Subgroups of Quantum Groups

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A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2015

To my parents.

Acknowledgements

Firstly, I would like to thank my supervisor Kobi Kremnizer, for giving me this project and introducing me to this beautiful area of research, for his optimism and guidance throughout my work, but also for his constant support and friendship during my entire time in Oxford. His impact on my life has been enormous – and will now be part of me. I am blessed to have met him.

I would also like to thank Michael Collins for being the first person here in Oxford who believed in me, Karin Erdmann for being my role model, Balázs Szendröi and Oren Ben-Bassat for their friendship and support. I am also grateful to my Professors in Athens Apostolis Giannopoulos and Mihalis Maliakas for inspiring me as an undergraduate student.

Many thanks to my office mates Robert Laugwitz and Teresa Conde. A special thanks to Emily Cliff for sharing an office, recipes and hugs, but also for her LaTeX advice and for helping me submit this thesis!

I am especially grateful to my favourite Jo French. She has been an office mate, a mathematical sister, a soulmate, a psychotherapist –at times– and many more things. I wouldn't have survived those last years without her. Meeting her was one of the best parts of my DPhil.

I also want to thank my two oldest friends Marina Iliopoulou and Myrto Manolaki. They have been in my life and made it more beautiful for many many many years now.

Huge thanks to my parents and my sister for innumerable reasons. They made me the person I am today. I obviously wouldn't be here today without them.

Finally, thanks to my husband and my daughter for keeping me sane and tolerating me during my studies.

My DPhil studies were funded by the Alexander S. Onassis Public Benefit Foundation of Greece. I am deeply grateful for its support.

Abstract

In this thesis we investigate the notion of a subgroup of a quantum group. We suggest a general definition, which takes into account the work that has been done for quantum homogeneous spaces. We further restrict our attention to reductive subgroups, where some faithful flatness conditions apply. We give examples of quantum subgroups, some known and some new, which are all part of the family of spherical subgroups. The ultimate goal would be to quantize all spherical subgroups. Furthermore, we proceed with a categorical approach to the problem of finding quantum subgroups. We translate all existing results into the language of module and monoidal categories and give another characterization of the notion of a quantum subgroup.

Contents

Introduction

0.1 The story

Our motivation to better understand the notion of a subgroup of a quantum group came from the idea to quantize the class of spherical varieties. Let X be a normal algebraic variety, G a reductive and connected algebraic group acting on X and B a Borel subgroup of G. We say that X is a spherical variety if X contains an open orbit under the action of B. In particular, the homogeneous space G/H is called spherical if it contains an open orbit under the action of a Borel subgroup B of G. Spherical varieties are very interesting. First of all they include some very popular spaces like toric varieties, flag varieties (complete and partial), symmetric spaces and wonderful varieties. Hence, using the general theory of spherical varieties one can study and obtain results for the above spaces simultaneously. In addition to this, spherical varieties are often used as a "test case" in the study of actions of reductive groups.

Conversely to the above, one can try to generalize nice properties of these specific examples in the case of spherical varieties. For example we can find analogues in the theory of spherical varieties of the Bruhat decomposition and the Borel-Weil-Bott theorem (inspired by flag varieties), the little Weyl group and the Harish-Chandra isomorphism (by symmetric spaces), the geometry of fans and convex polytopes (by toric varieties).

Following this philosophy, we can take inspiration from the quantum analogues that have been defined for toric and flag varieties as well as for symmetric spaces. It seems very natural to ask if a similar theory can be obtained for spherical varieties.

To do so, it became clear to us that we first had to restrict our attention to some specific cases of spherical varieties. First, it seemed natural to consider spherical embeddings, that is normal G-varieties X with an open G-orbit isomorphic to G/H for a spherical homogeneous space G/H. The reason for this is that spherical embeddings have been classified using some combinatorial data, similar to the ones we have for toric varieties. Very briefly this is done by first covering a spherical embedding by finitely many open simple spherical subembeddings. (A G/H embedding is called simple if it contains a unique closed G-orbit.) This covering is done using the orbits of the G-action and the fact that any G-variety can be covered by G-translates of open affine B-stable sets. Then, a cone is attached to every simple embedding (so that we get a cone-orbit correspondence as in toric varieties), and finally a fan glues together the cones and gives us the initial spherical embedding.

Since general spherical embeddings are glued from simple spherical ones, it is natural to try to quantize them first. And since simple spherical embeddings contain G/H as an open G-orbit, the first step is to understand how to quantize spherical homogeneous spaces G/H, or equivalently spherical subgroups H.

More than that we need to further restrict our attention to reductive subgroups H, or to the case where G/H is an affine space. In general for an affine spherical variety X, $\mathcal{O}(X)$ has the structure of a G-module. As such, and since G is considered to be reductive, $\mathcal{O}(X)$ can be written as a sum of simple G-modules, say $\mathcal{O}(X) = \oplus V_\lambda$ where the λ's belong to a submonoid Γ of the weight lattice of G. Now in general given such a submonoid Γ we can obtain several affine spherical varieties which are not equivariantly isomorphic. We need more data, the system of spherical roots of X, in order to distinguish them. The different affine spherical varieties correspond to different multiplications on the G-algebra $\mathcal{O}(X)$. However when X is smooth then X is uniquely determined by the submonoid Γ. This means that the smooth and affine case is the easiest one to consider. In particular, when G is a reductive group and H is a closed subgroup, then the homogeneous space is always smooth. Moreover G/H is affine if and only if H is a reductive subgroup. In this case the spherical variety G/H is determined by $\mathcal{O}(G/H)$ in a unique way. In particular $\mathcal{O}(G/H) = \mathcal{O}(G)^H$, the H-invariant functions in $\mathcal{O}(G)$.

This is how we decided to start looking at different notions of subgroups of quantum groups, and especially those for which the corresponding classical quotient space is affine. In doing so we quickly realized that although quantum groups have been defined and understood for several years now, the notion of a quantum subgroup is still a bit unclear in the literature. Consider a general reductive group G. We know that both the universal eneveloping algebra $U(\mathfrak{g})$ and the coordinate ring $\mathcal{O}(G)$ have the structure of a Hopf algebra. The same is true for their quantum counterparts $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$. Now, given a subgroup M of G we would like to define its quantum version $\mathcal{O}_q(M)$. The first guess is to require for this to give rise to a Hopf algebraic structure as well, and therefore to look at the "correct" (with respect to M) Hopf subalgebra of $U_q(\mathfrak{g})$ or quotient Hopf agebra of $\mathcal{O}_q(G)$. This is what happens when M is for instance a Borel subgroup of G. However, it has been clear for some time now that this definition is too restrictive in general. The theory that has been developed for quantum homogeneous spaces shows that Hopf subalgebras of a given Hopf algebra are not enough. Indeed, quantum homogeneous spaces are often associated to coideal subalgebras of the quantized coordinate algebras $\mathcal{O}_q(G)$ (or the quantized universal enveloping algebras) and cannot be associated to any Hopf algebra. In addition to this, $\mathcal{O}_q(G)$ is required to be faithfully flat over the defined coideal subalgebra.

Let us see how the language of coideal subalgebras together with the faithful flatness condition can give us a definition for a quantum subgroup. Given a closed subgroup M of G we can consider its corresponding Lie subalgebra $\mathfrak{m} \subset \mathfrak{g}$. Then $U(\mathfrak{m}) \subset U(\mathfrak{g})$ is a left coideal subalgebra. In the quantum case, if we can find a coideal of $U_q(\mathfrak{g})$ to be the quantum analogue of $U(\mathfrak{m})$ then using the general theory of Hopf algebras we can construct a right coideal subalgebra A of $\mathcal{O}_q(G)$ but also a quotient left coalgebra B. Moreover it is known that $\mathcal{O}_q(G)$ is faithfully flat over A if and only if it is faithfully coflat over B. Moreover we can consider the category $\mathcal{M}_A^{\mathcal{O}_q(G)}$ whose objects are $\mathcal{O}_q(G)$-comodules which are also A-modules with the compatibility condition that the $\mathcal{O}_q(G)$-comodule map is a morphism of A-modules. Then, under the faithfully flat condition, $\mathcal{M}_A^{\mathcal{O}_q(G)}$ is equivalent to the category of B-comodules. We use this quotient coalgebra B as the definition of the quantum subgroup corresponding to M. Our intuition is that if we think of A as the quantum coordinate algebra $\mathcal{O}_q(G/M)$ corresponding to the classical homogeneous space G/M, then the category $\mathcal{M}_A^{\mathcal{O}_q(G)}$ corresponds to vector spaces with an action of the (unknown) quantum subgroup coorresponding to M. By the above, this is equivalent to the category of B-comodules. Hence, B carries the right representation theory with respect to the classical M, and therefore can be used as a definition for $\mathcal{O}_q(M)$. This is one way to look at quantum subgroups.

However, in addition to the above, there is an alternative, categorical approach to the problem of defining quantum subgroups. Classically, whenever M is a subgroup of G we can consider the categories of their representations. Rep(G) is a monoidal category, and Rep(M) becomes a module category over it. Moreover between the two categories can be defined a restriction and an induction functor. The same should be true for their quantum analogues. The category of $\mathcal{O}_q(G)$-comodules is again a monoidal category, and it is natural to expect that the category of comodules corresponding to a quantum subgroup of G should have the structure of a module category over it as happens in the classical case. So, in order to define a quantum subgroup it is enough to find the appropriate module category over the category of $\mathcal{O}_q(G)$-comodules. This approach gives a categorical characterization of the notion of a quantum subgroup and it provides a dictionary between the two worlds that gives us more flexibility when dealing with the theory of quantum subgroups.

0.2 Related research

Spherical varieties arose when Danilov [14] conjectured that an extension of the theory of toric varieties should be possible if one replaced the torus T by an arbitrary reductive group. A bit later Luna and Vust [30], found a way to describe all embeddings of homogeneous G-varieties. In the case of spherical embeddings their theory becomes simpler and more transparent and attaches to a spherical embedding some combinatorial data (called coloured fans) similar to the ones obtained in the case of toric varieties. Spherical homogeneous spaces have since been studied by Luna, Vust, Brion, Knop, Bravi, Pezzini, Losev, Cupit–Foutou and others. Many interesting results have been proved (for a nice exposition on these see Brion [6] and Akhiezer [3]).

Some examples of quantizations of spherical varieties have already been realized. The quantization of toric varieties was done by Ingalls and can be found in [22]. Quantum flag varieties were defined by Kremnizer and Backelin in [4]. Quantum symmetric spaces were constructed by Letzter in [29].

Concerning subgroups of quantum groups, the need for a Hopf algebra structure has been demonstrated in several cases (see for example [47], [43], or [53]) where by definition a quantum subgroup is first of all a Hopf algebra satisfying some extra conditions. However the work on quantum homogeneous spaces has, in our point of view, changed the picture. Podlés in [44] constructs a class of quantum homogeneous spaces with an $SU_q(2)$-action which correspond to the classical 2-sphere $SU(2)/SO(2)$. Dijkhuizen in [17] gives a survey regarding the construction of some compact quantum symmetric spaces such as $SU(n)/SO(n)$ or $SU(2n)/Sp(n)$. Letzter in [29] constructs quantum symmetric spaces. In all of the above cases, quantum homogeneous spaces are defined as coideal subalgebras. In addition to this, Müller and Schneider in [37] proved that in the cases of Podlés and Dijkhuizen above, $\mathcal{O}_q[G]$ is faithfully flat over the defined coideal subalgebra, enriching at the same time the theory with the notions of semisimplicity and cosemisimplicity.

Finally, the idea that module categories are related to quantum subgroups has been found in previous works as well. In [24] Kirillov and Ostrik classify the "finite subgroups in $U_q(\mathfrak{sl}_2)$" , where $q = e^{\pi i/l}$ is a root of unity. For them a subgroup in

$U_q(\mathfrak{sl}_2)$ is a commutative associative algebra in a tensor category $\mathcal{C}$. Similarly, in [20] Grossman and Snyder interpret quantum subgroups of finite groups as simple module categories over the category $\mathcal{C}$ of G-modules. Ocneanu in [39] uses similar definitions to classify quantum subgroups of $SU(n)$.

0.3 Outline of thesis

Let us now say a few words on how this work is organized. The first part of the thesis deals with the definition of a quantum subgroup using the language of Hopf algebras. In chapter 1 we recall some background material from quantum groups. In chapter 2 we give the main results from the theory of Hopf algebras which we think are the correct tools for the definition of a quantum subgroup. These are results that have been known for some time now, but not exactly used in this context. We look at both coideal subalgebras and quotient coalgebras, which are dual to each other and provide two equivalent perspectives. We see how this correspondence is achieved and obtain results for categories of representations over them. In section 2.3.1 we also discuss the notions of semisimplicity and cosemisimplicity and how they are related to the faithful flatness condition. The section ends with our definition in 2.4. We suggest that quantum subgroups should correspond to certain module quotient coalgebras (or equivalently coideal subalgebras) of a Hopf algebra. In particular if we restrict to subgroups for which the corresponding quotient space is affine, a faithful coflatness condition (respectively faithful flatness condition) is required.

In the second part of the thesis we focus on the categorical approach of the problem. In chapter 3 we recall the general theory of monoidal and module categories. In chapter 4 we formulate the problem presented in chapter 2 but using the language of module categories and give our alternative definitions. In particular, we start with a monoidal category $\mathcal{C}$ and a module category $\mathcal{M}$ over it. We prove in Theorem 4.2.1 that if there exist adjoint functors of module categories $Res : \mathcal{C} \to \mathcal{M}$ and $Ind : \mathcal{M} \to \mathcal{C}$ such that Ind is exact and faithful, then the module category $\mathcal{M}$ is equivalent to a module category $\mathrm{Mod}_{\mathcal{C}}(A)$ for an algebra $A \in \mathcal{C}$. We investigate the properties of such module categories when $\mathcal{C}$ is taken to be the category of comodules over a Hopf algebra H. In chapter 5 we proceed one step further. In our main theorem of this chapter, Theorem 5.2.5, we prove that if $\mathcal{C}$ is $\mathcal{M}^H$, namely the category of right comodules for a Hopf algebra H with bijective antipode, $\mathcal{M}$ is $\mathcal{M}^B$, the category of right comodules for a coalgebra B, and if Res in the adjunction above is moreover a functor of module categories that carries the forgetful functor to the forgetful functor then B has the structure of a quotient H-module coalgebra and H is faithfully coflat over B. Furthermore there exists a coideal subalgebra A' of H such that $\mathcal{M} \simeq \mathrm{Mod}_{\mathcal{C}}(A')$ and H is faithfully flat over A'. This gives us the categorical characterization of a quantum subgroup. We also show in 5.2.1 that quantum subgroups satisfy the conditions of theorem 5. We are grateful to Uli Krähmer for suggesting to add this section to the thesis and also for pointing out which isomorphisms to use in the proof.

Finally, in the last part we look at an application of our work. In the last chapter, chapter 6, we give a presentation of the project of quantizing all spherical subgroups. We briefly recall the classification of spherical subgroups. We then move to Letzter's construction of quantum symmetric pairs, which covers many examples of spherical subgroups. We finish by quantizing some new examples of spherical subgroups.

Part I

Subgroups of quantum groups

Chapter 1

Quantum groups – Some background

1.1 The quantized enveloping algebra

For a proper reference dealing with the theory of quantum groups we suggest [23], [11] and [31].

We fix a ground field k and an element $q \in k$, $q \neq 0$. Let $\mathfrak{g}$ be a semisimple Lie algebra, let Π be a basis of the root system Φ with respect to a Cartan subalgebra and let Λ be the weight lattice of Φ. Then the Lie algebra has a presentation with $3|\Pi|$ generators $x_\alpha, y_\alpha, h_\alpha$ where $\alpha \in \Pi$ satisfying the following relations:

$$[h_\alpha, h_\beta] = 0 \qquad [x_\alpha, y_\beta] = \delta_{\alpha\beta} h_\alpha$$

$$[h_\alpha, x_\beta] = a_{\alpha\beta} x_\beta \qquad [h_\alpha, y_\beta] = -a_{\alpha\beta} y_\beta$$

and for all $\alpha \neq \beta$:

$$\sum_{i=0}^{1-a_{\alpha\beta}} (-1)^i \binom{1-a_{\alpha\beta}}{i} x_\alpha^{1-a_{\alpha\beta}-i} x_\beta {x_\alpha}^i = 0$$

$$\sum_{i=0}^{1-a_{\alpha\beta}} (-1)^i \binom{1-a_{\alpha\beta}}{i} y_\alpha^{1-a_{\alpha\beta}-i} y_\beta {y_\alpha}^i = 0$$

where $a_{\alpha\beta} = 2(\alpha,\beta)/(\alpha,\alpha)$ denote the entries of the Cartan matrix.

Having remembered this, the definition of the quantized enveloping algebra comes quite naturally:

Definition 1.1.1. The quantized enveloping algebra $U_q(\mathfrak{g})$ is defined as the k-algebra with generators $E_\alpha, F_\alpha, K_\alpha$ and ${K_\alpha}^{-1}$ for all $\alpha \in \Pi$ satisfying the following relations:

1. $K_\alpha {K_\alpha}^{-1} = {K_\alpha}^{-1} K_\alpha = 1$, $K_\alpha K_\beta = K_\beta K_\alpha$

2. $K_\alpha E_\beta {K_\alpha}^{-1} = q^{\langle\alpha,\beta\rangle} E_\beta$

3. $K_\alpha F_\beta {K_\alpha}^{-1} = q^{-\langle\alpha,\beta\rangle} F_\beta$

4. $E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - {K_\alpha}^{-1}}{q-q^{-1}}$

5. $\sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \binom{1-a_{\alpha\beta}}{s}_\alpha E_\alpha^{1-a_{\alpha\beta}-s} E_\beta {E_\alpha}^s = 0$

6. $\sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \binom{1-a_{\alpha\beta}}{s}_\alpha F_\alpha^{1-a_{\alpha\beta}-s} F_\beta {F_\alpha}^s = 0$

where $[a]_\alpha = \frac{{q_\alpha}^a - {q_\alpha}^{-a}}{q_\alpha - {q_\alpha}^{-1}}$, $q_\alpha = q^{(\alpha,\alpha)/2}$ and $[n]_\alpha!$ and $\binom{n}{k}_\alpha$ are defined accordingly.

With this definition in hand, one can prove the following proposition.

