure as claimed in the statement.
Next, let $\mathcal{M} \in \mathcal{B}-\mathrm{Mod}$ be a free functorially cyclic module and let $\hat{\Theta}''$ be its the associated indecomposable split Frobenius object on $\mathcal{B}$, see Proposition 4.3.20. We obtain its associated indecomposable split Frobenius object $\hat{\Theta}'$ in $\mathcal{B}$ by evaluating $\hat{\Theta}''$ at the unit object. Since $\mathcal{B}^{\mathrm{rev}} \to \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{B},\mathcal{B})$ is quasi-inverse to evaluation at the unit object and since $\hat{\Theta}'$ is isomorphic to $\hat{\Theta}$ by assumption, $\hat{\Theta}''$ is isomorphic to $-\hat{\Theta}$ as well. The claim that $\mathcal{M} \cong \mathcal{B}_{\hat{\Theta}}$ follows now from Proposition 4.3.57. ☐

We can now generalise Corollary 3.3.53:

Theorem 4.5.10. *The assignment $\hat{\Theta} \mapsto \mathcal{B}_{\hat{\Theta}}$ extends to a 2-functor $\mathfrak{F}rob(\mathcal{B})^{\mathrm{rev}} \to \mathcal{B}-\mathrm{Mod}$ that is a locally an equivalence and that has an essential image given by all non-trivial free functorially cyclic modules.*

Proof. In Lemma 4.5.11, we specify an equivalence ${}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2} \cong {}_{\hat{\Theta}_1}\left(\mathcal{B}_{\hat{\Theta}_2}\right)$. By Corollary 4.5.9, we also have ${}_{\hat{\Theta}_1}\left(\mathcal{B}_{\hat{\Theta}_2}\right) \cong \mathcal{H}\mathrm{om}_{\mathcal{B}}\left(\mathcal{B}_{\hat{\Theta}_1}, \mathcal{B}_{\hat{\Theta}_2}\right)$. We now can define the 2-functor $\mathfrak{F}rob(\mathcal{B})^{\mathrm{rev}} \to \mathcal{B}-\mathrm{Mod}$ using the thus available functors ${}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2} \to \mathcal{H}\mathrm{om}_{\mathcal{B}}\left(\mathcal{B}_{\hat{\Theta}_1}, \mathcal{B}_{\hat{\Theta}_2}\right)$. Note that by convention $F \otimes G = G \circ F$ for $\mathcal{B}$-functors which results in the need for taking the reverse category one one side of the equivalence. Since for both 2-categories the associator is trivial, we ought to specify the corresponding 2-arrows using identity morphisms as well. We leave it to check that the unit transformations of $\mathfrak{F}rob(\mathcal{B})$ define the necessary 2-arrows for the compatibility with unit objects. It should be obvious to the reader that we have made Definition 4.3.63 exactly so that this definition of the 2-functor $\mathfrak{F}rob(\mathcal{B})^{\mathrm{rev}} \to \mathcal{B}-\mathrm{Mod}$ is well defined. Since we have build this 2-functor from equivalences, it is clearly a local equivalence. By Corollary 4.5.9, the image of this 2-functor on objects is contained in the set of all functorially cyclic modules (here we use that the objects of $\mathfrak{F}rob(\mathcal{B})$ are right indecomposable

Frobenius objects). Together with Proposition 4.3.20, Corollary 4.5.9 also shows that every non-trivial free functorially cyclic modules lies in the essential image of the 2-functor. An essential step in checking all the details is provided in Lemma 4.5.12 below that sufficiently unravels all the constructions involved. □

Lemma 4.5.11. *The assignment* $(B, e) \mapsto ((B, id_{B\Theta_2}), e)$ *defines an equivalence of additive categories* ${}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2} \cong {}_{\hat{\Theta}_1}\left(\mathcal{B}_{\hat{\Theta}_2}\right)$.

Proof. We leave it to the reader to specify the functor on morphisms and to check that it indeed an additive functor. The assignment $((B, e_1), e_2) \mapsto (B, e_2)$ specifies a quasi-inverse. □

Lemma 4.5.12. *Let* $\hat{\Theta} = (\Theta, \Delta, \mu)$ *be a split Frobenius object in* $\mathcal{B}$ *and let* $\mathcal{N} \in \mathcal{B} - \mathrm{Mod}$. *The functor that establish the equivalence* ${}_{\hat{\Theta}}\mathcal{N} \cong \mathcal{H}om_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}}, \mathcal{N})$ *maps* $(M, f) \in {}_{\hat{\Theta}}\mathcal{N}$ *to the* $\mathcal{B}$*-functor given on objects by* $(B, e) \mapsto (B\Theta M, (eM) \circ (Bf))$.

Proof. This is a further concretion of the toolbox we developed previously. Chasing through the proofs (in particular Proposition 4.3.48 with $F : \mathrm{For} \to \mathcal{H}om_{\mathcal{B}}(\mathcal{B}, -)$ the morphism of 2-functors given by the 2-Yoneda Lemma 3.3.49), leads us to $\hat{F}$ for which $\hat{F}_{\mathcal{N}}$ is given by the composition ${}_{\hat{\Theta}}\mathcal{N} \to \mathcal{N} \xrightarrow{F_{\mathcal{N}}} \mathcal{H}om_{\mathcal{B}}(\mathcal{B}, \mathcal{M}) \to \mathcal{H}om_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}}, \mathcal{M})$ which maps $(M, f) \mapsto (\Theta M, f) \mapsto (B \mapsto B.(\Theta M, f)) \mapsto ((B, e) \mapsto (B\Theta, e).(\Theta M, f))$. To obtain the desired functor, we still have to apply an idempotent e_F to $\hat{F}_{\mathcal{N}}$, see Proposition 4.3.48. Evaluated at $(M, f) \in {}_{\hat{\Theta}}\mathcal{N}$, the idempotent e_F is the idempotent of $(B\Theta, e).(\Theta M, f)$ given by $(e_F)_{(M,f)} = (B\Theta\mu M) \circ (B\Delta\Theta M)$, where we used that $(B\Theta\Theta, e\Theta).(\Theta M, f) = (B\Theta, e).(\Theta\Theta M, \Theta f)$. Now using $\mu \circ \Delta = id_\Theta$, it is easy to see that the summand of $(B\Theta, e).(\Theta M, f) = (B\Theta\Theta M, ef)$ associated to the idempotent $(e_F)_{(M,f)}$ is isomorphic to $(B\Theta M, (eM) \circ (Bf))$, thus establishing the proposition. □

4.5.3 Equivalences and weak equivalences

In this section, we discuss the representation theoretic meaning of the preorders on split Frobenius objects introduced in Section 4.3.6. Theorem 4.5.14 states that free functorially cyclic modules are determined up to weak equivalence by equivalence classes of split Frobenius objects under the Frobenius preorder. Furthermore, this theorem also states that strong equivalence between Frobenius objects corresponds to an equivalence between the associated functorially cyclic modules.

Before we turn to Theorem 4.5.14, let us highlight that even strongly equivalent split Frobenius objects can have rather different underlying objects. A simple way to obtain examples is as follows:

Example 4.5.13. Let $\mathcal{B}_{\hat{\Theta}}$ be a free (F, G, α, β)-cyclic $\mathcal{B}$-module with associated Frobenius object $\hat{\Theta} = (GF, G\alpha F, G\beta F)$ on $\mathcal{B}$, and let $G' \in \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}}, \mathcal{B})$ be arbitrary. Clearly, there exist $\mathcal{B}$-transformations $id_{\mathcal{B}_{\hat{\Theta}}} \overset{\alpha'}{\Longrightarrow} F(G \oplus G') \overset{\beta'}{\Longrightarrow} id_{\mathcal{B}_{\hat{\Theta}}}$ with $\beta' \circ \alpha' = id_{\mathcal{B}_{\hat{\Theta}}}$. It follows easily that $\hat{\Theta}' := ((G \oplus G')F, (G \oplus G')\alpha' F, (G \oplus G')\beta' F)$ is a split Frobenius object on $\mathcal{B}$. Proposition 4.3.20 implies that $\hat{\Theta}'$ is indecomposable. Further, Proposition 4.3.57 implies that $\hat{\Theta}' \simeq \hat{\Theta}$. Furthermore for all $\mathcal{B}$-functors $\mathcal{B} \xrightarrow{\widetilde{F}} \mathcal{B}_{\hat{\Theta}} \xrightarrow{\widetilde{G}} \mathcal{B}$ where $\widetilde{F}$ is strongly indecomposable and all $\mathcal{B}$-transformations $id_{\mathcal{B}_{\hat{\Theta}}} \overset{\widetilde{\alpha}}{\Longrightarrow} \widetilde{F}\widetilde{G} \overset{\widetilde{\beta}}{\Longrightarrow} id_{\mathcal{B}_{\hat{\Theta}}}$ with $\widetilde{\beta} \circ \widetilde{\alpha} = id_{\mathcal{B}_{\hat{\Theta}}}$, we have $\hat{\Theta} \simeq (\widetilde{G}\widetilde{F}, \widetilde{G}\widetilde{\alpha}\widetilde{F}, \widetilde{G}\widetilde{\beta}\widetilde{F})$.

Theorem 4.5.14. *Let $\hat{\Theta}_1 = (\Theta_1, \Delta_1, \mu_1)$ and $\hat{\Theta}_2 = (\Theta_2, \Delta_2, \mu_2)$ be right indecomposable split Frobenius objects in $\mathcal{B}$. Then, we have $\hat{\Theta}_1 \leq \hat{\Theta}_2$ if and only if $\mathcal{B}_{\hat{\Theta}_2}$ weakly dominates $\mathcal{B}_{\hat{\Theta}_1}$. Furthermore, $\hat{\Theta}_1 \simeq \hat{\Theta}_2$ if and only if $\mathcal{B}_{\hat{\Theta}_1}$ and $\mathcal{B}_{\hat{\Theta}_2}$ are equivalent.*

Proof. The theorem follows by transferring Definition 4.3.70 of $\leq$ and Definition 4.3.72 of $\simeq$ along the equivalence ${}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_2} \cong \mathcal{H}om_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}_1}, \mathcal{B}_{\hat{\Theta}_2})$ using the methods developed in Section 4.3.5. We start by proving that $\hat{\Theta}_1 \simeq \hat{\Theta}_2$ if and only if $\mathcal{B}_{\hat{\Theta}_1} \cong \mathcal{B}_{\hat{\Theta}_2}$. The pair (A, e_A), see Definition 4.3.72, corresponds to a $\mathcal{B}$-functor $F : \mathcal{B}_{\hat{\Theta}_1} \to \mathcal{B}_{\hat{\Theta}_2}$ and the pair (B, e_B) corresponds to a $\mathcal{B}$-functor $G : \mathcal{B}_{\hat{\Theta}_2} \to \mathcal{B}_{\hat{\Theta}_1}$. The compositions GF and FG correspond to the pairs $(A\Theta_2 B, (e_A B\Theta_2) \circ (\Theta_1 A e_B))$ and $(B\Theta_1 A, (e_B A\Theta_1) \circ (\Theta_2 B e_A))$ respectively. The $\mathcal{B}$-functors $id_{\mathcal{B}_{\hat{\Theta}_1}}$ and $id_{\mathcal{B}_{\hat{\Theta}_2}}$ correspond to the pairs $(I, \Delta_1 \circ \mu_1)$ and $(I, \Delta_2 \circ \mu_2)$, where I is the identity object of $\mathcal{B}$. The identity $\mathcal{B}$-transformations on these four $\mathcal{B}$-functors correspond to the respective second entry of the four pairs. Furthermore, the morphisms ι_1, π_1, ι_2 and π_2 corresponds to $\mathcal{B}$-transformations $id_{\mathcal{B}_{\hat{\Theta}_1}} \Rightarrow GF$, $GF \Rightarrow id_{\mathcal{B}_{\hat{\Theta}_1}}$, $id_{\mathcal{B}_{\hat{\Theta}_2}} \Rightarrow FG$ and $FG \Rightarrow id_{\mathcal{B}_{\hat{\Theta}_2}}$. Since composition of $\mathcal{B}$-transformations corresponds to composition of morphisms, the last two bullet points in the theorem are equivalent to $id_{\mathcal{B}_{\hat{\Theta}_1}} \cong GF$ and $id_{\mathcal{B}_{\hat{\Theta}_2}} \cong FG$. The characterisation of $\simeq$ follows.

Turning to the first part of the theorem, assume that $\hat{\Theta}_1 \leq \hat{\Theta}_2$. Using the same notation as above, we have $id_{\mathcal{B}_{\hat{\Theta}_1}} \subset^{\oplus} GF$ and thus existence of ι_1 and π_1 with $\pi_1 \circ \iota_1 = \Delta_1 \circ \mu_1$. Since $\Delta_1 \circ \mu_1$ is the idempotent that factors over Θ_1, we have shown that $\Theta_1 \subset^{\oplus} \Theta_1 A \Theta_2 B \Theta_1$, *i.e.* $\Theta_1 \leq_{LR} \Theta_2$. Conversely, let us assume that $\Theta_1 \leq_{LR} \Theta_2$. Thus, there exist objects $A, B \in \mathcal{B}$ and morphisms $j \in \mathcal{B}(\Theta_1, A\Theta_2 B)$, $\tau \in \mathcal{B}(A\Theta_2 B, \Theta_1)$ such that $\tau \circ j = id_{\Theta_1}$. Now, define $\iota := (\Theta_1 j \Theta_1) \circ (\Theta_1 \Delta_1) \circ \Delta_1 \circ \mu_1$ and $\pi := \Delta_1 \circ \mu_1 \circ (\Theta_1 \mu_1) \circ (\Theta_1 \tau \Theta_1)$. Since $\hat{\Theta}_1$ is split, it follows that $\pi \circ \iota = \mu_1 \circ \Delta_1$. Using the Frobenius relations of $\hat{\Theta}_1$ as well as (co)associativity of μ (and Δ), one obtains that π and ι satisfy two-sided $\hat{\Theta}_1$-descent and thus define morphisms in ${}_{\hat{\Theta}_1}\mathcal{B}_{\hat{\Theta}_1}$. The relation $\Theta_1 \leq \Theta_2$ follows. □

Remark 4.5.15. It remains an open task to classify strong equivalence classes. As a first step, one can strengthen the above theorem towards this direction using the result of Proposition 4.5.45. Note however that the orders $\leq_L$ and $\leq_R$ are not sufficient for the purpose of determining strong equivalence classes, see Remark 5.3.9 for a concrete example.

Remark 4.5.16. Given a right indecomposable split Frobenius object (Θ, Δ, μ) in $\mathcal{B}$ and an object $B \in \mathcal{B}$ with $B \leq_{LR} \Theta \leq_{LR} B$, in general there does not exists a split Frobenius object with underlying object B. See Section 5.5 for a Soergel bimodule example of type B_2 in which B is (strongly) indecomposable.

Remark 4.5.17. The classification of free functorially cyclic modules presented in this section extends naturally to semi-free functorially cyclic modules (and even for larger classes). For this purpose, one has to extend the preorders on split Frobenius objects in $\mathcal{B}$ to corresponding preorders on split Frobenius objects in monoidal quotients of $\mathcal{B}$ (or to split Frobenius objects on $\mathcal{B}$-quotients of $\mathcal{B}$). However, the reader making such generalisations should keep in mind that Frobenius objects on different $\mathcal{B}$-quotients can be associated to the same module.

4.5.4 Semi-free functorially cyclic modules

In this section, we discuss functorially cyclic modules that arise as restrictions of free functorially cyclic modules. The techniques developed in Section 4.4 are used both in the definition of semi-free functorially cyclic modules, as well as in the study of the relationship between such modules (and their associated Frobenius objects) that arise from different quotients. Further, we obtain a clean description of the centre of semi-free functorially cyclic modules.

For notational convenience, we make the following definition.

Definition 4.5.18. *Let $\mathcal{B}$ be an additive monoidal category. A* cyclic object for $\mathcal{B}$ *is a pair $\hat{\Theta} = (\hat{\Theta}, \pi)$, where $\pi : \mathcal{B} \to \overline{\mathcal{B}}$ is a strict additive monoidal quotient functor, $\overline{\mathcal{B}}$ is idempotent closed and $\hat{\Theta}$ is a right indecomposable split Frobenius object in $\overline{\mathcal{B}}$. We say that $\hat{\Theta}$ is a* cyclic object for $\mathcal{B}$ that lives in $\overline{\mathcal{B}}$.

By Corollary 4.5.9, we know how to associate a free functorially cyclic module $\overline{\mathcal{B}}_{\hat{\Theta}} \in \overline{\mathcal{B}} - \mathrm{Mod}$ to a right indecomposable split Frobenius object in an additive monoidal category $\overline{\mathcal{B}}$. This is the basis of the following definition:

Definition 4.5.19. *Let $\hat{\Theta} = (\hat{\Theta}, \pi)$ be a cyclic object for $\mathcal{B}$ that lives in $\overline{\mathcal{B}}$. We call $\mathcal{C}_{\hat{\Theta}} := \operatorname{Res}_\pi \left(\overline{\mathcal{B}}_{\hat{\Theta}}\right)$ the* functorially cyclic module associated to the cyclic object $\hat{\Theta}$.

Proof. Since π is a monoidal quotient functor, we have an equivalence $\operatorname{Ind}_\pi(\mathcal{B}) \cong \overline{\mathcal{B}}$ in $\overline{\mathcal{B}} - \operatorname{Mod}$. In Proposition 4.4.10, we discussed that $\mathcal{B} \to \operatorname{Res}_\pi \operatorname{Ind}_\pi(\mathcal{B})$ is a quotient $\mathcal{B}$-functor. Thus, we have quotient $\mathcal{B}$-functor $\mathcal{B} \to \operatorname{Res}_\pi(\overline{\mathcal{B}})$. Applying the 2-functor Res_π to the $\overline{\mathcal{B}}$-functors and $\overline{\mathcal{B}}$-transformations that come with the free functorially cyclic module $\overline{\mathcal{B}}_{\hat{\Theta}}$ yields the required structure that turns $\operatorname{Res}_\pi \left(\overline{\mathcal{B}}_{\hat{\Theta}}\right)$ into a functorially cyclic $\mathcal{B}$-module. It is important to note that Ind_π splits Res_π. Therefore, Res_π is locally faithful and strong indecomposability is thus preserved under Res_π. ☐

Remark 4.5.20. By Definition 4.2.23, $\mathcal{C}_{\hat{\Theta}}$ is semi-free if $\mathfrak{J}(\pi) \trianglelefteq \mathcal{B}$ is thick.

In preparation for Section 4.5.5, we also make the following definition:

Definition 4.5.21. A cell object for $\mathcal{B}$ *is a cyclic object $(\hat{\Theta}, \pi : \mathcal{B} \to \overline{\mathcal{B}})$ for $\mathcal{B}$ such that the associated functorially cyclic module $\mathcal{C}_{\hat{\Theta}}$ is a cell module.*

Remark 4.5.22. If a cyclic object actually lives in a lowest left cell of $\overline{\mathcal{B}}$ with respect to $\leq_{LR}$, then it is a cell object. In general, the object underlying the Frobenius object of a cell object can be a direct sum of indecomposable objects that are associated to different left cells.

Proposition 4.5.23. *Let $\hat{\Theta} = (\hat{\Theta}, \pi)$ be a cyclic object for $\mathcal{B}$. Then, $\mathcal{C}_{\hat{\Theta}}$ represents the 2-functor ${}_{\hat{\Theta}}(\operatorname{Coind}_\pi)$.*

Proof. Definition/Proposition 4.4.5 and Corollary 4.5.9 say it all. ☐

Remark 4.5.24. Let $(\hat{\Theta}, \pi : \mathcal{B} \to \overline{\mathcal{B}})$ be a cyclic object for $\mathcal{B}$. Denote by (Θ, Δ, μ) the indecomposable split Frobenius object on $\overline{\mathcal{B}} \in \overline{\mathcal{B}} - \operatorname{Mod}$ obtained from $\hat{\Theta}$ via transport along the equivalence $\overline{\mathcal{B}}^{\mathrm{rev}} \cong \mathcal{H}om_{\overline{\mathcal{B}}}(\overline{\mathcal{B}}, \overline{\mathcal{B}})$, see Lemma 4.3.56 for the construction. Since Res_π is a strict 2-functor, we obtain an indecomposable split Frobenius object $(\operatorname{Res}_\pi(\Theta), \operatorname{Res}_\pi(\Delta), \operatorname{Res}_\pi(\mu))$ on $\operatorname{Res}_\pi(\overline{\mathcal{B}}) \in \mathcal{B} - \operatorname{Mod}$. This is precisely the split Frobenius object that we typically would associated to $\mathcal{C}_{\hat{\Theta}}$ following Proposition 4.3.20.

Let us discuss the relation between Frobenius objects on different quotients:

Proposition 4.5.25. *Let $\hat{\Theta} = (\hat{\Theta}, \pi : \mathcal{B} \to \overline{\mathcal{B}})$ be a cyclic object for $\mathcal{B}$, and let $\psi : \overline{\mathcal{B}} \to \widetilde{\mathcal{B}}$ be a monoidal quotient functor. Assume that $\psi(\hat{\Theta}) = (\psi(\hat{\Theta}), \psi\pi)$ is a cyclic object for $\mathcal{B}$. Then, we have equivalences $\mathcal{C}_{\psi(\hat{\Theta})} \cong \operatorname{Res}_\pi \left(\mathcal{C}_{(\psi(\hat{\Theta}), \psi)}\right)$ in $\mathcal{B} - \operatorname{Mod}$ and $\operatorname{Ind}_{\psi\pi}(\mathcal{C}_{\hat{\Theta}}) \cong \mathcal{C}_{(\psi(\hat{\Theta}), id_{\widetilde{\mathcal{B}}})}$ in $\widetilde{\mathcal{B}} - \operatorname{Mod}$. Furthermore, we have a canonical quotient $\mathcal{B}$-functor $\mathcal{C}_{\hat{\Theta}} \to \mathcal{C}_{\psi(\hat{\Theta})}$.*

Proof. The first claim follows directly from the fact that $\operatorname{Res}_{\psi\pi} = \operatorname{Res}_{\pi}\operatorname{Res}_{\psi}$. By standard adjunction arguments, this fact also implies an equivalence $\operatorname{Ind}_{\psi\pi} \cong \operatorname{Ind}_{\psi}\operatorname{Ind}_{\pi}$ of 2-functors. Recall also that $\operatorname{Ind}_{\psi}\operatorname{Res}_{\psi} \cong id_{\widetilde{\mathcal{B}}-\mathrm{Mod}}$. Thus, $\operatorname{Ind}_{\psi\pi}(\mathcal{C}_{\hat{\Theta}}) \cong \operatorname{Ind}_{\psi}(\mathcal{C}_{(\hat{\Theta},id_{\overline{\mathcal{B}}})})$. By Definition/Proposition 4.4.8 and Corollary 4.5.9, the latter module represents the 2-functor ${}_{\hat{\Theta}}\operatorname{Res}_{\psi} : \widetilde{\mathcal{B}} - \mathrm{Mod}^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$. Note however that $\operatorname{Res}_{\psi}$ does not change the underlying additive category, it only makes $\hat{\Theta}$ act on this underlying additive category by $\pi(\hat{\Theta})$. Thus, ${}_{\hat{\Theta}}\operatorname{Res}_{\psi} = {}_{\psi(\hat{\Theta})}(-)$ as 2-functors. The latter is clearly represented by $\left(\overline{\mathcal{B}}_1\right)_{\psi(\hat{\Theta})} = \mathcal{C}_{(\psi(\hat{\Theta}),id_{\widetilde{\mathcal{B}}})}$, see Corollary 4.5.9. By the 2-Yoneda Lemma 3.3.49, $\operatorname{Ind}_{\psi\pi}(\mathcal{C}_{\hat{\Theta}}) \cong \mathcal{C}_{(\psi(\hat{\Theta}),id_{\widetilde{\mathcal{B}}})}$ follows. Furthermore, the B-functor $\mathcal{C}_{\hat{\Theta}} \to \operatorname{Res}_{\psi\pi}\operatorname{Ind}_{\psi\pi}(\mathcal{C}_{\hat{\Theta}})$ coming from the adjunction is full and essentially surjective, see Definition/Proposition 4.4.8. The last claim thus follows using the already proven equivalence and $\operatorname{Res}_{\psi\pi}\left(\mathcal{C}_{(\psi(\hat{\Theta}),id_{\widetilde{\mathcal{B}}})}\right) = \mathcal{C}_{\psi(\hat{\Theta})}$. ☐

For future reference, we state the following proposition. Note that it holds subject to existence of $\operatorname{Ind}_{\phi}$ only. Aside this detail, it provides backing for the statement made in the introduction that right singular Soergel bimodules are induced from the trivial representation. Also note that we do not assume indecomposability of the Frobenius object nor that its image under ϕ is non-trivial. In the latter case, $\operatorname{Ind}_{\phi}(\mathcal{B}_{\hat{\Theta}}) = 0$.

Proposition 4.5.26. *Let $\phi : \mathcal{B} \to \mathcal{B}_1$ be an additive strict monoidal functor between idempotent closed categories and let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a split Frobenius object in $\mathcal{B}$. Then, we have an equivalence $\operatorname{Ind}_{\phi}(\mathcal{B}_{\hat{\Theta}}) \cong (\mathcal{B}_1)_{\phi(\hat{\Theta})}$ in $\mathcal{B}_1 - \mathrm{Mod}$.*

Proof. The only ingredient that we have not yet mentioned is that $\operatorname{Res}_{\phi}$ preserves idempotent closed modules. The statement follows by adjunction and uniqueness of representing objects of 2-functors. ☐

Homomorphism categories between functorially cyclic modules associated to cyclic objects are very explicit:

Proposition 4.5.27. *Let $\hat{\Theta}_1 = (\hat{\Theta}_1, \pi_1)$ and $\hat{\Theta}_2 = (\hat{\Theta}_2, \pi_2 : \mathcal{B} \to \overline{\mathcal{B}})$ be cyclic objects for $\mathcal{B}$. Assume that π_2 factors over π_1, i.e. there exists π such that $\pi_2 = \pi\pi_1$. Then, we have an equivalence $\mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{C}_{\hat{\Theta}_1}, \mathcal{C}_{\hat{\Theta}_2}) \cong {}_{\pi(\hat{\Theta}_1)}\overline{\mathcal{B}}_{\hat{\Theta}_2}$ of additive categories.*

Proof. This follows as above together with Lemma 4.5.11 (or the Remark 4.3.68) in the final step. ☐

Corollary 4.5.28. *Let $\hat{\Theta} = (\hat{\Theta}, \pi : \mathcal{B} \to \overline{\mathcal{B}})$ be a cyclic object for $\mathcal{B}$. Then,*

$$\mathcal{Z}_{\mathcal{B}}(\mathcal{C}_{\hat{\Theta}}) \cong \left\{ b \in \overline{\mathcal{B}}(\Theta, \Theta) \,\middle|\, b \text{ satisfies two-sided } \hat{\Theta}\text{-descent} \right\}.$$

Proof. Let (Θ, Δ, μ) denote the underlying split Frobenius object of $\hat{\Theta}$. By Definition 4.1.1, $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}_{\hat{\Theta}})$ is given by the transformations of the identity $\mathcal{B}$-functor in $\mathcal{E}nd_{\mathcal{B}}(\mathcal{C}_{\hat{\Theta}})$. By Proposition 4.5.27, $\mathcal{E}nd_{\mathcal{B}}(\mathcal{C}_{\hat{\Theta}}) \cong {}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}$. Thus, it remains to calculate the endomorphisms of the identity object $I_{\hat{\Theta}} = (I, \Delta \circ \mu)$ of ${}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}$. By definition, we have

$$ {}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}\left(I_{\hat{\Theta}}, I_{\hat{\Theta}}\right) = \left\{ f \in \overline{\mathcal{B}}(\Theta\Theta, \Theta\Theta) \;\middle|\; \begin{array}{l} f = \mu \circ \Delta \circ f = f \circ \mu \circ \Delta \\ \quad = (\Theta\mu) \circ (f\Theta) \circ (\Theta\Delta) \\ \quad = (\mu\Theta) \circ (\Theta f) \circ (\Delta\Theta) \end{array} \right\}. $$

The assignments $f \mapsto \mu \circ f \circ \Delta$ and $b \mapsto \mu \circ b \circ \Delta$ are easily shown to be mutually inverse isomorphism that establish the claim. □

Remark 4.5.29. The identity functor of a functorially cyclic module that is restricted from a free module is strongly indecomposable. This follows from Corollary 4.5.28, since any non-trivial idempotent of the identity functor of a functorially cyclic module corresponds to an idempotent of Θ that satisfies two-sided $\hat{\Theta}$-decent, and we know that $\hat{\Theta}$ is right indecomposable (which is a stronger property).

4.5.5 Universal cell modules

In this section, we introduce universal cell modules and strongly universal cell modules in the context of additive higher representation theory. In Propositions 4.5.41 and 4.5.42, we will show that the functorially cyclic cell module $\mathcal{C}_{\hat{\Theta}}$ associated to a cell object $\hat{\Theta}$ as introduced in Section 4.5.4 is strongly universal provided a further combinatorial condition. Then for weakly regular additive 2-categories, Theorem 4.5.34 shows that existence of such a cell object is equivalent to the existence of strongly universal cell modules. Further, Proposition 4.5.37 shows some independence of functorially cyclic cell modules from the quotient they live on, which we already observed classically. Furthermore, Proposition **??** establishes that a weak equivalence which is a retract is an equivalence.

Let us start by generalising Definitions 2.5.1 and 2.5.2.

Definition 4.5.30. *Let $\mathfrak{A}$ be an additive 2-category and let $\mathcal{C} \in \mathfrak{A} - \mathrm{Mod}$ be a cell module. Then, we call $\mathcal{C}$ a* universal cell module *if $\mathcal{H}om_{\mathfrak{A}}(\mathcal{C}, \mathcal{C}') \neq 0$ for all cell modules $\mathcal{C}' \in \mathfrak{A} - \mathrm{Mod}$ for which the maximal thick subideals of $\mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(\mathcal{C})$ and $\mathfrak{A}\mathrm{nn}_{\mathfrak{A}}(\mathcal{C}')$ agree. Further, we call a universal cell module $\mathcal{C}$* strongly universal cell *if $\mathcal{E}nd_{\mathfrak{A}}(\mathcal{C})$ has a single left cell.*

Definition 4.5.31. *Let $\mathfrak{A}$ be an additive 2-category and let $\underline{c} \subset W(\mathfrak{A})$ be a 2-sided cell. We say that $\mathfrak{A}$* has $\underline{c}$-cell modules, *if there exists a cell module $\mathcal{C} \in \mathfrak{A} - \mathrm{Mod}$ such that the maximal thick subideal of $\mathfrak{A}\mathrm{nn}_A(C)$ is generated by $\{B_x\}_{x \not\geq_{LR} \underline{c}}$. In this situation, we also say that $\mathcal{C}$* belongs to $\underline{c}$.

Remark 4.5.32. From the classical perspective, it may be surprising that maximal thick subideals appear in both Definitions 4.5.30 and 4.5.30. A first justification for this will be given in Proposition 4.5.35. Furthermore, we will discuss other ways to state Definition 4.5.30 in Section 4.9.1, which may be regarded as more canonical.

Let us restate Theorem 3 of the introduction:

Theorem 4.5.33. *Let $\mathfrak{A}$ be a weakly regular additive 2-category and let $\underline{c}$ be a 2-sided cell of $\mathfrak{A}$. Then, there exists an idempotent closed, strongly universal cell module that belongs to $\underline{c}$ if and only if there exist $a \in \mathfrak{A}$ and a right indecomposable, split Frobenius object $\hat{\Theta}$ in $\mathfrak{A}/\mathfrak{J}_{\not\geq_{LR}\underline{c}}(a,a)$ whose underlying object lies in a left cell minimal with respect to $\leq_L$ and such that ${}_{\hat{\Theta}}\mathcal{P}_a^{\mathcal{I}}$ is a cell module, where $\mathcal{I} := \mathfrak{J}_{\not\geq_{LR}}(a,-)$.*

For notational simplicity, we present the proof only for an additive monoidal category $\mathcal{B}$ instead of a 2-category $\mathfrak{A}$. Let us restate the theorem in this situation using the terminology not available in the introduction.

Theorem 4.5.34. *Let $\mathcal{B}$ be a weakly regular additive 2-category and let $\underline{c}$ be a 2-sided cell of $\mathcal{B}$. Then, there exists an idempotent closed, strongly universal cell module that belongs to $\underline{c}$ if and only if there exist a cell object $\hat{\Theta}$ that lives in $\mathcal{B}/\mathfrak{J}_{\not\geq_{LR}\underline{c}}$ whose underlying object lies in a left cell minimal with respect to $\leq_L$.*

Proof. Let $\mathcal{C}$ be an idempotent closed strongly universal cell module that belongs to $\underline{c}$. Since $\mathcal{B}$ is weakly regular by assumption, there exists a minimal left cell $c \subset \underline{c}$. Let us denote by $\mathcal{C}_c$ the full submodule of $\mathcal{B}/\mathfrak{J}_{\not\geq_{LR}}$ that contains the indecomposable objects associated to c. Note that $\mathcal{C}_c$ is 1-cyclic only and thus does not satisfy a good universal property. Since $\mathfrak{A}\mathrm{nn}_{\mathcal{B}}(\mathcal{C})$ does not contain the indecomposable objects associated to $\underline{c}$ by assumption, there exists $x \in c$ and $C \in \mathcal{C}$ such that $B_x.C \neq 0$. Evaluation at C yields a $\mathcal{B}$-functor $\mathcal{B} \to \mathcal{C}$ which factors over the quotient $\mathcal{B}$-functor $\mathcal{B} \to \mathcal{B}/\mathfrak{J}_{\not\geq_{LR}}$. Thus, we have constructed a non-trivial $\mathcal{B}$-functor $\mathcal{C}_c \to \mathcal{C}$. Since $\mathcal{C}$ is universal, we also have a non-trivial $\mathcal{B}$-functor $\mathcal{C} \to \mathcal{C}_c$. Since $\mathcal{C}_c$ is a cell module, B_x is a summand of an object in the image of $\mathcal{C} \to \mathcal{C}_c$, and it follows that the composition $\mathcal{C} \to \mathcal{C}_c \to \mathcal{C}$ is non-trivial. Since $\mathcal{C}$ is strongly universal, we can find there exists a $\mathcal{B}$-functor $\mathcal{C} \to \mathcal{C}$ such that the composition $\mathcal{C} \to \mathcal{C} \to \mathcal{C}_c \to \mathcal{C}$ contains the identity-$\mathcal{B}$-functor of $\mathcal{C}$ as a summand. Note further that the $\mathcal{B}$-functor $\mathcal{B}/\mathfrak{J}_{\not\geq_{LR}} \to \mathcal{C}$ is strongly indecomposable, since C is strongly indecomposable, where we used that $\mathcal{C}$ is indecomposable. Thus, $\mathcal{C}$ is functorially cyclic. By the fourth bijection in Theorem 4.5.4, $\mathcal{C}$ is thus equivalent to $\mathcal{C}_{\hat{\Theta}}$, where $\hat{\Theta}$ is the associated right indecomposable split Frobenius object in $\mathcal{B}/\mathfrak{J}_{\not\geq_{LR}}$. Since $\mathcal{C}$ is cyclic, $\hat{\Theta}$ is thus a cell object that lives in $\mathcal{B}/\mathfrak{J}_{\not\geq_{LR}}$. It remains to check the properties of the object underlying $\hat{\Theta}$. Again by Theorem 4.5.4, this underlying object is given by the evaluation of $\mathcal{B} \to \mathcal{B}/\mathfrak{J}_{\not\geq_{LR}} \to \mathcal{C} \to \mathcal{C} \to \mathcal{C}_c \to \mathcal{B}/\mathfrak{J}_{\not\geq_{LR}}$ at the unit object $I \in \mathcal{B}$. By construction, this object lies in $\mathcal{C}_c$ and thus is a direct sum of indecomposable objects associated to c.

On the other hand, let $\hat{\Theta}$ be a cell object that lives in $\mathcal{B}/\mathfrak{J}_{\not\geq_{LR}}$, *i.e.* a right indecomposable, split Frobenius object in $\mathcal{B}/\mathfrak{J}_{\not\geq_{LR\underline{c}}}$ such that $\mathcal{C}_{\hat{\Theta}}$ is a cell module. We will show in Proposition 4.5.41 that $\mathcal{C}_{\hat{\Theta}}$ is universal. Finally, Proposition 4.5.42 states that $\mathcal{C}_{\hat{\Theta}}$ is indeed strongly universal and the proof is complete. □

The following proposition justifies the way we generalised Definition 2.5.1:

Proposition 4.5.35. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a non-trivial $\mathcal{B}$-functor between cell modules. Then, $\mathfrak{Ann}_{\mathcal{B}}(\mathcal{C}) \subset \mathfrak{Ann}_{\mathcal{B}}(\mathcal{C}')$ and the maximal thick subideals of the annihilators agree.*

Proof. Since $\mathcal{C}'$ is a cell module, $\operatorname{coker}(F) = 0$ and thus $\mathfrak{Ann}_{\mathcal{B}}(\mathcal{C}) \subset \mathfrak{Ann}_{\mathcal{B}}(\mathcal{C}')$ by Lemma 3.3.35. On the other hand for all $B \in \mathcal{B}$ such that $id_B \in \mathfrak{Ann}_{\mathcal{B}}(\mathcal{C}')$, we have $0 \cong B.F(C) \cong F(B.C)$ for all $C \in \mathcal{C}$. Thus for all $C \in \mathcal{C}$, $B.C$ lies in $\ker(F)$. However, $\mathcal{C}$ is a cell module and thus $\ker(F) = 0$, where we used $F \neq 0$. Thus $B.C = 0$ for all $C \in \mathcal{C}$, *i.e.* $id_B \in \mathfrak{Ann}_{\mathcal{B}}(\mathcal{C})$. □

Remark 4.5.36. Therefore, a universal cell module has the minimal annihilator amongst all cell modules that it is linked to by non-trivial morphisms. We do not expect that the annihilator of a universal cell module is thick in general nor that thickness of the annihilator of a cell module implies universality.

Recall that classically universal cell modules arise as submodules of different quotients of the base ring, see Remark 2.3.15. Even though there is an additional layer of 2-arrows that suggest that we ought to use the largest quotient, an analogous statements holds 2-categorically:

Proposition 4.5.37. *Let $\underline{c}$ be a 2-sided cell of $\mathcal{B}$, let $\mathfrak{J} \trianglelefteq \mathfrak{J}_{\not\geq_{LR\underline{c}}}$ be a subideal that contains $\{id_{B_x}\}_{x \lneq_{LR\underline{c}}}$, let $\hat{\Theta}$ be a cell object for $\mathcal{B}$ that lives in $\mathcal{B}/\mathfrak{J}$, and denote by $\pi : \mathcal{B}/\mathfrak{J} \twoheadrightarrow \mathcal{B}/\mathfrak{J}_{\not\geq_{LR\underline{c}}}$ the canonical monoidal quotient functor. Then, we have an equivalence $\mathcal{C}_{\hat{\Theta}} \cong \mathcal{C}_{\pi(\hat{\Theta})}$.*

Proof. To reduce the notational burden, we introduce the abbreviation $\mathfrak{K} := \mathfrak{J}_{\not\geq_{LR}\underline{c}}$. Proposition 4.5.25 implies that we have an equivalence $\mathrm{Ind}_\pi\left((\mathcal{B}/\mathfrak{J})_{\hat{\Theta}}\right) \cong (\mathcal{B}/\mathfrak{K})_{\pi(\hat{\Theta})}$ in $\mathcal{B}/\mathfrak{K} - \mathrm{Mod}$. Now we apply $\mathrm{Res}_{\pi_{\mathfrak{K}}}$, use $\mathrm{Res}_{\pi_{\mathfrak{K}}} = \mathrm{Res}_{\pi_{\mathfrak{J}}} \mathrm{Res}_\pi$ and the adjunction morphism $id_{\mathcal{B}/\mathfrak{J}-\mathrm{Mod}} \to \mathrm{Res}_\pi \mathrm{Ind}_\pi$ of Proposition 4.4.10, to obtain a $\mathcal{B}$-functor $\mathrm{Res}_{\pi_{\mathfrak{J}}}\left((\mathcal{B}/\mathfrak{J})_{\hat{\Theta}}\right) \to \mathrm{Res}_{\pi_{\mathfrak{K}}}\left((\mathcal{B}/\mathfrak{K})_{\pi(\hat{\Theta})}\right)$. Further by the construction of Ind_π in Definition/Proposition 4.4.8, the $(\mathcal{B}/\mathfrak{J})$-functor $(\mathcal{B}/\mathfrak{J})_{\hat{\Theta}} \to \mathrm{Res}_\pi\left(\mathrm{Ind}_\pi((\mathcal{B}/\mathfrak{J})_{\hat{\Theta}})\right)$ is full and essentially surjective. Thus the same holds for the above constructed $\mathcal{B}$-functor. Furthermore, we also know which morphisms are send to zero under this quotient: All morphisms that factor over a morphism obtained from a morphism of $\mathrm{Res}_{\pi_{\mathfrak{J}}}\left((\mathcal{B}/\mathfrak{J})_{\hat{\Theta}}\right)$ by acting on it with a morphism in $\mathfrak{K}$. On the other hand, the composition of $\mathcal{B}_{\mathfrak{K}}$-functors $(\mathcal{B}/\mathfrak{K})_{\pi(\hat{\Theta})} \to \mathcal{B}/\mathfrak{K} \to (\mathcal{B}/\mathfrak{K})_{\pi(\hat{\Theta})}$ contains the identity functor as a summand and thus $\mathrm{Res}_{\pi_{\mathfrak{K}}}\left((\mathcal{B}/\mathfrak{K})_{\pi(\hat{\Theta})}\right) \to \mathrm{Res}_{\pi_{\mathfrak{K}}}(\mathcal{B}/\mathfrak{K})$ is faithful. Hence, Lemma 3.3.35 implies $\mathfrak{Ann}_{\mathcal{B}}\left(\mathrm{Res}_{\pi_{\mathfrak{K}}}\left((\mathcal{B}/\mathfrak{K})_{\pi(\hat{\Theta})}\right)\right) \supset \mathfrak{Ann}_{\mathcal{B}}\left(\mathrm{Res}_{\pi_{\mathfrak{K}}}(\mathcal{B}/\mathfrak{K})\right) \overset{\text{Rem 3.3.58}}{=} \mathfrak{K}$. Further since $\mathfrak{K}$ is a thick ideal and because we have constructed a non-trivial morphism between the two cell modules, Proposition 4.5.35 implies that $\mathfrak{K} \subset \mathfrak{Ann}_{\mathcal{B}}\left(\mathrm{Res}_{\pi_{\mathfrak{J}}}\left((\mathcal{B}/\mathfrak{J})_{\hat{\Theta}}\right)\right)$, as well. Hence, all the morphisms by which the quotient is taken are already zero, *i.e.* the quotient $\mathcal{B}$-functor induces an equivalence $\mathrm{Res}_{\pi_{\mathfrak{J}}}\left((\mathcal{B}/\mathfrak{J})_{\hat{\Theta}}\right) \cong \mathrm{Res}_{\pi_{\mathfrak{K}}}\left((\mathcal{B}/\mathfrak{K})_{\pi(\hat{\Theta})}\right)$. □

Remark 4.5.38. A posteriori, one can construct any universal cell module $\mathcal{C}$ that arises from a cell object using a split Frobenius object in $\mathcal{B}/\mathfrak{Ann}_{\mathcal{B}}(\mathcal{C})$. However, *a priori* we have no access to $\mathfrak{Ann}_{\mathcal{B}}(\mathcal{C})$ and a posteriori, we do not know whether the ideals $\mathfrak{K} \subset \mathfrak{Ann}_{\mathcal{B}}(\mathcal{C})$ agree.

Remark 4.5.39. Regardless of the equivalence above, we do in general not expect that the quotient $\mathcal{B}$-functor $\mathcal{B}/\mathfrak{J} \twoheadrightarrow \mathcal{B}/\mathfrak{K}$ induces an equivalence on either the full subcategories associated to $\underline{c}$ or any of the full sub-cell modules associated to left cells in $\underline{c}$.

Remark 4.5.40. Proposition 4.5.37 can be viewed as a more concrete version of the effect discussed in Proposition 4.2.39. Let us denote the $\mathcal{B}$-functor $\mathcal{B}/\mathfrak{J} \to \mathcal{C}_{\hat{\Theta}}$ implicitly present in Proposition 4.5.37 by F. Then, $\operatorname{im} F$ is certainly a $\mathcal{B}$-quotient of $\mathcal{B}/\mathfrak{K}$. It is not clear to us whether there is relation between $\operatorname{im} F$ and $\mathcal{B}/\mathfrak{Ann}_{\mathcal{B}}(\mathcal{C})$ that holds in full generality.

Proposition 4.5.41. *Let $\hat{\Theta} = (\hat{\Theta}, \pi : \mathcal{B} \to \overline{\mathcal{B}})$ be a cell object and let $\underline{c}$ be a 2-sided cell of $\mathcal{B}$. Assume that $\mathfrak{J}_{\leq_{LR}\underline{c}} \subset \mathfrak{J}(\pi) \subset \mathfrak{J}_{\not\geq_{LR}\underline{c}}$ and that the object underlying $\hat{\Theta}$ allows for a split injection of an indecomposable object associated to a minimal left cell in $\underline{c}$ with respect to $\leq_L$. Then, $\mathcal{C}_{\hat{\Theta}}$ is a universal cell module that belongs to $\underline{c}$.*

Proof. By Proposition 4.5.37, we can assume w.l.o.g that $\overline{B} = \mathcal{B}/\mathfrak{K}$, where we again abbreviate $\mathfrak{K} := \mathfrak{J}_{\not\geq_{LR}\underline{c}}$. Furthermore, we have seen in the proof of Proposition 4.5.37 that $\mathfrak{K} \subset \mathrm{Ann}_{\mathcal{B}}(\mathcal{C}_{\hat{\Theta}})$. By assumption there exists an indecomposable object B_x with $x \in \underline{c}$ (that is contained in a minimal left cell). Since annihilators are ideals, it follows that $\mathfrak{K}$ is the maximal thick subideal of $\mathrm{Ann}_{\mathcal{B}}(\mathcal{C}_{\hat{\Theta}})$ and thus that $\mathcal{C}_{\hat{\Theta}}$ belongs to $\underline{c}$. To show that $\mathcal{C}_{\hat{\Theta}}$ is universal, let $\mathcal{C} \in \mathcal{B}-\mathrm{Mod}$ be an arbitrary cell module that belongs to $\underline{c}$. Using $\mathcal{H}om_{\mathcal{B}}(\mathcal{B}, \mathcal{C}) \cong \mathcal{C}$, for every $C \in \mathcal{C}$ we have a $\mathcal{B}$-functor $\mathcal{B} \to \mathcal{C}$ that sends $B \mapsto B.C$. Since $\mathfrak{K} \subset \mathfrak{Ann}_{\mathcal{B}}(\mathcal{C})$, each such functor factors over $\mathrm{Res}_{\pi_{\mathfrak{K}}}(\mathcal{B}/\mathfrak{K})$. Furthermore since $\mathfrak{K} \subset \mathfrak{Ann}_{\mathcal{B}}(\mathcal{C})$ is the maximal thick subideal, for each indecomposable object $B \in \mathcal{B}/\mathfrak{K}$ there exists $C \in \mathcal{C}$ such that $B.C \neq 0$. Recall from the proof of Proposition 4.5.37 that we have a non-trivial $\mathcal{B}$-functor $\mathcal{C}_{\hat{\Theta}} \to \mathrm{Res}_{\pi_{\mathfrak{K}}}(\mathcal{B}/\mathfrak{K})$. Now we can pick any object in the image of this $\mathcal{B}$-functor and will find by the above discussion a $\mathcal{B}$-functor $\mathrm{Res}_{\pi_{\mathfrak{K}}}(\mathcal{B}/\mathfrak{K}) \to \mathcal{C}$ that does not annihilate this object. It follows that the composition $\mathcal{C}_{\hat{\Theta}} \to \mathrm{Res}_{\pi_{\mathfrak{K}}}(\mathcal{B}/\mathfrak{K}) \to \mathcal{C}$ is also non-trivial and the proof is complete. $\square$

Proposition 4.5.42. *Let $\hat{\Theta}$ be a cell object for $\mathcal{B}$ and let $\mathcal{C}_{\hat{\Theta}}$ be the associated functorially cyclic cell module. Then, $\mathcal{E}nd_{\mathcal{B}}(\mathcal{C}_{\hat{\Theta}})$ consists of a single left cell.*

Proof. Let $\hat{\Theta} = (\hat{\Theta}, \pi)$ be a cell object of $\mathcal{B}$, where $\pi : \mathcal{B} \twoheadrightarrow \overline{\mathcal{B}}$ is a monoidal quotient functor and $\hat{\Theta} = (\Theta, \Delta, \mu)$ is a right indecomposable split Frobenius object in $\overline{\mathcal{B}}$, and let $\mathcal{C}_{\hat{\Theta}} = \mathrm{Res}_{\pi}(\overline{\mathcal{B}}_{\hat{\Theta}})$ be the associated functorially cyclic cell module. We have $\mathrm{End}_{\mathcal{B}}(\mathcal{C}_{\hat{\Theta}}) \overset{\text{Pro 4.4.10}}{\cong} \mathcal{H}om_{\overline{\mathcal{B}}}(\mathrm{Ind}_{\pi}(\mathrm{Res}_{\pi}(\overline{\mathcal{B}}_{\hat{\Theta}})), \overline{\mathcal{B}}_{\hat{\Theta}}) \overset{\text{Pro 4.4.9}}{\cong} \mathcal{E}nd_{\overline{\mathcal{B}}}(\overline{\mathcal{B}}_{\hat{\Theta}}) \overset{\text{Thm 4.5.10}}{\cong} {}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}$. It remains to show that ${}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}$ consists of a single left cell. Recall that $(I, \Delta \circ \mu)$ is the unit object of ${}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}$, and note that it is indecomposable by Remark 4.5.29. It is thus sufficient to show that $(I, \Delta \circ \mu) \leq_L (B, e)$ for all $(B, e) \in {}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}$, since the opposite always holds in a monoidal category. Recall the functors ${}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}} \xrightarrow{{}_{\hat{\Theta}}\vartheta} \overline{\mathcal{B}}_{\hat{\Theta}} \xrightarrow{{}^{\hat{\Theta}}\vartheta} {}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}$ which (replacing $\mathcal{B}$ by $\overline{\mathcal{B}}$) we introduced in Remark 4.3.67. We have ${}_{\hat{\Theta}}\vartheta\,((B, e)) = (\Theta B, e)$. Note that due to the Frobenius relations, Δ and μ satisfy right (resp. 2-side) $\hat{\Theta}$-descent and thus any morphism between objects of $\overline{\mathcal{B}}_{\hat{\Theta}}$ (resp. ${}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}$) constructed from Δ, μ and identity morphisms will be a morphism in the respective categories. In particular, $(\Theta, \Delta \circ \mu) \in \overline{\mathcal{B}}_{\hat{\Theta}}$, $f_1 := (\Theta\Delta) \circ \Delta \circ \mu \in {}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}\,((\Theta, \Theta(\Delta \circ \mu)), (I, \Delta \circ \mu))$ and $f_2 := \Delta \circ \mu \circ (\Theta\mu) \in {}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}\,((I, \Delta \circ \mu), (\Theta, \Theta(\Delta \circ \mu)))$. Since $\overline{\mathcal{B}}_{\hat{\Theta}}$ is a cell $\overline{\mathcal{B}}$-module, there exists $M \in \overline{\mathcal{B}}$ such that $(\Theta, \Delta \circ \mu) \subset^{\oplus} M.(\Theta B, e) = (M\Theta B, Me)$. Applying ${}^{\hat{\Theta}}\vartheta$, we obtain $(\Theta, \Theta(\Delta \circ \mu)) \subset^{\oplus} (M\Theta B, \Theta Me)$. Further using $\mu \circ \Delta = id_{\Theta}$, we obtain $f_1 \circ f_2 = \Delta \circ \mu = id_{(I, \Delta\circ\mu)}$, *i.e.* $(I, \Delta \circ \mu) \subset^{\oplus} (\Theta, \Theta(\Delta \circ \mu)$. Finally, by Definition 4.3.63, we have $(M, id_{\Theta M \Theta}) \otimes (B, e) = (M\Theta B, \Theta Me)$. Hence, $(I, \Delta \circ \mu) \subset^{\oplus} (M, id_{\Theta M \Theta}) \otimes (B, e)$ and the proof is complete. $\square$

Remark 4.5.43. The same argument is not sufficient to show that that ${}_{\hat{\Theta}}\overline{\mathcal{B}}_{\hat{\Theta}}$ consists of a single right cell, since, similar to the fact that right indecomposablility of $\hat{\Theta}$ does not imply that ${}_{\hat{\Theta}}\mathcal{B}$ is a functorially cyclic right-$\mathcal{B}$-module, it is not guaranteed that ${}_{\hat{\Theta}}\mathcal{B}$ is cyclic.

