HONG KONG PRESERVICE TEACHERS' BELIEFS AND ATTITUDES TOWARDS TEACHING PROOF IN SCHOOL MATHEMATICS: A DESIGN-BASED RESEARCH

by

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Abstract

Researchers have suggested that proof is essential for the learning of mathematics and should be placed at the centre of school mathematics. Yet, many studies suggest that teachers have counterproductive beliefs and attitudes towards teaching proof in school mathematics, which are constraints on making proof central to school mathematics. This study developed an intervention of short duration that promotes preservice teachers' more productive beliefs and attitudes towards teaching proof, which in turn can support preservice teachers' future teaching of proof.

Design-based research methodology was used to develop the intervention, which involved four research cycles of implementation, evaluation, and revision. The intervention comprised three weekly two-hour workshops. Preservice teachers who enrolled in different initial teacher training programmes of different universities in Hong Kong were recruited to participate in the intervention. Data were collected from multiple sources, including a questionnaire, individual semi-structured interviews, and fieldnotes and recordings of the intervention. In Research Cycles 1–3, data were used to assess changes in preservice teachers' beliefs and attitudes towards teaching proof in school mathematics and evaluate early versions of the intervention, which in turn informed refinements of the intervention. In this thesis, I report the findings of Research Cycle 4.

The results indicated that, after the intervention, the preservice teachers developed more productive beliefs and attitudes towards teaching proof in school mathematics, namely, enhanced beliefs about the importance of proof in school mathematics, greater enjoyment and interest in proof, less anxiety about proof, and less negative responses when failing to prove, and became more open and felt more prepared to implement proof in their future teaching. Components of

the intervention that facilitated the change in the preservice teachers' beliefs and attitudes towards teaching proof have also been investigated.

This study can extend our understanding of multifaceted beliefs and attitudes which influence whether and how preservice teachers plan to implement proof in their future teaching, particularly from an East-Asian perspective, and the design principles for facilitating change in beliefs and attitudes towards teaching proof in school mathematics. In practice, the intervention developed in this study can promote preservice teachers' more productive beliefs and attitudes towards teaching proof in school mathematics in a relatively short period.

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Chapter 1: Research Background

1.1 Proof as a central feature of learning mathematics

Proof and mathematical reasoning are essential to both the discipline of mathematics and learning mathematics with understanding. Proof not only plays an important role in justifying and establishing mathematical knowledge, but also can have explanatory power; through proof, mathematical knowledge and values are communicated and preserved (e.g., Hanna & Barbeau, 2008; Siu, 2008; Zaslavsky et al., 2012). In addition, as it involves applications of logical rules of inference and deduction, proof is also associated with the development of skills in logical and deductive thinking (Rips, 1994).

In recent decades, not only have experts in mathematics and mathematics education suggested that proof (and proving) should be a central feature of all students' experiences of learning mathematics, as early as primary school mathematics (e.g., Hanna & Barbeau, 2008; Schoenfeld, 1994; G. J. Stylianides et al., 2017; Stylianou et al., 2009), but the development of students' skills of mathematical reasoning and proof has also become a focus across different curricula (e.g., Curriculum Development Council, 2017; Department for Education, 2014; National Council of Teachers of Mathematics, 2000). For example, in Hong Kong, school mathematics aims at developing students' mathematical skills: to use basic logical concepts (e.g., "and", "or", "because") to perform simple deductions in lower primary mathematics (Years 1–3), the ability to use syllogism and provide counterexamples in upper primary mathematics (Years 4–6), the ability to perform proofs and check their validity in junior secondary mathematics (Years 7–9) and the ability to evaluate proofs to avoid committing logical fallacies in senior secondary mathematics (Years 10–12) (Curriculum Development Council, 2017, pp. 221–223).

In the following, I briefly discuss Hong Kong students' learning opportunities of proof in school mathematics (Section 1.2) and the relationships between teachers' beliefs and attitudes related to the teaching of proof and students' opportunities to learn proof (Section 1.3). As this introduction serves to give an overview of this study, the definitions of the key concepts, such as proof, beliefs and attitudes, and relevant literature are discussed in Chapter 2.

1.2 Learning opportunities of proof – Hong Kong as an example

Researchers have conducted various investigations into students' opportunities to learn proof and proof-related abilities, such as proof construction and proof comprehension, identifying different challenges of learning and teaching proof (e.g., Healy & Hoyles, 2000; Knuth et al., 2009; Lin & Yang, 2007). They have also made an effort to address these challenges (e.g., Jahnke & Wambach, 2013; Marrades & Gutiérrez, 2000; Miyazaki et al., 2017). In general, students have limited opportunities to learn proof, and hence often apply inappropriate approaches, such as empirical arguments, to justify mathematical statements. Having said that, students' learning opportunities can vary between and within contexts. I use Hong Kong as an example; in this context, the situation of the learning and teaching of proof in school mathematics seems ambivalent, as explained below.

On the one hand, Hong Kong students appear to have more opportunities to be exposed to proof than their counterparts in some other countries. TIMSS 1999 Video Study showed that in Hong Kong about 12% of lessons contained at least one proof-related problem, compared with less than 5% of tasks in Australia, Czech Republic, the Netherlands and the United States, whereas 11% in Switzerland and 39% in Japan (Hiebert et al., 2003; F. K.-S. Leung, 2005). Also, Hong Kong textbooks introduce proof, such as proofs of theorems, how to perform simple geometric proofs, to students as early as junior secondary mathematics, namely, under the topics of Pythagorean theorem and deductive geometry in Year 8, which align with the state curriculum (Curriculum Development Council, 2017; Mullis et al., 2016; Oates, 2014).

On the other hand, there are indications that proof has a marginal place in Hong Kong classrooms. Linking teachers' work and textbooks, such as teaching contents based on the textbooks, using exercises provided in the textbooks for homework setting (Foxman, 1999; Remillard et al., 2009), Wong and colleagues analysed a series of Hong Kong textbooks in senior secondary mathematics, and reported that over 80% of the exercise tasks, in both the strands of geometry and algebra, in the textbooks are non-proof and focus on developing students' routine procedures (K.-C. Wong, 2017; K.-C. Wong & Sutherland, 2018). Moreover, another investigation into public examination papers indicated that about 20% of the problems are proof-related (i.e., requiring candidates to prove a statement or make and explain an argument) but only 10% of the total marks of the examination papers are given to candidates who demonstrate some *argument-oriented* ability to apply and/or interpret prior calculation result(s) or statement(s) to make an argument, compared with the fact that most of the remaining marks are given for *calculation-oriented* ability to perform routine procedures, such as use of formula, substitution and intermediate step(s) during calculation, correct calculation result(s) (C.-Y. Lee, 2021). In Table 1.2.1, I use an adapted question to illustrate the two types of problem (proof-related vs. non-proof related) and the two types of marks (argument-oriented vs. calculation-oriented).

Table 1.2.1
An example of a problem in an examination in Hong Kong

Problem		
Given $A(3, 4)$ and $B(5, 6)$.		
A is translated 12 units down to A'.		
B' is the reflection image of B with respect to the x-axis.		
(a)	Write down the coordinates of A' and B'.	Non-proof related
(b)	Prove that AB is parallel to $A'B'$.	Proof related

Marking		
(a)	The coordinates of *A*' are (3, –8).	1 mark – calculation-oriented [correct answer]
	The coordinates of *B*' are (5, –6).	1 mark – calculation-oriented [correct answer]
(b)	The slope of *AB* = (6 – 4)/(5 – 3) = 1 The slope of *A*'*B*' = (–6 – (–8))/(5 – 3) = 1	1 mark – calculation-oriented [use of formula: slope = $(y_2 - y_1)/(x_2 - x_1)$]
	Since the slope of *AB* is equal to the slope of *A*'*B*', *AB* is parallel to *A*'*B*'.	1 mark – argument-oriented [interpretation of prior calculation results]

Note: Adapted from Question 6 in the 2017 examination paper

As Hong Kong teachers often link their decisions around teaching to the content of public exams that they perceive to be true, some teachers prioritised the development of students' ability to perform routine procedures over proof and reported that they would prefer to skip proof (e.g., proofs of Pythagoras' theorem) or replace it with activities of 'discovering' theorems (e.g., identifying patterns only) in teaching (Huang & Leung, 2004; C.-Y. Lee, 2019; Yan, 2020). The research findings of the state of proof, in terms of the proportion of proof tasks in textbooks and examination papers, and the limited class time spent on proof, in Hong Kong echo the findings in other countries (e.g., Bergwall & Hemmi, 2017; Davis, 2012; Fan et al., 2018; Frasier & Panasuk, 2013; Kotelawala, 2016; Nyaumwe & Buzuzi, 2007; Otten et al., 2014; Sears & Chávez, 2014; G. J. Stylianides, 2009; D. R. Thompson et al., 2012).

1.3 Teachers' beliefs and attitudes and students' opportunities to learn proof

There are different potential factors, such as state curriculum, textbooks and school policies, that can have influence on the contents of mathematics, including proof, being taught in class (P. Morris & Adamson, 2010; N.-Y. Wong et al., 2009). Among them, teachers are presumably major decision makers of what contents and materials are presented in their own teaching (Choi et al., 2012; Goldin et al., 2009; Schoenfeld, 2011), thereby influencing students' experience of learning mathematics (N.-Y. Wong et al., 2002, 2016), including proof. For this reason, in order to improve students' opportunities to learn proof, I argue that it is vital to investigate the factors influencing teachers' decisions around teaching proof and address the issues that make them hesitate in spending time on teaching proof in their own teaching.

Based on the close relationships among knowledge, beliefs and attitudes, and decision-making and teaching (Ernest, 1989b; Schoenfeld, 2011; Wilkins, 2008; N.-Y. Wong et al., 2016), researchers have carried out studies on teachers' proof-related knowledge, beliefs and attitudes (e.g., Corleis et al., 2008; Furinghetti & Morselli, 2011; Knuth, 2002a; Kotelawala, 2016; Mingus & Grassl, 1999; G. J. Stylianides et al., 2007; Varghese, 2009). These studies identified teachers' knowledge, beliefs and attitudes that were inadequate to make proof a central feature of school mathematics, thereby providing insight into the importance of teacher training and professional development that help teachers become competent in integrating proof into their teaching.

Assuming that teachers should be knowledgeable in constructing and evaluating proofs, the results of the studies concerning preservice teachers' (PTs') proof-related knowledge have been ambivalent. On one hand, some studies reported that, in different contexts, more than half of PTs were aware that empirical arguments do not warrant the truth of mathematical statements (Bleiler et al., 2014; Demiray & Işıksal Bostan, 2017) and could correctly produce proofs for mathematical statements that often appear in textbooks, such as the formula of the sum of the first n consecutive positive integers (A. K. Morris, 2007; Sun, 2013). On the other hand, studies also reported that many PTs had difficulty constructing proofs correctly – even if some proofs required only lower secondary mathematical contents (Schwarz et al., 2008; Stavrou, 2014; G. J. Stylianides et al., 2007). Also, a number of PTs would accept empirical or incorrect arguments as being mathematically correct proofs – particularly, when they perceived the arguments as students' products (Bleiler et al., 2014; Martin & Harel, 1989). G. J. Stylianides and Stylianides (2009) indicated that some PTs were aware of the aforementioned limitation of empirical arguments but when they had difficulty proving some mathematical statements, they would provide empirical arguments to the statements.

Moreover, researchers have also found that practising teachers and PTs have ambivalent beliefs and attitudes towards proof, and particularly, the teaching of proof. For example, whilst PTs generally have positive views about the importance of proof in doing mathematics, many of them have anxiety and relatively low confidence in proof (Almeida, 2000; Doruk & Güler, 2014; Uzel & Ozdemir, 2009), which may therefore result in their indecisiveness or limited time spent on teaching proof (Ersen, 2016; Kotelawala, 2016; Varghese, 2009). I discuss practising teachers and PTs' beliefs and attitudes related to (the teaching of) proof and their relationship with teachers' decision-making and teaching in more detail in Chapter 2.

Acknowledging the aforementioned issues that may make PTs hesitate to integrate proof in their future teaching, many researchers have conducted studies involving interventions – where the term 'intervention' is used here to refer to "action taken to improve a situation in relation to the teaching and learning of mathematics" (G. J. Stylianides et al., 2017, p. 121). Their interventions addressed different aspects of PTs' proof-related knowledge, namely, understanding of inductive and deductive modes of reasoning in relation to the power of establishing the truth or falsity of mathematical statements, ability to use definitions and proof when doing mathematics, ability to integrate proof into lesson plans of different mathematical topics at different grade levels, knowledge of proof and validity of inferences in the context of history and community and ability to produce proofs that are appropriate for a given context, and some touched on some aspects of proof-related beliefs and attitudes (Buchbinder & McCrone, 2020; Guala & Boero, 2017; Kempen & Biehler, 2019; G. J. Stylianides & Stylianides, 2009). However, only few studies so far focused on promoting PTs' proof-related beliefs and attitudes (Conner et al., 2011; Yoo, 2008; Zengin, 2017). I review these interventional studies in more detail in Chapter 2.

1.4 Purpose of the study

After reviewing the literature on (changing) PTs' proof-related beliefs and attitudes (Chapter 2), I found that (i) almost all existing studies were conducted in Western (e.g., the United States) or Middle Eastern (e.g., Turkey) countries, and any East Asian perspective is limited; (ii) when an intervention was involved, the intervention often lasted at least nine weeks, and the number of studies that involved intervention(s) of shorter duration is also limited; and (iii) whilst other studies have pursued changing broader mathematical beliefs and attitudes (e.g., Liljedahl et al., 2007, 2012), questions regarding how to facilitate change in beliefs and attitudes towards teaching proof (e.g., beliefs about the nature of mathematics and proof) remain open (Conner et al., 2011). In order to make progress in addressing these gaps, this thesis reports a design-based research study (the present study) that aimed to address a question (*Whether and how PTs' beliefs and attitudes towards teaching proof can be changed in a short period?*) by developing a learning environment of relatively short duration (2 hours per week × 3 weeks = 6 hours), which developed Hong Kong PTs' beliefs and attitudes towards implementing and making proof central to their future teaching. The research questions (RQs) that guided the present study are:

RQ 1. What were Hong Kong preservice teachers' pre-intervention beliefs and attitudes towards teaching proof in school mathematics?

RQ 2. How did Hong Kong preservice teachers' beliefs and attitudes towards teaching proof change after participating in a specifically designed intervention of short duration?

RQ 3. What components of the intervention were perceived by the preservice teachers as contributing to changes in their beliefs and attitudes (as identified in RQ 2)?

1.5 Description of the study

An intervention was developed using design-based research (DBR) methodology. The process of development consisted of:

(1) A draft of the intervention was developed based on existing literature on proof, including studies on practising teachers and PTs in different countries, and knowledge of the Hong Kong context and Hong Kong PTs.

(2) The intervention was conducted (by me) with Hong Kong PTs (a few groups per research cycle, 12 groups in total).

(3) Data were collected to evaluate the effects of the intervention on the PTs' beliefs and attitudes towards teaching proof.

(4) The results of the evaluation were used to inform improvement in the intervention.

(5) After repeating (2) to (4) in multiple research cycles (Research Cycles 1–3) until the final (Research Cycle 4), the final version of the intervention was conducted with another few groups of PTs for answering RQs 1–3.

A questionnaire, which consisted of Likert items and open-ended questions, individual semi-structured interviews, and audio-recordings and fieldnotes of the intervention were used to collect data. The questionnaire was distributed to the PTs who participated in the intervention, before and after the intervention. The PTs' pre-intervention responses to the questionnaire were used to address RQ 1 and their pre-post responses were used to address RQ 2. A subgroup of the PTs were interviewed after the intervention. The interviews allowed the PTs to elaborate their (pre-post) responses to the questionnaire, which supplemented the analysis of the questionnaire for RQs 1–2, and to report their own experiences in the intervention, which were used to address RQ 3. Every session of the intervention was audio-recorded and fieldnotes were made, in order to supplement the analysis for RQ 3.

I discuss the methods used in the present study (including the development of the intervention, the methods of data collection and analysis) in more detail in Chapter 3. Results of the analyses are reported in Chapter 4, followed by a discussion about implications of the present study in Chapter 5.

Chapter 2: Literature review

This chapter is divided into four main parts. In Section 2.1, I describe the working definition of proof for the present study and discuss the importance of research into the learning and teaching of proof in school mathematics, particularly regarding beliefs and attitudes that PTs hold related to the learning and teaching of proof in school mathematics. Then I shift my discussion to literature on teachers' beliefs and attitudes, the key concepts in the present study, in Section 2.2. I also discuss the literature on change in beliefs and attitudes (Section 2.2.4), which informed the design of the intervention of the present study. In Section 2.3, I discuss the current research community's understanding of teachers' beliefs and attitudes towards teaching proof in school mathematics, and review intervention studies that involved PTs concerning their beliefs and attitudes towards teaching proof. The chapter ends by revisiting the research questions of the present study (RQs 1–3, stated in Section 1.4).

2.1 Proof in mathematics education

The term 'proof' may mean different things to different people, often depending on their mathematical experience and philosophies (Almeida, 1996; CadwalladerOlsker, 2011; Hersh, 1993; Reid, 2005). In the present study, the knowledge of mathematics, and proof particularly, is situated in the context of school mathematics, and I consider a *proof* to be a connected sequence of true statements for, or against, a mathematical statement, which includes three essential components: *sets of accepted statements*, *modes of argumentation* and *modes of argument representation* that are appropriate, valid and known to (or within the capacity of) the students who are communicating mathematics; and *proving* is an act of searching for a proof (A. J. Stylianides, 2007, pp. 290–291). Although the aforementioned statements indicate clearly proof as an object and proving as an act, I simplify the phrase ‘proof and proving’ into ‘proof’ in this thesis for the ease of communication; for example, the phrase ‘teaching proof’ refers to an act (of a teacher) in order to help students learn proofs of certain theorems and how to prove.

The use of A. J. Stylianides’ conceptualisation of proof suited the present study for two reasons. Theoretically, it allows me to describe proofs across different levels and topics of school mathematics without omitting important aspects of proof that are valued by mathematicians and experts of mathematics education, such as the recognition of proof as a logical deduction which prevents invalid empirical arguments from being considered as proofs and the social aspects of proof (Herbst & Balacheff, 2009; Mariotti et al., 2018; A. J. Stylianides, 2007). Practically, its usefulness has been shown by its appearance in various classroom-based research studies (e.g., Bleiler et al., 2014; Morselli, 2013; Reid & Vargas, 2017), particularly, as the basis for developing and/or guiding proof-related interventions at different levels of mathematics (e.g., Komatsu, 2010; Mata-Pereira & da Ponte, 2017; G. J. Stylianides & Stylianides, 2009).

A related phrase ‘reasoning-and-proving’ refers to a set of classroom activities that are intended to become a vehicle to sense making and deep learning in school mathematics by promoting investigation and justification of mathematical statements (G. J. Stylianides et al., 2013). The activities include identifying patterns, making conjectures, providing non-proof arguments, and providing proofs (G. J. Stylianides, 2008). The usefulness of reasoning-and-proving in the present study has been shown by its appearance in various research studies, for example, as the basis for analysing textbook contents or written curriculum documents (e.g., Davis, 2012; Fujita & Jones, 2014; Otten et al., 2014), for analysing classroom episodes (e.g., Rogers & Steele, 2016; G. J. Stylianides & Silver, 2009) and, particularly, for designing activities involved in a proof-related intervention (G. J. Stylianides & Stylianides, 2009).

In the present study, the conceptualisations of proof and reasoning-and-proving were used as the basis for developing an intervention that developed PTs' beliefs and attitudes towards teaching proof in school mathematics. A detailed description is provided in Chapter 3.

In school mathematics, the importance of proof and proving largely lies in its link to doing and knowing mathematics – proof is essential for developing, establishing and communicating mathematical knowledge. Proof is used to justify (or falsify) mathematical statements and convince oneself and others of the truth (or falsity) of a mathematical statement, and can also be used to explain mathematical ideas and discover *new* (to students) ideas, skills and strategies (CadwalladerOlsker, 2011; de Villiers, 1990; Hanna, 1990; Hersh, 1993; G. J. Stylianides, 2009). It follows that one need for teaching proof in school mathematics is to promote students' conceptual understanding of mathematical ideas and acquisition of mathematical practice, skills, strategies and values (Hanna & Barbeau, 2008; Siu, 2008).

Moreover, as it involves application of logical inference and deduction, teaching proof is associated with the development of skills in logical inference and deductive reasoning (Rips, 1994; Zaslavsky et al., 2012). With careful design and proper implementation, proof can also be enjoyable and used for promoting students' awareness of the need and liking for proof in mathematics (Jahnke & Wambach, 2013; Waring, 2001a, 2001b). In his discussion on proof in different traditions, Siu (2012, 2009) points out that learning proof (and mathematics) is sometimes considered to be linked with the development of personal characters and moral ideas, such as a sense of virtue, justice and dignity. Some researchers in mathematics and mathematics education also comment that students should have more opportunities to learn proof as a key concept in school mathematics (e.g., Hanna & de Villiers, 2008; Stylianou et al., 2009), for a better understanding of school mathematics (Hanna & Barbeau, 2008) and for a better transition to university mathematics (Thomas et al., 2015).

However, in reality, the place of proof in school mathematics differs from this vision. Proof is considered to be “hard for teachers to teach and difficult for students to learn” (G. J. Stylianides et al., 2017, p. 237).

***Proof is difficult for students to learn*.** The learning of proof involves proof construction, proof comprehension, and more. Large-scale studies have found that students at all levels often have difficulty constructing proofs. These studies investigated *how well students can write proofs* in different mathematical domains such as geometry (e.g., Senk, 1985) and algebra (e.g., Healy & Hoyles, 2000). For example, in Senk’s (1985) study, 1520 students, between grades 9 and 11 (Years 10–12) from five different states in the US, were asked to produce four geometry proofs. Senk found that whilst 31% of the students were able to produce at least three of four proofs correctly, 29% could not produce any correct proof.

Later, Knuth, Choppin and Bieda (2009) conducted another large-scale study with approximately 400 US students between grades 6 and 8 (Years 7–9). Six proving tasks, on the topic of numbers, were used for investigating the students’ production of justifications. The results indicated that on one hand, there was an overall increase in attempts to produce a general argument as the students progressed from Year 7 to Year 9. On the other hand, empirical arguments were the predominant means of justification used by the students. Knuth and colleagues noted that a significant proportion of students made a response which was “typically nonsensical […], based on a misinterpretation of the item […], or based on computational errors” (ibid., p. 156), or did not provide a mathematical justification for the proving tasks.

In the UK, Healy and Hoyles (2000) investigated the characteristics of arguments that high-attaining students recognised as proofs in the domain of numbers and algebra. They defined high-attaining students as the top 20–25% of the UK student population as determined by scores in national tests (Key Stage 3). In their study, 2,459 high-attaining students, aged between 14 and 15, were tested. Similar to the other studies, Healy and Hoyles' study showed that the majority of the high-attaining students were not able to produce correct proofs.

Apart from students' difficulty in producing proof, students also have difficulty reading and comprehending proofs. In Taiwan, Lin and Yang (2007) conducted a study to investigate students' reading comprehension of proofs in geometry. They developed a test to assess students' reading comprehension of a geometric proof – they divided the idea of comprehending a geometric proof into four hierarchical levels, from (i) Surface, (ii) Recognising Elements and (iii) Chaining Elements, to (iv) Encapsulation (Yang & Lin, 2008). 601 students, between Year 10 and Year 11, were tested. The results showed that only about 38% of the Year-11 students reached beyond the level of chaining elements, and a significant proportion of the students could not reach beyond the level of recognising elements – indicating that those students not only had difficulty understanding geometric knowledge such as the definition/meaning and properties of a geometric object involved in the proof, but also had difficulty understanding the logical relations between the arguments in the proof and some critical proof ideas.

Based on Lin and Yang's work, researchers developed their own tests of reading comprehension of proofs and tested students in different counties such as Indonesia, Japan and Turkey, and at different grade levels. The results are consistent with each other. On the one hand, most of the senior students in their studies were able to give reasons for the steps of a proof and knew the meanings of the terms, statements and symbols used in the proof (Herizal et al., 2019;

İnam et al., 2018; Lin & Yang, 2007); many senior students could also recognise and understand the immediate consequence(s) of a given statement and the deduction between two consecutive steps in a proof (İnam et al., 2018; Miyazaki et al., 2017). On the other hand, the students had difficulties in validating if there was an error in a step of a proof, applying and extending a proof or proving methods into other unfamiliar situations, even if they were similar to the original proof, and lacked understanding of the logical principle of transitivity ("$P \rightarrow Q$, $Q \rightarrow R$ implies $P \rightarrow R$") (Herizal et al., 2019; İnam et al., 2018; Komatsu et al., 2017; Miyazaki et al., 2017).

Apart from school students, research has also indicated different university students' struggles with proof (Iannone & Inglis, 2010; Kolahdouz et al., 2020; A. J. Stylianides et al., 2004; Zazkis & Zazkis, 2016). The results have revealed various difficulties that the students, at different grade levels, often encounter when learning proof.

However, there are grounds for optimism. Some students value general and explanatory arguments, and are aware that empirical arguments are not general that checking a few examples does not guarantee the truth of a mathematical statement (Ball & Bass, 2003; Chazan, 1993; Healy & Hoyles, 2000). On the other hand, younger students often consider examples as a means of convincing people of the truth of a mathematical statement (Bell, 1976; Healy & Hoyles, 2000) and limit their arguments to empirical arguments due to lack of knowledge of proof (K. S. Lee, 2016; A. J. Stylianides & Stylianides, 2009).

Moreover, students tend to develop better knowledge of proof as they progress to higher grade levels. In the study of Knuth et al. (2009), the proportion of students who attempted to produce a general argument increased from grade 6 to grade 8 (Years 7–9). In the study of Lin and Yang (2007), the proportion of students who reached the top level of the reading comprehension of geometry proofs increased from Year 10 to Year 11. Having said that, one's knowledge of proof does not grow naturally. As people do not necessarily think deductively initially (Tall et al., 2012), the development of students' knowledge of proof requires more learning opportunities for proof (Healy & Hoyles, 2000; Knuth et al., 2009) created by teachers. However, research has shown that teachers are constrained in teaching proof.

Proof is hard for teachers to teach. Teachers' knowledge and beliefs about proof are influential in how they treat proof in their teaching (Furinghetti & Morselli, 2011; Schoenfeld, 2011; Wilkins, 2008). Whilst research on proof construction has indicated that many practising teachers and PTs are able to produce at least one proof to a given mathematical statement (particularly, one that they are familiar with) (Demiray & Işıksal Bostan, 2017; I. K.-C. Leung & Lee, 2017; Sun, 2013), successful attempts to produce proofs do not necessarily imply strong knowledge about proof (Jones, 2000). Various studies have indicated that many practising teachers and PTs have inadequate knowledge about proof (e.g., Martin & Harel, 1989; Schwarz et al., 2008; Stavrou, 2014; G. J. Stylianides et al., 2007) and counterproductive beliefs about proof and roles of proof in the learning of mathematics (e.g., Furinghetti & Morselli, 2011; Knuth, 2002a; Ugurel, 2012; Varghese, 2009). This discourages them from implementing proof in their teaching.

Observing the homework of 97 PTs majoring or minoring mathematics in Canada that involved proof in basic number theory and abstract algebra, Stavros (2014) found four common errors that the PTs often made in proof, namely, replacing proofs with examples, making circular arguments, misusing definitions, and proving only one direction for bidirectional statements. Also, after introducing the common errors to another group of 91 PTs, Stavros observed that the second group left more homework exercises unanswered. The observation revealed that the PTs had difficulties with how to start a proof. Similar findings were also observed in other studies (e.g., Creager, 2016; İmamoğlu & Toğrol, 2015).

Apart from difficulties with producing arguments and proofs, practising teachers and PTs also had difficulties with evaluating students' arguments and proofs. For example, in Martin and Harel's (1989) study, 101 PTs in primary mathematics in the US were given two mathematical statements: "If the sum of the digits of a whole number is divisible by 3, then the number is divisible by 3", and "If a divides b, and b divides c, then a divides c" (ibid., p. 43), and empirical and deductive arguments for the statements, and were asked to judge which of the arguments were mathematically correct. Martin and Harel found that over 50% of the PTs accepted an empirical argument as a proof, 38–52% (depending on the statements) thought that an incorrect deductive argument was mathematically correct, and over 30% accepted both empirical and (correct) deductive arguments to be mathematically valid.

Similar results were also found in Gomes et al. (2015), in which 66 PTs in primary mathematics in Portugal were tested with arguments for a geometrical statement ("Diagonals of a rhombus are perpendicular") and an algebraic statement ("$a, b \in \mathbb{R}$, $(a + b)^2 = a^2 + 2ab + b^2$"). Over 50% of the PTs accepted empirical arguments to be proofs of the two statements, because they believed that the use of particular cases allows generalisation and a formula to be checked, whereas the PTs who rejected empirical arguments tended to believe the opposite – that particular cases do not guarantee generalisation in algebra and geometry.

A. K. Morris (2007) investigated whether PTs evaluate students' arguments differently in different contexts. Thirty-four PTs in K-8 (Years 1–9) mathematics in the US were interviewed. She identified two tendencies of beliefs about mathematical justification that the PTs held – the first focused on a justification being explanatory that helps in understanding why a mathematical statement is true, whereas the second was based on the PTs' belief that inductive reasoning and testing examples can justify a general statement. She also observed that whilst the PTs could apply a range of criteria to evaluate students' arguments, they seldom used logical validity as a criterion and many of the PTs tended to fill in the holes of students' arguments or change them, instead of attending only to what was written. This resulted in inappropriate evaluation.

In Imamoglu and Togrol's (2015) study, 113 Turkish PTs were asked to evaluate different types of arguments for two statements: "If the square of a natural number is even, then that number must be even" and "The equality $1 + 3 + 5 + \ldots + 2n–1 = n^2$ is true for all integers $n \geq 1$", arranging from empirical arguments to proof. Imamoglu and Togrol found that many of the PTs had difficulties with arguments in which there was no obvious mistake but some steps were missing or left unjustified. They also found that whilst many of the PTs were aware that empirical arguments are not accepted to be proofs, contrasting to the findings of Martin and Harel (1989) and Gomes et al. (2015), many PTs, particularly those in primary mathematics, had difficulties with conditional statements, particularly in distinguishing between a statement and its converse.

Knuth (2002a) interviewed 16 experienced teachers in secondary mathematics in the US, to investigate their knowledge about proof. He found that the teachers thought that a proof and a counterexample of a mathematical statement could coexist, implying their beliefs that a general statement is true for most, but not all, cases and that a proof is to demonstrate that the statement is 'usually' true. This may explain why many practising teachers and PTs accepted empirical arguments to be proofs (see also Gomes et al., 2015; Martin & Harel, 1989), presumably because they adopted inductive reasoning to generate a general statement and treated counterexamples as some exceptions.

The above studies have indicated not only the difficulties that PTs often have when evaluating arguments, but also that their inappropriate evaluation of arguments often depends on their beliefs about proof. Before discussing the literature on teachers' beliefs and attitudes related to proof and its teaching, I will discuss the notions of (teachers') beliefs and attitudes and their relationships with teaching, and literature on change in beliefs and attitudes.

2.2 Teachers' beliefs and attitudes

Extensive research has shown that teachers' thought processes and behaviour are connected to their affective domains, especially to beliefs and attitudes (e.g., Beswick, 2006; Buehl et al., 2014; Furinghetti & Morselli, 2011; Schoenfeld, 2011). This is the rationale for developing an intervention that aimed to develop PTs' productive beliefs and attitudes towards teaching proof in school mathematics. In this thesis, the adjective 'productive' is used to describe beliefs and attitudes that are linked to the encouragement of the teaching of proof, whereas the adjective 'counterproductive' is used to describe beliefs and attitudes that discourage PTs from implementing proof in their future teaching. In the following, I first discuss the concept of emotions because the concepts of beliefs and attitudes are linked to emotions (Goldin et al., 2017; McLeod, 1992), and emotions are seen as important for the development or change in beliefs and attitudes (Bekdemir, 2010; Beswick, 2018; Di Martino & Zan, 2015; Hannula, 2014).

2.2.1 Teachers' emotions

In mathematics education, researchers have been interested in students' emotions (e.g., anxiety) and their relationship with learning and doing mathematics since the 1970s (McLeod, 1992; Hannula, 2014; Philipp, 2007). According to Mandler (1989), emotions are the result of how individuals interpret experience, which indicates the relationship between emotions and beliefs (Hannula, 2014). Emotions can function on an individual's physiological (e.g., tears), psychological (e.g., choice of goals) and social (e.g., communication with people) responses (Hannula, 2012; Power & Dalgleish, 2016), implying that emotions can influence teachers' choices of instruction (Frasier, 2010; Kotelawala, 2016). For example, a teacher can be anxious about some educational innovation, which is novel to them, even if such an innovation is useful for students' learning; as a result, they may hesitate to implement the innovation into their instructional practice (Hills, 2007; Kotelawala, 2016).

2.2.2 Teachers' beliefs

A teacher's instructional practice is often considered to be a function of beliefs together with other affective and cognitive domains (e.g., Eccles & Wigfield, 2002; Ernest, 1989b; Schoenfeld, 2011; A. G. Thompson, 1992; N.-Y. Wong et al., 2016). Among different available definitions of beliefs, there is a common feature that a belief refers to a proposition, or statement of relation among ideas and/or objects that an individual holds to be true (e.g., Beswick, 2018; Goldin, 2002; Philipp, 2007; Wilson & Cooney, 2002). That is, 'a teacher believes statement P' means 'a teacher thinks statement P is true', implying that beliefs are attached to objects of belief (e.g., 'a belief that …' or 'beliefs about …') (Goldin et al., 2009).

According to Green (1971, cited in Beswick, 2018; Goldin et al., 2009; A. G. Thompson, 1992), there are three features of beliefs. First, beliefs may have a quasi-causal relationship, where beliefs can be divided into primary and derivative beliefs. For example, based on Ernest's (1989b) proposition that "a teacher's view of knowledge as integrated can lead to teaching …" (p. 21), an example of a primary belief can be a teacher's belief that mathematics is a field that involves human inquiry and problem solving, and its associated derivative belief might be that problem solving should be focused on in classrooms.

Second, beliefs can be distinguished by how strongly they are held. A central belief is strongly held and more resistant to change, whereas a peripheral belief is less strongly held and more likely to be influenced. The resistance of central beliefs to change is often linked to how long these beliefs have been developed and held by individuals, and how intensely they are intertwined with other beliefs and emotions (Pajares, 1992; Rokeach, 1968; Swan, 2007).

The third feature of beliefs is that beliefs cluster together whereas some beliefs are more isolated from others. For example, beliefs about the nature of mathematics (Ernest, 1989a) refer to a clustered set of propositions relating to the philosophy of mathematics (e.g., the role of mathematics, how mathematical knowledge is developed) that an individual holds to be true. In addition, beliefs about learning and teaching mathematics are sometimes considered as derivative beliefs of or ‘inseparable’ from beliefs about the nature of mathematics (Barkatsas & Malone, 2005; Ernest, 1989b), and the consistency between the two clusters can be affected by teachers’ beliefs about students (Beswick, 2004; Forgasz & Leder, 2008).

Moreover, in relation to variation in research methods and definitions, self-reported beliefs (beliefs revealed by self-report instruments, such as interviews and questionnaires) and enacted beliefs (beliefs revealed through observing and interpreting one’s actions, e.g., classroom observation) are used to categorise and compare beliefs that are identified from different data sources (Lerman, 2002; Speer, 2005). There are consistencies and inconsistencies found between teachers’ self-reported and enacted beliefs, and between data collected from different methods (Cross, 2009; Liljedahl, 2009; Wilkins, 2008). Looking into such consistencies and inconsistencies may lead to a better understanding of the broader set of beliefs (e.g., beliefs relating to the contexts) that are influential in teachers' decision making and practices (Cross Francis, 2015; Furinghetti & Morselli, 2011; Philipp, 2007; G. J. Stylianides et al., 2013).

In the present study, I particularly focused on the detection of PTs' self-reported beliefs about whether, how and why proof should be implemented in their future teaching (and their changes after participating in an intervention, see Section 3.2). Assuming that beliefs and attitudes play an essential role in shaping teachers' decision making and teaching practices, a PT's *beliefs about teaching proof* were defined, in the present study, to be a set of self-reported statements that a PT held to be true and were related to their ideas about whether, how and why proof should be implemented in their teaching (Furinghetti & Morselli, 2011; Goldin, 2002; Philipp, 2007; Voss et al., 2013). This working definition hinted that any inconsistencies between what one said, as inconsistencies could be captured by different data collection methods (e.g., responses to Likert items, open-ended questions and interviews; see Section 3.3), could lead to investigations into a broader set of beliefs which better explained such inconsistencies.

Beliefs and emotions are interrelated. Beliefs can arouse emotions as a result of a combination of cognitive analyses and physiological responses, particularly when situations deviate from an individual's expectations (beliefs) (Di Martino & Zan, 2011; Mandler, 1989). Depending on beliefs, and, of course, on the actual situations, emotions aroused can range from positive (delight) to negative (anxiety), from mild (pleasure) to intense (frustration), and from basic (anger) to complex (a mixture of fear and joy) (Di Martino & Zan, 2011; Silvia, 2008).

Emotions can influence beliefs. For example, fear and anxiety can direct one's attention towards threatening information and depression, and bias memories towards less positive beliefs about past experiences, which, in turn can affect the formation of subsequent beliefs (Beswick, 2018; Eccles & Wigfield, 2002; Hannula, 2014; Power & Dalgleish, 2016). A teacher's beliefs about the teaching of proof can be seen as an outcome of their previous beliefs and emotions in mathematics and proof as a learner of mathematics.

The close relationship between beliefs and emotions informed the design of the present study: items related to PTs' emotional dispositions towards proof were used to collect data (see Section 3.3.1); PTs' emotions (e.g., of proof, of different activities) were also considered when I developed the intervention (see Section 3.2.2).

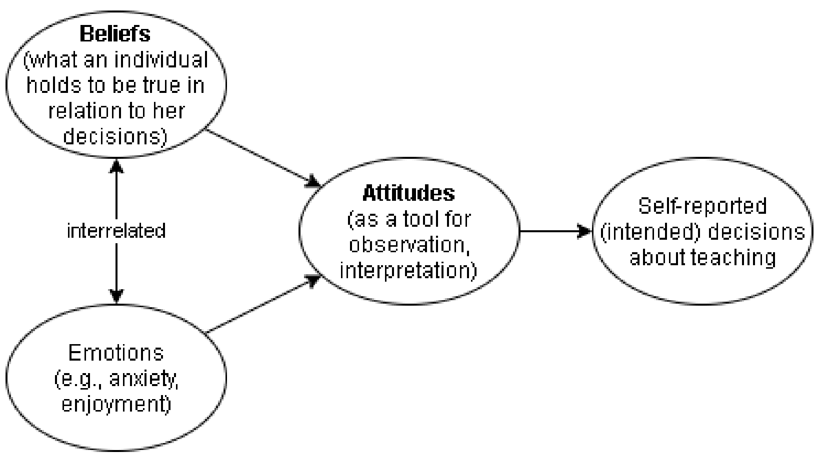
2.2.3 Teachers' attitudes

Depending on the foci of their studies, researchers use the term *attitudes* differently. For Allport (1935), attitude refers to "a mental state of readiness" (p. 810), which is shaped by experiences and influences one's responses to related objects and situations. Some other researchers consider attitudes as a general term to include beliefs, or to address different constructs, such as preference, interest, and motivation (Di Martino & Zan, 2015; McLeod, 1992), whereas some consider attitudes as a subset of beliefs (Rokeach, 1968). For McLeod (1992), attitudes refer to "affective responses that involve positive or negative feelings of moderate intensity and reasonable stability. Examples of attitudes […] include liking […] disliking […] being curious […] being bored […]" (ibid., p. 581). Yet, some researchers critique McLeod's construct of attitudes, and seek to revise them (e.g., Di Martino & Zan, 2011, 2015; Hannula, 2012). For Di Martino and Zan (2015), attitude is seen as "a trait of an individual that has a direct influence upon his/her behaviour" (p. 52). In general, researchers link attitudes to beliefs, emotions and behaviour. In mathematics education, Di Martino and Zan (2011) and Hannula (2002) propose that attitudes have a cognitive (beliefs), an affective (emotions), and a behavioural component.

In the present study, I defined a PT's attitude towards teaching proof to be a mental state of readiness for implementing proof in their future teaching. This working definition allowed me to consider attitudes as a tool to interpret a PT's self-reported decisions (Di Martino & Zan, 2011, 2015) about whether proof should be implemented in their teaching, through looking into their emotions and beliefs about proof through their self-reported responses to (the teaching of) proof (Figure 2.2.1). Particularly, emotions (statements of how PTs felt; e.g., anxiety, enjoyment) and beliefs about the importance of proof and about relevance of proof to school curriculum have been found influential in teachers' decisions about the place of proof in teaching (e.g., time spent on proof) (Frasier & Panasuk, 2013; Kotelawala, 2016; Nyaumwe & Buzuzi, 2007).

Figure 2.2.1

Relationships between teachers' beliefs, attitudes and emotions, and their decisions about teaching



It is worth noting that apart from beliefs and attitudes, *values* are often considered to be influential in teachers' instructional decisions (e.g., Bishop et al., 2003; DeBellis & Goldin, 2006). Values refer to "the deep, 'personal truths' or commitments cherished by individuals. They help motivate long-term choices and shorter-term priorities" (DeBellis & Goldin, 2006, p. 135). Bishop et al. (2003) discussed similarities and distinctions between values, beliefs and attitudes. Having said that, I do not highlight values in the present study, not because they are unimportant, but I argue that the construct of value had been covered by the working definitions of beliefs and attitudes, for two reasons. First, values and beliefs have several similarities, and values can be considered as a subset of beliefs (centrally held beliefs), regarding truth-values of statements (Bishop et al., 2003; Philipp, 2007). Second, in the present study, PTs' attitudes towards teaching proof are considered as their willingness and intentions to implement proof-related activities in their future teaching, which include their perception of "desirability […], preference […], worthiness […], priority […], moral rightness […], or the potential benefit" (components of values) (Bishop et al., 2003, p. 723) of the inclusion of proof in school mathematics.

2.2.4 Changing beliefs and attitudes

Teachers' beliefs (and attitudes) are often considered to be difficult or slow to change, and such change is considered as "a complex, arduous, and long-term process" (Ashton, 2014, p. 43). Several studies indicated few changes in PTs' beliefs over a programme, a course or an intervention in teacher education (Philipp, 2007; Richardson, 2003), whereas patterns of PTs' emotional responses, such as anxiety about mathematics, remained stable (Hannula, 2012; Philippou & Christou, 1998). The stability of beliefs and attitudes is often attributed to a strongly connected cluster of related beliefs and attitudes, often related to PTs' past experiences as a learner (Bekdemir, 2010; Beswick, 2018; Swan, 2007).

Having said that, under appropriate settings, beliefs and attitudes can change (e.g., Liljedahl et al., 2007, 2012; Shilling-Traina & Stylianides, 2013), even in a short period of time (e.g., A. J. Stylianides & Stylianides, 2014). For example, research has shown that providing innovative learning experiences (e.g., use of history, problem-based instruction, video study) can facilitate change (e.g., Liljedahl, 2005; Philippou & Christou, 1998; Schram et al., 1988).

Looking into the change mechanisms, researchers propose the notions of ‘rejection’, ‘replacement’ and ‘cognitive conflict’ (Liljedahl et al., 2021; Zaslavsky et al., 2012).

For beliefs and attitudes to change, for example, an individual has to be aware that their existing belief is no longer favourable or suitable to be held true (rejection), and look for ‘better’ information to replace the existing belief (replacement). Having said that, it is possible that an individual experiences rejection, but not replacement, because alternative experiences are not available, and hence, their existing ‘rejected’ beliefs and related practices remain (Liljedahl et al., 2021). Cognitive conflict is considered as a framework for triggering rejection and replacement: as learners are introduced to alternative experiences, their existing beliefs and attitudes are challenged, and change in beliefs and attitudes occurs when the conflict is resolved (Limón, 2001; Rolka & Liljedahl, 2007; Zaslavsky et al., 2012).

Some opportunities to increase the likelihood of successful changes in beliefs and attitudes have been discussed in the literature (e.g., Liljedahl et al., 2021; Limón, 2001; G. J. Stylianides & Stylianides, 2009). Liljedahl et al. (2021) studied change in PTs’ beliefs about mathematics after attending a pedagogy course for primary mathematics, and concluded that (1) any change in beliefs (belief replacement) is preceded by belief rejection, and any case of belief rejection is preceded by some phenomenon of cognitive conflict, and that (2) belief rejection does not automatically translate into belief replacement, and alternative experiences (of learning mathematics) should be provided for PTs to develop new beliefs.

Limon (2001) reviewed literature on cognitive conflict that had been used as an instructional strategy to promote conceptual change in classroom contexts. She analysed studies, reported in literature, in which the use of cognitive conflict did not work well, and argued that conceptual change starts with a cognitive conflict that is meaningful for students (and lack of meaningfulness leads to unsuccessful change). Limon also identified different factors important for the development of meaningful cognitive conflict, including students' (and teachers') motivation and interests, existing knowledge, epistemological beliefs (about mathematics and its learning and teaching), values and attitudes, learning strategies and cognitive engagement, reasoning abilities, and peer interactions. To increase the likelihood of meaningful cognitive conflict, it is important to create a learning environment in which students are motivated and interested in the topic, and can apply their existing knowledge, beliefs and reasoning abilities.

G. J. Stylianides and Stylianides (2009) reported an intervention that helped PTs change from reliance on empirical arguments to awareness of need for proof in mathematics, especially when validating mathematical generalisations. They identified that presenting alternative experiences (in their case, using counterexamples to introduce limitations of empirical arguments) is not sufficient to help PTs replace undesirable understanding of proof (accepting empirical arguments as proof) with awareness of need for proof when validating mathematical generalisations, and argued that it is important for PTs to be aware of their existing (undesirable) understanding of proof, particularly that is intended to be challenged (in their case, belief that empirical arguments can be accepted as proof). G. J. Stylianides and Stylianides (2009) introduced the notion of 'conceptual awareness pillars', which are "instructional activities that aim to direct students' attention to their conceptions about a particular mathematical topic" (p. 322), and argued that the use of such activities helped the participants in their study become aware of their understanding of proof, thereby preparing them for cognitive conflict and change.

Based upon the work of G. J. Stylianides and Stylianides (2009), Shilling-Traina and Stylianides (2013) reported an intervention that helped PTs develop problem-solving beliefs about mathematics. They used the data collected from a sample of 25 PTs to argue that the use of conceptual awareness pillars encouraged the PTs to “critically reconsider and reflect on [their instrumentalist and Platonist] beliefs” (p. 402), resulting in many of the PTs moving towards problem-solving beliefs and away from instrumentalist and Platonist beliefs. In addition, Shilling-Traina and Stylianides highlighted that the use of materials based on the work of teaching and the role that the instructor play in an intervention are also important for changing PTs’ beliefs about mathematics.

The above literature formed the basis for the development of the intervention in the present study, and five design principles emerged as key features of the intervention. The first feature concerned the position of proof in the intervention. As PTs often lack meaningful past experiences with proof (Mingus & Grassl, 1999), particularly in school mathematics, proof was positioned in the intervention to play a central role in learning and teaching mathematics (G. J. Stylianides & Stylianides, 2009), providing PTs with meaningful experiences with the learning and teaching of proof. These experiences were designed to challenge PTs’ existing beliefs and act as alternatives for PTs to develop new beliefs (Liljedahl et al., 2021).

The second feature concerned the use of cognitive conflict as an instructional strategy for facilitating belief and attitude changes (Liljedahl et al., 2021; Shilling-Traina & Stylianides, 2013). The intervention was designed to create a learning environment in which PTs were provided with opportunities to reflect on their existing (counterproductive) beliefs and attitudes towards teaching proof in school mathematics. It is worth noting that challenging PTs’ existing beliefs only does not necessarily result in successful belief and attitude changes, as supporting the resolution to the cognitive conflict (e.g., adapting new information, replacing existing beliefs) is essential (Liljedahl et al., 2021; G. J. Stylianides & Stylianides, 2009).

The third feature concerned the importance of social interaction in supporting belief and attitude changes. The intervention provided PTs with many opportunities to discuss and collaborate with each other on proof, for example, working on mathematical tasks that involve proof and argumentation and designing instructional tasks that involve proof and argumentation. With these opportunities to interact with peers, PTs were allowed to explore and experiment with proof from different perspectives (Schram et al., 1988), and might look at the activities in the intervention as meaningful (Limón, 2001), increasing the likelihood of successful belief and attitude changes.

The fourth feature concerned the importance of using materials based on the work of teaching. The intervention used activities and materials relevant to or derived from the teaching of school mathematics. The use of materials based on the work of teaching allowed PTs to make connections between the activities of the intervention and their future teaching, and reflect on their existing beliefs, particularly about teaching (Shilling-Traina & Stylianides, 2013).

Last but not least, the fifth feature concerned the role of the instructor of the intervention. Research has indicated that apart from peers, the instructor also play an important role in facilitating cognitive conflict and belief and attitude changes (Limón, 2001; Shilling-Traina & Stylianides, 2013; G. J. Stylianides & Stylianides, 2009). In the present study, the instructor had specific roles to play in order to create a supportive environment that engaged PTs in the activities and facilitated meaningful cognitive conflict.

The five features guided the development of the intervention in the present study. These features played essential roles in facilitating changes in beliefs and attitudes towards teaching proof in school mathematics, and were interrelated with each other. I revisit the features and explain how they guided the development of the intervention in Chapter 3.

2.3 Research on teachers' beliefs and attitudes towards teaching proof

A range of research has investigated teachers' beliefs and attitudes towards teaching proof in school mathematics. Qualitative and quantitative studies have been conducted to identify components of practising teachers' beliefs and attitudes that link to their (intended) decisions about including proof in their teaching (Frasier & Panasuk, 2013; Furinghetti & Morselli, 2011; Knuth, 2002b; Kotelawala, 2016). In the following, I summarise the key findings of selected studies.

Knuth (Knuth, 2002b) interviewed 17 experienced secondary teachers in the US. He found that (i) most of the teachers did not consider proof to be a central idea in secondary mathematics but encouraged students' empirical arguments, (ii) they believed that proof seemed to be appropriate only for students who enrolled or pursued advanced mathematics, and (iii) some believed that if proof is introduced to students, it should be introduced in geometry. His findings imply that teachers' instructional decisions regarding proof were influenced by their beliefs about proof, their beliefs about students' needs and expectations, and their beliefs about the context (e.g., topics or grade levels).

By using a questionnaire with 91 items, Frasier and Panasuk (2013) investigated beliefs related to proof that 374 US high-school teachers held. He found that (i) most of the teachers believed that proof can help students develop logical thinking ability, (ii) whilst most of the teachers believed that proof is important for senior school students, only half of the teachers believed that proof is essential for the learning of mathematics – particularly, at junior levels, (iii) the teachers tended to spend more time on proof during senior-level geometry, (iv) the teachers who believed the importance of proof were more likely to spend more time on proof in their teaching than those who did not, and (v) students’ capacity for proof, and appearance of proof in examinations and topics were the teachers’ concerns about the teaching of proof.

Similarly, Kotelawala (2016) conducted a questionnaire study with 78 secondary teachers in the US. She found that (i) whilst the teachers were aware of the importance of proof (e.g., developing students’ logical thinking ability), many valued instruction that focuses on students’ procedural skills more than proof-related activities, and (ii) the teachers’ past experiences and anxiety about proof had influence on their prioritisation of proof (over procedural practices).

Furinghetti and Morselli (2011) interviewed ten experienced upper-secondary teachers in Italy. They found that (i) most of the teachers reported that they included proof in their teaching (contrasting to the findings of the previous studies), (ii) the teachers’ decisions on the teaching of proof could be influenced by their beliefs about the classroom context, their beliefs about mathematics, and their beliefs about students’ needs and expectations, and (iii) when making instructional decisions about proof, the teachers weighted different beliefs differently. They used the notion of ‘leading belief’ – “a belief that prevails over other beliefs in influencing the way teachers act in their teaching context” (ibid., p. 598) – to explain the relationship between teachers’ beliefs and instructional practices regarding proof.

The results of the previous studies have indicated the close relationship between practising teachers’ decisions about the teaching of proof and their beliefs (and emotions). The results have also suggested major components of teachers’ beliefs and attitudes towards teaching proof, including: beliefs about mathematics; beliefs about the importance of proof and its teaching; beliefs about students’ capacity and interest in proof; and interest and anxiety about proof. The components are interrelated with each other and have been found to be influential in teachers’ decisions about the teaching of proof. Having said that, the relationship between teachers' beliefs and the teaching of proof is not merely linear nor static, but rather can be complex and context-dependent. In the following, I discuss each component and its relationship with other components and teachers' decisions about the teaching of proof.

***Beliefs about mathematics*.** Teachers' beliefs about mathematics often refer to highly-related theoretical components: beliefs about the nature of mathematics, beliefs about learning mathematics, and beliefs about teaching mathematics (Barkatsas & Malone, 2005; Ernest, 1989a; Voss et al., 2013; Q.-T. Wong et al., 2009). Beliefs about the nature of mathematics concern issues related to philosophies of mathematics (e.g., "What is mathematical knowledge for?" "How is mathematical knowledge developed?"). Beliefs about learning mathematics concern issues related to models of learning mathematics (e.g., "What role do students play in classroom instruction?" "What kinds of learning activities are appropriate for students?"). Beliefs about teaching mathematics concern issues related to models of teaching mathematics (e.g., "What role does a teacher play in classroom instruction?" "What are the intended outcomes in teaching school mathematics?"). The close relationships among beliefs about the nature of mathematics, beliefs about learning mathematics, and beliefs about teaching mathematics have been verified by studies using correlation analyses of Likert items (e.g., Barkatsas & Malone, 2005; Grigutsch et al., 1998; Voss et al., 2013). For example, teachers who believe that mathematical knowledge is dynamic construction of human creation tend to consider students as active, capable learners and apply more collaborative, discussion-based instruction, whereas teachers who believe that mathematical knowledge is static accumulation of rules and facts for applications tend to consider students as receptive learners and apply more drilling instruction.

Research has also indicated some observations that link teachers' beliefs about mathematics and their (intended) decisions about the teaching of proof. For example, teachers who believe that mathematical knowledge is dynamic construction of human creation might aim to develop students' ability in using proofs when doing mathematics (Barkatsas & Malone, 2005), whereas teachers who believe that mathematical knowledge is the static accumulation of rules and facts for applications might devalue proof and consider it to be irrelevant, unattainable and excludable (Solomon, 2006; Törner & Pehkonen, 1999), implying a reduction in the amount of proof in teaching (Knuth, 2002b). Apart from showing the relationship between teachers' beliefs about mathematics and their (intended) decisions about the teaching of proof, the results have also suggested common models of teaching that teachers often hold.

Yoo (2008) proposed two contrasting models of the teaching of proof for analysing PTs' beliefs about approaches for teaching mathematics and proof, based on literature (e.g., de Villiers, 1999; Ernest, 1989a, 1989b; Hanna, 1991; Hersh, 1993): product-oriented and process-oriented. Teachers who hold product-oriented beliefs consider mathematics as static accumulation of rules and facts, and proof as a means to verify mathematical results. They consider teaching to be knowledge transmission, including facts, rules and proofs, and students' learning to be receipt of knowledge and drill practice. They would value and apply more teacher-centred, content-focused teaching. In contrast, teachers who hold process-oriented beliefs consider mathematics as dynamic construction of human creation, and proof indispensable for doing and understanding mathematics. They consider that learning is a process of students' cognitive construction (via interactions with people and/or learning objects) and teachers play roles in facilitating such construction process (e.g., encouraging students to discuss and work on mathematics through argumentation and proof).

The two proposed, contrasting models are consistent with the findings of Furinghetti and Morselli (2011). Furinghetti and Morselli (2011) found two common ways of how the teachers of their study used proof in teaching: In the first way, the teacher showed a mathematical statement together with its proof to students, whereas in the second, students were actively engaged in proof construction. Therefore, it appears to be reasonable to categorise teachers' common models of the teaching of proof into: (i) that they reject or hesitate to include proof in their teaching, (ii) that they are willing to include proof in their teaching and tend to apply the product-oriented model, (iii) that they are willing to include proof in their teaching and tend to apply the process-oriented model, and (iv) that they are willing to include proof in their teaching and mix both models when teaching.

The previous studies have shown how teachers' beliefs about mathematics, including beliefs about approaches for teaching mathematics and proof, have influence on the teaching of proof. Apart from beliefs about mathematics, research has also shown that teachers' beliefs about the importance of proof, and positive (e.g., enjoyment, interest, likes) and negative (e.g., anxiety, dislikes, stress) emotions about proof have influence on the amount of proof (e.g., time spent on proof) that they used (or planned to use) in their teaching (e.g., Frasier & Panasuk, 2013; Kotelawala, 2016; Nyaumwe & Buzuzi, 2007), implying the inclusion of these beliefs and emotions in the major components of teachers' attitudes towards teaching proof.

***Beliefs about the importance of proof and its teaching*.** Research has investigated teachers' beliefs about the importance of proof, often, by asking practising teachers and PTs to describe the importance of proof from their perspectives and their reasons behind the implementation of proof in school mathematics in interviews or questionnaires. The results have suggested that practising teachers and PTs are often aware that proof can be used to justify mathematical statements (e.g., Lesseig et al., 2019; Varghese, 2009), and often believe that proof can help students develop logical thinking and other mathematical skills (Frasier & Panasuk, 2013; Knuth, 2002b; Kotelawala, 2016; Lesseig et al., 2019; Moralı et al., 2006; Nyaumwe & Buzuzi, 2007) and deepen mathematical understanding (e.g., Lesseig et al., 2019; Mingus & Grassl, 1999; Nyaumwe & Buzuzi, 2007). Apart from those functions of proof, (some) practising teachers and PTs also believe that proof can help in discovering, explaining, communicating and systematising mathematical knowledge, as well as uncovering students' learning and thought processes and enhancing student agency (Knuth, 2002b; Lesseig et al., 2019; Mingus & Grassl, 1999; Varghese, 2009). Particularly, developing students' logical thinking and mathematical skills and promoting students' mathematical understanding and meaningful learning are the most common reasons for practising teachers and PTs to include proof in their teaching (Knuth, 2002b; Lesseig et al., 2019; Mingus & Grassl, 1999). The stronger practising teachers and PTs believe that proof is important in mathematics, the stronger they believe that proof is important in school mathematics and its teaching, and the more they spend time on proof (Frasier & Panasuk, 2013), implying more positive attitudes towards teaching proof.

Having said that, beliefs that proof can help student develop logical thinking and mathematical skills and that proof can promote students' mathematical understanding and meaningful learning do not necessarily, automatically translate into strong belief that proof is important in school mathematics and positive attitudes towards teaching proof. Research has identified constraints on the teaching of proof. For example, there is a common concern by PTs that students are not ready nor interested in (learning) proof. Varghese (2009) interviewed 17 PTs (in Canada), and found that although most of the PTs believed that proof should be implemented in school mathematics, they hesitated to introduce it to all students and some believed that proof is beyond the abilities of most students. Lesseig et al. (2019) used responses of 34 PTs (in Australia, the US and South Korea) to a questionnaire and found that a number of the PTs believed that proof is too complicated or abstract for students and students do not have adequate abilities for proof, and some believed that students are not interested and have no intention to learn proof. These findings are consistent with other studies on practising teachers (e.g., Kotelawala, 2007; Nyaumwe & Buzuzi, 2007), particularly, the indication that many teachers believed proof is only suitable for students who are at senior levels and/or plan to pursue advanced mathematics and proof can be removed from other students' learning (Frasier & Panasuk, 2013; Knuth, 2002b).

Moreover, other common concerns that practising teachers and PTs often have regarding the teaching of proof are related to *time* and *curriculum*. For concerns related to time, research has shown that a number of practising teachers and PTs consider proof to be time-consuming (e.g., Nyaumwe & Buzuzi, 2007). To these teachers, a considerable amount of time is required to (plan to) teach and learn proof properly (Frasier, 2010; Lesseig et al., 2019). Believing that class time is fixed and limited, some practising teachers and PTs are unsure whether there is enough class time to implement proof in their teaching (Lesseig et al., 2019) or whether it is worthwhile teaching proof in school mathematics (Furinghetti & Morselli, 2011), limiting their attitudes towards teaching proof.

In relation to beliefs about school curriculum, a number of practising teachers and PTs believe that proof should be taught in school geometry only rather than in other areas of school mathematics, such as school algebra (Knuth, 2002b; Varghese, 2009). Such belief can influence teachers' decisions about the time spent on proof in different areas of school mathematics, resulting that the teaching of proof is limited to school geometry (Frasier & Panasuk, 2013; Knuth, 2002b). To these teachers, proof tends to appear explicitly and exclusively in school geometry in school mathematics curriculum (Frasier & Panasuk, 2013; Knuth, 2002b; Varghese, 2007), presumably due to their past experiences of proof being explicit in school geometry and lack of experiences of proof in other areas of school mathematics (Mingus & Grassl, 1999).

In addition, belief that proof is not relevant to school curriculum can deter the teaching of proof. Research has indicated that in a range of countries, most of the tasks in textbooks are not designed to provide students with meaningful opportunities of the learning of proof (e.g., Bergwall & Hemmi, 2017; Davis, 2012; Fujita & Jones, 2014; Otten et al., 2014; G. J. Stylianides, 2009; D. R. Thompson et al., 2012; K.-C. Wong & Sutherland, 2018) and most of the tasks in examinations (particularly, those designed by teachers) require students to recall and apply procedural skills only rather than skills in argumentation and proof (C.-Y. Lee, 2021; Palm et al., 2011). Presumably in relation to their experiences with the aforementioned textbooks and examinations and their beliefs about what proof is, a number of teachers believe that textbooks and examinations do not have many items related to proof, but rather focus on recall and procedural application of mathematical ideas (Frasier & Panasuk, 2013; Kotelawala, 2016; Nyaumwe & Buzuzi, 2007). In addition, some PTs believe that school mathematics curriculum does not emphasise proof nor its relationship with the learning of mathematics (Lesseig et al., 2019; Varghese, 2009). To these teachers, proof might become less prioritised in their teaching (Kotelawala, 2016; Nyaumwe & Buzuzi, 2007).