Proposition 1.1.2 ([23], Proposition 4.11)**.** *There is a unique Hopf algebra structure on $U_q(\mathfrak{g})$ such that for all $\alpha \in \Pi$:*

$$\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \ \ \epsilon(E_\alpha) = 0, \ \ S(E_\alpha) = -{K_\alpha}^{-1} E_\alpha$$

$$\Delta(F_\alpha) = F_\alpha \otimes {K_\alpha}^{-1} + 1 \otimes F_\alpha, \ \ \epsilon(F_\alpha) = 0, \ \ S(F_\alpha) = -F_\alpha K_\alpha$$

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \ \ \epsilon(K_\alpha) = 1, \ \ S(K_\alpha) = {K_\alpha}^{-1}$$

1.2 Representations of $U_q(\mathfrak{g})$

In this chapter we will briefly recall how the representation theory of $U_q(\mathfrak{g})$ looks. We assume throughout that q is not a root of unity. We will see that in this case the theory of finite-dimensional representations of $U_q(\mathfrak{g})$ is very similar to that of the classical enveloping algebra $U(\mathfrak{g})$ (and thus of $\mathfrak{g}$). Let us begin with a definition.

Definition 1.2.1. Let $\lambda \in \Lambda$ and let $\sigma : \mathbb{Z}\Phi \to \{1, -1\}$ be a group homomorphism. Let also M be a $U_q(\mathfrak{g})$-module. A **weight space** of M is defined as follows :

$$M_{\lambda,\sigma} = \{m \in M | K_\mu m = \sigma(\mu) q^{(\lambda,\mu)} m \text{ for all}\, \mu \in \mathbb{Z}\Phi\}.$$

Proposition 1.2.2 ([23], Proposition 5.1)**.** *Let M be a finite-dimensional $U_q(\mathfrak{g})$-module. Then M is the direct sum of all $M_{\lambda,\sigma}$. We have for all λ and σ:*

$$E_\alpha M_{\lambda,\sigma} \subset M_{\lambda+\alpha,\sigma} \text{ and } F_\alpha M_{\lambda,\sigma} \subset M_{\lambda-\alpha,\sigma}$$

for all $\alpha \in \Pi$. Each E_α, F_α acts nilpotently on M.

The above proposition means that if M is a finite-dimensional $U_q(\mathfrak{g})$-module then $M = \oplus_\sigma M^\sigma$ where $M^\sigma = \oplus_\lambda M_{\lambda,\sigma}$.

Definition 1.2.3. We say that M is of **type** σ if $M = M^\sigma$. We say that M is of **type 1** if it is of type σ with $\sigma(\beta) = 1$ for all $\beta \in \mathbb{Z}\Phi$.

Remark 1.2.4 ([23], 5.2)**.** *There is an equivalence of categories between the category of all finite-dimensional $U_q(\mathfrak{g})$-modules of type 1 and those of type σ.*

Lemma 1.2.5 ([23], 5.4)**.** *The category of all finite-dimensional $U_q(\mathfrak{g})$-modules of type 1 is closed under taking tensor products, dual modules and Hom spaces.*

From now on we restrict to the type 1 representations. The main result regarding the representation theory of $U_q(\mathfrak{g})$ is the following:

Theorem 1.2.6. *Suppose that char(k) =0 and that q is not a root of unity. Then every finite-dimensional $U_q(\mathfrak{g})$-module is semisimple.*

Moreover, it is known that the simple $U_q(\mathfrak{g})$-modules are highest weight modules of highest weights parametrized by the dominant weights. This shows exactly how the representation theory of $U_q(\mathfrak{g})$ for generic q is similar to the classical case.

1.3 The quantized coordinate algebra $\mathcal{O}_q(G)$

We finish this section by recalling the quantum counterpart of the algebra $\mathcal{O}(G)$ of regular functions on the simply connected, semisimple algebraic group G with Lie algebra $\mathfrak{g}$.

Definition 1.3.1. Let M be a finite-dimensional $U_q(\mathfrak{g})$-module. For all $m \in M$, $f \in M^*$ we denote by $c_{f,m} \in U_q(\mathfrak{g})^*$ the linear form with $c_{f,m}(u) = f(um)$ for all $u \in U_q(\mathfrak{g})$ and call it a **matrix coefficient**.

Lemma 1.3.2 ([23], Lemma 7.10)**.** *Let M and N be finite-dimensional $U_q(\mathfrak{g})$-modules. Then we have*

$$c_{f,m}c_{g,n} = c_{f\otimes g, m\otimes n}$$

for all $m \in M$, $n \in N$, $f \in M^*$ and $g \in N^*$.

Lemma 1.3.2 implies that the subspace of $U_q(\mathfrak{g})^*$ generated by all matrix coefficients is closed under multiplication. Moreover, it contains the identity $\epsilon \in U_q(\mathfrak{g})^*$ since ϵ is a matrix coefficient for the trivial one dimensional $U_q(\mathfrak{g})$-module. So this subspace is a subalgebra.

Definition 1.3.3. We denote the subalgebra generated by the matrix coefficients of the finite-dimensional type 1 $U_q(\mathfrak{g})$-modules by $\mathcal{O}_q(G)$ and call it the **quantized coordinate algebra**.

Lemma 1.3.4 ([23], 7.11)**.** *There is a Hopf algebra structure on $\mathcal{O}_q(G)$.*

Indeed, it can be shown using Lemma 1.2.5 that for each $\phi \in \mathcal{O}_q(G)$ there exists a unique $\Delta^*(\phi) \in \mathcal{O}_q(G) \otimes \mathcal{O}_q(G)$ such that :

$$\Delta^*(\phi)(u \otimes u') = \phi(uu') \text{ for all } u, u' \in U_q(\mathfrak{g}).$$

This map Δ^* is proved to be an algebra homomorphism. It is the comultiplication on $\mathcal{O}_q(G)$. Moreover the map $\epsilon^* : \mathcal{O}_q(G) \to k$ with $\epsilon^*(\phi) = \phi(1)$ is the counit and it satisfies $\epsilon^*(c_{f,m}) = f(m)$. Finally the antipode S^* on $\mathcal{O}_q(G)$ is given by $S^*(\phi) = \phi \circ S$.

Chapter 2

Quantum subgroups

In this chapter we suggest a way to construct and define quantum subgroups. To do so we consider quantum groups as Hopf algebras. We recall the main results from the theory of Hopf algebras that deal with coideal subalgebras and quotient coalgebras. Following the work that has been done for quantum homogeneous spaces, we conclude that quantum subgroups correspond to certain quotient H-module coalgebras.

2.1 Quantum groups as Hopf algebras

If $\mathfrak{g}$ is a Lie algebra, then its universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra. As we saw in chapter 1 the same is true for the quantized enveloping algebra $U_q(\mathfrak{g})$ corresponding to $\mathfrak{g}$. Moreover if G is the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ then for each type 1 finite–dimensional representation, we can define matrix coefficients and define the quantized coordinate algebra $\mathcal{O}_q(G)$ which is again a Hopf algebra. There is a natural pairing $(\,,\,) : U_q(\mathfrak{g}) \times \mathcal{O}_q(G) \to k$.

By the above we can conclude that the theory of Hopf algebras is fundamental in the study of quantum groups. In particular a quantum group is often identified with a Hopf algebra U (the quantized enveloping algebra), or its dual H (the quantized coordinate algebra). Throughout the work that follows, we state and prove several results that lie in the abstract world of Hopf algebras and their categories of comodules. However, we always have in mind, and actually we are aiming at, the corresponding applications to quantum groups.

2.2 Definitions and notation

Throughout this section, we let H be a Hopf algebra over an algebraically closed field k. We denote by Δ_H the comultiplication and by ϵ_H the counit of the coalgebraic structure. S_H will denote the antipode and 1_H the unit. We also adopt the Sweedler notation, therefore $\Delta_H(h) = h_{(1)} \otimes h_{(2)}$. In the case of coactions, if $\rho : N \to N \otimes H$, then $\rho(n) = n_{(0)} \otimes n_{(1)}$. Similarly if N is left comodule with $\rho : N \to H \otimes N$, then $\rho(n) = n_{(-1)} \otimes n_{(0)}$.

We will be working with categories of modules and comodules, over algebras and coalgebras respectively. We will use the following general rules for the notation:

$\mathcal{M}_B$ denotes the category of right B-modules

$\mathcal{M}^D$ denotes the category of right D-comodules

Similarly:

${}_B\mathcal{M}$ denotes the category of left B-modules

${}^D\mathcal{M}$ denotes the category of left D-comodules.

Definition 2.2.1. We say that A is a **right coideal subalgebra** of H if A is a subalgebra which is also a right coideal of H, namely $\Delta_H(A) \subset A \otimes H$.

Notice that this gives A the structure of a right H-comodule. Given a Hopf algebra H and a coideal subalgebra A we can consider categories carrying both a module and a comodule structure with some compatibility conditions. Following the notation that we used above, we can make the following definition :

Definition 2.2.2. By $\mathcal{M}_A^H$ we will denote the category of right A-modules, right H-comodules which satisfy the compatibility condition 3 below. In particular, the objects in this category are vector spaces M such that

1. M is a right A-module, i.e. there is a module map $a : M \otimes A \to M$.

2. M is a right H-comodule, i.e there is a comodule map $\rho : M \to M \otimes H$.

3. The map $a : M \otimes A \to M$ is a map of H-comodules.

The morphisms are A-linear, H-colinear maps.

For condition 3 to make sense, $M \otimes A$ must carry the structure of an H-comodule. This is not always true but it is always the case in our situation, namely when H is a Hopf algebra and A a coideal subalgebra (and in particular an H-comodule).

We would like to point out that in some texts instead of condition 3 they use the alternative condition

3.(a) The map $\rho : M \to M \otimes H$ is a map of A-modules.

It is easy to see that conditions 3 and 3(a) are equivalent in our case and that they define the same category denoted by $\mathcal{M}_A^H$. Similarly we can define the category ${}_A\mathcal{M}^H$.

Definition 2.2.3. We say that C is a **quotient left H-module coalgebra** if C is the quotient of H by a coideal and left ideal I.

Notice that then the induced map $p : H \otimes C \to C$ gives C the structure of an H-module. Given the projection map $\pi : H \to H/I = C$, we will usually denote $\pi(h)$ by $\bar{h}$.

As in the case of coideal subalgebras, given a Hopf algebra H and a quotient left H-module coalgebra C we can define the following :

Definition 2.2.4. We let ${}_H^C\mathcal{M}$ be the category of left H-modules, left C-comodules which satisfy the compatibility condition 3 below. In particular the objects in this category are vector spaces M such that

1. There is a module map $a : H \otimes M \to M$
2. There is a comodule map $\rho : M \to C \otimes M$
3. The map $\rho : M \to C \otimes M$ is a map of H-modules.

The morphisms are H-linear and C-colinear maps.

Again, for this definition to make sense, $C \otimes M$ must be an H-module, which in our case is true. In particular we are using the left action induced by the comultiplication on H. Similarly, we can define ${}^C\mathcal{M}_H$.

Finally we would like to recall the notions of a faithfully flat module and a faithfully coflat comodule.

Definition 2.2.5. A right module M over an algebra B is **flat** (respectively **faithfully flat**) if and only if the functor $M \otimes_B - :\ {}_B\mathcal{M} \to {}_k\mathcal{M}$ preserves (respectively preserves and reflects) short exact sequences.

Definition 2.2.6. Let $V \in \mathcal{M}^D$ and $W \in {}^D\mathcal{M}$. The **cotensor product** $V \underset{D}{\square} W$ is defined to be the kernel of

$$\rho_V \otimes id - id \otimes \rho_W : V \otimes W \to V \otimes D \otimes W.$$

Definition 2.2.7. A right comodule V over a coalgebra D is **coflat** (respectively **faithfully coflat**) if and only if the functor $V \underset{D}{\square} - : {}^D\mathcal{M} \to {}_k\mathcal{M}$ preserves (respectively preserves and reflects) short exact sequences.

2.3 General results

In this section we will recall some known facts about coideal subalgebras and quotient coalgebras of Hopf algebras. We will see how the induced module categories defined previously are related when we add the necessary conditions of faithful flatness. Our main references for this are [49], [50], [12], [37].

Let us start by describing a notion of duality for Hopf algebras. We use here the approach that can be found in [37]. If U is a Hopf algebra then its linear dual $U^* = \mathrm{Hom}(U, k)$ is not necessarily a Hopf algebra (it may not be a coalgebra). This is because $U^* \otimes U^*$ is a proper subspace of $(U \otimes U)^*$ when U is not finite-dimensional and therefore the image of $m^* : U^* \to (U \otimes U)^*$ might not lie in $U^* \otimes U^*$. However, we can define a dual Hopf algebra as follows:

First, in the case that U is an algebra, the dual coalgebra $U^o \subset U^*$ is spanned by the matrix coefficients of all finite-dimensional U-modules. If $\rho : U \to \mathrm{End}(V)$ is a representation of U, we denote by C^V the image of the dual coalgebra $(\mathrm{End}(V))^*$ under $\rho^* : (\mathrm{End}(V))^* \to U^*$. That means that C^V is the k-linear span of all matrix coefficients $c_{f,v} \in U^*$, $f \in V^*, v \in V$, where $c_{f,v}(u) = f(uv)$ for all $u \in U$. The coalgebra structure on C^V is given by $\Delta(c_{f,v}) = \sum_i c_{f,v_i} \otimes c_{f_i,v}$ where $(v_i), (f_i)$ are dual bases of V, V^*. Then U^o is the sum of all the subcoalgebras C^V. Now, each C^V has a natural (U, U)-bimodule structure given by

$$(x \cdot a)(u) = a(ux) \text{ and } (a \cdot x)(u) = a(xu)$$

where $x, u \in U$, $a \in U^*$.

Assume now that U is a Hopf algebra. We define a tensor category $\mathcal{C}$ of finite-dimensional left U-modules to be a class of finite-dimensional left U-modules such that:

- $k \in \mathcal{C}$ as the trivial U-module via ϵ_U.

- If $V, W \in \mathcal{C}$ then $V \oplus W$ and $V \otimes W$ (with the diagonal action) are in $\mathcal{C}$ as well.
- If $V \in \mathcal{C}$, then $V^* \in \mathcal{C}$ where $(uf)(v) = f(S(u)v)$ for $f \in V^*, v \in V, u \in U$ and S the antipode of U.

Similarly, we can define a tensor category of finite-dimensional right U-modules.

We define the dual Hopf algebra of U with respect to $\mathcal{C}$ to be $H := \sum C^V$ for all $V \in \mathcal{C}$. See [37] for more details on this construction.

Notice that when U is the quantized enveloping algebra of a Lie algebra, and $\mathcal{C}$ is the category of finite-dimensional type 1 representations then H is the quantized coordinate algebra $\mathcal{O}_q(G)$.

We will now see that any right (or left) coideal in U gives rise to a right (or left) coideal subalgebra of H. We use the proof found in [29].

Proposition 2.3.1 ([29], Theorem 3.1)**.** *Let U be a Hopf algebra with bijective antipode, $\mathcal{C}$ a tensor category of finite-dimensional right U-modules and let H be the dual of U with respect to $\mathcal{C}$. If $Z \subset U$ is a right coideal of U, then $A \subset H$ defined by $A := \{h \in H | h \cdot z = \epsilon_U(z)h \text{ for all} z \in Z\}$ where $h \cdot z$ is the restriction of the natural action of U on H described above, is a right coideal subalgebra.*

Proof. First we will show that if $h, h' \in A$ then their product hh' is also in A. To see this, let $z \in Z$. Then $(hh') \cdot z = \sum (h \cdot z_{(1)})(h' \cdot z_{(2)}) = \sum (\epsilon(z_{(1)})h)(h' \cdot z_{(2)})$, since Z is a right coideal. Now, $\sum (\epsilon(z_{(1)})h)(h' \cdot z_{(2)}) = \sum h(h'\epsilon(z_{(1)})z_{(2)}) = h(h' \cdot z) = h\epsilon(z)h' = \epsilon(z)(hh')$.

It remains to show that A is also a right coideal, namely that $\Delta(A) \subset A \otimes H$. Let $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for some h in A. We want to show that $h_{(1)} \in A$, namely that $h_{(1)} \cdot z = \epsilon(z)h_{(1)}$ $\forall z \in Z$. But since $\Delta(h)$ is such that $\Delta(h)(u \otimes u') = h(uu')$ it follows that:

$$\Delta(h \cdot z) = \Delta(h)(z \otimes 1) = \sum h_{(1)} \cdot z \otimes h_{(2)}.$$

On the other hand

$$\Delta(h \cdot z) = \Delta(\epsilon(z)h) = \sum \epsilon(z)h_{(1)} \otimes h_{(2)}.$$

We can assume that the $h_{(2)}$ are linearly independent and thus it follows that $h_{(1)} \cdot z = \epsilon(z)h_{(1)}$ as we wanted.

□

Moreover, there is a one-to-one correspondence between right coideal subalgebras of a Hopf algebra H and quotient left module coalgebras C.

Proposition 2.3.2 ([49], Proposition 1)**.** *Let A be a right coideal subalgebra of H and denote $A^+ = A \cap ker\,\epsilon_H$; then $H_A = H/HA^+$ is a quotient left module coalgebra of H. Dually, if $\pi : H \to C$ is a quotient left module coalgebra, then ${}^C H = \{h \in H | \pi(h_{(1)}) \otimes h_{(2)} = \pi(1) \otimes h\} = {}^{coC}H$ is a right coideal subalgebra of H.*

Proof. HA^+ is a coideal and a left ideal making H/HA^+ a quotient coalgebra, which is also a left H-module by left multiplication on H. For the other direction, ${}^C H$ is clearly a right coideal. We will show that it is also a subalgebra. Let a, b in ${}^C H$; then $\pi((ab)_{(1)}) \otimes (ab)_{(2)} = \pi(a_{(1)}b_{(1)}) \otimes a_{(2)}b_{(2)}$. Now notice that C is also an H-module via π, therefore $\pi(a_{(1)}b_{(1)}) = a_{(1)}\pi(b_{(1)})$ giving us the following: $\pi(a_{(1)}b_{(1)}) \otimes a_{(2)}b_{(2)} = a_{(1)}\pi(b_{(1)}) \otimes a_{(2)}b_{(2)} = a_{(1)}\pi(1) \otimes a_{(2)}b = \pi(a_{(1)}) \otimes a_{(2)}b = \pi(1) \otimes ab$ which shows that $ab \in {}^C H$.

□

Combining Propositions 2.3.1 and 2.3.2 we can deduce that whenever we are given a coideal in U we can define a coideal subalgebra of its dual Hopf algebra H or equivalently a quotient left H-module coalgebra. Let us see now how the conditions of faithful flatness fit into the picture. The following theorems can be found in [37], [49], [34] and [33].

