Robust Optimisation with Applications in Supply Chain Management



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Abstract

We consider a production planning problem in supply chain management, namely the lot sizing problem under uncertainty. Lot sizing problems contain a fairly high degree of uncertainty in real life. Thus pre-decided optimal production plans are no longer of practical use. Current methods to address the uncertainty in the lot sizing problem either make strong assumptions or increase the complexity of the deterministic model significantly. In this study, we propose lot sizing models that use robust optimization to deal with uncertainty in demand and production costs. These models provide a good trade-off between the robustness of the solution, i.e., to what extent the solution is feasible, and the quality of the objective function value.

Four robust methodologies are used, namely, the minimax regret criterion, the uncertainty budget, the uncertainty range, and the bw-robustness criterion. With the minimax regret criterion, the robust lot sizing model aims to achieve a production plan that minimizes the maximum surplus between the total cost of the production plan and that of the optimal production plan after revealing the uncertain data. With the uncertainty budget and the uncertainty range, the robust lot sizing model minimizes the maximum total cost that could occur due to the uncertain data. However, the size of the total uncertainty is bounded by the uncertainty budget and the scope of the allocation of the uncertainty data is bounded by the uncertainty range. The bw-robust lot sizing model identifies a production plan that guarantees an objective function value of at most w in all data circumstances and maximizes the probability that the objective function is below b.

While the main priority of most classic robust optimization approaches is to deal with feasibility, the quality of the objective function value is equally important to the decision maker in the process of production planning. In this thesis, we focus more on the quality of the objective function and, besides being robust, we model explicitly how the uncertain parameters affect the total production cost in the lot sizing problem. As far as the author

is aware, the minimax regret lot sizing study is the first to consider uncertainty in both the coefficients of the objective function and a parameter that defines the feasible region, and the *bw*-robust lot sizing study is the first to deal with a parameter that defines the feasible region. With the popular idea of uncertainty budget, we redefine the worst case caused by data uncertainty as when it harms the objective function value the most, and extend this idea by further proposing the uncertainty range parameter which helps to provide a better trade-off between robustness and the quality of the objective function value.

Efficient polynomial time algorithms are proposed for the minimax regret uncapacitated lot sizing model, the robust lot sizing model with uncertainty budget, and the robust lot sizing model with uncertainty range. For the *bw*-robust lot sizing model with demand uncertainty, we propose Mixed Integer Nonlinear Programming formulations, which are reformulated as Mixed Integer Linear Programming formulations and solved with a commercial package. Numerical results are provided with various settings of parameters to show the effectiveness of the solutions of the proposed robust lot sizing models, in terms of feasibility and quality of the objective function.

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List of Abbreviations

CCLS	Constant Capacitated Lot Sizing model
CCLSUB	Robust Constant Capacitated Lot Sizing Model with Uncertainty Budget
CLS	Capacitated Lot Sizing model
DP	Dynamic Programming
EOQ	Economic Order Quantity
FK	Florian-Klein
MLLS	Multi-Level Lot Sizing Model
MILP	Mixed Integer Linear Programming
MINLP	Mixed Integer Nonlinear Programming
MRULS	Minimax Regret Robust Uncapacitated Lot Sizing Model
ROCC	Robust Optimization Model with Cardinality Constrained Uncertainty Set
SCM	Supply Chain Management
ULS	Uncapacitated Lot Sizing Model
ULSUB	Robust Uncapacitated Lot Sizing Model with Uncertainty Budget
ULSUR	Robust Uncapacitated Lot Sizing Model with Uncertainty Range
WW	Wagner-Whitin
ZIO	Zero Inventory Ordering

Contents

1 Introduction **1**

1.1 Motivation 1

1.2 The absolute robust optimization approach 3

1.3 Efforts to reduce over-conservatism 5

1.3.1 Uncertainty budget 5

1.3.2 The minimax regret criterion 11

1.3.3 The *bw*-robustness criterion 12

1.4 Structure of the thesis 12

2 Lot Sizing Models **15**

2.1 Introduction 15

2.2 The formulation 16

2.3 The Wagner-Whitin algorithm for the ULS model 22

2.4 The Florian-Klein algorithm for the CCLS model 24

2.5 Lot sizing models with uncertainty 26

2.6 Summary 30

3 The Minimax Regret ULS Model **31**

3.1 Introduction 31

3.2 The formulation 34

3.3 The solution approach 37

3.3.1 The dynamic programming algorithm 37

3.3.2 Subplan properties 38

3.3.3 Minimax regret of a subplan 41

3.3.4 The running time 44

3.4 Single type uncertainty 44

3.4.1 Demand uncertainty 44

3.4.2 Production cost function uncertainty . . . 47
3.5 Summary . . . 47

4 The Lot Sizing Model with Uncertainty Budget 49
4.1 Introduction . . . 49
4.2 The formulation . . . 52
4.3 The algorithm . . . 54
4.4 Minmax cost of a subplan . . . 56
4.4.1 The ULSUB model . . . 56
4.4.2 The CCLSUB model . . . 57
4.4.3 The running time . . . 60
4.5 Single type uncertainty . . . 60
4.5.1 Production cost function uncertainty . . . 60
4.5.2 Demand uncertainty . . . 62
4.6 Summary . . . 63

5 The Lot Sizing Model with Uncertainty Range 65
5.1 Introduction . . . 65
5.2 The formulation . . . 67
5.3 Properties . . . 73
5.4 The solution approach . . . 76
5.5 Numerical results . . . 82
5.5.1 Experiment setup . . . 82
5.5.2 The base experiment . . . 84
5.5.3 Time-invariant unit production cost . . . 85
5.5.4 Non-increasing unit production cost . . . 85
5.5.5 Non-decreasing unit production cost . . . 86
5.6 Summary . . . 87

6 The *bw*-Robust Uncapacitated Lot Sizing Model 103
6.1 Introduction . . . 103
6.2 The formulation . . . 105
6.2.1 The *bw*-robust ULS with scenario-represented uncertain demand . . . 105
6.2.2 The *bw*-robust ULS with interval uncertain demand . . . 106
6.3 Numerical results . . . 113
6.3.1 Experiment setup . . . 113

6.3.2 The scenario case 116
6.3.3 The interval case 118
6.4 Summary 121

7 Conclusions and Future Research Directions **137**

Bibliography **141**

Chapter 1

Introduction

1.1 Motivation

According to Simchi-Levi et al. [157], Supply Chain Management (SCM) is a set of approaches utilised to effectively integrate suppliers, manufacturers, warehouses, and stores, so that merchandise is produced and distributed at the right quantities, to the right locations, and at the right time, in order to minimize system-wide cost while satisfying service level requirements. In its simplest form, many SCM applications involve making decisions of when and how much of a product to produce such that demand over a planning horizon is satisfied and the trade-off between production costs and inventory holding costs is optimized [143]. This is usually referred as the lot sizing problem. The lot sizing problem is embedded in many practical production planning problems. Many more complex production planning problems, involving multiple products, production locations, production stages crossing multiple levels of supply chain, contain the lot sizing problem as a substructure. Therefore, a good understanding of the lot sizing model has immediate implications for many practical production problems, and therefore for SCM [143]. However, variability in demand and production cost affect these decisions on two major service dimensions: reliability of the production plan, and cost of production. Therefore, there is a need to develop lot sizing models that directly take account of this uncertainty.

The early developments in the lot sizing problem have their roots in the Economic Order Quantity (EOQ) model developed by Harris [96]. The EOQ model proposed by Harris [96] is based on a continuous-review scheme and applies only when demand for the product is continuous and constant over the planning horizon. A few decades later, Wagner and Whitin [175] proposed a model that assumes a discrete and finite planning horizon. Unlike the EOQ

model, the Wagner-Whitin model considers discrete demands and allows the demands to change over time. Since then, the lot sizing problem and a wealth of extensions have been the subject of a wide body of research. For a thorough review of lot sizing models, we refer the reader to Chapter 2 and to e.g. [48, 67, 105, 106, 111]. In this thesis, we consider the lot sizing problem as a discrete Mixed Integer Linear Programming (MILP) model over a planning horizon of n periods.

In the literature on models and algorithms that take into account uncertainty in lot sizing problems, stochastic programming [44, 108, 145, 152] has been the most popular approach. The most studied area has been stochastic lot sizing problems with uncertain demand [6, 49, 64, 87, 88, 90, 86, 92, 97, 100, 149]. Other areas include, but are not limited to, stochastic lot sizing problems uncertain production cost [64, 87, 88, 89, 101, 186], stochastic lot sizing problems with uncertain lead times [50, 125, 134], etc. Generally speaking, stochastic programming models, to address the uncertainty in the lot sizing problem, consider one of the following strategies (or both): they either make strong assumptions regarding the distribution of the uncertain parameters, or the methods approximate the lot sizing problem with a much larger deterministic model in which the uncertainty is represented through scenarios. Moreover, the objective of these models is to obtain a solution that minimizes the expected value. The resulting solution is potentially sensitive to the actual data that occurs in the problem. Considering that the lot sizing solution in a given application will only face a single realization of the uncertainty, a more reasonable approach might be to obtain a "robust" solution, i.e., a "good" solution for all possible data realizations. This is the goal of this thesis.

One advantage of robust optimization is that the approach does not make restrictive probabilistic or distributional assumptions on uncertain input data. Very often, there is not a solid statistical basis to back up the choice of a particular distribution [32]. In fact, in some cases there may be no statistical data available about the phenomenon that is to be modeled. A typical example of this is when the uncertain data refers to the future behaviour of other companies, agencies or governments, and changes of public attitudes and priorities [117]. In other situations, the frequently used probabilistic assumptions are inappropriate for decision environments where a few factors determine the uncertainty of many elements in the data set, thus inducing strong, but hard-to-specify, correlations among the associated probability distribution [117]. Robust optimization considers the uncertain input data as

deterministic and set-based. Using uncertainty sets means that no underlying stochastic model of the data is assumed to be known (or even to exist), although such knowledge may be of use to obtain reasonable uncertainty sets [32].

Lot sizing problems involve integer decisions with respect to production timing, and real number decisions with respect to production and inventory quantities. This environment imposes significant computational barriers to solve robust lot sizing models. In this thesis, we construct efficient models and algorithms to overcome this difficulty. In addition, while the main priority of most classic robust optimization approaches is to deal with feasibility, rewriting the feasible region of the original problem to account for possible deviations of parameters and thus focusing on the feasibility level of the problem instances, the quality of the objective function value is equally important to the decision maker in the process of production planning. In this thesis, we focus more on the quality of the objective function and, besides being robust, we model explicitly how the uncertain parameters affect the total production cost in the lot sizing problem.

In the remainder of this chapter, we will discuss robust optimization in more detail in Sections 1.2 and 1.3. In Section 1.4, we give the layout of the thesis.

1.2 The absolute robust optimization approach

The origins of robust optimization date back to the establishment of modern decision theory in the 1950s and the use of worst case analysis and Wald's maximin model [176] as a tool for the treatment of severe uncertainty. It is a methodology capable of detecting cases when data uncertainty can heavily affect the quality of the nominal solution, and in these cases to generate a solution, that is aimed to immunise against the effect of data uncertainty when accurate statistical information is not available. We refer the reader to [117] for earlier developments in robust optimization. Robust optimization approaches have been widely applied in many areas, for instance, in portfolio optimization [28, 60, 72, 85, 91, 127, 167, 178], in statistical learning and estimation [52, 69, 71, 114, 119, 179, 180], in numerical linear algebra [68], in supply chain management [20, 2, 26, 40, 42, 95, 104, 116, 139, 138, 177, 184], in engineering [29, 41, 47, 99, 126, 128, 137, 185]. For a thorough review of absolute robust

models, we refer the reader to [34, 36, 78, 160].

The classic format of a robust optimization model employs the so-called absolute robust criterion, which is to minimize the maximum total cost among all feasible solutions over all realisable input data. As this approach is based on the anticipation that the worst case might well happen, it is also referred as the worst-case approach. Soyster [159] is among the earliest researchers to initiate the theoretical analysis of robust optimization with this criterion. We introduce the Soyster's model by considering the following linear optimization problem:

$$\begin{aligned} & \min_{\mathbf{x}} \ \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{b}_l \leq \mathbf{x} \leq \mathbf{b}_u, \end{aligned}$$

where $\mathbf{x}$ is a decision variable vector, $\mathbf{c}$ the coefficient vector of the objective function, $\mathbf{A}$ the constraint matrix, and $\mathbf{b}_l$ and $\mathbf{b}_u$ the lower and upper bounds of the decision variables. In the rest of this thesis, we use bold letters to denote vectors of parameters and decision variables.

In the above formulation, we assume without loss of generality that data uncertainty only affects the elements in matrix $\mathbf{A}$. Indeed, if there is uncertainty in $\mathbf{c}$, we can define an auxiliary variable z that will be minimized, and introduce a new constraint $\mathbf{c}^\top \mathbf{x} \leq z$, which can be easily integrated into $\mathbf{A}\mathbf{x} \leq \mathbf{b}$. Similarly, if there is uncertainty in $\mathbf{b}$, we can define an auxiliary variable $\mathbf{y}$ with zero costs and both upper and lower bounds equal to 1. Then by letting $\mathbf{x}' = (\mathbf{x}, \mathbf{y})^\top$ and $\mathbf{A}' = (\mathbf{A}, -\mathbf{b})$, we get $\mathbf{A}'\mathbf{x}' \leq \mathbf{0}$, in which the uncertainty is removed from $\mathbf{b}$.

Consider a particular row i of the matrix $\mathbf{A}$ and let J_i represent the set of coefficients in row i that are subject to uncertainty. Soyster's model assumes an interval representation of the uncertainty. That is, each entry a_{ij}, $j \in J_i$, is assumed to take values in a bounded interval $[\tilde{a}_{ij} - \hat{a}_{ij}, \tilde{a}_{ij} + \hat{a}_{ij}]$, which is centered at nominal values $\tilde{a}_{ij}$. The interval representation of uncertainty is introduced by Moore [133] and Zadeh [181]. In the rest of the thesis, we call a deterministic model with all data input taking the nominal values the nominal model. The robust formulation of the linear optimization proposed in [159] is equivalent to a deterministic problem with all uncertain a_{ij}'s reaching the upper bounds $\tilde{a}_{ij} + \hat{a}_{ij}$, which is formulated as

follows:

$$\begin{aligned} & \max_{\mathbf{x}} \ \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & \sum_j \tilde{a}_{ij} x_j + \sum_{j \in J_i} \hat{a}_{ij} |x_j| \leq b_i, \ \forall i \\ & \mathbf{b}_l \leq \mathbf{x} \leq \mathbf{b}_u. \end{aligned} \tag{1.1}$$

The model can be justified in some engineering applications, such as robust control theory, but is less advisable in operations research, where adverse events such as low customer demand do not produce the high-profile repercussions that engineering failures can have, such as a doomed satellite launch or a destroyed unmanned robot [78]. Hence by considering all uncertain parameters reaching upper bounds, Soyster's model could produce a solution that is too conservative in production planning.

1.3 Efforts to reduce over-conservatism

The issue of over-conservatism hindered the adoption of absolute robust optimization techniques. Researchers have been making efforts to address this issue with various approaches.

