

Ricci Flow in Milnor Frames



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Abstract

In this work, we are going to find sufficient conditions on the initial metric on some 4-dimensional manifolds foliated by homogeneous S^3 for a Type I singularity to occur when it is flowed under the Ricci flow. This work generalises the study on rotationally symmetric manifolds done by Angenent and Isenberg as well as the work of Isenberg, Knopf, and Šešum, in which they introduced some ansatz for the problem setup. In the latter study, a global frame for the tangent bundle called the Milnor frame was used to set up the problem. We begin with some discussions on the symmetries of the manifold and its ansatz metric derived from Lie groups as well as the global tangent bundle frame developed by Milnor. The curvature quantities and the Ricci flow equation of this ansatz metric will then be explicitly computed. Numerical simulations of the Ricci flow on these manifolds are done, which provides insight and conjectures for the main problems. Some analytic results will be proven for the manifolds $S^1 \times S^3$ and S^4 using maximum principles from parabolic PDE theory and sufficiency conditions for a neckpinch singularity will be provided. Finally, a problem from general relativity with similar metric symmetries but endowed on manifolds homeomorphic to $\mathbb{CP}^2 \setminus B^4$ and $\bar{B}^4$ will be discussed, which will provide a possible direction for future work.

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Statement of Originality

I declare that the work in this thesis is, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known. This thesis has not been submitted for a degree at another university.

M.S.Johar

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Chapter 1

Introduction

The Ricci flow is an evolution equation of the metric on a Riemannian manifold $(M, g(t))$ introduced by Richard Hamilton in 1982 for the study of the famous Poincaré conjecture. This metric evolution equation is given by:

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t)), \\ g(0) = g_0, \end{cases} \quad \begin{matrix} (1.1) \\ (1.2) \end{matrix}$$

where Ric is the Ricci curvature of the metric $g(t)$ at time $t \geq 0$.

This geometric flow has been important in the study of topological properties of manifolds. In particular, it was instrumental in the proof of the Poincaré Conjecture and the Thurston Geometrisation Conjecture by Perelman in early 2000s. [75] The Poincaré Conjecture claimed that every simply-connected closed 3-manifold is homeomorphic to the 3-sphere. Additionally, [92] Thurston's Geometrisation Conjecture pushes this further by claiming that the interior of every compact 3-manifold has a canonical decomposition into pieces which have model geometric structures.

These two conjectures are generalisations of the Uniformisation Theorem for 2-dimensional manifolds, which states that every connected 2-dimensional Riemann surface is conformally equivalent to one of the following model geometries of homogeneous spaces: the spherical geometry, the Euclidean geometry, or the hyperbolic geometry.

However, instead of just three geometries as in the 2-dimensional case, there is a total of 8 types of model geometries for 3-dimensional manifolds, which are called the Thurston geometries. They are: the Euclidean geometry $E(3)$, the spherical geometry S^3 , the geometry of the universal cover $\widetilde{SL}(2, \mathbb{R})$, the Heisenberg group or Nil geometry Heis , the Sol geometry $E(1, 1)$, the hyperbolic geometry $H(3)$, the geometry of $S^2 \times \mathbb{R}$, and the geometry of $H(2) \times \mathbb{R}$.

The first five of these can be realised as left-invariant metrics on Lie groups. We shall elaborate more on this in Chapter 3.

The ground-breaking proof by Perelman (see [73] and [74]) for both of the conjectures by Poincaré and Thurston was based on a program developed by Hamilton [42] in which he proposed a way to improve the metrics on closed 3-manifolds to make them more uniform by using the Ricci flow, which is a “heat-type” equation.

Intuitively speaking, as a heat-type equation on the metric, the Ricci flow will try to diffuse the metric over the Riemannian manifold. Therefore, it will aim to distribute the infinitesimal distances (and hence the curvature) evenly throughout the manifold over time. Hence, we expect the manifold will become more homogeneous as time progresses. In particular, if the manifold is simply connected and compact, we can expect the metric to get rounder and approach the round metric as time progresses.

It has been shown that some initial metrics converge to a steady state of the Ricci flow, which are obtained by equating the Ricci flow equation (1.1) with 0. This gives us the Ricci-flat metrics. As an example, see the work by Hamilton where he first proposed the Ricci flow [42].

However, as seen in the works of Berger [8], Bergery [7], Böhm [12], and Hitchin [49], not all manifolds admit an Einstein metric. See the book by Besse [9] for a survey on this. The Ricci flow would not converge on these manifolds, in the sense that a singularity would develop during the flow. This is due to the non-linear nature of the Ricci flow that would only guarantee the short time existence of the solution to this flow from any initial data. The finite-time singularities of the Ricci flow can be classified into two types, which are called the Type I and Type II singularities, depending on the curvature asymptotics.

Studying the singularities of the Ricci flow has been a long-standing subject of research in differential geometry. There has been an interest in constructing explicit initial data which develop singularities and modelling these singularities with self-similar Ricci flow solutions, which are called the Ricci solitons.

1.1 Aims

In this work, we are going to study the Ricci flow on some families of Riemannian manifolds and their singularities using a global frame on the manifold. More specifically, we are going to construct initial metrics which develop Type I singularity using an ansatz in the form of foliations of a parallelisable manifold (see Definition A.1.14).

Due to the difficulty with working in coordinate patches on general manifolds, the parallelisability assumption simplifies the Ricci flow considerably by allowing us to write it in a global manner. Furthermore, since the Ricci tensor is diffeomorphism-invariant, any initial symmetry persists as time progresses during the Ricci flow. We will assume the Riemannian manifolds that we are investigating to have some symmetry at initial time. Combined, these two features enable us to write down the problem as an explicit system of partial differential equations.

The first non-trivial concrete example for an initial metric which leads to the Type I singularity has been constructed by Angenent and Knopf in 2004 [6]. In the paper, they constructed this metric by considering a warped product metric on S^3 of the form:

$$g_0 = ds^2 + \psi(s)^2 \hat{g}_{S^2}, \quad (1.3)$$

where $\hat{g}_{S^2}$ is the canonical metric on the unit 2-sphere and $\psi(s)$ is a function on the interval $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. The function $\psi(s)$ describes the radius of the S^2 spheres which foliate the S^3 manifold over the interval I such that ψ vanishes at the boundaries of the interval I and $\psi'(s) \rightarrow \pm 1$ as $s \rightarrow \mp \frac{\pi}{2}$. This way, they reduced the Ricci flow equation to a one-dimensional parabolic PDE, which simplifies the problem considerably. They then showed that the function:

$$\psi(s) = \begin{cases} \sqrt{A + Bs^2} & \text{for } |s| \leq s_{A,B}, \\ \cos s & \text{for } s_{A,B} < |s| < \frac{\pi}{2}, \end{cases} \quad (1.4)$$

where $s_{A,B}$ is the unique positive solution for $\cos s = \sqrt{A + Bs^2}$ for some $A > 0$ and $0 < B < \frac{1}{2}$, once smoothed out in a neighbourhood of the point $s_{A,B}$, yields a family of initial metrics $g_{A,B}$ that results in a neckpinch singularity at finite time.

This family of metrics $g_{A,B}$ with the choice of $\psi(s)$ is special, in the sense that they satisfy some sufficiency conditions for a neckpinch, which are:

1. the metrics have positive scalar curvature,
2. we have the bound $|\partial_s \psi| \leq 1$,
3. the manifolds have the neck radius $r_{\min}(0) = \sqrt{A}$, where:

$$r_{\min}(t) = \min_s \{ \psi(s, t) : \partial_s \psi(s, t) = 0 \},$$

4. the manifolds have a bump of height at least height $\hat{\psi}(\alpha) > \sqrt{B}\alpha$.

In fact, this also holds true for any n -dimensional spheres for $n > 3$ by considering $0 < B < 1$. Furthermore, they showed that the neckpinch is asymptotically a cylinder and forms a Type I singularity. More specifically, they proved the following theorem:

Theorem 1.1.1. [6] If $n \geq 2$, there exists an open subset of the family of metrics on S^{n+1} possessing $\text{SO}(n+1)$ symmetries such that the Ricci flow starting at any metric in this set develops a neckpinch at some time $T < \infty$. The singularity is rapidly-forming (Type I) and any sequence of parabolic dilations formed at the developing singularity converges to a shrinking cylinder soliton. This convergence takes place uniformly in any ball of radius $o\left(\sqrt{-(T-t)\log(T-t)}\right)$ centered at the neck.

Furthermore, there exist constants $0 < \delta$ and $C < \infty$ such that the radius ψ of the sphere at distance σ from the neck pinch is bounded from above by:

$$\psi \leq \sqrt{2(n-1)(T-t)} + \frac{C\sigma^2}{|\log(T-t)|\sqrt{T-t}},$$

for $|\sigma| \leq 2\sqrt{-(T-t)\log(T-t)}$ and:

$$\psi \leq \frac{C\sigma}{\sqrt{-\log(T-t)}} \log\left(\frac{\sigma}{\sqrt{-(T-t)\log(T-t)}}\right),$$

for $2\sqrt{-(T-t)\log(T-t)} \leq \sigma \leq (T-t)^{\frac{1}{2}-\delta}$. ■

This family of explicit examples was generalised further by Isenberg, Knopf, and Šešum [54]. Instead of considering the warped product metric on S^3 of the form (1.3), they considered a warped product metric on $S^1 \times S^3$. The base of the foliation S^1 simplifies the analysis in [6] considerably by doing away the technical analysis at the poles of the manifold, which are special orbits of the symmetry. Furthermore, the fibre manifold S^3 is parallelisable, which means that there exists a global vector field frame $\{E_1, E_2, E_3\}$ on this manifold. Therefore, a more general form for the metric on the fibre is given by:

$$g = a^2\omega^1 \otimes \omega^1 + b^2\omega^2 \otimes \omega^2 + c^2\omega^3 \otimes \omega^3,$$

for some positive constants a, b , and c such that $\{\omega^1, \omega^2, \omega^3\}$ are dual frames to the vector fields $\{E_1, E_2, E_3\}$. Instead of this general form, they studied the biaxial case for simplicity; that is, they set $b \equiv c$. In short, they studied the Ricci flow of metrics on $S^1 \times S^3$ of the form:

$$g = \phi(z)^2 ds^2 + a(z)^2 \omega^1 \otimes \omega^1 + c(z)^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

where $a(z)$ and $c(z)$ are 2π -periodic positive functions of z .

Under assumptions on the initial ordering of the metric components a and c as well as curvature assumptions, they managed to find sufficiency conditions for a Type I singularity to occur. Furthermore, with additional conditions, they proved that the singularity resembles a shrinking cylinder after scaling, which is similar to what was done by Angenent and Knopf for the S^3 case. More specifically, first they proved:

Theorem 1.1.2. [54] The eccentricity of every warped Berger solution of Ricci flow is uniformly bounded: there exists a constant $C_0 > 0$ depending only on the initial data such that the estimate:

$$|a - c| \leq C_0 \min(a, c),$$

holds pointwise for as long as the solution exists. ■

Furthermore, by assuming these three increasingly stronger assumptions:

1. Assumptions 1:

- (a) $0 < a \leq c$ at $t = 0$,
- (b) $\min_{s,t=0}(S) \cdot \max_{s,t=0}(c^2) > -3$ at $t = 0$,
- (c) there exists $T < \infty$ such that $\limsup_{t \rightarrow T} \max_s |\text{Ric}(s, t)|$,

2. Assumptions 2:

- (a) the metric g satisfies Assumptions 1,
- (b) $a \geq (1 - \varepsilon)c$ for some ε small enough ($\varepsilon = \frac{1}{4}$ is sufficient) that $2(1 - \varepsilon)^5 + 4(1 - \varepsilon)^4 > \frac{4}{3}$,
- (c) $|\partial_s a| < 1$,

3. Assumptions 3:

- (a) the metric g satisfies Assumptions 2,
- (b) the metric g is reflection symmetric at $t = 0$, with its smallest neck located at a fixed point $\xi_* \in S^1$,

the authors concluded the following theorems:

Theorem 1.1.3. [54] There exist open sets of warped Berger metrics satisfying Assumptions 1 such that all solutions originating in these sets develop local neckpinch singularities at some $T < \infty$. Each such solution has the properties that:

- 1. the ordering $a \leq c$ is preserved,
- 2. the singularity is Type I with $|\text{Ric}| \leq \frac{C}{\min_s(a^2)}$ and $\frac{\sqrt{T-t}}{C} \leq \min_s(a) \leq C\sqrt{T-t}$,
- 3. the diameter of (M, g) is bounded as $t \rightarrow T$. ■

Theorem 1.1.4. [54] There exist open sets of warped Berger metrics satisfying Assumptions 2 such that, as solutions originating in these sets become singular, they become asymptotically round at rates that break scale invariance. Specifically, in addition to the properties above, they satisfy the following bounds at the neck:

$$\begin{aligned}(T-t)^{-\frac{1}{2}}|a-c| &\leq C\sqrt{T-t}, \\ (T-t)(K_{12}-K_{23}) &\leq C\sqrt{T-t}, \\ (T-t)(K_{01}-K_{02}) &\leq C\sqrt{T-t}.\end{aligned}$$

In a neighbourhood of each smallest neck where $K_{01} < 0$ there is the further bound:

$$(T-t)(|K_{01}| + |K_{02}|) \leq \frac{C}{|\log(T-t)|}.$$

The radius of a smallest neck is $(1+o(1))2\sqrt{T-t}$. Type I blowups $\tilde{g} = (T-t)^{-1}g$ of the solution converge near each neck to the shrinking cylinder soliton. If $\tilde{s}$ is the arclength from a smallest neck and $\sigma := \frac{\tilde{s}}{\sqrt{T-t}}$, then there exist constants $C, D < \infty$ independent of time such that as $t \rightarrow T$ the estimates:

$$\begin{aligned}1+o(1) &\leq \frac{a}{2\sqrt{T-t}} \leq 1+C\frac{\sigma^2}{|\log(T-t)|}, \\ 1+o(1) &\leq \frac{C}{2\sqrt{T-t}} \leq (1+o(1))\left(1+C\frac{\sigma^2}{|\log(T-t)|}\right),\end{aligned}$$

hold for $|\sigma| \leq D\sqrt{|\log(T-t)|}$ and the estimate:

$$\frac{a+c}{\sqrt{T-t}} \leq \frac{C|\sigma|}{\sqrt{|\log(T-t)|}} \sqrt{\log\left(\frac{|\sigma|}{\sqrt{|\log(T-t)|}}\right)},$$

holds for $D\sqrt{|\log(T-t)|} \leq |\sigma| \leq (T-t)^{-\frac{\varepsilon}{2}}$ for $\varepsilon \in (0, 1)$. ■

Theorem 1.1.5. [54] There exist open sets of reflection-symmetric warped Berger metrics satisfying Assumptions 3 such that any solution originating in these sets has the following property: for any small δ and large Σ , there exist $T^* < T$ and $C > 0$ such that the stronger estimate:

$$(T-t)^{-\frac{1}{2}}|a-c| \leq C(T-t)^{1+\delta},$$

holds for all $|\sigma| \leq \Sigma$ and $T^* < t < T$. ■

In this work, we are going to elaborate more on the existence of the dual frames $\{\omega^1, \omega^2, \omega^3\}$ from the work of [65], their properties, and other possible manifolds which admit this family of

dual frames. Then we are going to generalise Theorems 1.1.2 and 1.1.3 to manifolds different from $S^1 \times S^3$. The main difference to these theorems are the topologies of the manifolds, which give rise to differences in the analysis. The generalisations of these theorems, which are the main results of this work, are given by Theorems 6.0.1, 6.0.2, 7.0.1, and 7.0.2.

1.2 Organisation of Thesis

The key step to the setup in [54] is the use of a special tangent bundle frame on the manifold. This special frame, known as the Milnor frame, exists on certain Lie groups G and enables us to write down a bi-invariant Riemannian metric on G as a diagonal 2-tensor globally. We shall employ this tool in this work.

In Chapter 2, we are going to review the Ricci flow, briefly going through the short time existence proof due to DeTurck [26] and give some basic examples of initial metrics that develops a singularity after some finite time. We shall define the two different types of singularities as well as some of the previous observations and constructions of initial metrics which lead to these singularities done in the literature which, among others, are [4], [5], [6], [35], [39], [44], [54], and [97]. We shall also briefly touch upon the technique of blow-up on a sequence of pointed Riemannian manifolds and its convergence due to Cheeger and Gromov, which is used in modelling the singularities with self-similar solutions. This blow-up method was used in [6] and will be used in this work in Chapter 7.

In Chapter 3, we demonstrate that closed 3-dimensional spheres are parallelisable; that is, they have trivial tangent bundles. This means that the tangent bundle can be written as (TS^3, S^3, π) where TS^3 is a global direct product instead of a local one. Furthermore, in a suitably chosen frame, any bi-invariant metric on S^3 can be written as a diagonal 2-tensor. This can be realised using the Milnor frame, which was developed in [10], [11], and [65], via the theory of Lie groups and Lie algebras. We will review the classification of all such homogeneous 3-dimensional manifolds and focus on one of them, the 3-sphere S^3 , for the rest of the thesis. Some known results on the normalised and unnormalised Ricci flow of this family of homogeneous manifolds from [53] and [59] will be mentioned. A new proof of these results relevant to our studies will be shown along with a simple numerical simulation.

Note that the preliminary background and notations for the material in the first two chapters are included in Appendices A and B.

In Chapter 4, we formulate the problem that we are studying in this thesis and set up an

ansatz for the metric by using the material from the previous chapters. More explicitly, we shall consider $S^1 \times S^3$ and S^4 via foliations of S^3 along a circle and an open interval with each end collapsed to a point. We also calculate the curvature tensors and explicitly write down the Ricci flow equation along with the evolution equations of other relevant quantities for the analysis in Chapters 6 and 7.

In Chapter 5, we numerically simulate the Ricci flow equation and discuss the behaviour of the evolution. Due to the weakly parabolic nature of the equation, we use the DeTurck-Ricci flow, introduced in Chapter 2, instead of the Ricci flow. The numerical simulations were done in MATLAB and allow us to pose conjectures for the behaviour of the metric under the Ricci flow. The numerical methods will employ the finite difference scheme inspired by the work of [35] and [81]. Plots of the evolution of the metric components and some ratios of them will be included.

In Chapters 6 and 7, we carry out the analysis of the singularity formation on the manifolds $S^1 \times S^3$ and S^4 equipped with the special form of metrics discussed in Chapter 4. This original work utilises the methods used in [6] and [54]. In particular, we shall generalise Theorems 1.1.2 and 1.1.3. The theorems that we are going to prove are:

Theorem 6.0.3. Suppose that we run a Ricci flow on the manifold $(S^1 \times S^3, g_0)$ where the initial metric g_0 is of the form:

$$g_0 = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + b(z)^2 \omega^2 \otimes \omega^2 + c(z)^2 \omega^3 \otimes \omega^3,$$

for $z \in S^1 = [0, 2\pi)$ and $0 \leq a(z) \leq b(z) \leq c(z)$. Then the Ricci flow preserves the form of the metric, the ordering of the metric components, and there exist constants $\tilde{C}_0, \bar{C}_0 > 0$ depending on the data on the initial data such that for all $(z, t) \in S_T^1 = S^1 \times [0, T)$, we have:

$$|b(z, t) - c(z, t)| \leq \tilde{C}_0 \min(b(z, t), c(z, t)),$$

$$|a(z, t) - c(z, t)| \leq \bar{C}_0 \min(a(z, t), c(z, t)). \quad \blacksquare$$

Theorem 6.0.4. There exist open sets of warped metrics on $S^1 \times S^3$ of the form:

$$g = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + b(z)^2 \omega^2 \otimes \omega^2 + c(z)^2 \omega^3 \otimes \omega^3,$$

for $z \in S^1 = [0, 2\pi)$, satisfying:

1. $0 < a \leq b \leq c < 2a$ at $t = 0$,
2. $\min_{s, t=0} (S) \geq 0$ at $t = 0$,

3. there exists a $T < \infty$ such that $\limsup_{t \rightarrow T} \max_s |\text{Ric}(s, t)| = \infty$,

such that all solutions of the Ricci flow originating in these sets develop local neckpinch singularities at some $T < \infty$. Each such solution has the properties that:

1. the ordering $a \leq b \leq c$ is preserved,
2. the singularity is of Type I with $|\text{Rm}| \leq \frac{C}{\min_s(a^2)}$ and $D\sqrt{T-t} \leq \min_s(a) \leq C\sqrt{T-t}$ for some constants $C, D > 0$. ■

The above theorems will be proven in Chapter 6. These theorems generalise Theorems 1.1.2 and 1.1.3 by considering a triaxial Bianchi IX metric on $S^1 \times S^3$. This generalisation will provide an extra difficulty in analysis when we are handling the third metric quantity which will be discussed in more depth later. In Chapter 7, we shall prove the following theorems:

Theorem 7.0.4. Suppose that we run a Ricci flow on the manifold (S^4, g_0) where the initial metric g_0 is of the form:

$$g_0 = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + c(z)^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

for $z \in I = (-1, 1)$ and $0 < a(z) \leq c(z)$. Then the Ricci flow preserves the form of the metric, the ordering of the metric components, and there exists a constant $\tilde{C}_0 > 0$ depending on the data on the parabolic boundary $\mathcal{P}I_T$ such that for all $(z, t) \in (-1, 1) \times [0, T)$ we have:

$$|a(z, t) - c(z, t)| \leq \tilde{C}_0 \min(a(z, t), c(z, t)). \quad \blacksquare$$

Theorem 7.0.5. There exist open sets of warped metrics on S^4 of the form:

$$g = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + c(z)^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

for $z \in I = (-1, 1)$, satisfying:

1. $0 < a \leq b \equiv c \leq 2a$ at $t = 0$,
2. $|c'(s, 0)| \leq \sqrt{\frac{23}{8}}$,
3. $\min_{s, t=0}(S) \geq -\frac{1}{4}$ at $t = 0$,
4. there exists a $T < \infty$ such that $\limsup_{t \rightarrow T} \max_s |\text{Ric}(s, t)| = \infty$,

such that all solutions originating in these sets develop local neckpinch singularities at some $T < \infty$. Each such solution has the properties that:

1. the ordering $a \leq c$ is preserved,
2. the singularity is Type I with $|\text{Rm}| \leq \frac{C}{\min_s(a^2)}$ and $\frac{\sqrt{T-t}}{C} \leq \min_s(a) \leq C\sqrt{T-t}$. ■

These two theorems differ from Theorems 1.1.2 and 1.1.3 since we are considering the problem on a different topology of S^4 . We shall need a stronger version of the maximum principle from parabolic theory, which will be proven in Chapter 4. Some techniques from [55] are also utilised in the estimations. We shall show that, under certain sufficiency conditions, the initial metric on these manifolds will develop a singularity. Moreover, we shall conclude that the singularities are of Type I by analysing the curvature asymptotics.

Furthermore, towards the end of Chapter 7, we shall provide some attempts and ideas to prove that singularities would not occur at the poles of the manifold S^4 given a certain condition on the behaviour of the metric components close to the singular time T . This was proven in [6], but the solution is still out of reach in our case due to the multiple warping functions involved in our metrics.

Finally, in Chapter 8, we state a similar problem to the cases studied in the previous chapters but on a different topological setting. This will involve the Ricci flow on manifolds with boundaries studied in [51], [77], and [86]. The setting would come from the Taub-Bolt and Taub-Nut metrics in a box, which have the same form as our ansatz metric from the previous chapters. However, their topology is different; the metrics are endowed on manifolds homeomorphic to $\mathbb{CP}^2 \setminus B^4$ and $\bar{B}^4$ respectively. The manifolds with boundaries are realised by considering foliations of S^3 over the interval $(0, R]$ for some constant $R > 0$ and extended smoothly to 0. Both of these manifolds have S^3 boundaries at R . The difference in topology for both of these Riemannian manifolds comes from the smoothness at 0 at which the manifold either collapses to a point (for the Taub-Nut metric) or an S^2 factor (for the Taub-Bolt metric). These metrics arise as Ricci-flat solutions to the Einstein equation as discussed in, among others, [47], [69], [72], and [91].

The Ricci flow of perturbations of these Ricci-flat metrics have been studied numerically in [48] and [51]. These studies contain some numerical evidence showing that one of the Taub-Bolt metrics (which is called the small bolt), when perturbed in a certain directions, will either converge to the other Taub-Bolt metric (which is called the large bolt) or the Taub-Nut metric. In the latter case, there would be a change in topology during the flow. This indicates that a singularity would occur during the flow at the tip of the manifold. We shall review the previous studies, replicate some of the numerical simulations, prove several analytic results, and provide

some discussions that pave towards a possible future research direction, which may tie in with [55] and [30].

Chapter 2

Ricci Flow

In order to make sense of the Ricci flow equation (1.1)-(1.2), we first have to define the Ricci curvature tensor. This is derived from the Riemann curvature tensor from manifold theory. For a brief background, notations, and preliminaries on manifold theory leading to Riemannian geometry, see Appendix A.

2.1 Curvatures of Riemannian Manifolds

We first define the Riemann curvature on a Riemannian metric (M, g) . This quantity measures the non-commutativity of covariant derivatives on the manifold.

Definition 2.1.1 (Riemann curvature). Given a Riemannian manifold (M, g) , the Riemann curvature of the Levi-Civita connection ∇ of g on the tangent bundle TM is the tensor $\text{Rm} \in \Gamma(T^*M \otimes T^*M \otimes T^*M \otimes TM)$ such that for any $X, Y, V \in \Gamma(TM)$:

$$\begin{aligned}\text{Rm}(X, Y)V &= \nabla^2 V(X, Y) - \nabla^2 V(Y, X) \\ &= \nabla_X(\nabla_Y V) - \nabla_Y(\nabla_X V) - \nabla_{[X, Y]}V,\end{aligned}$$

where ∇^2 is the second covariant derivative defined in Definition A.3.20. ■

Remark 2.1.2. Some authors use an opposite sign convention to the one used here. From now on, we set the sign convention to be as in Definition 2.1.1. ■

This tensor is also called the $\binom{3}{1}$ -Riemann curvature tensor. In local coordinates $\{x^i\}_{i=1}^m$ on an m -dimensional manifold M , we can write the Riemann curvature tensor as:

$$\text{Rm} = \text{Rm}_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l.$$

In fact, the quantity $\text{Rm}_{ijk}{}^l$ can be calculated explicitly since we know all the Christoffel symbols of a given Riemannian metric g from the identity (A.1). We have:

$$\text{Rm}_{ijk}{}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{ih}^l \Gamma_{jk}^h - \Gamma_{jh}^l \Gamma_{ik}^h. \quad (2.1)$$

We also have the $\binom{4}{0}$ -Riemann curvature tensor, which is denoted by Rm as well. This tensor is a contracted version of the $\binom{3}{1}$ -tensor, which is defined as:

$$\text{Rm}(X, Y, V, W) = g(\text{Rm}(X, Y)V, W),$$

for all $V, W, X, Y \in \Gamma(TM)$. In local coordinates, this tensor can be written as:

$$\text{Rm} = \text{Rm}_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

where $\text{Rm}_{ijkl} = g_{lh} \text{Rm}_{ijk}{}^h$. Furthermore, the Riemannian curvature tensor has several symmetry properties, which can be useful in calculations:

Theorem 2.1.3 (Symmetries of Riemann curvature tensor). Suppose that we have vector fields $V, W, X, Y \in \Gamma(TM)$. The Riemann curvature tensor has the following symmetries:

1. (anti-symmetry in first pair of arguments): $\text{Rm}(X, Y, V, W) = -\text{Rm}(Y, X, V, W)$,
2. (anti-symmetry in second pair of arguments): $\text{Rm}(X, Y, V, W) = -\text{Rm}(X, Y, W, V)$,
3. (symmetry of first and second pairs of arguments): $\text{Rm}(X, Y, V, W) = \text{Rm}(V, W, X, Y)$. ■

A simpler way to study the curvature is to define the sectional curvature $K(X, Y)$, which is the curvature of the manifold in relation to the 2-plane spanned by X and Y . This is reminiscent of the sectional curvature as studied in 2-dimensional surfaces, which measures the deviation of the surface from a flat plane. In the Riemannian setting, it is defined as:

$$K(X, Y) = \frac{g(\text{Rm}(X, Y)Y, X)}{\|X\|^2 \|Y\|^2 - g(X, Y)^2}.$$

Another way of simplifying the Riemannian curvature tensor is by contracting the 4-tensor Rm with itself to get a 2-tensor, which is called the Ricci curvature tensor.

Definition 2.1.4 (Ricci curvature). The trace of the Riemann curvature tensor over the second and third (or first and fourth, by the symmetries of the Riemann curvature tensor) indices is called the Ricci curvature tensor. In coordinates, the 2-tensor can be written as:

$$\text{Ric}_{ij} = \text{Rm}_{kij}{}^k = \text{Rm}_i{}^k{}_{kj} = g^{kl} \text{Rm}_{iklj}. \quad \blacksquare$$

By tracing the expression in (2.1), explicitly we have:

$$\text{Ric}_{ij} = \text{Rm}_{kij}{}^k = \partial_k \Gamma_{ij}^k - \partial_i \Gamma_{kj}^k + \Gamma_{ij}^k \Gamma_{kl}^l - \Gamma_{ik}^l \Gamma_{jl}^k. \quad (2.2)$$

There is a close relationship between the sectional curvatures and the Ricci curvature tensors. Explicitly, the Ricci curvature at $p \in M$ is given by:

$$\text{Ric}(X, Y)_p = \sum_{i=1}^m g(\text{Rm}(X, E_i)E_i, Y)_p,$$

for some $X, Y \in T_p M$ and $\{E_1, E_2, \dots, E_m\}$ is any orthonormal basis of $T_p M$. By choosing $E_1 = X$, we can interpret the quantity $\text{Ric}(X, X)$ as the sum of the sectional curvatures of the planes spanned by X and E_i for $i = 2, 3, \dots, m$ since:

$$\text{Ric}(X, X) = \sum_{i=2}^m g(\text{Rm}(X, E_i)E_i, X) = \sum_{i=2}^n K(X, E_i).$$

Finally, we can contract the Ricci tensor further, which gives us a scalar function on M called the scalar curvature.

Definition 2.1.5 (Scalar curvature). The scalar curvature is a 0-tensor that is obtained by further tracing the Ricci curvature with g :

$$S = g^{ij} \text{Ric}_{ij}. \quad \blacksquare$$

We note that all of the geometric quantities depend on the metric g and its derivatives. Therefore, one might wonder how would the lengths and curvatures change if we were to scale the original metric by some constant $C > 0$. Elementary computations show:

Proposition 2.1.6. Let (M, g) be a Riemannian manifold with Riemann curvature tensor, Ricci curvature tensor, scalar curvature, and sectional curvatures denoted by Rm , Ric , S , and K respectively. Let $(M, \tilde{g})$ be another Riemannian manifold obtained by scaling the metric g by a positive constant; that is, $\tilde{g} = Cg$ for some constant $C > 0$. Then we have:

1. $\tilde{\text{Rm}}_{ijkl} = C \text{Rm}_{ijkl}$,
2. $\tilde{\text{Ric}}_{ij} = \text{Ric}_{ij}$,
3. $\tilde{S} = C^{-1}S$,
4. $\tilde{K}_{ij} = CK_{ij}$,

where $\tilde{\text{Rm}}$, $\tilde{\text{Ric}}$, $\tilde{S}$, and $\tilde{K}$ are the Riemann, Ricci, scalar, and sectional curvatures for the metric $\tilde{g}$ respectively. Furthermore, given a curve $\gamma : [0, 1] \rightarrow M$ in M , its lengths with respect to the metric g and $\tilde{g}$ are related by $L(\gamma) = \sqrt{C} \tilde{L}(\gamma)$. $\blacksquare$

We end this section by defining the Laplacian operator in geometric analysis, which is a generalisation of the Laplacian operator in the study of partial differential equations on the Euclidean space.

Definition 2.1.7 (Rough Laplacian). The rough Laplacian Δ on a vector bundle $\mathcal{E}$ is defined as the trace of the second covariant derivative tensor on the vector bundle $\mathcal{E}$. In local coordinates, for $\varsigma \in \Gamma(\mathcal{E})$, we have:

$$\Delta\varsigma = g^{ij} D_{\partial_i}^2 D_{\partial_j} \varsigma,$$

where D^2 is the second covariant derivative on the vector bundle $\mathcal{E}$ defined in Definition A.3.20. If $\mathcal{E}$ is the 0-tensor bundle and D Levi-Civita connection for a metric g , then we denote the Laplacian as Δ_g , which we call the Laplace-Beltrami operator. ■

The Riemann curvature tensor involves the first derivatives of the Christoffel symbols, which in turn involve the first derivatives of the metric components. Thus, the Ricci curvature tensor and the scalar curvature involve second derivatives of the metric tensor components. Therefore, it is natural to consider elliptic or parabolic partial differential equations of the metric which involve these terms. In particular, one type of equation that has been well studied is the Ricci flow equation.

2.2 Ricci Flow

We recall from Chapter 1 that the Ricci flow is defined to be the evolution equation for the metric of a Riemannian manifold (M, g) given by:

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t)), \\ g(0) = g_0, \end{cases}$$

where Ric is the Ricci curvature of the metric $g(t)$ at time t . The Ricci flow is considered a “heat-type” evolution equation because we can show that the Ricci curvature, which involve second derivatives of the metric, can be written down as a Laplacian of the metric quantity g plus some lower order terms in harmonic coordinates. Therefore, locally, the equation resembles the heat equation.

Harmonic coordinates of an m -dimensional Riemannian manifold (M, g) is a set of local coordinates $\{x^i\}_{i=1}^m$ such that $\Delta_g x^i = 0$ for all $i = 1, 2, \dots, m$. As a consequence of this choice of

coordinates, from Definition 2.1.7, we have that:

$$\begin{aligned} 0 &= \Delta_g x^k = g^{ij} (\nabla_{\partial_i} (\nabla_{\partial_j} x^k) - \nabla_{\nabla_{\partial_i} \partial_j} x^k) \\ &= g^{ij} (\partial_i (\delta_j^k) - \Gamma_{ij}^l \partial_l (x^k)) = -g^{ij} \Gamma_{ij}^l \delta_l^k, \end{aligned}$$

and thus $g^{ij} \Gamma_{ij}^k = 0$ for all $k = 1, 2, \dots, m$.

In these local coordinates, by using the expression in equation (2.2) and collecting the quadratic terms in g^{-1} and ∂g , the Ricci tensor Ric_{ij} can be written locally as:

$$\begin{aligned} \text{Ric}_{ij} &= \partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{ij}^k \Gamma_{kl}^l - \Gamma_{ik}^l \Gamma_{jl}^k \\ &= \frac{1}{2} g^{kr} (\partial_k (\partial_j g_{ri} + \partial_i g_{rj} - \partial_r g_{ij}) - \partial_j (\partial_k g_{ri} + \partial_i g_{rk} - \partial_r g_{ik})) + Q_{ij}(g^{-1}, \partial g) \\ &= \frac{1}{2} (-g^{kr} \partial_{kr} g_{ij} + g^{kr} (\partial_{ki} g_{rj} + \partial_{jr} g_{ik} - \partial_{ij} g_{rk})) + Q_{ij}(g^{-1}, \partial g) \\ &= \frac{1}{2} (-g^{kr} \partial_{kr} g_{ij} + g_{is} \partial_j (\cancel{g^{kr} \Gamma_{kr}^s}) + g_{js} \partial_i (\cancel{g^{kr} \Gamma_{kr}^s})) + Q_{ij}(g^{-1}, \partial g) \\ &= -\frac{1}{2} g^{kr} \partial_{kr} g_{ij} + Q_{ij}(g^{-1}, \partial g), \end{aligned}$$

where $Q_{ij}(g^{-1}, \partial g)$ are the collected quadratic terms in g^{-1} and derivatives of g . Thus, the Ricci flow in harmonic coordinates is given by:

$$\partial_t g_{ij} = g^{kr} \partial_{kr} g_{ij} + Q_{ij}(g^{-1}, \partial g).$$

This is reminiscent of the classical heat equation on the components of the metric g with some lower order derivatives, which is what we have claimed earlier.

However, this only gives us a formal intuition of the behaviour of the equation. We cannot guarantee that the chosen harmonic coordinates for the initial metric $g(0)$ will still be harmonic with respect to the flowed metric $g(t)$, even if it is flowed for a small amount of time $t > 0$.

In fact, this formula can be quite misleading: the Ricci flow is actually not strongly parabolic. To see this, we linearise the Ricci curvature tensor by considering it as a map between the infinite-dimensional manifolds of smooth sections of vector bundles. Since these are infinite-dimensional Fréchet manifolds (whose charts map patches of the manifold to Fréchet spaces instead of Euclidean spaces), to calculate its pushforward, we appeal to the method of finding pushforwards via curves as in Remark A.1.8.

Definition 2.2.1 (Linearisation). Let $\mathcal{P} : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{F})$ be a non-linear differential operator where $\mathcal{E}$ and $\mathcal{F}$ are vector bundles over a manifold M . The linearisation of the operator $\mathcal{P}$ at $u \in \Gamma(\mathcal{E})$ is the linear operator $D\mathcal{P}_u : T_u \Gamma(\mathcal{E}) \rightarrow T_{\mathcal{P}(u)} \Gamma(\mathcal{F})$ such that for a curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow$

$\Gamma(\mathcal{E})$ defined for some sufficiently small $\varepsilon > 0$ with $\gamma(0) = u$ and $\gamma'(0) = v \in T_u\Gamma(\mathcal{E})$ we have:

$$D\mathcal{P}_u(v) = \left. \frac{d}{ds} \right|_{s=0} \mathcal{P}(\gamma(s)). \quad \blacksquare$$

From Remark A.3.8, we can see that the tangent space of the space of Riemannian metrics $\mathfrak{Met}$ is the space of sections of symmetric 2-tensors $\Gamma(\text{Sym}^2(T^*M))$. We calculate the linearisation of the Ricci tensor, which is a second order differential operator on the Riemannian metric g , by picking an arbitrary $h \in \Gamma(\text{Sym}^2(T^*M))$ and carry out the calculation in Definition 2.2.1, so that we have the following lemma.

Lemma 2.2.2. [21] [22] Let (M, g) be a Riemannian manifold. In local coordinates, the linearisation of the Ricci tensor is given by the linear operator:

$$(DRic_g(h))_{ij} = \frac{1}{2}g^{kl}(\nabla_k \nabla_j h_{li} + \nabla_i \nabla_l h_{jk} - \nabla_k \nabla_l h_{ij}) - \frac{1}{2}\nabla_i \nabla_j (g^{kl} h_{kl}), \quad (2.3)$$

for any $h \in \Gamma(\text{Sym}^2(T^*M))$. $\blacksquare$

Proof. Define a curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathfrak{Met}$ to be $\gamma(s) = g + sh$. This is well defined for a sufficiently small $\varepsilon > 0$ as the space of Riemannian metrics on a manifold forms an open set in the vector space $\Gamma(\text{Sym}^2(T^*M))$. We denote the Christoffel symbols for the metric $\gamma(s)$ by $\hat{\Gamma}$.

In local normal coordinates with respect to the metric g around an arbitrary point $p \in M$, we have the relations $\Gamma_{ij}^k(p) = 0$, $\partial_i g_{jk}(p) = 0$, and $\partial_i h_{jk}(p) = \nabla_i h_{jk}(p)$ (see Proposition A.3.19). We note first that the linearisations of the Christoffel symbols, written down in the normal coordinates centred at p , are given by:

$$\begin{aligned} (D\Gamma(h))_{ij}^k &= \left. \frac{\partial}{\partial s} \right|_{s=0} \hat{\Gamma}_{ij}^k \\ &= \left. \frac{\partial}{\partial s} \right|_{s=0} \frac{1}{2}(g + sh)^{kl}(\partial_i(g + sh)_{jl} + \partial_j(g + sh)_{il} - \partial_l(g + sh)_{ij}) \\ &= \frac{1}{2}h^{kl}(\cancel{\partial_i g_{jl}} + \cancel{\partial_j g_{il}} - \cancel{\partial_l g_{ij}}) + \frac{1}{2}g^{kl}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}) \\ &= \frac{1}{2}g^{kl}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}). \end{aligned} \quad (2.4)$$

Thus, by using normal coordinates around the point p , the fact that $\Gamma_{ij}^k(p) = 0$, and the

linearisation above, we compute:

$$\begin{aligned}
 (DRic_g(h))_{ij} &= \frac{\partial}{\partial s} \Big|_{s=0} Ric(g + sh)_{ij} \\
 &= \frac{\partial}{\partial s} \Big|_{s=0} (\partial_k \hat{\Gamma}_{ij}^k - \partial_i \hat{\Gamma}_{kj}^k + \hat{\Gamma}_{ij}^l \hat{\Gamma}_{kl}^k - \hat{\Gamma}_{ik}^l \hat{\Gamma}_{jl}^k) \\
 &= \frac{1}{2} g^{kl} \partial_k (\partial_j h_{li} + \partial_i h_{lj} - \partial_l h_{ij}) - \frac{1}{2} g^{kl} \partial_i (\partial_k h_{lj} + \partial_j h_{kl} - \partial_l h_{jk}) \\
 &= \frac{1}{2} g^{kl} (\partial_{kj} h_{li} + \partial_{il} h_{jk} - \partial_{kl} h_{ij} - \partial_{ij} h_{kl}) \\
 &= \frac{1}{2} g^{kl} (\nabla_k \nabla_j h_{li} + \nabla_i \nabla_l h_{jk} - \nabla_k \nabla_l h_{ij} - \nabla_i \nabla_j (g^{kl} h_{kl})).
 \end{aligned}$$

Since $p \in M$ is arbitrary, this holds everywhere on M , which is what we wanted to prove. $\square$

In fact, if we define the divergence of a 2-tensor h by $\delta h_k = -g^{ij} \nabla_i h_{jk}$, we can write the linearisation of the Ricci tensor in terms of the Lichnerowicz Laplacian, which is defined as follows:

Definition 2.2.3 (Lichnerowicz Laplacian). The Lichnerowicz Laplacian on the Riemannian manifold (M, g) is the operator $\Delta_g^L : \Gamma(\text{Sym}^2(T^*M)) \rightarrow \Gamma(\text{Sym}^2(T^*M))$ which satisfies:

$$\Delta_g^L h_{ij} = \Delta_g h_{ij} - g^{kl} (\text{Ric}_{ik} h_{lj} + \text{Ric}_{jk} h_{li}) + 2g^{kp} g^{lq} \text{Rm}_{iklj} h_{pq},$$

where Δ_g is the usual Laplacian operator with respect to the metric g . $\blacksquare$

Thus, by commuting derivatives, we can write the linearisation of the Ricci tensor as:

$$(DRic_g(h))_{ij} = -\frac{1}{2} (\Delta_g^L h_{ij} + \nabla_i (\delta h)_j + \nabla_j (\delta h)_i + \nabla_i \nabla_j (g^{kl} h_{kl})).$$

To check the parabolicity of the Ricci flow, we calculate the principal symbol of the non-linear differential operator $-2\text{Ric}(g(t))$. The symbol is defined only for linear differential operators but the definition carries over to non-linear operators by considering their linearisations.

Definition 2.2.4 (Symbol of a linear differential operator). Let $\mathcal{P} : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{F})$ be a linear differential operator of order k . In local coordinates, the operator $\mathcal{P}$ can be written as:

$$\mathcal{P}(u) = \sum_{|\alpha| \leq k} a^\alpha \nabla^\alpha u,$$

where $a^\alpha \in \text{Hom}(\mathcal{E}, \mathcal{F})$ are bundle homomorphisms (or fibrewise linear maps between the vector bundles). We define the symbol $\sigma_{\mathcal{P}}$ of the operator $\mathcal{P}$ as a map $\sigma_{\mathcal{P}} : T^*M \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F})$ by:

$$\sigma_{\mathcal{P}}(x, v) = \sum_{|\alpha| \leq k} a^\alpha v_\alpha,$$

where (x, v) is a local trivialisation of the cotangent bundle. $\blacksquare$

This is well defined since it is independent of local coordinates. Most of the time, we are interested in the principal symbol $\hat{\sigma}_{\mathcal{P}}(x, v)$ of the operator $\mathcal{P}$, which is the leading term in $\sigma_{\mathcal{P}}(x, v)$:

$$\hat{\sigma}_{\mathcal{P}}(x, v) = \sum_{|\alpha|=k} a^{\alpha} v_{\alpha}.$$

Principal symbols are important in the theory of PDEs on vector bundles because they allow us to determine the type of the PDE. Let us look at the operator $-2\text{Ric}(g(t))$. This operator is a non-linear map from $\mathfrak{Met}$ to $\Gamma(\text{Sym}^2(T^*M))$. From equation (2.3), for a fixed covector $v \in T^*M$, the principal symbol for its linearised operator acting on an arbitrary element of $h \in \text{Sym}^2(T^*M)$ is given by:

$$(\hat{\sigma}_{-2D\text{Ric}_g}(x, v)h)_{ij} = g^{kl}(v_k v_l h_{ij} + v_i v_j h_{kl} - v_k v_i h_{lj} - v_l v_j h_{ik}). \quad (2.5)$$

The linearisation of the operator at g is strongly elliptic if there exists a positive function $\lambda : M \rightarrow \mathbb{R}$ such that for all $h \in \text{Sym}^2(T^*M)$ and non-zero $v \in T^*M$, we have:

$$\langle \hat{\sigma}_{-2D\text{Ric}_g}(x, v)h, h \rangle \geq \lambda(x) \|v\|^2 \|h\|^2,$$

where $\langle \cdot, \cdot \rangle$ is the induced inner product on the vector bundle $\text{Sym}^2(T^*M)$. If this ellipticity condition holds, then the associated non-linear flow is then called strongly parabolic.

However, if we choose $h_{ij} = v_i v_j$ (so that h is symmetric), we can see that the left hand-side of equation (2.5) vanishes. Therefore, the Ricci flow equation is not strongly parabolic. It is only weakly parabolic because the value 0 is attained by some v and h .

This is somehow to be expected: if we assume that the equation is parabolic, then the stationary solution of the Ricci flow on a compact manifold solves an elliptic equation, namely $\text{Ric}(g(t)) = 0$ for all $t > 0$. We know that the space of solutions of elliptic equation on a compact manifold is finite-dimensional, so we expect the solution space for the stationary Ricci flow to be finite-dimensional. However, the Ricci tensor is invariant under the full diffeomorphism group of the manifold M (which is infinite-dimensional). So, given a stationary solution of the Ricci flow we can construct another linearly independent solution by pulling it back by any diffeomorphism from M to itself, which is a contradiction.

2.2.1 Short Time Existence of Solutions

Since the Ricci flow is only weakly parabolic, we cannot utilise the existence theory in parabolic PDE theory as it requires the equation to be strongly parabolic. However, Hamilton proved the short-time existence of the Ricci flow if the manifold is compact and the initial metric is

smooth using the Nash-Moser inverse function theorem [42]. Later, DeTurck managed to give a short-time existence result in a much simpler way which is known as DeTurck's Trick [26] today.

DeTurck's Trick enables us to express the weakly parabolic Ricci flow in a strongly parabolic manner. Thus, this allows us to exploit the theories of existence and regularity developed for strongly parabolic equations. The crux of the trick is that, as we have noted before, the Ricci tensor is invariant under the full diffeomorphism group.

At the initial time, we can find a coordinate system that makes the Ricci flow strongly parabolic, which is the harmonic coordinates. To keep the flow strongly parabolic at all time, we try to construct a family of time dependent diffeomorphism ϕ_t of the manifold onto itself that evolves along with the Ricci flow and preserves the parabolicity of the equation. By pulling the solution of this modified flow back by ϕ_t , we get our original Ricci flow.

The equation for this modified flow is often called the DeTurck-Ricci flow and it is given by:

$$\begin{cases} \frac{\partial}{\partial t} \bar{g}(t) = -2\text{Ric}(\bar{g}(t)) + \mathcal{L}_{X(t)} \bar{g}(t), \\ \bar{g}(0) = \bar{g}_0, \end{cases} \quad (2.6)$$

where $\mathcal{L}_{X(t)}$ is the Lie derivative in the direction of $X(t)$, which is the vector field generating the diffeomorphism ϕ_t such that $\phi_0 = \text{Id}_M$.

If $\bar{g}$ solves the DeTurck-Ricci flow (2.6)-(2.7), then it can be shown that the pullback of this metric by the diffeomorphisms ϕ_t , namely $g(t) = \phi_t^* \bar{g}(t)$, solves the Ricci flow (1.1)-(1.2) with $g_0 = \bar{g}_0$. Thus, the two problems are equivalent, despite the fact that the DeTurck-Ricci flow has the advantage of being strongly parabolic and enjoys the parabolic theory of partial differential equations.

To construct the vector field $X(t)$ explicitly, first, we are going to inspect the linearisation of the Ricci tensor (2.3) and see which terms are the “bad” terms that make the equation weakly parabolic. By symmetry of the metric tensor g and the symmetric tensor h , we can write the linearisation of the tensor $-2\text{Ric}(g(t))$ from Lemma 2.3 as:

$$-2(D\text{Ric}_g(h))_{ij} = g^{kl} (\nabla_k \nabla_l h_{ij} + \nabla_i \nabla_j h_{kl} - \nabla_k \nabla_j h_{li} - \nabla_i \nabla_l h_{jk}). \quad (2.8)$$

To make the expression simpler, we define a covector $\omega \in \Gamma(T^*M)$ on M by the local coordinates expression:

$$\omega = g^{kl} \left(\frac{1}{2} \nabla_j h_{kl} - \nabla_l h_{kj} \right) dx^j = \frac{1}{2} g^{kl} (\nabla_j h_{kl} - \nabla_l h_{kj} - \nabla_k h_{lj}) dx^j.$$

Remark 2.2.5. A different expression for the covector component ω_i is $-g^{kl} g_{ij} (D\Gamma(h))_{kl}^j$ where Γ is the Christoffel symbol for g . This expression will be useful later. ■

Since covariant derivatives commute with trace operation, we can write (2.8) as:

$$-2(D\text{Ric}_g)(h)_{ij} = g^{kl}\nabla_k\nabla_l h_{ij} + \nabla_i\omega_j + \nabla_j\omega_i + \dots, \quad (2.9)$$

where we omit the terms with no h derivatives. The first term in the expression above is a “good” term because it is essentially the Laplacian of the tensor h , which is strongly parabolic. The remaining second order terms are the “bad” terms. We now aim to get rid of these terms by introducing a time-dependent cancelling diffeomorphism ϕ_t .

We define a new metric $\bar{g}$ such that $g(t) = \phi_t^*\bar{g}(t)$ for some time-dependent diffeomorphism ϕ_t that is to be determined. We are going to construct a parabolic equation for $\bar{g}(t)$. Since $g(t)$ satisfies the Ricci flow and the Ricci tensor is diffeomorphically invariant, we have the following:

$$\begin{aligned} \frac{\partial}{\partial t}(\phi_t^*\bar{g}(t)) &= \frac{\partial}{\partial t}g(t) \quad \Rightarrow \quad \phi_t^*\left(\frac{\partial}{\partial t}\bar{g}(t)\right) + \phi_t^*(\mathcal{L}_{X(t)}\bar{g}(t)) = -2\text{Ric}(g(t)) \\ &= -2\text{Ric}(\phi_t^*\bar{g}(t)) \\ &= -2\phi_t^*\text{Ric}(\bar{g}(t)), \end{aligned}$$

where $X(t)$ is a time-dependent vector field defined by:

$$\begin{cases} \frac{\partial}{\partial t}\phi_t = X(t), \\ \phi_0 = \text{Id}_M. \end{cases}$$

Thus, the metric $\bar{g}(t)$ satisfies the equation:

$$\frac{\partial}{\partial t}\bar{g}(t) = -2\text{Ric}(\bar{g}(t)) - \mathcal{L}_{X(t)}\bar{g}(t) := \mathcal{P}(\bar{g}).$$

If we linearise the operator $\mathcal{P}(\bar{g})$ in the direction h , we would obtain a similar term to (2.9) with an extra term coming from the linearisation of $\mathcal{L}_{X(t)}\bar{g}$. Namely:

$$(D\mathcal{P}_{\bar{g}}(h))_{ij} = \bar{g}^{kl}\nabla_k\nabla_l h_{ij} + \nabla_i\omega_j + \nabla_j\omega_i - (D\mathcal{L}_{X(t)}\bar{g}(h))_{ij} + \dots,$$

Now, we want to define the vector field $X(t)$ such that the linearisation of $\mathcal{L}_{X(t)}\bar{g}$ is equal to the “bad” term in (2.9). In fact, we can explicitly find a suitable time-dependent vector field $X(t) \in \Gamma(TM)$ to make the equation parabolic. Based on the observation in Remark 2.2.5, we note that the linearisation of the Christoffel symbols of $\bar{g}$, which was calculated in equation (2.4), is given by:

$$(D\bar{\Gamma}(h))_{ij}^k = \frac{1}{2}\bar{g}^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}).$$

This almost looks like the covector ω except for the index k . Therefore, we choose the vector field $X(t)$ such that its related 1-form is given by the covector:

$$X(t)^b = -\bar{g}_{ij}\bar{g}^{kl}(\bar{\Gamma}_{kl}^j - \hat{\Gamma}_{kl}^j)dx^i,$$

where $\bar{\Gamma}_{kl}^j$ is the Christoffel symbol of the metric $\bar{g}$ and $\hat{\Gamma}_{kl}^j$ is the Christoffel symbol of some fixed background metric on the manifold. The background metric is required to ensure that this vector field transforms tensorially. Thus, with this choice of X , we can calculate:

$$(D\mathcal{L}_{X(t)}\bar{g}(h))_{ij} = \nabla_i\omega_j + \nabla_j\omega_i + \dots,$$

where we omitted the lower order terms of h . This choice for $X(t)$ ensures that the “bad” terms disappear, making the flow (2.6)-(2.7) strongly parabolic. Thus, by virtue of parabolic theory on compact domains, a unique smooth solution $\bar{g}(t)$ exists for some small time interval.

Pulling back this metric $\bar{g}$ by the diffeomorphism ϕ_t generated by this choice of X gives us a solution for the Ricci flow (1.1)-(1.2). Furthermore, this solution is unique. This can be shown by using a different type of flow called the harmonic map flow. See [2], [21], and [22] for details.

Remark 2.2.6. The above argument gives us a short-time existence of the Ricci flow equation for a given initial metric on compact manifolds. For non-compact cases, the proof gets significantly more difficult and some geometric conditions must be imposed to deal with the metric’s behaviour at infinity. Shi proved that one only requires a bounded geometry globally at initial time (that is, $\sup_M |\text{Rm}|_{g_0} < \infty$) for a short-time existence of smooth solution to the Ricci flow (1.1)-(1.2) on non-compact Riemannian manifolds [89]. The uniqueness of such flows was proven by Chen and Zhu in [19].

The bounded geometry condition has been generalised by Xu by utilising the idea of local Ricci flow which was introduced by Yang [100]. In his work [99], Xu guarantees a short-time existence of the solution to the Ricci flow on non-compact m -dimensional manifolds by requiring that for the initial metric g_0 there exist positive constants $K_1, K_2, r > 0$ and $q > \frac{m}{2}$ such that:

$$\begin{aligned} \text{Ric}(g_0) &\geq -K_1 g_0, \\ \left(\frac{1}{\text{Vol}_{g_0}(B_r^{g_0}(p))} \int_{B_r^{g_0}(p)} |\text{Rm}(g_0)|_{g_0}^q d\text{Vol} \right)^{\frac{1}{q}} &\leq K_2, \end{aligned}$$

for any $p \in M$.

2.3 Singularities in Ricci Flow

Since the Ricci flow describes a partial differential equation, it would be difficult to obtain explicit solutions in general. However, there are some special forms of the metrics which can give rise to an explicit solution. For example, if the initial metric is an Einstein metric, then we can easily find an explicit solution of the Ricci flow.

Example 2.3.1. Suppose that the initial metric g_0 is an Einstein metric; that is, $\text{Ric}(g_0) = \lambda g_0$ for some $\lambda \in \mathbb{R}$. Then a solution of the Ricci flow with this initial metric is given by $g(t) = (1 - 2\lambda t)g_0$ for an appropriate range of t depending on λ . Einstein metrics form an important class of metrics since the solution is self-similar by scaling under the Ricci flow.

Einstein manifolds have also been studied extensively due to their geometrical properties as well as applications in physics. The Einstein condition provides information about the topology of the manifold: for example, Myers theorem [66] states that n -dimensional Einstein manifolds with positive Einstein factor λ are always compact, have finite fundamental group, and have diameter bounded from above by $\pi\sqrt{\frac{n-1}{\lambda}}$.

Furthermore, coupled with the cohomogeneity one condition on the manifold, which we are going to consider in this work, many interesting curvature results can be shown. See the paper by Bérard Bergery in which the author demonstrated some of these curvature properties and constructed examples of Einstein metrics [7].

Einstein metrics generalise to Ricci solitons (see Definition 2.5.1), which are solutions that evolve only via scaling and diffeomorphisms of the manifold onto itself under the Ricci flow. This is an important family of metrics, which we shall see later at the end of this chapter. ■

Unfortunately, some obstructions may happen during the flow. This is due to the non-linear nature of the Ricci flow equation. Note that from the example above, for $\lambda > 0$ the coefficient $1 - 2\lambda t$ will vanish at some finite time $T = \frac{1}{2\lambda}$. At this time, the metric will be degenerate and so the solution of the Ricci flow with this Einstein initial metric can only exist up to this finite singular time T .

Therefore, we can only expect the existence of solutions for a short interval of time. In particular, singularities can occur in the metric at some finite time. In fact, one can derive the evolution equation for the scalar curvature S under the Ricci flow, which is given by:

$$\partial_t S = \Delta S + 2|\text{Ric}|_g^2 \geq \Delta S + \frac{2}{n}S^2. \quad (2.10)$$

For a compact manifold, if the scalar curvature of the metric at initial time is strictly positive (that is, $S(p, 0) \geq s_0 > 0$), using a version of the maximum (or minimum) principle for parabolic equations (see [93] or Theorem 4.5.3), we can show that:

$$S(p, t) \geq \phi(t) \quad \text{where } \phi \text{ satisfies } \frac{d\phi}{dt} = \frac{2}{n}\phi^2 \quad \text{with } \phi(0) = s_0,$$

which implies that $S(p, t) \geq \frac{1}{\frac{1}{s_0} - \frac{2t}{n}}$. Thus, there exists some finite time $T \leq \frac{n}{2s_0}$ such that $\lim_{t \rightarrow T} S(p, t) = \infty$, which implies that the metric becomes singular at this time T . We now look at the two different types of finite-time singularities which may occur during the Ricci flow.

2.3.1 Type I Singularity

The simplest example for a metric singularity can be obtained by starting with an n -dimensional unit sphere S^n with the canonical round metric. This is an Einstein manifold as in Example 2.3.1 with $\lambda = n - 1$. As we have noted earlier, for any $n > 1$, the metric $g(t)$ contracts with time. Thus, the sphere will shrink and at a finite time $T = \frac{1}{2(n-1)}$ it will degenerate to a point as depicted in Figure 2.1.

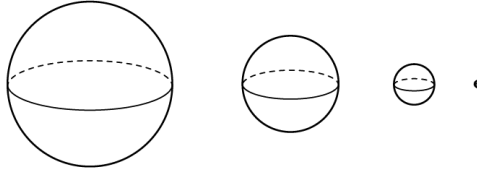


Figure 2.1: A round sphere degenerates under the Ricci flow.

This type of singularity falls under the first category, called the Type I singularity (also known as rapidly forming singularity), for which the singularity occurs at a finite time $T < \infty$ such that we have the following curvature asymptotics:

$$\sup_{M \times [0, T)} (T - t) |\text{Rm}(\cdot, t)|_g < \infty, \quad (2.11)$$

where Rm is the Riemann curvature tensor and $|\cdot|_g$ is the norm on 4-tensors induced from the metric g .

Another simple explicit example for this type of singularity is the standard cylinder $\mathbb{R} \times S^n$ with the metric inherited from the usual embedding in $\mathbb{R}^{n+2}$. This cylinder shrinks in size radially and evolves under the Ricci flow over time as follows:

$$g(t) = ds^2 + 2(n-1)(T-t)g_{S^n}.$$

Furthermore, we can create more examples from this. Hamilton conjectured that a neckpinch singularity, which is another Type I singularity, can be achieved for a specially chosen initial metric on S^3 for the flow [44]. This “dumbbell” metric is constructed by joining two round spheres with a neck that looks like a cylinder $I \times S^2$ where I is an open connected interval in $\mathbb{R}$. See Figure 2.2 for an illustration on 2-dimensional manifold. As a word of caution, this illustration is only to be taken as a guide as this singularity only occurs for manifolds of dimensions greater than or equal to 3.

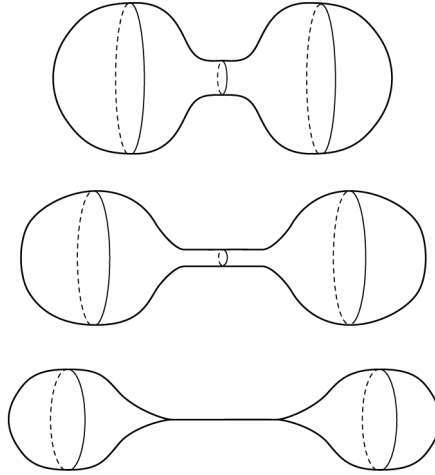


Figure 2.2: A neckpinch singularity. The neck behaves like a shrinking cylinder.

If we endow the standard cylindrical metric on this neck, we expect that it will pinch off at some finite time before the ends of the “dumbbell” shrink to the size of the neck. This is done intuitively by considering the curvatures on each part of the manifold separately. If the neck is thin and long enough, it will pinch off before the metric can be distributed evenly throughout the manifold. However, despite this intuition, there are very few examples of this singularity. Two notable non-trivial examples were constructed by Angenent and Knopf [6] as well as Isenberg, Knopf, and Šešum [54] which were discussed in Chapter 1.

2.3.2 Type II Singularity

Apart from the Type I singularity, by dichotomy, there also exists the Type II singularity (also known as slowly forming singularity). Like the Type I singularity, this singularity occurs at finite time. However, the defining property of the Type II singularity is such that the curvature asymptotics behave as:

$$\sup_{M \times [0, T)} (T - t) |\text{Rm}(\cdot, t)|_g = \infty.$$

Remark 2.3.2. In some literature (for example in [44] and [21]), this singularity is called the Type IIa singularity, if we take into account the infinite time singularity of the same curvature asymptotics which is distinguished as the Type IIb singularity. ■

In the survey paper by Hamilton [44], he also considered the Ricci flow from a non-symmetrical dumbbell initial metric. Graphically, he described that for a suitable size of the sphere on one end of the dumbbell we can make the sphere shrink to a round point before the neck disap-

pears. This results in a singularity called the degenerate neckpinch, as shown in Figure 2.3. The discussion by Hamilton was just a conjecture which was explained informally.

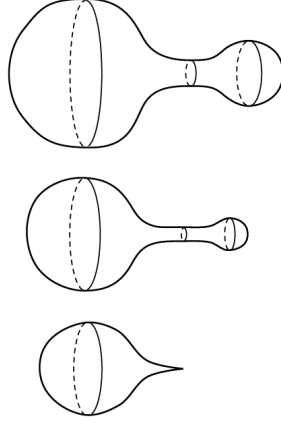


Figure 2.3: A degenerate neckpinch singularity.

Garfinkle and Isenberg ran numerical simulations to predict the existence of initial metrics that result in a degenerate neckpinch singularity at finite time [35]. They considered the corseted sphere geometries on S^3 which are metrics of the form:

$$g = e^{2X}(e^{-2W}d\psi^2 + e^{2W}\sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)),$$

where (ψ, θ, ϕ) is the standard angular coordinate system on the 3-sphere such that $\psi, \theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, and W, X are functions of ψ only. By setting $W = X$ and choosing X such that $4e^{4X}\sin^2\psi = \sin^2 2\psi$ for $\cos^2\psi \geq \frac{1}{2}$ and $4e^{4X}\sin^2\psi = \sin^2 2\psi + 4\lambda \cos^2 2\psi$ for $\cos^2\psi \leq \frac{1}{2}$, we obtain a family of initial metrics g_λ for which $\lambda \geq 0$ is a parametrising variable.

For $\lambda = 0$ the geometry corresponds to two spheres meeting at a point, so singularity under the Ricci flow occurs immediately. As λ increases, the waist of the sphere gets bigger and we expect that the curvature will be positive everywhere on the manifold for large enough λ .

By solving the normalised Ricci flow (under which the metric evolves under the Ricci flow whilst keeping their volume constant) numerically and plotting the curvatures of $(S^3, g_\lambda(t))$ for some $\lambda > 0$ and a suitable range of t , they demonstrated the behaviour of the curvature evolutions of some choice of initial metrics.

They considered the normalised Ricci flow because if the initial metric converges to the round point under the Ricci flow, then the solution for the corresponding normalised Ricci flow will exist for all time and tends to a metric with uniform curvature (see Theorem 3.3.1). Otherwise, a neckpinch singularity will occur eventually and thus the normalised flow can only exist up to some finite time.

By interpolating between $\lambda = 0$ and some large $\lambda > 0$, we can numerically detect the transitional value of λ for which the degenerate neckpinch singularity is expected to occur, thus strengthening the conjecture by Hamilton.