Theorem 2.3.3 ([49], Theorem 2)**.** *Let H be a Hopf algebra and C a quotient left module coalgebra; then ${}^C H$ as defined in Proposition 2.3.2 is a right coideal subalgebra. Suppose that H is faithfully coflat as a right C-comodule. Then ${}_{{}^C H}\mathcal{M} \simeq^C_H \mathcal{M}$ and H is faithfully flat as a right A-module.*

Theorem 2.3.4 ([34], Theorem 2.1)**.** *Let H be a Hopf algebra with bijective antipode and $A \subset H$ a right coideal subalgebra. Let H_A be (as in Proposition 2.3.2) the corresponding quotient left module coalgebra. Then the following are equivalent:*

1. *H is faithfully coflat as a left H_A-comodule.*
2. *H is faithfully flat as a left A-module.*
3. *There is an equivalence of categories $\mathcal{M}_A^H \simeq \mathcal{M}^{H_A}$.*

From the above we conclude the following:

Corollary 2.3.5. *[[33], Theorem 1.11] : Let H be a Hopf algebra with bijective antipode. Then $A \mapsto H_A$, $C \mapsto {}^C H$ give a one-to-one correspondence between the right coideal subalgebras of H over which H is left faithfully flat and the quotient left H-module coalgebras of H over which H is left faithfully coflat.*

2.3.1 Semisimplicity and cosemisimplicity

The notion of faithful flatness (respectively faithful coflatness) is an important condition when we want to work with the quantum analogue of an affine homogeneous space.
In this direction, we will look at some results from [37] that connect some semisimplicity (respectively cosemisimplicity) conditions with the faithful flatness (respectively faithful coflatness) notion that we are looking for. In their work Müller and Schneider use a slightly different approach which we briefly recall here:

Let us start with a Hopf algebra U with bijective antipode and a left coideal subalgebra K. Let $\mathcal{C}$ be the tensor category of finite-dimensional left U-modules and let H be the dual of U with respect to $\mathcal{C}$. Then

$$A := \{h \in H | h \cdot K^+ = 0\}$$

is a right coideal subalgebra of H. Then (as before) $C = H/HA^+$ is a left module quotient coalgebra and the results from the previous section hold.

Definition 2.3.6. Let $\mathcal{C}$ be a tensor category of left U-modules. A subalgebra $K \subset U$ is called **$\mathcal{C}$-semisimple** if all $V \in \mathcal{C}$ are semisimple as left K-modules (by restriction).

Theorem 2.3.7 ([37], Theorem 2.2)**.** *Let U be a Hopf algebra, $K \subset U$ a left coideal subalgebra and $\mathcal{C}$ a tensor category of finite-dimensional left U-modules. Let H be the dual Hopf algebra with respect to $\mathcal{C}$, $A := \{h \in H | h \cdot K^+ = 0\}$ and $C = H/HA^+$. Assume that the antipode of H is bijective. Then:*

1. *$A \subset H$ is a right coideal subalgebra with $A = ^{coC} H$.*

2. *If K is $\mathcal{C}$-semisimple then C is cosemisimple and H is faithfully flat as a left and right A-module.*

For the full statement of the above theorem we refer to [37], Theorem 2.2.

2.4 Quantum subgroups – A definition

Definition 2.4.1. Let H be a Hopf algebra with bijective antipode. A **quantum homogeneous space** is a right coideal subalgebra A of H over which H is faithfully flat.

This definition agrees with the classical picture. Indeed, it is known that a flat morphism of commutative rings $f : R \to S$ is faithfully flat if and only if the dual map $f* : \text{Spec}S \to \text{Spec}R$ is surjective (see for example [35] Theorem 7.3). In the same spirit is Lemma 2.2 of [45]:

Lemma 2.4.2. *Let $\phi : X \to Y$ be a surjective morphism of smooth irreducible affine algebraic varieties and let $s = dimX - dimY$. Assume that for every $y \in Y$, each irreducible component of the fibre $\phi^{-1}(y)$ is s-dimensional. Then $\mathcal{O}(X)$ is a fuithfully flat $\mathcal{O}(Y)$-module.*

Keeping in mind the above definition and the previous results we can proceed to the following construction: Let I be a right coideal of U. By Proposition 2.3.1 this gives rise to a right coideal subalgebra A of the dual Hopf algebra H. By Proposition 2.3.2 this gives us a quotient left module coalgebra H_A of H. If moreover H is faithfully flat over A (that is, if A corresponds to a quantum homogeneous space), by Theorem 2.3.4 we conclude that there is an equivalence of categories $\mathcal{M}_A^H \simeq \mathcal{M}^{H_A}$. Therefore if the coideal I with which we started is associated to a classical subgroup M of a group G, then the quotient coalgebra H_A can be used as the definition of the quantum subgroup corresponding to M. In this setting U is the quantized universal enveloping algebra of the Lie algebra of G and H is the quantum coordinate algebra $\mathcal{O}_q(G)$.

This leads to the following definition:

Definition 2.4.3. Let H be a Hopf algebra with bijective antipode. A **quantum subgroup** is a quotient left H-module coalgebra C of H such that H is faithfully coflat over C.

We would like to point out here, complementing the remarks about the homogeneous spaces above, that with the above definition we aim to quantize subgroups H of a group G such that G/H is affine. As we saw, classically, the faithfully flatness condition corresponds exactly to these subgroups for which the quotient space is affine. It is also worth mentioning that if we restrict to the case where the ambient group is reductive (as is the case for spherical subgroups for example) then by Matsushima's criterion we know that G/H is affine if and only if H is reductive. So in this case the two notions coincide.

In this latter case the following can be useful:

Remark 2.4.4. *If H is cosemisimple, which is the case when we work with the quantum coordinate algebra $\mathcal{O}_q(G)$ of a reductive group for generic q, and if A is not only a coideal subalgebra, but a Hopf subalgebra of H, then by the main result in [12] H is faithfully flat over A.*

Part II

A categorical approach to Quantum Subgroups

Chapter 3

Module categories

We start the second part by introducing the main objects of interest. These are monoidal and module categories. Our main references for this chapter are [5], [24] and [40]. We will assume throughout that all categories are abelian over an algebraically closed field k. All functors are assumed to be additive.

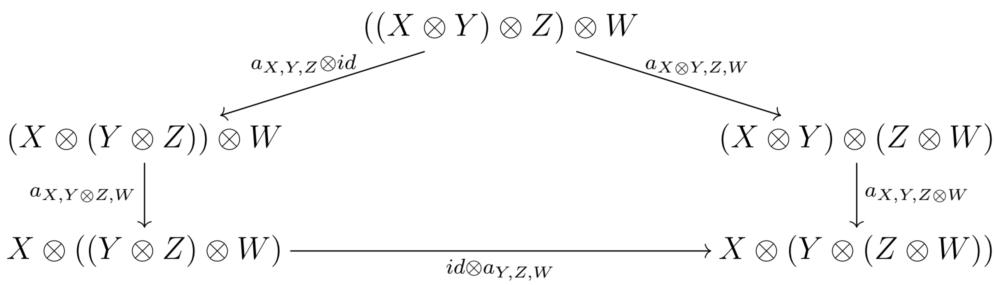
3.1 Monoidal categories

We start with the definition of a monoidal category.

Definition 3.1.1. A **monoidal category** consists of the following:

1. A category $\mathcal{C}$,
2. a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
3. functorial isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,
4. a unit object $\mathbf{1} \in \mathcal{C}$, and
5. functorial isomorphisms $r_X : X \otimes \mathbf{1} \to X$ and $l_X : \mathbf{1} \otimes X \to X$,

such that the following diagrams:



and

$$\begin{array}{ccc} (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{a_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\ & {}_{r_X \otimes id}\searrow \quad \swarrow_{id \otimes l_Y} & \\ & X \otimes Y & \end{array}$$

commute.

Definition 3.1.2. Let $\mathcal{C}_1, \mathcal{C}_2$ be two monoidal categories. A **monoidal functor** between $\mathcal{C}_1$ and $\mathcal{C}_2$ is a triple (F, b, u) consisting of a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, functorial isomorphisms $b_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)$ and isomorphism $u : F(\mathbf{1}) \to \mathbf{1}$ such that the following diagrams:

$$\begin{array}{ccccc} F((X \otimes Y) \otimes Z) & \xrightarrow{b_{X \otimes Y,Z}} & F(X \otimes Y) \otimes F(Z) & \xrightarrow{b_{X,Z} \otimes id} & (F(X) \otimes F(Y)) \otimes F(Z) \\ \downarrow{\scriptstyle F a_{X,Y,Z}} & & & & \downarrow{\scriptstyle a_{F(X),F(Y),F(Z)}} \\ F(X \otimes (Y \otimes Z)) & \xrightarrow{b_{X,Y \otimes Z}} & F(X) \otimes F(Y \otimes Z) & \xrightarrow{id \otimes b_{Y,Z}} & F(X) \otimes (F(Y) \otimes F(Z)) \end{array}$$

and

$$\begin{array}{ccc} F(\mathbf{1} \otimes X) & \xrightarrow{b_{\mathbf{1} \otimes X}} & F(\mathbf{1}) \otimes F(X) \\ \downarrow{\scriptstyle F(l_X)} & & \downarrow{\scriptstyle u \otimes id} \\ F(X) & \xrightarrow{l_{F(X)}} & \mathbf{1} \otimes F(X) \end{array} \qquad \begin{array}{ccc} F(X \otimes \mathbf{1}) & \xrightarrow{b_{X,\mathbf{1}}} & F(X) \otimes F(\mathbf{1}) \\ \downarrow{\scriptstyle F(r_X)} & & \downarrow{\scriptstyle id \otimes u} \\ F(X) & \xrightarrow{r_{F(X)}} & F(X) \otimes \mathbf{1} \end{array}$$

commute.

Notice that in some other texts, a functor of the form defined above, for which b and u are required to be isomorphisms, is often called a strict monoidal functor.

Example 3.1.3. *If H is a Hopf algebra, then the category of H-comodules is a monoidal category. Indeed, given two H-comodules V and W, we can consider $V \otimes W$ in $\mathcal{M}^H$ with comodule structure given as follows:*

$$V \otimes W \xrightarrow{\rho_V \otimes \rho_W} V \otimes H \otimes W \otimes H \xrightarrow{id \otimes flip \otimes id} V \otimes W \otimes H \otimes H \xrightarrow{id \otimes id \otimes m_H} V \otimes W \otimes H$$

We finish this section by stating a theorem that will be useful later in order to simplify arguments in some proofs. First, we need a definition.

Definition 3.1.4. A monoidal category $\mathcal{C}$ is called **strict** if for all objects $X, Y, Z \in \mathcal{C}$ the functorial isomorphisms (from Definition 3.1.1) $a_{X,Y,Z}, r_X$ and l_X are the identity isomorphisms. In this case we have $X \otimes \mathbf{1} = X$, $\mathbf{1} \otimes X = X$ and $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$, and similarly for multiple tensor products.

Theorem 3.1.5. *Every monoidal category is equivalent to a strict one.*

A proof of this can be found in [32].

3.2 Module categories

Definition 3.2.1. A **module category** over a monoidal category $\mathcal{C}$ consists of the following:

1. A category $\mathcal{M}$,
2. an exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$,
3. functorial isomorphisms $m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$, for $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$ and
4. functorial isomorphisms $l_M : \mathbf{1} \otimes M \to M$,

such that the following diagrams:

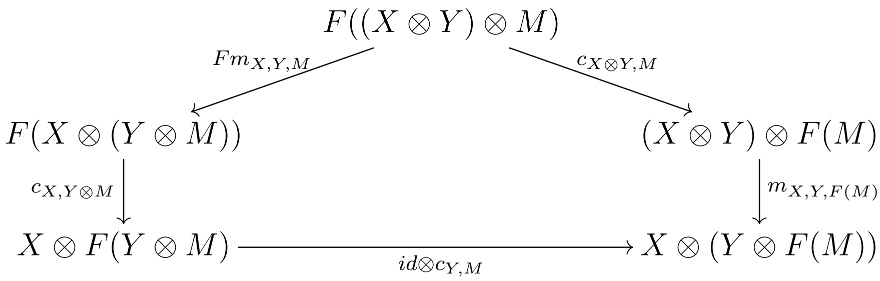
$$\begin{array}{ccccc} & & ((X \otimes Y) \otimes Z) \otimes M & & \\ & \swarrow^{a_{X,Y,Z} \otimes id} & & \searrow^{m_{X \otimes Y, Z, M}} & \\ (X \otimes (Y \otimes Z)) \otimes M & & & & (X \otimes Y) \otimes (Z \otimes M) \\ \downarrow{\scriptstyle m_{X, Y \otimes Z, M}} & & & & \downarrow{\scriptstyle m_{X,Y,Z \otimes M}} \\ X \otimes ((Y \otimes Z) \otimes M) & & \xrightarrow[id \otimes m_{Y,Z,M}]{} & & X \otimes (Y \otimes (Z \otimes M)) \end{array}$$

and

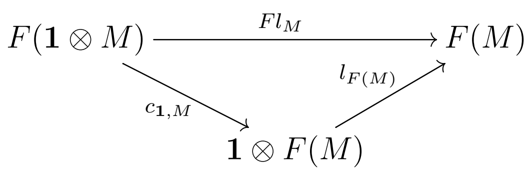
$$\begin{array}{ccc} (X \otimes \mathbf{1}) \otimes M & \xrightarrow{m_{X,\mathbf{1},M}} & X \otimes (\mathbf{1} \otimes M) \\ & {\scriptstyle r_X \otimes id} \searrow \quad \swarrow {\scriptstyle id \otimes l_M} & \\ & X \otimes M & \end{array}$$

commute.

Definition 3.2.2. Let $\mathcal{M}_1, \mathcal{M}_2$ be two module categories over a monoidal category $\mathcal{C}$. A **module functor** from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ together with functorial isomorphisms $c_{X,M} : F(X \otimes M) \to X \otimes F(M)$ for every $X \in \mathcal{C}$ and $M \in \mathcal{M}_1$, such that the following diagrams:



and



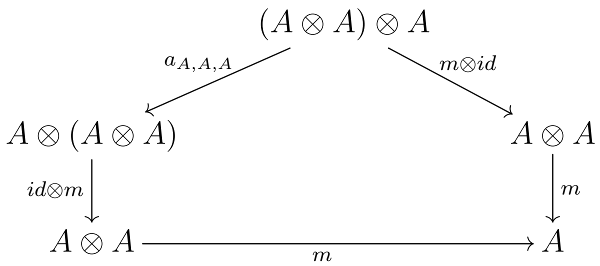
commute.

Notice that what we have defined is often called a strict module functor in other texts, because we are requiring that the morphisms $c_{X,M}$ are isomorphisms.

Example 3.2.3. *Any monoidal category $\mathcal{C}$ can be viewed as a module category over itself, with associativity and unit isomorphisms given by the ones from the monoidal structure.*

3.3 Algebras in monoidal categories

Definition 3.3.1. An **algebra** in a monoidal category $\mathcal{C}$ is an object $A \in \mathcal{C}$ together with a multiplication morphism $m : A \otimes A \to A$ and a unit morphism $e : \mathbf{1} \to A$ such that the following diagrams:



and

$$\begin{array}{ccc} & \mathbf{1}\otimes A & \\ {\scriptstyle l_A}\swarrow & & \searrow{\scriptstyle e\otimes id} \\ A & \xleftarrow[m]{\qquad} & A\otimes A \end{array} \qquad \begin{array}{ccc} & A\otimes \mathbf{1} & \\ {\scriptstyle r_a}\swarrow & & \searrow{\scriptstyle id\otimes e} \\ A & \xleftarrow[m]{\qquad} & A\otimes A \end{array}$$

commute.

Example 3.3.2. *If $\mathcal{C}$ is the monoidal category of H-comodules for a Hopf algebra H, and if A is a coideal subalgebra of H, then A is an algebra for the monoidal category $\mathcal{C}$.*

Definition 3.3.3. A **right module** over an algebra A in a monoidal category $\mathcal{C}$ is an object $M \in \mathcal{C}$ together with an action morphism $a : M \otimes A \to M$ such that the following diagrams:

$$\begin{array}{ccc} M\otimes A\otimes A & \xrightarrow{id\otimes m} & M\otimes A \\ \downarrow{\scriptstyle a\otimes id} & & \downarrow{\scriptstyle a} \\ M\otimes A & \xrightarrow{a} & M \end{array} \qquad \begin{array}{ccc} & M\otimes \mathbf{1} & \\ {\scriptstyle r_M}\swarrow & & \searrow{\scriptstyle id\otimes e} \\ M & \xleftarrow[a]{\qquad} & M\otimes A \end{array}$$

commute. Similarly we can define the notion of a left module.

Finally,

Definition 3.3.4. A **morphism** between two right modules M_1, M_2 over A is a morphism in $\mathcal{C}$ such that the diagram commutes:

$$\begin{array}{ccc} M_1\otimes A & \xrightarrow{a\otimes id} & M_2\otimes A \\ \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_2} \\ M_1 & \xrightarrow{a} & M_2 \end{array}$$

There is a similar definition for left modules.

If A is an algebra in a monoidal category $\mathcal{C}$, we will denote the category of right A-modules by $\mathrm{Mod}_{\mathcal{C}}(A)$. This is an abelian category.

Remark 3.3.5. *Using the above notations, $Mod_{\mathcal{C}}(A)$ can be endowed with the structure of a module category over $\mathcal{C}$.*

Indeed, let M be a right A-module and let $X \in \mathcal{C}$. We want to define a functor $\otimes : \mathcal{C} \times \mathrm{Mod}_{\mathcal{C}}(A) \to \mathrm{Mod}_{\mathcal{C}}(A)$. Since $X, M \in \mathcal{C}$, we can consider the object $X \otimes M \in \mathcal{C}$ (using the structure of $\mathcal{C}$). We see that $X \otimes M$ is also a right A-module with action morphism given by $id \otimes a_M$. All necessary properties of the module category $\mathrm{Mod}_{\mathcal{C}}(A)$ now follow from the monoidal structure of $\mathcal{C}$.