Enjoyment and anxiety about proof. Apart from beliefs, some research has also investigated teachers' emotions related to proof and relationships between emotions and attitudes towards teaching proof. Research has indicated that many practising teachers and PTs had positive emotions related to proof, whereas a number had negative emotions. For example, on the one hand, Frasier (2010) found, in a sample of 374 US secondary mathematics teachers, over 70% reported that they were interested in proof, and over 80% reported that they felt comfortable and considered themselves to be capable of working on and presenting proof, and evaluating students' proofs, expressing their confidence in proof. In another sample of 78 US secondary mathematics teachers, Kotelawala (2007) found that about 40% reported they enjoyed working on proof, and over 40% reported they felt confident about proving mathematical results themselves. Nyaumwe and Buzuzi (2007) found, in a sample of 34 Zimbabwean secondary mathematics teachers, over 70% reported they enjoyed proving mathematical results. Similarly, in a sample of 337 Turkish PTs in both primary and secondary mathematics, Morali et al. (2006) found over 40% reported that they liked working on proof, and about a quarter reported that they had confidence in proof. Ersen (2016) analysed the written responses of another sample of 192 Turkish PTs and found that about 40% used positive vocabulary, such as "proof gives joy" and "it's beneficial", to complete the sentence "Mathematical proof is … because …", indicating their positive opinions and emotions about proof (p. 94).

On the other hand, Frasier (2010) found about 10% of his sample reported that they disliked proof, although about three quarters rejected the statement. Nyaumwe and Buzuzi (2007) found over 10% of their sample considered proof to be boring to teach and about one third considered proof to be time-consuming. Kotelawala (2007) found about a quarter of her sample reported that they had frequent anxiety about proof. Similarly, Morali et al. (2006) found about one third of their sample considered proof to be hard, about 40% considered proof to be unnecessary, and about 40% expressed their anxiety about proof. Ersen (2016) found about one third of her sample used negative vocabulary to describe mathematical proving, such as "difficult", "aggravation", "terrifying" and "unnecessary", expressing their negative opinions and emotions about proof.

Research has also shown that teachers who are more interested in proof tend to have stronger beliefs about the importance of proof and are willing to spend more time on proof in teaching (Frasier, 2010), whereas teachers who experienced more frequent anxiety about proof tend to reduce the amount of proof in teaching (Kotelawala, 2016). Moreover, since emotions can bias beliefs about the past, thereby influencing subsequent beliefs and attitudes (Beswick, 2018; Hannula, 2014; P. Hughes et al., 2019; Valcke et al., 2010), emotions (particularly, anxiety) related to proof might influence beliefs about difficulty in learning and teaching proof and its demands on students, and in turn, beliefs about students' capacity and interest in proof and about self-capacity for proof and the teaching of proof. For example, a small number of practising teachers and PTs are concerned that they do not have the knowledge and skills necessary for the teaching of proof (Frasier & Panasuk, 2013; Lesseig et al., 2019), which appear to be related to their (lack of) past experiences with proof (particularly, in school mathematics) (Mingus & Grassl, 1999; Varghese, 2009), limiting their attitudes towards teaching proof in school mathematics (Kotelawala, 2016).

In summary, beliefs and attitudes towards teaching proof in school mathematics are multifaceted, and existing research has investigated different components of beliefs and attitudes towards teaching proof in school mathematics (and their relationships with the teaching of proof). Major components include beliefs about (the importance of) proof and its teaching, beliefs about students' capacity and interest in proof, and interest and anxiety about proof. Different models of the teaching of proof (in terms of whether, how often and how) have also been observed within and between different practising teachers and PTs, depending on not only their beliefs that have been developed through their experiences with the teaching and learning of mathematics and proof, but also their beliefs about the contexts (e.g., students or the curriculum) that they often encounter when teaching.

Many studies have surveyed the beliefs and attitudes related to proof of practising teachers and PTs in a range of countries and cultures (e.g., Canada, Turkey, the US). In general, many practising teachers and PTs believe that proof is important, and enjoy proof. However, such productive belief and emotion do not necessarily translate into a strong belief that proof should be made central to school mathematics and a regular teaching of proof, as other counterproductive beliefs and emotions that are often related to contexts and limited experiences with proof might play a part. A number of practising teachers and PTs do not consider proof to be a central idea in school mathematics and believe that proof should be introduced to selected groups of students (e.g., students who are at senior levels and/or (plan to) pursue advanced mathematics) or under certain topics only (e.g., school geometry), and some (but not very few) practising teachers and PTs have anxiety about proof and believe that proof is unnecessary in school mathematics and is less important than the development of students' procedural skills in solving (routine) problems, particularly for examinations.

The results of the existing studies on PTs appear to be consistent with each other, implying that beliefs and attitudes towards teaching proof in school mathematics held by PTs in a range of countries and cultures are similar. Having said that, it is worth noting that almost all existing studies on PTs (except Lesseig et al. (2019)) were conducted in Western (e.g., Canada, the US) or Middle Eastern (e.g., Turkey) countries, and any East Asian perspective is limited. Moreover, despite the similarities in beliefs and attitudes towards teaching proof in school mathematics held by their sample PTs, Lesseig et al. (2019) identified that the sample of PTs from South Korea had concerns about maintaining students' interest in mathematics whereas the samples of PTs from Australia and the US did not, implying possible cultural differences. As a result, there is still a need for studying beliefs and attitudes towards teaching proof in school mathematics held by PTs from East Asia (in the present study, Hong Kong PTs).

Intervention studies on beliefs and attitudes towards teaching proof. Longitudinal studies, including intervention studies, are essential to answering questions about the development of PTs' beliefs and attitudes related to proof and the teaching of proof. The questions include (but are not limited to): Whether and how do PTs develop their current beliefs and attitudes? Whether and how can a training programme facilitate change in PTs' beliefs and attitudes? What feature(s) of the programme are essential for the change? The existing literature on intervention studies that involved PTs in the area of proof (Buchbinder & McCrone, 2020; Conner et al., 2011; Kempen & Biehler, 2019; Koichu, 2012; Shilling-Traina & Stylianides, 2013; G. J. Stylianides & Stylianides, 2009; Yoo, 2008; Zengin, 2017) underpins the research questions and the intervention design of the present study.

Kempen and Biehler (2019), Koichu (2012) and G. J. Stylianides and Stylianides (2009) each taught a mathematics education course involving PTs who had limited experiences with university mathematics. In their articles, they reported instructional sequences that aroused the PTs' awareness of the need for proof when doing mathematics, resulting in that the PTs successfully progressed from experimenting with empirical arguments to using definitions and proof when doing mathematics. Moreover, further investigation into the intervention developed by G. J. Stylianides and Stylianides (2009) indicated some changes in PTs' beliefs and attitudes: The PTs developed more positive self-efficacy beliefs about mathematical problem-solving (A. J. Stylianides & Stylianides, 2014), and in a later re-run, the PTs developed more problem-solving beliefs about mathematics (Shilling-Traina & Stylianides, 2013).

Buchbinder and McCrone (2020) taught a mathematics education course involving senior PTs. In their one-semester intervention, Buchbinder and McCrone aimed to promote the PTs' knowledge for teaching proof by engaging them in identifying opportunities to implement proof-related tasks in school mathematics, drafting lesson plans with peer and instructor's feedback, and experiencing different models of proof-related tasks. After the intervention, the PTs became more able to develop lesson plans that integrated proof with different topics and at different grade levels. Whilst Buchbinder and McCrone (2020) focused on PTs' knowledge for teaching proof (particularly their ability to develop lesson plans that integrate proof), they used the comments from two PTs to touch on beliefs and attitudes towards teaching proof, and remarked that there were "positive shifts in [the PTs'] perceptions of the feasibility of teaching proof at the secondary level and their increased confidence in doing so" (p. 16).

The studies discussed mainly focused on improving PTs' proof-related knowledge, and touched on their proof-related beliefs and attitudes. However, there are a few intervention studies investigating PTs' proof-related beliefs and attitudes. In the following, I discuss these studies.

Yoo (2008) compared the effects of lecture-based instruction and problem-based instruction on the changes in PTs' beliefs. She found that problem-based instruction provided opportunities for the PTs to develop more humanist beliefs about the nature and role of proof and more process-oriented beliefs about approaches for teaching mathematics and proof than lecture-based instruction. The PTs in problem-based instruction saw the purpose of proof as to promote students' understanding and explanation of what and why mathematical statements are true, which is often argued to be the main reason for making proof central to school mathematics (Hanna, 1990; Hersh, 1993; Zaslavsky et al., 2012). As problem-based instruction usually involves students discussing mathematics a great deal (Goos, 2004; Smith et al., 2009), Yoo's findings also suggested that involving the PTs discussing and working on mathematics and proof not only actively helped them develop more humanist and process-oriented beliefs, but also a connection between process-oriented beliefs (e.g., about involving students in participating in mathematical discussion and collaboration) and decisions about the teaching of proof.

Unlike Yoo's findings, Conner et al. (2011) reported that after a two-semester intervention, the PTs' beliefs about the nature of mathematics and proof were stable, whereas their beliefs about teaching were changed from more teacher-centred beliefs to beliefs that value the activities and understandings of students. Conner et al. (2011) attributed the stability of beliefs about the nature of mathematics and proof to the facts that the PTs had experienced university mathematics for several years and had developed appreciation of the need for proof and that "there was no attempt to directly confront [the PTs'] mathematical beliefs" (p. 498).

Zengin (2017) developed a nine-week intervention that involved PTs in a dynamic geometry environment by using GeoGebra (a dynamic mathematics software). He investigated the effect of GeoGebra on the PTs' attitudes towards working on proof. After his intervention, the PTs developed more positive attitudes towards working on proof: appreciating need for proof, liking and having more confidence in proof, and reducing anxiety about proof. Zengin attributed the changes in attitudes to the enjoyable and visual learning environment created by the software. Having said that, the study appeared to have some limitations. First, Zengin (2017) remarked, "[s]ince there was no control group, it was difficult to attribute the changes to only the intervention" (p. 1019). Second, his argument about the changes in the PTs' attitudes towards working on proof was limited by the fact that only written data were collected (via Likert items and open-ended questions) and analysed. Moreover, whilst Zengin discussed and made some connection between attitudes towards working on proof and attitudes towards teaching proof, whether the PTs developed more productive beliefs and attitudes towards teaching proof remains open, because the data did not involve the situations of 'teaching' or 'school mathematics', but the perspective of PTs as provers. It is also worth noting that Zengin's intervention was restricted to geometric proof only, rather than involving proof in both school algebra and school geometry.

In summary, there are some studies that do assess changes in different aspects of PTs' proof-related beliefs and attitudes, but more research is needed (Conner et al., 2011; Pajares, 1992). The reviewed intervention studies indicated some changes in some aspects of beliefs and attitudes towards teaching proof, for example, beliefs about approaches for teaching mathematics and proof (Conner et al., 2011; Shilling-Traina & Stylianides, 2013; Yoo, 2008), and attitudes towards working on proof (Zengin, 2017). However, some other aspects were not investigated (e.g., beliefs about the importance of proof in school mathematics), and questions regarding how to facilitate change in beliefs and attitudes towards teaching proof (e.g., beliefs about the nature and role of proof) remain open (Conner et al., 2011).

Moreover, the reviewed studies that focused on PTs' proof-related beliefs and attitudes involved interventions of 'relatively long' duration ranging from nine weeks (Zengin, 2017) to two semesters (Conner et al., 2011), except the study of A. J. Stylianides and Stylianides (2014) that involved a 75-min intervention. Such interventions of long duration may not be easily adopted by practitioners into existing curricular structures or used at scale due to the time demands of the interventions (A. J. Stylianides & Stylianides, 2013); such limitation inspired the need for the present study that aimed to develop an effective intervention of relatively short duration (three weekly 2-hour workshops) that promotes PTs' productive beliefs and attitudes towards teaching proof in school mathematics. As beliefs and attitudes towards teaching proof are often considered to be difficult to change, probing into 'what an intervention of short duration can work and how it works' can also address the overarching question of the present study: ***Whether and how PTs' beliefs and attitudes towards teaching proof can be changed in a short period?*** This question could be considered as a "challenging but important question for mathematics education researchers: *Would it be possible to design classroom-based interventions of short duration in mathematics classrooms that could help alleviate significant problems of students' learning in mathematics?*" (A. J. Stylianides & Stylianides, 2013, p. 339).

2.4 Revisiting research questions

I revisit the research questions (RQs 1–3) that guided the present study to discuss how they are linked to the overarching question of the present study.

RQ 1. What were Hong Kong preservice teachers' pre-intervention beliefs and attitudes towards teaching proof in school mathematics?

RQ 1 served two purposes. First, it allowed me to document the initial state of beliefs and attitudes towards teaching proof in school mathematics held by a sample of Hong Kong PTs before they participated in the intervention (Research Cycle 4) of the present study, providing a baseline for the analysis of post-intervention belief and attitude changes of the PTs (focus of RQ 2). Second, supplemented with data from Research Cycles 1–3, the analysis gave a more general description of beliefs and attitudes towards teaching proof held by a larger sample of Hong Kong PTs. I also compared the results of the Hong Kong sample and other relevant studies, exploring possible cultural differences (discussed in Chapter 5).

RQ 2. How did Hong Kong preservice teachers' beliefs and attitudes towards teaching proof change after participating in a specifically designed intervention of short duration?

I used RQ 2 to guide the investigation that aimed at addressing the ‘whether’ part of the overarching question, by identifying different changes (or not) in different components of beliefs and attitudes towards teaching proof. The analysis addressed the need for developing an intervention of short duration that can promote PTs’ productive beliefs and attitudes towards teaching proof in school mathematics and studying the change in their beliefs and attitudes towards teaching proof (Conner et al., 2011; A. J. Stylianides & Stylianides, 2013; G. J. Stylianides et al., 2017). It also formed basis for the subsequent analysis of links between components of the intervention and the PTs’ post-intervention belief and attitude changes (focus of RQ 3). Here ‘a component’ of the intervention refers to an activity in the intervention or a design principle that guided the development of the intervention (a feature) (to be discussed in Section 3.2.2).

RQ 3. What components of the intervention were perceived by the preservice teachers as contributing to changes in their beliefs and attitudes (as identified in RQ 2)?

I used RQ 3 to guide the investigation that aimed at addressing the ‘how’ part of the overarching question, by identifying links between components of the intervention and the PTs’ post-intervention belief and attitude changes, from the PTs’ perspectives. The analysis allowed me to investigate *why* and *how* the intervention worked, providing support for design principles to guide belief and attitude changes, particularly related to (the teaching of) proof, and in turn, investigating the potential of the intervention to be adapted and applied in teacher education (G. J. Stylianides et al., 2017).

Chapter 3: Methodology

Design-based research (DBR) formed the basis of the present study's research design. The present study went through four research cycles (Research Cycles 1–4) of implementation, evaluation and revision of an intervention that aimed to promote preservice teachers' (PTs') beliefs and attitudes towards teaching proof in school mathematics. Data were collected by using a questionnaire (before and after the intervention), individual semi-structured interviews (after the intervention), and audio-recordings and fieldnotes of the intervention. Between two research cycles, the intervention design underwent formative evaluation to inspect what parts of the intervention worked or did not (and how), informing revisions of the design and/or refinements of the implementation. To do so, I looked at the intervention as a whole, as well as breaking it down into different activities and tasks. "A part works" when it was found from the data (e.g., interviews, audio-recordings, fieldnotes) to have a contribution to the development of PTs' more productive beliefs and attitudes towards teaching proof (e.g., degree of participation). In this thesis, Chapter 4 (the results chapter) focuses on the reporting of the data collected from Research Cycle 4, whilst in Section 3.2.2, I give a description of how the intervention evolved over research cycles.

In this chapter, I first discuss the context in which the participants were recruited, and the recruitment methods used. This includes how this context and the participants influenced the format and design of the intervention (Section 3.1). Then, I discuss the rationale and use of DBR and the evolution of the intervention design (Section 3.2). Next, I describe data collection process, and procedures for analysing the data (Sections 3.3–3.4). This chapter ends with a discussion on ethical considerations (Section 3.5).

3.1 The context and participants

The present study took place in Hong Kong in 2018–2019. In Hong Kong, one can become a mathematics teacher by obtaining an undergraduate (UG) degree from a five-year mathematics education programme, or obtaining an UG degree in a mathematics-related subject[1] followed by a one-year full-time postgraduate (PG) training in teaching mathematics. Although one can also be employed (as a teaching assistant or a teacher) by schools and given a permit by Education Bureau[2] to teach mathematics in specified school(s) after obtaining a mathematics-related degree, it is very common for them to complete a two-year part-time PG teacher training in their first few years of teaching. The part-time training does not differ from its full-time counterpart, regarding programme structures, except that those who enrol in part-time PG training have to be a practising teacher as an entry requirement[3]. In this sense, those who enrol in part-time PG training are considered as practising teachers, whereas those who enrol in full-time PG training or study in an UG mathematics education programme are considered as PTs, the target participants of the present study.

There are five universities offering mathematics teachers training programmes in Hong Kong. Three of the universities are considered as the major universities because over 90% of Hong Kong PTs enrol in one of the programmes offered by the three universities. In academic years 2018/2019 and 2019/2020, University A offered two UG programmes in primary and secondary mathematics respectively, having around 300 PTs in total; University B offered one

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[1] Mathematics-related subjects include, but are not limited to, mathematics, physics, engineering and computer sciences.
[2] https://www.edb.gov.hk/en/teacher/qualification-training-development/qualification/teacher-registration/index.html
[3] https://www.eduhk.hk/acadprog/pgde/admission.htm; https://www.fed.cuhk.edu.hk/pgde/cuhk_admission_requirements.html; https://web.edu.hku.hk/programme/pgde/admissions

UG programme in secondary mathematics and two PG programmes in primary and secondary mathematics respectively, having around 130 PTs; University C offered one PG programme in secondary mathematics, having around 15 PTs. Among different universities, the corresponding programmes have similar graduation requirements, in terms of courses related to mathematics education. The UG programmes require PTs to complete at least 14 university mathematics courses and at least 10 education courses, and the PG programmes require PTs to complete a two-semester course in pedagogy.

Hong Kong PTs from the three major universities, regardless of the programmes and years of study, were recruited, for two reasons. First, such inclusion allowed me to understand the general status of Hong Kong PTs' beliefs and attitudes towards teaching proof. By examining their beliefs and attitudes, I could identify the major counterproductive beliefs and attitudes that made them hesitate to make proof central to their future teaching. Second, a study finding that PTs from different mathematical backgrounds and at different years of study could develop more positive problem-solving beliefs about mathematics within a university mathematics course (Shilling-Traina & Stylianides, 2013) provided optimism that it was possible for Hong Kong PTs from different programmes and at different years of study to develop more productive beliefs and attitudes towards teaching proof within a carefully designed and implemented intervention.

Before recruiting participants, I sought for approvals for class visits and recruitment from mathematics education departments of the three major universities and course instructors. Also, invitations were sent to the PTs via internal emails thanks to the support of the departments and instructors. Interested PTs replied to the invitation and registered for a scheduled intervention.

Unlike other intervention studies on PTs and proof (see Section 2.3), in which interventions were implemented in regular mathematics education courses and PTs took part in the interventions compulsorily[4], the intervention of the present study was designed as extracurricular workshops, for several reasons. First, there was a need for developing a new intervention of short duration that could promote PTs' beliefs and attitudes towards teaching proof in school mathematics (Section 2.3). The extracurricular format allowed the design of the intervention of short duration to be tested and revised multiple times within a reasonably short period of time (a year), resulting in a manageable research project.

Second, being not an instructor of any mathematics teacher training programme in Hong Kong, I did not have access to any regular course to implement my intervention. Third, there was limited room and incentive for course instructors to revise the course contents and fit my intervention design into their courses, because such change of a programme needs to be audited by the university and an external body.

Fourth, if the intervention of the present study was implemented in a regular course, the participants would have been limited to a single programme and years of study per implementation due to curricular settings. Last but not least, designing the intervention as extracurricular workshops could reduce burdens that PTs often have in a regular course (e.g., deadlines, grading). PTs also had more autonomy to exercise their right to opt out if they do not find the intervention suitable (British Educational Research Association, 2018). If the intervention was implemented in a regular course and the PTs found it unsuitable, they would have less freedom to drop out of the intervention except for very limited options (e.g., voicing their concerns, refusing to allow their data to be used).

[4] A PT's participation in an intervention does not necessarily translate into their participation in a study, because they can exercise their right to refuse to allow their data to be used in the study.

The decision of designing and implementing the intervention as extracurricular workshops were made based on the research, practical and ethical issues as discussed above, but this decision could result in a sample of PTs who had specific (positive) beliefs and particular interest in the area of proof. Having said that, the data indicated that there were some participating PTs who reported that they did not find proof important and had negative emotions about mathematics and proof. They stated that they participated in the intervention because they wanted to gain knowledge of teaching mathematics at school. This suggests that the decision of having the intervention as extracurricular workshops did not ultimately result in a sample of PTs who only held positive beliefs and/or had particular interest in proof, but allowed the inclusion of PTs who had different beliefs and emotions about mathematics and proof to learn the teaching of school mathematics using proof and argumentation.

The intervention of the present study was designed and took place in the form of three weekly workshops. Each workshop lasted two hours. Due to university policies and unforeseen situations in Hong Kong[5], the intervention could only take place at University A. Consequently, most of the participants were recruited from University A (Table 3.1.1).

Table 3.1.1
Distribution of the participants who took part in the intervention

	Research Cycle 1	Research Cycle 2	Research Cycle 3	Research Cycle 4
University A	8	12	3	23
University B	3	0	0	3
University C	0	2	1	1

[5] Since June 2019, Hong Kong had undergone numbers of pro-democracy protests against the government and police violence, causing interruptions to travel and different degrees of suspension of universities (see also Section 3.5).

3.2 Design-based research

Design-based research (DBR), as a methodology, aims to improve educational issues by systematically developing and studying an intervention design (often in naturalistic settings) (van den Akker et al., 2013). This methodology does not follow the strict requirements of a traditional educational experiment (e.g., randomised controlled trials) (Cohen et al., 2011), but is similar to an experiment in a sense that it is focused on domain-specific, intentional interventions (Cobb et al., 2003; Schoenfeld, 2006).

DBR is appropriate for open or difficult-to-solve problems. DBR is suggested when the initial situation is unclear or complex, the goal situation is unclear or difficult to be attained (e.g., due to complex external factors that may negatively affect progress), and/or the transition process from the initial situation to the goal situation is unknown or unclear (Kelly, 2013). If research is situated in a complex, unclear context, it is difficult for researchers to design an intervention and instruments for traditional experiments that have tightly controlled conditions (Schoenfeld, 2006).

DBR suits the present study for two main reasons. First, although there are quite extensive cross-sectional studies on beliefs and attitudes related to the teaching of proof in other countries (Section 2.3) and some indication that Hong Kong PTs and PTs in other countries might have similar beliefs and attitudes towards teaching proof, the status of Hong Kong PTs' beliefs and attitudes towards teaching proof had remained unclear and complex (see Section 1.2). Second, changing PTs' beliefs and attitudes (towards teaching proof) is considered to be difficult and complex (Conner et al., 2011; Kagan, 1992; Richardson, 2003), presumably because of the complexity of beliefs and attitudes and the difficulty in assessing them (Ashton, 2014; Hoffman & Seidel, 2014; Schraw & Olafson, 2014). Therefore, the use of DBR enabled me to refine my data collection instruments and to investigate Hong Kong PTs' initial beliefs and attitudes towards teaching proof, thereby developing an adequate intervention and investigating how to create a learning environment that can facilitate change in beliefs and attitudes in a relatively short period.

There are four agreed characteristics of DBR suggested by various researchers (Cobb et al., 2003; Collins, 1992; McKenney & Reeves, 2012; van den Akker et al., 2006). First, an educational DBR aims to develop an instructional design (intervention) that can address educational issues in real-world situations. Second, DBR focuses on the evolution of the design. By implementing the design with hypotheses of interest, researchers can reflect on existing theoretical propositions and findings from multiple try-outs. Different parts of the design are evaluated concerning what and how they work, such that improvement can be achieved (Cobb et al., 2003). Third, DBR not only serves practical need for a design that addresses educational issues in real-world situation, but also allows researchers to contribute to theory building, validation and/or development by testing hypotheses and theoretical propositions in try-outs of the design. Last but not least, DBR involves research cycles that feature invention, implementation, evaluation and revision of the design.

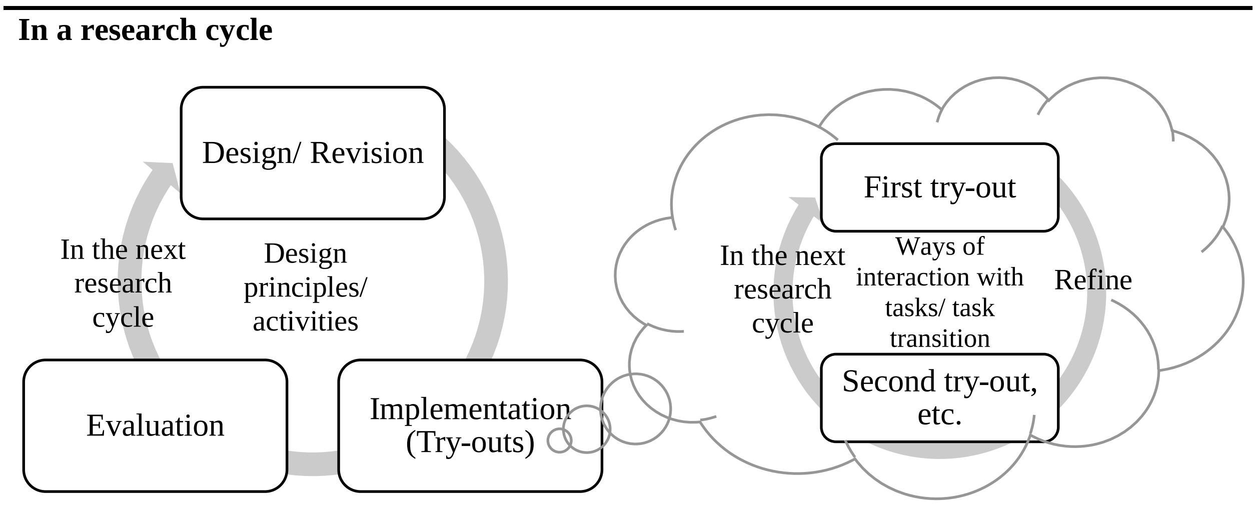
The study reported in this thesis adhered to these four interrelated characteristics. First, the present study aimed to develop an effective intervention that promotes PTs' more productive beliefs and attitudes towards teaching proof in school mathematics. Second, based on literature and other relevant studies on changing beliefs and attitudes and data from early research cycles, I improved the design. Third, apart from developing an intervention that served practical need to promote Hong Kong PTs' beliefs and attitudes towards teaching proof, the present study also made contributions to research progression. In the present study, I adapted several existing principles of designing promising interventions that encourage PTs' learning (to teach) from findings of different studies (see Section 3.2.2) and tested them in an innovative intervention from an East Asian perspective and of a short duration. Lastly, the present study consisted of four research cycles of designing, implementing, evaluating and revising the intervention design. After implemented with groups of PTs in a research cycle, the design was evaluated and revised for new groups of PTs in the next cycle.

DBR can be generally characterised into two distinct, but not necessarily separate, orientations: development and validation studies (van den Akker et al., 2013). A development study mainly serves the development of a usable, effective intervention and the construction of design principles that resolve certain educational problems, whereas a validation study focuses on the design of learning environments or trajectories with the purpose of developing and validating theories about the process of learning and how learning environments can be designed. The present study could be identified as a hybrid of development and validation studies, as I was not only developing a new intervention and searching for design principles that could help PTs develop more productive beliefs and attitudes towards teaching proof in a relatively short period, but at the same time investigating if some design principles and theories developed in other contexts could be applied in the present study.

DBR can involve qualitative, quantitative, or both types of data and methods (Collins et al., 2004; van den Akker et al., 2013). In the present study, I used a questionnaire, individual semi-structured interviews, fieldnotes and audio-recordings of the intervention as sources to gather both qualitative and quantitative data. These data were used to evaluate the design, regarding what worked, and how and why it worked, which in turn informed revisions of the design: I investigated the effects of the design on changing the participants' beliefs and attitudes towards teaching proof by analysing pre-post data, and examined how the design worked by analysing the participants' comments and comparing different try-outs of the design as well as their outcomes. Details of data collection and analysis, for the final research cycle in particular, are discussed in Sections 3.3 and 3.4, respectively.

Apart from revisions of the intervention design between research cycles, the design was also refined within each research cycle. In each research cycle, I conducted the design with several different groups of PTs. Here, I use the term "try-out" to refer to the implementation of the intervention design with a group of PTs within a research cycle. Between and within try-outs, the design was refined concerning the ways how the participants and I, the instructor, interacted and transition of different tasks (Figure 3.2.1). Details of the research cycles and the evolution of the design are discussed in the following Sections 3.2.1 and 3.2.2, respectively.

Figure 3.2.1
Cycles of design, implementation, evaluation and revision



3.2.1 Multiple research cycles

The present study consisted of four research cycles (Research Cycles 1–4) of designing, implementing, evaluating and revising an intervention. Each research cycle had different foci regarding the development of the intervention; in general, Research Cycles 1–3 aimed to investigate the 'general state' of Hong Kong PTs' beliefs and attitudes towards teaching proof and to improve and refine the intervention design by implementing and evaluating the design with different PTs, and Research Cycle 4 focused on evaluating the final version. In the following, I give a description of each research cycle. As there were different versions of the intervention design throughout the study, I hereafter used 'the design' to refer to early versions of the intervention design in Research Cycles 1–3 and 'the intervention' to refer to its final version in Research Cycle 4. Table 3.2.1 shows a summary of my actions in each research cycle.

Research Cycle 1 (June 2018–May 2019). This cycle had two main purposes. First, this cycle aimed at designing, implementing and evaluating the first design and data collection instruments (i.e., a questionnaire, semi-structured interviews; hereafter referred to as the instruments). Since adequate assumptions of the initial state of participants are essential to the development of a meaningful intervention but the status of Hong Kong PTs' beliefs and attitudes towards teaching proof had not been widely studied, it was difficult to make assumptions most suitable to develop the intervention design of the present study. Therefore, the second purpose was to preliminarily study Hong Kong PTs' beliefs and attitudes. The adverb 'preliminarily' is used because, in this cycle, the first version of the questionnaire was administered to two classes of PTs (n = 57) – the sample size was small and the questionnaire had not been tested to be adequate for the present study.

The first design was developed based on my prior knowledge of Hong Kong practising teachers and PTs, and existing research with PTs in other countries. Before conducting the intervention design with PTs, I piloted a part of the design with four practising teachers, who had taught mathematics for at most five years, in order to seek for their feedback on the activities and to study their beliefs and attitudes towards teaching proof, reported in C.-Y. Lee (2019). The pilot was conducted in parallel with the preliminary questionnaire study on PTs' beliefs and attitudes towards teaching proof, between July and September 2018. The results of the pilot and the preliminary questionnaire study enabled me to refine the design and the instruments before conducting the design with PTs.

The refined design and the refined instruments were then conducted with five PTs in secondary mathematics (Try-out 1) in January 2019 and six PTs in primary mathematics (Try-out 2) in April 2019. Then, both the design and the instruments were formatively evaluated. In this cycle, I developed better understanding of Hong Kong PTs' beliefs and attitudes towards teaching proof, and the basis of the instruments was formed. Some activities of the design were also revised (Section 3.2.2).

Research Cycle 2 (June–July 2019). This cycle focused on improving the design. The design was conducted with 10 PTs in primary mathematics (Try-out 3), four PTs in secondary mathematics (Try-out 4) and one group of seven experienced secondary mathematics teachers, in parallel, between June and July 2019. I compared the effects of the design on different groups of PTs, and explored which activities worked in one group but did not work as expected in another. I also sought the experienced teachers' feedback on the intervention, concerning whether the contents were relevant to their teaching. Based on the data from Research Cycles 1–2, I was convinced that the design could have some positive effects on PTs' beliefs and attitudes towards teaching proof, and its contents were relevant to the work of teaching in Hong Kong, despite the incomplete data of Try-out 3[6].

[6] In Try-out 3, whilst five PTs completed the questionnaire before the intervention, post-intervention data were missing – only one PT partly completed the questionnaire again and none attended the semi-structured interview.

Research Cycle 3 (August–September 2019). This cycle aimed to prepare the design and the instruments (particularly the questionnaire) for summative evaluation in the successive Research Cycle 4. To do this, I conducted the design with four PTs (Try-out 5) in August 2019 and administered two parts of the questionnaire (Likert items and two open-ended questions about the teaching of proof) to 184 PTs for evaluation (see Section 3.3.1) in September 2019. The data from Research Cycles 1–3 were used to finalise the design and the instruments.

Research Cycle 4 (October–November 2019). This cycle aimed to conduct summative evaluation of the intervention. I conducted the intervention with six groups (Try-outs 6–11), a total of 27 PTs, between October and November 2019. Pre-intervention data were used to address the question about the PTs' initial beliefs and attitudes towards teaching proof (RQ 1). Pre- and post-intervention data were compared to address the questions about effects and effectiveness of the intervention (RQ 2). Interview data of the PTs' views about their experiences of the intervention were used to investigate what features of the intervention (and how) facilitated change in PTs' beliefs and attitudes towards teaching proof (RQ 3).

Table 3.2.1
Summary of the research cycles

Research Cycle		Actions
1	Jun – Sep 2018	• Preliminary research on the status of PTs' beliefs and attitudes • Develop and pilot the instruments • Develop the design and pilot a part of it; seek for teachers' feedback
	Oct – Dec 2018	• Reflect on the pilot and feedback • Revise the instruments
	Jan – Apr 2019	• Conduct the design • Research on the status of PTs' beliefs and attitudes
	May 2019	• Reflect and revise the design • Revise the instruments
2	Jun – Jul 2019	• Conduct the design • Research on the status of PTs' beliefs and attitudes • Seek for teachers' feedback • Reflect and revise the design • Revise the instruments
3	Aug – Sep 2019	• Research on the status of PTs' beliefs and attitudes

		• Conduct the design • Reflect and revise the design (into the intervention) • Finalise and evaluate the instruments
4	Sep – Nov 2019 Dec 2019 – 2021	• Conduct the intervention • Evaluate the intervention

More try-outs and data collection processes (e.g., interviews) were originally planned in December 2019. However, suspension of universities in Hong Kong (since late November 2019) resulted in cancellation of the plans and a reduction in the amount of data collected. I discuss this issue in detail in Section 3.5.

3.2.2 Evolution of intervention

As research cycles advanced, the intervention design evolved concerning the sequence and purposes of the activities and the composition of the features of the intervention. In this section, I first discuss how the existing studies and theories were incorporated in the design, and how the try-outs of the design in Research Cycles 1–3 improved my understanding of the existing studies and theories that I used, and improved the design. I then describe the framework of the intervention conducted in Research Cycle 4.

In Research Cycle 1, the design was developed based on a temporary assumption that Hong Kong PTs' beliefs and attitudes towards teaching proof are similar to their counterparts in other countries, in terms of the structure and status of beliefs and attitudes (Sections 1.2 & 1.4). The design was intended to make participants aware of four statements. First, proof can be satisfying and enjoyable (Frasier, 2015; Waring, 2001b). Second, there are educational needs for proof in school mathematics (Zaslavsky et al., 2012). Third, 'difficult' tasks of proof can be solvable and understood within students' capacity (Nardi & Knuth, 2017; Waring, 2001a). Fourth, proof is not restricted to systematised mathematical referents. For example, the notions of preformal proofs (Blum & Kirsch, 1991; van Asch, 1993) and generic examples (Balacheff, 1988; Mason & Pimm, 1984) can be counted as proof in school mathematics (A. J. Stylianides, 2007).

Relevant research has suggested that change in PTs' beliefs (about mathematics) can be facilitated by incorporating three design principles in the design, namely, using activities based on the work of teaching (Shilling-Traina & Stylianides, 2013), using problem-solving tasks (Liljedahl, 2005; Schram et al., 1988) and emphasising social interactions (e.g., involving group collaboration and explanation) (Goos & Geiger, 2010; Schram et al., 1988; Shilling-Traina & Stylianides, 2013). The principles were intended to stimulate cognitive conflict (Limón, 2001; Rolka & Liljedahl, 2007; Zaslavsky, 2005) for changing PTs' beliefs and attitudes. The design consisted of selected excerpts from two videos (HK3 Polygons and NL2 Pythagorean theorem) of TIMSS 1999 Video Study[7] (TIMSS videos), a direct lecture about the operational definition and educational roles of proof in school mathematics as suggested by experts (Section 2.1), a range of problem-solving tasks, and mini-teaching with some of these tasks.

The design consisted of three weekly workshops (Sessions 1–3), each lasting two hours. At Session 1, the TIMSS videos were shown to PTs, and the PTs were asked to compare and discuss the use of proof in the videos. The operational definition and educational roles of proof and proving in school mathematics were then introduced through direct lecture, aiming to raise PTs' awareness of the importance of proof in school mathematics. Later, the PTs were asked to produce (multiple) solutions to a few problem-solving tasks; they were encouraged to produce and present their solutions in different representations, aiming to stimulate their reflection on the use of proof in school mathematics as well as their awareness of difficulties that students may

[7] http://www.timssvideo.com/ According to the website, the videos are available to public.

encounter when learning proof. At Session 2, the PTs were engaged in more problem-solving tasks and discussions on strengths and weaknesses of different representations of a proof in terms of learning mathematics, aiming to make them more familiar with the notion of multiple representations, thereby helping them become aware that proof is not restricted to systematised mathematical referents. Session 3 was intended to help the PTs become familiar with integrating proof into the teaching of mathematics, by asking each of them to prepare and demonstrate a mini-teaching[8] on a mathematical topic. The use of mini-teaching aimed to help the PTs develop teaching skills and gain confidence to teach proof (Allen, 1967; Bilen, 2015).

After implementing the design in Research Cycle 1, I had a number of observations. I present five observations to demonstrate how they guided my revision of the design. First, direct lecture could transmit the information, but did not work well on facilitating discussions on the operational definition and roles of proof. Second, the participating PTs' initial engagement was passive and silent, consistent with the findings of studies on characteristics of Hong Kong students' learning styles (Huang, 2002; Sit, 2013), and a few hesitated about engaging in discussions throughout the design try-out. Third, after the try-out, some PTs remained hesitant to teach proof and worried about students' incapability and lack of interest in learning proof. I use Participant D's interview excerpt to illustrate the third observation:

Interviewer:	[…] You mentioned that if facing different students… you assumed the weaker students […] you would have different [teaching] approaches. What are the differences?
Participant D:	The differences are … [I] would not explain [the motivation behind the idea], and directly state the formula. […] Then, proof would be removed too. [And say,] "now you should learn this formula, remember it, [and] complete the exercise." Because if they are weak [in mathematics], they would not be interested in learning [proof]. They probably have this thought. Then, if they don't want to learn but [I] talk too much, it won't

[8] As the term *micro-teaching* can mean that a PT gives a (short) lesson to a group of target students or, sometimes, peers, I use *mini-teaching* to indicate a micro-teaching that the audience are peer PTs, to avoid confusion.

> work. That is, they would be confused, then you ask them to do maths exercises – which is not good. So [I would] directly, from the start, state the formula [and] ask them to start working [on exercises].

Fourth, a few problem-solving tasks were left incomplete and skipped during the design try-outs. Lastly, some PTs found the mini-teaching activity helpful and wanted to spend more time discussing different approaches for teaching proof. This observation did not contradict the first and third, but provided some optimism about transition from a more passive, silent learning situation to a more active, discussion-based setting of the design. I use Participant E's interview excerpt to illustrate the fifth observation:

> Interviewer: What have you experienced or learnt in [the intervention]?
>
> Participant E: In the workshops, we were asked to share how we would teach the proof. I think the advantage is [to provide an opportunity] to see how different people would use different methods for treating the same proof, which is very interesting. Especially, when facing another group of students, a different proof may help them learn better. […] [This activity] helped us to think about how to design [our instruction]. That is, in the classroom, how to present a proof. […] I think the direction [of my instruction] became clear.

The first and second observations reflected a common learning culture in Hong Kong. In Hong Kong, lecturing is common in mathematics lessons (F. K.-S. Leung, 1995; Mok, 2009). It is possible that some PTs hesitated about sharing their opinions because of their past learning experiences, that they had developed reliance on instructor's lecturing and devalued their own opinions when being involved in any educational programme. This interpretation seemed to be reasonable because, in subsequent research cycles, the second observation remained at the beginning of the try-outs. For example, in Research Cycle 4, Participant 7 expressed, "[…] at school, my expectation seemed not to be valued. Whatever the teacher told me, I listened."

As the third observation suggested, first, one thing that made PTs hesitant to teach proof was their concerns about students' incapability and lack of interest in learning proof. Second, the design did not work well on helping the PTs develop the intended ideas, that is, that proof can be satisfying and enjoyable for students and that ‘difficult’ tasks of proof can be solvable and understood within students’ capacity.

To address the first three observations, I made use of notions of social interaction and pre-class preparation. Social interaction (e.g., discussion, group work) can encourage change in beliefs and attitudes (Goos & Geiger, 2010; Shilling-Traina & Stylianides, 2013; Wilcox et al., 1991), and pre-class preparation can strengthen sense of preparedness and reduce anxiety about expressing opinions (Abdous, 2019; Hsu, 2015) which in turn, facilitate active participation in discussion and collaboration (Alayont, 2014; Strayer, 2012).

Therefore, I included a self-introduction activity at the beginning of the try-outs – the PTs were asked to introduce themselves to each other and express their opinions about the teaching of mathematics, about what they expected to experience in the intervention and about roles of proof. This activity was intended to introduce an idea that all opinions were welcomed and the PTs could contribute to collective understanding of the teaching of mathematics and proof.

To further facilitate active participation in discussion and collaboration, I also used take-home tasks as pre-class preparation to strengthen the PTs’ sense of preparedness and reduce their anxiety about expressing opinions. At the end of Sessions 1 and 2, I gave two sets of worksheets to the PTs respectively. The worksheets were intended to prepare the PTs for Sessions 2 and 3, particularly preparing them to discuss notion of multiple representations and methods of proving at Session 2 and to develop, demonstrate and discuss their lesson plans at Session 3.

To create more opportunities for the PTs to discuss in the intervention, I also replaced the lecture on the operational definition and roles of proof in school mathematics at Session 1 with activities and follow-up discussions, in which the information was embedded. For example, I used two videos of Deborah Ball's lessons with primary students (Ball's videos)[9] (The Mathematics Teaching and Learning to Teach Project, 2010a, 2010b) to encourage discussion on roles of proof (e.g., proof and argumentation can facilitate meaningful dialogues in mathematics classrooms and promote students' conceptual understanding) and to challenge the PTs' beliefs such as "proof is too abstract and difficult for students" and "proof is restricted to systematised mathematical referents".

To address the fourth and fifth observations, I reduced the number of problem-solving tasks in the intervention, and redesigned the tasks that remained as models of learning activities with proof. The reduction in the number of tasks allowed more time to be spent on discussing multiple representations and methods of proving and exploring how a task and its proofs can be applied or generalised to other mathematical ideas. The redesigned tasks were intended to create an environment in which the PTs took part as learners of mathematics and explored mathematical ideas through proof and argumentation (Goos & Geiger, 2010; Shilling-Traina & Stylianides, 2013), thereby demonstrating models of different approaches for integrating proof into teaching (Buchbinder & McCrone, 2020).

The revised intervention design was then subjected to iterations of try-outs, evaluation and revision in Research Cycles 2 and 3. In Research Cycle 2, apart from conducting the design with PTs, I also conducted the design with seven experienced teachers and interviewed three of them to seek for their feedback. They reported that the activities in the intervention were related to their work of teaching and inspired them to integrate proof and inquiry-based instruction, such as the model activities of learning proof, into their own teaching.

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[9] The use of the two videos in the intervention was approved by the University of Michigan.

Drawing from the data of Research Cycles 1–3, I observed some development of the PTs' more productive beliefs and attitudes towards teaching proof. Analysis of pre-post data showed that after the design try-outs, more PTs agreed that proof is necessary to the development of one's logical thinking, and reported reduced anxiety about proof. After graduating from their teacher training[10], two PTs shared with me how they integrated proof and inquiry-based instruction into their own teaching. These observations convinced me that the revised design was adequate to help the PTs develop more productive beliefs and attitudes towards teaching proof, so I finalised the design for the intervention study in Research Cycle 4. In the following, I report the finalised intervention in detail.

The intervention in Research Cycle 4. The intervention aimed to help participants see that with careful planning and appropriate implementation, (i) proof is *enjoyable*, (ii) proof is worthwhile and accessible to *all* students, and (iii) proof is *relevant* to student learning in school mathematics (particularly fostering conceptual understanding, preparing students for examinations). To achieve these intended goals, the intervention incorporated featuring activities including videos of classroom instruction that involved proof and argumentation, learning activities that involved proof and were intended as models of different approaches for integrating proof into teaching, and mini-teaching. The work of G. J. Stylianides and Stylianides (2009) inspired the design of the intervention in the present study – due to the similarities between their study and the present study (i.e., to develop interventions in the area of proof), the design principles that guided their intervention formed the basis for the design of the intervention in the present study (Features 1–5). Due to different foci of the two studies, I designed my own

[10] Seven PTs who took part in Research Cycles 1–3 had become practising teachers in September 2019 after graduating from teacher training.

activities, and adapted the features of their intervention for the intervention in the present study, based upon the existing literature on belief and attitude changes (Section 2.2.4). In the following, I discuss each feature of the intervention in the present study, and then the sequence and functions of the activities.

Feature 1 concerned the place of proof and its teaching, as the key elements of the present study, in the intervention. Feature 2 concerned the main instructional strategy to facilitate belief and attitude changes (i.e., cognitive conflict). Features 3–5 concerned the highlighted elements of the intervention that played important roles in supporting belief and attitude changes (via cognitive conflict).

Feature 1 (The central place of proof). The place of proof, including its operational definition and educational roles, in the intervention was aligned with the aforementioned goals of the intervention. Research has indicated that proof is often perceived by teachers as a difficult topic in which only some students are capable of learning proof and sometimes as a topic separate from other topics of school mathematics (e.g., Conner et al., 2011; Lesseig et al., 2019; Varghese, 2009) presumably because of textbooks' presentation of proof (e.g., Kwong, 2013) and teachers' past experiences with and/or beliefs about proof as symbolic manipulation (e.g., Ozdemir & Ovez, 2012; Varghese, 2009). In contrast, in the intervention, proof was positioned to play a central role in learning and teaching mathematics through inquiry-based instruction (G. J. Stylianides & Stylianides, 2009; Stylianou et al., 2009; Zaslavsky et al., 2012) and to involve different representations and methods of proving (Balacheff, 1988; Levav-Waynberg & Leikin, 2012; A. J. Stylianides, 2007).

In the intervention, PTs were engaged in activities that were designed based on the notion of reasoning-and-proving (G. J. Stylianides, 2008) – activities that are "frequently part of the process of making sense and establishing mathematical knowledge” (p. 9) through making mathematical generalisations and providing arguments for/against mathematical claims – creating and providing the PTs with opportunities to learn proof that differed from their previous experiences with proof (e.g., teacher-centred direct lecturing/demonstration of proof(s) of a theorem). After experiencing the activities in relation to the learning of proof, the PTs were also asked to develop and try out a lesson based on reasoning-and-proving as mini-teaching, in order to offer an opportunity for PTs to apply the teaching of proof (Allen, 1967; Bilen, 2015). The activities were intended to motivate changes in the PTs’ beliefs and attitudes towards teaching proof using cognitive conflict as an instructional strategy (Feature 2) and using discussion and collaboration (Feature 3), using materials based on the work of teaching (Feature 4) and making use of the instructor’s actions (Feature 5) as strategies that increased the likelihood of successful change.

Feature 2 (The use of cognitive conflict). Cognitive conflict (Liljedahl et al., 2021; G. J. Stylianides & Stylianides, 2009; Zaslavsky et al., 2012) was used as an instructional strategy and played a central role in motivating belief and attitude changes of the PTs in the intervention. Cognitive conflict occurs when there is a mismatch of information that a new situation cannot be explained by one’s existing information. The journey of resolving the conflict leads to change. One way of resolving the conflict is to gather more information to modify one’s existing information and its assumptions to explain the new situation:

For example, when a PT who believes “proof is difficult for students to learn” is presented with the Ball’s videos that show a group of primary students taking part in mathematical investigation and trying to develop arguments and proofs, the situation in the videos does not fit into the PT’s existing beliefs and experiences with proof. To accommodate the videos, the PT compares different types of proof and argumentation, particularly between the proofs that were constructed by the primary students in the videos and formal proofs that are experienced by the PT in university mathematics and often involve the use of technical language, abstract assumptions and axioms that make the PT struggle. The PT might then resolve this conflict by recognising that the notion of proof depends on the community involved, in which different sets of accepted statements and modes of argument representation take part. The PT finally accepts that proof, with appropriate sets of accepted statements and modes of argument representation, can be accessible to different students.

To facilitate successful changes, it is essential for the PTs in the intervention to see it meaningful to modify their existing (counterproductive) beliefs to resolve cognitive conflicts (Liljedahl et al., 2021; Limón, 2001). Otherwise, they might tend to treat the conflicts as exceptions (Liljedahl, 2010). In their work, G. J. Stylianides and Stylianides (2009) introduced the notion of conceptual awareness pillars to the use of cognitive conflict as an instructional strategy to facilitate conceptual change. Conceptual awareness pillars refer to “instructional activities that aim to direct students’ attention to their understandings or conceptions of a particular mathematical topic or idea” (ibid., p. 322) and have been tested to be useful in facilitating changes in PTs’ beliefs about mathematics (Shilling-Traina & Stylianides, 2013).

In the present study, the intervention incorporated activities and the instructor’s actions that were intended to make the PTs aware of mismatch between their existing beliefs and learning issues (Features 1 & 5) and materials that were intended to make the PTs aware of the connection between the learning issues and the work of teaching (Feature 4). As social interactions allow PTs to explore and exchange ideas about the teaching of proof from different perspectives and look at the activities as meaningful (Limón, 2001; Schram et al., 1988), the PTs were also encouraged to discuss and collaborate with each other during the activities (Feature 3).

Feature 3 (The use of discussion and collaboration). Discussion and collaboration with peers can help participating PTs consider activities in the intervention meaningful, thereby increasing the likelihood of successful cognitive conflict (Limón, 2001; Shilling-Traina & Stylianides, 2013) and change in their beliefs and attitudes towards teaching proof. During discussion and collaboration, the PTs were allowed to exchange ideas (Wilcox et al., 1991), to explore and experiment with new information and situations about the learning and teaching of proof that they encountered in the intervention.

To facilitate quality discussion and collaboration, it is crucial to ensure a supportive environment that the PTs are ready and welcomed to exchange their opinions. Important to note is that, without appropriate guidance or instruction, discussion and collaboration might lose focus on learning issues (Clark, 2001), so the instructor plays a role in directing the discussion and collaboration to the intended learning issues (Feature 5). Moreover, since senses of preparedness are conducive to one's motivation and participation (Eccles & Wigfield, 2002; Muir, 2017), I found take-home tasks helpful (in Research Cycles 1–3 and from literature, e.g., Alayont, 2014; Strayer, 2012) and included them as pre-class preparation in the intervention to enhance the engagement and quality of discussion and collaboration.

Feature 4 (The use of materials based on the work of teaching). Shilling-Traina and Stylianides' (2013) work, in which their participants were engaged in pedagogy-related mathematics tasks (G. J. Stylianides & Stylianides, 2010) and demonstrated changes in beliefs about mathematics, provides optimism that the use of materials based on the work of teaching might facilitate changes in PTs' beliefs and attitudes towards teaching proof. As the defining aspect, a pedagogy-related mathematics task has to have an pedagogical element that is "substantial" and "situates the mathematical object of the task in a particular pedagogical space, which relates to school mathematics and may derive from actual or fictional classroom records" (G. J. Stylianides & Stylianides, 2010, p. 164), allowing PTs' engagement in the intervention of the present study from the perspective of a mathematics teacher(-to-be). In the present study, the intervention incorporated different materials based on the work of teaching, for example, videos of classroom instruction involving proof and argumentation (The Mathematics Teaching and Learning to Teach Project, 2010a, 2010b; TIMSS, 1999) and activities involving the learning of school mathematics through proof (Feature 1) and mini-teaching.

Materials based on the work of teaching played two roles in the intervention. Since belief change can be observed when PTs are involved as learners of mathematics and pedagogy (Feiman-Nemser & Featherstone, 1992; Liljedahl et al., 2007; Shilling-Traina & Stylianides, 2013), materials used in the intervention were intended to create an environment that allowed participating PTs to experience active, innovative learning in school mathematics through proof. This environment might have been absent from the PTs' previous learning experiences. The intervention aimed at involving the PTs in this environment and encouraging them to compare their existing beliefs and attitudes towards teaching proof and the environment, to identify mismatches between the two, then to facilitate cognitive conflict and belief change.

Moreover, the use of materials based on the work of teaching are conducive to PTs' awareness of the connection between learning issues and their future teaching, and in turn, conducive to successful cognitive conflict and belief change (Feature 2). Apart from involving the PTs as mathematics learners, the intervention also involved materials that reflect situations of the teaching of school mathematics (G. J. Stylianides & Stylianides, 2010), for example, using problems in senior primary and junior secondary mathematics and videos of classroom instruction. These materials were intended to help the PTs to relate what they experienced and learnt from the intervention to the work of teaching, thereby facilitating changes in the PTs' beliefs and attitudes towards teaching proof.

Feature 5 (The instructor's actions). Instructors play important roles in conducting effective interventions. For example, the instructor can challenge participants on their existing beliefs and direct them to intended learning issues and occasionally offer participants direct access to knowledge that is needed but the participants cannot develop on their own (Shilling-Traina & Stylianides, 2013; G. J. Stylianides & Stylianides, 2009), and can create and maintain a supportive and collaborative environment in which the participants are encouraged to express their opinions and discuss the learning issues (Smith et al., 2009).

As the instructor of the intervention, I intentionally directed participating PTs' attention to their beliefs and attitudes towards teaching proof. For example, before playing Ball's videos, I asked the PTs to describe their impression of learning and teaching proof. This was intended to make the PTs more aware of their existing beliefs (G. J. Stylianides & Stylianides, 2009), such as "teaching proof is less feasible in school mathematics" and "students are not capable of producing, understanding and appreciating proofs", thereby preparing them for the videos and discussion. Then I challenged the PTs' existing beliefs by asking, "Does this classroom (in the videos) differ in any way from other classrooms you have experienced? If so, how?" and "When would students be ready for proof?" When realising that proof was feasible and accessible to the primary students shown in the videos, the PTs might experience cognitive conflict.

Apart from the above, I was also responsible for creating and maintaining a supportive environment that encouraged the PTs to express their opinions and to collaborate with others in completing the tasks in the intervention (Feature 3). Since Hong Kong classrooms often develop a norm of silent, passive learning culture (Mok, 2009; Sit, 2013) and PTs tend to hesitate about engaging in discussions, particularly at the beginning of the intervention (Research Cycles 1–3), I made use of questioning (Ellis, 1993) to motivate the PTs' participation. For example, since senses of preparedness and sureness of answers and responses are conductive to participation (Hsu, 2015; Muir, 2017), I initiated discussions about the videos of classroom instruction by asking for the PTs' factual observations and feelings, such as "In the videos, what do you find interesting or different from your previous mathematics lessons?" and "How old were the students shown in the videos?" These questions also helped the PTs rehearse subsequent questions that asked the PTs to evaluate and discuss different teaching approaches.

The sequence of activities in the intervention. The intervention consisted of three weekly workshops (Sessions 1–3). Each of which lasted two hours. Activities of the intervention were designed based on Features 1–5. Table 3.2.2 presents the activities, their main characteristics and intended functions.

Table 3.2.2
Activities of the intervention

Activity[a]	Characteristics	Intended functions
1-1	• Discuss ideas of proof • Ask "What role(s) does proof play in school mathematics?" "What do you think and feel about proof?"	○ Gather information about PTs' existing beliefs and attitudes towards teaching proof ○ Direct the PTs to different roles of proof and their existing beliefs about proof
1-2	• Include excerpts of two videos of classroom instruction involving proof (Ball's videos) (The Mathematics Teaching and Learning to Teach Project, 2010a, b); the videos show two lessons of Grade 3 (Year 4) students in which the students learnt mathematics through discussing ideas and making arguments and proofs • Ask "In the videos, what do you find interesting or different from your previous mathematics lessons?" "How old were the students in the videos?" "What are the strengths (and weaknesses) of this teaching?"	○ Allow PTs to explore an alternative teaching approach (learning mathematics through proof, questions & dialogues) and to reflect on roles of proof ○ Introduce ideas "Primary students can actively discuss and learn mathematics through producing pictorial arguments and proofs" ○ Stimulate cognitive conflict, regarding "Proof can foster mathematical understanding through inquiry-based instruction" "Proof can be accessible to all students"

1-3	• Justify a conjecture “the sum of two odds is even” made by the students in the second video in Activity 1-2 • Ask for different solutions and approaches for presenting them • Ask “What do you think about your solutions?” “Do you encounter any difficulty when producing and presenting your solutions?” “How can the difficulties be resolved?”	○ Allow PTs to explore different approaches for solving a problem and presenting the solution(s) ○ Stimulate cognitive conflict, regarding “There can be multiple solutions to a proving task” “Depending on the communities engaged, modes of representations can be different”
2-1	• Introduce at Session 1 as take-home task; revisit at Session 2 • Include problems that the PTs find familiar in school mathematics “Why is the sum of first n positive integers $n(n+1)/2$?” (Sum of numbers problem) “Why are two triangles on equal bases and between the same parallel lines equal in areas?” (Triangles problem) • Explore opportunities to extend the problems into more generalised ideas	○ Direct the PTs to the notions of multiple representations and methods of proving, and their relations with the teaching of mathematics through proof ○ Stimulate cognitive conflict, regarding “Proof can foster mathematical understanding through inquiry-based instruction” “There can be multiple solutions to a proving task” “Depending on the communities engaged, modes of representations can be different”
2-2	• Include an excerpt of a video of classroom instruction involving proof (HK3 Polygons of TIMSS 1999 Video Study) (TIMSS, 1999); the video shows that a Grade 8 (Year 9) lesson focused on deriving the sum of the interior of a polygon from triangles, to quadrilaterals then pentagons (and potentially, *n*-sided polygons). • Ask “In the video, what do you find interesting or different from your previous mathematics lessons?” “How was proof involved in the lessons presented in the video?”	○ Allow PTs to explore an alternative teaching approach (learning mathematics through extending mathematical ideas and proof) and to reflect on roles of proof ○ Direct PTs’ attention to a sequence of teaching mathematics (through reasoning-and-proving): Making a mathematical generalisation and providing a proof (or a non-proof argument) to a mathematical claim (G. J. Stylianides & Stylianides, 2009)
2-3	• Include a model lesson based on the notion of reasoning-and-proving • Justify “the sum of three consecutive integers is a multiple of three” • Explore “the sum of *n* consecutive integers is a multiple *n*” considering proof(s) of the case $n = 3$ (and a few cases more)	○ Introduce PTs to the notion of reasoning-and-proving as a framework for the learning and teaching of mathematics (through proof) ○ Stimulate cognitive conflict, regarding “Proof can foster mathematical understanding through inquiry-based instruction” “The teaching of proof can be achieved through careful lesson plans (and implementation)”
3-1	• Introduce at Session 2 as take-home tasks; revisit at Session 3 as mini-teaching • Prepare a lesson plan on the topic of introducing the area of a circle, based on the notion of reasoning-and-proving; PT could also choose their own topic • Demonstrate the plan; PT was encouraged to try out the plan, but could also talk through the plan if they found suitable • Discuss the mini-teaching regarding PTs’ experiences and difficulties when demonstrating	○ Direct PTs’ attention to the teaching of proof and the notion of reasoning-and-proving ○ Prepare PTs to adopt reasoning-and-proving ○ Stimulate cognitive conflict, regarding “Difficulties with the teaching of proof can be resolved through peer exchanges and careful lesson plans and implementation”

	their lesson plans, and possible ways to resolve the difficulties	
3-2	• Include a model lesson based on the notion of reasoning-and-proving • Explore two key topics of school mathematics in Hong Kong: Area of circle, Quadrilaterals	○ Allow PTs to explore an alternative teaching approach (learning mathematics through proof, questions and dialogues)
3-3	• Revisit ideas of proof • Ask "What role(s) does proof play in school mathematics?" "What do you think and feel about proof?"	○ Allow PTs to reflect on their experiences in the intervention, regarding The relevance of proof to the learning of mathematics The accessibility of proof to students at different grade levels and mathematical backgrounds The possibilities of resolving students' and teachers' difficulties with proof and its learning and teaching

[a]X-Y represents the Y-th activity at Session X.

3.2.3 Summary of Section 3.2

DBR was used in the present study. Through four research cycles incorporating design, implementation, evaluation and revision, an intervention was developed, aiming to help participating PTs see that proof is worthwhile and accessible to all students, is enjoyable and is relevant to student learning in school mathematics. To achieve the goals, the intervention incorporated five interrelated features to create an innovative environment that the PTs can learn mathematics and pedagogy of proof, from more active approaches (e.g., inquiry-based learning, as opposed to passive learning, such as teacher-directed lecturing and demonstration of proofs).

3.3 Data collection

Data were collected from different sources in the present study. The use of multiple data sources is guided by the notion of triangulation, in order to enhance reliability and internal validity of the study (Cohen et al., 2011; Patton, 1999). Its logic is based on the premise that the analysis of data collected from different sources can reveal different aspects of the phenomenon under study and provide cross-data validity checks. Since beliefs and attitudes are multifaceted and complex, and inconsistencies may be found in comparisons between data collected from different sources (Cross, 2009; Liljedahl, 2009), the use of data collected from different sources and triangulation in the present study are intended to explain potential inconsistencies and to elucidate complementary aspects of belief change (Cross Francis, 2015; Furinghetti & Morselli, 2009; Patton, 1999).