Finally, in 2008, Gu and Zhu published a paper proving the existence of an initial metric which leads to this degenerate neckpinch [39]. Their proof relied on the estimates and results in [6]. This is done by finding a family of initial metrics g_ν on S^{n+1} which vary smoothly in the C^3 topology with parameter $\nu \in [0, 1]$.

This parameter interpolates between the metrics g_0 and g_1 , where g_0 is the metric defined in (1.3)-(1.4) and g_1 is some initial metric which will shrink to a round point at some finite time (for example, the round sphere metric). They then showed that the set of parameter ν for which the initial metrics g_ν develop a neckpinch singularity forms a connected open set in $[0, 1]$, say $(\bar{\nu}, 1]$. With some analysis, they concluded that the initial metric $g_{\bar{\nu}}$ is the metric that leads to the degenerate neckpinch singularity, which they call a Type II singularity. Whilst the existence of such singularity is proven, not much information can be gleaned about the characteristics of this singularity from that study.

In 2011, Angenent, Isenberg, and Knopf [4] collaborated on a paper detailing a conjecture for an asymptotic behaviour of the degenerate Ricci flow neckpinch on S^{n+1} . In this paper, they utilised the method of matched asymptotics to predict the behaviour of the singularity by extending the work done by Angenent and Knopf in their 2004 paper.

They constructed an ansatz profile of the solutions $\{(S^{n+1}, g_k(t))\}_{k \in \mathbb{N}}$ for the Ricci flow in four different regions (namely: outer, parabolic, intermediate, and tip regions) and formally patched these regions together using matched asymptotics. For each $k \geq 3$, at the tip region the curvature of manifold $(S^{n+1}, g_k(t))$ satisfies:

$$\sup_M |\text{Rm}(\cdot, t)|_g = \frac{C}{(T - t)^{2-2/k}},$$

where C is a constant depending only on n and k . In each case, the singularity at the tip is slowly forming, hence these manifolds form Type II singularities. In their work, they conjectures that there exists a initial metrics on S^{n+1} that develops a Type II singularity under the Ricci flow and their profiles near the singular time are given by the constructed matched ansatz profile.

This conjecture was confirmed in 2015 when the authors proved the existence of Ricci flow solutions on S^{n+1} for $n \geq 2$ that behave as described above using barrier methods. Furthermore, they showed that the tip region, after rescaling, converges uniformly to the steady Bryant soliton [14] (see Definition 2.5.1 for the definition of a Ricci soliton). For more details, see [5].

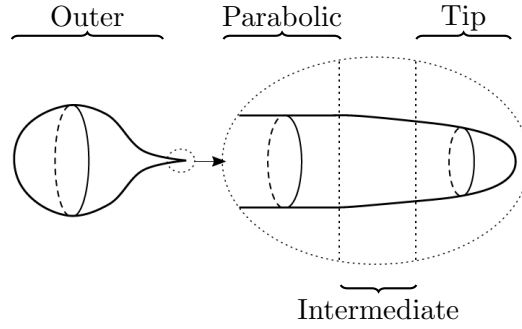


Figure 2.4: Glueing the Bryant soliton to a shrinking Ricci flow solution.

On a separate but related note, in 2014, Wu constructed several non-compact examples of initial metrics which lead to the Type II singularity on $\mathbb{R}^n$ for $n \geq 3$ [97]. However, an explicit compact example is still yet to be discovered.

2.4 Blow-Up Sequence

As commented in the examples, we note that the neckpinch constructed by Angenent and Knopf asymptotically tends to a shrinking cylinder whilst the degenerate neckpinch is modelled by the Bryant soliton. These two types of Riemannian manifolds are examples of a special class of manifolds called the Ricci solitons. They are generalisations of Einstein manifolds as noted in Example 2.3.1.

These two are examples of behaviour of blow-ups of the Ricci flow near the singularity time, which were obtained by “zooming-in” at certain points on the manifold. We want to be able to study this blow-up limit in order to understand the behaviour of the Ricci flow near the singularity. First, we want to define the notion of sequences of Riemannian manifolds in order to define a blow-up sequence of the Ricci flow. As we have noted, the “zooming-in” process has to be done within specific regions on the manifold, so we must have a set of reference points on the manifold. Hence we define:

Definition 2.4.1 (Cheeger-Gromov Convergence). [20] A sequence (M_i, g_i, p_i) of smooth, complete pointed Riemannian manifolds converges smoothly to a pointed manifold $(M_\infty, g_\infty, p_\infty)$ as $i \rightarrow \infty$ if there exists:

1. a sequence of domains $\Omega_i \subset M_\infty$ such that any compact set $K \subset M_\infty$ is contained in some Ω_i for large enough i with $p_\infty \in \Omega_i$ for each i ,

2. a sequence of smooth maps $\phi_i : \Omega_i \rightarrow M_i$ which are diffeomorphic onto their images and $\phi_i(p_\infty) = p_i$ for all i ,

such that the pullback metric $\phi_i^* g_i \rightarrow g_\infty$ smoothly locally on M_∞ as $i \rightarrow \infty$. ■

Of course, given a sequence of smooth complete pointed m -dimensional Riemannian manifolds (M_i, g_i, p_i) , it need not converge. However, a remarkable theorem states that given some conditions on the injectivity radii of these manifolds at the points p_i and the curvature in neighbourhoods of these points, we can extract a subsequence from (M_i, g_i, p_i) that converges to some limit manifold $(M_\infty, g_\infty, p_\infty)$ which is of the same dimension as M_i .

Theorem 2.4.2 (Compactness). [18] [20] Suppose that (M_i, g_i, p_i) is a sequence of smooth complete pointed m -dimensional Riemannian manifolds such that:

1. (curvature bound): for all $r > 0$ and $k \in \mathbb{N} \cup \{0\}$ we have $\sup_{i \in \mathbb{N}} \sup_{B_r^{g_i}(p_i)} |\nabla^k \text{Rm}|_{g_i} < \infty$, where $B_r^{g_i}(p_i)$ are the geodesic balls of radius r in metric g_i centred at the point p_i ,
2. (injectivity radius bound): $\inf_{i \in \mathbb{N}} \text{inj}(M_i, g_i, p_i) > 0$, where $\text{inj}(M_i, g_i, p_i)$ is the injectivity radius of the manifold (M_i, g_i) at the point p_i as defined in Definition A.3.18.

Then there exists a smooth complete pointed Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ of dimension m such that after passing to a subsequence in i , we have $(M_i, g_i, p_i) \rightarrow (M_\infty, g_\infty, p_\infty)$ as $i \rightarrow \infty$ in the sense of Definition 2.4.1. ■

We can modify Definition 2.4.1 and Theorem 2.4.2 to include flows and this is the result known as Hamilton's Compactness Theorem for Ricci flows.

Definition 2.4.3 (Cheeger-Gromov Convergence of flows). [20] [45] Let $(M_i, g_i(t))$ be a sequence of flows on some time interval $I_i \subset \mathbb{R}$ such that $\inf_i |I_i| > 0$ and let $p_i \in M_i$ for each i . We say that the flow converges smoothly to a pointed flow $(M_\infty, g_\infty(t), p_\infty)$ where g_∞ is defined on some interval $I_\infty \subset \mathbb{R}$ as $i \rightarrow \infty$ if there exists:

1. a sequence of domains $\Omega_i \subset M_\infty$ such that any compact set $K \subset M_\infty$ is contained in some Ω_i for large enough i with $p_\infty \in \Omega_i$ for each i ,
2. a sequence of smooth maps $\phi_i : \Omega_i \rightarrow M_i$ which are diffeomorphic onto their images and $\phi_i(p_\infty) = p_i$ for all i ,

such that the pullback metric $\phi_i^* g_i(t) \rightarrow g_\infty(t)$ smoothly locally on $M_\infty \times I_\infty$ as $i \rightarrow \infty$. ■

Theorem 2.4.4 (Hamilton’s Compactness Theorem). [45] Let (M_i, p_i) be a sequence of pointed m -dimensional manifolds with $p_i \in M_i$ for each i . Suppose that $g_i(t)$ is a sequence of complete Ricci flows on $M_i \times I_i$ for some intervals I_i containing $t = 0$ such that $\inf_i |I_i| > 0$. Suppose further that:

1. (curvature bound): for all $r > 0$ we have $\sup_{i \in \mathbb{N}} \sup_{B_r^{g_i}(p_i), t \in I} |\text{Rm}|_{g_i(t)} < \infty$, where $B_r^{g_i}(p_i)$ are the geodesic balls of radius r in metric g_i centred at the point p_i ,
2. (injectivity radius bound): $\inf_{i \in \mathbb{N}} \text{inj}(M_i, g_i(0), p_i) > 0$, where $\text{inj}(M_i, g_i(0), p_i)$ is the injectivity radius of the manifold $(M_i, g_i(0))$ at the point p_i .

Then there exists a smooth complete manifold M_∞ of dimension m , a complete Ricci flow $g_\infty(t)$ on M_∞ defined on some interval I_∞ , and a point $p_\infty \in M_\infty$ such that after passing to a subsequence in i , we have $(M_i, g_i(t), p_i) \rightarrow (M_\infty, g_\infty(t), p_\infty)$ as $i \rightarrow \infty$ in the sense of Definition 2.4.3. ■

Note the difference between the first condition in Theorem 2.4.2 and Theorem 2.4.4. We only require the condition for the first derivative of the Riemann curvature tensor in the flow case since the Ricci flow, which is a “heat-type” equation, has good regularity properties. Based on the Bernstein-Bando-Shi derivative estimates [88], a uniform bound on the Riemann curvature tensor is sufficient to bound all of its derivatives.

Also note that, in presence of the uniform bound of the Riemann curvature tensor, we can modify the condition on the injectivity radius by replacing it with an equivalent statement which requires the volume of geodesic balls to satisfy the bound $\inf_{i \in \mathbb{N}} \text{Vol}_{g_i(0)}(B_1^{g_i(0)}(p_i)) > 0$. We note that this is important in the study of the blow-up sequence, augmented with the celebrated theorem by Perelman:

Theorem 2.4.5 (Perelman’s No Local Collapsing Theorem). [73] Suppose that M is a closed manifold and $g(t)$ is a Ricci flow on M for $t \in [0, T)$ for some $T > 0$. Then for all $p \in M$ and $t \in [0, T)$, if $r \in (0, 1]$ is sufficiently small so that $|\text{Rm}|_{g(t)} \leq r^{-2}$ in $B_r^{g(t)}(p)$, then there exists some $\kappa > 0$ depending only on the dimension of M , the initial data $g(0)$, and T such that:

$$\frac{\text{Vol}_{g(t)}(B_r^{g(t)}(p))}{r^m} \geq \kappa > 0. \quad \blacksquare$$

Therefore, by carefully constructing a sequence of Ricci flows such that the Riemann curvature remains uniformly bounded for all i , we can deduce the volume bound in order to satisfy the sufficient conditions in Theorem 2.4.4. This construction can be done by rescaling and shifting of the metrics, which will be crucial in one of the proofs later.

2.5 Singularity Models

It is useful to study these singularities to determine the topological behaviour of the manifold near the time of singularity. One method of studying these singularities is the blow-up method by a sequence of parabolic dilations on the metric. Here, the singular solutions are rescaled by dilations about a sequence of points and times approaching the singular time. The limits of these dilations are called the singularity models. This is where the Ricci solitons that we have discussed earlier in Example 2.3.1 come into the picture.

After this sequence of scalings, we expect the singularity to converge to a self-similar solution of the Ricci flow, which exists for an infinite amount of time. Thus, these singularities can be modelled by some long-time solutions of the Ricci flow, namely the ancient, immortal, or eternal solutions [20]. It has been proven that there exists a sequence of dilations of the solution which converges to a singularity model for a solution of the Ricci flow satisfying a certain injectivity radius estimate. See [20], [44], and [93] for more details on the injectivity radius estimate as well as the proof for this statement.

As an example, the round sphere S^n , which exhibits the Type I singularity under the Ricci flow, can be modelled with an ancient solution metric that exists for the time interval $(-\infty, T)$ for some finite T by extending the solution of the Ricci flow backwards in time from $t = 0$.

For the neckpinch singularity as in Figure 2.2, it can be modelled as a shrinking cylinder $S^1 \times \mathbb{R}$, which is another ancient solution. This model arises in the analysis done in [6] and [54] as the blow-up limit of their example manifolds near the singularity time and space, which is also a Type I singularity.

The singularity models described above are both self-similar solutions, in the sense that they only evolve via scaling without change in the shape up to diffeomorphisms. Recall from Example 2.3.1 that these type of manifolds are called the Ricci solitons. Formally, they are defined by:

Definition 2.5.1 (Ricci solitons). A solution $g(t)$ of the Ricci flow on M is a Ricci soliton if there exists a positive function $\sigma(t)$ and a 1-parameter family of diffeomorphism $\varphi(t) : M \rightarrow M$ such that:

$$g(t) = \sigma(t)\varphi(t)^* g_0,$$

or equivalently, if the metric g satisfies the equation:

$$-2\text{Ric}(g) = \varepsilon g + \mathcal{L}_X g,$$

for some vector field $X \in \Gamma(TM)$ and constant $\varepsilon \in \mathbb{R}$. We say that the soliton is expanding, steady, or shrinking if $\varepsilon > 0$, $\varepsilon = 0$, or $\varepsilon < 0$ respectively. A Ricci soliton is called a gradient Ricci soliton if $X = \nabla f$ for some $f \in C^\infty(M)$. ■

We note that the Type I singularities mentioned above are both modelled by shrinking solitons. Hamilton conjectured that if the Ricci flow develops a Type I singularity at some time $T < \infty$, then a suitable blow-up sequence of the flow converges to a non-trivial shrinking gradient soliton. This was confirmed for compact blow-up limits by Šešum [84] and for the general case in any dimension by Naber [67]. The latter result was refined by Enders, Müller, and Topping [28].

For the degenerate neckpinch, which is a Type II singularity, it has been speculated that this singularity can be modelled by the steady Ricci soliton constructed by Bryant [14]. Wu showed that his non-compact examples indeed converge uniformly to the Bryant soliton near the origin. In the paper by Garfinkle and Isenberg [35], they showed that their example of the degenerate neckpinch matches the profile of the Bryant soliton numerically. The ansatz solution approximated by Angenent, Isenberg, and Knopf in [4] and [5], by construction, does converge to the Bryant soliton at the tip region after rescaling. These provide more evidence that Type II singularities can indeed be modelled by the steady Ricci solitons.

However, it is still unclear whether the same holds true for any Type II singularity.

The singularity model is a very useful tool in studying the shape of Riemannian manifolds, particularly in the work for the Poincaré Conjecture and the Thurston Geometrisation Conjecture by Hamilton and Perelman. These singularity models would reflect the topological connected sum decomposition of the manifold. For example, the neckpinch, which is a singularity modelled by the shrinking cylinder, indicates that the manifold can be decomposed into a connected sum of two smaller components.

To further decompose the manifold, one applies a surgery to excise this neck near the singular time. Suitable spherical caps are put in their place in order to get rid of the singularity, thus allowing the Ricci flow to continue beyond this singular time for further decomposition. See [20] and [44] for more details on this.

Chapter 3

Milnor Frames

The key method employed in this work is the construction of a global frame for the tangent bundle TM on our manifold M and deriving explicit PDEs with respect to this frame. This construction can only be done on some special manifolds. The manifolds that we are going to consider are manifolds which have $SU(2)$ -symmetry. The two examples we are going to investigate are $S^1 \times S^3$ and S^4 . The latter is a special manifold which can be realised by a foliation of S^3 on an open interval with the ends collapsed, which give rise to two special orbits of the $SU(2)$ action at the poles.

The reason why we chose S^3 as the fibres of the foliation is because it is an example of a special family of manifolds called the Lie groups that, as we shall see in this chapter, has trivial tangent bundle; that is, it admits a global tangent bundle frame. This fact would simplify the computations considerably since it allows us to work on just one coordinate chart whose complement is at most a finite number of points. For a brief background, notations, and preliminaries on Lie groups and Lie algebras, see Appendix B.

3.1 Left-Invariant Metrics

As Lie groups have the structure of a smooth manifold, we can equip them with Riemannian metrics in order to define curvatures and lengths. First, we define the left-invariant metrics on the tangent bundle TG of G . These form a family of Riemannian metrics on the Lie group G such that they remain the same under the pullback of the left action $L_g : G \rightarrow G$ of any $g \in G$. Thus, intuitively, a Lie group equipped with a left-invariant metric geometrically looks similar locally everywhere. In order to avoid notational ambiguity between Riemannian metrics and elements of a Lie group, we denote the usual Riemannian metric $g(\cdot, \cdot)$ by $\langle\langle \cdot, \cdot \rangle\rangle$ instead.

Definition 3.1.1 (Left-invariant metric). A Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle \in \mathfrak{Met}$ (see Definition A.3.6) on a Lie group G is called left-invariant if for all $g, h \in G$ and $X, Y \in T_h G$ we have:

$$\langle\langle X, Y \rangle\rangle_h = \langle\langle (DL_g)_h X, (DL_g)_h Y \rangle\rangle_{gh}.$$

This is equivalent to saying that L_g is an isometry of $(G, \langle\langle \cdot, \cdot \rangle\rangle)$ onto itself. ■

Similarly, right-invariant metrics are defined by replacing L_g in the definition above with R_g , which is the right action of g on G .

Definition 3.1.2 (Homogeneous manifold). A Riemannian manifold $(M, \langle\langle \cdot, \cdot \rangle\rangle)$ is called homogeneous if its group of isometries act transitively. ■

Similar to the way we constructed left-invariant vector fields on G in Section B.1, we can also construct a left-invariant Riemannian metric. Indeed, suppose that we have an inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$ of G (an inner product on $T_e G$), then we can translate this inner product over the whole of G , thus inducing a Riemannian metric on G , by defining:

$$\langle\langle X, Y \rangle\rangle_g = \langle (DL_{g^{-1}})_g X, (DL_{g^{-1}})_g Y \rangle,$$

for $X, Y \in \Gamma(TG)$ and $g \in G$. Conversely, we can define an inner product on the Lie algebra $\mathfrak{g}$ of G from a left-invariant Riemannian metric by restricting it to the tangent space at $e \in G$. This proves the following proposition:

Proposition 3.1.3. There is a one-to-one correspondence between the inner products on the Lie algebra $\mathfrak{g}$ and the left-invariant metrics on G . ■

3.1.1 Haar Measure and Unimodular Lie Groups

Recall that a volume form on an m -dimensional manifold M is a nowhere vanishing section of the top form $\wedge^m(T^*M) = \Omega^m(M)$. These volume forms then give rise to measures on the manifold M . Of course, there are infinitely many of volume form and hence measures on a given manifold. One class of measures that we are interested in is the Haar measure.

Definition 3.1.4 (Left Haar measure). A measure μ on G is called left Haar measure if it is a Radon measure such that:

1. μ is left-invariant; that is, $\mu(U) = \mu(gU)$ for all $g \in G$ and U is a Borel subset of G ,
2. $\mu(U) < \infty$ for all compact Borel subset U of G . ■

Similar to the left Haar measure, we define the right Haar measure by requiring μ to be right-invariant. Up to multiplication with a positive constant, there is a unique non-trivial left (and right) Haar measure on G . Thus, we can find a unique normalised (that is, we set the measure of the whole group to be $\mu(G) = 1$) Haar measure on each compact Lie group.

We are now going to construct a left Haar measure on an n -dimensional Lie group G . The idea is to define a non-vanishing alternating multilinear n -form at $e \in G$ and left-translate this top form to the whole of G . This gives us a nowhere vanishing volume form which is left-invariant by construction. A particular way of doing this is to endow the Lie group G with a left-invariant Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$. Using this metric, we can induce a top form given by the volume form on G . This top form is given explicitly in local coordinates by:

$$d\mu = \sqrt{\det(\langle\langle \cdot, \cdot \rangle\rangle)} dx^1 \wedge dx^2 \wedge \dots \wedge dx^n,$$

on n -dimensional subsets of G . This measure inherits left-invariance from the isometry structure of the metric. More explicitly, for any $h \in G$ we have:

$$\int_G f(hg) d\mu = \int_G f(g) d\mu,$$

giving us a left Haar measure. This measure is unique up to constant multiplication: this can be seen by considering two left Haar measures $d\mu$ and $d\tilde{\mu}$ which, by the 1-dimensionality of the top form, are related via $d\tilde{\mu} = \phi(g)d\mu$ for some positive function $\phi : G \rightarrow (0, \infty)$. By left invariance, we have $\phi(hg) = \phi(g)$ for all $h \in G$, which implies that ϕ is a constant function.

Remark 3.1.5. This only works for measures arising from volume forms as we have used the one-dimensionality of the volume form in the argument.

For each $h \in G$, we now consider a left Haar measure $d\mu$ and construct a new measure $d\mu^h$ defined by:

$$\int_G f(g) d\mu^h := \int_G f(gh) d\mu.$$

Clearly, for a fixed $h \in G$, $d\mu^h$ is also a left Haar measure. By uniqueness of the left Haar measure, we have the relation:

$$d\mu^h = \Delta_G(h) d\mu \quad \Leftrightarrow \quad d\mu(g) = \Delta_G(h) d\mu(gh^{-1})$$

where $\Delta_G : G \rightarrow (0, \infty)$ is a group homomorphism between G and $(\mathbb{R}_+, \times)$. This homomorphism Δ_G is called the modular function of G .

If $\Delta_G \equiv 1$ is the trivial homomorphism, we call the group G unimodular. In other words, the measure $d\mu$ is also a right Haar measure since $d\mu(g) = d\mu(gh^{-1}) \Rightarrow d\mu(hk) = d\mu(k)$ for all $g, h, k \in G$.

Definition 3.1.6 (Unimodular Lie group). A Lie group G is called unimodular if it has a left invariant Haar measure $d\mu$ which is also a right Haar measure. ■

Theorem 3.1.7. [65] A Lie group G is unimodular if and only if $|\det(\text{Ad}(g))| = 1$ for all $g \in G$, where $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ is the adjoint representation on the Lie group G . ■

Proof. ($\Rightarrow$): Since G is unimodular, for any $h \in G$ the pullback of a left Haar measure $d\mu$ on G is given by:

$$d\mu(g) = d\mu(gh^{-1}) = d\mu(R_{h^{-1}}(g)) = |\det(DR_{h^{-1}})_g| d\mu(g),$$

which implies that $|\det(DR_h)_g| = 1$ for all $h, g \in G$. Similarly, since $d\mu$ is right-invariant, we can show that $|\det(DL_h)_g| = 1$. Thus, denoting C_g as the conjugation action of g on G , we deduce:

$$|\det(\text{Ad}(g))| = |\det(DC_g)_e| = |\det(DL_g)_e| |\det(DR_{g^{-1}})_e| = 1.$$

($\Leftarrow$): Suppose that $|\det(\text{Ad}(g))| = 1$. Let $d\mu$ be a left Haar measure on G . Then, for all $g, h \in G$ we have:

$$d\mu(hgh^{-1}) = |\det(\text{Ad}(g))| d\mu(g) = d\mu(g).$$

On the other hand, by left-invariance, we also have $d\mu(hgh^{-1}) = d\mu(gh^{-1})$. By putting these two facts together, we conclude that $d\mu(gh^{-1}) = d\mu(g)$ for all $h \in G$, and this implies the right-invariance of the measure $d\mu$. □

3.1.2 Compact Lie Groups

In this work, we are going to work with compact Lie groups, which have many more additional properties. Firstly, we define the notion of bi-invariant metrics. As the name suggests, bi-invariant metrics are both left- and right-invariant.

Definition 3.1.8 (Bi-invariant metrics). A Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle \in \mathfrak{Met}$ on a Lie group G is called bi-invariant if for all $g, h \in G$ and $X, Y \in T_h G$ we have:

$$\langle\langle X, Y \rangle\rangle_h = \langle\langle (DL_g)_h X, (DL_g)_h Y \rangle\rangle = \langle\langle (DR_g)_h X, (DR_g)_h Y \rangle\rangle.$$

This is equivalent to saying that L_g and R_g are isometries for $(G, \langle\langle \cdot, \cdot \rangle\rangle)$ onto itself. ■

For compact Lie groups, we can construct a bi-invariant metric by a useful averaging trick, commonly employed in the study of representation theory:

Lemma 3.1.9. [50] Every compact Lie group G admits a bi-invariant metric. ■

Proof. We have seen that we can construct a left-invariant metric $\langle\langle \cdot, \cdot \rangle\rangle$ on G . We now want to define a metric which is bi-invariant from this. Let $d\mu$ be a left Haar measure for the metric $\langle\langle \cdot, \cdot \rangle\rangle$ on G . Since G is compact, its volume $\text{Vol}_{d\mu}(G)$ is finite. For any $h \in G$ and $X, Y \in T_h G$, we define inner products $\langle \cdot, \cdot \rangle_h$ at $T_h G$ by:

$$\langle X, Y \rangle_h = \frac{1}{\text{Vol}_{d\mu}(G)} \int_G \langle\langle (DR_g)_h X, (DR_g)_h Y \rangle\rangle d\mu(g).$$

This fibrewise inner product extends to a metric on G . Clearly, the resulting metric is left-invariant as well. $\square$

For a bi-invariant metric, we have another useful property:

Lemma 3.1.10. Let $\langle\langle \cdot, \cdot \rangle\rangle$ be a bi-invariant metric on G with an induced inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Then for any $x, y, z \in \mathfrak{g}$ we have $\langle [x, y], z \rangle = \langle x, [y, z] \rangle$. $\blacksquare$

Proof. By the bi-invariance of $\langle\langle \cdot, \cdot \rangle\rangle$, for any $g \in G$ and $V, W \in \Gamma(TG)$ we have $\langle\langle DL_g V, W \rangle\rangle = \langle\langle V, DL_{g^{-1}} W \rangle\rangle$ and $\langle\langle DR_g V, W \rangle\rangle = \langle\langle V, DR_{g^{-1}} W \rangle\rangle$. Thus, by denoting $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ as the adjoint representation on the Lie algebra $\mathfrak{g}$, we derive:

$$\langle [x, y], z \rangle = -\langle [y, x], z \rangle = -\langle \text{ad}(y)x, z \rangle = -\frac{d}{dt} \Big|_{t=0} \langle\langle (\text{Ad} \circ \exp(ty))x, Z \rangle\rangle,$$

where $Z \in \Gamma(TG)$ is the left-invariant vector field generated by z and $\exp : \mathfrak{g} \rightarrow G$ is the exponential map. By definition of the Ad representation, we compute:

$$\begin{aligned} \langle [x, y], z \rangle &= -\frac{d}{dt} \Big|_{t=0} \langle\langle DC_{\exp(ty)}x, Z \rangle\rangle = -\frac{d}{dt} \Big|_{t=0} \langle\langle (DL_{\exp(ty)} \circ DR_{\exp(-ty)})x, Z \rangle\rangle \\ &= -\frac{d}{dt} \Big|_{t=0} \langle\langle X, (DR_{\exp(ty)} \circ DL_{\exp(-ty)})z \rangle\rangle \\ &= -\frac{d}{dt} \Big|_{t=0} \langle\langle X, DC_{\exp(-ty)}z \rangle\rangle = \langle x, \text{ad}(y)z \rangle = \langle x, [y, z] \rangle, \end{aligned}$$

where $X \in \Gamma(TG)$ is the left-invariant vector field generated by x . $\square$

Lemma 3.1.11. [83] Let G be a compact Lie group with a bi-invariant Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$. Any geodesic from any point $g \in G$ must be of the form $\gamma(t) = g \exp(tx)$ for some $x \in \mathfrak{g}$. $\blacksquare$

Proof. Suppose that $\gamma : [\alpha, \omega] \rightarrow G$ is a geodesic on G with $\alpha < 0 < \omega$ and $\gamma(0) = g$. Let $y : [\alpha, \omega] \rightarrow \mathfrak{g}$ be an arbitrary curve in $\mathfrak{g}$ such that $y(\alpha) = y(\omega) = 0$. Consider a perturbation of γ given by:

$$\begin{aligned} \bar{\gamma} : [\alpha, \omega] \times [\alpha, \omega] &\rightarrow G, \\ (t, s) &\mapsto \gamma(t) \exp(sy(t)). \end{aligned}$$

We denote $\nabla_t \bar{\gamma} = \partial_t \bar{\gamma} \in \Gamma(TG|_{\bar{\gamma}})$ and $\nabla_s \bar{\gamma} = \partial_s \bar{\gamma} \in \Gamma(TG|_{\bar{\gamma}})$. Since γ is a geodesic, the first variation of γ , obtained by taking the derivative of a length functional for $\bar{\gamma}$ in the s direction at $s = 0$, vanishes. Here, by bi-invariance of the metric, we introduce the $DL_{\bar{\gamma}^{-1}}$ term in the metric to map the tangent field on the curve γ to $e \in G$ for all t , hence we use $\langle \cdot, \cdot \rangle$ instead of $\langle\langle \cdot, \cdot \rangle\rangle$:

$$\begin{aligned} 0 &= \frac{1}{2} \frac{d}{ds} \Big|_{s=0} \int_{\alpha}^{\omega} \langle (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma}, (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma} \rangle dt \\ &= \int_{\alpha}^{\omega} \langle \nabla_s ((DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma}), (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma} \rangle \Big|_{s=0} dt. \end{aligned} \quad (3.1)$$

Note that $\nabla_s ((DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma}) = [(DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_s \bar{\gamma}, (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma}] + \nabla_t ((DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_s \bar{\gamma})$ from the definition of Lie brackets and Proposition A.1.17. Thus, the integrand in equation (3.1), by virtue of Lemma 3.1.10, is given by:

$$\begin{aligned} &\langle [(DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_s \bar{\gamma}, (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma}] + \nabla_t ((DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_s \bar{\gamma}), (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma} \rangle \\ &= \langle (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_s \bar{\gamma}, [(DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma}, (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma}] \rangle + \langle \nabla_t ((DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_s \bar{\gamma}), (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma} \rangle \\ &= \langle \nabla_t ((DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_s \bar{\gamma}), (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma} \rangle. \end{aligned} \quad (3.2)$$

Thus, by substituting equation (3.2) into (3.1) and using integration by parts, we have:

$$\begin{aligned} 0 &= \int_{\alpha}^{\omega} \langle \nabla_t ((DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_s \bar{\gamma}), (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma} \rangle \Big|_{s=0} dt \\ &= \langle (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_s \bar{\gamma}, (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma} \rangle \Big|_{t=\alpha, s=0}^{t=\omega, s=0} - \int_{\alpha}^{\omega} \langle (DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_s \bar{\gamma}, \nabla_t ((DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_t \bar{\gamma}) \rangle \Big|_{s=0} dt. \end{aligned}$$

But at $s = 0$ we have $(DL_{\bar{\gamma}^{-1}})_{\bar{\gamma}} \nabla_s \bar{\gamma} = y(t)$, which is zero for $t = \alpha, \omega$. So the first term vanishes. Thus, we have:

$$0 = \int_{\alpha}^{\omega} \langle y(t), \nabla_t ((DL_{\gamma^{-1}})_{\gamma} \nabla_t \gamma) \rangle dt.$$

Since $y(t)$ is an arbitrary perturbation in $\mathfrak{g}$, necessarily $\nabla_t ((DL_{\gamma^{-1}})_{\gamma} \nabla_t \gamma) \equiv 0 \in \mathfrak{g}$. Thus, we must have $(DL_{\gamma^{-1}})_{\gamma} \nabla_t \gamma = x \in \mathfrak{g}$, which implies that $\gamma(t) = g \exp(tx)$ by the condition $\gamma(0) = g$. $\square$

From the above, we have seen that the geodesics on a compact Lie group G are equivalent to the exponential mapping of a line in $\mathfrak{g}$. An alternative proof can be seen in [34].

This characterisation of the exponential mapping as geodesics allows us to use the classical Hopf-Rinow theorem [52], which states that for any two points $p, q \in M$ on a compact and connected Riemannian manifold M , there exists a length minimising geodesic connecting them. By letting $M = G$, $p = e$, and $q = g \in G$ be arbitrary, we have a geodesic connecting e and g .

Furthermore, by Theorem 3.1.11, this geodesic is of the form $\gamma(t) = \exp(tx)$ for some $x \in \mathfrak{g}$. Hence, we have the following corollary:

Corollary 3.1.12. Let G be a compact and connected Lie group with Lie algebra $\mathfrak{g}$. The exponential map $\exp : \mathfrak{g} \rightarrow G$ is surjective. ■

Remark 3.1.13. For a matrix Lie group, which is a closed Lie subgroup of $GL(n, \mathbb{R})$ or a Lie group that can be realised as a matrix group, there is a simpler way of showing the surjectivity of the exponential map by using the maximal torus theorem (see [41]). This theorem states that for a compact and connected matrix Lie group G , every element $g \in G$ is contained in some maximal torus.

A torus $\mathbb{T} \subset G$ is a Lie subgroup of G which is isomorphic to the direct product of finite copies of S^1 . A torus $\mathbb{T}$ is called maximal in G if it is not properly contained in any other torus in G . Clearly, the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$ is contained in the Lie algebra $\mathfrak{g}$ of G ; that is, $\mathfrak{t} \subset \mathfrak{g}$. Furthermore, note that the exponential map on a torus is surjective. Since the maximal torus theorem asserts that any element $g \in G$ is contained in some maximal torus $\mathbb{T} \subset G$, there exists some $x \in \mathfrak{t} \subset \mathfrak{g}$ such that $g = \exp(x)$, thus proving the surjectivity of the exponential map for compact and connected matrix Lie groups. ■

With this corollary, if we are working on compact, connected Lie groups, we can have a stronger version of Theorem 3.1.7, which is:

Theorem 3.1.14. [65] If G is a compact connected Lie group, then G is unimodular if and only if $\text{tr}(\text{ad}(x)) = 0$ for all $x \in \mathfrak{g}$. ■

Proof. Since G is compact and connected, by Corollary 3.1.12, the exponential map is surjective. Theorem 3.1.7 states that G is unimodular if and only if $|\det(\text{Ad}(g))| = 1$ for all $g \in G$. Fix an arbitrary $g \in G$. By surjectivity of the exponential map, there exists an $x \in \mathfrak{g}$ such that $g = \exp(x)$. By considering Proposition B.2.6 with $H = \text{End}(\mathfrak{g})$ and $\phi = \text{Ad}$ the adjoint representation on G , we deduce:

$$1 = |\det(\text{Ad}(g))| = |\det(\text{Ad}(\exp(x)))| = |\det(\exp(D\text{Ad}_e(x)))| = |\det(\exp(\text{ad}(x)))|.$$

Upon choosing a basis for $T_e G = \mathfrak{g}$, we can write $\text{ad}(x) \in \text{End}(\mathfrak{g})$ as a matrix $A \in \text{Mat}_{n \times n}(\mathbb{R})$. The exponential map of this matrix is:

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i.$$

By the Jacobi identity of matrix determinants, we have $\det(\exp(A)) = \exp(\text{tr}(A))$. Thus, we have $\exp(\text{tr}(\text{ad}(x))) = 1$, which is true if and only if $\text{tr}(\text{ad}(x)) = 0$. □

3.1.3 Curvatures of Homogeneous Manifolds

The curvature tensors of the Riemannian metric on a Lie group G can be expressed in terms of the adjoint maps of adjoint representations. Let us first digress from geometric analysis to the definitions of the adjoint of a linear map. Note that this is different from the adjoint maps we have seen earlier (it is an unfortunate coincidence in their naming similar to how the terms such as kernel, distribution, and torsion mean different things in different areas of mathematics).

Definition 3.1.15 (Adjoint linear map). Suppose that V is a vector space with an inner product $\langle \cdot, \cdot \rangle$ and $L : V \rightarrow V$ is a vector space endomorphism. The adjoint of the map L is defined as $L^* : V \rightarrow V$ such that for all $v, w \in V$ we have:

$$\langle L^*v, w \rangle = \langle v, Lw \rangle.$$

The map L is called self-adjoint if $L = L^*$. ■

Thus, for each $x \in \mathfrak{g}$ the vector space endomorphism $\text{ad}(x) \in \text{End}(\mathfrak{g})$ has the adjoint $(\text{ad}(x))^* \in \text{End}(\mathfrak{g})$. This can be extended over the left-invariant vector field by defining the metric $\langle \cdot, \cdot \rangle$ at any $g \in G$ via translations of the inner product $\langle \cdot, \cdot \rangle$ for $\mathfrak{g}$.

We recall the Levi-Civita connection from Section 2.1 and try to write them in a succinct way as discussed in [21]. Suppose that $X, Y, Z \in \Gamma(TG)$ are vector fields on G such that $\langle X, Y \rangle, \langle X, Z \rangle$, and $\langle Y, Z \rangle$ are constants. Then, by using Proposition A.3.15, we have:

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} (\langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle - \langle X, [Y, Z] \rangle). \quad (3.3)$$

Let X, Y , and Z be left-invariant vector fields obtained from $x, y, z \in \mathfrak{g}$ respectively. Since the induced Riemannian metric is an isometry, equation (3.3) can be simplified to:

$$\nabla_X Y = \frac{1}{2} ([x, y] - \text{ad}(x)^*y - \text{ad}(y)^*x),$$

By recalling its definition from Section 2.1, the sectional curvatures are defined by:

$$K(X, Y) = \frac{\langle \text{Rm}(X, Y)Y, X \rangle}{\|X\|^2\|Y\|^2 - \langle X, Y \rangle^2}.$$

So, if X and Y are left-invariant orthonormal vector fields, we have:

$$\begin{aligned} K(X, Y) = \langle \text{Rm}(X, Y)Y, X \rangle &= \frac{1}{4} |\text{ad}(x)^*y + \text{ad}(y)^*x|^2 - \langle \text{ad}(x)^*x, \text{ad}(y)^*y \rangle \\ &\quad - \frac{3}{4} |[x, y]|^2 - \frac{1}{2} \langle x, [[x, y], y] \rangle - \frac{1}{2} \langle y, [[y, x], x] \rangle. \end{aligned} \quad (3.4)$$

Note that these just depend on the elements $x, y \in \mathfrak{g}$ and the inner product on $\mathfrak{g}$; that is, the metric of G at e . This is why these manifolds are called homogeneous in Definition 3.1.2: the curvatures look the same locally everywhere.

3.2 Milnor Frames in Dimension 3

[65] Suppose that we have a Lie group G of dimension 3. We assume that G is connected and unimodular (see Definition A.1.14). Let $\mathfrak{g}$ be its Lie algebra with inner product $\langle \cdot, \cdot \rangle$. Since G is parallelisable, there exists a left-invariant moving frame $\{F_1, F_2, F_3\}$ for TG which is orthonormal with respect to the left-invariant metric $\langle \cdot, \cdot \rangle$ on G induced by $\langle \cdot, \cdot \rangle$. This moving frame can be identified with the Lie algebra elements $\{f_1, f_2, f_3\}$. [21] We define a vector space homomorphism:

$$\begin{aligned}\phi : \mathfrak{g} &\rightarrow \wedge^2 \mathfrak{g}, \\ f_i &\mapsto f_{i+1(\text{mod } 3)} \wedge f_{i+2(\text{mod } 3)}.\end{aligned}$$

By composing this map with the Lie bracket, we have a Lie algebra endomorphism φ :

$$\begin{aligned}\varphi &= [\cdot, \cdot] \circ \phi : \mathfrak{g} \rightarrow \mathfrak{g}, \\ f_i &\mapsto [f_{i+1(\text{mod } 3)}, f_{i+2(\text{mod } 3)}].\end{aligned}$$

Note that, by using the structure constants of $\mathfrak{g}$ defined in Definition B.2.3, we have $[f_i, f_j] = c_{ij}^k f_k$. So, with respect to the basis $B_1 = \{f_1, f_2, f_3\}$ of $\mathfrak{g}$, we can write this endomorphism as a matrix φ_{B_1} given as:

$$\varphi_{B_1} = \begin{pmatrix} c_{23}^1 & c_{31}^1 & c_{12}^1 \\ c_{23}^2 & c_{31}^2 & c_{12}^2 \\ c_{23}^3 & c_{31}^3 & c_{12}^3 \end{pmatrix},$$

where $c_{ij}^k \in \mathbb{R}$.

From Remark B.3.5, the adjoint maps can be represented as a linear combination of the adjoint maps on the basis of $\mathfrak{g}$. In a fixed basis, these are just 3×3 matrices with entries in $\mathbb{R}$. Here, with respect to the basis $\{f_1, f_2, f_3\}$, the matrices are given by:

$$\text{ad}(f_1) = \begin{pmatrix} 0 & c_{12}^1 & c_{13}^1 \\ 0 & c_{12}^2 & c_{13}^2 \\ 0 & c_{12}^3 & c_{13}^3 \end{pmatrix}, \quad \text{ad}(f_2) = \begin{pmatrix} c_{21}^1 & 0 & c_{23}^1 \\ c_{21}^2 & 0 & c_{23}^2 \\ c_{21}^3 & 0 & c_{23}^3 \end{pmatrix}, \quad \text{ad}(f_3) = \begin{pmatrix} c_{31}^1 & c_{32}^1 & 0 \\ c_{31}^2 & c_{32}^2 & 0 \\ c_{31}^3 & c_{32}^3 & 0 \end{pmatrix}.$$

By connectedness and unimodularity of G , from Theorem 3.1.14, we have that the trace of each of these matrices is 0; that is:

$$c_{12}^2 + c_{13}^3 = 0, \quad c_{12}^1 + c_{23}^3 = 0, \quad c_{31}^1 + c_{32}^2 = 0.$$

Together with the anti-symmetry of the structure constants of a Lie algebra mentioned in Remark B.2.4, this implies that φ_{B_1} is a symmetric matrix. Hence, the map φ is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$.

Thus, we can choose a new basis $B_2 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathfrak{g}$ by an orthogonal transformation of B_1 such that φ_{B_2} is the diagonal matrix. Without loss of generality, by choosing a suitable sign for $\mathbf{e}_1$, we assume that the basis B_1 is positively oriented and the diagonal matrix is given by $\varphi_{B_2} = \text{Diag}(\alpha, \beta, \gamma)$ such that $\alpha \leq \beta \leq \gamma$. More precisely, in the induced non-vanishing left-invariant vector field (and hence tangent bundle frame), we have the identities:

$$[E_2, E_3] = \alpha E_1, \quad [E_3, E_1] = \beta E_2, \quad [E_1, E_2] = \gamma E_3. \quad (3.5)$$

In fact, by suitable scalings of the frame, we can reduce the diagonal elements α, β , and γ to be in the set $\{-1, 0, +1\}$. However, note that the rescaled frame is no longer normal, merely orthogonal.

The orthogonal moving frame $\{E_1, E_2, E_3\}$ generated by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called the Milnor frame. In this frame, the left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on the unimodular Lie group G can be written with respect to the set of 1-forms $\{\omega^1, \omega^2, \omega^3\}$ dual to the vector fields $\{E_1, E_2, E_3\}$ in a nice form:

$$\langle \cdot, \cdot \rangle = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3,$$

where A, B , and C are some positive constants.

3.2.1 Classification of 3-Dimensional Homogeneous Manifolds

By applying the previous procedures, one is able to classify all of the 3-dimensional unimodular Lie groups by using the diagonalisation of the Lie algebra endomorphism ϕ . In fact, this classification was the motivating factor for the construction of Milnor frames.

In 1974, John Milnor showed that any simply connected 3-dimensional unimodular Lie group is isomorphic to either $\text{SU}(2)$, $\text{E}(2)$, $\text{SL}(2, \mathbb{R})$, Nil , Sol , or $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ [65]. This is done by noticing that the elements α, β , and γ in the matrix φ_{B_2} , which were defined in (3.5), can be scaled with positive integers while keeping the underlying Lie algebra and Lie group unchanged.

Therefore, only the signs of these quantities matter. Hence, there are only six possible types of configurations, which can be summarised in Table 3.1 according to the signatures of the quantities $\alpha \leq \beta \leq \gamma$:

Signatures	Lie Group	Description	Bianchi Class
$(-, -, -)$ $(+, +, +)$	$SU(2)$	Special unitary group.	IX
$(-, -, 0)$ $(0, +, +)$	$E(2)$	The Euclidean group, which is the group of isometries of $\mathbb{R}^3$.	VIIIb
$(-, -, +)$ $(-, +, +)$	$SL(2, \mathbb{R})$	Orientation preserving isometries of the Poincaré half-space.	VIII
$(-, 0, 0)$ $(0, 0, +)$	Heis or Nil	The Heisenberg group, which is the group of real matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$.	II
$(-, 0, +)$	$E(1,1)$ or Sol	Isometries of the Minkowski 2-space.	VIb
$(0, 0, 0)$	$\mathbb{R}^3$	Euclidean space with addition.	I

Table 3.1: List of 3-dimensional homogeneous manifolds.

In fact, this mirrors the much earlier work (1898) done by Bianchi in classifying the 3-dimensional Lie algebras into 11 classes in [10] (see [11] for the English translation). Hence, sometimes the Lie groups above are referred to their Bianchi classification, ranging from I to IX (with two types of VI and VII). Note that not all of the Bianchi classes appear in Table 3.1 since the rest of them are not unimodular.

3.3 Ricci Flow in Milnor Frames

Another useful application of the Milnor frames is that they enable us to study the Ricci flow as an ODE instead of a PDE. The Ricci flow of a metric in this form has been studied by Isenberg and Jackson [53] in their paper which examines the behaviour of the homogeneous geometries listed in Table 3.1 and three other geometries: $H(3), S^2 \times \mathbb{R}$, and $H(2) \times \mathbb{R}$, where $H(n)$ is the group of isometries of the n -dimensional hyperbolic space.

These nine geometries are the same as the eight Thurston geometries (with the geometries of $\mathbb{R}^3$ and $E(2)$ both represented by the geometry of $E(3)$ in the list by Thurston since he only considered metrics with maximal symmetry). In their work, Isenberg and Jackson showed that the metrics on these manifolds usually tend to approach (but not necessarily converge) to the metric of maximal symmetry in each class.

We note here that for different metrics $\langle\langle \cdot, \cdot \rangle\rangle$ in any one of the nine types of homogeneous manifolds above, the diagonalising vector field $\{E_1, E_2, E_3\}$ may be different. However, via

calculations which we are going to demonstrate below, Isenberg and Jackson showed that the choice of the basis is preserved under the Ricci flow.

Recall that for any $X, Y \in \Gamma(TM)$ the Ricci curvature tensor on a manifold M with Riemannian metric $\langle \cdot, \cdot \rangle$ is given by:

$$\text{Ric}(X, Y)_p = \sum_{i=1}^n \langle \text{Rm}(X, E_i)E_i, Y \rangle_p,$$

for any $\{E_1, \dots, E_n\}$ orthonormal basis of $T_p M$.

Consider a unimodular, connected 3-dimensional Lie group equipped with a global frame for TG . From this global frame, we can find a corresponding orthogonal (but not orthonormal) Milnor frame for TG , which we denote as $\{E_1, E_2, E_3\}$, such that $[E_2, E_3] = \alpha E_1$, $[E_3, E_1] = \beta E_2$, and $[E_1, E_2] = \gamma E_3$ for some constants $\alpha, \beta, \gamma \in \mathbb{R}$.

The sectional curvatures determine all the components in the Ricci curvature tensor, so we have, for example, $\text{Ric}(E_1, E_1) = \|E_1\|^2(K(E_1, E_2) + K(E_1, E_3))$, and the sectional curvature $K(E_1, E_2)$ from (3.4) is given by the formula:

$$\begin{aligned} K(E_1, E_2) = \frac{1}{|e_1|^2 |e_2|^2} & \left(\frac{1}{4} |\text{ad}(e_1)^* e_2 + \text{ad}(e_2)^* e_1|^2 - \langle \text{ad}(e_1)^* e_1, \text{ad}(e_2)^* e_2 \rangle \right. \\ & \left. - \frac{3}{4} |[e_1, e_2]|^2 - \frac{1}{2} \langle e_1, [[e_1, e_2], e_2] \rangle - \frac{1}{2} \langle e_2, [[e_2, e_1], e_1] \rangle \right). \end{aligned}$$

To calculate the adjoint of the adjoint map, we use:

$$\text{ad}(e_i)^* e_j = \sum_{k=1}^3 \langle \text{ad}(e_i)^* e_j, \hat{e}_k \rangle \hat{e}_k = \sum_{k=1}^3 \langle e_j, [e_i, \hat{e}_k] \rangle \hat{e}_k,$$

where $\hat{\cdot}$ denotes the normalised vector. Thus, we have:

$$\begin{aligned} \text{ad}(e_1)^* e_2 &= -\frac{\beta B}{C} e_3, \\ \text{ad}(e_2)^* e_1 &= \frac{\alpha A}{C} e_3, \\ \text{ad}(e_1)^* e_1 &= (\text{ad}(e_2))^* e_2 = 0. \end{aligned}$$

Repeating over all pairs, the sectional curvatures are given by:

$$K_{12} = K(E_1, E_2) = \frac{(\alpha A - \beta B)^2}{4ABC} - \frac{3\gamma^2 C}{4AB} + \frac{\gamma}{2} \left(\frac{\alpha}{B} + \frac{\beta}{A} \right), \quad (3.6)$$

$$K_{13} = K(E_1, E_3) = \frac{(\gamma C - \alpha A)^2}{4ABC} - \frac{3\gamma^2 B}{4AC} + \frac{\beta}{2} \left(\frac{\alpha}{C} + \frac{\gamma}{A} \right), \quad (3.7)$$

$$K_{23} = K(E_2, E_3) = \frac{(\beta B - \gamma C)^2}{4ABC} - \frac{3\gamma^2 A}{4BC} + \frac{\alpha}{2} \left(\frac{\beta}{C} + \frac{\gamma}{B} \right). \quad (3.8)$$

Thus, the Ricci curvature tensor, written with respect to the coframe $\{\omega^1, \omega^2, \omega^3\}$, is given by a diagonal matrix with the diagonal elements:

$$\begin{aligned} \text{Ric}(E_1, E_1) &= A \left(\frac{(\alpha A - \beta B)^2}{4ABC} - \frac{3\gamma^2 C}{4AB} + \frac{\gamma}{2} \left(\frac{\alpha}{B} + \frac{\beta}{A} \right) + \frac{(\gamma C - \alpha A)^2}{4ABC} - \frac{3\gamma^2 B}{4AC} + \frac{\beta}{2} \left(\frac{\alpha}{C} + \frac{\gamma}{A} \right) \right), \\ \text{Ric}(E_2, E_2) &= B \left(\frac{(\alpha A - \beta B)^2}{4ABC} - \frac{3\gamma^2 C}{4AB} + \frac{\gamma}{2} \left(\frac{\alpha}{B} + \frac{\beta}{A} \right) + \frac{(\beta B - \gamma C)^2}{4ABC} - \frac{3\gamma^2 A}{4BC} + \frac{\alpha}{2} \left(\frac{\beta}{C} + \frac{\gamma}{B} \right) \right), \\ \text{Ric}(E_3, E_3) &= C \left(\frac{(\gamma C - \alpha A)^2}{4ABC} - \frac{3\gamma^2 B}{4AC} + \frac{\beta}{2} \left(\frac{\alpha}{C} + \frac{\gamma}{A} \right) + \frac{(\beta B - \gamma C)^2}{4ABC} - \frac{3\gamma^2 A}{4BC} + \frac{\alpha}{2} \left(\frac{\beta}{C} + \frac{\gamma}{B} \right) \right). \end{aligned}$$

Finally, the Ricci flow equation $\partial_t g = -2\text{Ric}(g(t))$ can be written the following system:

$$\begin{cases} \frac{dA}{dt} = -2A \left(\frac{(\alpha A - \beta B)^2}{4ABC} - \frac{3\gamma^2 C}{4AB} + \frac{\gamma}{2} \left(\frac{\alpha}{B} + \frac{\beta}{A} \right) + \frac{(\gamma C - \alpha A)^2}{4ABC} - \frac{3\gamma^2 B}{4AC} + \frac{\beta}{2} \left(\frac{\alpha}{C} + \frac{\gamma}{A} \right) \right), & (3.9) \\ \frac{dB}{dt} = -2B \left(\frac{(\alpha A - \beta B)^2}{4ABC} - \frac{3\gamma^2 C}{4AB} + \frac{\gamma}{2} \left(\frac{\alpha}{B} + \frac{\beta}{A} \right) + \frac{(\beta B - \gamma C)^2}{4ABC} - \frac{3\gamma^2 A}{4BC} + \frac{\alpha}{2} \left(\frac{\beta}{C} + \frac{\gamma}{B} \right) \right), & (3.10) \\ \frac{dC}{dt} = -2C \left(\frac{(\gamma C - \alpha A)^2}{4ABC} - \frac{3\gamma^2 B}{4AC} + \frac{\beta}{2} \left(\frac{\alpha}{C} + \frac{\gamma}{A} \right) + \frac{(\beta B - \gamma C)^2}{4ABC} - \frac{3\gamma^2 A}{4BC} + \frac{\alpha}{2} \left(\frac{\beta}{C} + \frac{\gamma}{B} \right) \right). & (3.11) \end{cases}$$

Consequently, since the mixed terms do not appear in the equations above, the symmetry (and hence form) of the metric is preserved for as long as the flow exists.

In this work, we are going to study the singularity of the Ricci flow using Milnor frames. The basis of the study is to use one of the model geometries, namely $SU(2)$ or S^3 described above, and put it in a suitable setting.

We note that in the paper by Isenberg and Jackson, they studied the normalised Ricci flow for all the six homogeneous 3-manifolds listed in Table 3.1 plus three more remaining geometry from the list by Thurston. They obtained, among others, the following result which is relevant to our study:

Theorem 3.3.1. [53] Suppose that (S^3, g_0) is a Riemannian manifold where g_0 is of the form $g_0 = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$. Then the normalised Ricci flow, given by the equation:

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t)) + \frac{2}{3} \frac{\int_{S^3} S d\mu}{\text{Vol}_g(S^3)} g, \\ g(0) = g_0, \end{cases}$$

on S^3 starting with the initial metric g_0 converges to the standard round sphere metric g_{S^3} of some radius $R > 0$ exponentially in t . ■

This was done by studying the system of ODEs (3.9)-(3.11) by setting $\alpha = \beta = \gamma = -2$ and imposing the product ABC to be a constant (say, $ABC = 1$) for all $t \geq 0$ to keep the volume

constant, hence normalising the flow. This results in the following set of ODEs:

$$\begin{cases} \frac{dA}{dt} = \frac{2}{3}(-A^2(2A - B - C) + A(B - C)^2), \\ \frac{dB}{dt} = \frac{2}{3}(-B^2(2B - A - C) + B(A - C)^2), \\ \frac{dC}{dt} = \frac{2}{3}(-C^2(2C - A - B) + C(A - B)^2). \end{cases}$$

Under the assumption that the initial data A_0, B_0 , and C_0 satisfies $A_0 \leq B_0 \leq C_0$, by studying the equations for $\frac{d}{dt}(B - A)$ and $\frac{d}{dt}(C - B)$, this ordering remains preserved under the normalised flow and the difference $C - A$ decays exponentially as:

$$C - A \leq (C_0 - A_0)e^{-2C_0^2 t}.$$

By assuming $ABC = 1$, this simplifies the required analysis considerably. In this work, we are dealing with the unnormalised flow and we wish to find a similar bound for the metric quantities. The unnormalised Ricci flow on a Riemannian manifold with the $SU(2)$ geometry is given by:

$$\frac{dA}{dt} = -8 + 4 \frac{B^2 + C^2 - A^2}{BC}, \quad (3.12)$$

$$\frac{dB}{dt} = -8 + 4 \frac{A^2 + C^2 - B^2}{AC}, \quad (3.13)$$

$$\frac{dC}{dt} = -8 + 4 \frac{A^2 + B^2 - C^2}{AB}. \quad (3.14)$$

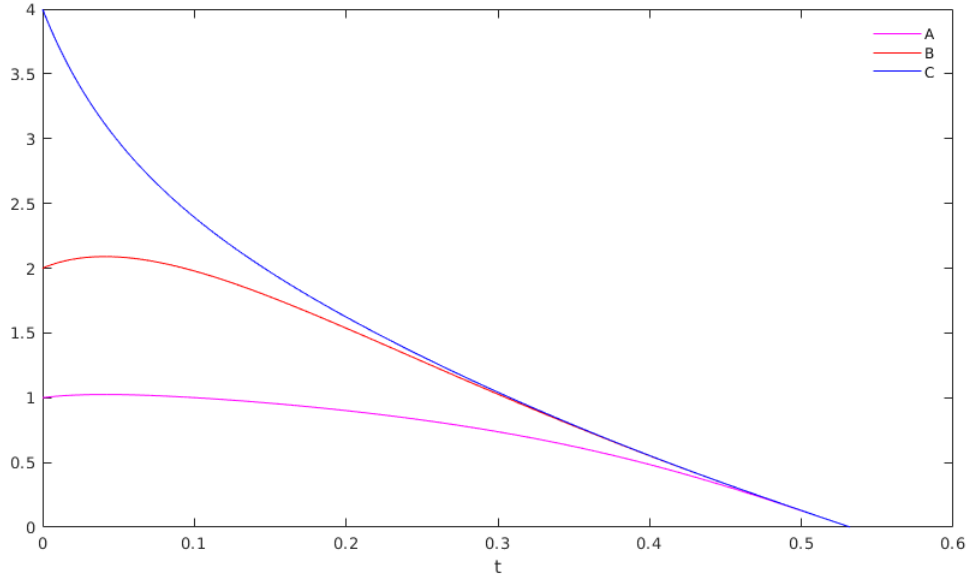
By using the finite difference scheme, we numerically plot the unnormalised Ricci flow evolution of the quantities A, B , and C for a choice of initial data A_0, B_0 , and C_0 satisfying the ordering $A_0 \leq B_0 \leq C_0$ in Figure 3.1.

We note from the plot that the ratio $\frac{C}{A}$ tends to 1 as t tends to the singular time where all three of the metric quantities vanish. This implies that upon parabolic rescaling, the metric resembles the round sphere. Thus, the singularity is expected to be of Type I. In fact, we shall prove the following theorem:

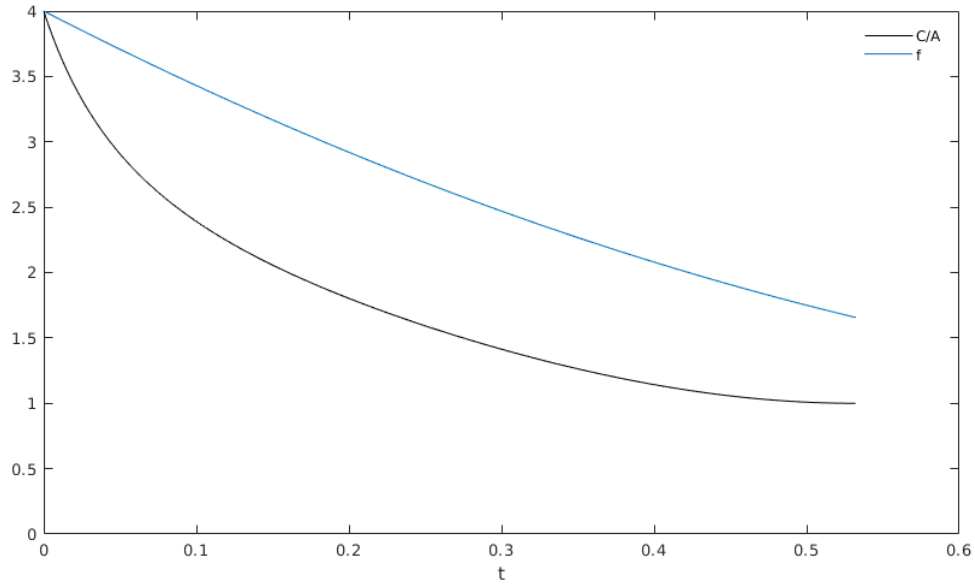
Theorem 3.3.2. Suppose that (S^3, g) is a Riemannian manifold such that g is of the form:

$$g = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3,$$

where A, B , and C are positive constants. Then under the Ricci flow, the metric develops a Type I singularity at a finite time. ■



(a) Ricci flow evolution of quantities A (magenta), B (red), and C (blue) for initial data $A_0 = 1$, $B_0 = 2$, and $C_0 = 4$.



(b) Evolution for the ratio $\frac{C}{A}$ from this initial data. The blue curve represents the function $f(t) = 3(1-t)^2 + 1$, which is the theoretical bound for the ratio $\frac{C}{A}$ obtained from Proposition 3.3.3.

Figure 3.1: Ricci flow simulation for S^3 .

Indeed, a more general fact has been proven by Hamilton for 3-dimensional Riemannian manifolds with positive Ricci curvature [42]. By invoking this celebrated result, Theorem 3.3.2 is immediate. Furthermore, finite time existence for the solution to the system (3.12)-(3.14) was discussed in [59]. However, in this work we shall present a different proof for Theorem 3.3.2.

Analytically, we can first prove the following decay estimate:

Proposition 3.3.3. Under the Ricci flow, supposing that $A_0 \leq B_0 \leq C_0$ at initial time, the ordering remains preserved and we have the following estimate:

$$\frac{C}{A} \leq \left(\frac{C_0}{A_0} - 1 \right) \left(1 - \frac{4t}{C_0} \right)^2 + 1. \quad \blacksquare$$

Proof. According to equations (3.13) and (3.14), the quantity $B - C$ evolves according to:

$$\frac{d}{dt}(B - C) = -4 \frac{(B - C)(C - A + B)(A + B + C)}{ABC}.$$

By applying Grönwall's inequality to this equation, we prove that the inequality $B \leq C$ remains true as long as the flow exists. By repeating this procedure, we can deduce that the ordering $A \leq B$ also persists.

Next, the ordering implies the following bound:

$$\frac{dC}{dt} = -4 \left(\frac{C^2 - (A - B)^2}{AB} \right) = -4 \frac{(C - A + B)(C - B + A)}{AB} \leq -4.$$

This implies that C is strictly decreasing for all time and $C \leq C_0 - 4t$. Furthermore, the quantity $\frac{C-A}{A}$ evolves according to:

$$\begin{aligned} \frac{d}{dt} \left(\frac{C - A}{A} \right) &= -8 \left(\frac{C - A}{A} \right) \frac{A - B + C}{AB} \leq -\frac{8}{B} \left(\frac{C - A}{A} \right) \\ &\leq -\frac{8}{C} \left(\frac{C - A}{A} \right) \leq -\frac{8}{C_0 - 4t} \left(\frac{C - A}{A} \right), \end{aligned}$$

which implies that the ratio $\frac{C}{A}$ is non-increasing. Integrating this inequality and rearranging the terms yield the desired result. $\square$

A direct corollary of this is that the ratio $\frac{C}{A}$ is bounded by its initial value. Thus, for all time, we have the ordering $A \leq B \leq C \leq \lambda A$ where $\lambda = \frac{C_0}{A_0} \geq 1$. This also implies that the singularity time T is finite since after at most time $t = \frac{C_0}{4}$, all the metric quantities A, B , and C become equal; that is, the metric becomes a round sphere metric, which can only exist for a finite time.

Another new result which we can glean from this analysis is the concavity and convexity of the smallest and largest metric components respectively.

Proposition 3.3.4. Under the Ricci flow, supposing that $A_0 \leq B_0 \leq C_0$ at initial time, we have that A is a concave function and C is a convex function of t . ■

Proof. By simple computations, from equations (3.12)-(3.14), the second derivative of the quantity A satisfies the following equation:

$$\begin{aligned} \frac{d^2 A}{dt^2} &= \frac{4}{B^2 C^2} \left(2BC \left(B \frac{dB}{dt} + C \frac{dC}{dt} - A \frac{dA}{dt} \right) - (B^2 + C^2 - A^2) \left(C \frac{dB}{dt} + B \frac{dC}{dt} \right) \right) \\ &= \frac{32}{AB^2 C^2} (2A^4 - A^2(B-C)^2 - A^3(B+C) + A(B+C)(B-C)^2 - (B-C)^2(B+C)^2). \end{aligned}$$

However, we note that $2A^4 - A^3(B+C) \leq 0$ and $A(B+C)(B-C)^2 - (B-C)^2(B+C)^2 \leq 0$ by the ordering $A \leq B \leq C$ from the previous theorem. Thus, $\frac{d^2 A}{dt^2} \leq 0$ for all $t \in [0, T)$.

On the other hand, we have:

$$\begin{aligned} \frac{d^2 C}{dt^2} &= \frac{32}{A^2 B^2 C} (2C^4 - C^2(B-A)^2 - C^3(B+A) + C(B+A)(B-A)^2 - (B-A)^2(B+A)^2) \\ &= \frac{32C^3}{A^2 B^2} \left(2 - \left(\frac{B-A}{C} \right)^2 - \left(\frac{B+A}{C} \right) + \left(\frac{B+A}{C} \right) \left(\frac{B-A}{C} \right)^2 - \left(\frac{B-A}{C} \right)^2 \left(\frac{B+A}{C} \right)^2 \right). \end{aligned}$$

Thus, we study the function $P(x, y) = 2 - (x-y)^2 - (x+y) + (x+y)(x-y)^2 - (x-y)^2(x+y)^2$ for $x, y \geq 0$. Note that since we have $A \leq B \leq C \leq \lambda A$ for some $\lambda \geq 1$ and we are interested in the sign of $P\left(\frac{A}{C}, \frac{B}{C}\right)$, we may restrict our attention to the domain $\Omega = \{(x, y) : 0 < x \leq y \leq 1\}$, which is depicted in Figure 3.2.