Example 3.3.6. *Suppose that $\mathcal{C}$ is the monoidal category of right H-comodules for a Hopf algebra H, and that A is a coideal subalgebra. Then $Mod_{\mathcal{C}}(A)$ is equal to the category $\mathcal{M}_A^H$ defined in 2.2.2.*

Using Remark 3.3.5 and Example 3.3.6 above we can state the following

Corollary 3.3.7. *Any coideal subalgebra A of a Hopf algebra H gives rise to a module category over the category of H-comodules.*

Chapter 4

Module categories of the form $\mathbf{Mod}_{\mathcal{C}}(A)$

Chapter 3 finished with the interesting Corollary 3.3.7. It is natural to ask under what conditions we could have a converse version of it. Specifically, we need to find answers to two questions. Firstly, given a monoidal category $\mathcal{C}$, and a module category $\mathcal{M}$ over it, when is $\mathcal{M}$ equivalent to $\mathrm{Mod}_{\mathcal{C}}(A)$ for an algebra A in $\mathcal{C}$? And secondly, in the specific case where $\mathcal{C}$ is considered to be the category of H-comodules for a Hopf algebra H, when does an algebra A in $\mathcal{C}$ correspond to a coideal subalgebra of H? In the following sections, we will try to find an answer to these two questions.

4.1 Background material

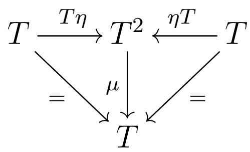
We start by recalling the theory of monads and comonads.

4.1.1 Monads

Definition 4.1.1. A **monad** T on a category $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\mu : T^2 \to T$ and $\eta : id_{\mathcal{C}} \to T$ such that the following diagrams:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \downarrow{\scriptstyle \mu T} & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

and



commute.

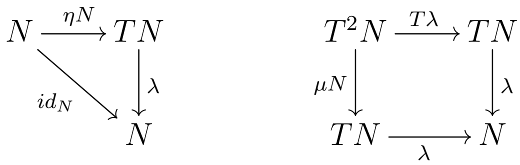
Remark 4.1.2. *In the above definition, by $T\mu$ at an object X we mean $T(\mu_X)$, and by μT at an object X we mean $\mu_{T(X)}$, where in general μ_Y denotes the morphism $\mu : T^2(Y) \to T(Y)$. The same notation is being used for η.*

Monads are closely related to adjoint functors. Specifically, it is known that a pair of adjoint functors gives rise to a monad.

Theorem 4.1.3. *Let $G : \mathcal{C} \to \mathcal{D}$ be a functor between two categories which admits a right adjoint $U : \mathcal{D} \to \mathcal{C}$ with adjunction morphisms $\eta : id \to UG$ and $\epsilon : GU \to id$. Then $T = (UG, \eta, U\epsilon G)$ is a monad on $\mathcal{C}$.*

For a proof of this very well known fact, see for example [9].

Definition 4.1.4. Let (T, μ, η) be a monad on a category $\mathcal{C}$. A T**-algebra** is a pair (N, λ) where $N \in \mathcal{C}$ and $\lambda : TN \to N$ is a morphism in $\mathcal{C}$ such that the following two diagrams:



commute. A map $f : (N, \lambda) \to (B, \lambda')$ is a map $f : N \to B$ in $\mathcal{C}$ such that $f \circ \lambda = \lambda' \circ Tf$.

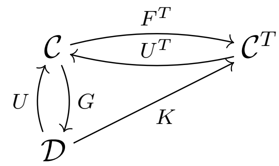
We denote the category of all T-algebras for a monad T in a category $\mathcal{C}$ by $\mathcal{C}^T$.

Remark 4.1.5. *We remark that every object $X \in \mathcal{C}$ gives rise to a T-algebra $(T(X), \mu_X)$.*

The assignment $X \mapsto (T(X), \mu_X)$ yields a functor $F^T : \mathcal{C} \to \mathcal{C}^T$ with $F^T f = Tf$. On the other hand every T-algebra (N, λ) can be seen as an object in $\mathcal{C}$ by considering the underlying object N and forgetting the structure given by λ. This gives a functor $U^T : \mathcal{C}^T \to \mathcal{C}$ with $U^T f = f$.

$(F^T, U^T, \eta, \epsilon_T)$ is a pair of adjoint functors and it is easy to see that it defines the given monad (T, μ, η). Moreover it satisfies a universal property:

For every adjoint pair (G, U, η, ϵ) between two categories $\mathcal{C}$ and $\mathcal{D}$ that defines the monad (T, μ, η) there is a unique functor $K : \mathcal{D} \to \mathcal{C}^T$, called the **comparison functor**, such that $KG = F^T$ and $U^T K = U$:



(See also [32]).

Definition 4.1.6. We say that an adjoint pair (G, U, η, ϵ) is **monadic** if the functor $K : \mathcal{D} \to \mathcal{C}^T$ is an equivalence.

The very well known Barr-Beck Monadicity theorem gives conditions under which an adjoint pair is monadic. Here, we give a version of the Barr-Beck theorem for abelian categories which can be found in [10].

Theorem 4.1.7. *Let $G : \mathcal{C} \to \mathcal{D}$ be an additive functor which admits a right adjoint $U : \mathcal{D} \to \mathcal{C}$ which is exact. Denote the defined monad by T. Then the comparison functor $K : \mathcal{D} \to \mathcal{C}^T$ is an equivalence if and only if U is faithful.*

4.1.2 Comonads

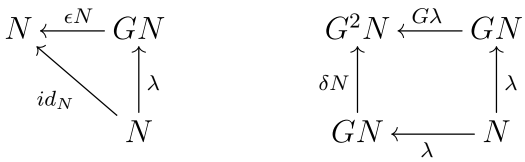
Similarly, we can define comonads and coalgebras for a comonad and also give a dual version of the Barr-Beck theorem.

Definition 4.1.8. A **comonad** in a category $\mathcal{B}$ is a monad in $\mathcal{B}^{op}$. Thus (G, ϵ, δ) is a comonad in $\mathcal{B}$ if G is an endofunctor of $\mathcal{B}$, and $\epsilon : G \to id$, $\delta : G \to G^2$ are natural transformations satisfying the dual versions of the diagrams of Definition 4.1.1 above.

As in the case of monads the following theorem holds:

Theorem 4.1.9. *Let $U : \mathcal{B} \to \mathcal{C}$ be a functor that admits a left adjoint $F : \mathcal{C} \to \mathcal{B}$ with adjunction morphisms $\eta : id \to UF$ and $\epsilon : FU \to id$. Then $(FU, \epsilon, F\eta U)$ is a comonad on $\mathcal{B}$.*

Definition 4.1.10. Let (G, ϵ, δ) be a comonad on a category $\mathcal{B}$. A G**-coalgebra** is a pair (N, λ) where $N \in \mathcal{B}$ and $\lambda : N \to GN$ is a morphism in $\mathcal{B}$ such that the following two diagrams:



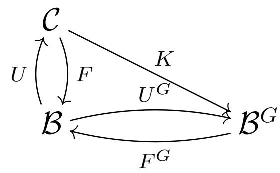
commute. A map $f : (N, \lambda) \to (B, \lambda')$ is a map $f : N \to B$ in $\mathcal{B}$ such that $Gf \circ \lambda = \lambda' \circ f$.

Remark 4.1.11. *We remark that every object $X \in \mathcal{B}$ gives rise to a G-coalgebra $(G(X), \delta_X)$.*

The assignment $X \mapsto (G(X), \delta_X)$ yields a functor $U^G : \mathcal{B} \to \mathcal{B}^G$ with $U^G f = Gf$. On the other hand every G-coalgebra (N, λ) can be seen as an object in $\mathcal{B}$ by considering the underlying object N and forgetting the structure given by λ. This gives a functor $F^G : \mathcal{B}^G \to \mathcal{B}$ with $F^G f = f$.

(F^G, U^G) is a pair of adjoint functors and it is easy to see that it defines the given comonad (G, ϵ, δ). Moreover it satisfies a universal property :

For every adjoint pair (F, U, η, ϵ) between two categories $\mathcal{C}$ and $\mathcal{B}$ that defines the comonad (G, ϵ, δ) there is a unique functor $K : \mathcal{C} \to \mathcal{B}^G$, called the **comparison functor**, such that $F = F^G K$ and $KU = U^G$:



Finally the Barr-Beck theorem for comonads has the following form :

Theorem 4.1.12. *Let $F : \mathcal{C} \to \mathcal{B}$ be an additive functor which is exact and faithful. Assume also that it admits a right adjoint $U : \mathcal{B} \to \mathcal{C}$. Denote the defined comonad on $\mathcal{B}$ by G. Then $\mathcal{C}$ is equivalent to the category $\mathcal{M}^G$ of G-coalgebras.*

4.2 Main theorem

Theorem 4.2.1. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category and let $\mathcal{M}$ be a module category over $\mathcal{C}$. Assume that there is a pair of adjoint functors $Res : \mathcal{C} \to \mathcal{M}$ and $Ind : \mathcal{M} \to \mathcal{C}$*

such that both Res and Ind are morphisms of module categories. Assume further that Ind is exact and faithful. Then $\mathcal{M}$ is equivalent to $Mod_{\mathcal{C}}(A)$ for an algebra $A \in \mathcal{C}$.

Proof. Without loss of generality (see also Theorem 3.1.5), we can assume that $\mathcal{C}$ is a strict monoidal category.

We start by noticing that Res and Ind define a monad $T = Ind \circ Res$ in $\mathcal{C}$.

We will split the proof in steps.

Step 1. *We show that $T(V) = V \otimes T(I)$ for all $V \in \mathcal{C}$.*

Since both Ind and Res are functors of module categories we have the following:

$$T(M \otimes N) \simeq Ind(Res(M \otimes N)) \simeq Ind(M \otimes Res(N)) \simeq M \otimes Ind(Res(N)) =$$

$$= M \otimes T(N), \text{ for every } M, N \in \mathcal{C}.$$

In particular, for every $V \in \mathcal{C}$ we have $V = V \otimes I$, hence $T(V) = T(V \otimes I) \simeq V \otimes T(I)$. Let us define $G : \mathcal{C} \to \mathcal{C}$ by $G = id \otimes T(I)$. Then by what we did above we see that T is naturally equivalent to G but to make notation simpler we will assume that $T = G$. We will use this notation later.

Step 2. *We show that $T(I)$ is an algebra in $\mathcal{C}$.*

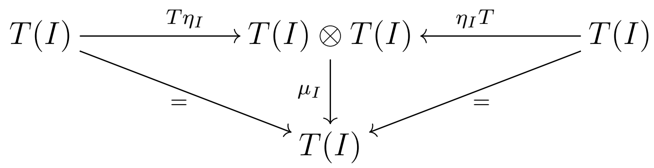
Since T is a monad, there are natural transformations $\mu : T^2 \to T$ and $\eta : id \to T$ which, as we saw before (Remark 4.1.5), endow every object $T(V)$ with the structure of a T-algebra. In particular, the pair $(T(I), \mu_I)$ is a T-algebra. This means that there exists a morphism $\mu_I : T^2(I) \to T(I)$. By Step 1, $T^2(I) \simeq T(I) \otimes T(I)$ hence $\mu_I : T(I) \otimes T(I) \to T(I)$. Moreover, by η we get a morphism $\eta_I : I \to T(I)$. It remains to show that $(T(I), \mu_I, \eta_I)$ satisfies the commutative diagrams from Definition 3.3.1. For the associativity diagram, we use the first diagram of Definition 4.1.1 of a monad at the object I. This gives us the following:

$$\begin{array}{ccc} T(I) \otimes T(I) \otimes T(I) & \xrightarrow{T\mu_I} & T(I) \otimes T(I) \\ \downarrow{\scriptstyle \mu_{T(I)}} & & \downarrow{\scriptstyle \mu_I} \\ T(I) \otimes T(I) & \xrightarrow{\mu_I} & T(I) \end{array}$$

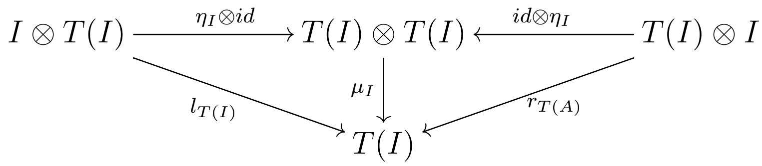
Recall now the functor G that we defined at at the end of Step 1. The natural transformations μ and η induce natural transformations $\tilde{\mu} : G^2 \to G$ and $\tilde{\eta} : id \to G$, so that $\tilde{\mu} = id \otimes \mu_I$ and $\tilde{\eta} = id \otimes \eta_I$. But $G = T$ and therefore $\mu = id \otimes \mu_I$ and similarly $\eta = id \otimes \eta_I$.

In particular $\mu_{T(I)} = id \otimes \mu_I$. Furthermore, $T\mu_I = \mu_I \otimes id$. So the diagram above gives us exactly what we wanted.

Similarly, the commutativity of the diagrams for η_I follows directly from the second diagram in Definition 4.1.1, which at the object $T(I)$ gives:



But $\mathcal{C}$ is strict ($T(I) = T(I) \otimes I = I \otimes T(I)$ and $id = r_{T(I)} = l_{T(I)}$). Moreover $T(\eta_I) = \eta_I \otimes id$ and, as we noticed above, $\eta_{T(I)} = id \otimes \eta_I$, which gives us :

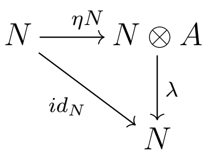


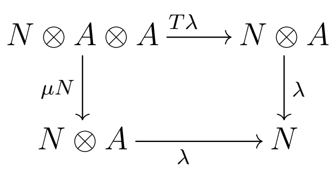
Again, this is exactly what we wanted.

- From now on we will denote $T(I)$ by $\mathbf{A}$.

Step 3. *We show that $\mathcal{M}$ is equivalent to the category of T-algebras.*
Since Ind is exact and faithful, by the Barr-Beck theorem for abelian categories, Theorem 4.1.7, we can conclude that $\mathcal{M}$ is equivalent to the category of T-algebras.

Step 4. *We show that the category of T-algebras is equivalent to $Mod_{\mathcal{C}}(A)$.*
Recall the diagrams from the definition of a T-algebra. For $T(N) = N \otimes A$ they give:





Using the identities from Step 2 (and recalling that $N = N \otimes I$ and $r_N = id$) we can rewrite this as:

$$\begin{array}{ccc} N \otimes I & \xrightarrow{id \otimes \eta_I} & N \otimes A \\ & \searrow^{r_N} & \downarrow \lambda \\ & & N \end{array} \qquad \begin{array}{ccc} N \otimes A \otimes A & \xrightarrow{\lambda \otimes id} & N \otimes A \\ \downarrow id \otimes \mu_I & & \downarrow \lambda \\ N \otimes A & \xrightarrow[\lambda]{} & N \end{array}$$

Similarly, morphisms between T-algebras are maps $f : N \to B$ in $\mathcal{C}$ such that $f \circ \lambda = \lambda' \circ Tf$. Since $T(N) = N \otimes A$ and $Tf = f \otimes id$ this gives us:

$$\begin{array}{ccc} N \otimes A & \xrightarrow{f \otimes id} & B \otimes A \\ \downarrow \lambda & & \downarrow \lambda' \\ N & \xrightarrow{a} & B \end{array}$$

But these are exactly the definitions of right A-modules and their morphsims. Therefore we have indeed shown that the category of T-algebras is the same as the category $\mathrm{Mod}_{\mathcal{C}}(A)$ of A-modules for the algebra A in $\mathcal{C}$.

This completes our proof.

□

Remark 4.2.2. *We notice that the converse of Theorem 4.2.1 is also true. Indeed, let $\mathcal{C}$ be a monoidal category and $Mod_{\mathcal{C}}(A)$ a module category of $\mathcal{C}$ with the action defined in 3.3.5. We can view $\mathcal{C}$ as a module category over itself using its monoidal structure. Then the functors $Res : \mathcal{C} \to Mod_{\mathcal{C}}(A)$ with $Res(X) = X \otimes A$ and $Ind : Mod_{\mathcal{C}}(A) \to \mathcal{C}$ with Ind being the forgetful functor (forgetting the A-module structure of objects) are clearly functors of module categories. Moreover Ind is then exact and faithful.*

4.3 Properties of $\mathrm{Mod}_{\mathcal{C}}(A)$ when $\mathcal{C}$ is $\mathcal{M}^H$ for a Hopf algebra H

We will investigate some properties of the module categories $\mathrm{Mod}_{\mathcal{C}}(A)$ when $\mathcal{C}$ is the category of H-comodules for a Hopf algebra H.

Theorem 4.3.1. *Let $\mathcal{D}$ be an abelian k-category which is cocomplete and locally presentable. Let F be an additive, cocontinuous, exact and faithful functor to the category Vect of vector spaces. Suppose moreover that the right adjoint of F is cocontinuous. Then $\mathcal{D}$ is equivalent to the category of C-comodules for a k-coalgebra C.*

Proof. First we notice that F has a right adjoint by the adjoint functor theorem for locally presentable categories. Let us denote this adjoint by Q and assume that Q is cocontinuous.

We let $G = F \circ Q$ be the corresponding comonad on Vect. G is cocontinuous as a composition of cocontinuous functors. It is moreover an endofunctor on Vect. We will show that G is isomorphic to tensoring with $G(\mathbb{C})$. Indeed, let V be a vector space. V is then the colimit of finite-dimensional vector spaces V_i and each such V_i can be assumed to be isomorphic to $\mathbb{C}^{n_i}$ for some $n_i \in \mathbb{N}$. Therefore $G(V) = G(\varinjlim \mathbb{C}^{n_i}) = \varinjlim G(\mathbb{C}^{n_i})$ since G is cocontinuous. But $G(\mathbb{C}^{n_i}) = G(\mathbb{C})^{n_i}$ since G is additive. If we let $C = G(\mathbb{C})$, we get that $G(\mathbb{C})^{n_i} = C^{n_i} = \mathbb{C}^{n_i} \otimes C$. Therefore, $\varinjlim G(\mathbb{C}^{n_i}) = \varinjlim(\mathbb{C}^{n_i} \otimes C) = \varinjlim \mathbb{C}^{n_i} \otimes C = V \otimes C$.

Now, as in the proof of Theorem 4.2.1, we can show that the comonad G endows C with the structure of a G-coalgebra, which in particular makes C a k-coalgebra. Also, it is then easy to see (following the same arguments) that the category of G-coalgebras is equivalent to the category of C-comodules. By the Barr-Beck theorem for comonads, since F is exact and faithful we can deduce that $\mathcal{D}$ is equivalent to G-coalgebras, and therefore to $\mathcal{M}^C$ for the coalgebra C.