In this section, I first elaborate the data sources for the present study (Sections 3.3.1–3.3.3), then describe the ways in which data were used to answer the RQs (Section 3.3.4). The data sources consisted of a questionnaire (Section 3.3.1), individual semi-structured interviews (Section 3.3.2) and fieldnotes and recordings of the intervention (Section 3.3.3). The questionnaire, consisting of Likert items, open-ended questions and two vignettes, was used to assess participating PTs' beliefs and attitudes towards teaching proof before the intervention and changes in the PTs' beliefs and attitudes after the intervention. The interviews were used to obtain more specific information about the PTs' beliefs and attitudes, their changes, and whether and how the PTs' experiences in the intervention affected their beliefs and attitudes towards teaching proof. In this thesis, I report the data collected in Research Cycle 4.

To supplement my analysis, I also used the data collected from a group of PTs who did not participate in the intervention. They responded to the questionnaire twice in September and October 2019 respectively. Their data provided additional information to assist in interpreting the analysis of the data on the PTs who participated in the intervention.

3.3.1 Questionnaire

Questionnaires have been used extensively to gather information in different areas of research, including educational research in mathematics and particularly research on (self-reported) beliefs and attitudes related to the teaching of proof (e.g., Frasier & Panasuk, 2013; Kotelawala, 2016).

To gather information about PTs' beliefs and attitudes towards teaching proof in school mathematics, I developed a questionnaire, which consisted of Likert items, and open-ended questions about the teaching of proof and two vignettes of classroom instruction (Appendix A). An anonymised, completed questionnaire is presented as an illustrative example in Appendix E.

Likert items. The Likert items were adapted from a Turkish survey developed by Keceli-Bozdag et al. (2014). The Turkish survey was selected for two reasons. First, it was designed to assess PTs' beliefs about the importance of proof, their enjoyment of proof, their dislike of proof and their anxiety about proving, which were found to be influential in teachers' time spent on proof in their teaching (Frasier, 2010; Kotelawala, 2016), thereby assessing dimensions of PTs' beliefs and attitudes towards teaching proof in school mathematics and suiting the present study.

Second, some aspects of the quality (e.g., content validity, construct validity, internal consistency) of the Turkish survey had been tested. The survey was developed by adapting items from other studies on teachers' beliefs and attitudes related to (the teaching of) proof (Almeida, 2000; Kotelawala, 2007; Moralı et al., 2006; Nyaumwe & Buzuzi, 2007; Uzel & Ozdemir, 2009). The survey was reviewed by experts, then validated by principal component analysis using varimax rotation with data on 727 Turkish PTs in primary and secondary mathematics. The final survey consisted of 22 Likert items concerning the four aforementioned dimensions – the first dimension explained 17.79% of the total variance related to the survey items, the second explained 16.44%, the third explained 13.47% and the fourth explained 7.36%. The alpha coefficient of the survey was found to be .89.

Having said that, there were still concerns raised by adapting items from different cultures and languages, particularly whether the items were cross-culturally valid after translation (Andrews & Diego-Mantecón, 2015). To test and ensure the quality of translation, the survey items were first translated from Turkish into English using online translation software. The product was then compared with their possible English origins (as some Turkish survey items were adapted from English studies), followed by being reviewed by a Turkish-English bilingual, who is a doctoral student in education and has a background in English education. To perform a back translation check, the English-reviewed product was also translated to Turkish using online translation software and compared with the original items. After piloting and revising the English items (six items were added) in Research Cycle 1 (Section 3.2.1), a Chinese translation (reviewed by two Chinese-English bilinguals) was added in parallel with the English items. Although Hong Kong PTs were assumed to have competent English ability[11], the additional Chinese translation allowed Hong Kong PTs to choose their preferred language for completing the items, which in turn, reduced burdens.

To examine the factor structure and internal consistency of the translated, adapted Likert items, the items were administered to 184 Hong Kong PTs, in Research Cycle 3. I used confirmatory factor analysis (CFA) to test whether the Hong Kong PTs' responses to the translated items had the same factor structure as the Turkish PTs' responses to the original survey, reported by Keceli-Bozdag et al. (2014). Whilst the CFA results indicated discrepancies between the factor structures of the translated items and the original survey, I gained understanding of dimensions of Hong Kong PTs' beliefs and attitudes towards teaching proof in

[11] As an admission requirement: https://www.studyinhongkong.edu.hk/en/apply-to-study/admission-requirement.php

school mathematics. Details of the development and validation of the Likert items are presented in Appendix B.

The Likert items consisted of 28 statements. Each item was measured on a 5-point scale (**1 = Strongly Disagree**; **2 = Disagree**; **3 = Neutral**; **4 = Agree**; **5 = Strongly Agree**). Twenty four of them were used to assess five dimensions of PTs' beliefs and attitudes towards teaching proof in school mathematics. Eight items (Items I1, I2, I3, I4, I5, I6, I7 & D5) provided information regarding PTs' beliefs about the importance of proof in learning mathematics, six items (Items E1, E2, E4, E5, E6, E7 & D3) to their enjoyment and interest in proof, four items (Items E3, D1, A2 & A4) to their anxiety about proof, two items (Items A1 & A3) to their negative responses when failing to prove, and three items (Items T2, T3 & T5) to their beliefs about relevance of proof to examinations.

In addition to the factor structure of the Likert items established by CFA, internal consistency tests indicated that the items measured the same general construct. Moreover, a subsample of 27 PTs, who did not participate in the intervention, completed the Likert items once again, around one month after the first time (replicating the duration of the intervention), and the data were used to inspect test-retest reliability. The results indicated that the PTs had consistent responses to the Likert items between two time points in general (Appendix B).

Open-ended questions**.** Apart from the Likert items, the questionnaire included two open-ended questions about the teaching of proof and six questions about two vignettes of classroom instruction, to gather information about PTs' intended decisions about whether and how proof should be implemented in school mathematics, which represented their beliefs about the place of proof in school mathematics and approaches for teaching mathematics and proof. The open-ended questions about the teaching of proof focused on PTs' intended decisions about implementing proof in school mathematics *in a general context*, by asking "should we implement…" and "when and how…", whereas the questions about the vignettes gathered information about PTs' intended decisions about teaching mathematics and proof *in a specific teaching context* (Appendix A), as vignettes are useful tools for eliciting PTs' beliefs (R. Hughes & Huby, 2002; Skilling & Stylianides, 2020; N.-Y. Wong et al., 2016).

Drawing on the work of Hughes and Huby (2004) and Skilling and Stylianides (2020) that guides the construction of vignettes (Table 3.3.1), I carefully constructed two vignettes based on literature and my own experience of teaching mathematics at secondary schools. The vignettes situated in the topic of factorisation and identities of polynomials – a fundamental topic of school algebra in Hong Kong (Curriculum Development Council, 2017; Mok, 2015), and using proof in this topic. The vignettes illustrated two contrasting models of beliefs about approaches for teaching mathematics and proof (product-oriented vs. process-oriented) developed by Yoo (2008). The first vignette was constructed from an exemplary Hong Kong lesson (Mok, 2009) and characteristics of the product-oriented beliefs. Likewise, the second vignette was developed based on characteristics of the process-oriented beliefs and the notion of reasoning-and-proving (G. J. Stylianides, 2008). The following excerpt describes the two contrasting models:

> In a typical day of class, teachers [of product-oriented beliefs] introduce mathematical concepts and then demonstrate and explain methods of solving problems or proving theorems. Students are encouraged to learn a single, correct method given by a teacher and practice routine problems for accurate performance (Ernest, 1989a). […] [T]eachers [of process-oriented beliefs] guide students' engagement in the learning process by exploring ideas, solving problems or constructing proofs, and discussing the results. Students are encouraged to produce their own solutions or proofs, discuss different methods, justify arguments or reasoning, and develop their understandings of the mathematical results while participating in classroom discourse. (Yoo, 2008, pp. 16–17)

After reading the two vignettes, PTs were asked (i) to compare and identify the elements that they considered to be (not) important in learning and teaching the selected topic in the vignettes, and (ii) to provide a brief description of how they would teach the selected topic (Appendix A). The former type of questions collected information about what the PTs focused on teaching mathematics, and the latter collected information about their intended decisions about teaching mathematics and proof.

Table 3.3.1
The construction of the two vignettes

Elements of framework	Description
Are the vignettes relevant to the research topic?	I drew on literature on models of teaching mathematics and proof (e.g., Ernest, 1989a; G. J. Stylianides, 2008; Yoo, 2008) and on an exemplary Hong Kong lesson (Mok, 2009) to construct the two vignette narratives (scenarios of Teachers A and B's lessons). I also drew on my personal experiences of teaching secondary mathematics and working with mathematics teachers. The narratives (and questions) were reviewed by an experienced researcher. By doing so, I was able to align the vignettes to the aims of my research.
Are the vignettes relevant to the participants?	The vignettes were written to be similar to classroom activities and teaching approaches that Hong Kong preservice teachers would be familiar with. The context of the vignettes was the topic of factorisation of polynomials. This topic is fundamental to school algebra in Hong Kong (Curriculum Development Council, 2017; Mok, 2015) and also a topic that both Hong Kong primary and secondary preservice teachers are familiar with.
What purpose/function do the vignettes have?	The vignettes were made as a phase of data collection to provide a specific teaching context for six open-ended questions. Particularly, participants were asked to describe their teaching approaches to the topic of factorisation of polynomials and their identification to one of the two approaches depicted. The vignettes also enabled the participants to further unpack their responses and gain insights into their beliefs during follow-up individual interviews (R. Hughes & Huby, 2002; N.-Y. Wong et al., 2016). The vignettes and other data sources (e.g., Likert items, individual interviews) elicited rich data for analysis, and also enabled data triangulation to examine and ensure the credibility and reliability of the findings (Cohen et al., 2011; Patton, 1999).
How are the vignettes presented?	The vignettes were presented in a written format and depicted two parallel examples of classroom instruction on the topic of factorisation of polynomials, depicting two contrasting approaches for involving proof and argumentation in teaching. The two vignettes were concise (less than 250 words) and similar in length (8 lines each) and included similar details about learning activities, so that participants would not be overwhelmed and were provided with equal opportunities to respond to both vignettes. The vignettes also provided necessary information for participants so that the depicted teaching scenarios involving proof and argumentation enabled interpretation by the participants (R. Hughes & Huby, 2004; Skilling & Stylianides, 2020).
Are the settings and terminology familiar to the participants?	The vignette narratives aimed to show a typical junior secondary (the age of 13–14) mathematics classroom and applied the terminology that participants were familiar with (Bradbury-Jones et al., 2014; Skilling & Stylianides, 2020). I used 'verify', instead of 'prove', to enable participants' interpretation and further elaboration during interviews, rather than drawing their (over)attention

	to proof, influencing their responses regarding the importance of proof in teaching mathematics.
Are the questions closed or open? From what perspective are the participants being asked?	Six open-ended questions were used, providing participants with opportunities to reflect on the situations depicted in the vignettes (R. Hughes & Huby, 2004). Participants were asked to respond to the vignettes from their own perspective. The first four questions asked participants to make comments on the choices and practices of Teachers A and B. The remaining two questions asked about their own teaching approaches to the same topic and how these were similar to or different from those depicted in the vignettes. The responses provided data on the participants' thinking, judgement and intended practices for addressing the research topic (R. Hughes & Huby, 2004; Skilling & Stylianides, 2020).
Have the vignettes been piloted?	Apart from reviewed by an experienced researcher, the vignettes were piloted with preservice and inservice teachers in Research Cycles 1–3. This assisted in assessing the representativeness of the vignettes and the clarity of the instructions (R. Hughes & Huby, 2004; Skilling & Stylianides, 2020).
Are clear instructions provided?	An introductory paragraph provided instructions for reading and responding to the vignettes, and brief information about the context of the vignettes.
When and how are the vignettes used?	The vignettes were part of the questionnaire described in this section. The administration of the questionnaire is discussed in Section 3.3.4.

The construction framework was adapted from Hughes and Huby (2004) and Skilling and Stylianides (2020).

3.3.2 Semi-structured interviews

Semi-structured interviews represent the second data source used in the present study – although the questionnaire had gathered essential information about PTs' self-reported beliefs and attitudes towards teaching proof in school mathematics, using semi-structured interviews was another fruitful method of gathering more qualitative data that complemented the former data source (Cohen et al., 2011; Drew et al., 2008). After the intervention, every participant was invited to attend a semi-structured interview. A set of pre-set guiding questions structured the interviews (Appendix C), and made data more comprehensive and systematic. The interviews enabled more opportunities of following up for understanding how the PTs interpreted the questionnaire, and allowed flexibility for probing into their beliefs and attitudes towards teaching proof (mainly post-intervention) and how they perceived their experiences in the intervention, for elaboration and clarification (Cohen et al., 2011; Maykut & Morehouse, 1994).

The interviews consisted of five parts. In the first part, I explained the structure of the interviews, attempted to make the interviewees feel at ease, and obtained oral consent from them. Although written and oral consent had already been obtained from each PT before the intervention, I ensured, at the beginning of the interviews, that each interviewee understood the conduct of the interview and their rights to participate in and withdraw from the interview or the whole study, which were essential for ensuring the present study was ethical and minimising the effects of participant bias on PTs' responses to the interview questions (British Educational Research Association, 2018; Drew et al., 2008). In the second part, two questions guided my probes into the PTs' backgrounds in mathematics education, concerning their reasons for teaching mathematics and their prior experiences in learning mathematics, which are related to their beliefs and attitudes towards teaching proof in school mathematics (Beswick, 2018; Varghese, 2009). In the third part, three questions concerned the PTs' intended general approaches for implementing proof in school mathematics, to gather information about their beliefs and attitudes towards teaching proof, with limited reference to their responses to the questionnaire (before next part). In the fourth part, I used two questions to probe into the PTs' elaboration of their responses to the questionnaire and any 'noticeable' changes that they made before and after the intervention. The vignettes presented in the questionnaire also acted as a stimulus for discourse on the PTs' intended decisions and beliefs about approaches for teaching mathematics and proof in a specific context of classroom instruction (R. Hughes & Huby, 2002; N.-Y. Wong et al., 2016). Data collected from the second, third and fourth parts were used for cross-data checks with the questionnaire data to minimise the risk of misinterpretation (Cohen et al., 2011; Patton, 1999) of (changes in) PTs' beliefs and attitudes towards teaching proof. In the final part, the PTs were asked to express their experiences in the intervention and give feedback about the intervention, thereby providing information to identify possible components (i.e., activities, features; see also Section 2.4) of the intervention that facilitated the PTs' changes and learnings.

All interviews were conducted by me in Cantonese, the first language of the interviewees, which reduced burdens. Each interview lasted, on average, 70 minutes. All interviews were audio-recorded, and supplementary observations and notes were also made during and after the interviews. A transcribed excerpt is presented as an illustrative example in Appendix E.

3.3.3 Audio-recordings and fieldnotes of the intervention

To provide observational data about the intervention, every try-out of the intervention was audio-recorded, and fieldnotes were made to describe some highlighted events and my observations of the participants' engagement in each session (Cohen et al., 2011; Schoenfeld, 2008). These observational data of the PTs' engagement in the intervention (the intervention data) were used to analyse plausible activities and discussions in the intervention, and features of the intervention that facilitated change in the PTs' beliefs and attitudes towards teaching proof in school mathematics.

Everything that occurred in the intervention was audio-recorded as faithfully and fully as possible (Cohen et al., 2011; Schoenfeld, 2008). During the intervention, two audio-recorders were placed near me (the instructor) and the PTs, respectively (Altrichter et al., 2005). The recorder near me recorded all of my words and vocal expressions. The PTs' statements from those within a few metres of the recorder near the PTs were picked up less consistently depending on the volume of the PTs' voices; many of their statements were audible. A transcribed excerpt is presented as an illustrative example in Appendix E.

Fieldnotes were made by me, during or shortly after each session of the intervention. The notes provided observational data, from my perspective as a participant observer, including date, time, attendance, sitting places of the attendees, depiction of the attendees' engagement, description of activities and discussions that caught my attention when I conducted the intervention, and my reflection on how I conducted the intervention (Cohen et al., 2011; van den Akker et al., 2013). Illustrative examples are presented in Appendix E.

3.3.4 Data sources used for different research questions

Data collected from different sources were used for addressing different research questions. Before discussing the ways in which data were used to answer RQs 1–3 respectively, I re-state the research questions in the following.

RQ 1. Where were Hong Kong preservice teachers' pre-intervention beliefs and attitudes towards teaching proof in school mathematics?

RQ 2. How did Hong Kong preservice teachers' beliefs and attitudes towards teaching proof change after participating in a specifically designed intervention of short duration?

RQ 3. What components of the intervention were perceived by the preservice teachers as contributing to changes in their beliefs and attitudes (as identified in RQ 2)?

Data used for RQ 1. After they accepted the invitation to participate in the intervention, the questionnaire was administered to the PTs one week before the intervention via email. This administration gathered pre-intervention written information about the PTs' self-reported beliefs and attitudes towards teaching proof, concerning the five dimensions of the Likert items (Section

3.3.1) and their beliefs about the place of proof and approaches for teaching mathematics and proof (Section 3.3.2)

Moreover, some of the interview data supplemented my analysis of the questionnaire data. As my interpretation of written information was involved in data analysis, and some responses were too short and vague for meaningful interpretation, I consulted the interviews with the PTs (if available) in which the PTs elaborated their responses to the questionnaire (Guiding Question 4a, Appendix C). Yet, I am aware that the PTs' elaboration might not necessarily reflect their pre-intervention beliefs and attitudes towards teaching proof because the interviews were conducted after the intervention.

Twenty-three of the PTs who participated in the intervention completed the pre-intervention questionnaire and 12 of them attended the interviews after the intervention. I analysed these data to address RQ 1. Table 3.3.2 presents the participation of the PTs.

Table 3.3.2
Participation of the PTs

	Intervention				Questionnaire		
Participant	What Try-out were they in?	Session 1	Session 2	Session 3	Pre	Post	Interview
1[1,2]	6	√	√	√	√		√
2[1,2,3]	6	√	√	√	√	√	√
3[1,2]	6	√	√	√	√		√
4[1,2]	6	√	√		√	√	√
5[1,2,3]	6	√	√	√	√	√	√
6[1]	6		√		√	√	
7[1,2,3]	6	√	√	√	√	√	√
8[1]	6	√	√	√	√		
9[1]	6	√			√		
10[1,2]	7	√	√	√	√		√
11[1,2]	7	√	√		√		√
12[1]	7	√	√	√	√		
13[1]	7	√	√	√	√		
14	7	√	√	√			
15	7	√					
16[1,2,3]	8	√	√	√	√	√	√
17[1,2]	8	√	√	√	√		√
18[1,2,3]	9	√	√	√	√	√	√

19[1,2,3]	9	√	√	√	√	√	√
20	9	√	√	√			
21[1]	10	√			√		
22	10	√	√	√			
23[1]	10	√			√		
24[1]	10	√	√	√	√		
25[1]	10	√	√	√	√		
26[1]	11	√			√		
27[1]	11	√	√		√		

[1]Data of the corresponding PTs used to analyse their pre-intervention beliefs and attitudes towards teaching proof in school mathematics (focus of RQ 1); [2]data of the corresponding PTs used to analyse post-intervention changes in their beliefs and attitudes (focus of RQ 2) and links between the changes and the intervention (focus of RQ 3); [3] 'complete' data

Data used for RQ 2. The questionnaire was administered to the PTs once again at the end of the intervention. They were asked to complete the questionnaire, and return it via email or in the interviews (if they agreed to attend). The PTs' post-intervention responses to the questionnaire and elaboration of their written responses (Guiding Question 4a), and their interviews (Guiding Questions 2a–b & 3a–c) were analysed to identify their post-intervention (self-reported) beliefs and attitudes towards teaching proof.

In this analysis, I included only data of the PTs who attended at least two sessions of the intervention (Condition 1), completed the pre-intervention questionnaire and either completed the post-intervention questionnaire or attended an interview (Condition 2), for two main reasons. First, inclusion of data with Condition 2 allowed me to compare the PTs' post-intervention data with their corresponding pre-intervention data, in order to identify differences, which represented changes in the PTs' beliefs and attitudes towards teaching proof. Second, as most of the contents of the intervention had been covered in two sessions, the PTs who attended at least two sessions (Condition 1) could be considered that they had reasonable experiences with the intervention, whereas changes observed in whom attended only one session were less likely associated with it.

Consequently, data on 12 of the PTs who participated in the intervention were used to address RQ 2. They produced seven sets of pre-post responses to the Likert items and 12 sets of pre-post responses to the open-ended questions and interviews. The difference between the total number of the PTs who participated in the intervention and the amount of data used was a result of the fact that many of the PTs did not return the questionnaire nor attend an interview after the intervention due to unforeseen situations in Hong Kong (to be discussed in Section 3.5).

Comparing pre-post data provided information about changes in the PTs' beliefs and attitudes towards teaching proof between two time points. Yet changes inferred by comparing the pre-post data did not necessarily, automatically translate into outcomes of the intervention (Cohen et al., 2011). To reduce other possible attributions of the belief changes, I utilised two additional data sources to supplement the pre-post comparison.

First, I utilised test-retest questionnaire responses of 27 PTs who did not participate in the intervention, as a *comparison group*. They respond to a shorter version of the questionnaire (consisting of Likert items and two open-ended questions only) twice in September and October 2019 respectively, producing test-retest data (Appendix B). I used this comparison group to increase my confidence about the findings of pre-post data on the PTs who participated in the intervention (the intervention group) – differences in the pre-post data of the intervention group were compared to those in the test-retest data of the comparison group to identify differences between the two groups in terms of changes in their responses, inferring effects of the intervention on the PTs' beliefs and attitudes towards teaching proof (Cohen et al., 2011; E. Marsden & Torgerson, 2012; O'Donnell, 2004).

The comparison group completed the questionnaire twice, with a one-month interval (replicating the duration of the intervention), in Research Cycle 3 (see also Section 3.3.1). This group produced 27 sets of test-retest Likert data and 22 sets of test-retest responses to the two open-ended questions about the teaching of proof. Here, the term 'comparison' is used to indicate that this group did not meet some requirements of being a control group as in a randomised controlled trial – the group was not randomly selected, that the variation in the programmes which the PTs had enrolled was not controlled. Nonetheless, data collected at the first time point suggested that both the intervention and comparison groups were alike (see Sections 4.1 & 4.2).

Second, when interviewing the PTs, I probed into their elaboration of differences in their pre-post responses to the questionnaire (Guiding Question 4b) and their learnings from the intervention (Guiding Question 5b) (Willis, 1999). The former question allowed the PTs to report what a change in pre-post responses to the questionnaire was meant and whether it was meaningful to them, whereas the latter question allowed the PTs to report what they had learnt from the intervention. Their responses not only supplemented the analysis of the PTs' pre-post responses to the questionnaire, but also indicated their self-reported changes and learnings that were attributed to the intervention.

Data used for RQ 3. Semi-structured interviews (Section 3.3.2) and audio-recordings and fieldnotes of the intervention (Section 3.3.3) were used to address RQ 3. During interviews, the PTs were asked to identify and elaborate changes in their pre-post responses to the questionnaire and learnings from the intervention (Guiding Questions 4b & 5b), and I also probed into their views about what facilitated their changes and learnings in the intervention. Their responses to the questions enabled me to identify what components (i.e., activities, features) of the intervention the PTs perceived to be helpful in developing their beliefs and attitudes towards teaching proof in school mathematics. Additional information was also gathered when the PTs were asked about how they felt about the intervention (Guiding Questions 5c–d) for evaluating the intervention.

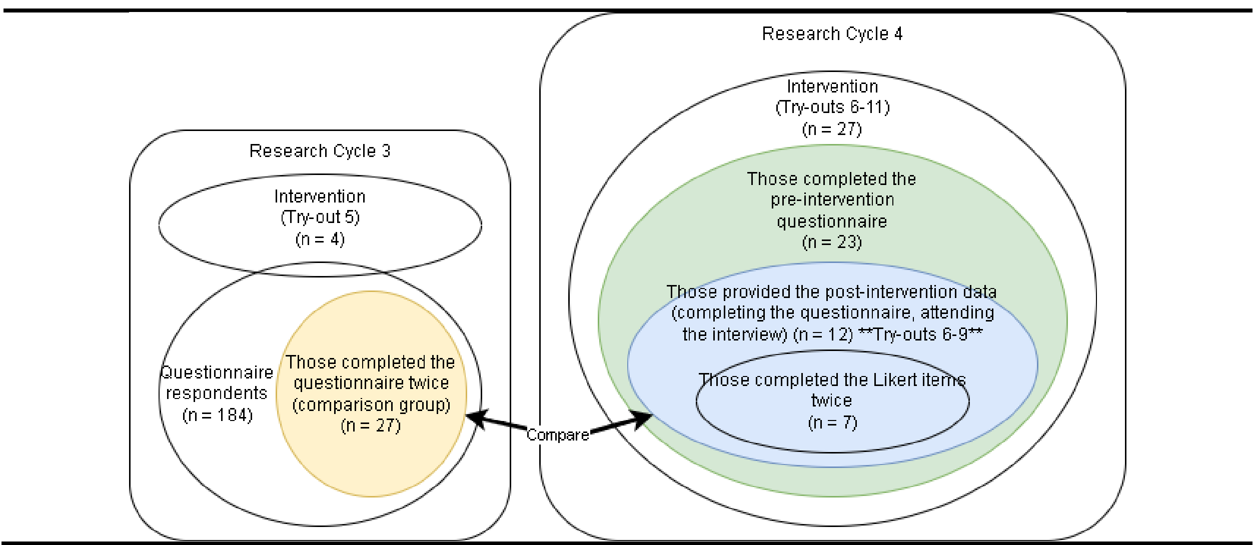
After identifying potential components of the intervention that facilitated changes in the PTs' beliefs and attitudes towards teaching proof from the interviews, I reviewed the corresponding audio-recordings in order to investigate how the changes were facilitated, thereby identifying and evaluating the relevant activities and/or features of the intervention. As the audio-recordings did not capture visual information nor my observations and reflection on the potential components, fieldnotes of the intervention were also used to supplement the review of the components.

As post-intervention data on the PTs in Try-outs 10–11 were missing (Table 3.3.2) resulting in exclusion of their data from the analyses for RQ 2, it is less appropriate to link the analyses for RQ 2 to Try-outs 10–11. Therefore, I focused on the audio-recordings and fieldnotes of Try-outs 6–9, whilst those of Try-outs 10–11 were used to supplement analysis of the former.

3.3.5 Summary of Section 3.3

Figure 3.3.1 illustrates the data sources for the analysis. Table 3.3.3 also summarises the RQs, the instruments, and data used for answering each RQ.

Figure 3.3.1
Data sources used



Note: Coloured regions are the data sources used for analysis

Table 3.3.3
Research questions, instruments and data used for analyses

Research question	Analyses	Instruments	Data[2]
RQ 1 What were Hong Kong PTs' pre-intervention beliefs and attitudes towards teaching proof in school mathematics?	• Analysis of responses to Likert items (Section 4.1.1) • Analysis of responses to open-ended questions (Section 4.1.2)	• Likert items of the questionnaire (Section 3.3.1) • Open-ended questions of the questionnaire Section 3.3.1) • Individual semi-structured interviews[1] (Section 3.3.2)	• 23 IPTs who completed the pre-intervention questionnaire • 12 IPTs who attended an interview[1]
RQ 2 How did Hong Kong preservice teachers' beliefs and attitudes towards teaching proof change after participating in a specifically designed intervention of short duration?	• Analysis of changes in responses to Likert items (Section 4.2.1)	• Likert items of the questionnaire	• 7 IPTs who attended at least two sessions of the intervention and completed pre-post Likert items • 27 CPTs who completed test-retest Likert items[1]
	• Analysis of changes in responses to open-ended questions (Section 4.2.2)	• Open-ended questions of the questionnaire	• 12 IPTs who either completed pre-post questions or completed pre-intervention questions and attended a post-intervention interview • 22 CPTs who completed test-retest questions[1]
	• Analysis of data on interviews (Section 4.2.3)	• Individual semi-structured interviews	• 12 IPTs who attended an interview
RQ 3 What components of the intervention were perceived by the preservice teachers as contributing to changes in their beliefs and attitudes?	• Identification of activities and features of the intervention that were linked to changes in the PTs' beliefs and attitudes towards teaching proof (Sections 4.3.1–4.3.2)	• Individual semi-structured interviews • Audio-recordings and fieldnotes of the intervention[1] (Section 3.3.3)	• 12 IPTs who attended an interview • Try-outs 6–9 of the intervention • Other try-outs[1]

[1]Used to supplement the corresponding analysis; [2]IPTs refer to PTs who participated in the intervention, CPTs refer to PTs who did not participate in it

3.4 Data analysis

I structure this section by discussing analyses relating to each of the data sources described previously (Sections 3.3.1–3.3.3) in Sections 3.4.1–3.4.3. In Section 3.4.4, I discuss methods that I used to examine and ensure the validity and reliability of the present study.

3.4.1 Analysis of responses to the questionnaire

***Analysis of responses to the Likert items*.** All the PTs' pre-post responses to the Likert items were entered into a spreadsheet as the first step of analysing the data. I grouped the data according to the five dimensions of the items (Section 3.3.1) and explored for interesting patterns of individual items, at individual and group levels. Then, for each PT, I computed a score for each dimension (dimensional score) using the factor loadings of the CFA results of item validation (Appendix B); for the whole group, I computed arithmetic means of the scores.

The use of factor loadings for computing dimensional scores recognised the strength (or lack of strength) for each item on its corresponding latent dimension; that is, items with higher factor loadings would have a larger effect on the score (Armor, 1974; DiStefano et al., 2009). Apart from using weighted sum scores by factor loadings to analyse the data, I also used unweighted sum scores and weighted sum scores by regression (DiStefano et al., 2009; Field, 2013) for triangulation of analysis methods (Cohen et al., 2011; Patton, 1999). I present the analysis of using the scores by factor loadings in Chapter 4 as there was little difference in the findings among those methods.

I compared the pre-post data to examine changes. First, I investigated item-wise and dimensional changes in data on each PT of the intervention group, and probed into interesting patterns of these pre-post differences for the whole group. Then, I compared the means of the group’s pre-post responses. By comparing the findings of the item-wise and dimensional analyses, differences between the intervention group’s pre-post Likert data were inferred. As discussed in Section 3.3.4, 12 of the PTs were interviewed and asked to elaborate (differences in) their responses. The findings were also compared with data on the comparison group. The analysis of the pre-intervention Likert responses was used to address RQ 1 (Section 4.1.1), and the analysis of the pre-post Likert responses was used to address RQ 2 (Section 4.1.2).

***Analysis of responses to the open-ended questions*.** When analysing responses to the open-ended questions (Section 3.3.1), I applied coding to the responses (Cohen et al., 2011; Mayring, 2014) for organising and quantifying the data. A set of coding schemes were developed based on relevant research and literature on teachers’ beliefs about teaching proof (e.g., Knuth, 2002a; Lesseig et al., 2019; Varghese, 2009) (Section 2.3), then piloted by using data from Research Cycles 1–3 (Section 3.2.1). When the coding process uncovered data that did not fit into any existing code in the schemes, new codes were added inductively until no new information was discovered (Cohen et al., 2011; R. Weber, 1990). After cycles of implementation and revision in Research Cycles 1–3, the coding schemes were finalised and applied to analyse the PTs’ responses to the open-ended questions in Research Cycle 4. Tables 3.4.1–3.4.4 present the finalised codes used to analyse the PTs’ responses to the open-ended questions (Columns 1–2) and illustrative examples of each code (Column 3). The development of the coding schemes is discussed in more detail in Appendix D, due to space limitations and its irrelevance.

In Table 3.4.1, I present the codes that described whether and how the PTs believed that proof should be implemented in school mathematics. Drawing on research findings that some practising teachers and PTs rejected the needs for proof in school mathematics (e.g., Ersen, 2016;

I. K.-C. Leung & Lee, 2017) and some believed proof should only be needed for certain groups of students (e.g., high-achieving students, students who plan to pursue advanced mathematics) and should be treated as a supplementary topic (e.g., Knuth, 2002a; Varghese, 2009), five distinct codes were defined (Table 3.4.1).

Table 3.4.1
Codes of PTs' beliefs about the place of proof in school mathematics

Code	Definition	Illustrative examples
No place	The PT rejects the learning and teaching of proof in school mathematics.	"No, because it is difficult to understand"
Hesitating	The PT is unsure whether proof should be implemented in school mathematics.	"Yes in a mathematical standpoint. No if related to exam/assessment. "
Unclear	The PT generally agrees with the importance of proof in school mathematics, but their approaches are unclear.	"Yes, raise interest." "Yes, it can improve students' logical thinking."
Supplementary	The PT generally agrees with the importance of proof in school mathematics, whilst positing that proof should be treated as supplementary materials for some (but not all) students.	"Yes but only when the students' performance are better than average since students can seldom understand proving and they are trained to be exam-oriented so they are not interested in it… enrichment."
Regular	The PT agrees that proof should be implemented in classroom instruction regularly.	"Yes. Making proofs helps students' understanding… After finish one topic […] Guide them to think from a different view. Introducing a new topic. Discovery learning => deductive conclusion => new knowledge. (proof)."

In Table 3.4.2, I present the codes that described the PTs' beliefs about approaches for teaching mathematics in general, and proof in particular. I adapted Yoo's (2008) framework of two contrasting models of beliefs about approaches for teaching mathematics and proof: **Product-oriented** and **Process-oriented**. Noting that (the centrality of) PTs' beliefs could vary with contexts (Beswick, 2018), what of interest was PTs' *overall tendencies* to the models. Whilst the two models are contrasting and considered to be two extremes of a spectrum of beliefs about approaches for teaching mathematics and proof, a PT could hold a combination of both models, mixing characteristics of the two models. Therefore, the term 'overall tendencies' was put into consideration during the coding process, and the code **Mixed** was added to describe data when a PT demonstrated a balanced view of the two models (see also Section 2.3).

Table 3.4.2
Codes of PTs' beliefs about approaches for teaching mathematics and proof

Code	Definition	Illustrative samples
Product-oriented	The PT generally *tends* to follow the product-oriented model of beliefs about approaches for teaching mathematics and proof. For example, • She values teacher-centred, content-focused teaching; • Her goal of teaching is to explain and transmit mathematical knowledge and procedures to students; and • She emphasises reception and mastery of (the relation among) concepts, skills and proofs (Yoo, 2008).	"After students acquire basic skills in arithmetic and reading comprehension, the teacher can guide and explain the examples [of proof], which help students form basis [in learning mathematics]." "After introducing the theorems, to exercise relevant theorems in proving certain questions as example demonstration."
Mixed	The PT presents a balanced/indecisive view of the product-oriented beliefs and the process-oriented beliefs.	"demonstrate proof: introducing the mathematical concepts… proving activities: after class as a fun task because it involves times and room for students to think and discuss."
Process-oriented	The PT generally *tends* to follow the process-oriented model of beliefs about approaches for teaching mathematics and proof. For example, • She values student-centred, construction-focused teaching; • Her goal of teaching is to guide students to construct mathematical knowledge and understanding; and • She emphasises involvement in problem-solving or proving process and the communication of the understanding of the results (Yoo, 2008).	"Some theorems can be proved by using other theorems that [students] learnt previously… [The teacher] can invite students to give opinions, and guide them to completion [of proving] as much as possible." "Group/individual presentations, because simply explaining solutions [proofs] cannot train students' reasoning."

In Tables 3.4.3–3.4.4, I present the codes that described the PTs' self-reported reasons for using proof and concerns about the use of proof in school mathematics, respectively. Research has shown that some PTs found the use of proof in school mathematics unnecessary or tended to reject it even if they recognised how students might benefit from proof (e.g., Doruk & Güler, 2014; Ersen, 2016; Lesseig et al., 2019; Varghese, 2009), so I focused on PTs' reasons behind the use or rejection of proof in teaching, rather than their awareness and/or knowledge of the educational roles of proof, challenges and constraints on implementing proof in their future class. In other words, I *only* coded statements of responses that indicated that the PTs related their awareness and/or knowledge of the educational roles of proof, challenges and constraints on implementing proof in teaching to their ideas about whether and how proof should be included in their future teaching.[12] Drawing on literature on (teachers' beliefs about) proof (e.g., Lesseig et al., 2019; Varghese, 2009) (see also Section 2.3), I used seven codes for describing PTs' reasons for using proof in school mathematics (Table 3.4.3), and four codes for describing their concerns about the use of proof in school mathematics (Table 3.4.4)

Table 3.4.3
Codes of PTs' reasons for using proof in school mathematics

Code	Definition	Illustrative samples
Justification	Proof is used for justifying mathematical claims and convincing students (Varghese, 2009; Zaslavsky et al., 2012).	"Yes, making proofs helps develop the smart image of mathematics and hence attract[s] students' interests. besides, it helps convince students to accept certain mathematical concepts."
Promotion of understanding	Proof is used for explaining mathematical claims (Hanna, 1990; Lesseig et al., 2019; Zaslavsky et al., 2012) and promoting students' conceptual understanding (N.-Y. Wong, 2007).	"Yes, the students will be more familiar with the theorem but not only the statement. It is the reason behind. At the end, they will 'know' the theorem."
Memorisation	Proof is used for helping students systematise mathematical knowledge and making connections (de Villiers, 1990; Zaslavsky et al., 2012) to memorise knowledge (N.-Y. Wong, 2007).	"Yes, it can help student to understand the basic and fundamental of Mathematic[s]. It can also help student to memorise some concept."
Generation of knowledge	Proof is used for enabling students to generate mathematical knowledge and ideas (de Villiers, 1990; Lesseig et al., 2019).	"Yes. The process of pro[v]ing gives the reason why something is correct and can show how a truth can lead to something new which is also true."
Development of reasoning	Proof is used for helping students develop ability and skills in problem	"Yes, to train the logic mind of students"

[12] This coding approach is similar (but not identical) to the notion of *leading beliefs* (Furinghetti & Morselli, 2011). Drawing on my previous research experience (C.-Y. Lee, 2019), this approach enabled me to reduce less-relevant data but keeping the analysis meaningful.

	solving, logical thinking and reasoning (Lesseig et al., 2019; Varghese, 2009; N.-Y. Wong, 2007).	"Yes, because making pro[of] can improve students' thinking."
Promotion of students' motivation	Proof is used for promoting students' motivation and interest in learning mathematics (Lesseig et al., 2019; Siu, 2008).	"Yes, I think it helps students better understand that topic. Also it arouse students' interest in maths because of their curiosity."
Self affection	Proof is used because of teacher's own affection for proof (Frasier, 2010; Kotelawala, 2016; Wilkins, 2008).	"Yes, it is fun and can help develop their logical thinking."

Table 3.4.4

Codes of PTs' concerns about the use of proof in school mathematics

Code	Definition	Illustrative samples
Students	Students do not have adequate knowledge, skills or interest in proof (Frasier & Panasuk, 2013; Lesseig et al., 2019; Varghese, 2009).	"Yes, but minimize it… bored to students and sometimes difficult" **(Reason behind the place of proof)**
Self	The teacher relates decisions about the teaching of proof to (lack of) autonomy, experiences, knowledge, confidence or interest in proof (Creager, 2016; Doruk & Güler, 2014; Lesseig et al., 2019).	"Depends on (1) school (policies) (2) teachers (degree of freedom) (3) students (interest, quality)" **(Reason behind the place of proof)**
Curriculum	Proof is not made regular (or not implemented) in school mathematics because of the curriculum (including examinations) (Frasier & Panasuk, 2013; Lesseig et al., 2019; Nyaumwe & Buzuzi, 2007).	"Yes in a mathematical standpoint. No if related to exam/assessment." **(Reason behind the place of proof)**
Time	Proof is time-demanding activities (Lesseig et al., 2019; Nyaumwe & Buzuzi, 2007).	"demonstrate proof: introducing the mathematical concepts… proving activities: after class as a fun task because it involves times and room for students to think and discuss." **(Reason behind approaches for teaching mathematics and proof)**

The codes presented in Tables 3.4.1–3.4.2 are mutually exclusive, whereas the codes presented in Tables 3.4.3–3.4.4 are not, because it was possible that one single response made by a PT would include several reasons. As a result, several codes were needed to capture them all, where each statement received corresponding code(s).

Additionally, my analysis of the data showed that the PTs might use the concerns stated in Table 3.4.4 to explain their responses to the place of proof in school mathematics (focus of Table 3.4.1) or to approaches for teaching mathematics and proof (focus of Table 3.4.2). To gather information about this occurrence, I introduced two non-mutually-exclusive notions: *reasons behind the place of proof* and *reasons behind approaches for teaching mathematics and proof*. The former refers to reasons that the PTs used to explain why they rejected or hesitated to use proof in teaching, or considered proof to be supplementary materials, and the latter refers to reasons that the PTs used to explain the choice of their (less desired) approaches for teaching mathematics and proof. For example, as shown in the last row of Table 3.4.4, a PT expressed that they would "demonstrate proofs" (a product-oriented approach) to students in class whilst "discussions and proving activities" (process-oriented approaches) would be placed after class because they considered time to be a factor of their decisions about teaching approaches.

When coding the data, I coded responses to the open-ended questions about the teaching proof and the vignettes (Section 3.3.1) separately, then compared the coded responses to assign a final code to each PT. Excerpts 3.4.1–3.4.4 are used to demonstrate the assignment of codes of PTs' beliefs about the place of proof in school mathematics (Table 3.4.1).

Excerpt 3.4.1
Yes, [we should implement tasks of proof in school mathematics]. Although [students] need to know how to apply formulae in examinations, the reasoning and thoughts during the process of proving help the development of thinking.

Excerpt 3.4.2
[I would use proof] when a new concept is being introduced.

Excerpt 3.4.3
[If time is not enough, I would remove] probably the proof. I think that some proof can be complicated and need some more knowledge to understand it thoroughly, which may scare the students.

Excerpt 3.4.4
[My teaching approach: …] (3) Using unknowns to prove, maybe something like this: (a pictorial argument) […]

Excerpts 3.4.1–3.4.2 are a PT's responses to the questions about the teaching proof whereas Excerpts 3.4.3–3.4.4 are their responses to the questions about the vignettes. Excerpts 3.4.1–3.4.2 were coded **Regular** whilst Excerpts 3.4.3–3.4.4 were coded **Supplementary**. Although the former excerpts show that the PT agreed on the importance of proof and wanted to include proof in their teaching, the latter excerpts show that they considered proof to be removable due to class-time limitation and their beliefs about students' lack of knowledge and ability, and their fear of proof. As a result, I assigned the code **Supplementary** to this PT.

After the coding process, all pre-intervention coded data were entered into a spreadsheet for analysis. I explored the descriptive statistics for patterns of individual codes (focus of RQ 1). When analysing post-intervention data, as some PTs did not complete the post-intervention questionnaire but answered the open-ended questions orally during interviews (Guiding Questions 3a–c & 4a, Section 3.3.2), I consulted the interviews to identify those PTs' post-intervention beliefs and attitudes towards teaching proof. Then, the pre-post data were compared to explore common themes of changes in individual PTs' beliefs about the place of proof in school mathematics and about approaches for teaching mathematics and proof (focus of RQ 2). When necessary to illustrate trends found in the data, the Cantonese/Chinese responses would be translated into English texts by me.

Akin to the analysis of responses to the Likert items, I compared the analysis of responses to the open-ended questions of the intervention group to those of the comparison group. The comparison group's responses to the open-ended questions were used to illustrate changes in the data that stood out in the intervention group (Cohen et al., 2011; E. Marsden & Torgerson, 2012).

3.4.2 Analysis of semi-structured interviews

Most of the interview data (Guiding Questions 2a–4b) were used to supplement the analyses of the questionnaire data (Section 3.4.1) and the PTs' responses to Guiding Questions 4b and 5b–d were used for identifying their changes and learnings from the intervention (focus of RQ 2), and to identify and evaluate the features of the intervention (see Section 3.2.2) that facilitated the PTs' changes (focus of RQ 3).

Twelve PTs were interviewed, on average, 17.6 days after the intervention, depending on their availability and the availability of transport. Each interview was audio-recorded and fully transcribed (Brenner, 2006; Cohen et al., 2011). I also listened to each interview in its entirety to review the conversation. I played back the recordings, coded the PTs' responses to Guiding Questions 2a–4b (using Tables 3.4.1–3.4.4) and kept summaries of the particular matters being discussed with time markers. This enabled me to become more familiar with the data and to highlight certain points in the interviews that were relevant to which I desired to capture for my analysis. These highlighted segments were interpreted into English, when necessary to illustrate trends of changes in the PTs' beliefs and attitudes towards teaching proof.

Moreover, I analysed the PTs' responses to Guiding Questions 4b & 5b to find (common themes of) their self-reported changes and learnings from the intervention. I read and listened to their responses in turn, coded and created a list of different changes and learning, and kept a frequency count for each change and learning (Table 4.2.10).

For the PTs' experiences in the intervention, I analysed their responses to Guiding Questions 5b–d to find common themes of whether and why activities of the intervention stood out to them (focus of RQ 3). I read and listened to the PTs' responses in turn, coded and created a list of each component (i.e., activities, features) of the intervention that stood out to the PTs (adding new categories that emerged), and kept a frequency count for each component (Table 4.3.1). I further examined the coded responses to identify whether and how the observed components were linked to different changes in the PTs' beliefs and attitudes towards teaching proof (as identified by the analyses for RQ 2) – I coded and created a list of each observed link, and kept a frequency count. Excerpts of the PTs' responses were used as illustrative examples of the emerging themes.

3.4.3 Analysis of audio-recordings and fieldnotes of the intervention

There are two major issues related to the process of transcribing audio-recordings of an intervention – the time it takes and the amount of data produced as its product. In a two-hour recording, it can take around 25 pages to capture most of what was said, but it is often only a small portion of the transcription that captures data that is relevant to the RQs under study. As a result, I applied partial transcriptions to the analysis of the audio-recordings of the intervention – I listened to the recordings, kept a summary of each activity or discussion occurred with time markers, identified and highlighted segments that I found important and relevant to RQ 3, and transcribed the highlighted segments into English when necessary to use them as illustrative examples (Ingram & Elliott, 2019).

The analysis of the audio-recordings (supplemented with fieldnotes) supplemented the analysis of the interviews (Section 3.4.2) (Patton, 1999; Suter, 2012; van den Akker et al., 2013) concerning the identification of components of the intervention that facilitated changes in the PTs' beliefs and attitudes towards teaching proof. I compared the list of components that stood out to the PTs (Table 4.3.1) to recordings of the corresponding components, to examine how the components were linked to the PTs' changes. Excerpts of the intervention recordings were used as illustrative examples of the themes found.

3.4.4 Strategies of ensuring validity and reliability

Various measures have been applied to ensure validity and reliability of the present study, and the notions of *transparency* and *triangulation* formed the basis. For transparency, I not only make the process of collecting and analysing the data, and how I ensured researcher bias to be minimal during data collection and analysis especially, explicit in Sections 3.3–3.4 (Atkins & Wallace, 2012; Cohen et al., 2011; Mayring, 2014), but also provide details of the development of the intervention and the questionnaire (e.g., frameworks, decisions on revision) in Sections 3.2–3.3. By doing so, each detail of the research design of the present study can be inspected for various forms of validity (e.g., construct validity, content validity, credibility, transferability) and reliability (e.g., internal consistency, stability, trustworthiness) of the present study (including the instruments used) by external parties (Akkerman et al., 2013; Cohen et al., 2011; McKenney & Reeves, 2012).

Triangulation was used to increase validity and reliability (e.g., credibility, trustworthiness) of collecting and analysing the data (Drew et al., 2008; Patton, 1999). First, I applied various methods to ensure the quality of the instruments used for data collection and analysis. For example, when developing and revising the instruments (e.g., the Likert items, the coding schemes), I compared different existing instruments (e.g., Keceli-Bozdag et al., 2014; Yoo, 2008), relevant literature on (Hong Kong) PTs' beliefs and attitudes (towards teaching proof) (e.g., Lesseig et al., 2019; Varghese, 2009; N.-Y. Wong et al., 2016) and data collected in Research Cycles 1–3, to ensure content validity of the instruments. When evaluating the Likert items, I used CFA to inspect and establish structural validity of the items (Brown, 2006; Harrington, 2009), inspected internal consistency of the items, and used various methods of analysing the test-retest data (e.g., Svensson's (1998) method, intraclass correlation coefficients with dimensional scores) to inspect and establish stability of the items (Appendix B).

Moreover, when evaluating the coding schemes, I invited a second coder to pilot the schemes in Research Cycle 3, in order to test how similar another observer and I interpreted the same data using the same set of coding schemes, which in turn, examined whether my subjectivity in developing and implementing the schemes was minimal (Cohen et al., 2011; Drew et al., 2008). She was given a copy of Tables 3.4.1–3.4.4 and received basic instruction and training about the coding process. The instruction and training that she received were *basic* in order to minimise any potential effects of my bias on them (e.g., influencing their coding decision) but to ensure their understanding of the definition of each code and the coding process. The second coder was asked to code the data collected from administering the two open-ended questions (Section 3.3.1) to PTs in Research Cycle 3 and create new codes if she discovered meaningful data that did not fit into any existing codes. This pilot suggested that the coding schemes guided an adequate coding process, increasing my confidence about the findings of the codes, and helped to finalise the coding schemes: adding a new code (**Unclear**) to Table 3.4.1 (see Appendix D for more information about the development of the coding schemes).

Second, triangulation was used to examine and ensure credibility and transferability of the findings of the present study (Cohen et al., 2011; Patton, 1999). For example, when analysing and interpreting the data to explore (change in) the PTs' beliefs and attitudes towards teaching proof, I compared various types of data collected from different sources (Likert items, open-ended questions, interviews), and analysed the data in different levels (e.g., individual, group), to identify agreements (and discrepancies) between different analyses, thereby minimising the risk of researcher bias and ensuring credibility of the analysis.

To link changes in the PTs' pre-post responses (representing changes in their beliefs and attitudes towards teaching proof) to the intervention, I used two methods (Section 3.3.4): First, I used data on a comparison group of PTs, who did not participate in the intervention, and compared them with data on the intervention group – the differences between the groups were used to infer the effects of the intervention on the intervention group (E. Marsden & Torgerson, 2012; O'Donnell, 2004). Second, I used semi-structured interviews, in which the PTs of the intervention group were invited to elaborate the changes in their responses to the questionnaire and their experiences in the intervention (Guiding Questions 4a–b & 5b), representing the effects of the intervention on the PTs from their own perspective. The use of the comparison group and the interviews aimed to ensure credibility of the analysis of the effects of the intervention.

Moreover, when examining links between the features of the intervention and changes in the PTs' beliefs and attitudes towards teaching proof (focus of RQ 3), I compared data collected from different research cycles, to evaluate and evolve the intervention (Section 3.2), thereby establishing credibility of the analysis of features of the intervention. I also probed into the generalisability of the intervention and its features, by comparing data on different try-outs (Try-outs 6–9 in particular) and features of other related interventions (e.g., A. J. Stylianides & Stylianides, 2014; Zengin, 2017), thereby evaluating transferability of the present study.

3.5 Ethical considerations

Following the ethical guidelines recommended by British Educational Research Association (2018), ethical clearance has been granted from the Central University Research

Ethics Committee of the University of Oxford and the Human Research Ethics Committee of The Education University of Hong Kong to conduct the present study. Informed consent, confidential handling of data (which is subjected to the General Data Protection Regulation and Data Protection Act 2018), de-identified data presentation, and the right to withdraw are key guiding ethical principles by which the present study has adhered. Details of the ethical approval and related documents are presented in Appendix F.

In the present study (particularly, during data collection), I played different roles: As a researcher, I was responsible for conducting a quality (e.g., ethical, valid, reliable) study on (changing) PTs' beliefs and attitudes towards teaching proof; as a developer, I was responsible for developing an effective intervention; as an instructor (of the intervention), I was responsible for ensuring that the PTs had quality learning experiences during the intervention. Table 3.5.1 summarises the strengths and issues about my multiple roles.

Table 3.5.1
Strengths and issues about my multiple roles

	Strengths	Issues
Researcher–developer–instructor	• Maintain control as a single decision maker who made all decisions about the research design and analysis of the data (Collins et al., 2004)	• Great effort required to maintain the balance of different roles and motives, and an ethical, quality study (Akkerman et al., 2013)
Researcher–developer	• Make use of research- and engineer-based knowledge to design and evolve an effective intervention that meets research and educational needs (McKenney & Brand-Gruwel, 2018)	• Risk of researcher bias (imposing developer's expectation of an effective intervention) (Cohen et al., 2011)
Researcher–instructor	• Provide an 'insider' perspective of how the intervention worked for research needs (e.g., reflection on each try-out) (Cohen et al., 2011; Tabach, 2011) • Ensure that the PTs would receive 'adequate' learning experiences (based on research and practices) in the intervention	• Risk of researcher bias (tendency to reject 'ineffective' teaching) (Cohen et al., 2011) • Risk of power differential (British Educational Research Association, 2018) • Tensions between research and educational needs (Tabach, 2011)
Developer–instructor	• Ensure that the intervention was conducted according to the design (and its rationale) (Collins et al., 2004)	• Risk of 'over-reliance' on the instructor perspective for developing the intervention

	• Provide a 'user' perspective of how the intervention could be improved	

In the following, I discuss five highlighted issues and measures that I took to mitigate them. First, my roles of the developer and instructor of the intervention could cause my (as the researcher) subjectivity in analysing the data and reporting the findings. In Section 3.4.4, I have discussed how I used the notions of transparency and triangulation to reduce the risk and influence of such subjectivity, thereby increasing validity and reliability of the analyses.

Second, issues of power differential, which could threaten the quality of participation, might be raised during the recruitment process by class visits and invitations through university emails, as PTs might misinterpret that they might benefit academically by participating in the present study. To mitigate such issues, on participant information sheet and invitations (class visits, emails), I informed PTs, as open and honest as possible without imposing my intention of developing and conducting the intervention (otherwise, issues of participant bias would develop), about the data collection process, the free choice to participate in or withdraw from the present study, and that the present study would not affect their academic outcomes (British Educational Research Association, 2018; Maykut & Morehouse, 1994).

Third, and related to the previous issue, the quality of data collected might be affected by participant bias. I imposed three measures to mitigate such issues: (i) Before collecting the data, from administering the questionnaire and conducting the interviews, I was open and honest with the PTs about the data collection process (e.g., the structure of interviews) and the use of data, without imposing any intended outcomes of the intervention. I also informed and reassured the PTs that there are no model answers to the questions, the questions are not used to assess whether they had the qualities of being a 'good' teacher and their data are truly confidential, but their own views and opinions are most valuable in the research process. By doing so, power differential between the PTs and me (as the interviewer) was reduced, thereby reducing social pressure and social desirability bias, and increasing the likelihood of collecting quality data (Drew et al., 2008; Krosnick & Presser, 2010; Maykut & Morehouse, 1994). (ii) During data collection, the questionnaire was first self-administered to the PTs without the presence of the researcher, to avoid pressure to participate and social desirability bias, whilst later, the PTs' responses were elaborated and clarified in the interviews (if they attended) (Cohen et al., 2011; Krosnick & Presser, 2010). I also maintained a stress-free interview environment, to increase the likelihood of collecting quality data (Maykut & Morehouse, 1994). (iii) During data analysis, data collected from different sources (Likert items, open-ended questions, interviews) were compared in order to inspect and reduce the effects of participant bias and possibilities of my (as the researcher) misinterpretation of the data (Section 3.4.4).

Fourth, I was aware that the PTs were situated in a discussion-based learning environment, where they might not be familiar with, and might develop discomfort, which might influence their participation (British Educational Research Association, 2018). To reduce such discomfort, I (as the instructor) was responsible for creating and maintaining a supportive environment in which the PTs were encouraged to engage in discussions (Section 3.2.2).

Fifth, there were tensions between my multiple roles that could affect the operation of the intervention (Tabach, 2011). For example, my perspective as the researcher and developer wanted the intervention to be conducted exactly according to plan (Section 3.2.2) for evaluation, but my perspective as the instructor decided some ad-hoc adjustment to the intervention (e.g., time spent on activities) when conducting the intervention, as I believed the PTs would benefit. Although try-outs of the intervention were not identical, I kept them as educational and consistent with what I intended as possible for evaluation. I also kept fieldnotes and audio-recorded every try-out as fully as possible (Section 3.3.3), such that my perspective as the researcher and developer who wanted to systematically study and evaluate the intervention could be fulfilled, and different try-outs of the intervention could be compared (in terms of their process and outcomes), which opened opportunities to inspect the generalisability of the intervention (McKenney & Reeves, 2014; The Design-Based Research Collective, 2003).

Moreover, other potential risks, for instance, field safety and research-related risks, had been considered and mitigated beforehand of my best to ensure the implementation of data collection (Risk Assessment Form, Appendix F). Lack of necessary research support or skills was another potential risk, which was mitigated by self-evaluation, attending research training and meetings, consultation with researchers, and regular supervisions.

However, despite careful considerations and measures against potential risks, unforeseen protests against political issues (Hong Kong government's controversial policies and police brutality) and clashes between police and protesters occurred in main roads and campuses of universities in Hong Kong. These events caused traffic interruptions, risks during transport and of conducting data collection, and increased withdrawals of participation. After consulting the escalating situations, planned class visits and try-outs of the intervention had been postponed, and eventually, cancelled from November 2019, to ensure safety. Although the amount of data collected was less than what was planned (to be discussed in Chapter 5), thanks to the support and advice from my supervisors and academics in Hong Kong, I was able to collect the necessary data for the present study.

Chapter 4: Results

In this chapter, I report the results of analyses of data collected from implementation of the intervention in Research Cycle 4, which were used for answering RQs 1–3. I first report the analysis of participating PTs' beliefs and attitudes towards teaching proof in school mathematics before the intervention (the focus of RQ 1; Section 4.1), then the analysis of changes in the PTs' beliefs and attitudes towards teaching proof after the intervention (the focus of RQ 2; Section 4.2), and finally, the analysis of the features of the intervention that facilitated the changes in the PTs' beliefs and attitudes towards teaching proof (the focus of RQ 3; Section 4.3).

4.1 Research Question 1: The PTs' pre-intervention beliefs and attitudes towards teaching proof in school mathematics

In this section, I report the analysis of the pre-intervention responses to the Likert items of the PTs who later participated in the intervention (the intervention group) (Section 4.1.1), then the analysis of their pre-intervention responses to the open-ended questions (Section 4.1.2). Lastly, I summarise the findings of both analyses (Section 4.1.3).

4.1.1 Analysis of responses to the Likert items

Descriptive statistics of the pre-intervention responses to the Likert items of the PTs in the intervention group are presented in Tables 4.1.1–4.1.5, regarding the PTs' beliefs about the importance of proof, their beliefs about relevance of proof to examinations, their enjoyment and interest in proof, their anxiety about proof and their negative responses when failing to prove (Section 3.3.1), respectively. In the following, when I talk about *agreeing responses* I will refer to responses to the items that lay in '**4 = Agree**' or '**5 = Strongly Agree**' and when I talk about *disagreeing responses* I will refer to responses that lay in '**2 = Disagree**' or '**1 = Strongly Disagree**'.

Table 4.1.1
The PTs' pre-intervention beliefs about the importance of proof

Item	Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree	Mean
I1. I think that theorems and proofs are the foundations of mathematics.	6	**14**	3	0	0	4.13
I2. Proofs are very important in understanding mathematical language.	5	**16**	2	0	0	4.13
I3. Proofs are indispensable for mathematics.	9	**11**	3	0	0	4.36
I4. Making proofs improves mathematical thinking.	**12**	9	2	0	0	4.43
I5. To me, it is important to make proofs in mathematics.	5	**17**	1	0	0	4.17
I6. To improve logical thinking, making proofs is necessary.	2	**14**	7	0	0	3.78
I7. In the development of reasoning, making proofs plays an important role.	5	**14**	4	0	0	4.04
D5. I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists.	1	4	**8**	**8**	2	2.74
Dimensional score	Mean	4.06		s.d.	.43	

n = 23; modes **bolded**

Before the intervention, most of the PTs reported that they believed proof is important for mathematics and personal development (Table 4.1.1). For Items I1–7, all but one item have means over 4 (**= Agree**), except that Item I6 has a mean of 3.78, slightly below 4. This pattern of the means is consistent with the pattern that most of the responses to the items are agreeing responses. Yet, for Item D5 "I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists", the PTs had diverse opinions. Although the item has a mean close to 3 (**= Neutral**) and eight out of 23 PTs chose '**3 = Neutral**', 10 PTs had disagreeing responses whereas five had agreeing responses. The dimensional scores reveal that the PTs generally agreed with the statements corresponding to the importance of proof, as evidenced by having a mean of 4.06 deviating from 3 (**= Neutral**) ($t(22) = 11.41$, $p < .001$).

Table 4.1.2
The PTs' pre-intervention beliefs about relevance of proof to examinations

Item	Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree	Mean
T2. Knowing how to make proofs is very important in excelling in an examination/a test in school mathematics.	0	**11**	9	2	1	3.30
T3. In an examination/a test in school mathematics, making proofs plays an important role.	0	**11**	7	4	1	3.22
T5. Practising making proofs does not help solve a task of an examination/a test in school mathematics.	0	5	**8**	**8**	2	2.70
Dimensional score	Mean	3.28		s.d.	.63	

n = 23; modes **bolded**

Most of the PTs either believed that proof is relevant to examinations or had neutral views about this relevance (Table 4.1.2). At the first glance, each item has a mean between 2.70 and 3.30, and the mean of the PTs' dimensional scores is 3.28, close to 3 (**= Neutral**) ($t(22) = 2.05$, $p = .052$). Yet the means do not agree with the modes: 11 PTs agreed that "knowing how to make proofs is very important in excelling in an examination/a test in school mathematics" (Item T2), 11 agreed that "in an examination/a test in school mathematics, making proofs plays an important role" (Item T3), and 10 rejected that "practising making proofs does not help solve a task of an examination/a test in school mathematics" (Item T5). Moreover, whilst 10 PTs had a dimensional score between 2.5 and 3.5, there were 10 PTs who had a dimensional score over 3.5. As a result, the analysis reveals two major clusters of the PTs: (a) who believed that proof is relevant to examinations, and (b) who had neutral views about this relevance.