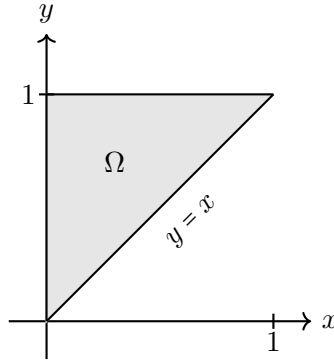


Figure 3.2: Domain Ω of the function $P(x, y)$.

Our aim now is to minimise the function P over Ω . By inspecting $\text{grad}(P) = (-1 - 2x + 3x^2 - 4x^3 + 2y - 2xy - y^2 + 4xy^2, -1 + 2x - x^2 - 2y - 2xy + 4x^2y + 3y^2 - 4y^3)$, we conclude that there are no global critical points of P over $\mathbb{R}^2$. Thus, necessarily the minimum of $P(x, y)$ on Ω lies on the boundary $\partial\Omega$.

Along the boundary of Ω , we have:

$$\begin{aligned} P(x, x) &= 2 - 2x, \\ P(x, 1) &= x^3 - x^4, \\ P(0, y) &= -y^4 + y^3 - y^2 - y + 2. \end{aligned}$$

Note that $P(x, x) \geq 0$ and $P(x, 1) \geq 0$ for $0 < x \leq 1$. Furthermore, $P(0, y) = -y^4 + y^3 - y^2 - y + 2 = (y^3 - y^4) + (1 - y^2) + (1 - y) \geq 0$ for any $y \in [0, 1]$. Thus, the minimum of $P(x, y)$ on Ω is 0, attained at the point $(1, 1)$. This implies that $\frac{d^2 C}{dt^2} \geq 0$ for all $t \in [0, T)$. $\square$

We now prove Theorem 3.3.2:

Proof of Theorem 3.3.2. Finite time existence of the metric under the Ricci flow is a corollary of Proposition 3.3.3. Indeed, Proposition 3.3.3 implies that either the solution becomes singular before some finite time or the solution becomes a round metric at some finite time. In the latter, we know that the solution will eventually shrink to a round point from the discussion in Section 2.3.1. We now wish to prove that the metric develops a Type I singularity using Proposition 3.3.4.

If we complete the function A , which is defined on $[0, T)$, to the interval $[0, T]$ by defining $A(T) = \lim_{t \rightarrow T} A(t) = 0$, we get a continuous concave function whose graph lies completely under the tangent line of the curve $A(t)$ at $t = T$. We note that this tangent line has finite gradient because:

$$-12 \leq -8 - 4 \frac{A^2}{BC} \leq \frac{dA}{dt} \leq -8 + 4 \frac{B^2 + C^2}{BC} \leq -4 + \lambda.$$

By concavity, the curve $A(t)$ lies completely above the secant line joining the points $(0, A(0))$ and $(T, A(T))$. This secant line has a strictly negative gradient since $A(0) > 0$. Thus, we conclude that $\mu(T - t) \leq A(t) \leq \nu(T - t)$ for some constants $\mu, \nu > 0$ depending only on λ .

Using this estimate and the fact that $A \leq B \leq C \leq \lambda A$ for all $t \in [0, T)$, the sectional curvature $|K_{12}|$ can be bounded by:

$$\begin{aligned} |K_{12}| &\leq \frac{A}{4BC} + \frac{B}{4AC} + \frac{3C}{4AB} + \frac{1}{2A} + \frac{1}{2B} + \frac{1}{2C} \\ &\leq \frac{1}{4C} + \frac{1}{4A} + \frac{3\lambda}{4B} + \frac{1}{2A} + \frac{1}{2B} + \frac{1}{2C} \leq \frac{8 + 3\lambda}{4A} \leq \frac{Q(\lambda)}{(T - t)}, \end{aligned}$$

for some finite constant $Q(\lambda) > 0$. A similar bound can be shown for $|K_{13}|$ and $|K_{23}|$. Thus, we deduce:

$$|\text{Rm}|_g^2 = 2(K_{12}^2 + K_{13}^2 + K_{23}^2) \quad \Rightarrow \quad |\text{Rm}|_g^2 \leq \frac{\bar{Q}(\lambda)^2}{(T - t)^2},$$

for some finite constant $\bar{Q}(\lambda) > 0$. By rearranging and taking the supremum over $S^3 \times [0, T)$, we obtain:

$$\sup_{S^3 \times [0, T)} (T - t) |\text{Rm}|_g \leq \bar{Q}(\lambda) < \infty,$$

which gives us a different proof for showing that the singularity is indeed of Type I. $\square$

Remark 3.3.5. In fact, studies for the Ricci flows on the other model geometries were done by Knopf and McLeod [59] as well as Onda in his unpublished work [70]. It would be interesting to apply the analysis done in the proof of Theorem 3.3.2 in these studies.

We expect that this behaviour also hold true for the Ricci flow on foliations of S^3 , which we now turn our attention to.

Chapter 4

Ricci Flow on Foliations of S^3

In the previous chapter, we have seen all the possible types of homogeneous 3-manifolds. From the list, only $SU(2)$, which can be identified with a 3-sphere, is compact. However, the other homogeneous 3-manifolds, whilst non-compact, may admit a compactification. For example $\mathbb{R}^3$ can be compactified to $\mathbb{T}^3$.

The 3-sphere is special since there exist non-vanishing vector fields on S^3 . Furthermore, Adams [1] showed that any n -sphere with $n = (2a + 1)2^b - 1$ and $b = c + 4d$ where $a, b, c, d \in \mathbb{Z}_{\geq 0}$ and $0 \leq c \leq 3$ admits $2^c + 8d - 1$ pointwise linearly independent vector fields. For $n = 3$, the 3-sphere admits a set of 3 vector fields $X_1, X_2, X_3 \in \Gamma(TM)$ such that for each $p \in S^3$ the vectors $X_1(p), X_2(p), X_3(p) \in T_p S^3$ are linearly independent. This implies that the 3-sphere is parallelisable.

Based on the work by Angenent and Knopf, we want to consider the product manifold $M \times N$. Let (M, g_M) and (N, g_N) be two Riemannian manifolds. On a Riemannian product manifold $M \times N$ there exists an induced metric given by $g = \pi^* g_M + \sigma^* g_N$ where π and σ are projections of $M \times N$ onto M and N respectively. A new class of metrics on $M \times N$ can be obtained by defining the metric on the product with an added scale factor to one of the constituent metrics. These metrics are called the warped product metrics.

Definition 4.0.1 (Warped products). [71] Let (B, g_B) and (F, g_F) be two Riemannian manifolds and let $\varphi : B \rightarrow \mathbb{R}$ be a smooth positive function. The warped product $B \times_{\varphi} F$ is the product manifold $B \times F$ with the metric $g = \pi^* g_B + (\varphi \circ \pi)^2 \sigma^* g_F$. The function φ is called the warping function and the manifolds B and F are called the base and fibre respectively. ■

In the work of Angenent and Knopf, as stated in (1.3), they considered $B = I$ where I is a bounded, open interval of $\mathbb{R}$ and $F = S^n$. For concreteness, we set $I = (-L, L)$ for some $L > 0$.

This cylindrical manifold is then equipped with a metric of the form:

$$g = \phi(z)^2 dz^2 + \psi(z)^2 g_{S^n}.$$

In order to construct a complete smooth metric from this cylindrical metric, we have the cylinder-to-sphere rule which collapses each end of the manifold to a single point smoothly, resulting in the manifold S^{n+1} .

Lemma 4.0.2 (Cylinder-to-sphere rule). [21] Let $0 < L \leq \infty$ and g be a cylindrical warped product metric on $I \times S^n$ where $I = (-L, L)$, that is $g = \phi(z)^2 dz^2 + \psi(z)^2 g_{S^n}$ where g_{S^n} is the round metric on the unit n -sphere S^n . The metric g extends to a smooth metric on S^{n+1} if and only if the following hold:

1. $\int_{-L}^L \phi(z) dz < \infty$,
2. $\lim_{z \rightarrow \pm L} \psi(z) = 0$,
3. $\lim_{z \rightarrow \pm L} \frac{d\psi}{ds}(z) = \mp 1$,
4. $\lim_{z \rightarrow \pm L} \frac{d^{2k}\psi}{ds^{2k}}(z) = 0$ for all $k \in \mathbb{N}$,

where ds is the arclength element induced by $\phi(z)$; that is, $s(z) = \int_0^z \phi(w) dw$. ■

The last three conditions in the cylinder-to-sphere rule imply that the metric component $\psi : (-L, L) \rightarrow \mathbb{R}$ extends to a periodic function on $\mathbb{R}$ with period $4L$ by defining for all $n \in \mathbb{Z}$ the following extended function:

$$\hat{\psi} : \mathbb{R} \rightarrow \mathbb{R},$$

$$z \mapsto \begin{cases} \psi(z - 4nL) & \text{for } z \in ((4n-1)L, (4n+1)L), \\ -\psi(z - (4n+2)L) & \text{for } z \in ((4n+1)L, (4n+3)L), \\ 0 & \text{otherwise.} \end{cases}$$

We note that this implies that the function $\hat{\psi}$ is locally an odd function about the points $z = (4n+1)L$ for each $n \in \mathbb{Z}$.

4.1 Metrics in Milnor Frame

4.1.1 Metric on $S^1 \times S^3$

We have seen that we can always endow the manifold S^3 with a metric of the form $g = a^2 \omega^1 \otimes \omega^1 + b^2 \omega^2 \otimes \omega^2 + c^2 \omega^3 \otimes \omega^3$ by fixing a Milnor frame $\{E_1, E_2, E_3\}$ dual to $\{\omega^1, \omega^2, \omega^3\}$ on this

manifold. We can then vary the constants a, b , and c as we foliate S^1 with a smoothly varying metric of S^3 . Thus, by the periodicity of the foliation, we can endow the manifold $N = S^1 \times S^3$ with a metric of the form:

$$g = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + b(z)^2 \omega^2 \otimes \omega^2 + c(z)^2 \omega^3 \otimes \omega^3, \quad (4.1)$$

where ϕ, a, b , and c are positive 2π -periodic functions on the base manifold $B = S^1$. This is exactly the type of Riemannian manifold studied by Isenberg, Knopf, and Šešum [54]. However, in their study they simplified the ansatz so that $b \equiv c$. This results in a biaxial Bianchi IX metric on $S^1 \times S^3$.

Let π be a projection of this manifold N onto the base space S^1 . Then, the fibres of each $z \in S^1$ are copies of S^3 with the same Milnor frame $\{E_1, E_2, E_3\}$ dual to $\{\omega^1, \omega^2, \omega^3\}$. This frame is chosen such that $[E_i, E_j] = -2E_k$ for any cyclic permutation of i, j, k from the ordered set $\{1, 2, 3\}$. In fact, this frame can be expressed explicitly if we embed S^3 in the standard Euclidean space $\mathbb{R}^4$ as the level set $\{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$. The frames are given by the vectors fields:

$$E_1 = -x_2 \partial_1 + x_1 \partial_2 - x_4 \partial_3 + x_3 \partial_4,$$

$$E_2 = x_4 \partial_1 + x_3 \partial_2 - x_2 \partial_3 - x_1 \partial_4,$$

$$E_3 = -x_3 \partial_1 + x_4 \partial_2 + x_1 \partial_3 - x_2 \partial_4.$$

Note that with this choice of frame, the round sphere metric is given by $g_{S^3} = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$. The fibre of each $z \in S^1$ is diffeomorphic to the Riemannian manifold $(S^3, \hat{g}_z)$ where $\hat{g}_z$ is the fibre metric:

$$\hat{g}_z = a(z)^2 \omega^1 \otimes \omega^1 + b(z)^2 \omega^2 \otimes \omega^2 + c(z)^2 \omega^3 \otimes \omega^3.$$

The upshot is that the Riemannian manifold N has a trivial tangent bundle with a global frame $\{\partial_z, E_1, E_2, E_3\}$ where ∂_z is the vector field on N perpendicular to the S^3 fibres. This would simplify the Ricci flow equation as we would not have to work on different coordinate patches to write the equations explicitly. In order to simplify our notations, we introduce the Levi-Civita symbols ϵ_{ijk} .

Definition 4.1.1 (Levi-Civita symbol). The Levi-Civita symbol $\epsilon_{ijk} \in \{-1, 0, 1\}$ is defined so that $i, j, k \in \{1, 2, 3\}$ with $\epsilon_{123} = 1$ satisfying $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$ and $\epsilon_{ijk} = -\epsilon_{jik}$. $\blacksquare$

Thus, the relation between our chosen Milnor frame vector fields can be succinctly represented as $[E_i, E_j] = -2\epsilon_{ijk} E_k$.

4.1.2 Metric on S^4

Furthermore, we can change the base B of the manifold N to an open interval $I = (-1, 1)$, resulting in the incomplete cylindrical manifold $N' = I \times S^3$ which is homeomorphic to S^4 with two points removed. This generalises the warped product metric considered by Angenent and Knopf in their work [6]. We equip this punctured sphere N' with a metric of the form in (4.1) but without the periodicity assumption on the metric components ϕ, a, b , and c

This punctured sphere N' would also have a global frame given by $\{\partial_z, E_1, E_2, E_3\}$ where ∂_z is the vector field on N' perpendicular to the S^3 fibres. Now we want to ensure that our metric g closes up smoothly to S^4 . In place of the periodicity assumption on the functions ϕ, a, b , and c , we impose an additional ansatz at the boundaries of the manifold by a collapsing condition and the cylinder-to-sphere rule in Lemma 4.0.2, namely:

1. $a, b, c \rightarrow 0$ as $z \rightarrow \pm 1$,
2. $\frac{a}{b}, \frac{b}{c} \rightarrow 1$ to ensure that it collapses smoothly at both ends,
3. $\partial_s a, \partial_s b, \partial_s c \rightarrow \mp 1$ as $z \rightarrow \pm 1$ and $\partial_s^{2k} a, \partial_s^{2k} b, \partial_s^{2k} c \rightarrow 0$ as $z \rightarrow \pm 1$ for all $k \in \mathbb{Z}$.

Remark 4.1.2. The last condition simply requires that the metric functions behave like odd functions locally around the points $z = \pm 1$. If $a(z) = b(z) = c(z)$ for all $z \in (-1, 1)$, then this metric is simply the warped product metric on $I \times_a S^3$. So, without loss of generality, we assume that they are ordered such that $0 < a \leq b \leq c$. It will be shown in Chapters 6 and 7 that this ordering is preserved under the Ricci flow as long as the flow exists. ■

However, contrasting with the previous case of $S^1 \times S^3$, this resulting manifold S^4 does not have a global frame as the global frame $\{\partial_z, E_1, E_2, E_3\}$ on N' does not extend to the whole of S^4 . This agrees with the Hairy Ball Theorem [13], which states that there are no non-vanishing vector fields on even dimensional spheres S^{2n} . Therefore, we work with the almost global frame on S^4 induced by N' . This is sufficient if we are careful enough with the analysis near the poles of this sphere at $z = \pm 1$.

4.2 Curvature Tensors

Now we wish to calculate the curvature tensors of the metric g in the frame $\{\partial_z = E_0, E_1, E_2, E_3\}$ for $M = B \times S^3$, where $B = S^1$ or I . The first thing we have to consider is the derivatives of the

frame vector fields. We write the derivatives of the frame vector fields using the symbols $\Sigma_{\alpha\beta}^\gamma$ for $\alpha, \beta, \gamma \in \{0, 1, 2, 3\}$ by:

$$\nabla_{E_\alpha} E_\beta = \Sigma_{\alpha\beta}^\gamma E_\gamma.$$

Recall that the Christoffel symbols Γ_{ij}^k are defined via local coordinates. Thus, the symbols $\Sigma_{\alpha\beta}^\gamma$ are not the same as Christoffel symbols because the vector fields $\{E_0, E_1, E_2, E_3\}$ on N and S^4 are not coordinate vector fields. We call $\Sigma_{\alpha\beta}^\gamma$ the frame symbols. Using the Koszul formula in Proposition A.3.15, we calculate each of the frame symbols. We have that:

$$\begin{aligned} 2g(E_\gamma, E_\gamma) \Sigma_{\alpha\beta}^\gamma &= 2g(\nabla_{E_\alpha} E_\beta, E_\gamma) \\ &= E_\alpha g(E_\beta, E_\gamma) + E_\beta g(E_\gamma, E_\alpha) - E_\gamma g(E_\alpha, E_\beta) \\ &\quad - g(E_\alpha, [E_\beta, E_\gamma]) + g(E_\beta, [E_\gamma, E_\alpha]) + g(E_\gamma, [E_\alpha, E_\beta]). \end{aligned} \quad (4.2)$$

Remark 4.2.1. Note that the frame symbols $\Sigma_{\alpha\beta}^\gamma$ may not be symmetric in the α and β indices since the (possibly non-zero) quantity $[E_\alpha, E_\beta]$ in the last term is anti-symmetric. This further shows us that the frame symbols are not the same as the Christoffel symbols. ■

Proposition 4.2.2. Let $i, j, k \in \{1, 2, 3\}$. The frame symbols for the frame $\{\partial_z = E_0, E_1, E_2, E_3\}$ on (M, g) are given by:

1. $\Sigma_{00}^0 = \frac{1}{2} g^{00} \partial_z g_{00},$
2. $\Sigma_{0i}^0 = \Sigma_{i0}^0 = \Sigma_{00}^i = 0,$
3. $\Sigma_{j0}^i = \Sigma_{0j}^i = \frac{1}{2} g^{ii} \partial_z (\delta_i^j g_{ij}),$
4. $\Sigma_{ij}^0 = -\frac{1}{2} g^{00} \partial_z (\delta_i^j g_{ij}),$
5. $\Sigma_{ij}^k = \hat{\Sigma}_{ij}^k = \epsilon_{ijk} g^{kk} (g_{ii} - g_{jj} - g_{kk})$ where $\hat{\Sigma}_{ij}^k$ is the frame symbol for the fibre metric $\hat{g}_z = a(z)^2 w^1 \otimes w^1 + b(z)^2 w^2 \otimes w^2 + c(z)^2 w^3 \otimes w^3.$ ■

Proof. Equation (4.2) can be simplified for certain combinations of i, j , and k . Note that $E_1, E_2, E_3 \in \mathcal{L}(TS^3)$ while $E_0 \in \mathcal{L}(TB)$, where $\mathcal{L}$ denotes the lift of the respective vector fields to $\Gamma(TM)$. From Proposition A.2.6, we have that $[E_0, E_i] = 0$ for $i = 1, 2, 3$. Furthermore, the frame is chosen to be orthogonal and $g(E_0, E_0) = \phi(z)^2 = g_{00}$. Thus, if all three indices are 0, then $2g_{00}\Sigma_{00}^0 = \partial_z g_{00}$. If exactly two of the indices in the frame symbol $\Sigma_{\alpha\beta}^\gamma$ are 0, then the frame symbol vanishes identically.

If exactly one of the indices α, β , or γ is 0, we have $\Sigma_{\alpha\beta}^\gamma = \Sigma_{\beta\alpha}^\gamma$ since the anti-symmetric term $g(E_\gamma, [E_\alpha, E_\beta])$ in (4.2) vanishes. By symmetry, there are two cases: Σ_{j0}^i or Σ_{ij}^0 . In both cases,

the last three terms of (4.2) vanish since $\{E_1, E_2, E_3\}$ is chosen so that $[E_i, E_j] = -2\epsilon_{ijk}E_k$. Furthermore, since $E_0 \perp E_i$ for any $i = 1, 2, 3$, only one of the remaining terms survives for both of these symbols. Respectively, we compute $2g_{ii}\Sigma_{j0}^i = \partial_z(\delta_i^j g_{ij})$ and $2\Sigma_{ij}^0 = -\partial_z(\delta_i^j g_{ij})$, which gives us the result.

Finally, for non-zero indices, if all three indices are the same, then clearly the frame symbols vanish. If exactly two of them are the same, there are three cases: Σ_{ii}^j , Σ_{ji}^i or Σ_{ij}^i . In all cases, the first three terms of (4.2) vanish since $a(z)$, $b(z)$, and $c(z)$ are constant in the fibres of each $z \in B$, giving us a multiple of (3.3). This implies that the frame symbol Σ_{ij}^k is the same as the frame symbol $\hat{\Sigma}_{ij}^k$ on the fibre $(S^3, \hat{g}_z)$. Furthermore, the last three terms also vanish since $[E_i, E_i] = 0$ and $[E_i, E_j] = -2\epsilon_{ijk}E_k \perp E_i$.

Thus, the only (possibly) non-vanishing frame symbol of this form is when i, j , and k are distinct. Substituting in all the Lie brackets of vector fields, we get $2g^{kk}\Sigma_{ij}^k = 2(\epsilon_{jki}g_{ii} - \epsilon_{kij}g_{jj} - \epsilon_{ijk}g_{kk})$. By permuting the indices of the Levi-Civita symbol, we deduce the result. $\square$

With these explicitly defined, we can calculate the Riemann curvature tensor components. Note here that we are calculating the tensor with respect to the frame $\{\partial_z = E_0, E_1, E_2, E_3\}$ instead of coordinate vector fields, so we denote $\text{Rm}_{\alpha\beta\gamma\delta} = g(\text{Rm}(E_\alpha, E_\beta)E_\gamma, E_\delta)$ for $\alpha, \beta, \gamma, \delta \in \{0, 1, 2, 3\}$.

Proposition 4.2.3. Let $i, j, k \in \{1, 2, 3\}$ be distinct indices. The Riemann curvature tensor components in the frame $\{\partial_z = E_0, E_1, E_2, E_3\}$ on (M, g) are given by:

1. $\text{Rm}_{000i} = \text{Rm}_{0000} = 0$,
2. $\text{Rm}_{i00j} = 0$ and $\text{Rm}_{i00i} = \text{Rm}_{0ii0} = \frac{1}{4}(g^{00}g^{ii}\partial_z g_{00}\partial_z g_{ii} + g^{ii}(\partial_z g_{ii})^2 - 2\partial_{zz}g_{ii})$,
3. $\text{Rm}_{ijjk} = \text{Rm}_{ijj0} = 0$ and $\text{Rm}_{ijji} = -\frac{1}{4}g^{00}\partial_z g_{jj}\partial_z g_{ii} + \hat{\text{Rm}}_{ijji}$ where $\hat{\text{Rm}}_{ijji} = -g^{kk}(g_{kk}^2 - (g_{ii} - g_{jj})^2) - 2(g_{kk} - g_{jj} - g_{ii})$ is the Riemann curvature tensor of the fibre metric $\hat{g}_z$. $\blacksquare$

Proof. From the previous proposition, we note that the frame symbols are only dependent on the variable z . The first identity is clear. For the second, we compute:

$$\begin{aligned}
 \nabla^2 E_0(E_i, E_0) - \nabla^2 E_0(E_0, E_i) &= \nabla_{E_i} \nabla_{E_0} E_0 - \nabla_{E_0} \nabla_{E_i} E_0 - \nabla_{[E_i, E_0]} E_0 \\
 &= \nabla_{E_i} (\Sigma_{00}^0 E_0) - \nabla_{E_0} (\Sigma_{i0}^i E_i) \\
 &= \Sigma_{00}^0 \Sigma_{i0}^i E_i - \partial_z \Sigma_{i0}^i E_i - \Sigma_{i0}^i \nabla_{E_0} E_i \\
 &= \Sigma_{00}^0 \Sigma_{i0}^i E_i - \partial_z \Sigma_{i0}^i E_i - \Sigma_{i0}^i \Sigma_{0i}^i E_i \\
 &= \frac{1}{4}g^{00}g^{ii}\partial_z g_{00}\partial_z g_{ii} E_i - \frac{1}{2}\partial_z (g^{ii}\partial_z g_{ii}) E_i - \frac{1}{4}(g^{ii}\partial_z g_{ii})^2 E_i.
 \end{aligned}$$

By using the fact that $0 = \partial_s(g^{ii}g_{ii}) = g_{ii}\partial_s g^{ii} + g^{ii}\partial_s g_{ii}$, we can simplify this to:

$$\nabla^2 E_0(E_i, E_0) - \nabla^2 E_0(E_0, E_i) = \left(\frac{1}{4}g^{00}g^{ii}\partial_z g_{00}\partial_z g_{ii} + \frac{1}{4}(g^{ii}\partial_z g_{ii})^2 - \frac{1}{2}g^{ii}\partial_{zz}g_{ii} \right) E_i.$$

By taking the inner product of this quantity with E_i and E_j and using the symmetries of the Riemann curvature tensor, we obtain the second relation. For the third, we calculate:

$$\begin{aligned} & \nabla^2 E_j(E_i, E_j) - \nabla^2 E_j(E_j, E_i) \\ &= \nabla_{E_i}\nabla_{E_j}E_j - \nabla_{E_j}\nabla_{E_i}E_j - \nabla_{[E_i, E_j]}E_j \\ &= \nabla_{E_i}(\Sigma_{jj}^0 E_0) - \nabla_{E_j}(\Sigma_{ij}^k E_k) + 2\epsilon_{ijk}\nabla_{E_k}E_j \\ &= \Sigma_{jj}^0 \Sigma_{i0}^i E_i - \Sigma_{ij}^k \Sigma_{jk}^i E_i + \epsilon_{ijk}\Sigma_{kj}^i E_i \\ &= -\frac{1}{4}g^{00}g^{ii}\partial_z g_{ii}\partial_z g_{jj}E_i - \epsilon_{ijk}\epsilon_{jki}g^{ii}g^{kk}(g_{ii} - g_{jj} - g_{kk})(g_{jj} - g_{kk} - g_{ii})E_i \\ &\quad + 2\epsilon_{ijk}\epsilon_{kji}g^{ii}(g_{kk} - g_{jj} - g_{ii})E_i \\ &= \left(-\frac{1}{4}g^{00}g^{ii}\partial_z g_{jj}\partial_z g_{ii} - g^{ii}g^{kk}(g_{kk}^2 - (g_{ii} - g_{jj})^2) - 2g^{ii}(g_{kk} - g_{jj} - g_{ii}) \right) E_i. \end{aligned}$$

Again, by taking the inner product with E_0, E_i , and E_k , and using the symmetries of the Riemann curvature tensor, we extract the results. Note that throughout the calculation above, the last two terms do not involve the vector E_0 at all, hence they are the same as the Riemann curvature tensor $\hat{\text{Rm}}_{ijjk}$ of the fibre metric $\hat{g}_z$. $\square$

We can also calculate the Ricci tensor components and the sectional curvatures by taking the appropriate traces of the Riemann curvature tensor from above.

Proposition 4.2.4. We have the following Ricci tensor components:

1. $\text{Ric}_{\alpha\beta} = 0$ for $\alpha \neq \beta$,
2. $\text{Ric}_{00} = \frac{1}{4} \sum_{i=1}^3 (g^{00}(g^{ii})^2 \partial_z g_{00} \partial_z g_{ii} + (g^{ii} \partial_z g_{ii})^2 - 2g^{ii} \partial_{zz} g_{ii}),$
3. $\text{Ric}_{ii} = \frac{1}{4} (g^{00}(g^{00}g^{ii}\partial_z g_{00}\partial_z g_{ii} + g^{ii}(\partial_z g_{ii})^2 - 2\partial_{zz}g_{ii}) - \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^3 g^{jj}(g^{00}\partial_z g_{jj}\partial_z g_{ii} + 4g^{kk}(g_{kk}^2 - (g_{ii} - g_{jj})^2) + 8(g_{kk} - g_{jj} - g_{ii}))).$ $\blacksquare$

Proposition 4.2.5. For $i, j \in \{1, 2, 3\}$, the sectional curvatures of the metric g are given by:

1. $K_{0i} = \frac{1}{4}g^{00}g^{ii}(g^{00}g^{ii}\partial_z g_{00}\partial_z g_{ii} + g^{ii}(\partial_z g_{ii})^2 - 2\partial_{zz}g_{ii}),$
2. $K_{ij} = -\frac{1}{4}g^{00}g^{ii}g^{jj}\partial_z g_{ii}\partial_z g_{jj} + \hat{K}_{ij}$ where $\hat{K}_{ij} = -g^{ii}g^{jj}(g^{kk}(g_{kk}^2 - (g_{ii} - g_{jj})^2) + (g_{kk} - g_{jj} - g_{ii}))$ for $k \neq i, j$ is the sectional curvature of the fibre metric $\hat{g}_z$ spanned by E_i and E_j . $\blacksquare$

4.2.1 Curvature Tensors in Arclength Coordinates

Recall that we have defined an arclength coordinate in the cylinder-to-sphere rule. The arclength element ds is induced from $\phi(z)$ by defining the arclength coordinate $s(z) = \int_0^z \phi(w) dw$ and using this coordinate in place of z . In fact, this is a more natural geometric quantity as the formulation for the Ricci flow equation written in this coordinate, as we shall see later, is strongly parabolic. Therefore, we would not have to resort to DeTurck's Trick to formulate a system of parabolic PDEs.

If we choose to calculate the symbols in the arclength coordinate frame (that is, $\{\partial_s = E_0, E_1, E_2, E_3\}$), we simply substitute $g^{00} = 1$ in Propositions 4.2.2-4.2.5 to get:

Corollary 4.2.6. In the arclength coordinate frame, the Christoffel symbols can be written explicitly as:

1. $\Sigma_{00}^0 = \Sigma_{0i}^0 = \Sigma_{i0}^0 = \Sigma_{00}^i = 0$,
2. $\Sigma_{j0}^i = \Sigma_{0j}^i = \frac{1}{2}g^{ii}\partial_s(\delta_i^j g_{ij})$,
3. $\Sigma_{ij}^0 = -\frac{1}{2}\partial_s(\delta_i^j g_{ij})$,
4. $\Sigma_{ij}^k = \hat{\Sigma}_{ij}^k = \epsilon_{ijk}g^{kk}(g_{ii} - g_{jj} - g_{kk})$ where $\hat{\Sigma}_{ij}^k$ is the frame symbol for the fibre metric $\hat{g}_s = a(s)^2\omega^1 \otimes \omega^1 + b(s)^2\omega^2 \otimes \omega^2 + c(s)^2\omega^3 \otimes \omega^3$.

The Riemann curvature tensor components for distinct indices $i, j, k \in \{1, 2, 3\}$ can be written explicitly as:

1. $\text{Rm}_{000i} = \text{Rm}_{0000} = 0$,
2. $\text{Rm}_{i00j} = 0$ and $\text{Rm}_{i00i} = \text{Rm}_{0ii0} = \frac{1}{4}(g^{ii}(\partial_s g_{ii})^2 - 2\partial_{ss}g_{ii})$,
3. $\text{Rm}_{ijjk} = \text{Rm}_{ijj0} = 0$ and $\text{Rm}_{ijji} = -\frac{1}{4}\partial_s g_{jj}\partial_s g_{ii} + \hat{\text{Rm}}_{ijji}$ where $\hat{\text{Rm}}_{ijji} = -g^{kk}(g_{kk}^2 - (g_{ii} - g_{jj})^2) - 2(g_{kk} - g_{jj} - g_{ii})$ is the Riemann curvature tensor of the fibre metric $\hat{g}_s$.

The Ricci curvature tensor components can be written explicitly as:

1. $\text{Ric}_{\alpha\beta} = 0$ for $\alpha \neq \beta$,
2. $\text{Ric}_{00} = \frac{1}{4} \sum_{i=1}^3 ((g^{ii}\partial_s g_{ii})^2 - 2g^{ii}\partial_{ss}g_{ii})$,
3. $\text{Ric}_{ii} = \frac{1}{4}(g^{ii}(\partial_s g_{ii})^2 - 2\partial_{ss}g_{ii} - \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^3 g^{jj}(\partial_s g_{jj}\partial_s g_{ii} + 4g^{kk}(g_{kk}^2 - (g_{ii} - g_{jj})^2) + 8(g_{kk} - g_{jj} - g_{ii})))$.

The sectional curvatures can be written explicitly as:

1. $K_{0i} = \frac{1}{4}((g^{ii}\partial_s g_{ii})^2 - 2g^{ii}\partial_{ss}g_{ii}),$
2. $K_{ij} = -\frac{1}{4}g^{ii}g^{jj}\partial_s g_{ii}\partial_s g_{jj} + \hat{K}_{ij}$ where $\hat{K}_{ij} = -g^{ii}g^{jj}(g^{kk}(g_{kk}^2 - (g_{ii} - g_{jj})^2) + (g_{kk} - g_{jj} - g_{ii}))$ for $k \neq i, j$ is the sectional curvature of the fibre metric $\hat{g}_s$ spanned by E_i and E_j . ■

From now on, unless otherwise stated, we are going to use the arclength coordinate frame, which we make distinct from the original coordinate frame ∂_z by writing ∂_s instead.

4.3 Ricci Flow Equation

Explicitly, in the arclength coordinate frame $\{\partial_s = E_0, E_1, E_2, E_3\}$, we can calculate the Ricci tensor components of the metric in (4.1) to get:

$$\text{Ric}_{00} = -\left(\frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c}\right), \quad (4.3)$$

$$\text{Ric}_{11} = -aa'' - aa'\left(\frac{b'}{b} + \frac{c'}{c}\right) + a^2(\hat{K}_{12} + \hat{K}_{13}), \quad (4.4)$$

$$\text{Ric}_{22} = -bb'' - bb'\left(\frac{a'}{a} + \frac{c'}{c}\right) + b^2(\hat{K}_{12} + \hat{K}_{23}), \quad (4.5)$$

$$\text{Ric}_{33} = -cc'' - cc'\left(\frac{a'}{a} + \frac{b'}{b}\right) + c^2(\hat{K}_{13} + \hat{K}_{23}), \quad (4.6)$$

$$\text{Ric}_{\alpha\beta} = 0 \quad \text{for } \alpha \neq \beta, \quad (4.7)$$

where:

$$\begin{aligned} \hat{K}_{12} &= \frac{(a^2 - b^2)^2 - 3c^4}{(abc)^2} + \frac{2}{a^2} + \frac{2}{b^2}, \\ \hat{K}_{13} &= \frac{(a^2 - c^2)^2 - 3b^4}{(abc)^2} + \frac{2}{a^2} + \frac{2}{c^2}, \\ \hat{K}_{23} &= \frac{(b^2 - c^2)^2 - 3a^4}{(abc)^2} + \frac{2}{b^2} + \frac{2}{c^2}, \end{aligned}$$

are the sectional curvature of the $SU(2)$ fibres. These were calculated in the previous chapter in equations (3.6)-(3.8) as well as in Proposition 4.2.5.

The $\dots'$ denotes derivative with respect to the arclength variable $s(z, t)$ via the change of variable $ds = \phi(z, t) dz$. From now on, we denote derivatives with respect to the original variable z and the new gauge $s(z, t)$ by $\dots$ and $\dots'$ respectively.

Remark 4.3.1. Note that by choosing this gauge, the new space variable s depends on both the original space and time variables. Furthermore, from the relation $ds = \phi(z, t)dz$, we have

the chain rule identity $\frac{\partial}{\partial s} = \frac{1}{\phi(z,t)} \frac{\partial}{\partial z}$. Thus, the derivatives in the t direction and s direction do not commute as the original variables z and t do. Therefore, we have the following commutator relation:

$$[\partial_t, \partial_s] = -\frac{\partial_t \phi}{\phi} \partial_s. \quad (4.8)$$

From now on, keeping this in mind, we suppress the dependence of the variable s on the variables z and t . ■

The sectional curvatures of the manifold $B \times S^3$ are given by:

$$\begin{aligned} K_{01} &= -\frac{a''}{a} \\ K_{02} &= -\frac{b''}{b}, \\ K_{03} &= -\frac{c''}{c}, \\ K_{12} &= -\frac{a'b'}{ab} + \frac{(a^2 - b^2)^2 - 3c^4}{(abc)^2} + \frac{2}{a^2} + \frac{2}{b^2} = -\frac{a'b'}{ab} + \hat{K}_{12}, \\ K_{13} &= -\frac{a'c'}{ac} + \frac{(a^2 - c^2)^2 - 3b^4}{(abc)^2} + \frac{2}{a^2} + \frac{2}{c^2} = -\frac{a'c'}{ac} + \hat{K}_{13}, \\ K_{23} &= -\frac{b'c'}{bc} + \frac{(b^2 - c^2)^2 - 3a^4}{(abc)^2} + \frac{2}{b^2} + \frac{2}{c^2} = -\frac{b'c'}{bc} + \hat{K}_{23}. \end{aligned}$$

Since the off-diagonal terms in the Ricci tensor vanish identically, the Ricci flow equation preserves the form of the initial metric; that is, for all time for which the solution exists, the metric $g(t)$ would be of the form (4.1) for some positive functions ϕ, a, b , and c which are all functions of z and t . Equating the first component in the Ricci flow equation gives us:

$$\begin{aligned} \partial_t(\phi^2) &= 2 \left(\frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} \right) \phi^2 \\ \Rightarrow \quad \partial_t(\log \phi) &= \frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} = -(K_{01} + K_{02} + K_{03}), \end{aligned} \quad (4.9)$$

and thus $[\partial_t, \partial_s] = (K_{01} + K_{02} + K_{03}) \partial_s$.

The remaining equations are:

$$\begin{aligned} \partial_t a &= a'' + a' \left(\frac{b'}{b} + \frac{c'}{c} \right) - a(\hat{K}_{12} + \hat{K}_{13}), \\ \partial_t b &= b'' + b' \left(\frac{a'}{a} + \frac{c'}{c} \right) - b(\hat{K}_{12} + \hat{K}_{23}), \\ \partial_t c &= c'' + c' \left(\frac{a'}{a} + \frac{b'}{b} \right) - c(\hat{K}_{13} + \hat{K}_{23}). \end{aligned}$$

In the arclength coordinate s , the Ricci flow is a semilinear parabolic system of equations:

$$\begin{cases} \partial_t a = a'' + a' \left(\frac{b'}{b} + \frac{c'}{c} \right) - 2a \left(\frac{a^4 - (b^2 - c^2)^2}{(abc)^2} \right), \end{cases} \quad (4.10)$$

$$\begin{cases} \partial_t b = b'' + b' \left(\frac{a'}{a} + \frac{c'}{c} \right) - 2b \left(\frac{b^4 - (a^2 - c^2)^2}{(abc)^2} \right), \end{cases} \quad (4.11)$$

$$\begin{cases} \partial_t c = c'' + c' \left(\frac{a'}{a} + \frac{b'}{b} \right) - 2c \left(\frac{c^4 - (a^2 - b^2)^2}{(abc)^2} \right). \end{cases} \quad (4.12)$$

This system, along with the commutator relation $[\partial_t, \partial_s] = -\left(\frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c}\right)\partial_s$, will be the system of equations that we will analyse in the next few chapters. Furthermore, from equations (4.3)-(4.7), we can calculate the scalar curvature S for g , which will be useful later:

$$\begin{aligned} S &= 2(K_{01} + K_{02} + K_{03} + K_{12} + K_{13} + K_{23}) \\ &= 2 \left(-\frac{a''}{a} - \frac{b''}{b} - \frac{c''}{c} - \frac{a'b'}{ab} - \frac{a'c'}{ac} - \frac{b'c'}{bc} + \frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}{a^2b^2c^2} \right). \end{aligned} \quad (4.13)$$

We also need the evolution equation for the Ricci tensor under the Ricci flow, which is given by (see, among others, [2], [21], and [42]):

$$\partial_t \text{Ric}_{\alpha\beta} = \Delta \text{Ric}_{\alpha\beta} + 2g^{\gamma\delta} g^{\eta\zeta} \text{Rm}_{\gamma\alpha\beta\eta} \text{Ric}_{\delta\zeta} - 2g^{\gamma\delta} \text{Ric}_{\alpha\gamma} \text{Ric}_{\beta\delta}. \quad (4.14)$$

4.4 Evolution Equations of Other Quantities

In order to proceed with the analysis in Chapters 6, 7, and 8, the evolution of the following quantities are also considered. First, the quantities a^2, b^2 , and c^2 evolve according to:

$$\partial_t(a^2) = 2aa'' + 2aa' \left(\frac{b'}{b} + \frac{c'}{c} \right) - 4 \left(\frac{a^4 - (b^2 - c^2)^2}{b^2c^2} \right), \quad (4.15)$$

$$\partial_t(b^2) = 2bb'' + 2bb' \left(\frac{a'}{a} + \frac{c'}{c} \right) - 4 \left(\frac{b^4 - (a^2 - c^2)^2}{a^2c^2} \right), \quad (4.16)$$

$$\partial_t(c^2) = 2cc'' + 2cc' \left(\frac{a'}{a} + \frac{b'}{b} \right) - 4 \left(\frac{c^4 - (a^2 - b^2)^2}{a^2b^2} \right). \quad (4.17)$$

We consider the quantity $\xi := a - b$. From equations (4.10) and (4.11), we can derive the evolution equation for ξ , which is given by:

$$\partial_t \xi = \xi'' + \frac{c'}{c} \xi' + \left(\frac{a'b'}{ab} - \frac{2(a^4 + b^4 - c^4)}{a^2b^2c^2} - \frac{4(a^2 + b^2 - c^2)}{abc^2} - \frac{4}{c^2} \right) \xi.$$

Similarly, the quantities $\zeta := b - c$ and $\chi := a - c$ satisfy the equations:

$$\partial_t \zeta = \zeta'' + \frac{a'}{a} \zeta' + \left(\frac{b'c'}{bc} - \frac{2(b^4 + c^4 - a^4)}{a^2b^2c^2} - \frac{4(b^2 + c^2 - a^2)}{a^2bc} - \frac{4}{a^2} \right) \zeta,$$

$$\partial_t \chi = \chi'' + \frac{b'}{b} \chi' + \left(\frac{a'c'}{ac} - \frac{2(a^4 + c^4 - b^4)}{a^2b^2c^2} - \frac{4(a^2 + c^2 - b^2)}{ab^2c} - \frac{4}{b^2} \right) \chi.$$

In the vein of Proposition 3.3.3 and the work in [54], we consider the quantities $\frac{\xi}{a} = \frac{a-b}{a}$ and $\frac{\xi}{b} = \frac{a-b}{b}$. These quantities can be thought of the measure for eccentricity of the fibre at each $z \in B$. These eccentricity quantities evolve according to the PDEs:

$$\partial_t \left(\frac{\xi}{a} \right) = \left(\frac{\xi}{a} \right)'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) \left(\frac{\xi}{a} \right)' - 4 \left(\frac{1}{c^2} - \frac{1}{a^2} + \frac{b^2}{a^2 c^2} + \frac{a^2 + b^2 - c^2}{abc^2} \right) \left(\frac{\xi}{a} \right), \quad (4.18)$$

$$\partial_t \left(\frac{\xi}{b} \right) = \left(\frac{\xi}{b} \right)'' + \left(2 \frac{b'}{b} + \frac{c'}{c} \right) \left(\frac{\xi}{b} \right)' - 4 \left(\frac{1}{c^2} - \frac{1}{b^2} + \frac{a^2}{b^2 c^2} + \frac{a^2 + b^2 - c^2}{abc^2} \right) \left(\frac{\xi}{b} \right). \quad (4.19)$$

Furthermore, the other eccentricity quantities, defined by $\frac{\zeta}{b} = \frac{b-c}{b}$, $\frac{\zeta}{c} = \frac{b-c}{c}$, $\frac{\chi}{a} = \frac{a-c}{a}$, and $\frac{\chi}{c} = \frac{a-c}{c}$ evolve according to the following PDEs:

$$\partial_t \left(\frac{\zeta}{b} \right) = \left(\frac{\zeta}{b} \right)'' + \left(2 \frac{b'}{b} + \frac{a'}{a} \right) \left(\frac{\zeta}{b} \right)' - 4 \left(\frac{1}{a^2} - \frac{1}{b^2} + \frac{c^2}{a^2 b^2} + \frac{b^2 + c^2 - a^2}{a^2 bc} \right) \left(\frac{\zeta}{b} \right), \quad (4.20)$$

$$\partial_t \left(\frac{\zeta}{c} \right) = \left(\frac{\zeta}{c} \right)'' + \left(2 \frac{c'}{c} + \frac{a'}{a} \right) \left(\frac{\zeta}{c} \right)' - 4 \left(\frac{1}{a^2} - \frac{1}{c^2} + \frac{b^2}{a^2 c^2} + \frac{b^2 + c^2 - a^2}{a^2 bc} \right) \left(\frac{\zeta}{c} \right), \quad (4.21)$$

$$\partial_t \left(\frac{\chi}{a} \right) = \left(\frac{\chi}{a} \right)'' + \left(2 \frac{a'}{a} + \frac{b'}{b} \right) \left(\frac{\chi}{a} \right)' - 4 \left(\frac{1}{b^2} - \frac{1}{a^2} + \frac{c^2}{a^2 b^2} + \frac{a^2 + c^2 - b^2}{ab^2 c} \right) \left(\frac{\chi}{a} \right), \quad (4.22)$$

$$\partial_t \left(\frac{\chi}{c} \right) = \left(\frac{\chi}{c} \right)'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) \left(\frac{\chi}{c} \right)' - 4 \left(\frac{1}{b^2} - \frac{1}{c^2} + \frac{a^2}{b^2 c^2} + \frac{a^2 + c^2 - b^2}{ab^2 c} \right) \left(\frac{\chi}{c} \right). \quad (4.23)$$

To calculate the evolution of the curvatures, we need to find the evolution equations of the first and second derivatives of the metric components. For the first derivatives, by using the commutator relation (4.8) to swap the order of the time and space derivatives ∂_t and ∂_s , we derive the following evolution equations:

$$\begin{aligned} \partial_t(a') &= (\partial_t a)' + [\partial_t, \partial_s]a' = (\partial_t a)' - \left(\frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} \right) a' \\ &= a''' + a'' \left(\frac{b'}{b} + \frac{c'}{c} - \frac{a'}{a} \right) - a' \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} + \frac{6a^4 + 2(b^2 - c^2)^2}{(abc)^2} \right) \\ &\quad + 4 \left(\frac{a^3 b'}{b^3 c^2} + \frac{a^3 c'}{b^2 c^3} - \frac{c^2 b'}{ab^3} - \frac{b^2 c'}{ac^3} + \frac{cc'}{ab^2} + \frac{bb'}{ac^2} \right), \end{aligned} \quad (4.24)$$

$$\begin{aligned} \partial_t(b') &= b''' + b'' \left(\frac{a'}{a} + \frac{c'}{c} - \frac{b'}{b} \right) - b' \left(\frac{(a')^2}{a^2} + \frac{(c')^2}{c^2} + \frac{6b^4 + 2(a^2 - c^2)^2}{(abc)^2} \right) \\ &\quad + 4 \left(\frac{b^3 a'}{a^3 c^2} + \frac{b^3 c'}{a^2 c^3} - \frac{c^2 a'}{ba^3} - \frac{a^2 c'}{bc^3} + \frac{cc'}{ba^2} + \frac{aa'}{bc^2} \right), \end{aligned} \quad (4.25)$$

$$\begin{aligned} \partial_t(c') &= c''' + c'' \left(\frac{a'}{a} + \frac{b'}{b} - \frac{c'}{c} \right) - c' \left(\frac{(a')^2}{a^2} + \frac{(b')^2}{b^2} + \frac{6c^4 + 2(a^2 - b^2)^2}{(abc)^2} \right) \\ &\quad + 4 \left(\frac{c^3 a'}{a^3 b^2} + \frac{c^3 b'}{a^2 b^3} - \frac{b^2 a'}{ca^3} - \frac{a^2 b'}{cb^3} + \frac{aa'}{cb^2} + \frac{bb'}{ca^2} \right), \end{aligned} \quad (4.26)$$

Furthermore, we have:

$$\begin{aligned} \partial_t \left(\frac{(a')^2}{a^2} \right) &= \Delta_g \left(\frac{(a')^2}{a^2} \right) - 2 \frac{(a')^2}{a^2} \left(2 \frac{(a')^2}{a^2} + \frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} + \frac{4(a^4 + (b^2 - c^2)^2)}{a^2 b^2 c^2} \right) \\ &\quad + 8 \frac{a'}{a^2} \left(\frac{a^3 b'}{b^3 c^2} + \frac{a^3 c'}{b^2 c^3} - \frac{c^2 b'}{a b^3} - \frac{b^2 c'}{a c^3} + \frac{c c'}{a b^2} + \frac{b b'}{a c^2} \right) - 2 K_{01}^2 - 4 \frac{(a')^2}{a^2} K_{01}, \end{aligned} \quad (4.27)$$

$$\begin{aligned} \partial_t \left(\frac{(b')^2}{b^2} \right) &= \Delta_g \left(\frac{(b')^2}{b^2} \right) - 2 \frac{(b')^2}{b^2} \left(2 \frac{(b')^2}{b^2} + \frac{(a')^2}{a^2} + \frac{(c')^2}{c^2} + \frac{4(b^4 + (a^2 - c^2)^2)}{a^2 b^2 c^2} \right) \\ &\quad + 8 \frac{b'}{b^2} \left(\frac{b^3 a'}{a^3 c^2} + \frac{b^3 c'}{a^2 c^3} - \frac{c^2 a'}{b a^3} - \frac{a^2 c'}{b c^3} + \frac{c c'}{b a^2} + \frac{a a'}{b c^2} \right) - 2 K_{02}^2 - 4 \frac{(b')^2}{b^2} K_{02}, \end{aligned} \quad (4.28)$$

$$\begin{aligned} \partial_t \left(\frac{(c')^2}{c^2} \right) &= \Delta_g \left(\frac{(c')^2}{c^2} \right) - 2 \frac{(c')^2}{c^2} \left(2 \frac{(c')^2}{c^2} + \frac{(a')^2}{a^2} + \frac{(b')^2}{b^2} + \frac{4(c^4 + (a^2 - b^2)^2)}{a^2 b^2 c^2} \right) \\ &\quad + 8 \frac{c'}{c^2} \left(\frac{c^3 a'}{a^3 b^2} + \frac{c^3 b'}{a^2 b^3} - \frac{b^2 a'}{c a^3} - \frac{a^2 b'}{c b^3} + \frac{a a'}{c b^2} + \frac{b b'}{c a^2} \right) - 2 K_{03}^2 - 4 \frac{(c')^2}{c^2} K_{03}, \end{aligned} \quad (4.29)$$

where Δ_g is the Laplacian operator, which is defined as:

$$\Delta_g f = f'' + \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} \right) f',$$

for any $f \in C^2(I)$. As for the second derivatives, we can calculate:

$$\begin{aligned} \partial_t(a'') &= a'''' + a''' \left(\frac{b'}{b} + \frac{c'}{c} - \frac{a'}{a} \right) + a'' \left(\frac{(a')^2}{a^2} - 2 \frac{(b')^2}{b^2} - 2 \frac{(c')^2}{c^2} - 2 \frac{a''}{a} - \frac{6a^4 + 2(b^2 - c^2)^2}{(abc)^2} \right) \\ &\quad - 2a' \left(\frac{b'b''}{b^2} - \frac{(b')^3}{b^3} + \frac{c'c''}{c^2} - \frac{(c')^3}{c^3} + \frac{6aa'}{b^2 c^2} + \frac{4bb'}{a^2 c^2} + \frac{4cc'}{a^2 b^2} - \frac{2a'c^2}{a^3 b^2} - \frac{2a'b^2}{a^3 c^2} + \frac{4a'}{a^3} \right. \\ &\quad \left. - \frac{12a^2 b'}{b^3 c^2} - \frac{12a^2 c'}{b^2 c^3} - \frac{4b^2 c'}{a^2 c^3} - \frac{4c^2 b'}{a^2 b^3} \right) + 4a^3 \left(\frac{b''}{b^3 c^2} - \frac{3(b')^2}{b^4 c^2} - \frac{4b'c'}{b^3 c^3} + \frac{c''}{b^2 c^3} - \frac{3(c')^2}{b^2 c^4} \right) \\ &\quad + 4 \left(\frac{(c')^2 + cc''}{ab^2} + \frac{(b')^2 + bb''}{ac^2} + \frac{3(b')^2 c^2}{ab^4} + \frac{3b^2 (c')^2}{ac^4} - \frac{4cc'b' + c^2 b''}{ab^3} - \frac{4bb'c' + b^2 c''}{ac^3} \right), \\ \partial_t(b'') &= b'''' + b''' \left(\frac{a'}{a} + \frac{c'}{c} - \frac{b'}{b} \right) + b'' \left(\frac{(b')^2}{b^2} - 2 \frac{(a')^2}{a^2} - 2 \frac{(c')^2}{c^2} - 2 \frac{b''}{b} - \frac{6b^4 + 2(a^2 - c^2)^2}{(abc)^2} \right) \\ &\quad - 2b' \left(\frac{a'a''}{a^2} - \frac{(a')^3}{a^3} + \frac{c'c''}{c^2} - \frac{(c')^3}{c^3} + \frac{6bb'}{a^2 c^2} + \frac{4aa'}{b^2 c^2} + \frac{4cc'}{a^2 b^2} - \frac{2b'c^2}{b^3 a^2} - \frac{2b'a^2}{b^3 c^2} + \frac{4b'}{b^3} \right. \\ &\quad \left. - \frac{12b^2 a'}{a^3 c^2} - \frac{12b^2 c'}{a^2 c^3} - \frac{4a^2 c'}{b^2 c^3} - \frac{4c^2 a'}{b^2 a^3} \right) + 4b^3 \left(\frac{a''}{a^3 c^2} - \frac{3(a')^2}{a^4 c^2} - \frac{4a'c'}{a^3 c^3} + \frac{c''}{a^2 c^3} - \frac{3(c')^2}{a^2 c^4} \right) \\ &\quad + 4 \left(\frac{(c')^2 + cc''}{ba^2} + \frac{(a')^2 + aa''}{bc^2} + \frac{3(a')^2 c^2}{ba^4} + \frac{3a^2 (c')^2}{bc^4} - \frac{4cc'a' + c^2 a''}{ba^3} - \frac{4aa'c' + a^2 c''}{bc^3} \right), \\ \partial_t(c'') &= c'''' + c''' \left(\frac{a'}{a} + \frac{b'}{b} - \frac{c'}{c} \right) + c'' \left(\frac{(c')^2}{c^2} - 2 \frac{(a')^2}{a^2} - 2 \frac{(b')^2}{b^2} - 2 \frac{c''}{c} - \frac{6c^4 + 2(a^2 - b^2)^2}{(abc)^2} \right) \\ &\quad - 2c' \left(\frac{a'a''}{a^2} - \frac{(a')^3}{a^3} + \frac{b'b''}{b^2} - \frac{(b')^3}{b^3} + \frac{6cc'}{a^2 b^2} + \frac{4aa'}{b^2 c^2} + \frac{4bb'}{a^2 b^2} - \frac{2c'b^2}{c^3 a^2} - \frac{2c'a^2}{c^3 b^2} + \frac{4c'}{c^3} \right. \\ &\quad \left. - \frac{12c^2 a'}{a^3 b^2} - \frac{12c^2 b'}{a^2 b^3} - \frac{4b^2 a'}{c^2 a^3} - \frac{4a^2 b'}{c^2 b^3} \right) + 4c^3 \left(\frac{a''}{a^3 b^2} - \frac{3(a')^2}{a^4 b^2} - \frac{4a'b'}{a^3 b^3} + \frac{b''}{a^2 b^3} - \frac{3(b')^2}{a^2 b^4} \right) \\ &\quad + 4 \left(\frac{(b')^2 + bb''}{ca^2} + \frac{(a')^2 + aa''}{cb^2} + \frac{3a^2 (b')^2}{cb^4} + \frac{3(a')^2 b^2}{ca^4} - \frac{4aa'b' + a^2 b''}{cb^3} - \frac{4bb'a' + b^2 a''}{ca^3} \right). \end{aligned}$$

Thus, from the above equations, the sectional curvatures K_{01} , K_{02} , and K_{03} evolve according to the following PDEs:

$$\begin{aligned}
 \partial_t K_{01} &= \partial_t \left(-\frac{a''}{a} \right) = -\frac{\partial_t(a'')}{a} + \frac{a'' \partial_t a}{a^2} \\
 &= \Delta_g K_{01} + 2K_{01}^2 - 2K_{01} \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} + \frac{2a^4 + 2(b^2 - c^2)^2}{(abc)^2} \right) \\
 &\quad + 2K_{02} \left(\frac{2a^2}{b^2 c^2} + \frac{2b^2}{a^2 c^2} - \frac{2c^2}{a^2 b^2} - \frac{a'b'}{ab} \right) + 2K_{03} \left(\frac{2a^2}{b^2 c^2} + \frac{2c^2}{a^2 b^2} - \frac{2b^2}{a^2 c^2} - \frac{a'c'}{ac} \right) \\
 &\quad + 2\frac{a'}{a} \left(-\frac{(b')^3}{b^3} - \frac{(c')^3}{c^3} + \frac{6aa'}{b^2 c^2} + \frac{4bb'}{a^2 c^2} + \frac{4cc'}{a^2 b^2} - \frac{2a'c^2}{a^3 b^2} - \frac{2a'b^2}{a^3 c^2} + \frac{4a'}{a^3} \right. \\
 &\quad \left. - \frac{12a^2 b'}{b^3 c^2} - \frac{12a^2 c'}{b^2 c^3} - \frac{4b^2 c'}{a^2 c^3} - \frac{4c^2 b'}{a^2 b^3} \right) + 4a^2 \left(\frac{3(b')^2}{b^4 c^2} + \frac{4b'c'}{b^3 c^3} + \frac{3(c')^2}{b^2 c^4} \right) \\
 &\quad - \frac{4}{a} \left(\frac{(c')^2}{ab^2} + \frac{(b')^2}{ac^2} + \frac{3(b')^2 c^2}{ab^4} + \frac{3b^2 (c')^2}{ac^4} - \frac{4cc'b'}{ab^3} - \frac{4bb'c'}{ac^3} \right), \tag{4.30}
 \end{aligned}$$

$$\begin{aligned}
 \partial_t K_{02} &= \Delta_g K_{02} + 2K_{02}^2 - 2K_{02} \left(\frac{(a')^2}{a^2} + \frac{(c')^2}{c^2} + \frac{2b^4 + 2(a^2 - c^2)^2}{(abc)^2} \right) \\
 &\quad + 2K_{01} \left(\frac{2a^2}{b^2 c^2} + \frac{2b^2}{a^2 c^2} - \frac{2c^2}{a^2 b^2} - \frac{a'b'}{ab} \right) + 2K_{03} \left(\frac{2b^2}{a^2 c^2} + \frac{2c^2}{a^2 b^2} - \frac{2a^2}{b^2 c^2} - \frac{b'c'}{bc} \right) \\
 &\quad + 2\frac{b'}{b} \left(-\frac{(a')^3}{a^3} - \frac{(c')^3}{c^3} + \frac{6bb'}{a^2 c^2} + \frac{4aa'}{b^2 c^2} + \frac{4cc'}{a^2 b^2} - \frac{2b'c^2}{b^3 a^2} - \frac{2b'a^2}{b^3 c^2} + \frac{4b'}{b^3} \right. \\
 &\quad \left. - \frac{12b^2 a'}{a^3 c^2} - \frac{12b^2 c'}{a^2 c^3} - \frac{4a^2 c'}{b^2 c^3} - \frac{4c^2 a'}{b^2 a^3} \right) + 4b^2 \left(\frac{3(a')^2}{a^4 c^2} + \frac{4a'c'}{a^3 c^3} + \frac{3(c')^2}{a^2 c^4} \right) \\
 &\quad - \frac{4}{b} \left(\frac{(c')^2}{ba^2} + \frac{(a')^2}{bc^2} + \frac{3(a')^2 c^2}{ba^4} + \frac{3a^2 (c')^2}{bc^4} - \frac{4cc'a'}{ba^3} - \frac{4aa'c'}{bc^3} \right), \tag{4.31}
 \end{aligned}$$

$$\begin{aligned}
 \partial_t K_{03} &= \Delta_g K_{03} + 2K_{03}^2 - 2K_{03} \left(\frac{(a')^2}{a^2} + \frac{(b')^2}{b^2} + \frac{2c^4 + 2(a^2 - b^2)^2}{(abc)^2} \right) \\
 &\quad + 2K_{01} \left(\frac{2a^2}{b^2 c^2} + \frac{2c^2}{a^2 b^2} - \frac{2b^2}{a^2 c^2} - \frac{a'c'}{ac} \right) + 2K_{02} \left(\frac{2b^2}{a^2 c^2} + \frac{2c^2}{a^2 b^2} - \frac{2a^2}{b^2 c^2} - \frac{b'c'}{bc} \right) \\
 &\quad + 2\frac{c'}{c} \left(-\frac{(a')^3}{a^3} - \frac{(b')^3}{b^3} + \frac{6cc'}{a^2 b^2} + \frac{4aa'}{b^2 c^2} + \frac{4bb'}{a^2 b^2} - \frac{2c'b^2}{c^3 a^2} - \frac{2c'a^2}{c^3 b^2} + \frac{4c'}{c^3} \right. \\
 &\quad \left. - \frac{12c^2 a'}{a^3 b^2} - \frac{12c^2 b'}{a^2 b^3} - \frac{4b^2 a'}{c^2 a^3} - \frac{4a^2 b'}{c^2 b^3} \right) + 4c^2 \left(\frac{3(a')^2}{a^4 b^2} + \frac{4a'b'}{a^3 b^3} + \frac{3(b')^2}{a^2 b^4} \right) \\
 &\quad - \frac{4}{c} \left(\frac{(b')^2}{ca^2} + \frac{(a')^2}{cb^2} + \frac{3a^2 (b')^2}{cb^4} + \frac{3(a')^2 b^2}{ca^4} - \frac{4aa'b'}{cb^3} - \frac{4bb'a'}{ca^3} \right). \tag{4.32}
 \end{aligned}$$

4.5 Maximum Principles

All of the equations computed in the previous section are of parabolic type. In order to study the behaviour of the metric components and the various other quantities computer earlier under the Ricci flow, we require some tools from parabolic theory, namely the maximum principles. Let I be a bounded open interval of $\mathbb{R}$ and define the initial-boundary value problem for a linear

parabolic equation:

$$\begin{cases} \mathcal{P}(u) = \partial_t u - u'' - f(x, t)u' - g(x, t)u = 0 & \text{on } I \times [0, T) \subset \mathbb{R} \times \mathbb{R}_+, \end{cases} \quad (4.33)$$

$$\begin{cases} u|_{I \times \{0\}}(x) = v(x), \end{cases} \quad (4.34)$$

$$\begin{cases} u|_{\partial I \times [0, T)} = w(x, t). \end{cases} \quad (4.35)$$

First, we define the parabolic boundary for the domain of the solution as follows:

Definition 4.5.1 (Parabolic domain, closure, interior, and boundary). Suppose that the solution of the parabolic problem (4.33)-(4.35) exists up to a time $T > 0$. For a given $\tau \leq T$, the parabolic domain I_τ for the equation is given by the set $I \times [0, \tau)$. The closure and interior of I_τ are defined as $\bar{I}_\tau = \bar{I} \times [0, \tau]$ and $\mathring{I}_\tau = \mathring{I} \times (0, \tau)$ respectively. Furthermore, the parabolic boundary of I_τ is given by the set $\mathcal{P}I_\tau = I \times \{0\} \cup \partial I \times [0, \tau)$. ■

We now state the parabolic maximum principles that we are going to employ in the proofs of the results in this work.

Theorem 4.5.2 (Maximum Principle I). [29] Suppose that I is a bounded and connected open domain in $\mathbb{R}$. Let $\mathcal{P}(u) = \partial_t u - u'' - f(x, t)u' - g(x, t)u$ be a parabolic operator such that $f, g : I \times \mathbb{R}_+ \rightarrow \mathbb{R}$ are some functions with $g(x, t) \leq 0$ in I_T where $T > 0$. Suppose that $u \in C^{2,1}(I_T) \cap C^{0,0}(\bar{I}_T)$. Then:

1. If u satisfies the parabolic inequality $\mathcal{P}(u) \leq 0$, then we have $\max_{I_T}(u) = \max_{\mathcal{P}I_T}(u^+)$ where $u^+ = \max(u, 0)$.
2. If u satisfies the parabolic inequality $\mathcal{P}(u) \geq 0$, then we have $\min_{I_T}(u) \geq -\min_{\mathcal{P}I_T}(u^-)$ where $u^- = -\min(u, 0)$.
3. In particular, if $\mathcal{P}(u) = 0$, then we have $\max_{I_T}(|u|) = \max_{\mathcal{P}I_T}(|u|)$ and consequently $|u| \leq \max_{\mathcal{P}I_T}(|u|)$. ■

Theorem 4.5.3 (Maximum Principle II). [93] Suppose that I is a bounded open interval in $\mathbb{R}$ and $u \in C^{2,1}(I_T) \cap C^{0,0}(\bar{I}_T)$ satisfies the parabolic inequality:

$$\partial_t u \leq u'' + f(x, t)u' + g(u, t),$$

for some functions $f : I \times \mathbb{R}_+ \rightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$. Assume that g is locally Lipschitz in the u variable and $u \leq C$ on $\mathcal{P}I_T$ where $T > 0$ is the maximal time for which the solution exists. If

v is the solution of the associated ODE:

$$\begin{cases} \frac{dv}{dt} = g(v, t), \\ v(0) = C, \end{cases}$$

then we have $u(x, t) \leq v(t)$ for all $(x, t) \in I_T$ for which v exists. $\blacksquare$

If we have non-positive data on the parabolic boundary, then we have a stronger result which does not require any sign conditions on the reaction term $g(x, t)$ as in Theorem 4.5.2. A classical result [76] simply requires this reaction term to be uniformly bounded. In fact, the uniformly bounded condition can be weakened to locally bounded since the analysis are all done locally. For our purposes, we provide the proof for a simpler case for which we have vanishing boundary data and the coefficients are allowed to blow up to infinity at the boundaries.

