

Abstract

In the paper we present the exact solutions of one-dimensional Non-linear Schrödinger Equation. The solutions correspond to the Bogoliubov excitations in Bose-gas with a local interaction. The obtained expression is used for evaluating the transmission coefficient of the excitations across a δ -functional potential barrier.

Exact form of the Bogoliubov excitations in one-dimensional nonlinear Shrödinger equation

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1 Introduction

Bose-Einstein condensation has been observed in trapped dilute alkali-metal atomic gases. The experimental developments have been impressive, and a number of fundamental properties of these many-body systems can be investigated due to the ideal possibilities for manipulation and monitoring of the condensates. Theoretically they are well treated by the simplest approximate theories of many-body physics, and they are therefore ideal for investigating properties of simple many-body systems [2]. These systems are well described by the Gross-Pitaevskii equation [4],[7] which transforms to Nonlinear Shrödinger Equation (NLS) in a local limit. Nonlinear Shrödinger Equation which is one of the fundamental equations of nonlinear physics [1] is widely used in many fields of modern science from nonlinear optics to elementary particle physics. Localized solutions of NLS which correspond to nonreflective potentials of scattering problem are well-investigated. It is used to employ Bogoliubov's method of decomposition of time-dependent wave function by wave functions of elementary excitations. One can find for the last a whole system of second order differential equations, which is known as a Bogoliubov-deGennes system. There are methods of exact investigation of such equations [1]. In the paper it is shown explicit form of exact solutions, which wasn't found in literature. As an example the obtained expression is used for evaluating the transmission coefficient of the excitations across a δ -functional potential barrier. The result is interesting in connection with investigations of Josephson-like effect in BEC.

2 Bogoliubov-de Gennes system of equations

We consider a zero-temperature Bose-Einstein condensate of atoms with mass m and chemical potential μ . The atoms interact by elastic collisions with s-wave scattering length a_{sc} and their low kinetic energies permit a replacement of their short-range interaction by a contact term $V_{int}(\mathbf{r}) \rightarrow U_0 \delta(\mathbf{r}) = 4\pi\hbar^2 a_{sc} \delta(\mathbf{r})/M$, so that the single particle wave functions obeys the Gross-Pitaevskii equation

[5]

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 - \mu + U_0 |\Psi(\mathbf{r}, t)|^2 \right] \Psi(\mathbf{r}, t). \quad (1)$$

Using proper units of measurement we will have in one dimension

$$i \frac{\partial \Psi(x, t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 \Psi(x, t)}{\partial x^2} - \Psi(x, t) + |\Psi(x, t)|^2 \Psi(x, t). \quad (2)$$

Equation (2) can be linearized using representation of $\Psi(x, t)$ as a sum of time-independent wave function $\Psi_0(x)$ which describes the ground state of the condensate and time-dependent wave function of the excitations $\psi(x, t)$,

$$\Psi(x, t) = \Psi_0(x) + \psi(x, t). \quad (3)$$

Substitute (3) into (2) and assume $\psi(x, t)$ is a small value in comparison with $\Psi_0(x)$ we will have the equation

$$-\frac{1}{2} \frac{d^2 \Psi_0(x)}{dx^2} - \Psi_0(x) + \Psi_0^3(x) = 0 \quad (4)$$

The exact well-known solution of this equation is the instanton [3]

$$\Psi_0(x) = \tanh x. \quad (5)$$

Time-dependent wave function $\psi(x, t)$ is described by the equation

$$i \frac{\partial \psi(x, t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi(x, t)}{\partial x^2} - \psi(x, t) + g(x) (2\psi(x, t) + \psi^*(x, t)) \quad (6)$$

where $g(x) = |\Psi_0(x)|^2 = \tanh^2 x$.

Let's find the solution of the equation (6), representing the wave function of the excited state as a superposition of wave functions of the elementary excitations,

$$\psi(x, t) = u(x) e^{-i\epsilon t} - v^*(x) e^{i\epsilon t} \quad (7)$$

Substituting (7) into (6), we will have Bogoliubov-de Gennes system of equations

$$\epsilon u(x) = -\frac{1}{2} \frac{\partial^2 u(x)}{\partial x^2} - u(x) + g(x) (2u(x) - v(x)) \quad (8)$$

$$-\epsilon v(x) = -\frac{1}{2} \frac{\partial^2 v(x)}{\partial x^2} - v(x) + g(x) (2v(x) - u(x)) \quad (9)$$