Remark 4.5.44. A particularly nice situation arises, if the indecomposable objects are stable under composition. This situation and more generally any group action on a module is the topic of the subsequent Section 4.6. It would be interesting to study whether there exists a unified framework between this and the split Frobenius construction. Remark 4.9.20 points into a similar direction. A definite result with regards to this topic is Corollary 4.6.19.

Proposition 4.5.45. *Let $\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}$ be $\mathcal{B}$-functors between cell modules such that $GF \cong id_{\mathcal{C}}$ and $FG \cong id_{\mathcal{C}'} \oplus R$. Then, $\mathcal{C}$ and $\mathcal{C}'$ are equivalent.*

Proof. We have $id_{\mathcal{C}} = id_{\mathcal{C}} id_{\mathcal{C}} \cong GFGF \cong G(id_{\mathcal{C}'} \oplus R)F \cong GF \oplus GRF \cong id_{\mathcal{C}} \oplus GRF$, and thus $GRF = 0$. Since F and G are non-trivial $\mathcal{B}$-functors between cell modules, they cannot annihilate any object. Thus, R has to annihilate all objects in the image of F. Since R is additive it also annihilates all direct summands of objects in the image of F, and thus $R = 0$. □

One might wonder whether it would be interesting to study couniversal cell modules, *i.e.* cell modules that admit non-trivial morphisms with domain any cell module that is associated to the same 2-sided cell. We do not believe that this would lead to a good notion, since there are exists a large class of examples for which this notion would be empty. Let us give an example:

Example 4.5.46. Let $\mathfrak{A}$ be such that $\mathcal{Z}^{\otimes}(\mathfrak{A})$ has two or more maximal ideals. Precisely one of the maximal ideals will act trivially on any module that has no non-trivial quotients. Between two such modules, for which the ideals that act trivially differ, there cannot exist any non-trivial $\mathfrak{A}$-functor.

Nevertheless, it is very fruitful to study modules on this other end of the spectrum:

Remark 4.5.47. [MaMi1]-[MaMi5] argues that the best substitute for the classical notion of irreducibility are so called simple transitive 2-representations. By definition these are idempotent closed cell modules that have no non-trivial quotient. Clearly, any module that has no non-trivial quotients is a cell module. In [MaMi1], certain Abelian cell 2-representations are constructed which reconstruct various categories of interest that arise in the context of category $\mathcal{O}$ by abstract means, see [MaSt, KhMaSt]. Amongst them is the principal block of the parabolic category $\mathcal{O}$, as noted in [MaSt, Remark 14]. This achievement lies at the heart of all the definitions found in [MaMi1]-[MaMi5]. In [MaMi2], the corresponding additive cell 2-representations are introduced and in [MaMi5] it is shown that these additive cell 2-representations have no non-trivial quotients. We will discuss the connection to universal cell modules in Remark 4.9.25.

4.6 Coinvariant theory

In this section, we discuss coinvariant theory. Given an additive higher representation that admits a group action, we construct a universal property, see Definition 4.6.2, and construct an explicit coinvariant module, see Definition 4.6.4, that satisfies this universal property, see Theorem 4.6.14. We provide simple Examples 4.6.6 and 4.6.17 of a trivial and of a non-trivial group action, which show that coinvariant modules are typically not idempotent closed, and postpone more involved examples to Section 5.5. Provided the module satisfies the Krull-Schmidt property, we will show in Proposition 4.6.16 that on the level of split Grothendieck group, this construction indeed coincides with the classical notion of coinvariant modules. Curiously, the coinvariant module is weakly equivalent to the module we started from, see Proposition 4.6.18. Finally in Corollary 4.6.19, we observe that functorially cyclic may have a functorially cyclic coinvariant module and calculate the associated right indecomposable split Frobenius object.

We start by describing an analogue statement in classical invariant theory. Intentionally, we use a non-standard formulation since the subsequent generalisation to higher representation theory is formulated in this way: Given a group G, a representation M of G and a set N, one might want to understand the set of all maps $f : M \to N$ which are invariant under the G-action, *i.e.* $f \circ (g.) = f$ for all $g \in G$. In this situation, the set of coinvariants $M_G := \{G.m | m \in M\}$, where $G.m := \{g.m | g \in G\}$, solves this question: The invariant maps are exactly the maps $M \to N$ that factor over the canonical quotient map $\pi_G : M \to M_G$. Another way of saying this is that M_G represents the functor $\phi_G : \mathcal{S}et \to \mathcal{S}et$ which is defined by $\phi_G(X) := \{f \in \mathcal{S}et(M, X) | f \circ (g.) = f \ \forall g \in G\}$ for all $X \in \mathcal{S}et$ on objects and in the obvious way on morphisms. Clearly, precomposition with π_G induces an isomorphism $\mathcal{S}et(M_G, -) \cong \phi_G$. Thus, ϕ_G is the universal property of M_G and the

Yoneda lemma states that M_G is unique with this property up to unique isomorphism.

To extend the above discussion to higher representation theory, we need to define group actions. For this purpose, recall that any associative unital monoid can be considered as a monoidal category, where the only non-empty homomorphism spaces are endomorphisms spaces, each containing a single (identity) morphism. Now, we can make the following definition:

Definition 4.6.1. *Let G be a finite group and let $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$. A* group action S of G on $\mathcal{M}$ *is a monoidal functor $S = (S, \{G_{g_1,g_2}\}_{(g_1,g_2)\in G\times G}, G_e) : G \to \mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$.*

For the remainder of this section, we fix $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ with a group action S of a finite group G. For all $g \in G$, we write $S_g := S(g)$. We can think about S_g as a symmetry of the $\mathcal{M}$. Note that since S is a monoidal functor between strict monoidal categories, the requirements on the iso-$\mathcal{B}$-transformations $G_{g_1,g_2} : S_{g_1}S_{g_2} \Rightarrow S_{g_1g_2}$ and $G_e : id_{\mathcal{M}} \Rightarrow S_e$ simplify to $G_{g_1,g_2g_3} \circ (S_{g_1}G_{g_2,g_3}) = G_{g_1g_2,g_3} \circ (G_{g_1,g_2}S_{g_3})$ and $G_{e,g_1} \circ (G_eS_{g_1}) = S_{g_1} = G_{g_1,e} \circ (S_{g_1}G_e)$.

Definition 4.6.2. *The 2-functor $\Phi_G : \mathcal{B} - \mathrm{Mod} \to \mathfrak{A}dd$ is defined on any $\mathcal{N} \in \mathcal{B} - \mathrm{Mod}$, $H \in \mathcal{H}om_{\mathcal{B}}(\mathcal{N}_1, \mathcal{N}_2)$ and $\beta \in \mathcal{H}om_{\mathcal{N}_1\to\mathcal{N}_2}(H_1, H_2)$, by setting*

- $$\Phi_G(\mathcal{N}) := \left\{ \begin{array}{ll} Obj: & (F, \{\nu_g\}_{g\in G}) \text{ for all} \\ & F \in \mathcal{H}om_{\mathcal{B}}(\mathcal{M}, \mathcal{N}), \nu_g \in \mathcal{H}om_{\mathcal{M}\to\mathcal{N}}(FS_g, F) \text{ s.t.} \\ & \nu_{g_2} \circ (\nu_{g_1}S_{g_2}) = \nu_{g_1g_2} \circ (FG_{g_1,g_2}) \text{ and} \\ & \nu_e = FG_e^{-1} \\ Mor: & \Phi_G(\mathcal{N})((F_1, \{\nu_g^1\}_{g\in G}), (F_2, \{\nu_g^2\}_{g\in G})) := \\ & \{\alpha \in \mathcal{H}om_{\mathcal{M},\mathcal{N}}(F_1, F_2) | \alpha \circ \nu_g^1 = \nu_g^2 \circ (\alpha S_g) \ \forall g \in G\} \end{array} \right\},$$
- $$\Phi_G(H) := \left\{ \begin{array}{ll} Obj: & \Phi_G(H)((F, \{\nu_g\}_{g\in G})) := (HF, \{H\nu_g\}_{g\in G}) \\ & \text{for all } (F, \{\nu_g\}_{g\in G}) \in \Phi_G(\mathcal{N}_1) \\ Mor: & \Phi_G(H)(\alpha) := H\alpha \text{ for all} \\ & \alpha \in \Phi_G(\mathcal{N}_1)((F_1, \{\nu_g^1\}_{g\in G}), (F_2, \{\nu_g^2\}_{g\in G})) \end{array} \right\} \text{and}$$
- $\Phi_G(\beta) := \big(\Phi_G(\beta)_{(F,\{\nu_g\}_{g\in G})}\big)_{(F,\{\nu_g\}_{g\in G})\in\Phi_G(\mathcal{N}_1)} := (\beta F)_{(F,\{\nu_g\}_{g\in G})\in\Phi_G(\mathcal{N}_1)}$.

Proposition 4.6.3. *Equipped with trivial structure maps, $\Phi_G : \mathcal{B} - \mathrm{Mod} \to \mathfrak{A}dd$ is a strict 2-functor.*

Proof. We restrict ourselves to listing the tasks and leave it to the reader to execute the details: First, one has to prove that $\Phi_G(\mathcal{N})$ is a category, that $\Phi_G(H)$ is a functor, and that $\Phi_G(\beta)$ is a transformation. Second, one needs to prove that Φ_G defines a functor $\mathcal{H}om_{\mathcal{B}}(\mathcal{N}_1, \mathcal{N}_2) \to \mathrm{Fun}(\Phi_G(\mathcal{N}_1), \Phi_G(\mathcal{N}_2))$, which follows from the interchange law for composition and juxtaposition. Third, the structure maps have to be natural isomorphisms $\Phi_G(H_2H_1) \xrightarrow{\sim} \Phi_G(H_2)\Phi_G(H_1)$ and can be clearly chosen to be the identity. Fourth, strictness says that structure map $id_{\Phi_G(\mathcal{N})} \Rightarrow \Phi_G(id_{\mathcal{N}})$ shall be the identity transformation, which is clearly possible. Last, the compatibility conditions for the structure maps with Φ_G are trivially satisfied because $\mathcal{B}-\mathrm{Mod}$ and $2-\mathcal{C}at$ are both strict 2-categories and the structure maps are given by identity morphisms. □

We will show in Theorem 4.6.14 that Φ_G is representable by a $\mathcal{B}$-module denoted $\mathcal{M}_G$. We start by constructing the underlying additive category:

Definition 4.6.4. *Let $\mathcal{M}$ be an additive category and let G be a group that acts on $\mathcal{M}$. We define the category $\mathcal{M}_G$ as the category with objects $\mathrm{Ob}(\mathcal{M}_G) := \mathrm{Ob}(\mathcal{M})$ and morphisms $\mathcal{M}_G(M_1, M_2) := \bigoplus_{g\in G} \mathcal{M}(M_1, S_g(M_2))$ for all $M_1, M_2 \in \mathrm{Ob}(\mathcal{M}_G)$. Given $f^1 = \left(f^1_g\right)_{g\in G} \in \mathcal{M}_G(M_1, M_2)$ and $f^2 = \left(f^2_g\right)_{g\in G} \in \mathcal{M}_G(M_2, M_3)$ the composition in $\mathcal{M}_G$ is defined by $(f^2 \circ f^1)_g := \sum_{k\in G}(G_{k,k^{-1}g})_{M_3} \circ S_k(f^2_{k^{-1}g}) \circ f^1_k$. The identity morphism of $M \in \mathcal{M}_G$ is given by $(\delta_{g,e}(G_e)_M)_{g\in G}$.*

Proof. First note that the morphism spaces of $\mathcal{M}_G$ are defined using a direct sum. Thus, the composition is well defined. To prove associativity let $f^3 = \left(f^3_g\right)_{g\in G} \in \mathcal{M}_G(M_3, M_4)$. Note that we can shift the summation by group elements. For example for $g, m \in G$, we have $(f^3 \circ f^2)_g := \sum_{k\in G}(G_{mk,k^{-1}m^{-1}g})_{M_4} \circ S_{mk}(f^3_{k^{-1}m^{-1}g}) \circ f^2_{mk}$. For all $g \in G$, we are now able to compute

$$
\begin{aligned}
&(f^3 \circ (f^2 \circ f^1))_g \\
&\quad= \sum_{k_1,k_2} (G_{k_1,k_1^{-1}g})_{M_4} \circ S_{k_1}(f^3_{k_1^{-1}g}) \circ (G_{k_2,k_2^{-1}k_1})_{M_3} \circ S_{k_2}(f^2_{k_2^{-1}k_1}) \circ f^1_{k_2} \\
&\quad= \sum_{k_1,k_2} (G_{k_1,k_1^{-1}g})_{M_4} \circ (G_{k_2,k_2^{-1}k_1} S_{k_1^{-1}g})_{M_4} \circ S_{k_2}(S_{k_2^{-1}k_1}(f^3_{k_1^{-1}g}) \circ f^2_{k_2^{-1}k_1}) \circ f^1_{k_2} \\
&\quad= \sum_{k_1,k_2} (G_{k_2,k_2^{-1}g})_{M_4} \circ S_{k_2}((G_{k_2^{-1}k_1,k_1^{-1}g})_{M_4} \circ S_{k_2^{-1}k_1}(f^3_{k_1^{-1}g}) \circ f^2_{k_2^{-1}k_1}) \circ f^1_{k_2} \\
&\quad= \sum_{k_2} (G_{k_2,k_2^{-1}g})_{M_4} \circ S_{k_2}((f^3 \circ f^2)_{k_2^{-1}g}) \circ f^1_{k_2} \\
&\quad= ((f^3 \circ f^2) \circ f^1)_g.
\end{aligned}
$$

In the third equation we used $G_{g_1,g_2g_3} \circ (S_{g_1}G_{g_2,g_3}) = G_{g_1g_2,g_3} \circ (G_{g_1,g_2}S_{g_3})$ and in the fourth composition rule the shifted by $m = k_2^{-1}$ as explained prior to the computation. Further, $(\delta_{g,e}(G_e)_M)_{g\in G}$ is an identity morphism on $M \in \mathcal{M}_G$. This holds due to $\left((\delta_{g,e}(G_e)_{M_2})_{g\in G} \circ f^1\right)_g = (G_{g,e})_{M_2} \circ S_g((G_e)_{M_2}) \circ f^1_g = f^1_g$ and furthermore due to $\left(f^1 \circ (\delta_{g,e}(G_e)_{M_1})_{g\in G}\right)_g = (G_{e,g})_{M_2} \circ S_e(f^1_g) \circ (G_e)_{M_1} = (G_{e,g})_{M_2} \circ (G_e)_{S_g(M_2)} \circ f^1_g = f^1_g$.

The last equation in both computations holds since $G_{e,g} \circ (G_e S_g) = S_g = G_{g,e} \circ (S_g G_e)$ by definition of a G-action. $\square$

Example 4.6.5. For all groups G and all $\mathcal{M} \in \mathcal{B}-\mathrm{Mod}$, we can defined a trivial group action by setting $S_g := id_{\mathcal{M}}$. Classically, the coinvariant module does agrees with the original module, *i.e.* $[\mathcal{M}]_G = [\mathcal{M}]$. The categorical coinvariant module $\mathcal{M}_G$ behaves in the same way, *i.e.* $[\mathcal{M}_G] = [\mathcal{M}]_G$, as we will show in Proposition 4.6.16 assuming the Krull-Schmidt property for $\mathcal{M}$. If $G = \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}_0$, the coinvariant module is equivalent to a base change. In formulas, we have $\mathcal{M}_G \cong \mathcal{M} \otimes_{\mathcal{Z}} \mathcal{Z}_n$ where $\mathcal{Z} := \mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ and $\mathcal{Z}_n := \mathcal{Z}[x]/\langle x^n = 1\rangle$.

Example 4.6.6. Let $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$, let $G = \mathbb{Z}/2\mathbb{Z}$ act trivially on $\mathcal{M}$ and let $2 \in \mathcal{Z} = \mathcal{Z}_{\mathcal{B}}(\mathcal{M})$ be invertible. Then, $\mathcal{M}_G^{\mathrm{idem}}$ is equivalent to $\mathcal{M} \oplus \mathcal{M}$, since $\frac{1+x}{2}$ and $\frac{1-x}{2}$ are orthogonal idempotents in $\mathcal{Z}_2$.

Example 4.6.7. Recall from Example 4.5.7 the k-linear monoidal category $\mathcal{B}_2$ and its higher representations $\mathcal{A}$ and $\mathcal{A} \oplus \mathcal{A}$ and the endo-$\mathcal{B}_2$-transformations I and S of $\mathcal{A} \oplus \mathcal{A}$. Now, the symmetric group $S_2 = \{e, s\}$ acts on $\mathcal{A} \oplus \mathcal{A}$ through $S_e := I$ and $S_s := S$. It is not too hard to see that $(\mathcal{A} \oplus \mathcal{A})_{S_2} \cong \mathcal{A}$. In the following proposition, we will discuss a canonical functor $\mathcal{A} \oplus \mathcal{A} \to (\mathcal{A} \oplus \mathcal{A})_{S_2}$ (and show later that this functor can be equipped with the structure of a $\mathcal{B}_2$-functor). It is not hard to see that under the identification $(\mathcal{A} \oplus \mathcal{A})_{S_2} \cong \mathcal{A}$, this functor corresponds to the $\mathcal{B}_2$-functor $\mathcal{A} \oplus \mathcal{A} \to \mathcal{A}$ already discussed in Example 4.5.7. We will provide an abstract reason that show that $\mathcal{A}$ and $\mathcal{A} \oplus \mathcal{A}$ are weakly equivalent in Proposition 4.6.18.

Proposition 4.6.8. *The category $\mathcal{M}_G$ is an additive category and we have an additive functor $\iota_G : \mathcal{M} \to \mathcal{M}_G$ which is faithful and essentially surjective.*

Proof. First of all, it is rather obvious that $\mathcal{M}_G$ is a category enriched in Abelian groups. The zero object of $\mathcal{M}$ is clearly an initial object of $\mathcal{M}_G$. Since all S_g for $g \in G$ are additive functors, this initial object is terminal as well. That $\mathcal{M}_G$ has finite direct sums, follows once we construct ι_G and show that it is essentially surjective functor and enriched in Abelian groups.
On objects, we define ι_G to be the identity map. On morphisms $f \in \mathcal{M}(M_1, M_2)$, we set $\iota_G f := (\delta_{g,e}(G_e)_{M_2} \circ f)_{g \in G}$. Clearly, ι_G is essentially surjective and it is faithful, since G_e is an iso-($\mathcal{B}$-)transformation. Also, G_e is a transformation between additive functors, thus ι_G is enriched in Abelian groups. Now, we define (finite) direct sums in $\mathcal{M}_G$ by transporting the corresponding structure from $\mathcal{M}$ via ι_G. Clearly, this makes ι_G into an additive functor. $\square$

Remark 4.6.9. The motivation at the beginning of the chapter suggests that we should denote the functor ι_G by π_G. However, we decided to use the notation ι_G to emphasise that ι_G is faithful, rather than highlighting that it is essentially surjective.

Lemma 4.6.10. *Let $M \in \mathcal{M}$ and $g \in G$. Then, $M \cong S_g(M)$ in $\mathcal{M}_G$.*

Proof. One calculates that $\left(\left(G^{-1}_{g^{-1},g} \circ G_e\right)_M \delta_{h,g^{-1}}\right)_{h \in G} \in \mathcal{M}_G(M, S_g(M))$ and shows similarly that $(id_M \delta_{g,h})_{h \in G} \in \mathcal{M}_G(S_g(M), M)$. These morphisms are inverse to each other, one composition being straight forward, the other requiring both axioms of the monoidal functor S. $\square$

Corollary 4.6.11. *The homomorphism of Abelian groups $[\iota_G] : [\mathcal{M}] \to [\mathcal{M}_G]$ factors over the homomorphism $[\mathcal{M}] \twoheadrightarrow G \setminus [\mathcal{M}]$.*

We will improve this statement in Proposition 4.6.16, where we show that $[\iota_G]$ is an isomorphism provided that $\mathcal{M}$ satisfies the Krull-Schmidt property. For now, let us however proceed working towards Theorem 4.6.14.

Proposition 4.6.12. *There exist an operation of $\mathcal{B}$ on $\mathcal{M}_G$ such that ι_G becomes a $\mathcal{B}$-functor.*

Proof. First, we define an additive monoidal functor $\rho : \mathcal{B} \to \mathrm{End}(\mathcal{M}_G)$. Let $B \in \mathcal{B}$. Since $\mathrm{Ob}(\mathcal{M}_G) = \mathrm{Ob}(\mathcal{M})$, we can define $\rho(B)$ on objects by setting $\rho(B)(M) := B.M$ for all $M \in \mathcal{M}_G$. Now let $f = (f_g)_{g \in G} \in \mathcal{M}_G(M_1, M_2)$, and define $\rho(B)(f)$ by setting $(\rho(B)(f))_g := (\mathrm{can}_{B,S_g})_{M_2} \circ (B.(f_g))$ for all $g \in G$, where $\mathrm{can}_{B,S_g} : (B.)S_g \overset{\sim}{\Longrightarrow} S_g(B.)$ is the natural transformation which is part of the datum of the $\mathcal{B}$-functor S_g. Since $id_M \Rightarrow S_e$ is a transformation of $\mathcal{B}$-functors and since the structure as a $\mathcal{B}$-functor is trivial, it follows that $(\rho(B)(id_M))_e = (id_M)_e$. In addition, $\rho(B)$ is defined using additive functors, hence sends zero morphisms to zero morphisms which implies that $\rho(B)(id_M) = id_M$. Let $f^1 \in \mathcal{M}_G(M_1, M_2)$ and let $f^2 \in \mathcal{M}_G(M_2, M_3)$. The last step in showing that $\rho(B)$ is an additive functor, we have to show that $\rho(B)(f^2) \circ \rho(B)(f^1) = \rho(B)(f^2 \circ f^1)$. For all $g \in G$, we have

$$
\begin{aligned}
\big(\rho(B)(f^2)\circ\rho(B)(f^1)\big)_g &= \sum_{k\in G}(G_{k,k^{-1}g})_{B.M_3}\circ S_k((\rho(B)(f^2))_{k^{-1}g})\circ(\rho(B)(f^1))_k\\
&= \sum_{k\in G}(G_{k,k^{-1}g})_{B.M_3}\circ S_k((\mathrm{can}_{B,S_{k^{-1}g}})_{M_3}\circ(B.(f^2_{k^{-1}g})))\\
&\qquad\circ(\mathrm{can}_{B,S_k})_{M_2}\circ(B.(f^1_k))\\
&= \sum_{k\in G}(G_{k,k^{-1}g})_{B.M_3}\circ S_k((\mathrm{can}_{B,S_{k^{-1}g}})_{M_3})\circ(\mathrm{can}_{B,S_k})_{S_{k^{-1}g}M_3}\\
&\qquad\circ(B.S_k(f^2_{k^{-1}g}))\circ(B.(f^1_k))\\
&= \sum_{k\in G}(G_{k,k^{-1}g})_{B.M_3}\circ(\mathrm{can}_{B,S_kS_{k^{-1}g}})_{M_3}\circ(B.S_k(f^2_{k^{-1}g}))\\
&\qquad\circ(B.(f^1_k))\\
&= \sum_{k\in G}(\mathrm{can}_{B,S_g})_{M_3}\circ(B.(G_{k,k^{-1}g})_{M_3})\circ(B.S_k(f^2_{k^{-1}g}))\\
&\qquad\circ(B.f^1_k)\\
&= \sum_{k\in G}(\mathrm{can}_{B,S_g})_{M_3}\circ(B.((G_{k,k^{-1}g})_{M_3}\circ S_k(f^2_{k^{-1}g})\circ f^1_k))\\
&= (\mathrm{can}_{B,S_g})_{M_3}\circ\left(B.\left(\sum_{k\in G}(G_{k,k^{-1}g})_{M_3}\circ S_k(f^2_{k^{-1}g})\circ f^1_k\right)\right)\\
&= (\mathrm{can}_{B,S_g})_{M_3}\circ(B.((f^2\circ f^1)_g))\\
&= \big(\rho(B)(f^2\circ f^1)\big)_g\,,
\end{aligned}
$$

where we used that can_{B,S_k} is a natural transformation, that $\mathrm{can}_{B,GF}=(G\ \mathrm{can}_{B,F})\circ(\mathrm{can}_{B,G}\,F)$ for all $\mathcal{B}$-functors F and G, and that $B.$ is a functor.

Next for all $b\in\mathcal{B}(B_1,B_2)$, we define a transformation $\rho(b):\rho(B_1)\Rightarrow\rho(B_2)$. For all $M\in\mathcal{M}_G$, we define $\rho(b)_M$ by setting $\left(\rho(b)_M\right)_g:=\delta_{g,e}\left((\mathrm{can}_{B_2,S_e})_M\circ(b.((G_e)_M))\right)=\delta_{g,e}\left((G_e)_{B_2.M}\circ(b.id_M)\right)$ for all $g\in G$. To show that $\rho(b)$ is indeed a transformation, we have to show that for all $f\in\mathcal{M}_G(M_1,M_2)$ we have $\rho(b)_{M_2}\circ\rho(B_1)(f)=\rho(B_2)(f)\circ\rho(b)_{M_1}$. For all $g\in G$, we have

$$
\begin{aligned}
\left(\rho(b)_{M_2} \circ \rho(B_1)(f)\right)_g &= \sum_{k\in G} (G_{k,k^{-1}g})_{B_2.M_2} \circ S_k((\rho(b)_{M_2})_{k^{-1}g}) \circ (\rho(B_1)(f))_k \\
&= \sum_{k\in G} (G_{k,k^{-1}g})_{B_2.M_2} \circ S_k(b.id_{M_2}\delta_{e,k^{-1}g}) \circ (\rho(B_1)(f))_k \\
&= (G_{g,e})_{B_2.M_2} \circ S_g(((G_e)_{B_2.M_2}) \circ (b.id_{M_2})) \circ (\rho(B_1)(f))_g \\
&= (G_{g,e} \circ (S_g G_e))_{B_2.M_2} \circ S_g(b.id_{M_2}) \circ (\mathrm{can}_{B_1,S_g})_{M_2} \circ (B_1.f_g) \\
&= S_g(b.id_{M_2}) \circ (\mathrm{can}_{B_1,S_g})_{M_2} \circ (B_1.f_g) \\
&= (\mathrm{can}_{B_2,S_g})_{M_2} \circ (b.id_{S_g(M_2)}) \circ (B_1.f_g) \\
&= (\mathrm{can}_{B_2,S_g})_{M_2} \circ (b.f_g) \\
&= (G_{e,g})_{B_2.M_2} \circ (G_e)_{B_2.M_2} \circ (\mathrm{can}_{B_2,S_g})_{M_2} \circ (B_2.f_g)) \circ (b.id_{M_1}) \\
&= (G_{e,g})_{B_2.M_2} \circ S_e((\rho(B_2)(f))_g) \circ (G_e)_{B_2.M_1} \circ (b.id_{M_1}) \\
&= \sum_{k\in G} (G_{k,k^{-1}g})_{B_2.M_2} \circ S_k((\rho(B_2)(f))_{k^{-1}g}) \circ (\rho(b)_{M_1})_k \\
&= \left(\rho(B_2)(f) \circ \rho(b)_{M_1}\right)_g .
\end{aligned}
$$

Second, we show that ρ is a functor. Since $\rho(id_B)_M = (G_e)_{B.M}$ which is the identity morphism of $\rho(B(M)$ in $\mathcal{M}_G$, we have $\rho(id_B) = id_{\rho(B)}$. Last for all $b_1 \in \mathcal{B}(B_1, B_2)$ and $b_2 \in \mathcal{B}(B_2, B_3)$, we show that $\rho(b_2) \circ \rho(b_1) = \rho(b_2 \circ b_1)$. For all $M \in \mathcal{M}_G$ and $g \in G$, we have

$$
\begin{aligned}
\left(\rho(b_2)_M \circ \rho(b_1)_M\right)_g &= \sum_{k\in G} (G_{k,k^{-1}g})_{B_3.M} \circ S_k((\rho(b_2)_M)_{k^{-1}g}) \circ (\rho(b_1)_M)_k \\
&= \delta_{k^{-1}g,e}\delta_{k,e}(G_{e,e})_{B_3.M} \circ S_e((G_e)_{B_3.M} \circ (b_2.id_M)) \\
&\qquad \circ (G_e)_{B_2.M} \circ (b_1.id_M) \\
&= \delta_{g,e} S_e(b_2.id_M) \circ (G_e)_{B_2.M} \circ (b_1.id_M) \\
&= \delta_{g,e}(G_e)_{B_2.M} \circ (b_2.id_M) \circ (b_1.id_M) \\
&= \delta_{g,e}(G_e)_{B_2.M} \circ ((b_2 \circ b_1).id_M) \\
&= \left(\rho(b_2 \circ b_1)_M\right)_g .
\end{aligned}
$$

Third, we show that ρ is a monoidal functor and thus show that $\mathcal{M}_G \in \mathcal{B} - \mathrm{Mod}$. Since $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$, for all $B_1, B_2 \in \mathcal{B}$ we a have natural isotransformation $\psi_{B_1,B_2} : B_1. \circ B_2. \xrightarrow{\sim} (B_1 \otimes B_2).$ in $\mathrm{End}_{\mathcal{B}}(\mathcal{M})$. We define a natural isotransformation $\phi_{B_1,B_2} : \rho(B_1)\rho(B_2) \xrightarrow{\sim} \rho(B_1 \otimes B_2)$ by defining $(\phi_{B_1,B_2})_M$ for all $M \in \mathcal{M}_G$ through setting $\left((\phi_{B_1,B_2})_M\right)_g := \delta_{g,e}\left((G_e)_{(B_1\otimes B_2).M} \circ (\psi_{B_1,B_2})_M\right)$ for all $g \in G$. For all $f \in \mathcal{M}_G(M_1, M_2)$, we have $(\phi_{B_1,B_2})_{M_2} \circ \rho(B_1)(\rho(B_2)(f)) = \rho(B_1 \otimes B_2)(f) \circ (\phi_{B_1,B_2})_{M_1}$, since for all $g \in G$

$$
\left((\phi_{B_1,B_2})_{M_2} \circ \rho(B_1)(\rho(B_2)(f))\right)_g
$$

$$
\begin{aligned}
&= \sum_{k\in G} (G_{k,k^{-1}g})_{(B_1\otimes B_2).M_2} \circ S_k(((\phi_{B_1,B_2})_{M_2})_{k^{-1}g}) \circ (\rho(B_1)(\rho(B_2)(f)))_k \\
&= (G_{g,e})_{(B_1\otimes B_2).M_2} \circ S_g((G_e)_{(B_1\otimes B_2).M_2} \circ (\psi_{B_1,B_2})_{M_2}) \circ (\rho(B_1)(\rho(B_2)(f)))_g \\
&= S_g((\psi_{B_1,B_2})_{M_2}) \circ ((\mathrm{can}_{B_1,S_g})_{\rho(B_2)(M_2)} \circ (B_1.((\rho(B_2)(f))_g)) \\
&= S_g((\psi_{B_1,B_2})_{M_2}) \circ (\mathrm{can}_{B_1,S_g})_{B_2.M_2} \circ (B_1.((\mathrm{can}_{B_2,S_g})_{M_2} \circ (B_2.(f_g))) \\
&= S_g((\psi_{B_1,B_2})_{M_2}) \circ (\mathrm{can}_{B_1,S_g})_{B_2.M_2} \circ (B_1.(\mathrm{can}_{B_2,S_g})_{M_2}) \circ (B_1.(B_2.(f_g))) \\
&= (\mathrm{can}_{B_1\otimes B_2,S_g})_{M_2} \circ (\psi_{B_1,B_2})_{S_g(M_2)} \circ (B_1.(B_2.(f_g))) \\
&= (\mathrm{can}_{B_1\otimes B_2,S_g})_{M_2} \circ ((B_1\otimes B_2).(f_g)) \circ (\psi_{B_1,B_2})_{M_1} \\
&= (G_{e,g})_{(B_1\otimes B_2).M_2} \circ (G_e)_{S_g((B_1\otimes B_2).M_2)} \circ (\mathrm{can}_{B_1\otimes B_2,S_g})_{M_2} \\
&\qquad \circ ((B_1\otimes B_2).(f_g)) \circ (\psi_{B_1,B_2})_{M_1} \\
&= (G_{e,g})_{(B_1\otimes B_2).M_2} \circ S_e((\mathrm{can}_{B_1\otimes B_2,S_g})_{M_2} \circ ((B_1\otimes B_2).(f_g))) \\
&\qquad \circ (G_e)_{(B_1\otimes B_2).M_1} \circ (\psi_{B_1,B_2})_{M_1} \\
&= (G_{e,g})_{(B_1\otimes B_2).M_2} \circ S_e((\rho(B_1\otimes B_2)(f))_g) \circ (G_e)_{(B_1\otimes B_2).M_1} \circ (\psi_{B_1,B_2})_{M_1} \\
&= \sum_{k\in G} (G_{k,k^{-1}g})_{(B_1\otimes B_2).M_2} \circ S_k((\rho(B_1\otimes B_2)(f))_{k^{-1}g}) \circ ((\phi_{B_1,B_2})_{M_1})_k \\
&= (\rho(B_1\otimes B_2)(f) \circ (\phi_{B_1,B_2})_{M_1})_g
\end{aligned}
$$

holds, where we have used that S_g is a $\mathcal{B}$-functor and in particular the relation $\mathrm{can}_{B_1\otimes B_2,S_g} \circ (\psi_{B_1,B_2} S_g) = (S_g \psi_{B_1,B_2}) \circ (\mathrm{can}_{B_1,S_g}(B_2.)) \circ ((B_1.)\,\mathrm{can}_{B_2,S_g})$. Thus, ϕ_{B_1,B_2} is a natural transformation. For all $M \in \mathcal{M}_G$ and $g \in G$, we set $((\phi^{-1}_{B_1,B_2})_M)_g := \delta_{e,g}\left(S_e((\psi_{B_1,B_2})^{-1}_M) \circ (G_e)_{(B_1\otimes B_2).M}\right)$, which defines an inverse for ϕ_{B_1,B_2}. Hence, we have shown that ϕ_{B_1,B_2} is a natural isotransformation. Note the following useful relation $((\phi^{-1}_{B_1,B_2})_M)_g = \delta_{g,e}\left((G_e)_{B_1.(B_2.M)} \circ (\psi_{B_1,B_2})^{-1}_M\right)$. Furthermore, we need to show that ϕ defined through ϕ_{B_1,B_2} for all $B_1, B_2 \in \mathcal{B}$ is a natural transformation. The technicality being very similar, we only provide it for naturality in the second argument of ϕ. For all $b \in \mathcal{B}(B_2, B_3)$, we have $\phi_{B_1,B_3} \circ (\rho(B_1)\rho(b)) = (\rho(id_{B_1} \otimes b)) \circ \phi_{B_1,B_2}$, since for all $M \in \mathcal{M}_G$ and $g \in G$ we have
$((\phi_{B_1,B_3} \circ (\rho(B_1)\rho(b)))_M)_g$

$$
\begin{aligned}
&= ((\phi_{B_1,B_3})_M \circ (\rho(B_1)\rho(b))_M)_g \\
&= \sum_{k\in G}(G_{k,k^{-1}g})_{(B_1\otimes B_3).M} \circ S_k((\phi_{B_1,B_3})_M)_{k^{-1}g}) \circ ((\rho(B_1)\rho(b))_M)_k \\
&= (G_{g,e})_M \circ S_g((G_e)_{(B_1\otimes B_3).M} \circ (\psi_{B_1,B_3})_M) \circ ((\rho(B_1)\rho(b))_M)_g \\
&= S_g((\psi_{B_1,B_3})_M) \circ (\rho(B_1)(\rho(b)_M))_g \\
&= S_g((\psi_{B_1,B_3})_M) \circ (\mathrm{can}_{B_1,S_g})_{B_3.M} \circ (B_1.((\rho(b)_M)_g)) \\
&= S_g((\psi_{B_1,B_3})_M) \circ (\mathrm{can}_{B_1,S_g})_{B_3.M} \circ (B_1.(\delta_{g,e}(\mathrm{can}_{B_3,S_e})_M \circ (b.((G_e)_M)))) \\
&= \delta_{g,e}\left(S_e((\psi_{B_1,B_3})_M) \circ (\mathrm{can}_{B_1,S_e})_{B_3.M} \circ (B_1.(\mathrm{can}_{B_3,S_e})_M) \circ (B_1.(b.(G_e)_M))\right) \\
&= \delta_{g,e}\left((\mathrm{can}_{B_1\otimes B_3,S_e})_M \circ (\psi_{B_1,B_3})_{S_e(M)} \circ (B_1.(b.(G_e)_M))\right) \\
&= \delta_{g,e}\left((\mathrm{can}_{B_1\otimes B_3,S_e})_M \circ ((id_{B_1}\otimes b).((G_e)_M)) \circ (\psi_{B_1,B_2})_M\right) \\
&= (\rho(id_{B_1}\otimes b)_M)_g \circ (\psi_{B_1,B_2})_M \\
&= (G_{e,g})_{(B_1\otimes B_3).M} \circ (G_e)_{(B_1\otimes B_3).M} \circ (\rho(id_{B_1}\otimes b)_M)_g \circ (\psi_{B_1,B_2})_M \\
&= (G_{e,g})_{(B_1\otimes B_3).M} \circ S_e((\rho(id_{B_1}\otimes b)_{(B_1\otimes B_2).M})_g) \circ (G_e)_{(B_1\otimes B_2).M} \circ (\psi_{B_1,B_2})_M \\
&= \sum_{k\in G}(G_{k,k^{-1}g})_{(B_1\otimes B_3).M} \circ S_k((\rho(id_{B_1}\otimes b)_{(B_1\otimes B_2).M})_{k^{-1}g}) \circ ((\phi_{B_1,B_2})_M)_k \\
&= ((\rho(id_{B_1}\otimes b)_{(B_1\otimes B_2).M} \circ (\phi_{B_1,B_2})_M)_g \\
&= (((\rho(id_{B_1}\otimes b)) \circ \phi_{B_1,B_2})_M)_g.
\end{aligned}
$$

Next, we provide the unit transformation for $\mathcal{M}_G$. Let us denote the unit transformation of $\mathcal{M}$ by $u : id_{\mathcal{M}} \Rightarrow I$. where $I \in \mathcal{B}$ is the unit object. We define the unit transformation $\widetilde{u} : id_{\mathcal{M}_G} \Rightarrow \rho(I)$ of $\mathcal{M}_G$ by setting $(\widetilde{u}_M)_g := \delta_{g,e}\left((G_e)_{I.M} \circ u_M\right)$ for all $g \in G$ and $M \in \mathcal{M}_G$. For all $f \in \mathcal{M}_G(M_1, M_2)$, we have

$$
\begin{aligned}
(\widetilde{u}_{M_2} \circ f)_g &= \sum_{k\in G}(G_{k,k^{-1}g})_{I.M_2} \circ S_k((\widetilde{u}_{M_2})_{k^{-1}g}) \circ f_k \\
&= (G_{g,e})_{I.M_2} \circ S_g((G_e)_{I.M_2} \circ u_{M_2}) \circ f_g \\
&= S_g(u_{M_2}) \circ f_g \\
&= (\mathrm{can}_{I,S_g})_{M_2} \circ u_{S_g(M_2)} \circ (I.(f_g)) \\
&= (\mathrm{can}_{I,S_g})_{M_2} \circ (I.(f_g)) \circ u_{M_1} \\
&= (G_{e,g})_{I.M_2} \circ (G_e)_{S_g(I.M_2)} \circ (\mathrm{can}_{I,S_g})_{M_2} \circ (I.(f_g)) \circ u_{M_1} \\
&= (G_{e,g})_{I.M_2} \circ S_e((\mathrm{can}_{I,S_g})_{M_2} \circ (I.(f_g))) \circ (G_e)_{I.M_1} \circ u_{M_1} \\
&= (G_{e,g})_{I.M_2} \circ S_e((\rho(I)(f))_g) \circ (G_e)_{I.M_1} \circ u_{M_1} \\
&= \sum_{k\in G}(G_{k,k^{-1}g})_{I.M_2} \circ S_k((\rho(I)(f))_{k^{-1}g}) \circ (\widetilde{u}_{M_1})_k \\
&= ((\rho(I)(f)) \circ \widetilde{u}_{M_1})_g
\end{aligned}
$$

for all $g \in G$, where we used that S_g is a $\mathcal{B}$-functor and in particular $S_g u =$

$\mathrm{can}_{(I.),S_g} \circ (uS_g)$. Thus, $\widetilde{u}$ is indeed a natural transformation. Note that $\widetilde{u}$ is invertible with inverse defined by $(\widetilde{u}_M)_g := \delta_{g,e}\left((G_e)_M \circ u_M^{-1}\right)$. Thus, $\widetilde{u}$ is a natural isotransformation. We leave it to the reader to check that ϕ is compatible with the associator in $\mathcal{B}$ (and the trivial associator of $\mathrm{End}_{\mathcal{B}}(\mathcal{M}_G)$), as well as with the left (and right) unit transformation of $\mathcal{B}$ and $\mathcal{M}_G$ (and the trivial unit transformation in $\mathrm{End}_{\mathcal{B}}(\mathcal{M}_G)$).

Fourth, we show that ι_G is a $\mathcal{B}$-functor. With all the preparation, this is now rather straight forward. We restrict ourselves to providing the morphisms that define the canonical isotransformation can_{B,ι_G}. Note that for $B \in \mathcal{B}$, $M \in \mathcal{M}_G$, we have $\rho(B.)(\iota_G(M)) = \iota_G(B.M)$ and thus we can use the identity morphism on this object and the canonical map. Thus, we set $((\mathrm{can}_{B,\iota_G})_M)_g := \delta_{e,g}\left((G_e)_{B.M}\right)$ for all $g \in G$. Clearly, can_{B,ι_G} is a natural isotransformation. Furthermore, can_{B,ι_G} is obviously natural in B as well. Again, we leave it to the reader to check that the can_{B,ι_G} for $B \in \mathcal{B}$ satisfy the necessary compatibility with the monoidal structure of $\mathcal{B}$, ρ and the action on $\mathcal{M}$, as well as with the compatibility with the u and $\widetilde{u}$. $\square$

The $\mathcal{B}$-functor ι_G allows for the following additional structure which will be essential in establishing representability of Φ_G:

Proposition 4.6.13. *There exist iso-$\mathcal{B}$-transformations $\nu_g^G : \iota_G S_g \xRightarrow{\sim} \iota_G$ satisfying $\nu_{g_2}^G \circ (\nu_{g_1}^G S_{g_2}) = \nu_{g_1 g_2}^G \circ (\iota_G G_{g_1,g_2})$ and $\nu_e^G = \iota_G G_e^{-1}$.*

Proof. Let us define ν_g^G for all $g \in G$ by setting $\left((\nu_g^G)_M\right)_h := \delta_{h,g}\left(id_{S_g(M)}\right)$ for all $M \in \mathcal{M}$ and $h \in G$. For all $f \in \mathcal{M}(M_1, M_2)$, we have

$$\begin{aligned}
(\iota_G(f) \circ (\nu_g^G)_{M_1})_h &= \sum_{k\in G}(G_{k,k^{-1}h})_{M_2} \circ S_k(\delta_{e,k^{-1}h}\left((G_e)_{M_2} \circ f)\right) \circ (\delta_{k,g}\left(id_{S_g(M_1)}\right)) \\
&= \delta_{e,g^{-1}h}\left((G_{g,g^{-1}h})_{M_2} \circ S_g((G_e)_{M_2} \circ f) \circ id_{S_g(M_1)}\right) \\
&= \delta_{g,h}\left((G_{g,e})_{M_2} \circ S_g((G_e)_{M_2}) \circ S_g(f) \circ id_{S_g(M_1)}\right) \\
&= \delta_{g,h}\left(S_g(f)\right) \\
&= \delta_{h,g}\left((G_{e,h})_{M_2} \circ S_e(id_{S_g(M_2)}) \circ (G_e)_{S_g(M_2)} \circ S_g(f)\right) \\
&= \sum_{k\in G}(G_{k,k^{-1}h})_{M_2} \circ S_k(\delta_{k^{-1}h,g}\left(id_{S_g(M_2)}\right)) \circ (\delta_{e,k}\left((G_e)_{S_g(M_2)} \circ S_g(f)\right)) \\
&= ((\nu_g^G)_{M_2} \circ \iota_G(S_g(f)))_h
\end{aligned}$$

for all $h \in G$ and thus $\iota_G(f) \circ (\nu_g^G)_{M_1} = (\nu_g^G)_{M_2} \circ \iota_G(S_g(f))$. Thus, ν_g^G is a transformation. By Lemma 4.6.10 $(\nu_g^G)_M$ is invertible, and thus ν_g^G is an isotransformation. The relation $\nu_{g_2}^G \circ (\nu_{g_1}^G S_{g_2}) = \nu_{g_1 g_2}^G \circ (\iota_G G_{g_1,g_2})$ holds, since we have

$$
\begin{aligned}
\left((\nu^G_{g_2} \circ \nu^G_{g_1} S_{g_2})_M\right)_h &= \sum_{k \in G} (G_{k,k^{-1}h})_M \circ S_k(((\nu^G_{g_2})_M)_{k^{-1}h}) \circ ((\nu^G_{g_1})_{S_{g_2}(M)})_k \\
&= \sum_{k \in G} (G_{k,k^{-1}h})_M \circ S_k(\delta_{k^{-1}h,g_2}(id_{S_{g_2}(M)})) \circ (\delta_{k,g_1}(id_{S_{g_1}(S_{g_2}(M))})) \\
&= \delta_{h,g_1g_2}\left((G_{g_1,g_2})_M \circ S_{g_1}(id_{S_{g_2}(M)}) \circ id_{S_{g_1}(S_{g_2}(M))}\right) \\
&= \delta_{h,g_1g_2}(G_{g_1,g_2})_M \\
&= \delta_{g_1g_2,h}\left((G_{e,h})_M \circ ((G_e)_{S_{g_1g_2}(M)} \circ (G_{g_1,g_2})_M\right) \\
&= \sum_{k \in G} (G_{k,k^{-1}h})_M \circ S_k\left(\delta_{g_1g_2,k^{-1}h}\left(id_{S_{g_1g_2}(M)}\right)\right) \\
&\qquad \circ \left(\delta_{k,e}((G_e)_{S_{g_1g_2}(M)} \circ (G_{g_1,g_2})_M)\right) \\
&= \sum_{k \in G} (G_{k,k^{-1}h})_M \circ S_k((\nu^G_{g_1g_2})_M)_{k^{-1}h}) \circ (\iota_G((G_{g_1,g_2})_M))_k \\
&= \left((\nu^G_{g_1g_2} \circ (\iota_G G_{g_1,g_2})_M\right)_h
\end{aligned}
$$

for all $M \in \mathcal{M}$ and $h \in G$. Further, the relation $\nu^G_e = \iota_G G_e^{-1}$ also holds, since $((\iota_G G_e^{-1})_M)_h = (\iota_G((G_e^{-1})_M))_h = \delta_{h,e}((G_e)_M \circ (G_e^{-1})_M) = \delta_{h,e}\left(id_{S_e(M)}\right) = \left((\nu^G_e)_M\right)_h$.

□

Theorem 4.6.14. *The $\mathcal{B}$-module $\mathcal{M}_G$ represents the 2-functor Φ_G.*

Proof. First, we have to provide morphisms of 2-functors $\sigma^1 : \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{M}_G, -) \to \Phi_G$ and $\sigma^2 : \Phi_G \to \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{M}_G, -)$. That is for all $\mathcal{N} \in \mathcal{B} - \mathrm{Mod}$ we have to provide additive functors $\sigma^1_{\mathcal{N}} : \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{M}_G, \mathcal{N}) \to \Phi_G(\mathcal{N})$ and $\sigma^2_{\mathcal{N}} : \Phi_G(\mathcal{N}) \to \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{M}_G, \mathcal{N})$ together with certain isotransformations (like $\sigma^1_{\mathcal{N}_2} \circ \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{M}_G, F) \cong \Phi_G(F) \circ \sigma^1_{\mathcal{N}_1}$ for all $F \in \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{N}_1, \mathcal{N}_2)$) satisfying axioms. Next we will provide these additive functors in such a way that the above mentioned isotransformations can be chosen as identity transformations. Thus, the required axioms are satisfied automatically.

Let $\mathcal{N} \in \mathcal{B} - \mathrm{Mod}$. We set $\sigma^1_{\mathcal{N}}(\tilde{F}) := (\tilde{F}\iota_G, \{\tilde{F}\nu^G_g\}_{g \in G})$ for all $\tilde{F} \in \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{M}_G, \mathcal{N})$ and $\sigma^1_{\mathcal{N}}(\tilde{\alpha}) := \tilde{\alpha}\iota_G$ for all morphism in $\mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{M}_G, \mathcal{N})$. Proposition 4.6.13 together with the interchange law (Remark 3.1.3) implies that $\sigma^1_{\mathcal{N}}$ is well defined. Thus, we have fully defined σ^1.