Theorem 4.5.4. Suppose that $I \subset \mathbb{R}$ is a bounded open interval and $u \in C^{2,1}(I_T) \cap C^{0,0}(\bar{I}_T)$ is a solution to the parabolic inequality $\mathcal{P}(u) = \partial_t u - u'' - f(x, t)u' - g(x, t)u \leq 0$ where $f, g : \mathring{I}_T \rightarrow \mathbb{R}$ are continuous functions in I_T . If $u \leq 0$ on $\mathcal{P}I_T$, then $u \leq 0$ in I_T . $\blacksquare$

Proof. If we have the strict inequality $\partial_t u < u'' + fu' + gu$, then at the first time u reaches 0 at an interior point, say at the point $(x_0, t_0) \in \mathring{I}_T$, we must have $0 \leq \partial_t u < u'' + fu' + gu \leq 0$ since $u'' \leq 0$ and $u' = u = 0$ at this point. This is a contradiction.

Now we want to prove the more general case for which we have $\partial_t u \leq u'' + fu' + gu$. Fix $\varepsilon > 0$. Consider the quantity $u_\varepsilon = u - \varepsilon e^{Kt} \leq u$ for some constant $K > 0$ which is to be determined. Note that $u_\varepsilon \leq -\varepsilon < 0$ on $\mathcal{P}I_T$. Therefore, consider the set $\Omega_\varepsilon = \{(x, t) \in I_T : u_\varepsilon \geq -\frac{\varepsilon}{2}\}$. This is a compact set and since g is continuous in I_T , it must attain its bounds in Ω_ε . Set $K = \max_{\Omega_\varepsilon} |g|$.

Suppose now that u_ε reaches 0 at some $(x_1, t_1) \in I_T$. Necessarily it must be attained in Ω_ε . Then at this point (x_1, t_1) we have the following inequality:

$$\begin{aligned} 0 &\leq \partial_t u_\varepsilon = \partial_t u - K\varepsilon e^{Kt} \\ &\leq u'' + fu' + gu - K\varepsilon e^{Kt} \\ &= u''_\varepsilon + fu'_\varepsilon + g(u_\varepsilon + \varepsilon e^{Kt}) - K\varepsilon e^{Kt} \\ &\leq (g - K)\varepsilon e^{Kt}, \end{aligned}$$

since $u''_\varepsilon \leq 0$ and $u'_\varepsilon = u'_\varepsilon = 0$. However, by the choice of K , we have $|g| < K$ in Ω_ε . Thus $0 \leq \partial_t u_\varepsilon < 0$, which gives us a contradiction. So $u_\varepsilon < 0$ in Ω_ε and hence in I_T . Taking the limit as $\varepsilon \rightarrow 0$, we have $u \leq 0$ in I_T , which proves the result. $\square$

We would also need the tensor version of the maximum principle, which was first introduced by Hamilton in his studies of Ricci flow on 3-dimensional manifolds with positive Ricci curvature:

Theorem 4.5.5 (Tensor Maximum Principle I). [42] Let $\{(M_t, g(t)) : 0 \leq t < T\}$ be a smooth 1-parameter family of compact Riemannian manifolds. Let E be a symmetric 2-tensor field on M_t such that E evolves by:

$$\partial_t E \geq \Delta E + \langle X, \nabla E \rangle + P(E, g),$$

where X is a time-dependent vector field on M_t and $P(E, g)$ is a polynomial in E formed by contracting products of E with itself using the metric g . Assume that this polynomial satisfies the null-eigenvector condition; that is, if $E(p, t)_{ij} V^j = 0$, then $P(E, g)_{ij} V^j = 0$. If $E(p, 0) \geq 0$ for all $p \in M_0$, then it remains so for all $0 \leq t < T$. ■

A modified version of this maximum principle was introduced in [6] by introducing boundaries on the manifold and a special form for the reaction term in the equation as follows:

Theorem 4.5.6 (Tensor Maximum Principle II). [6] Let $\{(M_t, \partial M_t, g(t)) : 0 \leq t < T\}$ be a smooth 1-parameter family of compact Riemannian manifolds with boundary. Let E and F be symmetric 2-tensor fields on M_t such that E evolves by:

$$\partial_t E \geq \Delta E + \langle X, \nabla E \rangle + F * E, \tag{4.36}$$

where X is a time-dependent vector field on M_t and $F * E$ denotes the symmetrised tensor product defined as $(F * E)_{ij} = F_i^k E_{kj} + E_i^k F_{kj}$. Assume that the tensor $F * E$ satisfies the null-eigenvector condition; that is, if $E(p, t)_{ij} V^j = 0$, then $(F * E)(p, t)_{ij} V^j \geq 0$. If $E(p, 0) \geq 0$ for all $p \in M_0$ and $E(p, t) \geq 0$ for all $p \in \partial M_t$ and $t \in [0, T)$, then it remains so for all $0 \leq t < T$. ■

Chapter 5

Numerics

Now we are going to have a brief glimpse at the numerical analysis of parabolic equations. This will be useful for getting a profile of the solution to the Ricci flow since it is very difficult to solve the equations analytically in general. Thus, by having graphical evidence, conjectures may be made regarding the behaviour of the Ricci flow.

In this work, we are going to use the finite difference method which was done by Rubinstein and Sinclair [81] as well as Garfinkle and Isenberg [35]. First, let us consider the following scalar parabolic problem on a finite closed interval $[\alpha, \beta]$ and time interval $[0, T]$:

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} = a(x, t) \frac{\partial^2 u}{\partial x^2} + 2b(x, t) \frac{\partial u}{\partial x} + c(x, t)u & \text{for } x \in (\alpha, \beta), \\ u(x, 0) = v(x) & \text{for } x \in [\alpha, \beta], \\ u(\alpha, t) = f(t) & \text{for } t > 0, \\ u(\beta, t) = g(t) & \text{for } t > 0, \end{array} \right. \quad \begin{array}{l} (5.1) \\ (5.2) \\ (5.3) \\ (5.4) \end{array}$$

with ellipticity condition on a , bounded coefficients b and c , as well as some functions v, f , and g . For more details on these conditions, see [29]. We want to express this problem as a finite dimensional problem which is easier to solve and can be done numerically.

5.1 Finite Difference Scheme

The finite difference method is done by discretising the space-time variables and finding the solution $u(\xi, \tau)$ at any discrete point $(x, t) = (\xi, \tau)$ by reducing the problem to a linear equation in terms of the values of u, a, b , and c at points (x, t) such that $t < \tau$.

In order to discretise the domain, we choose a fixed grid on the space $[\alpha, \beta] \times \mathbb{R}^+$. We set the number of intervals in the space coordinate as m and the number of intervals in the time

coordinate as n .

Thus, we have an array of coordinate points (x_i, t_j) for $i = 0, \dots, m$ and $j = 0, \dots, n$. Suppose that the space step and the time step are given by $\Delta x = \frac{(\beta - \alpha)}{m} = h$ and $\Delta t = k$. Since the grid is equispaced, we can find the coordinates of these point, which are given by $x_i = \alpha + ih$ for $i = 0, \dots, m$ and $t_j = jk$ for $j = 0, \dots, n$.

We denote the value of the solution u of the parabolic equation (5.1)-(5.4) at the point (x_i, t_j) by $u_i^j := u(x_i, t_j)$. From the initial condition (5.2), we have the values u_i^0 for $i = 0, \dots, m$, which are given by $u_i^0 = v(\alpha + ih)$. Furthermore, the boundary conditions (5.3)-(5.4) give us the values of u_0^j and u_m^j for $j = 0, \dots, n$ which are $u_0^j = f(jk)$ and $u_m^j = g(jk)$ respectively.

In order to determine the values of the function u at the grid points (x_i, t_j) for $0 < i < m$ and $j > 0$, we appeal to the partial differential equations (5.1). We need to approximate the derivatives of u with respect to x and t at the points x_i and t_j using difference quotients. Using a forward-in-time and central-in-space explicit Euler scheme [90], we define the following difference quotients for the derivatives at the interior points $i = 1, \dots, m - 1$ and $j = 0, \dots, n - 1$:

$$\frac{\partial u}{\partial x}(x_i, t_j) := \frac{u_{i+1}^j - u_{i-1}^j}{2h}, \quad (5.5)$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) := \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}, \quad (5.6)$$

$$\frac{\partial u}{\partial t}(x_i, t_j) := \frac{u_i^{j+1} - u_i^j}{k}. \quad (5.7)$$

Figure 5.1 summarises the whole discussion on the grid size and difference quotients of the quantities that we are considering. Note that the relative grid sizes in the x and t directions are not drawn to scale.

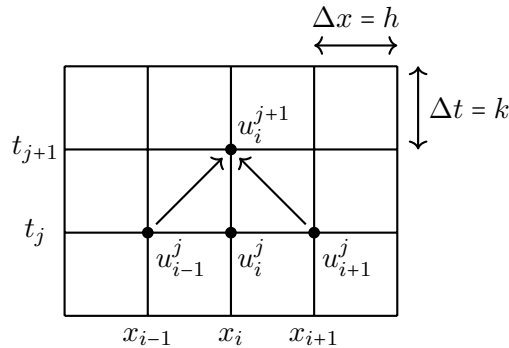


Figure 5.1: Forward-in-time and central-in-space Euler scheme.

By substituting the approximations (5.5)-(5.7) into equations (5.1)-(5.4), we derive the fol-

lowing finite difference problem:

$$\begin{cases} h^2 w_i^{j+1} = k u_{i+1}^j (a_i^j + b_i^j h) + w_i^j (h^2 + k h^2 c_i^j - 2k a_i^j) + k u_{i-1}^j (a_i^j - b_i^j h), \end{cases} \quad (5.8)$$

$$\begin{cases} u_i^0 = v(\alpha + i h) \end{cases} \quad \text{for } i = 0, \dots, m, \quad (5.9)$$

$$\begin{cases} w_0^j = f(jk) \end{cases} \quad \text{for } j = 0, \dots, n, \quad (5.10)$$

$$\begin{cases} u_m^j = g(jk) \end{cases} \quad \text{for } j = 0, \dots, n, \quad (5.11)$$

where equation (5.8) holds for $i = 1, \dots, m-1$ and $j = 0, \dots, n-1$ with $a_i^j = a(x_i, t_j)$, $b_i^j = b(x_i, t_j)$, and $c_i^j = c(x_i, t_j)$. For a general linear heat equation with constant coefficients depending only on x and t , the finite difference scheme is stable in the sense of Courant-Friedrichs-Lewy provided that:

$$\sup_{x,t} a(x, t) \frac{k}{h^2} < \frac{1}{2}.$$

Here, stability means that the approximation errors accrued would not be amplified as the computation progresses. For more details on the stability condition of the numerics for parabolic equation see [80].

5.2 DeTurck-Ricci Flow Formulation

We now focus on the formulation of our problem. Instead of implementing the numerical scheme on the Ricci flow equation, we are going to run our numerical simulation on the DeTurck-Ricci flow in equation (2.7). This formulation allows us to express the Ricci flow as a strongly parabolic equation, thus all properties of parabolic partial differential equations as well as numerics hold true for the Ricci flow. This is important in our numerical study because in order to have stability in the numerical simulation, it is important to consider the strongly parabolic problem rather than the system (1.1)-(1.2).

Garfinkle and Isenberg [35] considered the strongly parabolic version of the Ricci flow in their numerical study using the finite difference method and managed to get good numerical profiles for the solutions to the Ricci flow. Some of their profiles are comparable to the Bryant soliton profile. In fact, the DeTurck-Ricci flow has been investigated in depth numerically and many sophisticated models have been developed. For an example, see the work by Fritz [32] for a finite element method approach.

Remark 5.2.1. The method introduced by Fritz involves embedding the initial manifold (M, g_0) in some fixed ambient Euclidean space $\mathbb{R}^n$ for some large enough n using the Nash

Embedding Theorem and extending the metric g_0 on the manifold M to a δ -neighbourhood of the embedding. This is done in order to discretise the manifold into simplices in $\mathbb{R}^n$.

Whilst this allows us to model manifolds with more complicated symmetry (or asymmetry), it is not very economical to implement for manifolds of high dimensions since the Nash Embedding Theorem allows one to embed any m -dimensional Riemannian manifold (M, g) isometrically into a Euclidean space $\mathbb{R}^n$ for some n such that $n \leq \frac{1}{2}m(3m + 11)$ for compact M and $n \leq \frac{1}{2}m(m + 1)(3m + 11)$ for non-compact M [68]. Therefore, in order to utilise the method introduced by Fritz, one has to *a priori* embed the initial Riemannian manifold (M, g_0) in the Euclidean space with the largest such dimension, which might be far from necessary.

For example, consider the 2-torus $S^1 \times S^1$ with the metric g_0 inherited from the standard embedding in $\mathbb{R}^3$. Under the Ricci flow, the solution exists for all time and this manifold converges to the flat torus which can be isometrically embedded in $\mathbb{R}^4$. Therefore, we just need to embed the 2-torus $(S^1 \times S^1, g_0)$ in $\mathbb{R}^4$ in order for this flow to make sense for all time, even though the *a priori* Euclidean space that would guarantee isometric embedding for all time from Nash Embedding Theorem is $\mathbb{R}^{17}$, which would not be economical from a numerical perspective. For general cases, it is difficult to determine the optimal dimension of the ambient Euclidean space n required for this method beforehand. ■

On the other hand, it does not stop one from working with the original Ricci flow equation as Rubinstein and Sinclair did in their study [81]. In their paper, they used the finite difference method on the weakly parabolic Ricci flow equation. However, as noted by the authors, this is numerically unstable and care must be taken with the results as they are only qualitatively accurate. In other words, they are only useful for graphical visualisation purposes.

In the same paper, a more accurate result is obtained by them using a method of asymptotic scaling near the singularity. Here, they expanded the metric quantities that they considered into a power series expansion up to the tenth order and constructed a system of partial differential equations which then can be solved numerically.

5.2.1 Numerics for $S^1 \times S^3$

Recall that the DeTurck-Ricci flow is given by the equation:

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t)) + \mathcal{L}_{X(t)}g(t), \\ g(0) = g_0, \end{cases}$$

where $\mathcal{L}_{X(t)}$ is the Lie derivative in the direction of $X(t)$. The vector field $X(t)$ is chosen such that its related 1-form is given by the covector:

$$X^\flat = g_{ij}g^{kl}(\Gamma_{kl}^j - \bar{\Gamma}_{kl}^j)dx^i, \quad (5.12)$$

where $\bar{\Gamma}_{kl}^j$ is the Christoffel symbol of some fixed background metric $\bar{g}$ on the manifold. For simplicity, we choose the background metric to be:

$$\bar{g} = dz^2 + w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3,$$

for $z \in S^1$. We wish to run our numerical simulation in this coordinate rather than the arclength coordinate $s(z, t)$ since the latter varies with time. As a result, the correction term $\mathcal{L}_{X(t)}g(t)$ is necessary in order to formulate a parabolic system.

We recall that in Proposition 4.2.2 we have calculated the frame symbols $\Sigma_{\beta\gamma}^\alpha$ in the frame $\{\partial_z = E_0, E_1, E_2, E_3\}$. Even though these are not the same as Christoffel symbols, we shall see later that by using the frame symbols in place of Christoffel symbols in the DeTurck covector field (5.12) the resulting formulation is still parabolic.

Based on our ansatz metric in (4.1) (in particular, $g_{\alpha\beta} = 0$ for $\alpha \neq \beta$), we deduce that $X^\flat = g_{\alpha\alpha}g^{\beta\beta}(\Sigma_{\beta\beta}^\alpha - \bar{\Sigma}_{\beta\beta}^\alpha)\omega^\alpha$ would only depend on the frame symbols of the form $\Sigma_{\beta\beta}^\alpha$ and $\bar{\Sigma}_{\beta\beta}^\alpha$ for $\alpha, \beta \in \{0, 1, 2, 3\}$. Note that the metric components of $\bar{g}$ are all independent of z . Therefore, the only non-vanishing frame symbols are of the form $\bar{\Sigma}_{ii}^j$ for $i, j \in \{1, 2, 3\}$.

Using the formula $\bar{\Sigma}_{ij}^k = \epsilon_{ijk}\bar{g}^{kk}(\bar{g}_{ii} - \bar{g}_{jj} - \bar{g}_{kk})$ for the metric $\bar{g}$, we compute $\bar{\Sigma}_{ii}^j = 0$ for all $i, j \in \{1, 2, 3\}$. So, all the frame symbols for $\bar{g}$ vanish and thus our DeTurck covector $X^\flat$ defined in (5.12) is given by:

$$X^\flat = (\Sigma_{00}^0 + g_{00}g^{ii}(\Sigma_{ii}^0 - \bar{\Sigma}_{ii}^0))dz = \left(\frac{\dot{\phi}}{\phi} - \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right)\right)dz.$$

Furthermore, by using the formula $(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$ and equating components in the DeTurck-Ricci flow, we acquire the following system of parabolic equations:

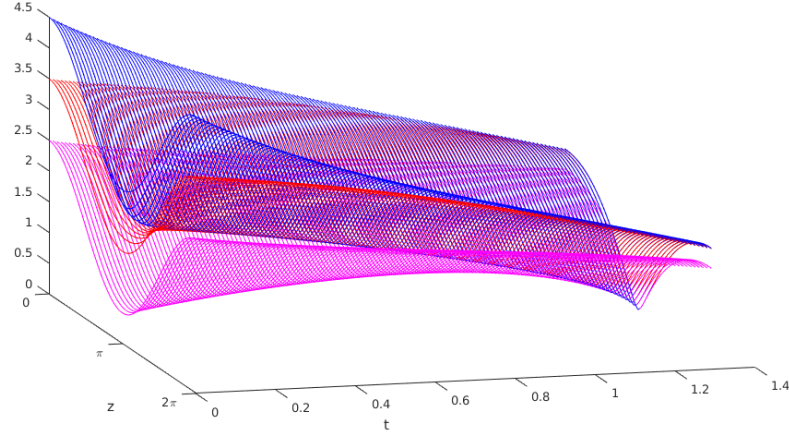
$$\begin{cases} \partial_t \phi = \frac{\ddot{\phi}}{\phi^2} - \frac{\dot{\phi}^2}{\phi^3} + \frac{\dot{a}^2}{a^2\phi} + \frac{\dot{b}^2}{b^2\phi} + \frac{\dot{c}^2}{c^2\phi} - \frac{\dot{a}\dot{\phi}}{a\phi^2} - \frac{\dot{b}\dot{\phi}}{b\phi^2} - \frac{\dot{c}\dot{\phi}}{c\phi^2}, \end{cases} \quad (5.13)$$

$$\begin{cases} \partial_t a = \frac{\ddot{a}}{\phi^2} - \frac{\dot{a}\dot{\phi}}{\phi^3} + \frac{\dot{a}}{\phi^2} \left(\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) - 2a \left(\frac{a^4 - (b^2 - c^2)^2}{(abc)^2}\right), \end{cases} \quad (5.14)$$

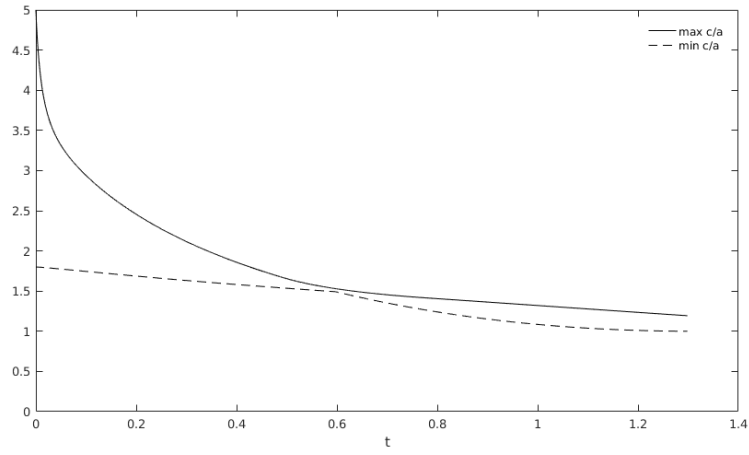
$$\begin{cases} \partial_t b = \frac{\ddot{b}}{\phi^2} - \frac{\dot{b}\dot{\phi}}{\phi^3} + \frac{\dot{b}}{\phi^2} \left(\frac{\dot{a}}{a} + \frac{\dot{c}}{c}\right) - 2b \left(\frac{b^4 - (a^2 - c^2)^2}{(abc)^2}\right), \end{cases} \quad (5.15)$$

$$\begin{cases} \partial_t c = \frac{\ddot{c}}{\phi^2} - \frac{\dot{c}\dot{\phi}}{\phi^3} + \frac{\dot{c}}{\phi^2} \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right) - 2c \left(\frac{c^4 - (a^2 - b^2)^2}{(abc)^2}\right). \end{cases} \quad (5.16)$$

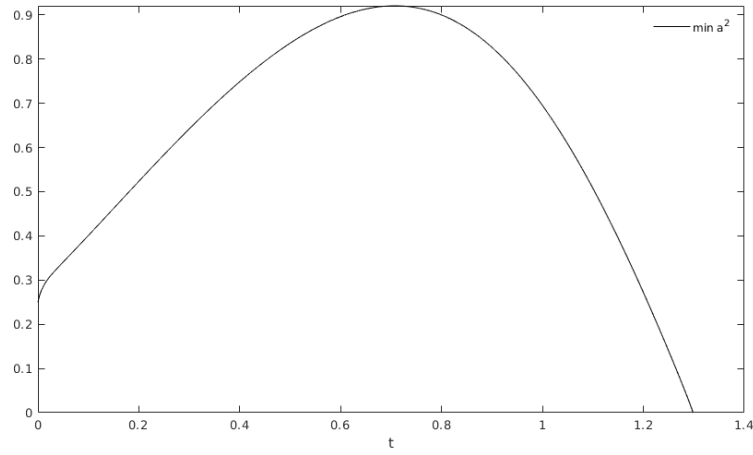
By using the finite difference scheme defined by the definitions (5.5)-(5.7) to numerically simulate the evolution profiles of the system (5.13)-(5.16), we plot the Ricci flow evolution of the quantities a , b , and c for some choice of initial data ϕ_0, a_0, b_0 , and c_0 satisfying the ordering $a_0 \leq b_0 \leq c_0$. We also plot the quantities $\max\left(\frac{c}{a}\right)$ and $\min\left(\frac{c}{a}\right)$ which will be studied analytically later. Note that if $\max\left(\frac{c}{a}\right) = \min\left(\frac{c}{a}\right)$, the Bianchi IX metric becomes uniaxial. The space-step used in the simulations was $h = \frac{2\pi}{200}$ and the time-step used was $k = 0.0000001$. Therefore, the Courant-Friedrichs-Lewy stability condition is satisfied in each of the simulations.



(a) Evolution of a (purple), b (red), and c (blue) for initial data $\phi_0 = 1$, $a_0 = \cos(z) + 1.5$, $b_0 = \cos(z) + 2.5$, and $c_0 = \cos(z) + 3.5$.

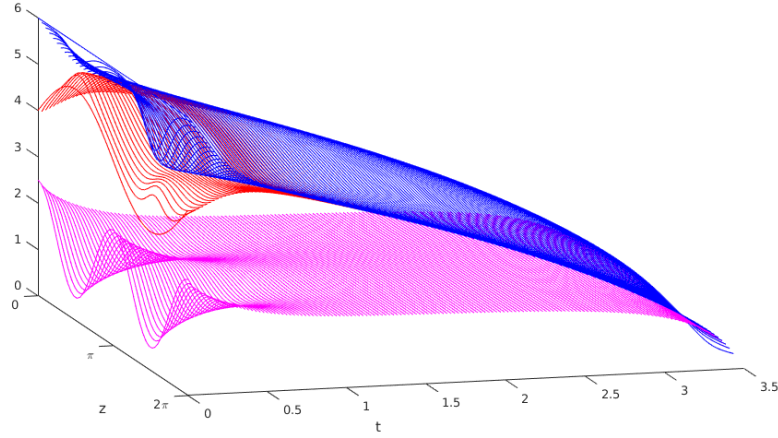


(b) Evolution of maximum and minimum for the ratio $\frac{c}{a}$.

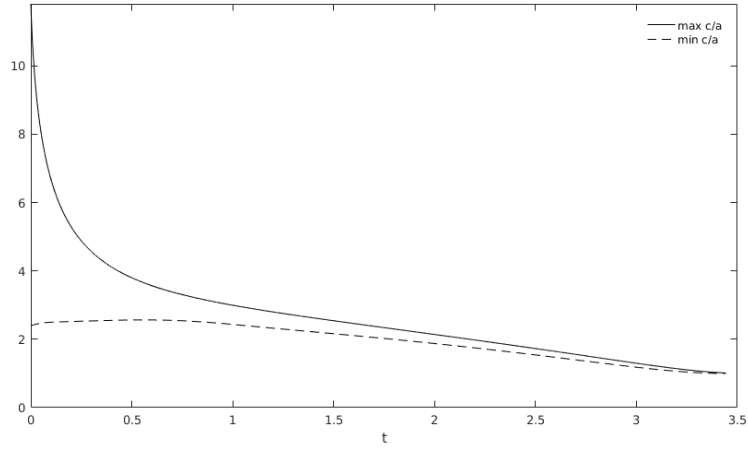


(c) Evolution of minimum of a^2 .

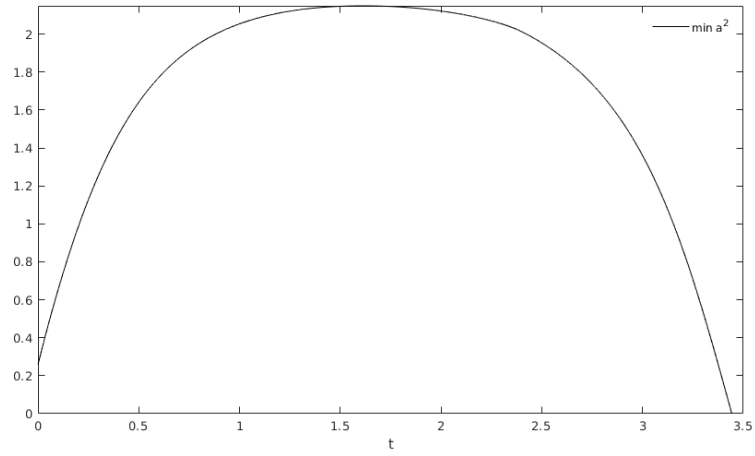
Figure 5.2: Ricci flow simulation for $S^1 \times S^3$.



(a) Evolution of a (purple), b (red), and c (blue) for initial data $\phi_0 = 1$, $a_0 = \cos(2z) + 1.5$, $b_0 = \sin(z) + 4$, and $c_0 = 6$.

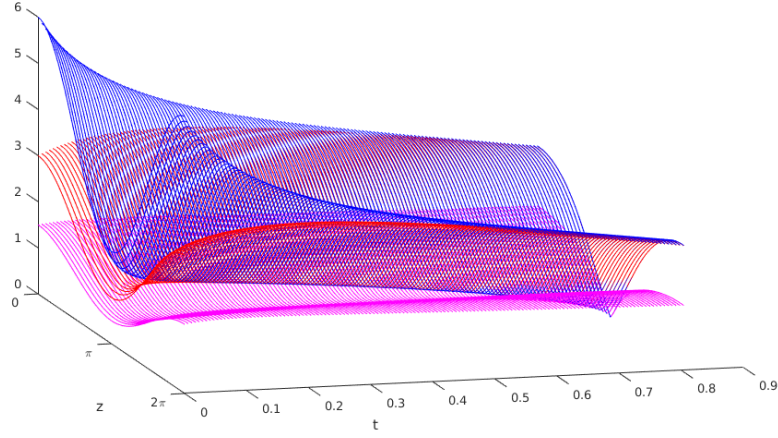


(b) Evolution of maximum and minimum for the ratio $\frac{c}{a}$.

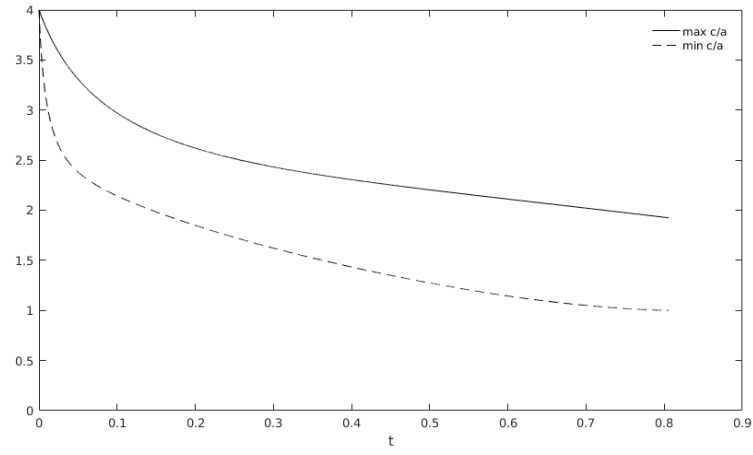


(c) Evolution of minimum of a^2 .

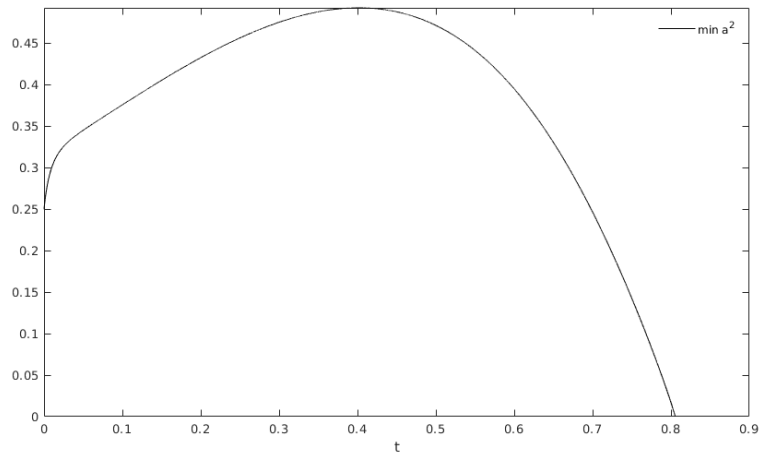
Figure 5.3: Ricci flow simulation for $S^1 \times S^3$.



(a) Evolution of a (purple), b (red), and c (blue) for initial data $\phi_0 = 1$, $a_0 = \frac{1}{2} \cos(z) + 1$, $b_0 = \cos(z) + 2$, and $c_0 = 2 \cos(z) + 4$.



(b) Evolution of maximum and minimum for the ratio $\frac{c}{a}$.



(c) Evolution of minimum of a^2 .

Figure 5.4: Ricci flow simulation for $S^1 \times S^3$.

From the plots in Figures 5.2-5.4, we can see that the ordering $0 < a \leq b \leq c$ is preserved for all time under the Ricci flow. This has been noted and proven in [54], which forms a basis of the analysis. Furthermore, we see that the ratios of $\max(\frac{c}{a})$ and $\min(\frac{c}{a})$ are continuous functions of t and they decay with time.

We note that the maximum decays very rapidly near $t = 0$, which echoes the analysis on the homogeneous manifold that we have seen in Chapter 3. Furthermore, we note that these ratios both tend to 1 as $t \rightarrow T$, which implies that the fibres get rounder as time progresses. However, for the plots in Figures 5.2 and 5.3, the maximum of the ratio is strictly greater than the minimum of the ratios at the singular time. This implies that only some of the fibres shrink to the round point before the others get round. Therefore, in these cases, the singularity does not occur uniformly over the manifold.

Finally, we note that the plots for the minimum of a^2 look like concave functions of $t \in [0, T]$ which finally vanishes at $t = T$. This echoes the result in Proposition 3.3.4. Therefore, it may be possible to bound the quantity $\min_z(a^2)$ from above and below by two linear functions; that is, $D(T - t) \leq \min_z(a^2) \leq C(T - t)$ for some constants $C, D > 0$. This was the crucial step in the work of [54] in order to deduce Type I singularity.

5.2.2 Numerics for S^4

For the S^4 case, we need to impose closure conditions on our fixed background metric $\bar{g}$. This can be done by multiplying the fibre metric with the cosine function and rescaling the z variable appropriately. Thus, we choose:

$$\bar{g} = \left(\frac{\pi}{2}\right)^2 dz^2 + \cos^2\left(\frac{\pi}{2}z\right)(\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

for $z \in (-1, 1)$, which is the standard round metric on S^4 .

Again, based on our ansatz metric in (4.1), we deduce that $X^b = g_{\alpha\alpha}g^{\beta\beta}(\Sigma_{\beta\beta}^\alpha - \bar{\Sigma}_{\beta\beta}^\alpha)\omega^\alpha$ would only depend on the frame symbols of the form $\Sigma_{\beta\beta}^\alpha$ and $\bar{\Sigma}_{\beta\beta}^\alpha$ for $\alpha, \beta \in \{0, 1, 2, 3\}$. Using this formula for the metric $\bar{g}$, we calculate $\bar{\Sigma}_{00}^0 = \bar{\Sigma}_{\beta\beta}^i = 0$ and $\bar{\Sigma}_{ii}^0 = \frac{1}{\pi} \sin(\pi z)$ for all $i \in \{1, 2, 3\}$ and $\beta \in \{0, 1, 2, 3\}$. Since $\Sigma_{\beta\beta}^i$ and $\bar{\Sigma}_{\beta\beta}^i$ vanish for all such i and β , the only non-zero term in X^b is the component of dz . Thus, X^b is given by:

$$X^b = (\Sigma_{00}^0 + g_{00}g^{ii}(\Sigma_{ii}^0 - \bar{\Sigma}_{ii}^0))dz = \left(\frac{\dot{\phi}}{\phi} - \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) - \frac{1}{\pi} \sin(\pi z)\phi^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\right)dz.$$

Furthermore, using the formula $(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$ and equating components in the

DeTurck-Ricci flow, we have the parabolic system:

$$\partial_t \phi = \frac{\ddot{\phi}}{\phi^2} - \frac{\dot{\phi}^2}{\phi^3} + \frac{\dot{a}^2}{a^2 \phi} + \frac{\dot{b}^2}{b^2 \phi} + \frac{\dot{c}^2}{c^2 \phi} - \frac{\dot{a}\dot{\phi}}{a\phi^2} - \frac{\dot{b}\dot{\phi}}{b\phi^2} - \frac{\dot{c}\dot{\phi}}{c\phi^2} - \cos(\pi z) \phi \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) + \frac{2}{\pi} \sin(\pi z) \left(\frac{\dot{a}\phi}{a^3} + \frac{\dot{b}\phi}{b^3} + \frac{\dot{c}\phi}{c^3} - \frac{\dot{\phi}}{a^2} - \frac{\dot{\phi}}{b^2} - \frac{\dot{\phi}}{c^2} \right), \quad (5.17)$$

$$\partial_t a = \frac{\ddot{a}}{\phi^2} - \frac{\dot{a}\dot{\phi}}{\phi^3} + \frac{\dot{a}}{\phi^2} \left(\frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right) - 2a \left(\frac{a^4 - (b^2 - c^2)^2}{(abc)^2} \right), \quad (5.18)$$

$$\partial_t b = \frac{\ddot{b}}{\phi^2} - \frac{\dot{b}\dot{\phi}}{\phi^3} + \frac{\dot{b}}{\phi^2} \left(\frac{\dot{a}}{a} + \frac{\dot{c}}{c} \right) - 2b \left(\frac{b^4 - (a^2 - c^2)^2}{(abc)^2} \right), \quad (5.19)$$

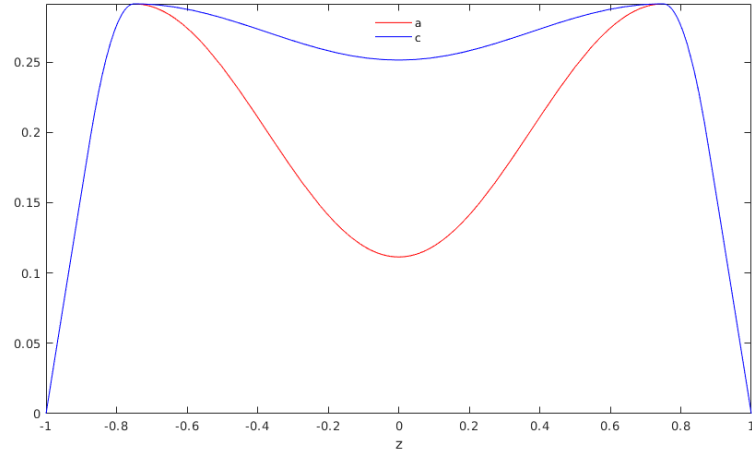
$$\partial_t c = \frac{\ddot{c}}{\phi^2} - \frac{\dot{c}\dot{\phi}}{\phi^3} + \frac{\dot{c}}{\phi^2} \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) - 2c \left(\frac{c^4 - (a^2 - b^2)^2}{(abc)^2} \right). \quad (5.20)$$

As for the initial data for the system (5.17)-(5.20), we construct a corseted sphere by defining piecewise smooth functions with continuous first derivative by interpolating a quadratic polynomial in between two cosine functions. This interpolation is inspired by the work by Buzano, Haslhofer, and Hershkovits [15] on the mean curvature flow.

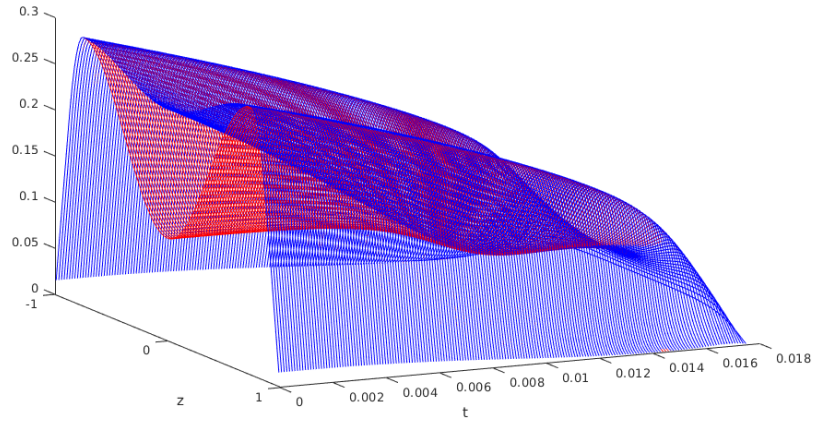
From now on, for simplicity we consider the biaxial metric by setting $b \equiv c$. We start with the usual round metric to ensure smoothness at the poles. That is, we start with $\phi_0(z) = \frac{\pi}{2}$ and $a(z) = c(z) = \cos\left(\frac{\pi z}{2}\right)$ for $z \in [-1, 1]$. Then, we excise the cosine function from $z \in \left[-\frac{7}{8}, \frac{7}{8}\right]$ which is to be filled in with the neck. For $z \in \left[-\frac{3}{4}, \frac{3}{4}\right]$, we want to define a negative cosine function with period $\frac{3}{2}$ of the form $-\lambda \cos\left(\frac{4\pi z}{3}\right) + C$ for some constants $\lambda, C > 0$ so that its minimum at $z = 0$ is strictly positive and its derivatives are 0 at the endpoints.

For $z \in \left[\frac{3}{4}, \frac{7}{8}\right]$, we want to interpolate the two cosine functions by a smooth quadratic curve with derivative 0 at $z = \frac{3}{4}$ and derivative $-\frac{\pi}{2} \sin\left(\frac{7\pi}{16}\right)$ at $z = \frac{7}{8}$. We interpolate a straight line between the two points $\left(\frac{3}{4}, 0\right)$ and $\left(\frac{7}{8}, -\frac{\pi}{2} \sin\left(\frac{7\pi}{16}\right)\right)$ and integrate this line to give us a quadratic polynomial with the desired property. This polynomial is approximately given by: $p(z) = -6.1624z^2 + 9.24368z - 3.1750$ for $z \in \left[\frac{3}{4}, \frac{7}{8}\right]$. As for the interval $\left[-\frac{7}{8}, -\frac{3}{4}\right]$, we fit the polynomial $q(z) = p(-z)$ in this interval. Finally, we are able to define in the cosine function in the middle of the interval. The amplitude λ must be at most $\frac{p(\frac{3}{4})}{2}$ ($\lambda \in (0, 0.1457)$ approximately) and the vertical shift is $C = p\left(\frac{3}{4}\right) - \lambda$ so that it matches the interpolating quadratics. So, we define $r_\lambda(z) = -\lambda \cos\left(\frac{4\pi z}{3}\right) + 0.2914 - \lambda$. This is a one parameter family of cosine functions parametrised by $\lambda \in (0, 0.1457)$, which defines the degree of corseting for our sphere.

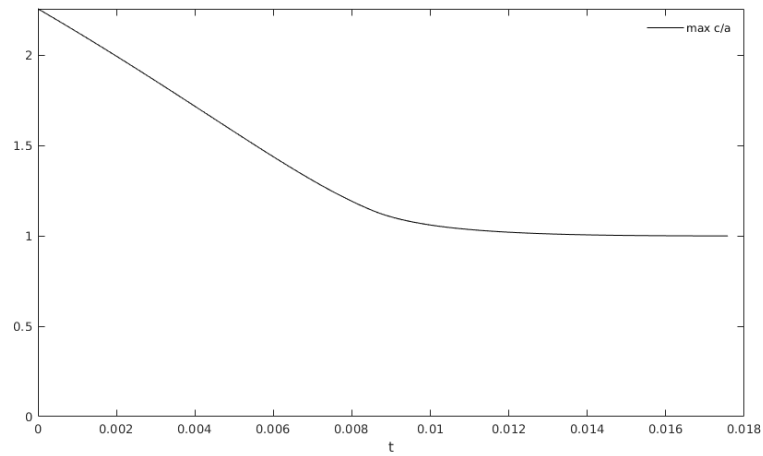
Several plots of the quantities a and c are obtained for some choices for λ and μ . The DeTurck-Ricci flow from these initial data are then plotted in Figures 5.5 to 5.8. The space-step used in the simulations was $h = \frac{2}{99}$ and the time-step used was $k = 0.0000001$. Therefore, the Courant-Friedrichs-Lewy stability condition is satisfied in each of the simulations.



(a) Initial data a_λ (red) and c_μ (blue) for $\lambda = 0.09$ and $\mu = 0.02$.

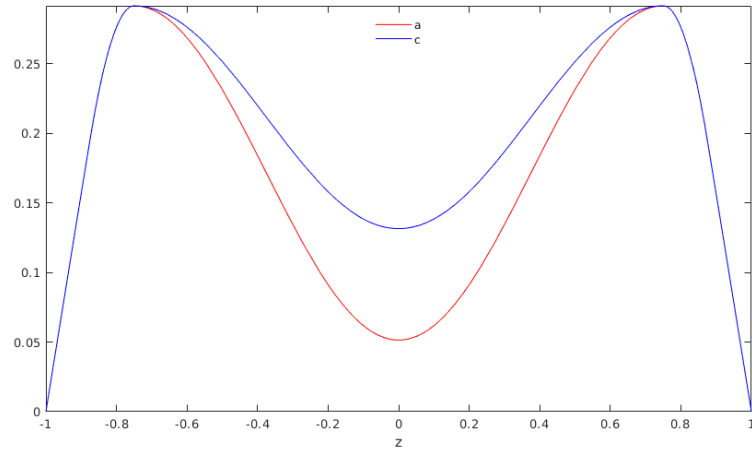


(b) Evolution of a (red) and c (blue) for initial data above.

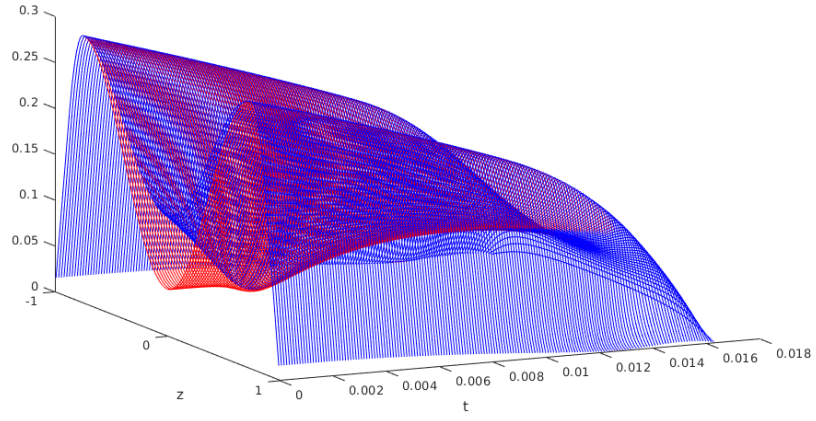


(c) Evolution of maximum for the ratio $\frac{c}{a}$.

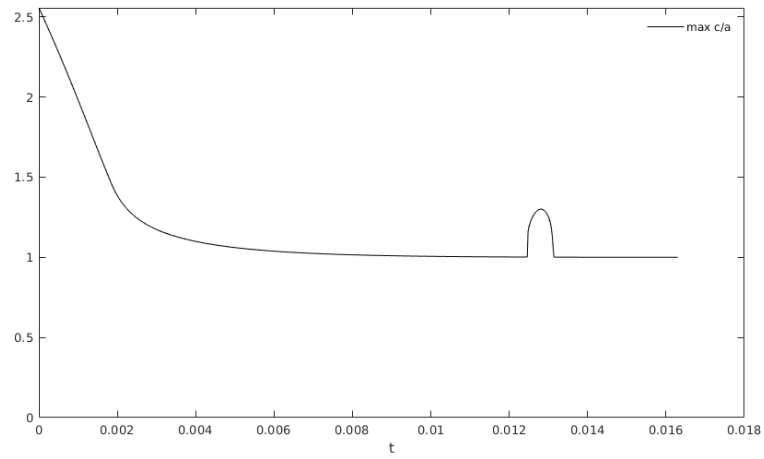
Figure 5.5: Ricci flow simulation for S^4 .



(a) Initial data a_λ (red) and c_μ (blue) for $\lambda = 0.12$ and $\mu = 0.08$.

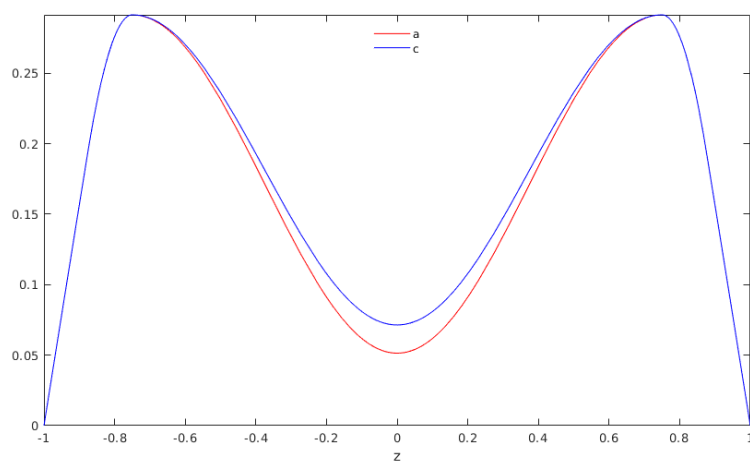


(b) Evolution of a (red) and c (blue) for initial data above.

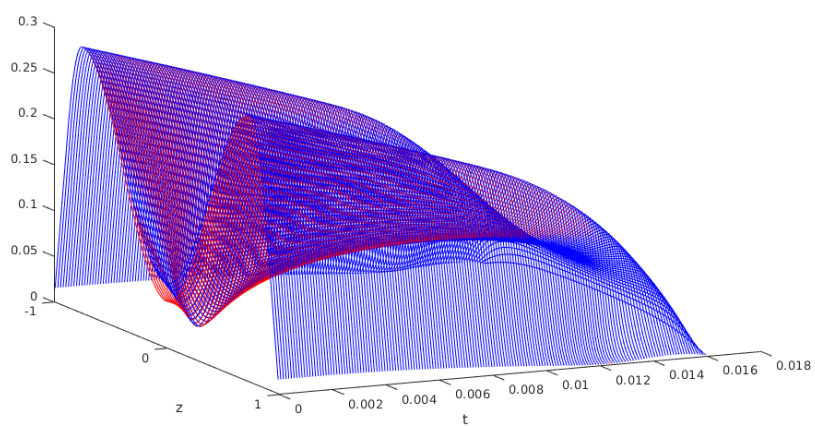


(c) Evolution of maximum for the ratio $\frac{c}{a}$.

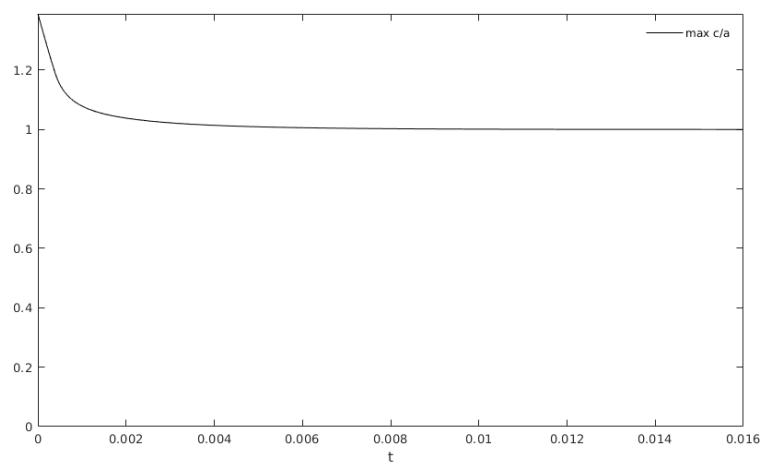
Figure 5.6: Ricci flow simulation for S^4 .



(a) Initial data a_λ (red) and c_μ (blue) for $\lambda = 0.12$ and $\mu = 0.11$.

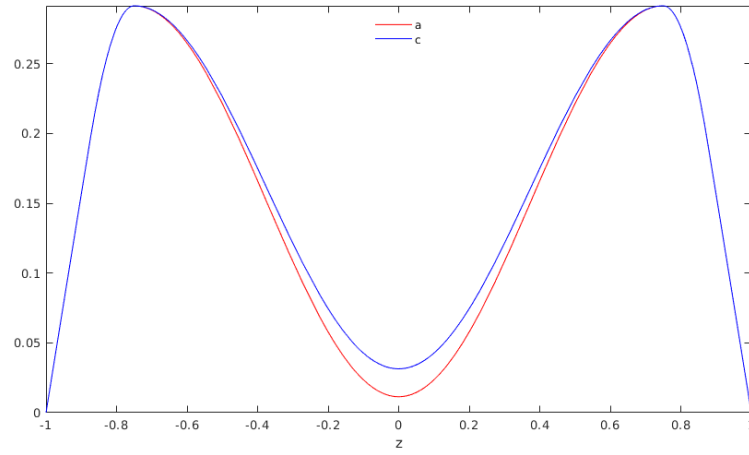


(b) Evolution of a (red) and c (blue) for initial data above.

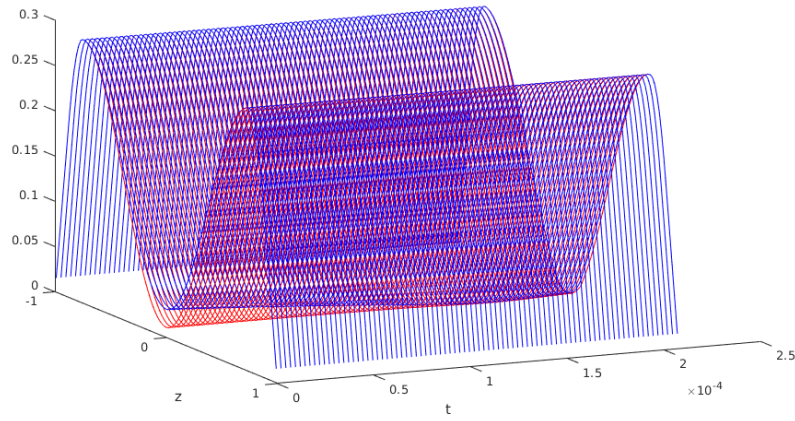


(c) Evolution of maximum for the ratio $\frac{c}{a}$.

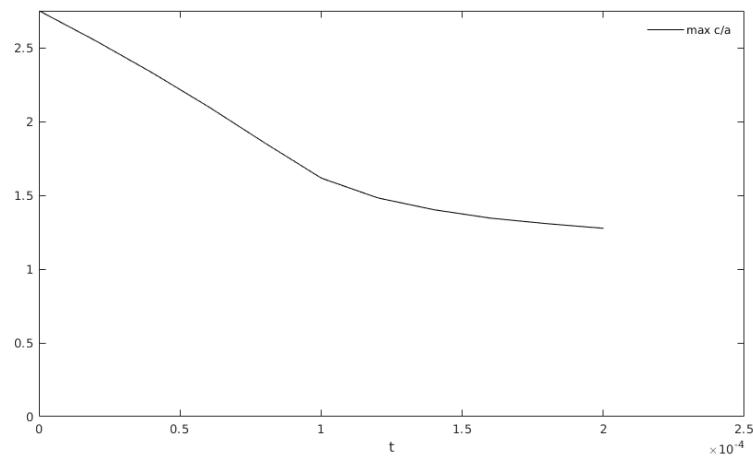
Figure 5.7: Ricci flow simulation for S^4 .



(a) Initial data a_λ (red) and c_μ (blue) for $\lambda = 0.14$ and $\mu = 0.13$



(b) Evolution of a (red) and c (blue) for initial data above.



(c) Evolution of maximum for the ratio $\frac{c}{a}$.

Figure 5.8: Ricci flow simulation for S^4 .

As in the previous case, from the plots above we can see that, given the ordering $0 < a_0(z) \leq b_0(z) \leq c_0(z)$ for $z \in [0, 2\pi)$ at the initial time, the ordering of the metric components is preserved under the Ricci flow. Furthermore, we see that the ratio of $\max\left(\frac{c}{a}\right)$ are continuous functions of t and they approach 1 as t increases. The only exception is Figure 5.6 where a small bump occurs around the time $t = 0.013$. It is unclear whether this is an anomaly in the numerical simulation or a singularity. From the plot of the evolution of the quantities, it seems like there is a pinching of the quantities at the poles of the manifold at $z = \pm 1$.

We observe that this ratio converges to 1 as $t \rightarrow T$ in Figures 5.5, 5.6, and 5.7, implying that the fibres get rounder as time progresses. We note from the main plots that the quantities a and c in these plots converge to 0 over the whole interval. This implies that the manifolds shrink to the round point at the singular time.

However, for the plots in Figure 5.8, the maximum of the ratio is strictly greater than 1 at the singular time. This implies that only some of the fibres shrink to the round point before the others get round. By inspecting the plot, we can see that only one of the fibres shrink to a point at the singular time. Therefore, this produces a neckpinch.

Remark 5.2.2. We can change the length of the neck of these manifolds by varying the metric component $\phi(z)$ since this quantity corresponds to the infinitesimal lengths transverse to the foliating S^3 . We note that intuitively, as noted by Hamilton [44], the longer and thinner the neck, the more probable it is that a neckpinch will occur. ■

Chapter 6

Analysis on $S^1 \times S^3$

We now proceed towards generalising the work done by Isenberg, Knopf, and Šešum in [54]. In their work, they studied the formation of a Type I singularity for the manifold $S^1 \times S^3$ equipped with the biaxial Bianchi IX metric:

$$g = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + b(z)^2 \omega^2 \otimes \omega^2 + c(z)^2 \omega^3 \otimes \omega^3,$$

where $b(z) \equiv c(z)$.

Here we shall extend their result by considering the triaxial metric in which $b(z) \leq c(z)$. In this case, the Riemannian manifold possesses an $SU(2)$ symmetry since the 1-forms ω^i are invariant by definition. We first prove that initial ordering of the metric components is preserved in the course of the evolution (as seen in the numerical simulations in Chapter 5) and some eccentricity bounds, which were also shown in [54]. This shall be the first main result of this chapter. We shall also estimate the singular time and the decay rate of the quantity $\max(\frac{c}{a})$ which were also observed in the numerical simulations.

An important result in this chapter will be the upper and lower bounds of the smallest metric component a close to the singular time. We shall show that the decay rate for the spatial minima of this metric component, under certain sufficiency conditions on the scalar curvature at the initial time, behaves as $\sqrt{T-t}$.

Finally, we want to estimate the curvature behaviour close to the singular time. This is done by bounding the first and second derivatives of the metric components and, by the explicit formula for the curvatures computed earlier, this is sufficient to bound all the curvatures. Coupled with the estimates on the smallest component of the metric, we can show that this singularity is in fact of Type I. Thus, the theorems that we wish to prove in this chapter are:

Theorem 6.0.1. Suppose that we run a Ricci flow on the manifold $(S^1 \times S^3, g_0)$ where the initial metric g_0 is of the form:

$$g_0 = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + b(z)^2 \omega^2 \otimes \omega^2 + c(z)^2 \omega^3 \otimes \omega^3,$$

for $z \in S^1 = [0, 2\pi)$ and $0 \leq a(z) \leq b(z) \leq c(z)$. Then the Ricci flow preserves the form of the metric, the ordering of the metric components, and there exist constants $\tilde{C}_0, \bar{C}_0 > 0$ depending on the data on the initial data such that for all $(z, t) \in S_T^1 = S^1 \times [0, T)$, we have:

$$|b(z, t) - c(z, t)| \leq \tilde{C}_0 \min(b(z, t), c(z, t)),$$

$$|a(z, t) - c(z, t)| \leq \bar{C}_0 \min(a(z, t), c(z, t)). \quad \blacksquare$$

Theorem 6.0.2. There exist open sets of warped metrics on $S^1 \times S^3$ of the form:

$$g = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + b(z)^2 \omega^2 \otimes \omega^2 + c(z)^2 \omega^3 \otimes \omega^3,$$

for $z \in S^1 = [0, 2\pi)$, satisfying:

1. $0 < a \leq b \leq c < 2a$ at $t = 0$,
2. $\min_{s, t=0}(\mathbf{S}) \geq 0$ at $t = 0$,
3. there exists a $T < \infty$ such that $\limsup_{t \rightarrow T} \max_s |\text{Ric}(s, t)| = \infty$,

such that all solutions of the Ricci flow originating in these sets develop local neckpinch singularities at some $T < \infty$. Each such solution has the properties that:

1. the ordering $a \leq b \leq c$ is preserved,
2. the singularity is of Type I with $|\text{Rm}| \leq \frac{C}{\min_s(a^2)}$ and $D\sqrt{T-t} \leq \min_s(a) \leq C\sqrt{T-t}$ for some constants $C, D > 0$. ■

These theorems are generalisations of Theorems 1.1.2 and 1.1.3, which were proven in [54]. Here, we assume that the metric quantity b is not identically equal to c . The main difficulty here is that we have to control three quantities instead of just two. In some of the equations that we are going to study, these quantities and their derivatives might be coupled. These will be dealt by utilising Young's inequality to separate the various terms. Furthermore, in order to deal with the resulting inequalities, we have to impose the extra condition such that $a \leq c < 2a$ at initial time, which was not present in Theorem 1.1.3.

6.1 Analysis of the Metric Components

Recall that our ansatz metric is of the form:

$$g = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + b(z)^2 \omega^2 \otimes \omega^2 + c(z)^2 \omega^3 \otimes \omega^3,$$

where $z \in S^1 = [0, 2\pi)$. To ensure parabolicity, we choose the gauge $s(z, t)$ by the change of variables $s(z, t) = \int_0^z \phi(w, t) dw$. We denote the derivatives with respect to the new gauge $s(z, t)$ by $\dots'$.

We have seen that the Ricci flow preserves the symmetry of the metric. We now prove that the initial ordering of the metric components is also preserved under the Ricci flow.

Lemma 6.1.1. [54] If $0 < a(z, 0) \leq b(z, 0) \leq c(z, 0)$, then under the Ricci flow this ordering is preserved for all $t > 0$ for which the flow exists. $\blacksquare$

Proof. Consider the parabolic PDEs satisfied by the quantities $\frac{\xi}{a} = \frac{a-b}{a}$ and $\frac{\zeta}{b} = \frac{b-c}{b}$. They evolve according to the equations (4.18) and (4.20). Thus, the maximum principles hold here where the parabolic boundary $\mathcal{P}S_T^1 = S^1 \times \{0\}$ is simply the base S^1 at initial time. Note that $a \leq b \leq c$ initially. By applying Theorem 4.5.4 to the equation for $\frac{\xi}{a}$ and Theorem 4.5.2 to the equation for $\frac{\zeta}{b}$, we deduce that $\frac{\xi}{a} \leq 0$ and $\frac{\zeta}{b} \leq 0$ in S_T^1 and hence we have $0 < a \leq b \leq c$ for all $(z, t) \in S_T^1$. $\square$

Furthermore, by using Theorem 4.5.2 on equations (4.20)-(4.23), in the same vein as the proof by Isenberg, Knopf, and Šešum, we can deduce the eccentricity result in Theorem 6.0.1.

Proof of Theorem 6.0.1. We consider equations (4.20) and (4.21). Since $0 < a \leq c$ for all time $t \in [0, T)$, we can deduce the following inequalities:

$$\begin{aligned} -4 \left(\frac{1}{a^2} - \frac{1}{c^2} + \frac{b^2}{a^2 c^2} + \frac{b^2 + c^2 - a^2}{a^2 b c} \right) &= -4(b^2 + c^2 - a^2) \left(\frac{1}{a^2 c^2} + \frac{1}{a^2 b c} \right) \leq 0, \\ -4 \left(\frac{1}{a^2} - \frac{1}{b^2} + \frac{c^2}{a^2 b^2} + \frac{b^2 + c^2 - a^2}{a^2 b c} \right) &= -4(b^2 + c^2 - a^2) \left(\frac{1}{a^2 b^2} + \frac{1}{a^2 b c} \right) \leq 0. \end{aligned}$$

By applying Theorem 4.5.2, there exist some constants $C_1, C_2 > 0$ which depend only on the data on $\mathcal{P}S_T^1$ such that $|\frac{\zeta}{c}| \leq C_1$ and $|\frac{\xi}{b}| \leq C_2$. If we choose $\tilde{C}_0 = \max(C_1, C_2)$, then $|b - c| \leq \tilde{C}_0 \min(b, c)$ for all $(z, t) \in S_T^1$. Similarly, by applying Theorem 4.5.2 to equations (4.22) and (4.23) as well as noting that $c \geq b$, there exists a constant $\hat{C}_0 > 0$ such that $|a - c| \leq \hat{C}_0 \min(a, c)$ for all $(z, t) \in S_T^1$. $\square$

In fact, these equations tell us more:

Lemma 6.1.2. Suppose that at time $t = 0$ we have $0 < a$ and $1 \leq \max_s \left(\frac{c}{a} \right) \leq \lambda$ for some constant $\lambda \geq 1$. Then $1 \leq \frac{c}{a} \leq \lambda$ for all $(z, t) \in S_T^1$. ■

Proof. Since $\frac{\chi}{a} = \frac{a-c}{a} = 1 - \frac{c}{a}$, the evolution equation for $\frac{c}{a}$ is obtained from equation (4.22) as:

$$\begin{aligned} \partial_t \left(\frac{c}{a} \right) &= -\partial_t \left(\frac{\chi}{a} \right) = -\left(\frac{\chi}{a} \right)'' - \left(2\frac{a'}{a} + \frac{b'}{b} \right) \left(\frac{\chi}{a} \right)' + 4(a^2 + c^2 - b^2) \left(\frac{1}{a^2 b^2} + \frac{1}{ab^2 c} \right) \left(\frac{\chi}{a} \right) \\ &= \left(\frac{c}{a} \right)'' + \left(2\frac{a'}{a} + \frac{c'}{c} \right) \left(\frac{c}{a} \right)' + 4(a^2 + c^2 - b^2) \left(\frac{1}{a^2 b^2} + \frac{1}{ab^2 c} \right) \left(1 - \frac{c}{a} \right) \\ &\leq \left(\frac{c}{a} \right)'' + \left(2\frac{a'}{a} + \frac{c'}{c} \right) \left(\frac{c}{a} \right)', \end{aligned}$$

since $1 - \frac{c}{a} \leq 0$. By applying Theorem 4.5.2, we conclude that $\frac{c}{a} \leq \max_{s, t=0} \left(\frac{c}{a} \right) = \lambda$. □

Next, since we have the ordering $a(z, t) \leq b(z, t) \leq c(z, t)$ at all points $(z, t) \in S_T^1$, we wish to study the behaviour of the minima of the function a . Intuitively, one can think of the quantity a as the smallest “radius” of the $SU(2)$ fibre at z . We define the quantity $\check{a}$ that will be used in our study for the neckpinching of the manifold by:

$$\check{a}(t) = \min_s (a(s, t)).$$

Necessarily, we have $\check{a}(t) > 0$ as long as the Ricci flow exists and at this point, $a'(s, t) = 0$ and $a''(s, t) \geq 0$. By picturing a neckpinch, we expect the quantity a would vanish at the singular time; that is, $\check{a}(T) = 0$. Furthermore, the spatial global minima, denoted by $\check{a}(t)$, is a Lipschitz continuous function of time [43].