□

Lemma 4.3.2. *Let $\mathcal{C}$ be the category of H-comodules for a Hopf algebra H. Then $Mod_{\mathcal{C}}(A)$, for an algebra $A \in \mathcal{C}$, is a locally presentable and cocomplete category.*

Proof. $\mathrm{Mod}_{\mathcal{C}}(A)$ is obviously cocomplete. So we only need to prove that it is locally presentable. Let $M \in \mathrm{Mod}_{\mathcal{C}}(A)$. This is an H-comodule endowed with an action map $a : M \otimes A \to M$ which is also a map of comodules. We take M_i a finite-dimensional H-subcomodule and consider the A-submodule it generates. We denote this by $< M_i >$. Notice that $M_i \subseteq < M_i >$ since A has a unit. The restriction of the action map a to $M_i \otimes A$ gives a surjection to $< M_i >$ and therefore $< M_i >$ is also an H-subcomodule as it is the image of a subcomodule. Now we consider a directed set I and to each $i \in I$ we associate $< M_i >$; whenever $i \leqslant j$ we consider the inclusion $\theta_i^j :< M_i > \to < M_j >$. Since $M = \varinjlim M_i$ as H-comodules it follows that $M = \varinjlim < M_i >$. Now we turn our attention to the $< M_i >$s in $\mathrm{Mod}_{\mathcal{C}}(A)$. First we notice that any object of the form $M_i \otimes A$ for a finite-dimensional H-subcomodule M_i is compact in $\mathrm{Mod}_{\mathcal{C}}(A)$. Indeed, by the adjunction (Res, Ind) between $\mathrm{Mod}_{\mathcal{C}}(A)$ and $\mathcal{M}^H$ we have that

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathcal{C}}(A)}(M_i \otimes A, N) \simeq \mathrm{Hom}_{\mathcal{M}^H}(M_i, N).$$

Moreover the finite dimensional H-comodules are compact objects in $\mathcal{M}^H$. Indeed, let V be a finite-dimensional H-comodule and consider $\mathrm{Hom}_{\mathcal{M}^H}(V, \varinjlim N_j)$. Let f be in $\mathrm{Hom}_{\mathcal{M}^H}(V, \varinjlim N_j)$. We will show that f factors through one of the inclusions $N_j \to \varinjlim N_j$. Since f is an H-comodule map it is also k-linear. Therefore $f \in \mathrm{Hom}_{\mathrm{Vect}}(V, \varinjlim N_j)$. But now V is a finite-dimensional vector space and hence compact, which means that f factors through some N_j. Since N_j is an H-comodule by assumption we conclude that V is compact in $\mathcal{M}^H$ as well. So in particular the finite-dimensional H-subcomodule M_i is compact and therefore $\mathrm{Hom}_{\mathcal{M}^H}(M_i, \varinjlim N_d) \simeq \varinjlim \mathrm{Hom}_{\mathcal{M}^H}(M_i, N_d)$ for every colimit $\varinjlim N_d$ over a filtered category D. So the same holds for the objects $M_i \otimes A$.

Now, consider again the restriction of a to $M_i \otimes A$ and let us denote it by $\tilde{a}$. Notice that $M_i \otimes A$ has the structure of a right A-module (where A acts by multiplication on itself). Then $\tilde{a} : M_i \otimes A \to < M_i >$ can be seen as a map of A-modules, and its kernel $ker(\tilde{a})$ is again an object in $\mathrm{Mod}_{\mathcal{C}}(A)$ such that $< M_i > \simeq (M_i \otimes A)/ker(\tilde{a})$. Let us take a closer look at the kernel $ker(\tilde{a})$ and let W_j be a finite-dimensional H-subcomodule of it. As before there exists a map $W_j \otimes A \to ker(\tilde{a})$. Again, for a directed set J we can consider inclusions $\theta_j^k : W_j \otimes A \to W_k \otimes A$, whenever $j \leq k$. The colimit of this diagram surjects to $Ker(\tilde{a})$; therefore for fixed i, $< M_i > \simeq M_i \otimes A / \varinjlim W_j \otimes A = \varinjlim (M_i \otimes A / W_j \otimes A)$. But now the quotients $(M_i \otimes A / W_j \otimes A)$ are compact objects since finite colimits of compact objects are compact. We have shown that each $< M_i >$ is a colimit of compact objects, therefore every $M \in \mathrm{Mod}_{\mathcal{C}}(A)$ is a colimit of compact objects.

□

Proposition 4.3.3. *Let $\mathcal{C}$ be the category of H-comodules for a Hopf algebra H. Consider the module category $Mod_{\mathcal{C}}(A)$ for an algebra $A \in \mathcal{C}$. Consider also the functor Ind: $Mod_{\mathcal{C}}(A) \to \mathcal{M}^H$. Then Ind has a right adjoint functor $\tilde{Q}$. Moreover if $\tilde{Q}$ is cocontinuous, then $Mod_{\mathcal{C}}(A)$ is equivalent to $\mathcal{M}^C$ for a k-coalgebra C.*

Proof. Since $\mathcal{C}$ is the category of right H-comodules, it is equipped with a forgetful functor Forget : $\mathcal{C} \to$ Vect to the category of vector spaces. The forgetful functor is the one that considers the underlying vector space and forgets the comodule structure, and is clearly exact and faithful. Now, the composition $F :=$ (Forget $\circ$ Ind) : $\mathrm{Mod}_{\mathcal{C}}(A) \to$ Vect is cocontinuous, exact and faithful, being the composition of two such functors. Therefore using Lemma 4.3.2 it can be deduced that F has a right adjoint functor Q. In this setting, Q can be computed and it is equal to

$Q(V) = \widetilde{\mathrm{HOM}}_k(A, V \otimes H)$ (which will be defined below). Indeed, we will show that

$$\mathrm{Hom}_{\mathrm{Vect}}(F(M), V) \simeq \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{C}}(A)}(M, \widetilde{\mathrm{HOM}}_k(A, V \otimes H)).$$

Since $- \otimes H$ is the right adjoint to Forget : $\mathcal{M}^H \to \mathrm{Vect}$ it is enough to show that the functor $\tilde{Q} : \mathcal{M}^H \to \mathrm{Mod}_{\mathcal{C}}(A)$ given by $N \mapsto \widetilde{\mathrm{HOM}}_k(A, N)$ is right adjoint to Ind. We recall the definition of HOM which can be found in [52] or [42]. Let A, N be H-comodules. The space $\mathrm{Hom}_k(A, N)$ is not necessarily an H-comodule. However there are maps as follows:

$$\omega : \mathrm{Hom}_k(A, N) \to \mathrm{Hom}_k(A, N \otimes H)$$

given by

$$\omega(f)(m) = f(a_{(0)})_{(0)} \otimes f(a_{(0)})_{(1)} S(a_{(1)})$$

and

$$\nu : \mathrm{Hom}_k(A, N) \otimes H \to \mathrm{Hom}_k(A, N \otimes H)$$

given by

$$\nu(f \otimes h) = f(m) \otimes h.$$

It is proved that ν is injective and the definition of HOM is given by:

$$\mathrm{HOM}_k(A, N) = \{f \in \mathrm{Hom}_k(A, N) | \omega(f) \in \mathrm{Im}(\nu)\}.$$

It is then shown in [52] that $\nu^{-1} \circ \omega$ defines a comodule structure on $\mathrm{HOM}_k(A, N)$.

We moreover notice that $\mathrm{HOM}_k(A, N)$ has a natural right A-module structure, where the A-action is given by $(f \cdot b)(a) := f(ba)$. We now define $\widetilde{\mathrm{HOM}}_k(A, N)$ as follows :

$$\widetilde{\mathrm{HOM}}_k(A, N) := \{f \in \mathrm{HOM}_k(A, N) | ((f \cdot b)_{(0)} \otimes (f \cdot b)_{(1)})(a) = (f_{(0)} b_{(0)} \otimes f_{(1)} b_{(1)})(a)\}$$

Then $\widetilde{\mathrm{HOM}}_k(A, N)$ is an object in $\mathrm{Mod}_{\mathcal{C}}(A)$ by its definition. In particular we defined $\widetilde{\mathrm{HOM}}_k(A, N)$ to consist of the objects in $\mathrm{HOM}_k(A, N)$ that are compatible with the natural right A-action.

We can now proceed to prove the adjunction.

Let $\Delta : \mathrm{Hom}_{\mathcal{M}^H}(Ind(M), N) \to \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{C}}(A)}(M, \widetilde{\mathrm{HOM}}_k(A, N))$ be defined as follows: If $\phi : M \to N$ is a morphism of H-comodules, $\Delta(\phi)$ is defined to be the morphism : $m \mapsto \tilde{\phi}_m$ where $\tilde{\phi}_m(a) = \phi(ma)$.

1. $\Delta(\phi)(m) \in \widetilde{\mathrm{HOM}}_k(A,N)$ for every $m \in M$. Indeed,

$$(\tilde{\phi}_m(a_{(0)}))_{(0)} \otimes (\tilde{\phi}_m(a_{(0)}))_{(1)} S(a_{(1)})$$
$$= (\phi(ma_{(0)}))_{(0)} \otimes (\phi(ma_{(0)}))_{(1)} S(a_{(1)})$$
$$= (\phi(ma_{(0)}))_{(0)} \otimes m_{(1)}(a_{(0)})_{(1)} S(a_{(1)}) \text{ (since } \phi \text{ is an H-comodule map)}$$
$$= \phi(m_{(0)}(a_{(0)})_{(0)}) \otimes m_{(1)}(a_{(0)})_{(1)} S(a_{(1)})$$
$$= (\phi(m_{(0)}a_{(0)})) \otimes m_{(1)}\epsilon(a_{(1)})$$
$$= (\phi(m_{(0)}a)) \otimes m_{(1)}$$
$$= (\tilde{\phi}_m(a))_{(0)} \otimes m_{(1)}.$$

Since $m_{(1)}$ is stable in $\tilde{\phi}_m$ it can be concluded that $\Delta(\phi)(m)$ lies in $\mathrm{HOM}_k(A,N)$. Moreover,

$$((\tilde{\phi}_m \cdot b)_{(0)} \otimes (\tilde{\phi}_m \cdot b)_{(1)})(a)$$
$$= ((\tilde{\phi}_m \cdot b)(a_{(0)}))_{(0)} \otimes ((\tilde{\phi}_m \cdot b)(a_{(0)}))_{(1)} S(a_{(1)})$$
$$= (\phi(mba_{(0)}))_{(0)} \otimes (\phi(mba_{(0)}))_{(1)} S(a_{(1)})$$
$$= (\phi(mba_{(0)}))_{(0)} \otimes m_{(1)}b_{(1)}(a_{(0)})_{(1)} S(a_{(1)})$$
$$= \phi(m_{(0)}b_{(0)}(a_{(0)})_{(0)}) \otimes m_{(1)}b_{(1)}(a_{(0)})_{(1)} S(a_{(1)})$$
$$= \phi(m_{(0)}b_{(0)}a_{(0)}) \otimes m_{(1)}b_{(1)}\epsilon(a_{(1)})$$
$$= \phi(m_{(0)}b_{(0)}a) \otimes m_{(1)}b_{(1)}$$
$$= (\tilde{\phi}_m(a))_{(0)}b_{(0)} \otimes m_{(1)}b_{(1)}$$
$$= ((\tilde{\phi}_m)_{(0)}b_{(0)} \otimes (\tilde{\phi}_m)_{(1)}b_{(1)})(a).$$

Therefore $\Delta(\phi)(m)$ lies in $\widetilde{HOM}_k(A,N)$.

2. $\Delta(\phi) : M \to \widetilde{\mathrm{HOM}}_k(A,N)$ is A-linear. Indeed, $mb \to \tilde{\phi}_{(mb)}$ and $\tilde{\phi}_{(mb)}(a) = \phi(mba) = (\tilde{\phi}_m \cdot b)(a)$.

3. $\Delta(\phi)$ is H-colinear. It is enough to show that $\tilde{\phi}_{m_{(0)}}(a) \otimes m_{(1)} = (\tilde{\phi}_m(a_{(0)}))_{(0)} \otimes (\tilde{\phi}_m(a_{(0)}))_{(1)} S(a_{(a)})$. But by 1 above we know that the right hand side is equal to $(\phi(ma))_{(0)} \otimes m_{(1)} = (\tilde{\phi}_m(a))_{(0)} \otimes m_{(1)}$. Since moreover $\phi(m)_{(0)} = \phi(m_{(0)})$, ϕ being a morphism of H-comodules, the result follows.

Finally, it is easy to see that Δ is one-to-one. Moreover, in the opposite direction, given an element in $\mathrm{Hom}_{\mathrm{Mod}_{\mathcal{C}}(A)}(M, \widetilde{\mathrm{HOM}}_k(A,N))$ we define an object in $\mathrm{Hom}_{\mathcal{M}^H}(Ind(M), N)$ by $\phi(m) := \tilde{\phi}_m(1_A)$. This is a well-defined map since the morphism $M \to \widetilde{\mathrm{HOM}}_k(A,N)$ is H-colinear. It is also a one-to-one map (if $\tilde{\phi}_m \neq \bar{\phi}_m$ then there exists an $a \in A$ such that $\tilde{\phi}_m(a) \neq \bar{\phi}_m(a)$ but $\tilde{\phi}_m(a) = \tilde{\phi}_m(1_A) \cdot a$ and the same is true for $\bar{\phi}_m(a)$). This means that there is a bijection between the two Hom sets and that the adjunction holds.

Finally we notice that if $\tilde{Q}$ is cocontinuous, then Q is also cocontinuous and by Theorem 4.3.1 we can conclude that $\mathrm{Mod}_{\mathcal{C}}(A)$ is equivalent to $\mathcal{M}^C$ for a k-coalgebra C.

□

Remark 4.3.4. *It would be interesting to find explicit conditions under which this adjoint functor is cocontinuous. In the above example it seems that some restrictions on the algebra A would be necessary.*

Chapter 5

Module categories corresponding to coideal subalgebras

In this chapter we are going to try to find the necessary conditions that a module category $\mathcal{M}$ over $\mathcal{M}^H$ must satisfy in order to be of the form $\mathrm{Mod}_{\mathcal{M}^H}(A)$ for a coideal subalgebra A of the Hopf algebra H.

5.1 Background material

We start by recalling some background material concerning categories of comodules.

5.1.1 Categories of comodules

The basic reference in this section is [51].

Definition 5.1.1. A comodule X in $\mathcal{M}^D$ for a coalgebra D is **quasi-finite** if $\mathrm{Hom}_{\mathcal{M}^D}(N, X)$ is finite-dimensional for all finite-dimensional comodules N.

Proposition 5.1.2 ([51], 1.3). *For a comodule $X \in \mathcal{M}^D$, X is quasi-finite if and only if the functor Vect $\to \mathcal{M}^D$, $W \mapsto W \otimes X$ has a left adjoint.*

When X is quasi-finite, the left adjoint of the functor $W \mapsto W \otimes X$ is called the Cohom functor. It is denoted by h_D and is written as $Y \mapsto h_D(X, Y)$ for a D-comodule Y. This functor has a behaviour similar to the behaviour of the Hom functor for algebras. If now X is a (Γ, D)-bicomodule and it is also quasi-finite, then $h_D(X, -)$ has the structure of a right Γ-comodule and the following proposition holds:

Proposition 5.1.3 ([51], 1.10). *For a (Γ, D)- bicomodule X the following are equivalent:*

1. *X is quasi-finite.*

2. *The functor $\mathcal{M}^\Gamma \to \mathcal{M}^D$, $Z \mapsto Z \underset{\Gamma}{\square} X$ has a left adjoint.*

In this case the left adjoint is the Cohom functor h_D.

Remark 5.1.4. *The set $Coend_D(M) = h_D(M, M)$ has the structure of a coalgebra. Thus M becomes a $(Coend_D(M), D)$-bicomodule (see also [51]) .*

The following proposition explains how "nice" functors between categories of comodules look and will be used heavily in the sequel.

Proposition 5.1.5 ([51], 2.1)**.** *Let $F : \mathcal{M}^\Gamma \to \mathcal{M}^D$ be a k-linear functor. If F is left exact and preserves direct sums, then there exists a (Γ, D)-bicomodule P such that $F(Z) = Z \underset{\Gamma}{\square} P$ as a functor of $Z \in \mathcal{M}^\Gamma$.*

5.1.2 Morita equivalence for categories of comodules

We are now in a position to state the Morita equivalence theorem for categories of comodules.

Definition 5.1.6. A set of **pre-equivalence data** $(\Gamma, D, {}_\Gamma P_D, {}_D Q_\Gamma, f, g)$ consists of coalgebras Γ and D, bicomodules ${}_\Gamma P_D$ and ${}_D Q_\Gamma$ and bicolinear maps $f : \Gamma \to P \underset{D}{\square} Q$ and $g : D \to Q \underset{\Gamma}{\square} P$ making the following diagrams commute:

$$\begin{array}{ccc} P & \xrightarrow{\simeq} & P \underset{D}{\square} D \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle id \square g} \\ \Gamma \underset{\Gamma}{\square} P & \xrightarrow[f \square id]{} & P \underset{D}{\square} Q \underset{\Gamma}{\square} P \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{\simeq} & Q \underset{\Gamma}{\square} \Gamma \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle id \square f} \\ D \underset{D}{\square} Q & \xrightarrow[g \square id]{} & Q \underset{\Gamma}{\square} P \underset{D}{\square} Q \end{array}$$

If f and g are isomorphisms then Γ and D are said to be Morita-Takeuchi equivalent and their categories of comodules are equivalent. The functors of the equivalence are given by $- \underset{\Gamma}{\square} P : \mathcal{M}^\Gamma \to \mathcal{M}^D$ and $- \underset{D}{\square} Q : \mathcal{M}^D \to \mathcal{M}^\Gamma$.

We finish this section by stating a theorem which characterizes all categories which are equivalent to a category of comodules over a coalgebra.

Definition 5.1.7. An abelian category $\mathcal{A}$ is **locally finite** if

1. $\mathcal{A}$ has direct sums.

2. For every directed family P_i of subobjects of P, the canonical map $\varinjlim P_i \to P$ induces an isomorphism $\varinjlim P_i \simeq \cup P_i$.

3. There is a set of generators M_i of $\mathcal{A}$ where each M_i is of finite length.

Definition 5.1.8. Let $\mathcal{A}$ be a locally finite abelian category. $\mathcal{A}$ is of **finite type** if $\mathrm{Hom}_{\mathcal{A}}(M, N)$ is finite-dimensional for all M, N in $\mathcal{A}$ of finite length.