1.3.1 Uncertainty budget

One stream of the studies introduce a parameter, the uncertainty budget, that controls the size of total data uncertainty. The by adjusting the size of the uncertainty budget, one obtain a trade-off between robustness and optimality. In the mid-1990s, Ben-Tal et al. [30, 31, 32], El Ghaoui and Lebret [69] and El Ghaoui et al. [70] started investigating models where feasibility is guaranteed with high probability. In their studies, the authors consider the so-called "random symmetric" ellipsoidal uncertainty

$$\begin{aligned} a_{ij} &= \tilde{a}_{ij} + \xi_{ij} \hat{a}_{ij} \\ \sum_{j \in J_i} \xi_{ij}^2 &\leq \Lambda^2, \end{aligned}$$

where ξ_{ij} are random variables distributed symmetrically in the interval $[-1, 1]$ if $j \in J_i$ and $\xi_{ij} = 0$ otherwise, and $\Lambda > 0$ is a positive parameter. Instead of finding solutions that are feasible for all possible realisations of the uncertain data, the model finds an "almost robust" $\mathbf{x}$ such that $\mathbf{x}$ is feasible for the nominal problem, i.e., $a_{ij} = \tilde{a}_{ij}$, and the probability of violating the i-th constraint is bounded by $\exp(-\Lambda^2/2)$, i.e.,

$$\mathbb{P}[\sum_j a_{ij} x_j > b_i] \leq \exp(-\Lambda^2/2).$$

Hence one can control the trade-off between robustness and optimality by adjusting the size of the ellipsoidal uncertainty set, i.e., adjusting the uncertainty budget parameter Λ. The proposed model involves solving the robust counterparts of the nominal problem in the form of conic quadratic problems. With properly chosen ellipsoids, such a formulation can be used as a reasonable approximation to more complicated uncertainty sets. However, a practical drawback of such an approach is that it leads to nonlinear, although convex, models, which are more demanding computationally than the earlier linear models by Soyster [159]. In the original work of Ben-Tal and Nemirovski [30, 31, 32], El Ghaoui and Lebret [69] and El Ghaoui et al. [70], it is shown that under the ellipsoidal uncertainty sets, the robust (mixed-integer) linear optimization problems become second-order (mixed-integer) cone optimization problems, which significantly increases the complexity of the original deterministic problem.

When the uncertainty satisfies that

$$\sum_{j \in J_i} \kappa_{ij} a_{ij} \leq \Gamma_i,$$

where $\kappa_{ij} \neq 0$ and Γ_i are parameters controlling the size of the uncertainty sets, we have the so-called polyhedral uncertainty sets [32]. One advantage of using polyhedral uncertainty sets is that robust (mixed-integer) linear optimization problems remain linear. Atamtürk [14] provided stronger formulations of the robust $0-1$ optimization problem with polyhedral uncertainty in the objective function, which are shown empirically to significantly improve computation time. The approach fails when considered in a general integer optimization problem [156].

Bertsimas and Sim [38, 39] further investigate the polyhedral uncertainty set by considering a family of polyhedral sets that encode a budget of uncertainty in terms of cardinality constraints, as opposed to size of an ellipsoid. In their study, the authors proposed the

cardinality constrained uncertainty set, as follows,

$$a_{ij} = \tilde{a}_{ij} + \xi_{ij}\hat{a}_{ij} \tag{1.2}$$

$$\sum_{j \in J_i} |\xi_{ij}| \leq \Gamma_i, \tag{1.3}$$

where ξ_{ij} is defined as in the ellipsoidal uncertainty sets. Then constraint (1.3) works effectively as a budget of uncertainty, i.e., the total scaled variation of the parameters in constraint i cannot exceed some threshold Γ_i. It can be shown that (1.3) is equivalent to constraint (1.1) where up to $\lfloor \Gamma_i \rfloor$ of the parameters are allowed to change, and one parameter a_{ik} changes by $(\Gamma_i - \lfloor \Gamma_i \rfloor)\hat{a}_{ik}$. The cardinality constrained uncertainty set is a special case of the polyhedral uncertainty set, hence robust linear MILPs remain linear.

Bertsimas and Sim [39, 40] propose a robust optimization model with the cardinality constrained uncertain sets. In their original work [39, 40], Bertsimas and Sim studied a robust optimization model with uncertainty sets formulated by (1.2) and (1.3), in which Γ_i, not necessarily integer, takes values in the interval $[0, |J_i|]$. The goal is to be protected against all cases in which up to $\lfloor \Gamma_i \rfloor$ (the integer part of Γ_i) of these coefficients are allowed to change, and one coefficient a_{it} changes by $(\Gamma_i - \lfloor \Gamma_i \rfloor)\hat{a}_{it}$. In other words, only a subset of the coefficients will change in order to adversely affect the solution. The original form of their robust optimization model with cardinality constrained uncertainty set (ROCC) is nonlinear, and it reads as follows:

$$\begin{aligned}
ROCC: \quad & \min_{\mathbf{x}} \ \mathbf{c}^\top \mathbf{x} \\
\text{s.t.} \quad & \tilde{a}_{ij}x_j + \max_{S_i \cup \{t_i\} \,|\, S_i \subset J_i, |S_i| = \lfloor \Gamma_i \rfloor, t_i \in J_i \setminus S_i} \{\hat{a}_{ij}x'_j + (\Gamma_i - \lfloor \Gamma_i \rfloor)\hat{a}_{it_i}x'_{t_i}\} \leq b_i, \ \forall i \\
& -x'_j \leq x_j \leq x'_j, \ \forall j \\
& \mathbf{b}_l \leq \mathbf{x} \leq \mathbf{b}_u, \\
& \mathbf{x}' \geq \mathbf{0}.
\end{aligned}$$

If Γ_i is chosen as an integer, the i-th constraint is protected by the inner maximisation problem $\max_{S_i \,|\, S_i \subset J_i, |S_i| = \Gamma_i}\{\hat{a}_{ij}x'_j\}$. When $\Gamma_i = 0$, the inner maximisation problem is equal to zero, the constraints are equivalent to that of the nominal problem. Likewise, if $\Gamma_i = |J_i|$, the model turns to be Soyster's model. Therefore, by varying $\Gamma_i \in [0, |J_i|]$, one has the flexibility of adjusting the robustness of the method against the level of conservatism of the solution.

In order to formulate the above as a linear optimization model, Bertsimas and Sim show that, for a given vector $\mathbf{x}$, the inner maximisation problem in the i-th constraint is equivalent to the following,

$$\max_{S_i \cup \{t_i\} \,|\, S_i \subset J_i, |S_i| = \lfloor \Gamma_i \rfloor, t_i \in J_i \setminus S_i} \{\hat{a}_{ij}|x_j| + (\Gamma_i - \lfloor \Gamma_i \rfloor)\hat{a}_{it_i}|x_j|\},$$

i.e., $x'_j = |x_j|$. Then they further show that it can be reformulated as the following linear optimization problem:

$$\begin{aligned} & \max_{\mathbf{x}} \sum_{j \in J_i} \hat{a}_{ij}|x_j|z_{ij} \\ \text{s.t.} \quad & \sum_{j \in J_i} z_{ij} \leq \Gamma_i \\ & 0 \leq z_{ij} \leq 1, \ \forall j \in J_i. \end{aligned}$$

Then by considering the dual of the above problem, the ROCC model has an equivalent linear formulation as follows:

$$\begin{aligned} DROCC: \quad & \max_{\mathbf{x}} \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & \sum_j a_{ij}x_j + z_i\Gamma_i + \sum_{j \in J_i} q_{ij} \leq b_i, \ \forall i \\ & z_i + q_{ij} \geq \hat{a}_{ij}x'_j, \ \forall i, j \in J_i \\ & -y_j \leq x_j \leq y_j, \ \forall j \\ & b_{l_j} \leq x_j \leq b_{u_j}, \ \forall j \\ & q_{ij} \geq 0, \ \forall i, j \in J_i \\ & y_j \geq 0, \ \forall j \\ & z_i \geq 0, \ \forall i. \end{aligned}$$

Bertsimas and Sim [38] provide a probabilistic guarantee that even if more than Γ_i uncertain parameters change in constraint i, the robust solution will remain feasible with high probability, i.e., for an optimal solution $\mathbf{x}^\star$ of the robust model with uncertainty budget

equal to Γ_i,

$$\mathbb{P}[\sum_j a_{ij} x_j^\star > b_i] \leq \mathbf{B}(|J_i|, \Gamma_i),$$

where

$$\mathbf{B}(|J_i|, \Gamma_i) = \frac{1}{2^{|J_i|}} \left\{ (1-\mu)\binom{|J_i|}{\lfloor v \rfloor} + \sum_{l=\lfloor v \rfloor + 1}^{|J_i|} \binom{|J_i|}{l} \right\},$$

in which $v = (\Gamma_i + |J_i|)/2$ and $\mu = v - \lfloor v \rfloor$.

With uncertainty budget, Bertsimas and Sim [39] give an explicit formulation which treats uncertainty in the objective function directly. The authors show that when only the cost coefficients are subject to uncertainty and the problem is a $0-1$ optimization problem with n variables, the robust model with uncertainty budget can be solved by considering at most $n+1$ instances of the original deterministic problem. Thus, a polynomially solvable $0-1$ optimization problem remains polynomially solvable. In the study, the authors consider the following class of $0-1$ optimization problems:

$$\begin{aligned} & \min_{\mathbf{x}} \ \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & x_i \in \{0,1\}, \ \forall i, \end{aligned} \tag{1.4}$$

and it is assumed that uncertainty only affects the coefficients in the objective function. Each entry c_j, $j \in \{1, \ldots, n\}$, takes values in $[\tilde{c}_j, \tilde{c}_j + \hat{c}_j]$, with $\hat{c}_j \geq 0$ and $\hat{c}_1 \geq \hat{c}_2 \geq \ldots \geq \hat{c}_n$. Suppose that at most Γ^o of the coefficients c_j are allowed to change, the robust $0-1$ optimization problem with constrained cardinality is of the form:

$$\begin{aligned} & \min_{\mathbf{x}} \{\mathbf{c}^\top \mathbf{x} + \max_{S^o \,|\, S^o \subset \{1,\ldots,n\}, |S^o| \leq \Gamma^o} \sum_{j \in S^o} \hat{c}_j x_j\} \\ \text{s.t.} \quad & x_i \in \{0,1\}, \ \forall i. \end{aligned} \tag{1.5}$$

By going through the similar procedure above, i.e., considering the inner maximisation problem as a linear problem and constructing the dual, (1.5) can be formulated as the following

MILP problem

$$\begin{aligned} & \min_{\mathbf{x},\theta,\mathbf{y}} \{\mathbf{c}^\top \mathbf{x} + \Gamma^o \theta + \sum_{j=1}^{n} y_j\} \\ \text{s.t.} \quad & y_j + \theta \geq \hat{c}_j x_j, \ \forall j \\ & y_j, \theta \geq 0, \ \forall j \\ & x_i \in \{0,1\}, \ \forall i. \end{aligned} \tag{1.6}$$

Then, one can eliminate $\mathbf{y}$ by using constraints (1.6), and get the following equivalent model

$$\min_{\mathbf{x}\subset\{0,1\}^n, \theta\geq 0} \{\mathbf{c}^\top \mathbf{x} + \Gamma^o \theta + \sum_{j=1}^{n} \max(\hat{c}_j x_j - \theta, 0)\}. \tag{1.7}$$

For $\mathbf{x} \in \{0,1\}^n$, we have

$$\max(\hat{c}_j x_j - \theta, 0) = \max(\hat{c}_j - \theta, 0) x_j. \tag{1.8}$$

This implies that (1.7) can be solved by purely considering the ordering of $\hat{c}_j$. Bertsimas and Sim then decompose $\mathbb{R}^+$ into intervals $[0, \hat{c}_n], [\hat{c}_n, \hat{c}_{n-1}], \ldots, [\hat{c}_2, \hat{c}_1], [\hat{c}_1, \infty]$ and prove that

$$\min_{l=1,\ldots,n+1} Z^l,$$

where

$$\begin{aligned} Z^l &= \min_{\mathbf{x}\in\{0,1\}^n, \theta\in[\hat{c}_l, \hat{c}_{l-1}]} \{\Gamma^o \theta + \mathbf{c}^\top \mathbf{x} + \sum_{j=1}^{l-1} (\hat{c}_j - \theta) x_j\} \\ &= \Gamma^o \hat{c}_l + \min_{\mathbf{x}\in\{0,1\}^n} \mathbf{c}^\top \mathbf{x} + \sum_{j=1}^{l-1} (\hat{c}_j - \hat{c}_{l-1}) x_j. \end{aligned}$$

Therefore, (1.7) can be solved by solving at most $n+1$ instances of original deterministic problem (1.4). Clearly, this approach is not valid when $\mathbb{X} \subset \mathbb{Z}^n$ since (1.8) does not hold in that case.

Bertsimas et al. [37] provide insights of constructing uncertainty sets by using general norms. They show that robust linear optimization problems with uncertainty sets described by more general norms lead to convex problems with constraints related to the dual norm.

In short, for many choices of the uncertainty sets, robust linear optimization problems are tractable.

1.3.2 The minimax regret criterion

An alternative direction to handle the issue of over-conservatism in robust optimization problems is to use different robust criteria. The relative robust criteria [117] aims to minimize the worst relative surplus cost per unit of cost if there was no uncertainty. Note that the relative cost surplus measures the excessive cost as a percentage of the cost that could occur if we knew the actual data. Or one may prefer the robust deviation objective [117]. That is to minimize the worst surplus cost over all realisations of the input data. To be more precise, the criterion needs a solution that exhibits the smallest worst case deviation from optimality over all potential realisations. So the robust deviation criterion is also referred as the minimax regret criterion. Both criteria allow the benchmarking of the performance of the decisions against the best possible outcome under any data realisation. In most circumstances, the two criteria can be used interchangeably and provide solutions that are less conservative than those by Soyster's absolute robust criteria [117].

The relative robust and the robust deviation criteria are suitable in situations where the decision maker is evaluated ex-post [117]. In these situations, a decision maker may be concerned not only with how a production plan's performance varies with the actual realisations of the task parameters, but also with how actual performance compares with the optimal performance that could have been achieved if perfect information had been available. It has been shown that, in the presence of uncertainty on prices and yields, the behaviour of some economical agents (farmers) could sometimes be better predicted using a relative robust or a robust deviation criterion, rather than a classical profit maximisation criterion [113]. Another justification of the two criteria is as follows. In uncertain decision contexts, a reasonable objective is to find a solution with performances as close as possible to the optimal values under all scenarios. This amounts to setting a threshold and looking for a solution such that the difference between the performance of the threshold and the solution is as small as possible, which is equivalent to determining a relative robust or robust deviation solution [117].

1.3.3 The *bw*-robustness criterion

Recent development in robust optimization leads to the introduction of new robustness criteria that focus more directly on the objective function. Roy [151] criticizes the worst-case approach for representing the robustness and proposes the *bw*-robustness criterion where the uncertainty is modeled by means of a set of scenarios. According to Roy, a solution is robust if it provides a good evaluation, in terms of feasibility and objective function value, in most scenarios without being too bad in any scenario. With uncertain objective cost coefficient parameter represented by scenarios, Roy [151] proposes a new robustness criterion, the *bw*-robustness, which minimizes the importance according to the worst case scenario and incorporates a trade-off between risk taking and guaranteed value. The *bw*-robustness criterion involves two parameters, denoted b and w. These parameters, set by the decision maker, are such that:

- b is a value that the decision maker wants to achieve under the largest number (or proportion) of scenarios,
- w is a value that the decision maker does not want to exceed under any scenario, this value is guaranteed.

This criterion allows the decision-maker to identify a solution which guarantees an objective function value, in a minimization problem, of at most w in all scenarios, and maximizes the probability of giving an objective function value that is less than b ($b > w$). Gabrel et al. [79] extend this criterion from a finite set of scenarios to the case of uncertainty intervals.

For more information of recent developments in robust optimization, we refer the reader to [34, 36, 78].

1.4 Structure of the thesis

In the remainder of the thesis, we study robust lot sizing models with four robust methodologies, namely, the minimax regret criterion, the uncertainty budget, the uncertainty range,

and the *bw*-robustness criterion. We consider two types of uncertain data. For the minimax regret lot sizing model and the robust lot sizing model with uncertainty budget, we assume that both the production cost function, which consists of the setup cost the unit production cost, and the demand are subject to uncertainty. For the robust lot sizing model with uncertainty range and the *bw*-robust lot sizing model, we assume the demand is subject to uncertainty, though the former can be readily generalized to take account of uncertain production cost function together with uncertain demand, and the later as well under some distributional assumptions. The proposed studies can be generalized when only the unit inventory holding cost is uncertain. In the remainder of this section we summarize the contents of Chapters 2-7.

The literature review in Chapter 2 provides a broader discussion of the lot sizing problem and its extensions. In particular, we formulate the lot sizing model, capacitated and uncapacitated, and present the corresponding solution approaches. In addition, we give a review on studies of uncertain lot sizing models.

In Chapter 3, we study the minimax regret version of the uncapacitated lot sizing model. We propose a polynomial time algorithm which solves the model in $\mathcal{O}(n^6)$ time. We improve this running time to $\mathcal{O}(n^5)$ when only the demand is uncertain, and to $\mathcal{O}(n^4)$ when only the production cost function is uncertain.