Table 4.1.3
The PTs' pre-intervention enjoyment and interest in proof

Item	Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree	Mean
E1. It's fun for me to make proofs.	3	9	**10**	1	0	3.61
E2. One of the things I love about mathematics is that proofs of theorems are done in the lessons.	3	**9**	6	4	1	3.39
E4. Making proofs arouses my curiosity.	1	**14**	5	3	0	3.57
E5. I find it interesting to make proofs.	1	**14**	6	2	0	3.61
E6. I enjoy proving mathematical results.	1	**17**	5	0	0	3.83
E7. I feel enthusiastic to prove a theorem when I see it.	1	3	**14**	5	0	3.00
D3. I find it boring to make proofs.	0	3	4	**15**	1	2.39
Dimensional score	Mean	3.52		s.d.	.61	

n = 23; modes **bolded**

Most of the PTs reported that they had enjoyment and interest in proof (Table 4.1.3). Most of the PTs agreed that proof aroused their curiosity (Item E4) and interested them (Item E5), and expressed that they enjoyed proof (Item E6) and did not find proof boring (Item D3), but they did not agree nor disagree about "feel[ing] enthusiastic to prove a theorem when [they] see it" (Item E7). The dimensional scores reveal that the PTs generally enjoyed and were interested in proof, as evidenced by having a mean of 3.52 deviating from 3 (**= Neutral**) ($t(22) = 4.01$, $p = .001$).

Table 4.1.4
The PTs' pre-intervention anxiety about proof

Item	Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree	Mean
D1. I feel myself under pressure when I make proofs in mathematics lessons.	0	**10**	8	5	0	3.22
E3. I do not worry when making proofs.	0	7	**8**	**8**	0	2.96
A2. Working on a proof in front of the class frightens me.	0	7	**11**	5	0	3.09
A4. It scares me to work on a proof.	0	5	**9**	8	1	2.78
Dimensional score	Mean	3.01		s.d.	.56	

n = 23; modes **bolded**

Most of the PTs reported that they were neutral about whether they felt anxious about proof (Table 4.1.4). Whilst 10 PTs expressed "[feeling themselves] under pressure when making proofs in mathematics lessons" (Item D1), means and modes reveal that the PTs tended to choose '**3 = Neutral**' in the items concerning whether they were worried (Item E3), frightened (Item A2) and scared (Item A4) by proof. Moreover, 15 PTs had a dimensional score between 2.5 and 3.5 with a mean of 3.01 close to 3 (**= Neutral**) ($t(22) = .08$, $p = .938$), indicating that the PTs generally did not agree nor disagree about feeling anxious about proof.

Table 4.1.5
The PTs' pre-intervention negative responses when failing to prove

Item	Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree	Mean
A1. My self-confidence diminishes when I cannot prove a theorem.	2	**12**	8	1	0	3.65
A3. Not being able to prove upsets me.	2	**12**	6	2	1	3.52
Dimensional score	Mean	3.59		s.d.	.69	

n = 23; modes **bolded**

Whilst the PTs were neutral about whether they felt anxious about proof, most of them reported that they would have negative responses when failing to prove (Table 4.1.5). These PTs expressed that, if they could not prove a mathematical claim, their confidence in their proof and proving ability would diminish (Item A1) and they would be upset (Item A3). The dimensional scores indicate that failure of the PTs' expectation about their ability in proof might result in negative effects on them affectively, as evidenced by having a mean of 3.59 deviating from 3 (**= Neutral**) ($t(22) = 4.04$, $p = .001$).

To summarise, on the one hand, the PTs generally agreed with the importance of proof for learning mathematics in general, and had enjoyment and interest in proof. On the other hand, they expressed negative responses when failing to prove, and were neutral about whether they felt anxious about proof. It is also worth noting that the PTs formed two clusters concerning beliefs about relevance of proof to examinations: (a) those who believed this relevance and (b) those who had neutral views.

4.1.2 Analysis of responses to the open-ended questions

Using the descriptions of Tables 3.4.1–3.4.4 (Section 3.4.1), I coded the PTs' responses to the open-ended questions about the teaching of proof and the two vignettes (written data; Section 3.3.1), in order to gather information about their beliefs about the place of proof in school mathematics, and about approaches for teaching mathematics and proof. Tables 4.1.6–4.1.9 show the analysis of the pre-intervention written data. In the tables, the first column shows the codes (for the descriptions of the codes, see Tables 3.4.1–3.4.4), and the second column reports the frequency counts of the PTs corresponding to the codes.

Table 4.1.6
The PTs' pre-intervention beliefs about the place of proof in school mathematics

Code	Number of PTs[1]	*Importance of proof*[2]	*Relevance of proof to examinations*[2]	*Enjoyment & interest in proof*[2]	*Anxiety about proof*[2]	*Negative responses when failing to prove*[2]
No place	1	4.29[c]	1.78[c]	5.00[c]	2.45[c]	4.45[c]
Hesitating	1	3.90[c]	3.74[c]	3.72[c]	3.19[c]	4.00[c]
Unclear	3	3.45[c]	3.49[c]	3.42[c]	2.86[c]	2.97[c]
Supplementary	7	4.24	3.10	3.49	3.15	3.51
Regular	11	4.10	3.43	3.41	3.00	3.70

n = 23; [1]The number of the PTs who made statements relating to a certain code; [2]The five dimensions of the Likert items (see Table 4.1.1–4.1.5); [2]sample size too small (n < 5)

Overall, most of the PTs agreed that proof has a place in school mathematics (**Unclear**, **Supplementary**, **Regular**; Table 4.1.6). Eleven believed that proof should be implemented in classroom instruction regularly (**Regular**), whereas seven believed that proof serves as supplementary materials (or enrichment activities) for learning mathematics (**Supplementary**). I use Excerpts 4.1.1–4.1.2 to exemplify the codes **Supplementary** and **Regular** respectively.

> **Excerpt 4.1.1** (Participant 3, coded **Supplementary**, **Development of reasoning**)
> Yes, because making prove can improve students' thinking.
> [Proof should be implemented at] extra time, [to] motivate [students'] interest.

> **Excerpt 4.1.2** (Participant 22, coded **Regular**, **Promotion of understanding**)
> Yes. [Proving] the identities or any mathematics concept with what students have learnt before can help them to understand the concept more thoroughly:
> Proof and proving should be done at the beginning of each chapter or any new identities appear.

For the PT who rejected proof in school mathematics (**No place**), although he stated that when teaching mathematics, he would "emphasise the logic behind the steps of solving practice problems" in the open-ended questions (Section 3.3.1) and, according to his dimensional scores of the Likert items, believed in the importance of proof in learning mathematics (= 4.29) and enjoyed proof (= 5.00), he believed that proof is not relevant to examinations (= 1.78).

Table 4.1.7
The PTs' pre-intervention beliefs about approaches for teaching mathematics and proof

Code	Number of PTs[1]	*Importance of proof*[2]	*Relevance of proof to examinations*[2]	*Enjoyment & interest in proof*[2]	*Anxiety about proof*[2]	*Negative responses when failing to prove*[2]
Product-oriented	15	3.99	3.51	3.04	3.67	3.30
Mixed	5	4.28	3.31	2.86	3.04	3.34
Process-oriented	1	3.83[c]	4.26[c]	2.00[c]	4.45[c]	3.00[c]
No place/Hesitating	2	4.09[c]	2.76[c]	4.36[c]	2.82[c]	4.22[c]

n = 23; [1]The number of the PTs who made statements relating to a certain code; [2]The five dimensions of the Likert items (see Table 4.1.1–4.1.5); [2]sample size too small (n < 5)

Except for two PTs who rejected the teaching of proof or hesitated to teach proof in school mathematics (**No place/Hesitating**; see also Table 4.1.6), three different categories concerning the PTs' beliefs about approaches for teaching mathematics and proof (**Product-oriented**, **Mixed**, **Process-oriented**) were observed in their responses to the open-ended questions (Section 3.3.1). Overall, product-oriented approaches for teaching mathematics and proof were most favoured, for 15 PTs had responses coded as **Product-oriented** and five had responses coded as **Mixed** (Table 4.17). When being asked about their approaches for teaching mathematics and proof and their opinions about teaching approaches that are important for student learning (Section 3.3.1), these PTs indicated that more product-oriented approaches, such as teacher-centred lecturing and demonstration, are more effective in teaching mathematics and proof, and some adhered to the practices upholding ideas, for example, teachers as a explainer of mathematical concepts and an instructor who ensures students' correct mathematical performance, and the learning of mathematics as passive reception of knowledge. Some PTs also related their choices of approaches to their past experiences with school mathematics. I use Excerpts 4.1.3–4.1.4 to exemplify the codes **Product-oriented** and **Mixed**, respectively.

> **Excerpt 4.1.3** (Participant 5, coded **Product-oriented**)
> Lecture, as I can directly tell students the most important concept. Demonstration, as it helps students understand the concept easier.
>
> **Excerpt 4.1.4** (Participant 9, coded **Mixed**)
> [Her approach] I learn it myself by the method of [product-oriented approaches], so this is the approach I'm more familiar with, but I also see the importance of student involvement in a lesson so I agree with [process-oriented approaches].

For the only PT who adhered to process-oriented approaches for teaching mathematics and proof, she valued activities that involve proof and enable students' active engagement in mathematical exploration (**Process-oriented**). She also had strong enjoyment and interest in proof (= 4.26) and low anxiety about proof (= 2.00) according to their dimensional scores of the Likert items.

Table 4.1.8
The PTs' pre-intervention reasons for implementing proof in school mathematics

Code	Number of PTs[1]
Justification	2
Promotion of understanding	17
Memorisation	7
Generation of knowledge	1
Development of reasoning	9
Promotion of students' motivation	2
Self affection	2

n = 23; [1]The number of the PTs who made statements relating to a certain code

Overall, most of the PTs reported they believed that proof can promote students' conceptual understanding in mathematics (**Promotion of understanding**,17PTs; Table 4.1.8). The second and third popular reasons for implementing proof in school mathematics were the statements that proof can help students develop ability and skills in problem solving, logical thinking and reasoning (**Development of reasoning**, 9 PTs) and that proof can promote students' memorisation of mathematical knowledge (**Memorisation**, 7 PTs), respectively. I use Excerpts 4.1.1, 4.1.2 and 4.1.5 to exemplify the codes **Promotion of understanding**, **Development of reasoning** and **Memorisation**, respectively.

> **Excerpt 4.1.5** (Participant 13, coded **Memorisation**)
> 1. [Proof and proving] can help students memorise relevant formulae. 2. can make students become able to deduce [formulae] themselves, [and] consolidate [their] understanding of the formulae

Table 4.1.9
The PTs' pre-intervention concerns about the use of proof in school mathematics

Code	Number of PTs[1]	P[1]	A[1]
Students (Concerns about students' ability, interest, etc.)	6	5	1
Self (Concerns about themselves, as a teacher, regarding knowledge, etc.)	5	0	5
Curriculum (Concerns about curriculum, (school) policy and/or examinations)	7	5	2
Time (Concerns about class time)	4	0	4

n = 23; P = Reasons behind the place of proof; A = Reasons behind approaches for teaching mathematics and proof; [1]The number of the PTs who made statements relating to a certain code

Table 4.1.9 shows that the PTs expressed different concerns about the use of proof in school mathematics. Six PTs stated that students are not capable of proof because they do not have adequate knowledge, skills or interest in proof (**Students**), and seven stated that proof is scarcely related to curriculum and examinations of school mathematics (**Curriculum**). Some of the PTs also expressed concern that proof is time-demanding (**Time**, 4 PTs) and related their decisions about the teaching of proof to their (lack of) autonomy, experiences, knowledge and/or interest in proof (**Self**, 5 PTs). I use Excerpts 4.1.6–4.1.7 to exemplify the codes **Curriculum** and **Self**.

> **Excerpt 4.1.6** (Participant 4, coded **Curriculum–Reasons behind the place of proof**)
> Proof, in my view, is not a very important thing to secondary school students. It is not so important in examinations.

> **Excerpt 4.1.7** (Participant 8, coded **Self–Reasons behind approaches for teaching mathematics and proof**)
> When I'm a student, [more product-oriented approaches were what] I was taught by my teachers and it works well on me.

Table 4.1.9 also shows the findings of reasons behind the place of proof in school mathematics and behind approaches for teaching mathematics and proof in the third and fourth columns, respectively (Section 3.3.1). Beliefs that students are neither capable nor interested in proof (**Students**, 5 PTs) and that proof is not relevant to the curriculum and examinations (**Curriculum**, 5 PTs) were the major reasons behind the PTs' decisions about the place of proof in school mathematics, whereas concerns that the PTs, themselves, are not ready and not confident about proof, or they are more familiar with product-oriented approaches for teaching mathematics and proof (**Self**, 5 PTs) and belief that proof is time-demanding (**Time**, 4 PTs) were the major reasons behind the PTs' decisions about approaches for teaching mathematics and proof.

To summarise, pre-intervention written data indicate that all but two PTs agreed with the importance of proof in school mathematics (Table 4.1.6) and seven PTs expressed that the curriculum and examinations influenced their decisions about the use of proof in their teaching (Table 4.1.9). The data also provided information about the PTs' beliefs about approaches for teaching mathematics and proof and their reasons for implementing proof in school mathematics. The majority held product-oriented beliefs (Table 4.1.7), and believed that proof can promote students' conceptual understanding in mathematics (Table 4.1.8).

4.1.3 Summary of Section 4.1

The findings of the analyses of the PTs' responses to the Likert items and the open-ended questions (Section 3.4.1) concerning their pre-intervention beliefs and attitudes towards teaching proof in school mathematics are consistent with each other. Both productive and counterproductive beliefs and attitudes were identified.

On the one hand, the PTs generally agreed that it is important to include proof in school mathematics. They indicated that they enjoyed and were interested in proof (Tables 4.1.1 and 4.1.3). They also acknowledged the explanatory role of proof (see "explanatory proof", Hanna, 1990) and agreed that in school mathematics, proof can promote students' conceptual understanding and memorisation of mathematical knowledge and can help students develop skills in problem solving and logical reasoning (Table 4.1.8).

On the other hand, the PTs expressed concerns about the use of proof in their future teaching. Particularly, they reported that to them proof is difficult and mentally demanding and that not all students are able to develop, evaluate and/or appreciate (Table 4.1.9), and the uncertainty about whether proof is relevant to curriculum materials, particularly examinations (Table 4.1.2), influenced the PTs' (pre-intervention) decisions around proof in school mathematics (Table 4.1.9).

Regarding beliefs about approaches for teaching mathematics and proof, the data indicate that most of the PTs tended to follow product-oriented approaches for teaching mathematics and proof (Table 4.1.7) when being asked about their general teaching approaches (Section 3.3.1). It is worth noting that some PTs related their product-oriented ideas to their past experiences with school mathematics and proof.

4.2 Research Question 2: Post-intervention changes in the PTs' beliefs and attitudes towards teaching proof in school mathematics

In this section, I report the analysis of changes in the PTs' pre-post responses to the Likert items (Section 4.2.1) and the analysis of changes in their pre-post responses to the open-ended questions (Section 4.2.2). I also report the analysis of the PTs' individual interviews concerning changes in their pre-post responses to the questionnaire and their learnings from the intervention (Interview Questions 4b & 5b, Section 3.3.2) from their own perspective (Section 4.2.3). Lastly, I summarise the findings of the analyses (Section 4.2.4). Since I considered only PTs who participated in at least two sessions of the intervention (Section 3.4) and provided both pre- and post-intervention data (Table 3.3.2) for the analyses in this section, the total number of the PTs included in these analyses is less than that in the analyses reported in Section 4.1. It is also worth noting that due to incomplete data, 16 PTs were not included in the analyses in Section 4.2.1 (Tables 4.2.1–4.2.5) and 11 PTs were not included in the analyses in Section 4.2.2 (Tables 4.2.6–4.2.9). These include the PTs who rejected the teaching of proof (**No place**) or hesitated to teach proof (**Hesitating**) or agreed with the teaching of proof but were unclear about the place of proof (**Unclear**) before the intervention (Table 4.1.6). Subsequent limitations are discussed in Chapter 5.

4.2.1 Analysis of changes in responses to the Likert items

When analysing the data from the Likert items, I defined a *notable change* in response(s) to an item or dimensional score(s) of several items as a shift (in the mean) from **Agreeing** (over 3.5) or **Neutral** (between 2.5 and 3.5) or **Disagreeing** (below 2.5) to another category. This change is considered to be notable because it demonstrates a modification in the way PT(s) thought about particular item(s), going from **Agreeing** to **Neutral** or **Disagreeing**, **Neutral** to **Agreeing** or **Disagreeing**, or **Disagreeing** to **Agreeing** to **Neutral**, depending on the initial state. A shift, for example, from ‘**3 = Neutral**’ to ‘**4 = Agree**’, might be an important shift to the PTs, as the interview excerpt below (Excerpt 4.2.1) exemplifies.

> **Excerpt 4.2.1** (Participant 2)
> Even if it is a 1-point shift, [I think] it is substantial. […] I think [I am] more positive [in the post-intervention questionnaire], [for example,] from having pressure to neutral. I

think [I am] a lot more positive; from ‘**2 = Disagree**’ to ‘**3 = Neutral**’ is already more positive [on item D1].

Tables 4.2.1–4.2.5 present descriptive statistics of differences in the PTs’ pre-post responses to the Likert items regarding their beliefs about the importance of proof, their beliefs about relevance of proof to examinations, their enjoyment and interest in proof, their anxiety about proof, and their negative responses when failing to prove, respectively. In the tables, the first column shows the item statements, the second to fourth columns report the means of pre-post responses of PTs who participated in the intervention (the intervention group) and their differences respectively, and the fifth to seventh columns report the means of test-retest responses of PTs who did not participate in the intervention (the comparison group) and their differences respectively.

Table 4.2.1
Changes in the PTs’ beliefs about the importance of proof

	Intervention (n = 7)			Comparison (n = 27)		
Item	X	Y	Δ	X	Y	Δ
I1. I think that theorems and proofs are the foundations of mathematics.	4.29	4.43	.14	4.22	4.33	.11
I2. Proofs are very important in understanding mathematical language.	4.43	4.71	.28	4.26	4.15	-.09
I3. Proofs are indispensable for mathematics.	4.57	4.71	.14	4.52	4.33	-.19
I4. Making proofs improves mathematical thinking.	4.86	4.57	-.29	4.41	4.33	-.08
I5. To me, it is important to make proofs in mathematics.	4.43	4.14	-.29	4.37	4.11	-.26
I6. To improve logical thinking, making proofs is necessary.	3.71	3.71	.00	3.89	3.93	.04
I7. In the development of reasoning, making proofs plays an important role.	4.29	4.43	.14	4.00	4.07	.07
D5. I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists.[1]	2.57	1.86	-.71	2.22	2.00	-.22
Dimensional score	4.29	4.37	.08	4.20	4.16	-.04

X = pre-intervention or first time point; Y = post-intervention or second time point; Δ = Y – X; [1]negatively phrased

Table 4.2.1 shows a notable change in the intervention group’s responses to the statement “I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists” (Item D5). After the intervention, the PTs generally disagreed with the statement as evidenced by a decrease in the mean from 2.57 (**Neutral**) to 1.86 (**Disagreeing**), and all seven PTs reported that they could understand why making proofs that have already been proven. Particularly, three of them changed their views from '4 = **Agree**' or '**3 = Neutral**' to '**2 = Disagree**'. There was little change in the means of the intervention group's responses to other items. The intervention group maintained an agreement with the statements in Items I1–7, as well as with the importance of proof, as evidenced by the mean of dimensional scores of 4.37.

Comparatively, there was also little change in the comparison group's beliefs about the importance of proof, as evidenced by no changes in the response categories of the means of all item responses. The comparison group maintained an agreement with the statements in Items I1–7 and a disagreement with the statement in Item D5, thereby maintaining an agreement with the beliefs about the importance of proof.

Table 4.2.2
Changes in the PTs' beliefs about relevance of proof to examinations

	Intervention (n = 7)			Comparison (n = 27)		
Item	X	Y	Δ	X	Y	Δ
T2. Knowing how to make proofs is very important in excelling in an examination/a test in school mathematics.	3.14	3.43	.29	3.44	3.22	-.22
T3. In an examination/a test in school mathematics, making proofs plays an important role.	3.14	3.43	.29	3.26	3.26	.00
T5. Practising making proofs does not help solve a task of an examination/a test in school mathematics.[1]	2.14	2.00	-.14	2.52	2.70	.18
Dimensional score	3.33	3.58	.25	3.40	3.25	-.15

X = pre-intervention or first time point; Y = post-intervention or second time point; Δ = Y – X; [1]negatively phrased

Table 4.2.2 shows that there was little change in the means of the responses of the intervention group to each item. Regarding the means, Items T2 and T3 maintained **Neutral** and Item T5 maintained **Disagreeing**, but there was a notable change in the mean of dimensional scores, from 3.33 (**Neutral**) to 3.58 (**Agreeing**). An investigation into the dimensional score for each PT provided an explanation of this change. Before the intervention, four PTs had a dimensional score over 3.5, two had a score between 2.5 and 3.5, and one had a score below 2.5. However, after the intervention, the three PTs who had dimensional scores below 3.5 had increases in their scores by over .5, resulting in a shift in a PT's dimensional score from **Disagreeing** to **Neutral** and another shift from **Neutral** to **Agreeing**. At the same time, the remaining four PTs maintained an agreement with relevance of proof to examinations and changes in their dimensional scores were little.

Comparatively, there was little change in the comparison group's beliefs about relevance of proof to examinations, as evidenced by no changes in the response categories of the means of all item responses and dimensional scores. The PTs remained neutral about the statements of Items T2, T3 and T5, and hence, neutral about relevance of proof to examinations. These indicate that these PTs remained uncertain whether proof is relevant to examinations.

Table 4.2.3
Changes in the PTs' enjoyment and interest in proof

	Intervention (n = 7)			Comparison (n = 27)		
Item	X	Y	Δ	X	Y	Δ
E1. It's fun for me to make proofs.	3.57	3.86	.29	3.93	3.81	-.12
E2. One of the things I love about mathematics is that proofs of theorems are done in the lessons.	3.29	3.57	.28	3.48	3.63	.15
E4. Making proofs arouses my curiosity.	3.71	4.14	.43	3.78	3.81	.03
E5. I find it interesting to make proofs.	3.57	3.86	.29	3.78	3.74	-.04
E6. I enjoy proving mathematical results.	3.86	4.00	.14	3.81	3.85	.04
E7. I feel enthusiastic to prove a theorem when I see it.	2.86	3.29	.43	3.30	3.44	.14
D3. I find it boring to make proofs.[1]	2.29	2.00	-.29	2.44	2.30	-.14
Dimensional score	3.52	3.83	.31	3.67	3.72	.05

X = pre-intervention or first time point; Y = post-intervention or second time point; Δ = Y – X; [1]negatively phrased

Table 4.2.3 shows a notable change in the mean of responses to the statement "One of the things I love about mathematics is that proofs of theorems are done in the lessons" (item E2), from 3.29 (**Neutral**) to 3.57 (**Agreeing**). The PTs generally developed more enjoyment and interest in proof as supported by both the mean of changes in the means of the item responses and the mean change in dimensional scores exceeding .3. Particularly, more PTs reported that proof aroused their curiosity (Item E4), and felt enthusiastic to prove a theorem (Item E7), with the mean differences exceeding .4.

Comparatively, there were very few changes in the comparison group's enjoyment and interest in proof as evidenced by differences in the means of all item responses found to be less than .2 as well as no changes in the response categories of the means of all (but one) item responses and dimensional scores. Most of the PTs in the comparison group stayed enjoying and interested in proof, as the means of most item responses (Items E1, 2, 4–6) and dimensional scores stayed over 3.5 (**Agreeing**) at second time point.

Table 4.2.4
Changes in the PTs' anxiety about proof

	Intervention (n = 7)			Comparison (n = 27)		
Item	X	Y	Δ	X	Y	Δ
D1. I feel myself under pressure when I make proofs in mathematics lessons.	3.43	2.86	-.57	3.07	3.04	-.03
E3. I do not worry when making proofs.[1]	2.57	2.86	.29	3.41	3.56	.15
A2. Working on a proof in front of the class frightens me.	3.29	2.43	-.86	2.74	2.56	-.18
A4. It scares me to work on a proof.	2.43	2.29	-.14	2.48	2.44	-.04
Dimensional score	3.09	2.65	-.44	2.69	2.59	-.10

X = pre-intervention or first time point; Y = post-intervention or second time point; Δ = Y – X; [1]negatively phrased

Table 4.2.4 shows a notable change in the intervention group's responses to the statement "Working on a proof in front of the class frightens me" (Item A2). After the intervention, the mean of responses to Item A2 decreased from 3.29 (**Neutral**) to 2.43 (**Disagreeing**) by .86, indicating that the PTs no longer found working on a proof in front of the class frightening or they found it less frightening. Particularly, whilst three of the PTs chose '**4 = Agree**' before the intervention, all chose either '**2 = Disagree**' or '**3 = Neutral**' in response to Item A2 after the intervention. Also, fewer PTs expressed that they felt themselves under pressure when they produced proofs in a lesson, as supported by a decrease in the mean of Item D1 from 3.43

(**Neutral**) to 2.86 (**Neutral**). Altogether these changes contributed to a decrease in the mean of dimensional scores of the intervention group by .44.

Comparatively, there was little change in the comparison group's anxiety about proof as evidenced by differences in the means of all item responses found to be less than .2 as well as no changes in the response categories of the means of all (but one) item responses and dimensional scores. Most of the PTs remained neutral about whether they felt anxious about proof, with the dimensional score mean of 2.59 at second time point.

Table 4.2.5
Changes in the PTs' negative responses when failing to prove

	Intervention (n = 7)			Comparison (n = 27)		
Item	X	Y	Δ	X	Y	Δ
A1. My self-confidence diminishes when I cannot prove a theorem.	3.71	3.43	-.28	3.26	3.22	-.04
A3. Not being able to prove upsets me.	3.43	3.00	-.43	3.33	3.26	-.07
Dimensional score	3.59	3.24	-.35	3.29	3.24	-.05

X = pre-intervention or first time point; Y = post-intervention or second time point; Δ = Y – X

Table 4.2.5 shows a reduction in the PTs' negative responses when failing to prove after the intervention. Particularly, there was a notable change in the mean of their dimensional scores, from 3.59 (**Agreeing**) to 3.24 (**Neutral**). After the intervention, fewer PTs reported that their self-confidence would diminish (Item A1) and they would get upset (Item A3) if they are not able to prove a theorem, as evidenced by the decreases in the mean of Item A1 from 3.71 (**Agreeing**) to 3.43 (**Neutral**) by .28 and that of Item A3 from 3.43 (**Neutral**) to 3.00 (**Neutral**) by .43.

Comparatively, there was little change in the comparison group's negative responses when failing to prove, as evidenced by differences in the means of all item responses found to be less than .1 as well as no changes in the response categories of the means of all item responses and dimensional scores. Most of the PTs remained neutral about whether they had negative responses when failing to prove, with the dimensional score mean of 3.24 at second time point.

To summarise, there were changes in the intervention group's responses to the Likert items corresponding to their beliefs about relevance of proof to examinations, their enjoyment and interest in proof, their anxiety about proof, and their negative responses when failing to prove (Section 3.3.1). These changes were more substantial than those of the comparison group.

For example, the intervention group's dimensional scores corresponding to their beliefs about relevance of proof to examinations improved from a mean of 3.33 (**Neutral**) to a mean of 3.58 (**Agreeing**) and those corresponding to their negative responses when failing to prove shifted from a mean of 3.59 (**Agreeing**) to 3.24 (**Neutral**). I use two interview excerpts (Excerpts 4.2.2–4.2.3) to illustrate the changes. On the contrary, the comparison group maintained their beliefs and attitudes towards teaching proof in school mathematics, as the mean of differences in the means of the comparison group's dimensional scores was less than .1.

> **Excerpt 4.2.2** (Participant 2)
> [After the intervention] I think I am different, from having little feelings to more positive [about proof]… I always agree with the roles that proof plays in mathematics, but the ways how I feel proof are different… [For example, before the intervention, in Item D5] I thought [proof] was [too] abstract… If I had to prove [the theorems] again, it would demand a lot of time… I thought, perhaps, we could ignore the proofs. I think it's about attitudes [towards proof].

> **Excerpt 4.2.3** (Participant 7)
> [Before the intervention] I thought proof was not relevant, because I, like [Item D1], I thought proof was something lengthy and difficult. This kind of proof… so I thought, '[I] don't know how to do [a proof] anyway,' then of course [proof] is not relevant to my examination. But [after the intervention, I] think if ideas of deduction can be proof, as an area in which we learn something [and proof] helps reasoning, so [it] is better (I have better feelings about proof).

4.2.2 Analysis of changes in responses to the open-ended questions

Table 4.2.6–4.2.9 show the results of the analysis of the PTs' pre-post responses to the open-ended questions (Section 3.3.1). Similar to Tables 4.1.6–4.1.9, the first column of each table presents the codes corresponding to the theme of the table. For example, the codes in Table 4.2.6 concern the PTs' beliefs about the place of proof in school mathematics. In Tables 4.2.6–4.2.8 the second and third columns report the number of the PTs (frequency) in the intervention group corresponding to each of the pre- and post-intervention codes, respectively, and the fourth and fifth columns for the comparison group at the two time points, accordingly. In Table 4.2.9, I include two additional columns with italicised headings after each frequency column. The additional columns report the numbers of the PTs who related their concerns about the use of proof in school mathematics to reasons behind their decisions about the place of proof and behind their approaches for teaching mathematics and proof, respectively (Section 3.3.1).

Table 4.2.6
Changes in the PTs' beliefs about the place of proof in school mathematics

	Intervention group (n = 12)		Comparison group (n = 22)	
Code	X[1]	Y[1]	X[1]	Y[1]
No place	0	0	0	0
Hesitating	0	0	5	4
Unclear	0	0	2	3
Supplementary	6	2	4	4
Regular	6	10	11	11

X = pre-intervention or first time point; Y = post-intervention or second time point; [1]The number of the PTs who made statements relating to a certain code

Table 4.2.6 shows a change in the intervention group's beliefs about the place of proof in school mathematics, towards the belief that proof should be implemented in classroom instruction regularly (**Regular**). Before the intervention, six PTs agreed with the importance of proof in school mathematics but treated proof as supplementary materials (**Supplementary**) whilst the others stated that proof should be implemented in classroom instruction regularly. After the intervention, four of the **Supplementary** PTs changed their beliefs, resulting in 10 **Regular** PTs. I use two interview excerpts (Excerpts 4.2.4–4.2.5) to illustrate the change.

> **Excerpt 4.2.4** (Participant 3)
> [Before, I thought proof should be treated as extra materials for motivating students' interest only.] But now, I think we can, in regular lessons, guide students and include

activities of proof to allow students to think. On the contrary, 'extra time' should be used to help weaker students out [if necessary]. […] [In the intervention,] I discover that teaching mathematics is not just 'listening to the teachers'. There are discussions between students. [On the contrary] what I experienced in learning mathematics was 'working on myself [only]'.

Excerpt 4.2.5 (Participant 7)
Now, I don't [remove proof even if time is limited], because [in the pre-intervention questionnaire] I wrote proof [as removable], but now I think it is very important.

Comparatively, there was little change in the comparison group's beliefs about the place of proof in school mathematics. Most of the PTs stayed believing that proof is important and should have a place in school mathematics. Eleven of the PTs stated that proof should be implemented in classroom instruction regularly at second time point, and 14 did not change their statements about the place of proof between the two time points (Table E.6, Appendix E).

Table 4.2.7
Changes in the PTs' beliefs about approaches for teaching mathematics and proof

	Intervention group (n = 12)		Comparison group (n = 22)	
Code	X[1]	Y[1]	X[1]	Y[1]
Product-oriented	8	0	11	13
Mixed	4	7	2	3
Process-oriented	0	5	4	2
No place/Hesitating	0	0	5	4

X = pre-intervention or first time point; Y = post-intervention or second time point; [1]The number of the PTs who made statements relating to a certain code

Table 4.2.7 shows a change in the intervention group's beliefs about approaches for teaching mathematics and proof, towards **Process-oriented** beliefs. Before the intervention, eight of the PTs provided statements about their intended teaching approaches aligned with product-oriented approaches for teaching mathematics and proof (**Product-oriented**) and four provided statements aligned with a balanced view of both product-oriented and process-oriented approaches (**Mixed**) whilst none made statements aligned with process-oriented approaches. After the intervention, all 12 PTs indicated that they would include some process-oriented approaches (**Mixed** or **Process-oriented**) into their approaches for teaching mathematics and proof, and particularly five made statements aligned with process-oriented approaches. I use Excerpts 4.2.6–4.2.7 to illustrate the change.

> **Excerpt 4.2.6** (Pre-intervention, Participant 16, coded **Product-oriented**)
> Because the class time is limited, activities of proof and proving should be led by the teacher. During the demonstration of the deduction process, [the teacher] should keep reminding students about what the conclusion of the proof should be.

> **Excerpt 4.2.7** (Post-intervention, Participant 16, coded **Process-oriented**)
> After the teacher introduces the topic, [the teacher] could use group work to carry out exploration of proof [of a mathematical claim]. Use students' presentation and teacher's summary to conclude [the lesson].

Comparatively, there was little change in the comparison group's beliefs about approaches for teaching mathematics and proof. Ten PTs maintained their statements about same approaches at the two time points (Table E.7, Appendix E) and 13 PTs made statements aligned with product-oriented approaches (e.g., teacher-centred lecturing and demonstration of proofs, student's practice in proving) at second time point.

Table 4.2.8
Changes in the PTs' reasons for implementing proof in school mathematics

	Intervention group (n = 12)		Comparison group (n = 22)	
Code	X[1]	Y[1]	X[1]	Y[1]
Justification	2	2	2	1
Promotion of understanding	9	12	10	9
Memorisation	3	6	1	1
Generation of knowledge	1	1	0	0
Development of reasoning	7	5	9	4
Promotion of students' motivation	1	2	2	3
Self affection	1	0	1	1

X = pre-intervention or first time point; Y = post-intervention or second time point; [1]The number of the PTs who made statements relating to a certain code

Table 4.2.8 shows some changes in the intervention group's reasons for implementing proof in school mathematics. The number of the PTs who made statements that proof can promote students' conceptual understanding in mathematics (**Promotion of understanding**) increased from nine to 12, and the number of the PTs who made statements that proof can promote students' memorisation of mathematical knowledge (**Memorisation**) increased from three to six. There was little change in the number of the PTs who provided responses relating to other reasons. I use Excerpt 4.2.8 to illustrate the change (of Participant 3).

> **Excerpt 4.2.8** (Post-intervention, Participant 3, coded **Promotion of understanding**)
> I think proof can make students understand more about 'what is happening', instead of simply using a formula to calculate… I seldom experienced proof when I was a student.

Comparatively, there was little change in the comparison group's reasons for implementing proof in school mathematics between two time points, except for the reason relating to the belief that proof can help students develop skills in problem solving, logical thinking and reasoning (**Development of reasoning**). The number of the PTs stating this reason decreased from nine to four. Overall, most of the PTs made statements that proof can promote students' conceptual understanding in mathematics (**Promotion of understanding**) and this belief continued to be considered as the comparison group's primary reason for implementing proof in school mathematics.

Table 4.2.9
Changes in the PTs' concerns about the use of proof in school mathematics

	Intervention group (n = 12)						Comparison group (n = 22)					
Code	X[1]	*P*[1]	*A*[1]	Y[1]	*P*[1]	*A*[1]	X[1]	*P*[1]	*A*[1]	Y[1]	*P*[1]	*A*[1]
Students	6	5	1	5	3	3	9	7	2	8	8	1
Self	3	0	3	4	0	4	2	2	0	0	0	0
Curriculum	6	4	2	2	2	0	4	4	0	2	0	2
Time	3	0	3	3	1	2	1	0	1	0	0	0

P = Reasons behind the place of proof; A = Reasons behind approaches for teaching mathematics and proof; X = pre-intervention or first time point; Y = post-intervention or second time point; [1]The number of the PTs who made statements relating to a certain code

Table 4.2.9 shows both maintenance of and changes in the intervention group's concerns about the use of proof in school mathematics. On one hand, the concerns about students' ability and interest in proof (**Students**, 5 PTs) stayed being considered as the primary concerns about the use of proof in school mathematics stated by the PTs. On the other hand, there was a decrease in the number of the PTs who made statements about considering curriculum and examinations to be one of their concerns (**Curriculum**), from six to two. Moreover, there were decreases in the number of the PTs who attributed students' ability, knowledge and interest (**Students**) and curriculum and examinations (**Curriculum**) to reasons behind their decisions about the place of proof in school mathematics, from five to three and from four to two, respectively.

Comparatively, there was little change in the comparison group's concerns about the use of proof in school mathematics between two time points. Particularly, they continued to consider **Students** as their primary concerns about the use of proof in school mathematics, similar to the intervention group. However, most of the PTs in the comparison group continued to consider **Students** to be reasons behind their decisions about the place of proof in school mathematics at the second time point, contrasting to the decrease in the number of the PTs in the intervention who attributed **Students** to reasons behind their decisions about the place of proof.

To summarise, the analysis of responses to the open-ended questions (Sections 3.3.1 & 3.4.1) indicate changes in the intervention group's beliefs about whether and how proof should be implemented in school mathematics. After the intervention, most of the PTs agreed that proof should be implemented in classroom instruction regularly. All PTs in the intervention group also adopted process-oriented approaches for teaching mathematics and proof in their intended teaching, and made statements that proof can promote students' conceptual understanding in mathematics. The influence of the PTs' concern that proof is not relevant to curriculum and examinations on their decisions about the place of proof in school mathematics appeared to be reduced, as suggested by the reduction in the number of the PTs who made such statements. Similarly, the influence of the PTs' concern that students are incapable of or not interested in learning proof on their decisions about the place of proof in school mathematics was slightly reduced. Having said that, this concern remained as the primary issues that influenced the PTs' decisions about the use of proof in school mathematics, in general.

4.2.3 Analysis of data on the interviews

Table 4.2.10 presents the analysis of the PTs' elaborations of changes in their responses to the questionnaire and learnings from the intervention (Section 3.4.2) during the interviews (Interview Questions 4b & 5b, Section 3.3.2). In this table, the first column shows a list of the PTs' changes and learnings. The second column shows the number of the PTs who made statements relating to each observed change and learning, and the third column shows the total number of separate statements that the PTs made and were captured by the corresponding change and learning. A PT could make statements relating to different changes and learnings, and several statements about one specific change and learning.

Table 4.2.10
Analysis of the PTs' response changes and learnings (n = 12)

Change and learning	Number of PTs[1,3]		Frequency[2,3]	
Different views about proof (**DifProof**)	10		22	
Different emotional dispositions towards proof (**DifProofEmo**)		(5)		(7)
Different views about the nature and roles of proof (**DifProofNR**)		(7)		(13)
Different views about students (**DifStudent**)	6		10	
Different views about teaching (**DifTeach**)	12		29	
Adoption of discussion and exploration (**DifTeachDE**)		(7)		(16)
Adoption of reasoning-and-proving (**DifTeachRP**)		(5)		(6)
Awareness of different teaching aids (**DifTeachTA**)		(3)		(3)
Hesitation in adopting ideas in the intervention (**HesitIdea**)	5		7	

[1]The number of the PTs who made statements relating to a certain code; [2]The number of separate statements identified; [3]Subcodes in brackets

Overall, the PTs' elaboration of their response changes and learnings from the intervention could be categorised into (i) development of different views about proof (**DifProof**), (ii) development of different views about students (**DifStudent**) and (iii) development of different views about teaching (**DifTeach**). Ten PTs reported that after the intervention, they had developed different (or broadened) views about proof (**DifProof**) which included emotional dispositions to (teach) proof (**DifProofEmo**; 5 PTs) and views about the nature and educational roles of proof (**DifProofNR**; 7 PTs). Six PTs reported that after the intervention, they had developed different (or broadened) views about students' ability to learn mathematics and proof and their roles in classroom (e.g., "students can play more active roles in classroom instruction") (**DifStudent**). And all twelve PTs reported that after the intervention, they had developed different (or broadened) views about teaching (**DifTeach**), particularly adopting different approaches for teaching such as discussion and exploration (**DifTeachDE**, 7 PTs) and reasoning-and-proving (**DifTeachRP**, 5 PTs), and becoming aware of different teaching aids (e.g., pictorial representations of proof, paper-folding) for teaching (**DifTeachTA**, 3 PTs). Despite the changes and learnings, five PTs reported that in different degrees, they had difficulty in adopting some ideas (particularly, of teaching) that were conveyed in the intervention (**HesitIdea**). I use Excerpts 4.2.9–4.2.12 to exemplify some of the changes and learnings.

> **Excerpt 4.2.9** (Participant 3, coded **DifProof**, **DifStudent**)
> Before, I only thought a maths lesson should consist of what students need in examinations. Proof seemed to be more advanced activities, which should be implemented at extra time for students who are interested in mathematics… But [now,] in a maths lesson, I think every students should experience proof, instead of problems present in examinations only.

> **Excerpt 4.2.10** (Participant 18, coded **DifProof(NR)**, **DifTeach(RP)**)
> I learned that there is actually proof in primary [mathematics]. Be[fore] I often thought that proof is, like [in] secondary [mathematics], demonstration of every step, but [pictorial arguments] might actually be considered as a proving process. I also learned the framework [of reasoning-and-proving] which was easily ignored because quite often, "to prove" means "to give you something [and ask] you [to write] a proof [of it]"; that is, before, [I] did not value this, [the use of] pattern, very much.

> **Excerpt 4.2.11** (Participant 16 coded **DifProofEmo**, **DifProofNR**)
> [Participant 16 elaborated the change in her responses to Likert Item A2 "Working on a proof in front of the class frightens me" from **Agree** to **Disagree**] I think [my] idea [of proof] has been changing because when doing this [pre-intervention questionnaire] I considered proofs as those [algebraic proofs] given by secondary teachers […] I did not know if I could do it well. [However,] when doing this [post-intervention questionnaire] I chose that I no longer feel worried because I think […] if having proof in primary mathematics lessons [proof] is not what I thought, that proof [has to be] complicated.

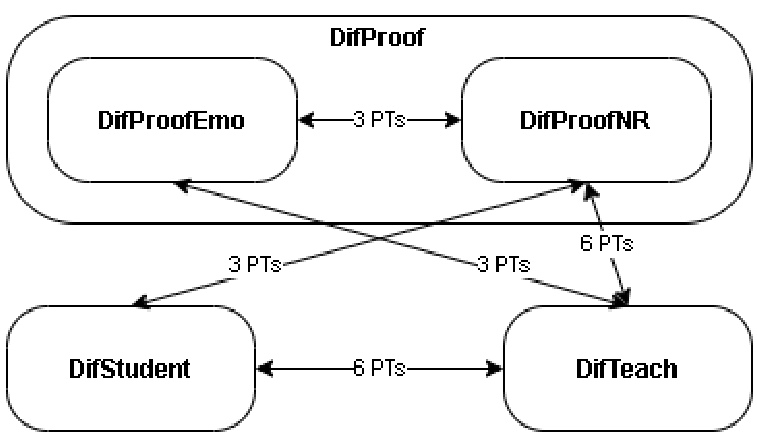
> **Excerpt 4.2.12** (Participant 10, coded **DifStudent**, **DifTeach(DE)**)
> [After the intervention] I think [I should] give more spaces to students, because they also have opportunities to learn through the process of [mathematical] exploration. I won't say: because [a student's] method is incorrect or partly incorrect [and] they are taking a

longer route [, they do not learn anything]. However, if [I] can provide more suitable guidance to them, I think they are actually learning something.

Excerpt 4.2.13 (Participant 1, coded **HesitIdea**)
The intervention gave me different ideas to think about. For example, the videos [in which] the primary students discussed [mathematical ideas] [made me think about] "Why could they have a [supportive] environment for discussion?" From [the videos I think] this [idea of involving students in mathematical discussion], [in my] previous mathematics lessons, nearly never happened – even if there was, there would not be many. [I] don't think [I] can *sacrifice* a lesson, 35 minutes, to let [students] think this (discuss a mathematical idea). I use the term "sacrifice" because [in] reality, teaching hours are not enough […] [Due to] these many factors combined, it is actually difficult to achieve [the idea of involving students in mathematical discussion].

Whilst Excerpts 4.2.9–4.2.10 demonstrate that different codes observed in a statement can be separate, for example, in the form of "I learned (Code X). I also learned (Code Y)", Excerpts 4.2.10–4.2.11 show occasions when two codes (representing different changes and learnings) were connected. Particularly, Excerpt 4.2.11 exemplifies a direct relationship between **DifProofEmo** and **DifProofNR**, that a PT changed their emotional dispositions towards proof (e.g., anxiety about proof) when their beliefs about the nature and educational roles of proof was changed. Subsequently, I observed various links between **DifProofEmo**, **DifProofNR**, **DifStudent** and **DifTeach**, except that no link was observed between **DifProofEmo** and **DifStudent**. Each link was supported by at least 3 PTs' statements (Figure 4.2.1).

Figure 4.2.1
Observed links between different changes and learnings



4.2.4 Summary of Section 4.2

In summary, the findings of the analyses of the PTs' responses to the Likert items and the open-ended questions (Section 3.4.1) concerning changes in their beliefs and attitudes towards teaching proof in school mathematics are consistent with each other. For example, the increased number of the PTs who made statements that proof can promote students' conceptual understanding in mathematics in the open-ended questions (Table 4.2.8) resonated with the increased number of the PTs who disagreed with the statement "I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists" (Item D5) in Likert items (Table 4.2.1). And the decreased number of the PTs who made statements that proof is not relevant to school mathematics curriculum or examinations in the open-ended questions (Table 4.2.9) resonated with the increased number of the PTs who agreed with the relevance of proof to examinations in Likert items (Table 4.2.2). The analysis of interview data on the PTs (Section 3.4.2) also supplemented the identification of changes in the PTs' beliefs and attitudes towards teaching proof (Table 4.2.10).

Overall, the PTs developed more productive beliefs and attitudes towards teaching proof in school mathematics after the intervention, regarding (i) their beliefs about the nature and educational roles of proof, (ii) their emotional dispositions towards proof, (iii) their beliefs about approaches for teaching mathematics and proof, and (iv) their beliefs about students. More PTs reported that they agreed with the educational roles of and the needs for proof when doing mathematics (Table 4.2.1) and with the relevance of proof to school mathematics curriculum (Tables 4.2.2 & 4.2.9) and related these ideas to their reasons for implementing proof in their teaching (Table 4.2.8). The PTs also reported that they developed stronger enjoyment and interest in proof and reduced anxiety about proof (Tables 4.2.3–4.2.5) and adopted process-oriented approaches for teaching mathematics and proof into their teaching (Table 4.2.7). Some PTs also expressed changes in their views about students, such as developing beliefs that students are able to learn proof or that students can play more active roles in learning (Table 4.2.10).

Having said that, apart from the encouraging outcomes of the intervention, some PTs expressed their hesitation in adopting some information conveyed in the intervention (Table 4.2.10), such as ideas of involving students in mathematical discussion (Excerpt 4.2.13), which resulted in maintenance of some of their counterproductive beliefs and attitudes towards teaching proof. There was little change in the number of the PTs who expressed concerns about class-time shortages, students' mathematical knowledge and ability, and their own teaching ability (Table 4.2.9), suggesting that these concerns remained influential in the PTs' beliefs about approaches for teaching mathematics and proof. For example, if a PT believed that class-time was not enough and students' mathematical knowledge and ability did not meet their expectations, they tended to reduce time spent on mathematical exploration and discussion, but replace these more process-oriented approaches with more product-oriented approaches such as teacher-centred lecturing and demonstration of proofs (e.g., Excerpt 4.2.13).

4.3 Research Question 3: Components of the intervention that facilitated belief changes

In this section, I present my analysis of the components of the intervention that facilitated changes in the PTs' beliefs and attitudes towards teaching proof in school mathematics. The design of the intervention, including the activities and features, was discussed in Section 3.2.2, and the PTs' changes were identified and reported in the analysis in Section 4.2.

In this analysis, I used the interview data from 12 PTs. During the interviews, the PTs were asked about their experiences and learning from the intervention (Interview Questions 4–5, see Section 3.3). In their responses, as a result of the analysis described in Section 3.4, what components of the intervention stood out to them and the links between these components and the changes in the PTs' beliefs and attitudes towards teaching proof were identified.

In the following, I first report which *components* of the intervention stood out to the PTs relating to what they experienced and learned in the intervention (and potentially, their changes in beliefs and attitudes towards teaching proof) (Section 4.3.1). Here, I use the term 'components' to avoid confusion with 'features', which I used in Section 3.2.2 for describing the design principles that I found in literature and used for designing the intervention. Then, I report the links identified between the components of the intervention and different changes in the PTs' beliefs and attitudes towards teaching proof (Section 4.3.2). Lastly, I summarise the overall findings relating to the evaluation of the features of the intervention (Section 4.3.3).

4.3.1 Components of the intervention that stood out to the PTs

The PTs made statements relating to the activities and features of the intervention that they experienced when being probed into reasons behind differences between their pre- and post-intervention responses to the questionnaire (Interview Question 4b), what they experienced and had learned in the intervention (Interview Question 5b) and what they perceived that the intervention wanted to achieve (Interview Question 5c) (Section 3.3.1.2). Components of the intervention that stood out to the PTs are summarised in Table 4.3.1, which shows the codes for analysis and their frequencies. In the following, I discuss the overall findings. Then, I include, for each component, some data extracts to illustrate my analysis.

Table 4.3.1
Components of the intervention that stood out to the PTs (n = 12)

Code	Number of PTs[1,3]	Frequency[2,3]
Proving activities (**ProvAct**)	12	19
That involved multiple representations and methods of proving (**ProvAct–MultiRepMet**)	(9)	(15)
That involved reasoning-and-proving framework (**ProvAct–RP**)	(3)	(4)
Discussion and collaboration (**DiscCol**)	8	11
That were encouraged by supportive environment for exchanging ideas (**DiscCol–SuppEnv**)	(4)	(5)
That were encouraged by pre-class activities (**DiscCol–PreCAct**)	(1)	(1)
Videos that reflect classroom instruction involving proof and reasoning (**Video**)	6	8
Mini-teaching that allowed the PTs to practise what they learned earlier in the intervention (**MiniTeach**)	5	9
The instructor's actions to encourage the PTs' learning in the intervention (**InstAct**)	5	6
Actions that linked the contents of the intervention to the work of teaching (**InstAct–LinkTeach**)	(4)	(4)
Questions that facilitated the PTs' cognitive conflict (**InstAct–CogConf**)	(2)	(2)

[1]The number of the PTs who made statements relating to a certain code; [2]The number of separate statements identified; [3]Subcomponents in brackets

In Table 4.3.1 there are three columns. The first column shows a list of the codes that I used for the analysis. The list consisted of 5 main codes (**ProvAct**, **DiscCol**, **Video**, **MiniTeach**, **InstAct**) and 6 subcodes (**ProvAct–MultiRepMet**, **ProvAct–RP**, **DiscCol–SuppEnv**, **DiscCol–PreCAct**, **InstAct–LinkTeach**. **InstAct–CogConf**). The second column shows the number of the PTs who made statements relating to a component of the intervention, and the third column shows the total number of separate statements that the PTs made and were captured by the corresponding code. A PT could make statements relating to different components, and several statements about one specific component. For example, the second row of Table 4.3.1 shows that 12 PTs made a total of 19 separate statements about the proving activities in which the PTs were asked to solve mathematical problems and produce proofs (**ProvAct**).

In Table 4.3.1 the numbers shown in each subcode row (e.g., **DiscCol–SuppEnv**) are included in its corresponding main code (e.g., **DiscCol**) since a statement about a subcomponent is naturally related to the corresponding main component. However, a PT could make a statement about a component in general, but that is not necessarily specific to any of its subcomponents. As a result, the numbers in a subcode row are always less than those in the corresponding main code row, but the sum of the subcode rows is not necessarily greater than the corresponding main code row. For example, the fifth to seventh rows of Table 4.3.1 show that 8 PTs considered discussion and collaboration (**DiscCol**) as a component that stood out to them, and among them, 4 PTs discussed how discussion and collaboration were encouraged by the environment in the intervention (**DiscCol–SuppEnv**) and 1 PT discussed how she was encouraged to take part in discussion and collaboration in the intervention by the pre-class activities (**DiscCol–PreCAct**).

From the PTs' perspective, five main components of the intervention stood out to them; no other components were found in the data. The components were: (a) the proving activities in which the PTs were asked to solve mathematical problems and produce proofs during the intervention (**ProvAct**), (b) the learning environment that emphasised discussion and collaboration in different activities of the intervention (**DiscCol**), (c) the videos that were played and discussed during the intervention (**Video**), (d) the mini-teaching activity that asked the PTs to plan and demonstrate a lesson of a mathematical topic at the end of the intervention (**MiniTeach**), and (e) the actions that the instructor did to encourage the PTs' learning during the intervention (**InstAct**). The proving activities stood out to all PTs, and the other components stood out to at least five PTs.

Furthermore, six subcomponents were found among three of the five main components. The PTs made statements highlighting their awareness of (a) proofs that were produced by using multiple representations and different methods of proving during the proving activities

(**ProvAct–MultiRepMet**), (b) ideas of the reasoning-and-proving framework that were embedded and conveyed in the proving activities (**ProvAct–RP**), (c) the supportive environment that was created and maintained for encouraging discussion and collaboration (**DiscCol–SuppEnv**), (d) pre-class activities that were used for facilitating in-intervention discussion and collaboration (**DiscCol–PreCAct**), (e) the actions that the instructor did to link the contents of the intervention to the work of teaching (**InstAct–LinkTeach**), and (f) the questions that the instructor asked which facilitated cognitive conflict that the PTs experienced during the intervention (**InstAct–CogConf**). The inclusion of the subcomponents in the analysis not only showed which specific part(s) of a main component stood out to the PTs, but also revealed that within a main component, different subcomponents were linked to different changes in the PTs' beliefs and attitudes towards teaching proof, which is discussed in Section 4.3.2.

In the following, I use extracts of the interview data to illustrate each (sub)component that stood out to the PTs. The statements that the PTs made vary in the degrees of detail and explicitness. Some statements are direct and descriptive, but others required more interpretation drawing on other parts of the interview – I choose examples that represented these two situations where possible.

The use of proving activities **(ProvAct).** Excerpts 4.3.1–4.3.3 are used for exemplifying the PTs' awareness of the proving activities during the intervention (Activities 1-3, 2-1, 2-3 & 3-2, Table 3.2.2) – particularly, the activities that involved multiple representations and methods of proving (**ProvAct–MultiRepMet**). Excerpt 4.3.4 is used for exemplifying the PTs' awareness of the proving activities in which ideas of the reasoning-and-proving framework were embedded and conveyed in the intervention (**ProvAct–RP**).

Excerpt 4.3.1

[Interviewer: What did you experience and learn in {the intervention}?]
Participant 11: Drawing figures is very helpful.

Participant 11 referred to "drawing figures" as something that he experienced in the intervention. However, it is not obvious that "drawing figures" was linked with the proving activities *that aimed to demonstrate that a mathematical claim can be proved using different methods and representations* (see Table 3.2.2). Therefore, other parts of his interview were consulted. When asked about his expectation before attending the intervention, he said:

> **Excerpt 4.3.2** (Participant 11) I wanted to learn more, from different angles [methods], how to prove one thing, or a thinking process – to tell oneself [students'] by thinking – that there are more than one method to explain one thing. […]

Excerpt 4.3.2 shows Participant 11's motivation for learning different methods to prove and to represent an argument – which also hints that he paid attention to the occasions during the intervention when a mathematical claim was proved using different methods and representations. Regarding Excerpt 4.3.1, the phrase "drawing figures" shows Participant 11's awareness that pictorial arguments were presented in the intervention as an alternative to arguments based on algebraic expressions when communicating mathematics, and the adjective "helpful" indicates that he was positive or learned about the strength of pictorial representations in explaining and communicating mathematics – particularly, to students during classroom instruction.

Excerpt 4.3.3 explicitly shows the link between what Participant 17 learned in the intervention and a proving activity (Activity 2-3, Table 3.2.2) that was used for *demonstrating that a mathematical claim can be proved using different representations*:

Excerpt 4.3.3

[Interviewer:	From your point of view, what was the objective(s) of {the intervention}?]
Participant 17:	[To convey that] proof does not have to be *written* in a clear, standard step-by-step [procedure]. [Instead, it] can be integrated into classroom activities, and even during the teaching process.
Interviewer:	[…] Why [do you] think so? What did you experience [in the intervention]?
Participant 17:	I really remember that there was an activity involving number cubes in the intervention. It was about divisibility, [which involved] joining many number cubes together.

Interviewer: Are you talking about [the activity that involved proofs of] "*the sum of three consecutive numbers is a multiple of three*"?

Participant 17: Yes, yes. [The use of] number cubes does not [require] writ[ing] down steps to explain "$(a - 1) + a + (a + 1) = 3a$", but [using] number cubes can also prove that thing [mathematical claim].

Without using the specific term, Participant 5 and two other PTs related what they experienced and learned in the intervention to the reasoning-and-proving framework (Table 4.3.1), featuring a learning sequence that involves identifying patterns, making conjectures, providing non-proof arguments, and providing proofs (G. J. Stylianides, 2008).

Excerpt 4.3.4

[Interviewer: What did you experience and learn in {the intervention}?]

Participant 5: […] What I experienced is… I learned that when they [students start to] learn proof, [it is possible to allow them to] explore [the mathematical claim]. Perhaps, [I can] provide them with some simple things [patterns] to explore what the proof is doing, then allow them to write down the conjecture themselves. And then, see if they can really prove [it]. This flow [learning sequence] is what I roughly experienced and is "okay" [suitable for students], which allows them to experience this *flow* [learning sequence]. […]

Excerpt 4.3.4 explicitly shows Participant 5's learning and idea of applying the reasoning-and-proving framework when teaching proof – by providing students with opportunities to investigate and justify mathematical claims. The first part of the sentence "[t]his flow is what I roughly experienced […]" indicates that he experienced a learning sequence of reasoning-and-proving during the intervention (see Table 3.2.2). And the second part indicates that he had a positive disposition to apply reasoning-and-proving into teaching mathematics.

The separation between proving activities that involved multiple representations and methods of proving (**ProvAct–MultiRepMet**) and that involved ideas of the reasoning-and-proving framework (**ProvAct–RP**) is driven by the data. **ProvAct–MultiRepMet** was linked to different aspects of change in the PTs' beliefs and attitudes towards teaching proof – for example, their beliefs about the nature and educational roles of proof, and beliefs about approaches for teaching mathematics and proof – whereas **ProvAct–RP** was linked to the change in the PTs' beliefs about approaches for teaching mathematics and proof only. I discuss this analysis in Section 4.3.2.

The use of discussion and collaboration **(DiscCol).** Excerpts 4.3.5–4.3.6 are used for exemplifying the PTs' awareness of the learning environment that emphasised discussion and collaboration during the intervention, particularly the elements that facilitated their engagement in discussion and collaboration during the intervention, namely: (a) maintaining a supportive environment for exchanging ideas (**DiscCol–SuppEnv**) and (b) using pre-class activities as preparations (**DiscCol–PreCAct**).

> **Excerpt 4.3.5** (Participant 1)
> In the intervention, I enjoyed the parts when we have discussions the most. During the process of thinking [generating ideas], there was not much pressure. The environment [for discussion] was comfortable. [Although] there were not many people, everyone had the same motivation – for learning how proof can be used in mathematics class. If it was a credit course of mathematics, this situation might hardly be done. The most difficult thing, what I think, is the competitive environment – [everyone] is living in such [competitive] environment. […] But now, [the intervention] was non-credit[-bearing] and there were not many people, and everyone was relatively passionate about education – wanting to learn more. [I felt] much comfortable sitting together with everyone.

In Excerpt 4.3.5, Participant 1 explicitly stated that she enjoyed the supportive environment for exchanging ideas in the intervention. Particularly, she felt that during the intervention, the environment was supportive (**DiscCol–SuppEnv**). Making a comparison with other more competitive learning environment (e.g., regular, credit-bearing courses), Participant 1 expressed and valued that the environment of the intervention allowed her to comfortably exchange ideas about teaching proof and mathematics. Three other PTs also reported that they valued the discussions and collaborations during the intervention which allowed them to exchange ideas, especially relating to teaching proof (Table 4.3.1).

Despite being unique, Excerpt 4.3.6 shows that the use of pre-class activities helped Participant 7 engage in discussion and collaboration in the intervention (**DiscCol–PreCAct**):

Excerpt 4.3.6 (Participant 7)
I think the worksheets [the pre-class activities] were interesting. Like, during the workshops [the intervention], with the worksheets, I could expect what would be taught, and I would have something prepared for discussion. Also, [I would] know that others would do the same for discussion, and could learn others' new [creative] thoughts, which made everything interesting.

According to Participant 7, the pre-class activities directed her attention to the contents which were going to be introduced, which prompted her to get prepared for discussion during the intervention. The sense of preparedness that Participant 7 developed from the pre-class activities also boosted her confidence to engage in discussion and collaboration. Having said that, no other PTs made similar statements about the pre-class activities during the interview – they did not make statements about the pre-class activities relating to discussion and collaboration, relating to other components, and/or that stood out alone.

The use of videos of classroom instruction **(Video).** Excerpts 4.3.7–4.3.9 are used as examples of the PTs' awareness of the videos (i.e., Ball's lessons, HK3 Polygons of TIMSS 1999 Video Study) played and discussed during the intervention (Activities 1-2 & 2-2, Table 3.2.2). Excerpt 4.3.7 exemplifies the PTs' statements that are relatively explicit, whereas Excerpts 4.3.8–4.3.9 illustrates those that required more interpretation in the analysis.