Theorem 5.1.9 ([51], Theorem 5.1)**.** *Let $\mathcal{A}$ be a k abelian category. $\mathcal{A}$ is k-linearly equivalent to $\mathcal{M}^C$ for a coalgebra C if and only if $\mathcal{A}$ is of finite type.*

5.2 Main theorem

We are now in the position to prove the second main theorem of the categorical characterization. We start with the following lemma. The statement of this result is also found in [16] (after Example 2.15) where a dual version of this theorem is proved. Here we give a different and direct proof.

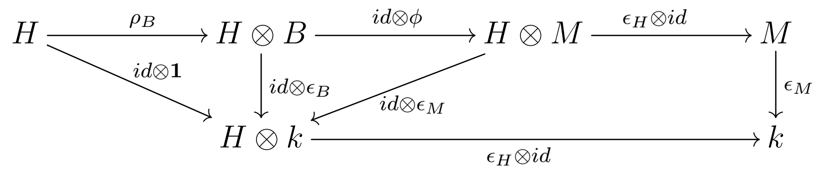
Lemma 5.2.1. *Let H, B be coalgebras and $\mathcal{M}^H, \mathcal{M}^B$ the corresponding categories of comodules. Assume that there exists a functor $\Psi : \mathcal{M}^H \to \mathcal{M}^B$ which carries the forgetful functor to the forgetful functor. Then there exists a map of coalgebras $\psi : H \to B$.*

Proof. Since Ψ carries the forgetful functor $f : \mathcal{M}^B \to \mathrm{Vect}$ to the forgetful functor $\bar{f} : \mathcal{M}^H \to \mathrm{Vect}$ we see that $\Psi(H) = H$ as a vector space, but it is now endowed with a B-comodule structure. Therefore H becomes an (H, B)-bicomodule via Ψ. We will show that there exists a bijection between (H, B)-bicomodule structures on H and coalgebra morphisms $H \to B$. It is obvious that indeed, every coalgebra morphism $\phi : H \to B$ endows H with the structure of a B-comodule via $(id \otimes \phi) \circ (\Delta_H) : H \to H \otimes B$. For the other direction, let H be an (H, B)-bicomodule. Using the Yoneda Lemma, we know that $\mathrm{Nat}(\mathrm{Hom}_{\mathrm{Coalg}}(B, -), \mathrm{Hom}_{\mathrm{Coalg}}(H, -)) \simeq \mathrm{Hom}_{\mathrm{Coalg}}(H, B)$. So it is enough to show that every (H, B)-bicomodule structure on H gives rise to a natural transformation $\tau : \mathrm{Hom}_{\mathrm{Coalg}}(B, -) \to \mathrm{Hom}_{\mathrm{Coalg}}(H, -)$. Let M be a coalgebra and assume that $\phi : B \to M$ is a coalgebra map. We will show that, using the (H, B)-bicomodule structure of H, we can induce a coalgebra map $\Phi : H \to M$. We define Φ as the composition of the following maps:

$$H \xrightarrow{\rho_B} H \otimes B \xrightarrow{id \otimes \phi} H \otimes M \xrightarrow{\epsilon_H \otimes id} M$$

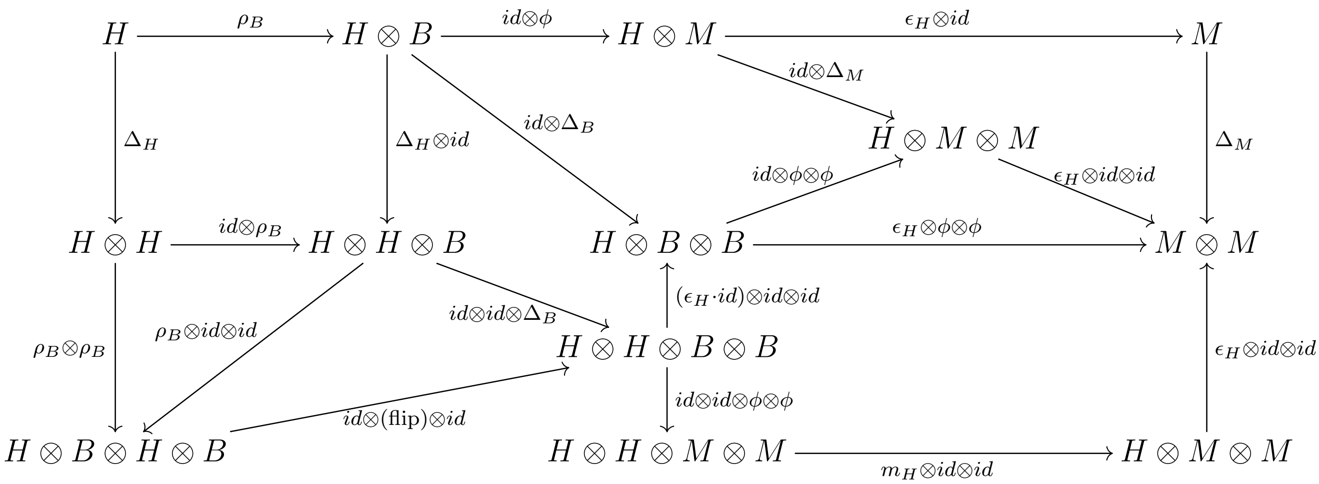
where ρ_B is the right B-comodule map on H.

First we will show that $\epsilon_H = \epsilon_M \circ \Phi$:



The first row is the map Φ; the last row, namely the composition $(\epsilon_H \otimes id) \circ (id \otimes \mathbf{1})$, is ϵ_H. Moreover, the first triangle is commutative because H is a right B-comodule; the second triangle because $\phi : B \to M$ is a coalgebra map; and the last square is commutative by its definition. Therefore it follows that $\epsilon_H = \epsilon_M \circ \Phi$.

Now we will show that $\Delta_M \circ \Phi = (\Phi \otimes \Phi)\Delta_H$:



All smaller diagrams in the above diagram are commutative. We will explain why by considering the diagrams at the top from left to right first. The first diagram is commutative by the definition of the (H, B)-bicomodule H which tells us that the comodule map $\rho_B : H \to H \otimes B$ is a map of left H-comodules. The second diagram is commutative because for every $h \in H$, we know that $\epsilon_H(h_{(1)})h_{(2)} = h$. The third diagram is commutative because $\phi : B \to M$ is a coalgebra map. Finally the two last diagrams at the top commute by their definitions. For the diagrams at the bottom, the first one commutes by its definition and the second one by the fact that H is a B-comodule. The last one is commutative because $\epsilon_H(\epsilon_H(h_{(1)})h_{(2)}) = \epsilon_H(h_{(1)})\epsilon_H(h_{(2)}) = \epsilon_H(h_{(1)}h_{(2)})$ since ϵ_H is k-linear and an algebra morphism. The top row is Φ. The composition of the last row $(\epsilon_H \otimes id \otimes id) \circ (m_H \otimes id \otimes id) \circ (id \otimes id \otimes \phi \otimes \phi) \circ (id \otimes (\text{flip}) \otimes id) \circ (\rho_B \otimes \rho_B) : H \otimes H \to M \otimes M$ is $\Phi \otimes \Phi$. Therefore we can deduce that $\Delta_M \circ \Phi = (\Phi \otimes \Phi)\Delta_H$.

□

Now by [51] we know that in a category of comodules $\mathcal{M}^H$, a comodule Y is called quasi-finite if the functor Vect $\to \mathcal{M}^H$ given by $V \mapsto V \otimes Y$ has a left adjoint denoted by h_Y . Therefore H is itself quasi-finite in $\mathcal{M}^H$ and in this case h_H is the forgetful functor. Moreover, it is also proved in [51] (Proposition 1.10), that when H is a (B, H)- bicomodule for a coalgebra B then h_H factors through the category $\mathcal{M}^B$ and the factorization map is left adjoint to the cotensor functor $-\underset{B}{\square} H : \mathcal{M}^B \to \mathcal{M}^H$.

This leads to the following:

Corollary 5.2.2. *Let H, B be coalgebras and assume that there exists a functor $\Psi : \mathcal{M}^H \to \mathcal{M}^B$ carrying the forgetful functor to the forgetful functor. Then Ψ is left adjoint to the cotensor functor $-\underset{B}{\square} H : \mathcal{M}^B \to \mathcal{M}^H$.*

Proof. Indeed, as we saw in 5.2.1 Ψ induces a unique coalgebra map $\psi : H \to B$.

On the other hand we notice also the following: Whenever we have a map of coalgebras $\psi : H \to B$, this induces a functor $\psi^* : \mathcal{M}^H \to \mathcal{M}^B$ as follows:

$$M \xrightarrow{\rho_M} M \otimes H \xrightarrow{id \otimes \psi} M \otimes B$$

This functor ψ^* obviously commutes with the two forgetful functors to Vect. Therefore we can identify Ψ with ψ^*. Moreover, ψ^* is known to be left adjoint to the cotensor functor $-\underset{B}{\square} H : \mathcal{M}^B \to \mathcal{M}^H$ (a proof of this appears in [8] (22.12)), and therefore Ψ is also left adjoint to the cotensor functor.

□

Lemma 5.2.3. *Assume that the coalgebra map $\psi : H \to B$ makes H left faithfully coflat over B. Then ψ is a surjective map.*

Proof. We first notice that from [38] (Theorem 3.1) we know that epimorphisms in the category of coalgebras are surjective maps.

The coalgebra map $\psi : H \to B$ yields the map $\psi \square id : H \underset{B}{\square} H \to B \underset{B}{\square} H \simeq H$ of H-comodules. By [8], 11.8 we know that the image of Δ_H is contained in $H \underset{B}{\square} H$ and that the composition of the maps

$$H \xrightarrow{\Delta_H} H \underset{B}{\square} H \xrightarrow{\psi \square id} B \underset{B}{\square} H \simeq H$$

yields the identity. Therefore, $\psi \Box id$ is a surjective map. Since the cotensor functor is exact and faithful it reflects epimorphisms, and this means that ψ is surjective.

$\Box$

It remains to see under which conditions the map ψ is also left H-linear.

Lemma 5.2.4. *Let $\mathcal{M}^B$ be a module category over $\mathcal{M}^H$. Suppose that the action of $\mathcal{M}^H$ on $\mathcal{M}^B$ commutes with the two forgetful functors to Vect, i.e. suppose that $X \otimes M$ is mapped to a B-comodule whose underlying vector space is equal to $X \otimes M$. Then the functor $\otimes : \mathcal{M}^H \times \mathcal{M}^B \to \mathcal{M}^B$ defining the module category $\mathcal{M}^B$ comes from a coalgebra map $\sigma : H \otimes B \to B$ which endows B with the structure of a left H-module.*

Proof. By its definition a module category $\mathcal{M}$ over a monoidal category $\mathcal{C}$ is a category together with an exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$. In particular $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is a functor from the Deligne tensor product of abelian categories $\mathcal{C} \boxtimes \mathcal{M}$ to $\mathcal{M}$. Recall here that the tensor product of abelian categories $\mathcal{A} \boxtimes \mathcal{B}$ is defined by the following requirement: That for each abelian category $\mathcal{D}$, the category of right exact functors $\mathcal{A} \boxtimes \mathcal{B} \to \mathcal{D}$ is equivalent to the category of functors $\mathcal{A} \times \mathcal{B} \to \mathcal{D}$ that are right exact in each variable (see also [15]). Consider now the case where $\mathcal{M}^B$ is a module category over $\mathcal{M}^H$. We first notice that $\mathcal{M}^H = \varinjlim \mathcal{M}^{H_i}$ where $H = \varinjlim H_i$ as a coalgebra. Indeed, every object $V \in \mathcal{M}^H$ is the colimit of its finite-dimensional vector spaces, and each such finite-dimensional vector space is a comodule over a finite-dimensional subcoalgebra of H. The same is true for the coalgebra $B = \varinjlim B_j$ and its category of comodules $\mathcal{M}^B = \varinjlim \mathcal{M}^{B_j}$. Therefore it can be concluded that $\mathcal{M}^H \boxtimes \mathcal{M}^B = \varinjlim \mathcal{M}^{H_i} \boxtimes \varinjlim \mathcal{M}^{B_j} = \varinjlim \varinjlim (\mathcal{M}^{H_i} \boxtimes \mathcal{M}^{B_j})$ (see also [15], 5.1 for this property of the tensor product of abelian categories). Now, it is known again by [15] that $\mathcal{M}^{H_i} \boxtimes \mathcal{M}^{B_j} \simeq \mathcal{M}^{H_i \otimes B_j}$ for finite-dimensional coalgebras. On the other hand, $\mathcal{M}^{H \otimes B} = \varinjlim \varinjlim \mathcal{M}^{H_i \otimes B_j}$ since every finite-dimensional subcoalgebra of $H \otimes B$ is of the form $H_i \otimes B_j$ for finite-dimensional subcoalgebras H_i, B_j of H and B respectively. Therefore $\mathcal{M}^H \boxtimes \mathcal{M}^B \simeq \mathcal{M}^{H \otimes B}$. This means that the exact bifunctor required from the definition of $\mathcal{M}^B$ as a module category over $\mathcal{M}^H$ is actually a functor from $\mathcal{M}^{H \otimes B} \to \mathcal{M}^B$. Moreover, this functor commutes with the forgetful functors to Vect (by assumption), and therefore, by Lemma 5.2.1 it comes from a coalgebra map $\sigma : H \otimes B \to B$. By the associativity axiom in the definition of a module category applied to the objects $H \otimes H \otimes B$, it moreover follows that this map σ satisfies the properties of a module map, endowing B with the structure of a left H-module. As a result of the above, the action of an object $X \in \mathcal{M}^H$ on an object $M \in \mathcal{M}^B$ can be interpreted as follows: The object in $\mathcal{M}^B$ obtained by this action is equal to $X \otimes M$ as a vector space, and the B-comodule structure is given by the following:

$$X \otimes M \xrightarrow{\rho_H \otimes \rho_B} X \otimes H \otimes M \otimes B \xrightarrow{id \otimes \text{flip} \otimes id} X \otimes M \otimes H \otimes B \xrightarrow{id \otimes id \otimes \sigma} X \otimes M \otimes B$$

□

Theorem 5.2.5. *Let $\mathcal{C}$ be the category of H-comodules for a Hopf algebra H with bijective antipode. Let $\mathcal{M}^B$ be a module category over $\mathcal{C}$ and assume that there exists a functor of module categories $\Psi : \mathcal{C} \to \mathcal{M}^B$ which commutes with the two forgetful functors and which has a right adjoint functor Ω that is exact and faithful. Then $\mathcal{M}^B \simeq Mod_{\mathcal{C}}(A)$ for a coideal subalgebra A of H, and H is faithfully flat over A.*

Proof. Notice that by Lemma 5.2.1 and Corollary 5.2.2 we can deduce that there exists a map of coalgebras $\psi : H \to B$ and that $\Psi = \psi^*$ is left adjoint to the cotensor functor $-\underset{B}{\square} H$. But Ψ is also left adjoint to Ω. Since Ω is exact and faithful, $-\underset{B}{\square} H$ must also be exact and faithful. But then by Lemma 5.2.3 we know that the coalgebra map $\psi : H \to B$ is a surjection. Now, we can consider the right coideal ${}^{\mathrm{co}B}H$ in H defined as follows: ${}^{\mathrm{co}B}H = \{h \in H | \psi(h_{(1)}) \otimes h_{(2)} = \psi(1) \otimes h\}$. We will show that this is also a subalgebra of H.

To do this we will use the arguments of Theorem 4.2.1. Indeed, the adjunction $(\Psi, -\underset{B}{\square} H)$ satisfies all the conditions of 4.2.1 and the composition of the two functors $\tilde{T} := (-)\underset{B}{\square} H \circ \Psi$ defines a monad on $\mathcal{C}$. But now the image of $I \simeq k$ under $\tilde{T}$ is $\tilde{T}(I) = \Psi(I)\underset{B}{\square} H$ which is isomorphic to ${}^{\mathrm{co}B}H$. Therefore ${}^{\mathrm{co}B}H$ is not only a right coideal of H but also an algebra in $\mathcal{M}^H$. Moreover, $\mathcal{M}^B \simeq \mathrm{Mod}_{\mathcal{M}^H}({}^{\mathrm{co}B}H)$ as module categories over $\mathcal{M}^H$. We need to show that the multiplication map μ_I on ${}^{\mathrm{co}B}H$ induced by the monad $\tilde{T}$ coincides with the multiplication on H. This is equivalent to showing that the coalgebra map $\psi : H \to B$ is left H-linear. By Lemma 5.2.4 we know that the action of $\mathcal{M}^H$ is induced by an H-module structure $\sigma : H \otimes B \to B$ on B. Moreover, since Ψ is a functor of module categories it follows that $\Psi(M \otimes V) \simeq M \otimes \Psi(V)$. Now, $M \otimes \Psi(V)$ is the B-comodule with underlying vector space $M \otimes V$ and comodule structure given by σ, therefore $m \otimes v \mapsto m_{(0)} \otimes v_{(0)} \otimes \sigma(m_{(1)}, \psi(v_{(1)}))$. On the other hand, $\Psi(M \otimes V)$ has comodule structure given by $m \otimes v \mapsto m_{(0)} \otimes v_{(0)} \otimes \psi(m_{(1)} v_{(1)})$. Therefore ψ must be compatible with the H-module structure on B.

This means that B has the structure of a left H-module quotient coalgebra. Moreover H is faithfully coflat over B. Using Corollary 2.3.5 it can be deduced that then ${}^{\mathrm{co}B}H$ is a right coideal subalgebra of H and that H is faithfully flat over ${}^{\mathrm{co}B}H$.

□

5.2.1 The converse of Theorem 5.2.5

In this section we prove that a quantum subgroup satisfies the assumptions of Theorem 5.2.5. This can be seen as a converse statement of the theorem. It also demonstrates how quantum subgroups fit into the categorical picture presented in this chapter.

The isomorphism maps γ and $\tilde{\gamma}$ used in the proof were pointed out to us by Uli Krähmer. The idea to add this section to the thesis was also his. I am grateful for his suggestions.