We may obtain a simpler equation for the excitations, if we rewrite the Bogoliubov-de Gennes equations,

$$\epsilon T(x) = -\frac{1}{2} \frac{d^2 S(x)}{dx^2} + (g(x) - 1) S(x) \quad (10)$$

$$\epsilon S(x) = -\frac{1}{2} \frac{d^2 T(x)}{dx^2} + (3g(x) - 1) T(x) \quad (11)$$

where we have introduced [5]

$$S(x) = u(x) + v(x), \quad T(x) = u(x) - v(x). \quad (12)$$

The system (11,10) can be transformed to the fourth order differential equation

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + 3 \tanh^2 x - 1\right) \left(-\frac{1}{2}\frac{d^2}{dx^2} + \tanh^2 x - 1\right) S(x) = \varepsilon^2 S(x) \quad (13)$$

Let's find a solution of the equation (13) by Bargman's method [1] as a composition of plane wave and a some function, which is the polynomial of k

$$S(x) = F(k, x) e^{ikx}. \quad (14)$$

The simplest nontrivial example has the form

$$S(x) = (\beta k + a(x)) e^{ikx}, \quad (15)$$

where $a(x)$ – is the arbitrary sufficiently smooth function. Substituting (15) into equation (13) and equating the components at the same powers of k , we will find the solution,

$$S(x) = \left(-\frac{1}{2}ik + \tanh x\right) e^{ikx}. \quad (16)$$

Where k is one of the roots of equation

$$\varepsilon = \frac{1}{2}k\sqrt{k^2 + 4}, \quad (17)$$

that determines the dispersion law of Bogoliubov's excitations. With small k (17) gives the sound spectrum $\varepsilon = k$. The equation (13) is the fourth order linear differential equation and it must have four linearly independent solutions. The equation (17) also has four different solutions,

$$k_{1,2} = \pm\sqrt{2}\sqrt{-1 + \sqrt{1 + \varepsilon^2}}, \quad (18)$$

$$k_{3,4} = \pm i\sqrt{2}\sqrt{1 + \sqrt{1 + \varepsilon^2}}, \quad (19)$$

that determines k with given ε . With $\varepsilon \ll 1$ the solutions (18) $k_{1,2} \approx \pm\varepsilon$, but (19) are pure imaginary $k_{3,4} \approx \pm 2i$. The general solution of (13) is a linear superposition of solutions (16) with arbitrary coefficients C_n

$$S(x) = \sum_{n=1}^4 C_n \left(-\frac{1}{2}ik_n + \tanh x\right) e^{ik_n x}. \quad (20)$$

Substituting (20) into (10) we will have for $T(x)$

$$T(x) = \sum_{n=1}^4 C_n \frac{k_n}{2\varepsilon} \left(k_n \left(-\frac{1}{2}ik_n + \tanh x\right) - i \operatorname{sech}^2 x\right) e^{ik_n x}. \quad (21)$$

Expressions (20) and (21) are total system of solutions for (13). When the excitation moves in a Bose-condensate there is no reflection from a kink because the wave function of the excitation is an eigenfunction in a system with kink.

3 Excitations scattering on a δ - functional potential barrier

As an example let's use the obtained solutions to calculate the transmission coefficient of excitations scattering on a δ -functional barrier. The problem is interesting in connection of the Josephson-like effect in BEC. It was solved at small k in [6]. The ground state of the equation, corresponding to a Bose-gas with a local interaction in one dimension with an external δ -functional potential

$$-\frac{1}{2} \frac{d^2 \Psi}{dx^2} - \Psi + V_0 \delta(x) \Psi + |\Psi|^2 \Psi = 0 \quad (22)$$

we will search in the form

$$\Psi(x) = \tanh(|x| + x_0). \quad (23)$$

The wave function must be continuous at $x = 0$ but a derivative has a break,

$$\Psi'(+0) - \Psi'(-0) = 2V_0 \Psi(0).$$

It follows that

$$1 - \xi^2 = V_0 \xi, \quad \xi = \tanh x_0 \quad (24)$$

The problem of excitations scattering on a local potential is described by the solution of the equation (20), which at large negative values of coordinate must transform to a superposition of incident and reflected wave, but at large positive values of x — to transmitted wave. Let's write the solutions at $x < 0$ and $x > 0$, satisfying necessary boundary conditions