Excerpt 4.3.7

[Interviewer:	What did you experience and learn in {the intervention}?]
Participant 7:	I think, [about] the video[s] played at the beginning of the first session [Ball's lessons], I had already found [it] very impacting, [that] children come out and talk everywhere.
Interviewer:	Why would [you find it] impacting?
Participant 7:	Because I think they tried very hard [and] had so many ideas to talk about something that I thought unnecessary to think. But the reason for "unnecessary to think" may be because everyone always told me that it is true; that is, [I] have already known the result, so [I found] nothing good to think about. But I think [when I] saw that they are serious to think some relatively simple(?)… not simple [but] "a bit" abstract ideas. But [they] are serious to say, together, express different thoughts. [In the video] "you think [it] wrong", like this, I find it, for me, quite *new and surprising*.
Interviewer:	"New and surprising" because…?
Participant 7:	I could not imagine [it].

Interviewer: [You] could not imagine [a lesson and students] can be like this situation?
Participant 7: Yes, I could not imagine that is possible; that is, starting from the teacher, [the students] investigated the topic. I could not think that this [topic] could actually be discussed [like] this long.

Excerpt 4.3.7 indicates that Participant 7 was surprised by the active interactions among the students shown in the videos of Ball's lessons, which stood out to her during the intervention. In the lessons, a class of Year 4 students discussed mathematical ideas and produced arguments and proofs whilst the teacher did not lecture but played a role in facilitating the discussions and proof production (Activity 1-2). As Participant 7 continued to elaborate her response, more evidence could be gathered to understand how this experience was linked to changes in her beliefs and attitudes towards teaching proof – the analysis of links between the use of videos of classroom instruction and different changes of the PTs is discussed in Section 4.3.2.

Next, I discuss why Excerpt 4.3.8, which required more interpretation as Participant 2 did not explicitly state the videos, indicates his awareness of the videos during the intervention:

Excerpt 4.3.8
[Interviewer: What did you experience and learn in {the intervention}?]
Participant 2: [In] the first session, [I] learned about discussion – discussion can [be used] to activate the atmosphere in class [motivate students' learning], or [facilitate] more exchanges of ideas – which is not *unidirectional*.
Interviewer: *Unidirectional* means…?
Participant 2: Only the teacher talking [to students]. This is [what I learned from] the first session.

I argue that *Excerpt 4.3.8 shows Participant 2's awareness of the Ball's videos played and discussed in the intervention*, for two reasons. First, according to this excerpt, "the first session" of the intervention stood out to Participant 2. The first session (Session 1) consisted of (a) introduction activities (Activity 1-1), and (b) two Ball's lessons videos and a follow-up proving activity (Activity 1-2–1-3) (Table 3.2.2). The former lasted about 20 minutes and the latter lasted about 65 minutes. As the discussions on and the proving activity relating to the videos were the *prime* content of Session 1, in terms of time taken, it is reasonable to interpret that “the first session” in Excerpt 4.3.8 *included* the videos.

Second, an analysis of recording of the intervention indicated a link between Participant 2’s statement about his learning in “the first session” and what he actually experienced in (b) at Session 1. Participant 2 actively participated in Session 1, including (b), by contributing his opinions to different parts of the session. Excerpt 4.3.9 summarises (b) and illustrates Participant 2’s participation in (b):

Excerpt 4.3.9

[Instructor paused the first video.]

Instructor: What happened in the classroom? What do you consider important or interesting? What has the teacher [in the video] done?

[The PTs compared the lesson in the video to their ideas of a typical Hong Kong lesson. They were not sure if the lesson in the video (particularly, the active in-class discussion) could appear in a typical Hong Kong lesson, except for lessons at international schools. They believed that teachers would prefer stating mathematical claims directly over spending time on discussion, because of limited class time, tight curriculum and other constraints. Participant 2 added his opinion:]

Participant 2: I think the difficulty lies in, if they (teacher) want to have a class in this format, in fact (that is), the dominance lies in students. Maybe, the student says the conclusion quickly (too early), or maybe [the students] have (reach) the outcome quickly (too early). That means [when] they (teacher) plan a lesson, time allocation is (becomes) more difficult. Perhaps there should be a few solutions, then it may work. It takes more to prepare than usual. Usually if only you (teacher) will talk, you can control your progress. But for this type (the lesson in the video), it is more difficult to control.

[The PTs continued to discuss the lesson in the video. Later, they were shown a second video and continued to discuss the role of allowing students to express their thoughts and reasoning. The discussion on the videos ended with ideas that discussion allows teachers to observe students’ understanding and ability and that discussion allows students to exchange and explore mathematical ideas.]

According to Excerpt 4.3.9, an idea “discussion allows students to exchange and explore mathematical ideas” appeared at the end of (b). This idea, as what Participant 2 (and other PTs) experienced in (b), is consistent with his interview statement about what he learned in Session 1 (“Discussion can [motivate students’ learning] or [facilitate] more exchanges of ideas”, Excerpt

4.3.8). Therefore, I considered the videos of Ball's lessons (**Video**) to be a component of the intervention that stood out to Participant 2.

The use of mini-teaching **(MiniTeach).** At Session 3 of the intervention, the PTs were asked to plan and demonstrate a lesson of a selected topic, either of their choice or of introducing the area of a circle. The activity provided the PTs with opportunities to apply ideas of teaching proof presented in the intervention, such as reasoning-and-proving, and to discuss the strengths and limitations of their plans with each other after trying out the plans (Activity 3-1, Table 3.2.2). Excerpts 4.3.10–4.3.12 are used as examples of the PTs' awareness of the mini-teaching activity. To be more specific, Excerpt 4.3.10 exemplify the PTs' statements that directly mentioned mini-teaching, whereas Excerpts 4.3.11–4.3.12 illustrate the statements that did not mention mini-teaching directly but indicated the PTs' need for mini-teaching.

Excerpt 4.3.10

Interviewer:	From your point of view, what was the objective(s) of [the intervention]?
Participant 16:	I think [that it] was [about], in mathematics lessons, how to make the contents of teaching changed into being able to strengthen students and effective.
Interviewer:	[Would you] elaborate [the meaning of] "effective"?
Participant 16:	"Effective" is – At the first stage, students can grasp its knowledge. The second stage is to develop their mathematical reasoning – which is also to cater different students' need – because some students might have thoughts about investigating how the knowledge came. And such lessons can provide more students, who have these thoughts, with more opportunities.
Interviewer:	[…] Is there any what you experienced or an example, of [your] experience of the intervention, [that] makes you think [that this is] the message of the workshops [the objective of the intervention]?
Participant 16:	The last – last week, that try-out teaching [mini-teaching]. That [made me] feel different from "the normal" – not "the normal", but what I thought previously. Then [so I] feel like, think that [it] should be so.
Interviewer:	That is [to say that] your experience of the third session, in which [you] tried to teach [tried out your lesson plan], started [made you] find that such ["effective", different] teaching can be done in class – which made you have the previous [aforementioned] idea?
Participant 16:	Yes.

Excerpt 4.3.10 shows a conversation between Participant 16 and me (the interviewer). Participant 16 was asked, "*From your point of view, what was the objective(s) of the intervention?*" She perceived that the objective of the intervention was to develop the PTs' competencies in fostering students' mathematical understanding and reasoning. When being asked to elaborate on her statement, she linked it to her experience with the mini-teaching, as evidenced also by a paraphrase that I made during the interview – paraphrasing was used for providing checks on the accuracy of my in-the-moment interpretations during interviews (see Section 3.3.2).

Unlike Participant 16 who experienced the mini-teaching at Session 3, Participant 4 did not attend Session 3. However, he experienced another mini-teaching activity during his teacher training.

Excerpt 4.3.11

Participant 4:	In fact, I, [in] previous micro-teaching [mini-teaching], used a similar approach – I taught [the topic of] infinite sum of G.S. [geometric sequence]. [Description of Participant 4's lesson plan] Then, my peers' feedback and professor['s] feedback concerned [the plan] would make students confused. […] I think this is what they worried, but my idea was – exactly because the conflict made [the lesson/topic] interesting.
Interviewer:	[…] Did they give any advice?
Participant 4:	I had thought of using another [approach], [and] they also said something similar. [Description of another approach]
Interviewer:	Do their comments help you resolve [your difficulties]?
Participant 4:	At least there won't not be confusing situations existing. […]

Participant 4 reported his previous experience of mini-teaching and that after discussing his lesson plan with his peers, he could improve the plan and teaching by resolving the difficulties that he had when trying out the plan at mini-teaching. Although Excerpt 4.3.11 does not show any direct link between the mini-teaching *as a component of the intervention* and Participant 4's experiences in the intervention (and/or changes in his beliefs and attitudes towards teaching proof), this excerpt shows his perception that mini-teaching can help him improve his teaching. This excerpt also supports my interpretation of Excerpt 4.3.12, which appeared at a later part of the interview with Participant 4:

Excerpt 4.3.12

[After Participant 4 commented on what he experienced and learned in the intervention]

Interviewer: Did what you experienced [in the intervention] satisfy [your] expectations [about attending the intervention]?

Participant 4: I think so – at least, [the intervention] gave me [inspired me with] some ideas of teaching proof. But I think, anyway, in practice – however [I] think – it depends on students, to see if [these ideas] suit students.

Interviewer: [Do you mean that you want] to have opportunities to try [these ideas] out […]?

Participant 4: Yes.

Participant 4's comments on the intervention reveal his need for more experience in *practising* his (new) ideas of teaching proof and mathematics. Combined with Excerpt 4.3.11 about his experience of mini-teaching *outside* the intervention, Excerpt 4.3.12 reveals an indirect link between mini-teaching *as a component of the intervention* and what the PTs experienced and learned in the intervention. Detailed analysis of this link is discussed in Section 4.3.2.

The instructor's actions **(InstAct).** Excerpts 4.3.13–4.3.14 are used for exemplifying the PTs' awareness of my (the instructor's) actions that encouraged their learning during the intervention, particularly actions that were done to link the contents of the intervention to the work of teaching (**InstAct–LinkTeach**) and questions that were used for facilitating cognitive conflict that the PTs experienced during the intervention (**InstAct–CogConf**).

Excerpt 4.3.13

Interviewer: From your point of view, what was the objective(s) of [the intervention]?

Participant 5: [I think it] at least is to make us, when teaching [mathematics] in the future, not afraid of proof, and to make us, apart from being not afraid of proof, [being aware of] some 'beginning' approaches for teaching. […]

Interviewer: Why would [you] think that these were the objectives?

Participant 5: […] Because when you taught [during the intervention], [you] tried to – at the first two sessions – say [convey a message]: "what I am doing is like an [authentic] lesson" […] You tried to link [the contents of the intervention] to how we could teach students at secondary school, so these helped us think more [about teaching]. […]

Excerpt 4.3.13 shows that Participant 5 was aware of my actions during the intervention. Using the connective “because”, he explicitly attributed his perception of the objectives of the intervention (“to make [the PTs] not afraid of [teaching] proof” and “to make [the PTs being aware of] some ‘beginning’ approaches for teaching [proof]”) to the instructor’s actions of linking the contents of the intervention and the teaching of mathematics, particularly proof (**InstAct–LinkTeach**).

Apart from **InstAct–LinkTeach**, questions that were used for facilitating cognitive conflict (**InstAct–CogConf)** were another type of the instructor’s actions as a component of the intervention that stood out to the PTs and potentially fostered changes in their beliefs and attitudes towards teaching proof. I use Excerpt 4.3.14 to exemplify it:

Excerpt 4.3.14

[Interviewer:	What did you experience and learn in {the intervention}?]
Participant 19:	For me, the most memorable one is [mini-]teaching – this is [what] I memorise the most. [This was] the third session. The first session was about odds and evens [the Ball’s lessons]. The second session was [about] A.S. (arithmetic sequences). I think the most interesting thing was that you [the instructor] kept asking questions – the questions that I have never thought about – that is, [when] I read books, as usual, whatever the books say, I follow. However, you asked: “Is it so? How about that?” Your questions made me… inspired me to consider more possibilities, to consider how to prove this such that [it] is 100% correct.
Interviewer:	Are these what you experienced in the intervention?
Participant 19:	I think, during the intervention, [it was] like having a storm in [my] brain. Many unexpected questions popped out – I am a person who finds difficult problems interesting – so I specially find [those unexpected questions] interesting, which looked like simple but not so easy to comprehend.

To Participant 19, the questions that I asked to challenge the PTs’ existing beliefs and attitudes relating to proof and its teaching during the intervention stood out to her as the most memorable component of the intervention. The questions surprised her and stimulated her to (re)consider the mathematical ideas that she found simple in the past. The terms “storm” and “unexpected” indicate that Participant 19 was challenged by the questions and experienced cognitive conflict. In other words, by asking challenging questions that the PTs may not have expected or considered, the instructor could create opportunities for the PTs to reflect on their previous experiences of learning and teaching.

Similar to the separation between the subcomponents of the use of proving activities (**ProvAct**), the separation between the instructor's actions that linked the contents of the intervention to the work of teaching (**InstAct–LinkTeach**) and the instructor's questions that were intended to facilitate cognitive conflict (**InstAct–CogConf**) is driven by the data. **InstAct–LinkTeach** was linked to the change in the PTs' beliefs about approaches for teaching mathematics and proof and emotional dispositions towards proof, whereas **InstAct–CogConf** was linked to the change in the PTs' beliefs about nature and educational roles of proof and emotional dispositions towards proof. I discuss this analysis in Section 4.3.2.

In summary, I investigated the interviews with the PTs to study which components of the intervention stood out to the PTs. Several components were identified, namely: (a) proving activities, (b) discussion and collaboration, (c) videos that reflect classroom instruction involving proof and reasoning, (d) mini-teaching that allowed the PTs to practise their learning, and (e) the instructor's actions to encourage the PTs' learning. These components identified from the interview data are consistent with the proposed features of the intervention (Section 3.2.2). This analysis formed the basis of the subsequent analysis, in which the list of these components was compared with the list of different changes in the PTs' beliefs and attitudes towards teaching proof (see Section 4.2), to identify links between two lists (Section 4.3.2).

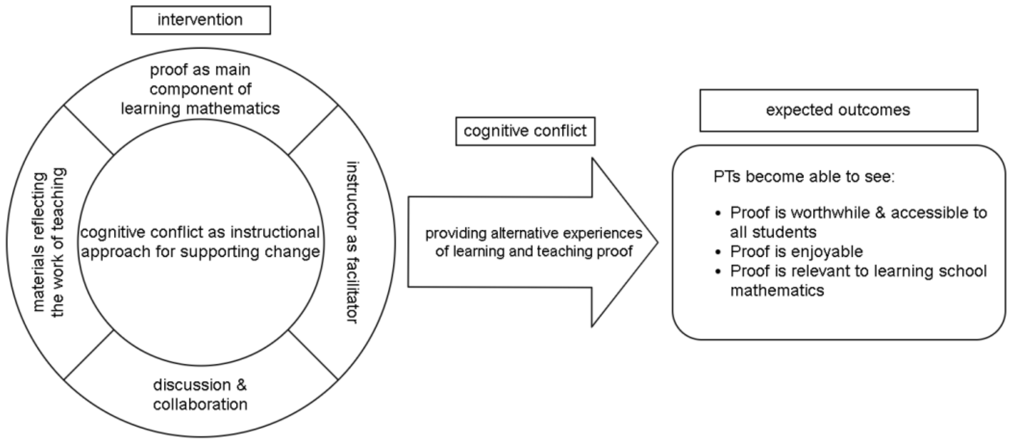
4.3.2 Links between different components of the intervention and different changes in the PTs' beliefs and attitudes towards teaching proof

Before reporting the analysis of links between the features of the intervention and the change in the PTs' beliefs and attitudes towards teaching proof, I recap (i) the design and the expected outcomes of the intervention, and (ii) the actual changes in the PTs' beliefs and attitudes towards teaching proof after the intervention. For details, see Sections 3.3.2 and 4.2, respectively.

***Design and expected outcomes of the intervention*.** The intervention focused on the learning and teaching of proof in school mathematics, and aimed to develop the PTs' more productive beliefs and attitudes towards teaching proof in school mathematics. To do so, I considered (i) cognitive conflict as an instructional strategy for motivating and supporting the PTs' change (Feature 2, Section 3.2.2). Alternative experiences in learning and teaching proof were intentionally created, by using (ii) activities that were designed to emphasise learning mathematics through proof (Feature 1), (iii) discussion and collaboration (Feature 3), (iv) materials based on the work of teaching (Feature 4) and (v) the instructor's actions (Feature 5), to support the stimulation and resolution of cognitive conflict.

Based on the analysis of early versions of the intervention in Research Cycles 1–3 (see Section 3.2.2), it was expected that after participating in the intervention in Research Cycle 4, PTs could develop more productive beliefs and attitudes towards teaching proof – seeing that proof is worthwhile and accessible to all students, enjoyable, and relevant to students' learning in school mathematics. Figure 4.3.1 illustrates the above description.

Figure 4.3.1
Proposed link between the features and the expected outcomes of the intervention



Changes in the PTs' beliefs and attitudes towards teaching proof. As reported in Section 4.2, the general post-intervention changes in the PTs' beliefs and attitudes towards teaching proof in school mathematics consisted of (i) change in the PTs' emotional dispositions towards proof, (ii) change in their beliefs about the nature and educational roles of proof, (iii) change in their beliefs about approaches for teaching mathematics and proof, and (iv) change in their beliefs about students (e.g., students' ability to learn proof, their roles in classroom). The PTs' hesitation in adopting (some) information conveyed in the intervention was also observed in interviews.

Next, after the recap, I report my analysis of the links between the components of the intervention that stood out to the PTs and the changes in the PTs' beliefs and attitudes towards teaching proof. Apart from showing which components of the intervention stood out to the PTs, the interview data (Interview Questions 4b, 5b & 5c) also reveal the links between different components and different changes in the PTs' beliefs and attitudes towards teaching proof. I paid attention to the PTs' explanation of their experiences and learning in the intervention in relation to the components that stood out to them. Using the interview data, the analysis consisted of two sequential parts: (1) identifying which components were related to which changes (Identification of links), and (2) identifying common aspects among the relationships between the components and the changes observed in (1) (Identification of common aspects of the links). For each part of the analysis, I first report the overall findings, then use data excerpts to exemplify the analysis.

Analysis Part (1): Identification of links. A link between Component X of the intervention and Change Y in beliefs and attitudes towards teaching proof was observed if a PT mentioned and connected both Component X and Change Y during interview (e.g., "through Component X, I learned Change Y"). Table 4.3.2 shows a contingency table that displays the frequency distribution of the observed links – A $\text{Cell}_{(Y,\ X)}$ indicates the number of the PTs who made statements that connected Component X and Change Y. For example, $\text{Cell}_{(\text{Column 3, Row 1})}$ indicates that eight PTs made statements that connected the change in their beliefs about approaches for teaching mathematics and proof to the proving activities (**ProvAct**), whereas $\text{Cell}_{(\text{Column 1, Row 5})}$ indicates that no link was observed between the change in the PTs' emotional dispositions towards proof and the mini-teaching (**MiniTeach**) according to the interview data. Apart from the PTs' changes, I also observed some links between the components of the intervention and the PTs' hesitation in adopting information conveyed in the intervention, which are shown in Column 5.

Table 4.3.2
Distribution of observed links between the components of the intervention and the changes in the PTs' beliefs and attitudes towards teaching proof

	Change in emotional dispositions towards proof	Change in beliefs about the nature and educational roles of proof	Change in beliefs about approaches for teaching mathematics and proof	Change in beliefs about students	Hesitation in adopting information
ProvAct	2	6	8	1	1
ProvAct–MultiRepMet	(1)	(6)	(6)	(1)	(1)
ProvAct–RP	(--)	(--)	(3)	(--)	(--)
DiscCol	--	--	6	3	1
Video	--	1	5	4	1
MiniTeach	--	--	2	--	2
InstAct	2	1	3	--	--

InstAct–LinkTeach	(1)	(--)	(3)	(--)	(--)
InstAct–CogConf	(2)	(1)	(--)	(--)	(--)

Note: Subcomponents in brackets

In total, 13 links were observed between the five main components of the intervention and the four changes in the PTs' beliefs and attitudes towards teaching proof (Columns 1–4). Different components had different links with the PTs' changes. For example, from the PTs' perspective, **ProvAct** had links to *all four* changes in the PTs' beliefs and attitudes towards teaching proof, whereas **MiniTeach** was linked to the change in the PTs' beliefs about approaches for teaching mathematics and proof *only*.

Based upon the PTs' elaboration, **ProvAct** was divided into (a) proving activities that involved multiple representations and methods of proving (**ProvAct–MultiRepMet**) and (b) those in which ideas of reasoning-and-proving framework were embedded and conveyed in the intervention (**ProvAct–RP**). **ProvAct–MultiRepMet** was linked to all four changes of the PTs, whereas **ProvAct–RP** was linked to the change in the PTs' beliefs about approaches for teaching mathematics and proof only. Similarly, the instructor's actions that encouraged the PTs' learning during the intervention (**InstAct**) were divided into (a) actions that were done to link the contents of the intervention to the work of teaching (**InstAct–LinkTeach**) and (b) questions that were used for facilitating cognitive conflict that the PTs experienced during the intervention (**InstAct–CogConf**). **InstAct–LinkTeach** was linked to the change in their emotional dispositions towards proof and the change in the PTs' beliefs about approaches for teaching mathematics and proof, whereas **InstAct–CogConf** was linked to the change in their emotional dispositions towards proof and the change in their beliefs about the nature and educational roles of proof.

Excluding the observed links that were reported by at most two PTs, the PTs attributed the change in their beliefs about the nature and educational roles of proof to **ProvAct** (**ProvAct–**

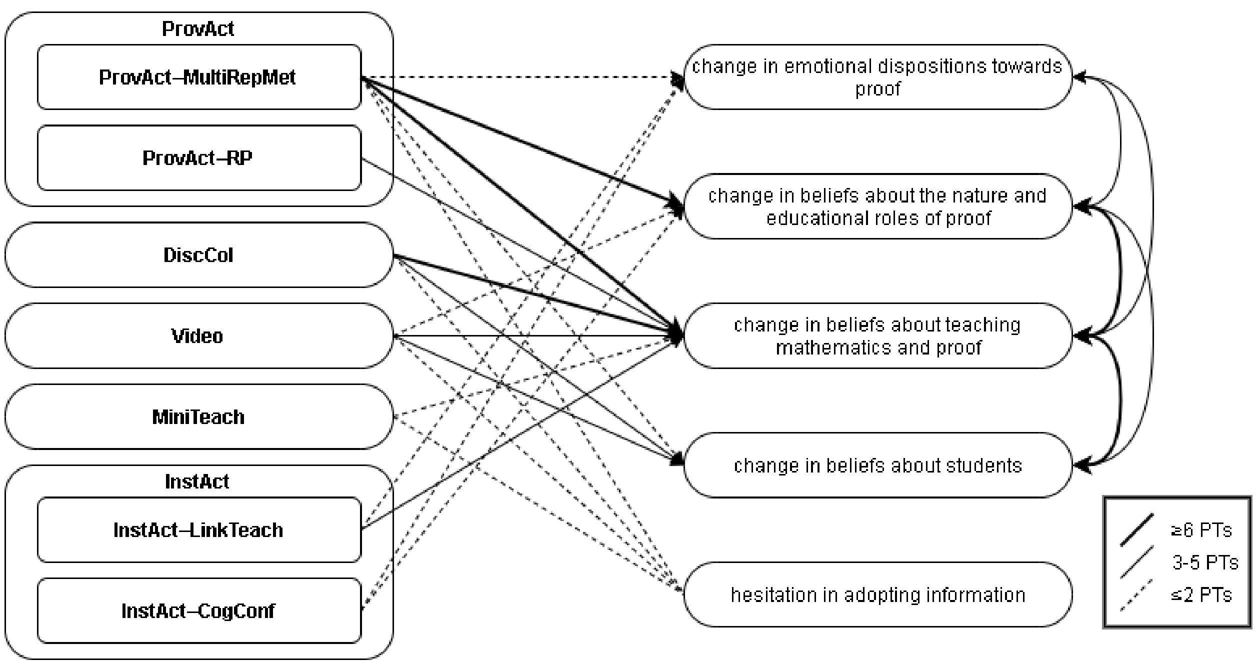
MultiRepMet only); the change in their beliefs about approaches for teaching mathematics and proof was attributed to **ProvAct** (**ProvAct–MultiRepMet** & **ProvAct–RP**), the learning environment that emphasised discussion and collaboration (**DiscCol**), the videos involving classroom instruction of proof and proving that were played and discussed (**Video**) and **InstAct** (**InstAct–LinkTeach** only); and the change in their beliefs about students was attributed to **DiscCol** and **Video**.

Interestingly, every observed link between the change in emotional dispositions towards proof and some component of the intervention was reported by at most two PTs. The low number of observed links that directly connected the change in the PTs' emotional dispositions towards proof to some component of the intervention was unexpected, because this emotional change was a prominent change of the PTs after the intervention (Section 4.2). However, the direct relationship between (the change in) the PTs' emotional dispositions towards proof and (the change in) their beliefs about the nature and educational roles of proof (Section 4.2) hints that there was some indirect link between the components of the intervention and the emotional change.

Additionally, some links between the PTs' hesitation in adopting information conveyed in the intervention and four components **(ProvAct(–MultiRepMet)**, **DiscCol**, **Video**, **MiniTeach**) were observed. Every link between the hesitation and each of the four components was reported by at most two PTs only. Despite the small number of PTs reporting the links, these links provided valuable information for me to study the PTs' hesitation in Analysis Part (2) in which I searched for common aspects among the links observed in Analysis Part (1).

Before reporting Analysis Part (2), I use Figure 4.3.2 to complement Table 4.3.2 and visualise the findings of Analysis Part (1). In Figure 4.3.2, lines and arrows are used for representing the observed links. Lines represent the links between the PTs' *hesitation* and the components, in which causal relationships were not observed. Arrows represent the links between the PTs' *changes* and the components, in which Change Y was attributed (by the PTs) to Component X. A dotted line or arrow represents an observed link that was reported by at most two PTs; a solid line arrow represents an observed link that was reported by 3–5 PTs; a **bold** line or arrow represents an observed link that was reported by at least six PTs (Table 4.3.2). Double ended arrows are also added to indicate the relationships among different changes in the PTs' beliefs and attitudes towards teaching proof, which were identified and reported in Section 4.2.

Figure 4.3.2
Observed links between the components of the intervention and the changes in the PTs' beliefs and attitudes towards teaching proof



In the following, I use interview excerpts to exemplify the links observed in Analysis Part (1). Excerpt 4.3.15 shows the link between the change in Participant 10's beliefs about the nature and educational roles of proof and **ProvAct–MultiRepMet**:

Excerpt 4.3.15

[Interviewer:	What are the roles of proof in school mathematics?]
Participant 10:	Proof can visualise a concept – that is, it can [help people] know the “why” behind – it is the most important to know “why”.
Interviewer:	About “why”, is there any example to share?
Participant 10:	[…] [The proving activity about the formula of] the area of a circle is quite memorable, or something [else] during the workshops [the intervention] – I previously did not know “why”.
Interviewer:	Are you talking about the area formula of a circle?
Participant 10:	Yes, I think, through that activity, I know [learn] why [the are formula of a circle is] πr^2 – you [I] had a piece of paper to fold, you [I] finally could see that [idea]. Then you [I] really knew why the formula comes like this.

Participant 10 was surprised by the use of paper-folding, as an alternative representation for communicating and understanding an argument for the area formula of a circle, presented in a proving activity during the intervention. This experience helped Participant 10 develop not only understanding of the formula, but also the beliefs that “proof can visualise a [mathematical] concept” and that proof can foster students’ *mathematical understanding* which she valued.

Excerpts 4.3.4 and 4.3.10 exemplify the link between the change in the PTs’ beliefs about approaches for teaching mathematics and proof and **ProvAct–MultiRepMet**, and the link between the change in the PTs’ beliefs about approaches for teaching mathematics and proof and **MiniTeach**, respectively. Participant 5 experienced a proving activity that involved paper-folding (Excerpt 4.3.4), which was intended as an example of alternative representations for communicating an argument of the area formula of a circle (**ProvAct–MultiRepMet**). Participant 16 attributed the development of her belief that teaching mathematics is to foster students’ mathematical understanding and reasoning to her experience of the mini-teaching (Excerpt 4.3.10).

To exemplify the link between the change in the PTs’ emotional dispositions towards proof (e.g., developing stronger enjoyment and interest in proof, reducing anxiety about proof) and the instructor's action in the intervention, I use two Participant 19's interview excerpts – Excerpt 4.3.14 and below:

Excerpt 4.3.16

Participant 19:	I find it unexpected – "unexpected" means that I did not know proof can actually be interesting. I had already laid down mathematics for many years. I gave it up for long.
Interviewer:	[What do you mean by] "laying down mathematics for many years"?
Participant 19:	That means – I, since high school, had already given up [mathematics], laid down [it]. When it came to algebra I had already given it up.
Interviewer:	[So you mean] during high school, [you] thought [your] goal was to pass [examinations], instead of…
Participant 19:	I thought it was not [my] companion; that is, I completely did not understand what mathematics was saying. However, starting from now (after the intervention), I take up [mathematics], I think.

According to Excerpt 4.3.14, during the intervention, Participant 19 was challenged by questions that were intended to challenge the PTs' existing beliefs and attitudes towards teaching proof and encourage cognitive conflict (**InstAct–CogConf**). The challenging questions surprised Participant 19 and aroused her interest in doing mathematics and producing proofs. Participant 19 further elaborated her change in emotional dispositions. According to Excerpt 4.3.16, the intervention helped her develop more positive emotional dispositions towards proof and re-engage with (learning) mathematics after "giv[ing] up" learning mathematics for several years.

Excerpt 4.3.7 exemplifies the link between the change in the PTs' beliefs about students (e.g., developing the belief that (primary) students can understand and produce proof) and **Video**. Participant 7 used "new and surprising" to describe her experience of watching the videos of Ball's lessons. The videos not only surprised Participant 7 by showing what the students in Ball's lessons did and could do, but also helped her develop an idea (but not yet a stable belief; see below) that (primary) students can (be motivated to) discuss mathematical ideas which seem to be abstract in nature.

Lastly, I use Excerpt 4.3.17 to exemplify the PTs' hesitation in adopting information conveyed in the intervention and its links with the components of the intervention, particularly **ProvAct–MultiRepMet**:

Excerpt 4.3.17

Participant 7:	I am doubting, a little, if there are many different methods. […] I saw someone did it [used a pictorial argument to prove "$a^2 - b^2 = (a + b)(a - b)$"], on the internet, but I think this [pictorial representation] is "forced" [unconvincing] – I think – although it looks like correct.
Interviewer:	Why do [you] think [it's] "forced"?
Participant 7:	I won't relate a formula [a^2] to a square, out of nowhere. […] I think they have known the result before forming it, so for me, I won't understand: "a teacher teaches this formula today, but why do they draw squares?" …[Although] I think they *have told* me why this is true – which might convince me.
Interviewer:	So for you, numbers [mathematical notations] and figures, are [they] fairly separate [ideas]?
Participant 7:	[They] can be connected, but for this identity, for me, [I] think: "it is more numerical or algebraic." Or I think "drawing figures" is *a* technique, but won't be used in practice. […] However, previously [proof of] "$n(n - 1)/2$" [during the intervention] also required drawing figures. So I think: if drawing figures is a comparatively more acceptable and understandable method, then it's worth trying. However, I think if asking students or asking myself to think, [we] won't suddenly form an idea to draw a square [to represent a^2] so I am not sure if everyone can think of different methods. I think if it is something that everyone cannot understand, or difficult, I would prefer not to do [teach it or use it in teaching], or [I would] let myself to talk [use teacher talk approach].

Participant 7 remained in doubt about whether students (as well as herself) are able to produce different representations (particularly pictorial representations) or methods to prove mathematical claims (beliefs about students), and whether visualisation of mathematical ideas can be applied practically in teaching (beliefs about approaches for teaching mathematics and proof). Having said that, Participant 7 was aware and valued that some visualisation of proof can foster understanding of mathematics. She also developed an idea that students can be engaged in meaningful discussions about mathematical ideas, as discussed previously.

The observed hesitations of the PTs reveal that the links between the components of the intervention and the change in the PTs' beliefs and attitudes towards teaching proof were not direct or simple – that is, the PTs did not change their beliefs and attitudes towards teaching proof *simply* because of the existence of those components. Instead, it is more possible to assume that there were some hidden mechanisms behind (or that can be used to explain) how to encourage the change in the PTs' beliefs and attitudes towards teaching proof. Therefore, after identifying the links presented in Figure 4.3.2, I looked into the observed links and searched for common aspects of how different components of the intervention encouraged the PTs' changes and learning in the intervention (Analysis Part (2)).

Analysis Part (2): Identification of common aspects of the links. Table 4.3.3 shows the overall findings of the common aspects of the links observed in Analysis Part (1), and consists of three columns. The first column shows a list of the common aspects. The second column shows the number of the PTs who made statements relating to an observed common aspect, and the third column shows the total number of separate statements that the PTs made and were captured by the corresponding common aspect. A PT could make statements relating to different common aspects, and several statements about one specific common aspect. For example, the second row of Table 4.3.3 shows that all 12 PTs made a total of 27 separate statements about how the components of the intervention encouraged the change in the PTs' beliefs and attitudes towards teaching proof.

In general, all PTs reported that different components of the intervention provided alternative experiences with mathematics and proof, as shown previously and in Section 4.3.1. These experiences, in turn, stimulated the change in their beliefs and attitudes towards teaching proof. There were two common comments, made by the PTs, about how the alternative opportunities provided by the intervention encouraged the changes. First, all PTs attributed their changes and learning to their experiences which they considered novel and positive in the intervention. Here, the adjectives ‘novel and positive’ are used for summarising how the PTs felt about their experiences (in relation to their changes and learning) in the intervention (Table 4.3.4). For example, the terms “storm” and “unexpected” stated by Participant 19 (Excerpt 4.3.14) exemplify the PTs’ feeling about the experiences that were novel and expanded their horizons.

Table 4.3.3
Common aspects relating to how the components of the intervention encouraged the change in the PTs’ beliefs and attitudes towards teaching proof

Common aspect	Number of PTs[1,3]	Frequency[2,3]
Alternative experiences provided in the intervention encouraged the PTs’ changes	**12**	**27**
Experiences were considered novel and positive	12	25
Opportunities were provided for the PTs to practise and actualise the ideas of teaching	9	14
Hesitations in adopting the alternative experiences and ideas of teaching proof presented in the intervention	**5**	**8**
(Feeling about) lack of practice in applying the ideas existed	3	6
Worries about students, schools and class time existed	3	3

[1]The number of the PTs who made statements relating to a certain common aspect; [2]The number of separate statements identified; [3]Overarching idea in **bold**

Table 4.3.4
The PTs’ feeling about their experiences in the intervention

Feeling	Number of PTs[1]	Frequency[2]
The experiences were novel and expanded the PTs’ horizons	8	15
The experiences were attractive	4	5
The experiences were satisfying	4	4
The experiences were pressure/worry-free	3	4
The experiences were meaningful	2	3
Other (e.g., encouraging, impacting)	1	2

[1]The number of the PTs who made statements relating to a certain feeling;
[2]The number of separate statements identified

About how the alternative opportunities encouraged the PTs’ changes, the second common comment was related to the degree of how practical the PTs perceived the ideas of teaching mathematics and proof to be, as conveyed in the intervention. Nine PTs reported that during the intervention, they had opportunities to practise the newly-learned ideas of teaching mathematics and proof, and/or believed that the ideas can be actualised. Excerpt 4.3.10 is an example of this type of comments.

Apart from the statements about how the intervention encouraged the PTs' changes and learning, five PTs also made different statements about their hesitations in adopting some contents of the intervention (Table 4.3.3). These hesitations, in turn, limited the change in their beliefs and attitudes towards teaching proof. Two common comments about how the PTs hesitated to adopt some ideas of teaching mathematics and proof conveyed in the intervention were identified. First, three PTs expressed their (feelings about) lack of practice in applying the ideas. Second, three PTs expressed their worries that the (future) students, schools and class time may prevent themselves from actualising the ideas. In other words, both types of hesitations were related to the degree of the PTs' beliefs that the ideas of teaching mathematics and proof, conveyed in the intervention, were less practical – the former reflected the PTs' personal factors (e.g., lack of self-efficacy beliefs about using the ideas) whereas the latter reflected the (PT-perceived) external factors (e.g., unsuitability of the ideas for classroom instruction that the PTs may have in the future).

In the following, I use Excerpts 4.3.18–4.3.20 to exemplify my analysis of the links between the PTs' changes (as well as hesitations) and the intervention reported in Table 4.3.3.

Excerpt 4.3.18

Interviewer:	Any other experience [in the intervention]?
Participant 5:	I experienced [and learned] that there are many things that can have different approaches and methods [of teaching], which is what I did not think of. […] The workshops [the intervention] made [helped] me learn that an activity can be applied in many different areas [topics], and I also learned many different activities – how to use paper-folding, how to use quadrilaterals, or how to use the problem of 'farmers redistributing lands' […].

The above excerpt shows that Participant 5 learned different ideas of teaching mathematics and proof from different experiences (e.g., paper-folding) he had during the intervention (**ProvAct–MultiRepMet**). The clause "which is what I did not think of" indicates that the experiences were novel to him and provided alternative experiences of learning and teaching mathematics and proof. The subsequent sentence "[the intervention helped] me learn that […]" indicates that the intervention were positive to Participant 5 as it helped him gain knowledge about different methods of teaching mathematics and proof.

However, seeing an idea as novel and positive did not automatically translate into belief change (e.g., change in beliefs about approaches for teaching mathematics and proof). According to Excerpt 4.3.19, Participant 5 doubted whether involving students in discussing mathematics, as a characteristic of process-oriented approaches for teaching mathematics and proof (Goos, 2004; Smith et al., 2009; Yoo, 2008), could be actualised because of lack of experience of the approach and his worries that the approach was not suitable for students and mathematics class with tight class time:

Excerpt 4.3.19

Participant 5: […] I think [what I] most often see is direct lecturing, this [teaching] approach. […] I am not sure if it is because my former secondary teachers liked this approach very much – because they rarely let us discuss in groups, or [included] many different approaches. I think [what I] most often see is this approach [direct lecturing]. Also an advantage of this approach is, like what I just said, to directly tell the key points to them [students]. Another advantage is that regarding time, they [the teacher] can handle [the lesson] more easily – because the class time, itself, is limited, and [let] the students to… if let them discuss freely, the time management would be hardly handled – that is, whenever to cut, to stop them, they might have discussed halfway through [but] couldn't discuss the other half. [They] wouldn't completely handle [learn] what [I] want them to learn in the topic. Therefore, this direct lecturing approach itself is important – Is it the most important? This [question], I cannot make a judgement [answer], but this [approach], from my perspective, is important, a very important approach. […] Apart from this, it may also include, as [I] previously said: when [I] let them [students] group discussion, they might come up with new ideas, [and] even some of them may be some methods and different ideas that I have never thought of when prep[aring for teaching the topic]. So I think I have rarely seen [teaching through discussion and collaboration], to be honest.

Interviewer: As the difficulties you talked about previously, you would think that lecturing or direct lecturing would be easier [for you to handle].

Participant 5: [Yes, direct lecturing] would be easier to handle. [I] would not say [it's] the most important, but at least [it] should be a more common, more often approach. At least this was an approach that I often saw [experienced] previously.

After the intervention, Participant 5 maintained his preference for the use of direct lecturing, and hesitated to adopt the idea of involving students in discussing proof (and mathematics). In Excerpt 4.3.19, Participant 5 repeatedly reported that direct lecturing is a very important teaching approach and he was more familiar with this approach. He was also able to describe several advantages of this approach, indicating that his belief and confidence to apply direct lecturing to the teaching of mathematics and proof were solid. The solid belief and confidence can be seen as both an outcome of the intervention and a factor that led to the outcome: (a) as an outcome, Participant 5 maintained his belief that mathematics and proof should be taught through direct lecturing; (b) as a factor, this belief, together with other related beliefs, hindered Participant 5 from (fully) adopting (beliefs about) process-oriented approaches for teaching mathematics and proof. Claim (a) can be justified by the repeated statements about Participant 5's preference for direct lecturing, and Claim (b) requires more interpretation of this excerpt and the previous one.

To justify Claim (b), I argue that the two excerpts indicate that (i) there were some changes in Participant 5's beliefs and attitudes towards teaching proof after the intervention, and (ii) he was aware of involving students in mathematical discussion as an approach for teaching mathematics and proof but maintained his preference for direct lecturing (Claim (a)). Excerpt 4.3.18 shows that Participant 5 experienced proving activities that helped him develop an idea of providing students with opportunities to investigate and justify mathematical claims using different methods (**ProvAct**). This indicates some change in Participant 5's belief about approaches for teaching mathematics and proof – towards an idea that students can learn through different approaches for teaching mathematics and proof (as opposed to learning by rote and repeated practice drills of single approach and method for solving problems).

Moreover, in Excerpt 4.3.19, Participant 5 reported that he was aware that students might develop new, innovative ideas through discussion, indicating that he was aware of involving students in mathematical discussion, as a characteristic of process-oriented approaches, for teaching mathematics and proof. However, his worries about class time and students indicate his beliefs (1) that the class time (for teaching a topic) is fixed, (2) that it is important for him as a teacher to ensure that he "directly tell[s] the key points to [students]" – considering learning mathematics as transmitting knowledge – and to ensure that students are able to grasp all intended contents of a topic by the end of a lesson, and (3) that involving students in mathematical discussion is time-demanding and he was not familiar with or confident about using this approach to encourage students' learning. Participant 5's worries about class time and students, and his (lack of) familiarity and the state of confidence to apply different teaching approaches mediated the comparison that he made between direct lecturing and involving students in mathematical discussion. As a result, his belief about the advantages of direct lecturing was strengthened, and in turn, he hesitated to adopt the idea of involving students in mathematical discussion and proof and maintained his preference for direct lecturing.

In contrast, the analysis of Excerpt 4.3.20 exemplifies a finding that by providing the PTs with more opportunities to practise their (new) ideas of teaching mathematics and proof (e.g., by using mini-teaching) and/or with evidence that showed the practicality of the ideas in their (future) teaching (e.g., by using videos of classroom instruction), the PTs developed some belief about the actualisation of the newly-learned ideas of teaching mathematics and proof. And in turn, they were more likely to change their beliefs and attitudes towards teaching proof:

Excerpt 4.3.20

[Interviewer: What are the differences between your responses to the questionnaire before and after {the intervention}?]

[Participant 19 discussed her responses]

Interviewer: Would [you] talk more about – because your responses to a few items apparently differ from [what you had] in the past [pre-intervention questionnaire] – have you experienced anything that makes you [change your responses]?

Participant 19: Perhaps, [it's because] your talk in class [the proving activities of the intervention] was very interesting. Normally, teachers in class only [give] example[s], but your instruments [teaching aids] – after [you] used these instruments, I think this thing [proof] becomes interesting. First, [I] think this thing becomes interesting, [and] not unpleasant.

Interviewer: So you, yourself, for proof, think [it] becomes interesting?

Participant 19: I think [proof] becomes interesting. Then, [when] you [proof] becomes interesting, you [I] would not [find it] unpleasant. Then, you [I] would want to know a little more. Then, after you [I] know a little more, you [I] would find it [proof] is actually – that is, it is something useful and we would do [learn proof].

Interviewer: Would [you] talk more about "proof seems to be useful"?

Participant 19: I… Previously, about that [mini-teaching of] circle area [the area formula of a circle], you asked me to design a lesson. I had never thought of how the formula came. When seeing π I would know [only] to [memorise] it [the formula] by rote. However, when I [started to] think [it] over [and found] "[it] actually came from this", [my] brain suddenly became clearer. So, [when] a simple topic of circle area [can] make you [me] know many mathematical foundations, I think that harder identity can be done with the same effect [other topics, like identities, can be learned through proof, yielding the same learning outcome]. So [I] think [(teaching) proof] is useful.

Excerpt 4.3.20 shows a successful change in Participant 19's emotional disposition towards proof, and more broadly, attitudes towards teaching proof, after the intervention. She reported that proof "became interesting" and "useful" to her after the intervention, and linked this change to her experience of the intervention. The statement "Normally, teachers in class only [give] example[s], but your [teaching aids] […]" shows that the intervention provided Participant 19 with an alternative experience of learning (and teaching) proof, which she found positive. Such positive alternative experience aroused her curiosity about proof and mathematical ideas, which eventually changed her belief about the usefulness of proof.

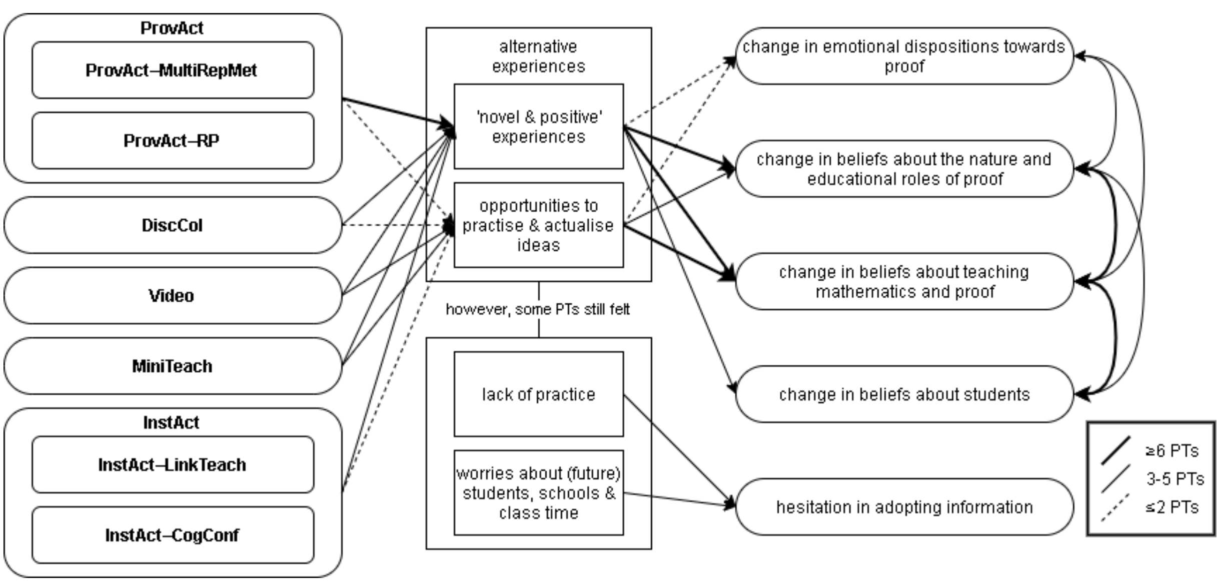
The second half of this excerpt shows the link between Participant 19's experience of the mini-teaching (**MiniTeach**) and her revised belief "proof is useful". First, **MiniTeach** provided an opportunity for her to develop a belief that teaching proof can be actualised in teaching the area formula of a circle. Then, she generalised her experience in certain topic (i.e., the area formula of a circle) into other topics (e.g., identities), and eventually, into the idea that proof is useful, particularly for developing one's better understanding of mathematical ideas. In other words, this excerpt shows that the successful change in Participant 19's (attitude and) belief about (teaching) proof was linked to the intervention that provided alternative experiences of learning proof, which convinced her of the practicality of teaching proof, through **ProvAct** and **MiniTeach**.

After reporting Table 4.3.3 and the examples (Excerpts 4.3.18–4.3.20), I use Figure 4.3.3 to visualise the common aspects of different links between the components of the intervention and the changes in the PTs' beliefs and attitudes towards teaching proof. Figure 4.3.3 is derived from Figure 4.3.2 by introducing and considering the boxes of 'alternative experiences' and its subcomponents (also, 'lack of practice' and 'worries about (future) students, schools & class time') to be intermediate between the components of the intervention and the changes in the PTs' beliefs and attitudes towards teaching proof (as well as the PTs' hesitation in adopting information conveyed in the intervention).

In summary, the analysis in this section indicates several links between different components of the intervention standing out to the PTs (reported in Section 4.3.1) and different changes in the PTs' beliefs and attitudes towards teaching proof (reported in Section 4.2). It also identifies common aspects of how the components encouraged the changes and why some PTs hesitated to adopt some information conveyed in the intervention from the PTs' own perspectives. To be more specific, first, the PTs reported that the components, which correspond to the intended features of the intervention (discussed in Section 3.2.2), created alternative experiences of learning and teaching mathematics and proof for the PTs. Second, the PTs reported that they were willing to adopt information conveyed in the intervention and changed their beliefs and attitudes towards teaching proof when they (1) found the particular experience of the intervention novel and positive and (2) could foresee that the ideas that they developed from the intervention can be actualised in their future teaching. However, some PTs (particularly those who did not experience the mini-teaching) reported that they hesitated to adopt the intended ideas of teaching mathematics and proof in the intervention, and had little change in their beliefs about approaches for teaching mathematics and proof. The PTs attributed their hesitations to (1) their lack of practice and experience and (2) their worries about (future) students, schools and class time. The data on their hesitations show that they developed or upheld some belief that the intended ideas of teaching mathematics and proof can hardly be actualised in their (future) teaching.

Figure 4.3.3

Alternative experiences to be intermediate between the components of the intervention and the changes in the PTs' beliefs and attitudes towards teaching proof



4.3.3 Summary of Section 4.3

In Section 4.3, I reported my analysis of the interview data, in order to identify which components of the intervention stood out to the PTs (Section 4.3.1) and study how the intervention encouraged the changes in the PTs' beliefs and attitudes towards teaching proof, which were identified in Section 4.2 (Section 4.3.2). I also analysed the self-reported data to identify the reasons why some PTs had little change in their beliefs, particularly about approaches for teaching mathematics and proof (Section 4.3.2).

There were different changes in the PTs' beliefs and attitudes towards teaching proof after the intervention in which alternative experiences of mathematics and proof were created. Successful changes were observed, when (1) the PTs found the contents of and the intended ideas conveyed in the intervention novel and had positive emotional dispositions towards the experience of the intervention and (2) they developed the belief that their ideas (particularly of teaching mathematics and proof) derived from the intervention can be actualised in their (future) teaching. On the one hand, the PTs reported that their confidence in actualising the ideas can be strengthened through trying out and/or practising the ideas (e.g., mini-teaching). On the other hand, this sense can be reduced by (their feeling about) lack of practice and their worries about some external factors (e.g., limited class time, students being unable to learn mathematics and proof through discussion), resulting in their hesitations in adopting their ideas derived from the intervention and an enhanced preference for their initial (product-oriented) ideas of teaching (e.g., direct lecturing, practice drills).

Chapter 5: Discussion

In this chapter, I first summarise the results of the present study, and compare the results with other related studies concerning teachers' beliefs and attitudes towards teaching proof in school mathematics (Section 5.1). Then, I discuss potential implications of the present study for teacher education (Section 5.2). Lastly, I conclude this thesis with discussions on limitations of the present study and potential future research directions (Section 5.3).

5.1 Summary of results

The present study aimed to examine changes in PTs' beliefs and attitudes towards teaching proof in school mathematics after an intervention of relatively short duration. In particular, I have developed a learning environment that facilitated the development of Hong Kong PTs' productive beliefs and attitudes towards teaching proof in the form of three weekly, two-hour extracurricular workshops. This intervention consisted of activities that involved PTs actively discussing the learning and teaching of school mathematics through argumentation and proof and involved them developing and trialling a lesson involving the teaching of proof. The present study assessed whether and how the intervention, in which 27 Hong Kong PTs participated, affected the beliefs and attitudes towards teaching proof held by 12 Hong Kong PTs. Three research questions guided the present study:

RQ 1. *What were Hong Kong preservice teachers' pre-intervention beliefs and attitudes towards teaching proof in school mathematics?*

RQ 2. *How did Hong Kong preservice teachers' beliefs and attitudes towards teaching proof change after participating in a specifically designed intervention of short duration?*

RQ 3. What components of the intervention were perceived by the preservice teachers as contributing to changes in their beliefs and attitudes (as identified in RQ 2)?

In the present study, data were collected by means of a questionnaire, individual semi-structured interviews, and fieldnotes and audio-recordings of the intervention. The PTs' pre-post responses to the questionnaire provided numerical evidence of the effects of the intervention on the PTs' beliefs and attitudes towards teaching proof in school mathematics. The interview data not only assisted in understanding and interpreting the analysis of the questionnaire data, but also provided a rich description of the 12 PTs' beliefs and attitudes towards teaching proof, as well as their experiences and opinions of the intervention, providing insight into how their beliefs and attitudes towards teaching proof changed in response to the intervention. The fieldnotes and audio-recordings of the intervention captured very rich information about the PTs' responses in the intervention. The use of different data sources not only provided data to answer different research questions, but also allowed comparisons of different types of data to deepen understanding of the analysis and increase validity of the findings via triangulation. The analysis process is also reported as detailed as possible for transparency.

5.1.1 PTs' existing beliefs and attitudes towards teaching proof

Before the intervention, the PTs generally agreed with the importance of proof for learning mathematics – they reported they believed that proof helps develop logical and mathematical thinking and promote mathematical understanding, consistent with Knuth (Knuth, 2002b) and Kotelawala (2016) in which most of the practising secondary mathematics teachers considered the development of logical thinking and reasoning skills as one of the major roles that proof plays in school mathematics, and with Lesseig et al. (2019) in which a number of the PTs of secondary mathematics acknowledged that proof has a role in explaining mathematics and that proof could be used as a venue for students to build and communicate understanding in mathematics. Most of the PTs in the present study also reported that they enjoyed and were interested in proof, consistent with Frasier (2010) and Nyaumwe and Buzuzi (2007) in which most of the practising secondary mathematics teachers disagreed that they disliked proof and expressed that they enjoyed and were interested in proof.

Having said that, a considerable proportion of the PTs in the present study expressed anxiety about proof, consistent with Kotelawala (2016) in which a number of the practising secondary mathematics teachers reported that they had frequent anxiety and struggle with proof in school and university mathematics. The PTs in the present study also reported concerns about the teaching of proof, revealing their counterproductive beliefs that their future students might not have adequate capacity for proof, that proof is not relevant to school mathematics curriculum and examinations, and that the teaching of proof is time-demanding and the curriculum is tight. These concerns were also reported by other practising teachers and PTs in previous research (e.g., Frasier & Panasuk, 2013; Lesseig et al., 2019; Nyaumwe & Buzuzi, 2007).

According to previous research (e.g., Kotelawala, 2016; Lesseig et al., 2019; Varghese, 2009), teachers' anxiety and counterproductive beliefs about proof have the potential to limit their opinions on, or even decisions about, when and how proof should be included in school mathematics. Such potential might explain the preference of the PTs in the present study for instructional activities that encourage students to practise procedural skills, and product-oriented approaches for teaching (e.g., direct lecturing, teacher's demonstration of proof, practice drills). In the present study, however, the PTs' anxiety and counterproductive beliefs about proof were not considered as a deficit but rather as a common standing point (Lesseig et al., 2019) for the subsequent intervention.

The data collected from the Hong Kong PTs before they participated in the intervention in the present study are generally aligned with the data collected from another sample of Hong Kong PTs who did not participate in the intervention as well as a larger sample of Hong Kong PTs in Research Cycles 1–3, and with previous research on practising teachers (Frasier & Panasuk, 2013; Knuth, 2002b; Kotelawala, 2016; Nyaumwe & Buzuzi, 2007) and PTs (e.g., Cyr, 2004; Ersen, 2016; Lesseig et al., 2019; Mingus & Grassl, 1999; Moralı et al., 2006; Varghese, 2009) in different countries. Having said that, there are variations in proportions of practising teachers and PTs who expressed different views about the teaching of proof. For example, about half of the PTs in the present study reported that they believed proof should appear regularly in classroom instruction, whereas most of practising teachers in Knuth (Knuth, 2002b) and most of the PTs and Varghese (2009) reported that they believed proof should only be introduced to certain selected groups of students (e.g., who plan to study advanced mathematics) but only few suggested proof be taught in every mathematics class.

The findings of beliefs and attitudes towards teaching proof in school mathematics the PTs held before the intervention in the present study, consistent with previous research, indicate that PTs often held at least some counterproductive beliefs about proof and/or had anxiety about proof when entering and studying at a teacher training programme whilst having productive beliefs, enjoyment and interest in proof. This observation that PTs hold some counterproductive beliefs and anxiety about proof, when entering a teacher training programme, is useful for developing appropriate measures (e.g., interventions, curriculum design) to enhance students' experiences with proof in school mathematics. I discuss the possible implications of the present study in Section 5.2.

5.1.2 Post-intervention change in PTs' beliefs and attitudes towards teaching proof

After the intervention, there were several changes in the PTs' beliefs and attitudes towards teaching proof in school mathematics, as evidenced by their pre-post responses to the questionnaire and their elaboration during interviews. The PTs who reported they were not sure about the relevance of proof to examinations in school mathematics became more aware that teaching proof can benefit students in terms of examination preparation. The PTs who reported they did not enjoy proof very much reported they developed enjoyment and interest in proof and reduced anxiety about proof. Those who considered proof as a supplement to the learning of school mathematics became aware that there are needs for including proof in classroom instruction regularly, and adopted at least some process-oriented approaches for teaching mathematics and proof (Section 2.3). Additionally, the PTs who expressed productive beliefs about the importance of proof and about the relevance of proof to examinations, and enjoyment and interest in proof, maintained their self-reported productive beliefs and enjoyment in proof.

The above results of the present study are consistent with other intervention studies on PTs concerning the area of proof, indicating that different beliefs and attitudes of PTs related to proof are possible to change, even over a course of short duration (i.e., three weeks, in the present study), in contrast with the idea that beliefs and attitudes are resistant to change (Kagan, 1992; Pajares, 1992). The fact that the results concerning the development of the PTs' enjoyment and interest in proof and the reduction of their anxiety about proof in the present study are consistent with Zengin (2017) indicates that PTs could develop enjoyment and interest in proof and reduce their anxiety about proof over an educational course. The results that the PTs in the present study reported they adopted process-oriented approaches for teaching and developed more process-oriented beliefs about approaches for teaching mathematics and proof are consistent with Buchbinder and McCrone (2020) and Yoo (2008), indicating that PTs could adopt more student-centred, inquiry-based instruction over an educational course. Furthermore, the observation that some PTs in the present study reported change in their beliefs about students' capacity for proof is consistent with the preliminary results of Buchbinder and McCrone (2018), suggesting that PTs could also become aware that students are able to learn proof and develop confidence in students' capacity for proof over an educational course. The results that the PTs in the present study reported a positive change in their beliefs about the importance of proof are consistent with studies involving interventions in the transition to proof (e.g., Koichu, 2012; G. J. Stylianides & Stylianides, 2009), showing that PTs could become aware of the need for using proof when doing mathematics.

Apart from being consistent with other studies, the results of the present study also reveal the connection between the PTs' beliefs about examinations in school mathematics and their beliefs and attitudes towards teaching proof (e.g., beliefs about relevance of proof to examinations), and indicate a possibility of changing PTs' beliefs about relevance of proof to examinations, from believing that the teaching of proof reduces the time spent on preparing students' procedural skills for examinations to believing that proof could develop students' skills in solving different types of problems and hence prepare students for examinations. According to previous research (e.g., Frasier, 2010; Kotelawala, 2016), teachers' productive beliefs about the importance of proof and enjoyment and interest in proof have the potential to increase the proportion of proof in classroom instruction, whereas their anxiety about proof has the potential to reduce the time spent on proof in their teaching. In the present study, the development of the PTs' enjoyment and productive beliefs about proof and the reduction of their anxiety about proof after the intervention had the potential to develop their productive attitudes towards teaching proof in school mathematics.

Having said that, the fact that the sample size was too small for conducting statistical tests of significance leads to a limitation. The fact that post-intervention data on PTs who rejected or were not sure about the teaching of proof in school mathematics before the intervention were missing from the analysis (discussed in Section 3.5) might reduce the validity of the analysis of pre-post responses to the questionnaire. However, apart from arguing that I ensured the validity of the present study by applying triangulation and maintaining transparency (Section 3.4.4), I also consider such limitation an opportunity for future research, which is discussed in Section 5.3.

5.1.3 Components of the intervention that facilitated PTs' belief change

The analysis of the interview data not only indicated that the intervention influenced the PTs' beliefs and attitudes towards teaching proof in school mathematics, but also allowed investigation into links between the change in the PTs' beliefs and attitudes towards teaching proof and the intervention, supplemented with the use of fieldnotes and audio-recordings of the intervention. Several components of the intervention were identified as elements that facilitated the PTs' belief change, including different learning experiences with proof (e.g., learning mathematics through proof and argumentation, using different representations), opportunities to discuss and collaborate with peers, videos of classroom instruction involving proof and argumentation, mini-teaching, and actions that the instructor did to support the PTs' learning.

The PTs reported that these components and activities of the intervention provided them with novel and positive experiences with the learning and teaching of proof. Such experiences with proof, as a critical element of the intervention, made the PTs realise the needs and potential for the learning and teaching of proof in school mathematics. The case of Participant 19 illustrated the need for novel and positive experiences to facilitate belief change. Her experiences in the intervention not only aroused her positive emotions (“interesting”) and motivated her to take part in proof (Excerpts 4.3.14 & 4.3.16), but also enabled her to compare these experiences with her past experiences, and contested her existing beliefs about the need for proof in school mathematics (Excerpt 4.3.20). Her case also suggested a sequence of her belief change: change in emotions about proof leads to change in beliefs about the need for proof, consistent with previous research showing that arousing PTs’ positive emotions about mathematics has the potential to change their negative beliefs about the learning and teaching of mathematics (e.g., Egloff & Souvignier, 2020; Liljedahl, 2005).

Having said that, even if the intervention provided the PTs with novel and positive experiences with mathematics and proof, as a critical element in the facilitation of belief change, not all of the PTs developed very productive beliefs and attitudes towards teaching proof in school mathematics after the intervention – some remained hesitant to use proof regularly in their future teaching, and some remained hesitant to fully adopt process-oriented approaches for teaching (even if they might have developed belief that proof should be included in classroom instruction regularly). The analysis of the interview data indicated that such resistance to change appeared to be attributed to the PTs’ worries about external factors (e.g., students’ capacity, school context, class time) and their sense of lack of practice and confidence to adopt their ideal images of teaching (i.e., making proof central to the learning and teaching of school mathematics, process-oriented instruction) into their future teaching. The results are consistent with previous research on challenges that PTs might face when bringing recommended images of teaching proof, (often newly) developed from teacher education, to actual work of teaching, where might not favour the idea of making proof central to school mathematics (e.g., G. J. Stylianides et al., 2013).