Defining σ^2 requires more work: We define $\sigma^2_{\mathcal{N}}\left((F, \{\nu^F_g\}_{g \in G})\right)$ for all $(F, \{\nu^F_g\}_{g \in G}) \in \Phi_G(\mathcal{N})$ to be the $\mathcal{B}$-functor that sends $M \in \mathcal{M}_G$ to $F(M)$ (where we used $\mathrm{Ob}(\mathcal{M}_G) = \mathrm{Ob}(\mathcal{M})$) and $(f_g)_{g \in G} \in \mathcal{M}_G(M_1, M_2)$ to $\sum_{g \in G} (\nu^F_g)_{M_2} \circ F(f_g)$. For convenience, we abbreviate this assignment by $\check{F}$. We have $\check{F}(id_M) = \sum_{g \in G} (\nu^F_g)_M \circ F(\delta_{e,g}((G_e)_M)) = (\nu^F_e)_M \circ F((G_e)_M) = \left(\nu^F_e \circ (FG_e)\right)_M = (id_F)_M = id_{F(M)} = id_{\check{F}(M)}$ for all $M \in \mathcal{M}_G$, where we used that $\nu^F_e = FG_e^{-1}$ which holds by definition of $\Phi_G(\mathcal{N})$. Also,

$$
\begin{aligned}
\check{F}(f^2 \circ f^1) &= \check{F}\left((\sum_{k \in G} (G_{k,k^{-1}g})_{M_3} \circ S_k(f^2_{k^{-1}g}) \circ f^1_k)_{g \in G} \right) \\
&= \sum_{k \in G} \sum_{g \in G} (\nu^F_g)_{M_3} \circ F\left((G_{k,k^{-1}g})_{M_3} \circ S_k(f^2_{k^{-1}g}) \circ f^1_k \right) \\
&= \sum_{k,g \in G} (\nu^F_{kg})_{M_3} \circ F((G_{k,g})_{M_3}) \circ F(S_k(f^2_g)) \circ F(f^1_k) \\
&= \sum_{g,k \in G} (\nu^F_g)_{M_3} \circ (\nu^F_k)_{S_k(M_3)} \circ F(S_k(f^2_g)) \circ F(f^1_k) \\
&= \left(\sum_{g \in G} (\nu^F_g)_{M_3} \circ F(f^2_g) \right) \circ \left(\sum_{k \in G} (\nu^F_k)_{M_2} \circ F(f^1_k) \right) \\
&= \check{F}(f^2) \circ \check{F}(f^1)
\end{aligned}
$$

for all $f^1 = (f^1_g)_{g \in G} \in \mathcal{M}_G(M_1, M_2)$ and $f^2 = (f^2_g)_{g \in G} \in \mathcal{M}_G(M_2, M_3)$, where we used that G is a group, that F is a functor and that $\nu^F_{kg} \circ (FG_{k,g}) = \nu^F_g \circ (\nu^F_k S_g)$ which holds by definition of $\Phi_G(\mathcal{N})$. Thus, $\check{F}$ is indeed a functor and it is additive since F is additive and since composition is bilinear. Next, we have to equipped $\check{F}$ with the structure of a $\mathcal{B}$-functor. Since F is a $\mathcal{B}$-functor, we have an isotransformation $\mathrm{can}_{B,F} : (B.)F \overset{\sim}{\Longrightarrow} F(B.)$ for all $B \in \mathcal{B}$. For all $B \in \mathcal{B}$, we define $\mathrm{can}_{B,\check{F}} : (B.)\check{F} \overset{\sim}{\Longrightarrow} \check{F}\rho(B)$ by setting $(\mathrm{can}_{B,\check{F}})_M := (\mathrm{can}_{B,\check{F}})_M$ for all $M \in \mathcal{M}_G$. This is possible since $\mathrm{Ob}(\mathcal{M}_G) = \mathrm{Ob}(\mathcal{M})$, $B.(\check{F}(M)) = B.(F(M))$ and $\check{F}(\rho(B)M) = \check{F}(B.M) = F(B.M)$. For all $B \in \mathcal{B}$, $\mathrm{can}_{B,\check{F}}$ is indeed a transformation since

$$
\begin{aligned}
(\mathrm{can}_{B,\check{F}})_{M_2} \circ (B.(\check{F}(f))) &= (\mathrm{can}_{B,F})_{M_2} \circ (B.(\sum_{g \in G} (\nu^F_g)_{M_2} \circ F(f_g))) \\
&= \sum_{g \in G} (\mathrm{can}_{B,F})_{M_2} \circ ((B.)\nu^F_g)_{M_2}) \circ B.(F(f_g)) \\
&= \sum_{g \in G} (\nu^F_g)_{B.M_2} \circ (\mathrm{can}_{B,FS_g})_{M_2}) \circ B.(F(f_g)) \\
&= \sum_{g \in G} (\nu^F_g)_{B.M_2} \circ F((\mathrm{can}_{B,S_g})_{M_2}) \circ (\mathrm{can}_{B,F})_{S_g(M_2)}) \circ B.(F(f_g)) \\
&= \sum_{g \in G} (\nu^F_g)_{B.M_2} \circ F((\mathrm{can}_{B,S_g})_{M_2}) \circ F(B.(f_g)) \circ (\mathrm{can}_{B,F})_{M_1}) \\
&= \check{F}(((\mathrm{can}_{B,S_g})_{M_2} \circ (B.(f_g)))_{g \in G}) \circ (\mathrm{can}_{B,F})_{M_1} \\
&= \check{F}(\rho(B)(f)) \circ (\mathrm{can}_{B,\check{F}})_{M_1}
\end{aligned}
$$

for all $f = (f_g)_{g \in G} \in \mathcal{M}_G(M_1, M_2)$, where we have used that ν^F_g is a $\mathcal{B}$-transformation and in particular that $\mathrm{can}_{B,f} \circ ((B.)\nu^F_g) = (\nu^F_g(B.)) \circ \mathrm{can}_{B,FS_g}$, as well as the definition of compositions of $\mathcal{B}$-functors. Since $\mathrm{can}_{B,\check{F}}$ is build from isomorphisms, it is an isotransformation. Further, $\mathrm{can}_{B,\check{F}}$ is natural in B: For all $b \in \mathcal{B}(B_1, B_2)$, we have

$$
\begin{aligned}
(\mathrm{can}_{B_2,\check{F}} \circ (b.\check{F}))_M &= (\mathrm{can}_{B_2,F})_M \circ (b.(F(id_M)))\\
&= F(b.id_M) \circ (\mathrm{can}_{B_1,F})_M\\
&= (\nu_e^F)_{B_2.M} \circ F((G_e)_{B_2.M} \circ (b.id_M)) \circ \mathrm{can}_{B_1,F})_M\\
&= \sum_{g\in G} (\nu_g^F)_{B_2.M} \circ F(\delta_{g,e}\,((G_e)_{B_2.M} \circ (b.id_M))) \circ (\mathrm{can}_{B_1,F})_M\\
&= \check{F}(\delta_{g,e}\,((G_e)_{B_2.M} \circ (b.id_M))) \circ \mathrm{can}_{B_1,F})_M\\
&= \check{F}((\rho(b)_M) \circ (\mathrm{can}_{B_1,F})_M\\
&= ((\check{F}\rho(b)) \circ \mathrm{can}_{B_1,\check{F}})_M
\end{aligned}
$$

for all $M \in \mathcal{M}_G$, where we used that $\nu_e^F = FG_e^{-1}$ by definition of $\Phi_G(\mathcal{N})$. The last step in equipping $\check{F}$ with the structure of a $\mathcal{B}$-functor is to show that the isotransformations $\mathrm{can}_{B,\check{F}}$ are compatible with the monoidal structure of $\mathcal{B}$, ρ and the action on $\mathcal{N}$. These hold trivially, since we have defined the monoidal structure of ρ using the isotransformations that define $\mathcal{M}$, they are mapped by $\hat{F}$ and F respectively to the same isotransformations in $\mathcal{N}$.
Finally, we return to the definition of $\sigma^2_{\mathcal{N}}$. For $\alpha \in \Phi_G(\mathcal{N})((F_1, \{\nu_g^1\}_{g\in G}), (F_2, \{\nu_g^2\}_{g\in G}))$, we define $\sigma^2_{\mathcal{N}}(\alpha) : \check{F}_1 \Rightarrow \check{F}_2$ to be the $\mathcal{B}$-transformation with $(\sigma^2_{\mathcal{N}}(\alpha))_M := \alpha_M$ for all $M \in \mathcal{M}_G$, where we used that $\mathrm{Ob}(\mathcal{M}_G) = \mathrm{Ob}(\mathcal{M})$. For notational convenience, we abbreviate $\check{\alpha} := \sigma^2_{\mathcal{N}}(\alpha)$. We have

$$
\begin{aligned}
\check{F}_2(f) \circ \check{\alpha}_{M_1} &= \sum_{g\in G} (\nu_g^2)_{M_2} \circ F_2(f_g) \circ \alpha_{M_1}\\
&= \sum_{g\in G} (\nu_g^2)_{M_2} \circ \alpha_{S_g(M_2)} \circ F_1(f_g)\\
&= \sum_{g\in G} \alpha_{M_2} \circ (\nu_g^1)_{M_2} \circ F_1(f_g)\\
&= \alpha_{M_2} \circ \left(\sum_{g\in G} (\nu_g^1)_{M_2} \circ F_1(f_g) \right)\\
&= \check{\alpha}_{M_2} \circ \check{F}_1(f)
\end{aligned}
$$

for all $f = (f_g) \in \mathcal{M}(M_1, M_2)$, where we used that α is a transformation and that $\alpha \circ \nu_g^1 = \nu_g^2 \circ (\alpha S_g)$ by definition of $\Phi_G(\mathcal{N})$. Thus, $\check{\alpha}$ is indeed a transformation. Further, $\check{\alpha}$ is a $\mathcal{B}$-transformation, since

$$
\begin{aligned}
\left(\mathrm{can}_{B,\check{F}_2} \circ ((B.)\check{\alpha})\right)_M &= (\mathrm{can}_{B,\check{F}_2})_M \circ (B.(\check{\alpha}_M))\\
&= (\mathrm{can}_{B,F_2})_M \circ (B.(\alpha_M))\\
&= (\alpha_{B.M}) \circ (\mathrm{can}_{B,F_2})_M\\
&= (\check{\alpha})_{\rho(B)(M)} \circ (\mathrm{can}_{B,\check{F}_2})_M\\
&= \left((\check{\alpha}\rho(B)) \circ \mathrm{can}_{B,\check{F}_2}\right)_M
\end{aligned}
$$

for all $B \in \mathcal{B}$ and $M \in \mathcal{M}_G$, where we used that α is a $\mathcal{B}$-transformation.
Next, we have to show that $\sigma^2_{\mathcal{N}}$ is an additive functor. Note that $id_{(F,\{\nu^F_g\}_{g\in G})} = id_F$. Thus, $\sigma^2_{\mathcal{N}}(id_{(F,\{\nu^F_g\}_{g\in G})}) = id_{\sigma^2_{\mathcal{N}}((F,\{\nu^F_g\}_{g\in G}))}$ follows. Further for all composable morphisms α_1 and α_2 in $\Phi_G(\mathcal{N})$, we have

$$\begin{aligned}\big(\sigma^2_{\mathcal{N}}(\alpha_2 \circ \alpha_1)\big)_M &= (\alpha_2 \circ \alpha_1)_M \\ &= (\alpha_2)_M \circ (\alpha_1)_M \\ &= \big(\sigma^2_{\mathcal{N}}(\alpha_2)\big)_M \circ \big(\sigma^2_{\mathcal{N}}(\alpha_1)\big)_M \\ &= \big(\sigma^2_{\mathcal{N}}(\alpha_2) \circ \sigma^2_{\mathcal{N}}(\alpha_1)\big)_M\end{aligned}$$

for all $M \in \mathcal{M}_G$. The definition of σ^2 is complete.

We now claim that $\sigma^1 \circ \sigma^2 = id_{\mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{M}_G,-)}$ and $\sigma^2 \circ \sigma^1 = id_{\Phi_G}$, which would complete the proof. We will restrict ourselves to showing that these equations hold at any $\mathcal{N} \in \mathcal{B} - \mathrm{Mod}$. We leave it to the reader to check isotransformation induced on the compositions actually agree with the identity transformations as claimed.
We fix $\mathcal{N} \in \mathcal{B} - \mathrm{Mod}$ and let $(F, \{\nu_g\}_{g\in G}) \in \Phi_G(\mathcal{N})$. Denoting as before $\check{F} := \sigma^1_{\mathcal{N}}((F, \{\nu_g\}_{g\in G}))$, we have $\sigma^1_{\mathcal{N}}(\check{F}) := (\check{F}\iota_G, \{\check{F}\nu^G_g\}_{g\in G})$. For all $M \in \mathcal{M}$, we have $(\check{F}\iota_G)(M) = \check{F}(\iota_G(M)) = \check{F}(M) = F(M)$ by definition of $\sigma^1_{\mathcal{N}}$. Further, we have

$$\begin{aligned}&(\check{F}\iota_G)(f) \\ &\quad= \check{F}((\delta_{g,e}((G_e)_{M_2} \circ f))_{g\in G}) \\ &\quad= \sum_{h\in G} (\nu_h)_{M_2} \circ F(\delta_{h,e}((G_e)_{M_2} \circ f)) \\ &\quad= (\nu_e)_{M_2} \circ F((G_e)_{M_2}) \circ F(f)) \\ &\quad= F(f)\end{aligned}$$

for all $f \in \mathcal{M}(M_1, M_2)$. Also, we have

$$\begin{aligned}(\mathrm{can}_{B,\check{F}\iota_G})_M &= ((\check{F}\,\mathrm{can}_{B,\iota_G}) \circ (\mathrm{can}_{B,\check{F}}\,\iota_G))_M \\ &= \check{F}((\mathrm{can}_{B,\iota_G})_M) \circ (\mathrm{can}_{B,\check{F}})_M \\ &= \check{F}((\delta_{g,e}((G_e)_{B.M}))_{g\in G}) \circ (\mathrm{can}_{B,F})_M \\ &= \sum_{h\in G} (\nu_h)_{B.M} \circ F(\delta_{h,e}((G_e)_{B.M}))) \circ (\mathrm{can}_{B,F})_M \\ &= (\nu_e)_{B.M} \circ F((G_e)_{B.M})) \circ (\mathrm{can}_{B,F})_M \\ &= (\mathrm{can}_{B,F})_M\end{aligned}$$

for all $B \in \mathcal{B}$ and all $M \in \mathcal{M}$. We conclude $F = \check{F}\iota_G$ as $\mathcal{B}$-functors. For all $\alpha \in \Phi_G(\mathcal{N})\big((F_1, \{\nu^1_g\}_{g\in G}), (F_2, \{\nu^2_g\}_{g\in G})\big)$, we denote as before $\check{\alpha} := \sigma^1_{\mathcal{N}}(\alpha)$. For all $M \in \mathcal{M}$, we have $(\sigma^2_{\mathcal{N}}(\check{\alpha}))_M = (\check{\alpha})_M = \alpha_M$. Thus, we have shown that $\sigma^2_{\mathcal{N}} \circ \sigma^1_{\mathcal{N}} = id_{\Phi_G(\mathcal{N})}$.
Next, let $\tilde{F} \in \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{M}_G, \mathcal{N})$. Then, $(F, \{\nu^F_g\}_{g\in G}) := \sigma^2_{\mathcal{N}}(\tilde{F}) = (\tilde{F}\iota_G, \{\tilde{F}\nu^G_g\}_{g\in G})$.

Denote $\check{F} = \sigma^1_{\mathcal{N}}((F, \{\nu^F_g\}_{g\in G}))$. For all $M \in \mathcal{M}_G$, we have $\check{F}(M) = F(M) = \tilde{F}(M)$. Further for all $f = (f_g)_{g\in G} \in \mathcal{M}_G(M_1, M_2)$, we have

$$\begin{aligned}
\check{F}(f) &= \sum_{g\in G} (\nu^F_g)_{M_2} \circ F(f_g) \\
&= \sum_{g\in G} \tilde{F}((\nu^G_g)_{M_2} \circ \iota_G(f_g)) \\
&= \sum_{g\in G} \tilde{F}\left(\left(\sum_{h\in G} (G_{h,h^{-1}k})_{M_2} \circ S_h(\delta_{h^{-1}k,g}(id_{S_g(M_2)})) \circ (\delta_{h,e}((G_e)_{S_g(M_2)} \circ f_g))\right)_{k\in G}\right) \\
&= \sum_{g\in G} \tilde{F}\left(\left(\delta_{k,g}((G_{e,k})_{M_2} \circ (G_e)_{S_g(M_2)} \circ f_g)\right)_{k\in G}\right) \\
&= \sum_{g\in G} \tilde{F}\left((\delta_{k,g}(f_g))_{k\in G}\right) \\
&= \tilde{F}\left(((f_k)_{k\in G}\right) \\
&= \tilde{F}(f).
\end{aligned}$$

Also for all $B \in \mathcal{B}$ and all $M \in \mathcal{M}_G$, we have

$$\begin{aligned}
(\mathrm{can}_{B,\check{F}})_M &= (\mathrm{can}_{B,F})_M \\
&= (\mathrm{can}_{B,\tilde{F}\iota_G})_M \\
&= ((\tilde{F}\,\mathrm{can}_{B,\iota_G}) \circ (\mathrm{can}_{B,\tilde{F}}\,\iota_G))_M \\
&= \tilde{F}((\delta_{g,e}((G_e)_{B.M}))_{g\in G}) \circ (\mathrm{can}_{B,\tilde{F}})_M \\
&= (\mathrm{can}_{B,\tilde{F}})_M,
\end{aligned}$$

where we used that $(\delta_{g,e}((G_e)_{B.M}))_{g\in G}$ is the identity morphism on $B.M$ in $\mathcal{M}_G$. Thus, $\tilde{F} = \check{F}$ as $\mathcal{B}$-functors. Finally for all $\alpha \in \mathcal{B} - \mathrm{Mod}_{\mathcal{M}_G \to \mathcal{N}}(F_1, F_2)$, we have $(\sigma^2_{\mathcal{N}} \circ \sigma^1_{\mathcal{N}}(\alpha))_M = (\sigma^2_{\mathcal{N}}(\alpha\iota_G))_M = (\alpha\iota_G)_M = \alpha_{\iota_G(M)} = \alpha_M$ for all $M \in \mathcal{M}_G$. Thus, we have shown that $\sigma^2_{\mathcal{N}} \circ \sigma^1_{\mathcal{N}} = id_{\Phi_G(\mathcal{N})}$. The proof is complete. □

For the remainder of the section, we study further properties of $\mathcal{M}_G$. We start with a criterion which ensures that indecomposable objects of $\mathcal{M}$ stay indecomposable in $\mathcal{M}$. Then, we will give a simple example that shows that this statement does not hold in this generality for strong indecomposability.

Lemma 4.6.15. *Let $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ and assume that $\mathcal{M}$ has the Krull-Schmidt property. Then, ι_G sends indecomposable objects to indecomposable objects.*

Proof. Let $M \in \mathcal{M}$ be indecomposable and assume that we are given a decomposition $\iota_G(M) \cong \iota_G(N) \oplus \iota_G(L)$ in $\mathcal{M}_G$, where we have used that ι_G is surjective on objects by assumptions. W.l.o.g, we can assume that N is non-trivial, and thus we have to show that $L = 0$. Further using the Krull-Schmidt property of $\mathcal{M}$ and the fact that

ι_G is additive, we can assume that N is indecomposable in $\mathcal{M}$. Abusing notation, we denote $G := \bigoplus_{g\in G} S_g \in \mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$. Clearly, G satisfies the necessary properties and we obtain a $\mathcal{B}$-functor $\hat{G} : \mathcal{M}_G \to \mathcal{M}$ such that $\hat{G} \circ \iota_G = G$. Since $\iota_G(N) \oplus \iota_G(L) \cong \iota_G(N \oplus L)$, we deduce that $\bigoplus_{g\in G} S_g N \subset^{\oplus} \bigoplus_{g\in G} S_g(M)$ in $\mathcal{M}$. Note that since M is indecomposable in $\mathcal{M}$, so is $S_g(M)$ for all $g \in G$. Using once more the Krull-Schmidt property, it follows that there exists $g \in G$ such that $N \cong S_g(M)$ in $\mathcal{M}$. By Lemma 4.6.10, it follows that $\iota_G(N) \cong \iota_G(M)$. Applying $\hat{G}$, we obtain that $G(N) \cong G(M)$. On the other hand, we obtain $G(M) \cong G(N) \oplus G(L)$ in the same way. A final application of the Krull-Schmidt property implies $G(L) = 0$. Since $id_{\mathcal{M}} \subset^{\oplus} G$, it follows that $L \cong 0$ in $\mathcal{M}$ and thus $\iota_G(L) \cong 0$ in $\mathcal{M}_G$. □

Proposition 4.6.16. *Let $\mathcal{M} \in \mathcal{B} - \mathrm{Mod}$ and assume that $\mathcal{M}$ satisfies the Krull-Schmidt property. Then, $\mathcal{M}_G$ satisfies the Krull-Schmidt property and the canonical homomorphism $G \setminus [\mathcal{M}] \to [\mathcal{M}_G]$ is an isomorphism.*

Proof. Let $\iota_G(M) \in \mathcal{M}_G$, and assume that $\iota_G(M) \cong \bigoplus_{i\in I} \iota_G(M_i) \cong \bigoplus_{j\in J} \iota_G(N_j)$ are two decompositions in indecomposable objects in $\mathcal{M}_G$. By the same arguments as above, all M_i and N_j have to be indecomposable objects in $\mathcal{M}$. Applying $\hat{G}$ to the two decompositions of M and using the Krull-Schmidt property of $\mathcal{M}$, we obtain $|I||G| = |J||G|$ and thus $|I| = |J|$. Also with analogous arguments as above, it follows that for all $i \in I$ there exists $j(i) \in J$ and $g_i \in G$ such that $M_i \cong S_{g_i} N_{g_{j(i)}}$ in $\mathcal{M}$. In particular, there exists $i_0 \in I$ and $j_0 \in J$ such that $G(M_{i_0}) \cong G(N_{j_0})$ in $\mathcal{M}$. Applying $\hat{G}$ to the two decomposition of M, we obtain $G(M_{i_0}) \oplus \bigoplus_{i\in I\setminus\{i_0\}} G(M_i) \cong \hat{G}(M) \cong G(N_{j_0}) \oplus \bigoplus_{j\in J\setminus\{j_0\}} G(N_j)$ in $\mathcal{M}$. The Krull-Schmidt property implies that $\bigoplus_{i\in I\setminus\{i_0\}} G(M_i) \bigoplus_{j\in J\setminus\{j_0\}} G(N_j)$. By an induction on $|I|$ (which is a finite number, since $\mathcal{M}$ has the Krull-Schmidt property), we thus obtain a bijection $\phi : I \xrightarrow{\sim} J$ with the property that $G(M_i) \cong G(N_{\phi(i)})$ in $\mathcal{M}$ for all $i \in I$. More concretely, for all $i \in I$ there exists $g_i \in G$ such that $M_i \cong S_{g_i}(N_{\phi(i)})$. By Lemma 4.6.10, we thus obtain $\iota_G(M_i) \cong \iota_G(N_{\phi(i)})$ in $\mathcal{M}_G$. Thus, $\mathcal{M}_G$ satisfies the Krull-Schmidt property.

To deduce that $G \setminus [\mathcal{M}] \to [\mathcal{M}_G]$ is an isomorphisms, it suffices to observe that for indecomposable objects $\iota_G(M), \iota_G(N) \in \mathcal{M}_G$ with $[\iota_G(M)] = [\iota_G(N)]$ in $[\mathcal{M}_G]$ there exists a $g \in G$ such that $M \cong S_g(N)$ in $\mathcal{M}$. Since $\mathcal{M}_G$ satisfies the Krull-Schmidt property, a standard argument implies that $\iota_G(M) \cong \iota_G(N)$ in $\mathcal{M}_G$ and the same arguments as above imply $M \cong S_g(N)$ in $\mathcal{M}$ for some $g \in G$. □

Next, we show through an example that indecomposable objects (and even strongly indecomposable objects) in $\mathcal{M}$ might no longer be strongly indecomposable in $\mathcal{M}_G$ in general. This also implies that $\mathcal{M}_G$ is not idempotent closed in general:

Example 4.6.17. Assume that $2 \in \mathcal{B}(I,I)$ is invertible, that $G = S_2 = \{e,s\}$ and that $S_s^2 = Id$. Further assume that there exists $\phi \in \mathcal{M}(M, S_s(M))$ such that $S_s(\phi) \circ \phi = id_M$. Then, $e_1 := \frac{1}{2}(id_M, \phi)$ and $e_2 := \frac{1}{2}(id_M, -\phi)$ are non-trivial idempotents of $\iota_G(M) \in \mathcal{M}_G$ for which $e_1 e_2 = e_2 \circ e_1 = 0$ and $id_{\iota_G(M)} = e_1 + e_2$ holds.

Even though we have increased the size of morphism spaces by roughly a factor $|G|$, we have the following result:

Proposition 4.6.18. *The $\mathcal{B}$-modules $\mathcal{M}_G$ and $\mathcal{M}$ are weakly equivalent.*

Proof. Since $id_{\mathcal{M}} \cong S_e \subset^{\oplus} G = \hat{G}\iota_G$, half the work is done. Let us denote the associated inclusion and projection $\mathcal{B}$-transformations by τ and γ. It remains to show that $id_{\mathcal{M}_G} \subset^{\oplus} \iota_G \hat{G}$. By Theorem 4.6.14, we can show equivalently that $(\iota_G, \{\nu_g^G\}_{g\in G}) \subset^{\oplus} (\iota_G\hat{G}\iota_G, \{\iota_G\hat{G}\nu_g^G\}_{g\in G})$ in $\Phi_G(\mathcal{M}_G)$. By definition of $\hat{G}$, we have $(\iota_G\hat{G}\iota_G, \{\iota_G\hat{G}\nu_g^G\}_{g\in G}) = (\iota_G G, \{\iota_G \oplus_{h\in G} G_{h,g}\}_{g\in G})$. It is easy to calculate that $\iota\tau$ and $\iota\gamma$ (for which $(\iota\gamma)\circ(\iota\tau) = \iota(\gamma\tau) = \iota id_{\mathcal{M}} = \iota$) are morphisms in $\Phi_G(\mathcal{M}_G)$. It follows that $(\iota_G, \{\nu_g^G\}_{g\in G})$ is a direct summand of $(\iota_G G, \{\iota_G \oplus_{h\in G} G_{h,g}\}_{g\in G})$. The proof is complete. $\square$

We end this section with an application to functorially cyclic modules. It is obvious that weak dominance is a transitive notion by simply composing the structure functors. Dominance however is not necessarily preserved, since strong indecomposability is not preserved under composition. Further, the above constructed $\mathcal{B}$-functor ι_G is not strongly indecomposable in general, see Example 4.6.6. Nevertheless, there exists example in which both statement are valid, and thus we describe the associated structure in this case.

Corollary 4.6.19. *Let $\mathcal{C} \in \mathcal{B} - \mathrm{Mod}$ be (F,K,α,β)-cyclic, and let G be a finite group that acts on $\mathcal{C}$. Denote by $\mathcal{C} \xrightarrow{\iota_G} \mathcal{C}_G \xrightarrow{\hat{G}} \mathcal{C}$ and $id_{\mathcal{C}} \overset{\alpha_G}{\Longrightarrow} \iota_G\hat{G} \overset{\beta_G}{\Longrightarrow} id_{\mathcal{C}}$ the above constructed $\mathcal{B}$-functors and $\mathcal{B}$-transformations for which $\beta_G \circ \alpha_G = id_{\mathcal{C}}$ holds. Provided that $\iota_G F$ is strongly indecomposable, $\mathcal{C}_G$ is functorially cyclic. More precisely, $\mathcal{C}_G$ is $\Big(\iota_G F, K\hat{G}, (K\hat{G}\iota_G\alpha\hat{G}\iota_G F) \circ (K\hat{G}\alpha_G\iota_G F), (K\hat{G}\beta_G\iota_G F) \circ (K\hat{G}\iota_G\beta\hat{G}\iota_G F)\Big)$-cyclic.*

Remark 4.6.20. The assumptions of Corollary 4.6.19 imply that ι_G is strongly indecomposable. Further, the underlying object of the associated indecomposable split Frobenius object on the implicit $\mathcal{B}$-quotient of $\mathcal{B}$ is given by $KGF = K\hat{G}\iota_G F$. This object is closely related to the indecomposable split Frobenius object associated to $\mathcal{C}$, since its underlying object is KF which is clearly a summand of KGF. Furthermore, the reader should note that $\mathcal{C}_G$ is not the functorially cyclic module that we usually would associate to the right indecomposable split Frobenius object of the statement of Corollary 4.6.19. Instead this is $\mathcal{C}_G^{\mathrm{idem}}$.

Remark 4.6.21. Let $\mathcal{C}$ be the functorially cyclic module of Corollary 4.6.19. Then, we have an associated universal property ${}_{\hat{\Theta}}\Phi : \mathcal{B}-\mathrm{Mod}^{\mathrm{idem}} \to \mathfrak{A}dd^{\mathrm{idem}}$. One can show that the G action on $\mathcal{C}$ induces a G action on ${}_{\hat{\Theta}}\Phi$. Similarly to the construction of $\mathcal{M}_G$ one can then construct a universal property ${}_{\hat{\Theta}}\Phi_G$. In the same way as ${}_{\hat{\Theta}}\Phi$ can be calculated using $\mathcal{C}$ as $\mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{C}, -)$, we can calculate ${}_{\hat{\Theta}}\Phi_G$ using $\mathcal{C}_G$. However, the 2-functor ${}_{\hat{\Theta}}\Phi$ is truly the universal property of $\mathcal{C}^{\mathrm{idem}}$, and in the same way ${}_{\hat{\Theta}}\Phi_G$ is represented by $\mathcal{C}_G^{\mathrm{idem}}$.

Remark 4.6.22. A reason for the lack of a genuine universal property for the coinvariant module is provided in [MaMi5, Proposition 21]. In a natural setting, it is shown there that it impossible to categorify the 2-dimensional simple representation which we had discussed in Section 2.6.3 using an idempotent closed module. Note that being idempotent closed is part of the notion of a finitary module. While one can construct a categorification in an *ad hoc* fashion by selecting certain submodules, $\mathcal{C}_G$ provides a conceptual way to construct a categorification which has the properties discussed in this section. See Section 5.5 for more details on this example.

4.7 Compatibility with quotients

In this section, we study our construction based on split Frobenius objects when applied along monoidal quotient functors. We show that the induced functors remain full. Assuming Frobenius descent, we also proof that they remain essential surjective.

Remark 4.7.1. This section is technical and will be applied in Section 5.7 only. Compared to 4.5.25, it reaches a similar conclusion, but provides more broadly applicable criteria.

Definition 4.7.2. *Let* $\hat{\Theta}_{\mathcal{C}} = (\Theta_{\mathcal{C}}, \Delta_{\mathcal{C}}, \mu_{\mathcal{C}}), \hat{\Theta}_{\mathcal{D}} = (\Theta_{\mathcal{D}}, \Delta_{\mathcal{D}}, \mu_{\mathcal{D}})$ *be Frobenius objects on* $\mathcal{C}, \mathcal{D} \in \mathfrak{Cat}$ *respectively and let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor that is compatible with the Frobenius objects via* $\mathrm{can}_F : F\Theta_{\mathcal{C}} \xRightarrow{\sim} \Theta_{\mathcal{D}}F$. *We say* F lifts idempotents with descent, *if and only if for all* $C \in \mathcal{C}$ *and for all idempotents* $e \in \mathcal{D}(\Theta_{\mathcal{D}}FC, \Theta_{\mathcal{D}}FC)$ *that satisfies* $\hat{\Theta}_{\mathcal{D}}$*-descent, there exists an idempotent* $\hat{e} \in \mathcal{C}(\Theta_{\mathcal{C}}C, \Theta_{\mathcal{C}}C)$ *that satisfies* $\hat{\Theta}_{\mathcal{C}}$*-descent and for which* $(\mathrm{can}_F)_C \circ F(\hat{e}) \circ \left(\mathrm{can}_F^{-1}\right)_C = e$ *holds.*

Proposition 4.7.3. *Let* $\mathcal{M}, \overline{\mathcal{M}} \in \mathcal{A}dd^{\mathrm{idem}}$, *let* $\hat{\Theta} = (\Theta, \Delta, \mu)$, $\hat{\overline{\Theta}} = (\overline{\Theta}, \overline{\Delta}, \overline{\mu})$ *be split Frobenius objects on* $\mathcal{M}$, $\overline{\mathcal{M}}$ *respectively and let* $\pi : \mathcal{M} \twoheadrightarrow \overline{\mathcal{M}}$ *be an additive quotient functor that is compatible with the Frobenius objects. Then, the induced functor* $\widetilde{\pi} : {}_{\hat{\Theta}}\mathcal{M} \twoheadrightarrow {}_{\hat{\overline{\Theta}}}\overline{\mathcal{M}}$ *is full and* $\mathcal{I}(\widetilde{\pi}) = \mathcal{I}(\pi\vartheta_{\hat{\Theta}})$. *Provided that* π *lifts idempotents with descent,* π *is a quotient functor.*

Proof. First, note that ${}_{\hat{\Theta}}\mathcal{M}$ and ${}_{\hat{\overline{\Theta}}}\overline{\mathcal{M}}$ exists due to Proposition 4.3.58, together with certain functors and transformations used below. Note that this proposition is stated for $\mathcal{B}-\mathrm{Mod}$, but holds of course as well for additive categories (simply consider the trivial additive monoidal category as $\mathcal{B}$ which acts on any additive category).

Second, let us discuss the existence of $\widetilde{\pi}$. Instead of constructing $\widetilde{\pi}$ explicitly and check that it satisfies all the necessary properties, we use the previously developed theory. We plug all the given structure into the first argument of $\mathfrak{A}dd^{\mathrm{idem}}(-,-)$. In this way, we obtain 2-functors with Frobenius objects acting on them and maps between 2-functors that are compatible with the Frobenius objects, see Proposition 4.3.57. This proposition (together with Proposition 4.3.33) also tells us that by applying $\mathfrak{A}dd^{\mathrm{idem}}(-,-)$, we have reached the setup of Proposition 4.3.48 and that all the assumptions are satisfied, Thus giving us a morphism of 2-functors $\mathfrak{A}dd^{\mathrm{idem}}({}_{\hat{\overline{\Theta}}}\overline{\mathcal{M}},-) \to \mathfrak{A}dd^{\mathrm{idem}}({}_{\hat{\Theta}}\mathcal{M},-)$. Appealing to the 2-Yoneda Lemma 3.3.49, we get the desired functor $\widetilde{\pi} : {}_{\hat{\Theta}}\mathcal{M} \to {}_{\hat{\overline{\Theta}}}\overline{\mathcal{M}}$. Going through all the above mentioned references, we obtain the following diagram (which could also have been constructed explicitly):

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\vartheta^{\hat{\Theta}}"] \arrow[d, "\pi"] & {}_{\hat{\Theta}}\mathcal{M} \arrow[r, "\vartheta_{\hat{\Theta}}"] \arrow[d, "\widetilde{\pi}"] & \mathcal{M} \arrow[d, "\pi"] \\
\overline{\mathcal{M}} \arrow[r, "\vartheta^{\hat{\overline{\Theta}}}"] & {}_{\hat{\overline{\Theta}}}\overline{\mathcal{M}} \arrow[r, "\vartheta_{\hat{\overline{\Theta}}}"] & \overline{\mathcal{M}}
\end{tikzcd}$$

We denote by $\phi : \pi\vartheta_{\hat{\Theta}} \overset{\sim}{\Longrightarrow} \vartheta_{\hat{\overline{\Theta}}}\widetilde{\pi}$ and $\psi : \widetilde{\pi}\vartheta^{\hat{\Theta}} \overset{\sim}{\Longrightarrow} \vartheta^{\hat{\overline{\Theta}}}\pi$ the isotransformations up to which the above squares commute. By assumption, we have an isotransformation $\mathrm{can}_{\pi} : \pi\Theta \overset{\sim}{\Longrightarrow} \overline{\Theta}\pi$ which expresses the compatibility of π with the Frobenius objects. The relation between can_{π} and the two transformations above is given by $\mathrm{can}_{\pi} = (\vartheta_{\hat{\overline{\Theta}}}\psi) \circ (\phi\vartheta^{\hat{\Theta}})$ holds. Furthermore, we have transformations $\alpha : id_{\mathcal{M}} \Rightarrow \vartheta^{\hat{\Theta}}\vartheta_{\hat{\Theta}}$ and $\beta : \vartheta\hat{\Theta}\vartheta_{\hat{\Theta}} \Rightarrow id_{\mathcal{M}}$ with $\beta \circ \alpha = id_{\mathcal{M}}$ and $\hat{\Theta} = (\vartheta_{\hat{\Theta}}\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\alpha\vartheta^{\hat{\Theta}}, \vartheta_{\hat{\Theta}}\beta\vartheta^{\hat{\Theta}})$. The analogous structure is of course present for the morphisms involving bars, as well. Finally, the compatibility conditions $(\vartheta^{\hat{\overline{\Theta}}}\phi)\circ(\psi\vartheta_{\hat{\Theta}})\circ(\widetilde{\pi}\alpha) = \overline{\alpha}\widetilde{\pi}$ and $(\widetilde{\pi}\beta)\circ(\psi^{-1}\vartheta_{\hat{\Theta}})\circ(\vartheta^{\hat{\overline{\Theta}}}\phi^{-1}) = \overline{\beta}\widetilde{\pi}$ are satisfied.

Third, we show that with π being full, so is $\widetilde{\pi}$. Take any objects $M_1, M_2 \in \mathrm{Ob}({}_{\hat{\Theta}}\mathcal{M})$ and any morphism $f \in {}_{\hat{\overline{\Theta}}}\overline{\mathcal{M}}(\widetilde{\pi}(M_1), \widetilde{\pi}(M_2))$. Since π is full, there exists a morphism $\widetilde{g}$ in $\mathcal{M}$ such that $\pi(\tilde{g}) = \phi_{M_2}^{-1} \circ \vartheta_{\hat{\overline{\Theta}}}(f) \circ \phi_{M_1}$. Now, define $g := \beta_{M_2} \circ \vartheta^{\hat{\Theta}}(\widetilde{g}) \circ \alpha_{M_1}$. Note that g is a morphism in ${}_{\hat{\Theta}}\mathcal{M}$, since $\vartheta^{\hat{\Theta}}$ is a functor and α and β are transformation of endofunctors of ${}_{\hat{\Theta}}\mathcal{M}$. We conclude that $\widetilde{\pi}$ is full by computing

$$
\begin{aligned}
\widetilde{\pi}(g) &= \widetilde{\pi}(\beta_{M_2}) \circ \widetilde{\pi}(\vartheta^{\hat{\Theta}}(\widetilde{g})) \circ \widetilde{\pi}(\alpha_{M_1}) \\
&= \widetilde{\pi}(\beta_{M_2}) \circ \psi^{-1}_{\vartheta_{\hat{\Theta}}(M_2)} \circ \vartheta^{\hat{\overline{\Theta}}}(\pi(\widetilde{g})) \circ \psi_{\vartheta_{\hat{\Theta}}(M_1)} \circ \widetilde{\pi}(\alpha_{M_1}) \\
&= \widetilde{\pi}(\beta_{M_2}) \circ \psi^{-1}_{\vartheta_{\hat{\Theta}}(M_2)} \circ \vartheta^{\hat{\overline{\Theta}}}(\phi^{-1}_{M_2}) \circ \vartheta^{\hat{\overline{\Theta}}}(\vartheta_{\hat{\overline{\Theta}}}(f)) \circ \vartheta^{\hat{\overline{\Theta}}}(\phi_{M_2}) \circ \psi_{\vartheta_{\hat{\Theta}}(M_1)} \circ \widetilde{\pi}(\alpha_{M_1}) \\
&= \overline{\beta}_{\widetilde{\pi}(M_2)} \circ \vartheta^{\hat{\overline{\Theta}}}(\vartheta_{\hat{\overline{\Theta}}}(f)) \circ \overline{\alpha}_{\widetilde{\pi}(M_1)} \\
&= \overline{\beta}_{\widetilde{\pi}(M_2)} \circ \overline{\alpha}_{\widetilde{\pi}(M_2)} \circ f \\
&= f.
\end{aligned}
$$

Fourth, we describe $\mathcal{I}(\widetilde{\pi})$. Since $id_{{}_{\hat{\overline{\Theta}}}\overline{\mathcal{M}}} \subset^{\oplus} \vartheta^{\hat{\overline{\Theta}}} \circ \vartheta_{\hat{\overline{\Theta}}}$, it follows that $\vartheta_{\hat{\overline{\Theta}}}$ is faithful. Thus $\mathcal{I}(\widetilde{\pi}) = \mathcal{I}(\vartheta_{\hat{\overline{\Theta}}}\widetilde{\pi})$. Using the isotransformation ϕ, we obtain $\mathcal{I}(\vartheta_{\hat{\overline{\Theta}}}\widetilde{\pi}) = \mathcal{I}(\pi\vartheta_{\hat{\Theta}})$ and thus $\mathcal{I}(\widetilde{\pi}) = \mathcal{I}(\pi\vartheta_{\hat{\Theta}})$.

Last, we show that $\widetilde{\pi}$ is essentially surjective provided that π lifts idempotents with descent. Let $M \in {}_{\hat{\overline{\Theta}}}\overline{\mathcal{M}}$. Since π is essentially surjective, there exists $N \in \mathcal{M}$ and an isomorphism $h : \vartheta_{\hat{\overline{\Theta}}}(M) \xrightarrow{\sim} \pi(N)$ in ${}_{\hat{\overline{\Theta}}}\overline{\mathcal{M}}$. With all categories involved being idempotent closed, we have $M \cong (\vartheta^{\hat{\overline{\Theta}}} \circ \vartheta_{\hat{\overline{\Theta}}}(M), e) \cong (\vartheta^{\hat{\overline{\Theta}}} \circ \pi(N), e') \cong (\widetilde{\pi} \circ \vartheta^{\hat{\Theta}}(N), e'')$. Here, $e := \overline{\alpha}_M \circ \overline{\beta}_M$ is an idempotent since $\hat{\overline{\Theta}}$ is split, and $e' := \vartheta^{\hat{\overline{\Theta}}}(h) \circ e \circ (\vartheta^{\hat{\overline{\Theta}}})^{-1}(h)$ and $e'' := \psi_N^{-1} \circ e' \circ \psi_N$ are idempotents because they are conjugate to an idempotent. Since $\widetilde{\pi}$ is full there exists a morphisms $e''' \in {}_{\hat{\Theta}}\mathcal{M}(\vartheta^{\hat{\Theta}}N, \vartheta^{\hat{\Theta}}N)$ with $\widetilde{\pi}(e''') = e''$ and thus $(\widetilde{\pi} \circ \vartheta^{\Theta}(N), e'') = (\widetilde{\pi} \circ \vartheta^{\Theta}(N), \widetilde{\pi}(e'''))$. If we could show that there is a choice of e''' which is an idempotent, then we would have shown that $M \cong (\widetilde{\pi} \circ \vartheta^{\Theta}(N), \widetilde{\pi}(e''')) = \widetilde{\pi}\left((\vartheta^{\Theta}(N), e''')\right)$, *i.e.* essential surjectivity of $\widetilde{\pi}$. Using the construction employed to prove that $\widetilde{\pi}$ is full, we can set $e''' := \beta_{\vartheta^{\hat{\Theta}}N} \circ \vartheta^{\hat{\Theta}}(\tilde{e}) \circ \alpha_{\vartheta^{\hat{\Theta}}N}$ where $\tilde{e}$ is any preimage under π of $\phi^{-1}_{\vartheta^{\hat{\Theta}}N} \circ \vartheta_{\hat{\overline{\Theta}}}(e'') \circ \phi_{\vartheta^{\hat{\Theta}}N}$. By assumption, there exists another idempotent $\hat{e} \in {}_{\hat{\Theta}}\mathcal{M}(\Theta N, \Theta N)$ that satisfies $\hat{\Theta}$-descent and $(\mathrm{can}_{\pi}^{-1})_N \circ \pi(\hat{e}) \circ (\mathrm{can}_{\pi})_N = \vartheta_{\hat{\overline{\Theta}}}(e')$. This holds since $\vartheta_{\hat{\overline{\Theta}}}(e')$ satisfies $\hat{\overline{\Theta}}$-descent, which is true for all morphisms in the image of $\vartheta_{\hat{\overline{\Theta}}}$. Hence, we can calculate that $\pi(\hat{e}) = (\mathrm{can}_{\pi})_N \circ \vartheta_{\hat{\overline{\Theta}}}(e') \circ (\mathrm{can}_{\pi}^{-1})_N = \phi^{-1}_{\vartheta^{\hat{\Theta}}N} \circ (\vartheta_{\hat{\overline{\Theta}}}\psi_N^{-1}) \circ \vartheta_{\hat{\overline{\Theta}}}(e') \circ (\vartheta_{\hat{\overline{\Theta}}}\psi_N) \circ \phi_{\vartheta^{\hat{\Theta}}N} = \phi^{-1}_{\vartheta^{\hat{\Theta}}N} \circ \vartheta_{\hat{\overline{\Theta}}}(e'') \circ \phi_{\vartheta^{\hat{\Theta}}N}$. We can thus choose $\widetilde{e} = \hat{e}$. Note that $\vartheta_{\hat{\Theta}}(e''') = \vartheta_{\hat{\Theta}}(\beta_{\vartheta^{\hat{\Theta}}N}) \circ \vartheta_{\hat{\Theta}}(\vartheta^{\hat{\Theta}}(\hat{e})) \circ \vartheta_{\hat{\Theta}}(\alpha_{\vartheta^{\hat{\Theta}}N}) = \mu_N \circ \hat{\Theta}(\hat{e}) \circ \Delta_N = \hat{e}$ is an idempotent, where the last equation holds due to the assumption that the lift $\hat{e}$ satisfies $\hat{\Theta}$-descent. Now we only need to remember that $\vartheta_{\hat{\Theta}}$ is faithful, to conclude that e''' is indeed an idempotent. $\square$

Remark 4.7.4. The property $\mathcal{I}(\widetilde{\pi}) = \mathcal{I}(\pi\vartheta_{\hat{\Theta}})$ means that the morphism that $\tilde{\pi}$ sends to zero are exactly those that are send to zero under π.

Proposition 4.7.3 generalises to $\mathcal{B} - \mathrm{Mod}$ by the same argument, since the statement is in this case simply a statement about the underlying additive categories.

Moreover, we have the following result:

Corollary 4.7.5. *Let $\pi : \mathcal{M} \to \overline{\mathcal{M}}$ be a quotient $\mathcal{B}$-functor between idempotent closed categories and let $\hat{\Theta}$ be a split Frobenius object in $\mathcal{B}$. Then, we have an additive quotient functor $\pi : \mathcal{M} \to \overline{\mathcal{M}}$ which is compatible with the split Frobenius objects on $\mathcal{M}$ and $\overline{\mathcal{M}}$ induced by $\hat{\Theta}$ through the respective $\mathcal{B}$-actions. Further provided that π lifts idempotents that satisfy $\hat{\Theta}$-descent, the induced additive functor ${}_{\hat{\Theta}}\mathcal{M} \to {}_{\hat{\Theta}}\overline{\mathcal{M}}$ is a quotient functor.*

Proof. Given any Frobenius object in $\mathcal{B}$ and a $\mathcal{B}$-functor $\mathcal{M}_1 \to \mathcal{M}_2$, we obtain induced Frobenius objects on the underlying additive categories $\mathcal{M}_1$ and $\mathcal{M}_2$ by applying the respective structure functors to the Frobenius object. That the underlying additive functor of F is compatible with these Frobenius object follows using the structure of F as a $\mathcal{B}$-functor. Thus, Proposition 4.7.3 applies. $\square$

4.8 Induced tensor products on functorially cyclic modules

In this section, we show in Theorem 4.8.3 that certain functorially cyclic modules can be equipped with a natural monoidal structure. This result assumes that the associated right indecomposable split Frobenius object is symmetrically central, which we define now.

Definition 4.8.1. *Let $\mathcal{B}$ be a strict additive monoidal category. A Frobenius object (Θ, Δ, μ) in $\mathcal{B}$ is* central, *if for all $B \in \mathcal{B}$ there exists a natural isomorphisms $s_B : B\Theta \xrightarrow{\sim} \Theta B$ in $\mathcal{B}$ such that $(\Delta B) \circ s_B = (\Theta s_B) \circ (s_B \Theta) \circ (B\Delta)$, $(\mu B) \circ (\Theta s_B) \circ (s_B \Theta) = s_B \circ (B\mu)$ and $s_{AB} = (s_A B) \circ (A s_B)$ hold for all A, $B \in \mathcal{B}$. A central Frobenius object is* symmetric *(also referred to as a symmetrically central Frobenius object) if $s_\Theta \circ \Delta = \Delta$ and $\mu \circ s_\Theta = \mu$.*

Remark 4.8.2. A central Frobenius object can equivalently be described as a Frobenius object in the Drinfeld centre $\mathcal{Z}^{\mathrm{Dr}}(\mathcal{B})$.

The above definition and all proofs are stated for a strict additive monoidal category $\mathcal{B}$ only. However, the generalisation to the non-strict case is straightforward, and thus we allow ourselves to state Theorem 4.8.3 for general additive monoidal categories. Recall the additive functors ${}^{\hat{\Theta}}\vartheta$ and ${}_{\hat{\Theta}}\vartheta$ introduced in Remark 4.3.67.

Theorem 4.8.3. *Let $\mathcal{B}$ be an additive monoidal category and let $\hat{\Theta}$ be a symmetrically central Frobenius object in $\mathcal{B}$. Then, there exists an idempotent e of ${}^{\hat{\Theta}}\vartheta$ such that the composition*

$$\mathcal{B}_{\hat{\Theta}} \xhookrightarrow{({}^{\hat{\Theta}}\vartheta, e)} {}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}} \xrightarrow{{}_{\hat{\Theta}}\vartheta} \mathcal{B}_{\hat{\Theta}} \; .$$

is isomorphic to the identity functor, thereby equipping $\mathcal{B}_{\hat{\Theta}}$ with the structure of a monoidal category. Further, the natural functor $\vartheta^{\hat{\Theta}} : \mathcal{B} \to \mathcal{B}_{\hat{\Theta}}$ carries the structure of a monoidal functor and the module $\mathrm{Res}_{\vartheta^{\hat{\Theta}}}(\mathcal{B}_{\hat{\Theta}})$ agrees with the free functorially cyclic module associated to $\hat{\Theta}$. Furthermore, the functor $\mathcal{B}_{\hat{\Theta}}^{\mathrm{rev}} \to \mathcal{E}nd_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}})$ induced from tensoring on the right agrees up to the equivalence $\mathcal{E}nd_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}}) \cong {}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}}$ with $({}^{\hat{\Theta}}\vartheta, e)$.

Proof. Recall from Remark 4.3.67 that we have additive functors $\mathcal{B}_{\hat{\Theta}} \xrightarrow{{}^{\hat{\Theta}}\vartheta} {}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}} \xrightarrow{{}_{\hat{\Theta}}\vartheta} \mathcal{B}_{\hat{\Theta}}$ and recall as well that ${}^{\hat{\Theta}}\vartheta\,{}_{\hat{\Theta}}\vartheta$ has the identity functor as a summand. Assuming that $\hat{\Theta}$ is a symmetrically central, split Frobenius object, we will show in Proposition 4.8.9 that ${}^{\hat{\Theta}}\vartheta$ carries an idempotent e. Then, Proposition 4.8.10 establishes that $({}_{\hat{\Theta}}\vartheta\,{}^{\hat{\Theta}}\vartheta, {}_{\hat{\Theta}}\vartheta e)$ is isomorphic to the identity functor. Note that without the symmetrical centrality assumption it is not true in general that ${}_{\hat{\Theta}}\vartheta\,{}^{\hat{\Theta}}\vartheta$ contains the identity functor as a summand. Furthermore in Proposition 4.8.13, we will show that the natural monoidal structure on ${}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}}$ introduced in Section 4.3.5 is compatible with $({}^{\hat{\Theta}}\vartheta, e)$. This induces an explicit monoidal structure on $\mathcal{B}_{\hat{\Theta}}$, which is uniquely determined by the property that $({}^{\hat{\Theta}}\vartheta, e)$ is a monoidal functor with trivial structure maps. Proposition 4.8.17 will establish that the additive functor underlying the $\mathcal{B}$-functor $\mathcal{B} \xrightarrow{\vartheta^{\hat{\Theta}}} \mathcal{B}_{\hat{\Theta}}$ can also be equipped with the structure of a monoidal functor. Thus, $\mathrm{Res}_{\vartheta^{\hat{\Theta}}}(\mathcal{B}_{\hat{\Theta}})$ agrees with $\mathcal{B}_{\hat{\Theta}} \in \mathcal{B}-\mathrm{Mod}$, which is the functorially cyclic modules associated to $\hat{\Theta}$. Now, the monoidal functor $\mathcal{B}_{\hat{\Theta}}^{\mathrm{rev}} \to \mathfrak{A}dd(\mathcal{B}_{\hat{\Theta}}, \mathcal{B}_{\hat{\Theta}})$ induced by taking tensor products on the right extends to a monoidal functor $\mathcal{B}_{\hat{\Theta}}^{\mathrm{rev}} \to \mathcal{H}om_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}}, \mathcal{B}_{\hat{\Theta}})$, see Proposition 4.8.20. Once composed with the equivalence $\mathcal{H}om_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}}, \mathcal{B}_{\hat{\Theta}}) \cong {}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}}$, this functor becomes isomorphic to $({}^{\hat{\Theta}}\vartheta, e)$, see Proposition 4.8.21. □

Remark 4.8.4. Note that it is not possible to equip $\mathcal{B}_{\hat{\Theta}} \xrightarrow{\vartheta_{\hat{\Theta}}} \mathcal{B}$ with a monoidal structure. However, we provide a push-pull formula in Corollary 4.8.22.

For the rest of the section, we fix a symmetrically central, split Frobenius object $\hat{\Theta} = (\Theta, \Delta, \mu)$ in a strict additive monoidal category $\mathcal{B}$, and denote the natural isomorphisms that make $\hat{\Theta}$ central by s_B for all $B \in \mathcal{B}$.

Notation 4.8.5. For all $B \in \mathcal{B}$, denote by $i_B := s_{B\Theta} \circ (B\Delta) : B\Theta \to \Theta B\Theta$, similarly $\pi_B := (B\mu) \circ s_{B\Theta}^{-1} : \Theta B\Theta \to B\Theta$ and $e_B := i_B \circ \pi_B$.