Lemma 6.1.3. If there exists a $T < \infty$ such that $\check{a}(T) = 0$, then $\check{a}(t)^2 \leq 4(T - t)$. ■

Proof. We use the original coordinate $z \in [0, 2\pi)$. Since the global spatial minima of the quantity a is a Lipschitz continuous function of time, we may define the derivative $\frac{d}{dt}\check{a}(t)$ in the sense of forward difference quotients as described in [43]. Defining $z(t)$ to be a position of the global spatial minima of a at time t , we have:

$$\begin{aligned} \frac{d}{dt}\check{a}(t) &= \partial_t a(z(t), t) = a'' + a' \left(\frac{b'}{b} + \frac{c'}{c} \right) - 2a \left(\frac{a^4 - (b^2 - c^2)^2}{(abc)^2} \right) \\ &\geq -2a \left(\frac{a^4 - (b^2 - c^2)^2}{(abc)^2} \right) \geq -\frac{2a^3}{b^2 c^2} = -\frac{2}{a} \frac{a^2}{b^2} \frac{a^2}{c^2}. \end{aligned} \tag{6.1}$$

Next, by the ordering $a \leq b \leq c$ we have $\frac{a^2}{b^2} \frac{a^2}{c^2} \leq 1$. Along with the inequality (6.1), we deduce:

$$\frac{d}{dt}\check{a} \geq -\frac{2}{\check{a}} \quad \Rightarrow \quad \frac{d}{dt}(\check{a}^2) \geq -4.$$

By integrating the differential inequality from t to T and using the fact that $\check{a}(T) = 0$, we have $\check{a}(t)^2 \leq 4(T - t)$ for almost every $t \in [0, T)$. □

The lemma above gives us a corollary:

Corollary 6.1.4. The singular time T for the Ricci flow is bounded below by $T \geq \frac{\check{a}(0)^2}{4}$. ■

Apart from the upper bound for $\check{a}(t)$, we can also find a similar bound for $\hat{c}(t) = \max_s(c(s, t))$. Similar as before, this quantity is a Lipschitz continuous function of time [43]. From equation (4.12), along the path for the maximum of c we have:

$$\frac{d}{dt}\hat{c} \leq -2c \left(\frac{c^4 - (a^2 - b^2)^2}{(abc)^2} \right) = -\frac{2}{c} \left(\frac{(c^4 - b^4) + a^2(2b^2 - a^2)}{(ab)^2} \right) \leq 0, \quad (6.2)$$

which implies that the maximum of c over S^1 decreases over time. In fact, we have the following estimate:

Lemma 6.1.5. The maximum of c over S^1 is bounded above by $\hat{c}(t)^2 \leq -4t + \hat{c}(0)^2$. ■

Proof. From inequality (6.2), we study the function $P(x, y) = 2 + \frac{1}{x^2 y^2} - \frac{x^2}{y^2} - \frac{y^2}{x^2}$ since the left hand side of (6.2) is $-\frac{2}{c}P\left(\frac{a}{c}, \frac{b}{c}\right)$. Let $c \leq \lambda a$ at the initial time. Thus, by Lemmas 6.1.1 and 6.1.2, we have the ordering $a \leq b \leq c \leq \lambda a$ at all time. Thus, it is sufficient to restrict the domain of the function $P(x, y)$ to $x \in [\lambda^{-1}, y]$ and $y \in [x, 1]$. We call this domain $\Pi_\lambda \subset \mathbb{R}^2$.

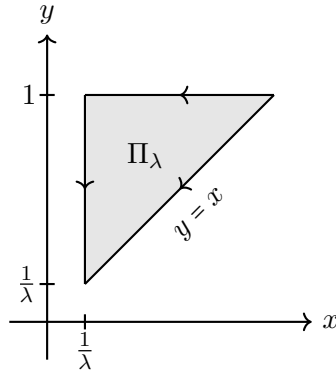


Figure 6.1: Domain Π_λ of the function $P(x, y)$.

The arrows indicate the direction $P(x, y)$ increases.

Our aim now is to minimise the function P over Π_λ . Since $\text{grad}(P) = \frac{2}{x^2 y^2} \left(\frac{y^4 - x^4 - 1}{x}, \frac{x^2 - y^2 - 1}{y} \right)$, we conclude that there are no global critical points of P over $\mathbb{R}^2$. Necessarily, the minimum of $P(x, y)$ on Π_λ lies on $\partial\Pi_\lambda$. Along the boundary of Π_λ we have:

$$\begin{aligned} P(x, x) &= \frac{1}{x^4}, \\ P(x, 1) &= 2 - x^2, \\ P(\lambda^{-1}, y) &= \frac{\lambda^2}{y^2} - \frac{1}{\lambda^2 y^2} - \lambda^2 y^2 + 2. \end{aligned}$$

We note that the functions $P(x, x)$ and $P(x, 1)$ are strictly decreasing as x increases from λ^{-1} to 1. Furthermore, since $\lambda \geq 1$, $\partial_y P(\lambda^{-1}, y) = \frac{2}{\lambda^2 y^3} (1 - \lambda^4 y^4 - \lambda^4) < 0$ for any $y \in [\lambda^{-1}, 1]$. So $P(\lambda^{-1}, y)$ is a decreasing function of y . Thus, the minimum of $P(x, y)$ on Π_λ is at the point $(1, 1)$ which gives us $P(x, y) \geq P(1, 1) = 1$. By using this in inequality (6.2), we get:

$$\frac{d}{dt} \hat{c} \leq -\frac{2}{\hat{c}} \quad \Rightarrow \quad \frac{d}{dt} (\hat{c}^2) \leq -4,$$

for all but a finite number of $t \in [0, T)$. By integrating the differential inequality from 0 to t , we have $\hat{c}(t)^2 - \hat{c}(0)^2 \leq -4t$ for almost every $t \in [0, T)$, hence the result. $\square$

This inequality is sharp and is attained when $a = b = c$; that is, it is attained by the shrinking sphere. The lemma above gives us another corollary:

Corollary 6.1.6. The singular time T for the Ricci flow is bounded above by $T \leq \frac{\hat{c}(0)^2}{4}$. $\blacksquare$

A consequence of the lemma above is a refinement of Lemma 6.1.2 using Maximum Principle II in Theorem 4.5.3. This is reminiscent of Proposition 3.3.3 we proved earlier.

Lemma 6.1.7. Suppose that at time $t = 0$ we have $1 \leq \max_s \left(\frac{c}{a} \right) \leq \lambda$ for some constant $\lambda \geq 1$. Then for all $(z, t) \in S_T^1$ we have the following estimate:

$$1 \leq \frac{c^2}{a^2} \leq e^{(\lambda^2-1)} (\lambda^2 - 1) \left(1 - \frac{4t}{\hat{c}(0)^2} \right)^2 + 1. \quad \blacksquare$$

Proof. Consider the evolution equation for $\frac{c}{a}$ from equation (4.22):

$$\partial_t \left(\frac{c}{a} \right) = \left(\frac{c}{a} \right)'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) \left(\frac{c}{a} \right)' + 4(a^2 + c^2 - b^2) \left(\frac{1}{a^2 b^2} + \frac{1}{ab^2 c} \right) \left(1 - \frac{c}{a} \right).$$

By defining $u := \frac{c}{a}$, the equation above can be rewritten as:

$$\partial_t u = u'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) u' + \frac{4}{a^2} \left(\frac{a^2}{b^2} - 1 + \frac{c^2}{b^2} + \frac{a^3}{b^2 c} + \frac{ac}{b^2} - \frac{a}{c} \right) (1 - u).$$

Since $1 - u < 0$, we want to minimise the coefficient of $1 - u$ in the equation above. Using the ordering $a \leq b \leq c$ we have:

$$\begin{aligned} \partial_t u &\leq u'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) u' + \frac{4}{a^2} \left(\frac{u+1}{u^3} \right) (1 - u) \\ &\leq u'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) u' + \frac{4}{c^2} \left(\frac{u+1}{u^3} \right) (1 - u) \\ &\leq u'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) u' + \frac{4}{\hat{c}^2} \left(\frac{u+1}{u^3} \right) (1 - u) \\ &\leq u'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) u' + \frac{4}{\hat{c}(0)^2 - 4t} \left(\frac{u+1}{u^3} \right) (1 - u), \end{aligned}$$

where we used Lemma 6.1.5 in the last line. We now appeal to Maximum Principle II by solving the following associated ODE:

$$\begin{aligned} \frac{dv}{dt} &= \frac{4}{\hat{c}(0)^2 - 4t} \left(\frac{1 - v^2}{v^3} \right), \\ v(0) &= \lambda. \end{aligned}$$

This is a separable ODE, which we can solve to get:

$$v^2 - \lambda^2 + \log \left(\frac{v^2 - 1}{\lambda^2 - 1} \right) = 2 \log \left(1 - \frac{4t}{\hat{c}(0)^2} \right) \Rightarrow e^{v^2 - \lambda^2} \left(\frac{v^2 - 1}{\lambda^2 - 1} \right) = \left(1 - \frac{4t}{\hat{c}(0)^2} \right)^2. \quad (6.3)$$

Maximum Principle II states that u is bounded above by the function v that satisfies equation (6.3). We can solve equation (6.3) for v using the Lambert-W function but this is not expressible in terms of elementary functions. For an explicit bound, since $1 \leq u \leq v$, we have:

$$e^{1 - \lambda^2} \left(\frac{v^2 - 1}{\lambda^2 - 1} \right) \leq e^{v^2 - \lambda^2} \left(\frac{v^2 - 1}{\lambda^2 - 1} \right) = \left(1 - \frac{4t}{\hat{c}(0)^2} \right)^2,$$

from which we can conclude the result $u^2 \leq v^2 \leq e^{(\lambda^2 - 1)} (\lambda^2 - 1) \left(1 - \frac{4t}{\hat{c}(0)^2} \right)^2 + 1$. $\square$

Now we want to find a lower bound for the quantity $\check{a}(t)$. Recall the proof of Theorem 3.3.2 in which we have obtained a lower linear bound for the smallest metric component A on the model geometry S^3 by considering the concavity of this quantity. The concavity of the function $\check{a}(t)^2$ was supported by the numerical simulations shown in Figures 5.2-5.4 in the previous chapter.

However, demonstrating the concavity of $\check{a}(t)^2$ by considering its second derivative would be complicated in our setting since the first and second derivatives of the other metric quantities along the minima of a^2 would appear in the analysis and we have no control over the sign of these quantities. One way of proceeding with this is to consider the definition of concavity from first principles; that is, by showing that the following inequality holds for any $t_1, t_2 \in [0, T)$ and $p \in [0, 1]$:

$$\check{a}(pt_1 + (1 - p)t_2)^2 \geq p\check{a}(t_1)^2 + (1 - p)\check{a}(t_2)^2.$$

This is still out of reach of our analysis. However, following the work in [54], by setting an assumption on the initial scalar curvature of the manifold as well as the ratio $\frac{c}{a}$, we can prove the following proposition:

Proposition 6.1.8. Suppose that at time $t = 0$ we have $1 \leq \frac{c}{a} < 2$ and $\min_{s, t=0}(\mathcal{S}) \geq 0$. If there exists a finite time $T > 0$ such that $\check{a}(T) = 0$, then there exists a uniform constant $D > 0$ such that $\check{a}(t)^2 \geq D(T - t)$. $\blacksquare$

Proof. Consider the quantity $\log(abc)$. This quantity evolves according to the PDE:

$$\begin{aligned}\partial_t \log(abc) &= \frac{\partial_t a}{a} + \frac{\partial_t b}{b} + \frac{\partial_t c}{c} \\ &= \frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} + 2 \left(\frac{a'b'}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc} \right) - 2(\hat{K}_{12} + \hat{K}_{13} + \hat{K}_{23}) \\ &= \frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} + 2 \left(\frac{a'b'}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc} \right) - 2 \left(\frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}{a^2b^2c^2} \right).\end{aligned}$$

Since the scalar curvature S satisfies $\partial_t S = \Delta S + \frac{n}{2}S^2 \geq \Delta S$ from equation (2.10), we have $S \geq 0$ by minimum principle. From this and the expression for the scalar curvature S , we deduce:

$$\begin{aligned}0 &\leq - \left(\frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} + \frac{a'b'}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc} - \frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}{a^2b^2c^2} \right) \\ \Rightarrow \quad \frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} &\leq - \left(\frac{a'b'}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc} \right) + \frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}{a^2b^2c^2}.\end{aligned}$$

Thus, we have:

$$\partial_t \log(abc) \leq \frac{a'b'}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc} + \frac{a^2}{b^2c^2} + \frac{b^2}{a^2c^2} + \frac{c^2}{a^2b^2} - \frac{2}{a^2} - \frac{2}{b^2} - \frac{2}{c^2}.$$

Furthermore, we have:

$$\begin{aligned}\frac{a'b'}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc} &= \frac{1}{2}(\log(abc))' \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} \right) - \frac{1}{2} \left(\left(\frac{a'}{a} \right)^2 + \left(\frac{b'}{b} \right)^2 + \left(\frac{c'}{c} \right)^2 \right) \\ &\leq \frac{1}{2}(\log(abc))' \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} \right).\end{aligned}$$

So, we acquire the inequality:

$$\begin{aligned}\partial_t \log(abc) &\leq \frac{1}{2}(\log(abc))' \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} \right) + \frac{a^2}{b^2c^2} + \frac{b^2}{a^2c^2} + \frac{c^2}{a^2b^2} - \frac{2}{a^2} - \frac{2}{b^2} - \frac{2}{c^2} \\ \Leftrightarrow \quad \partial_t(abc) &\leq \left(\frac{(abc)'}{2abc} \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} \right) + \frac{a^2}{b^2c^2} + \frac{b^2}{a^2c^2} + \frac{c^2}{a^2b^2} - \frac{2}{a^2} - \frac{2}{b^2} - \frac{2}{c^2} \right) abc.\end{aligned}$$

Since S^1 is compact, necessarily the global minima of abc over s satisfies $(abc)' = 0$. By using the same argument as in Lemma 6.1.3, along the path $(z(t), t)$ for the global minima of abc over S^1 , we have:

$$\frac{d}{dt} abc \leq \left(\frac{(abc)'}{2abc} \left(\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} \right) + \frac{a^2}{b^2c^2} + \frac{b^2}{a^2c^2} + \frac{c^2}{a^2b^2} - \frac{2}{a^2} - \frac{2}{b^2} - \frac{2}{c^2} \right) abc, \quad (6.4)$$

for all but a finite number of $t \in [0, T)$.

Following the proof by Isenberg, Knopf, and Šešum [54], we wish to bound the term $\frac{a^2}{b^2c^2} + \frac{b^2}{a^2c^2} + \frac{c^2}{a^2b^2} - \frac{2}{a^2} - \frac{2}{b^2} - \frac{2}{c^2}$ by a negative multiple of $\frac{1}{c^2}$; that is, for some $\mathfrak{C} > 0$, we want:

$$\frac{a^2}{b^2c^2} \left(1 + \left(\frac{b^2}{a^2} - \frac{c^2}{a^2} \right)^2 - 2\frac{b^2}{a^2} - 2\frac{c^2}{a^2} \right) \leq -\frac{\mathfrak{C}}{c^2}. \quad (6.5)$$

This motivates us to study the polynomial $P(x, y) = 1 + (x^2 - y^2)^2 - 2x^2 - 2y^2 = x^4 + y^4 - 2x^2y^2 - 2x^2 - 2y^2 + 1$ since the left hand side of (6.5) is a positive multiple of $P\left(\frac{b}{a}, \frac{c}{a}\right)$. As a result of the ordering $a \leq b \leq c \leq \lambda a$, we restrict the domain of the polynomial $P(x, y)$ to $x \in [1, y]$ and $y \in [x, \lambda]$. We call this domain $\Omega_\lambda \subset \mathbb{R}^2$.

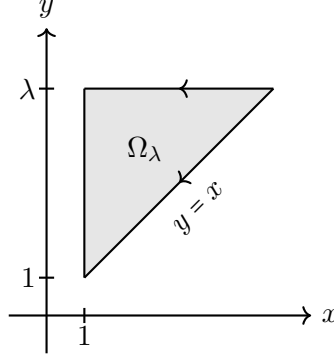


Figure 6.2: Domain Ω_λ of the function $P(x, y)$.

The arrows indicate the direction where $P(x, y)$ increases.

By examining $\text{grad}(P) = 4(x(x^2 - y^2 - 1), -y(x^2 - y^2 + 1))$, we conclude that the critical points of P are $(0, 0)$, $(0, \pm 1)$, and $(\pm 1, 0)$, all of which lie outside of Ω_λ . Thus, the maximum of $P(x, y)$ on Ω_λ lies on $\partial\Omega_\lambda$. Along the boundary of Ω_λ we have:

$$P(x, x) = 1 - 4x^2,$$

$$P(1, y) = y^4 - 4y^2,$$

$$P(x, \lambda) = 1 + x^4 + \lambda^4 - 2\lambda^2x^2 - 2x^2 - 2\lambda^2.$$

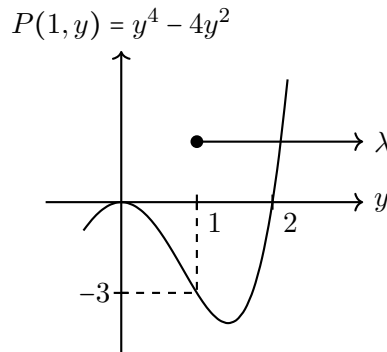


Figure 6.3: Plot of $P(1, y) = y^4 - 4y^2$.

For any $\lambda > 1$, the polynomials $P(x, x)$ and $P(x, \lambda)$ are strictly decreasing as x increases from 1 to λ . Thus, necessarily, the maximum of $P(x, y)$ on Ω_λ lies on the line segment connecting

$(1, 1)$ to $(1, \lambda)$. From the plot in Figure 6.3, we note that the maximum of $P(1, y)$ for $y \in [1, \lambda]$ is negative if $\lambda < 2$. Thus, a sufficient and necessary condition for the maximum of $P(x, y)$ to be negative in Ω_λ is $\lambda < 2$. Furthermore, $\max_{\Omega_\lambda}(P(x, y)) = \max(-3, \lambda^4 - 4\lambda^2) := -\lambda_0$ for some $\lambda_0 > 0$.

Hence, if we have $\frac{c}{a} < 2$ at time $t = 0$, by compactness of S^1 , there exists a $1 < \lambda < 2$ such that $\frac{c}{a} \leq \lambda$ at time $t = 0$. By virtue of Lemma 6.1.2, we have $\frac{c}{a} \leq \lambda$ for all S_T^1 and thus:

$$\begin{aligned} & 1 + \left(\frac{b^2}{a^2} - \frac{c^2}{a^2} \right)^2 - 2\frac{b^2}{a^2} - 2\frac{c^2}{a^2} = P\left(\frac{b}{a}, \frac{c}{a}\right) \leq -\lambda_0. \\ \Rightarrow & \frac{a^2}{b^2 c^2} \left(1 + \left(\frac{b^2}{a^2} - \frac{c^2}{a^2} \right)^2 - 2\frac{b^2}{a^2} - 2\frac{c^2}{a^2} \right) \leq -\lambda_0 \frac{a^2}{b^2 c^2} < -\frac{\lambda_0}{\lambda^2 c^2}, \end{aligned}$$

since the ordering $a \leq b \leq c \leq \lambda a < 2a$ is preserved under the Ricci flow. From inequality (6.4), along the minimum of abc we have:

$$\frac{d}{dt} abc \leq \left(-\frac{\lambda_0}{\lambda^2 c^2} \right) abc = -\frac{\lambda_0}{\lambda^2} \frac{ab}{c} \leq -\frac{\lambda_0}{\lambda^{\frac{10}{3}}} (abc)^{\frac{1}{3}},$$

where we used the fact that $a \leq b \leq c \leq \lambda a$ and Lemma 6.1.2 for the last inequality.

By setting $m(t) = abc(z(t), t)$, for all but a finite number of $t \in [0, T)$ we have:

$$\frac{d}{dt} m(z(t), t) \leq -D_1 m^{\frac{1}{3}},$$

where $D_1 = \frac{\lambda_0}{\lambda^{\frac{10}{3}}} > 0$ is some constant. Integrating this inequality from an arbitrary $t < T$ to T and noting that $m(T) = 0$ since $\check{a}(T) = 0$, we have $m(t)^{\frac{2}{3}} \geq \frac{2D_1}{3}(T - t)$. Furthermore, by using Lemma 6.1.2 we obtain the inequality:

$$\check{a}(t)^3 \geq \frac{1}{\lambda^2} abc(z(t), t) = \frac{1}{\lambda^2} m(t) \geq \frac{2\sqrt{6}D_1^{\frac{3}{2}}}{9\lambda^2} (T - t)^{\frac{3}{2}},$$

and thus $\check{a}(t)^2 \geq D(T - t)$ for the constant $D = \frac{2}{3} \frac{\lambda_0}{\lambda^{\frac{14}{3}}} > 0$ where $\lambda_0 = \min(3, 4\lambda^2 - \lambda^4)$. □

6.2 Curvature Behaviour

The next step is to find the bounds for the first and second derivatives of the metric components. This is required since they appear in the curvature quantities of the metric g . We prove:

Lemma 6.2.1. Suppose that $a \leq b \leq c \leq \lambda a$ for some $1 < \lambda < 2$ at time $t = 0$. Then for all $(s, t) \in S_T^1$ there exist constants $C_1(\lambda), C_2(\lambda), C_3(\lambda) > 0$ such that:

$$\begin{aligned} |a'(s, t)| &\leq C_1(\lambda) \leq \max\left(\frac{280\sqrt{3}}{9}, \max_s(|a'(s, 0)|)\right), \\ |b'(s, t)| &\leq C_2(\lambda) \leq \max\left(\frac{4\sqrt{57}}{3}, \max_s(|b'(s, 0)|)\right), \\ |c'(s, t)| &\leq C_3(\lambda) \leq \max\left(\frac{10\sqrt{93}}{9}, \max_s(|c'(s, 0)|)\right). \end{aligned} \quad \blacksquare$$

Proof. We recall the evolution equation for a' in (4.24):

$$\begin{aligned} \partial_t(a') &= a''' + a'' \left(\frac{b'}{b} + \frac{c'}{c} - \frac{a'}{a} \right) - a' \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} + \frac{6a^4 + 2(b^2 - c^2)^2}{(abc)^2} \right) \\ &\quad + 4 \left(\frac{a^3 b'}{b^3 c^2} + \frac{a^3 c'}{b^2 c^3} - \frac{c^2 b'}{ab^3} - \frac{b^2 c'}{ac^3} + \frac{cc'}{ab^2} + \frac{bb'}{ac^2} \right). \end{aligned} \quad (6.6)$$

We denote $\tilde{a}' = \max_s(a'(s, t))$. By using the same method as in Lemma 6.1.3, we have $\tilde{a}'' \leq 0$ and $\tilde{a}' = 0$ along the path $(z(t), t)$ for the maximum value of a' . We substitute this in (6.6):

$$\begin{aligned} \frac{d}{dt} \tilde{a}' &\leq -\tilde{a}' \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} + \frac{6a^4 + 2(b^2 - c^2)^2}{(abc)^2} \right) + 4 \left(\frac{a^3 b'}{b^3 c^2} + \frac{a^3 c'}{b^2 c^3} - \frac{c^2 b'}{ab^3} - \frac{b^2 c'}{ac^3} + \frac{cc'}{ab^2} + \frac{bb'}{ac^2} \right) \\ &= -\tilde{a}' \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} + \frac{6a^2}{b^2 c^2} + \frac{2b^2}{a^2 c^2} + \frac{2c^2}{a^2 b^2} \right) + \frac{4\tilde{a}'}{a^2} \\ &\quad + 4 \left(\frac{a^3 b'}{b^3 c^2} + \frac{a^3 c'}{b^2 c^3} - \frac{c^2 b'}{ab^3} - \frac{b^2 c'}{ac^3} + \frac{cc'}{ab^2} + \frac{bb'}{ac^2} \right). \end{aligned} \quad (6.7)$$

To deal with the terms in the second bracket on the right hand side of the inequality (6.7) we use Young's inequality on each of them. So, for any $\varepsilon_i > 0$ for $i = 1, 2, 3, 4, 5, 6$, we have:

$$\begin{aligned} \frac{a^3 b'}{b^3 c^2} &\leq \frac{\varepsilon_1}{2} \frac{(b')^2}{b^2} + \frac{1}{2\varepsilon_1} \frac{a^6}{b^4 c^4} \leq \frac{\varepsilon_1}{2} \frac{(b')^2}{b^2} + \frac{1}{2\varepsilon_1} \frac{a^2}{b^2 c^2}, \\ \frac{a^3 c'}{b^2 c^3} &\leq \frac{\varepsilon_2}{2} \frac{(c')^2}{c^2} + \frac{1}{2\varepsilon_2} \frac{a^6}{b^4 c^4} \leq \frac{\varepsilon_2}{2} \frac{(c')^2}{c^2} + \frac{1}{2\varepsilon_2} \frac{a^2}{b^2 c^2}, \\ -\frac{c^2 b'}{ab^3} &\leq \frac{\varepsilon_3}{2} \frac{(b')^2}{b^2} + \frac{1}{2\varepsilon_3} \frac{c^4}{a^2 b^4} \leq \frac{\varepsilon_3}{2} \frac{(b')^2}{b^2} + \frac{1}{2\varepsilon_3} \frac{\lambda^2 c^2}{a^2 b^2}, \\ -\frac{b^2 c'}{ac^3} &\leq \frac{\varepsilon_4}{2} \frac{(c')^2}{c^2} + \frac{1}{2\varepsilon_4} \frac{b^4}{a^2 c^4} \leq \frac{\varepsilon_4}{2} \frac{(c')^2}{c^2} + \frac{1}{2\varepsilon_4} \frac{b^2}{a^2 c^2}, \\ \frac{bb'}{ac^2} &\leq \frac{\varepsilon_5}{2} \frac{(b')^2}{b^2} + \frac{1}{2\varepsilon_5} \frac{b^4}{a^2 c^4} \leq \frac{\varepsilon_5}{2} \frac{(b')^2}{b^2} + \frac{1}{2\varepsilon_5} \frac{b^2}{a^2 c^2}, \\ \frac{cc'}{ab^2} &\leq \frac{\varepsilon_6}{2} \frac{(c')^2}{c^2} + \frac{1}{2\varepsilon_6} \frac{c^4}{a^2 b^4} \leq \frac{\varepsilon_6}{2} \frac{(c')^2}{c^2} + \frac{1}{2\varepsilon_6} \frac{\lambda^2 c^2}{a^2 b^2}, \end{aligned}$$

where we used the fact that the ordering $a \leq b \leq c \leq \lambda a$ is preserved for all $t > 0$. For a crude bound, we set $\varepsilon_1 = \varepsilon_2$, $\varepsilon_3 = \varepsilon_6$, and $\varepsilon_4 = \varepsilon_5$. Therefore, we deduce:

$$\begin{aligned} 4 \left(\frac{a^3 b'}{b^3 c^2} + \frac{a^3 c'}{b^2 c^3} - \frac{c^2 b'}{ab^3} - \frac{b^2 c'}{ac^3} + \frac{cc'}{ab^2} + \frac{bb'}{ac^2} \right) &\leq 2(\varepsilon_1 + \varepsilon_3 + \varepsilon_5) \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} \right) + \frac{1}{\varepsilon_1} \frac{4a^2}{b^2 c^2} \\ &\quad + \frac{1}{\varepsilon_5} \frac{4b^2}{a^2 c^2} + \frac{1}{\varepsilon_3} \frac{4\lambda^2 c^2}{a^2 b^2}. \end{aligned} \quad (6.8)$$

We want the right hand side of inequality (6.8) to be a positive multiple of the first bracket in the inequality (6.7). In order to choose the constants $\varepsilon_1, \varepsilon_3, \varepsilon_5 > 0$, we solve the following simultaneous equations for some constant $\mathfrak{C} > 0$:

$$2(\varepsilon_1 + \varepsilon_3 + \varepsilon_5) = \mathfrak{C}, \quad 4 = 6\mathfrak{C}\varepsilon_1, \quad 4 = 2\mathfrak{C}\varepsilon_5, \quad \text{and} \quad 4\lambda^2 = 2\mathfrak{C}\varepsilon_3.$$

The system above has a solution $\mathfrak{C} = \sqrt{\frac{16}{3} + 4\lambda^2}$ for some $\varepsilon_1, \varepsilon_3, \varepsilon_5 > 0$. Thus, we obtain:

$$\frac{d}{dt}\tilde{a}' \leq (\mathfrak{C} - \tilde{a}') \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} + \frac{6a^2}{b^2c^2} + \frac{2b^2}{a^2c^2} + \frac{2c^2}{a^2b^2} \right) + \frac{4\tilde{a}'}{a^2}.$$

Whenever $\tilde{a}' \geq \mathfrak{C}$, the first term would be negative whereas the second term will be positive. We want the right hand side to be negative, so we now aim to get rid of the positive term $\frac{4\tilde{a}'}{a^2}$ by adding a large enough negative term to it. By assuming $\tilde{a}' \geq \mathfrak{C}$, we have:

$$\begin{aligned} \frac{d}{dt}\tilde{a}' &\leq (\mathfrak{C} - \tilde{a}') \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} \right) + (\mathfrak{C} - \tilde{a}') \left(\frac{6a^2}{b^2c^2} + \frac{2b^2}{a^2c^2} + \frac{2c^2}{a^2b^2} \right) + \frac{4\tilde{a}'}{a^2} \\ &= (\mathfrak{C} - \tilde{a}') \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} \right) + \frac{4}{a^2} \left((\mathfrak{C} - \tilde{a}') \left(\frac{3a^4}{2b^2c^2} + \frac{b^2}{2c^2} + \frac{c^2}{2b^2} \right) + \tilde{a}' \right). \end{aligned}$$

We now look at the function of two variables $P(x, y) = \frac{3}{2x^2y^2} + \frac{x^2}{2y^2} + \frac{y^2}{2x^2}$ and determine its minimum value in the region Ω_λ as in Figure 6.2 in the proof for Proposition 6.1.8. Upon calculations, we note that the minimum of $P(x, y)$ in Ω_λ occurs at $(x, y) = (\lambda, \lambda)$, which gives us $\min_{\Omega_\lambda} P(x, y) = 1 + \frac{3}{2\lambda^4}$. Hence:

$$\begin{aligned} \frac{d}{dt}\tilde{a}' &\leq (\mathfrak{C} - \tilde{a}') \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} \right) + \frac{4}{a^2} \left((\mathfrak{C} - \tilde{a}') \left(1 + \frac{3}{2\lambda^4} \right) + \tilde{a}' \right), \\ &= (\mathfrak{C} - \tilde{a}') \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} \right) + \frac{4}{a^2} \left(\left(1 + \frac{3}{2\lambda^4} \right) \mathfrak{C} - \frac{3}{2\lambda^4} \tilde{a}' \right). \end{aligned}$$

Thus, we conclude that $\max_s(a'(s, t))$ cannot exceed $\mathfrak{C} \left(1 + \frac{2\lambda^4}{3} \right) \leq \frac{280\sqrt{3}}{9}$ and $\max_s(a'(s, 0))$. With a similar argument, by using the negative of a , we can show the lower bound $\min_s(a'(s, t)) \geq -\max \left(\frac{280\sqrt{3}}{9}, \max_s(|a'(s, 0)|) \right)$, which gives us the desired bound for a' .

Using arguments identical to the above and noting that the minimum of $P(x, y)$ on the triangles $\{(x, y) : \frac{1}{2} \leq x \leq 1, 1 \leq y \leq 2x\}$ and $\{(x, y) : \frac{1}{2} \leq x \leq 1, x \leq y \leq 1\}$ are $P(1, \sqrt{2}) = 2$ and $P(1, 1) = \frac{5}{2}$ respectively, similar bounds for b' and c' can be obtained. $\square$

[85] We note that a finite time singularity occurs at $T < \infty$ only if:

$$\limsup_{t \nearrow T} \max_{p \in M} |\text{Ric}(p, t)|_g = \infty.$$

Furthermore, this singularity is of Type I if we have:

$$\sup_{t \in [0, T)} \max_{p \in M} (T - t) |\text{Rm}(p, t)|_g \leq C < \infty,$$

for some constant $C > 0$. Thus, we now want to investigate the behaviour of the norm of the Riemann curvature tensor, which is given by:

$$|\text{Rm}|_g^2 = 2(K_{01}^2 + K_{02}^2 + K_{03}^2 + K_{12}^2 + K_{13}^2 + K_{23}^2).$$

Lemma 6.2.2. If $1 \leq \frac{c}{a} \leq \lambda$ for some $\lambda < 2$ at $t = 0$, then for all $p \in M$ we have:

$$|K_{12}(p, t)| + |K_{13}(p, t)| + |K_{23}(p, t)| \leq \frac{C(\lambda)}{a(\pi(p), t)^2}, \quad (6.9)$$

where $\pi : S^1 \times S^3 \rightarrow S^1$ is the projection map from the manifold $S^1 \times S^3$ to the S^1 base of the foliation. ■

Proof. The sectional curvatures K_{12} , K_{13} , and K_{23} are given by:

$$\begin{aligned} K_{12} &= -\frac{a'b'}{ab} + \frac{(a^2 - b^2)^2 - 3c^4}{(abc)^2} + \frac{2}{a^2} + \frac{2}{b^2}, \\ K_{13} &= -\frac{a'c'}{ac} + \frac{(a^2 - c^2)^2 - 3b^4}{(abc)^2} + \frac{2}{a^2} + \frac{2}{c^2}, \\ K_{23} &= -\frac{b'c'}{bc} + \frac{(b^2 - c^2)^2 - 3a^4}{(abc)^2} + \frac{2}{b^2} + \frac{2}{c^2}. \end{aligned}$$

Thus, by Lemma 6.2.1 and the fact that $a \leq b \leq c \leq \lambda a$ for all $t \geq 0$, we obtain the bound:

$$\begin{aligned} |K_{12}| + |K_{13}| + |K_{23}| &\leq \frac{|a'||b'|}{ab} + \frac{|a'||c'|}{ac} + \frac{|b'||c'|}{bc} + \frac{a^2}{b^2c^2} + \frac{b^2}{a^2c^2} + \frac{c^2}{a^2b^2} + \frac{2}{a^2} + \frac{2}{b^2} + \frac{2}{c^2} \\ &\leq \tilde{C}(\lambda) \left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} \right) + \frac{1}{c^2} + \frac{1}{a^2} + \frac{1}{\lambda^2 b^2} + \frac{6}{a^2} \leq \frac{C(\lambda)^2}{a^2}, \end{aligned}$$

for some constant $C(\lambda) > 0$. □

We now want to bound the sectional curvatures K_{01} , K_{02} , and K_{03} . These curvatures involve the second order derivatives of the quantities a, b , and c . A similar bound as in Lemma 6.2.2 may be obtained, but this requires a bit more work.

Lemma 6.2.3. If $1 \leq \frac{c}{a} \leq \lambda$ for some $\lambda < 2$ at $t = 0$, then for all $p \in M$ we have:

$$|K_{01}(p, t)| + |K_{02}(p, t)| + |K_{03}(p, t)| \leq \frac{C(\lambda)}{\tilde{a}(t)^2}. \quad \blacksquare$$

Proof. First we recall the evolution equation for K_{01} computed in (4.30):

$$\begin{aligned} \partial_t K_{01} &= \Delta_g K_{01} + 2K_{01}^2 - 2K_{01} \left(\frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} + \frac{2a^4 + 2(b^2 - c^2)^2}{(abc)^2} \right) \\ &\quad + 2K_{02} \left(\frac{2a^2}{b^2c^2} + \frac{2b^2}{a^2c^2} - \frac{2c^2}{a^2b^2} - \frac{a'b'}{ab} \right) + 2K_{03} \left(\frac{2a^2}{b^2c^2} + \frac{2c^2}{a^2b^2} - \frac{2b^2}{a^2c^2} - \frac{a'c'}{ac} \right) \\ &\quad + 2\frac{a'}{a} \left(-\frac{(b')^3}{b^3} - \frac{(c')^3}{c^3} + \frac{6aa'}{b^2c^2} + \frac{4bb'}{a^2c^2} + \frac{4cc'}{a^2b^2} - \frac{2a'c^2}{a^3b^2} - \frac{2a'b^2}{a^3c^2} + \frac{4a'}{a^3} \right. \\ &\quad \left. - \frac{12a^2b'}{b^3c^2} - \frac{12a^2c'}{b^2c^3} - \frac{4b^2c'}{a^2c^3} - \frac{4c^2b'}{a^2b^3} \right) + 4a^2 \left(\frac{3(b')^2}{b^4c^2} + \frac{4b'c'}{b^3c^3} + \frac{3(c')^2}{b^2c^4} \right) \\ &\quad - \frac{4}{a} \left(\frac{(c')^2}{ab^2} + \frac{(b')^2}{ac^2} + \frac{3(b')^2c^2}{ab^4} + \frac{3b^2(c')^2}{ac^4} - \frac{4cc'b'}{ab^3} - \frac{4bb'c'}{ac^3} \right). \end{aligned}$$

We can bound this quantity by using Lemma 6.2.1 and the fact that $a \leq b \leq c < 2a$ for all $t \geq 0$. Thus, for some positive constants $D_1, D_2, D_3, D_4 > 0$ depending only on $\lambda > 0$, we have:

$$\begin{aligned} \partial_t K_{01} &\leq \Delta_g K_{01} + 2K_{01}^2 + D_1 \frac{K_{01}}{a^2} + D_2 \frac{K_{02}}{a^2} + D_3 \frac{K_{03}}{a^2} + \frac{D_4}{a^4} \\ &\leq \Delta_g K_{01} + 3K_{01}^2 + K_{02}^2 + K_{03}^2 + \frac{D_0}{a^4}, \end{aligned}$$

where we applied the Cauchy-Schwarz inequality to get $D_0 := D_1^2 + D_2^2 + D_3^2 + D_4$. Furthermore, by using the same estimates and using Young's inequality for the last term in the evolution equation for the quantity $\frac{(a')^2}{a^2}$ obtained from (4.27), we obtain:

$$\begin{aligned} \partial_t \left(\frac{(a')^2}{a^2} \right) &= \Delta_g \left(\frac{(a')^2}{a^2} \right) - 2 \frac{(a')^2}{a^2} \left(2 \frac{(a')^2}{a^2} + \frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} + \frac{4(a^4 + (b^2 - c^2)^2)}{a^2 b^2 c^2} \right) \\ &\quad + 8 \frac{a'}{a^2} \left(\frac{a^3 b'}{b^3 c^2} + \frac{a^3 c'}{b^2 c^3} - \frac{c^2 b'}{ab^3} - \frac{b^2 c'}{ac^3} + \frac{cc'}{ab^2} + \frac{bb'}{ac^2} \right) - 2K_{01}^2 - 4 \frac{(a')^2}{a^2} K_{01}, \\ &\leq \Delta_g \left(\frac{(a')^2}{a^2} \right) + \frac{D_5}{a^4} - K_{01}^2, \end{aligned}$$

where $D_5 > 0$ is a positive constant. By repeating this for the evolution equations of the quantities $\frac{(b')^2}{b^2}$ and $\frac{(c')^2}{c^2}$ in (4.28) and (4.29) respectively, for some positive constants $D_6, D_7 > 0$ we have the inequalities:

$$\begin{aligned} \partial_t \left(\frac{(b')^2}{b^2} \right) &\leq \Delta_g \left(\frac{(b')^2}{b^2} \right) + \frac{D_6}{a^4} - K_{02}^2, \\ \partial_t \left(\frac{(c')^2}{c^2} \right) &\leq \Delta_g \left(\frac{(c')^2}{c^2} \right) + \frac{D_7}{a^4} - K_{03}^2. \end{aligned}$$

Let us define $P_+ = K_{01} + 3 \frac{(a')^2}{a^2} + \frac{(b')^2}{b^2} + \frac{(c')^2}{c^2} \geq K_{01}$. By linearity, the evolution of the quantity P_+ is governed by the inequality:

$$\partial_t P_+ \leq \Delta_g P_+ + \frac{D}{a^4},$$

where $D := D_0 + 3D_5 + D_6 + D_7 > 0$ is some positive constant. By tracking $\hat{P}_+ = \max_s (P_+(s, t))$ using the same argument as in Lemma 6.1.3, we have:

$$\frac{d}{dt} P_+(z(t), t) \leq \frac{D}{a^4} \leq \frac{\tilde{D}}{\tilde{a}^4}, \quad (6.10)$$

which implies that P_+ , and hence K_{01} , cannot blow up to $+\infty$ as long as the metric component a is bounded away from 0.

To bound the sectional curvature K_{01} from below, by using Lemma 6.2.1 and the fact that $a \leq b \leq c < 2a$ for all $t \geq 0$, we note that there exists a constant $C_0 > 0$ such that:

$$\partial_t K_{01} \geq \Delta_g K_{01} - 3K_{01}^2 - K_{02}^2 - K_{03}^2 - \frac{C_0}{a^4}.$$

We now define $P_- = K_{01} - 3\frac{(a')^2}{a^2} - \frac{(b')^2}{b^2} - \frac{(c')^2}{c^2} \leq K_{01}$. This quantity satisfies:

$$\partial_t P_- \geq \Delta_g P_- - \frac{C}{a^4},$$

for some constant $C := D + 3D_5 + D_6 + D_7 > 0$. Tracking the minimum of P_- , we obtain:

$$\frac{d}{dt} P_-(z(t), t) \geq -\frac{C}{a^4} \geq -\frac{C}{\check{a}^4}. \quad (6.11)$$

By a similar argument as before, this implies that P_- , and hence K_{01} , cannot approach $-\infty$ as long as the metric quantity a is bounded away from 0. Both of these arguments imply that the sectional curvature K_{01} is bounded on the manifold for times $[0, T)$ where T is the singular time when the quantity a reaches 0 somewhere.

Since we have shown that $D(T-t) \leq \check{a}(t)^2 \leq C(T-t)$ for some constants $C, D > 0$, by substituting the lower bound for $\check{a}$ in (6.10) and (6.11), we get the inequalities:

$$\begin{aligned} \frac{d}{dt} \hat{P}_+ &\leq \frac{D}{(T-t)^2}, \\ \frac{d}{dt} \check{P}_- &\geq -\frac{C}{(T-t)^2}, \end{aligned}$$

for some constants $C, D > 0$.

By integrating both of these, for some constants $\tilde{C}, \tilde{D} > 0$, we acquire:

$$|K_{01}| \leq \frac{\tilde{C}}{(T-t)} + \tilde{D} \leq \frac{\tilde{C} + T\tilde{D}}{(T-t)},$$

for all but a finite number of $t \in [0, T)$.

If we substitute the upper bound for $\check{a}^2$ in the inequality above, we get the required bound for $|K_{01}|$. A similar procedure can be used to bound $|K_{02}|$ and $|K_{03}|$ using equations (4.31) and (4.32). This proves the lemma. $\square$

By putting all of these together, we have:

$$|\text{Rm}|_g^2 = 2(K_{01}^2 + K_{02}^2 + K_{03}^2 + K_{12}^2 + K_{13}^2 + K_{23}^2) \leq \frac{3D^2}{\check{a}^4} + \frac{3C^2}{a^4} \leq \frac{\tilde{C}}{\check{a}^4},$$

for some finite constant $0 < \tilde{C}$. The lower bound for $\check{a}$ in Proposition 6.1.8 implies the bound $(T-t)^2 |\text{Rm}|_g^2 \leq C$ for some finite constant $C > 0$. If we take the the supremum over $S^1 \times S^3$ and $t \in [0, T)$ on the left hand side, we can show that the curvature asymptotics satisfy the Type I condition as in (2.11).

Finally, we note that the metric constructed by Isenberg, Knopf, and Šešum also satisfies the conditions required for Proposition 6.1.8, so the set of metrics satisfying the sufficient conditions in Theorem 6.0.2 is non-empty. Thus, this proves Theorem 6.0.2.

Chapter 7

Analysis on S^4

We now turn our attention to the study of the Ricci flow of the same type of metric but on a manifold with different topology to $S^1 \times S^3$. Instead of no boundary conditions as in the work by Isenberg, Knopf, and Šešum and in Chapter 6, we consider boundary conditions arising from the cylinder-to-sphere rule as discussed in Section 4.1.2, which results in a closed compact manifold with trivial fundamental group. For simplicity, we consider first the case for which the quantity $b \equiv c$. Thus, the metric is of the form:

$$g = ds^2 + a(s)^2 \omega^1 \otimes \omega^1 + c(s)^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3).$$

As in the previous chapter, this Riemannian manifold possesses an $SU(2)$ symmetry. Furthermore, by the condition $b(z) \equiv c(z)$, there is a right action by the subgroup $U(1) \subset SU(2)$ that acts by isometries. So, the manifold has an $SU(2)_L \times U(1)_R$ symmetry group.

We first prove that the initial ordering of the metric components is preserved and the eccentricity bounds. However, we need to adjust the maximum principle used in Chapter 6 as we are working on a bounded interval and some of the coefficients in the resulting parabolic equation blow up to $+\infty$ at the boundaries. Thus, we shall prove a version of the maximum principle which requires the coefficients to be merely locally bounded, which is a weaker assumption than the classical version.

Following the work in [54] and Chapter 6, the important result in this section will be the upper and lower bounds of the smallest metric component a close to the singular time. The analysis here would be slightly more complicated than the one in the previous chapter. The metric quantity a vanishes on the boundaries of the interval, so its minimum would be 0 over the whole of the domain, which is trivial. Instead, we focus on the minimum of the quantity a such that its space derivative is 0 as in the work by Angenent and Knopf [6]. Under certain

sufficiency conditions on the scalar curvature and the derivative of the c component at the initial time, we shall show that the decay of the local minimum for the metric component a behaves like $\sqrt{T-t}$.

Next, we want to estimate the curvature behaviour close to the singular time. This is done by bounding the first and second derivatives of the metric components by studying their evolution equation followed by using the explicit formula for the curvatures. This result is sufficient to bound all the curvatures of the metric. Coupled with the estimates on the smallest component of the metric, we can show that this singularity is in fact of Type I.

Finally, we shall present some attempts at showing that the poles of the S^4 manifold remain smooth under the condition that at the singular time the metric components do not vanish uniformly (which would have given us the shrinking round sphere instead). This was shown in the work by Angenent and Knopf for the uniaxial case. However, due to the presence of an extra warping function, it gets considerably more complicated due to the coupling of these two metric components. We shall prove the following theorems:

Theorem 7.0.1. Suppose that we run a Ricci flow on the manifold (S^4, g_0) where the initial metric g_0 is of the form:

$$g_0 = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + c(z)^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

for $z \in I = (-1, 1)$ and $0 < a(z) \leq c(z)$. Then the Ricci flow preserves the form of the metric, the ordering of the metric components, and there exists a constant $\tilde{C}_0 > 0$ depending on the data on the parabolic boundary $\mathcal{P}I_T$ such that for all $(z, t) \in (-1, 1) \times [0, T)$ we have:

$$|a(z, t) - c(z, t)| \leq \tilde{C}_0 \min(a(z, t), c(z, t)). \quad \blacksquare$$

Theorem 7.0.2. There exist open sets of warped metrics on S^4 of the form:

$$g = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + c(z)^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

for $z \in I = (-1, 1)$, satisfying:

1. $0 < a \leq b \equiv c \leq 2a$ at $t = 0$,
2. $|c'(s, 0)| \leq \sqrt{\frac{23}{8}}$,
3. $\min_{s, t=0}(S) \geq -\frac{1}{4}$ at $t = 0$,
4. there exists a $T < \infty$ such that $\limsup_{t \rightarrow T} \max_s |\text{Ric}(s, t)| = \infty$,

such that all solutions originating in these sets develop local neckpinch singularities at some $T < \infty$. Each such solution has the properties that:

1. the ordering $a \leq c$ is preserved,
2. the singularity is Type I with $|\text{Rm}| \leq \frac{C}{\min_s(a^2)}$ and $\frac{\sqrt{T-t}}{C} \leq \min_s(a) \leq C\sqrt{T-t}$. ■

These theorems are analogues of Theorems 1.1.2 and 1.1.3, which were proven in [54]. The major difference in our theorem is that we are working on a simply connected topology which requires the smooth collapsing of the metric quantities a and c at the boundaries of the interval I for all time for which the flow exists.

This difference introduces some complications during the analysis as we are now working on parabolic equations on a domain with boundaries. Therefore, care must be taken when we analyse the metric quantities close to the boundary and the maximum principles must be carefully applied. Furthermore, the neckpinch we need to consider is described by a local minima of the metric component a in the interior of the interval I rather than the global minima as seen in the previous chapter. This, as we shall see later, introduces some difficulties as we lose the Lipschitz property of the quantity.

7.1 Sectional Curvatures at the Poles

Recall that the sectional curvatures are given by:

$$\begin{aligned} K_{01} &= -\frac{a''}{a}, \\ K_{02} &= K_{03} = -\frac{c''}{c}, \\ K_{12} &= K_{13} = -\frac{a'c'}{ac} + \frac{a^2}{c^4} = \frac{-c^3a'c' + a^3}{ac^4}, \\ K_{23} &= -\frac{(c')^2}{c^2} - \frac{3a^2}{c^4} + \frac{4}{c^2} = \frac{-c^2(c')^2 + 4c^2 - 3a^2}{c^4}. \end{aligned}$$

From the equations above, it may seem that the Ricci curvature blows up near $z = \pm 1$ for any $t \geq 0$ since $a, c \rightarrow 0$ as $z \rightarrow \pm 1$. However, note that the last condition in the cylinder-to-sphere rule in Lemma 4.0.2 also implies that as long as the solution to the Ricci flow exists, the metric components are smooth and their even order derivatives with respect to the arclength coordinate s vanish as z approach the boundaries $z = \pm 1$.

This rule also implies that all the odd order derivatives of the metric components with respect to the variable $s(z, t)$ are finite as $z \rightarrow \pm 1$. Indeed, if any of the odd derivatives of

either a or c become unbounded as $z \rightarrow \pm 1$ (that is, either $\partial_s^{2k+1}a \rightarrow \pm\infty$ or $\partial_s^{2k+1}c \rightarrow \pm\infty$), then $\partial_s^{2k+2}a \rightarrow \pm\infty$ or $\partial_s^{2k+2}c \rightarrow \pm\infty$. This contradicts the cylinder-to-sphere rule and hence the smoothness of the solution to the Ricci flow.

We shall show that this apparent curvature blow-up near the boundaries of the foliation is a result of the coordinate choice rather than a genuine singularity. This would justify the choice of the coordinate patch that we are focusing our attention on.

By recalling the conditions on a and c near the boundaries, we immediately see that the limit of all of the sectional curvatures are of the indeterminate form $\frac{0}{0}$ as $z \rightarrow \pm 1$. This allows us to use the L'Hôpital's rule. Thus, for K_{01}, K_{02} , and K_{03} we have:

$$\begin{aligned} \lim_{z \rightarrow +1} K_{01} &= -\frac{\lim_{z \rightarrow +1} a'''}{\lim_{z \rightarrow +1} a'} = \lim_{z \rightarrow +1} a''' < \infty, \\ \lim_{z \rightarrow +1} K_{02} &= \lim_{z \rightarrow +1} K_{03} = -\frac{\lim_{z \rightarrow +1} c'''}{\lim_{z \rightarrow +1} c'} = \lim_{z \rightarrow +1} c''' < \infty. \end{aligned}$$

As for the sectional curvatures K_{12}, K_{13} , and K_{23} , finding their limits as $z \rightarrow \pm 1$ are not as straightforward. Note that the denominator for K_{12} and K_{13} is ac^4 . Thus, by Leibniz's rule, $\lim_{z \rightarrow +1} \partial_s^n(ac^4) = 0$ for $n = 0, 1, 2, 3, 4$ and the (possibly) non-vanishing term for the fifth derivative of ac^4 occurs when all the derivatives fall on each of the terms; that is:

$$\lim_{z \rightarrow +1} \partial_s^5(ac^4) = \lim_{z \rightarrow +1} \frac{5!}{(1!)^5} a'(c')^4 = -5! = -120.$$

We need to check whether the limit of the derivatives of the numerator of K_{12} and K_{13} vanish up to and including the fourth order. By tedious but straightforward calculations, we compute:

$$\begin{aligned} \partial_s(a^3 - c^3 a' c') &= 3a^2 a' - 3c^2 a'(c')^2 - c^3(a'' c' + a' c''), \\ \partial_s^2(a^3 - c^3 a' c') &= 3a^2 a'' + 6a(a')^2 - c(a'(c^2 c''' + 6(c')^3 + 9cc' c'') + c(a''' cc' + 2ca'' c'' + 6a''(c')^2)), \\ \partial_s^3(a^3 - c^3 a' c') &= 3a^2 a''' - a'(6(c')^4 + c^2(cc'''' + 9(c'')^2) + 12c^2 c''' c' + 36c(c')^2 c'') + 6(a')^3 \\ &\quad + 18aa' a'' - c(9a''' c(c')^2 + 18a''(c')^3 + cc'(a'''' c + 27a'' c'') + 3c^2(a''' c'' \\ &\quad + c''' a'')), \\ \partial_s^4(a^3 - c^3 a' c') &= 3a^2 a'''' - 36a''' c(c')^3 - 24a''(c')^4 - a'(60cc'''(c')^2 + 60(c')^3 c'' + c^2(cc'''' \\ &\quad + 30c''' c'') + 15cc'(cc'''' + 6(c'')^2)) - 12c(c')^2(a'''' c + 12a'' c'') + 36(a')^2 a'' \\ &\quad - c^2 c'(a'''' c + 54a''' c'' + 48c''' a'') - 2c^2(2a''(cc'''' + 9(c'')^2) + c(2a''' c'' \\ &\quad + 3a''' c''')) + 6a(3(a'')^2 + 4a''' a'), \end{aligned}$$

$$\begin{aligned}
 \partial_s^5(a^3 - c^3 a' c') &= 3a^2 a'''''' - 60a'''(c')^4 - 60(c')^3(a''''c + 5a''c'') + 60a'''(a')^2 + a'(90(a'')^2 \\
 &\quad - 120c'''(c')^3 - 90(c')^2(cc'''' + 3(c'')^2) - c(c^2 c'''''' + 30c(c''')^2 + 90(c'')^3 \\
 &\quad + 45cc''''c'') - 18cc'(cc'''' + 20c'''c'')) - 15c(c')^2(a''''c + 24a'''c'' \\
 &\quad + 20c'''a'') - cc'(75a''(cc'''' + 6(c'')^2) + c(a''''''c + 90a''''c'' + 120a'''c''')) \\
 &\quad - 5c^2(18a'''(c'')^2 + c''(a''''c + 30c'''a'')) + c(2a''''c''' + 2a'''c'''' + c''''''a'') \\
 &\quad + 30a(a''''a' + 2a'''a'').
 \end{aligned}$$

By recalling that all the even order (including 0) derivatives of a and c vanish and the first order derivatives of a and c tends to -1 as $z \rightarrow +1$, we have:

$$\begin{aligned}
 \lim_{z \rightarrow +1} \partial_s(a^3 - c^3 a' c') &= 0, \\
 \lim_{z \rightarrow +1} \partial_s^2(a^3 - c^3 a' c') &= 0, \\
 \lim_{z \rightarrow +1} \partial_s^3(a^3 - c^3 a' c') &= \lim_{z \rightarrow +1} (-6a'(c')^4 + 6(a')^3) = 0, \\
 \lim_{z \rightarrow +1} \partial_s^4(a^3 - c^3 a' c') &= 0, \\
 \lim_{z \rightarrow +1} \partial_s^5(a^3 - c^3 a' c') &= \lim_{z \rightarrow +1} (-60a'''(c')^4 + 60a'''(a')^2 - 120c'''a'(c')^3) = -120c''' < \infty.
 \end{aligned}$$

Thus, by L'Hôpital's rule, we have:

$$\lim_{z \rightarrow +1} K_{12} = \lim_{z \rightarrow +1} K_{13} = \frac{\lim_{z \rightarrow +1} \partial_s^5(a^3 - c^3 a' c')}{\lim_{z \rightarrow +1} \partial_s^5(ac^4)} = \frac{-120c'''}{-120} = c''' < \infty.$$

Similarly, for K_{23} we check whether the limit of the derivatives of $4c^3 - 3a^2 - c^2(c')^2$ vanish up to and including the third order. Therefore, by L'Hôpital's rule, we have:

$$\lim_{z \rightarrow +1} K_{23} = \frac{\lim_{z \rightarrow +1} \partial_s^4(4c^3 - 3a^2 - c^2(c')^2)}{\lim_{z \rightarrow +1} \partial_s^4(c^4)} = \frac{24a'''}{24} = a''' < \infty.$$

A similar method also yields finite curvature limit as $z \rightarrow -1$. Hence, we conclude that the curvatures remain bounded on the boundary and thus the Ricci flow equation makes sense up to the singular time T . Thus, at $t = T$ either the sectional curvatures blow up at some interior point of the interval $(-1, 1)$ or the third order derivatives of the metric quantities blow up to infinity near $z = \pm 1$ as $t \rightarrow T$. We shall now begin to analyse the behaviour of the metric quantities in the interior of the foliating interval I .

7.2 Analysis of Metric Components

Since we have assumed that $b \equiv c$ in this chapter, the Ricci flow equations (4.10)-(4.12) become:

$$\begin{cases} \partial_t a = a'' + 2a' \frac{c'}{c} - 2 \frac{a^3}{c^4}, \\ \partial_t c = c'' + c' \left(\frac{a'}{a} + \frac{c'}{c} \right) - 2 \left(\frac{2c^2 - a^2}{c^3} \right), \end{cases} \quad (7.1)$$

$$(7.2)$$

on $I = (-1, 1)$ with the cylinder-to-sphere conditions at the boundaries. First, we can check the following lemma:

Lemma 7.2.1. If $0 < a(z, 0) \leq c(z, 0)$ for all $z \in I$, then under the Ricci flow this ordering is preserved for all $t > 0$ for which the flow exists. $\blacksquare$

Proof. We recall the quantity $\chi = a - c$. On the parabolic boundary $\mathcal{P}I_T$ we have $0 \leq a \leq c$. Thus, $\frac{\chi}{a} \leq 0$ on the parabolic boundary. We now consider the evolution equation for $\frac{\chi}{a}$, which is deduced from equation (4.18) to be:

$$\partial_t \left(\frac{\chi}{a} \right) = \left(\frac{\chi}{a} \right)'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) \left(\frac{\chi}{a} \right)' - 4 \left(\frac{c+a}{c^3} \right) \left(\frac{\chi}{a} \right).$$

By applying the maximum principle in Theorem 4.5.4, we deduce that $\frac{\chi}{a} \leq 0$ in I_T and hence $a \leq c$ for all $(z, t) \in I_T$. $\square$

Furthermore, by using Theorem 4.5.4 on equations (4.18), we can provide the proof of Theorem 7.0.1.

Proof of Theorem 7.0.1. We consider the equations (4.22). Since $b = c \geq a$ for all time $t \in [0, T)$, we have:

$$\begin{aligned} -4 \left(\frac{1}{b^2} - \frac{1}{a^2} + \frac{c^2}{a^2 b^2} + \frac{a^2 + c^2 - b^2}{ab^2 c} \right) &= -4 \frac{(a+c)}{c^3} \leq 0, \\ -4 \left(\frac{1}{b^2} - \frac{1}{c^2} + \frac{a^2}{b^2 c^2} + \frac{a^2 + c^2 - b^2}{ab^2 c} \right) &= -4 \frac{(a^2 + ac)}{c^4} \leq 0. \end{aligned}$$

By applying Theorem 4.5.2, we deduce that there exist some constants $C_1, C_2 > 0$ which depend only on the data on $\mathcal{P}I_T$ such that $|\frac{\chi}{a}| \leq C_1$ and $|\frac{\chi}{c}| \leq C_2$. By choosing $\tilde{C}_0 = \max(C_1, C_2)$, we get the desired bound $|a - c| \leq \tilde{C}_0 \min(a, c)$. $\square$

In fact, this equation tells us more:

Lemma 7.2.2. Suppose that at time $t = 0$ we have $1 \leq \max_s \left(\frac{c}{a} \right) \leq \lambda$ for some constant $\lambda \geq 1$. Then $1 \leq \frac{c}{a} \leq \lambda$ for all $(z, t) \in [-1, 1] \times [0, T)$. $\blacksquare$

Proof. Since $\frac{\lambda}{a} = \frac{a-c}{a} = 1 - \frac{c}{a}$, equation (4.22) can be rewritten as:

$$\partial_t \left(\frac{c}{a} \right) = \left(\frac{c}{a} \right)'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) \left(\frac{c}{a} \right)' - 4 \frac{a+c}{c^3} \left(\frac{c-a}{a} \right) \leq \left(\frac{c}{a} \right)'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) \left(\frac{c}{a} \right)'.$$

Thus, the maximum principle tells us that $\frac{c}{a} \leq \max_{\mathcal{P}_{I_T}} \left(\frac{c}{a} \right) = \max(1, \lambda) = \lambda$ since $\frac{c}{a} \rightarrow 1$ as $z \rightarrow \pm 1$ and $\lambda \geq 1$. $\square$

Since we have the ordering $a(z, t) \leq c(z, t)$ at all points $(z, t) \in I_T$, we wish to study the behaviour of the local minima of the function a . Intuitively, one can think of the quantity a as the smaller “radii” of the $SU(2)$ fibre at z . We define the quantity $\check{a}$ that will be used in our study for the neckpinching of the manifold by:

$$\check{a}(t) = \min_s (a(s, t) : a'(s, t) = 0),$$

Necessarily we have $\check{a}(t) > 0$ as long as the Ricci flow exists. By picturing a neckpinch, we expect the quantity $\check{a}$ vanishes at the singular time T ; that is, $\check{a}(T) = 0$. Furthermore, we shall assume that for all but a finite number of times $t \in [0, T)$, the functions $a(\cdot, t)$ are Morse functions on $I = (-1, 1)$ and their critical values are smooth functions of time. Due to the existence of times at which $a(\cdot, t)$ are non-Morse functions, the minimum of these critical values $\check{a}(t)$ might have jumps. Therefore, it is a piecewise Lipschitz continuous function of time.

Lemma 7.2.3. Suppose that $a \leq c$ at time $t = 0$ and the functions $a(\cdot, t)$ are Morse functions for all but a finite number of times $t \in [0, T)$. If there exists a $T < \infty$ such that $\check{a}(T) = 0$, then $\check{a}(t)^2 \leq 4(T - t)$ for $t \geq t'$ where $t' \in [0, T)$ is the final time for which the function $a(\cdot, t)$ is not Morse. $\blacksquare$

Proof. We use the original coordinate $z \in (-1, 1)$ and copy the proof of Lemma 6.1.3. Let $t_0 \in [0, T)$ such that $a(z, t_0)$ is a Morse function on z and the smallest critical value of $a(z, t_0)$ is attained at some $z_0 \in (-1, 1)$. Since $a(\cdot, t_0)$ is a Morse function, by the Implicit Function Theorem, there is a smooth function $z(t)$ defined in a neighbourhood of $t = t_0$ such that $\partial_z a(z(t), t) = 0$. Thus at (z_0, t_0) , we have:

$$\begin{aligned} \frac{d}{dt} \check{a}(z(t), t) &= \partial_t a(z(t), t) + \cancel{\partial_z a(z(t), t)} \frac{dz(t)}{dt} = \partial_t a(z(t), t) \\ &= a'' + 2a' \left(\frac{c'}{c} \right) - a \left(\frac{2a^4}{a^2 c^4} \right) \geq -\frac{2a^3}{c^4}, \end{aligned} \tag{7.3}$$

since a attains a minimum at (z_0, t_0) . Next, from Lemma 7.2.1 we deduce $\frac{a^4}{c^4} \leq 1$ everywhere. This fact and the inequality (7.3) imply:

$$\frac{d}{dt} \check{a} \geq -\frac{2}{\check{a}} \quad \Rightarrow \quad \frac{d}{dt} (\check{a}^2) \geq -4,$$

for all but a finite number of $t \in [0, T)$. By integrating the differential inequality from $t \geq t'$ to the singular time T and using the fact that $\tilde{a}(T) = 0$, we deduce $\tilde{a}(t)^2 \leq 4(T - t)$ for every $t \in [t', T)$. $\square$

Remark 7.2.4. We note that the result above only holds for all t greater than the maximal time t' for which the function $a(\cdot, t)$ is not Morse. However, at earlier times the estimate is trivially true as the quantity a is bounded away from zero. Hence, similar estimate for the whole interval $[0, T)$ can be obtained with a different constant $C \geq 4$ by a compactness argument.