Using the idea of uncertainty budget as presented in Section 1.3.1, in Chapter 4 we study the robust lot sizing models in which the worst-case is defined as when uncertain data harms the quality of the objective function value the most. Polynomial time algorithms are proposed to solve the models. In Chapter 5, we extend the study in Chapter 4 by introducing two additional parameters, namely, the minimum number of periods in which one would like the demand to be protected against uncertainty, and the minimum scaled protection level per period. Again polynomial time algorithms are proposed to solve the models. Under various parameter settings, the numerical results show that the models proposed in Chapters 4 and 5 provide a good trade-off between the robustness of the solution, i.e., to what extent the solution is feasible, and the quality of the objective function value.

In Chapter 6, we apply the bw-robustness criterion to study the robust uncapacitated lot sizing problem with uncertain demand. The uncertain demand is modeled by means of scenarios as well as uncertainty intervals. We formulate the scenario model as an MILP problem. For the interval case, we propose a Mixed Integer Nonlinear Programming (MINLP) formulation, which we subsequently linearize. Numerical results are provided with various settings of parameters to show that, apart from controlling the quality of the objective function value through b and w, the solutions also achieve satisfactory feasibility levels when compared to the benchmark models.

Chapter 7 concludes the thesis with future research directions.

Chapter 2

Lot Sizing Models

In this chapter, we review the literature relating to deterministic lot sizing models, some theoretical results, and lot sizing models under uncertainty, revealing the need for more research around robust lot sizing models.

2.1 Introduction

In the lot sizing setting, we need to decide when and how much of a product to produce such that demand over a planning horizon is satisfied and the trade-off between production costs and inventory holding costs is optimized [143]. The early developments in this field have their roots in the Economic Order Quantity (EOQ) model developed by Harris [96]. The EOQ model proposed by Harris [96] is based on a continuous-review scheme and applies only when demand for the product is continuous and constant over the planning horizon. A few decades later, Wagner and Whitin [175] proposed a model that assumes a discrete and finite planning horizon which is subdivided into n periods. Unlike the EOQ model, the Wagner-Whitin model considers discrete demands and allows the demands to change over time.

In the original work of Wagner and Whitin [175], the authors assume that there are no constraints on production quantities, i.e., the model is uncapacitated. To solve the Uncapacitated Lot Sizing (ULS) model, Wagner and Whitin [175] proposed a dynamic programming

(DP) algorithm which runs in $\mathcal{O}(n^2)$ time. (Recall that the running time order of an algorithm is the domination term of an upper bound on the running time when the problem size goes to infinity [136].) Unlike the ULS, the Capacitated Lot Sizing (CLS) model explicitly states upper bounds on the production quantities. The CLS is $\mathcal{NP}$-Hard [46, 76], where $\mathcal{NP}$-Hard stands for non-deterministic polynomial-time hard [136]. However, the model is polynomially solvable when the capacities are constant throughout the planning horizon [75]. The algorithm proposed in Florian and Klein [75] solves the Constant Capacitated Lot Sizing (CCLS) model in $\mathcal{O}(n^4)$ time. In the following chapters, we will solve robust versions of the lot sizing model with either a Wagner-Whitin (WW) or a Florian-Klein (FK) type algorithm. These algorithms will be reviewed in this chapter, as well as extensions of the classical lot sizing model including the ones dealing with uncertainty.

The remainder of this chapter is structured as follows. In Section 2.2, we give the formulation of fundamental lot sizing models which involve a single item and a single production level. We then discuss extensions of these models. In Section 2.3, we introduce the DP algorithm proposed by Wagner and Whitin [175] for the ULS model. In Section 2.4, we discuss the DP algorithm proposed in Florian and Klein [75] for the CCLS model. In Section 2.5, studies of lot sizing models in an uncertain environment are reviewed, revealing the need for more research around robust lot sizing models. The summary of the chapter is given in Section 2.6.

2.2 The formulation

Recall that the planning horizon is split into n periods. The demand in period i is given by d_i, which must be satisfied by production during that period or by inventory carried from earlier periods. We introduce the notation $d_{ij} := \sum_{k=i}^{j} d_k$, $i \leq j$, and define period $n+1$ with $d_{n+1} = 0$ for future reference. If $i > j$, we let $d_{ij} = 0$. As it is common in the literature, we assume the production cost function to have a fixed-charge structure and the inventory holding function to be linear. That is to say, for each period i, there are unit production costs p_i, unit inventory costs h_i for the stock remaining at the end of the period i and a setup cost f_i if production takes place. We introduce the notation $h_{ij} := \sum_{k=i}^{j} h_k$ for $i \leq j$, and $h_{ij} = \infty$ if $i > j$. There is also an upper bound C_i on the production quantity in period i. For each period in the planning horizon, we must decide whether to produce, if so, how much to produce, and how much to keep in stock. The goal of the lot sizing model is to find

a production plan that minimizes the total production and inventory holding cost, and at the same time satisfies the demands. Assuming that the inventory at the beginning of the planning is equal to zero, the single-item single-level lot sizing model can be formulated as an MILP problem as follows:

$$LS: \quad \min_{\mathbf{y},\mathbf{x},\mathbf{I}} \sum_{i=1}^{n} (f_i y_i + p_i x_i + h_i I_i) \tag{2.1}$$

$$\text{s.t.} \quad I_{i-1} + x_i = I_i + d_i, \; \forall i = 1, \ldots, n \tag{2.2}$$

$$x_i \leq C_i y_i, \; \forall i = 1, \ldots, n \tag{2.3}$$

$$y_i \in \{0, 1\}, \; \forall i = 1, \ldots, n \tag{2.4}$$

$$x_i, I_i \geq 0, \; \forall i = 1, \ldots, n, \tag{2.5}$$

where x_i is the amount produced in period i, I_i the inventory level at the end of period i, and y_i is equal to 1 if $x_i > 0$ and 0 otherwise. Here constraints (2.2), the flow conservation constraints, ensure that the inventory of period $i-1$ plus the production at period i equal to the demand plus the inventory of period i. The capacity constraints are modeled by (2.3), which together with constraints (2.4), enforce that the setup variables y_i are well-defined. Constraints (2.5) impose the non-negativity of the decision variables x_i and I_i.

It is first noticeable that using the flow conservation constraints (2.2), one can eliminate the inventory holding variable I_i by equating I_i with $\sum_{k=1}^{n} x_k - d_{1i}$. The objective function (2.1) of the lot sizing model can be alternatively written as

$$\sum_{i=1}^{n} f_i \, y_i + \sum_{i=1}^{n} p'_i \, x_i - \sum_{i=1}^{n} h_i \, d_{1i},$$

where $p'_i = p_i + h_{in}$. After dropping out the constant term $\sum_{i=1}^{n} h_i \, d_{1i}$, the lot sizing model

can be written in an equivalent form

$$\min_{\mathbf{y},\mathbf{x}} \sum_{i=1}^{n} [f_i\, y_i + p_i'\, x_i] \tag{2.6}$$

$$\text{s.t.} \quad \sum_{k=1}^{i} x_k \geq d_{1i},\ \forall i = 1, \ldots, n \tag{2.7}$$

$$x_i \leq C_i y_i,\ \forall i = 1, \ldots, n \tag{2.8}$$

$$y_i \in \{0, 1\},\ \forall i = 1, \ldots, n \tag{2.9}$$

$$x_i \geq 0,\ \forall i = 1, \ldots, n. \tag{2.10}$$

Throughout this thesis, and where convenient, we will use this reformulation.

For the ULS model, we assume that C_i is high enough to never be binding in any optimal solution. That is, we assume $C_i = d_{in}$. Wagner and Whitin [175] proposed an $\mathcal{O}(n^2)$ time DP algorithm for the ULS model with WW costs, i.e., where p_i is constant over time. The algorithm is still valid for general p_i. In the early 90's, Federgruen and Tzur [73], Wagelmans et al. [174] and Aggarwal and Park [4] proposed algorithms with a reduction in computational complexity in comparison with the WW algorithm. The algorithms run in $\mathcal{O}(n\, logn)$ time for the fixed-charge production cost structure, as in (2.1)-(2.5). In addition, $\mathcal{O}(n)$ time algorithms are proposed in Federgruen and Tzur [73] when the setup costs are nondecreasing, and in Wagelmans et al. [174] in the absence of speculative motives to hold inventory, i.e., $p_i + h_i \geq p_{i+1}$, which means that, if given the choice, we will produce as late as possible. Please note that the WW costs are a special case of nonspeculative production costs since for constant p_i the condition $p_i + h_i \geq p_{i+1}$ boils down to $h_i \geq 0$.

When the capacities are binding, we have the CLS model given by (2.1)-(2.5). The computational complexity of the CLS model has attracted the attention of researchers. The CLS model with time-variant capacities has been shown to be $\mathcal{NP}$-hard by Florian et al. [76] and Bitran and Yanasse [46]. Based on these results, it is unlikely that one can develop any effective algorithm for this problem. Therefore, numerous heuristics approaches have been proposed. Examples include, but are not limited to, relaxation heuristics [43, 163, 166] which rely upon a relaxation of capacity constraints, branch and bound based heuristics [57, 65, 81, 98] and set partitioning and column generation heuristics [53, 55]. For a thorough review of heuristic approaches for the CLS, we refer the reader to e.g. [51, 111, 143]. There are also approximation methods for the CLS model. Examples include, but are not limited

to, pseudo-polynomial time approximation schemes [45], and fully polynomial approximation schemes [56, 80, 172].

A special and important case of the CLS model that does allow for a polynomial time algorithm is when the capacities are constant throughout the planning horizon, i.e. $C_i = C$. Florian and Klein [75] have shown that the CCLS model can be solved in $\mathcal{O}(n^4)$ time if the production and inventory holding costs are concave. Van Hoesel and Wagelmans [171] studied the CCLS model with fixed-charge production costs and linear inventory holding costs. The authors improved upon the FK algorithm by proposing an $\mathcal{O}(n^3)$ time algorithm. Studies on other polynomial solvable CLS models include Chuang and Lin [58] and van den Heuvel and Wagelmans [169] in which the setup and the unit production costs are nonincreasing and the capacities are nondecreasing, and Baker et al. [25] in which unit production costs and inventory holding costs are assumed to be time-invariant.

The fundamental lot sizing models formulated in (2.6)-(2.10) have been extended to model more accurately real life situations.

Backlogging was first considered by Zangwill [182]. Instead of assuming that the demand for a particular period is satisfied by production during that period or during earlier periods, Zangwill [182] extended the ULS model by allowing demand to go unsatisfied during some periods, provided it is satisfied eventually by production in some subsequent period. A stockout cost h_i^- is incurred for each unit backordered per time unit in period i. There are polynomial time algorithms for the ULS model with backlogging. Zangwill [182] assumed that the production cost, inventory holding cost and backlogging cost functions are all arbitrary concave functions and provided an $\mathcal{O}(n^3)$ time DP for computing an optimal solution. The running time of the algorithm is reduced to $\mathcal{O}(n^2)$ later by Zangwill [183]. Federgruen and Tzur [74] developed an $\mathcal{O}(n\,logn)$ algorithm for the same model discussed in Zangwill [182]. For special cases such as when $p_i + \sum_{k=1}^{i-1} h_k^-$ is nondecreasing or nonincreasing, the authors show that the algorithm can be implemented in $\mathcal{O}(n)$ time. Please note that if $p_i + \sum_{k=1}^{i-1} h_k^-$ is nondecreasing, there are no speculative motives to backlog production. Pochet and Wolsey [141] study the polyhedral structure of the ULS model with backlogging and derived cutting plane algorithms. Later, Pochet and Wolsey [142] improve the algorithm proposed in [141] and show that the problem can be solved as a linear program under the assumption of nonspeculative production costs.

The inventory can be bounded by upper and lower limits. Love [129] gave an $\mathcal{O}(n^3)$ algorithm that searched the extreme points of the solution space for the bounded inventory lot sizing model with piecewise concave production and inventory holding cost functions and backlogging. Gutiérrez et al. [94] proposed a new characterisation of an optimal production plan to reduce the computational effort of Love's procedure but with a constant inventory bound and no backlogging. The ULS model with nonspeculative production costs and inventory capacities is proved to be solvable in $\mathcal{O}(n^2)$ time by Toczylowski [165]. Atamtürk and Küçükyavuz [15] studied the polyhedral structure of the inventory bounded ULS model with fixed-charge production and inventory cost functions, and later in [16], the authors proposed an $\mathcal{O}(n^2)$ time algorithm to solve the model.

Assume now that the selling price is included in the model and that demand is assumed to be a function of the price. The goal of this joint production and pricing model is to find a production plan and selling price(s) that maximise the total profit. The joint production and pricing model with concave revenue functions was proposed by Thomas [164], who showed that the optimal solution consists of regeneration intervals as defined earlier. He further showed that for a given regeneration interval, prices can be determined by solving a nonlinear optimization problem in $\mathcal{O}(n\,logn)$ time. Kunreuther and Schrage [118] and Gilbert [83] studied the ULS model with a constant sale price. Kunreuther and Schrage [118] proposed a heuristic algorithm which provides upper and lower bounds on the optimal profit. Van den Heuvel and Wagelmans [170] derived an exact algorithm which runs in $\mathcal{O}(n^3\,logn)$ time for the model in [118]. Deng and Yano [63] and Geunes et al. [82] considered the problem with piecewise linear concave revenue functions and capacity constraints.

Items returned from the customers to the manufacturer can be remanufactured. Remanufacturing includes testing, repair, disassembly and reassembly operations. Richter and Sombrutzki [147] are among the first to extend the lot sizing model with remanufacturing. In their study, demand can be met either from newly manufactured products or from returned products which have been remanufactured. The authors provided an extension to the WW algorithm [175] for the particular case that the number of returns in the first period is sufficient to satisfy the total demand over the planning horizon. In Richter and Weber [148], the previous work is extended to include variable remanufacturing costs. The same problem with less restrictive assumptions in the return flows is analysed in Golany et al. [84]. The authors introduced a network flow formulation for the problem and prove that it is $\mathcal{NP}$-hard for the case of general concave cost functions. They also provide an exact

algorithm which runs in $\mathcal{O}(n^3)$ time for the case of linear costs. Van den Heuvel [168] proves that the lot sizing problem with remanufacturing is $\mathcal{NP}$-hard even when the production cost function has a fixed-charge structure and the inventory holding costs are linear. This last result is valid even when the setup and unit production costs are time-invariant. Teunter et al. [162] proposed several heuristics for the ULS problem with the remanufacturing option. Two versions of the problem are analysed with joint and separate setup costs for production and remanufacturing, respectively. For the case of joint setup costs, an $\mathcal{O}(n^4)$ time algorithm is provided. The authors conjecture that the problem with separate setup costs is $\mathcal{NP}$-hard.

An extension in another dimension leads to the Multi-Level Lot Sizing (MLLS) model. In single-level lot sizing problems, only independent demand, i.e., demand from outside the system is considered. The MLLS model is characterised by the fact that production of each item may either require a sequence of intermediate production or distribution processes, or generates dependent demand for its constituent components. The MLLS model is classified based on the production structure being used: serial, assembly and general. In terms of serial structure, Zangwill [183] studied the uncapacitated MLLS model with concave costs, and developed a DP algorithm that is polynomial in both the planning horizon and the number of levels. Melo and Wolsey proposed an $\mathcal{O}(n^2\, logn)$ time algorithm for the two-level ULS model with fixed-charge production costs. Kaminsky and Simchi-Levi [109] proposed a three-level model in which the first and third levels are capacitated production stages, and the second level is an uncapacitated transportation stage. For linear inventory holding costs, nonspeculative production costs and concave transportation costs the authors provided an $\mathcal{O}(n^8)$ time algorithm to solve the model in the presence of stationary production capacities. Van Hoesel et al. [173] developed an improved algorithm for the case of stationary production capacities that allows for general concave production, transportation and inventory holding cost functions and runs in $\mathcal{O}(n^7)$ time. In addition, they show that the running time can be further improved when inventory holding cost functions are linear and transportation cost functions have either a fixed charge structure and nonspeculative (yielding a running time of $\mathcal{O}(n^6)$) or are linear (for a running time of $\mathcal{O}(n^5)$). MLLS models with assembly and general production structures are in general more complex to solve, see e.g. [11, 62, 130]. Consequently, numerous studies concentrate on developing effective heuristics. Examples include, but are not limited to, genetic algorithms [62, 93], Lagrangean relaxation heuristics [43, 135, 161], meta-heuristics [153], linear programming based rounding heuristics [112], branch and bound based heuristics [3] and set partitioning and column generation heuristics [53].