Lemma 4.8.6. *For all $B \in \mathcal{B}$, $i_B = (s_B\Theta) \circ (B\Delta)$ and $\pi_B = (B\mu) \circ (s_B^{-1}\Theta)$, e_B is an idempotent and we have $s_\Theta^{-1} \circ \Delta \circ \mu \circ s_\Theta = \Delta \circ \mu$. Furthermore for all $f \in \mathcal{B}_{\hat{\Theta}}(B_1, B_2)$, we have $(\Theta f) \circ e_{B_1} = e_{B_2} \circ (\Theta f)$.*

Proof. Since $\hat{\Theta}$ is symmetrically central, $i_B = s_{B\Theta} \circ (B\Delta) = (s_B\Theta) \circ (Bs_\Theta) \circ (B\Delta) = (s_B\Theta) \circ (B\Delta)$ and the same argument establishes the expression given for π_B. The second statement holds since $\pi_B \circ i_B = (B\mu) \circ s_{B\Theta}^{-1} \circ s_{B\Theta} \circ (B\Delta) = (B\mu) \circ (B\Delta) = B(\mu \circ \Delta) = B(id_\Theta) = id_{B\Theta}$ and $e_B = i_B \circ \pi_B$. The third statement is a direct consequence of the definition of symmetric Frobenius objects. The fourth statement, follows from the more refined statement in the next lemma. □

Lemma 4.8.7. *For all morphisms $f \in \mathcal{B}_{\hat{\Theta}}(B_1, B_2)$, we have $(\Theta f) \circ i_{B_1} = i_{B_2} \circ f$ and $f \circ \pi_{B_1} = \pi_{B_2} \circ (\Theta f)$.*

Proof. The second statement being analogous to the first, we only show the first statement: $(\Theta f) \circ i_{B_1} = (\Theta f) \circ s_{B_1\Theta} \circ (B_1\Delta) = s_{B_2\Theta} \circ (f\Theta) \circ (B_1\Delta) = s_{B_2\Theta} \circ (B_2\Delta) \circ f = i_{B_2} \circ f$. The second equality holds by naturality of $s_{B\Theta}$. The third equality holds by the definition of $\mathcal{B}_{\hat{\Theta}}$ in Corollary 4.5.9, which is ultimately based on Definition 4.3.35. □

Lemma 4.8.8. *For all $B \in \mathcal{B}$, we have three equations $i_B = (\Theta s_B^{-1}) \circ (\Delta B) \circ s_B$, $\pi_B = s_B^{-1} \circ (\mu B) \circ (\Theta s_B)$ and $e_B = (\Theta s_B^{-1}) \circ (\Delta B) \circ (\mu B) \circ (\Theta s_B)$.*

Proof. We provide the computation for the third equation.

$$\begin{aligned}
e_B &= i_B \circ \pi_B \\
&= (s_B\Theta) \circ (B\Delta) \circ (B\mu) \circ (s_B^{-1}\Theta) \\
&= (\Theta s_B^{-1}) \circ (\Theta s_B) \circ (s_B\Theta) \circ (B\Delta) \circ (B\mu) \circ (s_B^{-1}\Theta) \circ (\Theta s_B^{-1}) \circ (\Theta s_B) \\
&= (\Theta s_B^{-1}) \circ (\Delta B) \circ s_B \circ s_B^{-1} \circ (\mu B) \circ (\Theta s_B) && (\hat{\Theta} \text{ central}) \\
&= (\Theta s_B^{-1}) \circ (\Delta B) \circ (\mu B) \circ (\Theta s_B)
\end{aligned}$$

The reader will notice that the other two equations are shown implicitly in this calculation, as well. □

Proposition 4.8.9. *For all $(B, f) \in \mathcal{B}_{\hat{\Theta}}$, $e_{(B,f)} := e_B \circ (\Theta f) = (\Theta f) \circ e_B$ defines an idempotent of ${}^{\hat{\Theta}}\vartheta\,((B, f))$. Further, this defines an idempotent e of ${}^{\hat{\Theta}}\vartheta : \mathcal{B}_{\hat{\Theta}} \to {}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}}$.*

Proof. First for all $(B, f) \in \mathcal{B}_{\hat{\Theta}}$, we have to show that $e_{(B,f)}$ is an idempotent of ${}^{\hat{\Theta}}\vartheta\,((B, f)) = (B, \Theta f)$ in ${}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}}$. Since Θf is an endomorphism of $(B, id_{\Theta B\Theta})$ in ${}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}}$ (say by construction of ${}^{\hat{\Theta}}\vartheta$), it is sufficient to show the following two claims: First, e_B is an endomorphism of $(B, id_{\Theta B\Theta})$ in ${}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}}$, and second $e_B \circ (\Theta f) = (\Theta f) \circ e_B$ holds.

The latter claim was shown Lemma 4.8.6 (this lemma also implies that $e^2_{(B,f)} = e_{(B,f)}$). To establish the former claim, we have to show that $e_B = (\mu B\Theta) \circ (\Theta e_B) \circ (\Delta B\Theta)$ and $e_B = (\Theta B\mu) \circ (e_B\Theta) \circ (\Theta B\Delta)$, according to the definition of ${}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}}$ at the start of Section 4.3.5. The first equation is a mirror image of the second, after replacing e_B by its representation found in Lemma 4.8.8. The second equation follows from the interchange law, from the Frobenius relations of $\hat{\Theta}$ and from $\hat{\Theta}$ being split:

$$\begin{aligned}(\Theta B\mu) \circ (e_B\Theta) \circ (\Theta B\Delta) &= (\Theta B\mu) \circ (s_B\Theta\Theta) \circ (B\Delta\Theta) \circ (B\mu\Theta) \circ (s_B^{-1}\Theta\Theta) \circ (\Theta B\Delta)\\ &= (s_B\Theta) \circ (B\Theta\mu) \circ (B\Delta\Theta) \circ (B\mu\Theta) \circ (B\Theta\Delta) \circ (s_B^{-1}\Theta)\\ &= (s_B\Theta) \circ (B\Delta) \circ (B\mu) \circ (B\Delta) \circ (B\mu) \circ (s_B^{-1}\Theta)\\ &= (s_B\Theta) \circ (B\Delta) \circ (B\mu) \circ (s_B^{-1}\Theta)\\ &= e_B.\end{aligned}$$

Next, we need to check e defines a natural endotransformation of $\vartheta^{\hat{\Theta}}$. Thus for all $g \in \mathcal{B}_{\hat{\Theta}}((B_1, f_1), (B_2, f_2))$, we need to establish that $e_{(B_2,f_2)} \circ \vartheta^{\hat{\Theta}}(g) = \vartheta^{\hat{\Theta}}(g) \circ e_{(B_1,f_1)}$. This follows directly from Lemma 4.8.6, since $f_1 \in \mathcal{B}_{\hat{\Theta}}(B_1, B_1)$, $g \in \mathcal{B}_{\hat{\Theta}}(B_1, B_2)$ and $f_2 \in \mathcal{B}_{\hat{\Theta}}(B_2, B_2)$. □

Proposition 4.8.10. *There exists an isotransformation* $({}_{\hat{\Theta}}\vartheta\,{}^{\hat{\Theta}}\vartheta,\ {}_{\hat{\Theta}}\vartheta e) \cong id_{\mathcal{B}_{\hat{\Theta}}}$.

Proof. We need to define natural transformations $id_{\mathcal{B}_{\hat{\Theta}}} \overset{i}{\Longrightarrow} ({}_{\hat{\Theta}}\vartheta\,{}^{\hat{\Theta}}\vartheta,\ {}_{\hat{\Theta}}\vartheta e) \overset{\pi}{\Longrightarrow} id_{\mathcal{B}_{\hat{\Theta}}}$ and show that they are mutually inverse. In the rest of the proof we will use that the composition ${}_{\hat{\Theta}}\vartheta\,{}^{\hat{\Theta}}\vartheta$ is given by left multiplication with Θ and that ${}_{\hat{\Theta}}\vartheta e$ becomes equal to e once evaluated on an object of $\mathcal{B}_{\hat{\Theta}}$.

For all $(B, f) \in \mathcal{B}_{\hat{\Theta}}$, we employ Lemma 4.8.7 and set $i_{(B,f)} := i_B \circ f = (\Theta f) \circ i_B$ as well as $\pi_{(B,f)} := f \circ \pi_B = \pi_B \circ (\Theta f)$. Both definitions yield morphisms in $\mathcal{B}_{\hat{\Theta}}$ since f (resp. Θf) is an endomorphism of $B \in \mathcal{B}_{\hat{\Theta}}$ (resp. $\Theta B \in \mathcal{B}_{\hat{\Theta}}$), provided that i_B and π_B are morphisms in $\mathcal{B}_{\hat{\Theta}}$. By Lemma 4.3.40, this holds if $i_B = (\Theta B\mu) \circ (i_B\Theta) \circ (B\Delta)$ and $\pi_B = (B\mu) \circ (\pi_B\Theta) \circ (\Theta B\Delta)$. These equations are easily computed and we provide the details for the first one: $(\Theta B\mu) \circ (i_B\Theta) \circ (B\Delta) = (\Theta B\mu) \circ (s_B\Theta\Theta) \circ (B\Delta\Theta) \circ (B\Delta) = (s_B\Theta) \circ (B\Theta\mu) \circ (B\Delta\Theta) \circ (B\Delta) \overset{\hat{\Theta}\,\mathrm{Fro}}{=} (s_B\Theta) \circ (B\Delta) \circ (B\mu) \circ (B\Delta) \overset{\hat{\Theta}\,\mathrm{spl}}{=} (s_B\Theta) \circ (B\Delta) = i_B$.

In order to show that i and π are indeed natural transformations, we need to probe whether $e_{(B_2,f_2)} \circ (\Theta g) \circ i_{(B_1,f_1)} = i_{(B_2,f_2)} \circ g$ and $\pi_{(B_2,f_2)} \circ e_{(B_2,f_2)} \circ (\Theta g) = g \circ \pi_{(B_1,f_1)}$ hold for all $g \in \mathcal{B}_{\hat{\Theta}}((B_1, f_1), (B_2, f_2))$. This is indeed the case, due to the Lemmas 4.8.7 and 4.8.6, the latter (and its proof) implying both that $e_{(B_2,f_2)} \circ i_{(B_2,f_2)} = i_{(B_2,f_2)}$ and $\pi_{(B_2,f_2)} \circ e_{(B_2,f_2)} = \pi_{(B_2,f_2)}$.

It remains to show that i and π are mutually inverse. Clearly, $\pi \circ i = id_{id_{\mathcal{B}_{\hat{\Theta}}}}$ as

$\pi_{(B,f)} \circ i_{(B,f)} = f \circ \pi_B \circ i_B \circ f = f \circ f = f = id_{(B,f)}$ for all $(B, f) \in \mathcal{B}_{\hat{\Theta}}$. Similarly for all $(B, f) \in \mathcal{B}_{\hat{\Theta}}$, $\pi_{(B,f)} \circ i_{(B,f)} = (\Theta f) \circ e_B \circ (\Theta f) = e_B \circ (\Theta f) \circ (\Theta f) = e_B \circ (\Theta f) = \left({}_{\hat{\Theta}}\vartheta\, {}^{\hat{\Theta}}\vartheta,\ {}_{\hat{\Theta}}\vartheta e \right)(f)$ holds and thus $i \circ \pi = id_{({}_{\hat{\Theta}}\vartheta\, {}^{\hat{\Theta}}\vartheta,\ {}_{\hat{\Theta}}\vartheta e)}$. $\square$

Remark 4.8.11. Typically, $({}^{\hat{\Theta}}\vartheta\, {}_{\hat{\Theta}}\vartheta, e\, {}_{\hat{\Theta}}\vartheta) \cong id_{{}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}}}$ does not hold, see for example Theorem 6.3.11. The functor $({}^{\hat{\Theta}}\vartheta\, {}_{\hat{\Theta}}\vartheta, e\, {}_{\hat{\Theta}}\vartheta)$ can however be regarded as a projection onto a monoidal sub-category. This follows from Proposition 4.8.13 below.

Lemma 4.8.12. *Let B_1, $B_2 \in \mathcal{B}$. Then,* $(e_{B_1} B_2 \Theta) \circ (\Theta B_1 e_{B_2}) = (\Theta B_1 e_{B_2}) \circ e_{B_1 \Theta B_2}$.

Proof. This requires nothing, but a lengthy calculation. We have
$(\Theta B_1 e_{B_2}) \circ e_{B_1 \Theta B_2}$

$$
\begin{aligned}
&= (\Theta B_1 e_{B_2}) \circ (s_{B_1\Theta B_2}\Theta) \circ (B_1\Theta B_2\Delta) \circ (B_1\Theta B_2\mu) \\
&\quad \circ (s^{-1}_{B_1\Theta B_2}\Theta) && \text{(Lemma 4.8.6)} \\
&= (\Theta B_1 e_{B_2}) \circ (s_{B_1}\Theta B_2\Theta) \circ (B_1 s_{\Theta B_2}\Theta) \circ (B_1\Theta B_2\Delta) \\
&\quad \circ (B_1\Theta B_2\mu) \circ (B_1 s_{\Theta B_2}\Theta)^{-1} \circ (s_{B_1}\Theta B_2\Theta)^{-1} && (\hat{\Theta}\text{ central}) \\
&= (s_{B_1} e_{B_2}) \circ (B_1 e_{\Theta B_2}) \circ (s^{-1}_{B_1}\Theta B_2\Theta) && \text{(Lemma 4.8.6)} \\
&= (s_{B_1} e_{B_2}) \circ (B_1\Theta s^{-1}_{\Theta B_2}) \circ (B_1\Delta\Theta B_2) \circ (B_1\mu\Theta B_2) \\
&\quad \circ (B_1\Theta s_{\Theta B_2}) \circ (s^{-1}_{B_1}\Theta B_2\Theta) && \text{(Lemma 4.8.8)} \\
&= (s_{B_1} e_{B_2}) \circ (B_1\Theta\Theta s^{-1}_{B_2}) \circ (B_1\Theta s^{-1}_{\Theta} B_2) \circ (B_1\Delta\Theta B_2) \\
&\quad \circ (B_1\mu\Theta B_2) \circ (s^{-1}_{B_1} s_{\Theta B_2}) && (\hat{\Theta}\text{ central}) \\
&= (s_{B_1}\Theta s^{-1}_{B_2}) \circ (B_1\Theta\Delta B_2) \circ (B_1\Theta\mu B_2) \circ (B_1\Theta s^{-1}_{\Theta} B_2) \\
&\quad \circ (B_1\Delta\Theta B_2) \circ (B_1\mu\Theta B_2) \circ (s^{-1}_{B_1} s_{\Theta B_2}) && \text{(Lemma 4.8.8)} \\
&= (s_{B_1}\Theta s^{-1}_{B_2}) \circ (B_1\Theta\Delta B_2) \circ (B_1\Theta\mu B_2) \circ (B_1\Delta\Theta B_2) \\
&\quad \circ (B_1\mu\Theta B_2) \circ (s^{-1}_{B_1} s_{\Theta B_2}) && (\hat{\Theta}\text{ symmetr. central}) \\
&= (s_{B_1}\Theta s^{-1}_{B_2}) \circ (B_1\Theta\Delta B_2) \circ (B_1\Delta B_2) \circ (B_1\mu B_2) \\
&\quad \circ (B_1\mu\Theta B_2) \circ (s^{-1}_{B_1} s_{\Theta B_2}) && (\hat{\Theta}\text{ Frobenius}) \\
&= (s_{B_1}\Theta s^{-1}_{B_2}) \circ (B_1\Delta\Theta B_2) \circ (B_1\Delta B_2) \circ (B_1\mu B_2) \\
&\quad \circ (B_1\Theta\mu B_2) \circ (s^{-1}_{B_1} s_{\Theta B_2}) && \text{(Lemma 4.3.7)} \\
&= (s_{B_1}\Theta s^{-1}_{B_2}) \circ (B_1\Delta\Theta B_2) \circ (B_1\Delta B_2) \circ (B_1\mu B_2) \\
&\quad \circ (B_1\Theta\mu B_2) \circ (B_1\Theta s_{\Theta} B_2) \circ (s^{-1}_{B_1}\Theta s_{B_2}) && (\hat{\Theta}\text{ central}) \\
&= (s_{B_1}\Theta s^{-1}_{B_2}) \circ (B_1\Delta\Theta B_2) \circ (B_1\Delta B_2) \circ (B_1\mu B_2) \\
&\quad \circ (B_1\Theta\mu B_2) \circ (s^{-1}_{B_1}\Theta s_{B_2}) && (\hat{\Theta}\text{ symmetr. central}) \\
&= (s_{B_1}\Theta s^{-1}_{B_2}) \circ (B_1\Delta\Theta B_2) \circ (B_1\mu\Theta B_2) \circ (B_1\Theta\Delta B_2) \\
&\quad \circ (B_1\Theta\mu B_2) \circ (s^{-1}_{B_1}\Theta s_{B_2}) && (\hat{\Theta}\text{ Frobenius}) \\
&= (s_{B_1}\Theta B_2\Theta) \circ (B_1\Delta B_2\Theta) \circ (B_1\mu B_2\Theta) \circ (s^{-1}_{B_1}\Theta s^{-1}_{B_2}) \\
&\quad \circ (\Theta B_1\Delta B_2) \circ (\Theta B_1\mu B_2) \circ (\Theta B_1\Theta s_{B_2}) && \text{(Remark 3.1.3)} \\
&= (e_{B_1} B_2\Theta) \circ (\Theta B_1 e_{B_2}). && \text{(Lemma 4.8.6 + 4.8.8)}
\end{aligned}
$$

and the lemma is established. □

In the following statement, we forget that $\mathcal{B}_{\hat{\Theta}}$ is a $\mathcal{B}$-module, and remember only that it is an additive category.

Proposition 4.8.13. *The additive category $\mathcal{B}_{\hat{\Theta}}$ carries a unique structure as a monoidal category, such that $({}^{\hat{\Theta}}\vartheta, e)$ equipped with trivial structure maps is a monoidal func-*

tor. More concretely, setting $(B_1, f_1) \otimes (B_2, f_2) := (B_1 \Theta B_2, (f_1 f_2) \circ (B_1 e_{B_2}))$ *for all* (B_1, f_1), $(B_2, f_2) \in \mathcal{B}_{\hat{\Theta}}$ *and specifying* $(I, id_{I\Theta})$ *to be the unit object is part of a well defined monoidal structure of* $\mathcal{B}_{\hat{\Theta}}$.

Proof. First, we have to show that the unit object of ${}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}}$ lies in the image of $({}^{\hat{\Theta}}\vartheta, e)$. Second, we have to show that the tensor product of any two objects in the image is isomorphic to an object in the image. Once these two statements are established, it remains to check that all necessary structure maps are available and that they satisfy the correct relations. This however holds automatically, since it follows from Proposition 4.8.10 that $({}^{\hat{\Theta}}\vartheta, e)$ is fully faithful.

We start by claiming that $(I, id_{I\Theta}) \in \mathcal{B}_{\hat{\Theta}}$ is mapped to the unit object provided in Definition 4.3.63. We compute $\left({}^{\hat{\Theta}}\vartheta, e\right)(I, id_{I\Theta}) = (I, (\Theta id_{I\Theta}) \circ e_I) = (I, e_I)$ and establish the claim by providing another representation of the idempotent e_I and thus the link to Definition 4.3.63: $e_I = s_{I\Theta} \circ (I\Delta) \circ (I\mu) \circ s_{I\Theta}^{-1} = s_{I\Theta} \circ (I\Delta) \circ l_{\Theta}^{-1} \circ l_{\Theta} \circ (I\mu) \circ s_{I\Theta}^{-1}$ $= s_{I\Theta} \circ (l_{\Theta}^{-1}\Theta) \circ \Delta \circ \mu \circ (l_{\Theta}\Theta) \circ s_{I\Theta}^{-1} = (\Theta l_{\Theta}^{-1}) \circ s_{\Theta} \circ \Delta \circ \mu \circ s_{\Theta}^{-1} \circ (l_{\Theta}\Theta)$, where the first equation holds by definition, the second inserts the left unit morphisms and its inverse (which are in fact the identity map in a strict monoidal category $\mathcal{B}$), the third uses the interchange law, the fourth uses that s_B is natural in the argument B and the fifth uses that $\hat{\Theta}$ is symmetrically central.

The second statement follows from the following claim: Given (B_1, f_1), $(B_2, f_2) \in \mathcal{B}_{\hat{\Theta}}$, we have $\left({}^{\hat{\Theta}}\vartheta, e\right)(B_1, f_1) \otimes \left({}^{\hat{\Theta}}\vartheta, e\right)(B_2, f_2) = \left({}^{\hat{\Theta}}\vartheta, e\right)(B_1 \Theta B_2, (f_1 f_2) \circ (B_1 e_{B_2}))$. Once we have established this claim, the proof is complete inclusive the uniqueness statement. This is because $({}^{\hat{\Theta}}\vartheta, e)$ is not only faithful, but reflects identities, *i.e.* $({}^{\hat{\Theta}}\vartheta, e)(B_1, f_1) = ({}^{\hat{\Theta}}\vartheta, e)(B_2, f_2)$ implies $(B_1, f_1) = (B_2, f_2)$. We leave the details to the reader, but mention that it is straight forward to show in this situation that $B_1 = B_2$, and that showing $f_1 = f_2$ requires a small computation using symmetrical centrality of $\hat{\Theta}$.

Two steps are required to establish this claim. We abbreviate $f := (f_1 f_2) \circ (B_1 e_{B_2})$. First, we need to show that $f \in \mathcal{B}_{\hat{\Theta}}((B_1 \Theta B_2, id_{B_1 \Theta B_2}), (B_1 \Theta B_2, id_{B_1 \Theta B_2}))$. We already know that f_2 and e_{B_2} are morphisms in $\mathcal{B}_{\hat{\Theta}}$. Due to the interchange law, left multiplication with morphisms in $\mathcal{B}$ does not change this (or less concretely since $\mathcal{B}_{\hat{\Theta}} \in \mathcal{B} - \mathrm{Mod}$), and a composition does of course not lead outside the category either. Second, we need to compute the claim. We have

$$\begin{aligned}
&\left({}^{\hat{\Theta}}\vartheta, e\right)(B_1, f_1) \otimes \left({}^{\hat{\Theta}}\vartheta, e\right)(B_2, f_2) \\
&\qquad = (B_1, (\Theta f_1) \circ e_{B_1}) \otimes (B_2, (\Theta f_2) \circ e_{B_2}) \\
&\qquad = (B_1 \Theta B_2, \{((\Theta f_1) \circ e_{B_1}) B_2 \Theta\} \circ \{\Theta B_1 ((\Theta f_2) \circ e_{B_2})\}) \quad \text{(Definition 4.3.63)}
\end{aligned}$$

and $\left({}^{\hat{\Theta}}\vartheta, e\right)(B_1\Theta B_2, f) = (B_1\Theta B_2, (\Theta f)\circ e_{B_1\Theta B_2})$. Thus, both objects agree, since

$$\begin{aligned}
&\{((\Theta f_1)\circ e_{B_1})\, B_2\Theta\}\circ\{\Theta B_1\,((\Theta f_2)\circ e_{B_2})\}\\
&\qquad = (\Theta f_1 f_2)\circ(e_{B_1}B_2\Theta)\circ(\Theta B_1 e_{B_2})\\
&\qquad = (\Theta f_1 f_2)\circ(\Theta B_1 e_{B_2})\circ e_{B_1\Theta B_2} && \text{(Lemma 4.8.12)}\\
&\qquad = (\Theta f)\circ e_{B_1\Theta B_2}.
\end{aligned}$$

This shows the claim and completes the proof. $\square$

Next, we begin to analyse the compatibility of the two structures of $\mathcal{B}_{\hat{\Theta}}$, namely the structure as a $\mathcal{B}$-module $\mathcal{B}_{\hat{\Theta}} = (\mathcal{B}_{\hat{\Theta}}, \rho)$ and the structure as a monoidal category $\mathcal{B}_{\hat{\Theta}} = (\mathcal{B}_{\hat{\Theta}}, \otimes)$:

Proposition 4.8.14. *The following diagram commutes up to an isotransformation:*

$$\begin{tikzcd}
\mathcal{B}\times\mathcal{B}_{\hat{\Theta}} \arrow[r, "\rho"] \arrow[d, "\vartheta^{\hat{\Theta}}\times id"'] & \mathcal{B}_{\hat{\Theta}} \\
\mathcal{B}_{\hat{\Theta}}\times\mathcal{B}_{\hat{\Theta}} \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

Proof. As in the diagram above, we denote $\otimes$ the newly constructed tensor product on $\mathcal{B}_{\hat{\Theta}}$ and by a dot the action of $\mathcal{B}$ on $\mathcal{B}_{\hat{\Theta}}$. Let $B\in\mathcal{B}$ and $(M,f)\in\mathcal{B}_{\hat{\Theta}}$. We have to show that $B.(M,f)\cong\vartheta^{\hat{\Theta}}(B)\otimes(M,f)$. By definition, $B.(M,f) = (BM, Bf)$ and $\vartheta^{\hat{\Theta}}(B)\otimes(M,f) = (B, id_{B\Theta})\otimes(M,f) = (B\Theta M, (id_{B\Theta}f)\circ(Be_M))$. An isomorphism between these two objects are given by Bi_M with inverse $B\pi_M$. That these two maps are indeed morphisms in $\mathcal{B}_{\hat{\Theta}}$ and that they induce transformations natural in B and M will be shown in the next proof. $\square$

Corollary 4.8.15. *Let $B_1, B_2\in\mathcal{B}$ and $(M,f)\in\mathcal{B}_{\hat{\Theta}}$. Then, we have a commutative diagram*

$$\begin{tikzcd}
B_1.(B_2.(M,f)) \arrow[r, "\sim"] \arrow[d, "\sim"] & B_1.(\vartheta^{\hat{\Theta}}(B_2)\otimes(M,f)) \arrow[d, "\sim"] \\
\vartheta^{\hat{\Theta}}(B_1)\otimes(B_2.(M,f)) \arrow[r, "\sim"] & \vartheta^{\hat{\Theta}}(B_1)\otimes(\vartheta^{\hat{\Theta}}(B_2)\otimes(M,f))
\end{tikzcd}.$$

Proof. This holds since the map $B.(M,f)\xrightarrow{\sim}\vartheta^{\hat{\Theta}}(B)\otimes(M,f)$ is natural in (M,f). $\square$

Remark 4.8.16. We will equip $\vartheta^{\hat{\Theta}}$ with the structure of a monoidal functor in Proposition 4.8.17. Then on can show that $\mathcal{B}_{\hat{\Theta}}\in\mathcal{B}-\mathrm{Mod}$ is equal to $\mathrm{Res}_{\vartheta^{\hat{\Theta}}}(\mathcal{B}_{\hat{\Theta}})$, where $\mathrm{Res}_{\vartheta^{\hat{\Theta}}}$ is the restriction 2-functor which we introduced in Section 4.4. Note that this is a stronger statement than Proposition 4.8.14. Most of the details needed to prove this statement can be found in the proof of Proposition 4.8.20, where we establish a similar result that is tailored to our needs.

Proposition 4.8.17. *The natural $\mathcal{B}$-functor $\vartheta^{\hat{\Theta}} : \mathcal{B} \to \mathcal{B}_{\hat{\Theta}}$ can be equipped with the structure of a monoidal functor.*

Proof. We need to provide isomorphisms $\phi_{B_1,B_2} : \vartheta^{\hat{\Theta}}(B_1)\vartheta^{\hat{\Theta}}(B_2) \to \vartheta^{\hat{\Theta}}(B_1B_2)$ for all $B_1, B_2 \in \mathcal{B}$ which are natural in both arguments, as well as a natural isomorphism $\vartheta^{\hat{\Theta}}(I) \to I_{\mathcal{B}_{\hat{\Theta}}}$. Further, we need to probe compatibility of these morphisms with the associators, as well as left and right units of the two monoidal categories $\mathcal{B}$ and $\mathcal{B}_{\hat{\Theta}}$. We start with the second isomorphism and define it to be the identity morphism. This is possible, since $\vartheta^{\hat{\Theta}}(I) = (I, id_{I\Theta})$ which is exactly the unit object of $\mathcal{B}_{\hat{\Theta}}$. Without doubt, this map is natural. Next for all B_1, $B_2 \in \mathcal{B}$, we define $\phi_{B_1,B_2} := B_1\pi_{B_2}$. We need to check that ϕ_{B_1,B_2} is indeed an isomorphisms between the correct objects and that it is natural in B_1 and B_2. We have $\vartheta^{\hat{\Theta}}(B_1)\vartheta^{\hat{\Theta}}(B_2) = (B_1, id_{B_1\Theta})(B_2, id_{B_2\Theta}) = (B_1\Theta B_2, B_1e_{B_2})$ and $\vartheta^{\hat{\Theta}}(B_1B_2) = (B_1B_2, id_{B_1B_2\Theta})$, hence it remains to show $\phi_{B_1,B_2} = (B_1B_2\mu)\circ(\phi_{B_1,B_2}\Theta)\circ(B_1\Theta B_2\Delta)$ and $\phi_{B_1,B_2}\circ(B_1e_{B_2}) = \phi_{B_1,B_2}$ to establish that ϕ_{B_1,B_2} is a morphisms between the correct objects. The latter holds by Lemma 4.8.6. To show the former, we compute

$$\begin{aligned}
\phi_{B_1,B_2} &= (B_1B_2\mu) \circ (B_1s_{B_2}^{-1}\Theta) && \text{(Lemma 4.8.6)}\\
&= (B_1B_2\mu) \circ (B_1s_{B_2}^{-1}\Theta) \circ (B_1\Theta B_2\mu) \circ (B_1\Theta B_2\Delta) && (\hat{\Theta} \text{ is split})\\
&= (B_1B_2\mu) \circ (B_1B_2\Theta\mu) \circ (B_1s_{B_2}^{-1}\Theta\Theta) \circ (B_1\Theta B_2\Delta) &&\\
&= (B_1B_2\mu) \circ (B_1B_2\mu\Theta) \circ (B_1s_{B_2}^{-1}\Theta\Theta) \circ (B_1\Theta B_2\Delta) && (\hat{\Theta} \text{ is associative})\\
&= (B_1B_2\mu) \circ (\phi_{B_1,B_2}\Theta) \circ (B_1\Theta B_2\Delta). && \text{(Lemma 4.8.6)}
\end{aligned}$$

Next, we show that ϕ_{B_1,B_2} is natural in B_1 and B_2: For all $f_1 \in \mathcal{B}(B_1^1, B_1^2)$ and $f_2 \in \mathcal{B}(B_2^1, B_2^2)$, we have $\left(\vartheta^{\hat{\Theta}}(f_1)\vartheta^{\hat{\Theta}}(f_2)\right)\circ\phi_{B_1^1,B_2^1} = (f_1\Theta f_2\Theta)\circ(B_1^1s_{B_2^1}\Theta)\circ(B_1^2B_2^2\Delta) = (B_1^2s_{B_2^2}\Theta)\circ(B_1^2B_2^2\Delta)\circ(f_1f_2\Theta) = \phi_{B_1^2B_2^2}\circ\vartheta^{\hat{\Theta}}(f_1f_2)$. It remains to show that the above defined structure maps are compatible with the unit maps and the associators of $\mathcal{B}$ and $\mathcal{B}_{\hat{\Theta}}$. Compatibility with unit maps is trivial as it involves only identity morphisms. Compatibility with the associator is more involved. Recall that we assume that $\mathcal{B}$ is strict and thus that the associator trivial. It follows that $\mathcal{B}_{\hat{\Theta}}$ is strict as well. Thus, we need to establish that $\phi_{B_1B_2,B_3} \circ (\phi_{B_1,B_2}\vartheta^{\hat{\Theta}}(B_3)) = \phi_{B_1,B_2B_3} \circ (\vartheta^{\hat{\Theta}}(B_1)\phi_{B_2,B_3})$. By Proposition 4.8.13, the left hand side is equal to

$$
\begin{aligned}
&(B_1B_2\pi_{B_3})\circ(B_1\pi_{B_2}B_3\Theta)\circ(B_1\Theta B_2e_{B_3})\\
&= (B_1B_2B_3\mu)\circ(B_1B_2s_{B_3}^{-1}\Theta)\circ(B_1B_2\mu B_3\Theta)\circ(B_1s_{B_2}^{-1}\Theta B_3\Theta)\\
&\qquad\circ(B_1\Theta B_2\Theta s_{B_3}^{-1})\circ(B_1\Theta B_2\Delta B_3)\circ(B_1\Theta B_2\mu B_3)\\
&\qquad\circ(B_1\Theta B_2\Theta s_{B_3}) && (\text{Lemma } 4.8.6+4.8.8)\\
&= (B_1B_2B_3\mu)\circ(B_1B_2s_{B_3}^{-1}\Theta)\circ(B_1B_2\Theta s_{B_3}^{-1})\circ(B_1B_2\mu\Theta B_3)\\
&\qquad\circ(B_1B_2\Theta\Delta B_3)\circ(B_1B_2\Theta\mu B_3)\circ(B_1s_{B_2}^{-1}\Theta s_{B_3})\\
&= (B_1B_2s_{B_3}^{-1})\circ(B_1B_2\mu B_3)\circ(B_1B_2\mu\Theta B_3)\circ(B_1B_2\Theta\Delta B_3)\\
&\qquad\circ(B_1B_2\Theta\mu B_3)\circ(B_1s_{B_2}^{-1}\Theta s_{B_3}). && (\hat{\Theta}\text{ symmetr. central})
\end{aligned}
$$

Similarly, the right hand side is equal to

$$
\begin{aligned}
&(B_1\pi_{B_2B_3})\circ(B_1\Theta B_2\pi_{B_3})\circ(B_1e_{B_2\Theta B_3})\\
&= (B_1B_2B_3\mu)\circ(B_1s_{B_2B_3}^{-1}\Theta)\circ(B_1\Theta B_2B_3\mu)\circ(B_1\Theta B_2s_{B_3}^{-1}\Theta)\\
&\qquad\circ(B_1\Theta s_{B_2\Theta B_3}^{-1})\circ(B_1\Delta B_2\Theta B_3)\circ(B_1\mu B_2\Theta B_3)\\
&\qquad\circ(B_1\Theta s_{B_2\Theta B_3}) && (\text{Lemma } 4.8.6+4.8.8)\\
&= (B_1B_2B_3\mu)\circ(B_1B_2s_{B_3}^{-1}\Theta)\circ(B_1s_{B_2}^{-1}B_3\Theta)\circ(B_1\Theta B_2B_3\mu)\\
&\qquad\circ(B_1\Theta B_2s_{B_3}^{-1}\Theta)\circ(B_1\Theta B_2s_{\Theta B_3}^{-1})\circ(B_1\Theta s_{B_2}^{-1}\Theta B_3)\\
&\qquad\circ(B_1\Delta B_2\Theta B_3)\circ(B_1\mu B_2\Theta B_3)\circ(B_1\Theta s_{B_2}\Theta B_3)\\
&\qquad\circ(B_1\Theta B_2s_{\Theta B_3}) && (\hat{\Theta}\text{ central})\\
&= (B_1B_2B_3\mu)\circ(B_1B_2s_{B_3}^{-1}\Theta)\circ(B_1B_2\Theta B_3\mu)\circ(B_1B_2\Theta s_{B_3}^{-1}\Theta)\\
&\qquad\circ(B_1s_{B_2}^{-1}s_{\Theta B_3}^{-1})\circ(B_1e_{B_2}\Theta B_3)\circ(B_1\Theta B_2s_{\Theta B_3}) && (\text{Lemma } 4.8.8)\\
&= (B_1B_2B_3\mu)\circ(B_1B_2s_{B_3}^{-1}\Theta)\circ(B_1B_2\Theta B_3\mu)\circ(B_1B_2\Theta s_{B_3}^{-1}\Theta)\\
&\qquad\circ(B_1B_2\Theta\Theta s_{B_3}^{-1})\circ(B_1s_{B_2}^{-1}s_{\Theta}^{-1}B_3)\circ(B_1e_{B_2}\Theta B_3)\\
&\qquad\circ(B_1\Theta B_2s_{\Theta B_3}) && (\hat{\Theta}\text{ central})\\
&= (B_1B_2B_3\mu)\circ(B_1B_2s_{B_3}^{-1}\Theta)\circ(B_1B_2\Theta s_{B_3}^{-1})\circ(B_1B_2\Theta\mu B_3)\\
&\qquad\circ(B_1s_{B_2}^{-1}s_{\Theta}^{-1}B_3)\circ(B_1e_{B_2}\Theta B_3)\circ(B_1\Theta B_2s_{\Theta B_3}) && (\hat{\Theta}\text{ central})\\
&= (B_1B_2s_{B_3}^{-1})\circ(B_1B_2\mu B_3)\circ(B_1B_2\Theta\mu B_3)\circ(B_1s_{B_2}^{-1}s_{\Theta}^{-1}B_3)\\
&\qquad\circ(B_1e_{B_2}\Theta B_3)\circ(B_1\Theta B_2s_{\Theta B_3}) && (\hat{\Theta}\text{ central})\\
&= (B_1B_2s_{B_3}^{-1})\circ(B_1B_2\mu B_3)\circ(B_1B_2\Theta\mu B_3)\circ(B_1B_2\Theta s_{\Theta}^{-1}B_3)\\
&\qquad\circ(B_1B_2\Delta\Theta B_3)\circ(B_1B_2\mu\Theta B_3)\circ(B_1s_{B_2}^{-1}s_{\Theta B_3}) && (\text{Lemma } 4.8.6)\\
&= (B_1B_2s_{B_3}^{-1})\circ(B_1B_2\mu B_3)\circ(B_1B_2\Theta\mu B_3)\circ(B_1B_2\Theta s_{\Theta}^{-1}B_3)\\
&\qquad\circ(B_1B_2\Delta\Theta B_3)\circ(B_1B_2\mu\Theta B_3)\circ(B_1B_2\Theta s_{\Theta}B_3)\\
&\qquad\circ(B_1s_{B_2}^{-1}\Theta s_{B_3}). && (\hat{\Theta}\text{ central})
\end{aligned}
$$

Equality of the left and the right hand side now follows, because

$$
\begin{aligned}
\mu \circ (\mu\Theta) \circ (\Theta\Delta) \circ (\Theta\mu) &= \mu \circ (\Theta\mu) \circ (\Theta\Delta) \circ (\Theta\mu) \\
&= \mu \circ (\Theta\mu) \\
&= \mu \circ (\Theta\mu) \circ (\Theta s_\Theta) \\
&= \mu \circ (\mu\Theta) \circ (\Theta s_\Theta) \\
&= \mu \circ (\mu\Theta) \circ (\Delta\Theta) \circ (\mu\Theta) \circ (\Theta s_\Theta) \\
&= \mu \circ (\Theta\mu) \circ (\Delta\Theta) \circ (\mu\Theta) \circ (\Theta s_\Theta) \\
&= \mu \circ (\Theta\mu) \circ (\Theta s_\Theta^{-1}) \circ (\Delta\Theta) \circ (\mu\Theta) \circ (\Theta s_\Theta),
\end{aligned}
$$

where we used repeatedly that $\hat{\Theta}$ is associative, symmetrically central and split. $\square$

Remark 4.8.18. The isotransformation ϕ defined in the proof of Proposition 4.8.17, can be obtained by composing the isotransformations in the following adjacent diagrams:

$$
\begin{tikzcd}
\mathcal{B} \times \mathcal{B} \arrow[r, "id \times \vartheta^{\hat{\Theta}}"] \arrow[d, "\otimes"] & \mathcal{B} \times \mathcal{B}_{\hat{\Theta}} \arrow[r, "\vartheta^{\hat{\Theta}} \times id"] \arrow[dr, "\rho"] & \mathcal{B}_{\hat{\Theta}} \times \mathcal{B}_{\hat{\Theta}} \arrow[d, "\otimes"] \\
\mathcal{B} \arrow[rr, "\vartheta^{\hat{\Theta}}"] & & \mathcal{B}_{\hat{\Theta}}
\end{tikzcd}
$$

The left part of the diagram commutes up to isotransformation since $\mathcal{B}_{\hat{\Theta}} \in \mathcal{B} - \mathrm{Mod}$, the right part commutes by Proposition 4.8.14.

Lemma 4.8.19. *Let $B_1, B_2 \in \mathcal{B}$ and $(M, f) \in \mathcal{B}_{\hat{\Theta}}$. We have a commutative diagram*

$$
\begin{tikzcd}
(B_1 \otimes B_2).(M,f) \arrow[r, equal] \arrow[d, "Pro\ 4.8.14"] & B_1.(B_2.(M,f)) \arrow[r, "Pro\ 4.8.14"] & B_1.(\vartheta^{\hat{\Theta}}(B_2) \otimes (M,f)) \arrow[d, "Pro\ 4.8.14"] \\
(\vartheta^{\hat{\Theta}}(B_1 \otimes B_2)) \otimes (M,f) \arrow[r, "Pro\ 4.8.17"] & (\vartheta^{\hat{\Theta}}(B_1) \otimes \vartheta^{\hat{\Theta}}(B_2)) \otimes (M,f) \arrow[r, equal] & \vartheta^{\hat{\Theta}}(B_1) \otimes (\vartheta^{\hat{\Theta}}(B_2) \otimes (M,f))
\end{tikzcd}
$$

Proof. Recall that by definition $B.(M, f) = (BM, Bf)$ for all $B \in \mathcal{B}$ and $(M, f) \in \mathcal{B}_{\hat{\Theta}}$. Together with the assumption of $\mathcal{B}$ being strict, this explains the horizontal identity arrow in the top left. The other identity arrow is due to the fact that the monoidal structure on $\mathcal{B}_{\hat{\Theta}}$ is strict as well.

Using the notation in the proofs of Propositions 4.8.14 and 4.8.17, the composition along the top right corner is given by $(B_1 i_{B_2} \otimes id_{(M,f)}) \circ (B_1 B_2 i_{(M,f)})$ and the composition along the bottom left corner is given by $(B_1 i_{\vartheta^{\hat{\Theta}}(B_2) \otimes (M,f)}) \circ (B_1 B_2 i_{(M,f)})$. First, note that we can prove equality of the two expressions without B_1 on the left. Second, we can also remove the idempotent f, since

$$
\begin{aligned}
&(i_{B_2} \otimes id_{(M,f)}) \circ (B_2 i_{(M,f)}) \\
&\qquad = (s_{B_2\Theta} M\Theta) \circ (B_2 \Delta f) \circ (B_2 s_{M\Theta}) \circ (B_2 M \Delta) \\
&\qquad\qquad \circ (B_2 f) \\
&\qquad = (s_{B_2\Theta} M\Theta) \circ (B_2 \Delta M\Theta) \circ (B_2 s_{M\Theta}) \circ (B_2 f\Theta) \\
&\qquad\qquad \circ (B_2 M\Delta) \circ (B_2 f) \\
&\qquad = (s_{B_2\Theta} M\Theta) \circ (B_2 \Delta M\Theta) \circ (B_2 s_{M\Theta}) \circ (B_2 M\Delta) \\
&\qquad\qquad \circ (B_2 f) \circ (B_2 f) && \text{(Proposition 4.3.38)} \\
&\qquad = (s_{B_2} \Theta M\Theta) \circ (B_2 s_\Theta M\Theta) \circ (B_2 \Delta M\Theta) \circ (B_2 s_{M\Theta}) \\
&\qquad\qquad \circ (B_2 M\Delta) \circ (B_2 f) && (\hat{\Theta} \text{ central})
\end{aligned}
$$

and

$$
\begin{aligned}
&(i_{\vartheta^{\hat{\Theta}}(B_2)\otimes(M,f)}) \circ (B_2 i_{(M,f)}) \\
&\qquad = (s_{B_2\Theta M\Theta}) \circ (B_2 \Theta M\Delta) \circ (B_2 \Theta f) \circ (B_2 s_{M\Theta}) \\
&\qquad\qquad \circ (B_2 M\Delta) \circ (B_2 f) \\
&\qquad = (s_{B_2} \Theta M\Theta) \circ (B_2 s_{\Theta M\Theta}) \circ (B_2 \Theta M\Delta) \circ (B_2 s_{M\Theta}) \\
&\qquad\qquad \circ (B_2 f\Theta) \circ (B_2 M\Delta) \circ (B_2 f) && (\hat{\Theta} \text{ central}) \\
&\qquad = (s_{B_2} \Theta M\Theta) \circ (B_2 s_{\Theta M\Theta}) \circ (B_2 \Theta M\Delta) \circ (B_2 s_{M\Theta}) \\
&\qquad\qquad \circ (B_2 M\Delta) \circ (B_2 f) \circ (B_2 f). && \text{(Proposition 4.3.38)}
\end{aligned}
$$

Note that we can now also remove B_2 since s_{B_2} is invertible. To complete the proof, we compare the remaining expressions

$$
\begin{aligned}
&(s_\Theta M\Theta) \circ (\Delta M\Theta) \circ (s_{M\Theta}) \circ (M\Delta) \\
&\qquad = (s_\Theta M\Theta) \circ (\Theta s_{M\Theta}) \circ (s_{M\Theta}\Theta) \circ (M\Theta\Delta) \circ (M\Delta) && (\hat{\Theta} \text{ central}) \\
&\qquad = (s_{\Theta M\Theta}) \circ (s_{M\Theta\Theta}) \circ (M\Theta s_\Theta^{-1}) \circ (M\Theta\Delta) \circ (M\Delta) && (\hat{\Theta} \text{ central}) \\
&\qquad = (s_{\Theta M\Theta}) \circ (s_{M\Theta\Theta}) \circ (M\Theta\Delta) \circ (M\Delta) && (\hat{\Theta} \text{ symmetr. central}) \\
&\qquad = (s_{\Theta M\Theta}) \circ (s_{M\Theta\Theta}) \circ (M\Delta\Theta) \circ (M\Delta) && \text{(Lemma 4.3.7)} \\
&\qquad = (s_{\Theta M\Theta}) \circ (\Theta M\Delta) \circ (s_{M\Theta}) \circ (M\Delta),
\end{aligned}
$$

where we used that $\hat{\Theta}$ is symmetrically central. $\square$

Proposition 4.8.20. *The functor $\mathcal{B}_{\hat{\Theta}}^{\mathrm{rev}} \to \mathfrak{A}dd(\mathcal{B}_{\hat{\Theta}}, \mathcal{B}_{\hat{\Theta}})$ induced by taking tensor products on the right factors over the forgetful functor $\mathcal{H}om_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}}, \mathcal{B}_{\hat{\Theta}}) \to \mathfrak{A}dd(\mathcal{B}_{\hat{\Theta}}, \mathcal{B}_{\hat{\Theta}})$*

Proof. Let $N \in \mathcal{B}_{\hat{\Theta}}$ and denote the associated additive functor by F. In other words for $M \in \mathcal{B}_{\hat{\Theta}}$, we have $F(M) = M \otimes N$, where $\otimes$ denotes the tensor product of $\mathcal{B}_{\hat{\Theta}}$. For all $B \in \mathcal{B}$, we need to provide an isomorphism $B.F(M) \cong F(B.M)$ which is natural in B. The composition $B.F(M) = B.(M \otimes N) \overset{\text{Pro 4.8.14}}{\cong} \vartheta^{\hat{\Theta}}(B) \otimes (M \otimes N) \cong$

$(\vartheta^{\hat{\Theta}}(B) \otimes M) \otimes N \overset{\text{Pro 4.8.14}}{\cong} (B.M) \otimes N = F(B.M)$ clearly satisfies this requirement, where the second isomorphism is provided by the associator of the monoidal category $\mathcal{B}_{\hat{\Theta}}$. The remaining requirements are summarised in Remark 3.3.4. First, since $\mathcal{B}$ and $\mathcal{B}_{\hat{\Theta}}$ are strict monoidal categories, we only need to show that the morphism $I.(M \otimes N) \to (I.M) \otimes N$ is the identity morphism. This is obvious. Second, the requirement involving the associator follows directly from the corresponding axiom for the associator of $\mathcal{B}$. □

Proposition 4.8.21. *The composition $\mathcal{B}_{\hat{\Theta}} \xrightarrow{(^{\hat{\Theta}}\vartheta, e)} {}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}}, \mathcal{B}_{\hat{\Theta}})$ is the same as the functor induced by right multiplication via the tensor product of $\mathcal{B}_{\hat{\Theta}}$.*

Proof. The composition $\mathcal{B}_{\hat{\Theta}} \xrightarrow{(^{\hat{\Theta}}\vartheta, e)} {}_{\hat{\Theta}}\mathcal{B}_{\hat{\Theta}} \xrightarrow{\text{Rem 4.3.67}} {}_{\hat{\Theta}}(\mathcal{B}_{\hat{\Theta}}) \xrightarrow{\text{Lem 4.5.12}} \mathcal{H}om_{\mathcal{B}}(\mathcal{B}_{\hat{\Theta}}, \mathcal{B}_{\hat{\Theta}})$ applied to $(B, f) \in \mathcal{B}_{\hat{\Theta}}$ is given by $(B, f) \mapsto (B, (\Theta f) \circ e_B) \mapsto ((B, id_{B\Theta}), (\Theta f) \circ e_B) \mapsto \big((C, g) \mapsto \big(C\Theta(B, id_{B\Theta}), (g\, id_{(B, id_{B\Theta})}) \circ (C(\Theta f \circ e_B))\big)\big)$. Since $\mathcal{B}_{\hat{\Theta}}$ is defined as an idempotent closure, the summand of an object (B, e_1) associated to an idempotent e_2 is given by $(B, e_1 \circ e_2)$. Since $C\Theta(B, id_{B\Theta}) = (C\Theta B, id_{C\Theta B\Theta})$ and $id_{(B, id_{B\Theta})} = id_{B\Theta}$ holds in $\mathcal{B}_{\hat{\Theta}}$, the above described image of (B, f) is equal to the functor given on objects by $((C, g) \mapsto (C\Theta B, id_{C\Theta B\Theta} \circ (gf) \circ (Ce_B)))$. To complete the proof, we have to show that this functor agree with the functor given by multiplication on right by (B, f), which is given on objects by $((C, g) \mapsto (C, g) \otimes (B, f))$. By Proposition 4.8.13, we have $(C, g) \otimes (B, f) = (C\Theta B, (gf) \circ (Ce_B))$ and the proof is complete. □

We end the section with the push-pull formula, which is usually stated in the form $\vartheta_{\hat{\Theta}}(\vartheta^{\hat{\Theta}}(B) \otimes M) \cong B \otimes \vartheta_{\hat{\Theta}}(M)$ for all $M \in \mathcal{B}_{\hat{\Theta}}$ and $B \in \mathcal{B}$:

Corollary 4.8.22. *The following diagram commutes up to isotransformation*

$$\begin{tikzcd}
\mathcal{B} \times \mathcal{B}_{\hat{\Theta}} \arrow[rr, "id \times \vartheta_{\hat{\Theta}}"] \arrow[d, "\vartheta^{\hat{\Theta}} \times id"] & & \mathcal{B} \times \mathcal{B} \arrow[d, "\otimes"] \\
\mathcal{B}_{\hat{\Theta}} \times \mathcal{B}_{\hat{\Theta}} \arrow[r, "\otimes"] & \mathcal{B}_{\hat{\Theta}} \arrow[r, "\vartheta_{\hat{\Theta}}"] & \mathcal{B}
\end{tikzcd}$$

Proof. Adding an arrow from the top left to the bottom middle, we recognise the triangle on the left from Proposition 4.8.14. The remaining square on the right commutes up to isotransformation as well, since $\vartheta_{\hat{\Theta}}$ is a $\mathcal{B}$-functor. □

Remark 4.8.23. Similarly to Corollary 4.8.22, the following diagram commutes up to an (explicitly definable) isotransformation

$$\begin{tikzcd} \mathcal{B}_{\hat{\Theta}} \times \mathcal{B} \arrow[rr, "\vartheta_{\hat{\Theta}} \times id"] \arrow[d, "id \times \vartheta^{\hat{\Theta}}"] & & \mathcal{B} \times \mathcal{B} \arrow[d, "\otimes"] \\ \mathcal{B}_{\hat{\Theta}} \times \mathcal{B}_{\hat{\Theta}} \arrow[r, "\otimes"] & \mathcal{B}_{\hat{\Theta}} \arrow[r, "\vartheta_{\hat{\Theta}}"] & \mathcal{B} \end{tikzcd}.$$

Remark 4.8.24. It would be interesting to try to simplify and possibly generalise this section using induction and restriction functors, and in particular Proposition 4.5.25. These techniques had not been available at the time of writing this section.

4.9 Towards a Jordan-Hölder theory

In this section, we provide an abstract framework in which Jordan-Hölder theory can be addressed. In Section 4.9.1, we introduce certain cofiltrations associated to a chosen refinement of the ordering of the set of 2-sided cells. This results in cosubquotients that effectively break up higher representations into smaller pieces associated to 2-sided cells. We also study how cyclic modules behave under these cofiltrations and introduce isotypic modules with respect to 2-sided cells. In Section 4.9.2, we then introduce isotypic covers. Using these covers, we formulate what one would expect from a Jordan-Hölder theory, and explain various problems that we have faced when trying to establish results.