Furthermore, we have the following bound for the derivatives of a and c :

Lemma 7.2.5. Suppose that $a \leq c$ at time $t = 0$. Then for all $(s, t) \in I_T$ we have:

$$\begin{aligned} |a'(s, t)| &\leq \max(2, \max_s(|a'(s, 0)|)), \\ |c'(s, t)| &\leq \max(2, \max_s(|c'(s, 0)|)). \end{aligned} \quad \blacksquare$$

Proof. By setting $b \equiv c$ in equation (4.24), we have:

$$\partial_t(a') = a''' + a'' \left(2 \frac{c'}{c} - \frac{a'}{a} \right) - a' \left(2 \frac{(c')^2}{c^2} + \frac{6a^2}{c^4} \right) + \frac{8a^3 c'}{c^5}.$$

At $\tilde{a}' = \max_s(a'(s, t)) \geq 2$, by using the same method as in the Lemma 6.1.3, along the path $(z(t), t)$ for the maximum value of a' , we have the inequality:

$$\frac{d}{dt} \tilde{a}' \leq -\tilde{a}' \left(2 \frac{(c')^2}{c^2} + \frac{6a^2}{c^4} \right) + \frac{8a^3 c'}{c^5}.$$

By using Young's inequality, for any $\varepsilon > 0$ we acquire:

$$\frac{8a^3 c'}{c^5} \leq 4\varepsilon \frac{(c')^2}{c^2} + \frac{4a^6}{\varepsilon c^8} \leq 4\varepsilon \frac{(c')^2}{c^2} + \frac{4a^2}{\varepsilon c^4},$$

where we used the fact that $a \leq c$ is preserved for all $t > 0$ in the second inequality. By choosing $\varepsilon = 1$, we have:

$$\begin{aligned} \frac{d}{dt} \tilde{a}' &\leq -\tilde{a}' \left(2 \frac{(c')^2}{c^2} + \frac{6a^2}{c^4} \right) + 4 \frac{(c')^2}{c^2} + \frac{12a^2}{c^4} - \frac{8a^2}{c^4} \\ &< (2 - \tilde{a}') \left(2 \frac{(c')^2}{c^2} + \frac{6a^2}{c^4} \right), \end{aligned}$$

and therefore whenever $\tilde{a}' \geq 2$, we have $\frac{d}{dt} \tilde{a}' < 0$. Thus, we conclude that $\max_s(a'(s, t))$ cannot exceed 2 and $\max_s(a'(s, 0))$. With a similar argument, by using the negative of a , we can show that $\min_s(a'(s, t)) \geq -\max(2, \max_s(|a'(s, 0)|))$.

From equation (4.26), by setting $b \equiv c$, we have:

$$\partial_t(c') = c''' + c''\left(\frac{a'}{a}\right) + c'\left(\frac{4}{c^2} - \frac{(a')^2}{a^2} - \frac{(c')^2}{c^2} - \frac{6a^2}{c^4}\right) + 4\frac{aa'}{c^3}.$$

At $\tilde{c}' = \max_s(c'(s, t)) \geq 1$, by using the same method as before, along the path $(z(t), t)$ for the maximum value of c' , we have the inequality:

$$\frac{d}{dt}\tilde{c}' \leq \tilde{c}'\left(\frac{4}{c^2} - \frac{(a')^2}{a^2} - \frac{(c')^2}{c^2} - \frac{6a^2}{c^4}\right) + \frac{4aa'}{c^3}. \quad (7.4)$$

Supposing $\tilde{c}' \geq 2$, then we have $4 - (c')^2 \leq 0$ and thus, by using Young's inequality on the term $\frac{4aa'}{c^3}$ with $\varepsilon = \frac{1}{\sqrt{6}}$, we obtain:

$$\begin{aligned} \frac{d}{dt}\tilde{c}' &\leq -\tilde{c}'\left(\frac{(a')^2}{a^2} + \frac{6a^2}{c^4}\right) + \frac{4aa'}{c^3} \leq -\tilde{c}'\left(\frac{(a')^2}{a^2} + \frac{6a^2}{c^4}\right) + \frac{2}{\sqrt{6}}\frac{(a')^2}{a^2} + 2\sqrt{6}\frac{a^4}{c^6} \\ &\leq \left(\frac{2}{\sqrt{6}} - \tilde{c}'\right)\left(\frac{(a')^2}{a^2} + \frac{6a^2}{c^4}\right) < 0, \end{aligned}$$

where we used the fact that $a \leq c$ is preserved. Thus, the quantity $\max_s(c'(s, t))$ cannot exceed 2 and $\max_s(c'(s, 0))$. With a similar argument, by using the negative of c , we can show that $\min_s(c'(s, t)) \geq -\max(2, \max_s(|c'(s, 0)|))$. $\square$

We can obtain a better bound for c' if we impose an extra condition on the ratio $\frac{c}{a}$ at $t = 0$.

Lemma 7.2.6. Suppose that $a \leq c$ and $\max_s\left(\frac{c}{a}\right) \leq 2$ at time $t = 0$. Then we have:

$$|c'(s, t)| \leq \max\left(\sqrt{\frac{23}{8}}, \max_s(|c'(s, 0)|)\right). \quad \blacksquare$$

Proof. Since $\max_s\left(\frac{c}{a}\right) \leq 2$ at time $t = 0$, by Lemma 7.2.2, we have $\frac{c}{a} \leq 2$ everywhere and thus $-\frac{a^2}{c^2} \leq -\frac{1}{4}$. By substituting this in the inequality (7.4), along the path $(z(t), t)$ for the maximum value of c' , we have the inequality:

$$\begin{aligned} \frac{d}{dt}\tilde{c}' &\leq \tilde{c}'\left(\frac{4}{c^2} - \frac{(a')^2}{a^2} - \frac{(c')^2}{c^2} - \left(\frac{9a^2}{2c^4} + \frac{3a^2}{2c^4}\right)\right) + \frac{4aa'}{c^3} \\ &\leq \tilde{c}'\left(\frac{4}{c^2} - \frac{9}{8c^2} - \frac{(a')^2}{a^2} - \frac{(c')^2}{c^2} - \frac{3a^2}{2c^4}\right) + \frac{4aa'}{c^3} \\ &= \tilde{c}'\left(\frac{23}{8c^2} - \frac{(a')^2}{a^2} - \frac{(c')^2}{c^2} - \frac{3a^2}{2c^4}\right) + \frac{4aa'}{c^3}. \end{aligned}$$

By supposing $\tilde{c}' \geq \sqrt{\frac{23}{8}}$, we have $\frac{23}{8} - (c')^2 \leq 0$. Thus, by using Young's inequality on the

term $\frac{4aa'}{c^3}$ with $\varepsilon = \sqrt{\frac{2}{3}}$, we obtain the bound:

$$\begin{aligned} \frac{d}{dt}\hat{c}' &\leq -\hat{c}'\left(\frac{(a')^2}{a^2} + \frac{3a^2}{2c^4}\right) + \frac{4aa'}{c^3} \leq -\hat{c}'\left(\frac{(a')^2}{a^2} + \frac{3a^2}{2c^4}\right) + \sqrt{\frac{8}{3}}\frac{(a')^2}{a^2} + \sqrt{6}\frac{a^4}{c^6} \\ &\leq -\hat{c}'\left(\frac{(a')^2}{a^2} + \frac{3a^2}{2c^4}\right) + \sqrt{\frac{8}{3}}\frac{(a')^2}{a^2} + \sqrt{6}\frac{a^2}{c^4} \\ &= \left(\sqrt{\frac{8}{3}} - \hat{c}'\right)\left(\frac{(a')^2}{a^2} + \frac{3a^2}{2c^4}\right) < 0, \end{aligned}$$

since $\hat{c}' \geq \sqrt{\frac{23}{8}} > \sqrt{\frac{8}{3}}$. Thus, we conclude that the quantity $\max_s(c'(s, t))$ cannot exceed $\sqrt{\frac{23}{8}}$ and $\max_s(c'(s, 0))$. With a similar argument, by using the negative of c , we can show that $\min_s(c'(s, t)) \geq -\max\left(\sqrt{\frac{23}{8}}, \max_s(|c'(s, 0)|)\right)$. $\square$

Thus, we have the following result, which is similar to the result in [6]:

Corollary 7.2.7. Suppose that at time $t = 0$ we have $1 \leq \frac{c}{a} \leq 2$ and $|c'(s, 0)| \leq \sqrt{\frac{23}{8}}$. Then $|c'(s, t)| \leq \sqrt{\frac{23}{8}}$ for all $(s, t) \in I_T$. $\blacksquare$

Now we want to find a lower bound for the quantity $\check{a}(t)$. By imposing an assumption on the initial scalar curvature of the manifold as well as the ratio $\frac{c}{a}$, we have the following proposition:

Proposition 7.2.8. Suppose that at time $t = 0$, we have $1 \leq \frac{c}{a} \leq 2$, $|c'(s, 0)| \leq \sqrt{\frac{23}{8}}$, and $\min_{s, t=0}(S) \cdot \max_{s, t=0}(c^2) > -\frac{1}{4}$. If there exists a finite $T > 0$ such that $\check{a}(T) = 0$, then there exists a uniform constant $D > 0$ such that $\check{a}(t)^2 \geq D(T - t)$. $\blacksquare$

Proof. Let us consider the quantity $\log(ac^2)$. This quantity evolves according to:

$$\begin{aligned} \partial_t \log(ac^2) &= \frac{\partial_t a}{a} + 2\frac{\partial_t c}{c} = \frac{a''}{a} + 2\frac{c''}{c} + 2\left(2\frac{a'c'}{ac} + \frac{(c')^2}{c^2}\right) - 2(\hat{K}_{12} + \hat{K}_{13} + \hat{K}_{23}) \\ &= \frac{a''}{a} + 2\frac{c''}{c} + 2\left(2\frac{a'c'}{ac} + \frac{(c')^2}{c^2}\right) - 2\left(\frac{4a^2c^2 + 2c^4 - a^4 - 2c^4}{a^2c^4}\right) \\ &= \frac{a''}{a} + 2\frac{c''}{c} + 2\left(2\frac{a'c'}{ac} + \frac{(c')^2}{c^2}\right) - 2\left(\frac{4c^2 - a^2}{c^4}\right). \end{aligned}$$

Note that since the scalar curvature S satisfies $\partial_t S = \Delta S + \frac{n}{2}S^2 \geq \Delta S$ from equation (2.10), by minimum principle, we have $S \geq 2s_0$ for some s_0 depending only on the data on $\mathcal{P}I_T$. From this and the expression for the scalar curvature S in (4.13), we deduce:

$$\begin{aligned} 2s_0 &\leq -2\left(\frac{a''}{a} + 2\frac{c''}{c} + 2\frac{a'c'}{ac} + \frac{(c')^2}{c^2} - \frac{4a^2c^2 + 2c^4 - a^4 - 2c^4}{a^2c^4}\right) \\ \Rightarrow \quad \frac{a''}{a} + 2\frac{c''}{c} &\leq -\left(2\frac{a'c'}{ac} + \frac{(c')^2}{c^2}\right) + \frac{4c^2 - a^2}{c^4} - s_0. \end{aligned}$$

Thus, we have:

$$\begin{aligned} \partial_t \log(ac^2) &\leq 2 \frac{a'c'}{ac} + \frac{(c')^2}{c^2} - \frac{4c^2 - a^2}{c^4} - s_0, \\ \Rightarrow \quad \partial_t(ac^2) &\leq \left(2 \frac{a'c'}{ac} + \frac{(c')^2}{c^2} - \frac{4c^2 - a^2}{c^4} - s_0 \right) ac^2 \\ &\leq \left(2 \frac{a'c'}{ac} + \frac{(c')^2}{c^2} - \frac{3}{c^2} - s_0 \right) ac^2, \end{aligned} \quad (7.5)$$

by using the fact that $a \leq c$. Furthermore, from our assumption on the ratio $\frac{c}{a}$, $|c'(s, 0)|$, and Corollary 7.2.7, we have $(c')^2 - 3 \leq \frac{23}{8} - 3 = -\frac{1}{8}$. Thus, along the path $(z(t), t)$ for which $a' = 0$ we have:

$$\frac{d}{dt}(ac^2) \leq \left(2 \frac{a'c'}{ac} - \frac{1}{8c^2} - s_0 \right) ac^2. \quad (7.6)$$

We note that $\hat{c}(t) = \max_s(c(s, t))$ is a decreasing function of time since equations (4.12) and (6.2) imply:

$$\frac{d}{dt}\hat{c} \leq -2a^2c \left(\frac{2c^2 - a^2}{c^4} \right) < 0.$$

Furthermore, by our assumption at $t = 0$ and compactness of the interval $I = [-1, 1]$, for some $0 < D_0 < \frac{1}{4}$ small enough, we have $\min_{s, t=0}(S) \cdot \max_{s, t=0}(c^2) \geq -\frac{1}{4} + D_0$. This implies:

$$2s_0 \geq \min_{s, t=0}(S) \geq \frac{1}{\max_{s, t=0}(c^2)} \left(-\frac{1}{4} + D_0 \right) \geq \frac{1}{c^2} \left(-\frac{1}{4} + D_0 \right) \Rightarrow -\frac{1}{8c^2} - s_0 \leq -\frac{D_0}{2c^2}.$$

By substituting this bound in the inequality (7.6), along the local minimum values of a , we have:

$$\frac{d}{dt}(ac^2)(z(t), t) \leq \left(-\frac{1}{8c^2} - s_0 \right) ac^2 \leq -\frac{D_0 a}{2}.$$

If we set $m(t) = ac^2(z(t), t)$ and use Lemma 7.2.2 with $C_0 = 2$, we obtain:

$$\frac{d}{dt}m(z(t), t) \leq -\frac{D_0 a}{2} \leq -\frac{D_0}{2^{\frac{5}{3}}}(ac^2)^{\frac{1}{3}} = -D_1 m^{\frac{1}{3}},$$

for all but a finite number of $t \in [0, T)$ where $D_1 > 0$ is some constant. By integrating this inequality from any $t < T$ to the singular time T and noting that $m(T) = 0$ (since $\check{a}(T) = 0$), we have $m(t)^{\frac{2}{3}} \geq \frac{2D_1}{3}(T - t)$. Furthermore, by using Lemma 7.2.2 with $C_0 = 2$ again, we have the inequality:

$$\check{a}(t)^3 \geq \frac{1}{4}ac^2(z(t), t) = \frac{1}{4}m(t) \geq \frac{D_1^{\frac{3}{2}}}{3\sqrt{6}}(T - t)^{\frac{3}{2}},$$

and thus, we conclude that $\check{a}(t)^2 \geq D(T - t)$ for some uniform constant $D > 0$. $\square$

Remark 7.2.9. The condition $1 \leq \frac{c}{a} \leq 2$ can be weakened but only very slightly. By considering $\frac{c}{a} \leq \lambda$ for some constant $\lambda \geq 1$, the inequality (7.4) can be written as:

$$\begin{aligned} \frac{d}{dt} \hat{c}' &\leq \hat{c}' \left(\frac{4}{c^2} - \frac{(a')^2}{a^2} - \frac{(c')^2}{c^2} - \frac{6a^2}{c^4} \right) + \frac{4aa'}{c^3} \\ &= \hat{c}' \left(\frac{4}{c^2} - \frac{(6-\kappa)a^2}{c^4} - \frac{(c')^2}{c^2} - \frac{(a')^2}{a^2} - \frac{\kappa a^2}{c^4} \right) + \frac{4aa'}{c^3} \\ &\leq \frac{\hat{c}'}{c^2} \left(4 - \frac{6-\kappa}{\lambda^2} - (c')^2 \right) - \hat{c}' \left(\frac{\kappa a^2}{c^4} + \frac{(a')^2}{a^2} \right) + 2\varepsilon \frac{(a')^2}{a^2} + \frac{2}{\varepsilon} \frac{a^2}{c^4} \\ &= \frac{\hat{c}'}{c^2} \left(4 - \frac{6-\kappa}{\lambda^2} - (c')^2 \right) + \left(\frac{2}{\sqrt{\kappa}} - \hat{c}' \right) \left(\frac{\kappa a^2}{c^4} + \frac{(a')^2}{a^2} \right), \end{aligned}$$

for some $0 < \kappa < 6$ and by choosing $\varepsilon = \frac{1}{\sqrt{\kappa}}$ in the application of Young's inequality. In order to conclude the bound for $\hat{c}'$, we require for the sake of contradiction $c' \geq \sqrt{4 - \frac{6-\kappa}{\lambda^2}} \geq \frac{2}{\sqrt{\kappa}}$. By contradiction, this would then imply that $c' \leq \sqrt{4 - \frac{6-\kappa}{\lambda^2}}$ supposing that the derivative at initial time also satisfies the same bound.

Furthermore, in order to utilise the inequality for c' in (7.5), we must have $\sqrt{3} > \sqrt{4 - \frac{6-\kappa}{\lambda^2}}$. Thus to find the admissible values of λ , we have to solve the inequality:

$$\frac{4}{\kappa} \leq 4 - \frac{6-\kappa}{\lambda^2} < 3,$$

which implies that $\kappa > \frac{4}{3}$ and $\lambda < \sqrt{\frac{14}{3}} \approx 2.16$. Therefore, the condition on the ratio $\frac{c}{a}$ at initial time can only be slightly improved. $\blacksquare$

7.3 Curvature Behaviour

[85] We note that a finite time singularity occurs at $T < \infty$ only if:

$$\limsup_{t \nearrow T} \max_{p \in M} |\text{Ric}(p, t)|_g = \infty.$$

Furthermore, this singularity is of Type I if we have:

$$\sup_{t \in [0, T)} \max_{p \in M} (T - t) |\text{Rm}(p, t)|_g \leq C < \infty,$$

for some constant $C > 0$. Thus, we now want to investigate the behaviour of the norm of the Riemann curvature tensor, which is given by:

$$|\text{Rm}|_g^2 = 2(K_{01}^2 + K_{02}^2 + K_{03}^2 + K_{12}^2 + K_{13}^2 + K_{23}^2).$$

Lemma 7.3.1. If $1 \leq \frac{c}{a} \leq 2$, $|a'(s, 0)| \leq \nu$, and $|c'(s, 0)| \leq \sqrt{\frac{23}{8}}$ at $t = 0$, then for all $p \in M$ we have:

$$|K_{12}(p, t)| + |K_{13}(p, t)| + |K_{23}(p, t)| \leq \frac{C(\nu)}{a(\pi(p), t)^2}. \quad \blacksquare$$

Proof. The sectional curvatures K_{12} , K_{13} , and K_{23} are given by:

$$\begin{aligned} K_{12} = K_{13} &= -\frac{a'c'}{ac} + \frac{a^2}{c^4}, \\ K_{23} &= -\frac{(c')^2}{c^2} + \frac{4}{c^2} - \frac{3a^2}{c^4}. \end{aligned}$$

Thus, by Corollary 7.2.7 and the fact that $a \leq c$ for all $t \geq 0$, we have the bound:

$$\begin{aligned} |K_{12}| + |K_{13}| + |K_{23}| &\leq \frac{|a'|c'|}{ac} + \frac{a^2}{c^4} + \frac{|c'|^2}{c^2} + \frac{4}{c^2} + \frac{3a^2}{c^4} \\ &\leq \frac{\nu\sqrt{23}}{\sqrt{8}ac} + \frac{1}{c^2} + \frac{23}{8c^2} + \frac{4}{c^2} + \frac{3}{c^2} \leq \left(\frac{\nu\sqrt{23}}{\sqrt{8}} + \frac{87}{8} \right) \frac{1}{a^2} = \frac{C(\nu)}{a^2}, \end{aligned}$$

for some constant $C(\nu) > 0$ depending only on the initial data ν . $\square$

We now want to bound the sectional curvatures K_{01} , K_{02} , and K_{03} . These curvatures involve the second order derivatives of the quantities a and c . We define the following quantities:

$$\begin{aligned} Q_{\pm} &= cc'' \pm (\mathfrak{A}(a')^2 + \mathfrak{B}(c')^2), \\ P_{\pm} &= aa'' \pm (\mathfrak{C}(a')^2 + \mathfrak{D}(c')^2), \end{aligned}$$

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} > 0$ are some arbitrary constants. Note that at the boundaries of the interval I , we have $Q_{\pm} = \pm(\mathfrak{A} + \mathfrak{B})$ and $P_{\pm} = \pm(\mathfrak{C} + \mathfrak{D})$. Our aim now is to bound $Q_{\pm}$ and $P_{\pm}$. By using the bounds for a' and c' from Lemma 7.2.5, the bounds for $Q_{\pm}$ and $P_{\pm}$ would imply the bounds of the quantities a'' and c'' which appear in the expressions of the sectional curvatures K_{01} , K_{02} , and K_{03} .

Lemma 7.3.2. If $1 \leq \frac{c}{a} \leq 2$, $|a'(s, 0)| \leq \nu$, and $|c'(s, 0)| \leq \sqrt{\frac{23}{8}}$ at $t = 0$, then for all $p \in M$ we have:

$$|K_{01}(p, t)| + |K_{02}(p, t)| + |K_{03}(p, t)| \leq \frac{C(\nu)}{a(\pi(p), t)^2}. \quad \blacksquare$$

Proof. The evolution equations of the quantities $Q_{\pm}$ can be computed to be:

$$\begin{aligned} \partial_t Q_{\pm} &= \Delta_g Q_{\pm} - \frac{4c'}{c} Q'_{\pm} + (\mp 2\mathfrak{B} - 2)(c'')^2 - c'' \left(\frac{2c(a')^2}{a^2} + \frac{4a^2}{c^3} \mp \frac{4\mathfrak{B}(c')^2}{c} \right) \\ &\quad \mp 2\mathfrak{A}(a'')^2 + a'' \left(\mp \frac{4\mathfrak{A}(a')^2}{a} + \frac{4a}{c^2} \pm \frac{8\mathfrak{A}a'c'}{c} - \frac{2ca'c'}{a^2} \right) \mp \frac{12\mathfrak{A}a^2(a')^2}{c^4} + \frac{4(a')^2}{c^2} \\ &\quad + (c')^4 \left(\mp \frac{2\mathfrak{B}}{c^2} + \frac{2}{c^2} \right) + (c')^2 \left(\mp \frac{4\mathfrak{A}(a')^2}{c^2} \mp \frac{2\mathfrak{B}(a')^2}{a^2} \mp \frac{12\mathfrak{B}a^2}{c^4} + \frac{24a^2}{c^4} \pm \frac{8\mathfrak{B}}{c^2} - \frac{8}{c^2} \right) \\ &\quad + c' \left(\pm \frac{16\mathfrak{A}a^3a'}{c^5} \pm \frac{8\mathfrak{B}aa'}{c^3} - \frac{24aa'}{c^3} + \frac{2c(a')^3}{a^3} \right). \end{aligned}$$

By a similar argument to Lemma 6.1.3, we study the maximum of Q_+ and minimum of Q_- by tracking their behaviour along the curves $(z(t), t)$ defined by the critical points of $Q_{\pm}$ in

the space coordinate. Along the maximum of Q_+ , which we denote by $\hat{Q}_+ = \max_s(Q_+(s, t))$, we have:

$$\begin{aligned} \frac{d}{dt}\hat{Q}_+(z(t), t) \leq & (-2\mathfrak{B} - 2)(c'')^2 - c'' \left(\frac{2c(a')^2}{a^2} + \frac{4a^2}{c^3} - \frac{4\mathfrak{B}(c')^2}{c} \right) \\ & - 2\mathfrak{A}(a'')^2 + a'' \left(-\frac{4\mathfrak{A}(a')^2}{a} + \frac{4a}{c^2} + \frac{8\mathfrak{A}a'c'}{c} - \frac{2ca'c'}{a^2} \right) - \frac{12\mathfrak{A}a^2(a')^2}{c^4} + \frac{4(a')^2}{c^2} \\ & + (c')^4 \left(-\frac{2\mathfrak{B}}{c^2} + \frac{2}{c^2} \right) + (c')^2 \left(-\frac{4\mathfrak{A}(a')^2}{c^2} - \frac{2\mathfrak{B}(a')^2}{a^2} - \frac{12\mathfrak{B}a^2}{c^4} + \frac{24a^2}{c^4} + \frac{8\mathfrak{B}}{c^2} - \frac{8}{c^2} \right) \\ & + c' \left(\frac{16\mathfrak{A}a^3a'}{c^5} + \frac{8\mathfrak{B}aa'}{c^3} - \frac{24aa'}{c^3} + \frac{2c(a')^3}{a^3} \right). \end{aligned} \quad (7.7)$$

Now we wish to estimate the terms which involve the second derivatives of a and c ; that is, we bound the first four terms in the right hand side of (7.7). By using the AM-GM inequality, we acquire the following inequalities:

$$\begin{aligned} & - (2\mathfrak{B} + 2)(c'')^2 - c'' \left(\frac{2c(a')^2}{a^2} + \frac{4a^2}{c^3} - \frac{4\mathfrak{B}(c')^2}{c} \right) \\ & \leq \left(-2\mathfrak{B} - 2 + \frac{3}{2} \right) (c'')^2 + \frac{2c^2(a')^2}{a^4} + \frac{8a^4}{c^6} + \frac{8\mathfrak{B}^2(c')^4}{c^2}, \end{aligned} \quad (7.8)$$

$$\begin{aligned} & - 2\mathfrak{A}(a'')^2 - a'' \left(\frac{4\mathfrak{A}(a')^2}{a} - \frac{4a}{c^2} - \frac{8\mathfrak{A}a'c'}{c} + \frac{2ca'c'}{a^2} \right) \\ & \leq (-2\mathfrak{A} + 2)(a'')^2 + \frac{8\mathfrak{A}^2(a')^4}{a^2} + \frac{8a^2}{c^4} + \frac{32\mathfrak{A}^2(a')^2(c')^2}{c^2} + \frac{2c^2(a')^2(c')^2}{a^4}. \end{aligned} \quad (7.9)$$

Hence, by choosing $\mathfrak{A} = 1$ and $\mathfrak{B} = \frac{1}{4}$, if we substitute the inequalities (7.8) and (7.9) in (7.7), we obtain:

$$\begin{aligned} \frac{d}{dt}\hat{Q}_+ \leq & -(c'')^2 + \frac{2c^2(a')^2}{a^4} + \frac{8a^4}{c^6} + \frac{(c')^4}{2c^2} + \frac{8(a')^4}{a^2} + \frac{8a^2}{c^4} + \frac{32(a')^2(c')^2}{c^2} + \frac{2c^2(a')^2(c')^2}{a^4} \\ & - \frac{12a^2(a')^2}{c^4} + \frac{4(a')^2}{c^2} + (c')^4 \left(-\frac{1}{2c^2} + \frac{2}{c^2} \right) + (c')^2 \left(-\frac{4(a')^2}{c^2} - \frac{(a')^2}{2a^2} + \frac{21a^2}{c^4} - \frac{6}{c^2} \right) \\ & + c' \left(\frac{16a^3a'}{c^5} - \frac{22aa'}{c^3} + \frac{2c(a')^3}{a^3} \right). \end{aligned}$$

Except for the first term, the other terms in inequality can be estimated using the hypotheses $1 \leq \frac{c}{a} \leq 2$, $|a'(s, 0)| \leq \nu$, and $|c'(s, 0)| \leq \sqrt{\frac{23}{8}}$ at $t = 0$ along with the fact that these conditions are preserved under the Ricci flow via Lemma 7.2.2, Lemma 7.2.5, and Corollary 7.2.7. Thus, we have:

$$\frac{d}{dt}\hat{Q}_+ \leq -(c'')^2 + \frac{\tilde{C}_1(\nu)}{c^2} \quad \Rightarrow \quad c^2 \frac{d}{dt}\hat{Q}_+ \leq -(cc'')^2 + \tilde{C}_1(\nu), \quad (7.10)$$

for some constant $\tilde{C}_1(\nu) > 0$ depending only on the initial data ν . Note that $Q_+ = cc'' + (a')^2 + \frac{(c')^2}{4} \leq cc'' + \nu^2 + \frac{23}{32}$. Hence, whenever $Q_+ \geq \nu^2 + \frac{23}{32}$ we have $(Q_+ - \nu^2 - \frac{23}{32})^2 \leq (cc'')^2$. By substituting this in (7.10), for large enough $\hat{Q}_+$ we have the inequality:

$$c^2 \frac{d}{dt} \hat{Q}_+ \leq - \left(\hat{Q}_+ - \nu^2 - \frac{23}{32} \right)^2 + \tilde{C}_1(\nu).$$

This implies that whenever $\hat{Q}_+$ is large enough (more specifically $\hat{Q}_+ > \tilde{C}_1(\nu)^{\frac{1}{2}} + \nu^2 + \frac{23}{32} := C_1(\nu)$), the maximum of Q_+ will be strictly decreasing. Thus, we conclude that:

$$Q_+ \leq \max \left(C_1(\nu), \max_s (Q_+(s, 0)) \right).$$

By using a similar argument, if we choose $\mathfrak{A} = 1$ and $\mathfrak{B} = \frac{9}{4}$ in Q_- , then by tracking the minimum value of Q_- , we can show that:

$$Q_- \geq \min \left(-C_2(\nu), \min_s (Q_-(s, 0)) \right),$$

for some constant $C_2(\nu) > 0$. Thus, since $Q_- \leq cc'' \leq Q_+$, we have that $|cc''| \leq C_0(\nu)$ for some constant $C_0(\nu) = \max \left(C_1(\nu), C_2(\nu), \max_s Q_+(s, 0), -\min_s Q_-(s, 0) \right)$. Hence, we conclude that $|K_{02}| = |K_{03}| \leq \frac{C_0(\nu)}{c^2} \leq \frac{C_0(\nu)}{a^2}$.

To bound the sectional curvature K_{01} , we repeat the same argument for the quantities $P_{\pm}$, which evolve according to the following equations:

$$\begin{aligned} \partial_t P_{\pm} = & \Delta_g P_{\pm} - \frac{4a'}{a} P'_{\pm} + (\mp 2\mathfrak{C} - 2)(a'')^2 + a'' \left(\frac{4(a')^2}{a} \pm \frac{4(a')^2 \mathfrak{C}}{a} - \frac{4a(c')^2}{c^2} - \frac{8a^3}{c^4} \right) \\ & \mp 2\mathfrak{D}(c'')^2 + c'' \left(\pm \frac{8\mathfrak{D}a'c'}{a} - \frac{4aa'c'}{c^2} + \frac{8a^4}{c^5} \mp \frac{4\mathfrak{D}(c')^2}{c} \right) \mp \frac{2\mathfrak{D}(c')^4}{c^2} \mp \frac{12\mathfrak{D}a^2(c')^2}{c^4} \\ & - \frac{40a^4(c')^2}{c^6} \pm \frac{8\mathfrak{D}(c')^2}{c^2} + (a')^2 \left(\mp \frac{2\mathfrak{D}(c')^2}{a^2} \mp \frac{12\mathfrak{C}a^2}{c^4} - \frac{12a^2}{c^4} \mp \frac{4\mathfrak{C}(c')^2}{c^2} \right) \\ & + a' \left(\pm \frac{16\mathfrak{C}a^3c'}{c^5} \pm \frac{8\mathfrak{D}ac'}{c^3} + \frac{48a^3c'}{c^5} + \frac{4a(c')^3}{c^3} \right) \end{aligned}$$

By choosing $\mathfrak{C} = \frac{1}{2}$ and $\mathfrak{D} = 1$, one can show that:

$$P_+ \leq \max \left(A_1(\nu), \max_s (P_+(s, 0)) \right),$$

for some constant $A_1(\nu) > 0$. Similarly, for the choices $\mathfrak{C} = \frac{5}{2}$ and $\mathfrak{D} = 1$, we have:

$$P_- \geq \min \left(-A_2(\nu), \min_s (P_-(s, 0)) \right),$$

for some constant $A_2(\nu) > 0$. Putting these together, we have $|K_{01}| \leq \frac{A_0(\nu)}{a^2}$ for some constant $A_0(\nu) = \max \left(A_1(\nu), A_2(\nu), \max_s P_+(s, 0), -\min_s P_-(s, 0) \right)$. Thus, we finally conclude that:

$$|K_{01}| + |K_{02}| + |K_{03}| \leq \frac{A_0(\nu)}{a^2} + \frac{2C_0(\nu)}{a^2} = \frac{C(\nu)}{a^2},$$

for some constant $C(\nu) := A_0(\nu) + 2C_0(\nu) > 0$. □

From the previous two lemmas and the fact that the norm of the Riemannian curvature tensor is given by:

$$|\mathrm{Rm}|_g^2 = 2(K_{01}^2 + K_{02}^2 + K_{03}^2 + K_{12}^2 + K_{13}^2 + K_{23}^2),$$

we conclude the following:

Corollary 7.3.3. If $1 \leq \frac{c}{a} \leq 2$, $|a'(s, 0)| \leq \nu$, and $|c'(s, 0)| \leq \sqrt{\frac{23}{8}}$ at $t = 0$, then for all $p \in M$ we have:

$$|\mathrm{Rm}(p, t)|_g \leq \frac{\tilde{C}(\nu)}{a(\pi(p), t)^2},$$

for some constant $\tilde{C}(\nu) > 0$ as long as the solution to the Ricci flow exists. ■

By using the lower bound for $\check{a}$ from Proposition 7.2.8, we obtain the inequality $(T - t)^2 |\mathrm{Rm}|_g^2 \leq C$ for some finite constant $C > 0$. If we take the supremum of the left hand side over $I \times S^3$ and $t \in [0, T)$, we can show that the curvature asymptotics satisfy the Type I condition as in (2.11).

Finally, we note that the metric constructed by Angenent and Knopf also satisfies the conditions required for Proposition 7.2.8, so the set of metrics satisfying the sufficient conditions in Theorem 7.0.2 is non-empty. Thus, this proves Theorem 7.0.2.

We have seen that the Riemann curvature tensor and the sectional curvatures blow up only if the quantity a vanishes. Thus, the only possibility for this to happen is at the poles of S^4 (that is, when $z \in \{\pm 1\}$) and at the singularity time $T < \infty$, when $\check{a}(T) = 0$.

7.4 Smoothness at the Poles

In the work [6] the authors showed that the singularity would not occur at the poles under certain conditions. By symmetry, it is sufficient to consider one of the poles. Without loss of generality, we focus our attention to the neighbourhood of $p_+ = \pi^{-1}(1) \in S^4$. Let us first introduce the following:

Definition 7.4.1. Let $\check{a}(t)$ be defined as the quantity $a(z_1(t))$ such that $z_1(t) = \max(z : a'(z, t) = 0)$. Similarly, let $\check{c}(t)$ be defined as the quantity $c(z_2(t))$ such that $z_2(t) = \max(z : c'(z, t) = 0)$. We call the quantities $\check{a}$ and $\check{c}$ as the right-most a -bump and right-most c -bump respectively. ■

Note that from equations (4.15)-(4.17), we have:

$$\begin{aligned}\partial_t(a^2) &= 2aa'' + 4aa'\frac{c'}{c} - 4\frac{a^4}{c^4}, \\ \partial_t(c^2) &= 2cc'' + 2cc'\left(\frac{a'}{a} + \frac{c'}{c}\right) + 4\frac{a^2 - 2c^2}{c^2}.\end{aligned}$$

Thus, assuming that at the initial time we have $1 \leq \frac{c}{a} \leq 2$, $|a'| \leq \nu$, and $|c'| \leq \sqrt{\frac{23}{8}}$, Lemma 7.3.2, implies:

$$|\partial_t(a^2)| \leq 2|P_+| + 4|a'||c'| + 4 \leq C_1(\nu),$$

for some constant $C_1(\nu) > 0$. Similarly, the time derivative of c^2 can be bound by:

$$|\partial_t(c^2)| \leq 2|Q_+| + 4|a'||c'| + 2|c'|^2 + 8 \leq C_2(\nu),$$

for some constant $C_2(\nu) > 0$. This shows that for all $z \in (-1, 1)$ the limits $\lim_{t \rightarrow T} a(z, t)$ and $\lim_{t \rightarrow T} c(z, t)$ exist. In particular, both $\lim_{t \rightarrow T} \check{a}(t)$ and $\lim_{t \rightarrow T} \check{c}(t)$ exist.

We denote $z_*(t) = \max(z_1(t), z_2(t))$ where $z_1(t)$ and $z_2(t)$ were defined in Definition 7.4.1 and $C_0 = \max(C_1(\nu), C_2(\nu))$. We define the cap region to be the closure of the set $\pi^{-1}([z_*(t), 1)) \subset S^4$. Note that in the cap region we have $a' \leq 0$ and $c' \leq 0$. Let $\lim_{t \rightarrow T} a(z_*(t)) = A$. Our aim now is to show that if $A > 0$, then the Riemann curvature remains bounded in the cap region. If we have this assumption, then necessarily $\sup_t z_*(t) < 1$.

The crucial step towards this goal in [6] is to show that the space derivative of a is bounded uniformly close to the pole after some time $t_1 > 0$; that is:

Assumption 7.4.2. There exist some $t_1 > 0$, $z_4 \in [\sup_{t \geq t_1} z_*(t), 1)$, and $\delta > 0$ such that $a'(z, t) < -\delta$ for all $z \in [z_4, 1)$ and $t \in [t_1, T)$. ■

By assuming this, we have the following theorem:

Theorem 7.4.3. Suppose that we run a Ricci flow on the manifold (S^4, g_0) where the initial metric g_0 is of the form:

$$g_0 = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + c(z)^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

for $z \in I = (-1, 1)$ and $0 < a(z) \leq c(z)$ such that $1 \leq \frac{c}{a} \leq 2$, $|a'(s, 0)| \leq \nu$, and $|c'(s, 0)| \leq \sqrt{\frac{23}{8}}$ at $t = 0$. Let $z_1(t) = \max(z : a'(z, t) = 0)$ and such $z_2(t) = \max(z : c'(z, t) = 0)$. Suppose also that $\lim_{t \rightarrow T} a(z_*(t)) = A > 0$ where $z_*(t) = \max(z_1(t), z_2(t))$. Assume that there exist $t_1 > 0$, $z_4 \in [\sup_{t \geq t_1} z_*(t), 1)$, and $\delta > 0$ such that $a'(z, t) < -\delta$ for all $z \in [z_4, 1)$ and $t \in [t_1, T)$. Then the curvature remains bounded at the poles of S^4 . ■

Proof. We prove that the curvature cannot blow up at the cap region by using a blow-up argument. For contradiction, let us assume that the Riemann curvature tensor is not bounded in the cap region $P = \pi^{-1}([z_4, 1))$ as $t \rightarrow T$; that is:

$$\limsup_{t \rightarrow T} \left(\sup_P |\text{Rm}(\cdot, t)| \right) = \infty. \quad (7.11)$$

For $i \geq 2$, we choose an increasing sequence of times $t_1 < t_i \rightarrow T$ as $i \rightarrow \infty$ such that:

$$\sup_{t \leq t_i} \left(\sup_P |\text{Rm}(\cdot, t)| \right) = |\text{Rm}(p_i, t_i)|,$$

for some $p_i \in P$. For this choice of points and times, by equation (7.11), we have a sequence $\varepsilon_i = |\text{Rm}(p_i, t_i)| \rightarrow \infty$ as $i \rightarrow \infty$.

We now define a sequence of parabolic dilations of the metric g , which we denote by $g_i(t) = \varepsilon_i g\left(t_i + \frac{t}{\varepsilon_i}\right)$ on the manifold M . Via the choice of the rescalings, we have that the norm of the Riemann curvature for each g_i is uniformly bounded by 1 for $t \in [-\varepsilon_i t_i, 0]$; that is, $|\text{Rm}_i| \leq 1$ for $t \in [-\varepsilon_i t_i, 0]$ with equality attained at the point p_i at $t = 0$ for each i . So for each i , $g_i(t)$ is a solution to the Ricci flow equation that exists for $t \in [-\varepsilon_i t_i, 0]$.

Moreover, let $\zeta_i = \pi(p_i) \in [z_4, 1)$ be the projection of the points p_i onto the base $I = (-1, 1)$ of the foliation. From Corollary 7.3.3, we know that there exists a uniform constant $C(\nu) > 0$ such that $|\text{Rm}(p, t)| \leq \frac{C(\nu)}{a(\pi(p), t)^2}$. Necessarily, we have $\zeta_i = \pi(p_i) \rightarrow 1$. Furthermore, at the space-time points (p_i, t_i) we have:

$$|\text{Rm}(p_i, t_i)| \leq \frac{C(\nu)}{a(\zeta_i, t_i)^2} \quad \Rightarrow \quad a(\zeta_i, t_i) \leq \sqrt{\frac{C(\nu)}{\varepsilon_i}}. \quad (7.12)$$

From Assumption 7.4.2 we have $-a'(z, t) \geq \delta$. Then, by denoting the distance between the point p_i and the pole $p^+ = \pi^{-1}(1)$ at time t_i measured with respect to the metric g by $d(p_i, p^+; t_i)$, we have:

$$\delta \leq \frac{a(\zeta_i, t_i) - a(1, t_i)}{s(1, t_i) - s(\zeta_i, t_i)} = \frac{a(\zeta_i, t_i)}{d(p_i, p^+; t_i)}.$$

By substituting (7.12) in this, we get:

$$d(p_i, p^+; t_i) \leq \frac{a(\zeta_i, t_i)}{\delta} \leq \frac{1}{\delta} \sqrt{\frac{C(\nu)}{\varepsilon_i}}.$$

If we denote the distance between the points p_i and the pole p_+ at time $t = 0$ measured with respect to the metric g_i by $d_i(p_i, p^+; 0)$, we have:

$$d_i(p_i, p^+; 0) \leq \frac{\sqrt{C(\nu)}}{\delta}.$$

Thus, we conclude that the points p_i at which the maximal curvature occurs are uniformly bounded from the pole. Hence, from the pointed sequence (M, g_i, p_i) of Riemannian manifolds, by virtue of Perelman's No Local Collapsing Theorem and Hamilton's Compactness Theorem (Theorems 2.4.5 and 2.4.4), we can extract a convergent subsequence whose limit is a complete ancient solution $(M_\infty, g_\infty, p_\infty)$ that exists for $t \in (-\infty, 0]$. We introduce radial coordinates for the metric g by shifting the s variable thus:

$$r_i(z, t) = \sqrt{\varepsilon_i} \left(s \left(1, t_i + \frac{t}{\varepsilon_i} \right) - s \left(z, t_i + \frac{t}{\varepsilon_i} \right) \right),$$

so that $r_i(p^+) = 0$ for all $i \in \mathbb{N}$. The limit Riemannian manifold has uniformly bounded sectional curvatures and non-zero sectional curvature occurring at time $t = 0$ at the point p_∞ which is bounded from the origin. Furthermore, the metric g_i can be written as:

$$g_i = dr_i^2 + \varepsilon_i a(r_i(z, t), t)^2 w^1 \otimes w^1 + \varepsilon_i c(r_i(z, t), t)^2 (w^2 \otimes w^2 + w^3 \otimes w^3). \quad (7.13)$$

If we define $A = \sqrt{\varepsilon_i} a$, $C = \sqrt{\varepsilon_i} c$, and $\tau = t_i + \frac{t}{\varepsilon_i}$, the Ricci flow equations (7.1)-(7.2) for the metric g_i becomes:

$$\frac{1}{\varepsilon_i^2} \partial_\tau A = \partial_{r_i}^2 A + 2 \frac{\partial_{r_i} A \partial_{r_i} C}{C} - 2 \frac{A^3}{C^4}, \quad (7.14)$$

$$\frac{1}{\varepsilon_i^2} \partial_\tau C = \partial_{r_i}^2 C + \frac{\partial_{r_i} A \partial_{r_i} C}{A} + \frac{(\partial_{r_i} C)^2}{C} - 2 \frac{(2C^2 - A^2)}{C^3}, \quad (7.15)$$

By taking the limit as $i \rightarrow \infty$, we note also that $\zeta_i \rightarrow 1$. By our ansatz $\frac{a}{c} \rightarrow 1$ and $\frac{\partial_s a}{\partial_s c} \rightarrow 1$ as $z \rightarrow 1$, we have $\frac{A}{C} \rightarrow 1$ and $\frac{\partial_{r_i} A}{\partial_{r_i} C} \rightarrow 1$. Thus, equations (7.14)-(7.15) yield the ODE:

$$A \frac{d^2 A}{dr^2} + 2 \left(\frac{dA}{dr} \right)^2 - 2 = 0, \quad (7.16)$$

where r is the limiting radial coordinate. Thus, our limit metric has the form:

$$g = dr^2 + A(r)^2 g_{S^3},$$

for some positive function $A(r) \geq 0$ satisfying (7.16). To explicitly find the function $A(r)$, we substitute $\alpha(A) = \frac{dA}{dr}$ in equation (7.16). We note that:

$$\frac{d^2 A}{dr^2} = \frac{d}{dr} \alpha(A) = \frac{d\alpha}{dA} \frac{dA}{dr} = \alpha \frac{d\alpha}{dA}.$$

Thus, equation (7.16) becomes the separable differential equation:

$$A\alpha \frac{d\alpha}{dA} + 2\alpha^2 - 2 = 0 \quad \Rightarrow \quad A \frac{d\alpha}{dA} = \frac{2 - 2\alpha^2}{\alpha}.$$

Solving this equation, we get:

$$\log(\alpha^2 - 1) = -4\log(A) + C \quad \Rightarrow \quad \alpha = \pm \frac{\sqrt{A^4 + D}}{A^2},$$

for some constants $C, D \in \mathbb{R}$.

For smoothness at $r = 0$, by the one-sided cylinder to-sphere rule in Lemma 4.0.2, we must have $A(0) = 0$, and $\frac{dA}{dr}(0) = 1$. This implies that $\alpha \rightarrow 1$ as $A \rightarrow 0$. With this, we conclude that $D = 0$ and $\alpha = \frac{dA}{dr} = 1$. Further solving this equation with the condition $A(0) = 0$ gives us $A(r) = r$. Thus, the limiting metric $g = dr^2 + r^2 g_{S^3}$ is a flat metric on $\mathbb{R}^4$. This contradicts the fact that there exists a point p_∞ close to the origin such that g has non-zero sectional curvature occurring at this point at $t = 0$. $\square$

We have proved a weak theorem by assuming Assumption 7.4.2. To strengthen it further, we need to prove Assumption 7.4.2. This is still out of reach with our analysis and here are two incomplete attempts to prove it.

Attempt 1

The original proof by Angenent and Knopf uses a lemma similar to the following, which we are going to adapt to the biaxial case. First we prove:

Lemma 7.4.4. If $a''(z, 0) < 0$ and $c''(z, 0) < 0$ for $z_*(0) < z < 1$, then $\text{Ric}_{00}, \text{Ric}_{22}, \text{Ric}_{33} \geq 0$ and $\text{Ric}_{11} > -1$ for $z_*(t) < z < 1$ and all $t \in [0, T)$. $\blacksquare$

Proof. We recall the evolution equation (4.14) for the Ricci tensor under the Ricci flow:

$$\partial_t \text{Ric}_{\alpha\beta} = \Delta \text{Ric}_{\alpha\beta} + 2g^{\gamma\delta} g^{\eta\zeta} \text{Rm}_{\gamma\alpha\beta\eta} \text{Ric}_{\delta\zeta} - 2g^{\gamma\delta} \text{Ric}_{\alpha\gamma} \text{Ric}_{\beta\delta}. \quad (7.17)$$

In order to apply Theorem 4.5.6 to equation (7.17), we have to study its reaction term $2(g^{\gamma\delta} g^{\eta\zeta} \text{Rm}_{\gamma\alpha\beta\eta} \text{Ric}_{\delta\zeta} - g^{\gamma\delta} \text{Ric}_{\alpha\gamma} \text{Ric}_{\beta\delta})$. Note that the symmetries of the metric and the Ricci tensor are preserved during the flow. So, the only (possibly) non-vanishing terms in (7.17) occur when $\gamma = \delta = \eta = \zeta$ and $\alpha = \beta$. Thus, for $\alpha \in \{0, 1, 2, 3\}$ we have:

$$\partial_t \text{Ric}_{\alpha\alpha} = \Delta \text{Ric}_{\alpha\alpha} + \sum_{\gamma=0}^3 (g^{\gamma\gamma})^2 \text{Rm}_{\gamma\alpha\alpha\gamma} \text{Ric}_{\gamma\gamma} - 2g^{\alpha\alpha} \text{Ric}_{\alpha\alpha}^2.$$

We define a new tensor quantity $T = \text{Ric} + \omega^1 \otimes \omega^1$. Since the tensor T has the same symmetries as the Ricci tensor, the evolution equation for the tensor T is given by:

$$\begin{aligned} \partial_t T_{\alpha\alpha} &= \partial_t \text{Ric}_{\alpha\alpha} = \Delta \text{Ric}_{\alpha\alpha} + \sum_{\gamma=0}^3 (g^{\gamma\gamma})^2 \text{Rm}_{\gamma\alpha\alpha\gamma} \text{Ric}_{\gamma\gamma} - 2g^{\alpha\alpha} \text{Ric}_{\alpha\alpha}^2 \\ &= \Delta T_{\alpha\alpha} + \sum_{\gamma=0}^3 (g^{\gamma\gamma})^2 \text{Rm}_{\gamma\alpha\alpha\gamma} (T - \delta_1^\gamma)_{\gamma\gamma} - 2g^{\alpha\alpha} (T - \delta_1^\alpha)_{\alpha\alpha}^2, \end{aligned}$$

where δ_1^α and δ_1^γ are the Kronecker deltas. Therefore, by straightforward but lengthy computations, the reaction terms for each index $\alpha \in \{0, 1, 2, 3\}$ are:

$$\alpha = 0 : -4(2K_{01}K_{02} + K_{02}^2 - K_{01}K_{12} - K_{02}K_{12} - K_{02}K_{23}), \quad (7.18)$$

$$\alpha = 1 : 4a^2(K_{01}K_{02} - 2K_{01}K_{12} + K_{02}K_{12} - K_{12}^2 + K_{12}K_{23}), \quad (7.19)$$

$$\alpha = 2, 3 : 2c^2(K_{01}K_{02} + K_{02}^2 + K_{01}K_{12} - 2K_{02}K_{12} + K_{12}^2 - K_{02}K_{23} - K_{12}K_{23}), \quad (7.20)$$

where $K_{\alpha\beta}$ for $\alpha, \beta \in \{0, 1, 2, 3\}$ are the sectional curvatures of the manifold.

We wish to write (7.17) in the form (4.36). Note that the metric and the tensor T are diagonal. So, by substituting them in the symmetric product $(F * T)_{\alpha\beta} = g^{\gamma\delta}(F_{\alpha\gamma}T_{\beta\delta} + T_{\alpha\gamma}F_{\beta\delta})$, we see that the only (possibly) non-vanishing terms are when $\gamma = \delta = \beta = \alpha$. In other words, they are $(F * T)_{\alpha\alpha} = 2g^{\alpha\alpha}F_{\alpha\alpha}T_{\alpha\alpha}$. Equating this expression with the reaction terms (7.18)-(7.20) for each $\alpha \in \{0, 1, 2, 3\}$, we get a diagonal 2-tensor F in terms of the sectional curvatures and the metric components. Furthermore, the 2-tensor $(F * T)$ satisfies the null-eigenvector condition trivially since both of the tensors F and T are diagonal. Therefore, Theorem 4.5.6 applies here.

To apply this theorem, we consider the cap region defined by the closure of $M_t = \pi^{-1}([z_*(t), 1))$. On ∂M_t , $a'', c'' < 0$ and either $a' = 0$ or $c' = 0$. So, we have:

$$\begin{aligned} T_{00} = \text{Ric}_{00} &= -\frac{a''}{a} - 2\frac{c''}{c} \geq 0, \\ T_{11} = \text{Ric}_{11} + 1 &= -aa'' - 2\frac{aa'c'}{c} + 2\frac{a^4}{c^4} + 1 > 0, \\ T_{22} = T_{33} = \text{Ric}_{22} = \text{Ric}_{33} &= -cc'' - (c')^2 - \frac{ca'c'}{a} - 2\frac{a^2}{c^2} + 4. \end{aligned}$$

Note that $\frac{a}{c} \leq 1$ and, by the Mean Value Theorem, since $c'' < 0$, we have $|c'| \leq 1$. So, $4 - \frac{2a^2}{c^2} - (c')^2 > 0$. Thus, $T_{22} = T_{33} > 0$ as well, which implies that $T \geq 0$ on ∂M_t .

Furthermore, in M_0 , $a', c' < 0$ and by assumption $a'', c'' < 0$ at $t = 0$. Furthermore, by Mean Value Theorem, we have $|a'| \leq 1$ and $|c'| \leq 1$ here. Thus:

$$T_{00} = \text{Ric}_{00} = -\frac{a''}{a} - 2\frac{c''}{c} \geq 0, \quad (7.21)$$

$$T_{11} = \text{Ric}_{11} + 1 = -aa'' - 2\frac{aa'c'}{c} + 2\frac{a^4}{c^4} + 1 \geq -aa'' + 2\left(\frac{a^4}{c^4} - \frac{a}{c}\right) + 1 > 0, \quad (7.22)$$

$$T_{22} = T_{33} = \text{Ric}_{22} = \text{Ric}_{33} = -cc'' - (c')^2 - \frac{ca'c'}{a} - 2\frac{a^2}{c^2} + 4 \geq -cc'' + 3 - \frac{c}{a} - 2\frac{a^2}{c^2} \geq 0, \quad (7.23)$$

where the inequalities (7.22) and (7.23) are obtained by considering the functions $f(x) = 2(x^4 - x) + 1$ and $g(x) = 3 - \frac{1}{x} - 2x^2$ in the range $\frac{1}{2} \leq x \leq 1$.

Applying Theorem 4.5.6, we conclude that for $z_*(t) < z < 1$ and all $t \in [0, T)$, we have that $T \geq 0$, which yields the result. $\square$

By using Lemma 7.4.4 and adding an extra condition of concavity for a near the pole, we can prove the following proposition:

Proposition 7.4.5. Let $z_*(t) = \max(z_1(t), z_2(t))$. Assume that $\lim_{t \rightarrow T} a(z_*(t)) = A > 0$ and $a(z, t)$ is concave for $z \in [z_*(t), 1)$ for all $t \in [0, T)$. Then Assumption 7.4.2 holds; that is, there exist $z_4 \in [z_*(t), 1)$, $t_1 > 0$, and $\delta > 0$ such that we have the uniform bound:

$$a'(z, t) < -\delta,$$

for all $z \in [z_4, 1)$ and $t \in [t_1, T)$. ■

Proof. Pick $t_1 \in [0, T)$ such that $C_0(T - t_1) \leq \frac{A^2}{8}$ where $C_0 = \max(C_1(\nu), C_2(\nu))$ is the maximum of the bounds $|\partial_t(a^2)| \leq C_1$ and $|\partial_t(c^2)| \leq C_2$. Then, by the fact that $|\partial_t(a^2)| \leq C_0$, integrating this from $t \in [t_1, T)$ to T along $z_*(t)$ we acquire:

$$a(z_*(t), t)^2 \geq A^2 - C_0(T - t) \geq \frac{7}{8}A^2 \quad \text{for all } t \in [t_1, T). \quad (7.24)$$

Next, we pick $z_3 \in [z_*(t), 1)$ to be the solution of $a(z_3, t_1)^2 = \frac{3}{4}A^2$. By integrating from t_1 to $t \in [t_1, T)$ and using the inequality (7.24), we have:

$$a(z_3, t)^2 \leq \frac{3}{4}A^2 + C_0(t - t_1) \leq \frac{3}{4}A^2 + C_0(T - t_1) \leq \frac{7}{8}A^2 \leq a(z_*(t), t)^2, \quad (7.25)$$

for all $t \in [t_1, T)$. Furthermore, by integrating the bound $-C_0 \leq \partial_t(a^2)$ at z_3 from t_1 to any $t \in [t_1, T)$, we obtain:

$$a(z_3, t)^2 \geq \frac{3}{4}A^2 - C_0(t - t_1) \geq \frac{3}{4}A^2 - C_0(T - t_1) \geq \frac{5}{8}A^2, \quad (7.26)$$

for all $t \in [t_1, T)$.

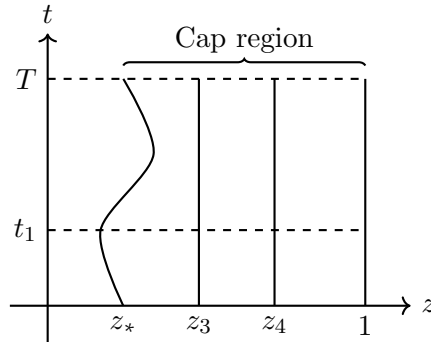


Figure 7.1: The positions for the curve of $z_*(t)$ as well as z_3 and z_4 .

Moreover, we note that the arclength distance $d_1(t) = s(1, t) - s(z_3, t)$ is non-increasing with time since:

$$d_1(t) = \int_{z_3}^1 \frac{\partial s}{\partial z} dz = \int_{z_3}^1 \phi dz \quad \Rightarrow \quad \partial_t d_1(t) = \int_{z_3}^1 \partial_t \phi dz = \int_{z_3}^1 \left(\frac{a''}{a} + 2 \frac{c''}{c} \right) \phi dz \leq 0,$$

by using equation (4.9) and the fact that $\text{Ric}_{00} \geq 0$ in the cap region from Lemma 7.4.4.

Let $z_4 \in [z_3, 1)$ be the solution of $a(z_4, t_1)^2 = \frac{3}{8}A^2$. Then, by integrating $\partial_t(a^2)$ at z_4 from t_1 to $t \in [t_1, T)$ and using the bound $-C_0 \leq \partial_t(a^2) \leq C_0$, we have the inequality:

$$a(z_4, t)^2 \leq \frac{3}{8}A^2 + C_0(t - t_1) \leq \frac{3}{8}A^2 + C_0(T - t_1) \leq \frac{1}{2}A^2. \quad (7.27)$$

By subtracting $a(z_4, t)^2$ from $a(z_3, t)^2$ and using inequalities (7.26) and (7.27), we obtain:

$$\frac{A^2}{8} \leq a(z_3, t)^2 - a(z_4, t)^2 = (a(z_3, t) + a(z_4, t))(a(z_3, t) - a(z_4, t)). \quad (7.28)$$

From inequalities (7.25) and (7.27), we have: $a(z_3, t) \leq \sqrt{\frac{7}{8}}A < A$ and $a(z_4, t) \leq \sqrt{\frac{1}{2}}A < A$ for all $t \in [t_1, T)$. Thus, inequality (7.28) implies:

$$\frac{A}{16} \leq a(z_3, t) - a(z_4, t).$$

Concavity of the quantity a in the hypothesis implies that for all $z \in [z_4, 1)$ and $t \in [t_1, T)$ we have:

$$a(z_3, t) \leq a(z, t) + a'(z, t)(s(z_3(t), t) - s(z(t), t)),$$

and thus:

$$-a'(z, t) \geq \frac{a(z_3, t) - a(z, t)}{s(z, t) - s(z_3, t)} \geq \frac{a(z_3, t) - a(z_4, t)}{s(1, t) - s(z_3, t)} \geq \frac{A}{16d_1(t)} \geq \frac{A}{16d_1(t_1)} = \delta > 0,$$

giving us the desired uniform bound. $\square$

In the work by Angenent and Knopf, the concavity of a comes from the corresponding version of Lemma 7.4.4 since in the uniaxial case $\text{Ric}_{00} = -3\frac{a''}{a} \geq 0$ immediately implies that $a'' \leq 0$. However, we could not reach the same conclusion for the biaxial case as there is a mix of second derivatives of a and c in Ric_{00} ; that is, $\text{Ric} = -\frac{a''}{a} - 2\frac{c''}{c} \geq 0$ in the cap region does not imply $a'' \leq 0$. Furthermore, the quantities $\text{Ric}_{11} \geq -1$ and $\text{Ric}_{33} \geq 0$ in the cap region do not help either since they only imply that $aa'' \leq 1 + 2\lambda^4$ and $cc'' \leq 4$. Therefore, we have to find a different way of deducing the concavity of a in the cap region for all $t \in [0, T)$.

Attempt 2

This original attempt wishes to prove Assumption 7.4.2 by considering the limit of the function a as $t \rightarrow T$. We have seen that for all $z \in (-1, 1)$ the limit $\lim_{t \rightarrow T} a(z, t)$ exists. We define this limiting function as $a(z, T)$

Furthermore, for any fixed $t \in [0, T)$, the function $a(z, t)$ is Lipschitz continuous in z since the derivative is bounded by a constant by Lemma 7.2.5. We choose an increasing sequence of times $t_k \in [0, T)$ such that $t_k \rightarrow T$ and define the sequence of functions $a_k(z) := a(z, t_k)$.

The sequence of functions $a_k(z)$ is equicontinuous. Applying the Arzelà-Ascoli Theorem, we can find a subsequence of functions that converge uniformly to some continuous function $\bar{a}(z)$. By uniqueness of limits, $a(z, T) = \bar{a}(z)$ which implies that $a(z, T)$ is a continuous function of $[-1, 1]$. Furthermore, since for all $t \in [0, T)$ the functions $a(z, t)$ are strictly decreasing in the interval $(z_*(t), 1)$, the limiting function $a(z, T)$ is also decreasing in the interval $(z_*(T), 1)$.

By supposing that the limiting function $a(z, T)$ is differentiable and $\lim_{t \rightarrow T} a'(z, t) = a'(z, T)$, we can deduce that $\lim_{z \rightarrow 1} a'(z, t) = -1$ for all $t \in [0, T]$. So, by continuity of a over space and time, for a fixed $t_0 \in [0, T]$ there exists a $\delta_0 \in (0, 1 - z_*(t_0))$ such that for all $(z, t) \in B_{\delta_0}^\infty((1, t_0)) = \{(z, t) : \|(z, t) - (1, t_0)\|_\infty < \delta_0\}$ we have $a'(z, t) < -\frac{1}{2}$.

Constructing $B_{\delta_0}^\infty((1, t_0))$ for each $t_0 \in [0, T]$ and denoting the projection of these open balls to the time coordinate by $p(B_{\delta_0}^\infty((1, t_0)))$ where $p : [z_*(t), 1] \times [0, T] \rightarrow [0, T]$ is the projection map, we have an open cover for the compact interval $[0, T]$ given by $\mathcal{C} = \{p(B_{\delta_0}^\infty((1, t_0)))\}_{t_0 \in [0, T]}$. Since this interval is compact, we have a finite subcover $\mathcal{C}' = \{p(B_{\delta_i}^\infty((1, t_i)))\}_{i=1}^k$ of open intervals in $[0, T]$. Thus, the preimage of these intervals under the projection map p correspond to a finite collection $\{B_{\delta_i}^\infty(1, t_i)\}_{i=1}^k$ in space-time. Thus, the quantity $\delta = \min_{i=1, \dots, k} (\delta_i)$ is well defined and non-zero.

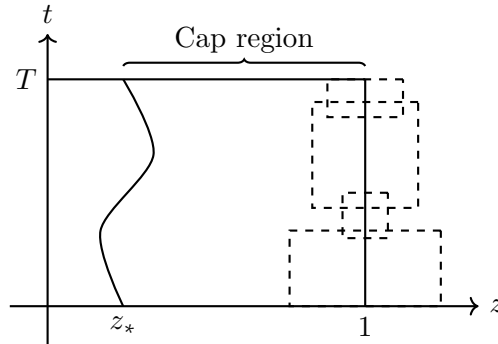


Figure 7.2: Preimage of the finite covering of $[0, T]$.

Thus, for all $z \in [1 - \delta, 1)$ and $t \in [0, T)$ we have $a'(z, t) \leq -\frac{1}{2}$, which implies Assumption 7.4.2.

However, in this argument, we have assumed that the limiting function $a(z, T)$ is differentiable and $\lim_{t \rightarrow T} a'(z, t) = a'(z, T)$. We could not prove this using the Arzelà-Ascoli Theorem as we did for the quantity a because we do not know whether:

1. $\lim_{t \rightarrow T} a'(z, t)$ exists for all $z \in (-1, 1)$,
2. the function $a'(z, t)$ is equicontinuous in space variable,
3. the limiting function $a(z, T)$ is differentiable.

In order to show the former, we have to bound the equation $\partial_t(a')$ which would then involve bounding the third order space derivative of a . Furthermore, an equicontinuity condition for a' is still out of reach since we have only have the bound $|a''| \leq \frac{C}{a}$ for some constant $C > 0$ instead of a uniform bound. Since $a \rightarrow 0$ as $z \rightarrow \pm 1$, the quantity a'' does not have a uniform bound.

However, note that since the limiting function $a(z, T)$ is monotone, we can conclude that it is differentiable almost everywhere. Therefore, we only have a weaker version of the third condition.

Chapter 8

Taub Metrics

In the final chapter, we are going to discuss some possible future directions of study by extending the work we have seen in the previous chapters. The bulk of the analysis we have seen previously hinges on the fact that the S^3 fibres of the base interval I possess the Milnor frame that fixes the symmetry of the metric at initial time. Furthermore, since the Ricci flow preserves symmetry, the diagonal form of the metric persists, which enables us to write down the Ricci flow explicitly.

In the study of general relativity, the Ricci flow appears as the “gradient flow” of the Riemannian Einstein-Hilbert action A on the space of Riemannian metric, which is given by:

$$A(g) = -\frac{c^4}{16\pi G} \int_M S \sqrt{\det(g)} d\mu - \frac{c^4}{8\pi G} \int_{\partial M} \sqrt{\det(\tilde{g})} d\mu,$$

where G is the gravitational constant, c is the speed of light in vacuum, and $\tilde{g}$ is the metric on ∂M induced by g [48]. By the principle of least action (variational principle), this yields the Einstein equation $\frac{1}{2}Sg - \text{Ric} = 0$. So, any Ricci-flat solutions to the Einstein equation are fixed points of the Ricci flow.