There are studies of lot sizing models that incorporate sustainability issues. Benjaafar et al. [35] are one of the first to include carbon emissions in single- and multi-level lot sizing models. Four regulatory policy settings are considered based respectively on a carbon cap, a tax on the amount of emissions, the cap-and-trade system and the possibility to invest on carbon offsets to mitigate carbon caps. More recently, Retel Helmrich et al. [146] and Absi et al. [1] have provided theoretical and algorithmic results on lot sizing models with emission constraints. Finally, Romeijn et al. [150] consider a bi-objective lot sizing model in which the total lot sizing costs including production and inventory holding costs and the maximum production and inventory emissions across blocks of given length are minimized.

There are other extensions available, for a thorough review of lot sizing models, we refer the reader to e.g. [48, 67, 105, 106, 111].

The seemingly basic LS model usually forms as a subproblem in more complex models. In addition, the solution approaches to those complex models integrate the technique of it to the LS model. Hence in the rest of this thesis, we focus our study on the LS model in the form of (2.6)-(2.10).

2.3 The Wagner-Whitin algorithm for the ULS model

Recall that the ULS model can be written as follows:

$$ULS: \quad \min_{\mathbf{y},\mathbf{x},\mathbf{I}} \sum_{i=1}^{n} (f_i y_i + p_i x_i + h_i I_i) \tag{2.11}$$

$$\text{s.t.} \quad I_{i-1} + x_i = I_i + d_i, \ \forall i = 1, \ldots, n \tag{2.12}$$

$$x_i \leq d_{in} y_i, \ \forall i = 1, \ldots, n \tag{2.13}$$

$$y_i \in \{0, 1\}, \ \forall i = 1, \ldots, n \tag{2.14}$$

$$x_i, I_i \geq 0, \ \forall i = 1, \ldots, n. \tag{2.15}$$

For the ULS model, Wagner and Whitin [175] showed that there exists an optimal solution that satisfies the Zero Inventory Ordering (ZIO) property. That is to say, for any period i within the planning horizon, either the inventory carried to period i will be zero or the

production quantity in that period will be zero, i.e.,

$$I_{i-1} \cdot x_i = 0. \tag{2.16}$$

The ZIO property implies that the planning horizon of the ULS problem can be split into a sequence of subplans, also known as regeneration intervals. No inventory enters or leaves the intervals and the inventory levels within the interval are positive.

We decompose the original DP proposed by Wagner and Whitin [175] into two parts, the outer and the inner DPs. The outer DP moves from node k to $t+1$, defining the subplan $[k,t]$, for $1 \leq k \leq t \leq n$. Note that the outer DP is in fact a shortest path problem in an acyclic network with nodes representing production periods and arcs representing the subplans with corresponding costs. The inner DP evaluates the optimal cost of the subplan.

Let $G(t+1)$ be the minimum cost of solving the ULS model over the first t periods. Then, the initial condition and the DP recursion are as follows:

$$G(1) = 0 \tag{2.17}$$

$$G(t+1) = \min_{k \leq t}\{G(k) + \phi(k,t)\}, \tag{2.18}$$

where $\phi(k,t)$ is the optimal cost of the subplan $[k,t]$. The optimal solution for the ULS is equal to $G(n+1)$. The minimum cost path from node 1 to node $n+1$ in this acyclic network can be found in linear time in the number of arcs [7]. Since we have an arc between k and $t+1$ where $1 \leq k \leq t \leq n$, it is trivial to see that we have $\mathcal{O}(n^2)$ arcs in total. Below we show that the inner DP calculates all $\phi(k,t)$ in $\mathcal{O}(n^2)$ time. Hence, the WW algorithm solves the ULS model in $\mathcal{O}(n^2)$ time.

For the ULS model, it is trivial to see that $\phi(k,t) = f_k + p_k d_{kt} + \sum_{r=k}^{t-1} h_r d_{r+1,t}$, and the cost of all subplans can be calculated in $\mathcal{O}(n^2)$ time. Indeed, one can calculate $\phi(k,t)$ by adding the constant term $p_k d_t + \sum_{r=k}^{t-1} h_r d_t$ to $\phi(k,t-1)$ for all $k \leq t \leq n$. By calculating $\sum_{r=1}^{n-1} h_r d_n$ first and the subtracting, $\sum_{r=k}^{t-1} h_r d_t$ can be calculated in constant time for $1 \leq k \leq tleqn$.

In future chapters, we will calculate in a preprocessing step the optimal objective function value of all ULS models from period i to j, for $1 < i \leq j \leq n$. We show here that this task can be performed in $\mathcal{O}(n^3)$ time. Given the structure of the DP recursion discussed above,

$G(j+1)$ is the optimal objective function value of the ULS model defined from period 1 to j. Hence for $i = 1$, we obtain the optimal objective function value of the ULS models from period 1 to j in $\mathcal{O}(n^2)$ time. The desired running time follows by changing the beginning of the planning horizon to i for $i = 1, \ldots, n$.

2.4 The Florian-Klein algorithm for the CCLS model

When the capacities of the CLS are all assumed to be constant, we have the CCLS, which reads as follows:

$$CCLS: \quad \min_{\mathbf{y},\mathbf{x},\mathbf{I}} \sum_{i=1}^{n} (f_i y_i + p_i x_i + h_i I_i) \tag{2.19}$$

$$\text{s.t.} \quad I_{i-1} + x_i = I_i + d_i, \, \forall i = 1, \ldots, n \tag{2.20}$$

$$x_i \leq C y_i, \, \forall i = 1, \ldots, n \tag{2.21}$$

$$y_i \in \{0, 1\}, \, \forall i = 1, \ldots, n \tag{2.22}$$

$$x_i, I_i \geq 0, \, \forall i = 1, \ldots, n. \tag{2.23}$$

Florian and Klein [75] characterised the extreme points of the feasible region of the CCLS model. The authors show that, similar to the ULS model, the planning horizon can also be split into a sequence of regeneration intervals. In each regeneration interval, there is at most one production period in which the quantity produced is below full capacity C. If it exists, we call this the fractional period. Similar to the WW algorithm, we decompose the FK algorithm into the outer and the inner DPs. The outer DP for the CCLS is exactly the same as the one for the ULS model, i.e., (2.17) and (2.18). Therefore we only present the inner DP to evaluate the optimal cost of the subplans. With an abuse of notation, we denote the costs again by $\phi(k, t)$. Recall that the outer DP runs in $\mathcal{O}(n^2)$ time. In the following, we show that the inner DP calculates all $\phi(k, t)$ in $\mathcal{O}(n^4)$ time. Therefore, in total, the FK algorithm solves the CCLS in $\mathcal{O}(n^4)$ time.

Below, we use the reformulation of the CCLS model, where the inventory variables are eliminated. Similar to (2.6)-(2.10), we have:

$$\min_{\mathbf{y},\mathbf{x}} \sum_{i=1}^{n} [f_i \, y_i + p_i' \, x_i] \tag{2.24}$$

$$\text{s.t.} \quad \sum_{k=1}^{i} x_k \geq d_{1i},\ \forall i = 1, \ldots, n \tag{2.25}$$

$$x_i \leq C y_i,\ \forall i = 1, \ldots, n \tag{2.26}$$

$$y_i \in \{0, 1\},\ \forall i = 1, \ldots, n \tag{2.27}$$

$$x_i \geq 0,\ \forall i = 1, \ldots, n. \tag{2.28}$$

Recall that for given periods k and t, the inner DP finds an optimal solution to the subproblem on the regeneration interval $[k, t]$. If $d_{kt} = 0$, the optimal solution is equal to zero. So we assume that $d_{kt} > 0$. From Florian and Klein [75], we know that there are $\lfloor \frac{d_{kt}}{C} \rfloor$ production periods in the interval $[k, t]$ with production to full capacity. Let $\pi = d_{kt} - \lfloor \frac{d_{kt}}{C} \rfloor C$ with $0 \leq \pi < C$. Then if $\pi > 0$, it is the production quantity in the fractional period. If $\pi = 0$, all production periods within the interval are to full capacity. We assume that $\pi > 0$ for now.

Let $\beta(l, \tau, \delta)$ be the objective function value of a minimum cost solution for periods k up to l, during which production occurs τ times, $\tau \in \{0, \ldots, l + 1 - k\}$, at full capacity and $\delta \in \{0, 1\}$ times at level π. Recall that we consider $[k, t]$ as a regeneration interval, hence the DP has to enforce $I_r > 0$ for $k \leq r \leq t - 1$. If $\tau C + \delta\pi \leq d_{kt}$, it is impossible to have $I_t > 0$ for these (τ, δ) values and $\beta(l, \tau, \delta) = \infty$. Also if $\tau + \delta > l - k + 1$, it is not possible to produce $\tau + \delta$ times in the interval $[k, l]$ and again $\beta(l, \tau, \delta) = \infty$. Then for $l = k$, we have the initial conditions

$$\beta(k, \tau, \delta) = \begin{cases} f_k + p'_k C & \text{if } \tau = 1, \delta = 0 \text{ and } d_k \leq C \\ f_k + p'_k \pi & \text{if } \tau = 0, \delta = 1 \text{ and } d_k \leq \pi \\ 0 & \text{if } \tau = 0, \delta = 0 \text{ and } d_k = 0 \\ \infty & \textit{otherwise.} \end{cases}$$

For $l = k + 1, \ldots, t$, $\tau = 0, \ldots, \lfloor \frac{d_{kt}}{C} \rfloor$, we have the following DP recursion:

$$\beta(l, \tau, 0) = \begin{cases} \infty & \text{if } \tau C \leq d_{kl} \text{ or } \tau > l - k + 1 \\ \min \begin{cases} \beta(l-1, \tau, 0) \\ \beta(l-1, \tau-1, 0) + f_l + p'_l C \end{cases} & \textit{otherwise} \end{cases}$$

and

$$\beta(l,\tau,1) = \begin{cases} \infty & \text{if } \tau C + \pi \leq d_{kl} \text{ or } \tau > l-k \\ \min \begin{cases} \beta(l-1,\tau,1) \\ \beta(l-1,\tau-1,1) + f_l + p'_l C \\ \beta(l-1,\tau,0) + f_l + p'_l \pi \end{cases} & \textit{otherwise.} \end{cases}$$

When $\pi > 0$, the optimal cost of the subplan $[k,t]$, $\phi(k,t)$, is equal to $\beta(t, \lfloor \frac{d_{kt}}{C} \rfloor, 1)$. The inner DP is again a shortest path problem in an acyclic network. The minimum cost path from node $(k,0,0)$ to node $(t, \lfloor \frac{d_{kt}}{C} \rfloor, 1)$ in this acyclic network can be found in linear time in the number of arcs [7], i.e., in $\mathcal{O}(n^2)$ time. When $\pi = 0$, it suffices to use the same recursion for $\beta(l,\tau,0)$ yielding $\phi(k,t) = \beta(t, \lfloor \frac{d_{kt}}{C} \rfloor, 0)$. Please note that this algorithm also works for general concave production cost [143].

2.5 Lot sizing models with uncertainty

The data used in the aforementioned lot sizing models is assumed to be known exactly. Realising the uncertain nature of the input data, several studies have been done in the field of lot sizing models under uncertainty. Stochastic programming [44, 108, 145, 152] has been the most popular approach. Robust optimisation models are also available. However, the literature is very limited compared to stochastic programming. We review here both stochastic programming and robust optimization studies in the framework of lot sizing models. In the rest of this section, the stochastic programming studies describe random parameters by scenario trees and algorithms are polynomial in the size of the tree.

One stream of these studies considers uncertain demand. Haugen et al. [97] studied the multi-stage ULS model with demand uncertainty and cast the progressive hedging algorithm [149] in a meta-heuristic framework in order to achieve a solution that is hedged against future uncertainty. Ahmed et al. [6] proposed a reformulation of the stochastic ULS model whose linear programming relaxation is tighter than the standard formulation and showed that the ZIO property does not hold for the stochastic case. Guan et al. [87] proposed a branch-and-cut algorithm to solve the stochastic ULS problem, using an MILP formulation of the problem. The authors develop a family of valid inequalities, called (Q, S_Q) inequalities, and carried out some computational experiments to demonstrate their effectiveness.

Later in Guan et al. [88], the authors showed that the (Q, S_Q) inequalities suffice to describe the convex hull of the set of solutions when there is just one period with uncertain outcomes. More recently, Guan and Miller [90] gave a polynomial time algorithm to solve the stochastic ULS model studied in Guan et al. [87]. Extensions such as two-stage ULS [90] and multi-item CLS problems with uncertain demand [49] are also studied later on. Huang and Ahmed [100] considered a multi-stage stochastic integer programming formulation for the stochastic ULS problem without setup costs and proposed an approximation algorithm to solve the problem. Di Summa and Wolsey [64] extended the results in [87, 88] to capacitated problems using an underlying model of lot sizing on a tree. The authors present a simple description of the (Q, S_Q) inequalities and then prove that these inequalities are all mixing inequalities [92], which play a central role in the development of strong cutting planes for MILP problems. Guan [86] proved the production path property for stochastic CLS with backlogging, a fundamental optimality condition analogous to the ZIO property. A polynomial time algorithm is proposed.

Production cost uncertainty forms another stream. Please note that some studies discussed earlier, i.e. [64, 87, 88], involve both demand and production cost uncertainty. Huang and Küçükyavuz [101] studied lot sizing problems where the costs, demands and order lead times are uncertain. A polynomial time algorithm for the case where orders do not cross in time is proposed. Zhou and Guan [186] considered a two-stage stochastic ULS model with deterministic demands and nonspeculative production costs. An extended formulation in a higher dimensional space that provides integral solutions is proposed by showing that its constraint matrix is totally unimodular. The authors also provided the integral polyhedron of the problem in the original space by projecting the extended formulation to the original space. Guan and Liu [89] proposed a polynomial time algorithm for the stochastic ULS model with inventory bounds and order capacities, in which demands, inventory bounds, and costs are uncertain and dependent on each other in consecutive periods.

Other sources of uncertainty have been studied in the literature. Burstein et al. [50] consider a multi-stage production process in series with deterministic demand, but uncertain lead times. The authors characterised the optimal solution and proposed a polynomial time algorithm for solving the problem. Liberatore [125] and Nevison and Burstein [134] give polynomial time algorithms for dynamic lot sizing problems with deterministic demands and stochastic lead times when orders do not cross in time.

Decisions in most of those studies are made based on expected performance. However, there are some circumstances in which decision makers would typically take a more cautious approach [154]. Only recently, there have been some attempts to develop solution methods for risk-averse stochastic lot sizing models. Alem and Morabito [9] studied a stochastic lot sizing problem with application in furniture plants. The authors considered two risk-averse formulations of the problem. The first formulation uses the conditional value-at-risk as the risk measure to avoid solutions influenced by a bad scenario with a low probability. The second strategy is a mean-risk model based on the upper partial mean that aggregates a risk term in the objective function. The last approach is a restricted recourse approach, in which the risk preferences are directly considered in the constraints. The authors used computational package to solve the problems and the numerical results indicate that it is possible to achieve significant risk reductions using the risk-averse strategies, without overly sacrificing average costs.

Robust lot sizing models have also been a focus of the studies in this direction. As discussed in Chapter 1, Soyster [159] showed that absolute robust linear optimization models with interval uncertainty sets are equivalent to deterministic models where the uncertain parameters are all to upper bounds of the intervals. This is too conservative in the sense that one gives up too much optimality of the nominal problem in order to ensure robustness.

One can achieve less conservative solutions by using interval uncertainty sets with the robust deviation criterion [117]. The robust deviation criterion is an alternative to the absolute one, where instead of constructing solutions having the best possible performance in the worst case, the aim is to obtain a solution that exhibits the least worst-case deviation from optimality, among all feasible solutions over all realisable input data. The deviation is often called regret and the robust deviation criterion is often regarded as the minimax regret criterion [155]. Under this criterion, the solutions will be less conservative since they take into account the magnitude of missed opportunities of a decision by benchmarking its performance with that of the optimal ex-post decision.