4.9.1 Cofiltrations

In this section, we construct cofiltrations and cosubquotients. The associated cosubquotient 2-functors come close to being projections, and we will give an criterion for this. The set of 2-sided cells being partially order, this is not a canonical construction. However, we will also discuss a maximal and a small version which are indeed canonical. We also define isotypic additive higher representations.

Let $\mathfrak{A}$ be an additive 2-category with countably many 2-sided cells. Recall Notation 3.3.6 and Definition 3.3.7. Now, choose $N \subset \mathbb{Z}$ and an enumeration $\{\underline{c}_i\}_{i \in N}$ of the 2-sided cells of $\mathfrak{A}$ such that $W(\mathfrak{A}) = \bigcup_{i \in N} \underline{c}_i$ is a disjoint union. We assume that for all $i, k \in N$ and $j \in \mathbb{Z}$ with $i < j < k$, we have $j \in N$. Further, we assume that $\underline{c}_i <_{LR} \underline{c}_j \Rightarrow i < j$ for all $i, j \in N$. For all $i \in N$, denote by $\mathfrak{J}_i \trianglelefteq \mathfrak{A}$ the ideal generated by all 1-arrows B_x where $x \in \bigcup_{j<i} \underline{c}_j$, and denote by $\pi_i : \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}_i$ the associated quotient 2-functor. Note that Definition 3.3.7 implies that $\pi_i(B_x) \neq 0$

for all $x \in \underline{c}_i$. Using Section 4.4, we define $\mathcal{F}_i := \operatorname{Res}_{\pi_i} \operatorname{Ind}_{\pi_i}$ for all $i \in N$, define $\mathcal{F}_i := id_{\mathfrak{A}-\mathrm{Mod}}$ for $i \in \mathbb{Z}$ with $i < \inf(N)$ and define $\mathcal{F}_i := 0$ for $i \in \mathbb{Z}$ with $i > \sup(N)$. For all $R \in \mathfrak{A} - \mathrm{Mod}$ and $i, j \in \mathbb{Z}$ such that $i \leq j$, Proposition 4.4.9 implies that we have a canonical quotient $\mathfrak{A}$-functor $\mathcal{F}_i(R) \to \mathcal{F}_j(R)$. Thus, we have constructed a cofiltration 2-functor $\mathcal{F} : \mathfrak{A} - \mathrm{Mod} \to \mathfrak{A} - \mathrm{Mod}(\mathfrak{A}dd^{\mathrm{cof}})$ where $\mathfrak{A}dd^{\mathrm{cof}}$ denotes the additive 2-category of $\mathbb{Z}$-cofiltered additive categories. Further for all $i \in \mathbb{Z}$, we define an endo-2-functor $\mathcal{Q}_i$ of $\mathfrak{A} - \mathrm{Mod}$ by setting $\mathcal{Q}_i := \ker(\mathcal{F}_i \to \mathcal{F}_{i+1})$. We call $\mathcal{Q}_i$ a $\underline{c}_i$-cosubquotient 2-functor. Note that $\mathcal{Q}_i \neq 0$ implies $i \in N$.

Clearly, this construction depends on the chosen refinement of the order $\leq_{LR}$. Let us fix a 2-sided cell $\underline{c}$ of $\mathfrak{A}$, and let us consider the extreme cases which can result from the above construction. Denote by $\mathfrak{J}_{\lneq_{LR}}$, $\mathfrak{J}_{\not\geq_{LR}}$, $\mathfrak{J}_{\leq_{LR}}$ and $\mathfrak{J}_{\not>_{LR}}$ the ideals of $\mathfrak{A}$ generated by all B_x where $x \in W(\mathfrak{A})$ is contained in $\bigcup_{\underline{c}' \lneq_{LR} \underline{c}} \underline{c}'$, $\bigcup_{\underline{c}' \not\geq_{LR} \underline{c}} \underline{c}'$, $\bigcup_{\underline{c}' \leq_{LR} \underline{c}} \underline{c}'$ and $\bigcup_{\underline{c}' \not>_{LR} \underline{c}} \underline{c}'$ respectively. Then, we have the following canonical commutative system of full 2-functors

$$\begin{tikzcd}
 & & \mathfrak{A} \arrow[dl, two heads, "\pi_{\underline{c}}"'] \arrow[dr, two heads, "\pi^{\underline{c}}"] \arrow[ddll, two heads, bend right, "\tau_{\underline{c}}"'] \arrow[ddrr, two heads, bend left, "\tau^{\underline{c}}"] & & \\
 & \mathfrak{A}/\mathfrak{J}_{\lneq_{LR}\underline{c}} \arrow[rr, two heads] \arrow[dl, two heads] & & \mathfrak{A}/\mathfrak{J}_{\leq_{LR}\underline{c}} \arrow[dr, two heads] & \\
\mathfrak{A}/\mathfrak{J}_{\not\geq_{LR}\underline{c}} \arrow[rrrr, two heads] & & & & \mathfrak{A}/\mathfrak{J}_{\not>_{LR}\underline{c}}
\end{tikzcd},$$

Now, we define $\Pi_{\underline{c}} := \operatorname{Res}_{\pi_{\underline{c}}} \operatorname{Ind}_{\pi_{\underline{c}}}$, $\Pi^{\underline{c}} := \operatorname{Res}_{\pi^{\underline{c}}} \operatorname{Ind}_{\pi^{\underline{c}}}$, $\mathcal{T}_{\underline{c}} := \operatorname{Res}_{\tau_{\underline{c}}} \operatorname{Ind}_{\tau_{\underline{c}}}$ and $\mathcal{T}^{\underline{c}} := \operatorname{Res}_{\tau^{\underline{c}}} \operatorname{Ind}_{\tau^{\underline{c}}}$.

Definition 4.9.1. *Let $\underline{c}$ be a 2-sided cell of $\mathfrak{A}$. We call $\mathcal{Q}_{\underline{c}} := \operatorname{Ker}(\Pi_{\underline{c}} \to \Pi^{\underline{c}})$ the* maximal $\underline{c}$-cosubquotient 2-functor *of $\mathfrak{A} - \mathrm{Mod}$. Further, we call $\mathcal{Q}^{\underline{c}} := \operatorname{Ker}(\mathcal{T}_{\underline{c}} \to \mathcal{T}^{\underline{c}})$ the* small $\underline{c}$-cosubquotient 2-functor *of $\mathfrak{A} - \mathrm{Mod}$.*

Remark 4.9.2. Using Proposition 4.4.9 and transitivity of the induction and restriction 2-functors, we obtain comparison morphisms $\mathcal{Q}_{\underline{c}_i} \to \mathcal{Q}_i \to \mathcal{Q}^{\underline{c}_i}$ all $i \in N$. Further given 2-sided cells $\underline{c}$, $\underline{c}'$ of $\mathfrak{A}$ and $R \in \mathfrak{A} - \mathrm{Mod}$ such that $\mathcal{Q}^{\underline{c}'}(R) \neq 0$ implies $\underline{c}' \not<_{LR} \underline{c}$, we have $\mathcal{Q}_{\underline{c}}(R) = \operatorname{Res}_{\pi_{\underline{c}}} \operatorname{Coind}_{\pi_{\underline{c}}}(R)$. This justifies calling $\mathcal{Q}_{\underline{c}}$ maximal. We do not call $\mathcal{Q}^{\underline{c}}$ minimal, since it is possible in examples to take quotients by further morphisms without annihilating additional objects. Compare Remark 4.9.17, where we discussed certain smallest quotients introduced in [MaMi5].

Remark 4.9.3. Let $\underline{c}$ be a 2-sided cell of $\mathfrak{A}$. Then, the sets of indecomposable objects up to isomorphism of $\mathcal{Q}_{\underline{c}}(\bigoplus_{a \in \mathfrak{A}} \mathcal{P}_a)$ and $\mathcal{Q}^{\underline{c}}(\bigoplus_{a \in \mathfrak{A}} \mathcal{P}_a)$ are in bijection with $\underline{c}$.

We now generalise the notion of λ-isotypic modules of [Rou5]. Note that we do not use $\mathcal{Q}^{\underline{c}}$, as we aim to retain as much structure as possible.

Definition 4.9.4. *Let $R \in \mathfrak{A} - \mathrm{Mod}$ and let $\underline{c}$ be a 2-sided cell of $\mathfrak{A}$. Then, we call R $\underline{c}$-*isotypic *if $\underline{c}$ is the unique 2-sided cell with $R = \mathcal{Q}_{\underline{c}}(R)$.*

Proposition 4.9.5. *We have a locally full comparison morphism $\mathcal{Q}_{\underline{c}} \to \mathcal{Q}^{\underline{c}}$ which is not necessarily locally essentially surjective.*

Proof. For all $R \in \mathfrak{A} - \mathrm{Mod}$, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{Q}_{\underline{c}}(R) \arrow[r, hook] \arrow[d, dashed, "\exists!"] & \Pi_{\underline{c}}(R) \arrow[r, two heads] \arrow[d, two heads] & \Pi^{\underline{c}}(R) \arrow[d, two heads] \\
\mathcal{Q}^{\underline{c}}(R) \arrow[r, hook] & \mathcal{T}_{\underline{c}}(R) \arrow[r, two heads] & \mathcal{T}^{\underline{c}}(R)
\end{tikzcd},$$

obtained from the adjunctions and the universal property of kernels. The vertical arrow on the left defines the comparison functor of the statement. By a standard diagram chasing argument, the comparison functor is locally full. In order to see that it is not locally essentially surjective in general, let us assume that there exist 2-sided cells $\underline{c}$ and $\underline{c}'$ of $\mathfrak{A}$ which are not $\leq_{LR}$-comparable. Further, assume that both 2-sided cells have cell modules. Let $a \in \mathfrak{A}$, let $x \in \underline{c}$ and let $y \in \underline{c}'$, such that B_x, $B_y \in \mathfrak{A}_a$. Consider the full sub-$\mathfrak{A}$-module R of $\mathcal{P}_a$ with objects isomorphic to $b.(B_x \oplus B_y)$ for all $b \in \mathfrak{A}(a, a')$. Clearly, $I_a.(B_x \oplus B_y)$ is a non-trivial object of both $\Pi_{\underline{c}}(R)$ and $\mathcal{T}_{\underline{c}}(R)$. Further, we claim that $I_a.(B_x \oplus B_y)$ is zero in $\mathcal{T}^{\underline{c}}(R)$, while being nonzero in $\Pi^{\underline{c}}(R)$. Thus, the above diagram shows that $I_a.(B_x \oplus B_y)$ is an object of $\mathcal{Q}^{\underline{c}}(R)$, whereas it not an object of $\mathcal{Q}_{\underline{c}}(R)$. Therefore, the comparison functor is not essentially surjective. To complete the proof, let us show the claim. Since we assumed that $\underline{c}'$ has cell modules, there exists $z \in \underline{c}'$ and a split embedding of B_y in $B_z B_y$. Thus, B_y also embeds into $B_z.(B_x \oplus B_y)$. Note that $z \not\geq_{LR} \underline{c}$ by assumption. By definition of $\mathcal{T}^{\underline{c}}$, it follows that $B_z.(B_x \oplus B_y)$ becomes zero in $\mathcal{T}^{\underline{c}}(R)$. Therefore, the idempotent corresponding to the second summand of $I_a.(B_x \oplus B_y)$ becomes zero in $\mathcal{T}^{\underline{c}}(R)$. The same argument applies to the idempotent associated to the first summand. Consequently, $I_a.(B_x \oplus B_y)$ is zero in $\mathcal{T}^{\underline{c}}(R)$. On the other hand, the identity morphisms of the objects that become zero in $\Pi^{\underline{c}}(R)$ factor over objects of the form $B_z.(B_x \oplus B_y)$ where $z \leq_{LR} \underline{c}$. Thus, the idempotent of $I_a.(B_x \oplus B_y)$ associated to B_y will not factor over the zero-object of $\Pi^{\underline{c}}(R)$. $\square$

Remark 4.9.6. Let F be an $\mathfrak{A}$-functor. If F is locally full (resp. locally faithful and locally essentially surjective) than $\mathcal{Q}_{\underline{c}}(F)$ is locally full (resp. locally faithful).

Definition 4.9.7. *Let $R \in \mathfrak{A}-\mathrm{Mod}$ and let $\underline{c}$ be a 2-sided cell. We say that $0 \neq N \in R$ has a* $\underline{c}$-source *if there exists $x \in \underline{c}$ and $M \in R$ such that $N \subset^{\oplus} B_x M$. In this situation, we call M a* $\underline{c}-$source*. We say that R* has $\underline{c}$-sources *provided that every object of R has a $\underline{c}$-source.*

Definition 4.9.8. *Let M_1, $M_2 \in R \in \mathfrak{A} - \mathrm{Mod}$ and assume that M_1 is a $\underline{c}-$source of M_2 and vice versa. Then, we call M_1 and M_2* equivalent $\underline{c}$-sources*.*

Remark 4.9.9. Let M_1, $M_2 \in R \in \mathfrak{A} - \mathrm{Mod}$ be equivalent $\underline{c}$-sources. Then, there exist x, y, $z \in \underline{c}$ and a split embedding of B_x into $B_y B_z$.

Proposition 4.9.10. *Let $\mathcal{C} \in \mathfrak{A} - \mathrm{Mod}$ be a cell module, $\underline{c}$ be a 2-sided cell of $\mathfrak{A}$, and assume that $\mathcal{Q}_{\underline{c}}(\mathcal{C}) \neq 0$. Then, $B_x C = 0$ for all $x <_{LR} \underline{c}$ and $C \in \mathcal{C}$, $\mathcal{C} \cong \mathcal{Q}_{\underline{c}}(\mathcal{C}) \cong \mathcal{Q}^{\underline{c}}(\mathcal{C})$, and $\mathcal{C}$ has $\underline{c}-$sources.*

Proof. The claim that $\mathcal{C} = \mathcal{Q}^{\underline{c}}(\mathcal{C})$ follows from $\mathcal{C}$ having $\underline{c}$-sources. The remaining claims are straightforward. $\square$

The next proposition applies for example to any cell module that has finitely many isomorphism classes of indecomposable objects.

Proposition 4.9.11. *Let $\underline{c}$ and $\underline{c}'$ be 2-sided cells of $\mathfrak{A}$, and let $\mathcal{C} \in \mathfrak{A} - \mathrm{Mod}$ be a non-trivial cell module such that $\mathcal{Q}_{\underline{c}}(\mathcal{C}) \cong \mathcal{Q}_{\underline{c}'}(\mathcal{C})$. Further, assume that $\mathfrak{A}$ has finitely many 2-sided cells or that $\mathcal{C}$ has equivalent $\underline{c}$-sources. Then, $\underline{c} = \underline{c}'$.*

Proof. The case, of finitely many 2-sided cells follows from the same arguments in in the classical case, see Corollary 2.4.10. Let us thus assume that $\mathcal{C}$ has equivalent $\underline{c}$-sources. Let $C \in \mathcal{C}$ be a $\underline{c}$-source that is equivalent to another $\underline{c}$-source. In particular, we have $x \in \underline{c}$ such that $C \subset^{\oplus} B_x C$. Since $\mathcal{Q}_{\underline{c}'}(\mathcal{C}) \neq 0$, there exists $y \in \underline{c}'$ and $N \in \mathcal{C}$ such that $C \subset^{\oplus} B_y N$. Thus, $C \subset^{\oplus} B_x C \subset^{\oplus} B_x B_y N$. However, $B_x B_y$ decompose into a direct sum of B_z with $z \leq_{LR} x$ and $z \leq_{LR} y$. If all such $z <_{LR} x$ (or $z \leq_{LR} y$), then $B_z N = 0$ and thus $B_x = 0$, a contradiction. Thus, $\underline{c} = \underline{c}'$. $\square$

If $R \in \mathfrak{A} - \mathrm{Mod}$ is not a cell module, $\mathcal{Q}_{\underline{c}}(R) \cong \mathcal{Q}_{\underline{c}'}(R)$ does no longer imply $\underline{c} = \underline{c}'$:

Example 4.9.12. Let $\mathcal{B}$ be the additive monoidal category with one generator X that satisfies the relation $XX = 0$. Then, we have two 2-sided cells $\{[I]\}$ and $\{[X]\}$ and $\mathcal{Q}_{\{[I]\}}(\mathcal{B}) = \mathcal{B}/X \cong X.\mathcal{B} = \mathcal{Q}_{\{[X]\}}(\mathcal{B})$.

The following definition provides an alternative to Definition 4.5.31:

Definition 4.9.13. *Let $\underline{c}$ be a 2-sided cells of $\mathfrak{A}$. We say $\mathfrak{A}$ has $\underline{c}$-*cell modules *if $\mathcal{Q}_{\underline{c}}$ is a projector,* i.e. *if $\mathcal{Q}_{\underline{c}}\mathcal{Q}_{\underline{c}} = \mathcal{Q}_{\underline{c}}$. We say $\mathfrak{A}$ has* cell modules*, if it has $\underline{c}$-cell modules for all 2-sided cells $\underline{c}$ of $\mathfrak{A}$.*

Proposition 4.9.14. *Let $\mathcal{C} \in \mathfrak{A} - \mathrm{Mod}$ be a cell module and let $\underline{c}$ be a 2-sided cell of $\mathfrak{A}$. Then, $\mathcal{C}$ belongs to $\underline{c}$ as in Definition 4.5.31 if an only if $\mathcal{C}$ belongs to $\underline{c}$ as in Definition 4.9.13.*

Proof. Let $\mathcal{C}$ be a cell module that belongs to $\underline{c}$ as in Definition 4.9.13. Then, Proposition 4.9.11 implies that $\{B_x | x \lneq_{LR} \underline{c}\} \subset \mathcal{A}\mathrm{nn}_{\mathcal{B}}(\mathcal{C})$. By construction of $\mathcal{Q}_{\underline{c}}$, $\{B_x | x \in \underline{c}\}$ cannot be contained in $\mathcal{A}\mathrm{nn}_{\mathcal{B}}(\mathcal{C})$. The assumptions of Proposition 2.4.9 are satisfied (which generalises directly to the categorical case), and it follows that the latter definition implies the former.

The other hand, assume that $\mathcal{C}$ belongs to $\underline{c}$ as in Definition 4.5.31. Then, it is easy to see that $\Pi^{\underline{c}}(\mathcal{C}) = 0$, while $\Pi^{\underline{c}}(\mathcal{C}) = \mathcal{C}$. Thus $\mathcal{Q}_{\underline{c}}(\mathcal{C}) = \mathcal{C}$. $\square$

We also have a statement analogous to Theorem 2.5.6. Let us give a part of the argument in the categorical situation:

Proposition 4.9.15. *Let $\underline{c}$ be a 2-sided cell of $\mathfrak{A}$. Assume that for all $x \in W(\mathfrak{A})$ there exists $y \in W(\mathfrak{A})$ such that $y \sim_{LR} x$ and such that the exists a split embedding of B_x into $B_y B_x$. Then, $\mathfrak{A}$ has $\underline{c}$-cell modules.*

Proof. Let $\underline{c}$ be a 2-sided cell of $\mathfrak{A}$. We have to show that $\mathcal{Q}_{\underline{c}}$ is a projector. It is easy to see that this holds provided that $\mathcal{Q}_{\underline{c}}(R)$ has $\underline{c}$-sources for all $R \in \mathfrak{A} - \mathrm{Mod}$. Now given any $N \in \mathcal{Q}_{\underline{c}}(R)$ there exists $x \in \underline{c}$ and $L \in R$ such that $N \subset^{\oplus} B_x L$ by definition of $\mathcal{Q}_{\underline{c}}(R)$. By assumption, we find $y \in \underline{c}$ such that $B_x \subset^{\oplus} B_y B_x$. Combining these two statements, we obtain $N \subset^{\oplus} B_x L \subset^{\oplus} B_y B_x L$. Thus, $B_x L$ is a $\underline{c}$-source for N. $\square$

Remark 4.9.16. The papers [Rou4, Rou5] construct filtrations of $\mathfrak{A}_{\mathfrak{g}}$-modules (or actually of subquotients thereof) under certain finiteness assumptions. This construction is inductive and uses $\mathfrak{A}_{\mathfrak{g}}$-functors from $\mathcal{L}(\lambda)$. More generally, instead of using the universal cell modules for this purpose, one could use the free versions ${}_{\hat{\Theta}}\mathcal{P}_a$ provided that the cell object $\hat{\Theta}$ has a lift in $\mathfrak{A}$. More conceptual, one can then construct filtrations of $\mathfrak{A} - \mathrm{Mod}$ using Coind in the similar fashion as we did above using Ind. Note however, this would not involve taking quotients of $\mathfrak{A}$, but $\mathfrak{A}$-functors between the ${}_{\hat{\Theta}}\mathcal{P}_a$ and thus requires an additional choice of a split Frobenius object per 2-sided cell.

Remark 4.9.17. Filtrations are also discussed in [MaMi5, Section 4.3]. The starting point is a maximal filtration of an idempotent closed module, *i.e.* an ascending chain of full idempotent closed submodules which cannot be refined. Then, the smallest quotients of the subquotients of this filtration are considered. By [MaMi5, Lemma 4], these smallest quotients exist due to the finiteness assumptions imposed. By construction, these subquotients have no non-trivial ideals, and are thus so called simple transitive modules. Note that taking smallest quotients practically ignores the initial filtration chosen. It appears unlikely, even under the finiteness assumptions of [MaMi5], that one can construct a canonical endo-2-functor of $\mathfrak{A} - \mathrm{Mod}$ that sends all $\mathfrak{A}$-modules to the direct sum of its smallest subquotients. This would require a better understanding of extension between simple transitive modules, as was obtained in [Rou5, Lemma 4.21] for 2-Kac-Moody algebras. If such a 2-functor would exists, precomposing it with $\bigoplus_{\underline{c}} \mathcal{Q}_{\underline{c}}$ would yield a direct sum decomposition. Also note that for all 2-sided cell $\underline{c}$, the comparison morphism $\mathcal{Q}_{\underline{c}} \to \mathcal{Q}^{\underline{c}}$ is locally a quotient functor on the modules considered in [MaMi5].

4.9.2 Covers and limitations

In this final section of the abstract part of this thesis, we introduce isotypic covers. One might hope that all isotypic modules are controlled in some sense by an isotypic cover. We will comment on why this cannot be true in the strong sense one might hope for in general, state difficulties that arise even in cases where one would expect this to be true and hint at a notion of generalised isotypic covers that might help to remedy these shortcomings in the future. We also comment on Jordan-Hölder theorems in the literature.

Definition 4.9.18. *Let $\underline{c}$ be a 2-sided cell of $\mathfrak{A}$, let $\mathcal{C} \in \mathfrak{A} - \mathrm{Mod}$ be a universal cell module associated to $\underline{c}$, and let R be a $\underline{c}$-isotypic $\mathfrak{A}$-module. Then, we call the evaluation $\mathfrak{A}$-functor $\mathcal{C} \otimes_{\mathcal{Z}_{\mathfrak{A}}(\mathcal{C})} \mathcal{H}om_{\mathfrak{A}}(\mathcal{C}, R) \to R$ a* $\underline{c}$-isotypic cover.

By Proposition 4.1.5, the evaluation $\mathfrak{A}$-functors used in this definition exist. Naively, one might hope that all $\underline{c}$-isotypic covers are equivalence for certain classes of acting additive 2-categories $\mathfrak{A}$, at least for certain locally full sub-2-categories of $\mathfrak{A} - \mathrm{Mod}$. Such a statement would be called Jordan-Hölder theorem. Indeed this was achieved for 2-Kac-Moody algebras in [Rou4], see Remark 4.9.24. A weaker, yet more broadly applicable statement was obtained in [MaMi5], see Remark 4.9.25. Next, we discuss situations in which $\underline{c}$-isotypic covers fail to be faithful, full or essentially surjective.

Remark 4.9.19. If $\mathfrak{A}$ is not strongly regular, cell $\mathfrak{A}$-modules can have non-trivial endomorphism categories. Section 5.5 gives several such examples. In these examples, there exist several cell modules associated to the same 2-sided cell $\underline{c}$ that have non-isomorphic different Grothendieck combinatorics. Thus for any fixed cell module associated to $\underline{c}$, the $\underline{c}$-isotypic covers will not be essentially surjective, not even when only applied to cell modules associated to $\underline{c}$. Furthermore, the construction of coinvariant modules in Section 4.6 shows that isotypic covers will not be full in this case. To see this, note that one can calculate $\mathcal{H}om_{\mathcal{B}}(\mathcal{M}, \mathcal{M}_G)$ using Theorem 4.2.19 for all 2-cyclic $\mathcal{B}$-modules $\mathcal{M}$ that are acted upon by a group G.

Remark 4.9.20. It might be possible to generalise the base change over the centre construction of Section 4.1, to allow for a base change over a monoidal category. For a 2-sided cell $\underline{c}$, a cell $\mathfrak{A}$-module $\mathcal{C}$ associated to $\underline{c}$ and $R \in \mathfrak{A} - \mathrm{Mod}$ that is $\underline{c}$-isotypic, this would give meaning to the expression $\mathcal{C} \otimes_{\mathcal{E}nd_{\mathfrak{A}}(\mathcal{C})} \mathcal{H}om_{\mathfrak{A}}(\mathcal{C}, R)$, and should also allow for an evaluation $\mathfrak{A}$-functor which we would call a *generalised* $\underline{c}$*-isotypic cover*. If all objects of $\mathcal{E}nd_{\mathfrak{A}}(\mathcal{C})$ are given by direct sums of the identity $\mathfrak{A}$-functor of $\mathcal{C}$, this construction would need to satisfy $\mathcal{C} \otimes_{\mathcal{E}nd_{\mathfrak{A}}(\mathcal{C})} \mathcal{H}om_{\mathfrak{A}}(\mathcal{C}, R) \cong \mathcal{C} \otimes_{\mathcal{Z}_{\mathfrak{A}}(\mathcal{C})} \mathcal{H}om_{\mathfrak{A}}(\mathcal{C}, R)$. Further, one might expect that $R_G \cong R \otimes_{\mathcal{E}nd_{\mathfrak{A}}(R)} \mathcal{H}\mathrm{om}_{\mathfrak{A}}(R, R_G)$ for all $R \in \mathfrak{A} - \mathrm{Mod}$ that carry a group action of a group G, see Section 4.6. Furthermore, we hope that such generalised *underlinec*-isotypic covers have better properties than $\underline{c}$-isotypic covers, and that they might even be equivalences for large classes of acting 2-categories and their modules.

Remark 4.9.21. Another obstacle is the non-trivial relationship between $\mathcal{Z}_{\mathfrak{A}}(\mathcal{C}_1)$ and $\mathcal{Z}_{\mathfrak{A}}(\mathcal{C}_2)$ for weakly equivalent (cell) $\mathfrak{A}$-modules $\mathcal{C}_1$ and $\mathcal{C}_2$. There exists a $\mathcal{Z}^{\otimes}(\mathfrak{A})$-module homomorphisms between theses centres, however this morphisms is not a ring homomorphisms in general, see Remark 5.4.5 which puts this statement into a concrete context. Given a $\mathcal{Z}_{\mathfrak{A}}(\mathcal{C}_1)$-algebra A_1, there is no obvious way to construct a $\mathcal{Z}_{\mathfrak{A}}(\mathcal{C}_2)$-algebra A_2 such that there exists a relationship (say a weak equivalence) between $\mathcal{C}_1 \otimes_{\mathcal{Z}_{\mathfrak{A}}(\mathcal{C}_1)} A_1$ and $\mathcal{C}_2 \otimes_{\mathcal{Z}_{\mathfrak{A}}(\mathcal{C}_2)} A_2$.

Remark 4.9.22. Even in the strongly regular case there may be obstacles. Say for a given 2-sided cell $\underline{c}$, there exists up to equivalence only a single universal cell modules. This does not imply that the Grothendieck groups of all cell modules associated to $\underline{c}$ have the same combinatorics. See Remark 5.1.17 for an example. Note however that this example does admit a cover by construction.

We end this section, by providing the argument of [Rou4]. For simplicity, we state it only for principal representations. Below, we comment why this is enough in the case of 2-Kac-Moody algebras $\mathfrak{A}_{\mathfrak{g}} - \mathrm{Mod}$.

Proposition 4.9.23. *Let $\mathcal{M} \in \mathfrak{A}-\mathrm{Mod}$, let F, $G \in \mathcal{H}\mathrm{om}_{\mathfrak{A}}(\mathcal{P}_a, \mathcal{M})$, let $\hat{\Theta} = (\Theta, \Delta, \mu)$ be a split Frobenius object on $a \in \mathfrak{A}$, and assume that $\mathfrak{A}$ has a duality. Then, the evaluation $\mathfrak{A}$-functor $\mathcal{P}_a \otimes_{\mathcal{Z}_{\mathfrak{A}}(\mathcal{P}_a)} \mathcal{H}om^{\mathfrak{A}}_{\mathcal{P}_a \to \mathcal{M}}(F, G) \to \mathcal{M}$ is locally fully faithful if and only if the evaluation functor $\mathcal{P}_a(a)(I_a, b) \otimes_{\mathcal{Z}_{\mathfrak{A}}(\mathcal{P}_a)} \mathcal{H}om^{\mathfrak{A}}_{\mathcal{P}_a \to \mathcal{M}}(F, G) \to \mathcal{M}(a)(F(I_a), G(b))$ is an isomorphism for all $b \in \mathfrak{A}(a, a)$.*

Proof. The $\mathfrak{A}$-functor $\mathcal{P}_a \otimes_{\mathcal{Z}_{\mathfrak{A}}(\mathcal{P}_a)} \mathcal{H}om^{\mathfrak{A}}_{\mathcal{P}_a \to \mathcal{M}}(F, G) \to \mathcal{M}$ is locally fully faithful if and only if for all $a' \in \mathfrak{A}$ and b_1, $b_2 \in \mathcal{P}_a(a')$ it induces an isomorphism between the morphism spaces $\mathcal{P}_a(a')(b_1, b_2) \otimes_{\mathcal{Z}_{\mathfrak{A}}(\mathcal{P}_a)} \mathcal{H}om^{\mathfrak{A}}_{\mathcal{P}_a \to \mathcal{M}}(F, G) \xrightarrow{\sim} \mathcal{M}(a')(F(b_1), G(b_2))$. Now, denoting the duality on $\mathfrak{A}$ by $\mathbb{D}$, we have $\mathbb{D}(b_1) \in \mathfrak{A}(a', a)$ and by functoriality of the action, $\mathcal{P}_a(a)(I_a, \mathbb{D}(b_1)b_2) \otimes_{\mathcal{Z}_{\mathfrak{A}}(\mathcal{P}_a)} \mathcal{H}om^{\mathfrak{A}}_{\mathcal{P}_a \to \mathcal{M}}(F, G) \xrightarrow{\sim} \mathcal{M}(a)(F(I_a), G(\mathbb{D}(b_1)b_2))$ is equivalent to the above isomorphism, where we used that I_a is the unit object in $\mathfrak{A}_a$. The claim follows. □

Remark 4.9.24. Proposition 4.9.23 was applied in [Rou4, Lemma 5.4] to the minimal categorifications $\mathcal{L}(\lambda) \in \mathfrak{A}_{\mathfrak{g}}$ where $\mathfrak{A}_{\mathfrak{g}}$ is a 2-Kac Moody algebra. What makes the situation easier, is that all objects in $\mathcal{L}(\lambda)(\lambda)$ are direct sums of the (image of) the unit object $I_\lambda \in \mathfrak{A}(\lambda, \lambda)$, and thus it is sufficient to consider the case $b = I_\lambda$ only. Since I_λ acts by definition (up to isomorphism) as the identity functor on $\mathcal{M}(\lambda)$ for all $\mathcal{M} \in \mathfrak{A}_{\mathfrak{g}} - \mathrm{Mod}$, and since $\mathcal{Z}_{\mathfrak{A}_{\mathfrak{g}}}(\mathcal{L}(\lambda)) = \mathcal{E}nd_{\mathcal{L}(\lambda)}(I_\lambda)$, fully faithfulness of the $\underline{c}$-isotypic cover becomes straight forward. Compare this with Theorem 4.2.19 which shows what remains true in the more general case. Note also that [Rou4, Theorem 5.23] already hints at an *isotypic case* and that λ-*isotypic modules* are discussed in more detail in [Rou5, Section 4.3.4]. In fact, our more broadly applicable notion of $\underline{c}$-isotypic modules, for $\underline{c}$ the 2-sided cell that contains λ, coincides for 2-Kac-Moody algebras with the notion of λ-isotypic modules.

Remark 4.9.25. The paper [MaMi5] proves a *weak Jordan-Hölder theorem.* The statement applies to modules with so called *weak Jordan-Hölder series.* This means one starts with a module that has a filtration that is determined on the level of Grothendieck groups, and that is as fine as possible. To a weak Jordan-Hölder series, a *weak composition subquotients* are constructed by taking the minimal quotients of the subquotients of the weak Jordan-Hölder series as discussed in Remark 4.9.17.

The weak Jordan-Hölder theorem then states that each weak composition subquotient is equivalent to a cell 2-representation as introduced in [MaMi2], and that the multiplicity of the cell 2-representation amongst the weak composition subquotients is independent of the filtration chosen. Thus, [MaMi5] are able to assign a cell module to each left cell, which is quite remarkable. Note that the cell associated to different left cell can be equivalent if the left cells lie in the same 2-sided cell. We now explain the relationship between our strongly universal cell modules and the cell 2-representations of [MaMi2]. Assume that $\mathfrak{A}$ has a finitary quotient $\pi : \mathfrak{A} \to \overline{\mathfrak{A}}$ that has the same 2-sided and left cells as $\mathfrak{A}$, and let c be a left cell. Denote by $\mathfrak{A}c$ (resp. $\overline{\mathfrak{A}}c$) the minimal thick sub-module of $\mathfrak{A}$ (resp. $\overline{\mathfrak{A}}$) that contains the indecomposable objects belonging to c. Note that $\mathrm{Ind}_\pi(\mathfrak{A}c) = \overline{\mathfrak{A}}c$. Following [MaMi5], there exists a maximal ideal $\mathcal{I}_c \trianglelefteq \overline{\mathfrak{A}}c$ that does not contain the indecomposable objects associated to c. The quotient $\mathbf{C}_c := \overline{\mathfrak{A}}c/\mathcal{I}_c$ is the 2-cell representation associated to c. Importantly, it has no non-trivial ideals. Now, assume that there exists a universal cell module $\mathcal{C}$ associated to the 2-sided cell that contains c. Thus, we have an $\mathfrak{A}$-functor $\mathcal{C} \to \mathrm{Res}_\pi(\mathbf{C}_c)$. Even though $\mathbf{C}_c$ has no non-trivial ideals, there is *a priori* no reason for this $\mathfrak{A}$-functor to be full or essentially surjective. Now, assume that the Duflo involution $d \in c$ can be equipped with the structure of a split Frobenius object $\hat{\Theta}_d$ in $\mathfrak{A}$. Similar to the discussion in Section 5.2, one can deduce from [MaMi1, Proposition 34] existence of the structure maps of $\hat{\Theta}_d$ in $\overline{\mathfrak{A}}$ which lift to $\mathfrak{A}$ by the above assumptions, while it can again only be conjectured that the Frobenius relation holds. Provided that $\mathfrak{A}$ satisfies right $\hat{\Theta}_d$-descent, $\mathcal{C}_{\hat{\Theta}_d}$ will be a thick sub-category of $\mathfrak{A}$ that contains the indecomposable object associated to d. The same holds for $\mathrm{Ind}_\pi(\mathcal{C}_{\hat{\Theta}_d})$. It follows that we have an essentially surjective $\mathfrak{A}$-functor $\mathcal{C}_{\hat{\Theta}_d} \to \mathrm{Res}_\pi(\mathbf{C}_c)$. In the regular case and due to the duality that is part of the finitary notion, $\mathbf{C}_c$ satisfies $\hat{\Theta}_d$-descent. It remains an open task to determine whether the ideal associated to this quotient is generated by elements in $\mathcal{Z}_d := \mathcal{Z}_{\mathfrak{A}}(\mathcal{C}_{\hat{\Theta}_d})$. We conjecture that this is the case and moreover we conjecture that we have an equivalence $\mathcal{C}_{\hat{\Theta}_d} \otimes_{\mathcal{Z}_d} \mathcal{H}om\left(\mathcal{C}_{\hat{\Theta}_d}, \mathrm{Res}_\pi(\mathbf{C}_c)\right) \to \mathrm{Res}_\pi(\mathbf{C}_c)$.

Chapter 5

Applications to Soergel Bimodules

This chapter is devoted to the higher representation theory of Soergel bimodules. Section 5.1 recalls the definition of Soergel bimodules $\mathcal{S}oe^\bullet$ as well as the refined version of singular Soergel bimodules. We also discuss the relevant literature, including the categorification theorem of the Hecke algebra and of the Schur algebroid, and Soergel's conjecture. Section 5.2 makes conjectures about existence and properties of split Frobenius objects in $\mathcal{S}oe^\bullet$ associated to Duflo involutions, and presents examples associated to parabolic longest elements. Section 5.3 provides a new perspective on singular Soergel bimodules and shows that they categorify the Schur algebroid in the representation theoretic sense. Furthermore, related centers are calculated. Section 5.4 studies universal the cell modules assuming the conjectures made in Section 5.2 and establishes several statements about their Grothendieck combinatorics. Section 5.5 applies the coinvariant theory developed in Section 4.6 to universal cell modules of Soergel bimodules for several Coxeter systems. Section 5.6 uses Conjecture 5.2.15 to show that morphisms of $\mathcal{S}oe$ have a strong impact on the structure of $\mathcal{S}oe - \mathrm{Mod}$. Finally, Section 5.7 generalises the main result of [Li] to singular Soergel bimodules.

5.1 Singular Soergel bimodules

In this section, we discuss singular Soergel bimodules and the most relevant related literature, including the categorification theorems and Soergel's conjecture.

For the remainder of this section, we fix a Coxeter system (W, S). We denote by $T := \bigcup_{w \in W} wSw^{-1}$ the set of reflections.

Definition 5.1.1. *Let k be an infinite field of characteristic $\neq 2$. A* reflection faithful representation *of (W, S) over k is a faithful, finite dimensional k-linear representation*

V of W with the property that $\operatorname{codim}_k(V^t) = 1$ *if and only if* $t \in T$ *for all* $t \in W$, *where* $V^t := \{v \in V | v = tv\}$.

Remark 5.1.2. It was shown in [Soe6] that reflection faithful representations exists over $\mathbb{R}$. The need for a reflection faithful representations as compared to a faithful representation is of a technical nature, and could at the time not be circumvented in certain proofs. However, [Li] showed that the category of Soergel bimodules constructed with a faithful representation retains the same properties. In Section 5.7, we will generalise this result to singular Soergel bimodules.

From now on fix a reflection faithful representation of (W, S) over an infinite field k of characteristic $\neq 2$ and denote by $R := \mathcal{O}(V) \cong k[V^*]$ the ring of regular functions on V. We consider R as a graded ring with linear functions concentrated in degree 2. This unusual grading stems from a connection between Soergel bimodules and equivariant hypercohomology of certain perverse sheaves. Also, it is necessary to categorify the root of q that is omnipresent in the literature on Hecke algebras.

The action of W on V induces an action of W on R, given by $(w.r)(v) := r(w^{-1}.v)$ for all $w \in W$, $r \in R$ and $v \in V$. Given $I \subset S$, we denote by $R^I := R^{W_I} \leq R$ the subring of W_I-invariant functions. We will make the following assumption about R, which is essential for the theory of Soergel bimodules, and is also key in our application:

$$\textbf{Assumption: } \text{For all finitary } I \subset J \subset S,\ R^I \text{ is a graded free } R^J\text{-module.} \tag{5.1}$$

Remark 5.1.3 ([Wi2, Remark 4.1.2, Lemma 4.1.3])**.** If k is of characteristic 0, then the above assumptions holds. If W is a finite Weyl group, then W acts on the weight lattice of the corresponding root system and one obtains a representation V over any field k by extension of scalars. In this case, the above assumption is true if the characteristic of k is not a torsion prime for W.

We denote by $R^I - R^J - \mathrm{Mod}^\bullet$ the graded additive category of graded R^I-R^J-bimodules, see Definition 3.1.5 for the category theoretic background.

Definition 5.1.4. *Assume that (5.1) is satisfied and let I, $J \subset S$ be finitary. We define* the category of (I, J)-singular Soergel bimodules*, denoted ${}^I\mathcal{S}oe^{\bullet J}$, to be the smallest full subcategory of* $R^I - R^J - \mathrm{Mod}^\bullet$ *which is closed under direct sums, summands and shifts and which contains all objects of the form*

$$R^{I_1} \otimes_{R^{J_1}} R^{I_2} \otimes_{R^{J_2}} \dots \otimes_{R^{J_{n-1}}} R^{I_n}$$

where $I = I_1 \subset J_1 \supset I_2 \subset J_2 \supset ... \subset J_{n-1} \supset I_n = J$ are finitary subsets of S.

Remark 5.1.5. Note that k acts on ${}^I\mathcal{S}oe^J$ from the left via R^I in the same way as from the right via R^J. More generally, this statement also holds for R^S, *i.e.* all functions that are invariant under W. Thus, we can regard objects of ${}^I\mathcal{S}oe^J$ as $R^I \otimes_{R^W} (R^J)^{\mathrm{op}}$-modules and ${}^I\mathcal{S}oe^J$ as a $R^I \otimes_{R^W} (R^J)^{\mathrm{op}}$-linear category. In the following, we will suppress the superscript op, since R is commutative.

Notation 5.1.6. We denote $\mathcal{S}oe^{\bullet} := {}^{\emptyset}\mathcal{S}oe^{\bullet\emptyset}$, $\mathcal{S}oe^{\bullet I} := {}^{\emptyset}\mathcal{S}oe^{\bullet I}$ and view $\mathcal{S}oe^{\bullet}$ as an idempotent closed, monoidal graded additive category. The monoidal structure is given by the tensor product $\otimes_R$ of $R - R - \mathrm{Mod}^{\bullet}$. Similarly, we denote by $\mathcal{S}oe$, $\mathcal{S}oe^I$, ${}^I\mathcal{S}oe^J$ the corresponding categories where we forget about both the shift and the grading of the homomorphism spaces.

Remark 5.1.7. Let $M \in \mathcal{S}oe^{\bullet}$. Then, we have an isomorphism $M \cong M[n]$ for all $n \in \mathbb{Z}$. Note that this isomorphism is of degree zero if and only if $n = 0$. Hence, we cannot distinguish M and $M[n]$ is $\mathcal{S}oe$, while we can distinguish them in $\mathcal{S}oe^{\bullet}$.

Remark 5.1.8. As presented above, $\mathcal{S}oe^{\bullet}$ appeared first in [Soe6]. The quotient $\mathcal{S}oe^{\bullet} \otimes_{R^W} R^W/(R^W)^+$ remains a graded monoidal category. Originally Soergel bimodules appeared in this form, see [Soe1]. This quotient is the prime example of a finitary 2-category, see [MaMi1].

Remark 5.1.9. We have an additive functor ${}^J\mathcal{S}oe^K \times {}^I\mathcal{S}oe^J \to {}^I\mathcal{S}oe^K$ defined by $(B_1, B_2) \mapsto B_2B_1 := B_2 \otimes_{R^J} B_1$ for all finitary $I, J, K \subset S$.

Definition 5.1.10. *Assume that (5.1) is satisfied. The additive 2-category $\mathfrak{S}oe$ of* singular Soergel bimodules *is defined as follows:*

- $\mathfrak{S}oe_0 := \{I \subset S | I \text{ is finitary}\}$
- $\mathfrak{S}oe(I, J) := {}^J\mathcal{S}oe^I$ *for all* $I, J \in \mathfrak{S}oe_0$
- $$\begin{array}{cccl} \otimes_{R^J}: & \mathfrak{S}oe(I,J) \times \mathfrak{S}oe(J,K) & \to & \mathfrak{S}oe(I,K) \quad \text{for all } I,J,K \in \mathfrak{S}oe_0 \\ & (B_1, B_2) & \mapsto & B_2B_1 \end{array}$$
- $R^I \in \mathfrak{S}oe(I, I)$ *is the unit object for all* $I \in \mathfrak{S}oe_0$
- *The associator and unit transformations are the obvious ones*

Using the same structure as in Definition 5.1.10, one can equip $\mathcal{S}oe^I$ with the structure of a $\mathcal{S}oe$-module. Thus, we have $\mathcal{S}oe^I \in \mathcal{S}oe - \mathrm{Mod}$. Alternatively, one may employ the theory developed in Section 4.4, and define $\mathcal{S}oe^I := \mathrm{Res}_\iota(\mathcal{P}_I^{\mathfrak{S}oe})$ as the restriction of a principal representation of $\mathfrak{S}oe$. Here, we have identified $\mathcal{S}oe$ with a 2-category with a single object and denoted by $\iota : \mathcal{S}oe \to \mathfrak{S}oe$ the inclusion 2-functor that sends the 0-cell to $\emptyset$ and is given by the monoidal identity functor on 1- and 2-arrows.

Remark 5.1.11. Given finitary subsets $I \subset K \subset S$ and $J \subset L \subset S$, there exists functors ${}^I\mathcal{S}oe^J \underset{{}^I_K Ind^J_L}{\overset{{}^I_K Res^J_L}{\rightleftarrows}} {}^K\mathcal{S}oe^L$, where ${}^I_K Ind^J_L$ (resp. ${}^I_K Res^J_L$) is defined on objects by ${}^I_K Ind^J_L(M) := R^I \otimes_{R^K} M \otimes_{R^L} R^J$ (resp. ${}^I_K Res^J_L(M) := {}_{R^K}|M|_{R^L}$) and analogously on morphisms. Up to isomorphism, these functors are part of the datum $\mathfrak{S}oe$. Under slightly stronger assumptions than (5.1), which are the topic of [ElSnWi], one can show that there exists a biadjunction between ${}^I_K Ind^J_L$ and ${}^I_K Res^J_L$.

Soergel's category has the graded Krull-Schmidt property, see [Soe1], and we have the following classification of indecomposable objects:

Theorem 5.1.12 ([Soe1],[Wi2]). *For all finitary I, $J \subset S$, there exists a canonical bijection*
$$\begin{array}{ccc} W_I \setminus W/W_J \times \mathbb{Z} & \overset{1:1}{\longleftrightarrow} & \left\{ \begin{array}{c} \text{indecomposable bimodules in } {}^I\mathcal{S}oe^{\bullet J} \\ \text{up to degree zero isomorphism} \end{array} \right\} \\ (\overline{w}, n) & \mapsto & {}^I B^J_{\overline{w}}[n] \end{array}$$
Further for all finitary $I \subset K \subset S$ and $J \subset L \subset S$, taking the longest element in the fibre of the quotient map $W_I \setminus W/W_J \to W_K \setminus W/W_L$ corresponds under this bijection to an application of ${}^I_K \mathrm{Ind}^J_L$. Furthermore, the indecomposable objects of ${}^I\mathcal{S}oe^{\bullet J}$ remain indecomposable in ${}^I\mathcal{S}oe^J$ and are thus up to isomorphism in bijection with $W_I \setminus W/W_J$.

If clear from the context, we abbreviate ${}^I B^J_{\overline{w}} = B_{\overline{w}}$ and write B_w if $I = J = \emptyset$. For any reduced expressions $s_1 ... s_{l(w)} = w$, the indecomposable object B_w occurs as the unique summand of $R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} ... \otimes_{R^{s_{l(w)}}} R[l(w)]$, which is not isomorphic to a shift of B_y for all $y < w$. In general, it is a hard and unsolved task to determine the idempotents explicitly. Contributions in this direction have been achieved by [Li2]. Which summands occur up to isomorphism, can however be determined by a Hecke algebra computation using the following two theorems. Recall the notation ${}^I\mathcal{H}^J$ which was introduced in Remark 2.6.20.

Theorem 5.1.13 ([Soe1],[Wi2]). *For all finitary $I,\ J \subset S$, there is an explicitly defined map* $\mathrm{ch} = {}^{I}\mathrm{ch}^{J} : \mathrm{Ob}\left({}^{I}\mathcal{S}oe^{\bullet J}\right) \to {}^{I}\mathcal{H}^{J}$. *It factors over the split Grothendieck group and induce an isomorphism of $\mathbb{Z}[v, v^{-1}]$-modules $[{}^{I}\mathcal{S}oe^{\bullet J}] \cong {}^{I}\mathcal{H}^{J}$, where v acts by the degree shift $[1]$ on the left. Furthermore, we have commutative diagrams*

$$\begin{tikzcd} {}^{I}\mathcal{S}oe^{\bullet J} \times {}^{J}\mathcal{S}oe^{\bullet K} \arrow[r, "- \otimes_J -"] \arrow[d, "\mathrm{ch} \times \mathrm{ch}"] & {}^{I}\mathcal{S}oe^{\bullet K} \arrow[d, "\mathrm{ch}"] \\ {}^{I}\mathcal{H}^{J} \times {}^{J}\mathcal{H}^{K} \arrow[r, "- *_J -"] & {}^{I}\mathcal{H}^{J} \end{tikzcd} \qquad \begin{tikzcd} {}^{I}\mathcal{S}oe^{\bullet J} \arrow[r, "-[1]"] \arrow[d, "\mathrm{ch}"] & {}^{I}\mathcal{S}oe^{\bullet J} \arrow[d, "\mathrm{ch}"] \\ {}^{I}\mathcal{H}^{J} \arrow[r, "v\cdot"] & {}^{I}\mathcal{H}^{J} \end{tikzcd} \qquad \begin{tikzcd} {}^{K}\mathcal{S}oe^{\bullet L} \arrow[r, "{}^{I}_{K}\mathrm{Ind}^{J}_{L}"] \arrow[d, "\mathrm{ch}"] & {}^{I}\mathcal{S}oe^{\bullet J} \arrow[d, "\mathrm{ch}"] \\ {}^{K}\mathcal{H}^{L} \arrow[r, hook] & {}^{I}\mathcal{H}^{J} \end{tikzcd},$$

and $\mathrm{ch}(B_s) = H_s + v$ *where* $B_s := R \otimes_{R^s} R[1] \in \mathcal{S}oe^{\bullet}$.

Soergel's Conjecture relates indecomposable objects to the famous Kazhdan-Lusztig basis. It was long expected to hold for V taken over any field of characteristic zero, and a proof over $\mathbb{R}$ has recently been given in [ElWi2].

Theorem 5.1.14 (Soergel's Conjecture, [ElWi2]). *Let $k = \mathbb{R}$ and W be an arbitrary Coxeter System. Then for all $w \in W$, we have* $\mathrm{ch}(B_w) = \underline{H}_w$.

Whenever we assume in the following that Soergel's conjecture holds for a field k, we mean that $\mathrm{ch}(B_w) = \underline{H}_w$ and do not necessarily require that the characteristic of k is zero.

Remark 5.1.15. Before [ElWi2], there was already plenty of evidence for Soergel's Conjecture 5.1.14. All proofs relied either on the geometry of flag varieties or on the comparably simple structure of the underlying Coxeter group. In [Soe5] and previously in [Soe1] in a non-equivariant setting, the conjecture is shown for finite Weyl groups over $\mathbb{C}$. [Hä] has used the same geometric techniques to establish the conjecture for affine Weyl groups. For a finite Weyl group of an algebraic group and characteristic larger than the Coxeter number, the conjecture is equivalent to Lusztig's conjectures on multiplicities of simple modules in from a Borel subgroup induced modules, see [Soe4]. Thus, it holds for sufficiently large characteristic by [AnJaSo]. Furthermore, [Fi] proves the conjecture for universal Coxeter groups using an alternative construction of Soergel bimodules via moment graphs. Also, for dihedral groups the indecomposable objects are constructed explicitly in [Soe6] and the conjecture is verified in this way.

Remark 5.1.16. By Theorem 5.1.13, $\mathcal{S}oe^{\bullet}$ categorifies the Hecke algebra equipped with the Kazhdan-Lusztig basis, whereas $\mathcal{S}oe$ categorifies the group algebra $\mathbb{Z}[W]$ equipped with the induced Kazhdan-Lusztig basis. These statements are based on

Remark 5.1.7 and Theorem 5.1.12, and in particular the fact that indecomposable Soergel bimodules remain indecomposable once we forget the grading. Summarizing, we have a commutative diagram of $(\mathbb{Z}, \mathbb{N}_0)$-ring homomorphisms

$$\begin{tikzcd} {[\mathcal{S}oe^\bullet]} \arrow[r] \arrow[d, "\sim"] & {[\mathcal{S}oe]} \arrow[d, "\sim"] \\ \mathcal{H} \arrow[r] & \mathbb{Z}[W] \end{tikzcd}$$

where $\mathcal{H} \to \mathbb{Z}[W]$ is the specialisation homomorphism that sends $v \mapsto 1$ and $H_w \mapsto w$. Thus, if one wants to study higher representation theory of the Hecke algebra, one thus has keep track of the grading, which is why we provided all the details about the graded case in chapter 3.