$$S(x) = e^{ikx} \left(-\frac{1}{2} ik - \tanh(|x| + x_0) \right) + G e^{-ikx} \left(\frac{1}{2} ik - \tanh(|x| + x_0) \right) + \quad (25)$$

$$+ P e^{qx} \left(-\frac{1}{2} q - \tanh(|x| + x_0) \right), x < 0 \quad (26)$$

$$T(x) = \frac{k}{2\varepsilon} e^{ikx} \left(k \left(-\frac{1}{2} ik - \tanh(|x| + x_0) \right) - i(1 - \tanh^2(|x| + x_0)) \right) + \quad (27)$$

$$+ G \frac{k}{2\varepsilon} e^{-ikx} \left(k \left(\frac{1}{2} ik - \tanh(|x| + x_0) \right) + i(1 - \tanh^2(|x| + x_0)) \right) - \quad (28)$$

$$- P \frac{q}{2\varepsilon} e^{qx} \left(q \left(-\frac{1}{2} q - \tanh(|x| + x_0) \right) + (1 - \tanh^2(|x| + x_0)) \right), x < 0$$

$$S(x) = F e^{ikx} \left(-\frac{1}{2} ik + \tanh(x + x_0) \right) + Q e^{-qx} \left(\frac{1}{2} q + \tanh(x + x_0) \right), x > 0$$

$$T(x) = F \frac{k}{2\varepsilon} e^{ikx} \left(k \left(-\frac{1}{2} k + \tanh(x + x_0) \right) - i(1 - \tanh^2(x + x_0)) \right) +$$

$$+ \frac{q}{2\varepsilon} e^{-qx} \left(-q \left(\frac{1}{2} q + \tanh(x + x_0) \right) + (1 - \tanh^2(x + x_0)) \right), x > 0$$

where,

$$k = \sqrt{2}\sqrt{-1 + \sqrt{1 + \varepsilon^2}} \approx \varepsilon$$

$$q = \sqrt{2}\sqrt{1 + \sqrt{1 + \varepsilon^2}} \approx 2$$

Using the lacing conditions of functions and its derivatives at zero and assuming (24) we will have the system of four linear equations with four indeterminate. Solution of the system in approach $q = 2$ gives the following for the transmission and reflection coefficients

$$D = |F|^2 = \xi^2 \frac{(4\xi^3 + 4\xi^2 + 2k^2\xi + 4\xi + 6k^2 + 4 + k^4)^2}{(k^2 + 4)(\xi^2 k^2 + (1 + \xi^2)^2)(k^4 + 4k^2 + 4k^2\xi + 4\xi^2 + 8\xi^3 + 4\xi^4)} \quad (29)$$

$$R = |G|^2 = k^2 \frac{(\xi^2 - 1)^2 (2\xi^2 + 2\xi + 4 + k^2)^2}{(k^2 + 4)(\xi^2 k^2 + (1 + \xi^2)^2)(k^4 + 4k^2 + 4k^2\xi + 4\xi^2 + 8\xi^3 + 4\xi^4)}$$

The transmission coefficient at $k \gg 1$ equals to 1, and at $k \rightarrow 0$ tends to 1 as

$$D = \frac{\xi^2}{\xi^2 + k^2}, \quad (30)$$

as in [6].

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References

- [1] G.L.Lamb, Elements of Soliton Theory, 1994 Dover Publications, Incorporated
- [2] Lev.P.Pitaevskii Theory of Bose-Einstein condensation in trapped gases. Review of Modern Physics, vol. 71, No.3, April 1999 p.463
- [3] A.M.Polyakov, Gauge Fields and Strings, Gordon & Breach Publishing Group, January 1987
- [4] L. D. Landau L. P. Pitaevskii Statistical Physics: Course of Theoretical Physics, 3rd Ed., Vol. 9 Butterworth-Heinemann, December 1990
- [5] K.Berg-Sørensen and K. Molmer Bose-Einstein condensates in spatially periodic potentials PRA Vol.58(2) 1998 pp.1480-1484
- [6] D.L.Kovrizhin, L.A. Maksimov "Transmission Coefficient for Excitations Passing through a One-Dimensional Barrier Described by the Nonlinear Schrödinger Equation" Doklady Physics 2001, Volume 46, Issue 5, pp. 328-330
- [7] D.L. Kovrizhin, L.A. Maksimov "Cherenkov radiation" of a sound in a Bose condensed gas. Physics Letters A, Vol. 282 (6) (2001) pp. 421-427