Zhang [184] studies the two-stage Minimax Regret robust ULS model (MRULS) with demand uncertainty. In her study, the setup decision vector $\mathbf{y}$ has to be made before revealing the uncertain demand, i.e., in the first stage. Because the model is uncapacitated, the ZIO property is satisfied. For a given first-stage setup decision vector $\mathbf{y}$, the demand periods

associated with each production period are known, and are independent of the uncertain data. It is clear that both $\mathbf{x}$ and $\mathbf{I}$ are functions of $\mathbf{y}$ and the uncertain demand $\mathbf{d}$. For convenience, with an abuse of notation, we drop this dependence in this section. We denote by $\hat{\mathbf{y}}$ the ex-post setup decision that one would have made after revealing the uncertain data. Then, for a given decision vector first-stage $\mathbf{y}$ and a realisation of uncertainty $\mathbf{u}$, the regret is defined as

$$\sum_{i=1}^{n}(f_i\, y_i + p_i\, x_i + h_i\, I_i) - \min_{\hat{\mathbf{y}}\in\{0,1\}^n} \sum_{i=1}^{n}(f_i\, \hat{y}_i + p_i\, \hat{x}_i + h_i\, \hat{I}_i),$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{I}}$ are the ex-post production quantities and inventory levels corresponding to $\hat{\mathbf{y}}$. Zhang [184] proposes an $\mathcal{O}(n^7)$ time to solve the MRULS model.

When backlogging is available, Bertsimas and Thiele [40] study the robust lot sizing model with uncertainty budget in which the demand is uncertain, which is formulated as follows,

$$\begin{aligned}
BTRULS: \quad & \min_{\mathbf{y},\mathbf{x},\mathbf{q}} \sum_{i=1}^{n}(f_i y_i + p_i x_i + q_i) && (2.29)\\
& \text{s.t. } q_i \geq h_i(x_{1i} - d^{\star}_{1i}),\ \forall i = 1,\ldots,n && (2.30)\\
& q_i \geq -h_i^{-}(x_{1i} - d^{\star}_{1i}),\ \forall i = 1,\ldots,n && (2.31)\\
& x_i \leq M\, y_i,\ \forall i = 1,\ldots,n && (2.32)\\
& y_i \in \{0,1\},\ \forall i = 1,\ldots,n && (2.33)\\
& x_i \geq 0,\ \forall i = 1,\ldots,n, && (2.34)
\end{aligned}$$

where $d^{\star}_{1i}$ is the optimal objective function value of the following maximisation problem,

$$\begin{aligned}
d^{\star}_{1i} \quad := \quad & \max_{\mathbf{w}}\{\tilde{d}_{1i} + \sum_{t=1}^{i} w_t \hat{d}_t\} && (2.35)\\
& \text{s.t. } 0 \leq w_t \leq 1,\ \forall t = 1,\ldots,i && (2.36)\\
& \sum_{t=1}^{i} w_t \leq \Gamma_i. && (2.37)
\end{aligned}$$

In the formulation of the BTRULS model, both the inventory level and the backlogging level are eliminated by using the difference between cumulative production quantity and the

cumulative demand, i.e., $x_{1i} - d^{\star}_{1i}$. The main idea of the BTRULS model is to identify the worst cumulative demand that could occur up to each period i, subject to the uncertainty budget Γ_i, and find the corresponding best production plan. This identification is obtained by the subproblem (2.35)-(2.37). The authors show that the problem can be transformed into a nominal lot sizing problem with adjusted demands if the cost parameters are time-invariant. With general cost parameter settings, Solyalı et al. [158] propose a reformulation of the BTRULS problem which can be solved in polynomial time.

Atamtürk and Zhang [17] study the two-stage robust optimization problems with uncertainty budget in the network framework. For BTRULS, the authors show a similar result to that in Bertsimas and Thiele [40]. In addition, they also analyse lot sizing models with inventory bounds as well as multi-level versions of those, and show that the separation problem they propose can be solved in polynomial time. Alem and Morabito [9, 10] apply the concept of uncertainty budget to study the robust CLS model with backlogging in the Brazil furniture industry. The problem is solved with a commercial optimization package. Agra et al. [5] propose a new dynamic programming algorithm to solve robust models with uncertainty budget. With the lot sizing problem, numerical results show the efficiency of the proposed algorithm by giving shorter computational times. Klabjan et al. [115] propose a robust lot sizing model that integrates data fitting which generalizes the Bayesian model. The model can be interpreted as minimizing history-dependent risk measures.

2.6 Summary

In this chapter, we have briefly discussed studies on the ULS and the CLS models. We have reviewed the structure of optimal solutions of the ULS and the CCLS models and presented polynomial time algorithms which solve the problems in $\mathcal{O}(n^2)$ and $\mathcal{O}(n^4)$ time respectively. We have also introduced extensions based on the ULS and the CLS models. Studies of lot sizing problems with uncertain data have been discussed, revealing the need for more research around robust lot sizing models. Robust lot sizing models are the focus of Chapters 3-6.

Chapter 3

The Minimax Regret ULS Model

In this chapter, we study a robust lot sizing model with both the uncertain production cost function $\mathbf{f}$ and $\mathbf{p}$, and the uncertain demand $\mathbf{d}$. We use the minimax regret criterion with two stages to achieve a production plan that minimizes the maximum surplus between the total cost of the fist-stage production plan and that of the optimal production plan after revealing the uncertain data. As far as the author is aware, this is the first minimax regret lot sizing study which considers the uncertainty in the coefficients of the objective function and in parameters that help define the feasible region. We give a theoretical analysis and propose an algorithm that solves the problem in polynomial time. When there is demand uncertainty only, the proposed algorithm reduces the running time in the existing literature. The study in this chapter can be found in [120].

3.1 Introduction

As discussed in Chapter 1, Soyster [159] showed that absolute robust linear optimization models with interval uncertainty sets are equivalent to deterministic models where the uncertain parameters are all to upper bounds of the intervals. This is too conservative in the sense that one gives up too much optimality of the nominal problem in order to ensure robustness. To address the issue of over-conservatism, absolute robust linear optimization models with parametric uncertainty sets have been proposed as alternatives to the model with interval uncertainty sets [159]. Examples of such parametric uncertainty sets include

polyhedral uncertainty sets [14, 32, 38, 39], and ellipsoidal uncertainty sets [30, 31, 32, 69, 70].

We can also achieve less conservative solutions by using interval uncertainty sets with the robust deviation criterion [117]. The robust deviation criterion is an alternative to the absolute one, where instead of constructing solutions having the best possible performance in the worst case, the aim is to obtain a solution that exhibits the least worst-case deviation from optimality, among all feasible solutions over all realisable input data. The deviation is often called regret and the robust deviation criterion is often regarded as the minimax regret criterion [155]. Under this criterion, the solutions will be less conservative since they take into account the magnitude of missed opportunities of a decision by benchmarking its performance with that of the optimal ex-post decision.

Minimax regret models with interval uncertainty sets have been studied extensively in recent years, e.g., in graph theory [12, 22, 110, 132], assignment problems [8], facility location [20, 54], resource allocation [21], portfolio selection [59], inventory control [184], energy and environmental management [66, 124], and many others. Two-stage minimax regret linear programs with uncertain objective function coefficients were addressed by Inuiguchi and Sakawa [102]. The authors showed that it is sufficient to consider the bounds of the uncertainty intervals. Based on this result, the authors proposed an algorithm which needs to consider up to 2^J scenarios of the uncertain parameters, where J is the total number of uncertain parameters. In Inuiguchi and Sakawa [103], the authors proposed a more sophisticated algorithm based on solving a mathematical programming model. Averbakh and Lebedev [24] showed that minimax regret linear programming problems with uncertain objective function are $\mathcal{NP}$-hard. The authors also showed that if uncertainty is represented by discrete scenarios, the problem is polynomially solvable in the number of scenarios and planning periods.

When it comes to (mixed) integer problems, Mausser and Laguna [131] studied the two-stage minimax regret MILP with uncertain objective function coefficients. The authors proposed a general algorithm which finds an optimal scenario by solving an MILP that adds a collection of binary variables and new constraints to the original model. The complexity of the resulting MILP limits the size of problems that can be addressed. Assavapokee et al. [13] studied the two-stage minimax regret MILP in which the uncertainty affects both the objective function and the constraint matrix. The authors proposed a three-stage algorithm that coordinates three mathematical programming formulations to iteratively solve

the overall problem. In general, the minimax regret versions of most polynomially solvable MILPs are $\mathcal{NP}$-hard in the case of interval representation of the uncertainty [18, 19, 23].

Averbakh [18, 19] showed that polynomial solvability is preserved for some problems such as selecting p elements of minimum total weight out of a set of m elements with interval uncertainty in the weights of the elements. In the lot sizing literature, which is the focus of this chapter, Zhang [184] applied the minimax regret approach to the ULS model with demand uncertainty. In her study, the author showed that it is sufficient to consider only the bounds of the uncertainty intervals. The author then decomposed the problem into a sequence of subproblems, in which one only needs to consider at most $\mathcal{O}(n)$ scenarios for the demand, where n is the length of the planning horizon. A polynomial time algorithm is proposed which solves the minimax regret ULS model with demand uncertainty in $\mathcal{O}(n^7)$ time.

In this chapter, we study a Minimax Regret ULS (MRULS) model with uncertain production cost function $\mathbf{f}$ and $\mathbf{p}$, and uncertain demand $\mathbf{d}$. The problem under consideration is a two-stage one. The setup decision vector $\mathbf{y}$ is to be decided before the uncertain data is disclosed at the first stage, while the production quantities $\mathbf{x}$ and inventory levels $\mathbf{I}$ can be adjusted at the second stage, after revealing the real demand. Two-stage robust optimization problems have received increasing attention over the last decade, see e.g., [17, 27, 36, 78]. Under interval demand uncertainty, Zhang [184] proposed an $\mathcal{O}(n^7)$ time algorithm for the two-stage MRULS model. In this chapter, we show that the MRULS model with both uncertain production cost function and uncertain demand can be solved in $\mathcal{O}(n^6)$ time. We improve this running time to $\mathcal{O}(n^5)$ when only the demand is uncertain and to $\mathcal{O}(n^4)$ when only the production cost function is uncertain.

The remainder of the chapter is organised as follows. We formulate the MRULS model in Section 3.2. In Section 3.3, some theoretical results proposed in Zhang [184] for the MRULS model with demand uncertainty (MRULS^d) are reviewed, theoretical results for the MRULS model with both uncertain production cost function and uncertain demand are derived, and an $\mathcal{O}(n^6)$ time algorithm is proposed. In Section 3.4, we show that the running time can be improved to $\mathcal{O}(n^5)$ when only the demand is uncertain and to $\mathcal{O}(n^4)$ when only the production cost function is uncertain. In Section 3.5, a summary of the chapter is given and lines of future research are proposed.

3.2 The formulation

Let $\mathbf{u} = (\mathbf{f}, \mathbf{p}, \mathbf{d}) \in \mathcal{U} := \mathcal{U}^f \times \mathcal{U}^p \times \mathcal{U}^d$ denote a realisation of the uncertain data. Then for each period i, we have $u_i = (f_i, p_i, d_i)$ as such that $f_i \in [\tilde{f}_i - \hat{f}_i, \tilde{f}_i + \hat{f}_i]$, $p_i \in [\tilde{p}_i - \hat{p}_i, \tilde{p}_i + \hat{p}_i]$, and $d_i \in [\tilde{d}_i - \hat{d}_i, \tilde{d}_i + \hat{d}_i]$. We denote by $\overline{f}_j = \tilde{f}_j + \hat{f}_j$, $\overline{p}_j = \tilde{p}_j + \hat{p}_j$, and $\overline{p}_j = \tilde{p}_j + \hat{p}_j$ the upper bounds of the intervals.

In the rest of the chapter, we assume that setup decision vector $\mathbf{y}$ has to be made before revealing the uncertain data, i.e., in the first stage. Because the model is uncapacitated, it is easy to show that the ZIO property is satisfied. For a given first-stage setup decision vector $\mathbf{y}$, the demand periods associated with each production period are known, and are independent of the uncertain data. It is clear that both $\mathbf{x}$ and $\mathbf{I}$ are functions of $\mathbf{y}$ and the uncertain demand $\mathbf{d}$. For convenience, with an abuse of notation, we drop this dependency throughout the chapter. We denote by $\hat{\mathbf{y}}$ the ex-post setup decision that one would have made after revealing the uncertain data. Then, for a given decision vector first-stage $\mathbf{y}$ and a realisation of uncertainty $\mathbf{u}$, the regret is defined as

$$\sum_{i=1}^{n}(f_i\, y_i + p_i\, x_i + h_i\, I_i) - \min_{\hat{\mathbf{y}}\in\{0,1\}^n} \sum_{i=1}^{n}(f_i\, \hat{y}_i + p_i\, \hat{x}_i + h_i\, \hat{I}_i),$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{I}}$ are the ex-post production quantities and inventory levels corresponding to $\hat{\mathbf{y}}$. Hence, the MRULS model is formulated as

$$\begin{aligned}\min_{\mathbf{y}\in\{0,1\}^n} \max_{\mathbf{u}\in\mathcal{U}} [&\sum_{i=1}^{n}(f_i\, y_i + p_i\, x_i + h_i\, I_i) - \\ &- \min_{\hat{\mathbf{y}}\in\{0,1\}^n} \sum_{i=1}^{n}(f_i\, \hat{y}_i + p_i\, \hat{x}_i + h_i\, \hat{I}_i)],\end{aligned}$$

which is equivalent to

$$\begin{aligned}\min_{\mathbf{y}\in\{0,1\}^n} \max_{\mathbf{u}\in\mathcal{U}} \max_{\hat{\mathbf{y}}\in\{0,1\}^n} \{&\sum_{i=1}^{n} f_i(y_i - \hat{y}_i) + \sum_{i=1}^{n} p_i\,(x_i - \hat{x}_i) + \\ &+ \sum_{i=1}^{n} h_i\,(I_i - \hat{I}_i)]\}.\end{aligned} \tag{3.1}$$

Note that the MRULS model consists of a two-stage ULS model with variables $\mathbf{y}$, $\mathbf{x}$, and $\mathbf{I}$,

and an ex-post ULS with variables $\hat{\mathbf{y}}$, $\hat{\mathbf{x}}$, and $\hat{\mathbf{I}}$. In the rest of the chapter, we refer to these as the "two-stage" and "ex-post" problems, respectively.

We introduce some notations that will be used throughout the chapter. We introduce the period $n+1$ with $d_{n+1} = 0$ and $f_{n+1} = p_{n+1} = h_{n+1} = 0$. Without loss of optimality, we assume that $y_{n+1} = 1 = \hat{y}_{n+1}$. In what follows, we define

- d_{ij} as the total demand from period i to j, i.e., $d_{ij} := \sum_{k=i}^{j} d_k$;
- h_{ij} as the total unit inventory holding cost from period i to j, i.e., $h_{ij} := \sum_{k=i}^{j} h_k$;
- α_{ij} as the unit cost of producing and delivering demand in period j from production in period i, i.e., $\alpha_{ij} := p_i + h_{i,j-1}$;
- $\underline{\alpha}_{ij} := \underline{p}_i + h_{i,j-1}$;
- $\overline{\alpha}_{ij} := \overline{p}_i + h_{i,j-1}$;
- m_i as the production period that satisfies the demand in period i for a given first-stage decision vector $\mathbf{y}$, i.e., $m_i := \max\{k : y_k = 1, 1 \le k \le i\}$;
- $\hat{m}_i$ as the production period that satisfies the demand in period i for a given ex-post decision vector $\hat{\mathbf{y}}$, i.e., $\hat{m}_i := \max\{k : \hat{y}_k = 1, 1 \le k \le i\}$;
- l_i as the first production period after period i for a given first-stage decision vector $\mathbf{y}$, i.e., $l_i := \min\{k : y_k = 1, i \le k \le n+1\}$, where if $l_i = n+1$, then there is no production in $[i, n]$;
- $\hat{l}_i$ as the first production period after period i for a given ex-post decision vector $\hat{\mathbf{y}}$, i.e., $\hat{l}_i := \min\{k : \hat{y}_k = 1, i \le k \le n+1\}$, where if $\hat{l}_i = n+1$, then there is no production in $[i, n]$.