Remark 5.1.17. By arguments related to [ElWi1, Remark 3.21], the canonical $\mathcal{S}oe$-functor $\mathcal{S}oe \to (\mathcal{S}oe \otimes_R Q)^{\text{idem}}$ almost categorifies the base change homomorphism of Example 2.1.8, where Q is the field of fractions (or another appropriate localisation) of R. For reasons explained there, this only works for $\mathcal{S}oe$, *i.e.* without the grading, and thus really categorifies Example 2.1.8 after evaluating v (or with care v^2) at 1. Further, the canonical monoidal functor $\mathcal{S}oe \to \left(\mathcal{S}oe_{\mathcal{Z}^{\otimes}(\mathcal{S}oe)} Q^W\right)^{\text{idem}}$ categorifies the same map considered as $(\mathbb{Z}, \mathbb{N}_0)$-ring homomorphism. Similarly for finite W, the $\mathcal{S}oe$-functor $\mathcal{S}oe^S \to \left(\mathcal{S}oe^S \otimes_{\mathcal{Z}_{\mathcal{S}oe}(\mathcal{S}oe^S)} Q^W\right)^{\text{idem}}$ categorifies the analogous base change for the regular $\mathbb{Z}[W]$-representation $\mathcal{H}^S|_{v=1} \cong \mathbb{Z}[W]$.

5.2 Conjectures on Duflo involutions and split Frobenius objects

In this section, we discuss split Frobenius objects in $\mathcal{S}oe^\bullet$, present examples associated to parabolic longest elements, and put forward conjectures about the existence of a larger class associated to arbitrary Duflo involutions. Furthermore, we make conjectures about compatibilities between morphisms of $\mathcal{S}oe^\bullet$ and the conjectural split Frobenius objects. For ease of notation, we pretend that $\mathcal{S}oe^\bullet$ is a strict monoidal category.

First, we need some notation which is based on *Lusztig's J-ring*, see [Lu6] for a comprehensive summary. The J-ring may be defined as the free $\mathbb{Z}$-algebra with basis $\{t_x\}_{x\in W}$ and structure coefficients $\{\gamma_{x,y,z}\}_{x,y,z\in W}$, *i.e.* we have $t_x t_y = \sum_{z\in W} \gamma_{x,y,z} t_z$ for all $x, y \in W$. The definition of the structure coefficients involves Lusztig's $\mathbf{a}$-function, which can be defined as $\mathbf{a}(z) := \min\{i \in \mathbb{N}_0 | h_{x,y,z} v^i \in \mathbb{N}_0[v] \ \forall x, y \in W\} \in \mathbb{N}_0 \coprod \{\emptyset\}$ for all $z \in W$. This definition agrees with Lusztig's, since the Kazhdan-Lusztig basis is self-dual and thus $h_{x,y,z} = \overline{h_{x,y,z}}$. Now, the structure coefficients can be defined as $\gamma_{x,y,z} := h_{x,y,z} v^{\mathbf{a}(z)}|_{v=0}$ for all $x, y, z \in W$.

In order to define Duflo involutions, we need another of Lusztig's functions, which is usually denoted by $\Delta : W \to \mathbb{N}_0$. Usually, the map Δ is defined in terms of Kazhdan-Lusztig polynomials. Alternatively whenever Soergel's Conjecture 5.1.14 holds, $\Delta(z)$ can be defined as the smallest non-trivial degree of $\mathrm{Hom}_{\mathcal{S}oe^\bullet}(B_z, B_e)$ for all $z \in W$.

Definition 5.2.1. *An element $d \in W$ is called a* Duflo involution *if $\Delta(d) = \boldsymbol{a}(d)$. We denote the set of Duflo involutions by $\mathcal{D} := \{z \in W | \Delta(z) = \boldsymbol{a}(z)\}$.*

We are going to use the following facts stated in [Lu6, Conjectures 14.2]. Note that these are shown to hold in [Lu6, Section 15] under positivity assumptions, which in turn hold due to Soergel's Conjecture 5.1.14: For all $x \in W$, there exists a unique $d \in c(x) \cap \mathcal{D}$. Further for all $d \in \mathcal{D}$, we have $d^2 = d$ which justifies the name Duflo involution. Furthermore, we have $\gamma_{x^{-1},x,d} = 1$ for all $x \in c(d)$.

Next, we construct two triples $\hat{\Theta}_d$ and $\overline{\Theta}_d$ for all Duflo involutions $d \in \mathcal{D}$. For this purpose, we assume that the underlying structure is chosen such that the statement of Soergel's Conjecture 5.1.14 holds. For the time being, fix $d \in \mathcal{D}$. Let us denote by $\hat{B}_d := B_d[-\mathbf{a}(d)]$ the indecomposable Soergel bimodule associated to d shifted upwards by the associated $\mathbf{a}$-function value. Since $\mathcal{D}$ consists of involutions, we have $\gamma_{d,d,d} = 1$ and thus, $B_d[\mathbf{a}(d)] \subset^\oplus B_d \otimes B_d$ or equivalently $\hat{B}_d \subset^\oplus \hat{B}_d \otimes \hat{B}_d$. We denote by $\Delta_d : \hat{B}_d \to \hat{B}_d \otimes \hat{B}_d$ and $\mu_d : \hat{B}_d \otimes \hat{B}_d \to \hat{B}_d$ the associated inclusion and projection morphism, and abbreviate $\hat{\Theta}_d := (\hat{B}_d, \Delta_d, \mu_d)$. Note that Δ_d and μ_d have degree zero and that $\mu_d \circ \Delta_d = id_{\hat{B}_d}$ holds. Further, denote by $\pi_d : \mathcal{S}oe^\bullet \to \overline{\mathcal{S}oe^\bullet}^d$ the strict monoidal quotient functor, where $\overline{\mathcal{S}oe^\bullet}^d := \mathcal{S}oe^\bullet / \langle B_x | x \not\geq_{LR} d \rangle$. Furthermore, denote $\overline{\Theta}_d := \pi_d(\hat{\Theta}_d) = (\pi_d(\hat{B}_d), \pi_d(\Delta), \pi_d(\mu))$. Due to the Krull-Schmidt property of $\mathcal{S}oe^\bullet$, the quotient $\overline{\mathcal{S}oe^\bullet}^d$ also has the Krull-Schmidt property and its split Grothendieck ring $\left[\overline{\mathcal{S}oe^\bullet}^d\right]$ is isomorphic to $\overline{\mathcal{H}}^d := \mathcal{H} / \{x \in W | x \not\geq_{LR} d\}$.

Now, we put forward four conjectures that involve the triples $\hat{\Theta}_d$ and $\overline{\Theta}_d$ where $d \in \mathcal{D}$. The first two conjectures ensure existence of free functorially cyclic and universal cell modules for $\mathcal{S}oe^\bullet$ respectively.

Conjecture 5.2.2. *Let $d \in \mathcal{D}$. Then, $\overline{\Theta}_d = (\overline{\Theta}_d, \pi_d)$ is a cell object for $\mathcal{S}oe^\bullet$.*

Conjecture 5.2.3. *Let $d \in \mathcal{D}$. Then, $\hat{\Theta}_d$ is a cyclic object for $\mathcal{S}oe^\bullet$.*

The second two conjectures ensure that the split Grothendieck groups of certain objects of $\mathcal{S}oe^\bullet$ − Mod behave as expected.

Conjecture 5.2.4. *Let d_1, $d_2 \in \mathcal{D}$ such that $d_1 \sim_{LR} d_2$. Then, $\overline{\mathcal{S}oe^\bullet}^{d_2}$ satisfies right $\overline{\Theta}_{d_2}$-descent and $\overline{\Theta}_{d_1}$-$\overline{\Theta}_{d_2}$-descent.*

Conjecture 5.2.5. *Let d_1, $d_2 \in \mathcal{D}$. Then, $\mathcal{S}oe^\bullet$ satisfies satisfies $\hat{\Theta}_{d_1}$-$\hat{\Theta}_{d_2}$-descent.*

Remark 5.2.6. We leave it to future research to tackle these conjectures, but would like to point out that the description of $\mathcal{S}oe^\bullet$ by generators and relations given in [ElWi1] (see references therein for earlier versions in special cases), provides an excellent starting point. It would be especially exciting, if it were possible to establish these conjectures up to positive scalars in the diagrammatic $\mathbb{Z}$-form of $\mathcal{S}oe^\bullet$. This would then also yield and answer to the following question and thus have implications to modular representation theory.

Question 5.2.7. *Let p be a prime. Do similar relations hold for the p-canonical basis elements associated to Duflo involutions? Clearly, this would yield analogous triples in $\mathcal{S}oe^\bullet$ defined over a field of characteristic p. Is it plausible to make the analogous conjectures in this case?*

Remark 5.2.8. For $d \in \mathcal{D}$, it holds that $\hat{\Theta}_d$ (resp. $\overline{\Theta}_d$) is right indecomposable since $\hat{B}_d$ (resp. $\pi_d(\hat{B}_d)$) is indecomposable. Thus, Conjecture 5.2.3 is equivalent to the equation $\Delta_d \circ \mu_d = (\hat{B}_d\mu_d) \circ (\Delta_d\hat{B}_d) = (\mu_d\hat{B}_d) \circ (\hat{B}_d\Delta_d)$, and Conjecture 5.2.2 is equivalent to equality of these three expressions after applying π_d. Hence, Conjecture 5.2.3 implies Conjecture 5.2.2. Similarly, Conjecture 5.2.5 implies Conjecture 5.2.4 due to the following two reasons. First, for all $B \in \mathcal{B}$ and for all idempotents e of $\pi_d(\Theta_d)\pi_d(B)$ there exists an idempotent $\widetilde{e}$ in the endomorphism ring of $\Theta_d B$ such that $\pi_d(\widetilde{e}) = e$. Second, right $\overline{\Theta}_{d_2}$-descent is the same as $\overline{\Theta}_e$-$\overline{\Theta}_{d_2}$-descent, since $\hat{\Theta}_e$ is based on the unit object and the unit transformations of $\mathcal{S}oe^\bullet$.

Conjecture 5.2.3 is known to hold in the following two examples:

Example 5.2.9. Let $I \subset S$ be finitary. Then it is well known and not hard to show that the parabolic longest element w_0^I is a Duflo involution. For finite Weyl groups and over the complex numbers, it was shown in [Soe2] that $\hat{B}_{w_0^I} = R \otimes_{R^I} R$, where we also used that $\mathbf{a}(w_0^I) = l(w_0^I)$ as stated for example in [Lu6]. For arbitrary Coxeter systems, $\hat{B}_{w_0^I} = R \otimes_{R^I} R$ follows from [Wi2, Theorem 3], [Wi2, Theorem 7.4.1] and the fact that

$R^I \in {}^I\mathcal{S}oe^{\bullet I}$ is indecomposable. Furthermore, the basic assumption (5.1) provides us with an R^I-linear splitting $\pi_{w_0^I}$ of the inclusion map $i_{w_0^I} : R^I \hookrightarrow R$. If $|W_I|$ is invertible in k, the splitting $\pi_{w_0^I}$ is given by $r \mapsto \frac{1}{|W_I|}\sum_{x\in W_I} x.r$. In general, the R-bimodule morphism $\Delta_{w_0^I} : \hat{B}_{w_0^I} \to \hat{B}_{w_0^I} \otimes_R \hat{B}_{w_0^I}$, given by $x \otimes_{R^I} y \mapsto (x \otimes_{R^I} 1) \otimes_R (1 \otimes_{R^I} y)$, thus has a splitting $\mu_{w_0^I}$, given by $(x \otimes_{R^I} y) \otimes_R (z \otimes_{R^I} w) \mapsto x\pi_{w_0^I}(yz) \otimes_{R^I} w$. Now, $(\mu_{w_0^I} \otimes_R \hat{B}_{w_0^I}) \circ \operatorname{can} \circ (\hat{B}_{w_0^I} \otimes \Delta_{w_0^I}) = \Delta_{w_0^I} \circ \mu_{w_0^I} = (\hat{B}_{w_0^I} \otimes_R \Delta_{w_0^I}) \circ \operatorname{can}^{-1} \circ (\mu_{w_0^I} \otimes \hat{B}_{w_0^I})$ can be checked easily and $\pi_{w_0^I} \circ i_{w_0^I} = id_{\hat{B}_{w_0^I}}$ hold by construction. Thus, we have constructed $\hat{\Theta}_d = (\hat{B}_{w_0^I}, \Delta_{w_0^I}, \mu_{w_0^I})$ for $d = w_0^I$ explicitly.

Example 5.2.10. We have checked Conjecture 5.2.3 by explicit calculation in the diagrammatic calculus for the Duflo involution $d = s_2s_1s_3s_2$ in type A_3. Note that $d \sim_{LR} s_1s_3$ and that $\mathbf{a}(s_1s_3) = l(s_1s_3) = 2 \neq 4 = l(d)$. Since $\mathbf{a}$ is constant on 2-sided cells and since $\mathbf{a}(w_0^I) = l(w_0^I)$ for all finitary $I \subset S$, d cannot be a parabolic longest element.

Conjecture 5.2.5 also holds for parabolic longest elements:

Example 5.2.11. For any $M \in \mathcal{S}oe^\bullet$, it is known that $\hat{B}_{w_0^I} M \hat{B}_{w_0^J}$ decomposes into a direct sum of shifts of indecomposable objects B_x with $x \leq_R w_0^I$ and $x \leq_L w_0^J$. In particular $L(x) \supset L(w_0^I) = I$ and $R(x) \supset J$. By [Wi2, Theorem 3], there exists $B_{\overline{x}} \in {}^I\mathcal{S}oe^{\bullet J}$ such that $B_x = R \otimes_{R^I} B_{\overline{x}} \otimes_{R^J} R$. In other words, there exist $\widetilde{M} \in {}^I\mathcal{S}oe^{\bullet J}$ such that $\hat{B}_{w_0^I} M \hat{B}_{w_0^J} \cong R \otimes_{R^I} \widetilde{M} \otimes_{R^J} R$. By [Wi2, Theorem 3] and [Wi2, Theorem 7.4.1], the functor $R \otimes_{R^I} - \otimes_{R^J} R : R \otimes_{R^I} {}^I\mathcal{S}oe^{\bullet J} \otimes_{R^J} R \to \mathcal{S}oe^\bullet$ preserves the the graded ranks of the homomorphism spaces involved and is thus fully faithful. Hence, all idempotents e of $\hat{B}_{w_0^I} M \hat{B}_{w_0^J}$ are up to an isomorphism given by $e = 1 \otimes_{R^I} \widetilde{e} \otimes_{R^J} 1$ where $\widetilde{e}$ is an idempotent of $\widetilde{M}$. With this preparation it is easy to compute that $e = (\Delta_{w_0^I} M) \circ (B_{w_0^I} e) \circ (\mu_{w_0^I} M) = (M \Delta_{w_0^J}) \circ (e B_{w_0^J}) \circ (M \mu_{w_0^I})$.

In order to construct a universal cell modules for each 2-sided cell, it is enough that Conjecture 5.2.2 holds for a single Duflo involution per 2-sided cell. Due to Example 5.2.9, it is thus interesting to know which 2-sided cells in a given Coxeter systems contain a parabolic longest element. For brevity, let us call a 2-sided cell that contains a parabolic longest element a *parabolic 2-sided cell*:

Remark 5.2.12. Since we have $\mathbf{a}(w_0^I) = l(w_0^I)$ for all finitary $I \subset S$, the survey statement [Sh, Remark 8.13(a)] implies the following: All 2-sided cells are parabolic for Weyl groups of type A_n, $\widetilde{A}_n$, $\widetilde{B}_3$, $\widetilde{C}_2$, $\widetilde{C}_3$ and $\widetilde{G}_2$, as well as the dihedral groups. In type A_n (resp. $\widetilde{A}_n$) this follows form the (generalised) Robinson-Schensted correspondence. Further for all affine Weyl groups, the lowest 2-sided cell is a parabolic 2-sided cell. Furthermore for $n \geq 5$, [Be] shows that the hyperbolic n-gon (universal) Coxeter groups P_n have three 2-sided cells, which are therefore parabolic. For $n \in \{1, 2, 3\}$, the associated group P_n also consists of parabolic 2-sided cells, since it is given by an n-fold product of the Weyl group of type A_1, and the same holds for P_4 which is equal to the product of the infinite dihedral group with itself. Further by Remark 5.2.13, type B_4 and C_4 consists of parabolic 2-sided cells, while type B_5 and C_5 both have 10 left cells of size 15 which lie in a 2-sided cell which is not parabolic. Further, type F_4 has 9 left cells of size 9 and 4 left cells of size 6 with this property. Similarly, type H_3 has 3 left cells of size 6 with this property. The remaining candidates amongst finite Coxeter groups and affine Weyl groups that might consist of parabolic 2-sided cell are of type $\widetilde{B}_4$ and $\widetilde{C}_4$. Note that this is an exhaustive list of potential candidates, since all other types contain a sub-Weyl that have a non-parabolic 2-sided cell. For example, type E_6, E_7 and E_8 contain type D_4, and type H_4 contains H_3. Remark 5.2.14 below provides more details about the failure in type D_4.

Remark 5.2.13. To establish the claims made in the previous Remark, we used [CHEVIE], the fact that the element-wise inverse of a left cell is a right cell, the fact that for finite Coxeter groups left and right cells intersect if and only if they lie in the same 2-sided cell. The latter holds by the same argument as in [LuXi, §3], as stated in [Ge, Theorem 4.1].

```
RequirePackage( "chevie" );

W := CoxeterGroup( "H", 3 );

generators := List( W.generators, i -> CoxeterWord( W, i )[1] );

parabolicGenerators := Combinations( generators );

parabolicLongestElements := List( parabolicGenerators,

      i -> LongestCoxeterWord( ReflectionSubgroup( W, i ) ) );

leftCellsAndMu := LeftCells( W );

leftCells := List( leftCellsAndMu, i -> i[ 1 ] );

Size( Set( leftCells ) );

containsParabolicLongestElement := function ( lc )

      local cple;

      cple:=false;

      for k in parabolicLongestElements do

            cple := cple or (k in lc);

      od;

      return cple;

      end;;
```

```
leftCellsWithParabolicLongestElement := Filtered( leftCells, containsParabolicLongestElement );
rightCellsWithParabolicLongestElement :=[];
for plc in leftCellsWithParabolicLongestElement do
      Append( rightCellsWithParabolicLongestElement,
           [ List( plc, j -> CoxeterWord( W, PermCoxeterWord( W, Reversed( j ) ) ) ) ] );
od;
leftCellsWithParabolicCounter := [];
for l in leftCells do
      lWithPCounter:=[ l, 0 ];
      for pr in rightCellsWithParabolicLongestElement do
            if Size( IntersectionSet( Set( pr ), l ) ) > 0 then
                  lWithPCounter[ 2 ] := lWithPCounter[ 2 ] + 1;
            fi;
      od;
      Append( leftCellsWithParabolicCounter, [ lWithPCounter ] );
od;
leftCellsNoParabInTwoCell:=Filtered( leftCellsWithParabolicCounter, i -> i[ 2 ] = 0 );
NumberOfleftCellsNoParabInTwoCell := Size( Set( leftCellsNoParabInTwoCell ) );
SizesOfleftCellsNoParabInTwoCell := List( leftCellsNoParabInTwoCell, i -> Size( Set( i[ 1 ] ) ) );
```

While one can can test the above conjectures on any Duflo involution that is not a parabolic longest element, the following case would be particularly interesting.

Remark 5.2.14. The Weyl group $W(D_4) = \langle \{s_i\}_{i=1}^4 | (s_i s_2)^3 = 1 = (s_i)^2 \ \ \forall i \rangle$ contains a parabolic longest element in all but the 2-sided cell with $\mathbf{a}$-function value 7. This follows from [DJ], which is surveyed in [Sh, Theorem 8.12]. This theorem describes the twelve 2-sided cells of the affine Weyl group $W(\widetilde{D}_4) = \langle \{s_i\}_{i=0}^4 | (s_i s_2)^3 = 1 = (s_i)^2 \ \ \forall i \rangle$ explicitly. In particular, it is stated that there exists a 2-sided cell with $\mathbf{a}$-value 7. Furthermore, [Sh, Remark 8.13(a)] states that the element $s_1 s_3 s_2 s_1 s_3 s_2 s_4 s_2 s_1$ has $\mathbf{a}$-value 7. We have verified using [CHEVIE] that $s_1 s_2 s_1 s_3 s_2 s_4 s_2 s_1 s_3 s_2 s_4$ is the Duflo involution which is left equivalent to the above element and thus also has $\mathbf{a}$-value 7. Clearly, both expressions do not contain the affine reflection s_0 and can thus be considered as an element of the finite Weyl group of type D_4, where the $\mathbf{a}$-value remains 7. The non-trivial parabolic subgroups of D_4 are of type A_1, $A_1 \times A_1$, $A_1 \times A_1 \times A_1$, A_2, A_3 and D_4. Thus, parabolic longest elements have $\mathbf{a}$-values 1, 2, 3, 3, 6 and 12, where we used once more that $\mathbf{a}(w_0^I) = l(w_0^I)$ for all finitary subsets $I \subset S$.

We proceed with a further conjecture, which impacts only Section 5.6. Soergel's Conjecture 5.1.14 paired with the homomorphism formula [Wi2, Theorem 7.4.1] implies that $\mathrm{Hom}_{\mathcal{S}oe^\bullet}(B_x, B_y) = \pi_e(\underline{H}_x\underline{H}_{y^{-1}})R$ as left R-modules, where denote by $\pi_e : \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ denotes map that picks the coefficient of H_e of any element $H \in \mathcal{H}$ when expressed in the standard basis. It follows that the graded rank of both $\mathrm{Hom}_{\mathcal{S}oe^\bullet}(B_d, B_e)$ and $\mathrm{Hom}_{\mathcal{S}oe^\bullet}(B_e, B_d)$ is given by $\pi_e(\underline{H}_d) = h_{e,d}$, *i.e.* a Kazhdan-Lusztig polynomial. Thus, there exist non-trivial morphisms of minimal degree $\mathbf{a}(d)$, say $f_{e,d} : R \to \hat{B}_d$ and $f_{d,e} : \hat{B}_d \to R$. Further since $h_{x,y,z} = \overline{h_{x,y,z}}$, we also have a highest degree summand $B_d[-\mathbf{a}(d)] \subset^{\oplus} B_d \otimes_R B_d$. Denote the projection morphism (after shifting by $2\mathbf{a}(d)$) by $\widetilde{\mu}_d$.

Conjecture 5.2.15. *For all $d \in \mathcal{D}$ there is a choice of $f_{d,e}$ and $f_{e,d}$ such that with $\alpha_d := f_{d,e} \circ f_{e,d}(1) \in R$, we have $\mu_d = \widetilde{\mu}_d \circ \left(\hat{B}_d \otimes_R \alpha_d \hat{B}_d\right)$.*

The conjecture holds for $d = w_0^I$ where $I \subset S$ is finitary, see Example 5.6.1, and we have checked it as well in type A_3 for $d = s_2s_1s_3s_2$.

Remark 5.2.16. For crystallographic Coxeter systems that are bounded in a suitable sense (thus for finite and affine Weyl groups as well as universal Coxeter systems), [Lu3] showed that the coefficient in $p_{e,d}$ is actually 1 and thus that the morphisms $f_{e,d}$ and $f_{d,e}$ are unique up to scalar.

We end this section, putting forward another question yet to be explored:

Question 5.2.17. *Recall the well known (left and right) duality $\mathbb{D}$ in the sense of [Ka, XIV.2] on $\mathcal{S}oe^\bullet$ for which $\mathbb{D}(B_x) \cong B_{x^{-1}}$ holds for all $x \in W$. In terms of say [ElWi1] this duality is given by a rotation of $180°$, see [Kl] for an exposition that covers (twisted) singular Soergel bimodules and effectively provides a 2-functor $\mathbb{D} : \mathfrak{S}oe^{\mathrm{rev,op}} \to \mathfrak{S}oe$ that is the identity on objects. Since all $d \in \mathcal{D}$ are involutions, B_d is a self-adjoint object and there are associated degree zero morphisms $B_e \to B_d \otimes B_d$ and $B_d \otimes B_d \to B_e$ for which the snake relation holds. Now, it would be interesting to know whether these two morphisms agree (up to a scalar) with $\Delta_d \circ f_{e,d}$ and $f_{d,e} \circ \mu_d$.*

5.3 Singular Soergel bimodules revisited

In this section, we provide a new perspective on singular Soergel bimodules. We will show that they can be defined as the precise categorical analogue of the Schur algebroid, and thus establish Theorem 4 of the introduction. We also calculate the centre of right singular Soergel bimodules considered as a $\mathcal{S}oe^\bullet$-module, as well as the endomorphism ring $\mathcal{Z}^{\otimes}(\mathcal{S}oe^\bullet)$ of the identity functor of $\mathcal{S}oe^\bullet$.

For finitary $I \subset S$, recall the split Frobenius object $\hat{\Theta}_{w_0^I}$ from Example 5.2.9.

Theorem 5.3.1. *Let I, $J \subset S$ be finitary. Then, $\mathcal{S}oe^{\bullet I}$ is a free functorially cyclic $\mathcal{S}oe^\bullet$-module and we have an equivalence ${}^I\mathcal{S}oe^{\bullet J} \cong \mathcal{H}om_{\mathcal{S}oe^\bullet}(\mathcal{S}oe^{\bullet I}, \mathcal{S}oe^{\bullet J})$ of additive categories. Further, we have an equivalence $\mathcal{S}oe^{\bullet I} \cong \mathcal{S}oe^\bullet{}_{\hat{\Theta}_{w_0^I}}$ in $\mathcal{S}oe^\bullet - \mathrm{Mod}$ and an equivalence of additive categories ${}^I\mathcal{S}oe^{\bullet J} \cong {}_{\hat{\Theta}_{w_0^I}}\mathcal{S}oe^\bullet{}_{\hat{\Theta}_{w_0^J}}$. The analogous statements hold in the context of $\mathcal{S}oe - \mathrm{Mod}$.*

Proof. Before we begin, let us note that all the $\mathcal{S}oe^\bullet$-functors and $\mathcal{S}oe^\bullet$-transformations that will be considered in the proof are of degree zero. Of course, these functors also yield $\mathcal{S}oe$-functors and $\mathcal{S}oe$-transformations with the same properties. Therefore, the same conclusions hold in $\mathcal{S}oe - \mathrm{Mod}$.

Let $I \subset S$ be finitary. Recall that $\mathcal{S}oe^\bullet$ and $\mathcal{S}oe^{\bullet I}$ are idempotent closed by definition. In Section 5.1, we mentioned the functors $\vartheta^I := {}^{\emptyset}_{\emptyset}\mathrm{Res}^{\emptyset}_I : \mathcal{S}oe^\bullet \to \mathcal{S}oe^{\bullet I}$ and $\vartheta_I := {}^{\emptyset}_{\emptyset}\mathrm{Ind}^{\emptyset}_I : \mathcal{S}oe^{\bullet I} \to \mathcal{S}oe^\bullet$. The functor ϑ_I (resp. the shifted functor $\vartheta^I[l(w_0^I)]$) agrees with the the translation out of the wall functor $- \cdot {}^I\vartheta^{\emptyset}$ (resp. onto the wall functor $- \cdot {}^{\emptyset}\vartheta^I$) of [Wi2, Definition 6.1.3]. Since these functors only involve operations on the right, one can easily equip ϑ_I and ϑ^I with the structure of $\mathcal{S}oe^\bullet$-functors. For all $M \in \mathcal{S}oe^\bullet$, we have $\vartheta_I\vartheta^I(M) = M|_{R^I} \otimes_{R^I} R \cong M \otimes_R (R \otimes_{R^I} R) = M \otimes_R \Theta_{w_0^I}$. These isomorphisms define an iso-$\mathcal{S}oe^\bullet$-transformation $\vartheta_I\vartheta^I \overset{\sim}{\Longrightarrow} -\Theta_{w_0^I}$. Further using the morphism $i_{w_0^I}$ introduced in Example 5.2.9, we obtain $\mathcal{S}oe^\bullet$-transformations $id_{\mathcal{S}oe^{\bullet I}} \cong - \otimes_{R^I} R^I \overset{-\otimes_{R^I} i_{w_0^I}}{\Longrightarrow} - \otimes_{R^I} R|_{R^I} \cong (- \otimes_{R^I} R) \otimes_R R|_{R^I} = \vartheta_I\vartheta^I$. We denote the composition by α_I. Similarly, one constructs a left inverse β_I of α_I by using $\pi_{w_0^I}$ which was also defined in Example 5.2.9. It is now easy to check that the above constructed iso-$\mathcal{S}oe^\bullet$-transformation induces an isomorphism between the Frobenius objects $(\vartheta^I\vartheta_I, \vartheta_I\alpha_I\vartheta^I, \vartheta_I\beta_I\vartheta^I)$ and $\hat{\Theta}_{w_0^I}$. Thus, we have shown that Corollary 4.5.9 applies, and this proof is complete. $\square$

Remark 5.3.2. More explicitly, one can deduce from our constructions that there exist idempotent $\mathcal{S}oe$-transformations e_1 on $\vartheta^{\hat{\Theta}_{w_0^I}}\vartheta_I$ and e_2 on $\vartheta^I\vartheta_{\hat{\Theta}_{w_0^I}}$, such that $(\vartheta^{\hat{\Theta}_{w_0^I}}\vartheta_I, e_1)$ and $(\vartheta^I\vartheta_{\hat{\Theta}_{w_0^I}}, e_2)$ establish the equivalence $\mathcal{S}oe^I \cong \mathcal{S}oe_{\hat{\Theta}_{w_0^I}}$.

Remark 5.3.3. Right multiplication defines a functor ${}^I\mathcal{S}oe^J \to \mathcal{H}om_{\mathcal{S}oe}(\mathcal{S}oe^I, \mathcal{S}oe^J)$ for all finitary I, $J \subset S$, which yields a 2-functor $\mathfrak{S}oe^{\mathrm{rev}} \to \mathcal{S}oe - \mathrm{Mod}$. Up to an isotransformation, this functor is part of the second equivalence of Theorem 5.3.1. In fact, it is not too hard to show directly that this functor is fully faithful. Therefore, the contribution made here is to prove essential surjectivity. Note that even in type A_1, we were not able to establish this result without factually using the same arguments that enter the proof presented above.

Since right singular Soergel bimodules are free functorially cyclic $\mathcal{S}oe$-modules, Theorem 4.5.10 yields the following statement:

Corollary 5.3.4. *We have a 1-2-full 2-functor $\mathfrak{Soe} \to \mathfrak{Frob}(\mathcal{S}oe)$.*

Remark 5.3.5. Theorem 4.5.14 implies that this 2-functor is essentially surjective up to weak equivalence if and only if all 2-sided cell of the underlying Coxeter group contain a parabolic longest element. Note that even in type A, we do not know whether the 2-functor $\mathfrak{Soe} \to \mathfrak{Frob}(\mathcal{S}oe)$ is an equivalence, since it is not clear whether all right indecomposable split Frobenius objects of $\mathcal{S}oe$ of type A are equivalent to $\hat{\Theta}_{w_0^I}$ for some finitary $I \subset S$.

Next, we calculate the centres introduced in Definitions 3.2.1 and 4.1.1 for singular Soergel bimodules, and relate them to one another:

Corollary 5.3.6. *Let $I \subset S$ be finitary. Then, $\mathcal{Z}_{\mathcal{S}oe}(\mathcal{S}oe^I) \cong R^I$. Further, if S is finitary and $J \subset S$, then the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{Z}_{\mathcal{B}}(\mathcal{S}oe^I) \arrow[r] & \mathcal{Z}(\mathcal{H}om_{\mathcal{S}oe}(\mathcal{S}oe^I,\mathcal{S}oe^J)) & \mathcal{Z}_{\mathcal{B}}(\mathcal{S}oe^J) \arrow[l] \\
 & \mathcal{Z}({}^I\mathcal{S}oe^J) \arrow[u, "\sim"] & \\
R^I \arrow[uu, "\sim"] \arrow[r] & R^I \otimes_{R^W} R^J \arrow[u, "\sim"] & R^J \arrow[l] \arrow[uu, "\sim"]
\end{tikzcd} .$$

Proof. The following statement is well known: For all unital rings A and all full subcategories $\mathcal{M}$ of $A-\mathrm{Mod}$ (or $A-\mathrm{mod}$) that contain all (finitely generated) free A-modules, we have $\mathcal{Z}(\mathcal{M}) = Z(A)$. This statement follows from the diagram

$$\begin{tikzcd}
A \arrow[r, hook, "i_m"] \arrow[d, "z_A"] & \bigoplus_{m' \in M} A \arrow[r, two heads, "\pi"] \arrow[d, "z_{\oplus A}"] & M \arrow[d, "z_M"] \\
A \arrow[r, hook, "i_m"] & \bigoplus_{m' \in M} A \arrow[r, two heads, "\pi"] & M
\end{tikzcd}$$

that commutes for all $m \in M \in \mathcal{M}$ and $z = (z_M)_{M \in \mathcal{M}} \in \mathcal{Z}(\mathcal{M})$, where π denotes the morphism that sends $1 \in A$ in the component associated to $m' \in M$ to m'.

By [Wi2, Definition 7.1.1], we have $R^I \otimes_{R^W} R^J \in {}^I\mathcal{S}oe^J \subset R^I \otimes_{R^W} R^J - \mathrm{mod}$, and thus $R^I \otimes_{R^W} R^J \cong \mathcal{Z}({}^I\mathcal{S}oe^J)$. The isomorphism $R^I \cong \mathcal{Z}_{\mathcal{S}oe}(\mathcal{S}oe^I)$ follows now from Corollary 4.5.28 using the explicit maps provided in 5.2. Commutativity of the diagram can be calculated explicitly. $\square$

Another centre is of interest. Note that we do not make any finiteness assumptions.

Proposition 5.3.7. *We have* $\mathcal{Z}^{\otimes}(\mathcal{S}oe) \cong R^W$.

Proof. Since monoidal transformations of monoidal identity functors are determined by their value at the unit object, the elements of $\mathcal{Z}^{\otimes}(\mathcal{S}oe)$ can be identified with all $\alpha \in \mathcal{S}oe(B_e, B_e)$ such that $B\alpha = \alpha B$ in $\mathcal{S}oe(B, B)$ for all $B \in \mathcal{S}oe$. Note that we pretend here for notational convenience that $\mathcal{S}oe$ is strict. For any $s \in S$ and $B = B_s$, this equation implies that $\alpha \in R^{\{s\}}$. Taken together this implies $\alpha \in R^S$. By the explicit definition of Soergel bimodules as summands of sums of Bott-Samelson bimodules, we know on the other hand that the equation holds for all $\alpha \in R^S = R^W$. We leave it as an exercise to check that all $\alpha \in R^W$ define an element $\mathcal{Z}^{\otimes}(\mathcal{S}oe)$. $\square$

We end this section with non-trivial examples of strong equivalences between split Frobenius objects, *i.e.* examples which cannot be explained by an isomorphisms. Above, we have used the $\mathcal{S}oe$-functors ϑ^I and ϑ_I to recover $\mathcal{S}oe^I$ from the associated split Frobenius objects. However, we could have chosen other $\mathcal{S}oe$-functors which correspond to the split Frobenius objects discussed in the following example:

Example 5.3.8. For simplicity, we consider $\mathcal{S}oe^{\{s\}}$ in type A_2 only. Also, we will pretend that $\mathcal{S}oe$ is strict and further replace some expressions that involve tensor products over R by simplified equivalent expressions. So far, we had considered $\hat{\Theta}_{w_0^{\{s\}}} = \hat{\Theta}_s = (\hat{B}_s, \Delta_s, \mu_s)$. Now define another right indecomposable split Frobenius object $\hat{\Theta}_{ts} := (\hat{B}_{ts}, \Delta_{ts}, \mu_{ts})$ as follows: We define $\hat{B}_{ts} := R \otimes_{R^{\{t\}}} R \otimes_{R^{\{s\}}} R$, we specify μ_{ts} and Δ_{ts} uniquely by

$$\begin{aligned}&\mu_{ts}\left((a \otimes_{R^{\{t\}}} b \otimes_{R^{\{s\}}} c) \otimes_R (d \otimes_{R^{\{t\}}} e \otimes_{R^{\{s\}}} f)\right)\\ &\qquad = -a \otimes_{R^{\{t\}}} b\partial_s(cde) \otimes_{R^{\{s\}}} f\end{aligned}$$

and

$$\begin{aligned}&\Delta_{ts}\left(x \otimes_{R^{\{t\}}} y \otimes_{R^{\{s\}}} z\right)\\ &\qquad = (x \otimes_{R^{\{t\}}} y \otimes_{R^{\{s\}}} 1) \otimes_R (1 \otimes_{R^{\{t\}}} \alpha_t \otimes_{R^{\{s\}}} z)\\ &\qquad + (x \otimes_{R^{\{t\}}} y \otimes_{R^{\{s\}}} 1) \otimes_R (\alpha_t \otimes_{R^{\{t\}}} 1 \otimes_{R^{\{s\}}} z).\end{aligned}$$

We leave it to the reader to check the Frobenius property and note that splitness holds since $\partial_s(\alpha_t) = -\frac{1}{2}$. Clearly, $\hat{\Theta}_s$ and $\hat{\Theta}_{ts}$ are not isomorphic, however we claim that there exists a strong equivalence $\hat{\Theta}_s \simeq \hat{\Theta}_{ts}$, see Definition 4.3.72. Let $A := B := B_e = R \in \mathcal{S}oe$, and let e_A, e_B, ι_1, π_1, ι_2 and π_2 be defined by

$$\begin{aligned}
&e_A\left(x \otimes_{R^{\{s\}}} y \otimes_{R^{\{t\}}} z \otimes_{R^{\{s\}}} w\right)\\
&\qquad = -x \otimes_{R^{\{s\}}} \partial_s(yz)\alpha_t \otimes_{R^{\{t\}}} 1 \otimes_{R^{\{s\}}} w - x \otimes_{R^{\{s\}}} 1 \otimes_{R^{\{t\}}} \partial_s(yz)\alpha_t \otimes_{R^{\{s\}}} w,\\
&e_B\left(x \otimes_{R^{\{t\}}} y \otimes_{R^{\{s\}}} z \otimes_{R^{\{s\}}} w\right)\\
&\qquad = x \otimes_{R^{\{t\}}} y \otimes_{R^{\{s\}}} \partial_s(\alpha_s z) \otimes_{R^{\{s\}}} w,\\
&\iota_1\left(x \otimes_{R^{\{s\}}} y \otimes_{R^{\{s\}}} z\right)\\
&\qquad = x \otimes_{R^{\{s\}}} \alpha_t \otimes_{R^{\{t\}}} 1 \otimes_{R^{\{s\}}} y \otimes_{R^{\{s\}}} z + x \otimes_{R^{\{s\}}} 1 \otimes_{R^{\{t\}}} \alpha_t \otimes_{R^{\{s\}}} y \otimes_{R^{\{s\}}} z,\\
&\pi_1\left(x \otimes_{R^{\{s\}}} y \otimes_{R^{\{t\}}} z \otimes_{R^{\{s\}}} w \otimes_{R^{\{s\}}} v\right)\\
&\qquad = -x \otimes_{R^{\{s\}}} \partial_s(yz)\partial_s(\alpha_s w) \otimes_{R^{\{s\}}} v,\\
&\iota_2\left(x \otimes_{R^{\{t\}}} y \otimes_{R^{\{s\}}} z \otimes_{R^{\{t\}}} w \otimes_{R^{\{s\}}} v\right)\\
&\qquad = x \otimes_{R^{\{t\}}} y \otimes_{R^{\{s\}}} 1 \otimes_{R^{\{s\}}} z \otimes_{R^{\{t\}}} w \otimes_{R^{\{s\}}} v
\end{aligned}$$

and

$$\begin{aligned}
&\pi_2\left(x \otimes_{R^{\{t\}}} y \otimes_{R^{\{s\}}} z \otimes_{R^{\{s\}}} w \otimes_{R^{\{t\}}} v \otimes_{R^{\{s\}}} u\right)\\
&\qquad = -x \otimes_{R^{\{t\}}} y \otimes_{R^{\{s\}}} \partial_s(\alpha_s z)\partial_s(wv) \otimes_{R^{\{t\}}} \alpha_t \otimes_{R^{\{s\}}} v\\
&\qquad\quad - x \otimes_{R^{\{t\}}} y \otimes_{R^{\{s\}}} \partial_s(\alpha_s z)\partial_s(wv)\alpha_t \otimes_{R^{\{t\}}} 1 \otimes_{R^{\{s\}}} v.
\end{aligned}$$

We leave it to the reader to check that the maps provided satisfy all relations and properties required in Definition 4.3.72.

Remark 5.3.9. The object $\hat{B}_{ts}$ can be equipped with two different split Frobenius structures. We had discussed $\hat{\Theta}_{ts}$ in Example 5.3.8, and shown that $\mathcal{S}oe_{\hat{\Theta}_{ts}} \cong \mathcal{S}oe^{\{s\}}$. In fact, there exists another structure $\widetilde{\Theta}_{ts}$, such that $\mathcal{S}oe_{\widetilde{\Theta}_{ts}} \cong \mathcal{S}oe^{\{t\}}$. Interestingly, one can show that $\mathcal{S}oe$ does not satisfy $\widetilde{\Theta}_{ts}$-descent, whereas it does satisfy $\hat{\Theta}_{ts}$-descent. The opposite is true for the respective right descent properties.

Remark 5.3.10. There exists another non-isomorphic, right indecomposable split Frobenius object $\hat{\Theta}_{s\oplus sts}$ that is associated to $\mathcal{S}oe^{\{s\}}$. It has the underlying object $R \otimes_{R^{\{s\}}} R \otimes_{R^{\{t\}}} R \otimes_{R^{\{s\}}} R[2] = \hat{B}_s \oplus \hat{B}_{sts}[2]$. Similarly to the above arguments, one can show explicitly that $\hat{\Theta}_s \simeq \hat{\Theta}_{s\oplus sts}$. Furthermore, one can show that $\hat{\Theta}_s$ and $\hat{\Theta}_{s\oplus sts}$ become isomorphic after passing to the quotient of $\mathcal{S}oe$ by $\hat{B}_{sts}$.

Taking Conjecture 5.2.5 into account, the preceding remarks raises the following question:

Question 5.3.11. *Let $\hat{\Theta}$ be a split Frobenius object in $\mathcal{S}oe$ with indecomposable underlying object B. Provided that $\mathcal{S}oe$ satisfies two-sided $\hat{\Theta}$-descent, does there exist*

$d \in \mathcal{D}$ such that $B \cong B_d[-\boldsymbol{a}(d)]$? Further, does this isomorphism induce an isomorphism between $\hat{\Theta}$ and $\hat{\Theta}_d$?

5.4 Universal cell modules

In this section, we study universal the cell modules of Soergel bimodules associated to Duflo involutions assuming the conjectures made in Section 5.2. Theorem 5.4.1 determines the Grothendieck group of morphism categories between the universal cell modules. Corollary 5.4.4 provides equivalences between cell modules in regular 2-sided cells that contain parabolic longest elements. Further, Corollary 5.4.6 derives a homorphism formula for the classical counterparts in case of Weyl groups.

Given a Duflo involution $d \in \mathcal{D}$, we denote by $\mathcal{C}_d := \mathcal{C}_{\overline{\Theta}_d}$ the universal cell module associated to d by virtue of Conjecture 5.2.3. To not overload the notation, we suppress denoting the gradings explicitly, however the statements made below do require the grading. The analogous statements in the non-graded case hold as well, where we have to replace $\mathcal{H}$ by the group algebra $\mathbb{Z}[W]$, compare Remark 5.1.16. Recall the notation $\overline{\mathcal{H}}^d := \mathcal{H}/\{x \in W | x \not\geq_{LR} d\}$ for all $d \in W$.

Theorem 5.4.1. *Let d_1, $d_2 \in \mathcal{D}$, assume Conjectures 5.2.2 and 5.2.4 for d_1 and d_2, and assume that $\mathcal{H}om_{\mathcal{S}oe}(\mathcal{C}_{d_1}, \mathcal{C}_{d_1})$ is non-trivial. Then, we have $d_1 \sim_{LR} d_2$, the morphism $\mathrm{ev}_{\underline{H}_{d_1}} : \mathrm{Hom}_{\mathcal{H}}(\overline{\mathcal{H}}.\underline{H}_{d_1}, \overline{\mathcal{H}}.\underline{H}_{d_2}) \to \overline{\mathcal{H}}.\underline{H}_{d_2}$ surjects onto its coimage $\underline{H}_{d_1}.\overline{\mathcal{H}}.\underline{H}_{d_2}$, where $\overline{\mathcal{H}} := \overline{\mathcal{H}}^{d_1} = \overline{\mathcal{H}}^{d_2}$, and this surjection splits. Further, the natural composition $[\mathcal{H}om_{\mathcal{S}oe}(\mathcal{C}_{d_1}, \mathcal{C}_{d_2})] \hookrightarrow \mathcal{H}om_{[\mathcal{S}oe]}([\mathcal{C}_{d_1}], [\mathcal{C}_{d_2}]) = \mathrm{Hom}_{\mathcal{H}}(\overline{\mathcal{H}}.\underline{H}_{d_1}, \overline{\mathcal{H}}.\underline{H}_{d_2}) \twoheadrightarrow \underline{H}_{d_1}.\overline{\mathcal{H}}.\underline{H}_{d_2}$ is an isomorphism.*

Proof. Let $d \in \mathcal{D}$. Then, Conjecture 5.2.2 for d ensures existence of the universal cell modules $\mathcal{C}_d$. Further, the statement about right descent in Conjecture 5.2.4 for d implies that $\mathcal{C}_d$ is an idempotent closed subcategory of $\overline{\mathcal{S}oe}^d$ containing $\pi_d(\Theta_d)$. Since $\mathcal{C}_d$ is 1-cyclic, we thus obtain $[\mathcal{C}_d] \cong \overline{\mathcal{H}}^d.\underline{H}_d$, where $\overline{\mathcal{H}}^d.\underline{H}_d := \mathrm{coim}\left(\cdot\underline{H}_d : \overline{\mathcal{H}}^d \to \overline{\mathcal{H}}^d\right)$. Now let d_1, $d_2 \in \mathcal{D}$, assume Conjectures 5.2.2 and 5.2.4 for d_1 and d_2, and assume that $\mathcal{H}om_{\mathcal{S}oe}(\mathcal{C}_{d_1}, \mathcal{C}_{d_1}) \neq 0$. By Proposition 4.5.35, we have $d_1 \sim_{LR} d_2$ and thus can write $\overline{\mathcal{H}} := \overline{\mathcal{H}}^{d_1} = \overline{\mathcal{H}}^{d_2}$. Denote by $\underline{H}_{d_1}.\overline{\mathcal{H}}.\underline{H}_{d_2} := \mathrm{coim}\left(\underline{H}_{d_2} - \underline{H}_{d_2} : \overline{\mathcal{H}} \to \overline{\mathcal{H}}\right)$. Clearly $\underline{H}_{d_1}.\overline{\mathcal{H}}.\underline{H}_{d_2} \hookrightarrow \overline{\mathcal{H}}.\underline{H}_{d_2}$ and the evaluation $\mathrm{ev}_{\underline{H}_{d_1}} : \mathrm{Hom}_{\mathcal{H}}\left(\overline{\mathcal{H}}.\underline{H}_{d_1}, \overline{\mathcal{H}}.\underline{H}_{d_2}\right) \to \overline{\mathcal{H}}.\underline{H}_{d_2}$ factors over this inclusion. Let us denote this factorisation by ev_{d_1}. Further, denote $\overline{\mathcal{S}oe} := \overline{\mathcal{S}oe}^{d_1} = \overline{\mathcal{S}oe}^{d_2}$, and recall that $\left[\overline{\mathcal{S}oe}\right] \cong \overline{\mathcal{H}}$. Conjecture 5.2.4 implies that

${}_{\overline{\Theta}_{d_1}}\overline{\mathcal{S}oe}_{\overline{\Theta}_{d_2}}$ is an idempotent closed subcategory of $\overline{\mathcal{S}oe}$ with objects all direct sums of $\pi_{d_2}(\hat{B}_{d_2}B\hat{B}_{d_1})$ for all $B \in \mathcal{S}oe$. The claim follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{H}}\big([\mathcal{C}_{d_1}],[\mathcal{C}_{d_2}]\big) \arrow[rr, "\sim"] \arrow[ddd, hook, "{[\vartheta_{\overline{\Theta}_{d_2}}]-[\vartheta^{\overline{\Theta}_{d_1}}]}"'] & & \mathrm{Hom}_{\mathcal{H}}\big(\overline{\mathcal{H}}.\underline{H}_{d_1},\overline{\mathcal{H}}.\underline{H}_{d_2}\big) \arrow[rr, "\mathrm{ev}_{d_1}"] & & \underline{H}_{d_1}.\overline{\mathcal{H}}.\underline{H}_{d_2} \arrow[ddd, hook] \\
& \big[\mathcal{H}om_{\mathcal{S}oe}(\mathcal{C}_{d_1},\mathcal{C}_{d_2})\big] \arrow[ul, "{[-]}"'] \arrow[rr, "\cong"] \arrow[d, hook, "{\big[\vartheta_{\overline{\Theta}_{d_2}}-\vartheta^{\overline{\Theta}_{d_1}}\big]}"] & & \Big[{}_{\overline{\Theta}_{d_1}}\overline{\mathcal{S}oe}_{\overline{\Theta}_{d_2}}\Big] \arrow[ur, dashed, "\sim"] \arrow[d, hook, "{\Big[{}_{\overline{\Theta}_{d_1}}\vartheta_{\overline{\Theta}_{d_2}}\Big]}"] & \\
& \big[\mathcal{H}om_{\mathcal{S}oe}(\overline{\mathcal{S}oe},\overline{\mathcal{S}oe})\big] \arrow[rr, "\cong"] \arrow[dl, "{[-]}"'] & & \big[\overline{\mathcal{S}oe}\big] \arrow[dr, "\sim"] & \\
\mathrm{Hom}_{\mathcal{H}}([\overline{\mathcal{S}oe}],[\overline{\mathcal{S}oe}]) \arrow[rr, "\sim"] & & \mathrm{Hom}_{\mathcal{H}}(\overline{\mathcal{H}},\overline{\mathcal{H}}) \arrow[rr, "\mathrm{ev}_{\underline{H}_e}"] & & \overline{\mathcal{H}}
\end{tikzcd},$$

in which we used $\hookrightarrow$ to denote inclusions and $\xrightarrow{\sim}$ (resp. $\xrightarrow{\cong}$) for isomorphisms induced from isomorphisms (resp. from equivalences). Note that the notation ${}_{\overline{\Theta}_{d_1}}\vartheta_{\overline{\Theta}_{d_2}}$ was introduced in Remark 4.3.67, which we apply here to the quotient $\overline{\mathcal{S}oe}$ of $\mathcal{S}oe$ and Frobenius objects that live there rather than to $\mathcal{S}oe$ itself. $\square$

Remark 5.4.2. Assuming the stronger Conjectures 5.2.3 and 5.2.5 (or intermediate variants), for any quotient π of $\mathcal{S}oe$ that factors π_d, we can replace $\mathcal{C}_{d_1}$ by $\mathcal{C}_{\left(\pi(\hat{\Theta}_{d_1}),\pi\right)}$ and obtain the same statement. Note that for $\pi = id_{\mathcal{S}oe}$, $\mathcal{C}_{\left(\pi(\hat{\Theta}_{d_1}),\pi\right)}$ categorifies the cyclic representation $\mathcal{H}.\underline{H}_{d_1}$ which in turn for $d = w_0^I$ is equal to $\mathcal{H}^I$. Further, $\mathcal{C}_{d_2}$ can be replaced by the full sub-$\mathcal{S}oe$-module of $\overline{\mathcal{S}oe}$ that contains $\hat{B}_{d_2}$.