Remark 8.0.1. Technically, the “gradient flow” mentioned above is different to the formulation done in [93]. The flow obtained by the variational principle is given by:

$$\partial_t g = \frac{1}{2}Sg - \text{Ric},$$

which is not solvable in general. In [73], the Ricci flow is shown to be the true gradient flow of the Perelman $\mathcal{F}$ -functional defined by:

$$\mathcal{F}(g, f) = \int_M (S + |\nabla f|^2) e^{-f} dV,$$

where $f : M \rightarrow \mathbb{R}$, along the variations that preserve the volume form $e^{-f} dV$. The “gradient flow” formulation mentioned in [48] is done by using a different definition of gradient in order to obtain the Ricci flow.

Of course, there are many known explicit solutions of the Einstein equation such as the Kerr, Schwarzschild, and Reissner-Nordström solutions. A direct way to solve the Einstein equation is to set an ansatz Riemannian manifold, plug in the ansatz in the Einstein equation, and solve the resulting explicit ODEs. We can set the ansatz manifold to be $I \times S^3$ where I is an interval of $\mathbb{R}$ with metrics of the same form as what we have been studying. The resulting solutions are called the Taub-Nut and Taub-Bolt metrics.

8.1 Ricci-Flat Taub Metrics

In this section, we review the derivation of the Ricci-flat Riemannian Taub-Nut and Taub-Bolt metrics. For the full details of the calculations below and comments see, among others, [25], [37], and [82]. We consider a cohomogeneity one ansatz metric by foliating an interval of $\mathbb{R}$ with a family of S^3 , which has the form:

$$g = ds^2 + a(s)^2 \omega^1 \otimes \omega^1 + b(s)^2 \omega^2 \otimes \omega^2 + c(s)^2 \omega^3 \otimes \omega^3,$$

where $\{\omega^1, \omega^2, \omega^3\}$ is a set of 1-forms dual to the Milnor frames $\{E_1, E_2, E_3\}$ on the S^3 fibres satisfying the relationship $[E_i, E_j] = -2\epsilon_{ijk}E_k$.

Remark 8.1.1. In most physics literature, the convention used for the frame is $[E_i, E_j] = \epsilon_{ijk}E_k$. For consistency, we adapt to the notation used in the previous sections. ■

By substituting this metric in the equation $\text{Ric}(g) = 0$ using equations (4.3)-(4.7) and eliminating the second derivatives, we can derive a system of first-order equations [25]:

$$\begin{cases} a' = \frac{b'c'}{2} - \left(\frac{c^2 - a^2 - b^2}{2ab}\right)b' - \left(\frac{b^2 - a^2 - c^2}{2ac}\right)c' - \frac{b^4 + c^4 - 3a^4 + 2(a^2b^2 + a^2c^2 - b^2c^2)}{2a^2bc}, \end{cases} \quad (8.1)$$

$$\begin{cases} b' = \frac{a'c'}{2} - \left(\frac{c^2 - b^2 - a^2}{2ab}\right)a' - \left(\frac{a^2 - b^2 - c^2}{2bc}\right)c' - \frac{a^4 + c^4 - 3b^4 + 2(a^2b^2 + b^2c^2 - a^2c^2)}{2ab^2c}, \end{cases} \quad (8.2)$$

$$\begin{cases} c' = \frac{a'b'}{2} - \left(\frac{b^2 - a^2 - c^2}{2ac}\right)a' - \left(\frac{a^2 - b^2 - c^2}{2bc}\right)b' - \frac{a^4 + b^4 - 3c^4 + 2(a^2c^2 + b^2c^2 - a^2b^2)}{2abc^2}. \end{cases} \quad (8.3)$$

We substitute equations (8.2) and (8.3) in (8.1) and factorise the resulting equation to get:

$$\left(a' - \frac{a^2 - (b+c)^2}{bc}\right) \left(a' - \frac{a^2 - (b-c)^2}{bc}\right) \left(a' - \frac{a^2 - b^2 - c^2}{bc}\right) = 0.$$

By solving this equation and back substituting the solution in equation (8.2) and (8.3), we either get the Atiyah-Hitchin first order system (by setting $a' - \frac{a^2 - (b+c)^2}{bc} = 0$ or $a' - \frac{a^2 - (b-c)^2}{bc} = 0$) or the BGPP (Belinskii, Gibbons, Page, and Pope) system (by setting $a' - \frac{a^2 - b^2 - c^2}{bc} = 0$). Note

that the Atiyah-Hitchin system gives rise to the Atiyah-Hitchin and self-dual Taub-Nut metrics as its solutions and the BGPP system gives rise to the BGPP and Eguchi-Hanson metrics as its solutions [25].

If $a = b = c$, equations (8.1)-(8.3) reduce to the simple ODE $(a')^2 = 1$. Solving this, with a shift in the gauge s if necessary, yields the (Ricci) flat Euclidean metric $g = ds^2 + s^2 g_{S^3}$ on $\mathbb{R}^4$ if I is the half line.

For a slightly more complicated case, we now specialise to the biaxial Bianchi IX metric by setting $b = c$. By using the factorising method as in the general case above, the equations (8.1)-(8.3) simplify to the following system:

$$\begin{cases} a' = \frac{(a')^2}{2} + \frac{a'c}{a} + \frac{6a^2}{c^2} - 2, \\ 0 = (cc' + a - 2c)(cc' + a + 2c)(cc' + a). \end{cases}$$

By setting $cc' + a - 2c = 0$ or $cc' + a + 2c = 0$, we obtain the self-dual Taub-Nut metrics whereas setting $cc' + a = 0$ yields the Eguchi-Hanson metric. Here, self-dual means that the curvature of the connection is equal to its Hodge star. By solving the former via a change of gauge s to z , we get the two-parameter metric solution:

$$g = \frac{z^2 - n^2}{z^2 - 2mz + n^2} dz^2 + 16 \frac{n^2(z^2 - 2mz + n^2)}{z^2 - n^2} \omega^1 \otimes \omega^1 + 4(z^2 - n^2)(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3), \quad (8.4)$$

where $m, n \in \mathbb{R}$ are the constants of integration.

This metric was discovered by Taub [91]. Physically, the parameters m and n refer to mass and charge respectively. We now analyse the metric to see its geometric behaviour. We assume now that $m, n > 0$ and factorise the rational function $u(z) = \frac{z^2 - 2mz + n^2}{z^2 - n^2} = \frac{(z - z_-)(z - z_+)}{(z - n)(z + n)}$ where $z_{\pm} = m \pm \sqrt{m^2 - n^2}$.

The zeroes of the function u are real if and only if $m \geq n$ and consequently $z_+ \geq z_-$. So, we want to find an open interval of $\mathbb{R}$ for z which do not contain either of these two singular points $z_{\pm}$. Note that for this choice of m , we have $n \leq m + \sqrt{m^2 - n^2} = z_+$. Thus $c(z_+)^2 = 4(z_+^2 - n^2) \geq 0$ and $c(z) > 0$ for $z \in (z_+, \infty)$. Hence, the metric (8.4) defines a complete metric for the domain $z \in (z_+, \infty)$.

In order to close up the metric at $z = z_+$ smoothly without any conical singularity, we appeal to the cylinder-to-sphere rule from Lemma 4.0.2 to impose an additional condition on the constants m and n . Thus, we need to solve $\partial_s a \rightarrow 1$ as $z \rightarrow z_+$. The term $\partial_s a$ is given by:

$$\partial_s a = 4 \left(\frac{z^2 - 2mz + n^2}{z^2 - n^2} \right)^{\frac{1}{2}} \frac{d}{dz} \left(\frac{n^2(z^2 - 2mz + n^2)}{(z^2 - n^2)} \right)^{\frac{1}{2}} = 4n \frac{m(n^2 + z^2) - 2n^2 z}{(n^2 - z^2)^2}.$$

By taking the limit as $z \rightarrow z_+$ and equating with 1, we get either $m = n$ or $m = \frac{5n}{4}$. For $m = n$, we get the following complete metric:

$$g = \frac{z+n}{z-n} dz^2 + 16n^2 \left(\frac{z-n}{z+n} \right) \omega^1 \otimes \omega^1 + 4(z^2 - n^2)(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

for $z \in (n, \infty)$. In this case, both of the a and c factors vanish as z tends to n . So, the manifold closes up to a point here, which we call the nut singularity. This results in a $\mathbb{R}^4$ global topology. After Taub found the underlying space, this metric was constructed by Newman, Tamburino, and Unti [69], for whom the acronym NUT stands.

On the other hand, setting $m = \frac{5n}{4}$ yields a different type of singularity at $z = n$. We obtain:

$$g = \frac{z^2 - n^2}{z^2 - \frac{5}{2}nz + n^2} dz^2 + 16n^2 \left(\frac{z^2 - \frac{5}{2}nz + n^2}{z^2 - n^2} \right) \omega^1 \otimes \omega^1 + 4(z^2 - n^2)(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

for $z \in (2n, \infty)$. Note that as z tends to $2n$, the a factor vanishes while the c factor does not. Hence, the manifold collapses to an S^2 factor at $z = 2n$, which results in a different singularity called the bolt singularity (in contrast to nut singularity). Thus, we obtain a global topology of $\mathbb{CP}^2 \setminus \{p_0\}$.

This metric was first constructed by Page [72]. However, instead of the cylinder-to-sphere rule as used here, Page used a different method by comparing the area and the radius of the geodesic spheres centred at $\pi^{-1}(z_+)$ in his analysis.

8.2 Taub Metrics in a Box

The metrics that we have constructed above are non-compact. In order to have a simpler object to study, we cut off the tails of the Taub metrics by restricting our interval $I = (z_+, \infty)$ to some bounded interval, essentially putting the metrics in a box. We then redefine the coordinates so that both of the resulting Taub-Nut and Taub-Bolt metrics are prescribed on a family of S^3 foliated along the interval $I = (0, 1]$.

8.2.1 Taub-Bolt Metric

Let g be the flat Taub-Bolt metric on $\mathbb{CP}^2 \setminus \{p_0\}$ given by:

$$g = \frac{x^2 - n^2}{x^2 - \frac{5}{2}nx + n^2} dx^2 + 16n^2 \left(\frac{x^2 - \frac{5}{2}nx + n^2}{x^2 - n^2} \right) \omega^1 \otimes \omega^1 + 4(x^2 - n^2)(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3), \quad (8.5)$$

for $x \in (2n, \infty)$. There is a bolt singularity at $x = 2n$. We restrict the radial component to $x \in (2n, R]$ for some $R > 2n$ at which we introduce a boundary to the manifold. The

resulting manifold would be homeomorphic to $\mathbb{CP}^2 \setminus B^4$. By carrying out a coordinate change $x = \frac{n(4R-2n-(R-2n)z^2)}{2R-n-2(R-2n)z^2}$ and introducing new parameters $\alpha = \sqrt{2 + \frac{3n}{R-2n}}$ and $\beta = \sqrt{3}n$, we can write the metric (8.5) as:

$$g = 8\alpha^2\beta^2 \frac{(\alpha^4 - z^4)}{(\alpha^2 - 2z^2)^4} dz^2 + 8\alpha^2\beta^2 \frac{z^2}{(\alpha^4 - z^4)} \omega^1 \otimes \omega^1 + 4\beta^2 \frac{(\alpha^4 - z^4)}{(\alpha^2 - 2z^2)^2} (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3), \quad (8.6)$$

where $z \in (0, 1]$. The metric components behave like $a(z)^2 \sim k^2 z^2 + \mathcal{O}(z^4)$ near $z = 0$ and $c(0) \neq 0$.

We fix the metric on the S^3 boundary $\pi^{-1}(1)$ to be of the form $g_{z=1} = \mu^2 \omega^1 \otimes \omega^1 + \nu^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3)$ for some $\mu, \nu > 0$. This boundary metric extends to a complete Taub-Bolt metric g on $\mathbb{CP}^2 \setminus B^4$ in (8.6) if and only if the ratio $\tau^2 = \frac{\mu^2}{\nu^2} = \frac{2\alpha^2(\alpha^2-2)^2}{(\alpha^4-1)^2}$, which we call the squashing parameter, is smaller than or equal to the critical value $\tau_c^2 = 1 - \frac{9\sqrt[3]{3}}{8} + \frac{3\sqrt[3]{9}}{8}$, which is the global maximum of τ^2 in the region $\alpha > \sqrt{2}$. See Figure 8.1 for the plot of the function $\tau(\alpha)$

For each $0 < \tau^2 < \tau_c^2$, there are two solutions for the Ricci-flat metrics of the form g as in (8.6) corresponding to two pairs of solution parameters (α_1, β_1) and (α_2, β_2) where $\alpha_1 < \alpha_2$. These solutions are called the small bolt and the large bolt respectively. For the value $\tau^2 = \tau_c^2$, the two pairs (α, β) coincide, which describes a unique Ricci-flat metric with this prescribed boundary metric $g_{z=1}$.

In the plot, we denote the critical point α_c (that is, the value of α that gives us the critical value τ_c) by the dashed black line. The blue and red dashed lines correspond to the two pairs of bolts with the same squashing parameter τ .

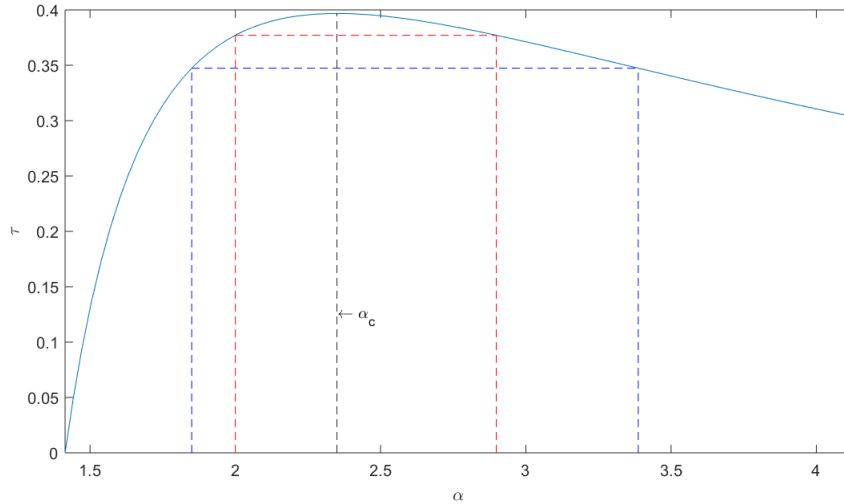
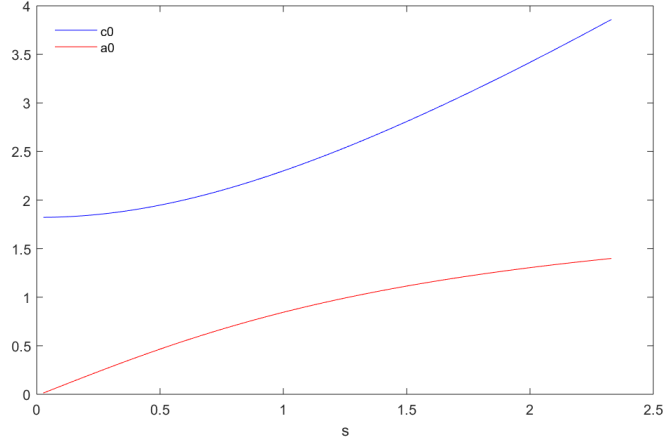


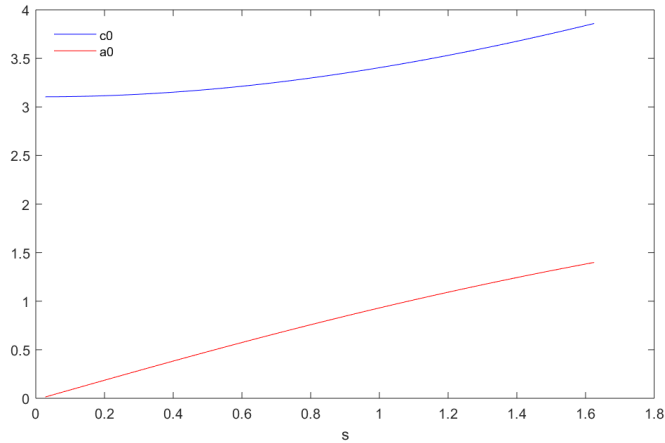
Figure 8.1: Plot of τ against α .

We note that for all $0 < \tau < \tau_c$, there are exactly two solutions of the Ricci-flat Taub-Bolt

metric. The parameter values $\alpha < \alpha_c$ correspond to the small bolts whereas the parameter values $\alpha > \alpha_c$ correspond to the big bolts. See Figure 8.2 for an example of a small and big bolt. The plots in Figure 8.2 display the metric quantities $a(s)$ and $c(s)$ along the arclength $s(z) = 2\sqrt{2}\alpha\beta \int_0^z \frac{\sqrt{\alpha^4 - w^4}}{(\alpha^2 - 2w^2)^2} dw$ for $z \in (0, 1]$.



(a) Small Taub-Bolt metric.



(b) Large Taub-Bolt metric.

Figure 8.2: Taub-Bolt metrics in a box for the same boundary metric $g_{z=1}$.

8.2.2 Taub-Nut Metric

Let g be the Ricci-flat metric on $\mathbb{R}^3$ given by:

$$g = \frac{z+n}{z-n} dz^2 + 16n^2 \left(\frac{z-n}{z+n} \right) \omega^1 \otimes \omega^1 + 4(z^2 - n^2)(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3), \quad (8.7)$$

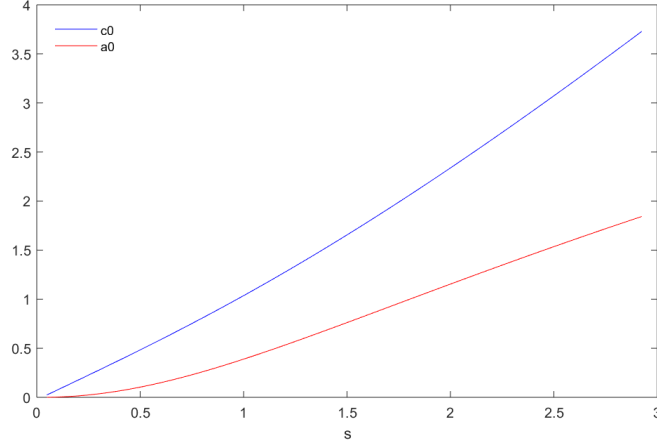
for $z \in (n, \infty)$. As in the previous case, we restrict the domain to $z \in (n, R]$ for some $R > n$ where we introduce a boundary to the manifold. This results in a manifold homeomorphic to

the closed ball $\bar{B}^4$. We introduce new parameters $\gamma = \frac{1}{R}\sqrt{2n}$ and $\delta = \sqrt{R}$, so the metric (8.7) can be written as:

$$g = 4\delta^2(z^2 + \gamma^2)dz^2 + 4\delta^2\gamma^2 \frac{z^2}{z^2 + \gamma^2} \omega^1 \otimes \omega^1 + 4\delta^2 z^2 (z^2 + \gamma^2) (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3), \quad (8.8)$$

where $z \in (0, 1]$. The metric components behave as $a(z)^2 \sim c(z)^2 \sim k^2 z^2 + \mathcal{O}(z^4)$ near $z = 0$.

As done previously, we fix the metric on the S^3 boundary $\pi^{-1}(1)$ to be of the form $g_{z=1} = \mu^2 \omega^1 \otimes \omega^1 + \nu^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3)$ for some $\mu, \nu > 0$. This boundary metric extends to a unique complete Taub-Nut metric g on $\bar{B}^4$ in (8.8) for any $\tau^2 \in (0, 1)$. See Figure 8.3 for an example of a Taub-Nut metric. The plot in Figure 8.3 displays the metric quantities $a(s)$ and $c(s)$ along the arclength $s(z) = 2\delta \int_0^z \sqrt{w^2 + \gamma^2} dw$ for $z \in (0, 1]$.



(a) Taub-Nut metric.

Figure 8.3: Taub-Nut metric in a box for the same boundary metric $g_{z=1}$ as the Taub-Bolt metrics in Figure 8.2.

8.3 Ricci Flow from Taub-Bolt Metric

We note that the Taub metrics are Ricci-flat and thus they are fixed points of the Ricci flow. Thus, they are critical points of the Riemannian Hilbert-Einstein action. In order to study their stability, we linearise the Ricci flow equation and study the corresponding eigenvalues of the linear system. As we have seen in Section 2.2, the linearised Ricci flow about some fixed metric g is the linear equation:

$$\partial_t h_{ij} = \Delta_g^L h_{ij} + \nabla_i (\delta h)_j + \nabla_j (\delta h)_i + \nabla_i \nabla_j (g^{kl} h_{kl}),$$

where Δ^L is the Lichnerowicz Laplacian (defined in Definition 2.2.3), which acts on the space $\Gamma(\text{Sym}^2(T^*M))$ by:

$$\Delta_g^L h_{ij} = \Delta_g h_{ij} - g^{kl}(\text{Ric}_{ik}h_{lj} + \text{Ric}_{jk}h_{li}) + 2g^{kp}g^{lq}\text{Rm}_{iklj}h_{pq}.$$

Therefore, in order to study the stability of the Ricci flow perturbed around the Ricci-flat Taub-Bolt or Taub-Nut metrics, it is sufficient to consider the sign of the eigenvalues of the Lichnerowicz Laplacian [46].

On a Riemannian manifold (M, g) , the trace and traceless decomposition of the space of symmetric 2-tensor is an orthogonal decomposition [60]; that is, any $h \in \Gamma(\text{Sym}^2(T^*M))$ can be written uniquely as the sum $h_{ij} = \overset{\circ}{h}_{ij} + \mu g_{ij}$ where $\overset{\circ}{h}_{ij}$ is the traceless part (that is, $g^{ij}\overset{\circ}{h}_{ij} = 0$) and $\mu \in \mathbb{R}$. When g is Ricci-flat, the Lichnerowicz Laplacian acts separately on the traceless and trace parts of h since:

$$\begin{aligned}\Delta_g^L \overset{\circ}{h}_{ij} &= \Delta_g \overset{\circ}{h}_{ij} + 2g^{kp}g^{lq}\text{Rm}_{iklj}\overset{\circ}{h}_{pq}, \\ \Delta_g^L \mu g_{ij} &= \Delta_g \mu g_{ij} + 2\mu g^{kp}g^{lq}\text{Rm}_{iklj}g_{pq} = \Delta_g \mu g_{ij} + 2\mu \cancel{\text{Ric}}_{ij} = \Delta_g \mu g_{ij}.\end{aligned}$$

Thus, the Lichnerowicz Laplacian on the trace part corresponds to the usual Laplacian on the tensor while it acts like the Schrödinger equation with potential on the traceless part. On a closed manifold with boundary, within the class of symmetric 2-tensors which vanish on the boundary and supposing we have an eigenvalue $\lambda \in \mathbb{R}$ of the Lichnerowicz Laplacian with eigentensor $T_{ij} = \mu g_{ij}$, we have:

$$\begin{aligned}-\Delta_g^L T &= -\Delta_g T = \lambda T, \\ \Rightarrow \int_M \langle -\Delta_g T, T \rangle_g d\mu &= \int_M \langle \lambda T, T \rangle_g d\mu \quad \Rightarrow \quad \int_M \|\nabla T\|_g^2 d\mu = \lambda \int_M \|T\|_g^2 d\mu,\end{aligned}$$

where $\langle \cdot, \cdot \rangle_g$ is the induced metric on $\text{Sym}^2(T^*M)$. This equation implies that the eigenvalues of the Laplacian on the trace part are non-negative.

On the traceless part, the Lichnerowicz Laplacian is similar to the Schrödinger's operator where the term involving the Riemann curvature tensor acts as the potential in the operator:

$$\Delta_g^L \overset{\circ}{h}_{ij} = \Delta_g \overset{\circ}{h}_{ij} + 2g^{kp}g^{lq}\text{Rm}_{iklj}\overset{\circ}{h}_{pq} \quad \sim \quad P(u) = -\Delta u(x) + V(x)u(x),$$

where P is the Schrödinger's operator with potential. At the regions with positive curvature, this corresponds to negative potential in the Schrödinger's operator. Negative potential in Schrödinger's operator corresponds to bound states, so the Lichnerowicz Laplacian may have

bound states, which are modes with negative eigenvalue. This would then imply instability of the Taub-Bolt or Taub-Nut metrics [48].

Actually finding the eigenvalues of the Lichnerowicz Laplacian about the Taub-Bolt metric is a very difficult problem which involves solving a set of ten coupled partial differential equations. Therefore, a more tangible approach would be to find a suitable approximation of the eigenvalues numerically. Indeed, Young showed that there exists such an eigenvalue within the interval $[-0.201, -0.200]$ by solving these equations numerically [101].

A more recent numerical approximation for the smallest eigenvalue of the Lichnerowicz Laplacian about the Taub-Bolt metric was done by Holzegel, Schmelzer, and Warnick in [51]. They plotted the minimum eigenvalue for various parameter α defined in (8.6). The plot is given in Figure 8.4.

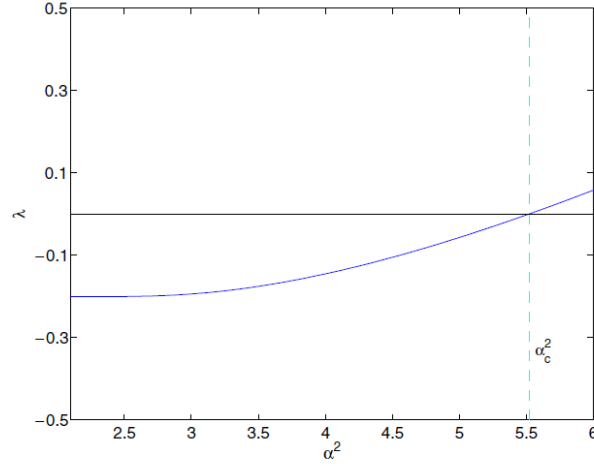


Figure 8.4: Plot of the minimum eigenvalues of the Lichnerowicz Laplacian about Taub-Bolt metric for various values of parameter α^2 [51].

Figure 8.4 shows that the minimum eigenvalue is negative for $\alpha < \alpha_c$ which correspond to the small bolt. This implies that the Taub-Bolt metric corresponding to the small bolts are unstable critical points. However, for $\alpha > \alpha_c$ the minimum eigenvalues are positive, which implies that the large bolts are stable critical points.

Remark 8.3.1. A similar study has been done for the Schwarzschild metrics in place of the Taub metrics by Headrick and Wiseman in [48]. The set up is as follows: they studied the Ricci flow on 4-manifolds with boundary $S^1 \times S^2$ where the metric on the boundary is fixed to be $S_\rho^1 \times S_R^2$, which is a product of two round metrics with radii ρ and R respectively.

The types of metric varies depending on the value $b = \frac{\rho}{R}$. For any value of the parameter b ,

a solution would be $S_r^1 \times \bar{B}_R^3$, which is called the hot flat space. For $b < b_c = \frac{4}{3\sqrt{3}}$ there are two more solutions, which are $\bar{B}^2 \times S^2$ equipped with the Schwarzschild metric:

$$g = 4r_0^2 \left(1 - \frac{r_0}{r}\right) dz^2 + \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2 g_{S^2},$$

with different values of r_0 , one smaller and the other greater than $\frac{2R}{3}$. These two solutions are called the small and large black hole respectively. Note that the variable r ranges from r_0 at the horizon to R at the boundary.

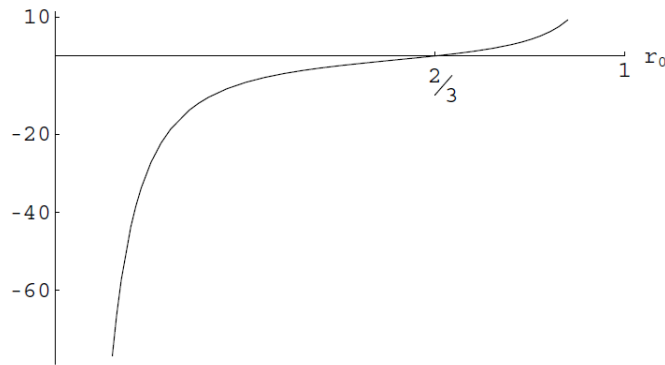


Figure 8.5: Plot of the minimum eigenvalues of the Lichnerowicz Laplacian about the Schwarzschild metric for various values of parameter b .

Note here that the value R is normalised to 1 [48].

In their work, they also computed the smallest eigenvalue of the Lichnerowicz Laplacian numerically. They showed that the smallest eigenvalue of the small black holes is negative while the smallest eigenvalue for the large black holes is positive. This implies that the small black holes are unstable critical points while the large black holes are stable critical points of the Ricci flow. This mirrors the investigation done in [51] for the Taub metrics. ■

To show the instability of the small bolt, the Ricci flow from the Taub-Bolt metric is simulated numerically. In [51] the Taub-Bolt metric ansatz is given by:

$$g = e^{2\Phi} dz^2 + z^2 e^{2A} \omega^1 \otimes \omega^1 + e^{2C} (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3), \quad (8.9)$$

and the DeTurck vector field was computed to be $X = -(2\dot{C} + \dot{A} - \dot{\Phi})dz$. The DeTurck-Ricci

flow equation is thus calculated to be:

$$\begin{cases} \partial_t \Phi = e^{-2\Phi} \left(\ddot{\Phi} - \frac{\dot{\Phi}}{z} + \frac{2\dot{A}}{z} + (\dot{A})^2 + 2(\dot{C})^2 - (\dot{\Phi})^2 \right), \end{cases} \quad (8.10)$$

$$\begin{cases} \partial_t A = e^{-2\Phi} \left(\ddot{A} + \frac{\dot{A}}{z} - 2z^2 e^{-4C+2A+2\Phi} \right), \end{cases} \quad (8.11)$$

$$\begin{cases} \partial_t C = e^{-2\Phi} \left(\ddot{C} + \frac{\dot{C}}{z} + 2z^2 e^{-4C+2A+2\Phi} - 4e^{-2C+2\Phi} \right), \end{cases} \quad (8.12)$$

with Neumann boundary conditions $\dot{\Phi}(t, 0) = \dot{A}(t, 0) = \dot{C}(t, 0) = 0$ and $(2\dot{C} + \dot{A} - \dot{\Phi})(t, 1) = 0$ as well as Dirichlet boundary conditions $A(t, 1) = A_0(1)$ and $C(t, 1) = C_0(1)$.

Remark 8.3.2. Of course, the main issue here is whether the Ricci flows on these manifolds with boundaries are well-posed. Currently, there are not much literature on this but [77], [86], and the appendix of [51] are a few sources that give sufficiency conditions on the boundary that ensure Ricci flows on manifolds with boundaries are well-posed. ■

For some perturbation of the small bolt, under the Ricci flow, there are two observed behaviours in the numerical plots. The plots in Figures 8.6-8.8 are the numerical simulations of equations (8.10)-(8.12) with initial condition coming from a perturbed Taub-Bolt metric carried out by Holzegel, Schmelzer, and Warnick. These plots are done for a specific choice of parameters $\alpha^2 = 2.5$ and $\beta = 1$ in the Taub-Bolt metric (8.6). However, the plots are done in the arclength coordinate gauge χ such that the metric in (8.9) is recast as the more geometric form:

$$g = d\chi^2 + \chi^2 V_2(\chi)^2 \omega^1 \otimes \omega^1 + V_1(\chi)^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3).$$

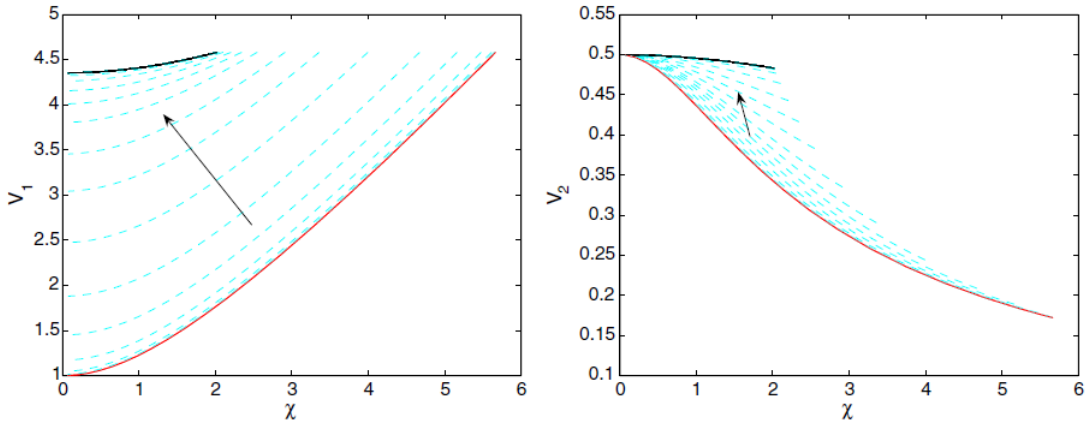


Figure 8.6: Perturbation of the small bolt flows to the large bolt under the Ricci flow [51].

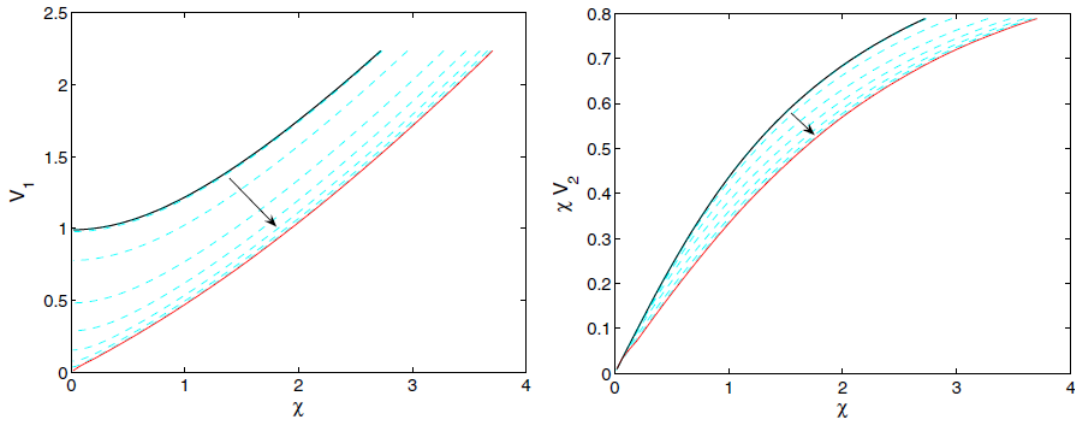


Figure 8.7: Perturbation of the small bolt flows to the nut under the Ricci flow [51].

In Figures 8.6, 8.7, and 8.8, the arrows denote the direction of the evolution under the Ricci flow. In Figures 8.6 and 8.7, the direction of perturbation are different and the plots show contrasting behaviours. In Figure 8.6, the value $V_1(0, t)$ increases as t increases, so the size of the S^2 bolt gets bigger and converges to what is assumed to be another Taub-Bolt metric with a larger bolt. Since the metric at $z = 1$ is fixed, the ratio τ remains fixed during the flow, so under the Ricci flow, the metric is expected to flow from $\alpha < \alpha_c$ horizontally under the graph of $\tau(\alpha)$ in Figure 8.1 along a constant value of τ until it reaches the graph again at $\alpha > \alpha_c$.

However, in Figure 8.7, the value $V_1(0, t)$ decreases under the Ricci flow as t increases, so the size of the S^2 bolt gets smaller and shrinks to a point. This is where a singularity occurs. At this singularity time T , the flow is stopped and a surgery is done by changing the values of $V_1(\chi, T)$ and $V_2(\chi, T)$ for all $0 < \chi < \varepsilon$ for a small ε to the profile of a small enough round S^4 cap metric. The simulation is resumed and is shown in Figure 8.8.

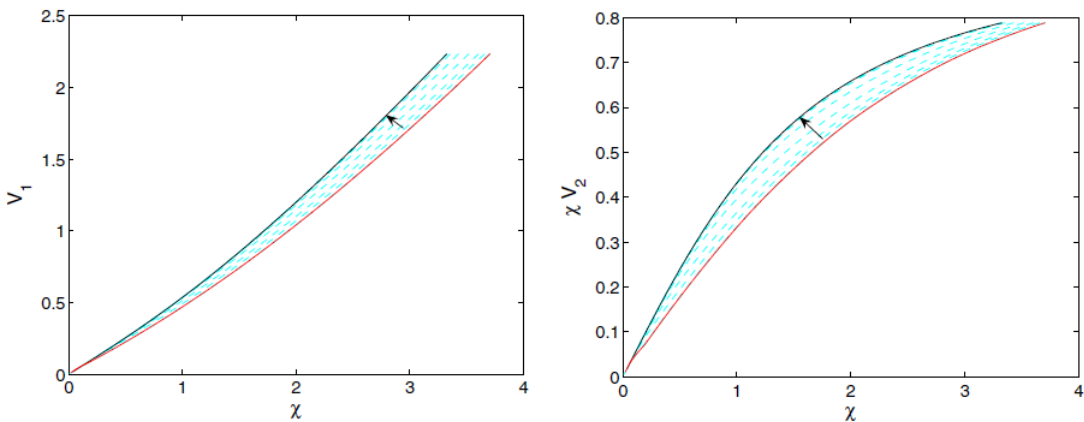


Figure 8.8: Continuation of the Ricci flow from Figure 8.7 which tends to a nut metric [51].

In Figure 8.8, the resulting metric converges to a fixed metric after some time under the Ricci flow. This metric is assumed to be one of the Taub-Nut family metric since both of the metric quantities vanish at $z = 0$.

Similar behaviour is also seen in the simulations done by Headrick and Wiseman for the perturbation of the small black hole which has the Schwarzschild metric. Under the Ricci flow, some perturbation will tend to the large black hole whereas some other perturbations will tend to the hot flat space [48]. Based on these numerical observations, Holzegel, Schmelzer, and Warnick provided two conjectures at the end of their paper:

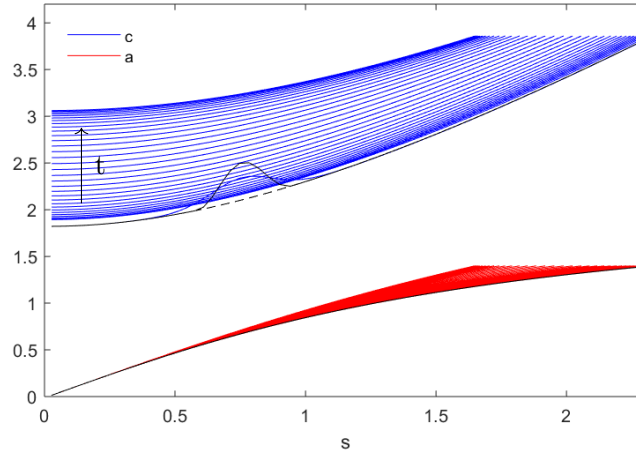
Conjecture 8.3.3. For any biaxial Bianchi IX symmetric initial data and fixed metric on the boundary S^3 given by $g_{z=1} = \mu^2 \omega^1 \otimes \omega^1 + \nu^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3)$ with $\tau = \frac{\mu}{\nu} \in (0, \tau_c)$, Ricci flow plus at most one surgery will converge to a member of either the Taub-Bolt or the Taub-Nut family. ■

Conjecture 8.3.4. For any biaxial Bianchi IX symmetric initial data with Taub-Nut topology $\bar{B}^4$, the Ricci flow admits long-time existence and converges to a member of the Taub-Nut family. ■

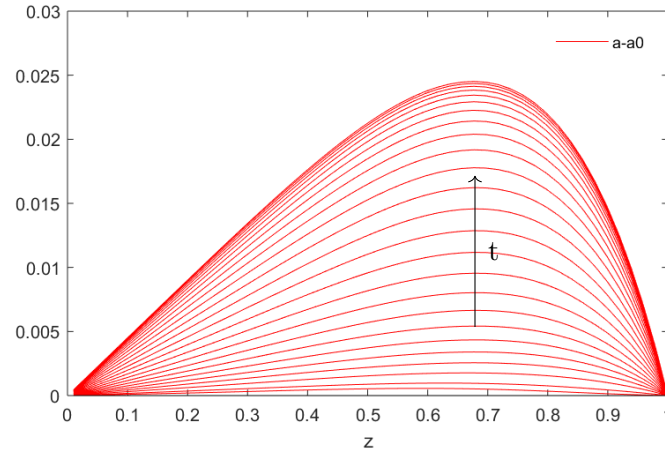
We repeat the numerical simulations that were done by Holzegel, Schmelzer, and Warnick but with compact perturbations. By translating their ansatz metric in (8.9) to our notation and our frame (that is, we change the variables in (8.9) via $\phi = e^\Phi, a = ze^A, c = e^C$ so that the metric will be of the more familiar form $g = \phi^2 dz^2 + a^2 \omega^1 \otimes \omega^1 + c^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3)$), we get the following system of strongly parabolic equations from the system (8.10)-(8.12):

$$\begin{cases} \partial_t \phi = \frac{1}{\phi} \left(\frac{\ddot{\phi}}{\phi} - \frac{2(\dot{\phi})^2}{\phi^2} - \frac{\dot{\phi}}{z\phi} - \frac{3\dot{a}}{za} + \frac{(\dot{a})^2}{a^2} + 2\frac{(\dot{c})^2}{c^2} + \frac{2}{z^2} \right), \\ \partial_t a = \frac{\ddot{a}}{\phi^2} - \frac{(\dot{a})^2}{\phi^2 a} + \frac{\dot{a}}{z\phi^2} - \frac{2a^3}{c^4}, \\ \partial_t c = \frac{\ddot{c}}{\phi^2} - \frac{(\dot{c})^2}{\phi^2 c} + \frac{\dot{c}}{z\phi^2} + \frac{2a^2}{c^3} - \frac{4}{c}. \end{cases}$$

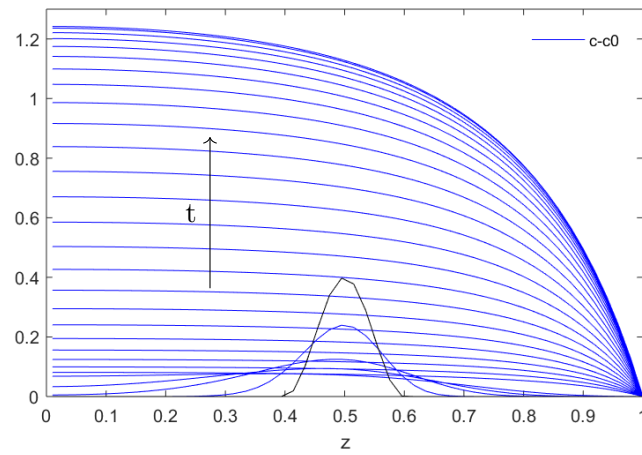
We suppose that $a = a_0 + \varphi$ and $c = c_0 + \psi$, where the pair (a_0, c_0) correspond to a flat small Taub-Bolt solution with φ and ψ some compact perturbations. We plot the evolution of the metric components and their perturbations with the initial data given by positive and negative perturbations of the flat Taub-Bolt metric (8.6) parametrised by $\alpha^2 = 3.65$ and $\beta = 0.91$ in Figures 8.9-8.12. These parameters are chosen simply to get a clear profile of the evolution. In these plots, the black curves denote the initial data of the evolution and the dashed black curves denote the flat metric components (a_0, c_0) .



(a) Evolution of a (red) and c (blue) for positive compact perturbation in the c component.

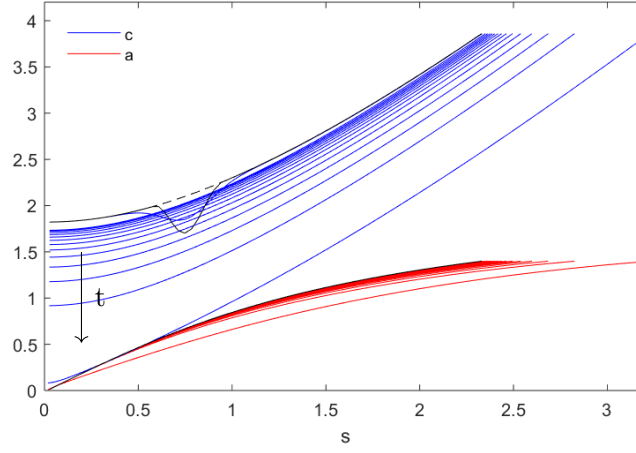


(b) Evolution of the perturbation $\varphi = a - a_0$.

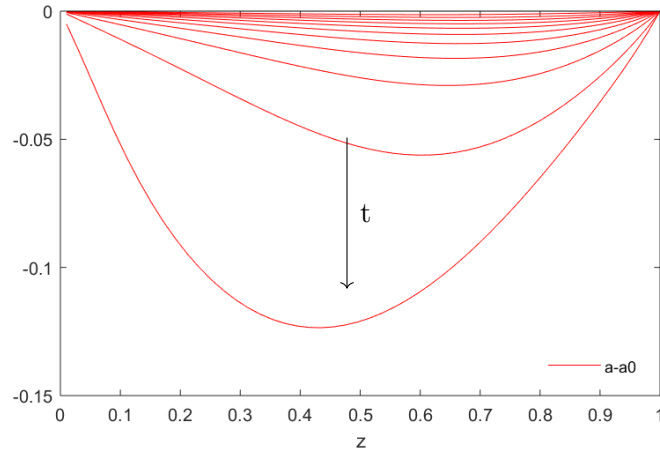


(c) Evolution of the perturbation $\psi = c - c_0$.

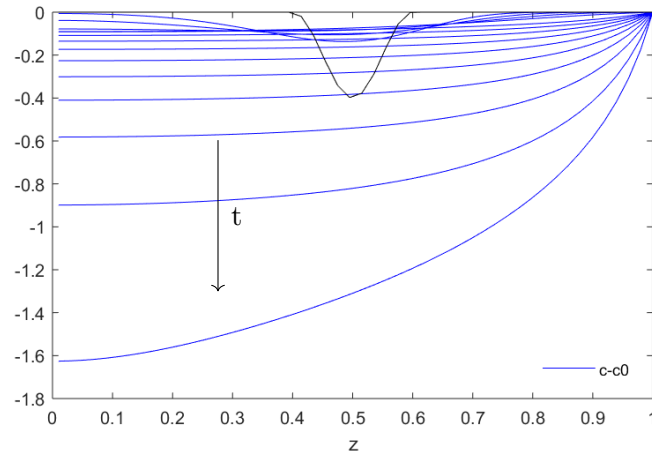
Figure 8.9: Ricci flow simulation for a perturbation of the small Taub-Bolt metric. Note that plot (a) is in the s coordinate whereas plots (b) and (c) are in the z coordinate.



(a) Evolution of a (red) and c (blue) for negative compact perturbation in the c component.

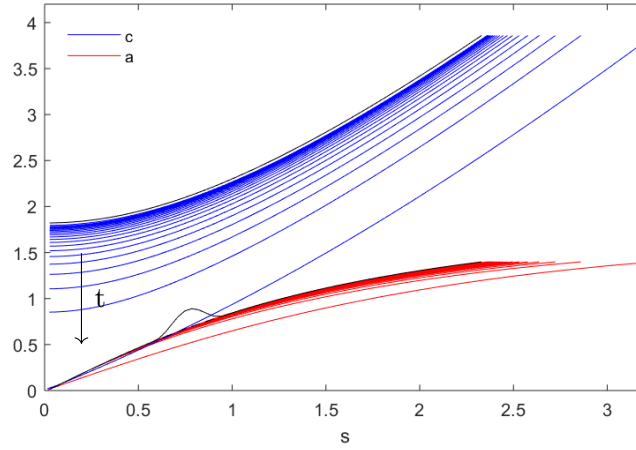


(b) Evolution of the perturbation $\varphi = a - a_0$.

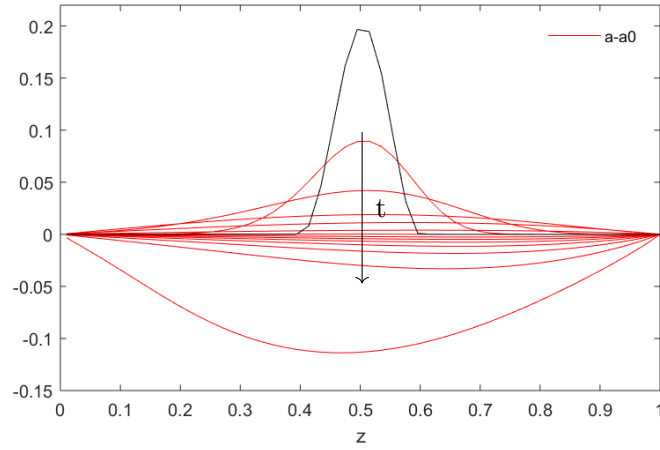


(c) Evolution of the perturbation $\psi = c - c_0$.

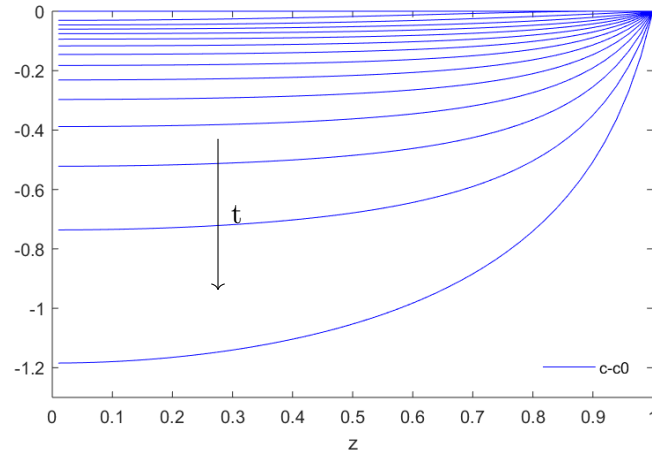
Figure 8.10: Ricci flow simulation for a perturbation of the small Taub-Bolt metric. Note that plot (a) is in the s coordinate whereas plots (b) and (c) are in the z coordinate.



(a) Evolution of a (red) and c (blue) for positive compact perturbation in the a component.

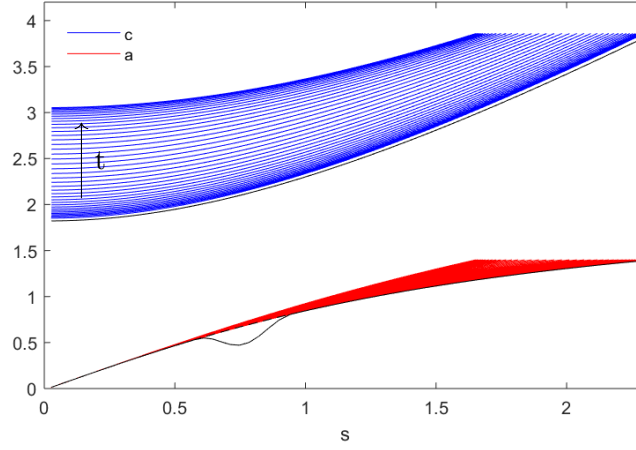


(b) Evolution of the perturbation $\varphi = a - a_0$.

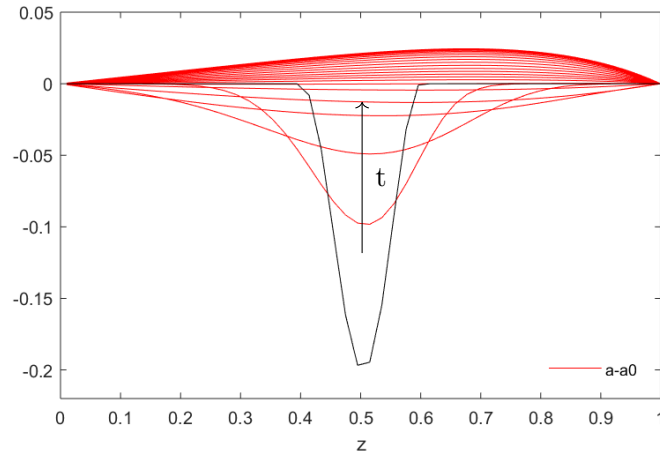


(c) Evolution of the perturbation $\psi = c - c_0$.

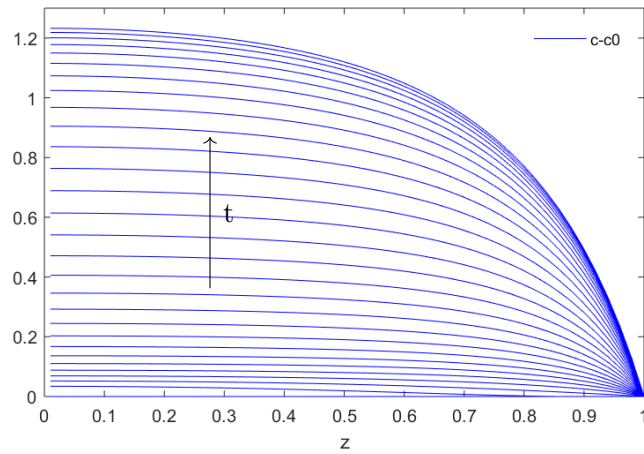
Figure 8.11: Ricci flow simulation for a perturbation of the small Taub-Bolt metric. Note that plot (a) is in the s coordinate whereas plots (b) and (c) are in the z coordinate.



(a) Evolution of a (red) and c (blue) for negative compact perturbation in the a component.



(b) Evolution of the perturbation $\varphi = a - a_0$.



(c) Evolution of the perturbation $\psi = c - c_0$.

Figure 8.12: Ricci flow simulation for a perturbation of the small Taub-Bolt metric. Note that plot (a) is in the s coordinate whereas plots (b) and (c) are in the z coordinate.

We note that from Figures 8.9 and 8.10, positive and negative perturbations of the quantity c are preserved under the Ricci flow. Furthermore, the evolution of the quantities a , c , and ψ for both of the cases satisfy the Sturmian property of parabolic equations; that is, the number of critical points for the quantities a , c , and ψ are non-increasing over time. As time progresses, the critical points are smoothed out and the quantities a and c converge to different profiles, which we assume to be the large Taub-Bolt or the Taub-Nut metric, with a surgery required prior to formation of the latter.

On the other hand, the quantity φ does not exhibit the Sturmian property as a critical point appears instantaneously in time. However, these behaviour indicate that there is some hope for the application of maximum or invariant principles in the study of these quantities. We pose the following conjecture:

Conjecture 8.3.5. If at $t = 0$, we have $\psi(z, 0) \geq 0$ and $\varphi(z, 0) = 0$ for $z \in (0, 1]$, then for all $t \in [0, T)$ and $z \in (0, 1]$ we have $\psi(z, t) \geq 0$. Similarly, if at $t = 0$ we have $\psi(z, 0) \leq 0$ and $\varphi(z, 0) = 0$ for $z \in (0, 1]$, then for all $t \in [0, T)$ and $z \in (0, 1]$ we have $\psi(z, t) \leq 0$. ■

This conjecture is a small step forward to prove Conjectures 8.3.3 and 8.3.4 by Holzegel, Schmelzer, and Warnick.

In contrast, for the perturbations of the a quantity, a reversed behaviour is observed in Figures 8.11 and 8.12. The negative perturbation of the quantity a in the Taub-Bolt metric results in the large bolt, where as the positive perturbation of this quantity results in the Taub-Nut metric. More precisely, according to the simulations, the signs of the perturbation of the quantity a is not preserved under the Ricci flow. Similar to the first two cases, the quantity φ does not exhibit the Sturmian property. A possible explanation for these phenomena is that they are artefacts of the numerical scheme. Therefore, it is not clear whether an analogue of Conjecture 8.3.5 with the roles of φ and ψ reversed exists here.

However, we note that all the quantities remain bounded during the flow. We now present the proofs for some observations above that can be obtained via the applications of the maximum principles which we have seen in Chapters 6 and 7.

8.4 Some Analytic Results

We recall that for a biaxial Bianchi IX initial metric of the form $g = \phi(z)^2 dz^2 + a(z)^2 \omega^1 \otimes \omega^1 + c(z)^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3)$ for $z \in I = (0, 1]$ the metric quantities a and c evolve under the Ricci

flow according to the following equations:

$$\begin{cases} \partial_t a = a'' + 2a' \frac{c'}{c} - 2 \frac{a^3}{c^4}, \\ \partial_t c = c'' + c' \left(\frac{a'}{a} + \frac{c'}{c} \right) - 2 \left(\frac{2c^2 - a^2}{c^3} \right). \end{cases} \quad (8.13)$$

$$(8.14)$$

where the $\dots'$ denote the spatial derivative with respect to the arclength quantity $s(z, t) = \int_0^z \phi(w, t) dw$. We furnish these equations with the Taub-Bolt boundary conditions: at $z = 0$ and all $t > 0$ for which the solution exists, we require $\lim_{z \rightarrow 0} c'(z, t) = 0$, $\lim_{z \rightarrow 0} a'(z, t) = 1$, and $\lim_{z \rightarrow 0} a(z, t) = 0$. On the other hand, at $z = 1$, for all $t > 0$ we fix $a(1, t) = a(1, 0)$ and $c(1, t) = c(1, 0)$. Note that since $\lim_{z \rightarrow 0} c'(z, t) = 0$, as long as the flow exists we must have $\lim_{z \rightarrow 0} c(z, t) > 0$ or otherwise the resulting manifold will have a conical singularity and hence not be smooth at $z = 0$.

Standard maximum and minimum principles from Theorem 4.5.2 allow us to deduce the bounds on the metric quantity a .

Lemma 8.4.1. Suppose that at time $t = 0$, we have $0 \leq a(z, 0) \leq a(1, 0)$ for $z \in (0, 1]$. Then for all $t \in [0, T)$ and $z \in (0, 1]$ we have $0 \leq a(z, t) \leq a(1, t)$. ■

Proof. Indeed, assume that a attains a value bigger than $a(1, 0)$ at some interior point $(z_0, t_0) \in \mathring{I}_T$ where $t_0 > 0$ is the earliest time it occurs. At this point we must have $\partial_t a \geq 0$, $a'' \leq 0$, and $a' = 0$. Thus, by putting these in equation (8.13), we get:

$$0 \leq \partial_t a = a'' + 2a' \frac{c'}{c} - 2 \frac{a^3}{c^4} \leq -2 \frac{a^3}{c^4} < 0,$$

which is a contradiction.

Similarly, if a becomes negative at some interior point $(z_0, t_0) \in \mathring{I}_T$, then at this point we have $\partial_t a \leq 0$, $a'' \geq 0$, and $a' = 0$. Again, by putting these in (8.13), we obtain:

$$0 \geq \partial_t a = a'' + 2a' \frac{c'}{c} - 2 \frac{a^3}{c^4} \geq -2 \frac{a^3}{c^4} > 0,$$

since $a^3 < 0$, which then yields a contradiction. □

We then prove that the initial ordering of the metric quantities a and c is preserved as long as the flow exists. We have to modify the proof of Theorem 4.5.2 since the original maximum principle assumes that the values of the evolving quantities are fixed at the boundaries. In our case here, the value of c is not fixed at one of the boundaries. We prove:

Lemma 8.4.2. If $a(z, 0) \leq c(z, 0)$, then this ordering is preserved under the Ricci flow for all $t > 0$ for which the flow exists. ■

Proof. At $z = 0$ we have $\lim_{z \rightarrow 0} c(z, t) > 0$ and $\lim_{z \rightarrow 0} a(z, t) = 0$ as long as the solution exists, so clearly this ordering is preserved at $z = 0$. Similarly, at $z = 1$ the values of a and c are fixed, so the ordering persists for all time. To prove the ordering in $\mathring{I}_T$, we consider the evolution of the quantity $\frac{\chi}{c} = \frac{a-c}{c}$ which was derived as equation (4.23):

$$\partial_t \left(\frac{\chi}{c} \right) = \left(\frac{\chi}{c} \right)'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) \left(\frac{\chi}{c} \right)' - 4 \left(\frac{a^2}{c^4} + \frac{a}{c^3} \right) \frac{\chi}{c}$$

Assume that a positive interior maximum of $\frac{\chi}{c}$ occurs at some interior point $(z_0, t_0) \in \mathring{I}_T$. Then at this point we have $\partial_t \left(\frac{\chi}{c} \right) \geq 0$, $\left(\frac{\chi}{c} \right)'' \leq 0$, and $\left(\frac{\chi}{c} \right)' = 0$. By putting this in the equation above, we obtain:

$$0 \leq \partial_t \left(\frac{\chi}{c} \right) = \left(\frac{\chi}{c} \right)'' + \left(2 \frac{a'}{a} + \frac{c'}{c} \right) \left(\frac{\chi}{c} \right)' - 4 \left(\frac{a^2}{c^4} + \frac{a}{c^3} \right) \frac{\chi}{c} \leq -4 \left(\frac{a^2}{c^4} + \frac{a}{c^3} \right) \frac{\chi}{c} < 0,$$

since $a > 0$ and $\frac{\chi}{c} > 0$. This is a contradiction. $\square$

The bound for the metric quantity c can also be obtained by an *ad hoc* maximum principle. Since the value $c(0, t)$ is not fixed, we have to keep track of its value at $z = 0$ as well. We prove:

Lemma 8.4.3. Suppose that at time $t = 0$ we have $a(z, 0) \leq c(z, 0) \leq c(1, 0)$ for $z \in (0, 1]$. Then for all $t \in [0, T)$ and $z \in (0, 1]$ we have $a(z, t) \leq c(z, t) \leq c(1, t)$. Furthermore, $\lim_{z \rightarrow 0} c(z, t) \leq c(1, t)$ for all $t \in [0, T)$. $\blacksquare$

Proof. The lower bound was obtained in Lemma 8.4.2. As for the upper bound, we have to check two cases. For the first case, assume that c attains a maximum at an interior point $(z_0, t_0) \in \mathring{I}_T$ where $t_0 > 0$ is the earliest time it occurs. Then at this point we must have $\partial_t c \geq 0$, $c'' \leq 0$, and $c' = 0$. Thus, by putting these in equation (8.14), we have:

$$0 \leq \partial_t c = c'' + c' \frac{a'}{a} + \frac{(c')^2}{c} - \frac{4}{c} + 2 \frac{a^2}{c^3} \leq -\frac{4}{c} + 2 \frac{a^2}{c^3} \leq -\frac{4}{c} + \frac{2}{c} < 0,$$

where we used the fact that $a \leq c$ from Lemma 8.4.2.

For the second case, assume for contradiction that there exists a time $t_0 \in (0, T)$ for which $\lim_{z \rightarrow 0} c(z, t_0) = c(1, t_0)$. At t_0 , we have $\lim_{z \rightarrow 0} \partial_t c \geq 0$, $\lim_{z \rightarrow 0} c' = 0$, and $\lim_{z \rightarrow 0} c'' \leq 0$. By taking the limits in equation (8.14), we obtain:

$$0 \leq \lim_{z \rightarrow 0} \partial_t c = \lim_{z \rightarrow 0} \left(c'' + \frac{c' a'}{a} + \frac{(c')^2}{c} - \frac{4}{c} + 2 \frac{a^2}{c^3} \right) = \lim_{z \rightarrow 0} c'' + \lim_{z \rightarrow 0} \frac{c' a'}{a} - \lim_{z \rightarrow 0} \frac{4}{c}.$$

We then appeal to L'Hôpital's rule for the second term to get:

$$0 \leq \lim_{z \rightarrow 0} 2c'' - \lim_{z \rightarrow 0} \frac{4}{c} < 0,$$

which is a contradiction. $\square$

In fact, we can deduce the evolution of c at $z = 0$ from the above proof.