Proposition 5.2.6. *Let H be a Hopf algebra with bijective antipode and B a left H-module quotient coalgebra of H.Assume further that H is faithfully coflat over B. Let us denote the projection map by $\pi : H \to B$. Then $\mathcal{M}^B$ is a module category over $\mathcal{M}^H$. Moreover $\pi^* : \mathcal{M}^H \to \mathcal{M}^B$ is a functor of module categories which commutes with the two forgetful functors to Vect. Its right adjoint $-\underset{B}{\square}H : \mathcal{M}^B \to \mathcal{M}^H$ is exact and faithful and a functor of module categories.*

Proof. We have already seen how the coalgebra map $\pi : H \to B$ induces a functor $\pi^* : \mathcal{M}^H \to \mathcal{M}^B$ which is left adjoint to the cotensor functor $-\underset{B}{\square}H : \mathcal{M}^B \to \mathcal{M}^H$. Moreover π^* commutes with the two forgetful functors to Vect by its definition and $-\underset{B}{\square}H$ is exact and faithful since H is assumed to be faithfully coflat over B. It remains to show that both functors are functors of module categories.

We begin by π^*. It is enough to show that $\pi^*(Y \otimes N) \simeq Y\hat{\otimes}\pi^*(N)$ for every $Y, N \in \mathcal{M}^H$. We notice that both $\pi^*(Y \otimes N)$ and $Y\hat{\otimes}\pi^*(N)$ are equal to $Y \otimes N$ as vector spaces. We claim that the identiy map is also a morphism of B-comodules and therefore that they are isomorphic in $\mathcal{M}^B$. Indeed, the B-comodule structure on $\pi^*(Y \otimes N)$ is given by $y \otimes n \mapsto y_{(0)} \otimes n_{(0)} \otimes \pi(y_{(1)}n_{(1)})$. On the other hand the B-comodule structure on $Y\hat{\otimes}\pi^*(N)$ is given by $y\otimes n \mapsto y_{(0)} \otimes n_{(0)} \otimes y_{(1)}\pi(n_{(1)})$. Since π is left H-linear by assumption, the isomorphism follows.

We now proceed with the proof that $-\underset{B}{\square}H : \mathcal{M}^B \to \mathcal{M}^H$ is a functor of module categories. It is enough to show that $X \otimes (M\underset{B}{\square}H) \simeq (X\hat{\otimes}M)\underset{B}{\square}H$ in $\mathcal{M}^H$ for every $X \in \mathcal{M}^H$, $M \in \mathcal{M}^B$. We claim that the map

$$\gamma : X \otimes (M\underset{B}{\square}H) \to (X\hat{\otimes}M)\underset{B}{\square}H$$

given by

$$x \otimes (m \otimes h) \mapsto (x_{(0)} \otimes m) \otimes x_{(1)}h$$

is an isomorphism with inverse map $\tilde{\gamma}$ given by

$$(x \otimes m) \otimes h \mapsto x_{(0)} \otimes (m \otimes S(x_{(1)})h).$$

We start by showing that if $x \otimes (m \otimes h)$ is an element of $X \otimes (M \underset{B}{\square} H)$ then $(x_{(0)} \otimes m) \otimes x_{(1)}h$ is in $(X \hat{\otimes} M) \underset{B}{\square} H$. Recall that

$$M \underset{B}{\square} H = \{m \otimes h | m_{(0)} \otimes m_{(1)} \otimes h = m \otimes \pi(h_{(1)}) \otimes h_{(2)}\}.$$

Also note that

$$(X \hat{\otimes} M) \underset{B}{\square} H = \{x \otimes m \otimes h | x_{(0)} \otimes m_{(0)} \otimes x_{(1)} m_{(1)} \otimes h = x \otimes m \otimes \pi(h_{(1)}) \otimes h_{(2)}\}.$$

Consider $(x_{(0)} \otimes m) \otimes x_{(1)}h$. We need to show that $(x_{(0)})_{(0)} \otimes m_{(0)} \otimes (x_{(0)})_{(1)} m_{(1)} \otimes x_{(1)}h = x_{(0)} \otimes m \otimes \pi((x_{(1)})_{(1)} h_{(1)}) \otimes (x_{(1)})_{(2)} h_{(2)}$. Indeed,

$$(x_{(0)})_{(0)} \otimes m_{(0)} \otimes (x_{(0)})_{(1)} m_{(1)} \otimes x_{(1)}h$$

$$= x_{(0)} \otimes m_{(0)} \otimes (x_{(1)})_{(1)} m_{(1)} \otimes (x_{(1)})_{(2)} h$$

$$= x_{(0)} \otimes m \otimes (x_{(1)})_{(1)} \pi(h_{(1)}) \otimes (x_{(1)})_{(2)} h_{(2)}$$

$$= x_{(0)} \otimes m \otimes \pi((x_{(1)})_{(1)} h_{(1)}) \otimes (x_{(1)})_{(2)} h_{(2)}.$$

We further need to check that this map is a morphism of H-comodules. The H-comodule structure on $x \otimes (m \otimes h)$ is given by $x \otimes m \otimes h \mapsto x_{(0)} \otimes m \otimes h_{(1)} \otimes x_{(1)} h_{(2)}$. On the other hand, the H-comodule structure on $(x_{(0)} \otimes m) \otimes x_{(1)}h$ is given by $x_{(0)} \otimes m \otimes x_{(1)}h \mapsto x_{(0)} \otimes m \otimes (x_{(1)})_{(1)} h_{(1)} \otimes (x_{(1)})_{(2)} h_{(2)}$. By the above isomorphism $x_{(0)} \otimes m \otimes h_{(1)} \otimes x_{(1)} h_{(2)} \mapsto (x_{(0)})_{(0)} \otimes m \otimes (x_{(0)})_{(1)} h_{(1)} \otimes x_{(1)} h_{(2)}$. But the last expression is equal to $x_{(0)} \otimes m \otimes (x_{(1)})_{(1)} h_{(1)} \otimes (x_{(1)})_{(2)} h_{(2)}$. This is exactly what we wanted.

We will show now that $\tilde{\gamma}$ is also well defined. Let $(x \otimes m) \otimes h \in (X \hat{\otimes} M) \underset{B}{\square} H$. We want to show that $\tilde{\gamma}((x \otimes m) \otimes h) = x_{(0)} \otimes m \otimes S(x_{(1)})h \in X \otimes (M \underset{B}{\square} H)$. For this it is enough to show that $m_{(0)} \otimes m_{(1)} \otimes S(x_{(1)})h = m \otimes \pi((S(x_{(1)}))_{(1)} h_{(1)}) \otimes (S(x_{(1)}))_{(2)} h_{(2)} = m \otimes S((x_{(1)})_{(2)}) \pi(h_{(1)}) \otimes S((x_{(1)})_{(1)}) h_{(2)}$.

Since $(x \otimes m) \otimes h \in (X \hat{\otimes} M) \underset{B}{\square} H$, we know that:

$$x_{(0)} \otimes m_{(0)} \otimes x_{(1)} m_{(1)} \otimes h = x \otimes m \otimes \pi(h_{(1)}) \otimes h_{(2)}$$

$$\Rightarrow x_{(0)} \otimes m_{(0)} \otimes S((x_{(1)})_{(2)}) x_{(1)} m_{(1)} \otimes h = x \otimes m \otimes S((x_{(1)})_{(2)}) \pi(h_{(1)}) \otimes h_{(2)}$$

$$\Rightarrow x_{(0)} \otimes m_{(0)} \otimes S((x_{(1)})_{(2)}) x_{(1)} m_{(1)} \otimes S((x_{(1)})_{(1)}) h = x \otimes m \otimes S((x_{(1)})_{(2)}) \pi(h_{(1)}) \otimes S((x_{(1)})_{(1)}) h_{(2)}.$$

We now consider only the left hand side of the above equation and have the following:

$$x_{(0)} \otimes m_{(0)} \otimes S((x_{(1)})_{(2)})x_{(1)}m_{(1)} \otimes S((x_{(1)})_{(1)})h$$
$$= (x_{(0)})_{(0)} \otimes m_{(0)} \otimes S(x_{(1)})x_{(1)}m_{(1)} \otimes S((x_{(0)})_{(1)})h$$
$$= x_{(0)} \otimes m_{(0)} \otimes S(x_{(1)})(x_{(1)})_{(2)}m_{(1)} \otimes S((x_{(1)})_{(1)})h$$
$$= x_{(0)} \otimes m_{(0)} \otimes S((x_{(1)})_{(1)}\epsilon((x_{(1)})_{(2)}))(x_{(1)})_{(2)}m_{(1)} \otimes S((x_{(1)})_{(1)})h$$
$$= x_{(0)} \otimes m_{(0)} \otimes \epsilon(x_{(1)})\epsilon((x_{(1)})_{(2)}))m_{(1)} \otimes S((x_{(1)})_{(1)})h$$
$$= x \otimes m_{(0)} \otimes m_{(1)} \otimes S(x_{(1)})h.$$

This means that $x \otimes m_{(0)} \otimes m_{(1)} \otimes S(x_{(1)})h = x \otimes m \otimes S((x_{(1)})_{(2)})\pi(h_{(1)}) \otimes S((x_{(1)})_{(1)})h_{(2)}$ and therefore we can conclude that $\tilde{\gamma}$ is indeed well defined.

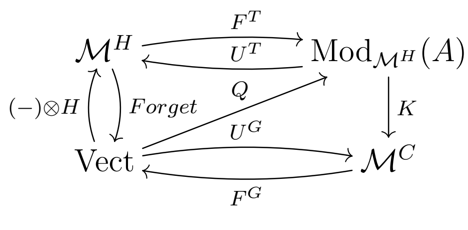
Finally, it is easy to see that γ is one-to-one. To show that it is also surjective, we consider the composition with the map $\tilde{\gamma}$ and show that it yields the identity map on $(X\hat{\otimes}M)\underset{B}{\square}H$. Indeed,

$$\gamma \circ \tilde{\gamma}(x \otimes m \otimes h)$$
$$= \gamma(x_{(0)} \otimes m \otimes S(x_{(1)})h)$$
$$= (x_{(0)})_{(0)} \otimes m \otimes (x_{(0)})_{(1)}S(x_{(1)})h$$
$$= x_{(0)} \otimes m \otimes (x_{(1)})_{(1)}S((x_{(1)})_{(2)})h$$
$$= x_{(0)} \otimes m \otimes \epsilon(x_{(1)})h$$
$$= x \otimes m \otimes h.$$

□

5.3 Morita-Takeuchi equivalence

Recall the results from 4.3.3 where we assumed that the right adjoint Q is cocontinuous. We are then in a situation as follows:



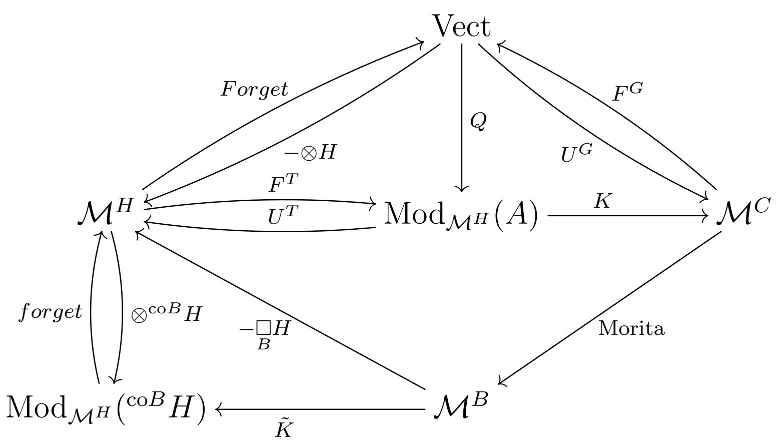
(F^T, U^T) is the monad adjunction for the monad defined in Theorem 4.2.1 using the functors Res and Ind. F^T is then the functor of tensoring an H-comodule with

A, and U^T is exact and faithful. Similarly, (F^G, U^G) is the comonad adjunction from Theorem 4.3.1 where F^G is the forgetful functor. K is an equivalence.

We would like to use Theorem 5.2.5 to deduce that C is actually a quotient left H-module coalgebra. However, the functor $F^T \circ K$ does not carry the forgetful functor to the forgetful functor. $F^T(V) = V \otimes A$ and $K(W) = W$.

We believe however that $\mathcal{M}^C$ is Morita-Takeuchi equivalent to a category of comodules over a coalgebra B such that the composition with the Morita equivalence yields a functor of module categories $\Psi : \mathcal{C} \to \mathcal{M}^B$ satisfying all conditions of Theorem 5.2.5.

This would yield the following diagram:



In this direction, we believe that an extra condition needed to proceed from the results of chapter 4 to the main theorem 5.2.5 of chapter 5 is that H is an object in $\mathrm{Mod}_{\mathcal{M}^H}(A)$.

Part III

An application: Quantizing spherical subgroups

Chapter 6

Quantizing spherical subgroups

Let us recall the definition of a spherical variety. We take X to be a normal algebraic variety, G a reductive and connected algebraic group acting on X and B a Borel subgroup of G. We say that X is a spherical variety, if X contains an open orbit under the action of B. Similarly, the homogeneous space G/H is called spherical if it contains an open orbit under the action of a Borel subgroup B of G.

As we already mentioned in the Introduction, spherical varieties have received a lot of attention. After the work of many researchers many interesting results have now been proved, the most important being the classification of all spherical homogeneous spaces, hence of all spherical subgroups.

In this chapter we use the classification results to define quantum counterparts for spherical subgroups. This project is still in progress, and will be completed in the future.

6.1 Classification of spherical subgroups

We already mentioned that a homogeneous space G/H is spherical if it contains an open B orbit. It would be useful, for our intuition, to see some equivalent versions of defining this.

Definition 6.1.1 (Definition-Theorem)**.** A homogeneous space G/H is spherical if any of the following equivalent properties holds:

1. B has an open orbit on G/H.

2. H has an open orbit on G/B.

3. For every irreducible G-module V and any character χ of H

$$\dim\{v \in V | hv = \chi(h)v \text{ for all } h \in H\} \leq 1.$$

4. Any G/H embedding contains finitely many G-orbits.

Let us recall briefly the classification results. The pairs (G, H) where G is simple and simply connected and H is reductive connected have been classified by M. Krämer in [27]. Brion in [7] classifies all spherical subgroups. To do this he uses an "inductive" process. First he proves that it is enough to reduce to the case where H is reductive. He then proceeds with the classification of pairs where G is simply connected and H a connected reductive subgroup. Then he finds all pairs (G, H) such that all subgroups between H and G are reductive. Finally he deals with the remaining cases. He remarks that it is enough to look at indecomposable spherical pairs (products of spherical pairs are spherical) and to assume that G is semisimple.

The classification of spherical pairs can be summarized in the following lists:

Theorem 6.1.2 ([7], Theoreme)**.** *Let (G, H) be an indecomposable spherical pair where G is a semisimple simply connected group and H a reductive connected subgroup. Then (G, H) is one of the following pairs:*

1. *G is simple and H appears in Krämer's list below.*

2. *H is simple and is diagonally embedded in $G = H \times H$.*

3. *$H \simeq SL(2)$ and $G = H \times H \times H$ (diagonally embedded).*

4. *There exist three reductive groups G', H', K' such that $G = H' \times G'$ and $H = H' \times K'$ and an embedding $i : H' \times K' \to G'$. We consider H to be embedded in G by*

$$H' \times K' \to H' \times G'$$

$$(u, v) \to (u, i(u, v)).$$

Moreover G', H', K' are one of the following triplets :

(a) $Sp(2n+2), SL(2), Sp(2n)$

(b) $Sp(2n+4), Sp(4), Sp(2n)$

(c) $SO(n+1), SO(n), \{1\}$

(d) $SO(8), Spin(7), \{1\}$.

5. $G = Sp(2m+2) \times Sp(2n+2)$ *and* $H = SL(2) \times Sp(2m) \times Sp(2n)$ *where* $(t,u,v) \to (t \oplus u, t \oplus v)$.

6. $G = Sp(4) \times Sp(2m+2) \times Sp(2n+2)$ *and* $H = SL(2) \times SL(2) \times Sp(2m) \times Sp(2n)$ *where* $(t,u,v,w) \to (t \oplus u, t \oplus v, t \oplus w)$.

7. $G = Sp(2l+2) \times Sp(2m+2) \times Sp(2n+2)$ *and* $H = SL(2) \times Sp(2l) \times Sp(2m) \times Sp(2n)$ *where* $(t,u,v,w) \to (t \oplus u, t \oplus v, t \oplus w)$.

8. $G = SL(n) \times SL(n+1)$ *and* $H = SL(n) \times \mathbb{G}_m$ *where* $(u,\lambda) \to (u, \lambda u \oplus \lambda^{-n})$.

9. $G = SL(m+2) \times Sp(2n+2)$ *and* $H = SL(2) \times SL(m) \times Sp(2n)$ *or* $H = SL(2) \times SL(m) \times Sp(2n) \times \mathbb{G}_m$ *where* $(u,v,w,\lambda) \to (\lambda^m u \oplus \lambda^{-2} v, u \oplus w)$.

Table 6.1: Krämer's List.

	G	H
1.	$SL(n), n \geq 2$	$SO(n)$
2.	$SL(m+n),\ m \geq n \geq 1$	$S(GL(m) \times GL(n))$
3.	$SL(m+n),\ m \geq n \geq 1$	$SL(m) \times SL(n)$
4.	$SL(2n),\ n \geq 2$	$Sp(2n)$
5.	$SL(2n+1),\ n \geq 1$	$\mathbb{G}_m \cdot Sp(2n)$
6.	$SL(2n+1),\ n \geq 1$	$Sp(2n)$
7.	$Sp(2n),\ n \geq 1$	$GL(n)$
8.	$Sp(2n),\ n \geq 2$	$\mathbb{G}_m \times Sp(2n-2)$
9.	$Sp(m+n),\ m,n \geq 2$ even	$Sp(m) \times Sp(n)$
10.	$SO(2n),\ n \geq 2$	$GL(n)$
11.	$SO(2n),\ n \geq 3$ odd	$SL(n)$
12.	$SO(8)$	$Sp(4) \otimes Sp(2)$
13.	$SO(8)$	$Spin(7)$
14.	$SO(8)$	G_2
15.	$SO(10)$	$SO(2) \times Spin(7)$
16.	$SO(2n+1),\ n \geq 2$	$GL(n)$
17.	$SO(m+n),\ m \geq n \geq 1$	$SO(m) \times SO(n)$
18.	$SO(9)$	$Spin(7)$
19.	$SO(7)$	G_2
20.	G_2	A_2
21.	G_2	$A_1 \times A_1$
22.	F_4	B_4
23.	F_4	$B_3 \times A_1$
24.	E_6	C_4
25.	E_6	F_4
26.	E_6	D_5
27.	E_6	$\mathbb{G}_m \cdot D_5$
28.	E_6	$A_5 \times A_1$
29.	E_7	$\mathbb{G}_m \cdot E_6$
30.	E_7	A_7
31.	E_7	$D_6 \times A_1$
32.	E_8	D_8
33.	E_8	$E_7 \times A_1$

6.2 Quantum symmetric spaces – Letzter's construction

In this section we recall the main steps of the construction of quantum symmetric spaces, as it appears in Letzter's work in [28]. We do this for two reasons. Firstly, Letzter's work is an excellent example of defining quantum subgroups using the language of coideal subalgebras. Secondly, symmetric spaces are an example of spherical homogeneous spaces. If we want to quantize all spherical subgroups, we must take into account the fact that the work for symmetric spaces has already been done.