Remark 5.4.3. The proof of Theorem 5.4.1 required intricate Frobenius descent properties. Recalling Remark 5.3.9, we note that the class in the Grothendieck group of the underlying object of an arbitrary right indecomposable split Frobenius object might not generate the correct submodules. This has two implications. First, the above categorification result does not generalise to $\mathfrak{F}rob(\mathcal{S}oe)$. Second, the good properties of the evaluation map $\mathrm{ev}_{\underline{H}_d}$ established above for $d \in \mathcal{D}$, do in general not hold for the corresponding evaluation maps for arbitrary members of the Kazhdan-Lusztig basis.

By Remark 5.2.12, the following corollary applies to all 2-sided cells of finite and affine Weyl groups of type A or of rank ≤ 3, as well as to the n-gon Coxeter groups P_n which are hyperbolic for $n \geq 5$.

Corollary 5.4.4. *Let $\underline{c}$ be a strongly regular 2-sided cell of $\mathcal{S}oe$ that contains a parabolic longest element, let d_1, $d_2 \in \underline{c} \cap \mathcal{D}$, and assume that Conjectures 5.2.3 and 5.2.5 hold for d_1 and d_2. Then, we have $\mathcal{C}_{d_1} \cong \mathcal{C}_{d_2}$.*

Proof. Let $\underline{c}$ be a strongly regular 2-sided cell, let $I \subset S$ be finitary such that $w_0^I \in \underline{c}$, and let $d \in \mathcal{D} \cap \underline{c}$. From the J-ring combinatoric (a matrix algebra in type A), we obtain $x \in W$ such that $\{w_0^I\} = L(x^{-1}) \cap R(x)$ and $\{d\} = L(x) \cap R(x^{-1})$. We thus have $\underline{H}_x \underline{H}_{x^{-1}} = p_1 \underline{H}_{w_0^I}$ and $\underline{H}_{x^{-1}} \underline{H}_x = p_2 \underline{H}_d$ for some $p_1, p_2 \in \mathbb{N}_0[v, v^{-1}]$ in $\overline{\mathcal{H}} = \overline{\mathcal{H}}^d$. By [Wi2, Section 2.3], we know that $\pi(I) := \sum_{x \in W_I} v^{2l(x) - l(w_0^I)}$ divides p_1. In the same time, the J-ring combinatoric implies that $p_1 v^{\pm l(w_0^I)}\Big|_{v=0} = 1$. Since the same property holds for $\pi(I)$, it follows that $p_1 = \pi(I)$. Clearly, there exist idempotents e_1 and e_2 such that $B_x \cong (\hat{B}_d B_x \hat{B}_{w_0^I}, e_1)$ and $B_{x^{-1}} \cong (\hat{B}_{w_0^I} B_{x^{-1}} \hat{B}_d, e_2)$ in $\overline{\mathcal{S}oe}$. Thus, we have defined objects in ${}_{\overline{\Theta}_{w_0^I}}\overline{\mathcal{S}oe}_{\overline{\Theta}_d}$ and ${}_{\overline{\Theta}_d}\overline{\mathcal{S}oe}_{\overline{\Theta}_{w_0^I}}$ whose tensor product has to be isomorphic to the unit object of ${}_{\overline{\Theta}_{w_0^I}}\overline{\mathcal{S}oe}_{\overline{\Theta}_{w_0^I}}$. The statement follows from Theorem 4.5.10 applied to $\mathcal{B} = \overline{\mathcal{S}oe}$ and Proposition 4.5.45. □

Remark 5.4.5. Let $\underline{c}$ be a 2-sided cell. It would be interesting to study the relationship between $\mathcal{Z}_{\mathcal{S}oe}(\mathcal{C}_{d_1})$ and $\mathcal{Z}_{\mathcal{S}oe}(\mathcal{C}_{d_1})$ for Duflo involutions $d_1, d_2 \in \mathcal{D} \cap \underline{c}$. In the situation of Corollary 5.4.4, these rings are of course isomorphic. Interestingly for finitary $I, J \subset S$ with $w_0^I \sim_{LR} w_0^J$, the weak equivalence of $\mathcal{S}oe^I$ and $\mathcal{S}oe^J$ induces a R^W-module homomorphism $R^I \to R^J$ between their respective centres as $\mathcal{S}oe$-modules, which in turn appears to induce the isomorphism of the centers of the cell modules. On the other hand, there exists an inner automorphism of W that induces an isomorphism $R^I \xrightarrow{\sim} R^J$, which also appears to induce the isomorphism of the centers of the cell modules. We have not checked this in detail, but if this statement holds, it would be interesting to understand the deeper reason behind this observation.

We end this section, by combining a result of Lusztig for the group algebra $\mathbb{C}[W]$ of a Weyl group W with Theorem 5.4.1 to deduce a statement about morphism spaces for cell modules of the Hecke algebra of W. Note that this statement clarifies how the behaviour changes beyond the case of strongly regular 2-sided cells, and also does not require that there exists a parabolic longest element in the 2-sided cell under consideration.

Corollary 5.4.6. *Let W be a Weyl group, let $d_1, d_2 \in W$ be Duflo involutions such that $d_1 \sim_{LR} d_2$, and let $c_1 = c(d_1)$ and $c_2 = c(d_2)$ be the associated left cells. Assume that Conjectures 5.2.3 and 5.2.5 hold for d_1 and d_2. Then, we have a $\mathbb{Z}[v, v^{-1}]$-linear isomorphism* $\mathrm{Hom}_{\mathcal{H}}(\overline{\mathcal{H}}.\underline{H}_{d_1}, \overline{\mathcal{H}}.\underline{H}_{d_2}) \cong \mathbb{Z}[v, v^{-1}][c_1^{-1} \cap c_2]$*, where* $\overline{\mathcal{H}} := \overline{\mathcal{H}}^{d_1} = \overline{\mathcal{H}}^{d_2}$.

Proof. Theorem 5.4.1 provides a split surjection $\mathrm{Hom}_{\mathcal{H}}(\overline{\mathcal{H}}.\underline{H}_{d_1}, \overline{\mathcal{H}}.\underline{H}_{d_2}) \twoheadrightarrow \underline{H}_{d_1}.\overline{\mathcal{H}}.\underline{H}_{d_2}$. The right hand side can easily be identified with $\mathbb{Z}[v, v^{-1}][c_1^{-1} \cap c_2]$. Specialising the splitting using $v \mapsto 1 \in \mathbb{C}$, we obtain a $\mathbb{C}$-linear embedding $\mathbb{C}[c_1^{-1} \cap c_2] \hookrightarrow \mathrm{Hom}_{\mathbb{C}[W]}(\overline{\mathcal{H}}.\underline{H}_{d_1} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}, \overline{\mathcal{H}}.\underline{H}_{d_2} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C})$. By [Lu2, Proposition 12.15] the dimensions agree, *i.e.* the embedding is an isomorphism. The canonical specialisation map $\mathrm{Hom}_{\mathcal{H}}(\overline{\mathcal{H}}.\underline{H}_{d_1}, \overline{\mathcal{H}}.\underline{H}_{d_2}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \twoheadrightarrow \mathrm{Hom}_{\mathbb{C}[W]}(\overline{\mathcal{H}}.\underline{H}_{d_1} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}, \overline{\mathcal{H}}.\underline{H}_{d_2} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C})$ is thus surjective. Now assume that there exists $f \in \mathrm{Hom}_{\mathcal{H}}(\overline{\mathcal{H}}.\underline{H}_{d_1}, \overline{\mathcal{H}}.\underline{H}_{d_2})$ which does not lie in the image of the splitting. W.l.o.g, we can assume that $f(\underline{H}_{d_1})$ has coefficients that specialise to zero, since we can otherwise subtract the image of $f(\underline{H}_{d_1})$ under the splitting from f. We can further assume w.l.o.g that $f(\underline{H}_{d_1}) = 0$, since otherwise we can separate the positive and the negative coefficients of $f(\underline{H}_{d_1})$ add another two morphisms in the image of the splitting to f. By linearity and positivity of the structure coefficients of $\mathcal{H}$, $f(\underline{H}_x) = 0$ for all $x \leq_L d_1$ and thus $f = 0$. This contradicts our assumption, and the proof is complete. □

Remark 5.4.7. *A priori* there is no to us apparent reason that would ensure that the basis of $\mathrm{Hom}_{\mathbb{C}[W]}(\overline{\mathcal{H}}.\underline{H}_{d_1} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}, \overline{\mathcal{H}}.\underline{H}_{d_2} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C})$, obtained from Lusztig's isomorphism in [Lu2, Proposition 12.15], has positive coefficients.

5.5 Coinvariant theory

In this section, we apply the tools developed in Section 4.6 to Soergel bimodules. We study examples in the case of dihedral groups, Weyl groups of types B_2, $\widetilde{A}_2$ and $\widetilde{C}_2$ and right-angled Coxeter groups P_n.

Remark 5.5.1. Let $d \in \mathcal{D}$ be a Duflo involution. Proposition 4.5.42 states that $\mathcal{E}nd_{\mathcal{S}oe}(\mathcal{C}_d)$ consists of a single left cell, which it does not ensure that this endomorphism category behaves like groups. Nevertheless, there may exist non-trivial group actions on $\mathcal{C}_d$. On the other hand, Corollary 5.4.6 implies that for Weyl groups it is only interesting to study group actions on $\mathcal{C}_d$ in case of 2-sided cells that are not regular.

Remark 5.5.2. The group actions that form the basis of Section 4.6, say on $\mathcal{C}_d$, require a monoidal functor $G \to \mathcal{E}nd_{\mathcal{S}oe}(\mathcal{C}_d)$. Fortunately, in the examples presented below it is sufficient to provide a map $g \mapsto S_g$ such that $S_{g_1}S_{g_1} \cong S_{g_1g_2}$. These isomorphisms can then be normalised using the following fact: Indecomposable Soergel bimodules can be constructed as direct summands of Bott-Samelson bimodules which have a distinguished element $1 \otimes ... \otimes 1$. Thus, choosing an isomorphism that preserves these distinguished elements results in a monoidal functor as required.

Remark 5.5.3. For arbitrary $d \in \mathcal{D}$, the indecomposable objects of $\mathcal{E}nd_{\mathcal{S}oe}(\mathcal{C}_d)$ do not behave like a group, as the dihedral groups $I_2(m)$ for $m \geq 5$ show. It is plausible that the symmetry group of the associated W-graph gives rise to a group action on cell modules. The periodic W-graphs of [Lu5] might also lead to symmetries. It would be interesting to study these examples in the future.

Type B_2 and dihedral groups

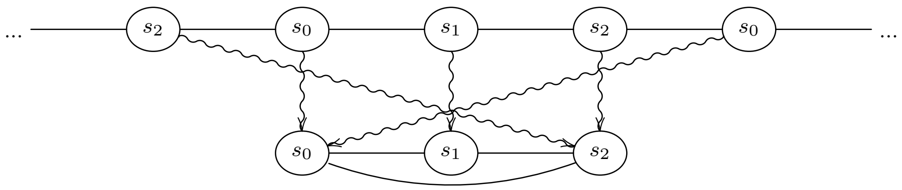
Recall the discussion of the classical case in Section 2.6.3. We have $W = W(B_2) = \langle s, t | s^2 = t^2 = (st)^4 \rangle$. The cyclic and the cell $\mathcal{S}oe$-module associated to the Duflo involution s are given by $\mathcal{C}_{\hat{\Theta}_s} \cong \mathcal{S}oe^{\{s\}}$ and $\mathcal{C}_s = \mathcal{C}_{\overline{\Theta}_s} = \mathcal{S}oe^{\{s\}} / \langle B_{\overline{stst}} \rangle$. The Grothendieck group of $\mathcal{C}_s$ has a basis parametrised by the left cell $\{s, ts, sts\}$. Further, $\mathcal{E}nd_{\mathcal{S}oe}(\mathcal{S}oe^{\{s\}}) \cong {}^{\{s\}}\mathcal{S}oe^{\{s\}}$ which has indecomposable objects $B_{\overline{s}}$, $B_{\overline{sts}}$ and $B_{\overline{stst}}$. Similarly, the indecomposable $\mathcal{S}oe$-functors, $id_{\mathcal{C}_s}$ and S say, in $\mathcal{E}nd_{\mathcal{S}oe}(\mathcal{C}_s) \cong {}^{\{s\}}\mathcal{S}oe^{\{s\}} / \langle B_{\overline{stst}} \rangle$ correspond to $B_{\overline{s}}$ and $B_{\overline{sts}}$. In this quotient, we have $B_{\overline{sts}} B_{\overline{sts}} \cong B_{\overline{s}}$. In particular, $B_{\overline{sts}}$ cannot be equipped with the structure of a split Frobenius object. On the Grothendieck group $[\mathcal{C}_s]$, these $\mathcal{S}oe$-functor induce exactly the two endomorphism provided in Section 2.6.3. It follows, that $SS \cong id_{\mathcal{C}_s}$ as $\mathcal{S}oe$-functors. Thus, we have an action $\mathbb{Z}/2\mathbb{Z} \to \mathcal{E}nd_{\mathcal{S}oe}(\mathcal{C}_s)$ and we obtain the coinvariant module $(\mathcal{C}_s)_{\mathbb{Z}/2\mathbb{Z}}$. Furthermore, $\mathcal{H}om_{\mathcal{S}oe}(\mathcal{C}_s, \mathcal{C}_t) \cong {}^{\{s\}}\mathcal{S}oe^{\{t\}} / \langle B_{\overline{stst}} \rangle$ which consist of direct sums of a unique indecomposable $\mathcal{S}oe$-functor, call it F. Hence, $F \cong FS$ holds and Theorem 4.6.14 yields a $\mathcal{S}oe$-functor $(\mathcal{C}_s)_{\mathbb{Z}/2\mathbb{Z}} \to \mathcal{C}_t$. Further, we have the canonical $\mathcal{S}oe$-functor $\mathcal{C}_s \to (\mathcal{C}_s)_{\mathbb{Z}/2\mathbb{Z}}$ of Proposition 4.6.12. Thus, we have completely categorified the situation described in Section 2.6.3. Also note similar patterns exists for all dihedral groups $I_2(m)$ for $m \geq 4$ even. Already the case $m = 5$ is quite intriguing, since the idempotent closure of the coinvariant module should admit S_3 (or $\mathbb{Z}_3$) as a symmetry group giving rise to a further rank 2 cell module. On the other hand, the case of odd m is trivial in terms of symmetries. However, the case $m = 4$ provides a simple example in which $\mathcal{E}nd_{\mathcal{S}oe}(\mathcal{C}_s)$ is not group like.

Simply laced and connected

We provide some preparatory facts for the upcoming examples. Assume that the Coxeter system (W, S) is connected and simply laced, *i.e.* for all s, $t \in S$ we have a sequence s_1, ..., $s_n \in S$ such that $s = s_1$, $t = s_n$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all $i \in \{1, .., n-1\}$. Let us assume that this sequence is chosen so that n is minimal. Then, $s_i s_j = s_j s_i$ for $|i-j| > 1$ and $s_i s_{i+1} s_i$ for all $i \in \{1, .., n-1\}$ are parabolic longest elements with **a**-value 2 and 3 respectively. There exists a W-graph that has the same shape as the Coxeter graph associated to (W, S). Furthermore, all $\mathcal{C}_{s_i}$ are equivalent, since $B_{\overline{s_i s_{i+1}}} \in {}^{\{s_i\}}\mathcal{S}oe^{\{s_{i+1}\}}$ induces a $\mathcal{S}oe$-functor $\mathcal{C}_{s_i} \to \mathcal{C}_{s_{i+1}}$ with inverse induced by $B_{\overline{s_{i+1} s_i}}$. To see this, recall that $B_{\overline{s_i s_{i+1}}} B_{\overline{s_{i+1} s_i}} \cong B_{\overline{s_i}} \oplus B_{\overline{s_i s_{i+1} s_i}}$ which corresponds to the identity $\mathcal{S}oe$-functor of $\mathcal{C}_{s_i}$, since the second summand lies in a lower 2-sided cell and is thus equal to the zero object in the quotient under consideration. Note that the combinatorics of $[\mathcal{C}_{s_i}]$ is in type A encoded in a W-graph that resembles the Coxeter graph, in general this is not the case.

Type $\widetilde{A}_2$

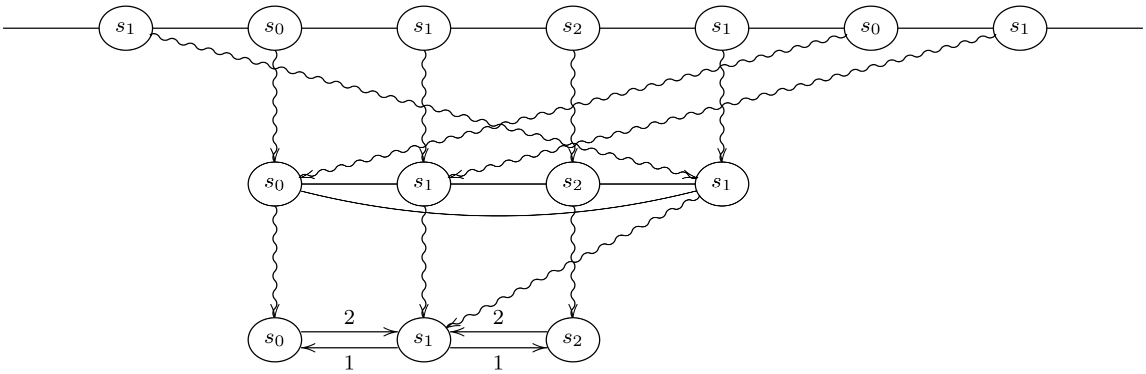
We have $W = W(\widetilde{A}_2) := \langle s_0, s_1, s_2 | s_i^2 = (s_i s_j)^3 = 1 \text{ for } \{i, j\} \subset \{0, 1, 2\}\rangle$, which is called the affine Weyl group of type $\widetilde{A}_2$. It is simply laced and connected. We will discuss cell modules associated to cells with **a**-function value 1. Classically we have the following W-graph combinatorics, [KaLu, Lu5] for more on periodic W-graphs:



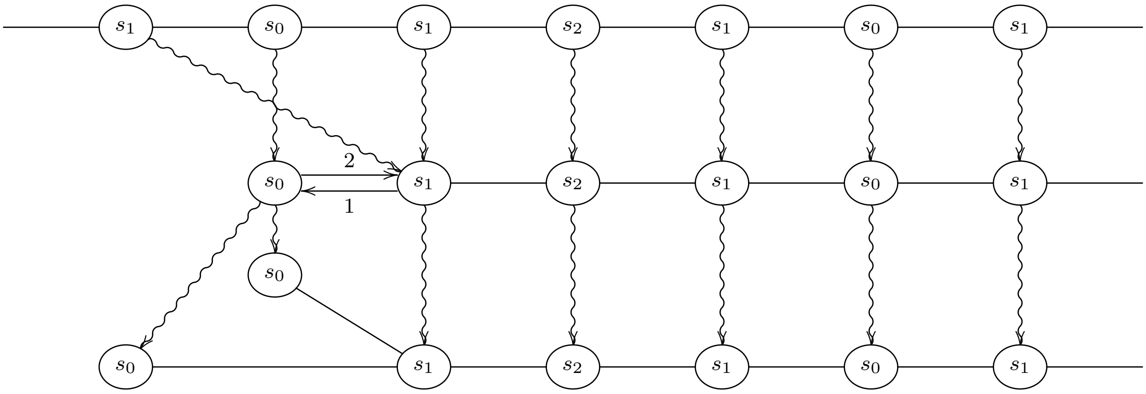
The upper W-graph describes the Grothendieck group of $\mathcal{C}_{s_0}$ (as well as of the equivalent modules $\mathcal{C}_{s_1}$ and $\mathcal{C}_{s_2}$). The lower graph can be described as a coinvariant module for the symmetry group $\mathbb{Z}$ generated by a translation by three nodes. We know that $\mathcal{E}nd_{\mathcal{S}oe}(\mathcal{C}_{s_0})$ can be described as a quotient of ${}^{\{s_0\}}\mathcal{S}oe^{\{s_0\}}$. The generating translation is given by $B_{s_0 s_1 s_2 s_0} \in {}^{\{s_0\}}\mathcal{S}oe^{\{s_0\}}$ with inverse $B_{s_0 s_2 s_1 s_0}$. Now the coinvariant module $(\mathcal{C}_{s_0})_{\mathbb{Z}}$ and the associated $\mathcal{S}oe$-functor $\mathcal{C}_{s_0} \to (\mathcal{C}_{s_0})_{\mathbb{Z}}$ categorifies the above picture. Also note that even though the morphism spaces in $\mathcal{C}_{s_0} \to (\mathcal{C}_{s_0})_{\mathbb{Z}}$ are given by infinite direct sums, each graded piece will still be finite dimensional which can be deduced from the homomorphism formula [Wi2, Theorem 7.4.1]. This statement also applies to the type $\widetilde{C}_2$ which we discuss next.

Type $\widetilde{C}_2$

We have $W = W(\widetilde{C}_2) := \langle s_0, s_1, s_2 | s_0^2 = s_1^2 = s_2^2 = (s_0s_1)^4 = (s_1s_2)^4 = 1\rangle$, which is the affine Weyl group of type $\widetilde{C}_2$. There exists the following morphisms of W-graphs:



The upper W-graph extends periodically to the left and right and corresponds to the Grothendieck group of $\mathcal{C}_{s_1}$. Its symmetry group is of type $\widetilde{A}_1$. The symmetry group is generated say by r_0, which reflects the graph at $[B_{\overline{s_0s_1}}]$ (the first node form the left with label s_0), and r_1, which reflects at $[B_{\overline{s_0s_2}}]$ (the first node form the left with label s_2). The middle graph describes the Grothendieck combinatorics of the coinvariant module associated to the subgroup generated by r_2r_0, which translates the graph four vertices to the right. The bottom graph is the coinvariant module associated to the entire symmetry group. Furthermore, we have the following W-graph morphisms:



The composition of the two morphisms shown in the figure above corresponds to the $\mathcal{S}oe$-functor $\mathcal{C}_{s_1} \to \mathcal{C}_{s_0}$ induced from $B_{\overline{s_1s_0}} \in {}^{\{s_1\}}\mathcal{S}oe^{\{s_0\}}$. It is clearly invariant under r_0, and thus factors over the coinvariant module $(\mathcal{C}_{s_1})_{\{e,r_0\}}$ whose Grothendieck group corresponds to the central W-graph just above. A more common depiction of the left cells in term of alcoves is given in the figure with title $r_1 = 1$ and $r_2 = 1$ in [Gu], where the left cells associated to s_0, s_1 and s_2 are shown in green.

Right-angled Coxeter groups

For $n \geq 5$, $P_n = \left\langle s_{\bar{1}}, s_{\bar{2}}, ..., s_{\bar{n}} | s_{\bar{i}}^2 = (s_{\bar{i}} s_{\overline{i+1}})^2 = 1 \text{ for all } \bar{i} \in \mathbb{Z}/n\mathbb{Z} \right\rangle$ is the group of isometries of the hyperbolic plane generated by reflections about the sides of a right-angled hyperbolic n-gon. The paper [Be] studies right-angled Coxeter groups and the groups P_n in particular. It is shown that P_n has three 2-sided cells and also describes the left cells of P_n explicitly. Further, it is stated that the left cells that contain a generator s_i are "equivalent" by cyclic permutations of the labels of the W-graph. For the associated $\mathcal{H}$-modules that means that they arise from one another by twisting them with an automorphism of $\mathcal{H}$. It should be noted that they are not isomorphic. However, they are weakly equivalent! The W-graph of $[\mathcal{C}_{s_1}]$ is a binomial tree and can be found in [Be, Figure 3]. We leave the intriguing exercise to the reader to draw the folding and unfolding operations in analogy to the preceding examples, which correspond to the $\mathcal{S}oe$-functors between universal cell modules and factorisations thereof over coinvariant modules.

5.6 Control from 2-morphisms

In this section, we use Conjecture 5.2.15 to show in Proposition 5.6.2 that morphisms of $\mathcal{S}oe$ have a strong impact on the structure of $\mathcal{S}oe - \text{Mod}$. Example 5.6.1, makes a particular morphism explicit that is associated to a parabolic longest elements.

Recall the elements $\alpha_d \in R$ associated to any $d \in \mathcal{D}$ in Conjecture 5.2.15. By construction is has degree $2\mathbf{a}(d)$. By abuse of notation, we can consider α_d as an endomorphisms $\alpha_d \in \mathcal{S}oe(B_e, B_e)$ with $\alpha_d(1) = \alpha_d$. The following example is based on facts that can be found in [Wi2, Section 3.1]:

Example 5.6.1. Let $I \subset S$ be finitary and abbreviate $\alpha_I := \alpha_{w_0^I}$. Recall that R is graded free as an R^I module. Further, the Demazure operator $\partial_I := \partial_{w_0^I}$ induces a non-degenerate pairing on R over R^I. Choose a R^I-basis $\{b_x\}_{x \in W_I}$ of R. For example, we may choose $b_e = \prod_{t \in W_I \cap T} \alpha_t$ where $T := \bigcup_{w \in W} wSw^{-1}$ is the set of reflections in W, and where α_t for all $t \in T$ is the root in $\mathfrak{h}^*$ associated with the reflection t. Then for all $x \in W_I$, we may define $b_x := \partial_x(b_e)$, where ∂_x is the Demazure operator discussed in [Wi2, Section 3.1]. Since ∂_I is non-degenerate, there exists a dual R^I-basis $\{b_x^*\}_{x \in W_I}$ of R. Thus, we have $\partial_I(b_x b_y^*) = \delta_{x,y}$ for all x, $y \in W_I$. Now, the morphism $R \to R \otimes_{R^I} R[l(w_0^I)]$ of minimal degree (which is unique up to scalar) is given by $r \mapsto \frac{r}{|W_I|} \sum_{x \in W_I} b_x \otimes b_x^*$ and the morphism $R \otimes_{R^I} R[l(w_0^I)] \to R$ of minimal degree (which is also unique up to scalar) is given by $x \otimes y \mapsto xy$. Thus, we have

$\alpha_I = \frac{1}{|W_I|}\sum_{x\in W_I} b_x b_x^*$, which is independent of the choice of basis made above. Note that we introduced the factor $|W_I|$ to ensure consistency of the two meanings of α_s. Finally, we have $\partial_I(\alpha_I r) = \frac{1}{|W_I|}\sum_{x\in W_I} x.r$. Thus, Conjecture 5.2.15 follows from the explicit definition of $\mu_{w_0^I}$ in Example 5.2.9.

Proposition 5.6.2. *Let $d \in \mathcal{D}$ be a Duflo involution, assume that Conjecture 5.2.15 holds for d, and let $\pi : \mathcal{S}oe \twoheadrightarrow \overline{\mathcal{S}oe}$ be an additive monoidal quotient functor. Then, $\pi(B_d) = 0$ if and only if $\pi(\alpha_d) = 0$.*

Proof. The implication $\pi(B_d) = 0 \Rightarrow \pi(\alpha_d) = 0$ is obvious. On the contrary, assume $\pi(\alpha_d) = 0$. Then, $\pi(\mu_d) = 0$ by Conjecture 5.2.15 and thus $\pi(\hat{B}_d) = \pi(\mu_d)\circ\pi(\Delta_d) = 0$. The result follows. □

Corollary 5.6.3. *Let $d \in \mathcal{D}$ be a Duflo involution, assume that Conjecture 5.2.15 holds for d, and let $\mathcal{M} = (\mathcal{M}, \rho) \in \mathcal{S}oe - \mathrm{Mod}$. Then, we have $\mathrm{Res}_{\pi_d}\mathrm{Ind}_{\pi_d}(\mathcal{M}) = \mathcal{M}$ if and only if $\rho(\alpha_d) = 0$.*

5.7 Reduction to non reflection faithful representations

In this section, we prove a generalisation of the main result of [Li]. More precisely, we show in Theorem 5.7.6 that Libedinsky's monoidal quotient functor $\mathcal{S}oe^\bullet \twoheadrightarrow \overline{\mathcal{S}oe}^\bullet$ between Soergel bimodules and Soergel bimodules constructed with a faithful, but not reflection faithful representation extends to a quotient 2-functor between the respective 2-categories of singular Soergel bimodules. For notational convenience, we suppress denoting the gradings explicitly in this section.

For the rest of the section, we fix a Coxeter system (W, S), a reflection faithful W-representation V and a subrepresentation V' of V. We denote by $R := \mathcal{O}(V)$ the ring of regular functions on V, and by $R^s := \{r \in R | s.r = r\}$ the ring of s-invariants for any $s \in S$. The following definition is due to Libedinsky:

Definition 5.7.1. *The pair (V, V') is called a* bonne paire *if and only if each $s \in S$ acts through reflections on both V and V', and given any sequence $\{s_i\}_{i=1}^n$ in S the homomorphism space $\mathrm{Hom}_{R-R}(R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} .... \otimes_{R^{s_n}} R, R)$ is graded free as a left R-module with graded rank given by the coefficient of the standard basis element $H_e \in \mathcal{H} = \mathcal{H}(W, S)$ when expressing the element $v^{-n}\underline{H}_{s_1} \cdot ... \cdot \underline{H}_{s_n} \in \mathcal{H}$ in the standard basis.*

The existence of bonne paires in the situation of interest had already been established in [Soe6]. We summarize some statements of this paper in the next proposition.

Proposition 5.7.2. *Over $\mathbb{R}$, there exists a reflection faithful representation V of (W,S) such that the geometric representation V' is a subrepresentation of V and (V,V') is a bonne paire. Furthermore, the minimal codimension of V' in V agrees with the number of indecomposable, infinite summands of (W,S).*

Denote by $\overline{\mathcal{S}oe}$ the monoidal category of Soergel bimodules constructed with V' (thus disregarding the part of the definition of Soergel bimodules which requires a reflection faithful representation) and by $\mathcal{S}oe$ the usual category of Soergel bimodules constructed using V. We denote $R := \mathcal{O}(V)$ and $R' := \mathcal{O}(V')$. Note that the inclusion $V' \hookrightarrow V$ induces an epimorphism $R \twoheadrightarrow R'$. Further, the functor $\mathcal{S}oe \to R' \otimes_{\mathbb{C}} R'-\mathrm{Mod}$ defined by $B \mapsto R' \otimes_R B \otimes_R R'$ factors over $\overline{\mathcal{S}oe}$, and we denote the induced functor by $\mathfrak{F} : \mathcal{S}oe \to \overline{\mathcal{S}oe}$.

Theorem 5.7.3 ([Li])**.** *Let M, $N \in \mathcal{S}oe$. The following statements hold:*

- $\mathfrak{F}(M \otimes_R N) \cong \mathfrak{F}(M) \otimes_{R'} \mathfrak{F}(N)$ *in* $\overline{\mathcal{S}oe}$.
- $R' \otimes_R \mathcal{S}oe(M,N) \cong \overline{\mathcal{S}oe}(\mathfrak{F}(M), \mathfrak{F}(N))$ *in* $R' - \mathrm{Mod}$.
- *M is indecomposable if and only if $\mathfrak{F}(M)$ is indecomposable.*
- *$\mathfrak{F}$ is essentially surjective.*
- *$\mathfrak{F}$ induces an isomorphism of graded split Grothendieck groups* $[\mathcal{S}oe] \xrightarrow{\sim} [\overline{\mathcal{S}oe}]$.

Libedinsky also states (without providing the details) that $\mathfrak{F}$ is a monoidal functor. In fact, it is not too hard to write down the isomorphism $\mathfrak{F}(M \otimes_R N) \cong \mathfrak{F}(M) \otimes_{R'} \mathfrak{F}(N)$ explicitly, as well as the associator and unit isomorphisms of $\mathcal{S}oe$ and $\overline{\mathcal{S}oe}$. Once this is done, a trivial computation establishes that $\mathfrak{F}$ carries indeed a monoidal structure. It follows that, $\overline{\mathcal{S}oe}$ also categorifies the Hecke algebra.

Let us reformulate Theorem 5.7.3. Following [Li], denote the quotient ring homomorphism $R \twoheadrightarrow R'$ by Q. The two-sided ideal of elements of R that vanish under Q is given by $I(Q) = R \cdot V'^{\perp}$ where $V'^{\perp} := \{r \in V^* | r(v) = 0 \;\; \forall v \in V'\}$. Since W acts trivially on V', we have $V'^{\perp} \subset R^W$. Thus, $R^W/(V'^{\perp} \cdot R^W) \otimes_{R^W} R \xrightarrow{\sim} R/I(Q)$ as R-right modules. Hence, $R^W/(V'^{\perp} \cdot R^W) \otimes_{R^W} M \xrightarrow{\sim} R/I(Q) \otimes_R M \cong R' \otimes_R M$ for any left R-module M. Denote $Z := \mathcal{Z}^{\otimes}(\mathcal{S}oe)$. By Proposition 5.3.7, we have $Z \cong R^W$, and we denote by $\overline{Z}$ the quotient ring of Z by the image of $V'^{\perp} \cdot R^W$. Clearly, we have a canonical monoidal quotient functor $\mathcal{S}oe \twoheadrightarrow \mathcal{S}oe \otimes_Z \overline{Z}$. By the above discussion, we also have $\left(\mathcal{S}oe \otimes_Z \overline{Z}\right)(M,N) = \mathcal{S}oe(M,N) \otimes_Z \overline{Z} \cong R' \otimes_R \mathcal{S}oe(M,N)$, which in turn is equivalent to $\overline{\mathcal{S}oe}(M,N)$ by Theorem 5.7.3.

Corollary 5.7.4. *The monoidal functor $\mathfrak{x}$ factors over the canonical monoidal quotient functor $\mathcal{S}oe \twoheadrightarrow \mathcal{S}oe \otimes_Z \overline{Z}$ and induces a monoidal equivalence $\mathcal{S}oe \otimes_Z \overline{Z} \cong \overline{\mathcal{S}oe}$.*

Combining this with Corollary 4.4.13, we obtain the following:

Corollary 5.7.5. *We have an equivalence* $\operatorname{Ind}_{\mathfrak{x}}(\mathcal{S}oe) \cong \overline{\mathcal{S}oe}$ *in* $\mathcal{S}oe - \mathrm{Mod}$.

Now, one can define singular Soergel bimodules $\overline{\mathfrak{S}oe}$ for $\overline{\mathcal{S}oe}$ in the same way as $\mathfrak{S}oe$ for $\mathcal{S}oe$. We now generalise Theorem 5.7.3 and show that $\overline{\mathfrak{S}oe}$ also categorifies the Schur algebroid:

Theorem 5.7.6. *The 2-functor* $\operatorname{Ind}_{\mathfrak{x}} : \mathcal{S}oe - \mathrm{Mod} \to \overline{\mathcal{S}oe} - \mathrm{Mod}$ *induces quotient functors on homomorphism categories of right singular Soergel bimodules. Expressed in terms of $\mathfrak{S}oe$ and $\overline{\mathfrak{S}oe}$ where $I, J, K \subset S$ are finitary, we have additive quotient functors ${}^I\pi^J : {}^I\mathcal{S}oe^J \twoheadrightarrow {}^I\overline{\mathcal{S}oe}^J$ such that the diagram*

$$\begin{tikzcd}
{}^I\mathcal{S}oe^J \times {}^J\mathcal{S}oe^K \arrow[r, "- \otimes_{R^J} -"] \arrow[d, "{}^I\pi^J \times {}^J\pi^K"] & {}^I\mathcal{S}oe^K \arrow[d, "{}^I\pi^K"] \\
{}^I\overline{\mathcal{S}oe}^J \times {}^J\overline{\mathcal{S}oe}^K \arrow[r, "- \otimes_{R^J} -"] & {}^I\overline{\mathcal{S}oe}^K
\end{tikzcd}$$

commutes up to an isotransformation. Furthermore each ${}^I\pi^J$ preserves and reflects indecomposable objects and induces an isomorphism on the graded split Grothendieck groups. In particular, Theorem 5.1.13 holds for $\overline{\mathcal{S}oe}$.

Proof. Recall from Section 4.4 the properties and explicit constructions of $\operatorname{Ind}_{\mathfrak{x}}$. In Section 5.3, we showed that $\mathcal{S}oe^I$ is the functorially cyclic module associated to the split Frobenius object $\hat{\Theta}_I := \hat{\Theta}_{w_0^I} = (\hat{B}_I, \Delta_I, \mu_I)$. The same arguments show that $\overline{\mathcal{S}oe}^I$ is associated to $\hat{\overline{\Theta}}_I := (\overline{B}_I, \Delta'_I, \mu'_I)$, where $\overline{B}_I := R' \otimes_{R'^I} R'$, $\Delta'_I : \overline{B_I} \to \overline{B_I} \otimes_{R'} \overline{B_I}$ is given by $x \otimes y \mapsto x \otimes 1 \otimes 1 \otimes y$ and $\mu'_I : \overline{B_I} \otimes_{R'} \overline{B_I} \to \overline{B_I}$ is given by $x \otimes y \otimes z \otimes w \mapsto \partial'_I(yz)x \otimes w$ with $\partial'_I(r) := \frac{1}{|W_I|} \sum_{x \in W_I} x.r$ for any $r \in R'$. Note that $\mathfrak{x}(B_I) = R' \otimes_R B_I \otimes_R R' \cong R' \otimes_{R^I} R' \cong R' \otimes_{R'^I} R' = \overline{B_I}$, where the first isomorphism is standard and the second follows, since the epimorphism $R \twoheadrightarrow R'$ induces and epimorphism $R^I \twoheadrightarrow R'^I$. Evaluating the composition $B'_I \xrightarrow{\sim} \mathfrak{x}(B_I) \xrightarrow{\mathfrak{x}(\Delta_I)} \mathfrak{x}(B_I \otimes_R B_I) \xrightarrow{\sim} \mathfrak{x}(B_I) \otimes_{R'} \mathfrak{x}(B_I) \cong B'_I \otimes_{R'} B'_I$ shows that it agrees with Δ'_I, and the same holds for the analogous composition involving $\mathfrak{X}(\mu_I)$ and μ'_I. Thus, $\mathfrak{X}(\hat{\Theta}_I)$ and $\hat{\overline{\Theta}}_I$ are compatible and $\mathrm{Ind}_{\mathfrak{X}}(\mathcal{S}oe^I) \cong \overline{\mathcal{S}oe}^I$ follows. Now, recall from Section 5.3 that ${}^I\mathcal{S}oe^J \cong \mathcal{H}om_{\mathcal{S}oe}(\mathcal{S}oe^I, \mathcal{S}oe^J)$. The same arguments show ${}^I\overline{\mathcal{S}oe}^J \cong \mathcal{H}om_{\overline{\mathcal{S}oe}}(\overline{\mathcal{S}oe}^I, \overline{\mathcal{S}oe}^J)$. Combining what we have shown thus far with the compatibility diagrams provided by the 2-functor Ind_π, we obtain the diagram claimed in the statement. In order to show that ${}^I\pi^J$ is a quotient functor in the case where both I and J are not the empty set, we have to work harder and invoke Section 4.8. It follows directly that ${}^I\pi^J$ is full, however it remains to check certain assumptions in order to deduce that ${}^I\pi^J$ is essentially surjective. Using the formulation of Corollary 4.7.5, it remains to check that the quotient functor π^J lifts idempotents with descent. The relevant Frobenius objects for this descent requirement are $\hat{\overline{\Theta}}^I$ on $\overline{\mathcal{S}oe}^J$ and $\hat{\Theta}_I$ on $\mathcal{S}oe^I$. By Conjecture 5.2.5 and its proof in Example 5.2.11 for the relevant case, $\mathcal{S}oe$ satisfies right $\hat{\Theta}_J$-descent, and by Theorem 5.7.3 the same holds for $\overline{\mathcal{S}oe}$ and right $\hat{\overline{\Theta}}^J$-descent. As already shown above this implies that π^J is a quotient functor. Furthermore, it also implies that the embeddings $\mathcal{S}oe^J \hookrightarrow \mathcal{S}oe$ and $\overline{\mathcal{S}oe}^J \hookrightarrow \mathcal{S}oe^J$ result in idempotent closed submodules. Thus, $[\pi^J]$ is an isomorphism on the level of split Grothendieck groups. By the same argument π lifts idempotents with descent for the left action of $\hat{\Theta}_I$ and $\hat{\overline{\Theta}}_I$. Now, every idempotent in $\overline{\mathcal{S}oe}^J$ that could satisfy $\hat{\overline{\Theta}}$-descent is mapped to an idempotent in $\overline{\mathcal{S}oe}$ where we know that it must satisfy $\hat{\overline{\Theta}}_I$-descent and thus has a lift in $\mathcal{S}oe$ that satisfies $\hat{\Theta}_I$-descent. By the homomorphism formula [Wi2, Theorem 7.4.1], this idempotent is of degree zero. Further [Wi2, Theorem 7.4.1] also implies that the functor $- \otimes_{R^J} R : \mathcal{S}oe^I \hookrightarrow \mathcal{S}oe$ induces a fully faithful embedding $\mathcal{S}oe^J \otimes_{R^J} R \hookrightarrow \mathcal{S}oe$, and we can deduce that the lifted idempotent actually comes from an idempotent in $\mathcal{S}oe^I$. Since we have a commutative diagram

$$\begin{tikzcd}
\mathcal{S}oe^J \arrow[r, hook] \arrow[d, "\pi^J"] & \mathcal{S}oe \arrow[d, "\pi"] \\
\overline{\mathcal{S}oe}^J \arrow[r, hook] & \overline{\mathcal{S}oe}
\end{tikzcd}$$

that comes from 2-functoriality of Ind_π where the horizontal arrow on the bottom is faithful, the idempotent is map to the corresponding idempotent in $\overline{\mathcal{S}oe}^J$ in a way compatible with the action of $\mathcal{S}oe$. Thus π^J lifts idempotents with descent and it follows that ${}^I\pi^J$ is essentially surjective. Furthermore using the same arguments, one obtains that $\mathcal{S}oe^J$ (resp. $\overline{\mathcal{S}oe}^J$) satisfies $\hat{\Theta}_I$ (resp. $\hat{\overline{\Theta}}^I$-descent). Therefore, ${}^I\mathcal{S}oe^J$ (resp. ${}^I\overline{\mathcal{S}oe}^J$) is an idempotent closed subcategory of $\mathcal{S}oe^J$ (resp. $\overline{\mathcal{S}oe}^J$). It follows that $\left[{}^I\pi^J\right]$ induces an isomorphism on the Grothendieck group. $\square$

The explicit construction of $\mathrm{Ind}_{\mathfrak{x}}$ implies the following:

Corollary 5.7.7. *For all finitary $I, J \subset S$, we have the following equivalences of additive categories* ${}^I\overline{\mathcal{S}oe}^J \cong {}^I\mathcal{S}oe^J \otimes_{R^J} R^W/(R^W)^+ \cong R^W/(R^W)^+ \otimes_{R^I} {}^I\mathcal{S}oe^J$.

Chapter 6

Applications to Group Theory

In this chapter, we discuss the higher representation theory of the additive monoidal category of representations of a group over a ring. Section 6.1 sets up the necessary notation. Section 6.2 discuses functorially cyclic modules and shows that there is only a single weak equivalence class of functorially cyclic modules provided an assumption that is satisfied for example if the order of the group is invertible in the base field. Independent of Section 6.2 and without invoking Tannakian formalism, Section 6.3 reconstructs the corresponding category of representations for all subgroups. Section 6.3 does not require us to work over a field, nor does it require any finiteness assumptions. Section 6.4 applies the additive Barr-Beck theorem to module categories that arise in the context of defect groups.

6.1 Basic notation

In this section, we fix the notation on which the remainder of Chapter 6 is based.

First, we fix a unital commutative ring k. Further, groups will typically be denoted by G, subgroups by $H \leq G$ and normal subgroups by $N \trianglelefteq G$.

Notation 6.1.1. Let G be a group. We denote by $\hat{G} = \hat{G}_k$ the *additive monoidal category of representations of* G *in* k*-modules.* An object of $\hat{G}$ is a k-module together with a group homomorphism from G into the k-module automorphisms of this k-module. A morphisms in $\hat{G}$ between two objects is a k-module homomorphisms between the underlying k-modules that intertwines the group homomorphisms. The tensor product of $\hat{G}$ will be denoted by $\otimes^G$. For V, $W \in \hat{G}$, the underlying k-module of $V \otimes^G W$ is given by the usual tensor product of modules over commutative rings $V \otimes_k W$ and is equipped with the diagonal G-action, *i.e.* $g(v \otimes_k w) := (gv) \otimes_k (gw)$ for all $g \in G$, $v \in V$ and $w \in W$. The unit object of $\hat{G}$ is given by the trivial representation k. The remaining structure of the monoidal category $\hat{G}$ is induced from the monoidal structure of $k-\mathrm{Mod}$.

Remark 6.1.2. We will consider $\hat{G}$ an additive monoidal category and nothing more. The definition given above suggests that as part of the structure we also have a forgetful functor to the category of k-modules (a so called fibre functor). We would like to point out that we will not use this structure. The explicit description of the category is however provided, so that we are able to establish the existence of certain morphisms and make calculations.

Remark 6.1.3. While the statements of in Section 6.2 require certain assumption on $\hat{G}$, Section 6.3 is completely independent of the cardinality of the groups involved. Also, both sections generalise to a groupoid G in which case $\hat{G}$ is a 2-category.

6.2 Functorially cyclic representations

In this section we discuses functorially cyclic modules for $\hat{G}$, where G is a group. Under the assumption that $\hat{G}$ has a single equivalence class for $\sim_{LR}$, Theorem 6.2.2 shows that there exists only a single weak equivalence class of functorially cyclic modules.

Example 6.2.1. Let G be a finite group, let k be a field, and assume that $|G|$ as well as the k-dimensions of all irreducible objects of $\hat{G}$ are invertible in k. Then for any $V \in \hat{G}$, we can pick a a non-trivial vector in V and consider the (irreducible) subrepresentation U that is generated by this element. Since $|G|$ is invertible, we can choose an invariant scalar product on V and deduce that U is a direct summand of V. Now, there exists a natural G-morphism $k \to U \otimes^G U^*$ (obtained from the identity morphism on U via the duality adjunction), as well as the G-morphism $U \otimes^G U^* \to k$ given by evaluation. The composition of these morphisms is equal to $\dim_k(U)$. Hence, $k \subset^{\oplus} V \otimes^G U^*$, and thus $\sim_{LR}$ has a single equivalence class.

Theorem 6.2.2. *Let G be a group and assume that $\hat{G}$ has a single equivalence class under $\sim_{LR}$. Then, all functorially cyclic modules of $\hat{G}$ are free and weakly equivalent to the principle representation $\hat{G} \in \hat{G} - \mathrm{Mod}$.*

Proof. Due to the assumption on $\sim_{LR}$, Lemma 6.2.3 implies that there exists no non-trivial $\hat{G}$-quotient of the principle representation of $\hat{G}$. Thus, any functorially cyclic module is free. Corollary 4.5.9 states that free functorially cyclic modules are completely determined by their associated split Frobenius object in $\hat{G}$. Theorem 4.5.14 completes the proof. $\square$

Lemma 6.2.3. *Let $\pi : \hat{G} \to \mathcal{M}$ be a $\hat{G}$-functor and let $V \in \hat{G}$ with $\pi(V) = 0$. If there exists $W \in \hat{G}$ and morphisms $k \to W \otimes^G V \to k$ with invertible composition, then $\pi = 0$.*

Proof. Let $V, W \in \hat{G}$ be as in the statement. Then, $\pi(W \otimes^G V) \cong W.\pi(V) = 0$, since π is a $\hat{G}$-functor and since the action of $\hat{G}$ on $\mathcal{M}$ is given by additive functors. Thus by functoriality of π, the identity morphism of $\pi(k)$ factors through a zero object. Thus, we have $\pi(k) = 0$. By the same argument for any $U \in \hat{G}$, we have $\pi(U) \cong \pi(U \otimes^G k) \cong U.\pi(k) = 0$ which completes the proof. $\square$

6.3 Subgroups

In this section we show that the additive monoidal category of representations of a subgroup of a group can be constructed directly from the additive monoidal category of representations of this group, see Theorem 6.3.6. Intuition for this equivalence can be obtained from Corollary 6.3.9 which provides a homomorphism formula for representation of the group that are restricted to the subgroup. Further in Theorem 6.3.11, we determine the endomorphism category of the category of representations of a normal subgroup when considered as a higher representation of the category of representations of the group. Comments on related research, namely Tannakian formalism and iso-categorical groups, are provided in Remarks 6.3.15 and 6.3.16.

Notation 6.3.1. Let G be a group and let $H \leq G$ be a subgroup. We denote by $\Theta_H := k[G/H] \in \hat{G}$ the permutation module on the G-set G/H. Note that Θ_H has a basis index by G/H, *i.e.* as k-modules we have $k[G/H] \cong \bigoplus_{\overline{g} \in G/H} k$. Denote by $\Delta_H : \Theta_H \to \Theta_H \otimes^G \Theta_H$ and $\mu_H : \Theta_H \otimes^G \Theta_H \to \Theta_H$ the morphisms in $\hat{G}$ determined by $\Delta_H(\overline{g}) := \overline{g} \otimes_k \overline{g}$ for all $g \in G$ and by $\mu_H(\overline{g}_1 \otimes_k \overline{g}_2) := \overline{g}_1$ if $g_2^{-1} g_1 \in H$ and $\mu_H(\overline{g}_1 \otimes_k \overline{g}_2) := 0$ otherwise for all $g_1, g_2 \in G$. Clearly, $\mu_H \circ \Delta_H = id_{\Theta_H}$ and it is an easy exercise to check that the Frobenius relation holds as well. Thus, we have constructed a split Frobenius object $\hat{\Theta}_H := (\Theta_H, \Delta_H, \mu_H)$ in $\hat{G}$.

Remark 6.3.2. Typically, Θ_H is not indecomposable. However, we will see below that $\hat{\Theta}_H$ is a right indecomposable Frobenius object.

Remark 6.3.3. Provided that H has finite index in G, Θ_H is finite dimensional. Under this assumption, all statements made in this section about $\hat{G}$ also apply to $k[G] - \text{mod}$, *i.e.* G-representations that are finitely generated over k.

Proposition 6.3.4. *Let G be a group and let $H \leq G$ be a subgroup. Then, $\hat{\Theta}_H$ is a symmetrically central Frobenius object.*

Proof. For all $V \in \hat{G}$, we have a morphism $s_V : V \otimes^G \Theta_H \to \Theta_H \otimes^G V$ in $\hat{G}$ that is determined by $v \otimes_k \overline{g} \mapsto \overline{g} \otimes_k v$ for all $g \in G$ and $v \in V$. All requirements of Definition 4.8.1 are now easily verified. $\square$

The following observation motivated this section as well as Section 4.8.

Remark 6.3.5. For all $V, W \in \hat{H}$, we have the following natural G-morphisms $\mathrm{Ind}_H^G(V \otimes^H W) \subset^{\oplus} \mathrm{Ind}_H^G(V \otimes^H \mathrm{Res}_H^G(\mathrm{Ind}_H^G(W))) \xrightarrow{\sim} \mathrm{Ind}_H^G(V) \otimes^G \mathrm{Ind}_H^G(W)$. The composition from the left to the right is given by $g \otimes_{k[H]} (v \otimes_k w) \mapsto (g \otimes_{k[H]} v) \otimes_k (g \otimes_{k[H]} w)$ and the map from the right to the left is given by $(g_1 \otimes_{k[H]} v) \otimes_k (g_2 \otimes_{k[H]} w) \mapsto g_1 \otimes_{k[H]} (v \otimes_k g_1^{-1} g_2 w) = g_2 \otimes_{k[H]} (g_2^{-1} g_1 v \otimes_k w)$ if $g_1^{-1} g_2 \in H$ and by 0 otherwise. The associated idempotent of $\mathrm{Ind}_H^G(V) \otimes^G \mathrm{Ind}_H^G(W)$ that factors over the direct summand $\mathrm{Ind}_H^G(V \otimes^H W)$ stabilises $(g_1 \otimes_{k[H]} v) \otimes_k (g_2 \otimes_{k[H]} w)$ if $g_1^{-1} g_2 \in H$ and annihilates it otherwise. Furthermore, it follows that $V \otimes^H W$ occurs as a summand of $\mathrm{Res}_H^G(\mathrm{Ind}_H^G(V) \otimes^G \mathrm{Ind}_H^G(W))$. Thus it is plausible that the tensor product of $\hat{H}$ can be recovered from the tensor product of $\hat{G}$.

We will now show that the additive monoidal category $\hat{H}$ is fully encoded in $\hat{\Theta}_H$. The general theory behind the following theorem was discussed in Section 4.8, and applies due to Proposition 6.3.4. The general versions of the functors $({}^{\hat{\Theta}_H}\vartheta, e)$ and ${}_{\hat{\Theta}_H}\vartheta$ used in the theorem below are also specified there.