Note that

$$\alpha_{ij} \le \alpha_{lj} \iff \alpha_{ir} \le \alpha_{lr} \quad \text{for } r \ge j, \tag{3.2}$$

while in any optimal solution of the second-stage ULS model,

$$\alpha_{\hat{m}_j,j} \le \alpha_{\hat{m}_i,j} \quad \text{for } j > i. \tag{3.3}$$

For given $\mathbf{y}$, $\mathbf{f}$, and $\mathbf{p}$, m_i and $\hat{m}_i$ are known and we can rewrite the MRULS model as follows:

$$\max_{\mathbf{d}\in\mathcal{U}^d}\max_{\hat{\mathbf{y}}\in\{0,1\}^n}[\sum_{i=1}^{n} f_i(y_i-\hat{y}_i)+\sum_{i=1}^{n}(\alpha_{m_i,i}-\alpha_{\hat{m}_i,i})d_i]. \tag{3.4}$$

Using (3.4), Zhang [184] shows that, without loss of optimality, we only need to consider the bounds of the uncertainty interval of $\mathbf{d}$ if $\mathbf{f}$ and $\mathbf{p}$ are given. We state the result here without a proof.

Lemma 3.1. *[184] For a given first-stage decision vector* $\mathbf{y}$*, realisations of* $\mathbf{f}$ *and* $\mathbf{p}$*, there exists an optimal scenario* $\mathbf{d}^\star$ *to (3.1), satisfying:*

$$d_i^\star=\begin{cases}\underline{d}_i & \text{if } \alpha_{m_i,i}<\alpha_{\hat{m}_i,i}\\ \overline{d}_i & \text{if } \alpha_{m_i,i}>\alpha_{\hat{m}_i,i}\\ \underline{d}_i \text{ or } \overline{d}_i & \text{if } \alpha_{m_i,i}=\alpha_{\hat{m}_i,i}.\end{cases}$$

We derive a similar result for the production cost function. For given $\mathbf{y}$ and $\mathbf{d}$, the MRULS model is equivalent to

$$\max_{\mathbf{f}\in\mathcal{U}^f}\max_{\mathbf{p}\in\mathcal{U}^p}\max_{\hat{\mathbf{y}}\in\{0,1\}^n}[\sum_{i=1}^{n} f_i\,(y_i-\hat{y}_i)+\sum_{i=1}^{n} p_i\,(x_i-\hat{x}_i)], \tag{3.5}$$

in which the total inventory holding costs are eliminated without loss of generality.

Lemma 3.2. *For a given first-stage decision vector* $\mathbf{y}$*, a realisation* $\mathbf{d}$*, and an ex-post decision vector* $\hat{\mathbf{y}}$*, there exist optimal scenarios* $\mathbf{f}^\star$ *and* $\mathbf{p}^\star$ *to (3.1), satisfying:*

$$(f_i^\star,p_i^\star)=\begin{cases}(\underline{f}_i,\underline{p}_i) & \text{if } x_i<\hat{x}_i^\star\\ (\overline{f}_i,\overline{p}_i) & \text{if } x_i>\hat{x}_i^\star\\ (\underline{f}_i,\underline{p}_i) \text{ or } (\overline{f}_i,\overline{p}_i) & \text{if } x_i=\hat{x}_i^\star.\end{cases}$$

Proof. If $x_i<\hat{x}_i^\star$, it is easy to see that the result holds for (3.5) since $(x_i-\hat{x}_i^\star)<0$ and $(y_i-\hat{y}_i^\star)\leq 0$. Similarly for the two other cases. □

3.3 The solution approach

Using the ZIO property, the vector $\mathbf{y}$ splits the planning horizon into a sequence of regeneration intervals [175], where no inventory enters or leaves the intervals and the inventory levels within the interval are positive. In this section we propose a DP algorithm that decomposes the MRULS problem into subplans that consists of two levels. The first level is defined by the regeneration intervals with respect to the two-stage ULS model with variables $\mathbf{y}$, $\mathbf{x}$, and $\mathbf{I}$. The second level is defined by the incoming and outgoing inventory levels of the ex-post ULS model with variables $\hat{\mathbf{y}}$, $\hat{\mathbf{x}}$, and $\hat{\mathbf{I}}$.

3.3.1 The dynamic programming algorithm

We construct a network with nodes $(k, \hat{m}_k, \hat{l}_k)$ in which the DP moves from node $(k, \hat{m}_k, \hat{l}_k)$ to $(t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$, defining the subplan $[(k, \hat{m}_k, \hat{l}_k), (t+1, \hat{m}_{t+1}, \hat{l}_{t+1})]$, for $1 \leq k \leq t \leq n$, $1 \leq \hat{m}_k \leq k \leq \hat{l}_k \leq \hat{l}_{t+1}$, and $\hat{m}_k \leq \hat{m}_{t+1} \leq t+1 \leq \hat{l}_{t+1} \leq n+1$. In this subplan, $[k, t]$ is a regeneration interval for the first-level ULS model, while in the second level, $\hat{I}_{k-1} = d_{k,\hat{l}_k-1}$ units enter the subplan and $\hat{I}_t = d_{t+1,\hat{l}_{t+1}-1}$ leave. The incoming inventory $\hat{I}_{k-1}$ is produced at period $\hat{m}_k$. If $\hat{m}_k$ is not a period in $[k, t]$, i.e., $\hat{m}_k < k$, then $x_{\hat{m}_k} \leq d_{\hat{m}_k,k-1} \leq d_{\hat{m}_k,\hat{l}_k-1} = \hat{x}_{\hat{m}_k}$, and by Lemma 3.2

$$\alpha^{\star}_{\hat{m}_k,i} = \underline{\alpha}_{\hat{m}_k,i} \quad \text{for } i = \hat{m}_k, \ldots, \hat{l}_k - 1. \tag{3.6}$$

In Figure 3.1, we give an illustration of the subplan $[(k, \hat{m}_k, \hat{l}_k), (t+1, \hat{m}_{t+1}, \hat{l}_{t+1})]$.

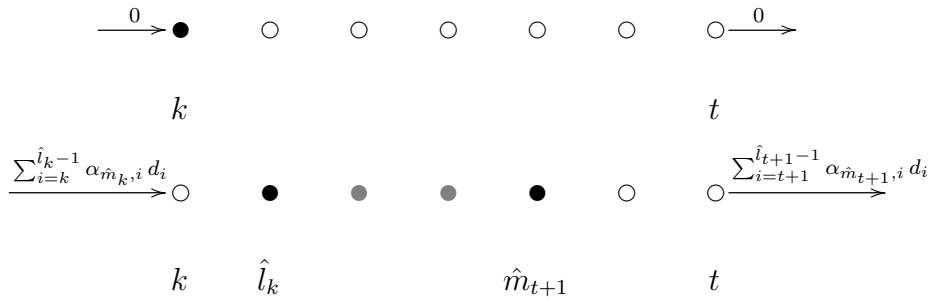


Figure 3.1: An illustration of the subplan $[(k, \hat{m}_k, \hat{l}_k), (t+1, \hat{m}_{t+1}, \hat{l}_{t+1})]$: ● zero incoming inventory production periods, ○ non-production periods, ● status unknown.

We denote by $\phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ the minimax regret for periods $[k, t]$ within the subplan $[(k, \hat{m}_k, \hat{l}_k), (t+1, \hat{m}_{t+1}, \hat{l}_{t+1})]$. Let $G(t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ be the minimax regret over the first t periods, where $\hat{m}_{t+1}$ is the last second-level production period in $[1, t+1]$, and $H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ be the minimax regret over the first t periods, where k is the last first-level production period in $[1, t+1]$ and $\hat{m}_{t+1}$ the last second-level production period. We can derive a recursion for $G(t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ and $H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$:

$$\begin{aligned} G(1,1,1) =& 0 \\ G(t+1, \hat{m}_{t+1}, \hat{l}_{t+1}) =& \min_{1 \le k \le t} H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1}) \\ H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1}) =& \max_{\substack{1 \le \hat{m}_k \le k \le \hat{l}_k \le \hat{l}_{t+1} \\ \hat{m}_k \le \hat{m}_{t+1}}} \{G(k, \hat{m}_k, \hat{l}_k) + \\ & + \phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})\}, \end{aligned}$$

for $1 \le \hat{m}_{t+1} \le t+1 \le \hat{l}_{t+1} \le n+1$.

If all values of $\phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ are known, we can show that the outer DP runs in $\mathcal{O}(n^6)$ time. Indeed, for given t, $\hat{m}_{t+1}$, and $\hat{l}_{t+1}$, calculating $G(t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ can be done in $\mathcal{O}(n)$ time due to the minimization over k. For each $H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$, we need to consider the maximisation over $\hat{m}_k$ and $\hat{l}_k$, which takes $\mathcal{O}(n^2)$ time. Therefore, all values of $H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ and $G(t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ can be calculated in $\mathcal{O}(n^6)$ time. Next, we will show that all costs $\phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ can also be obtained in $\mathcal{O}(n^6)$ time. Therefore, the DP solves the MRULS model in $\mathcal{O}(n^6)$ time.

3.3.2 Subplan properties

Below, we present two useful properties of the subplans. In Proposition 3.5, we derive the structure of an optimal $\mathbf{d}^\star$, and its proof uses two lemmas in Zhang [184], while Proposition 3.6 enables us to handle $\mathbf{f}$, $\mathbf{p}$, and $\mathbf{d}$ sequentially.

Lemma 3.3. *[184] For a given first-level regeneration interval $[k, t]$, a realisation of $\mathbf{p}$, and an ex-post decision vector $\hat{\mathbf{y}}$, if $\alpha_{kk} > \alpha_{\hat{m}_k, k}$, there exists an optimal scenario $\mathbf{d}^\star$ such that $d_i^\star = \overline{d}_i$ for $i = k, \ldots, t$.*

Lemma 3.4. *[184] For a given first-level regeneration interval $[k, t]$, a realisation of $\mathbf{p}$, and an ex-post decision vector $\hat{\mathbf{y}}$, if $\alpha_{kk} \le \alpha_{\hat{m}_k, k}$ and $\hat{y}_i = 0$ for $i = k+1, \ldots, t$, there exists an*

optimal scenario $\mathbf{d}^\star$ *such that* $d_i^\star = \underline{d}_i$ *for* $i = k, \ldots, t$.

Proposition 3.5. *Given the subplan* $[(k, \hat{m}_k, \hat{l}_k), (t+1, \hat{m}_{t+1}, \hat{l}_{t+1})]$ *with* $\hat{m}_{t+1} \leq t$*, a realisation of* $\mathbf{p}$*, and an ex-post decision vector* $\hat{\mathbf{y}}$*, there exists an optimal* $\mathbf{d}^\star$ *such that:*

If $\hat{m}_k < \hat{m}_{t+1}$,

$$(d_k^\star, \ldots, d_t^\star) = \begin{cases} b(i): \ (\overline{d}_k, \ldots, \overline{d}_t) & \text{if } \alpha_{kk} > \alpha_{\hat{m}_k,k} \\ b(ii): \ (\underline{d}_k, \ldots, \underline{d}_t) & \text{if } \alpha_{kk} \leq \alpha_{\hat{m}_k,k} \text{ and} \\ & \alpha_{k,\hat{m}_{t+1}} \leq \alpha_{\hat{m}_{t+1},\hat{m}_{t+1}} \\ b(iii): \begin{matrix}(\underline{d}_k, \ldots, \underline{d}_{\gamma-1}, \\ \overline{d}_\gamma, \ldots, \overline{d}_t)\end{matrix} & \begin{matrix}\text{if } \alpha_{kk} \leq \alpha_{\hat{m}_k,k} \text{ and} \\ \alpha_{k,\hat{m}_{t+1}} \leq \alpha_{\hat{m}_{t+1},\hat{m}_{t+1}},\end{matrix} \end{cases}$$

in which $\gamma = \min\{i : \hat{l}_k \leq i \leq \hat{m}_{t+1}, \hat{y}_i = 1, \alpha_{ki} > \alpha_{ii}\}$.

and if $\hat{m}_k = \hat{m}_{t+1}$,

$$(d_k^\star, \ldots, d_t^\star) = \begin{cases} a(i): \ (\overline{d}_k, \ldots, \overline{d}_t) & \text{if } \alpha_{kk} > \alpha_{\hat{m}_k,k} \\ a(ii): \ (\underline{d}_k, \ldots, \underline{d}_t) & \text{if } \alpha_{kk} \leq \alpha_{\hat{m}_k,k}; \end{cases}$$

Proof. Note that, by construction, $m_i = k$ for $i = k, \ldots, t$.

• Cases a(i) and b(i): The result follows directly from Lemma 3.3. Please note that the condition $\hat{m}_{t+1} \leq t$ is not required.

• Case a(ii): Since $\hat{m}_k = \hat{m}_{t+1}$, $\hat{y}_i = 0$ for $i = k+1, \ldots, t$, and the result follows from Lemma 3.4. Please note that the condition $\hat{m}_{t+1} \leq t$ is not required.

• Case b(ii): From Lemma 3.1, it will be enough to show that $\alpha_{ki} \leq \alpha_{\hat{m}_i,i}$ for $i = k, \ldots, t$. From the statement of the proposition, the inequality holds for $i = k$. For $i = \hat{m}_{t+1}, \ldots, t$, the inequality holds by observing that $\alpha_{ki} \leq \alpha_{\hat{m}_{t+1},i}$ from (3.2) and the fact that $\hat{m}_i = \hat{m}_{t+1}$. It remains to analyse $i = k+1, \ldots, \hat{m}_{t+1}-1$. By (3.3), $\alpha_{\hat{m}_{t+1},\hat{m}_{t+1}} \leq \alpha_{\hat{m}_i,\hat{m}_{t+1}}$. Combining this inequality with $\alpha_{k,\hat{m}_{t+1}} \leq \alpha_{\hat{m}_{t+1},\hat{m}_{t+1}}$, we obtain $\alpha_{k,\hat{m}_{t+1}} \leq \alpha_{\hat{m}_i,\hat{m}_{t+1}}$ and the desired inequality follows again using (3.2).

• Case b(iii): Using a similar argument as in Case b(ii), the result holds for periods $i = k$ and $i = \hat{m}_{t+1}, \ldots, t$. For $k+1, \ldots, \hat{m}_{t+1} - 1$, we need to consider whether there exists a period i, $k+1 \leq i \leq \hat{m}_{t+1} - 1$, such that $\hat{y}_i = 1$ and $\alpha_{ki} > \alpha_{ii}$. If such a period does not exists, then $\alpha_{ki} \leq \alpha_{\hat{m}_i,i}$ for $i = k+1, \ldots, \hat{m}_{t+1} - 1$, and the result follows by letting $\gamma = \hat{m}_{t+1}$. Otherwise, from the definition of γ, $\alpha_{k,\hat{m}_i} \leq \alpha_{\hat{m}_i,\hat{m}_i}$, and using (3.2), $\alpha_{ki} \leq \alpha_{\hat{m}_i,i}$ for $i = k+1, \ldots, \gamma - 1$. For $i = \gamma, \ldots, \hat{m}_{t+1} - 1$, we have $\alpha_{k\gamma} > \alpha_{\gamma\gamma}$, which implies $\alpha_{ki} > \alpha_{\gamma i}$ by (3.2), while $\alpha_{\gamma i} > \alpha_{\hat{m}_i,i}$ by (3.3). Combining the two inequalities, the result follows by Lemma 3.1. □

Proposition 3.6. *Given the subplan* $[(k, \hat{m}_k, \hat{l}_k), (t+1, \hat{m}_{t+1}, \hat{l}_{t+1})]$*, a realisation of* $\mathbf{d}$*, and an ex-post decision vector* $\hat{\mathbf{y}}$*, there exists an optimal* $\mathbf{p}^\star$ *such that:*

$$(p_k^\star, \ldots, p_t^\star) = \begin{cases} I: & (\overline{p}_k, \ldots, \overline{p}_t) & \text{if } \hat{l}_k = \hat{l}_{t+1} \\ II: & (\overline{p}_k, \ldots, \overline{p}_{\hat{l}_k - 1}, \underline{p}_{\hat{l}_k}, \ldots, \underline{p}_t) & \text{if } k < \hat{l}_k < \hat{l}_{t+1} \\ III: & (\overline{p}_k, \ldots, \overline{p}_{\theta-1}, \underline{p}_\theta, \ldots, \underline{p}_t) & \text{if } k = \hat{l}_k < \hat{l}_{t+1}, \end{cases}$$

where

$$\theta = \begin{cases} k & \text{if } \hat{y}_i = 0 \text{ for } i = k+1, \ldots, t, \\ \min\{i : k < i \leq t,\, \hat{y}_i = 1\} & \text{otherwise.} \end{cases}$$

Proof. We consider each case separately.