Accordingly, another critical element in the facilitation of belief change, particularly regarding beliefs about teaching, was the opportunities that enabled the PTs to observe how proof could be included in classroom instruction (e.g., videos of classroom instruction) and to trial the ideal teaching of proof (e.g., mini-teaching). These experiences helped the PTs develop a sense of practicality of the ideal images of teaching proof that they developed and/or learnt from the intervention. As the PTs' past experiences with proof were not ideal (e.g., proof was not made central to the learning of mathematics, the learning of proof relied heavily on teacher-centred, product-oriented instruction), they might have developed strong beliefs about teaching related to these past experiences for years (Beswick, 2018; Richardson, 1996), thereby hesitating to adopt ideas about teaching that they were not familiar with, particularly those that "appeared to" disagree with their existing beliefs. The sense of practicality increased the likelihood of successful belief change by developing the PTs' confidence that the ideas of teaching (Bilen, 2015), which contested their existing beliefs, could work. The cases of Participant 16 (Excerpt 4.3.10) and Participant 19 (Excerpt 4.3.20) showed how the mini-teaching activity in the intervention helped them develop confidence that their ideas of teaching proof could be practical, and the case of Participant 5 (Excerpt 4.3.13) illustrated that emphasis on the connection between the intervention materials and the work of teaching helped him reduce his concerns about the teaching of proof and realise some different approaches for teaching. In contrast, the case of

Participant 4 (Excerpt 4.3.12) demonstrated that without opportunities for him to practise or actualise his ideas of teaching proof (he did not participate in the mini-teaching activity and expressed that he wanted to trial his ideas), he remained uncertain about his ideas, resulting in hesitation in fully applying his ideal images of teaching and an ineffective change in his beliefs about teaching.

The above results from the analysis of the interview data provide additional empirical evidence to identify the usefulness of the design principles suggested by existing research findings and literature and used for the intervention of the present study. The results that the PTs attributed their belief change to their experiences with proof in the intervention are consistent with studies that involved creation of meaningful learning experiences of argumentation and proof (e.g., G. J. Stylianides & Stylianides, 2009; Zengin, 2017). Moreover, the observation that the PTs experienced cognitive conflict during the intervention is consistent with existing literature and studies that suggested the use of cognitive conflict for belief change (e.g., Gill et al., 2004; Liljedahl et al., 2021). The observation that the PTs related their belief change to their experiences of discussion and collaboration with peers in the intervention is consistent with existing studies that advocated the need for social interactions in the facilitation of belief change (e.g., Shilling-Traina & Stylianides, 2013; Wilcox et al., 1991). The observation that the PTs valued the videos and the mini-teaching in the intervention is consistent with the work of Shilling-Traina and Stylianides (2013) in which the materials and opportunities used allowed PTs to explore, from a teacher's perspective, the connection between the contents of the intervention and the work of teaching. Lastly, the results that the instructor's actions played a role in facilitating the PTs' belief change are consistent with literature and studies that discussed the importance of the instructor's role in facilitating an effective intervention (e.g., Shilling-Traina & Stylianides, 2013; Smith et al., 2009; G. J. Stylianides & Stylianides, 2009).

The results not only provide additional empirical evidence to support the usefulness of the aforementioned design principles, but also provide insights into the facilitation of belief change. According to Liljedahl et al. (2021), belief change or replacement is "preceded by belief rejection – and that any cases of belief rejection are, themselves, preceded by the phenomenon of cognitive conflict" (p. 28). The fact that the process of the PTs' belief change observed in the present study is consistent with such sequence of belief change suggests that by providing alternative experiences with proof, PTs could reflect on their existing beliefs about proof and the teaching of proof, and explore different roles that proof plays in the learning of mathematics, thereby changing their beliefs and attitudes towards teaching proof in school mathematics, towards more productive ones. Particularly, PTs could change beliefs when experiencing a learning environment that arouses their emotions, interest and motivation, as the first step to facilitate belief change (e.g., Liljedahl, 2005; Limón, 2001); such learning environment not only acts as a catalyst for cognitive conflict, and hence belief rejection, but also acts as an alternative idea of teaching that contests PTs' existing beliefs. Moreover, consistent with studies that investigated the effects of discussion on videos of classroom instruction and the effects of teaching practices on facilitating teacher change (e.g., Bilen, 2015; Buchbinder & McCrone, 2020; Hollingsworth & Clarke, 2017), the present study adds to literature by providing qualitative evidence that when some ideas of teaching are introduced in an intervention, it is important for PTs to have opportunities to develop a sense of practicality of the ideas; such sense of practicality strengthens PTs' confidence to adopt the ideas, thereby making the ideas into feasible alternatives that challenge and replace the PTs' existing beliefs.

5.1.4 Summary of Section 5.1

The results of the present study are consistent with existing studies on PTs' beliefs and attitudes related to the learning and teaching of proof. The present study adds to literature on proof by providing additional empirical findings, suggesting that it is reasonable to assume PTs often have at least some counterproductive beliefs and attitudes towards teaching proof in school mathematics when entering teacher training programmes and there is a need for the development of PTs' productive beliefs and attitudes towards teaching proof during teacher education. An intervention of short duration with a set of design principles has also been developed and tested to be effective in facilitating change in Hong Kong PTs' beliefs and attitudes towards teaching proof in school mathematics. As the design principles were adopted from literature and studies in other contexts, generalisation about the usefulness of the design principles could be made for implications for teacher education. The implications are discussed in Section 5.2.

5.2 Implications of the study

5.2.1 PTs have both productive and counterproductive beliefs and attitudes towards teaching proof in school mathematics

The data collected from the present study indicate that PTs' beliefs and attitudes towards teaching proof in school mathematics were multifaceted and they held both productive and counterproductive beliefs and attitudes towards teaching proof before the intervention. The findings of the present study, despite pertaining to Hong Kong PTs, support the findings of previous research on practising teachers and PTs (Ersen, 2016; Frasier & Panasuk, 2013; Knuth, 2002b; Kotelawala, 2016; Lesseig & Hine, 2021; Nyaumwe & Buzuzi, 2007; Varghese, 2009).

The common results that practising teachers and PTs often hold at least some counterproductive beliefs related to proof and concerns about the teaching of proof suggest that there is some room for improvement in mathematics teacher education to support the development of productive beliefs and attitudes towards teaching proof in school mathematics.

The data collected from the present study helped document a range of productive and counterproductive beliefs and attitudes towards teaching proof in school mathematics from an East Asian perspective, thereby highlighting areas for attention by teacher training programmes in supporting the development of PTs' more productive beliefs and attitudes towards teaching proof. The three most prominent counterproductive beliefs about the teaching of proof that the PTs held concerned their belief that proof is difficult and out of students' reach and sometimes confuses students, their lack of sureness that proof should be placed in a prominent position in the curriculum or relevant to the learning of school mathematics, particularly examination preparation, and their lack of experiences with process-oriented instruction (e.g., inquiry-based instruction) for the learning and teaching of mathematics and proof. Moreover, a considerable number of the PTs expressed anxiety about proof.

According to previous research (Furinghetti & Morselli, 2011; Knuth, 2002b; Kotelawala, 2016; Varghese, 2009), the counterproductive beliefs and attitudes towards teaching proof have the potential to limit teachers' perspectives on when and how proof should be used in school mathematics. In the present study, however, such counterproductive beliefs and attitudes towards teaching proof were not considered a deficit but rather, together with productive beliefs and attitudes towards teaching proof, a common starting point that PTs have when entering mathematics teacher training programmes (Lesseig et al., 2019). The PTs' productive beliefs and attitudes towards teaching proof, such as beliefs about the importance of proof in mathematics, could be used as a stepping stone that connects the role proof plays in the development of mathematics and its role in the learning of school mathematics (de Villiers, 1990; Hanna, 1990), whereas the counterproductive beliefs and attitudes towards teaching proof should be addressed by providing PTs with meaningful experiences with proof and its teaching.

5.2.2 PTs need meaningful experiences with proof in teacher education

The present study adds to the existing research, suggesting that providing PTs with meaningful experiences with proof during teacher education, even if it is in short duration, could help them develop more productive beliefs and attitudes towards teaching proof. According to previous research (e.g., Conner et al., 2011; Philipp, 2007; Yoo, 2008; Zengin, 2017), PTs' beliefs related to proof and its teaching are slow, sometimes difficult, to change and it requires a long period of time to change their beliefs. In the present study, however, the data collected from the intervention group indicate that the PTs changed their existing beliefs and attitudes towards teaching proof even within a month, after the intervention that created positive, meaningful experiences of proof for the PTs. In contrast, the data collected from the comparison group indicate that without having alternative experiences of proof and its different educational roles, the PTs appeared to maintain their existing beliefs and attitudes towards teaching proof which have been developed based on their past experiences of the learning of proof via traditional, product-oriented instruction.

The data collected in the present study suggest that PTs' beliefs and attitudes towards teaching proof are related to their experiences with proof. Particularly, their counterproductive beliefs and attitudes towards teaching proof appear to be developed from their lack of meaningful experiences with proof (Schram et al., 1988; Simon & Blume, 1996). During their school education, proof was rarely placed central to their learning – for example, proof was considered non-fundamental and removable (Frasier & Panasuk, 2013; Furinghetti & Morselli, 2011; C.-Y. Lee, 2019) and proof often appeared only in high-school geometry (Frasier & Panasuk, 2013; Varghese, 2009). Although proof appeared frequently in university mathematics, PTs often learnt mathematics and proof via product-oriented, teacher-centred instruction in which proof served merely the function of justification of mathematical knowledge (Dreyfus, 1999; Jones, 2000; K. Weber, 2004). According to previous research (Knuth, 2002b; Varghese, 2009), the view that proof is considered merely as the means for justifying mathematical knowledge has the potential to limit teachers' perspectives on when and how proof should be used in school mathematics. There were also limited opportunities for PTs to learn how to teach proof (He & An, 2018). These experiences are not ideal and are often attributed to PTs' difficulties with proof and, in turn, their hesitation in including proof in their future teaching.

When provided with opportunities to learn and discuss mathematics through proof and argumentation, PTs can not only make connections between different mathematical ideas and deepen understanding, but also realise the meaning of proof in school mathematics. The case of Participant 19 in the present study illustrates that even if a PT lacked positive experiences with proof in the past and was negative about proof and teaching proof, they could develop enjoyment and interest in proof and realise the importance of the learning and teaching of proof in school mathematics after having a positive, meaningful experience with proof in the intervention. A meaningful experience could make PTs become aware of their existing beliefs, compare the existing beliefs with the experience to identify limitations of the existing beliefs, and hence, search for available information to modify or replace the existing beliefs (Liljedahl et al., 2021).

Therefore, in order to develop practising teachers and PTs' more productive beliefs and attitudes towards teaching proof in school mathematics as a first step of making proof central to school mathematics, it is essential to provide practising teachers and PTs with more positive, meaningful experiences of different roles of proof in school mathematics. The next question is: How do teacher educators create an environment in which practising teachers and PTs could experience different roles of proof positively and meaningfully, thereby developing more productive beliefs and attitudes towards teaching proof? The data collected in the present study identify which components of the intervention, as design principles, were more prominent in the PTs' accounts, thereby highlighting areas for attention when developing a teacher education programme in supporting PTs to develop more productive beliefs and attitudes towards teaching proof. The three most prominent design principles concerned the opportunities provided to PTs to experience proof from an alternative perspective, the role of the instructor and its importance, and the opportunities provided to PTs to connect their ideal images of teaching proof and the work of teaching.

***Learning mathematics through discussing and working on proof and argumentation*.** The data collected in the present study indicate that the PTs' counterproductive beliefs and attitudes towards teaching proof in school mathematics seemed to derive from their past experiences in which proof was considered important in mathematics but dispensable from the learning of school mathematics. In many cases, during their school education, PTs were exposed to traditional, product-oriented instruction, in which they developed an impression that content (e.g., recalling and applying formulae and procedures) matters more than proof and argumentation. In the present study, the PTs expressed appreciation for the opportunities to discuss and work on mathematics and proof with peers in the intervention and considered such opportunities important for deepening their understanding of mathematical ideas and developing their productive beliefs about proof, suggesting a need to create an environment that makes proof as a vehicle for communicating and understanding mathematics and allows PTs to discuss and work on mathematics and proof with peers during teacher education.

According to previous research (e.g., Liljedahl et al., 2021; Limón, 2001; Shilling-Traina & Stylianides, 2013), exposing PTs to a learning environment that is different from how they learnt mathematics in the past (at school or at university) has the potential to stimulate the PTs to reflect on their existing beliefs about mathematics and its teaching. By creating an environment that allows them to discuss and work on mathematics and proof with peers, PTs become exposed to a different opportunity to learn mathematics and proof, and revisit school mathematics, from a perspective differing from their past experiences of traditional, product-oriented instruction that did not favour the teaching of proof (Yoo, 2008), and to exchange various related ideas and beliefs (Wilcox et al., 1991). The environment allows PTs to compare such opportunity and their past experiences, potentially leading to reflection on their beliefs related to proof and its teaching.

The data collected in the present study also reveal that presenting the aforementioned environment does not necessarily, automatically create a positive, meaningful experience which leads to successful belief change. In the present study, the PTs reported that they lacked experiences and confidence to express opinions and discuss mathematics and proof in class due to the fact that teacher talk was a major component of teaching the PTs experienced in the past (Mok, 2009). Having said that, such lack of experiences or confidence should not be considered a deficit or a constraint that limits PTs' experiences of proof, but a common starting point that PTs have when entering a teacher training programme. More importantly, the data collected in the present study indicate that the use of pre-class tasks helped the PTs develop a sense of preparedness for subsequent discussions and collaborative work, suggesting that with appropriate assistance it is possible for PTs to engage in and adopt an unfamiliar mode of learning.

***The instructor's role in facilitating belief change*.** Apart from presenting an environment that allows PTs to discuss and work on mathematics and proof with peers, the instructor of teacher education courses also has influences on PTs' beliefs and attitudes towards teaching proof. The data collected from the present study suggest that PTs value questions that the instructor asks to stimulate their discussion and reflection and consider the instructor's in-class actions as a model of teaching. In order to facilitate belief change, it is important for the instructor to facilitate PTs' engagement in discussion and other learning activities (Shilling-Traina & Stylianides, 2013; Smith et al., 2009) and to implement activities that direct PTs' awareness to their existing beliefs (G. J. Stylianides & Stylianides, 2009) such that they could compare their in-class experiences and their existing beliefs, potentially leading to cognitive conflict and belief rejection. For example, the instructor could challenge PTs' existing beliefs and encourage PTs to discuss limitations of their existing ideas about proof and teaching proof, and more importantly, how to mitigate the limitations and improve their ideas about teaching.

Moreover, the instructor's in-class actions also serve as an exemplar of the ideas about teaching mathematics and proof that are intended in teacher education. The actions help broadening PTs' library of ideas about teaching, and serve as alternative experiences of the learning and teaching of proof that stimulate PTs to compare their existing beliefs and the alternative experiences; belief replacement might be facilitated when the PTs reject the existing beliefs and accept the alternative experiences (Liljedahl et al., 2021). For example, when introducing an idea about teaching mathematics through proof and argumentation, the instructor could use questions and activities to guide PTs' exploration of proof and arguments of mathematical ideas, and connect their experiences to the work of teaching in school mathematics, as making connections between PTs' experiences in teacher education and the work of teaching appears to be essential to facilitating change in beliefs about ideas of teaching.

***Making connections between teacher education and the work of teaching*.** Using materials that are related to the work of teaching in school mathematics (e.g., developing activities based on tasks at school mathematics levels) could stimulate PTs' exploration of connections between the intended ideas conveyed in teacher education and their future work of teaching (Shilling-Traina & Stylianides, 2013). Connecting intended ideas about teaching conveyed in teacher education to the work of teaching enables PTs to develop a sense of practicality of the intended ideas about teaching. A sense of practicality of an idea about teaching refers to a belief that this idea could be actualised in a common classroom instruction (a pedagogical situation where a PT will often encounter in her future teaching), which is often related to different beliefs (e.g., beliefs about suitability of the idea for students and school curriculum).

The data collected from the present study suggest that using videos of classroom instruction and micro-teaching helps PTs acquaint themselves with ideas about teaching mathematics and proof that they often have limited experiences of. The use of videos of classroom instruction not only shows PTs how different ideas about teaching could be implemented in authentic classroom instruction, but also allows them to analyse the videos and improve their own instructional ideas (Hollingsworth & Clarke, 2017). For example, analysing the videos enables PTs to develop understanding of students' thinking (van Es & Sherin, 2010) and become aware that students have interesting ideas about mathematics and capacity for learning school mathematics through proof and argumentation, thereby facilitating the development of a belief that proof is suitable for students, and potentially, a sense of practicality of proof.

The use of micro-teaching improves PTs' confidence to apply intended ideas about teaching, conveyed in teacher education, to their teaching, and facilitates the adoption of the intended ideas (Allen, 1967; Bilen, 2015). When planning a lesson that involves proof and argumentation, PTs have to study and consider different instructional ideas that involve proof and argumentation, broadening their library of ideas about teaching. When trialling their planned lessons, PTs could develop positive emotions about their micro-teaching experiences and confidence to apply proof and argumentation via their planned lessons. Through peer evaluation, PTs could also improve their planned lessons. As their confidence develops, their sense of practicality of proof could also develop.

In the previous sections, I discussed the need to provide PTs with meaningful experiences of proof during teacher education and the design principles that were identified to be effective in the present study in creating meaningful experiences of proof and facilitating the development of PTs' more productive beliefs and attitudes towards teaching proof in school mathematics. The data collected in the present study also indicate conditions that were required when helping the PTs in the intervention adopt discussion-based, inquiry-based instruction for the teaching of proof which they had very limited experiences, adding to the existing research and suggesting two necessary conditions of changing PTs' beliefs about ideas of teaching.

5.2.3 There are two necessary conditions of change in PTs' beliefs about ideas of teaching

According to Liljedahl et al. (2021), belief change is preceded by belief rejection which is, in turn, preceded by cognitive conflict (p. 28). However, presenting an alternative idea about teaching mathematics and proof to PTs does not necessarily trigger cognitive conflict and belief rejection, nor translates into belief change. To increase the likelihood of successful belief change, particularly related to beliefs about ideas of teaching, the data collected from the present study suggest that it is essential to (i) develop PTs' positive emotions about the ideas of teaching, and (ii) develop their sense of the practicality of the ideas for school mathematics. The former condition is more personal to the PTs, whereas the latter is related to both personal and contextual factors.

Development of positive emotions about ideas of teaching. Emotions play an important role in shaping PTs' beliefs and attitudes (Hannula, 2014; Liljedahl, 2005): positive emotions often produce productive beliefs, whereas negative emotions often result in counterproductive beliefs. As PTs often had negative past experiences with proof during their school education, many hold relatively negative emotions (e.g., anxiety, boredom) and beliefs about the learning and teaching of proof when entering teacher education. It is therefore essential to create a learning environment that aims to facilitate the development of PTs' positive emotions about proof, as it serves as an alternative experience which directs PTs' awareness of their existing negative emotions and beliefs about proof for reflection. The positive experience and its subsequent positive emotions contest and often replace PTs' existing negative emotions, thereby triggering cognitive conflict and belief rejection, and potentially, forming the basis for the development of productive beliefs and attitudes towards teaching proof in school mathematics.

However, the data collected in the present study indicate there are PTs who have developed positive emotions about proof but express some hesitation in applying proof and argumentation and/or discussion-based, inquiry-based instruction to their future teaching. It implies that the development of positive emotions about proof is not sufficient for the adoption of the teaching of mathematics through proof and argumentation as well as the adoption of discussion-based, inquiry-based instruction. Due to unfamiliarity with the ideas and past experiences, PTs express scepticism about these ideas about teaching and beliefs about the learning and teaching of school mathematics. Such scepticism hinders PTs from rejecting their existing beliefs about product-oriented instruction and replacing the existing beliefs with new beliefs about teaching proof.

***Development of sense of practicality of ideas about teaching*.** Therefore, it is essential to develop PTs' sense of practicality of ideas about the teaching of proof and about discussion-based, inquiry-based instruction. According to previous research, if a proposed instructional idea is not considered practical in terms of classroom implementation, it will be rejected (Clarke, 1994; Doyle & Ponder, 1977). PTs' sense of practicality of an idea about teaching depends on not only their confidence to apply the idea to their future teaching, but also their opinions about the suitability of the idea related to, for example, students and school mathematics curriculum (Doyle & Ponder, 1977; Janssen et al., 2015).

Ideas about the teaching of proof and about discussion-based, inquiry-based instruction are often unfamiliar to PTs due to their past experiences with mathematics and proof during their school education. In the past, PTs often learnt school mathematics with little or no proof, via traditional instruction (e.g., teacher-centred lecturing, drill practice). Whilst many PTs are competent at school mathematics and at writing proofs at this level, and aware of the importance of proof in mathematics, they often lack experiences with ideas about the teaching of proof and discussion-based, inquiry-based instruction, and often have concerns and, possibly, doubts about how these ideas could be applied to usual classroom instruction.

According to Doyle and Ponder (1977) and Janssen et al. (2015), teachers tend to reject an instruction proposal if they do not recognise how to carry out the proposal in classroom instruction, or if they do not find that the proposal fits the classroom situations in which they work, or if they consider the proposal not cost-effective (e.g., requiring too much time or energy to prepare and implement the proposal). The present study provides empirical data showing that the development of the PTs' sense of practicality of the teaching of proof and discussion-based, inquiry-based instruction was related to the development of their confidence to apply the ideas and their beliefs about the need for the ideas and the suitability of the ideas for the learning of school mathematics. The development of PTs' self-confidence to apply the ideas, their knowledge of students' capacity for proof (e.g., seeing that students are not unable to discuss mathematics through proof and argumentation) and their awareness of the relevance of proof to the work of teaching (e.g., seeing that proof helps students understand mathematics and prepare for examinations) forms the basis for an impression that the teaching of proof and discussion-based, inquiry-based instruction could be actualised. This impression increases the likelihood that the ideas about the teaching of proof and discussion-based, inquiry-based instruction contest PTs' existing beliefs about, and adherence to, traditional instruction. Potentially, it results in belief rejection and adoption of the ideas (i.e., belief change).

Whilst the data collected from the present study suggest that using videos of classroom instruction and micro-teaching could increase the likelihood of the development of PTs' sense of practicality of some ideas about teaching, the identification of hesitation in adopting the ideas about the teaching of proof and discussion-based, inquiry-based instruction reveals that some beliefs are more resistant to change. Beliefs that are more subjective and experiential appear to be easier to change, even if the beliefs might have been developed for years (e.g., the case of Participant 19; Excerpt 4.3.16). On the contrary, beliefs that are related to factual, quantitative information (e.g., class time, the proportion of proof-related tasks in examinations) appear to be more difficult to change. This observation suggests a potential research direction (Section 5.3).

5.2.4 Summary of Section 5.2

The present study adds to the literature on proof by describing PTs' beliefs and attitudes towards teaching proof in school mathematics, and their changes, from an East Asian (Hong Kong) perspective. Moreover, the present study adds to the literature by describing the development of an intervention that facilitated change in PTs' beliefs and attitudes towards teaching proof in a short period. The consistency in the results of different interventions in PTs' proof-related knowledge and/or beliefs suggests that providing PTs with positive, meaningful opportunities to experience the learning and teaching of proof (different from what they experienced at school) could facilitate PTs' change, even if the duration is quite short, and suggests three design principles for interventions in teachers' beliefs about the learning and teaching of proof – i.e., re-positioning proof in the learning and teaching of mathematics, making use of the instructor, and connecting the learning contents of an intervention and the work of teaching. Lastly, the present study also adds to the literature by providing additional empirical data that support the sequence of belief change (cognitive conflict → belief rejection → belief change/replacement) suggested by Liljedahl et al. (2021) and describing two conditions that increase the likelihood of successful change in PTs' beliefs about ideas of teaching – i.e., the development of positive emotions about ideas of teaching, and the development of a sense of practicality of ideas about teaching.

5.3 Future research directions

In the following, I discuss four directions for potential future research. The directions emerge from the data collected from the present study as well as its limitations. The first direction is related to the stability of PTs' post-intervention beliefs and attitudes towards teaching proof in school mathematics. The second direction is about further evaluation and improvement of the questionnaire. The third direction is related to the generalisability of the present study. The last direction is related to questions about changing beliefs that are experiential or connected to factual, quantitative information.

5.3.1 Studying long-term effects of the intervention

The present study involved an extracurricular intervention that lasted three weeks. Whilst the observed changes in PTs' responses related to the learning and teaching of proof indicated that PTs' beliefs and attitudes towards teaching proof in school mathematics could change to more productive and process-oriented views within a short period, the present study, which ended after the intervention, has raised an unanswered question: *Will PTs continue to hold* (*or advance*) *their more productive and process-oriented beliefs and attitudes after an intervention for a period of time?*

In order to answer this question, I propose including delayed "tests" in future research. The inclusion of delayed tests will enable investigations into questions about whether and how long PTs will continue to hold their post-intervention beliefs and attitudes, and whether there are any delayed effects of the intervention (Pournara, 2021). If the answer of the question is positive, it is also of interest to study whether and how PTs will translate their more productive and process-oriented beliefs and attitudes towards teaching proof into their actual teaching practices, given that the ideas of teaching proof and process-oriented instruction are not necessarily embedded into the schools where the PTs will be working at and it is likely that they will continue to experience traditional, product-oriented instruction in the future. Further studies can be conducted by observing their classroom instruction after they graduate from teacher education and start to practise as a teacher.

5.3.2 Improving the questionnaire

In earlier research cycles, I used various methods (e.g., expert review, piloting) for examining and ensuring whether the Likert items, as well as the open-ended questions, measured what they were designed to measure (content validity), and used approximately 200 PTs' responses for evaluating the factor structure of the Likert items (Section 3.3.1). Although the results suggested that, in general, it was appropriate and reasonable to use the items for the present study, some limitations which could lead to further improvement were identified. For example, some items did not receive any **Strongly Disagree** responses, raising questions such as "Is the sample size not large enough to have **Strong Disagree** responses to certain items?" or "Is there a risk of bias in certain items?"

When examining the factor structure of the items, a dimension consisted of two items only. Although it is merely a rule of thumb to have a minimum of three items in one dimension, not a statistical necessity (Cohen et al., 2011), the small number of items caused difficulty in interpreting the dimension. The small number of items could also raise a question about the internal consistency of the dimension – in the case of two-item dimension, the internal consistency coefficients (e.g., Cronbach's alpha, average inter-item correlation) depend mainly on the single correlation (coefficient) between the two items.

When testing their test-retest stability, the Likert items showed reasonable results in general (Appendix B). Having said that, a few items did not show a good level of test-retest stability, possibly due to individual variations. Moreover, although I planned to collect data from over 100 PTs from different teacher training programmes in Research Cycle 3 for examining the test-retest stability, unexpected political situations in Hong Kong (see Section 3.5) led to a reduction of amount of data collected and a small, non-randomised subsample (n = 27) were used for the analysis, suggesting larger samples are needed to confirm the results.

For future research, I plan to improve the instrument for longitudinal study with large samples, ideally from different populations (e.g., countries). A few more items can be added to dimensions that consisted of two or three items. Items that did not receive any **Strongly Disagree** responses or did not show a good level of test-retest stability can be revised. Items concerning the learning and teaching of proof at different school levels and/or in different situations can also be added such that proof-related situational beliefs (Goldin et al., 2017; Lesseig & Hine, 2021) may be included in the investigation. The results from the open-ended questions in the present study may also be used for revising the Likert items and/or creating new items.

Furthermore, I plan to revise the Likert scale from a 5-point scale to a 7-point scale. Using a 7-point scale will allow participants to make more fine-grained distinctions between **Slightly (Dis)Agree**, **Moderately (Dis)Agree** and **Strongly (Dis)Agree** (Krosnick & Presser, 2010). When comparing the pre- and post-intervention responses of the PTs in the intervention, I found that for each of the items, particularly those concerning their enjoyment and interest in proof (e.g., Items E4, E6 and D3, Appendix E), most of the PTs chose the same **(Dis)Agree** response before and after the intervention but they expressed and elaborated on some post-intervention change in their beliefs and enjoyment in proof during interviews. The use of a 7-point scale aims to replace **(Dis)Agree** by **Slightly (Dis)Agree** and **Moderately (Dis)Agree** which may increase the likelihood of capturing differences in participants' responses between two time points for pre-post comparison and interpretation.

5.3.3 Repeating and scaling up the intervention

In previous sections (e.g., Section 3.5), I discussed that a limitation of the present study was that the available data for pre-post analysis were fewer than what I expected, due to unexpected political situations during the data collection period – I planned to recruit 40 PTs to participate in the intervention in Research Cycle 4 and expected to receive both pre- and post-intervention data from at least 30 of them, but the political situations led to cancellation of planned interventions and interviews, resulting in a reduction of the amount of data, particularly post-intervention data. The reduced sample size limited the generalisability of the findings of the present study. Although the existing data were sufficient for the present study and the findings of the present study were also supported by the data collected from earlier research cycles and the results of other related studies, I posit that there is a need to repeat and scale up the intervention.

I conducted the present study by using design-based research (DBR) methodology and discussed why the present study could be identified as both a development study and a validation study (Section 3.2). The present study involved the development of an intervention of short duration, because there was no available intervention that aimed to study and facilitate change in

PTs' beliefs and attitudes towards teaching proof in school mathematics in a short duration. I adopted design principles and theories that were formulated for different research contexts in other related studies, for developing the intervention of the present study. Although the iterative nature of the present DBR study and the consistency between the data collected from the present study and other related studies suggest a possibility of repeating and generalising the findings of the present study, more research has to be conducted to confirm the generalisations.

For future research, I plan to repeat the intervention with different samples. Including larger sample in the intervention will allow more advanced statistical analyses for confirming the effects of the intervention on PTs' beliefs and attitudes towards teaching proof. Repeating the intervention with different samples will also allow investigations into questions such as "Will the intervention affect participants from different countries (or learning cultures) differently?" and "Will the intervention have influences on practising teachers?" By answering these questions, the generalisability and replicability of the findings of the present study can be examined.

5.3.4 Studying change in different types of beliefs

The data collected from the present study showed that some PTs expressed hesitation in adopting new beliefs because their beliefs about the school mathematics curriculum (e.g., that the proportion of proof is low in the curriculum, that the curriculum focuses on the development of students' routine procedural skills) remained unchanged. These beliefs appeared to be evidentially-held (T. F. Green, 1971, cited in Beswick, 2018; Liljedahl et al., 2021) and closely related to the composition of proof-related tasks and tasks that require routine procedural skills in examination papers (C.-Y. Lee, 2021; C.-Y. Lee & Wong, 2019).

Can these evidentially-held beliefs be changed if the composition of tasks in examination papers is not changed? If so, how? For future research, apart from repeating the intervention with large samples for statistical analyses, I also plan to conduct qualitative research to study the change in different types of beliefs, particularly evidentially-held and non-evidentially-held beliefs. By doing so, I aim to address the following questions: Regarding beliefs related to the learning and teaching of proof in school mathematics, which beliefs are (non-)evidentially-held beliefs? If non-evidentially-held beliefs are more likely than evidentially-held beliefs to be placed inside an individual at a deeper, tacit level and are resistant to change (T. F. Green, 1971, cited in Beswick, 2018; Liljedahl et al., 2021), can creating a different and meaningful learning experience that arouses positive emotions affect non-evidentially-held beliefs?

5.4 Closing remarks

There has been a growing interest in studying teachers' beliefs and knowledge related to the learning and teaching of proof, due to the presumed links between teachers' beliefs and knowledge and their teaching practices, and between teachers' teaching practices and students' learning experiences. However, little research has focused on whether and how PTs' beliefs about the learning and teaching of proof can be changed in a short period. The present study therefore adds to the literature by designing an innovative intervention of short duration that could promote PTs' more productive beliefs and attitudes towards teaching proof in school mathematics.

The present study was designed to develop a promising intervention through iterations of design, trial, evaluation and refinement. Data were collected from multiple sources for assessing PTs' beliefs and attitudes towards teaching proof in school mathematics and for gaining deeper insights into belief change. Findings from analysis of questionnaire data and interviews were reported and implications were discussed.

The highlighted findings are:

1. After the intervention, PTs had notable changes in their beliefs and attitudes towards teaching proof in school mathematics.
2. PTs who participated in the intervention developed more positive beliefs about the relationship between proof and school curriculum (including examinations), developed more enjoyment and interest in proof, reduced anxiety about proof, and developed more process-oriented beliefs about ideas of teaching, thereby developing more positive attitudes towards teaching proof in school mathematics.
3. The results of the present study provided evidence that belief change can be triggered by providing PTs with experiences of proof that are different from their past learning experiences. Apart from making proof central to the intervention activities, different strategies used to design the intervention included (i) encouraging peer discussion and collaboration, (ii) using videos of classroom instruction, (iii) using micro-teaching, and (iv) making use of instructor's questioning and actions.
4. The data collected from the present study suggest two conditions that increase the likelihood of successful change in PTs' beliefs, particularly about ideas of teaching. The first condition concerns whether PTs find the intervention novel and positive, and the second concerns whether PTs are confident about the practicality of the ideas of teaching that they learn and develop from the intervention.

Further research on change in different beliefs about the learning and teaching of proof held by practising teachers and PTs is needed. Such research will continue to add to the understanding of the complex process of belief change and how more productive beliefs about the learning and teaching of proof can be developed. Such understanding will help teacher educators develop appropriate teacher training programmes for helping practising teachers and PTs develop beliefs that favour effective teaching of proof in school mathematics.

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Appendices

Appendix A Questionnaire

PARTICIPANT CONSENT FORM

CUREC Approval Reference: ED-CIA-18-198

Purpose of Study: The aim of this study is to learn more about how different preservice teachers perceive learning and teaching mathematics.

		Please initial each box
1	I confirm that I have read and understand the information sheet for the above study. I have had the opportunity to consider the information, ask questions and have had these answered satisfactorily.	
2	I understand that my participation is voluntary and that I am free to withdraw at any time, without giving any reason, and without any adverse consequences or academic penalty.	
3	I understand that research data collected during the study may be looked at by designated individuals from the University of Oxford where it is relevant to my taking part in this study. I give permission for these individuals to access my data.	
4	I understand that this project has been reviewed by, and received ethics clearance through, the University of Oxford Central University Research Ethics Committee.	
5	I understand who will have access to personal data provided, how the data will be stored and what will happen to the data at the end of the project.	
6	I understand how this research will be written up and published.	
7	I understand how to raise a concern or make a complaint.	
8	I consent to being audio recorded.	
9	I understand how audio recordings will be used in research outputs.	
10	I agree to take part in the study	
11	I agree for research data collected in this study to be given to researchers, including those working outside of the EU, to be used in other research studies. I understand that any data that leave the research group will be fully anonymised so that I cannot be identified.	
12	I agree for my personal data to be kept in a secure database for the purpose of contacting me about future studies.	

Name of Participant ______ Date ______ Signature ______

After you complete this survey, I am also seeking out a group of preservice teachers to be interviewed and/or participate in a teacher development workshop. If you are interested, permission will be requested from you.

Please check the following:

[ ] I am interested in participating in the interview. Contact me for more information.
[ ] I am interested in participating in the teacher development workshop. Contact me for more information.
[ ] I am not interested in participating in any of the above.

If you checked at least one of the first two options, please complete the following and I will contact you for further information and permission.

University email ______ Phone numbers ______
(optional for arranging interviews/workshops)

1

Survey

Thank you for participating in this survey.

Background questionnaire
If this is your first time to participate in this survey, please complete this background questionnaire; otherwise, you may skip this part. The following information is being collected to meet research requirements. It is confidential and will not be released except in the form of statistical summaries.

1. Your university [ ]
   Your type of programme
   [ ] Undergraduate; Year [ ]
   [ ] Postgraduate; Year [ ]

2. Gender: Female [ ] or Male [ ]

3. Did you take extended mathematics in high school? [Please indicate if you took M1 or M2 in HKDSE, mathematics HL or further mathematics HL in IB; additional mathematics in HKCEE; applied mathematics, mathematics and statistics or pure mathematics in HKALE; or others]
   Yes [ ] specify: [ ];
   No [ ]

4. How many mathematics education methods courses have you taken at the undergraduate and graduate levels?
   [ ] courses.

5. How many mathematics courses have you taken at the undergraduate and graduate levels?
   [ ] courses.

6. Have you started or completed your teaching practicum
   Yes, [ ]
   No [ ]

7. Please indicate the Grade set that you have taught:
   P.1–P.3 [ ] P.4–P.6 [ ] S.1–S.3 [ ] S.4–S.6 [ ]

8. In **high school mathematics**, what was your overall mathematics performance? Please make your best estimate.
   [ ] HKAL: A / HKDSE: 5** / IB: 7 / etc.
   [ ] HKAL: B / HKDSE: 5* / IB: 6 / etc.
   [ ] HKAL: C / HKDSE: 5 / IB: 5 / etc.
   [ ] HKAL: D / HKDSE: 4 / IB: 4 / etc.
   [ ] HKAL: E / HKDSE: 3 / IB: 3 / etc.
   [ ] Less than above
   [ ] Grades were not given

9. In **university mathematics courses**, what was your overall mathematics GPA? Please make your best estimate.
   [ ] 3.70–4.00 (mostly A's)
   [ ] 3.40–3.69 (about half A's and half B's)
   [ ] 2.70–3.39 (mostly B's)
   [ ] 2.40–2.69 (mostly half B's and half C's)
   [ ] 1.70–2.40 (mostly C's)
   [ ] Less than 1.70
   [ ] Grades were not given

2

Section 1: Attitude Scale 態度量表

For the following items, tick the box that best describes your opinion.
於下列各項中，選出描述你的意見，並勾出最適合的空格。

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree
1. I feel myself under pressure when I make proofs in mathematics lessons. 當我在課堂中建立證明的時候，我感到壓力。					
2. I do not like dealing with mathematical proofs. 我不喜歡處理數學證明。					
3. It's fun for me to make proofs. 建立證明是有趣的。					
4. One of the things I like about mathematics is the proofs of theorems in the lessons. 我喜歡數學的其中一個原因是課堂中定理的證明。					
5. I do not worry when making proofs. 當建立證明的時候，我並不憂慮。					
6. Making proofs arouses my curiosity. 建立證明引起我的好奇心。					
7. I find it boring to make proofs. 我認為建立證明是沉悶的。					
8. I think that theorems and proofs are the foundations of mathematics. 我認為定理和證明是數學的根基。					
9. My self-confidence diminishes when I cannot prove a theorem. 當我未能證明一個定理的時候，我的自信減少。					
10. I find it interesting to make proofs. 我認為建立證明是有趣的。					
11. Proofs are very important in understanding mathematical language. 證明對於了解數學語言十分重要。					
12. Proofs are indispensable for mathematics. 證明於數學中是必不可少的。					
13. Making proofs improves mathematical thinking. 建立證明有助改善數學思維。					
14. To me, it is important to make proofs in mathematics. 於我而言，建立證明對於數學十分重要。					
15. Working on a proof in front of the class frightens me. 於班中建立一個證明使我害怕。					
16. I enjoy proving mathematical results. 我享受證明數學結果。					
17. To improve logical thinking, making proofs is necessary. 為了改善邏輯思維，建立證明是必須的。					
18. I feel enthusiastic to prove a theorem when I see it. 對於要去證明定理，我感到雀躍。					
19. I love mathematics, but I do not like making proofs. 我鍾愛數學，但我不喜歡建立證明。					
20. In the development of deductive reasoning, making proofs plays an important role. 在發展推理思維時，建立證明扮演著一個重要的角色。					

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree
21. I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists. 我無法理解為何需要證明一些已被數學家及科學家證實的事。					
22. Not being able to prove upsets me. 未能證明使我煩亂。					
23. It scares me to work on a proof. 建立一個證明使我恐懼。					
24. Logical thinking is not necessary for working on an examination/a test in school mathematics. 學校數學的考試/測驗並不需要邏輯思維。					
25. Knowing how to make proofs is very important in excelling an examination/a test in school mathematics. 對於於學校數學的考試/測驗獲取好成績，懂得如何建立證明十分重要。					
26. In an examination/a test in school mathematics, making proofs plays an important role. 在學校數學的考試/測驗中，建立證明扮演著一個重要的角色。					
27. During an examination/a test in school mathematics, how well to recall the procedures of solving a task is more important than how well to make a proof. 於學校數學的考試/測驗，回想解題的步驟的能力比建立證明的能力更重要。					
28. Practising making proofs does not help solve a task of an examination/a test in school mathematics. 練習建立證明無助於解答學校數學中考試/測驗的題目。					

1. Should we implement tasks of proof and proving into mathematics class? Please briefly **explain**.
老師應否於數學課堂中引入數學證明活動? 請簡單**說明**。

2. If we implement proof and proving in class, (a) **when** and (b) **how** should we? Please briefly **explain**.
若要於數學課堂中引入數學證明活動，(a) **何時**及 (b) **如何** 進行? 請簡單**說明**。

4

Section 2: Teacher Scenarios 教學情境

Context: Two junior secondary (Form 2; i.e., 13–14 years old) mathematics teachers at one school introduced the identity "$a^2 – b^2 = (a – b)(a + b)$" for factorisation of polynomials. Below are how the lessons played out in Teacher A and Teacher B's classrooms. Please read each scenario and respond to the questions at the end. The line numbers for each scenario can help to make references to the text.

處境：兩位初中（中二；13 至 14 歲）數學老師於同一學校任教，他們於課堂中引入恆等式「$a^2 – b^2 = (a – b)(a + b)$」以用於多項式的因式分解。以下為教師 A 及教師 B 的教學流程。請閱讀各情境並回答及後的問題。各情境的行號可協助引文。

Scenario involving Teacher A 教師 A 的情境

1. Teacher A recalled the previous lessons of "expansion of the product of polynomials" and "factorisation using extraction of common factors" to students and assessed their abilities to expand some simple products of polynomials and to use extraction of common factors accurately.
2. The teacher wrote down "$a^2 – b^2 = (a – b)(a + b)$", and asked students to verify the truth of the statement by expanding "$(a – b)(a + b)$" into "$a^2 – b^2$".
3. When the identity was verified, the teacher told students about the use of this identity (e.g. for computation of $49*51 = 50^2 – 1$, mathematical results that make use of this identity).
4. The teacher continued by examples of using this identity to factorise some pre-set polynomials, emphasising the common mistakes that students may make in assignments/tests/examinations.
5. The teacher then asked students to work on pre-set practice questions; in case students made mistakes on answering the questions, the teacher pointed out their misconceptions and explained the correct procedures for answering the questions.
6. After assigning homework questions, the teacher ended the lesson.
7. On the next day, after ensuring that students could apply the identity to factorise some polynomials, the teacher asked students to solve more complicated problems adapted from tests/examinations that require students to use this identity.
8. The teacher assigned some students to demonstrate their solutions on the board. After evaluating students' solutions and reminding students to their common mistakes, the teacher ended the lesson.

Scenario involving Teacher B 教師 B 的情境

1. Teacher B also recalled the previous lessons of "expansion of the product of polynomials" and "factorisation using extraction of common factors" to students.
2. The teacher asked students to compute the values of 49*51, 48*52, 47*53 and so on. Through probing, the teacher guided students to identify the pattern of the answers 2499, 2496, 2491 is $50^2 – x^2$ where x is 1, 2, 3, and so on.
3. After identifying the pattern of 49*51, 48*52, 47*53 is $(50 – x)(50 + x)$ where x is 1, 2, 3, and so on, the teacher asked students to expand the expression to arrive $50^2 – x^2$; based upon students' work, the teacher wrote down "$(50 – x)(50 + x)$ = (students' work) = $50^2 – x^2$" and initiated a discussion about the meaning of "=" sign.
4. The teacher further initiated a discussion about the possible outcomes of replacing 50 and x by other variables/numbers; the teacher guided students to formulate the statement "$a^2 – b^2 = (a – b)(a + b)$".
5. Students discussed how to verify and make sense with this statement.
6. Based upon students' contributions, the teacher compared different methods of verifying this statement and made connections among the methods; after assigning homework questions, the teacher ended the lesson.
7. On the next day, after responding to students' common misconceptions and questions about homework questions, the teacher asked students to solve the problem "for a rectangle with a perimeter of 100 cm, what is the maximum area?" in groups.
8. The teacher guided students to discuss their solutions, and formulate and verify the statement "given a rectangle with perimeter x, its possible maximum area is $x^2/16$". After summarising students' contributions, the teacher ended the lesson.

Questions 問題

1. **Suggest** TWO major similarities between Teacher A and Teacher B.
   寫出老師 A 及老師 B 之間的兩項相似之處。

   The 1st major similarity:
   第一項

   The 2nd major similarity:
   第二項

5

2. **Suggest** TWO major differences between Teacher A and Teacher B.
寫出老師 A 及老師 B 之間的兩項相異之處。
The 1st major difference:
第一項

The 2nd major difference:
第二項

3. **Suggest** and briefly **explain** TWO teaching approaches in the scenarios that you find the most important for student learning.
寫出並簡單**說明**兩項最重要的教學活動。
The 1st most important:
第一項

The 2nd most important:
第二項

4. **Suggest** and briefly **explain** TWO teaching approaches in the scenarios that you would remove if you do not have time to include all teaching approaches.
寫出並簡單**說明**若教學時間不足時你會移除的兩項教學活動。
The 1st to remove:
第一項

The 2nd to remove:
第二項

5. If you are preparing for the same topic, introduction of the identity "$a^2 - b^2 = (a - b)(a + b)$" for factorisation of polynomials, **describe** and briefly **explain** your teaching approach.
你正在準備此課題的教學，試**描述**及簡單**說明**你的教學設計。

6. Which of the two teachers in the scenarios would you **identify** more (0 = mostly Teacher A; 100 = mostly Teacher B)? Please briefly **explain**.
若 0 代表最接近老師 A 及 100 代表最接近老師 B，你會如何**自找評價**? 請簡單**說明**。

Thank you for giving your time to complete this survey!
感謝你抽空完成問卷。

Appendix B The development and evaluation of the Likert items

In the research on the learning and teaching of proof, there are ten sets of Likert items developed for beliefs and attitudes relating to proof available in the existing literature (Almeida, 2000; Aydoğdu-Iskenderoğlu & Baki, 2011; Frasier, 2010; Keceli-Bozdag et al., 2014; Kotelawala, 2007; W.-I. Lee, 1999; Moralı et al., 2006; Nyaumwe & Buzuzi, 2007; Uzel & Ozdemir, 2009; Yoo, 2008). To balance the research focus, time availability and research rigour, I selected *Proof and Proving Attitude Scale* (PPAS) developed by Keceli-Bozdag et al. (2014) for the skeleton of the Likert items used in the present study, because (1) PPAS had been validated by administering the items to a large sample of PTs (n = 727), (2) its factor structure had been explored, (3) all of its items could be reconstructed, (4) its items could cover the major components of attitudes towards teaching proof that have the potential to influence teachers' (self-reported[13]) time spent on proof in classroom instruction (Frasier & Panasuk, 2013; Kotelawala, 2016; Varghese, 2009), and it appeared to be able to identify belief and attitude changes as shown in Zengin (2017).

Having said that, some concerns needed to be resolved before PPAS was used for the present study, namely, the concern about the adaptation of PPAS from Turkish context to Hong Kong context and the subsequent concerns about its translation (from Turkish to Chinese and English) and construct validity in different cultures. Andrews and Diego-Mantecon (2015) has extensively discussed the need and procedures for a 'systematic and warranted' approach to

[13] I am aware that participant's self-report data do not necessarily equate her actual behaviour, due to various reasons, including (but are not limited to) reporting bias, attempt to please researchers and different interpretations of the same statement between participants and researchers. Whilst researchers find some consistencies between self-reporting and observation data about teacher practices, they also find discrepancies between two kinds of data (e.g., Cross, 2009; Wilkins, 2008).

cross-culturally adapting a measure of beliefs and attitudes. Within the scope of the present study, the adaptation was conducted through four steps. First, the Turkish-written PPAS was translated into English using *Google Translate* (Forward translation). The translation was then reviewed by a Turkish-English bilingual[14], and some English translated items were also compared with their possible English origins as PPAS items were adapted from items in Almeida (2000), Kotelawala (2007) and Nyaumwe and Buzuzi (2007) (Forward translation review). Second, the English translation was translated back to Turkish using *Google Translate* (Backward translation), and the product was compared with the original PPAS (Backward translation review). Third, although mathematics education programmes in Hong Kong are administered in English and Hong Kong PTs are assumed to have competent (if not fluent) English, a Chinese translation was added, and reviewed by two Chinese-English bilinguals[15], in parallel with the English translation such that the participants could choose their preferred language to complete the measure, in order to reduce burdens. Finally, the translated PPAS was administered to 66 PTs and practising teachers (Pilot test).

The pilot aimed to confirm that there is no potential conceptual or semantic confusion when participants are completing the items, and to explore if any additional item could be added to refine the measure. To achieve these aims, the translated PPAS was distributed to 52 PTs who enrolled in UG programmes at university A and 14 practising mathematics teachers between July and September 2018, in Research Cycle 1. Seven of the practising teachers attended the interviews to investigate the content validity and accessibility of the translated PPAS; they were

[14] A doctoral student who has a background in English education.
[15] One of them is a doctoral student who has a background in English education and applied linguistics, and another has a Master’s degree in communication with a background in English literature.

invited to rephrase the items in their own words and to complete the translated PPAS using think-aloud protocols (Andrews & Diego-Mantecón, 2015; Presser et al., 2004). The interview also served as an exploration of ways to refine the items, after consulting expert comments.

For any potential conceptual or semantic confusion, I looked into partially completed responses: one PT consistently missed eleven items; after reviewing the raw data, the missing responses could be explained that the participant accidentally skipped the pages of the items. No other substantial missing responses were observed. The interviews also indicated that the respondents did not have difficulty in understanding or answering the items. It was evident that there was no potential conceptual or semantic confusion due to the translation or word choices of the draft. Moreover, the interviews obtained results of similar studies on teachers' beliefs and attitudes related to proof that how much time teachers would (intend to) spend on proof in classroom instruction can be affected by how important they believe that the use of proof in school mathematics is, how much they enjoy and appreciate proof, whether they believe that their students are capable of understanding and appreciating proof, and how much they believe that proof is relevant to school mathematics curriculum and examinations (Frasier & Panasuk, 2013; Kotelawala, 2016; Nyaumwe & Buzuzi, 2007). After reviewing the translated PPAS, additional items related to the relevance of proof to examinations were suggested. An expert also suggested to add one more item concerning the respondents' negative emotions about proof.

Six more items were added to form a set of 28 Likert items. The revised items were administered to 184 Hong Kong PTs for summative evaluation regarding the construct validity, the internal consistency and the stability of the items over time. To guide the evaluation of the construct validity, confirmatory factor analysis (CFA) was used to test if the revised items had the same factor structure as the original PPAS did as described by Keceli-Bodzag et al. (2014), and to explore if another factor structure existed that better described the Hong Kong sample through model re-specification (Brown, 2006; Harrington, 2009). At first, a model (model A) was specified, based on the factor structure found in Keceli-Bodzag et al. (2014) and the pilot, as well as the expert comments. The items were divided into five latent variables accordingly. Table B.1 illustrates model A and the data collected from the Hong Kong sample.

Table B.1
Descriptive statistics of the revised items and the hypothesised model (model A)

	Item	Model A	SA	A	N	D	SD	M	Mean
I1	I think that theorems and proofs are the foundations of mathematics.	Importance	57	109	15	2	1	0	4.19
I2	Proofs are very important in understanding mathematical language.	Importance	45	118	18	3	0	0	4.11
I3	Proofs are indispensable for mathematics.	Importance	68	99	15	1	1	0	4.26
I4	Making proofs improves mathematical thinking.	Importance	65	103	10	5	1	0	4.23
I5	To me, it is important to make proofs in mathematics.	Importance	44	115	21	4	0	0	4.08
I6	To improve logical thinking, making proofs is necessary.	Importance	24	119	35	5	1	0	3.87
I7	In the development of reasoning, making proofs plays an important role.	Importance	30	108	43	3	0	0	3.90
E1	It's fun for me to make proofs.	Enjoyment	24	98	50	10	2	0	3.72
E2	One of the things I love about mathematics is that proofs of theorems are done in the lessons.	Enjoyment	16	77	57	29	5	0	3.38
E3	I do not worry when making proofs.	Enjoyment	7	76	59	36	6	0	3.23
E4	Making proofs arouses my curiosity.	Enjoyment	25	101	40	12	5	1	3.70
E5	I find it interesting to make proofs.	Enjoyment	17	105	47	11	3	1	3.67
E6	I enjoy proving mathematical results.	Enjoyment	20	112	42	7	3	0	3.76
E7	I feel enthusiastic to prove a theorem when I see it.	Enjoyment	11	57	80	29	6	1	3.21
D1	I feel myself under pressure when I make proofs in mathematics lessons.	Dislike	5	67	71	38	3	0	3.18
D2	I do not like dealing with mathematical proofs.	Dislike	4	29	51	83	17	0	2.57
D3	I find it boring to make proofs.	Dislike	8	27	46	88	14	1	2.60
D4	I love mathematics, but I do not like making proofs.	Dislike	11	33	75	56	9	0	2.90
D5	I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists.	Dislike	8	24	46	78	28	0	2.49

A1	My self-confidence diminishes when I cannot prove a theorem.	Anxiety	13	84	60	23	3	1	3.44
A2	Working on a proof in front of the class frightens me.	Anxiety	7	39	79	52	6	1	2.94
A3	Not being able to prove upsets me.	Anxiety	19	84	55	23	3	0	3.51
A4	It scares me to work on a proof.	Anxiety	4	34	66	72	7	1	2.76
T1	Logical thinking is not necessary for working on an examination/a test in school mathematics.	Assessment	4	13	34	101	32	0	2.22
T2	Knowing how to make proofs is very important in excelling in an examination /a test in school mathematics.	Assessment	16	90	56	18	4	0	3.52
T3	In an examination/a test in school mathematics, making proofs plays an important role.	Assessment	11	91	53	26	3	0	3.44
T4	During an examination/a test in school mathematics, how well to recall the procedures of solving a task is more important than how well to make a proof.	Assessment	21	92	55	15	1	0	3.64
T5	Practising making proofs does not help solve a task of an examination/a test in school mathematics.	Assessment	3	38	62	65	16	0	2.71

n = 184; SA = **Strongly Agree** = 5; A = **Agree** = 4; N = **Neutral** = 3; D = **Disagree** = 2; SD = **Strongly Disagree** = 1; M = missing values

Analysis of the factor structure of the items. Because the items were ordinal (5-point scaling), unweighted least squares (ULS) was used to compute the parameters. ULS can provide more accurate parameters and test statistics than diagonally weighted least squares, another model estimator for ordinal data, although ULS may encounter a lower model-convergence rate (Forero et al., 2009; Li, 2014, 2016; Rhemtulla et al., 2012). As values of standard errors may be biased and resulting model test statistics (e.g., chi-square (χ^2) statistic) of ULS may deviate from the theoretical values, its robust corrections and analyses of the convergent and discriminant validity and the internal consistency were not used to accept any hypothesised model, but to inspect the need to re-specify the model until a parsimonious model was found.

There are three approaches to 'correct' the test statistics: namely, the mean correction (ULSM) and the mean and variance corrections (ULSMV and ULSMVS). Simulation studies have shown different limitations of these corrections. On one hand, as the number of items and the number of item categories increase, ULSM may yield (χ^2) test statistics that are noticeably greater than the theoretical values and inflated Type I error rates (Shi et al., 2018); on the other hand, ULSMV and ULSMVS statistics may have Type I error rates lower than the theoretical values (Savalei & Rhemtulla, 2013; Shi et al., 2018). The test statistics served for inspecting the need to re-specify the model, so ULSM was used as simulation study suggests that it can perform better than ULSMV(S) in term of χ^2 test when n = 200, the number of items is 30 and the number of dimensions is 5 (Shi et al., 2018).

To evaluate the models, model test statistics, item-fit and dimension-fit were inspected. First, model A was compared based on the results of the corrected model test statistics, including model chi-square statistic (χ^2_M), robust comparative fit index (rCFI), and robust root mean square error of approximation (rRMSEA). Each index provides the model fit from different perspectives. χ^2_M tests whether there is any significant misfit between the theoretical and observed correlation matrices; having said that, it is worth noting that χ^2 test is often sensitive to sample size and model complexity. (r)RMSEA tests whether the model fits 'reasonably' well in the population; in general, the closer to 0 the (r)RMSEA value is, the better the model is. (r)CFI is often used to evaluate the model fit relative to a more restricted, nested baseline model, that an (r)CFI value closer to 1 implies good model fit (Brown, 2006; Harrington, 2009; Kline, 2005). Note that the conventional CFI and RMSEA rules of thumb may not be applicable to the corrected model test statistics, due to the reduced power to identify model misspecification (e.g., Xia & Yang, 2019). Therefore, the conventional rules of thumb were not used for accepting any model. Instead, they were used for inspecting the need to re-specify the model(s). That is, rCFI 'noticeably' less than .90 and rRMSEA 'noticeably' greater than .08 indicated a need for model re-specification, whereas whether a model was accepted was not a decision based on simply looking at numeric values of rCFI and rRMSEA, but rather a decision involving more theoretical and practical considerations.

Apart from test statistics, I also examined item-fit and dimension-fit to investigate the convergent and discriminant validity of the models. For item-fit, item communalities (r-squared's), residual correlations and cross-loadings were inspected. Item(s) with low communality, high residual correlations and high cross-loadings were removed. The number of items on one dimension was taken into account when a decision to remove an item was made. For dimension-fit, the values of average variance extracted (AVE, average of r-squared's per dimension) were computed for inspecting any potential of model misspecification. For example, √(AVE) of a dimension greater than the largest correlation with other dimensions indicates that items on that dimension are more related to itself than to other dimension(s), whereas the contrary may indicate that an item is misspecified onto a dimension or a 'broader' dimension is mistakenly split into two.

In addition, I used the factor loadings, item variances and inter-item (polychoric) correlations from the CFA outputs and computed various reliability coefficients (namely, alpha coefficients, squared multiple correlations, correlated factors reliability coefficients, average inter-item correlations) to investigate the internal consistency. The coefficients were compared to identify agreements and disagreements. The results of these analyses informed the model re-specification. Table B.2 describes the process and analysis of model re-specification (models A–J), Table B.3 shows the last three models (models H–J) with factor loadings, Table B.4 reports the interdimensional correlations of the final model (model J), and Table B.5 shows the results of internal consistency tests of the last two models (models I–J).

Table B.2

Model test statistics for the models for the revised items

Model	Description and rationale	χ^2_M(df)	df	*p* for χ^2_M	rCFI	rRMSEA	90%CI of rRMSEA
A	Model specified with 5 dimensions based on the results of Keceli-Bodzag et al. (2014) and the pilot.	1774.500	340	< .001	.909	.104	[.100, .109]
B	T4 removed because it measured an unidentified dimension, as evidenced by its uncorrelations with other items and the lowest communality.	1727.329	314	< .001	.914	.105	[.100, .110]
C	D1 and D3 re-specified on a new dimension 'Failure' because they measured an unidentified dimension "emotional consequence when failing to prove", as evidenced by the discrepancy in factor loadings between them and other 'Anxiety' items.	1460.273	309	< .001	.932	.094	[.089, .099]
D	'Dislike' and 'Anxiety' items combined to form a dimension 'Negative' because they measured "negative emotions about proof", as evidenced by item inspection, AVEs and correlations.	1517.212	314	< .001	.928	.096	[.091, .101]
E	E3 re-specified on 'Negative' because it measured "worries about proving", as evidenced by item inspection, AVEs and correlations.	1438.560	314	< .001	.933	.093	[.088, .098]
F	D5 re-specified on 'Importance' because it measured "needs for proving known mathematical facts", as evidenced by item inspection, AVEs and correlations.	1446.049	314	< .001	.933	.093	[.088, .098]
G	D3 re-specified on 'Enjoyment' because it measured "boredom of proving", as evidenced by item inspection, AVEs and correlations.	1403.167	314	< .001	.935	.092	[.087, .097]
H	D2 and D4 cross-loaded on 'Enjoyment' and 'Negative' because they measured "(dis)like of proof", as	1350.092	312	< .001	.939	.089	[.085, .094]

	evidenced by item inspection, AVEs and correlations.						
I	D2 and D4 removed such that the potential dimensions of ‘Enjoyment’ and ‘Negative’ can be distinguished, as evidenced by item inspection, AVEs and correlations.	1113.642	300	< .001	.937	.089	[.084, .095]
J (final)	T1 removed because it did not directly measure “beliefs about relevance of proof to examinations”, as evidenced by item inspection and internal consistency coefficients.	1024.829	242	< .001	944	.087	[.081, .092]

R package ‘lavaan’; model estimator = ULSM; n = 179 due to listwise deletion of missing data

Model (re-)specification, testing and evaluation formed multiple cycles that aimed to arrive at a solution that ‘reasonably’ described the empirical data; every re-specification decision was made based on empirical and theoretical evidences, and each cycle involved only one re-specification condition, for instance, removal of an item or re-specification of an item onto another dimension, to reduce the risk of misspecification. For example, when inspecting the CFA results of model A, a few issues were identified: (i) Items A2 and A4 had extremely high (and problematic) factor loadings (.848 and 1.021 respectively) for the dimension ‘Anxiety’, whereas items A1 and A3 had ‘unproportionally’ low factor loadings (.330 and .283 respectively), indicating that items A2 and A4 could explain 71.9% and over 100% (problematic!) of variances in ‘Anxiety’ respectively whereas items A1 and A3 could explain ‘only’ 10.9% and 8.0% of the variances respectively. The inspection of the items indicated that although all items were related to anxiety and struggle for proving, items A1 and A3 differed from items A2 and A4 by a potential element of ‘the emotional consequences of failure to prove’. (ii) Item T4 explained only 0.3% (= $(-.050)^2$) of variance in ‘Assessment’. The inspection of the items in ‘Assessment’ indicated that T4 had a potential additional element of ‘comparing the importance of recall and proof’. (iii) Items A1, A3 and T4 had item communalities close to less than .1. It suggested that these items were misspecified into less desirable dimensions of the model or they measured multiple or some unidentified dimension(s). (iv) Contrarily, the communality of item A4 could not be computed due to the problematic factor loading parameter. It might be due to its extremely high factor loading in ‘Anxiety’, suggesting a potential of dimension or model misspecification. (v) The correlation between ‘Enjoyment’ and ‘Dislike’ was -.808 and that between ‘Dislike’ and ‘Anxiety’ was .843, and that between ‘Enjoyment’ and ‘Anxiety’ was -.558. The first two correlations were noticeably higher than the last one, which indicated that ‘Dislike’ items were correlated to ‘Enjoyment items’ and ‘Anxiety’ items, whereas ‘Enjoyment’ items were less correlated to ‘Anxiety’ items. The results suggested that there might be two potential dimensions that one included ‘Enjoyment’ and ‘Dislike’ whereas another included ‘Dislike’ and ‘Anxiety,’ or ‘Dislike’ items should be split into ‘Enjoyment’ and ‘Anxiety.’ Therefore, model misspecification was evidenced, and the issues were dealt one by one.