Theorem 6.3.6. *Let G be a group, let $H \leq G$ be a subgroup. Then, $\hat{H}$ is a free functorially cyclic $\hat{G}$-module, $\hat{\Theta}_H$ is right indecomposable and we have a dominated equivalence $\hat{H} \cong \hat{G}_{\hat{\Theta}_H}$ in $\hat{G} - \mathrm{Mod}$. Moreover, we have natural additive inclusion and projection functors $\hat{G}_{\hat{\Theta}_H} \xhookrightarrow{({}^{\hat{\Theta}_H}\vartheta, e)} {}_{\hat{\Theta}_H}\hat{G}_{\hat{\Theta}_H} \xrightarrow{{}_{\hat{\Theta}_H}\vartheta} \hat{G}_{\hat{\Theta}_H}$ that identify $\hat{G}_{\hat{\Theta}_H}$ with a monoidal subcategory of ${}_{\hat{\Theta}_H}\hat{G}_{\hat{\Theta}_H}$. Thus, $\hat{G}_{\hat{\Theta}_H}$ carries a monoidal structure. Furthermore, the additive functors underlying the equivalence $\hat{H} \cong \hat{G}_{\hat{\Theta}_H}$ in $\hat{G} - \mathrm{Mod}$ can be equipped with the structure of a monoidal equivalence.*

Proof. First, we provide the structure of $\hat{H}$ as a free functorially cyclic $\hat{G}$-module. Second, we will show that associated right indecomposable split Frobenius object in $\hat{G}$ is isomorphic to $\hat{\Theta}_H$. By the third bijection in Theorem 4.5.4, it follows that there exists a dominated equivalence $\hat{H} \cong \hat{G}_{\hat{\Theta}_H}$ in $\hat{G} - \mathrm{Mod}$. Further by Proposition 6.3.4, Theorem 4.8.3 applies and we obtain the monoidal structure of $\hat{G}_{\hat{\Theta}}$. Third and last, write out the additive functor which is part of equivalence $\hat{G}_{\hat{\Theta}_H} \cong \hat{H}$ explicitly and observe that it is compatible with the monoidal structures.

We now go through the four remaining steps. Since $H \leq G$, restricting the action of G on any object of $\hat{G}$ to H yields an object in $\hat{H}$. Clearly, this defines a monoidal functor $\mathrm{Res}_H^G : \hat{G} \to \hat{H}$ with identity transformations as structure maps. Since we can regard $\hat{H}$ as the principal $\hat{H}$-representation, we obtain a $\hat{G}$-module $\mathrm{Res}_{\mathrm{Res}_H^G}(\hat{H})$, where we used the restriction 2-functor that was introduced in Section 4.4. By abuse of notation, we write $\hat{H} = \mathrm{Res}_{\mathrm{Res}_H^G}(\hat{H})$ and have already done so in the statement of the theorem. Let us denote the structure functor of this $\hat{G}$-module by ρ. Further, we have an additive functor $\mathrm{Ind}_H^G : \hat{H} \to \hat{G}$ that is given by $k[G] \otimes_{k[H]} U$ for all $U \in \hat{H}$, where G acts by multiplication on the left of the first tensor factor. Now, consider the diagram:

$$\begin{tikzcd}
\hat{G} \times \hat{H} \arrow[r, "\mathrm{Res}_H^G \times id_{\hat{H}}"] \arrow[rr, bend left, "\rho"] \arrow[d, "id_{\hat{G}} \times \mathrm{Ind}_H^G"] \arrow[dd, bend right, "id_{\hat{G}} \times id_{\hat{H}}"'] & \hat{H} \times \hat{H} \arrow[r, "\otimes^H"] & \hat{H} \arrow[d, "\mathrm{Ind}_H^G"'] \arrow[dd, bend left, "id_{\hat{H}}"] \\
\hat{G} \times \hat{G} \arrow[rr, "\otimes^G"] \arrow[d, "id_{\hat{G}} \times \mathrm{Res}_H^G"] & & \hat{G} \arrow[d, "\mathrm{Res}_H^G"'] \\
\hat{G} \times \hat{H} \arrow[r, "\mathrm{Res}_H^G \times id_{\hat{H}}"] \arrow[rr, bend right, "\rho"'] & \hat{H} \times \hat{H} \arrow[r, "\otimes^H"] & \hat{H}
\end{tikzcd}$$

The top rectangle commutes up to the isotransformation γ, which we provide for reference purposes as the separate Lemma 6.3.7 below. We obtain the isotransformations $V \otimes^G \mathrm{Ind}_H^G(-) \xrightarrow{\sim} \mathrm{Ind}_H^G(\mathrm{Res}_H^G(V) \otimes^H -)$ that are required to turn Ind_H^G into a $\hat{G}$-functor by evaluating the first factor of γ^{-1} at all $V \in \hat{G}$. These transformations are clearly natural in V. We have to work a little harder to show that these transformations are compatible with the structure functors of the $\hat{G}$-modules $\hat{G}$ and $\hat{H}$. As for the compatibility with the tensor products, consider the diagram

$$\begin{tikzcd}
\mathrm{Ind}_H^G(\mathrm{Res}_H^G(V_1 \otimes^G V_2) \otimes^H U) \arrow[r] \arrow[d, equal] & (V_1 \otimes^G V_2) \otimes^G \mathrm{Ind}_H^G(U) \arrow[r] & V_1 \otimes^G (V_2 \otimes^G \mathrm{Ind}_H^G(U)) \arrow[d] \\
\mathrm{Ind}_H^G((\mathrm{Res}_H^G(V_1) \otimes^H \mathrm{Res}_H^G(V_2)) \otimes^H U) \arrow[r] & \mathrm{Ind}_H^G(\mathrm{Res}_H^G(V_1) \otimes^H (\mathrm{Res}_H^G(V_2) \otimes^H U)) \arrow[r] & V_1 \otimes^G \mathrm{Ind}_H^G(\mathrm{Res}_H^G(V_2) \otimes^H U)
\end{tikzcd}$$

for all V_1, $V_2 \in \hat{G}$ and $U \in \hat{H}$. Given the explicit nature of Lemma 6.3.7, it is easily verified that this diagram commutes. As for the requirement related to the unit object, we leave this simpler check to the reader. Thus, we have defined the $\hat{G}$-functor Ind_H^G. Similar, yet much easier arguments show that there exists a $\hat{G}$-functor with underlying additive functor given by Res_H^G, amongst others because the bottom rectangle in the previous diagram commutes on the nose. Further into the right- (resp. left-) most regions of the previous diagram, we can fit $\hat{G}$-transformations $\eta : id_{\hat{H}} \Rightarrow \operatorname{Res}_H^G \operatorname{Ind}_H^G$ and $\zeta : \operatorname{Res}_H^G \operatorname{Ind}_H^G \Rightarrow id_{\hat{H}}$ (resp. $id_{\hat{G}} \times \eta$ and $id_{\hat{G}} \times \zeta$), which we discuss next. It will be obvious that $\zeta \circ \eta = id_{\hat{H}}$, and thus it will follow that $\hat{H}$ is a free functorially cyclic $\hat{G}$-module. Let $U \in \hat{H}$. We define $\eta_U : U \to k[G] \otimes_{k[H]} U$ by setting $\eta_U(u) := e \otimes_{k[H]} u$ for all $u \in U$, where e denotes the unit element of G. Clearly, η_U is a morphism in $\hat{H}$. Further, we define $\zeta_U : k[G] \otimes_{k[H]} U \to U$ by setting $\zeta_U(g \otimes_{k[H]} u) := gu$ if $g \in H$ and $\zeta_U(g \otimes_{k[H]} u) := 0$ otherwise for all $u \in U$ and $g \in G$. It is again obvious that ζ_U is a morphism in $\hat{H}$. It is also clear that both constructions are natural in U, since morphisms in $\hat{H}$ are $k[H]$-linear. Thus, we have constructed transformations η and ζ. It remains to show that η and ζ are $\hat{G}$-transformations. Since the argument for η is analogous and slightly easier, we only discuss ζ. For all $V \in \hat{G}$ and $U \in \hat{H}$, we have to show that $\zeta_{\operatorname{Res}_H^G(V) \otimes^H U} = (\operatorname{Res}_H^G(V) \otimes^H \zeta_U) \circ (\mathrm{can}_V)_U$, where can_V^{-1} denotes the transformation associated to V and the $\hat{G}$-functor $\operatorname{Res}_H^G \operatorname{Ind}_H^G$. Since the structure maps of Res_H^G are trivial, can_V^{-1} evaluated at U is given by $\operatorname{Res}_H^G(\gamma_{(V,U)})$, where we used the notation of Lemma 6.3.7. For all $g \in G$, $v \in V$ and $u \in U$, the left hand side sends $g \otimes_{k[H]} (v \otimes_k u)$ to $g(v \otimes_k u) = gv \otimes_k gu$ if $g \in H$ and to 0 otherwise. The right hand side sends the same element to $(\operatorname{Res}_H^G(V) \otimes^H \zeta_U)(gv \otimes_k (g \otimes_{k[H]} u)) = gv \otimes_k gu$ if $g \in H$ and 0 otherwise. We have completed the first step.

Second, we compare the Frobenius objects. Since $\hat{H}$ is functorially cyclic, $\hat{\Theta} := (\operatorname{Ind}_H^G \operatorname{Res}_H^G, \operatorname{Ind}_H^G \eta \operatorname{Res}_H^G, \operatorname{Ind}_H^G \zeta \operatorname{Res}_H^G)$ is an indecomposable split Frobenius object in $\hat{G}$. As usual, evaluation at the unit object $k \in \hat{G}$ yields a right indecomposable split Frobenius object with underlying object $\operatorname{Ind}_H^G \operatorname{Res}_H^G(k) = k[G] \otimes_{k[H]} k$. We have to be careful to identify the structure maps correctly. For this purpose, we have to write out where $\operatorname{Ind}_H^G \operatorname{Res}_H^G \operatorname{Ind}_H^G \operatorname{Res}_H^G$ gets send to (the finial object in the following chain), and keep track of the isomorphisms involve:

$$\begin{aligned}
\operatorname{Ind}_H^G \operatorname{Res}_H^G \operatorname{Ind}_H^G \operatorname{Res}_H^G(k) &\cong \operatorname{Ind}_H^G \operatorname{Res}_H^G \left(\operatorname{Ind}_H^G \operatorname{Res}_H^G(k) \otimes^G k\right) \\
&\cong \operatorname{Ind}_H^G \left(\operatorname{Res}_H^G \operatorname{Ind}_H^G \operatorname{Res}_H^G(k) \otimes^H \operatorname{Res}_H^G(k)\right) \\
&\cong \operatorname{Ind}_H^G \operatorname{Res}_H^G(k) \otimes^G \operatorname{Ind}_H^G \operatorname{Res}_H^G(k) && \text{(Lemma 6.3.7)}
\end{aligned}$$

For g_1, $g_2 \in G$, this chain acts on elements as follows:

$$\begin{aligned}
g_1 \otimes_{k[H]} (g_2 \otimes_{k[H]} 1) &\mapsto g_1 \otimes_{k[H]} ((g_2 \otimes_{k[H]} 1) \otimes_k 1) \\
&\mapsto g_1 \otimes_{k[H]} ((g_2 \otimes_{k[H]} 1) \otimes_k 1) \\
&\mapsto (g_1 g_2 \otimes_{k[H]} 1) \otimes_k (g_1 \otimes_{k[H]} 1).
\end{aligned}$$

Evaluating the first structure map $\operatorname{Ind}_H^G \eta \operatorname{Res}_H^G$ of $\hat{\Theta}$ at $k \in \hat{G}$ and composing it with this isomorphism, yields the diagonal map which we denote by Δ. Pre-composing the evaluation at $k \in \hat{G}$ of the second structure map $\operatorname{Ind}_H^G \zeta \operatorname{Res}_H^G$ of $\hat{\Theta}$ with this isomorphism, finally results in the associated right indecomposable split Frobenius object $(k[G] \otimes_{k[H]} k, \Delta, (g_1 \otimes_{k[H]} 1) \otimes_k (g_2 \otimes_{k[H]} 1) \mapsto g_2 \otimes_{k[H]} 1\delta_{g_2^{-1}g_1,H})$ in $\hat{G}$. Now, denote by τ the morphism $k[G] \otimes_{k[H]} k \to k[G/H]$ in $\hat{G}$, which is determined by $g \otimes_{k[H]} 1 \mapsto \overline{g}$ for all $g \in G$. In fact, τ is an isomorphism, since $gh \otimes_{k[H]} 1 = g \otimes_{k[H]} h1 = g \otimes_{k[H]} 1$ for all $g \in G$ and $h \in H$, where we used that $k = \operatorname{Res}_H^G(k)$ is the trivial H-module. Recalling Notation 6.3.1, it is now obvious that τ is also an isomorphism between the right indecomposable Frobenius object constructed above and $\hat{\Theta}_H$.

Third and last, we show that the equivalence $\hat{H} \cong \hat{G}_{\hat{\Theta}_H}$ in $\hat{G}-\mathrm{Mod}$ induces an equivalence of monoidal categories. Recall that in order to obtain the $\mathcal{B}$-functor that gives the equivalence $\hat{H} \xrightarrow{\sim} \hat{G}_{\hat{\Theta}_H}$, we need to apply an idempotent e to $\hat{H} \xrightarrow{\operatorname{Ind}_H^G} \hat{G} \xrightarrow{\vartheta^{\hat{\Theta}_H}} \hat{G}_{\hat{\Theta}_H}$. This idempotent is given by the vertical arrow on the left in the commutative diagram

$$\begin{tikzcd}
\vartheta^{\hat{\Theta}} \operatorname{Ind}_H^G \arrow[r, hook, "\alpha_H"] \arrow[d, "e"] & \vartheta^{\hat{\Theta}} \vartheta_{\hat{\Theta}} \vartheta^{\hat{\Theta}_H} \operatorname{Ind}_H^G \arrow[r, equal] & \vartheta^{\hat{\Theta}}(-\Theta_H) \operatorname{Ind}_H^G \arrow[r, "\sim"] & \vartheta^{\hat{\Theta}} \operatorname{Ind}_H^G \operatorname{Res}_H^G \operatorname{Ind}_H^G \arrow[d, "\vartheta^{\hat{\Theta}} \operatorname{Ind}_H^G e'"] \\
\vartheta^{\hat{\Theta}} \operatorname{Ind}_H^G & \vartheta^{\hat{\Theta}} \vartheta_{\hat{\Theta}} \vartheta^{\hat{\Theta}_H} \operatorname{Ind}_H^G \arrow[l, two heads, "\beta_H"'] \arrow[r, equal] & \vartheta^{\hat{\Theta}}(-\Theta_H) \operatorname{Ind}_H^G \arrow[r, "\sim"] & \vartheta^{\hat{\Theta}} \operatorname{Ind}_H^G \operatorname{Res}_H^G \operatorname{Ind}_H^G
\end{tikzcd}$$

where α_H and β_H denote the inclusion and projection $\hat{G}$-transformations associated the the free functorially cyclic module $\hat{G}_{\hat{\Theta}_H}$, and $e' = \beta \circ \alpha$ is the idempotent $\hat{G}$-transformation associated the the free functorially cyclic module $\hat{H}$. Calculating this composition explicitly, we obtain that $\hat{H} \xrightarrow{\sim} \hat{G}_{\hat{\Theta}_H}$ sends $U \in \hat{H}$ to $(\operatorname{Ind}_H^G(U), e_U)$ where $e_U((g_1 \otimes_{k[H]} u) \otimes_k \overline{g_2}) = (g_1 \otimes_{k[H]} u) \otimes_k \overline{g_2}\delta_{\overline{g_1},\overline{g_2}}$ for all $g_1,\, g_2 \in G$ and $u \in U$. By Proposition 4.8.13, $(\operatorname{Ind}_H^G(U_1), e_{U_1}) \otimes (\operatorname{Ind}_H^G(U_2), e_{U_2})$ is given by $\Big(\operatorname{Ind}_H^G(U_1) \otimes^G \Theta_H \otimes^G \operatorname{Ind}_H^G(U_2), (e_{U_1} \otimes_k e_{U_2}) \circ (id_{\operatorname{Ind}_H^G(U_2)} \widetilde{e}_{U_2})\Big)$ for all $U_1,\, U_2 \in \hat{H}$, where $\widetilde{e}_{U_2}(\overline{g_1} \otimes_k (g_2 \otimes_{k[H]} u) \otimes_k \overline{g_3}) = \overline{g_1} \otimes_k (g_2 \otimes_{k[H]} u) \otimes_k \overline{g_3}\delta_{\overline{g_1},\overline{g_3}}$ for all $g_1,\, g_2,\, g_3 \in G$ and $u \in U_2$. This idempotent maps $(g_1 \otimes_{k[H]} u_1) \otimes_k \overline{g_2} \otimes_k (g_3 \otimes_{k[H]} u_2) \otimes_k \overline{g_4} \mapsto (g_1 \otimes_{k[H]} u_1) \otimes_k \overline{g_2} \otimes_k (g_3 \otimes_{k[H]} u_2) \otimes_k \overline{g_4}\delta_{\overline{g_1},\overline{g_2}}\delta_{\overline{g_3},\overline{g_4}}\delta_{\overline{g_2},\overline{g_4}}$ for all $g_1,\, g_2,\, g_3 \in G$, $u_1 \in U_1$ and $u_2 \in U_2$. It is now easy to write down explicit isomorphisms using diagonal maps paired with Kronecker symbols from $\Big(\operatorname{Ind}_H^G(U_1) \otimes^G \Theta_H \otimes^G \operatorname{Ind}_H^G(U_2), (e_{U_1} \otimes_k e_{U_2}) \circ (id_{\operatorname{Ind}_H^G(U_2)} \widetilde{e}_{U_2})\Big)$ to $\big(\operatorname{Ind}_H^G(U_1) \otimes^G \operatorname{Ind}_H^G(U_2), e_{U_1} \otimes_k e_{U_2}\big)$ and further to $\big(\operatorname{Ind}_H^G(U_1 \otimes^H U_2), e_{U_1 \otimes^H U_2}\big)$ for all $U_1,\, U_2 \in \hat{H}$. It is easily verifiable that these isomorphisms actually satisfy the requirements imposed on morphisms in $\mathcal{B}_{\hat{\Theta}_H}$, and that they are natural in both arguments. The unit object k of $\hat{H}$ is send to $(\operatorname{Ind}_H^G(k), e_k)$. It is easily verified that this object is naturally isomorphic to $(\Theta_H, \Delta_H \circ \mu_H)$, which in turn is naturally isomorphic to $(k, id_{k\Theta_H})$ by means of Δ_H and μ_H. It is now fairly obvious that this structure equips the full faithful functor $\hat{H} \cong \hat{G}_{\hat{\Theta}_H}$ with the structure of a monoidal functor. This completes the proof. □

In the above proof, we made heavy use of the push-pull formula which we now recall. Note that we had already shown in Corollary 4.8.22, that this formula holds for $\hat{G}_{\hat{\Theta}_H}$. The reader may want to check that the concrete version below and the abstract version in Corollary 4.8.22 correspond to each other via the equivalence provided in Theorem 6.3.6.

Lemma 6.3.7. *For all $V \in \hat{G}$ and $U \in \hat{H}$, we assign an isomorphism in $\hat{G}$ denoted by $\gamma_{(V,U)} : \operatorname{Ind}_H^G(\operatorname{Res}_H^G(V) \otimes^H U) \xrightarrow{\sim} V \otimes^G \operatorname{Ind}_H^G(U)$ which is determined by $\gamma_{(V,U)}(g \otimes_{k[H]} (v \otimes_k u)) := gv \otimes_k (g \otimes_{k[H]} u)$ for all $g \in G$, $v \in V$ and $u \in U$. This assignment defines an isotransformation $\gamma : \operatorname{Ind}_H^G(\operatorname{Res}_H^G(-) \otimes^H -) \overset{\sim}{\Longrightarrow} - \otimes^G \operatorname{Ind}_H^G(-)$ of additive functors $\hat{G} \times \hat{H} \to \hat{G}$.*

Proof. Let $V \in \hat{G}$ and $U \in \hat{H}$. It is an easy exercise to check that $\gamma_{(V,U)}$ as introduced in the statement of this lemma is well defined. The (similarly well defined) inverse is determined by $\gamma_{(V,U)}^{-1}(v \otimes_k (g \otimes_{k[H]} u)) = g \otimes_{k[H]} (g^{-1}v \otimes_k u)$ for all $g \in G$, $v \in V$ and $u \in U$. Furthermore for $f_1 \in \hat{G}(V, V_2)$ and $f_2 \in \hat{H}(U, U_2)$, we have

$$
\begin{aligned}
&\big(\gamma_{(V_2,U_2)} \circ (\operatorname{Ind}_H^G(\operatorname{Res}_H^G(f_1) \otimes^H f_2))\big)\,(g \otimes_{k[H]} (v \otimes_k u)) \\
&\qquad = \gamma_{(V_2,U_2)}(g \otimes_{k[H]} (f_1(v) \otimes_k f_2(u))) \\
&\qquad = gf_1(v) \otimes_k (g \otimes_{k[H]} f_2(u))) \\
&\qquad = f_1(gv) \otimes_k (g \otimes_{k[H]} f_2(u))) \\
&\qquad = (f_1 \otimes^G \operatorname{Ind}_H^G(f_2))(gv \otimes_k (g \otimes_{k[H]} u))) \\
&\qquad = (f_1 \otimes^G \operatorname{Ind}_H^G(f_2)) \circ \gamma_{(V,U)}(g \otimes_{k[H]} (v \otimes_k u)).
\end{aligned}
$$

Thus, the isomorphisms indeed define a transformation as claimed. □

Remark 6.3.8. Instead of Ind_H^G, we could have attempted to use the coinduction functor $\operatorname{Coind}_H^G := \operatorname{Hom}_{k[H]}(k[G], -)$. However, it is not possible to equip Coind_H^G with the structure of a $\hat{G}$-functor. Nevertheless, it is possible to equip Coind_H^G with the structure of a lax-$\hat{G}$-functor. This relates to Remark 4.2.31 where we had explained that there is a larger class of 2-cyclic modules, which comprises functorially cyclic modules, and which may be called lax-functorially cyclic modules. The pair Coind_H^G and Res_H^G together with appropriate lax-$\hat{G}$-transformations shows that $\hat{H}$ is one such lax-functorially cyclic module.

Using the equivalence $\hat{H} \cong \hat{G}_{\hat{\Theta}}$, we can derive a new homomorphism formula for G-representations restricted to H. For this purpose, we equip $\prod_{G/H} \operatorname{Hom}_k(V_1, V_2)$

with the following G-action: For all $f = (f_{\overline{g'}})_{\overline{g'} \in G/H} \in \prod_{G/H} \mathrm{Hom}_k(V_1, V_2)$, we define $g.f = \left((g.f)_{\overline{g'}}\right)_{\overline{g'} \in G/H}$ by setting $(g.f)_{\overline{g'}}(v) := g f_{\overline{g^{-1}g'}}(g^{-1}v)$ for all g, $g' \in G$ and $v \in V_1$. Taking G-invariants yields the following formula:

Corollary 6.3.9. *Let* V_1, $V_2 \in \hat{G}$. *Then,* $\mathrm{Hom}_H(V_1, V_2) \cong \left(\prod_{G/H} \mathrm{Hom}_k(V_1, V_2)\right)^G$.

Proof. Let V_1, $V_2 \in \hat{G}$. Using our more detailed notation, we have $\mathrm{Hom}_H(V_1, V_2) = \hat{H}(\mathrm{Res}^G_H(V_1), \mathrm{Res}^G_H(V_2))$. Recall that we calculated a fully faithful functor $\hat{H} \xrightarrow{\sim} \hat{G}_{\hat{\Theta}_H}$ explicitly in the proof of Theorem 6.3.6. For $V \in \hat{G}$, it is easy to see that $\mathrm{Res}^G_H(V)$ is send to an object isomorphic to $(V \otimes^G \Theta_H, id_V \otimes^G e_H)$, where $e_H = \Delta_H \circ \mu_H$ is the idempotent associated to $\hat{\Theta}_H$. Further this object is isomorphic to $(V, id_{V \otimes^G \Theta_H})$. Thus, the statement follows from identifying $\hat{G}_{\hat{\Theta}_H}((V_1, id_{V_1 \otimes^G \Theta_H}), (V_2, id_{V \otimes^G \Theta_H}))$ with the right hand side in the statement. We will calculate such homomorphism spaces quite explicitly in the proof of Theorem 6.3.11. For now, it is sufficient to observe that right $\hat{\Theta}_H$-descent forces all morphisms f to stabilise the right factor of every element $V \otimes_k \overline{g}$ for all $v \in V_1$ and $\overline{g} \in G/H$. Thus for all $\overline{g} \in G/H$, there exists a k-linear map $f_{\overline{g}} : V_1 \to V_2$ such that $f(v \otimes_k \overline{g}) = f_{\overline{g}} \otimes_k \overline{g}$. Keeping track of the G-action, yields the result: Since $f_{\overline{g_1 g_2}}(g_1 v) \otimes_k \overline{g_1 g_2} = f(g_1 v \otimes_k \overline{g_1 g_2}) = f(g_1(v \otimes_k \overline{g_2})) = g_1 f(v \otimes_k \overline{g_2}) = g_1(f_{\overline{g_2}}(v) \otimes_k \overline{g_2}) = g_1 f_{\overline{g_2}}(v) \otimes_k \overline{g_1 g_2}$ and thus $f_{\overline{g_2}}(v) = g_1^{-1} f_{\overline{g_1 g_2}}(g_1 v)$ for all g_1, $g_2 \in G$ and $v \in V_1$, we obtain $(g.f)_{\overline{g'}}(v) = g f_{\overline{g^{-1}g'}}(g^{-1}v) = f_{\overline{g'}}(v)$ for all g, $g' \in G$ and $v \in V_1$ and thus $g.f = f$ for all $f \in \prod_{G/H} \mathrm{Hom}_k(V_1, V_2)$ that lie in the image of the assignment. We leave it to the reader to check that this assignment surjects onto the G-invariant morphism in $\mathrm{Hom}_k(V_1, V_2)$. $\square$

Let us explain the isomorphism of Corollary 6.3.9 in two examples:

Example 6.3.10. Let V_1, $V_2 \in \hat{G}$. For $H = \{e\}$, the action of G on G/H is free and transitive. Thus, all elements of $(\prod_G \mathrm{Hom}_k(V_1, V_2))^G$ are uniquely determined by their value on any factor $\mathrm{Hom}_k(V_1, V_2)$ and all morphisms in $\mathrm{Hom}_k(V_1, V_2)$ extend to an of $(\prod_G \mathrm{Hom}_k(V_1, V_2))^G$. For $H = G$, we have $(\prod_{G/H} \mathrm{Hom}_k(V_1, V_2))^G = (\mathrm{Hom}_k(V_1, V_2))^G$, and the subset of G-invariants of $(\prod_{G/H} \mathrm{Hom}_k(V_1, V_2))^G$ is is exactly the same as the definition of $\mathrm{Hom}_G(V_1, V_2)$ as a subset of $\mathrm{Hom}_k(V_1, V_2)$.

In the case of a normal subgroup $N \trianglelefteq G$, we can improve upon Theorem 6.3.6. Let us denote by $\bigoplus_{G/N} \hat{N}$ the monoidal category whose tensor product is defined using the tensor product of $\hat{N}$ twisted by the group multiplication of G/N. By this we mean that objects are given by $\mathrm{Ob}\left(\bigoplus_{G/N} \hat{N}\right) = \{(V, \overline{g}) | V \in \hat{N}, \overline{g} \in G/N\}$, $\bigoplus_{G/N} \hat{N}((V_1, \overline{g}_1), (V_2, \overline{g_2})) = \delta_{\overline{g}_1, \overline{g_2}} \hat{N}(V_1, V_2)$ and $(V_1, \overline{g}_1) \otimes (V_2, \overline{g_2}) = (V_1 \otimes^N V_2, \overline{g_1 g_2})$ for all $(V_1, \overline{g}_1)$, $(V_2, \overline{g}_2) \in \bigoplus_{G/N} \hat{N}$.

Theorem 6.3.11. *Let k be a field, let G be a group and let $N \trianglelefteq G$ be a normal subgroup. Then, we have an equivalence $\bigoplus_{G/N} \hat{N} \cong \mathcal{E}nd_{\hat{G}}(\hat{N})$ of additive monoidal categories.*

Proof. Let G be a group and let $N \trianglelefteq G$. By Theorem 6.3.6, we have $\hat{N} \cong \hat{G}_{\hat{\Theta}_H}$, and by Theorem 4.5.10 we have a monoidal equivalence $\mathcal{E}nd_{\hat{G}}(\hat{G}_{\hat{\Theta}_N}) \cong {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$. Thus, it suffice to show that we have an equivalence of additive monoidal categories $\bigoplus_{G/N} \hat{G}_{\hat{\Theta}_N} \cong {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$. For readability, we will omit the associator of, as well as the brackets in $\hat{G}$. The proof consists of five steps: First, we provide an explicit description of all morphisms in ${}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}((V, id_{\Theta_N V \Theta_N}), (U, id_{\Theta_N U \Theta_N}))$ for all V, $U \in \hat{G}$. Second, we define an idempotent ${}^{\overline{g}}_{V}e$ of $(V, id_{\Theta_N V \Theta_N}) \in {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$ for all $\overline{g} \in G/N$ and $V \in \hat{G}$. Third, we show that this defines a family indexed by G/N of orthogonal idempotents ${}^{\overline{g}}e \in \mathcal{Z}({}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N})$, where we consider ${}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N} \in \mathfrak{A}dd$ or in other words the transformations ${}^{\overline{g}}e$ are not monoidal transformations. Thus, we obtain a direct sum decomposition ${}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N} = \bigoplus_{\overline{g} \in G/N} {}^{\overline{g}}e\left({}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}\right)$. Fourth, we show that the exists a natural isomorphisms $(V_1, {}^{\overline{g_1}}_{V_1}e) \otimes (V_2, {}^{\overline{g_2}}_{V_2}e) \cong (V_1 \otimes^G V_2, {}^{\overline{g_1 g_2}}_{V_1 \otimes^G V_2}e)$. In particular, we can deduce that taking tensor product on the left with $(k, {}^{\overline{g_1}}_{k}e)$ induces an equivalence ${}^{\overline{g_2}}e\left({}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}\right) \to {}^{\overline{g_1 g_2}}e\left({}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}\right)$. Fifth and last, we show that the component associated to ${}^{\overline{e}}e$ is equivalent to $\hat{G}_{\hat{\Theta}_N}$, whereby the proof will be complete. In the remainder of this proof, we will use the Einstein summation convention.

In the the first step, let $n := |G/N|$ and let us fix an enumeration $\{\overline{g_1}\}_{i=1}^{n}$ of G/N. Also, let us encode the left (resp. right) action of G on G/N through this enumeration, *i.e.* we denote by gi and ig the unique natural number such that $g\overline{g_i} = \overline{g_{gi}}$ and $\overline{g_i}g = \overline{g_{ig}}$. Let $V, U \in \hat{G}$, and fix k-bases $\{v_m\}$ and $\{u_n\}$ of the respective underlying k-vector spaces. For all $f \in {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}((V, id_{\Theta_N V \Theta_N}), (U, id_{\Theta_N U \Theta_N}))$, we define coefficients $\{f_{i,m,j}^{k,n,l}\}_{i,j,k,l,m}$ in k, which are uniquely determined the equations $\overline{g_i} \otimes_k v_m \otimes_k \overline{g_j} = f_{i,m,j}^{k,n,l}\overline{g_k} \otimes_k u_n \otimes_k \overline{g_l}$ for all valid i, m, j. It is easy to verify that these coefficients satisfy the following two equations and that any family of elements in k that does satisfy these two equations defines in turn a morphism in ${}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}((V, id_{\Theta_N V \Theta_N}), (U, id_{\Theta_N U \Theta_N}))$. First, we have $f_{i,m,j}^{k,n,l} = f_{i,m,j}^{i,n,j}\delta_{i,k}\delta_{j,l}$ for all valid i, j, k, l, m, n, since f satisfies two sided $\hat{\Theta}_N$-descent. Second, we have $f_{i,m,j}^{k,n,l} = {}_V g_n^{n'} f_{gi,m,gj}^{gk,n',gl} {}_U(g^{-1})_{m'}^{m}$ for all valid i, j, k, l, m, n, since f is $k[G]$-linear, where we denoted by ${}_V g$ the action of g on V and expressed this linear map in terms of the basis of V, and analogously for U. Further for all $f \in {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}((V, id_{\Theta_N V \Theta_N}), (U, id_{\Theta_N U \Theta_N}))$ and $g \in {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}((U, id_{\Theta_N U \Theta_N}), (W, id_{\Theta_N W \Theta_N}))$, we have $(g \circ f)_{i,m,j}^{k,n,l} = g_{k',n',l'}^{k,n,l} \circ f_{i,m,j}^{k',n',l'}$ for all valid i, j, k, l, m, n.

In the second step for all $\overline{g} \in G/N$ and $V \in \hat{G}$, we define an idempotent ${}^{\overline{g}}_{V}e$ of $(V, id_{\Theta_N V \Theta_N}) \in {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$ by setting ${}^{\overline{g}}_{V}e^{k,m',l}_{i,m,j} := \delta_{k,i}\delta_{j,l}\delta_{m,m'}\delta_{ig,j}$. Note that the second index of the last Kronecker symbol is well defined since ig is independent of the representative $g \in G$ of $\overline{g}$ chosen. Further, note that it is this Kronecker symbol that ensures that the second equation above is satisfied. Furthermore, note that the composition rule easily implies the thus defined morphisms are idempotents. Finally, note that $\sum_{\overline{g} \in G/N} {}^{\overline{g}}_{V}e = id_{\Theta_N V \Theta_N}$, since $(id_{\Theta_N V \Theta_N})^{k,m',l}_{i,m,j} = \delta_{i,k}\delta_{j,l}\delta_{m,m'}$ for all valid i, j, k, l, m, m'.

In the third step for all $\overline{g} \in G/N$, we define orthogonal idempotents ${}^{\overline{g}}e \in \mathcal{Z}({}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N})$ by setting $({}^{\overline{g}}e)_{(V,f)} := {}^{\overline{g}}_{V}e \circ f$ for all $(V, f) \in {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$. This definition is well defined, since the above presented composition rule easily implies that ${}^{\overline{g}}_{U}e \circ f = f \circ {}^{\overline{g}}_{V}e$ for all $f \in {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}((V, id_{\Theta_N V \Theta_N}), (U, id_{\Theta_N U \Theta_N}))$. In particular for all $V \in \hat{G}$ and $\overline{g} \in G/N$, the idempotent ${}^{\overline{g}}_{V}e$ commutes with all endomorphisms (and importantly idempotents) of $(V, id_{\Theta_N V \Theta_N})$. Therefore the following notation is well defined: For all $f \in {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}((V, f_1), (U, f_2))$ and $\overline{g} \in G/N$, we denote ${}^{\overline{g}}f := f \circ {}^{\overline{g}}_{V}e = {}^{\overline{g}}_{U}e \circ f$. We have obtained for all $\overline{g} \in G/N$, the morphism $({}^{\overline{g}}e)_{(V,f)} = {}^{\overline{g}}f$ for all $(V, f) \in {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$, which is an idempotent, and thus shown that the idempotent transformation ${}^{\overline{g}}e$ is well defined. Further, it also follows from the composition rule that these idempotents are orthogonal, *i.e.* that ${}^{\overline{g_1}}e \circ {}^{\overline{g_2}}e = {}^{\overline{g_1}}e\, \delta_{\overline{g_1},\overline{g_2}}$ for all $\overline{g_1}$, $\overline{g_2} \in G/N$. Furthermore, it follows from the second step that $\sum_{\overline{g} \in G/N} {}^{\overline{g}}e = id_{{}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}}$ and therefore we obtain the decomposition ${}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N} = \bigoplus_{\overline{g} \in G/N} {}^{\overline{g}}e\left({}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}\right)$ of additive categories.

In the fourth step for all (V_1, f_1), $(V_2, f_2) \in {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$, we provide natural isomorphisms $(V_1, {}^{\overline{g_1}}f_1) \otimes (V_2, {}^{\overline{g_2}}f_2) \cong (V_1 \otimes^G V_2, {}^{\overline{g_1 g_2}}f_{12})$ in ${}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$, where f_{12} is defined by setting $(f_{12})^{k,(n',m'),l}_{i,(n,m),j} := (f_1)^{k,n',kg_2}_{i,n,kg_2}(f_2)^{kg_2,m',l}_{kg_2,m,l}\,\delta_{ig_1g_2,j}$ for all valid i, j, k, l, (n, m), (n', m'). Note that we have used the indices n, n' (resp. m, m') for the the basis of V_1 (resp. V_2) and that the basis of $V_1 \otimes^G V_2$ is therefore parametrised by pairs (n, m), (n', m'). By Proposition 4.8.13, we have $(V_1, {}^{\overline{g_1}}f_1) \otimes (V_2, {}^{\overline{g_2}}f_2) = \left(V_1 \otimes^G \Theta_N \otimes^G V_2, \widetilde{f_{12}}\right)$ for all (V_1, f_1), $(V_2, f_2) \in {}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$, where $\widetilde{f_{12}} := ({}^{\overline{g_1}}f_1 \otimes_k id_{V_2 \otimes^G \Theta_N}) \circ (id_{\Theta_N \otimes^G V_1} \otimes_k {}^{\overline{g_1}}f_2)$. Using the basis $\{\overline{g_i} \otimes_k v^1_n \otimes_k \overline{g_j} \otimes_k v^2_m \otimes_k \overline{g_k}\}_{i,(n,j,m),k}$ of $\Theta_N \otimes^G V_1 \otimes^G \Theta_N \otimes^G V_2 \otimes^G \Theta_N$, we obtain $\left(\widetilde{f_{12}}\right)^{i',(n',j',m'),k'}_{i,(n,j,m),k} = (f_1)^{i',n',j'}_{i,n,j}(f_2)^{j',m',k'}_{j,m,k}\,\delta_{ig_1,j}\,\delta_{jg_2,k}$ for all valid indices. Now, the isomorphism $(V_1 \otimes^G V_2, id_{\Theta_N \otimes^G V_1 \otimes^G V_2 \otimes^G \Theta_N}) \xrightarrow{\sim} (V_1 \otimes^G \Theta_N \otimes^G V_2, {}^{\overline{g_1}}_{V_1}e \otimes_k id_{V_2 \otimes^G \Theta_N})$ in ${}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$ given on the basis by $\overline{g_i} \otimes_k v^1_n \otimes_k v^2_m \otimes_k \overline{g_k} \mapsto \overline{g_i} \otimes_k v^1_n \otimes_k \overline{g_{ig_1^{-1}}} \otimes_k v^2_m \otimes_k \overline{g_k}$ intertwines $\widetilde{f_{12}}$ and ${}^{\overline{g_1 g_2}}f$. Thus, ${}^{\overline{g_1 g_2}}f$ is indeed a well defined morphism. We leave it to the reader to check that the thus provide family of isomorphisms is natural.

It now follows that taking tensor product on the left with $(k, {}^{\overline{g_1}}_{k}e)$ induces an equivalence ${}^{\overline{g_2}}e\left({}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}\right) \xrightarrow{\sim} {}^{\overline{g_1 g_2}}e\left({}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}\right)$. Now, recall the definition of the tensor product on $\bigoplus_{\overline{g}\in G/N}\hat{N}$ defined directly before stating this theorem. Applying same construction to ${}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$ rather than to $\hat{N}$, yields an additive monoidal category $\bigoplus_{\overline{g}\in G/N} {}^{\overline{e}}e\left({}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}\right)$, which is equivalent to ${}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}$ by the arguments just given.

In the fifth and last step, it remains to show that the component ${}^{\overline{e}}e\left({}_{\hat{\Theta}_N}\hat{G}_{\hat{\Theta}_N}\right)$ associated to ${}^{\overline{e}}e$ is equivalent to $\hat{G}_{\hat{\Theta}_N}$. This follows directly from the isotransformation $({}_{\hat{\Theta}}\vartheta\,{}^{\hat{\Theta}}\vartheta,\ {}_{\hat{\Theta}}\vartheta e) \cong id_{\mathcal{B}_{\hat{\Theta}}}$ of Proposition 4.8.10, since the idempotent $e\,{}_{\hat{\Theta}}\vartheta$ on the composition in the other order ${}^{\hat{\Theta}}\vartheta\,{}_{\hat{\Theta}}\vartheta$ agrees exactly with ${}^{\overline{e}}e$. This completes the proof. $\square$

Remark 6.3.12. One can think of Theorem 6.3.11 in the following way: Any endo-$\hat{N}$-functor of $\hat{G}$ is given by right multiplication with some representation $V \in \hat{N}$. The representations V can be embedded in $|G/N|$-ways into $k[G/H] \otimes^N V \cong \operatorname{Res}\operatorname{Ind}(V)$. These different embedding give rise to different structures of $- \otimes_H V$ as a $\hat{G}$-functor.

Let us comment on two tasks that remain open:

Remark 6.3.13. It would be interesting to study whether for all right indecomposable split Frobenius objects $\hat{\Theta}$ in $\hat{G}$, there exists a subgroup $H \leq G$ such that $\hat{\Theta}_H$ is isomorphic to $\hat{\Theta}$ or weaker that there exists a strong equivalence $\hat{\Theta}_H \simeq \hat{\Theta}$. We have not looked into this.

Remark 6.3.14. The symmetry of $\hat{G}$ has only been used to show that $\hat{\Theta}_H$ is symmetrically central. It should be possible to construct the symmetry of $\hat{H}$ from the symmetry of $\hat{G}$ either directly or in the more abstract setting of Section 4.8.

We conclude this section by providing some context for this section from the literature.

Remark 6.3.15. Recall that for finite groups (and more generally in the case of affine group schemes over a fixed base filed), the group can be recovered from its monoidal category of representations by virtue of Tannakian formalism, see [DeMi]. To obtain this result, either the fibre functor to vector spaces (the group algebra is isomorphic to monoidal autotransformations of the fibre functor) or the additional structure as symmetric monoidal category is required. In this section, we mainly consider the structure as an additive monoidal category and had no need to reconstruct the group. However, we used the groups involved and the symmetry of $\hat{G}$ in order to construct $\hat{\Theta}_H$ and show its properties in Proposition 6.3.4.

Remark 6.3.16. Another aspect to keep in mind when putting this section into context, is the work [EtGe]. Therein, two finite groups are defined to be iso-categorical if their respective monoidal categories of complex finite dimensional representations are equivalent as monoidal categories. For example, simple groups or groups of order either $2k+1$ or $2*(2k+1)$ for $k \in \mathbb{N}_0$ are isomorphic if an only if they are iso-categorical. More precisely [EtGe] shows that given two non-isomorphic iso-categorical groups, both groups have an Abelian normal subgroup of order 2^{2m} for some $m \in \mathbb{N}$ such that the second group cohomology of the quotient of the group by this Abelian normal subgroup with coefficients in this Abelian normal subgroup is non-trivial. Furthermore, iso-categorical groups can be constructed from one another by twisting the multiplication by some non-trivial element of the above mentioned cohomology group.

6.4 Defect groups

In this section, we merely note that the additive Barr-Beck theorem can be applied in the context of defect groups. This gives a new construction of certain additive categories involved. We hope that by noting this here, we will stipulate future research.

Theorem 6.4.1. *Let k be a field of positive characteristic, let G be a finite group, let $B = k[G]b$ be a block of $k[G]$ and let $D \leq G$ be a defect group of B. Then, there exists a split Frobenius object in $k[D]-mod$ such that the associated additive category is equivalent to $k[G]b-mod$. In other words, all information about the module category of the block is contained in the module category of the defect group.*

Proof. There are various places from which one can extract the necessary ingredient, see for example [Lin, Theorem 2.1]. A reference that makes it straightforward to apply our theory is [Rou3, Proposition 4.7]. Its proof contains the following statement: A defect group D of B is a (smallest) subgroup such that the identity functor of $B-\mathrm{mod}$ is a direct summand of $\mathrm{Ind}_D^G \circ \mathrm{Res}_D^G$. Now, Theorem 4.3.26 applies. $\square$

Remark 6.4.2. This theorem uses the additive Barr-Beck theorem only. There is however an obvious generalisation of this theorem for $k[D]-\mathrm{mod}$ and $k[G]b-\mathrm{mod}$ considered as objects in $k[G]-\mathrm{mod}$.

Chapter 7

Future Directions

In this chapter, we discuss two questions that we believe deserve the attention of future research. Let us also remind the reader of the conjectures in Section 5.2 that if true would strongly increase the relevance of this work, of the open questions related to Jordan-Hölder theory in Section 4.9.2 and particularly the envisaged base change over endomorphism categories in Remark 4.9.20.

7.1 Passage between monoidal and 2-categories

One can merge a given additive 2-category $\mathfrak{A}$ into a single monoidal category $\mathcal{B}$ by formally adding a unit object. If $\mathfrak{A}$ has is infinitely many objects, this results in an infinite rank endomorphism ring of the unit object. However, it should be possible to determine a finiteness conditions on $\mathcal{B}$-modules such that the associated 2-category $\mathcal{B} - \mathrm{Mod}^{\mathrm{fc}}$ is equivalent to $\mathfrak{A} - \mathrm{Mod} \cong \mathcal{B} - \mathrm{Mod}^{\mathrm{fc}}$.

More intriguingly, let $\mathfrak{A}$ be a 2-category with an object $0 \in \mathfrak{A}$ maximal with respect to $\leq_{LR}$, see Definition 4.2.12. Now define $\mathcal{B} := \mathfrak{A}(0,0)$. Clearly, we have an inclusion 2-functor $\iota : \mathcal{B} \to \mathfrak{A}$. It would be important to have a good understanding of Res_ι. Any answer is likely to involve the construction of Ind_ι and should also take Coind_ι into account. Further, the embedding $\mathcal{B} \to \mathfrak{Frob}^{\mathrm{rev}}(\mathcal{B})$ might factors over the inclusion $\mathcal{B} \to \mathfrak{A}$ and one would like to understand the precise relation between $\mathfrak{Frob}(\mathcal{B})^{\mathrm{rev}}$ and $\mathfrak{A}$ (as well as $\mathfrak{Frob}(\mathfrak{A})^{\mathrm{rev}}$). This question is plausible since we assumed that $0 \in \mathfrak{A}$ is a maximal object. A starting point would be to study the properties of the functor $\mathfrak{A}^{\mathrm{rev}}(a, a') \to \mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathfrak{A}(0,a), \mathfrak{A}(0,a'))$ given by right multiplication.

More generally, it would be interesting to extend $\mathfrak{Frob}$ to an endo-2-functor of $2 - \mathfrak{A}dd$ and to study the relation between this 2-functor and its square.

7.2 Twisted singular Soergel bimodules

In this section, we briefly discuss twisted singular Soergel bimodules ${}^I\mathcal{S}oe^J_G$. Twisted singular Soergel bimodules were introduced in [Kl] an extension of singular Soergel bimodules. The objective of this section is to introduce interesting categories that arise when putting twisted singular Soergel bimodules into the context of higher representation theory and thereby to stipulate future research.

According to the geometric Satake correspondence [Lu1, Gi, MiVi] and results known to experts, the subcategory of ${}^S\mathcal{S}oe^S$ of unshifted singular Soergel bimodules for the affine Weyl group $\widetilde{W}$, where S denotes the generators of the finite Weyl group, forms a monoidal category after shifting the tensor product. Furthermore, considering only degree zero morphisms one obtains a monoidal category that is equivalent to the category of finite dimensional representations of the associated semi-simple adjoint algebraic group. Since the geometric Satake correspondence also holds for the simply connected cover (and any intermediate quotients), there should exist an enlargement of ${}^S\mathcal{S}oe^S$ that is similarly related to representation theory. This task was put forward in [Wi1]. In [Kl], we conjectured that ${}^S\mathcal{S}oe^S_G$ provides the answer and established this result for PGL_2. Later, [Do] independently constructed ${}^S\mathcal{S}oe^S_G$ (for affine Weyl groups) and established the connection to geometry.

Let us provide some more details on twisted singular Soergel bimodules. Following [Kl], denote by G a subgroup of the automorphisms of W that stabilize S, *i.e.* $G \leq \mathrm{Aut}(W, S)$. In fact, we will see below that one should allow larger class of groups G which act on R by ring homomorphisms such that $g(R^J) = R^{g(J)}$ for all finnitary I, $J \subset S$. One can use this action to obtain equivalences $\vartheta_g : {}^I\mathcal{S}oe^J \to {}^{g(I)}\mathcal{S}oe^{g(J)}$. One now defines twisted singular Soergel bimodules analogously to [Wi2, Definition 7.1.1] as an idempotent closed additive subcategory of $R^I \otimes^{R^W} R^J - \mathrm{Mod}$, allowing additionally for twist on the left and right side that come from the group action. In the case that none of the elements of $G\backslash\{e\}$ act in the same way as a non-trivial inner automorphisms of W, it is shown in [Kl] that ${}^I\mathcal{S}oe^I_G \cong \bigoplus^I_{g\in G} \mathcal{S}oe^{g(I)}$. This equivalence is an additive monoidal equivalence, where the tensor product on the right is defined using the functors ϑ_g for $g \in G$. In the general case, we have an essentially surjective, faithful monoidal functor $\bigoplus_{g\in G}^{I} \mathcal{S}oe^{g(I)} \to {}^{I}\mathcal{S}oe^{I}_{G}$.

In this thesis, we have typically considered right singular Soergel bimodules as modules for the original Soergel biodules. This can be regarded as a global to local step. Now, we discuss the local to global direction. Let $I \subset S$ be finitary and assume that the complement $S \setminus I$ is finitary as well. We are interested in a particular endomorphism category in ${}^{I}\mathcal{S}oe^{I} - \mathrm{Mod}$: $\mathcal{E}nd_{{}^{I}\mathcal{S}oe^{I}}\left(\bigoplus_{g\in\overline{G}}^{I} \mathcal{S}oe^{g(S\setminus I)}\right)$ where $\overline{G} := G/Stab_G(S \setminus I)$ avoids double counting. Clearly, we have an faithful monoidal functor $\left({}^{I}\mathcal{S}oe^{I}_{G}\right)^{\mathrm{rev}} \to \mathcal{E}nd_{{}^{I}\mathcal{S}oe^{I}}\left(\bigoplus_{g\in\overline{G}}^{I} \mathcal{S}oe^{g(S\setminus I)}\right)$. We propose to study the properties of this comparison functor and to determine when it is full or essentially surjective. To promote interest into this question, let us present three special cases.

The first case is $I = \emptyset$ and thus has rather simple and in the context of this thesis very well understood right hand side. One the left hand side ${}^{S}\mathcal{S}oe^{S}_{G}$ has non-isomorphic objects provided that G contains non-trivial inner automorphisms of W. This does occur for type A_n. In this case the functor is not full. This example shows that we have to exclude certain group actions. However, one has to be cautions when drawing conclusions for the other cases.

The second case is $I = S$. Here, the right hand side is given by $\mathcal{E}nd_{{}^{S}\mathcal{S}oe^{S}}\left({}^{S}\mathcal{S}oe\right)$. Clearly, there is a faithful embedding from $\mathcal{S}oe$ into this category. This observation was inspired by [Os2] and prompts us to ask whether there is an equivalence with $\mathcal{S}oe$. Note that from a classical viewpoint this is a very surprising question, since $[{}^{S}\mathcal{S}oe^{S}] = [{}^{S}\mathcal{S}oe]$ cannot see this feature. Thus, we are dealing with a truly categorical question within higher representation theory. In type A_1, one quickly sees that the comparison functor is fully faithful, but does not attain the second indecomposable object. Now, S_2 acts non-trivially on R giving rise to a twist by -1. In this way $\mathcal{S}oe_{S_2}$ actually agrees with $\mathcal{E}nd_{{}^{S}\mathcal{S}oe^{S}}\left({}^{S}\mathcal{S}oe\right)$.

Third, denote s_0 the affine reflection in an affine Weyl group and let S denote the reflections generating the finite Weyl group. Then, $\mathcal{E}nd_{{}^{\{s_0\}}\mathcal{S}oe^{\{s_0\}}}\left(\bigoplus_{g\in\overline{G}}^{\{s_0\}} \mathcal{S}oe^{g(S)}\right)$ contains $|G|$ copies of our candidate for the Satake correspondence. Clearly, one wonders whether the comparison functor is essentially surjective. One might also draw further inspiration from a comparison with the multiple copies that arose in Theorem 6.3.11.

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