• Case I: In this case $\hat{l}_k \geq t+1$ and therefore $\hat{x}_i = 0 \leq x_i$ for $i = k, \ldots, t$. We can assume without loss of optimality that $p_i^\star = \overline{p}_i$ for $i = k, \ldots, t$ by Lemma 3.2.

• Case II: In this case $\hat{l}_k \leq t$. Since $k < \hat{l}_k$, we can use the same argument as in Case I to show that the result holds for $i = k, \ldots, \hat{l}_k - 1$. For $i = \hat{l}_k, \ldots, t$, the result follows by using $x_i = 0 \leq \hat{x}_i$.

• Case III: We need to consider whether $\hat{\mathbf{y}}$ has a production period in $[k+1, t]$. If such a period does not exist, $x_k = d_{kt} \leq d_{k,\hat{l}_{t+1}} = \hat{x}_k$ and $x_i = 0 = \hat{x}_i$ for $i = k+1, \ldots, t$. So, we can assume $p_i^\star = \underline{p}_i$ for $i = k, \ldots, t$, and the desired result follows by choosing $\theta = k$.

Now suppose that such a production period exists and $\theta = \min\{i : k < i \leq t, \hat{y}_i = 1\}$. By construction we have $x_k = d_{kt} \geq d_{k,\theta-1} = \hat{x}_k$ and $x_i = 0 = \hat{x}_i$ for $i = k+1, \ldots, \theta-1$, which implies that $p_i^\star = \overline{p}_i$ for $i = k, \ldots, \theta-1$. For $i = \theta, \ldots, t$, $x_i = 0 \leq \hat{x}_i$ implies that $p_i^\star = \underline{p}_i$ and the desired result follows. □

3.3.3 Minimax regret of a subplan

In this section we calculate $\phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$. We start with diagonal subplans, i.e., $\hat{m}_{t+1} \leq t$. For convenience, we let $\underline{\mathcal{L}}_i^j$ be the optimal objective function value of the ULS model defined between periods i and j with all uncertain parameters at lower bounds. We define $\overline{\mathcal{L}}_i^j$ in a similar fashion, but where the demands are at their upper bounds.

Case I: $\hat{l}_k = \hat{l}_{t+1}$**,** $\hat{m}_{t+1} \leq t$

Using Proposition 3.6, $(p_k^\star, \ldots, p_t^\star) = (\overline{p}_k, \ldots, \overline{p}_t)$, while $\alpha^\star_{\hat{m}_k, i} = \underline{\alpha}_{\hat{m}_k, i}$, $i = k, \ldots, t$, by (3.6). In addition, according to Proposition 3.5, we need to consider demand scenarios in Cases a(i)-a(ii). Therefore,

$$\phi = \begin{cases} \overline{f}_k + \sum_{i=k}^t \overline{\alpha}_{ki} \overline{d}_i - \sum_{i=k}^t \underline{\alpha}_{\hat{m}_k, i} \overline{d}_i \text{ if } \overline{\alpha}_{kk} > \underline{\alpha}_{\hat{m}_k, k} \\ \overline{f}_k + \sum_{i=k}^t \overline{\alpha}_{ki} \underline{d}_i - \sum_{i=k}^t \underline{\alpha}_{\hat{m}_k, i} \underline{d}_i \text{ if } \overline{\alpha}_{kk} \leq \underline{\alpha}_{\hat{m}_k, k}. \end{cases} \tag{3.7}$$

Case II: $k < \hat{l}_k < \hat{l}_{t+1}$**,** $\hat{m}_{t+1} \leq t$

We have $(p_k^\star, \ldots, p_t^\star) = (\overline{p}_k, \ldots, \overline{p}_{\hat{l}_k - 1}, \underline{p}_{\hat{l}_k}, \ldots, \underline{p}_t)$ by Proposition 3.6, while again $\alpha^\star_{\hat{m}_k, i} = \underline{\alpha}_{\hat{m}_k, i}$, $i = k, \ldots, \hat{l}_k - 1$. In addition, according to Proposition 3.5, we need to consider demand scenarios in Cases (b)(i)-(b)(iii).

- Case II(b)(i): $\overline{\alpha}_{kk} > \underline{\alpha}_{\hat{m}_k, k}$

In this case, $(d_k^\star, \ldots, d_t^\star) = (\overline{d}_k, \ldots, \overline{d}_t)$. In the second level, there is no production in

$[k, \hat{l}_k - 1]$ or $[\hat{m}_{t+1} + 1, t]$. Therefore,

$$\begin{aligned}\phi =& \overline{f}_k + \sum_{i=k}^{t} \overline{\alpha}_{ki} \overline{d}_i - \sum_{i=k}^{\hat{l}_k - 1} \underline{\alpha}_{\hat{m}_k, i} \overline{d}_i - \overline{\mathcal{L}}_{\hat{l}_k}^{\hat{m}_{t+1}-1} - \\ & - \underline{f}_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \underline{\alpha}_{\hat{m}_{t+1}, i} \overline{d}_i.\end{aligned} \tag{3.8}$$

• Case II(b)(ii): $\overline{\alpha}_{kk} \leq \underline{\alpha}_{\hat{m}_k, k}$ and $\overline{\alpha}_{k, \hat{m}_{t+1}} \leq \underline{\alpha}_{\hat{m}_{t+1}, \hat{m}_{t+1}}$

In this case, ϕ is identical to (3.8) with $(d_k^\star, \ldots, d_t^\star) = (\underline{d}_k, \ldots, \underline{d}_t)$, i.e.,

$$\begin{aligned}\phi =& \overline{f}_k + \sum_{i=k}^{t} \overline{\alpha}_{ki} \underline{d}_i - \sum_{i=k}^{\hat{l}_k - 1} \underline{\alpha}_{\hat{m}_k, i} \underline{d}_i - \underline{\mathcal{L}}_{\hat{l}_k}^{\hat{m}_{t+1}-1} - \\ & - \underline{f}_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \underline{\alpha}_{\hat{m}_{t+1}, i} \underline{d}_i.\end{aligned}$$

• Case II(b)(iii): $\overline{\alpha}_{kk} \leq \underline{\alpha}_{\hat{m}_k, k}$ and $\overline{\alpha}_{k, \hat{m}_{t+1}} > \underline{\alpha}_{\hat{m}_{t+1}, \hat{m}_{t+1}}$

We need to go through all candidates to γ, i.e., $\hat{l}_k \leq \gamma \leq \hat{m}_{t+1}$ where $\overline{\alpha}_{k\gamma} > \underline{\alpha}_{\gamma\gamma}$, and force $\hat{y}_\gamma = 1$ and $(d_k, \ldots, d_t) = (\underline{d}_k, \ldots, \underline{d}_{\gamma-1}, \overline{d}_\gamma, \ldots, \overline{d}_t)$. By optimizing over all these candidates we get

$$\begin{aligned}\phi =& \overline{f}_k + \max_{\substack{\hat{l}_k \leq \gamma \leq \hat{m}_{t+1} \\ \overline{\alpha}_{k\gamma} > \underline{\alpha}_{\gamma\gamma}}} [\sum_{i=k}^{\gamma-1} \overline{\alpha}_{ki} \underline{d}_i + \sum_{i=\gamma}^{t} \overline{\alpha}_{ki} \overline{d}_i - \sum_{i=k}^{\hat{l}_k - 1} \underline{\alpha}_{\hat{m}_k, i} \underline{d}_i - \\ & - \underline{\mathcal{L}}_{\hat{l}_k}^{\gamma-1} - \overline{\mathcal{L}}_{\gamma}^{\hat{m}_{t+1}-1}] - \underline{f}_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \underline{\alpha}_{\hat{m}_{t+1}, i} \overline{d}_i.\end{aligned}$$

Case III: $k = \hat{l}_k < \hat{l}_{t+1}$**,** $\hat{m}_{t+1} \leq t$

Since $\hat{m}_k = k$, we cannot have demand scenarios a(i) or b(i). If $\hat{m}_{t+1} = k$, the demands in both levels are satisfied at period k and $\phi = 0$. Otherwise, $k < \hat{m}_{t+1} \leq t$ and $\alpha^*_{k, \hat{m}_{t+1}} = \overline{\alpha}_{k, \hat{m}_{t+1}}$ while $\alpha^*_{\hat{m}_{t+1}, \hat{m}_{t+1}} = \underline{\alpha}_{\hat{m}_{t+1}, \hat{m}_{t+1}}$.

• Case III(a)(ii): $\hat{m}_k = \hat{m}_{t+1}$

As argued above, $\phi = 0$.

• Case III(b)(ii): $\hat{m}_k < \hat{m}_{t+1}$ and $\overline{\alpha}_{k,\hat{m}_{t+1}} \leq \underline{\alpha}_{\hat{m}_{t+1},\hat{m}_{t+1}}$

In this case $(d_k^\star, \ldots, d_t^\star) = (\underline{d}_k, \ldots, \underline{d}_t)$. In the second level, we need to go through all candidates for θ, $k < \theta \leq \hat{m}_{t+1}$. We force $\hat{y}_\theta = 1$, $\hat{y}_i = 0$ for $i = k, \ldots, \theta - 1$, and $(p_k, \ldots, p_t) = (\overline{p}_k, \ldots, \overline{p}_{\theta-1}, \underline{p}_\theta, \ldots, \underline{p}_t)$, and optimize over all these candidates. Therefore,

$$\phi = \max_{k<\theta\leq\hat{m}_{t+1}} [\sum_{i=\theta}^{t} \overline{\alpha}_{ki}\underline{d}_i - \underline{\mathcal{L}}_\theta^{\hat{m}_{t+1}-1}] - \underline{f}_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \underline{\alpha}_{\hat{m}_{t+1},i}\underline{d}_i.$$

• Case III(b)(iii): $\hat{m}_k < \hat{m}_{t+1}$ and $\overline{\alpha}_{k,\hat{m}_{t+1}} > \underline{\alpha}_{\hat{m}_{t+1},\hat{m}_{t+1}}$

Besides θ, we need to optimize over all the candidates for γ as in Case II(b)(iii):

$$\begin{aligned}\phi = \max_{\substack{k<\theta\leq\gamma \\ \theta\leq\gamma\leq\hat{m}_{t+1} \\ \overline{\alpha}_{k\gamma}>\underline{\alpha}_{\gamma\gamma}}} &[\sum_{i=\theta}^{\gamma-1} \overline{\alpha}_{ki}\underline{d}_i + \sum_{i=\gamma}^{t} \overline{\alpha}_{ki}\overline{d}_i - \underline{\mathcal{L}}_\theta^{\gamma-1} - \overline{\mathcal{L}}_\gamma^{\hat{m}_{t+1}-1}] - \\ &- \underline{f}_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \underline{\alpha}_{\hat{m}_{t+1},i}\overline{d}_i.\end{aligned}$$

Non-diagonal subplans

In this section, we focus on non-diagonal subplans, i.e., $\hat{m}_{t+1} = \hat{l}_{t+1} = t + 1$. Note that if $\hat{l}_k = \hat{l}_{t+1}$, then (3.7) still holds. Therefore, we will assume that $\hat{l}_k \leq t$, i.e., there is at least one second-level production period in $[k, t]$.

For given k, $\hat{m}_k$ and $\hat{l}_k$, let us define $\Phi(t) = \phi(k, \hat{m}_k, \hat{l}_k, t, t, t)$. If $\hat{m}_k < k$, we have

$$\begin{aligned}\Phi(\hat{l}_k) &= \overline{f}_k + \sum_{i=k}^{\hat{l}_k-1} \overline{\alpha}_{ki}d_i^\star - \sum_{i=k}^{\hat{l}_k-1} \underline{\alpha}_{\hat{m}_k,i}d_i^\star \\ \Phi(t+1) &= \max\{\max_{\hat{l}_k\leq\hat{m}_t<t} [\phi(k, \hat{m}_k, \hat{l}_k, t, \hat{m}_t, t+1) + \\ &\quad + (\overline{\alpha}_{kt} - \underline{\alpha}_{\hat{m}_t,t})d_t^\star],\ \Phi(t) - \overline{f}_t +\end{aligned}$$

$$+(\overline{\alpha}_{kt} - \underline{\alpha}_{tt})d_t^\star\},$$

for $t = \hat{l}_k, \ldots, n$, where the direction of bounds of α follows from $k < \hat{m}_t \leq t$, and $d_i^\star$ is determined using Lemma 3.1. If $\hat{m}_k = k$, the first period to be considered by the recursion is $\hat{l}_k + 1$, and the initial condition should be replaced by $\Phi(\hat{l}_k + 1) = 0$.

3.3.4 The running time

In this section we show that all values of ϕ can be calculated in $\mathcal{O}(n^6)$ time. Note that $\overline{\mathcal{L}}_i^j$, $\underline{\mathcal{L}}_i^j$ and $\sum_{s=i}^{j} \alpha_{rs}^\star d_s^\star$ can be calculated in $\mathcal{O}(n^3)$ time for all values of $r \leq i \leq j$ in a preprocessing procedure. For diagonal subplans, ϕ can be calculated in constant time, except for Cases II(b)(iii) and III(b)(ii), in which it takes $\mathcal{O}(n)$ time, and for Case III(b)(iii), in which it takes $\mathcal{O}(n^2)$ time. The desired running time follows by observing that there are $\mathcal{O}(n^4)$ Case III subplans, while for Case II(b)(iii) there are only $\mathcal{O}(n^5)$ relevant subplans since

$$\phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, a+1) = \phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, a), \tag{3.9}$$

with $a > t + 1$. Once the values of ϕ for all diagonal subplans are available, it is straightforward to see from Section 3.3.3 that all non-diagonal subplans can be handled in $\mathcal{O}(n^5)$ time.

3.4 Single type uncertainty

In this section, an $\mathcal{O}(n^5)$ time algorithm is proposed for demand uncertainty only (the MRULS^d model), and an $\mathcal{O}(n^4)$ time one for production cost function uncertainty only (the MRULS^p model).

3.4.1 Demand uncertainty

When only the demand is uncertain, the nodes of the network proposed in Section 3.3 can be simplified to $(k, \hat{m}_k)$ and the number of subplans is reduced to $\mathcal{O}(n^4)$. Let $\phi^d(k, \hat{m}_k, t+1, \hat{m}_{t+1})$ denote the minimax regret for periods $[k, t]$ within the subplan $[(k, \hat{m}_k), (t+1, \hat{m}_{t+1})]$. By including the $\mathcal{O}(n^3)$ time preprocessing procedure in Section 3.3, we show below that

any diagonal subplan can be computed in at most $\mathcal{O}(n^2)$ time. Noting that

$$\phi^d(k, \hat{m}_k, t+1, b) = \phi^d(k, \hat{m}_k, t, b) + (\alpha_{kt} - \alpha_{bt})d_t^\star,$$

for $k \leq b \leq t$, all diagonal subplans can be handled in $\mathcal{O}(n^5)$ time, and the desired running time for the MRULS^d model follows. To calculate $\phi^d(k, \hat{m}_k, t+1, \hat{m}_{t+1})$, $\hat{m}_{t+1} \leq t$, we use Proposition 3.5.