Table B.3
The revised items and the re-specified (models H and I) and final (model J) models with factor loadings

	Item	Model H	Model I	Model J
	Beliefs about the importance of proof			
I1	I think that theorems and proofs are the foundations of mathematics.	.513	.529	.526
I2	Proofs are very important in understanding mathematical language.	.715	.724	.722
I3	Proofs are indispensable for mathematics.	.809	.819	.818
I4	Making proofs improves mathematical thinking.	.835	.839	.840
I5	To me, it is important to make proofs in mathematics.	.838	.826	.834
I6	To improve logical thinking, making proofs is necessary.	.685	.696	.693
I7	In the development of reasoning, making proofs plays an important role.	.776	.774	.775
D5	I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists.	-.639	-.610	-.607
	Enjoyment and interest in proof			
E1	It’s fun for me to make proofs.	.745	.751	.753
E2	One of the things I love about mathematics is that proofs of theorems are done in the lessons.	.634	.647	.648
E4	Making proofs arouses my curiosity.	.857	.880	.881
E5	I find it interesting to make proofs.	.856	.847	.847

E6	I enjoy proving mathematical results.	.756	.742	.740
E7	I feel enthusiastic to prove a theorem when I see it.	.747	.743	.744
D3	I find it boring to make proofs.	-.775	-.752	-.749
D2	I do not like dealing with mathematical proofs.	-.447	NA	NA
D4	I love mathematics, but I do not like making proofs.	-.213	NA	NA
	Anxiety about proof			
E3	I do not worry when making proofs.	-.704	-.725	-.730
D1	I feel myself under pressure when I make proofs in mathematics lessons.	.565	.549	.548
A2	Working on a proof in front of the class frightens me.	.757	.752	.750
A4	It scares me to work on a proof.	.885	.871	.869
D2	I do not like dealing with mathematical proofs.	.362	NA	NA
D4	I love mathematics, but I do not like making proofs.	.547	NA	NA
	Negative responses when failing to prove			
A1	My self-confidence diminishes when I cannot prove a theorem.	.853	.864	.860
A3	Not being able to prove upsets me.	.706	.696	.700
	Beliefs about relevance of proof to examinations			
T2	Knowing how to make proofs is very important in excelling in an examination/a test in school mathematics.	.609	.636	.937
T3	In an examination/a test in school mathematics, making proofs plays an important role.	.408	.429	.650
T5	Practising making proofs does not help solve a task of an examination/a test in school mathematics.	-.633	-.612	-.558
T1	Logical thinking is not necessary for working on an examination/a test in school mathematics.	-.505	-.504	NA
	Stand-alone item			
T4	During an examination/a test in school mathematics, how well to recall the procedures of solving a task is more important than how well to make a proof.	NA	NA	NA

R package ‘lavaan’; model estimator = ULSM; n = 179 due to listwise deletion of missing data; NA = not applicable

The effort put in CFA allowed me to inspect what latent variables (dimensions) of Hong Kong PTs’ attitudes towards teaching proof in school mathematics were measured by the revised Likert items (factor structure; construct validity) and whether the same general construct was measured (internal consistency) among the items. For factor structure, the comparison between models A and J rejected the hypothesis that the revised items had the same internal structure as the original PPAS did; that is, the dimensions explored by Keceli-Bodzag et al. (2014) using a Turkish sample could not be completely recovered by a Hong Kong sample, suggesting that the hypothesised model A, or the translation of PPAS, might not be accurate in different cultures other than Turkey. However, the step-by-step model re-specification and item inspection in each re-specified model (including factor loadings and interdimensional correlations) allowed me to gain more understanding of what (subtle) latent variables the revised items captured and to revise my interpretation. For example, originally, Keceli-Bodzag et al. (2014) used "İspat Esnasındaki Olumsuz Tutumlar" ("Negative attitudes during proving", translated by Zengin, 2017; or "Anxiety about proving", translated by me) to interpret the dimension measured by items A1, A2 and A3 (and A4), focusing on the negative emotions that one could encounter *when working on proof*, but the difference in the factor loadings of the items and the correlations between item A2 (as well as item A4) and other items indicated that items A1 and A3 measured a different (but relevant) idea, which was "negative emotions (e.g., being upset) and reactions (e.g., reduction of self-confidence) when one *failed to* prove," and I used 'Negative responses when failing to prove' to interpret this dimension. Also, the re-specification of items E3, A2 and A3 into the dimension 'Dislike' and the removal of item D3 (which was re-specified into 'Enjoyment') allowed me to rethink and reinterpret this dimension (originally, Keceli-Bodzag et al. used "Genel Olumsuz Tutumlar" and Zengin translated it as "Common negative attitudes") into 'Anxiety about proof,' because the terms "worry" (item E3), "under pressure" (item D1), "frightens" (item A2) and "scares" (item A4) were used in the items.

Table B.4
Interdimensional correlations of model J

	Importance	*Enjoyment*	*Anxiety*	*Failure*
Enjoyment	.578***			
Anxiety	-.307***	-.709***		
Failure	.140	.101	.599***	
Assessment	.270***	.132*	-.114	.100

Note: R package 'lavaan'; model estimator = ULSM; n = 179 due to listwise deletion of missing data; * $p < .05$; ** $p < .01$; *** $p < .001$

The analysis of the internal consistency of the items discovered that item T1 lowered the internal consistency coefficients for the measure, particularly, the dimension 'beliefs about relevance of proof to examinations' because it measured 'the needs for logical thinking (instead of proof) in examinations'. By removing item T1, the internal consistency coefficients of model J and the subscales of its dimensions have all reached desired values (Table B.5). As the internal consistency coefficients indicated that all latent variables (dimensions) can explain over 50% (.7 * .7 = .49) of variation in item responses, I argue that the internal consistency of the 24 Likert items, as a quantitative measure, are established, and they measure five components of PTs' attitudes towards teaching proof in school mathematics: namely, beliefs about the importance of proof, beliefs about relevance of proof to examinations, enjoyment and interest in proof, anxiety about proof, and negative responses when failing to prove. Related studies have shown that these components are influential in teachers' time spent on proof in teaching (Frasier & Panasuk, 2013; Kotelawala, 2016). Yet, it is worth noting that the Likert items gather little information about PTs' beliefs about the nature of mathematics and its teaching and learning, and beliefs about students' capacity and interest in proof (e.g., Furinghetti & Morselli, 2011).

Table B.5
Results of internal consistency tests of models I and J

	Alpha[a]		Squared multiple correlation[a]		Correlated factors reliability[b]		Average inter-item correlation[a]	
Dimension	Model I	Model J	Model I	Model J	Model I	Model J	Model I	Model J
Importance	.90	.90	.91	.91	.90	.90	.53	.53
Enjoyment	.91	.91	.93	.93	.91	.91	.59	.59
Anxiety	.80	.80	.80	.80	.82	.82	.45	.45
Failing	.75	.75	.60	.60	.76	.76	.60	.60
Assessment	.63	.73	.68	.74	.63	.77	.30	.48
Overall	.90	.90	.96	.96	.93	.94	.26	.27

n = 179 due to listwise deletion of missing values; [a]use of polychoric correlations; [b]for details, see Cho (2016)

Test-retest analysis. Recall that the items were intended to capture changes in PTs' beliefs and attitudes towards teaching proof in school mathematics after participating in my intervention. If the data collected from the items were generally stable over a duration similar to that of my intervention (about a month), then it would be reasonable to interpret that the changes observed in the PTs in the intervention were contributed by the intervention (Cohen et al., 2011). To ensure this stability, I administered the revised items to PTs of three PG classes at the same university in September (Time 1) and October (Time 2) 2019[16]. The time interval was about one month, to replicate the time interval between pre- and post-intervention data collected in the present study study. Twenty nine PTs completed the questionnaire at Time 1, and 29 PTs completed the questionnaire at Time 2. Among them, 28 PTs completed the questionnaire twice. The paired data collected were analysed in item-wise and dimensional ways for triangulation (different methods of analysis).

For item-wise analysis, I used Svensson's (1998) method because it is dedicated to analysing paired ordinal data. Svensson developed this statistical method that avoids using arithmetic operations on the data and "considers the rank-invariant properties of ordinal data without any other structural assumptions" (p. 2923). Using this statistical method, I examined the agreements between the data collected in Times 1 and 2, and distinguished and analysed systematic disagreements and individual variations between the paired ordinal data. In this method, there are four types of numerical outputs: percentage agreement (PA), relative position (RP), relative concentration (RC) and relative rank variance (RV).

PA computes the proportion of identical pairs to all paired data, indicating the absolute agreement between paired data of two time points. Assuming the data order: **Strong Disagree** < **Disagree** < **Neutral** < **Agree** < **Strongly Agree**, RP estimates the difference between the probabilities of the responses at Time 2 (Y) being rated higher than at Time 1 (X) and the opposite, $P(X < Y) – P(Y < X)$, in other words, the difference between the probability of the

[16] I intended to collect more data from UG PTs but could not execute the plan successfully due to unexpected challenges as discussed in Section 3.5 and Section 5.3.2.

paired data lying in the upper triangle and that in the lower triangle of the contingency table (Table B.6). Possible values of RP lie between -1 and 1, and a positive value indicates that the respondents systematically rated higher categories at Time 2 than they did at Time 1. RC estimates the difference between the probabilities in how the measurements are concentrated on the ordinal categories, in other words, the difference between how likely the data collected at Time 2 are concentrated to ‘central categories’ (i.e., towards ‘3 = **Neutral**’) at Time 1 and how likely the data collected at Time 1 are concentrated to central categories at Time 2. RC has possible values ranging from -1 to 1, and a positive value indicates that the data collected at Time 2 are more likely concentrated to central categories than Time 1. RV measures the individual variances, with possible values ranged from 0 to 1. A non-zero RV indicates presence of individual variations, and the higher the value, the more dispersed are the paired data (Svensson, 1998). A non-zero, high systematic disagreement (RP or RC) may indicate some source of systematic bias (e.g., memory effect), and a non-zero, high RV may indicate that the measure is sensitive to some non-systematic, individual factors, which affect the responses. A measure with high degree of stability over time is expected to have a high PA but RP, RC and RV close to 0 (Svensson, 2012).

Table B.6
The frequency distributions of 28 pairs of measurements (X = Time 1, Y = Time 2)

Beliefs about the importance of proof

	SD	D	N	A	SA
SA	1			3	8
A				12	2
N				2	
D					
SD					

I think that theorems and proofs are the foundations of mathematics.

	SD	D	N	A	SA
SA			1	4	3
A				13	5
N				1	
D			1		
SD					

Proofs are very important in understanding mathematical language.

	SD	D	N	A	SA
SA				2	9
A				10	6
N				1	
D					
SD					

Proofs are indispensable for mathematics.

	SD	D	N	A	SA
SA				3	8
A				11	5
N			1		
D					
SD					

Making proofs improves mathematical thinking.

	SD	D	N	A	SA
SA				2	5
A				13	5
N				3	
D					
SD					

To me, it is important to make proofs in mathematics.

	SD	D	N	A	SA
SA			1	2	
A			5	12	3
N			1	3	1
D					
SD					

To improve logical thinking, making proofs is necessary.

	SD	D	N	A	SA
SA				4	1
A			2	15	3
N			2	1	
D					
SD					

In the development of reasoning, making proofs plays an important role.

	SD	D	N	A	SA
SA					
A					1
N		3	1		
D	3	10	4		
SD		5		1	

I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists.

Beliefs about relevance of proof to examinations

	SD	D	N	A	SA
SA					
A		2	1	6	2
N		2	7	4	
D			2	1	1
SD					

Knowing how to make proofs is very important in excelling in an examination/a test in school mathematics.

Beliefs about relevance of proof to examinations

	SD	D	N	A	SA
SA					
A		2	1	6	2
N		2	7	4	
D			2	1	1
SD					

Knowing how to make proofs is very important in excelling in an examination/a test in school mathematics.

Beliefs about relevance of proof to examinations

	SD	D	N	A	SA
SA					
A		2	1	6	2
N		2	7	4	
D			2	1	1
SD					

Knowing how to make proofs is very important in excelling in an examination/a test in school mathematics.

Enjoyment and interest in proof

	SD	D	N	A	SA
SA				1	1
A		1		18	2
N			4		
D					
SD					1

It's fun for me to make proofs.

	SD	D	N	A	SA
SA					2
A			5	12	
N	1	3	1	2	
D		1			
SD					1

One of the things I love about mathematics is that proofs of theorems are done in the lessons.

	SD	D	N	A	SA
SA				1	1
A		1	3	14	4
N		1	1		
D				1	
SD	1				

Making proofs arouses my curiosity.

	SD	D	N	A	SA
SA					1
A		1	2	15	3
N			2	1	
D			1		
SD	1				

I find it interesting to make proofs.[1]

	SD	D	N	A	SA
SA				2	1
A		1	2	14	3
N			2	2	
D					
SD	1				

I enjoy proving mathematical results.

	SD	D	N	A	SA
SA				1	
A			4	10	1
N		2	4	2	
D		3			
SD	1				

I feel enthusiastic to prove a theorem when I see it.

	SD	D	N	A	SA
SA					
A				1	1
N		3	3		
D	1	15		3	
SD	1				

I find it boring to make proofs.

Anxiety about proof

	SD	D	N	A	SA
SA				1	1
A		3	1	10	2
N		2	2	2	
D		2		1	
SD	1				

I do not worry when making proofs.

	SD	D	N	A	SA
SA					
A		2	3	4	1
N		4	3	1	
D		2	5	3	
SD					

I feel myself under pressure when I make proofs in mathematics lessons.

	SD	D	N	A	SA
SA					
A		1		1	1
N		2	5	3	
D	1	8	5		
SD			1		

Working on a proof in front of the class frightens me.

	SD	D	N	A	SA
SA					
A				1	1
N		4	3	1	
D	1	14	2	1	
SD					

It scares me to work on a proof.

Negative responses when failing to prove

	SD	D	N	A	SA
SA					1
A			3	6	1
N	1	2	4	5	
D		4	1		
SD					

My self-confidence diminishes when I cannot prove a theorem.

	SD	D	N	A	SA
SA				1	
A		1		6	2
N		2	6	5	1
D	1	3			
SD					

Not being able to prove upsets me.

n = 28; SA = **Strongly Agree**, A = **Agree**, N = **Neutral**, D = **Disagree**, SD = **Strongly Disagree**; [1]n = 27 due to missing values

The analysis (Table B.7) showed that the PAs ranged from 32% to 82% (on average, 58%), and if difference not greater than 1 was allowed, PA±1s increased to 82%–100% (on average, 94%). This indicated that, in most cases, the responses between two time points differed at most by 1 category. In the analyses of the disagreements, none of the items had a significant non-zero RP, indicating that the respondents did not rate higher or lower categories at Time 2 in general. RC showed that, when disagreeing, the responses to items D3 (disagreement = 29%) and A3 (disagreement = 46%) were more concentrated towards the central categories at Time 2 than at Time 1. RV indicated that there were individual variations in responses to items D1, T2 and T5, and these items had PA less than or equal to 50%. The results of Svensson's method indicated that the items were generally stable over time item-wise. Yet, a few items had significant RC and RV values indicating that some individual change might have occurred between two time points, although the group of respondents as a whole had not changed systematically. It suggested that, when inspecting participant's changes in responses to these items in the present study, more information (e.g., interview data) would be needed to explain the changes.

Table B.7
Outputs of Svensson's method for measuring systematic and individual disagreements of the revised items between two time points

	Item	PA	±1	RP	(SE)	RC	(SE)	RV	(SE)
Beliefs about the importance of proof									
I1	I think that theorems and proofs are the foundations of mathematics.	71%	96%	.049	.098	-.052	.080	.086	.075
I2	Proofs are very important in understanding mathematical language.	57%	96%	-.003	.116	-.013	.064	.164	.111
I3	Proofs are indispensable for mathematics.	68%	100%	-.159	.094	-.044	.042	.026	.024
I4	Making proofs improves mathematical thinking.	71%	100%	-.069	.095	.014	.025	.033	.028
I5	To me, it is important to make proofs in mathematics.	64%	100%	-.176	.090	-.128	.068	.019	.018
I6	To improve logical thinking, making proofs is necessary.	46%	93%	.034	.135	.077	.101	.276	.146
I7	In the development of reasoning, making proofs plays an important role.	64%	100%	.061	.095	.006	.087	.025	.021
D5	I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists.	39%	93%	-.015	.119	-.029	.118	.190	.124
Beliefs about relevance of proof to examinations									
T2	Knowing how to make proofs is very important in excelling in an examination/a test in school mathematics.	46%	86%	-.134	.128	.176	.111	**.328**	**.160**
T3	In an examination/a test in school mathematics, making proofs plays an important role.	46%	86%	.049	.118	.017	.129	.184	.099

T5	Practising making proofs does not help solve a task of an examination/a test in school mathematics.	50%	86%	.070	.124	.073	.098	**.291**	**.145**
Enjoyment and interest in proof									
E1	It's fun for me to make proofs.	82%	93%	-.060	.090	.058	.077	.172	.154
E2	One of the things I love about mathematics is that proofs of theorems are done in the lessons.	57%	96%	.068	.105	.134	.078	.197	.166
E4	Making proofs arouses my curiosity.	61%	93%	.004	.096	.130	.087	.018	.014
E5	I find it interesting to make proofs.[1]	70%	96%	-.025	.083	.113	.076	.009	.010
E6	I enjoy proving mathematical results.	64%	96%	.006	.092	.065	.081	.020	.013
E7	I feel enthusiastic to prove a theorem when I see it.	64%	100%	.087	.074	.036	.094	.014	.013
D3	I find it boring to make proofs.	71%	89%	-.008	.089	**.162**	**.050**	.044	.036
Anxiety about proof									
E3	I do not worry when making proofs.	57%	86%	.051	.095	.160	.096	.054	.038
D1	I feel myself under pressure when I make proofs in mathematics lessons.	32%	82%	-.036	.134	-.069	.135	**.337**	**.136**
A2	Working on a proof in front of the class frightens me.	50%	93%	-.129	.104	.061	.115	.130	.079
A4	It scares me to work on a proof.	64%	96%	.028	.091	.155	.075	.033	.027
Negative responses when failing to prove									
A1	My self-confidence diminishes when I cannot prove a theorem.	54%	96%	-.020	.087	.177	.108	.044	.027
A3	Not being able to prove upsets me.	54%	93%	-.098	.098	**.308**	**.109**	.065	.049

n = 28; **bold** when the value deviates from 0 significantly (regarding 95% C.I.); [1]n = 27 due to missing values

Apart from item-wise analysis using Svensson's method, I also analysed the test-retest responses 'dimensionally'. To do so, I first utilised the CFA solutions to estimate scores for each dimension, as well as the whole set, of each PT's responses, and computed intraclass correlation coefficients (ICCs) using a single-measurement, absolute-agreement, 2-way mixed-effects model. Since the evaluation of the stability of the revised items over time referred to a study of the extent to which the same measurement at Time 2 equalled to that at Time 1, correlations and tests of difference between two time points and other forms of ICCs were not suitable methods for assessing the stability of the items over time (Berchtold, 2016; Koo & Li, 2016). There are various ways to calculate scores to represent and quantify one's responses to a scale and its subscales (DiStefano et al., 2009). In the following, I present the methods of using the factor loadings obtained from the CFA solutions to compute weighted sum scores (Armor, 1974; Field, 2013), by assuming the item scores: **1 = Strongly Disagree**; **2 = Disagree**; **3 = Neutral**; **4 = Agree**; **5 = Strongly Agree**; if the item is negatively phrased, the scoring is reversed.

(sub)scale score = $\frac{\sum_i f_i x_i}{\sum_i f_i}$, where f_i is the factor loading and x_i is the item score

This score computation method recognised the strength (or lack of strength) for each item on the latent dimension; that is, items with higher factor loadings would have a larger effect on the score. I also used different methods (unweighted sum scores, weighted sum scores by regression) to compute the scores (DiStefano et al., 2009; Field, 2013), and hence, the corresponding ICCs to inspect if any inconsistent result existed (Cohen et al., 2011; Patton, 1999). I found that the different methods of computation yielded similar ICC results (Boateng et al., 2018), so I present the analysis of the use of computing the dimensional scores by factor loadings in the following (Table B.8).

Table B.8

ICCs using single-measurement, absolute-agreement, 2-way mixed-effects model

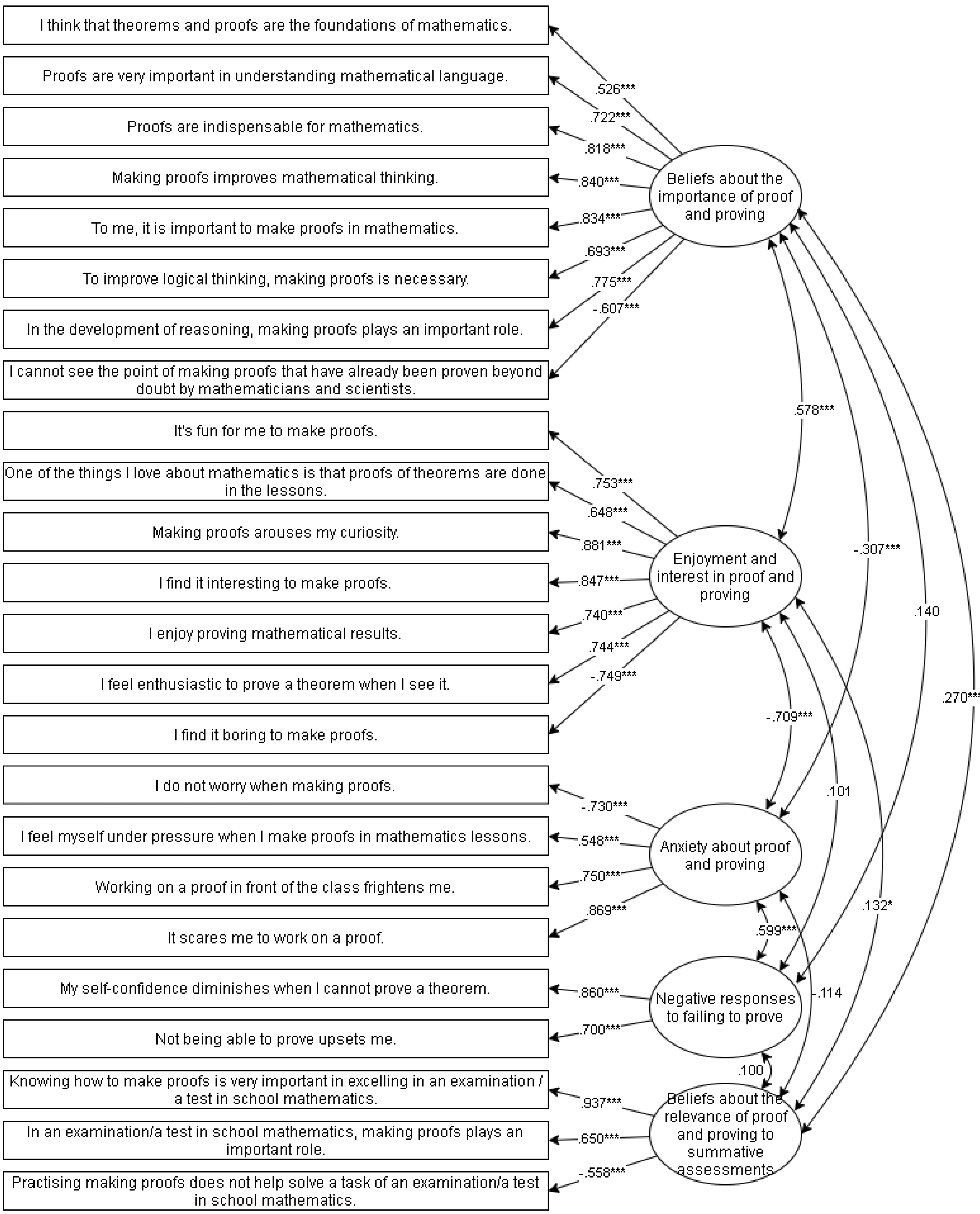
		95% C.I.		F test with true value 0			
Dimension	ICC	Lower	Upper	Value	df1	df2	sig
Importance	.545	.217	.761	3.315	27	27	.001
Assessment	.224	-.160	.548	1.568	27	27	.125
Enjoyment[1]	.790	.591	.898	8.324	26	26	< .001
Anxiety	.618	.328	.802	4.227	27	27	< .001
Failure	.725	.486	.863	6.115	27	27	< .001
Overall	.804	.618	.904	8.916	27	27	< .001

n = 28; [1]n = 27 due to missing values

The results (Table B.8) showed that ICC of the overall score was .804, and ICCs of the dimensional scores ranged from .545 to .790, indicating that the scores were “moderately” or “well” stable between two time points (Koo & Li, 2016), except for ‘Assessment’ (.224). The low ICC value might be the result of the weak item agreements and high individual variations observed, which could be because, between two time points, the respondents underwent changes in their beliefs about relevance of proof to examinations, or the items were sensitive to elements related to individual variations and time. More information would be needed to clarify any change in this dimension and its items when observed in the intervention study.

Figure B.1

The finalised measure and its factor structure



Summary. After the thorough design, revision, and evaluation, the revised Likert items have demonstrated its content and construct validity, internal consistency and stability in general, indicating its adequacy and readiness to gather information about (changes in) PTs' beliefs and attitudes towards teaching proof in school mathematics. Although in-depth evaluation revealed that some items concerning PTs' beliefs about relevance of proof to examinations lacked the stability over a month, the validity and reliability of data analysis could further be ensured by comparing the data collected from the Likert items and other sources (e.g., individual semi-structured interview) for triangulation (different data sources).

Further analysis. Ad hoc analyses of dimensional scores among PTs in different year levels and different universities were conducted to investigate whether they had similar responses. As the data collected were unbalanced (Table B.9), putting all variables (year of study, place of study) into one model might not be appropriate nor meaningful. So I first looked into the data collected from university A (Table B.10), to investigate whether there was any difference in dimensional scores across year of study, then compared the PTs in universities A, B and C (Table B.11).

Table B.9
Demographics of PTs

	University A[1]	University B	University C[2]	Total
UG-Year 1	68 (about 70%)	0 (0%)	---	68 (about 50%)
UG-Year 2	11 (about 20%)	2 (about 10%)	---	13 (about 20%)
UG-Year 3	27 (about 50%)	1 (about 5%)	---	28 (about 40%)
UG-Year 4	12 (about 20%)	3 (about 15%)	---	15 (about 20%)
UG-Year 5	8 (about 20%)	0 (0%)	---	8 (about 15%)
PG	---	36 (about 80%)	16 (100%)	52 (about 85%)
Total	126 (about 40%)	42 (about 30%)	16 (100%)	184 (about 40%)

[1]University A does not offer full-time PG programmes; [2]University C does not offer full-time UG programmes

The data collected from the PTs in University A indicate that there was no substantial difference in dimensional scores between the first- and fifth-year PTs. Among different year levels, there were some variations in scores (|value – overall| > .25): the fourth-year PTs reported stronger beliefs about the importance of proof; the third-year reported less enjoyment and interest in proof and weaker beliefs about relevance of proof to examinations, but stronger anxiety about proof. In general, the PTs in university A believed that proof is important and relevant to examinations, and expressed that they enjoyed and were interest in proof, but they did not agree nor disagree that they had anxiety about proof whilst reporting that they would have negative responses (e.g., being upset and reduced self-confidence) when they failed to prove. Further investigation (e.g., larger sample, longitudinal) is needed to confirm the outstanding values observed and examine the factors that caused such phenomenon.

Table B.10
Means of dimensional scores of PTs across year levels at university A

Year of study	*Importance*	*Assessment*	*Enjoyment*	*Anxiety*	*Failure*
UG-1 (n = 65)	3.922	3.605	3.609	2.750	3.428
UG-2 (n = 11)	3.994	3.507	3.611	2.988	3.646
UG-3 (n = 26)	3.872	**3.216**	**3.186**	**3.397**	3.598
UG-4 (n = 11)	**4.268**	3.546	3.349	3.178	3.545
UG-5 (n = 8)	3.955	3.506	3.621	2.764	3.470
Overall	3.952	3.501	3.495	2.951	3.498

n = 121; **bolded** if the value deviates from the overall value by .25.

Table B.11
Means of dimensional scores across different universities

University	*Importance*	*Assessment*	*Enjoyment*	*Anxiety*	*Failure*
A (n = 121)	3.952	3.501	3.495	2.951	3.498
B (n = 42)	4.216	3.277	3.736	2.801	3.447
C (n = 16)	4.256	3.473	3.632	2.664	3.375
Overall	4.041	3.446	3.564	2.890	3.475

n = 179; **bolded** if the value deviates from the overall value by .25.

The data indicate that there was no substantial differences in dimensional scores among different universities, whilst some variations in scores (highest – lowest > .25) can be observed: the PTs in university A tended to have slightly weaker beliefs about the importance of proof than the others did, whereas the PTs in university C tended to disagree that they had anxiety about proof. In general, the PTs believed that proof is important and relevant to examinations, and expressed that they enjoyed and were interest in proof, but they tended not to agree nor disagree that they had anxiety about proof and proving whilst reporting that they would have negative responses when they failed to prove, consistent with the data collected from the PTs in University A only.

To summarise the ad hoc analysis on the Likert data, Hong Kong PTs in different year levels and different universities did not have substantial difference in their responses to the Likert items. This provided some empirical evidence to suggest that Hong Kong PTs had somewhat similar beliefs and attitudes towards teaching proof in school mathematics across different year levels and different universities.

Table B.12

Polychoric (lower triangle) and Pearson's correlations (upper triangle) between the Likert items

	I1	I2	I3	I4	I5	I6	I7	E1	E2	E3	E4	E5	E6	E7	D1	D2	D3	D4	D5	A1	A2	A3	A4	T1	T2	T3	T4	T5
I1		.370	.382	.375	.389	.468	.354	.077	.169	-.108	.079	.084	.094	.052	-.042	-.014	.023	.038	-.155	.188	.030	.142	.059	-.210	.043	.052	-.011	-.004
I2	.499		.558	.491	.429	.364	.411	.246	.291	.042	.327	.160	.162	.204	-.037	-.182	-.172	-.023	-.343	.041	-.126	.029	-.224	-.170	.137	.042	.059	-.224
I3	.495	.687		.726	.472	.442	.511	.291	.295	.040	.377	.225	.253	.204	-.139	-.202	-.252	-.081	-.297	.127	-.168	.035	-.218	-.211	.149	.052	.071	-.219
I4	.497	.601	.810		.517	.445	.525	.352	.370	.130	.355	.267	.236	.256	-.166	-.248	-.293	-.106	-.313	.126	-.128	.003	-.252	-.206	.131	.044	.028	-.194
I5	.528	.530	.570	.626		.430	.542	.361	.355	.203	.415	.376	.347	.335	-.098	-.318	-.358	-.265	-.386	.037	-.123	.046	-.204	-.098	.172	.048	.020	-.189
I6	.598	.495	.522	.545	.515		.524	.242	.203	.100	.297	.194	.210	.278	-.100	-.133	-.133	-.112	-.241	.186	-.163	.093	-.116	-.198	.328	.233	.009	-.130
I7	.484	.514	.627	.656	.636	.662		.296	.306	.068	.406	.198	.242	.229	-.043	-.216	-.258	-.212	-.386	.048	-.113	-.004	-.208	-.148	.078	-.023	.100	-.220
E1	.119	.316	.339	.412	.420	.256	.352		.695	.353	.586	.559	.330	.506	-.110	-.452	-.452	-.342	-.247	-.076	-.312	-.007	-.319	.004	.117	.060	.098	-.181
E2	.211	.380	.275	.458	.427	.248	.379	.796		.338	.514	.432	.293	.443	-.054	-.349	-.261	-.294	-.217	-.008	-.191	.087	-.205	-.043	.119	.023	.171	-.066
E3	-.108	.060	.055	.166	.250	.117	.097	.401	.382		.490	.444	.345	.486	-.408	-.324	-.344	-.385	-.175	-.322	-.389	-.257	-.435	.041	.142	-.009	-.037	.053
E4	.164	.423	.497	.447	.495	.365	.489	.669	.584	.552		.620	.463	.513	-.167	-.423	-.587	-.398	-.336	-.155	-.347	-.037	-.404	-.010	.132	.005	.090	-.186
E5	.156	.249	.332	.348	.467	.241	.255	.652	.508	.518	.725		.669	.641	-.302	-.573	-.608	-.453	-.240	-.064	-.386	-.014	-.446	-.085	.072	.025	-.032	-.097
E6	.178	.259	.357	.337	.470	.287	.340	.425	.364	.429	.577	.746		.558	-.328	-.449	-.492	-.494	-.259	-.031	-.306	-.102	-.408	-.204	-.028	-.05	-.045	-.079
E7	.071	.255	.247	.294	.391	.327	.274	.578	.489	.545	.592	.734	.648		-.241	-.458	-.451	-.439	-.238	-.031	-.389	-.042	-.398	-.078	.028	-.014	.023	-.03
D1	-.046	-.044	-.147	-.170	-.111	-.099	-.037	-.121	-.044	-.464	-.215	-.344	-.388	-.286		.449	.326	.31	.16	.275	.451	.203	.462	.132	-.077	-.041	.146	-.002
D2	-.059	-.237	-.239	-.297	-.378	-.141	-.264	-.517	-.391	-.377	-.490	-.663	-.533	-.517	.521		.583	.531	.358	.16	.447	.118	.538	.089	-.008	.023	.126	.148
D3	-.033	-.237	-.343	-.378	-.442	-.154	-.307	-.521	-.292	-.378	-.662	-.685	-.564	-.491	.368	.653		.491	.419	.138	.451	.121	.58	.179	-.012	.048	.107	.24
D4	.015	-.052	-.114	-.152	-.310	-.127	-.257	-.417	-.349	-.446	-.471	-.526	-.574	-.498	.350	.601	.530		.424	.185	.49	.246	.572	-.034	.107	.142	.145	.089
D5	-.259	-.468	-.396	-.411	-.476	-.321	-.461	-.309	-.269	-.198	-.397	-.263	-.303	-.260	.170	.407	.478	.459		.142	.333	.162	.481	.167	-.032	.05	.044	.263
A1	.216	.062	.160	.170	.052	.220	.073	-.098	-.018	-.380	-.188	-.081	-.042	-.051	.326	.189	.157	.224	.156		.286	.53	.35	-.034	.053	.067	.031	-.052
A2	.011	-.160	-.207	-.166	-.158	-.187	-.141	-.370	-.216	-.463	-.424	-.463	-.352	-.438	.510	.501	.504	.556	.387	.333		.308	.616	.17	-.125	-.074	.039	.097
A3	.184	.071	.075	.052	.085	.129	.026	.013	.129	-.300	-.021	-.003	-.102	-.031	.244	.119	.113	.257	.158	.602	.345		.345	.078	.059	.006	.163	-.025
A4	.049	-.269	-.277	-.309	-.245	-.129	-.248	-.382	-.233	-.508	-.478	-.522	-.471	-.453	.538	.622	.642	.640	.529	.412	.699	.395		.185	-.035	.065	.131	.137
T1	-.302	-.248	-.304	-.305	-.143	-.298	-.234	.030	-.049	.027	-.010	-.083	-.215	-.082	.126	.089	.185	-.042	.215	-.080	.176	.066	.198		-.142	-.054	.103	.15
T2	.071	.200	.189	.139	.192	.370	.094	.133	.139	.169	.178	.097	-.012	.032	-.076	-.015	-.031	.120	-.028	.060	-.145	.074	-.048	-.143		.759	-.072	-.284
T3	.100	.088	.068	.036	.063	.273	-.017	.068	.013	-.019	.010	.028	-.058	-.019	-.033	.032	.047	.172	.057	.088	-.087	.013	.064	-.054	.831		-.077	-.219
T4	.008	.099	.118	.067	.034	.040	.139	.127	.203	-.044	.110	-.012	-.041	.032	.180	.141	.119	.153	.030	.048	.034	.211	.143	.081	-.054	-.058		.177
T5	-.047	-.271	-.266	-.212	-.216	-.158	-.261	-.213	-.077	.059	-.211	-.113	-.111	-.027	-.006	.169	.264	.086	.308	-.067	.100	-.035	.151	.182	-.324	-.278	.190	

n = 179 due to listwise deletion of missing values

Appendix C Protocol for individual semi-structured interview

Materials: clean white paper, several pens, completed questionnaire, two audio-recorders.

1. Discuss the conduct of the interviews and obtain oral consent (5 minutes).
2. Background (5 minutes):

    a. What do you want to achieve in teaching mathematics?

    b. [If past experiences in mathematics are mentioned] Would you describe your past experiences? How did you learn mathematics? How did your teachers teach it?

3. Proof and proving in school mathematics (15 minutes):

    a. What are the roles of proof and proving in school mathematics?

    b. What is a better way of the learning and teaching of proof and proving?

    c. What are the possible challenges of the learning and teaching of proof and proving? How can these challenges be resolved?

4. Written responses to the questionnaire (20 minutes):

    a. Would you explain your responses to the questionnaire?

    b. [If the questionnaire was completed twice] What are the differences between your responses to the questionnaire before and after the intervention?

5. Intervention (15 minutes):

    a. What was your expectation before attending [the intervention]?

    b. What did you experience and learn in [the intervention]?

    c. From your point of view, what were the objectives of [the intervention]?

    d. Do you have any feedback about [the intervention]?

Appendix D The development of coding schemes for analysing the questionnaire

I expand on Tables 3.4.1–3.4.4 in Chapter 3 and describe the development of the coding schemes for analysing PTs' responses to the open-ended questions in the questionnaire (Section 3.4.1) in order to capture information of PTs' beliefs about the teaching of proof. I first discuss the relevant theories and research that justify the basis for the codes, then describe how the coding schemes were piloted and revised and the comparison between the data collected from administering the open-ended questions and the Likert items to a sample of Hong Kong PTs.

The coding schemes. In analysing the data collected from the open-ended questions in the questionnaire, what of interest were whether, how and why PTs believe that proof is (or not) important and should (or not) be included in school mathematics instruction. The first and second target variables concerned PTs' intended decisions about whether and how proof should appear in their future teaching. The first concerned PTs' beliefs about whether and how frequent proof should appear in school mathematics. Whilst it is suggested that proof is essential for learning mathematics and should be made central in school mathematics (Hanna, 1997; National Council of Teachers of Mathematics, 2000; Stylianou et al., 2009), previous research indicates that some practising teachers and PTs rejected the need for proof and some believed proof and proving were only needed for a certain group of students and should be treated as a supplement to school mathematics (Ersen, 2016; Knuth, 2002b; I. K.-C. Leung & Lee, 2017; Varghese, 2009).

***Beliefs about the place of proof in school mathematics*.** Based on previous research, five distinct codes were defined (Table 3.4.1). The first code (**No place**) concerned the rejection of teaching proof. PTs' responses were coded into this when showing they tended to reject the teaching of proof in classroom instruction.

The second code (**Hesitating**) concerned PTs' hesitation in teaching proof. PTs could be confused about the need for teaching proof because, on the one hand, they agree that proof could benefit students, but, on the other hand, they hesitate to teach proof or skip proof due to opposite reasons (Huang & Leung, 2004; C.-Y. Lee, 2019; Yan, 2020). PTs' responses were coded into this when showing they could not decide whether proof should be included in school mathematics.

The third code (**Unclear**) was used when PTs' responses show they agreed that proof should be included in school mathematics but how they would place proof in their teaching was not clearly stated.

The four code (**Supplementary**) concerned PTs merely considering proof to be a supplementary material for learning mathematics. PTs' responses were coded into this when showing they agreed on the need for teaching proof but posited that proof is not appropriate or necessary for all students to learn and should be taught to only certain groups of students in spare time.

The fifth code (**Regular**) concerned PTs seeing that proof should permeate school mathematics at different grade levels and across different topics and students should be provided with opportunities of proof regularly. PTs' responses were coded into this when showing they agree that proof should be included in regular classroom instruction.

Beliefs about approaches for teaching mathematics and proof. The second target variable concerned PTs' beliefs about the models of how they teach mathematics and proof. Apart from their beliefs about the place of proof in school mathematics, PTs' ideas about how mathematics and proof should be taught and whether and how the ideas would change after the intervention were of interest in the present study. To guide my investigation on this component of PTs' beliefs and attitudes towards teaching proof in school mathematics, I adapted Yoo's (2008) framework of beliefs about teaching mathematics and proof. Based on ideas from Ernest (1989), Hersh (1993) and other researchers, Yoo (2008) proposed two contrasting models of beliefs about teaching mathematics and proof: product-oriented and process-oriented. According to Yoo (2008), product-oriented teachers consider the learning and teaching of mathematics as the transmission of mathematical facts, skills and formal proofs and value teacher-centred and content-focused teaching, that mathematics and proof should be learnt by rote practice and following teacher's demonstration, whereas process-oriented teachers consider the learning and teaching of mathematics as a construction process of students' own cognitive acts and emphasise the importance of engaging students in learning activities of collaborative work and discussion on exploring mathematical knowledge and proof. In the present study, the two models are considered to be two contrasting extremes of a spectrum of beliefs about approaches for teaching mathematics and proof, and it is possible for a PT to hold a combination of product-oriented and process-oriented beliefs.

Yoo's models guided my analysis on PTs' beliefs about approaches for teaching mathematics and proof. Although the centrality of PTs' beliefs may vary with contexts (Beswick, 2018; Furinghetti & Morselli, 2011), what of interest in the present study was PTs' overall adherence to the models. I defined three distinct codes (Table 3.4.2). The first code (**Product-oriented**) was used when PTs' responses showed they tended to value teacher-centred, content-focused teaching, emphasise reception and mastery of mathematical concepts, skills, proofs and their relation, and aim at explaining and transmitting mathematical knowledge and procedures to students when teaching. Here, the phrase 'tended to' was taken into account during the coding process because PTs could be aware of some characteristics of the process-oriented model but express in their responses that they preferred the product-oriented model when teaching.

The second code (**Mixed**) concerned a ‘balanced’ view of product-oriented and process-oriented models. PTs’ responses were coded into this when showing they did not have a strong preference for either model but a demonstration of a balanced view of the two models.

The third code (**Process-oriented**) was used when PTs’ responses showed they tended to value student-centred, construction-focused teaching, emphasise students’ engagement in problem-solving or proving process and the communication of the understanding of the results, and aim at guiding students to construct mathematical knowledge and understanding. Similar to the first code, the phrase ‘tended to’ was taken in account during the coding process.

The third and fourth target variables concerned the reasons behind PTs’ decisions about whether and how proof should (or not) be included in their future teaching. The third included PTs’ reasons for including proof in their teaching (Table 3.4.3), whereas the fourth included the reasons why they rejected proof or hesitated to make proof at the centre of school mathematics, and that influenced their approaches for teaching mathematics and proof (Table 3.4.4). As it is possible that PTs found the use of proof in school mathematics unnecessary or tended to reject it even if they recognised how students might benefit from proof (Doruk & Güler, 2014; Ersen, 2016; Lesseig et al., 2019; Varghese, 2009), I focused on PTs’ reasons behind the use or rejection of proof, rather than their awareness and knowledge of the roles, challenges and constraints on implementing proof in their future teaching. In other words, in this dimension, only statements of responses that indicated the relevance of PTs’ knowledge of the roles, challenges and constraints on the use of proof and proving to their ideas about whether and how proof should be included in their teaching were coded. The pre-defined codes were based on previous research (e.g., Knuth, 2002a, b; Lesseig et al., 2019). There were seven codes for the third target variable and four codes for the fourth.

***Reasons for implementing proof in school mathematics*.** The first code (**Justification**) concerned the justification power of proof. In research, proof is used to remove uncertainty of mathematical claims. In school mathematics, proof can be used to convince students of the (in)correctness of a mathematical claim (G. J. Stylianides, 2009; Varghese, 2009; Zaslavsky et al., 2012).

The second code (**Promotion of understanding**) concerned the explanatory power of proof. Hong Kong teachers and students often relate the understanding of mathematics to the knowledge of underlying principles and the clarification of mathematical ideas (N.-Y. Wong, 2002, 2007), which are relevant to the explanatory power of proof. Therefore, proof can be used to provide insight into why a mathematical claim is valid, thereby satisfying students' desire to ascertain the rationale for the claim (Hanna, 1990; Lesseig et al., 2019; Zaslavsky et al., 2012).

The third code (**Memorisation**) concerned the idea that proof is used for systematising mathematical knowledge and making connections to ease memorisation. Apart from conceptual understanding, Hong Kong teachers often value memorisation of mathematical knowledge (e.g., formulae) in learning mathematics (N.-Y. Wong, 2007). In research, proof is needed for structuring the mathematical knowledge into a deductive system of axioms, definitions and theorems developed by a mathematical community (e.g., the community of mathematicians) (de Villiers, 1990). In relation to the learning and teaching of mathematics, proof can help students (re)organise information into a logical structure (Zaslavsky et al., 2012). According to the data collected from Research Cycles 1–3, many PTs believed proof helps students memorise mathematical ideas and “re”-deduce a formula by themselves if the exact formula was not remembered.

The fourth code (**Generation of knowledge**) concerned the idea that proof is used for generating mathematical knowledge. Through exploration and analysis, proof can lead to discovery or invention of mathematical knowledge that is new to the community (de Villiers, 1990). According to previous research (e.g., Lesseig et al., 2019) and the data collected from Research Cycles 1–3, some PTs believed that proof enables students to generate their own mathematical knowledge.

The fifth code (**Development of reasoning**) concerned the idea that proof is used for developing students’ thinking skills. Many teachers and students agree that mathematics involves problem solving, logical thinking and reasoning (Frasier & Panasuk, 2013; N.-Y. Wong, 2002, 2007). As one utilises mathematical skills and ability to solve problems, reason and think logically in proof (Rips, 1994; K. Weber, 2005), many PTs relate the learning and teaching of proof to the development of students’ ability and skills in problem solving, logical thinking and reasoning (Lesseig et al., 2019; I. K.-C. Leung & Lee, 2017; Varghese, 2009).

The sixth code (**Promotion of students’ motivation**) concerned the idea that proof is used for developing students’ motivation and interest in mathematics. As proof is seen as one important element of mathematical pursuit (e.g., research) (Siu, 2008), some PTs believed that the learning and teaching of proof can be humanistic to serve as a means of developing students’ motivation and interest in mathematics (Lesseig et al., 2019).

The seventh code (**Self affection**) concerned the idea that proof is used because of teachers’ affection for proof. Teachers’ affection for proof is linked to their beliefs and practices related to the teaching of proof (e.g., time spent on proof in teaching) (Frasier, 2010; Kotelawala, 2016; Wilkins, 2008). It is possible that PTs posit the teaching of proof because they, as a teacher, enjoy proof and appreciate the values of proof in advanced mathematics and research.

***Concerns about the use of proof in school mathematics*.** The first code (**Student**) concerned the idea that students are not ready for learning proof. According to previous research, some practising teachers and PTs hesitated to include proof in their teaching, particularly in junior school mathematics, because they believed that proof is difficult and their (junior) students do not have the requisite mathematical knowledge and skills or interest in proof (Frasier & Panasuk, 2013; C.-Y. Lee, 2019; Lesseig et al., 2019; Varghese, 2009).

The second code (**Self**) concerned the idea that PTs worry they are not ready for teaching proof. According to previous research, PTs often have inadequate knowledge for proof and its teaching (e.g., I. K. C. Leung et al., 2013; Schwarz et al., 2008; G. J. Stylianides et al., 2007). Such inadequate knowledge may result in their anxiety and struggle of proof, and hence, poor self-confidence in using proof in their teaching (Creager, 2016; Doruk & Güler, 2014; Kotelawala, 2016; Lesseig et al., 2019).

The third code (**Curriculum**) concerned the idea that PTs do not believe proof is relevant to school mathematics curriculum and examinations. Apart from students' capacity, school mathematics curriculum (e.g., state standard, school curriculum and policy) and examinations are other major factors that discourage teachers from including proof in their teaching (Frasier & Panasuk, 2013; Lesseig et al., 2019; Nyaumwe & Buzuzi, 2007).

The fourth code (**Time**) concerned the idea that the use of proof is seen to be time-demanding. As the notion of proof and its learning and teaching often accompany by the notion of complexity and difficulty (see also Section 2.1), the teaching of proof is thought to take too much time (Lesseig et al., 2019; Nyaumwe & Buzuzi, 2007), underlying an idea that the benefits of teaching proof do not offset the time cost.

The development of the coding schemes. The coding schemes presented in Tables 3.4.1–3.4.4 in Chapter 3 are the final version, used in the present study. In an initial version of the schemes, there were fewer codes; I piloted the initial version, with the data collected from administering the two open-ended questions (Appendix A) in Research Cycle 3, and discovered and added more codes as my analysis advanced. I compared the initial and final versions, and describe the evolution of the coding schemes in the following.

Initially, the first target variable, beliefs about the place of proof in school mathematics (Table 3.4.1), had four codes only: **No place**, **Hesitating**, **Supplementary** and **Regular**. The code, **Unclear**, was added after observing some PTs agreed the need for proof in school mathematics, but did not state whether they would make proof regular and central to their teaching or consider proof as a supplementary material and/or removable from school mathematics.

In the initial version, the coding scheme for the fourth variable, concerns about the use of proof in school mathematics (Table 3.4.4), did not distinguish between PTs' reasons that influenced their decisions about the place of proof in school mathematics and reasons that influenced their decisions about approaches for teaching mathematics and proof. When analysing the data, I discovered that some PTs valued the importance of proof and the process-oriented model of teaching mathematics and proof, but they hesitated to apply the process-oriented model and preferred to follow the product-oriented model, because of reasons corresponding to the codes of Table 3.4.4. Therefore, I introduced the notions of *reasons behind the place of proof* and *reasons behind approaches for teaching mathematics and proof* to gather information about this occurrence.

In Research Cycle 3, I examined to what degree I had any subjectivity when applying the coding schemes to the analysis on the data collected from the open-ended questions and whether the coding schemes could exhaust the data. I invited a second coder to test whether another observer who uses the same coding schemes would interpret the same data in the same way (Cohen et al., 2011). She was given a copy of Tables 3.4.1–3.4.4 and received basic instruction and training about the coding process. The instruction and training that she received were basic in order to minimise any potential effects of my bias on her (e.g., influence on her coding decision) but to ensure her understanding of the definition of each code and the coding process. The second coder was asked to code the data collected from administering the two open-ended questions to PTs in Research Cycle 3, and was also reminded to create new codes if she discovered meaningful data that did not fit into any existing codes.

I compared the data coded by the second coder with mine, to examine the inter-rater reliability by percentage agreements. The percentage agreements for the codes ranged from 85% (**Promotion of understanding**) to 100% (**No place**), 94.3% on average. The results helped to finalise the coding schemes, that the code **Unclear** was created and added into the first target variable to exhaust the data, and also suggested that the schemes guided an adequate coding process and my researcher bias in the coding process was limited.

Comparing the data collected from the open-ended questions and the Likert items. I compared the data collected from administering the open-ended questions and the Likert items to 184 Hong Kong PTs to examine the relationships between their decisions about the place of proof in school mathematics and their intended approaches for teaching mathematics and proof, and the five dimensions of attitudes towards teaching proof in school mathematics (Appendix B). I grouped the PTs according to the codes of Tables 3.4.1–3.4.2 and compared the means of their dimensional scores accordingly (Tables D.1–2).

Table D.1
Relationship between beliefs about the place of proof in school mathematics and dimensional scores of the Likert items

Code	Frequency	*Importance*[1]	*Assessment*[1]	*Enjoyment*[1]	*Anxiety*[1]	*Failing*[1]
No place	8	3.62	3.20	2.75	3.26	3.56
Hesitating	14	3.99	2.82	3.64	3.08	3.36
Unclear	17	4.18	3.67	3.44	2.93	3.20
Supplementary	27	4.16	3.26	3.61	2.77	3.37
Regular	66	4.07	3.62	3.70	2.80	3.57

n = 132, due to missing values; [1]the five latent dimensions of the Likert items (Appendix B)

I compared the dimensional scores of the PTs who reported that they would place proof in their teaching differently. The PTs who rejected the teaching of proof in school mathematics (n = 8, mean = 3.62, s.d. = .38) tended to have lower scores on the items concerning their beliefs about the importance of proof than those who agreed to include proof in school mathematics (n = 110, mean = 4.11, s.d. = .49; $t(116) = 2.77$, $p < .01$). The former (n = 8, mean = 2.75, s.d. = .64) also tended to have lower scores on the items concerning their enjoyment and interest in proof than the latter (n = 108, mean = 3.64, s.d. = .64; $t(113) = 3.533$, $p < .01$). The PTs who rejected the teaching of proof or expressed their hesitation in teaching proof in school mathematics (n = 22, mean = 3.15, s.d. = .73) tended to have higher scores on the items concerning their anxiety about proof than those who agreed with the teaching of proof (n = 109, mean = 2.81, s.d. = .72; $t(129) = 2.00$, $p < .05$). Moreover, the PTs who reported that they would use proof in their teaching regularly (n = 66, mean = 3.63, s.d. = .69) tended to have higher scores on the items concerning their beliefs about relevance of proof to examinations than those who considered proof supplement to school mathematics (n = 27, mean = 3.26, s.d. = .78; $t(91) = 2.20$, $p < .05$), and those who rejected the teaching of proof or expressed their hesitation in teaching proof (n = 22, mean = 2.96, s.d. = .73; $t(78) = 3.849$, $p < .001$).

Table D.2
Relationship between beliefs about approaches for teaching mathematics and proof and dimensional scores of the Likert items

Code	Frequency	*Importance*[1]	*Assessment*[1]	*Enjoyment*[1]	*Anxiety*[1]	*Failing*[1]
Product-oriented	50	4.04	3.44	3.58	2.86	3.63
Mixed	13	4.09	3.47	3.64	2.96	3.51
Process-oriented	26	4.08	3.43	3.91	2.64	3.25

n = 89, due to missing values; [1]the five latent dimensions of the Likert items (Appendix B)

I compared the dimensional scores of the PTs who reported they would have different approaches for teaching mathematics and proof. The PTs who expressed that they preferred process-oriented approaches (n = 26, mean = 3.91, s.d. = .58) tended to have higher scores on the items concerning their enjoyment and interest in proof and proving than those who reported that they preferred product-oriented approaches (n = 49, mean = 3.58, s.d. = .68; $t(73) = 2.14$, $p < .05$). The former (n = 25, mean = 3.25, s.d. = .71) also tended to have lower scores on the items concerning their negative responses when failing to prove than the latter (n = 50, mean = 3.63, s.d. = 71; $t(73) = 2.17$, $p < .05$).

In summary, there seemed to have connections between the codes of Tables 3.4.1–3.4.2 and the dimensional scores of the Likert items. Particularly, the above results suggest that PTs' beliefs about the importance of proof, beliefs about relevance of proof to examinations, enjoyment and interest in proof, and anxiety about proof have the potential to influence their intended decisions about the place of proof in school mathematics, and their enjoyment and interest in proof, and negative responses to failing to prove have the potential to influence their intended decisions about approaches for teaching mathematics and proof.

Appendix E Supplementary information of the data collected in the present study

A completed questionnaire (identifier removed)

Attitude Scale 態度量表

For the following items, tick the box that best describes your opinion.
於下列各項中，選出描述你的意見，並勾出最適合的空格。

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree
1. I feel myself under pressure when I make proofs in mathematics lessons. 當我在課堂中建立證明的時候，我感到壓力。		✓			
2. I do not like dealing with mathematical proofs. 我不喜歡處理數學證明。					✓
3. It's fun for me to make proofs. 建立證明是有趣的。	✓				
4. One of the things I like about mathematics is the proofs of theorems in the lessons. 我喜歡數學的其中一個原因是課堂中定理的證明。	✓				
5. I do not worry when making proofs. 當建立證明的時候，我並不憂慮。				✓	
6. Making proofs arouses my curiosity. 建立證明引起我的好奇心。		✓			
7. I find it boring to make proofs. 我認為建立證明是沉悶的。				✓	
8. I think that theorems and proofs are the foundations of mathematics. 我認為定理和證明是數學的根基。	✓				
9. My self-confidence diminishes when I cannot prove a theorem. 當我未能證明一個定理的時候，我的自信減少。		✓			
10. I find it interesting to make proofs. 我認為建立證明是有趣的。		✓			
11. Proofs are very important in understanding mathematical language. 證明對於了解數學語言十分重要。	✓				
12. Proofs are indispensable for mathematics. 證明於數學中是必不可少的。	✓				
13. Making proofs improves mathematical thinking. 建立證明有助改善數學思維。	✓				
14. To me, it is important to make proofs in mathematics. 於我而言，建立證明對於數學十分重要。	✓				
15. Working on a proof in front of the class frightens me. 於班中建立一個證明使我害怕。			✓		
16. I enjoy proving mathematical results. 我享受證明數學結果。		✓			
17. To improve logical thinking, making proofs is necessary. 為了改善邏輯思維，建立證明是必須的。		✓			
18. I feel enthusiastic to prove a theorem when I see it. 對於要去證明定理，我感到雀躍。			✓		
19. I love mathematics, but I do not like making proofs. 我鍾愛數學，但我不喜歡建立證明。				✓	
20. In the development of deductive reasoning, making proofs plays an important role. 在發展推理思維時，建立證明扮演著一個重要的角色。	✓				

	Strongly agree	Agree	Neutral	Disagree	Strongly disagree
21. I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists. 我無法理解為何需要證明一些已被數學家及科學家證實的事。					✓
22. Not being able to prove upsets me. 未能證明使我煩亂。		✓			
23. It scares me to work on a proof. 建立一個證明使我恐懼。				✓	
24. Logical thinking is not necessary for working on an examination/a test in school mathematics. 學校數學的考試/測驗並不需要邏輯思維。					✓
25. Knowing how to make proofs is very important in excelling an examination/a test in school mathematics. 對於於學校數學的考試/測驗獲取好成績，懂得如何建立證明十分重要。			✓		
26. In an examination/a test in school mathematics, making proofs plays an important role. 在學校數學的考試/測驗中，建立證明扮演著一個重要的角色。		✓			
27. During an examination/a test in school mathematics, how well to recall the procedures of solving a task is more important than how well to make a proof. 於學校數學的考試/測驗，回想解題的步驟的能力比建立證明的能力更重要。				✓	
28. Practising making proofs does not help solve a task of an examination/a test in school mathematics. 練習建立證明無助於解答學校數學中考試/測驗的題目。					✓

1. Should we implement tasks of proof and proving into mathematics class? Please briefly **explain**.
   老師應否於數學課堂中引入數學證明活動? 請簡單**說明**。

應該。雖然最後應試可能直接用公式就可以,但證明途中的思維和想法很有助思路的應建立。

Yes. Although it would be enough [for students] to apply formulae to examination eventually, reasoning and thoughts involved in the process of proving help [students] develop ways of thinking.

2. If we implement proof and proving in class, (a) **when** and (b) **how** should we? Please briefly **explain**.
   若要於數學課堂中引入數學證明活動，(a) **何時**及 (b) **如何** 進行? 請簡單**說明**。

a) 剛開始講解新概念的時候

b) 展示步驟

a) At the beginning of lecturing new concepts.
b) Demonstrate procedures

Context: Two junior secondary (Form 2; i.e., 13–14 years old) mathematics teachers at one school introduced the identity "$a^2 - b^2 = (a - b)(a + b)$" for factorisation of polynomials. Below are the teaching flows by Teacher A and Teacher B. Please read each scenario and respond to the questions at the end. The line numbers for each scenario can help to make references to the text.
處境：兩位初中（中二；13 至 14 歲）數學老師於同一學校任教，他們於課堂中引入恆等式「$a^2 - b^2 = (a - b)(a + b)$」以用於多項式的因式分解。以下為教師 A 及教師 B 的教學流程。請閱讀各情境並回答及後的問題。各情境的行號可協助引文。

Scenario involving Teacher A 教師 A 的情境 $(x-a)(x+b)(x-c)$ $x^3+x^2+x = x(x^2+x+1)$

1. Teacher A recalled the previous lessons of "expansion of the product of polynomials" and "factorisation using extraction of common factors" to students and assessed their abilities to expand some simple products of polynomials and to use extraction of common factors accurately.
2. The teacher wrote down "$a^2 - b^2 = (a - b)(a + b)$", and asked students to verify the truth of the statement by expanding "$(a - b)(a + b)$" into "$a^2 - b^2$".
3. When the identity was verified, the teacher told students about the use of this identity (e.g. for computation of $49*51 = 50^2 - 1$, mathematical results that make use of this identity).
4. The teacher continued by examples of using this identity to factorise some pre-set polynomials, emphasising the common mistakes that students may make in assignments/tests/examinations.
5. The teacher then asked students to work on pre-set practice questions; in case students made mistakes on answering the questions, the teacher pointed out their misconceptions and explained the correct procedures for answering the questions.
6. After assigning homework questions, the teacher ended the lesson.
7. On the next day, after ensuring that students could apply the identity to factorise some polynomials, the teacher asked students to solve more complicated problems adapted from tests/examinations that require students to use this identity.
8. The teacher assigned some students to demonstrate their solutions on the board. After evaluating students' solutions and reminding students to their common mistakes, the teacher ended the lesson.