Lemma 8.4.4. We have $\lim_{z \rightarrow 0} \partial_t c > 0$ if and only if $\lim_{z \rightarrow 0} cc'' > 2$. Similarly, $\lim_{z \rightarrow 0} \partial_t c < 0$ if and only if $\lim_{z \rightarrow 0} cc'' < 2$. ■

Proof. By taking the limit as z tends to 0, the proof of Lemma 8.4.3 implies:

$$\lim_{z \rightarrow 0} \partial_t c = \lim_{z \rightarrow 0} \left(2c'' - \frac{4}{c} \right),$$

from which the conclusion follows. □

We wish to fix the initial condition for this flow with a perturbed Taub-Bolt metric on $\mathbb{CP}^2 \setminus B^4$. We recall from (8.6) that for fixed constants $\alpha > \sqrt{2}$ and $\beta > 0$ the Ricci-flat metrics are given by:

$$g = 8\alpha^2 \beta^2 \frac{(\alpha^4 - z^4)}{(\alpha^2 - 2z^2)^4} dz^2 + 8\alpha^2 \beta^2 \frac{z^2}{(\alpha^4 - z^4)} \omega^1 \otimes \omega^1 + 4\beta^2 \frac{(\alpha^4 - z^4)}{(\alpha^2 - 2z^2)^2} (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

for $z \in (0, 1]$. We denote the arclength coordinate $s(z) = \int_0^z 2\sqrt{2}\alpha\beta \frac{\sqrt{\alpha^4 - w^4}}{(\alpha^2 - 2w^2)^2} dw$. We have the following properties for the metric components of the Taub-Bolt metric :

Lemma 8.4.5. Let $a_0(z) = 2\sqrt{2}\alpha\beta \frac{z}{\sqrt{\alpha^4 - z^4}}$ and $c_0(z) = 2\beta \frac{\sqrt{\alpha^4 - z^4}}{(\alpha^2 - 2z^2)}$. Then we have that a_0 is a strictly increasing concave function and c_0 is a strictly increasing convex function of the arclength coordinate s on $z \in (0, 1]$. ■

Proof. We compute the first and second derivatives of a_0 and c_0 with respect to the arclength coordinate s as:

$$\begin{aligned} a'_0 &= \frac{(\alpha^2 - 2z^2)^2 (\alpha^4 + z^4)}{(\alpha^4 - z^4)^2}, \\ c'_0 &= \frac{\sqrt{2}\alpha z (2\alpha^2 - z^2)}{\alpha^4 - z^4}, \\ a''_0 &= -\frac{\sqrt{2}\alpha z (\alpha^2 - 2z^2)^3 (2\alpha^6 - 3\alpha^4 z^2 + 6\alpha^2 z^4 - z^6)}{\beta (\alpha^4 - z^4)^{\frac{7}{2}}}, \\ c''_0 &= \frac{(\alpha^2 - 2z^2)^2 (2\alpha^6 - 3\alpha^4 z^2 + 6\alpha^2 z^4 - z^6)}{2\beta (\alpha^4 - z^4)^{\frac{5}{2}}}. \end{aligned}$$

Since $\alpha > \sqrt{2}$ and $z \in (0, 1]$, we immediately see that a'_0 and c'_0 are strictly positive, which implies that a_0 and c_0 are strictly increasing. Next, we look at the function $f(z, \alpha) = 2\alpha^6 - 3\alpha^4 z^2 + 6\alpha^2 z^4 - z^6$. Note that $\frac{\partial f}{\partial \alpha} = 12\alpha(\alpha^4 - \alpha^2 z^2 + z^4) > 0$ since $\alpha > \sqrt{2}$ and $z \in (0, 1]$. Furthermore, $f(z, \sqrt{2}) = 16 - 12z^2 + 12z^4 - z^6 > 0$. Thus, $f(z, \alpha) > 0$ for all $\alpha > \sqrt{2}$ and $z \in (0, 1]$. We conclude that $a''_0 < 0$ and $c''_0 > 0$ for all $\alpha > \sqrt{2}$ and $z \in (0, 1]$. This finishes the proof. □

8.5 Discussion

We now discuss some difficulties and challenges that may arise in the study of the problems in this chapter. We will also relate these problems with the singularity formation in Ricci flow studied in [55] and state a conjecture for future studies.

Gradient Bounds

Since we are working on manifolds with boundaries, there are no smoothness conditions for the quantities a and c at the boundary $z = 1$. Therefore, gradient estimates of the quantities a and c similar to Lemmas 6.2.1, 7.2.5, and 7.2.6 would require boundary gradient estimates at $z = 1$. From our estimates, we know that at $z = 1$ we have $a' \leq 0$ and $c' \leq 0$. A usual method to find the lower bound for these quantities would be to find suitable lower barrier functions at $z = 1$, which can be seen in [38], [62], and [94].

However, since (8.13)-(8.14) is a system of non-linear PDEs, we expect that constructing the barriers for a and c simultaneously will be complicated. Instead, we study the evolution equation for $\frac{\chi}{c} = \frac{a-c}{c}$ that was computed in (4.23):

$$\partial_t \left(\frac{\chi}{c} \right) = \left(\frac{\chi}{c} \right)'' + 2 \left(\frac{a'}{a} + \frac{c'}{c} \right) \left(\frac{\chi}{c} \right)' - 4 \left(\frac{a^2}{c^4} + \frac{a}{c^3} \right) \left(\frac{\chi}{c} \right).$$

If we can find a lower barrier f to the solution $\frac{\chi}{c}$ at $z = 1$, then we can conclude that there exists a finite constant $\mathfrak{C}$ (which is equal to f' at $z = 1$) such that:

$$\left(\frac{\chi}{c} \right)' = \left(\frac{a}{c} - 1 \right)' = \frac{ca' - ac'}{c^2} \geq \mathfrak{C},$$

at $z = 1$. This would then imply the derivative bound $a' \geq \mathfrak{C}c^2 + \frac{a}{c}c' \geq \mathfrak{C}c^2 + c'$, which we can then substitute in the equations (8.13) and (8.14) to deduce the uniform bounds for a' and c' .

In order to construct the barrier function f , we need to solve the auxiliary equation:

$$\partial_t f \geq f'' + 2 \left(\frac{a'}{a} + \frac{c'}{c} \right) f' - 4 \left(\frac{a^2}{c^4} + \frac{a}{c^3} \right) f,$$

such that $f \left(\frac{\chi}{c} \right) = \frac{a}{c} - 1 \leq 0$ at $z = 1$ and $f \left(\frac{\chi}{c} \right) = \frac{a}{c} - 1 = -1$ at $z = 0$. By standard maximum principle, if $f \leq 0$ initially, then we must have that $f \leq 0$ whenever it exists. Therefore, it is sufficient to find the barrier f such that:

$$\partial_t f \geq f'' + 2 \left(\frac{a'}{a} + \frac{c'}{c} \right) f', \tag{8.15}$$

In [38], [62], and [94], the barrier function f is constructed by assuming that f is independent of t . Thus we have $\partial_t f = 0$, which reduces (8.15) to a second order ODE that can be

solved explicitly. However, this strategy would not work here since the function f is implicitly dependent on t : the derivatives of the barrier function f in (8.15) are taken with respect to the variable $s(z, t)$, which is a time-dependent gauge. Furthermore, the coefficient of f' in (8.15) depends on the quantities a' and c' , which are the quantities that we wish to bound in the first place. Without a uniform bound for this coefficient, this would also imply that the barrier function f is dependent on t . More careful analysis and construction of the barrier f would be a future direction for this study.

Invariance of Signs of Perturbations

Recall that we have defined the quantities φ and ψ such that $a = a_0 + \varphi$ and $c = c_0 + \psi$ prior to our numerical simulation where the pair (a_0, c_0) correspond to some Ricci-flat Taub-Bolt metric. The perturbations φ and ψ of a_0 and c_0 evolve according to the system of PDEs:

$$\begin{cases} \partial_t \varphi = \varphi'' + \frac{2(a'_0 + \varphi')(c'_0 + \psi')}{c_0 + \psi} - \frac{2a'_0 c'_0}{c_0} - \frac{2(a_0 + \varphi)^3}{(c_0 + \psi)^4} + \frac{2a_0^3}{c_0^4}, \\ \partial_t \psi = \psi'' + \frac{(a'_0 + \varphi')(c'_0 + \psi')}{a_0 + \varphi} - \frac{a'_0 c'_0}{a_0} + \frac{(c'_0 + \psi')^2}{c_0 + \psi} - \frac{(c'_0)^2}{c_0} + \frac{2(a_0 + \varphi)^2}{(c_0 + \psi)^3} - \frac{2a_0^2}{c_0^3} - \frac{4}{c_0 + \psi} + \frac{4}{c_0}. \end{cases}$$

However, the standard maximum principle for system of PDEs fails to apply here since the system is strongly coupled. As mentioned in [61], for a strongly coupled system as above, since invariant or maximum principles are generally unavailable, the boundedness of the solutions is already a hard problem. One usually needs to use *ad hoc* techniques on a case by case basis to show that the solution is bounded. We did manage to show that the metric quantities a and c remain bounded during the flow in Lemmas 8.4.1 and 8.4.3, which then implies that the perturbations φ and ψ remain bounded.

However, determining invariance of the signs of the perturbations φ and ψ is a difficult problem. In fact, we do not even have an invariance of sign for the quantity φ . Indeed, at initial time, assuming that $\varphi \equiv 0$ and $\psi \geq 0$ is a compact perturbation, we have:

$$\begin{aligned} \partial_t \varphi &= \frac{2a'_0(c'_0 + \psi')}{c_0 + \psi} - \frac{2a'_0 c'_0}{c_0} - \frac{2a_0^3}{(c_0 + \psi)^4} + \frac{2a_0^3}{c_0^4} \\ &\leq \frac{2a'_0 \psi'}{c_0 + \psi} + \underbrace{\left(\frac{2a_0^3}{c_0^4} - \frac{2a_0^3}{(c_0 + \psi)^4} \right)}_{\geq 0}, \end{aligned}$$

by using the fact that $a'_0 c'_0 \geq 0$ from Lemma 8.4.5.

If we choose a small initial data for ψ with very large negative gradient to cancel the positive term, this forces $\partial_t \varphi < 0$ at some point in $(0, 1)$. So, φ becomes negative instantaneously.

Therefore, a non-negativity result for φ might require some bound on the gradient of the initial data for ψ . For example $\left(\frac{\psi}{c_0}\right)' \geq 0$ is sufficient for this, but this perturbation ψ would not be a compact perturbation.

A possible *ad hoc* method to prove Conjecture 8.3.5, by following the arguments for the weakly coupled systems in [31], [63], [96], and [98], would be the following:

We assume for contradiction that there exists some (z_0, t_0) at which $\psi(z_0, t_0) = 0$ such that $t_0 > 0$ is minimal. Then, necessarily, we have $\partial_t \psi \leq 0$, $\psi' = 0$, and $\psi'' \geq 0$ at this point. By substituting these in the evolution equation for ψ , at (z_0, t_0) we have:

$$0 \geq \frac{(a'_0 + \varphi')c'_0}{a_0 + \varphi} - \frac{a'_0 c'_0}{a_0} + \frac{2(a_0 + \varphi)^2}{c_0^3} - \frac{2a_0^2}{c_0^3}.$$

By using Lemma 8.4.5, which implies that $a'_0 c'_0 \geq 0$, we deduce:

$$0 \geq \frac{\varphi' c'_0}{a_0 + \varphi} + \left(\frac{a'_0 c'_0}{a_0 + \varphi} - \frac{a'_0 c'_0}{a_0} + \frac{2(a_0 + \varphi)^2}{c_0^3} - \frac{2a_0^2}{c_0^3} \right). \quad (8.16)$$

However, we do not know the sign of the sum of the bracketed terms in (8.16) since we have seen that φ does not have a sign invariance. Furthermore, since φ is 0 at the boundaries, by the Mean Value Theorem, there is at least one interior point where $\varphi' > 0$ and at least one interior point where $\varphi' < 0$ for any fixed $t_0 > 0$ as seen in Figures 8.9 and 8.10. So this *ad hoc* maximum principle fails to prove anything.

Consequently, the proof for the Sturmian property of the quantity ψ is also out of reach here since in most literature such as [3], [24], [33], [64], and [95], a maximum principle is required to prove the Sturmian property of solutions to parabolic equations. A comparison or invariant principle for the strongly coupled system of parabolic equations along the lines of [23], [78], and [79] would also be of interest for future studies.

Singularity Formation

Whilst it may be very difficult to prove the conjectures posed by Holzegel, Schmelzer, and Warnick, one of the main points that we are interested in here is that prior to the surgery, when the bolt of the Taub-Bolt metric tends to a single point, what type of singularity does it encounter? It is expected that the analysis which were carried out in Chapters 6 and 7 may shed some light onto this problem.

Since the singularity occurs at the tip of the manifold, it would not look like a shrinking cylinder after parabolic rescaling. It has been conjectured (see [17] or [40]) that the only generic singularity models of the Ricci flow in dimension 4 are S^4 , $S^3 \times \mathbb{R}$, $S^2 \times \mathbb{R}^2$, $\mathbb{CP}^2$, and the

blowdown shrinking soliton $(\mathcal{L}_{-1}^2, h)$, which is also known as the FIK soliton. So, after rescaling of the singularity at the tip of the Taub-Bolt manifold, it is expected to resemble one of these singularity models. However, one might be able to rule out the manifold $\mathbb{CP}^2$ equipped with the Fubini-Study metric as a singularity model since it is dynamically unstable, as shown by Kröncke [58] as well as Knopf and Šešum [57]. Here is a digression describing the FIK soliton:

Remark 8.5.1. [16] A metric on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is called a Calabi metric if it is Kähler and $U(2)$ -invariant. Calabi observed that any Kähler and $U(2)$ -invariant metric on $\mathbb{C}^2 \setminus (0, 0)$ must be of the form:

$$h = (e^{-r} \phi \delta_{\alpha\beta} + e^{-2r} (\phi_r - \phi) \bar{z}^\alpha z^\beta) dz^\alpha \otimes d\bar{z}^\beta,$$

where $r = \log(|z_1|^2 + |z_2|^2)$ and $\phi(r) = \partial_r P(r)$ where P is a Kähler potential. This metric extends smoothly to $(0, 0)$ if and only if there exists positive constants $a_0, a_1 > 0$ such that $\phi(r) = a_0 + a_1 e^r + a_2 e^{2r} + \mathcal{O}(e^{3r})$ as $r \rightarrow \infty$.

At spatial infinity, the metric closes up smoothly if and only if these two conditions hold: $\phi_r > 0$ everywhere and there exist constants $b_0 > a_0$ and $b_1 < 0$ such that $\phi(r) = b_0 + b_1 e^{-r} + b_2 e^{-2r} + \mathcal{O}(e^{-3r})$ as $r \rightarrow \infty$. These two conditions then induce uniquely a smooth metric on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. Under the Kähler-Ricci flow, the evolution of an arbitrary Calabi metric is determined by:

$$\partial_t \phi = \frac{\phi_{rr}}{\phi} + \frac{\phi_r}{\phi} - 2. \quad (8.17)$$

The blowdown soliton $(\mathcal{L}_{-1}^2, h)$, which was constructed and studied in [30], is obtained by setting ϕ to be equal to a function φ which satisfies the separable first order ODE:

$$\varphi_r = \frac{\varphi}{\sqrt{2}} - (\sqrt{2} - 1) - \left(1 - \frac{1}{\sqrt{2}}\right) \varphi^{-1}.$$

A remark from [30] also states that φ satisfies the second order ODE:

$$\frac{\varphi_{rr}}{\varphi} + \frac{\varphi_r}{\varphi} - \sqrt{2} \varphi_r + \varphi - 2 = 0,$$

which, along with (8.17), implies that $\varphi_t = \sqrt{2} \varphi_r - \varphi$. Hence, the soliton $(\mathcal{L}_{-1}^2, h)$ evolves only by translation and scaling. ■

A similar problem has been studied in [55] in which Isenberg, Knopf, and Šešum studied the biaxial Bianchi IX metric of the form:

$$g = ds^2 + a(s)^2 \omega^1 \otimes \omega^1 + c(s)^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

for $s \in [-1, 1]$. This is similar to the one we studied in Chapter 7. However, instead of S^4 , the analysis was done on a closed manifold $S^2 \tilde{\times} S^2$ which is called the twisted bundle of S^2 over S^2 .

The difference in topology is manifested by the boundary conditions of the metric components a and c .

The conditions that were adopted in their work for a are similar to what we had for the S^4 case in Chapter 7; that is, a satisfies the cylinder-to-sphere rule at the tips. However, the metric component c satisfies the following conditions:

1. Define $\bar{c}(s) = c(-1 + s)$ for $0 \leq s \leq 2$. This function satisfies $\bar{c}(0) > 0$ and $\lim_{s \rightarrow 0} \frac{d^{2k+1}\bar{c}}{ds^{2k+1}} = 0$ for all $k \in \mathbb{N}$. In other words, locally around $s = -1$ the metric component c extends to an even function.
2. Define $\tilde{c}(s) = c(1 + s)$ for $-2 \leq s \leq 0$. This function satisfies $\tilde{c}(0) > 0$ and $\lim_{s \rightarrow 0} \frac{d^{2k+1}\tilde{c}}{ds^{2k+1}} = 0$ for all $k \in \mathbb{N}$. In other words, locally around $s = 1$ the metric component c extends to an even function.

At the tips of the manifold the topology becomes a disc bundle over S^2 locally; that is, locally the manifold looks like $D^2 \hookrightarrow D_1^4 \twoheadrightarrow S^2$ with boundary $\partial D_1^4 \approx S^3$. The main result obtained in [55] is the convergence of a non-Kähler Riemannian manifold to a Kähler-Ricci soliton at this tip. More precisely, they proved the following theorem:

Theorem 8.5.2. [55] There exists a non-empty open set of non-Kähler metrics g on $S^2 \tilde{\times} S^2$ of the form $g = ds^2 + a(s)^2 \omega^1 \otimes \omega^1 + c(s)^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3)$ satisfying the following conditions:

1. $a \leq c$,
2. $c|c'| \leq a$,
3. $|a'| \leq \frac{2}{\sqrt{3}}$,
4. $c(1)^2 - 3c(-1)^2 \geq \delta^2$ for some $\delta > 0$,
5. $c' \geq 0$ with strict inequality off the poles,

such that any Ricci flow solution originating from this set has the following properties:

1. the first four assumptions above are preserved by the flow,
2. the solution develops a Type I singularity at $T < \infty$ with $\lim_{t \rightarrow T} \text{Vol}(\pi^{-1}(-1)) = 0$,
3. every blow-up sequence $(S^2 \tilde{\times} S^2, g_k(t), p)$ with $p \in \pi^{-1}(-1)$ subconverges to a Kähler singularity model that is the blowdown shrinking soliton $(\mathcal{L}_{-1}^2, h)$. ■

Note that the conditions on the metric components a and c for the twisted bundle $S^2 \tilde{\times} S^2$ are similar to the Taub-Bolt metric components on $\mathbb{CP}^2 \setminus B^4$ with the boundary occurring at $z = 1$. However, we must have $a(1) > 0$ in the latter manifold so that the squashing parameter is not degenerate; that is, $\tau > 0$. However, we note that this is a local condition away from the singularity. Since the blow-up sequence is carried out at $p \in \pi^{-1}(-1)$, the difference between the geometries of $S^2 \tilde{\times} S^2$ and $\mathbb{CP}^2 \setminus B^4$ at $z = 1$ might not play a crucial role in the development of the singularity at the other end of the manifold.

The other differences are more subtle. For example, we have to show that the metric component c increases over the fibre and we have the ordering $a \leq c$ in the Taub-Bolt metric. Thus, we expect that the analysis will be similar, if not the same, to the one carried out by Isenberg, Knopf, and Šešum in [55]. Their analysis requires the bounds for a' and c' , which are still unavailable as discussed earlier in this section. Furthermore, we need to discount a neckpinch occurring within the interval $(0, 1)$; that is, we have to show that the quantity a cannot reach 0 for some $z > 0$ at some finite time. This has been observed in the numerical simulations but not proven analytically.

We end this thesis by stating the following conjecture, which is an augmented version of Conjecture 8.3.3:

Conjecture 8.5.3. For any biaxial Bianchi IX symmetric initial data and fixed metric on the boundary S^3 given by $g_{z=1} = \mu^2 \omega^1 \otimes \omega^1 + \nu^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3)$ with $\tau = \frac{\mu}{\nu} \in (0, \tau_c)$, Ricci flow plus at most one surgery will converge to a member of either the Taub-Bolt or the Taub-Nut family. Prior to the convergence to the latter, the solution develops a Type I singularity at $T < \infty$ such that every blow-up sequence $(\mathbb{CP}^2 \setminus B^4, g_k(t), p)$ with $p \in \pi^{-1}(\{-1\})$ subconverges to a Kähler singularity model that is the blowdown shrinking soliton $(\mathcal{L}_{-1}^2, h)$. ■

Appendices

Appendix A

Manifold Theory

A.1 Smooth Manifolds

Manifolds are generalisations of the Euclidean spaces: at a microscopic level, they look like Euclidean spaces. However, they themselves may not have a vector space structure of the Euclidean space, so we need to work around that in order to define the analogues of familiar notions of calculus on the Euclidean space.

Definition A.1.1 (Topological manifolds). A manifold of dimension m is a second countable, Hausdorff topological space which is locally Euclidean of the same dimension. Locally Euclidean means that for every point $p \in M$ there exists an open neighbourhood U of p such that U is homeomorphic to an open subset Ω of $\mathbb{R}^m$. ■

Such homeomorphism $\varphi : U \rightarrow \Omega \subset \mathbb{R}^m$ is called a coordinate chart or local coordinates around p . These homeomorphisms are usually denoted as:

$$\begin{aligned}\varphi : U &\rightarrow \Omega, \\ q &\mapsto (x_1(q), x_2(q), \dots, x_m(q)).\end{aligned}$$

We now define smooth functions on the manifold M . These are defined by using the charts.

Definition A.1.2 (Smooth functions). A function $f : M \rightarrow \mathbb{R}$ is called a smooth function if for each chart (U_i, φ_i) the pullbacks $f \circ \varphi_i^{-1} : \Omega_i \rightarrow \mathbb{R}$ are smooth functions. ■

Definition A.1.3 (Smooth maps). A map $F : M \rightarrow N$ between two manifolds of dimensions m and n respectively is called a smooth map if for each chart (U_i, φ_i) of M and $(\tilde{U}_j, \tilde{\varphi}_j)$ of N the compositions $\tilde{\varphi}_j \circ F \circ \varphi_i^{-1} : \Omega_i \rightarrow \tilde{\Omega}_j$ are smooth maps between subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$. A smooth bijective map with smooth inverse is called a diffeomorphism. ■

We want to be able to approximate these functions and maps by linear maps. Since manifolds themselves may not have a vector space structure and we need this structure in order to define linear maps, we need to approximate the manifolds locally by some vector space. This is done by attaching a vector space at each point $p \in M$ on the manifold.

Definition A.1.4 (Tangent vector). A tangent vector (p, X) , X_p , or X at a point $p \in M$ is a derivation, which is a linear map $X \in \mathfrak{D}_p$ such that for any smooth functions $f, g : M \rightarrow \mathbb{R}$ we have the Leibniz rule:

$$X(fg) = g(p)Xf + f(p)Xg. \quad \blacksquare$$

Definition A.1.5 (Tangent space). The tangent space $T_p M$ at the point $p \in M$ is the space of all derivations at p . More concretely, we can identify this space with $\{p\} \times E_p$ where E_p is an m dimensional vector space. If the local coordinates in a neighbourhood U of p is given by $(x_1, x_2, \dots, x_m)$, then the tangent spaces $T_q M|_U$ for all $q \in U$ are spanned by the vectors $\left\{ \frac{\partial}{\partial x^1} \Big|_q, \frac{\partial}{\partial x^2} \Big|_q, \dots, \frac{\partial}{\partial x^m} \Big|_q \right\}$. $\blacksquare$

Remark A.1.6. For brevity, in local coordinates we usually denote $\frac{\partial}{\partial x^i} \Big|_q$ as ∂_i by suppressing the point dependence. We also write ∂_{x_i} when there is an ambiguity in coordinate labels. $\blacksquare$

With this extra structure, we can now define the derivative of a map between two manifolds $f : M \rightarrow N$ at some point $p \in M$ since there exist approximating vector spaces allocated to the points p and $f(p)$. It is the induced linear map between tangent spaces of manifolds. In other words, it is the best linear approximation of the map f .

Definition A.1.7 (Derivative of a map). Let $f : M \rightarrow N$ be a map between manifolds. The derivative Df of the map f (also called the pushforward of f , denoted by f_*) at $p \in M$ is the induced linear map Df_p defined by:

$$Df_p \text{ or } f_* : T_p M \rightarrow T_{f(p)} N,$$

such that for any smooth function $g : N \rightarrow \mathbb{R}$ and $X \in T_p M$ we have:

$$(Df_p(X))(g) = X(g \circ f)(p). \quad \blacksquare$$

Remark A.1.8. By Definition A.1.7, derivatives can be calculated explicitly in local coordinates by expressing the vector X with respect to the basis of coordinate derivatives. In some cases, for example when we are dealing with infinite-dimensional manifolds, it is often more useful to consider the following formulation of pushforwards. In order to explicitly compute the

image of a given $X \in T_p M$ under the pushforward, we can consider any curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ such that $\gamma(0) = p$ and $\gamma'(0) = X$. Then the image of X is given explicitly by:

$$Df_p(X) = \left. \frac{d}{dt} \right|_{t=0} (f \circ \gamma) \in T_{f(p)} N. \quad \blacksquare$$

Proposition A.1.9 (Properties of the pushforward). Let M, N, P be manifolds and $f : M \rightarrow N$ and $g : N \rightarrow P$ are maps between them. Then the pushforward satisfies the following:

1. (linearity): $Df_p : T_p M \rightarrow T_{f(p)} N$ is a linear map; that is, for any scalar $\lambda \in \mathbb{R}$ and $X, Y \in T_p M$ we have $Df_p(X + \lambda Y) = Df_p X + \lambda Df_p Y$.
2. (chain rule): the composition $g \circ f : M \rightarrow P$ has a derivative at $p \in M$ which can be expressed as the map $D(g \circ f)_p : T_p M \rightarrow T_{g \circ f(p)} P$ defined by the composition:

$$D(g \circ f)_p = Dg_{f(p)} \circ Df_p. \quad \blacksquare$$

We can define the tangent spaces pointwise on the manifold. By collecting all the tangent spaces for each point on the manifold M , we have an object called the tangent bundle:

Definition A.1.10 (Tangent bundle). A tangent bundle on M is the triple (TM, M, π) where TM is called the total space and π is the projection map $\pi : TM \rightarrow M$ such that $\pi((p, X)) = p$ for all $(p, X) \in T_p M$. In other words, it is the disjoint union:

$$TM = \bigsqcup_{p \in M} T_p M. \quad \blacksquare$$

Another very important object that we are going to define from the tangent bundle is a smooth section of the tangent bundle, which is called a vector field. Intuitively speaking, it is a choice of one vector $X(p)$ in $T_p M$ for each $p \in M$ such that it varies smoothly over the manifold.

Definition A.1.11 (Vector field). Let TM be a tangent bundle. A vector field or a smooth section of the tangent bundle is a smooth map $X : M \rightarrow TM$ such that $\pi \circ X = \text{id}_M$. The space of vector fields is denoted $\Gamma(TM)$, $C^\infty(TM)$, or $\mathfrak{X}(M)$. $\blacksquare$

Therefore, the notion of derivative or a pushforward of a map at a point extends to the whole of M by considering it pointwise on any vector field.

At each point $p \in M$, a vector field points in a direction such that it varies smoothly in a neighbourhood of p . Given a vector field X on M we can define an integral curve and flow generated by the vector field X from a point $p \in M$ by following these “directions”. Essentially, these curves are paths $\gamma(t)$ in M such that the derivative of the curve coincides with X at each point on the path; that is, $\gamma'(t) = X(\gamma(t))$.

Definition A.1.12. The integral curve of a vector field X at a point $p \in M$ is the curve $\gamma : (\varepsilon, \varepsilon) \rightarrow M$ for some small enough $\varepsilon > 0$ such that $\gamma(0) = p$ and $\gamma'(t) = X(\gamma(t))$. ■

Furthermore, the solution map:

$$\Phi : (\Gamma(TM), I, M) \mapsto \Phi(X, t, p) = \{\gamma(t) \text{ such that } \gamma(0) = p \text{ and } \gamma'(t) = X(\gamma(t))\},$$

defines a local flow on the manifold M .

In local coordinates, we can choose the coordinate vector fields $\{\partial_1, \partial_2, \dots, \partial_m\}$ as the basis for the tangent spaces of all the points in a small neighbourhood U of p . However, a different choice of basis for each $q \in U$ might be more useful. This choice of basis may vary between points in U while still spanning the fibre at each point, which forms a collection of vector fields.

Definition A.1.13 (Frame). A frame of a tangent bundle $TM|_U$ is a collection of vector fields $\{E_1, E_2, \dots, E_m\}$ such that for each $p \in U$ the set of tangent vectors $\{E_1(p), E_2(p), \dots, E_m(p)\}$ is a basis for T_pM . ■

Such a frame always exist locally: for example, the coordinate vector field. However, it may not exist globally. For example, there does not exist a global frame on the sphere S^n for even n since any vector field on these manifolds must vanish somewhere by the Hairy Ball Theorem. If there does exist a global frame, we call the manifold parallelisable.

Definition A.1.14 (Parallelisable manifold). A manifold M is called parallelisable if there exists a global frame for T , that is there exist vector fields $\{X_1, X_2, \dots, X_m\}$ such that for all $p \in M$, the set $\{X_1(p), X_2(p), \dots, X_m(p)\}$ forms a basis of the tangent space T_pM . Equivalently, the tangent bundle is said to be trivial. ■

Since the fibres at any point $p \in M$ are distinct vector spaces and each vector space is linear over $\mathbb{R}$, the space $\Gamma(TM)$ forms a $C^\infty(M)$ -module. More concretely, given $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$, the linear combination $X + fY$ is also a vector field where the addition is considered fibrewise; that is:

$$X + fY = \bigsqcup_{p \in M} (X + fY)(p) = \bigsqcup_{p \in M} (X(p) + f(p)Y(p)) \quad \text{where} \quad X(p), Y(p) \in T_pM.$$

Another way of combining two vector fields is by the Lie brackets:

Definition A.1.15 (Lie bracket). Let $X, Y \in \Gamma(TM)$. Then the commutator element $[X, Y] = XY - YX$ is also in $\Gamma(TM)$. The operation $[\cdot, \cdot]$ is called the Lie bracket. ■

Here are some properties of the Lie bracket of vector fields:

Proposition A.1.16 (Properties of the Lie bracket). Let $X, Y, Z \in \Gamma(TM)$. Then we have the following properties:

1. (anti-symmetry): $[X, Y] = -[Y, X]$. In particular, $[X, X] = 0$,
2. (bi-linearity): for $\lambda \in \mathbb{R}$, we have $[\lambda X + Y, Z] = \lambda[X, Z] + [Y, Z]$,
3. (Jacobi identity): $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$,
4. If $\{\partial_1, \dots, \partial_m\}$ are coordinate vector fields on a coordinate patch $U \subset M$, then for all $p \in U$ and $i, j = 1, \dots, m$ we have $[\partial_i, \partial_j] \equiv 0$,
5. Suppose that we have $X = X^i \partial_i$ and $Y = Y^j \partial_j$ in local coordinates. Then in these coordinates we have $[X, Y] = (X^i \partial_i Y^j - Y^i \partial_i X^j) \partial_j$. ■

Since $[X, Y]$ is also a vector field on M , we can pushforward it under the diffeomorphism $f : M \rightarrow N$. In fact, we have the following result:

Proposition A.1.17. Suppose that $X, Y \in \Gamma(TM)$ and $f : M \rightarrow N$ is a diffeomorphism. Then for all $p \in M$ we have:

$$Df_p([X, Y]) = [Df_p(X), Df_p(Y)]_{f(p)}. \quad \blacksquare$$

Recall that the tangent space at each point is a vector space. With any vector space V , we can define a dual vector space V^* , which is the vector space of linear functionals on V ; that is:

$$V^* = \{f \text{ such that } f : V \rightarrow \mathbb{R} \text{ is a linear map}\} = \text{Hom}(V, \mathbb{R}).$$

So, at each point $p \in M$ there exists another vector space T_p^*M , which is dual to T_pM . This space is called the cotangent space at p . To see this space explicitly, pick a local coordinate (U, φ) on an m -dimensional manifold M . Then, as we have seen before, in local coordinates each tangent space in the tangent bundle $TM|_U$ is spanned by $\{\partial_1, \partial_2, \dots, \partial_m\}$. We construct a dual for this space by declaring a choice of m covectors $\{dx^1, dx^2, \dots, dx^m\}$ such that $dx^i(\partial_j) = \delta_j^i \in \mathbb{R}$ at all points $p \in U$.

In a similar fashion to what we did for the tangent spaces, we can define the cotangent bundle, cotangent field, and coframe.

Definition A.1.18 (Cotangent bundle). The cotangent bundle T^*M is the dual to the tangent bundle TM . In local coordinates it is spanned by $\{dx^1, \dots, dx^m\}$ such that $dx^i(\partial_j) = \delta_j^i$ for all $i, j = 1, 2, \dots, m$. ■

Definition A.1.19 (Covector field). Let T^*M be a cotangent bundle. A covector field or a smooth section of the cotangent bundle is a map $s : M \rightarrow T^*M$ such that $\pi \circ s = \text{id}_M$. The space of covector fields is denoted $\Gamma(T^*M)$ or $C^\infty(T^*M)$. ■

Definition A.1.20 (Coframe). Let $\{E_1, E_2, \dots, E_m\} \subset \Gamma(TM|_U)$ be a frame of the tangent bundle. We define the coframe to be an ordered collection of m smooth sections of the cotangent bundle $\{\omega^1, \omega^2, \dots, \omega^m\} \subset \Gamma(T^*M|_U)$ such that at each $p \in U$, $\omega^i(E_j) = \delta_j^i$ for all $i, j \in \{1, 2, \dots, m\}$. This also implies that $\{\omega^1(p), \omega^2(p), \dots, \omega^m(p)\}$ forms a basis of T_p^*M at every $p \in U$. ■

A.2 Product Manifolds and Lifts

From two manifolds M and N of dimensions m and n respectively, we can construct a new manifold by considering the product manifold $P = M \times N$. Conversely, suppose that we are given a product manifold, we can decouple the factors by projection operations.

Definition A.2.1 (Projections). Given a product manifold $P = M \times N$, the projections $\pi : P \rightarrow M$ and $\sigma : P \rightarrow N$ are the restriction of the product manifold to its factors. ■

Definition A.2.2 (Fibres and leaves). Suppose that $P = M \times N$ is a product manifold and $\pi : P \rightarrow M$ is a projection onto the first factor. The fibre of the projection at $p \in M$ is the preimage of this projection; that is, $\pi^{-1}(p) = \{p\} \times N$. The leaves of this projection are the manifolds $M \times \{q\}$ for each $q \in N$. ■

The projections are obviously smooth and surjective. Thus their pushforwards $D\pi$ and $D\sigma$ exist, and so vector fields on P induce vector fields on M and N . In fact, we have:

Proposition A.2.3. Let $(p, q) \in P$. The tangent space $T_{(p,q)}P$ is the direct sum of the tangent space of the fibre of π at $p \in N$ and its leaf at $q \in M$, which are $T_{(p,q)}(\{p\} \times N)$ and $T_{(p,q)}(M \times \{q\})$. In other words, each element of $T_{(p,q)}P$ has a unique expression $X + V$ where $X \in T_{(p,q)}(\{p\} \times N)$ and $V \in T_{(p,q)}(M \times \{q\})$. ■

Now we are going to define the reverse of this; that is, we are going to induce a unique vector field on the product manifold from each constituent manifolds.

Definition A.2.4 (Lifts of functions). Let $f : M \rightarrow \mathbb{R}$ be a function on the manifold M . The lift of f is the function $\tilde{f} = f \circ \pi$ defined on the product manifold P . ■

Definition A.2.5 (Lifts of vector fields). Given a vector field X on the manifold M , we define the lift $\tilde{X}$ of this vector field as the unique vector field on P such that $D\pi(\tilde{X}) = X$ and $D\sigma(\tilde{X}) = 0$. ■

These two definitions extend obviously to the factor N as well. We are going to denote the set of lifts of vector fields from M and N by $\mathcal{L}(TM), \mathcal{L}(TN) \subset \Gamma(TP)$ respectively. Sometimes these are called the horizontal and vertical lifts respectively. We have the following proposition:

Proposition A.2.6. [71] Suppose that $\tilde{X}, \tilde{Y} \in \mathcal{L}(TM)$ and $\tilde{V} \in \mathcal{L}(TN)$. Then we have:

1. $[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]} \in \mathcal{L}(TM)$,
2. $[\tilde{X}, \tilde{V}] = 0$. ■

Since a vector field and its lift have the same expression in local coordinates on their respective base manifolds, we utilise an abuse of notation in which the lift $\tilde{X}$ of a vector field is also denoted by X , without the $\widetilde{}$ notation.

A.3 Riemannian Manifolds

To generalise the notion of tangent and cotangent bundles further, their constructions can be done for general vector spaces as well; that is, we attach arbitrary vector spaces to each point on the manifold and ensure that they transition between different coordinate patches smoothly. This construction is called the vector bundle and it is defined as:

Definition A.3.1 (Vector bundles). An n -dimensional vector bundle over an m -dimensional manifold M is the triple $(\mathcal{E}, M, \pi)$ where $\mathcal{E}$ is the total space and π is the projection map $\pi : \mathcal{E} \rightarrow M$ such that:

1. Each fibre $\mathcal{E}_p = \pi^{-1}(\{p\})$ is an n -dimensional vector space,
2. For each $p \in M$ there exists a neighbourhood $U \subset M$ of p and a diffeomorphism $\psi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times \mathbb{R}^n \\ \pi \downarrow & & \downarrow \pi' \\ U & \xrightarrow{\text{Id}} & U \end{array}$$

where π' is the projection of $U \times \mathbb{R}^n$ to the first factor.

3. The restriction of ψ to each fibre $\psi|_{\mathcal{E}_p} : \mathcal{E}_p \rightarrow \{p\} \times \mathbb{R}^n$ is a linear isomorphism. ■

From what we have seen previously, the construction of a covector bundle is derived from the vector space structure of the fibres of a tangent bundle, namely the existence of duals of the

fibres. We can construct many more vector bundles out of the tangent and cotangent spaces via constructions of new vector spaces and attaching copies of these new vector spaces pointwise on M . In particular, one important construction is the tensor product $\otimes$ of the tangent and/or cotangent spaces. We recall that abstract tensor products are defined as thus:

Definition A.3.2 (Tensor product). A tensor product of two vector spaces V and W is the vector space $V \otimes W$ together with a bi-linear map $\psi : V \times W \rightarrow V \otimes W$ satisfying the universal property; that is, if $\phi : V \times W \rightarrow Z$ is any bi-linear map into a vector space Z , then there exists a unique linear map $\bar{\phi} : V \otimes W \rightarrow Z$ such that the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{\psi} & V \otimes W \\ & \searrow \phi & \swarrow \exists! \bar{\phi} \\ & Z & \end{array}$$

■

Of course, this is a very abstract definition and not very useful for explicit computations. Thus, usually the bi-linear map ψ is suppressed. In place of it, a basis for $V \otimes W$ is constructed thus giving us a more concrete definition of tensor products:

Lemma A.3.3. A tensor product of two vector spaces V and W of dimensions m and n , with bases $\{v_1, v_2, \dots, v_m\}$ and $\{w_1, w_2, \dots, w_n\}$ respectively is given by the vector space $V \otimes W$ spanned by the linear combinations of mn formal vectors $\{v_i \otimes w_j\}$. ■

In order to define lengths and angles on a manifold, we need a notion of infinitesimal distances on the manifold as in Euclidean spaces. We can find lengths and angles between two vectors in $\mathbb{R}^n$ by introducing an additional structure on $\mathbb{R}^n$ which is the inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$. We generalise this inner product as a symmetric tensor product of the cotangent vectors at each point $p \in M$, which is also a 2-tensor bundle.

Definition A.3.4 (2-tensor bundle). The space of 2-tensor bundle is the disjoint union of all the $T_p^* M \otimes T_p^* M$ over the points $p \in M$:

$$T^* M \otimes T^* M \cong \bigsqcup_{p \in M} \text{Hom}(T_p M \otimes T_p M, \mathbb{R}).$$

■

Definition A.3.5 (Metric tensor bundle). A metric tensor bundle $\mathcal{M}$ is a subset of the tensor bundle $T^* M \otimes T^* M$ such that any $g \in \mathcal{M}$ has the following properties:

1. (bi-linear): $g(X + fY, Z) = g(X, Z) + fg(Y, Z)$ for any $f \in C^\infty(M)$ and $X, Y, Z \in \Gamma(TM)$,
2. (symmetric): $g(X, Y) = g(Y, X)$ hence $\mathcal{M} \subset \text{Sym}^2(T^* M)$,

3. (non-degenerate): for all $p \in M$ and $g_p \in \mathcal{M}_p$ (that is, the fibre of the metric bundle at p) if $X, Y \in T_p M$, then $g_p(X, Y) \neq 0$ unless X or $Y = 0$. ■

Definition A.3.6 (Riemannian metric). A Riemannian metric is a smooth section of $\mathcal{M}$ such that for every $p \in M$, the metric g_p is positive definite; that is, for any non-zero $X \in T_p M$ we have $g_p(X, X) := \|X\|_p^2 > 0$. We denote the space of Riemannian metrics as $\mathfrak{Met}$. ■

This metric can be seen as the infinitesimal inner product on the manifold. In particular, for any $X \in T_p M$ the quantity $g(X, X)$ measures the length of the vector X in $T_p M$. From the curve-based definition of the tangent space $T_p M$, we can think of this quantity as the infinitesimal length of a curve in the direction of the tangent vector X at p . Thus, for a given curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ in M , we can define its length $L(\gamma)$ as the sum of these infinitesimal lengths, or more explicitly:

$$L(\gamma) = \int_{-\varepsilon}^{\varepsilon} \sqrt{g(\partial_t \gamma, \partial_t \gamma)} dt.$$

In essence, this is not dissimilar to the inner product on a Euclidean space. Locally, the metric can be expressed as a matrix of functions on $U \subset M$ which is symmetric and positive definite. In this local coordinate chart U we can write components of g as $g_{ij} = g(\partial_i, \partial_j) = g_{ji}$, which can be identified with a symmetric matrix. In local coordinates and language of tensors, a metric can be written locally as:

$$g = g_{ij}(x) dx^i \otimes dx^j.$$

Definition A.3.7 (Riemannian manifold). A Riemannian manifold is defined as a smooth manifold M equipped with the metric g denoted by (M, g) . ■

Remark A.3.8. The space of Riemannian metrics on a compact manifold $\mathfrak{Met}$ is an open subset of the space of sections of symmetric 2-tensors $\Gamma(\text{Sym}^2(T^*M))$. It is important to note that the superspace $\Gamma(\text{Sym}^2(T^*M))$ is a infinite-dimensional vector space over $\mathbb{R}$. Since $\mathfrak{Met}$ is an open subset of $\Gamma(\text{Sym}^2(T^*M))$, its tangent space is $\Gamma(\text{Sym}^2(T^*M))$ itself [36]. ■

Remark A.3.9. Given two Riemannian manifolds (M, g_M) and (N, g_N) we can furnish the product manifold $P = M \times N$ with a natural metric g called the product metric. This metric is defined as $g = \pi^* g_M + \sigma^* g_N$ or more concretely:

$$g(X, Y) = g_M(D\pi(X), D\pi(Y)) + g_N(D\sigma(X), D\sigma(Y)),$$

where $X, Y \in \Gamma(P)$ and π and σ are projection functions from P to M and N respectively. Of course, there are many other metrics that can be equipped to the manifold P . One important example is the warped product metric which was defined in Definition 4.0.1 in Chapter 4. ■

Definition A.3.10 (Affine connection). Let M be a differentiable manifold and $\mathcal{E}$ a vector bundle of dimension n over M . A connection or covariant derivative on $\mathcal{E}$ at a point $p \in M$ is a map $D : \Gamma(\mathcal{E}) \rightarrow \Gamma(T^*M \otimes \mathcal{E})$ such that for any $V, W \in T_p M$, $\varsigma, \tau \in \Gamma(\mathcal{E})$, $f \in C^\infty(M)$, and $\lambda \in \mathbb{R}$ we have that $D_V \varsigma \in \mathcal{E}_p$ with the following properties:

1. D is tensorial over $T_p M$: $D_{fV+W} \varsigma = f D_V \varsigma + D_W \varsigma$,
2. D is $\mathbb{R}$ -linear in $\Gamma(\mathcal{E})$: $D_V(\lambda \varsigma + \tau) = \lambda D_V \varsigma + D_V \tau$,
3. D satisfies the Leibniz rule over $\Gamma(\mathcal{E})$: $D_V(f \varsigma) = df(V) \cdot \varsigma + f D_V \varsigma$. ■

Remark A.3.11. Given connections $D^\mathcal{E}$ and $D^\mathcal{F}$ on vector bundles $\mathcal{E}$ and $\mathcal{F}$ respectively, we can define a connection $D^{\mathcal{E} \otimes \mathcal{F}}$ on the tensor bundle of $\mathcal{E} \otimes \mathcal{F}$ by:

$$D_X^{\mathcal{E} \otimes \mathcal{F}}(\varsigma \otimes \tau) = (D_X^\mathcal{E} \varsigma) \otimes \tau + \varsigma \otimes (D_X^\mathcal{F} \tau).$$

Similarly, a connection $D^\mathcal{E}$ on vector bundle $\mathcal{E}$ induces a connection $D^{\mathcal{E}^*}$ on $\mathcal{E}^*$ by using the product rule and connection on $\mathcal{E}$. Explicitly, if $\varsigma \in \mathcal{E}$ and $s \in \mathcal{E}^*$, we have:

$$X(s(\varsigma)) = (D_X^{\mathcal{E}^*} s)(\varsigma) + s(D_X^\mathcal{E} \varsigma). \quad \blacksquare$$

This extends to higher tensor products of $\mathcal{E}$ and $\mathcal{E}^*$ via the product rule. Note that for any tensor field $Y \in \Gamma(\mathcal{E})$ and $X \in \Gamma(TM)$ the covariant derivative commutes with trace for any pairs of indices; that is, $D_X(\text{tr}(Y)) = \text{tr}(D_X Y)$ since trace is a linear operator.

Concretely, any affine connection is fully defined by the values of $D_\varsigma \in \Gamma(T^*M \otimes \mathcal{E})$ for all $\varsigma \in \mathcal{E}$. Since the vector bundle is n -dimensional, at any point $p \in M$ we can choose a basis $\{\varsigma_i\}_{i=1}^n$ of $\mathcal{E}_p$ and define the values of D_{ς_i} for all $i = 1, 2, \dots, n$. Therefore, for a given vector bundle on M there are infinitely many choices of connections on it.

The definition of a connection can be defined prior to the introduction of a metric on the manifold. We can endow the vector bundle $\mathcal{E}$ with a bundle metric, which is a family of inner products on the fibres of $\mathcal{E}_p$ for all $p \in M$ which varies smoothly on M . If $\mathcal{E}$ is the tangent bundle TM , then this bundle metric can be chosen to be the Riemannian metric g . We can impose compatibility and torsion free conditions on the connection. We define:

Definition A.3.12 (Compatible connection). A connection D on a vector bundle $\mathcal{E}$ is compatible with the bundle metric $\langle \cdot, \cdot \rangle$ on $\mathcal{E}$ if for any $X \in \Gamma(TM)$ and $\varsigma, \tau \in \Gamma(\mathcal{E})$ we have:

$$D_X \langle \varsigma, \tau \rangle = X \langle \varsigma, \tau \rangle = \langle D_X \varsigma, \tau \rangle + \langle \varsigma, D_X \tau \rangle. \quad \blacksquare$$

Definition A.3.13 (Torsion tensor). The torsion tensor Tor is defined such that given two vector fields $X, Y \in \Gamma(TM)$ we have:

$$\text{Tor}(X, Y) = D_X Y - D_Y X - [X, Y]. \quad \blacksquare$$

By demanding the compatibility of the connection and vanishing of the torsion tensor, we reduce the number of possible connections on this manifold from infinitely many to just one.

Theorem A.3.14 (Fundamental theorem of Riemannian geometry). On each Riemannian manifold (M, g) , there exists a unique connection on TM which is compatible with the metric and is torsion-free; that is, $\nabla g \equiv 0$ and $\text{Tor} \equiv 0$. This connection is called the Levi-Civita connection and denoted by ∇ . $\blacksquare$

An outline of the proof of Theorem A.3.14 is the following. For vector fields $X, Y \in \Gamma(TM)$, in local coordinates $\{x^i\}$ we can write them explicitly as $X = X^i(x)\partial_i$ and $Y = Y^j(x)\partial_j$. The expression for $\nabla_X Y$ can be computed directly by the properties of a connection as follows:

$$\nabla_X Y = \nabla_{X^i \partial_i} (Y^j \partial_j) = X^i \nabla_{\partial_i} (Y^j \partial_j) = (X^i \partial_i Y^j) \partial_j + X^i Y^j (\nabla_{\partial_i} \partial_j).$$

In the above expression all of the terms are defined except for $\nabla_{\partial_i} \partial_j \in \Gamma(TM)$. These can be written in terms of a collection of objects called the Christoffel symbols Γ_{ij}^k where:

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k \quad \text{for all } i, j \in \{1, 2, \dots, m\}. \quad \blacksquare$$

Therefore, a Levi-Civita connection is fully characterised by these symbols. In local coordinates, the Christoffel symbols can be computed explicitly in terms of the metric g using the torsion free and compatibility properties as well as the Koszul formula:

Proposition A.3.15 (Koszul formula). Let $X, Y, Z \in \Gamma(TM)$. Then we have the identity:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

Using coordinates, we can calculate the Christoffel symbols using the Koszul formula with $X = \partial_i$, $Y = \partial_j$ and $Z = \partial_k$, which gives us an explicit formula:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij}). \quad (\text{A.1})$$

Note the symmetry between the indices i and j ; that is, $\Gamma_{ij}^k = \Gamma_{ji}^k$. Note also that by chain rule, the Christoffel symbols define covariant derivatives on covectors as well. We can compute:

$$\nabla_{\partial_i} dx^j = -\Gamma_{ik}^j dx^k. \quad (\text{A.2})$$

The covariant derivative allows us to define geodesics, which are the “straight lines” on a Riemannian manifold (M, g) . More precisely:

Definition A.3.16 (Geodesic). A geodesic on a smooth Riemannian manifold (M, g) is defined as a curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ such that $\nabla_{\partial_t \gamma} \partial_t \gamma = 0$ for all $t \in (-\varepsilon, \varepsilon)$. In local coordinates, γ satisfies the equation:

$$\frac{d^2 \gamma^k}{dt^2} + \Gamma_{ij}^k \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0,$$

where Γ_{ij}^k are the Christoffel symbols of the metric g . ■

Note that the geodesic equation is a second order ODE. Therefore, given the data $\gamma(0) = p \in M$ and $\partial_t \gamma(0) \in T_p M$, the geodesic is uniquely determined in some maximal interval $I \subset \mathbb{R}$ containing $t = 0$. Next, we define the exponential map which is a function from a subset of the tangent space at p to the manifold M .

Definition A.3.17 (Exponential map). Let $p \in M$ and $\mathcal{U}_p \subset T_p M$ be the subset of $T_p M$ such that geodesics with initial data $X \in \mathcal{U}_p$, which are denoted by γ_X , are defined on an interval containing $[0, 1]$. The exponential map is the map:

$$\exp : \mathcal{U}_p \rightarrow M,$$

$$X \mapsto \gamma_X(1). \quad \blacksquare$$

We note that the pushforward of the exponential map at p is invertible (it is the identity) and thus, by the Inverse Function Theorem, there is an open neighbourhood $\mathcal{U} \subset T_p M$ of 0 such that the exponential map restricted to $\mathcal{U}$ is a diffeomorphism onto its image. For small enough $\varepsilon > 0$ we call the image $B_\varepsilon^g(p) = \exp(B_\varepsilon(0))$ the geodesic ball of radius ε around p .

Definition A.3.18 (Injectivity radius). The injectivity radius $\text{inj}(M, g, p)$ of the Riemannian manifold (M, g) at the point $p \in M$ is the supremum of $r > 0$ such that the exponential map on $B_r(0) \in T_p M$ is a diffeomorphism onto its image. ■

Furthermore, by the local diffeomorphism property of the exponential map, we can define the map $\exp^{-1} : U \rightarrow T_p M$ on a small neighbourhood U of an arbitrary point $p \in M$. We can also define a vector space isomorphism $E : T_p M \rightarrow \mathbb{R}^m$ which identifies the orthonormal basis of $T_p M$ (with respect to the metric g) with an orthonormal basis of $\mathbb{R}^m$ (with respect to the Euclidean metric). Thus, the composition $\varphi := E \circ \exp^{-1} : U \rightarrow \mathbb{R}^m$ is a special local chart for the manifold M around the point p , which is called the normal coordinate chart centred at p .

Proposition A.3.19 (Properties of the normal coordinates). Let (U, φ) be a normal coordinate chart of the Riemannian manifold (M, g) centred at p . Then, we have the following:

1. the coordinate of p is $(0, 0, \dots, 0)$; that is, $\varphi(p) = (0, 0, \dots, 0)$,
2. at the point p , the Riemannian metric in the normal coordinates is given by $g_{ij}(p) = \delta_{ij}$,
3. the Christoffel symbols and the partial derivatives of the metric components vanish at the point p ; that is, $\Gamma_{ij}^k(p) = 0$ and $\partial_i g_{jk}(p) = 0$. Furthermore, $\partial_i h_{jk}(p) = \nabla_i h_{jk}(p)$. ■

With the covariant derivatives completely defined, we can define the second covariant derivatives, which will be used to define curvature of this connection. The second covariant derivative is simply the composition of two covariant derivatives on a section of a vector bundle.

Definition A.3.20 (Second covariant derivative). We define the second covariant derivative on a vector bundle $\mathcal{E}$ as the map $D^2 : \Gamma(\mathcal{E}) \rightarrow \Gamma(T^*M \otimes T^*M \otimes \mathcal{E})$ given by:

$$\begin{aligned} D_{XY}^2 \varsigma &= (D^2 \varsigma)(X, Y) = (D_X D_Y \varsigma)(Y) = D_X(D_Y \varsigma) - D_Y(D_X \varsigma) \\ &= D_X(D_Y \varsigma) - D_{\nabla_X Y} \varsigma, \end{aligned}$$

where $\varsigma \in \Gamma(\mathcal{E})$ and ∇ is the Levi-Civita connection. ■

Unlike the second order derivative in Euclidean space, the second covariant derivatives generally do not commute. In other words, the following expression may not vanish identically:

$$D^2 \varsigma(X, Y) - D^2 \varsigma(Y, X) = D_X(D_Y \varsigma) - D_Y(D_X \varsigma) - D_{[X, Y]} \varsigma.$$

The expression above measures the failure for the second covariant derivative to commute. We can easily show that this quantity is in fact a tensor. This quantity is called the curvature of the connection, which is the central object of study in Riemannian geometry.

Definition A.3.21 (Curvature). The curvature of the connection D on the vector bundle $\mathcal{E}$ is the tensor $\Omega \in \Gamma(T^*M \otimes T^*M \otimes \mathcal{E}^* \otimes \mathcal{E})$ such that:

$$\begin{aligned} \Omega(X, Y) \varsigma &= D^2 \varsigma(X, Y) - D^2 \varsigma(Y, X) \\ &= D_X(D_Y \varsigma) - D_Y(D_X \varsigma) - D_{[X, Y]} \varsigma \in \Gamma(\mathcal{E}), \end{aligned} \tag{A.3}$$

for any $X, Y \in \Gamma(TM)$ and $\varsigma \in \Gamma(\mathcal{E})$. ■

Along with the Levi-Civita connection ∇ of a Riemannian metric g , this definition leads to the Riemannian, Ricci, and scalar curvatures in Chapter 2.

Appendix B

Lie Groups and Lie Algebras

B.1 Lie Groups

Definition B.1.1 (Lie group). A Lie group is a group G that is also a finite-dimensional differentiable manifold, on which the multiplication map $(g, h) \mapsto gh$ and the inversion map $g \mapsto g^{-1}$ are smooth maps. ■

Since these groups are also manifolds, we can endow them with the structures of differentiable manifolds. Thus, we can have smooth maps between Lie groups:

Definition B.1.2 (Lie group homomorphism). Let G and H be Lie groups. A group homomorphism $\phi : G \rightarrow H$ is a Lie group homomorphism if the map ϕ is smooth with respect to the manifold structure of G and H . ■

The Lie groups G and H are called isomorphic if there exists a diffeomorphism that is also a group homomorphism between them. Furthermore, we can introduce tangent and cotangent bundles as well as vector and covector fields. An important family of vector fields on a Lie group is the left-invariant vector fields.

Definition B.1.3 (Left and right multiplication map). For every $g \in G$ we define the left multiplication map $L_g : G \rightarrow G$ as $L_g(h) = gh$ for all $h \in G$. Similarly, we define the right multiplication map $R_g : G \rightarrow G$ as $R_g(h) = hg$. By definition of Lie groups, these maps are smooth maps. ■

Definition B.1.4 (Left-invariant vector fields). A vector field $X \in \Gamma(G)$ is left-invariant if the pushforward of a left multiplication map fixes X for all $g \in G$; that is $(DL_g)(X) = X$ for all $g \in G$. More concretely, for all $h \in G$ we have: $(DL_g)_h(X_h) = X_{gh}$. ■

Of course, one can easily see that left invariant vector fields only depend on the value of the vector field at the identity element since we can determine the whole vector field by considering it fibrewise $(DL_g)_e(X_e) = X_g$ for all $g \in G$, that is:

$$X = \bigsqcup_{g \in G} ((DL_g)_e X_e). \quad (\text{B.1})$$

Therefore, by construction, for any Lie group G there exists always a left-invariant vector field on it. Furthermore, this implies that the space of left-invariant vector field is finite-dimensional and can be isomorphically identified with $T_e G$. Left-invariant vector fields are special in the way that we have the following result:

Proposition B.1.5. Given a Lie group G and a left-invariant vector field X on G , for any $g \in G$ the flow map generated by X satisfies $\Phi(X, t, g) = g \Phi(X, t, e)$. ■

A direct corollary of the above is that the flow map exists for all $t \in \mathbb{R}$. This is true because by short time existence we can always extend integral curves uniformly and indefinitely.

Corollary B.1.6. The flow map $\Phi(X, t, e)$ is defined for all $t \in \mathbb{R}$. ■

Furthermore, by Proposition A.1.17, we have:

Proposition B.1.7. The Lie bracket of two left-invariant vector fields is left-invariant; that is, if X and Y are left-invariant vector fields, then $[X, Y]$ is also left-invariant. ■

B.2 Lie Algebras

Definition B.2.1 (Lie algebra). A Lie algebra is a vector space V over some field F equipped with the multiplication map $\bullet : V \times V \rightarrow V$ such that it is bi-linear, anti-symmetric, and satisfies the Jacobi identity; that is, for any $u, v, w \in V$, we have:

$$u \bullet (v \bullet w) + v \bullet (w \bullet u) + w \bullet (u \bullet v) = 0. \quad \blacksquare$$

Since the left invariant vector fields can be identified to vectors in $T_e G$, we can define a binary “multiplication” operation $\bullet$ on $T_e G$ by $X_e \bullet Y_e := [X, Y]_e$ where X and Y are the left-invariant vector field generated by X_e and Y_e respectively.

The vector $[X, Y]$ is also left-invariant by Proposition B.1.7, thus it can be identified with an element of $T_e G$. Furthermore, from Proposition A.1.16, since the Lie bracket is bi-linear, anti-symmetric, and satisfies the Jacobi identity, the vector space $T_e G$ equipped with $\bullet$ is a Lie algebra, which we denote by $\mathfrak{g}$.

A Lie group homomorphism between G and H induces a Lie algebra homomorphism between $\mathfrak{g}$ and $\mathfrak{h}$. Note that an algebra homomorphism preserves the multiplicative structure of the algebra; that is, if $(A, \bullet)$ and $(B, \circ)$ are two algebras and F is an algebra homomorphism between them, then $F : A \rightarrow B$ implies that $F(a \bullet b) = F(a) \circ F(b)$ for all $a, b \in A$.

Proposition B.2.2. Suppose that $\phi : G \rightarrow H$ is a Lie group homomorphism. Then the push-forward of ϕ at the identity $D\phi_e : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism. ■

Like any algebra, the Lie algebra can be determined fully by its structure constants:

Definition B.2.3 (Structure constants). Given a set of basis vectors $\{e_1, \dots, e_n\}$ for an n -dimensional algebra $(V, \bullet)$ over a field F , the structure constants are defined as the collection of constants $\{c_{ij}^k \in F : i, j, k = 1, 2, \dots, n\}$ such that $e_i \bullet e_j = c_{ij}^k e_k$ for all $i, j, k = 1, 2, \dots, n$. ■

Remark B.2.4. For a Lie algebra $\mathfrak{g}$ with basis $\{x_1, \dots, x_n\}$, the structure constants are $c_{ij}^k \in \mathbb{R}$ such that $[x_i, x_j] = c_{ij}^k x_k$. Since the Lie bracket is anti-symmetric, we have $c_{ij}^k = -c_{ji}^k$. ■

Next, we can define an operation that maps $\mathfrak{g}$ to G , which is called the exponential map. This uses the concept of integral curves and flow that we have seen in Appendix A.

Definition B.2.5 (Exponential map). The exponential map on Lie algebras is the map:

$$\begin{aligned} \exp : \mathfrak{g} &\rightarrow G, \\ x &\mapsto \Phi(X, 1, e), \end{aligned}$$

where X is the left-invariant vector field on G generated by $x \in \mathfrak{g}$ and $\Phi(X, t, e)$ is the flow generated by X from the point e . ■

Of course, if we consider a family of vectors in a Lie algebra given by the line $L(x) = tx$ for $x \in \mathfrak{g}$ and $t \in [0, 1]$, the exponential map on this family gives us the unique curve:

$$\exp(tx) = \Phi(X, t, e) = \{\gamma(t) \in G : \gamma(0) = e, \gamma(1) = \exp(x)\}, \quad (\text{B.2})$$

by solving the one-parameter ODE $\gamma'(t) = (DL_{\gamma(t)})_e x$.

Finally, here is an important correspondence between Lie algebra homomorphisms and Lie group homomorphisms:

Proposition B.2.6. Let $\phi : G \rightarrow H$ be a Lie group homomorphism. Then we have $\phi \circ \exp = \exp \circ D\phi_e$. In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{D\phi_e} & \mathfrak{h} \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\phi} & H \end{array}$$

■

B.3 Adjoint Representations

One way of studying groups is to study their representations. This is a more concrete way of viewing groups as matrices in $\text{GL}(V)$ (more generally, as automorphisms $\text{Aut}(V)$) for some vector space V . Recall the definition of representation of a group:

Definition B.3.1 (Representation of a group). A representation of a group G is a vector space V along with a group homomorphism ρ :

$$\rho : G \rightarrow \text{Aut}(V). \quad \blacksquare$$

So, instead of studying the Lie groups themselves, we can deduce more information by studying their representations. Since V is arbitrary, there are many different representations of any given Lie group G .

For example, any Lie group action of G on an m -dimensional topological manifold M induces various representations of G , such as the infinite-dimensional representation on the vector space $C^\infty(M)$ defined by:

$$(\rho(g)(f))(p) = f(g^{-1} \cdot p),$$

for any $f \in C^\infty(M)$. Furthermore, if M has a G -invariant point $p \in M$ (that is $g \cdot p = p$ for all $g \in G$), then we can define an m -dimensional representation on the vector space $T_p M$ via the pushforward of the action of g at the point $p \in M$; that is, for any $X \in T_p M$ we define:

$$\rho(g)(X) = g_*(X) \in T_p M.$$

For a given Lie group G , there is a canonical action of the group onto itself with a G -invariant point, which is defined as follows:

Definition B.3.2 (Conjugation map). For each $g \in G$ we define the conjugation map $C_g : G \rightarrow G$ as the composition $L_g \circ R_{g^{-1}}$. More concretely, we have $C_g(h) = ghg^{-1}$. $\blacksquare$

A G -invariant point of the conjugation map is the identity $e \in G$. So, for any $g \in G$ the derivative of C_g at e is the linear map from $T_e G = \mathfrak{g}$ to itself, which gives us a representation of G . We call this canonical representation the adjoint representation of G .

Definition B.3.3 (Adjoint representation of Lie groups). The adjoint representation of a Lie group G is the vector space $\mathfrak{g}$ along with the homomorphism Ad defined as:

$$\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g}),$$

$$g \mapsto (DC_g)_e \in \text{Aut}(\mathfrak{g}). \quad \blacksquare$$

In a similar fashion, we can also define representations on algebras. We can define the adjoint representation for a Lie algebra $\mathfrak{g}$ of a Lie group G by taking the derivative of the adjoint map at $e \in G$, which is $D\text{Ad}_e$. This map sends an element $x \in T_e G = \mathfrak{g}$ to the tangent space of $\text{Aut}(\mathfrak{g})$ at Id , which is $\text{End}(\mathfrak{g})$ (by choosing a basis, we can identify $\text{Aut}(V)$ and $\text{End}(V)$ with $\text{GL}(n, \mathbb{R})$ and $\text{Mat}_{n \times n}(\mathbb{R})$ respectively). We denote this canonical representation of $\mathfrak{g}$ by ad , which is:

$$\begin{aligned} \text{ad} : \mathfrak{g} &\rightarrow \text{End}(\mathfrak{g}), \\ x &\mapsto D\text{Ad}_e(x) \in \text{End}(\mathfrak{g}). \end{aligned}$$

More concretely we have an explicit definition:

Definition B.3.4 (Adjoint representation of Lie algebras). The adjoint representation of a Lie algebra $\mathfrak{g}$ of a Lie group G is defined as the pushforward of the adjoint representation Ad on G at e . Explicitly, it is given by the map:

$$\begin{aligned} \text{ad} : \mathfrak{g} &\rightarrow \text{End}(\mathfrak{g}), \\ x &\mapsto [x, \cdot] \in \text{End}(\mathfrak{g}). \end{aligned} \quad \blacksquare$$

Remark B.3.5. This map is a linear map. Thus, if $\{x_1, x_2, \dots, x_n\}$ is a basis for $\mathfrak{g}$, by linearity of the adjoint representation of the Lie algebra, we can determine the map ad fully by its basis; that is, the maps $\text{ad}(x_i) = [x_i, \cdot]$ for all $i = 1, 2, \dots, n$ fully determine the map ad . This would give us a set of $n \times n$ matrices M_i such that the i -th column is the zero vector and the k -th column is $(c_{ik}^1, c_{ik}^2, \dots, c_{ik}^n)^T$ for all $k \neq i$. ■

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