6.2.1 Notation

Let us start by briefly recalling Letzter's notation:

Let $\mathfrak{g} = n^- \oplus h \oplus n^+$ be a semisimple Lie algebra over an algebraically closed field k of characteristic zero with Cartan matrix (α_{ij}). Let also $\pi = \{\alpha_1, ..., \alpha_l\}$ be the set of positive simple roots. We write e_i, f_i, h_i for the standard generators of $\mathfrak{g}$. Finally we define $\mathcal{Q} = \sum \mathbb{Z}\alpha_i$ and $\mathcal{Q}^+ = \sum \mathbb{N}\alpha_i$. Now if $x_1, ...x_l, y_1, ..., y_l, t_1, ..., t_l, t_1^{-1}, ..., t_l^{-1}$ are the generators of the quantized enveloping algebra $U_q(\mathfrak{g})$ (as was defined in 1.1.1) we denote by τ the isomorphism of abelian groups from $\mathcal{Q}$ to the group $< t_1, ..., t_l >$ given by $\tau(\alpha_i) = t_i$.

However Letzter uses an extension U of $U_q(\mathfrak{g})$ by adding the elements

$$t_1^{1/2}, ..., t_l^{1/2}, t_1^{-1/2}, ..., t_l^{-1/2}$$

and the scalars

$$q^{1/2}, q^{-1/2}.$$

In particular U is a $k(q^{1/2})$-algebra which satisfies the usual relations of $U_q(\mathfrak{g})$ with the following added relations: If $v \in U_q(\mathfrak{g})$ is of weight λ, then $t_i^{1/2} v t_i^{-1/2} = q^{(\alpha_i/2, \lambda)} v$ and $t_i^{-1/2} v t_i^{1/2} = q^{(-\alpha_i/2, \lambda)} v$. She denotes by T the group generated by $t_1^{1/2}, ..., t_l^{1/2}$. Notice that the Hopf structure can be extended to U for all elements of T.

6.2.2 Classical and quantum involutions

We are now ready to describe the main steps of the construction. Throughout the work we are working with a Lie algebra $\mathfrak{g}$ and a Lie agebra involution θ satisfying the following three conditions:

$$\theta(\mathfrak{h}) = \mathfrak{h} \tag{6.1}$$

$$\theta(e_i) = e_i \text{ and } \theta(f_i) = f_i \text{ when } \theta(h_i) = h_i \tag{6.2}$$

$$\theta(e_i)\,(\text{resp.}\,\theta(f_i)) \text{ is a nonzero root vector in } n^- \,(\text{resp.}\, n^+) \text{ if } \theta(h_i) \neq h_i \tag{6.3}$$

Then θ induces an automorphism of the root system of $\mathfrak{g}$ which we denote by Θ. We also define the equivalence class of the set of involutions inducing the same root automorphism and denote this class by $[\Theta]$. We notice that $\Theta(\pi)$ is a new base for the root system of $\mathfrak{g}$ and define the set

$$\pi_\Theta = \{\alpha_i \in \pi | \Theta(\alpha_i) = \alpha_i\}.$$

Since $\Theta(\pi)$ is a basis for the vector space spanned by the roots we must have a permutation σ on the set $\{i \,|\, \alpha_i \in \pi - \pi_\Theta\}$ such that for each $\alpha_i \in \pi - \pi_\Theta$

$$\Theta(\alpha_i) + \alpha_{\sigma(i)} \in \sum_{\alpha \in \pi_\Theta} \mathbb{Z}\alpha.$$

Letzter then in section 3 of [28] proceeds in order to lift this involution θ to the quantum case by quantizing the root automorphism Θ. She ends up with a quantized class $[\Theta]_q$ of k-algebra involutions of U. For this class she proves the theorem below.

Before stating the theorem we recall that by U^+ (respectively U^-) we denote the $k(q^{1/2})$ subalgebra generated by the x_i (respectively the y_i). Also, $G^- = \sum_{\beta \in \mathcal{Q}^+} U^-_{-\beta}\tau(\beta)$ and finally U^+_β denotes the weight space of U^+ of weight β (similarly for U^-). (Recall that both U^+ and U^- are direct sums of finite-dimensional weight spaces where the weights are elements of $\mathcal{Q}^+$ for the former and $-\mathcal{Q}^+$ for the latter).

Theorem 6.2.1 ([28], Theorem 3.1)**.** *Let θ be an involution of $\mathfrak{g}$ satisfying the relations (3.1), (3.2) and (3.3) and inducing the root system automorphism Θ. Then the set of involutions $[\Theta]_q$ specializes to the set $[\Theta]$. Moreover, for all $\tilde{\theta} \in [\Theta]_q$ we have the following:*

$$\tilde{\theta}(\tau(\lambda)) = \tau(-\Theta(\lambda)) \text{ for all } \tau(\lambda) \in T \tag{6.4}$$

$$\tilde{\theta}(q^{1/2}) = q^{-1/2} \tag{6.5}$$

$$\tilde{\theta}(x_i) = x_i \text{ and } \tilde{\theta}(y_i) = y_i \text{ for all } \alpha_i \in \pi_\Theta \tag{6.6}$$

$$\tilde{\theta}(x_i) \text{ in a nonzero element of } G^-_{\Theta(\alpha_i)} \text{ for all } \alpha_i \notin \pi_\Theta \tag{6.7}$$

$$\tilde{\theta}(y_i t_i) \text{ in a nonzero element of } U^+_{\Theta(-\alpha_i)} \text{ for all } \alpha_i \notin \pi_\Theta \tag{6.8}$$

6.2.3 Quantum $U(\mathfrak{g}^\theta)$

Let $[\Theta]$ be a fixed class of involutions of $\mathfrak{g}$ and $[\Theta]_q$ its quantum analogue. We are now in the position to define quantum analogues of the subalgebra $U(\mathfrak{g}^\theta)$ for all $\theta \in [\Theta]$. This is presented in section 4 of [28].

Let $\tilde{\theta}$ be an element in $[\Theta]_q$ and let $\theta \in [\Theta]$ be the involution such that $\tilde{\theta}$ specializes to θ. We define

$$T_\Theta = \{\tau(\lambda) \in T \,|\, \Theta(\lambda) = \lambda\}.$$

We let R denote the subalgebra of U generated by T_Θ and the set $\{x_i, y_i \,|\, \alpha_i \in \pi_\Theta\}$. We set also:

$$\tilde{x_i} = x_i t_i^{-1/2} \tau(\Theta(\alpha_i))^{1/2} \text{ and } \tilde{y_i} = y_i t_i^{1/2} \tau(\Theta(\alpha_i))^{1/2}$$

for each i such that $\alpha_i \notin \pi_\Theta$. Similarly:

$$\tilde{x_i} = x_i \text{ and } \tilde{y_i} = y_i$$

for each i such that $\alpha_i \in \pi_\Theta$. Finally we set:

$$B_i = \tilde{x_i} + \tilde{\theta}(\tilde{x_i})$$

for each $1 \leq i \leq l$.

Definition 6.2.2. We define $B_{\tilde{\theta}}$ to be the subalgebra of U generated by R and all the B_i where $\alpha_i \in \pi - \pi_\Theta$.

Letzter then continues to prove the following

Theorem 6.2.3. *$B_{\tilde{\theta}}$ is a right coideal subalgebra of U that specializes to $U(\mathfrak{g}^\theta)$.*

Finally, she proves a uniqueness result in section 5 of [28]. Specifically she shows that any maximal right coideal subalgebra specializing to $U(\mathfrak{g}^\theta)$ is isomorphic to one of the subalgebras $B_{\tilde{\theta}}$.

6.3 New examples

The work of Letzter presented in the previous section can be used to quantize all spherical pairs that are symmetric spaces. From Krämer's list on Table 3.1 most of the examples do actually correspond to the symmetric case. The pairs that are not symmetric spaces are the following: 3, 5, 6, 8, 11, 13, 14, 15, 16, 18, 19, 20 and 27. So in order to quantize all spherical subgroups we need to quantize the remaining ones. Moreover, we have to look at Brion's cases separately. In this section we present some of these quantizations, which, besides the more general project, provide new examples of quantum subgroups.

6.3.1 Brion's list

We first look at new examples from Brion's List in 6.1.2.

Case 2 : Quantum diagonal embedding

I am grateful to Uli Krähmer for suggesting to me the quantum double approach which makes this example work.

We let G be a reductive subgroup and consider its diagonal embedding $\delta : G \to G \times G$. At the level of the coordinate algebras, this corresponds to the multiplication map $m : \mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G)$. We would like to use the same idea to quantize the diagonal embedding. Unfortunately there is a serious problem with this. The multiplication map $m : \mathcal{O}_q(G) \otimes \mathcal{O}_q(G) \to \mathcal{O}_q(G)$ is indeed a surjective map of coalgebras, giving us a quotient coalgebra, but it is not an algebra map, therefore $\mathcal{O}_q(G)$ does not become a left $(\mathcal{O}_q(G) \otimes \mathcal{O}_q(G))$-module quotient coalgebra, as the definition of a quantum subgroup would require. However there is a nonstandard quantization of $\mathcal{O}(G \times G)$, the quantum double $\mathcal{O}_q(G) \bowtie \mathcal{O}_q(G)$, which makes the multiplication map a Hopf algebra map.

We will briefly recall the construction of this here and see how this gives us a quantum subgroup in the sense of our definition 2.4.3.

We start by giving the definition of $A \bowtie A$ found in [26].

Let A be a coquasitriangular Hopf algebra with universal r-form r. Then we can form the quantum double $A \bowtie A$, which is again a Hopf algebra but with a twisted product. In particular $A \bowtie A$ is the tensor product coalgebra $A \otimes A$ endowed with the following product:

$$(a \otimes b)(c \otimes d) = (ac_{(2)} \otimes b_{(2)}d)\bar{r}(b_{(1)} \otimes c_{(1)}r(b_{(3)} \otimes c_{(3)})$$

where $\bar{r}$ is the convolution inverse of r. The antipode is given by $S(a \otimes b) = (1 \otimes S(b))(S(a) \otimes 1)$.

For more details on the quantum double one can look [21], [19] and [26].

With this definition in hand we have the following:

Theorem 6.3.1 ([19], Proposition 3.1)**.** *The multiplication map $m : A \bowtie A \to A$ given by $(a \otimes b) \mapsto ab$ is a Hopf algebra homomorphism.*

In the case where $A = \mathcal{O}_q(G)$ a universal r-form is obtained in [21] using the Rosso form of $U_q(\mathfrak{g})$. This makes $\mathcal{O}_q(G)$ coquasitriangular. It is moreover mentioned in [21] that in this case the above mutliplication map $m : \mathcal{O}_q(G) \bowtie \mathcal{O}_q(G) \to \mathcal{O}_q(G)$ corresponds to the embedding of G into $G \times G$. This agrees completely with our definition, as in this case $\mathcal{O}_q(G)$ becomes through m a left $\mathcal{O}_q(G) \bowtie \mathcal{O}_q(G)$-module quotient coalgebra. Finally, since G is assumed to be reductive we know that $\mathcal{O}_q(G)$ is, for q generic, cosemisimple which implies that $\mathcal{O}_q(G) \bowtie \mathcal{O}_q(G)$ is faithfully coflat over $\mathcal{O}_q(G)$.

Case 4: $\mathbf{H}' \times \mathbf{K}' \to \mathbf{H}' \times \mathbf{G}'$, $(\mathbf{u}, \mathbf{v}) \to (\mathbf{u}, \mathbf{i}(\mathbf{u}, \mathbf{v}))$

In all these cases, it is enough to show that the embedding $i : U_q(\mathfrak{h}') \otimes U_q(\mathfrak{k}') \to U_q(\mathfrak{g}')$ gives a coideal subalgebra in $U_q(\mathfrak{g}')$. Indeed, then $U_q(\mathfrak{h}') \otimes \mathrm{Im}(i(U_q(\mathfrak{h}') \otimes U_q(\mathfrak{k}')))$ will be a coideal subalgebra of $U_q(\mathfrak{h}') \otimes U_q(\mathfrak{g}')$ as well.

- Case 4(a): $SL(2) \times Sp(2n) \leq Sp(2n+2)$
  This case is important because once quantized, it can be used for the quantization of some of the following cases in Brion's list. The first idea would be to send the generators of $U_q(\mathfrak{sl}_2)$ to the first three generators of $U_q(\mathfrak{sp}_{2n+2})$ and the generators of $U_q(\mathfrak{sp}_{2n})$ to the remaining generators of $U_q(\mathfrak{sp}_{2n+2})$. Then by the Hopf algebra structure on the quantized universal enveloping algebra, $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sp}_{2n})$ would indeed be a Hopf subalgebra of $U_q(\mathfrak{sp}_{2n+2})$, so in particular a coideal subalgebra. However there is a problem with this approach. The copies of $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sp}_{2n})$ would not commute inside $U_q(\mathfrak{sp}_{2n+2})$. It seems, that what needs to be done is to indeed send the generators of $U_q(\mathfrak{sp}_{2n})$ to the last $3n$ generators of $U_q(\mathfrak{sp}_{2n+2})$ and then try to find a copy of $U_q(\mathfrak{sl}_2)$ commuting with this. This copy of $U_q(\mathfrak{sl}_2)$ would not correspond to the generators E_1, F_1, K_1 of course, but probably to a combination of the E_1, E_2's and F_1, F_2's together with K_1.

- Case 4(b): $Sp(4) \times Sp(2n) \leq Sp(2n+4)$
  This is a symmetric space example, so a coideal subalgebra from Letzter's work.

- Case 4(c): $SO(n) \times \{1\} \leq SO(n+1)$
  $U_q(\mathfrak{so}_n)$ can be embedded in $U_q(\mathfrak{so}_n)$ as a Hopf subalgebra. It is a case of a Dynkin diagram inclusion (see below).

Now considering the tensor category of finite-dimensional type 1 representations $\mathcal{C}$, it is easy to see that the examples for which we get a Hopf subalgebra (so all apart from 4.(b)) are $\mathcal{C}$-semisimple, since restricting the action to any of the above

Hopf subalgebras still gives a finite-dimensional representation, which for q generic is always semisimple.

Cases 5, 6 and 7

All these cases are similar and follow from the case $SL(2) \times Sp(2n) \leq Sp(2n+2)$.

6.3.2 Krämer's list

Now we proceed with examples from Krämer's list.

Case 3: $SL(m) \times SL(n) \leq SL(m+n)$

We look at the corresponding quantized enveloping algebras and consider the embedding of $U_q(\mathfrak{sl}_m) \otimes U_q(\mathfrak{sl}_n)$ inside $U_q(\mathfrak{sl}_{m+n})$. Recall that $U_q(\mathfrak{sl}_m)$ has $3(m-1)$ generators. The above embedding corresponds to sending the $3(m-1)$ generators of $U_q(\mathfrak{sl}_m)$ to the first $3(m-1)$ generators of $U_q(\mathfrak{sl}_{m+n})$ and the $3(n-1)$ generators of $U_q(\mathfrak{sl}_n)$ to the last $3(n-1)$ generators of $U_q(\mathfrak{sl}_{m+n})$ (so that in the end, the generators corresponding to the $3m$-th simple root haven't got anything mapped to them). We make this choice to make sure that the the elements from the two copies of $U_q(\mathfrak{sl}_m)$ and $U_q(\mathfrak{sl}_n)$ inside $U_q(\mathfrak{sl}_{m+n})$ commute according to the relations of the quantized enveloping algebra between the E_i's, F_i's and K_i's. Indeed the Cartan matrix for $U_q(\mathfrak{sl}_n)$ is

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ & & & \ddots & & \\ & & & & \ddots & \\ 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix}$$

From this it follows that the E_i's (resp. F_i's) commute with the K_j's and the E_j's (resp. F_j's) when $|i-j| > 1$. It is also clear from the Hopf algebra structure of the quantized universal enveloping algebra that $U_q(\mathfrak{sl}_m) \otimes U_q(\mathfrak{sl}_n)$ is actually a Hopf subalgebra of $U_q(\mathfrak{sl}_{m+n})$ (so in particular a left coideal subalgebra). Moreover it is $\mathcal{C}$-semisimple for the category of finite-dimensional type 1 $U_q(\mathfrak{sl}_{m+n})$-representations. Indeed any finite-dimensional type 1 representation of $U_q(\mathfrak{sl}_{m+n})$ when restricted to the Hopf subalgebra $U_q(\mathfrak{sl}_m) \otimes U_q(\mathfrak{sl}_n)$ is again finite-dimensional and of type 1. But the category of finite-dimensional type 1 representations of $U_q(\mathfrak{sl}_m) \otimes U_q(\mathfrak{sl}_n)$ is semisimple for generic q. Therefore using the same arguments as the ones in Corollary **??** we conclude that this construction gives a quantum subgroup.

Inclusion of one Dynkin diagram into the other

The method that we used above ($SL(m) \times SL(n) \leq SL(m+n)$) can be used whenever we have a case of inclusion of the Dynkin diagram corresponding to the subgroup H into the Dynkin diagram corresponding to G. By "inclusion" here we mean that the Dynkin diagram of H is a (connected) subset of the nodes of the Dynkin diagram of G, with all edges between them. Any such inclusion gives an inclusion of the basis of the root system of the first Lie algebra into the second's. Therefore mapping the corresponding generators of $U_q(\mathfrak{h})$ in $U_q(\mathfrak{g})$ gives a Hopf subalgebra which is $\mathcal{C}$-semisimple exactly as in the case above. Case 11 falls in this category.

A few informal comments on other examples

Case 8: $\mathbb{G}_m \times Sp(2n-2) \leq Sp(2n)$ should be an easy modification of the case above (Inclusion of one Dynkin diagram to the other). The same holds for case 27, since the Dynkin diagram of D_5 can be included in the diagram of E_6. Also with regards to case 13: $Spin(7) \leq SO(8)$. Knop and Röhrle point out in [25] that using triality this embedding is equivalent to $SO(7) \leq SO(8)$.

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