49 = 50 − 1
51 = 50 + 1

Scenario involving Teacher B 教師 B 的情境

1. Teacher B also recalled the previous lessons of "expansion of the product of polynomials" and "factorisation using extraction of common factors" to students. 追問
2. The teacher asked students to compute the values of 49*51, 48*52, 47*53 and so on. Through probing, the teacher guided students to identify the pattern of the answers 2499, 2496, 2491 is $50^2 - x^2$ where x is 1, 2, 3, and so on. → 2500−1 2500−4 2500−9 →
3. After identifying the pattern of 49*51, 48*52, 47*53 is $(50 - x)(50 + x)$ where x is 1, 2, 3, and so on, the teacher asked students to expand the expression to arrive $50^2 - x^2$; based upon students' work, the teacher wrote down "$(50 - x)(50 + x)$ = (students' work) = $50^2 - x^2$" and initiated a discussion about the meaning of "=" sign.
4. The teacher further initiated a discussion about the possible outcomes of replacing 50 and x by other variables/numbers; the teacher guided students to formulate the statement "$a^2 - b^2 = (a - b)(a + b)$".
5. Students discussed how to verify and make sense with this statement.
6. Based upon students' contributions, the teacher compared different methods of verifying this statement and made connections among the methods; after assigning homework questions, the teacher ended the lesson.
7. On the next day, after responding to students' common misconceptions and questions about homework questions, the teacher asked students to solve the problem "for a rectangle with a perimeter of 100 cm, what is the maximum area?" in groups. ?
8. The teacher guided students to discuss their solutions, and formulate and verify the statement "given a rectangle with perimeter x, its possible maximum area is $x^2/16$". After summarising students' contributions, the teacher ended the lesson.

Questions 問題

1. **Suggest** TWO major similarities between Teacher A and Teacher B.
寫出老師 A 及老師 B 之間的兩項相似之處。

The 1st major similarity: 第一項 Both of them recalled the previous lessons at the beginning of introducing some new concepts.

The 2nd major similarity: 第二項 Both of them let their students know what their common misconceptions.

2. **Suggest** TWO major differences between Teacher A and Teacher B.
寫出老師 A 及老師 B 之間的兩項相異之處。

The 1st major difference: Teacher B raises questions to let the students think while teacher A provides the answers to them directly
第一項

The 2nd major difference: Teacher B shows that why we need the "new" formula (for easier thinking / faster manipulation). Under the teaching of A, students may only know that they learn that because they will be tested in tests / exams.
第二項

3. **Suggest** and briefly **explain** TWO teaching approaches in the scenarios that you find the most important for student learning.
寫出並簡單**說明**兩項最重要的教學活動。? I think people may easily lose attention / patient when they think sth is not important to them.

The 1st most important: Identify the purpose of doing something. Answering
第一項 the question why we can think in that way.

The 2nd most important: Raise the interest of students so that they can
第二項 develop the ability to think and search for the answers, which can be considered as active learning.

4. **Suggest** and briefly **explain** TWO teaching approaches in the scenarios that you would remove if you do not have time to include all teaching approaches.
寫出並簡單**說明**若教學時間不足時你會移除的兩項教學活動。

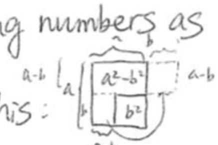
The 1st to remove: Practicing the exercises given by the book. I believe that
第一項 if the concept is clear and students understand when they are listening, they can try to do exercises at home.

The 2nd to remove:
第二項 Probably the proof. I think that some proof can be complicated and need some more knowledge to understand it thoroughly, which may scare the students.

5. If you are preparing for the same topic, introduction of the identity "$a^2 - b^2 = (a - b)(a + b)$" for factorisation of polynomials, **describe** and briefly **explain** your teaching approach.
你正在準備此課題的教學，試**描述**及簡單**說明**你的教學設計。

(Actually I have not much thoughts as it seems my math teachers in secondary school are type A, I will try my best to try what will be better.

1. States out what is the ~~new~~ tends to be topic that is going to discuss. (Write the formula on board)
2. Explain why we may need that. verify by using numbers as examples. (similar of 3 in B)
3. Using unknowns to prove, maybe something like this:



4. Apply the formula in different cases.
5. Let the students to do their own and check for the errors.

6. Which of the two teachers in the scenarios would you **identify** more (0 = mostly Teacher A; 100 = mostly Teacher B)? Please briefly **explain**.
若 0 代表最接近老師 A 及 100 代表最接近老師 B，你會如何**自我評價**? 請簡單**說明**。→ I will try my best to learn more before becoming a teacher!

70%. My knowledge does not support me to become so informative as B. B raises questions, connecting the past experiences with the new one, and explain something that ~~we~~ I may think obvious (such as the meaning of equal sign).

Thank you for giving your time to complete this survey!
感謝你抽空完成問卷。

6

A transcribed excerpt of a semi-structured interview

Date: 30/10/2019; Time: 12:00; Length: 1h55m30s

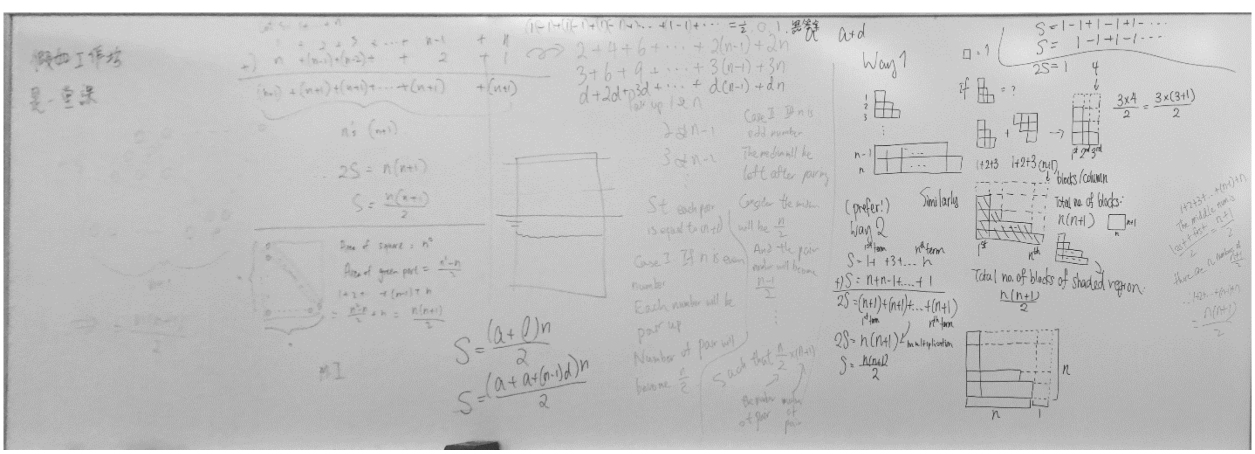
Time	Cantonese transcript	English translation
Interviewer		
Interviewee		
00:14:26	咁對於你嚟講, 諗返起以前嘅上堂, 又或者你將來你去到, 你話有衷境嘛, 去到將來喺實習嘅時候, 你嘅課堂…其實證明佢係有咩功能或者係作用呢? 喺你呢啲課堂入面.	After learning more about your thoughts and learning experiences, [my] second question is about our workshops and questionnaire, as you know, which are about proof and its relevance. The topic is about proof. To you, in the lessons in the past or your expected lessons in the future during teaching practicum, what is/are the function(s) or role(s) of proof?
00:15:49	我自己都係最主要覺得佢會話你知果樣嘢佢係啱, 或者點嚟囉 有時有啲嘢會好無原無故咁寫條公式出嚟, 咁你唔係好知道佢點用, 或者佢有咩用, 人哋為乜嘢要寫呢條嘢出嚟, 果個意思我會覺得係, 喺證明度講到囉	I, myself, mainly think it (proof) tells you that thing (a mathematical claim) is correct/true or how it comes. There is sometimes a formula written from nowhere, that you are not sure how to apply it (the formula), or about its application, why someone wrote (developed) this [formula]. This idea, I think, can be explained in proof.
	係, 即係你所講嘅點嚟其實係講緊, 某一個數學工具佢嘅起源? [嗯] 同個目的 [嗯] 咁就會涉及到嘅會係一啲歷史問題㗎喇啵? 唔係單單 [係] 證明喎, 會唔會? [嗯] 即係主要, 頭先講緊嘅就係, 果個數學嘅工具啱定唔啱? 用唔用得? 第二樣嘢就會係點解需要有呢個數學工具. 除此之外, 仲有無其他證明嘅功用畀你係會諗得到?	Yes. When you said "how it comes", you mean the origin of some mathematical tool? [Yes.] And its aim. [Yes.] Then it is related to some issues about history? It's not merely about proof. [Yes.] So, to summarise what you just said, [the first role of proof is for verifying] whether that mathematical tool is true or not true? Applicable or inapplicable? The second [role] is why this mathematical tool is needed. Apart from these, is there any other role of proof that you can think of?
00:17:00	我要諗一諗… 我會覺得證明係比較接近數學嘅源本, 即係原先, 因為我覺得, 以我上堂嘅感覺, 啲數字嘅題目係夾硬整出嚟, 即係為你要用呢條公式而做, 但係你呢條公式嘅果個推算係點, 佢個邏輯係點係要證明先體現到	Let me think… I think proof is close to the origin of mathematics, because I think, during lessons, those numbers in the tasks are artificial, merely for you to apply this formula. Yet, what the derivation of this formula is [and] what its logic is are illustrated in proof.

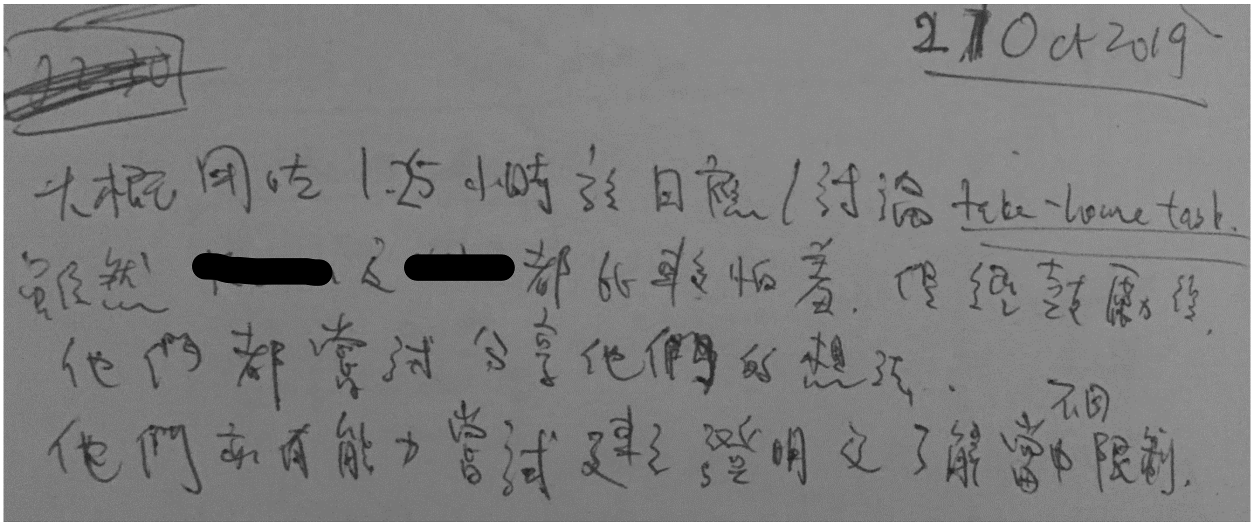
A transcribed excerpt of the audio-recordings of the intervention

Date: 26/09/2019; Time: 14:00; Length: 1h59m15s

	Cantonese transcript	English translation
1:14:18	問返個問題, 有無證明喺當中? 成條片, 成個過程有無證明喺入面?	Is there any proof in there? In the video, in the [teaching] process, is there any proof?
Some participants	有. 少少.	Yes… some…
Participant 4	佢證明咗個四邊形, 或者五邊形囉, 但係佢未去到 polygon 果個.	[The teacher] proved [the sum of interior angles of] quadrilateral, or pentagon. But he did not reach the [formula of the sum of interior angles of n-sided] polygon.
Participant 7	佢係咪想搵 pattern 跟著再去寫返, 即係 polygon 嘅式? 即係開頭果幾個四邊形, 五邊形, 佢都有嘗試去到證明, 但果個嘅內角和, 分明係 360 度, 同埋 540 度.	Didn't he want to find a pattern and then state… the formula of polygon? So, at the beginning, he tried to prove that the sums of interior angles of quadrilateral and pentagon are 360 degrees and 540 degrees [respectively].
Participant 2	我覺得似歸納多啲囉.	I think it was like induction.
Participant 1	搵個 pattern 出嚟, 跟著, 我諗佢最尾都係想自己搵個, 最尾果條公式出嚟… 自己去, 自己去搵; 填晒啲窿窿之後, 當三角形…四邊形係點樣, 五邊形係點樣, 你自己填晒啲 pattern 落去之後, 自己去搵出嚟. 但係呢樣, 我覺得似 build 出嚟, 即係唔係一個好實際嘅, 真係證明到係點樣.	Find the pattern. Then, I think [the teacher] eventually wanted [the students] to find the ultimate formula by themselves. Find it themselves. After completing the blanks, triangles… what would it be in quadrilaterals and pentagons. After [the students] filled the patterns into the blanks, find [the formula] themselves. However, this, I think, seemed to be inductive, which was not a proof.

Examples of fieldnotes of the intervention





"[We] spent around 1.25 hours to discuss the take-home tasks. Although [the participants] seemed to be shy, they tried to share their thoughts after encouragement. They had ability in constructing proofs and identifying limitations of different [representations and proving methods]."

Contingent tables of the PTs' responses to the Likert items. The investigation into individual responses to the Likert items provided explanations of the changes in the means of the responses related to the Likert items of the seven PTs in the intervention group. In the following, the contingent tables (Tables E.1–E.5) present the counts of the ordered pairs (X, Y) of individual PTs' responses to each Likert item, before and after the intervention. X represents the pre-intervention response, and Y represents the post-intervention response. For example, a PT agreed (**4 = Agree**) with the statement in item D1 before the intervention, and they disagreed (**2 = Disagree**) with this statement after the intervention, then the ordered pair representing their responses to the statement in item D1 are (4, 2). The same logic also applies to the contingent tables for the dimensional scores, in which the scores are rounded off.

Table E.1
Changes in the PTs' beliefs about the importance of proof

Beliefs about the importance of proof																	
	SD	D	N	A	SA		SD	D	N	A	SA		SD	D	N	A	SA
SA				1	2	SA				2	3	SA				1	4

A				4	
N					
D					
SD					

I1. I think that theorems and proofs are the foundations of mathematics.

A				2	
N					
D					
SD					

I2. Proofs are very important in understanding mathematical language.

A			1		1
N					
D					
SD					

I3. Proofs are indispensable for mathematics.

	SD	D	N	A	SA
SA					4
A				1	2
N					
D					
SD					

I4. Making proofs improves mathematical thinking.

	SD	D	N	A	SA
SA					2
A				3	1
N				1	
D					
SD					

I5. To me, it is important to make proofs in mathematics.

	SD	D	N	A	SA
SA				1	
A			1	3	
N				1	
D			1		
SD					

I6. To improve logical thinking, making proofs is necessary.

	SD	D	N	A	SA
SA					3
A			1	3	
N					
D					
SD					

I7. In the development of reasoning, making proofs plays an important role.

	SD	D	N	A	SA
SA					
A					
N					
D	1	2	1	2	
SD		1			

D5. I cannot see the point of making proofs that have already been proven beyond doubt by mathematicians and scientists.

	SD	D	N	A	SA
SA				1	2
A				4	
N					
D					
SD					

The dimensional scores

n = 7

Table E.2

Changes in the PTs' beliefs about relevance of proof to examinations

Beliefs about relevance of proof to examinations

	SD	D	N	A	SA
SA					
A			1	3	
N		1	1		
D		1			
SD					

T2. Knowing how to make proofs is very important in excelling in an examination/a test in school mathematics.

	SD	D	N	A	SA
SA					
A			2	2	
N		1		1	
D		1			
SD					

T3. In an examination/a test in school mathematics, making proofs plays an important role.

	SD	D	N	A	SA
SA					
A					
N					
D	2	3	1	1	
SD					

T5. Practising making proofs does not help solve a task of an examination/a test in school mathematics.

	SD	D	N	A	SA
SA					
A			1	4	
N		1	1		
D					
SD					

The dimensional scores

n = 7

Table E.3

Changes in the PTs' enjoyment and interest in proof

Enjoyment and interest in proof

	SD	D	N	A	SA
SA					
A			4	2	
N					1
D					
SD					

E1. It's fun for me to make proofs.

	SD	D	N	A	SA
SA					1
A		1	1	1	
N			1	1	
D		1			
SD					

E2. One of the things I love about mathematics is that proofs of theorems are done in the lessons.

	SD	D	N	A	SA
SA		1			
A				6	
N					
D					
SD					

E4. Making proofs arouses my curiosity.

	SD	D	N	A	SA
SA					
A		1	1	4	
N				1	
D					
SD					

E5. I find it interesting to make proofs.

	SD	D	N	A	SA
SA					
A			1	6	
N					
D					
SD					

E6. I enjoy proving mathematical claims.

	SD	D	N	A	SA
SA					
A		1	1		
N		1	3	1	
D					
SD					

E7. I feel enthusiastic to prove a theorem when I see it.

	SD	D	N	A	SA
SA					
A					
N					
D		6		1	
SD					

D3. I find it boring to make proofs.

	SD	D	N	A	SA
SA					
A		1	1	5	
N					
D					
SD					

The dimensional scores

n = 7

Table E.4

Changes in the PTs' anxiety about proof

Anxiety about proof

	SD	D	N	A	SA
SA					
A			1		
N			1	3	
D		1		1	
SD					

D1. I feel myself under pressure when I make proofs in mathematics lessons.

	SD	D	N	A	SA
SA					
A		1			
N		1	2	1	
D		2			
SD					

E3. I do not worry when making proofs.

	SD	D	N	A	SA
SA					
A					
N			1	2	
D		1	2	1	
SD					

A2. Working on a proof in front of the class frightens me.

	SD	D	N	A	SA
SA					
A					
N			1	1	
D	1	3	1		
SD					

	SD	D	N	A	SA
SA					
A					
N			2	2	
D		1	2		
SD					

The dimensional scores

A4. It scares me to work on a proof.

n = 7

Table E.5

Changes in the PTs' negative responses when failing to prove

Negative responses when failing to prove

	SD	D	N	A	SA
SA					
A				3	1
N			2		
D			1		
SD					

A1. My self-confidence diminishes when I cannot prove a theorem.

	SD	D	N	A	SA
SA					
A			1	2	
N				2	
D			1		
SD		1			

A3. Not being able to prove upsets me.

	SD	D	N	A	SA
SA					
A				3	1
N			2		
D			1		
SD					

The dimensional scores

n = 7

Contingent tables of the PTs' responses to the open-ended questions and the vignettes. The investigation into contingent tables of the responses to the open-ended questions and the vignettes provided descriptions of the changes in the overall percentages of the codes related to the data of the 12 PTs in the intervention group (as well as the 22 PTs in the comparison group). Similar to Tables E.1–E.5, the contingent tables below (Tables E.6–9) present the counts of the ordered pairs (X, Y) corresponded to the codes of individual PTs' responses to the open-ended questions and the vignettes, before and after the intervention. X represents the pre-intervention response, and Y represents the post-intervention response.

Table E.6

Changes in the PTs' beliefs about the place of proof in school mathematics

Intervention group (n = 12)

	N	H	U	S	R
R				4	6
S				2	
U					
H					
N					

Percentage agreement = 67%
Relative position = .33; 95% C.I. = [.08, .59]

Comparison group (n = 22)

	N	H	U	S	R
R			1	2	8
S			1	2	1
U		1			2
H		4			
N					

Percentage agreement = 64%
Relative position = .01; 95% C.I. = [-.17, .20]

N = No place; H = Hesitating; U = Unclear; S = Supplementary; R = Regular

Table E.7

Changes in the PTs' beliefs about approaches for teaching mathematics and proof

Intervention group (n = 12)					Comparison group (n = 22)				
	unclear	product	mixed	process		unclear	product	mixed	process
process		3	2		process			1	1
mixed		5	2		mixed	2	2		
product					product		9	1	3
unclear					unclear	4			
Percentage agreement = 17%					Percentage agreement = 64%				
Relative position = .81; 95% C.I. = [.64, .97]					Relative position = -.02; 95% C.I. = [-.23, .19]				

process = process-oriented beliefs; mixed = mixed beliefs; product = product-oriented beliefs; unclear = responses that did not indicate approaches for teaching mathematics and proof

Table E.8

Changes in the PTs' reasons for implementing proof in school mathematics

Code	Intervention group (n = 12)			Comparison group (n = 22)		
Justification		No	Yes		No	Yes
	Yes	1	1	Yes		1
	No	9	1	No	20	1
	PA = 83%; RP = .00; 95% C.I. = [-.22, .22]			PA = 95%; RP = -.05; 95% C.I. = [-.13, .04]		
Promotion of understanding		No	Yes		No	Yes
	Yes	3	9	Yes	3	6
	No			No	9	4
	PA = 75%; RP = .25; 95% C.I. = [.02, .48]			PA = 68%; RP = -.05; 95% C.I. = [-.28, .18]		
Memorisation		No	Yes		No	Yes
	Yes	3	3	Yes	1	
	No	6		No	20	1
	PA = 75%; RP = .25; 95% C.I. = [.02, .48]			PA = 91%; RP = .00; 95% C.I. = [-.12, .12]		
Generation of knowledge		No	Yes		No	Yes
	Yes	1		Yes		
	No	10	1	No	22	
	PA = 83%; RP = .00; 95% C.I. = [-.22, .22]			PA = 100%; RP = .00; 95% C.I. = [.00, .00]		
Development of reasoning		No	Yes		No	Yes
	Yes	2	3	Yes	1	3
	No	3	4	No	12	6
	PA = 50%; RP = -.17; 95% C.I. = [-.54, .21]			PA = 68%; RP = -.23; 95% C.I. = [-.44, -.02]		
Promotion of students' motivation		No	Yes		No	Yes
	Yes	2		Yes	2	1
	No	9	1	No	18	1
	PA = 75%; RP = .08; 95% C.I. = [-.18, .35]			PA = 86%; RP = .05; 95% C.I. = [-.10, .20]		
Self affection		No	Yes		No	Yes
	Yes	0	0	Yes	1	
	No	11	1	No	20	1
	PA = 92%; RP = -.08; 95% C.I. = [-.23, .07]			PA = 91%; RP = .00; 95% C.I. = [-.12, .12]		

PA = percentage agreement; RP = relative position

Table E.9

Changes in the PTs' concerns about the use of proof in school mathematics

Code	Intervention group (n = 12)			Comparison group (n = 22)		
Student		No	Yes		No	Yes
	Yes	2	3	Yes	2	6
	No	4	3	No	11	3
	PA = 58%; RP = -.08; 95% C.I. = [-.43, .26]			PA = 77%; RP = -.05; 95% C.I. = [-.24, .15]		
Self		No	Yes		No	Yes
	Yes	2	2	Yes		
	No	7	1	No	20	2
	PA = 75%; RP = .08; 95% C.I. = [-.18, .35]			PA = 91%; RP = -.09; 95% C.I. = [-.21, .03]		
Curriculum		No	Yes		No	Yes
	Yes	2		Yes	2	
	No	4	6	No	16	4
	PA = 33%; RP = -.33; 95% C.I. = [-.74, .07]			PA = 73%; RP = -.09; 95% C.I. = [-.30, .12]		
Time		No	Yes		No	Yes
	Yes	2	1	Yes		
	No	7	2	No	21	1
	PA = 67%; RP = .00; 95% C.I. = [-.31, .31]			PA = 95%; RP = -.05; 95% C.I. = [-.13, .04]		

PA = percentage agreement; RP = relative position

Appendix F CUREC application, participant information sheet and consent form

CENTRAL UNIVERSITY RESEARCH ETHICS COMMITTEE (CUREC)

Form CUREC 1A Checklist for the Social Sciences and Humanities

UNIVERSITY OF OXFORD

The University of Oxford places a high value on the knowledge, expertise, and integrity of its members and their ability to conduct research to high standards of scholarship and ethics. The research ethics clearance procedures have been established to ensure that the University is meeting its obligations as a responsible institution. They start from the presumption that all members of the University will take their responsibilities and obligations seriously and will ensure that their research involving human participants is conducted according to the established principles and good practice in their fields and in accordance, where appropriate, with legal requirements. Since the requirements of research ethics review will vary from field to field and from project to project, the University accepts that different guidelines and procedures will be appropriate.

- Please check **"Where and how to apply for ethical review"** and the **CUREC flowchart** first to see if you need ethics approval.
- Please complete this form using a word processor and email it, together with your supporting documents, to your **Departmental Research Ethics Committee (DREC)** (**if applicable**). If you don't have a DREC please email this form to **ethics@socsci.ox.ac.uk** using your official **ox.ac.uk email** address. **Only emailed applications will be accepted.**

WHAT THIS CHECKLIST IS DESIGNED FOR		
This **CUREC 1A checklist** is designed largely for research that falls within the Divisions of Social Sciences and Humanities where ethical issues are relatively few and straightforward. Interviews, field work and oral history are also included in the CUREC process. The **full CUREC 2 application** is only required where certain project characteristics (e.g. type of participants, or procedures) result in a more complex set of ethical issues. It is expected that only in a limited number of cases will it be necessary for researchers to complete a CUREC 2 application. The checklist below will direct you to a CUREC 2 application if needed.		
WHAT THIS CHECKLIST WILL NOT ASSESS		
This checklist does not cover research governance, satisfactory methodology, or compliance with the requirements of publishers when administering their tests or questionnaires. As **principal researcher (i.e. principal investigator)**, it is your responsibility to ensure that requirements in these areas are met. CUREC does not review studies classed as **audit** (see Glossary and Decision Flowchart for CUREC on our website). If your study involves **NHS patients, NHS staff / data / facilities, or human tissue**, please check the Decision Flowchart for NHS approval and contact the Clinical Trials and Research Governance (CTRG) team in the first instance. Further information on the University's research ethics procedures is available from the CUREC website.		
SECTION A: Filter for CUREC2 application This section determines whether your study raises more complex issues which require the completion of a full application for ethical review, known as the CUREC 2 application. **(Please mark 'X' in the Yes/No column as appropriate to indicate your response.)**		
1. Are research participants classed as **people whose ability to give free and informed consent is in question**? (This may include those under 18 (though see "competent youths"), prisoners, or adults "at risk".) Your attention is drawn to the University's **Safeguarding Code of Practice** and its implications for researchers involving children or adults at risk, including the need for the work to be risk assessed and for researchers to undertake related training. (**Note**: If any of your participants are aged 16 or under, please answer 'Yes' here **and** also answer question 5 below.)	Yes	No X
2. By taking part in the research, will participants be at serious risk of criminal prosecution (e.g. by providing information on drug abuse or child abuse)?	Yes	No X

3. Does the research involve the deception of participants?	Yes	No X
4. Does your research raise issues relevant to the **Counter-Terrorism and Security Act (the Prevent duty)**, which seeks to prevent people from being drawn into terrorism? Please see advice on this on our **Best Practice Guidance web page**.	Yes	No X
If you have answered 'No' to all of the questions above please go to **Section B**. If you have answered 'Yes' to any question above continue to question 5 below.		
5. Is your project covered by a CUREC **approved procedure** (formerly known as "CUREC Protocols")?	Yes	No
If yes, please give research procedure number(s):		
If you answered '**Yes'** to **ANY** of questions **1-4**, **and** answered '**No'** to question **5**, **please stop completing this checklist and do not submit it for ethical review**. Instead, please complete the CUREC 2 application form from the CUREC website. Then submit the CUREC 2 form for ethical review. If you answered 'Yes' to ANY of questions 1-3, **and** answered 'Yes' to question 5, please go on to **Section B**.		

SECTION B: Contact details and project description (NB: must be typed not handwritten)	
Contact details:	
1. supervisor (if student research (title and full name):	Gabriel Stylianides; Jenni Ingram
2. Name of student (if student research):	Chun-Yeung Lee
3. Degree programme, e.g. DPhil, BA, MPhil, BSc, MSc (if student research):	DPhil
4. Department or Institute name:	Department of Education
5. Address for correspondence (if different from above):	
6. University e-mail (**not** private email) and telephone:	chun.lee@education.ox.ac.uk; +447537883330
7. Name and status of others taking part in the project, e.g. third year undergraduate; postdoctoral research assistant:	Not applicable
SECTION B continued	
Project description:	
8. Title of research project:	How to promote proving-and-reasoning in teacher education? A design research on Hong Kong secondary mathematics preservice teachers
9. List of location(s) where project will be conducted:	Hong Kong
10. If your research involves overseas travel or fieldwork and your department requires a travel risk assessment, will you have completed and returned a risk assessment form beforehand? (This has to be approved by your department before you travel. If you are travelling overseas, you are strongly advised to take out University travel insurance.)	Yes X No Not required in this instance
11. Anticipated duration of research project overall:	4 years (maximum 5)
12. Anticipated start and end dates of the research project involving human participants:	From: (01/07/18) To: (31/12/19) Please note that you will need ethics approval **before** you start your research. CUREC 1As may take up to 30 days to process.

13. External organisation funding the research (if applicable):
14. Title and very brief and simple lay description of **research** (about 150 words), plus description (about 200 words) of the nature of participants.
a) Title, brief description of research (150 words) in lay language. When describing the research, please include your methodology, how you are applying professional guidelines, and the use to which results/data will be put. **Please also declare any conflicts of interest here.**
How to promote reasoning-and-proving in teacher education? A design research on Hong Kong secondary mathematics preservice teachers To address the situation suggested by recent research that Hong Kong (HK) preservice teachers (PSTs) were not ready for teaching reasoning-and-proving (RP), this design-research-based study aims to (1) document the status of HK PSTs' orientations towards teaching RP, that represents the effects of HK teacher education programmes on learnirpnng arpnd teaching RP in HK, and (2) develop an intervention that promotes teachers' orientations toward teaching RP. Through a longitudinal study, multiple data sources are proposed to capture PSTs' orientations towards teaching RP by triangulation, and the data serve for the basis of developing and evaluating the intervention. Throughout iterations of this design research, information of PSTs' change in orientations towards teaching RP, engagements and feedback are collected for refining the interventions. The findings of this study may serve, practically and theoretically, for (a) finalising an intervention that is designated for developing teachers' orientations towards teaching RP, and (b) providing empirical data for theorising the elements and mechanisms that support teacher change (in orientations), when designing an intervention. Data sources include questionnaires (Appendix A, pp.12-16), interviews (Appendix B, pp. 17-21), participants' reflective journals/presentations, researcher's fieldnotes, and audio-recordings of the intervention workshop.
SECTION B continued
b) Description of participants and **obtaining informed consent** (200 words). When describing participants, please include • criteria for inclusion/exclusion • method of recruitment • processes for consent to participate Please ensure you attach as separate documents (if applicable, in English translation): • your recruitment and advertisement material e.g. a poster or brief invitation letter/ email • information for participants to read (or hear) before they agree to take part e.g. written information sheets or (only if applicable) oral information scripts. • a document to record informed consent. Templates for written consent forms and/or oral information scripts (in case of an oral consent process) are available from the CUREC website • a guide to interview questions (this may be a list of questions to be asked, or a preliminary scope of questions), or a sample of other instruments (such as a sample questionnaire) • (if relevant) debriefing document after participants have taken part
The nature of participants In Hong Kong, three universities, namely, Chinese University of Hong Kong (CUHK), Education University of Hong Kong (EdUHK), and The University of Hong Kong (HKU) offer teacher education programmes for potential secondary mathematics teachers, including undergraduate programmes and postgraduate programmes. Any student who has enrolled in one of these programmes is identified as a PST. Every postgraduate PST is tartgeted, whereas undergraduate PST is targeted if he/she is in the third year or above. Therefore, the target participants are adults who are receiving or received tertiary education, that it is reasonable to assume that they are able to make consent themselves. Method of recruitment and how to obtain consent? A few lecturers have been contacted via emails/calls for discussing the possibilities of conducting this study in their classes; in July and August 2018, the programme coordinators and lecturers will be visited again for obtaining final consents of the brief presentations of advertisement and recruitment in their classes (in September 2018 for pilot study/March 2019 the main study). During the brief presentations of advertisement and recruitment, PSTs will receive a copy of participant information sheets and consent forms of this study. There are three main types of consent needed to be done by PSTs: (1) consent of participating the study by completing questionnaires (the consent will be

made before PSTs start working on the questionnaires), (2) consent of participating the interviews (the consent form is attached at the end of the questionnaires, asking PSTs' university emails for scheduling interviews), and (3) consent of participating the intervention (the consent form is attached at the end of the questionnaires, asking PSTs' university emails for contacts the details of intervention workshop). [Note that, for Cycle 3 of this study, the recruitment and consent are made in September 2019.]
15. What are the ethical issues connected with your research and what steps have you taken to address them? Please do not answer 'none'. The committee needs to see evidence that you have identified potential ethical issues with respect to your research and have taken steps to address them. These issues could relate to: • your own physical and psychological safety as a researcher (please see the University's and Social Science Division's Safety in Fieldwork guidance • participant burdens and/or risks, and • data protection/ confidentiality (please also see section 18). For more guidance on ethical issues, please see http://researchsupport.admin.ox.ac.uk/governance/ethics/resources
Depending on the number of participants, the PSTs may be divided into two groups: control and intervention groups. The intervention group will receive treatment from the intervention workshop, whereas the control group will receive no treatment. The intervention workshop will take place on Saturdays when the PSTs have no time-conflict with the usual teacher education programmes, but the intervention group spending more time on the intervention workshop may be a potential cost. The participants are well informed before their participation that there is no direct benefit on them by participating in this study. For the control group, they are invited to a debriefing workshop after data collection; the debriefing workshop is optional. When interviewed, the participants may encounter the unfamiliar tasks/questions that simulate a classroom situation, and they may feel uncomfortable to these tasks/questions. Before and during interviews, the researcher should inform the participants that despite the fact that the tasks/questions may be unfamiliar to the participants, the interviews do not aim to create any examination (i.e., it is not about correct or wrong answers), but to understand how a preservice teacher reacts when encountering the simulated situation. The researcher should try to create an environment that the participants feel safe to share their opinion. The participants are encouraged to respond to the questions/tasks in their best knowledge, and also informed when they give their consent that they may withdraw themselves from the study at any time. To store the data, the written data are scanned into electronic copies, and the audio data are transcibed by the researcher into texts; all data are stored in a new drive (harddisk/flashdisk) that is accessible by the researcher and supervisors only; data are **not** stored in any online drive (iCloud/dropbox/GoogleDrive) except the secure university server. Data that may include identifiable information of any PST are anonymised by codes. Lecturers and programme coordinators are not allowed to access any information of the data (The researcher will explain in details to the lecturers and programme coordinators in July/August 2018 when visiting them before collecting any data). The computer and drive for storing data and processing data analysis will be performed virus-scans regularly (once a month) to ensure that data won't be damaged or infected, but securely stored.

Section B continued		
16. Will you obtain informed consent according to CUREC guidelines and good practice in your discipline before participation?	Yes X	No
If you have marked '**No**', please give a brief explanation and justification for this decision here:		
17. Will your research involve discussing sensitive issues? This could be information relating to race or ethnic origin, political opinions, religious beliefs, physical/mental health, trade union membership, sexual life or criminal activities.	Yes	No X
If you have marked '**Yes**', please make sure that you have included some **supporting information** (as directed in question 14 of this section) showing the range of questions covering these issues.		

18. **Management and handling of personal and other research data** Your management and handling of personal data and special category data of human participants, either directly or via a third party, will need to comply with the requirements of the General Data Protection Regulation (GDPR) and the new Data Protection Act, as set out in the University's Guidance on Data Protection and Research. In answering the questions below, please also consider the points raised in the Data Protection Checklist. For advice on research data management and security, please consult with the University's Research Data Team (**researchdata@ox.ac.uk)** and/or your local IT department and the University's web pages on research data management.		
a) Will your research involve the collection of **records of consent** (e.g. written forms, audio-recorded, or other recorded consent)? **If 'Yes', these will be classed as fully identifiable personal data (directly linked to an individual).**	Yes X	No
b) Will your research involve the collection of **other personal data?** **If 'Yes'**, specify in what form(s) this will be stored:	Yes X	No
• Fully identifiable (directly linked to an individual)	Yes	No X
• Pseudonymised (potentially identifiable as data may be attributed to an individual if linkage information can be accessed elsewhere by researchers)	Yes	No X
• Fully anonymised (i.e. cannot be linked to an individual)	Yes X	No
c) Will your research involve the collection of **special category** data? **If 'Yes',** in what form(s) will this be stored:	Yes	No X
• Fully identifiable (directly linked to an individual)	Yes	No
• Pseudonymised (potentially identifiable as data may be attributed to an individual if linkage information can be accessed elsewhere by researchers)	Yes	No
• Fully anonymised (i.e. cannot be linked to an individual)	Yes	No
d) How will any personally identifiable data be collected, transferred and backed up? Please describe the arrangements for any physical transfer of personal data (including paper records and data captured electronically via portable media) from where it is collected to local storage.		
To store the data, the written data are scanned into electronic copies, and the audio data are transcibed by the researcher into texts; all data are stored in a new drive (harddisk/flashdisk) that is accessible by the researcher and supervisors only; data are **not** stored in any online drive (iCloud/dropbox/GoogleDrive) except the secure university server. Data that may include identifiable information of any PST are anonymised by codes. Lecturers and programme coordinators are not allowed to access any information of the data (The researcher will explain in details to the lecturers and programme coordinators in July/August 2018 when visiting them before collecting any data). Any physical data is confidential stored that only the researcher can access. Labelling participants with anonymised codes which are stored separately to other identifying details; and removing potentially identifying details in publicly disseminated materials (thus complying with the DPA). The data will be stored for as long as they are of academic value.		
e) Where, and for how long, will participants' personally identifiable data be stored during and after the study? (Please outline the procedures for ensuring confidentiality, eg security arrangements, anonymisation or pseudonymisation of such data. Please distinguish between records of consent and other forms of personally identifiable data stored)		
The researcher will adhere to the Data Protection Act and will ensure that disseminated results are anonymous, and do not include any information that allows individuals, local authorities, agencies or services to be identified. Consent forms and any other hard copies of personal information will be stored in a lockable filing cabinet in the research office. Audio-recordings will be encrypted and uploaded to the secure university server as soon as possible, and after the recordings have been transcribed, they can then be deleted. The researcher will then screen the transcript for identifiable data and ensure that it is anonymised. The participants will also be labelled with a code and identifiable data will be stored separately. The researcher will also adhere to a clear desk policy to ensure that all information is stored safely and securely.		
f) If storing pseudonymised data, please confirm that identifiers will be held separately from the research data and linked through a unique study number. Specify how and at what point the pseudonymisation will occur, how the linkage information will be stored and state whether or not (and when) the linkage will be destroyed.		
Not applicable.		

g) Who will have access to the personally identifiable data? If personally identifiable data is to be shared with another organisation, how will it be transferred/disclosed securely?
Only the researcher has access to the personally identifiable data. The data will not be shared with another organisation.
h) When and how will personally identifiable data be destroyed? (NB. Personally identifiable data should be destroyed when no longer required.)
The personally identifiable data will be destroyed – 3 months after interviews and workshops (data collection) finish.
i) How, where and for how long will other research data be stored after the study has finished? For more information about University and research funder retention policies, please see the University's web pages on research data management.
The data will be stored for as long as they are of academic value –at least 3 years after last publication.

SECTION C: Methods and procedures to be used

Method used: Please ensure you have addressed any potential ethical issues related to these methods in Section 14 and in your Participant Information Sheet	**Please mark 'X'**
1. Analysis of existing records	
2. Snowball sampling (recruiting through contacts of existing participants)	X
3. Use of casual or local workers e.g. interpreters	
4. Participant observation	
5. Covert observation	
6. Observation of specific organisational practices	
7. Participant completes questionnaire in hard copy	X
8. Participant completes online questionnaire or other online task	
9. Using social media	
10. Participant performs paper and pencil task	
11. Participant performs verbal or aural task (e.g. for linguistic study)	
12. Focus group	
13. Interview	X
14. Audio-recording of participant (you will generally need specific consent from participants for this)	X
15. Video recording of participant (you will generally need specific consent from participants for this)	
16. Photography of participant (you will generally need specific consent from participants for this)	
17. Others (please specify): Researcher's fieldnotes	X

SECTION D: Professional guidelines and training		
In this section, please mark 'X' against at least one of the following professional guidelines you aim to adhere to. You should use the principles listed in your chosen guideline(s) in conducting your own research. **Note**: this is not an exhaustive list.		**Please mark 'X'**
Research specialism/ methodology	**Association and guidance document**	
Anthropology	Association of Social Anthropologists of the UK and Commonwealth	
Criminology	http://www.britsoccrim.org/ethics/	
Education	British Educational Research Association Ethical Guidelines for Educational Research	X
Geography	Association of American Geographers Statement on Professional Ethics	
History	Oral History Society of the UK Ethical Guidelines	
Internet-based Research	British Psychological Society: Conducting Research on the Internet Association of Internet Researchers Ethics Guide Also see our Best Practice Guidance on internet-based research	
Law (Socio-Legal)	Socio-Legal Studies Association: Statement of Principles of Ethical Research	
Management	Academy of Management's Professional Code of Ethics	
Political Science	American Political Science Association (APSA) Guide to Professional Ethics in Political Science	
Politics	Political Studies Association. Guidelines for Good Professional Conduct	
Psychology	British Psychological Society Code of Ethics and Conduct	
Social Research	Social Research Association: Ethical Guidelines	
Sociology	The British Sociological Association: Statement of Ethical Practice	
Visual Research	ESRC National Centre for Research Methods Review Paper: Visual Ethics: Ethical Issues in Visual Research	
Other professional guidelines. Please specify the other guidelines used here*:*		
Please indicate what training in research ethics the researchers involved with this study have received, e.g. the title of the course and date completed (online training available at http://researchsupport.admin.ox.ac.uk/support/training/ethics). If no formal training has been undertaken, please indicate any discussions of research methodology between researchers and supervisors here.		
Research integrity online training (social and behavioural sciences) and passed the quiz (20 out of 20, 16 Jan 2018) Reading on Ethical guidelines for educational research https://www.bera.ac.uk/wp-content/uploads/2014/02/BERA-Ethical-Guidelines-2011.pdf?noredirect=1		

SECTION E: Signatures (The SSH IDREC Secretariat accepts either option below. If you have a DREC, check which signature option it prefers.)

- **Option 1:** 'Electronic signatures', i.e. email confirmations from a University of Oxford email address, can be accepted. Separate emails should come from each of the relevant signatories as outlined below, indicating acceptance of the relevant responsibilities. **Pasted images of signatures <u>cannot</u> be accepted in the sections below.**
- **Option 2:** Handwritten (wet-ink) signatures. Please scan them and the rest of the checklist pages to create a single PDF document and email through.

Please ensure this checklist is signed by:

For staff research:	**For student research**:
1. Principal investigator	1. Principal investigator **(project supervisor)**
2. **Head of Department (or nominee)**	2. **Head of Department (or nominee)**
	3. **Student researcher**

1. **Principal investigator signature/supervisor signature** (if student research)

I understand my responsibilities as **principal investigator** as outlined in the CUREC glossary and guidance on the CUREC website.

I declare that the answers above accurately describe the research as presently designed, and that a new checklist will be submitted should the research design change in a way which would alter any of the above responses so as to require completion of CUREC 2 (involving full scrutiny by an IDREC). I will inform the relevant IDREC if I cease to be the principal investigator on this project and supply the name and contact details of my successor if appropriate.

Signature: ……………………………………………………………………..

Print name (block capitals): J.INGRAM…………………………… **Date**: 9th May 2018…………

2. **Departmental endorsement signature**

I have read the research project application named above. On the basis of the information available to me, I:

(i) consider the principal investigator to be aware of her/his ethical responsibilities in regard to this research;
(ii) consider that any ethical issues raised have been satisfactorily resolved or are covered by relevant professional guidelines and/or CUREC approved procedures, and that it is appropriate for the research to proceed (noting the principal investigator's obligation to report should the design of the research change in a way which would alter any of the above responses so as to require completion of a CUREC 2 full application);
(iii) am satisfied that: the proposed project design and scientific methodology is sound; the project has been/will be subject to appropriate **peer review**; and is likely to contribute to existing knowledge and/or to the education and training of the researcher(s) and that it is in the **public interest**.

Signed by Head of Department or nominee *(example nominees for student research include the Director of Graduate Studies/ Director of Undergraduate Studies)*:

Signature: ……………………………………………………………………..

Print name (block capitals): …………………………….................................. **Date**: ……………………………………...

3. **Student signature** (if student research)

I understand the questions and answers that have been entered above describing the research, and I will ensure that my practice in this research complies with these answers, subject to any modifications made by the principal investigator properly authorised by the CUREC system.

Signed by student:

Print name (block capitals): CHUN-YEUNG LEE.......... **Date**: …8th May 2018

Fieldwork Form

Declaration

Please note: electronic signatures are not acceptable

Please remember to email this form to Louise.Mireh@education.ox.ac.uk and submit the hard copy to Jackie Bridges in the HDO

Student Declaration

- [x] I have completed the risk assessment relevant to my fieldwork
- [x] The information given on this form is correct to the best of my knowledge and I will ensure that it is updated as necessary (referring to the FCO website)
- [x] I have familiarised myself with current university Health & Safety Policies (and university statement of safety organisation)
- [x] I have agreed with my supervisor a regular check-in time and contact method by which we will keep in touch.
- [x] I am aware that it is departmental policy that I must check in with my supervisor at least once per month.
- [x] I have also signed the travel insurance form!

Signed   Date: 29 April 2018

Supervisor Declaration

- [x] I have read this risk assessment form and discussed it with the student
- [x] I have completed the supervisor check list with comments as appropriate
- [x] I have agreed with my student r a regular check-in time and contact method by which we will keep in touch.
- [x] I am aware that it is departmental policy that I must check in with my student at least once per month.

Signed ______________________   Date: 3/5/18

Departmental Administrator

I have signed the insurance form! (please tick to confirm) [ ]

DGS / HoD Approval
(HoD Approval required when region/area to be visited is deemed 'inadvisable' by the FCO)

Signed ______________________   Date: 14/5/18

Full details of travel insurance are available at www.admin.ox.ac.uk/finance/insurance/travel

Risk Assessment Form

Fieldwork Safety, Risk Assessment and Insurance

- This **entire** form **must** be completed by **all** students intending to carry out fieldwork.
- The Head of Department or their representative is required to approve all overseas fieldwork.
- It is mandatory to submit this form to the Higher Degrees Office **at least five weeks** in advance of carrying out your research.
- No travel should be booked until approval has been obtained by your supervisor and the Head of Department (or their representative).
- Please note that even if you're travelling to your home country on fieldwork, it still falls under university business and **must** be treated with equal consideration.
- **All signatures must be original.**
- Carefully consider your travel plans, the nature of activity with which you will be engaged and its location. Consider any associated risks or anyone else's personal safety and health. Consider the likelihood of any risks occurring and the severity of outcome if they were to occur. **Discuss these risks with your supervisor.**
- Check the FCO Website to see if the country/countries to which you are travelling have any advice/warnings posted. And sign up to the FCO email alert for updates while you are travelling.
- **Please note: if the FCO advises AGAINST all or all but essential travel to the country/countries you are visiting you must contact Louise Mireh immediately and be prepared to complete a full statement of intent and high risk FCO travel assessment.**

1. General information

First name Chun-Yeung	**Surname** Lee
Nationality Hong Kong	**Supervisor** Gabriel Stylianides; Jenni Ingram
Contact Address prior to fieldwork: St Antony's College, 62 Woodstock Road, Oxford OX2 6JF	
Contact email prior to fieldwork chun.lee@education.ox.ac.uk	**Contact phone number prior to fieldwork** [edited]
Please note: it is vital that we are able to contact you while you are on fieldwork in case of an emergency. Please make sure the contact details below are up-to-date.	
Contact Address (or addresses) while on fieldwork: [edited]	
Contact email while on fieldwork chun.lee@education.ox.ac.uk	**Contact phone number while on fieldwork** [edited]
Have you declared a disability?	**No**

2. Itinerary

Where is the fieldwork taking place? *Please be specific if you are visiting particular areas within a city, or region*. (For example, Cowley, not just Oxford).	Universities in Hong Kong, namely, The Education University of Hong Kong, Chinese University of Hong Kong, and University of Hong Kong
What activities will you be undertaking?	Fieldwork: organising teacher development workshop, interviews, visiting lectures

Please list specific sites of activity, along with the nature of work taking place, including topics of research, e.g. interviewing students with specific learning difficulties at The Durmstrang Institute, Sweden. If relevant please attach full itinerary as separate document	
Will you be travelling alone at night, and what precautions will you take?	No.
According to the FCO Website, the area I am visiting has been identified as:	**Low Risk https://www.gov.uk/foreign-travel-advice/hong-kong**
I have signed up to FCO email alert for this country	☑
If you are not a British national, does your own government advise against travel to the area you propose to visit	**No**
Date(s) of visit *Please note, this form can only cover you for a single period. If you plan to return at a later date, you will have to submit another form.*	1 Jan 2019–31 Dec 2019
Host family / Local Contacts *Please give as much as information as possible (address, contact name, phone, mobile phone, fax, email)*	RM3210, Yee Leung House, Yee Nga Court, Tai Po, Hong Kong +852 63248600

3. Health/medical and emergency procedures

Have you taken advice from the University Occupational Health Service on the health risks associated with travel to your field site (http://www.admin.ox.ac.uk/uohs/at-work/travel/)?	**Yes**
If no please state why:	Click here to enter text.
All accidents, incidents or near misses must be reported to Erica Oakes (erica.oakes@education.ox.ac.uk) at the Department of Education and your supervisor. Please check the box to confirm that you will ensure this takes place while you are on fieldwork: ☑	
Next of Kin: *Please state full name, address, telephone number, email and their relationship to you*	[edited]
Passport / visa numbers (overseas travel only)	**[edited]**

What first aid training have you received or what first aid is available to you?	I have received some basic first aid training in 2008, and emergency medical assistance is available by phone in Hong Kong.
Will you need to take any first aid provisions?	No.
If you're visiting an area that is fairly isolated, what access do you have to emergency services and medical help? If relevant, please also provide details of emergency plans in place (e.g. for evacuation from the research area, or if there is an injury to a worker in a small team)? If I need emergency medical assistance during my trip, dial 999 and ask for an ambulance. https://www.gov.uk/foreign-travel-advice/hong-kong/health	

4. Training and or experience

Have you received any training for your overseas travel? Please give details. No.	Have you experience in similar work? Please give details. Yes, I worked as a research assistant in Hong Kong.
Training Available from: https://www.admin.ox.ac.uk/safety/safetytraining/safetytraining/course/?crsID=104	

5. Supervision

- A fieldworker will still be under supervision with regards to health and safety, even though the supervisor may not be present.
- Please provide details of your supervisor, or other nominated person, who you will contact on a regular basis to ensure your safety.
- The nominated person should be provided with the contact details of a person in the Department, in case of emergency etc. Specify contact period times (e.g. daily/weekly).
- **Please note Departmental policy which specifies that the contact period or a student out on fieldwork cannot be less than one month.**

Name Gabriel Stylianides; Jenni Ingram	**Contact Period (when and how often you will contact your supervisor)** For email: weekly; For Skype: monthly, if necessary
Address Department of Education, University of Oxford, 15 Norham Gardens, Oxford OX2 6PY	**Telephone** +44 (0)1865 274024
Email gabriel.stylianides@education.ox.ac.uk; jenni.ingram@education.ox.ac.uk	
How will contact be made? You must ensure that this is possible (e.g. if reliance is put on a mobile phone, you must ensure there is adequate signal): Emails and Skype calls.	

6. Risk Assessment

- A risk assessment must be carried out in order to comply with University Safety Policy. You will need to read UPS S3/07 Overseas Travel and UPS S5/07 Safety in Fieldwork.
- Please remember to consider not only possible risks to your own health and safety, but also the effect that your work may have on other people or the environment.
- **The competency of external agencies hosting a fieldworker must be considered. Arrangements and responsibilities for the fieldwork must be agreed and documented as part of the risk assessment.**

Hazard identified (please specify) (see checklists in appendix to UPS S5/07)	**Risk(s) involved and control measures to be put in place to minimise risk**	**Estimated level of risk**
Safety and Security (e.g. lone work, first aid, violence, crime, travel, handling cash, transport)	transport	Low Risk
Physical health and natural disasters (e.g. extreme weather, fitness and medical risks)	Cyclones in summertime (If happens, stay indoor)	Low Risk
Equipment (will you be taking a laptop, mobile, cash, internet access etc? These should be specified in your insurance form)	Laptop and mobile	Low Risk
Environmental and surroundings (e.g. pollution, waste, language barriers, local culture and awareness of this)	Air pollution	Low Risk
Other and entry requirements	Nil	Low Risk

7. Supervisor Risk Assessment check list

Supervisor to complete, then sign the front page of the form.

		Additional comments if relevant
I have discussed with the student the general risks associated with the planned fieldwork.	x	Click here to enter text.
I have discussed with the student the potential additional specific safety issues and risks associated with this fieldwork, and appropriate measures to reduce them.	x	Click here to enter text.
I believe that travel arrangements discussed are satisfactory and safe.	x	Click here to enter text.
I have discussed with the student any specific health issues associated with the	x	Click here to enter text.

area they are travelling to and any issues regarding their research.		
We have identified specific requirements for additional guidance and advice (indicate as necessary) and this is being sought by the student/researcher	x	Click here to enter text.
I am satisfied that the student has adequately assessed the risks associated with the planned work, and agrees to carry out the work in a manner that reduces risks to health and safety to a satisfactory level.	x	Click here to enter text.
I have agreed to regular (Bi-Weekly) check-in dates with my students and have a means (Email) to contact them. If you have select 'other' for the above please clarify below: Click here to enter text.		

A design research on Hong Kong preservice secondary mathematics teachers
PARTICIPANT INFORMATION SHEET

Ethics Approval Reference: [ED-CIA-18-198]

1. ***What is the purpose of this research?***

   The aim of this study is to learn more about how different preservice teachers perceive learning and teaching mathematics.

2. ***Why have I been invited to take part?***

   You have been invited because you are a preservice teacher (i.e., an undergraduate student at the third or above year, or a postgraduate student, who has enrolled in a mathematics teacher education programme).

3. ***Do I have to take part?***

   No. You can ask questions about the research before deciding whether or not to participate. If you do agree to participate, you may withdraw yourself from the study at any time, without giving any reason, by advising the researchers of this decision.

4. ***What will happen to me if I take part in the research?***

   If you are happy to take part in the research, you will be asked to complete a survey questionnaire at multiple timepoints: September 2018, March 2019, May 2019, and July 2019. For each survey, it should take approximately 30 minutes. After completing the questionnaire, you may be invited to participate an interview, which is **optional**. The interview aims at (1) clarifying your responses on the questionnaire, and (2) understanding how preservice teachers encounter the situation about learning and teaching mathematics that may happen in a classroom setting. It should take approximately 1 hour. We will talk you through the procedures of the interview session and give you the chance to ask any questions before the interview is conducted. In late-June/early-July 2019, you may also be invited to attend a teacher development workshop, which is **optional**. The workshop aims at promoting further development of preservice competency of learning and teaching mathematics. The workshop will take place on three Saturdays in late-June/early-July 2019; for each session, it should take approximately 2 hours. Any arrangement of the workshop (e.g., place and time) will be informed later through emails. Both interviews and workshop are **audio-recorded**: The audio-recordings will be transcribed into texts for data analysing, and any direct-quote of the interviews will be anonymised, i.e., you will **not be identified** when materials are published. If you are still happy to take part, you will then be asked to sign a consent form.

5. ***Are there any potential risks in taking part?***

   Please kindly be aware that there are no right or wrong answers. You will not be asked any questions that are expected to cause distress or discomfort or be forced to answering any questions you do not want to answer. Hence, we do not anticipate any distress while answering these type of questions. If, however, you would feel uncomfortable at any point, please contact the researcher immediately.

6. ***Are there any benefits in taking part?***

   There will be no direct benefit to you from taking part in this research.

7. ***What happens to the data provided?*** [17]

[17] Please refer to CUREC's Best Practice Guidance on Data Collection and Management (BPG 09)

The information you provide as part of the study is the **research data**. Any research data from which you can be identified (e.g. your name, email address, audio-recording), is known as **personal data**. It does not include data where the identity has been removed (anonymous data). We will minimise our use of personal data in the study as much as possible. The **research and personal data** will be stored confidentially using a physical hard-drive with encryption, and will **not be stored** using any online storage. Your responses will be anonymised. The researcher and supervisors will have access to research data, whereas only the researcher will have access to personal data. We would also like your permission to use direct quotes. All research data and records will be stored for at least 3 years after publication or public release of the work of the research. We may retain and store your personal data for an additional period of time as necessary for the purposes of the study, and for further research.

8. ***Will the research be published?***

The University of Oxford is committed to the dissemination of its research for the benefit of society and the economy and, in support of this commitment, has established an online archive of research materials. This archive includes digital copies of student theses successfully submitted as part of a University of Oxford postgraduate degree programme. Holding the archive online gives easy access for researchers to the full text of freely available theses, thereby increasing the likely impact and use of that research. The research will be written up as a thesis. On successful submission of the thesis, it will be deposited both in print and online in the University archives, to facilitate its use in future research. The thesis will be openly accessible.

9. ***Who is organising and funding the research?***

This study is organised by Department of Education, University of Oxford.

10. ***Who has reviewed this study?***

This study has been reviewed by, and received ethics clearance through, the University of Oxford Central University Research Ethics Committee (Reference number: ED-CIA-18-198).

11. ***Who do I contact if I have a concern about the study or I wish to complain?***

If you have a concern about any aspect of this study, please speak to Mr Chun Yeung (Gabriel) Lee by chun.lee@education.ox.ac.uk or the Department of Education, University of Oxford general.enquiries@education.ox.ac.uk, who will do their best to answer your query. The researcher should acknowledge your concern within 10 working days and give you an indication of how they intend to deal with it. If you remain unhappy or wish to make a formal complaint, please contact the relevant chair of the Research Ethics Committee at the University of Oxford who will seek to resolve the matter in a reasonably expeditious manner: Chair, **Social Sciences & Humanities Inter-Divisional Research Ethics Committee**; Email: ethics@socsci.ox.ac.uk; Address: Research Services, University of Oxford, Wellington Square, Oxford OX1 2JD

12. ***Data Protection***

The University of Oxford is the data controller with respect to your personal data, and as such will determine how your personal data is used in the study. The University will process your personal data for the purpose of the research outlined above. Research is a task that we perform in the public interest. Further information about your rights with respect to your personal data is available from http://www.admin.ox.ac.uk/councilsec/compliance/gdpr/individualrights/.

13. ***Further Information and Contact Details***

If you would like to discuss the research with someone before (or if you have questions afterwards), please contact:

Mr Chun-Yeung (Gabriel) Lee

Department of Education, University of Oxford, 15 Norham Gardens, Oxford OX2 6PY

Tel: [edited]; Email: chun.lee@education.ox.ac.uk

PARTICIPANT CONSENT FORM

CUREC Approval Reference: ED-CIA-18-198

Purpose of Study: The aim of this study is to learn more about how different preservice teachers perceive learning and teaching mathematics.

		Please initial each box
1	I confirm that I have read and understand the information sheet for the above study. I have had the opportunity to consider the information, ask questions and have had these answered satisfactorily.	
2	I understand that my participation is voluntary and that I am free to withdraw at any time, without giving any reason, and without any adverse consequences or academic penalty.	
3	I understand that research data collected during the study may be looked at by designated individuals from the University of Oxford where it is relevant to my taking part in this study. I give permission for these individuals to access my data.	
4	I understand that this project has been reviewed by, and received ethics clearance through, the University of Oxford Central University Research Ethics Committee.	
5	I understand who will have access to personal data provided, how the data will be stored and what will happen to the data at the end of the project.	
6	I understand how this research will be written up and published.	
7	I understand how to raise a concern or make a complaint.	
8	I consent to being audio recorded.	
9	I understand how audio-recordings will be used in research outputs.	
10	I agree to take part in the study	
11	I agree for research data collected in this study to be given to researchers, including those working outside of the EU, to be used in other research studies. I understand that any data that leave the research group will be fully anonymised so that I cannot be identified.	
12	I agree for my personal data to be kept in a secure database for the purpose of contacting me about future studies.	

Name of Participant   Date   Signature

Ethical approval by The Education University of Hong Kong

香港教育大學
The Education University of Hong Kong

9 May 2019

Mr LEE Chun Yeung
Project Assistant
Department of Mathematics and Information Technology

Dear Mr Lee,

Application for Ethical Review <Ref. no. 2018-2019-0323>

I am pleased to inform you that approval has been given by the Human Research Ethics Committee (HREC) for your research project:

Project title: How to Promote Reasoning-and-proving in Teacher Education? A Design Research Involving Hong Kong Preservice Mathematics Teachers

Ethical approval is granted for the project period from 9 May 2019 to 31 December 2019. If a project extension is applied for lasting more than 3 months, HREC should be contacted with information regarding the nature of and the reason for the extension. If any substantial changes have been made to the project, a new HREC application will be required.

Please note that you are responsible for informing the HREC in advance of any proposed substantive changes to the research proposal or procedures which may affect the validity of this ethical approval. You will receive separate notification should a fresh approval be required.

Thank you for your kind attention and we wish you well with your research.

Yours sincerely,

Patsy

Patsy Chung (Ms)
Secretary
Human Research Ethics Committee

c.c. Prof CHOU Kee Lee, Chairperson, Human Research Ethics Committee

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