Case a: $\hat{m}_k = \hat{m}_{t+1}$

Please note that $\hat{m}_k = \hat{m}_{t+1}$ implies $\hat{m}_{t+1} \leq t$. We have

$$\phi^d = \begin{cases} f_k + \sum_{i=k}^{t} \alpha_{ki}\overline{d}_i - \sum_{i=k}^{t} \alpha_{\hat{m}_k,i}\overline{d}_i & \text{if } \alpha_{kk} > \alpha_{\hat{m}_k,k} \\ f_k + \sum_{i=k}^{t} \alpha_{ki}\underline{d}_i - \sum_{i=k}^{t} \alpha_{\hat{m}_k,i}\underline{d}_i & \text{if } \alpha_{kk} \leq \alpha_{\hat{m}_k,k} \\ & \text{and } \hat{m}_k < k \\ 0 & \text{if } \alpha_{kk} \leq \alpha_{\hat{m}_k,k} \\ & \text{and } \hat{m}_k = k. \end{cases}$$

Case b: $\hat{m}_k < \hat{m}_{t+1} \leq t$

• Case b(i): $\alpha_{kk} > \alpha_{\hat{m}_k,k}$

In this case, $\hat{m}_k < k$. If η is the first second-level production period in $[k+1, \hat{m}_{t+1}]$, then

$$\begin{aligned} \phi^d = & f_k + \sum_{i=k}^{t} \alpha_{ki}\overline{d}_i - \min_{k<\eta\leq\hat{m}_{t+1}} [\sum_{i=k}^{\eta-1} \alpha_{\hat{m}_k,i}\overline{d}_i + \overline{\mathcal{L}}_\eta^{\hat{m}_{t+1}-1}] - \\ & - f_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \alpha_{\hat{m}_{t+1},i}\overline{d}_i. \end{aligned}$$

• Case b(ii): $\alpha_{kk} \leq \alpha_{\hat{m}_k,k}$ and $\alpha_{k,\hat{m}_{t+1}} \leq \alpha_{\hat{m}_{t+1},\hat{m}_{t+1}}$

In this case, ϕ^d is similar to that in Case b(i), where $(d_k^\star, \ldots, d_t^\star) = (\underline{d}_k, \ldots, \underline{d}_t)$. Thus,

$$\begin{aligned}\phi^d =& f_k + \sum_{i=k}^{t} \alpha_{ki}\underline{d}_i - \min_{k<\eta\leq\hat{m}_{t+1}} [\sum_{i=k}^{\eta-1} \alpha_{\hat{m}_k,i}\underline{d}_i + \underline{\mathcal{L}}_\eta^{\hat{m}_{t+1}-1}] - \\ & - f_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \alpha_{\hat{m}_{t+1},i}\underline{d}_i\end{aligned}$$

if $\hat{m}_k < k$, and

$$\phi^d = f_k + \sum_{i=k}^{t} \alpha_{ki}\underline{d}_i - \underline{\mathcal{L}}_k^{\hat{m}_{t+1}-1} - f_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \alpha_{\hat{m}_{t+1},i}\underline{d}_i$$

if $\hat{m}_k = k$.

- Case b(iii): $\alpha_{kk} \leq \alpha_{\hat{m}_k,k}$ and $\alpha_{k,\hat{m}_{t+1}} > \alpha_{\hat{m}_{t+1},\hat{m}_{t+1}}$

In addition to η, and similarly to Cases IIb(iii) and IIIb(iii) in Section 3.3.3, we need to optimize over candidates for γ. Therefore,

$$\begin{aligned}\phi^d =& f_k + \max_{\substack{k<\eta\leq\gamma \\ \eta\leq\gamma\leq\hat{m}_{t+1} \\ \alpha_{k\gamma}>\alpha_{\gamma\gamma}}} [\sum_{i=k}^{\gamma-1} \alpha_{ki}\underline{d}_i + \sum_{i=\gamma}^{t} \alpha_{ki}\overline{d}_i - \sum_{i=k}^{\eta-1} \alpha_{\hat{m}_k,i}\underline{d}_i - \\ & - \underline{\mathcal{L}}_\eta^{\gamma-1} - \overline{\mathcal{L}}_\gamma^{\hat{m}_{t+1}-1}] - f_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \alpha_{\hat{m}_{t+1},i}\overline{d}_i\end{aligned}$$

if $\hat{m}_k < k$, and

$$\begin{aligned}\phi^d =& f_k + \max_{\substack{k\leq\gamma\leq\hat{m}_{t+1} \\ \alpha_{k\gamma}>\alpha_{\gamma\gamma}}} [\sum_{i=k}^{\gamma-1} \alpha_{ki}\underline{d}_i + \sum_{i=\gamma}^{t} \alpha_{ki}\overline{d}_i - \underline{\mathcal{L}}_k^{\gamma-1} - \overline{\mathcal{L}}_\gamma^{\hat{m}_{t+1}-1}] - \\ & - f_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \alpha_{\hat{m}_{t+1},i}\overline{d}_i\end{aligned}$$

if $\hat{m}_k = k$.

3.4.2 Production cost function uncertainty

The nodes of the network for the MRULS^p model can be simplified to $(k, \hat{l}_k)$ and again the number of subplans is reduced to $\mathcal{O}(n^4)$. Let $\phi^p(k, \hat{l}_k, t+1, \hat{l}_{t+1})$ denote the minimax regret for periods $[k, t]$ within the subplan $[(k, \hat{l}_k), (t+1, \hat{l}_{t+1})]$, and $\mathcal{L}^p(i, t, j)$, $i \leq t \leq j$, denote the optimal objective function value of a ULS model from periods i to t with demands $(d_i, \ldots, d_{t-1}, d_{t,j-1})$ and setup and unit production costs at their lower bounds. We have

$$\phi^p = \begin{cases} \overline{f}_k + \overline{p}_k d_{kt} & \text{if } \hat{l}_k = \hat{l}_{t+1} \\ \overline{f}_k + \overline{p}_k d_{kt} - \mathcal{L}^p(\hat{l}_k, t, \hat{l}_{t+1}) & \text{if } k < \hat{l}_k < \hat{l}_{t+1} \\ \max\{\underline{p}_k d_{t+1,\hat{l}_{t+1}-1}, \max_{k<\theta\leq t}[\overline{p}_k d_{\theta t} - & \\ -\mathcal{L}^p(\theta, t, \hat{l}_{t+1})]\} & \text{if } k = \hat{l}_k < \hat{l}_{t+1}. \end{cases}$$

For a given value of j, $\mathcal{L}^p(i, t, j)$ can be calculated in $\mathcal{O}(n^3)$ time for all values of i and t. Hence we can calculate all $\mathcal{L}^p(i, t, j)$ in $\mathcal{O}(n^4)$ time in a preprocessing procedure. It is then straightforward to see that $\phi^p(k, \hat{l}_k, t+1, \hat{l}_{t+1})$ can be calculated in constant time for $\hat{l}_k = \hat{l}_{t+1}$ and $k < \hat{l}_k < \hat{l}_{t+1}$. For $k = \hat{l}_k < \hat{l}_{t+1}$, the running time is $\mathcal{O}(n)$, but there are only $\mathcal{O}(n^3)$ of these subplans. Therefore, the overall running time remains at $\mathcal{O}(n^4)$, and the desired running time for the MRULS^d model follows.

3.5 Summary

In this chapter, we have formulated the minimax regret ULS model with both production cost function and demand uncertainty. We gave a theoretical analysis to reduce the number of optimal uncertainty scenarios and proposed an algorithm that solves the problem in $\mathcal{O}(n^6)$ time. Lower running times were found for the case in which only one single type of uncertainty is present. In particular, an $\mathcal{O}(n^4)$ time algorithm is proposed for production cost function uncertainty only, and an $\mathcal{O}(n^5)$ time one for demand uncertainty only. The latter reduces the running time of the $\mathcal{O}(n^7)$ time algorithm proposed in [184]. If the two-stage assumption is not realistic, and instead, the uncertain data is realised in multiple stages, we can still derive polynomial algorithms for the multi-stage MRULS model by solving a collection of two-stage models. These new algorithms are therefore linear in the number of stages and polynomial in the number of periods.

Chapter 4

The Lot Sizing Model with Uncertainty Budget

In this chapter, we study a robust lot sizing model where the robustness is modeled using the absolute criterion [159]. As explained in Chapter 1, robust optimization models with an uncertainty budget effectively reduce the over-conservatism of the classic absolute robust approach proposed by Soyster [159]. We adopt the idea of uncertainty budget and propose a robust lot sizing model that minimizes the worst-case total cost due to uncertainty. Unlike in the classic robust optimization literature, in this study we define the worst case that could occur, due to the uncertain data, as when it harms the objective function value the most. This reflects the cost-concern attitude in the production planning process in real life and, at the same time provides a production plan that is robust for most data realizations. Polynomial time algorithms are proposed and numerical results are presented in Chapter 5. The study in this chapter can be found in [123].

4.1 Introduction

Recall from Chapter 2 that when backlogging is available, Bertsimas and Thiele [40] study the robust lot sizing model with uncertainty budget in which demand is uncertain, which is

formulated in (2.29)-(2.34). For convenience, we give the formulation here again,

$$
\begin{aligned}
BTRULS: \quad & \min_{\mathbf{y},\mathbf{x},\mathbf{q}} \sum_{i=1}^{n} (f_i y_i + p_i x_i + q_i) \\
\text{s.t.} \quad & q_i \geq h_i(x_{1i} - d^{\star}_{1i}),\ \forall i = 1, \ldots, n \\
& q_i \geq -h_i^{-}(x_{1i} - d^{\star}_{1i}),\ \forall i = 1, \ldots, n \\
& x_i \leq M\, y_i,\ \forall i = 1, \ldots, n \\
& y_i \in \{0, 1\},\ \forall i = 1, \ldots, n \\
& x_i \geq 0,\ \forall i = 1, \ldots, n,
\end{aligned}
\tag{4.1}
$$

where $d^{\star}_{1i}$ is the optimal objective function value of the following maximisation problem,

$$
\begin{aligned}
d^{\star}_{1i} \quad := \quad & \max_{\mathbf{w}} \{\tilde{d}_{1i} + \sum_{t=1}^{i} w_t \hat{d}_t\} \\
& \text{s.t. } 0 \leq w_t \leq 1,\ \forall t = 1, \ldots, i \\
& \sum_{t=1}^{i} w_t \leq \Gamma_i.
\end{aligned}
$$

For the maximisation problem, $d^{\star}_{1i}$ is the worst-case cumulative demand that could occur subject to Γ_i, the cumulative uncertainty budget up to period i. When cost parameters are time-invariant, the authors show that the optimal policy obtained in this robust approach is identical to the optimal policy obtained in the deterministic case for a modified and explicitly computable demand vector. This result is in agreement with that in Atamtürk and Zhang [17]. This property does not hold for general cost parameters.

Bertsimas and Thiele [40] examine the relative performance of the lot sizing model with budget of uncertainty compared to the stochastic programming lot sizing model. They consider various values for the time-invariant cost parameters of the system, and study several (assumed and realised) distributions of the demand. The stochastic policy is first computed using a binomial distribution, and then using an approximation to the Gaussian distribution. The experiment shows that whether the stochastic policy or the robust policy performs better is by and large not influenced by the unit production costs $\mathbf{p}$, but strongly affected by the size of the unit inventory holding costs $\mathbf{h}$. In addition, the authors notice that there is no significant difference between the policies when the assumed input distribution of the stochastic policy is Gaussian, and that the robust policy leads to significant cost

reduction when the assumed input distribution of the stochastic policy is binomial. Alem and Morabito [10, 9] study an extension of the BTRULS model by considering capacity and storage constraints, and overtime. When applied in the Brazil furniture industry, the authors provide some insights into the relationship between the budgets of uncertainty, fill rates (the fraction of satisfied demand), and total cost. In addition, the authors compare the performance of the lot sizing model with budget of uncertainty with that of Soyster's absolute robust model in several parameter settings. Numerical evidence indicates the uncertainty budget model outperforms Soyster's model in both fill rates and the total cost if the storage constraints are removed. Both models benefit from expanding the capacities by giving very good fill rates with reduced total cost. However, it is noticeable that level of backlogged demand may reach a very high level with multiple parameter settings. In particular, if overtime is not allowed, both robust models fail to provide solutions with fill rates above 40%.

In this chapter, we adopt the appealing idea of uncertainty budget and study a robust lot sizing model. We aim to find a production plan that minimizes the total cost, over the most costly data realisations, i.e., we guide the allocation of worst-case periods by objective cost function. We consider two types of uncertain parameters, namely the production cost function and the demand. For each of them, we impose an uncertainty budget which controls the overall uncertainty level. We first study the ULS model with both production cost function and demand uncertainty budget (ULSUB). An $\mathcal{O}((\Gamma^d)^2 n^2)$ time algorithm, is proposed. Then we extend the study by considering constant capacity (the CCLSUB model), and propose an $\mathcal{O}(\Gamma^p(\Gamma^d)^2 n^4)$ time algorithm to solve the problem with the assumption that the deviation components $\hat{d}_i$ are time-invariant, i.e., $\hat{d}_i = \hat{d}$. Finally we show that when only a single type of uncertainty exists, i.e., either the production cost function or the demand, the running times for both the ULSUB model and the CCLSUB model can be improved.

The remainder of the chapter is organised as follows. We formulate the robust lot sizing model with uncertainty budget in Section 4.2. To solve the problem, we propose a Dynamic Programming (DP) solution approach in Section 4.3. In Section 4.4, we first solve the ULSUB model with both production cost function and demand uncertainty in $\mathcal{O}((\Gamma^d)^2 n^2)$ time. Then an $\mathcal{O}(\Gamma^p(\Gamma^d)^2 n^4)$ time algorithm is proposed for the CCLSUB model. In Section 4.5, we show that when only a single type of uncertainty exists, the running times for both models can be improved. In Section 4.6, a summary of the paper is given and lines of future research are proposed.

4.2 The formulation

In this section, we give the formulations for our robust lot sizing model. We consider the case in which both the production cost function and the demand are uncertain, i.e., $u_i = (f_i, p_i, d_i)$ such that $f_i \in [\tilde{f}_i - \hat{f}_i, \tilde{f}_i + \hat{f}_i]$, $p_i \in [\tilde{p}_i - \hat{p}_i, \tilde{p}_i + \hat{p}_i]$, and $d_i \in [\tilde{d}_i - \hat{d}_i, \tilde{d}_i + \hat{d}_i]$. We denote by $\overline{f}_j = \tilde{f}_j + \hat{f}_j$, $\overline{p}_j = \tilde{p}_j + \hat{p}_j$, and $\overline{p}_j = \tilde{p}_j + \hat{p}_j$ the upper bounds of the intervals. Let $\mathbf{\Gamma} = (\Gamma^p, \Gamma^d)$ be the uncertainty budget vector, in which $\Gamma^p \in \{1, \ldots, n\}$, as in Bertsimas and Sim [39], is the uncertainty budget for the production cost function $(\mathbf{f}, \mathbf{p})$, and $\Gamma^d \in \mathbb{R}$ and $0 \leq \Gamma^d \leq n$, as in Bertsimas and Thiele [40], is the uncertainty budget for the demand.

The robust lot sizing model with uncertainty budget is a minmax one, in which we aim to find a production plan that minimizes the total cost, over the worst case of possible data realisations. The problem under consideration is a two-stage one. The decision $\mathbf{y}$ is made at the first stage before we know the demand realisation. The production and the inventory decisions ($\mathbf{x}$, $\mathbf{I}$) are made at the second stage, based on the demand realisation. Hence the model reads as follows,

$$LSUB: \quad \min_{\mathbf{y},\mathbf{x},\mathbf{I}} \max_{\mathbf{z}^p,\mathbf{z}^d} \sum_{i=1}^{n} [(\tilde{f}_i + \hat{f}_i z_i^p) y_i + (\tilde{p}_i + \hat{p}_i z_i^p) x_i(\mathbf{y}, \mathbf{d}) + h_i I_i(\mathbf{y}, \mathbf{d})] \tag{4.2}$$

$$\text{s.t.} \quad I_{i-1} + x_i = I_i + \tilde{d}_i + \hat{d}_i z_i^d, \, \forall i = 1, \ldots, n \tag{4.3}$$

$$x_i \leq Cap\, y_i, \, \forall i = 1, \ldots, n \tag{4.4}$$

$$\sum_{i=1}^{n} z_i^p \leq \Gamma^p, \tag{4.5}$$

$$\sum_{i=1}^{n} z_i^d \leq \Gamma^d, \tag{4.6}$$

$$y_i \in \{0, 1\}, \, \forall i = 1, \ldots, n \tag{4.7}$$

$$x_i \geq 0, \, \forall i = 1, \ldots, n \tag{4.8}$$

$$z_i^p \in \{0, 1\}, \, \forall i = 1, \ldots, n \tag{4.9}$$

$$0 \leq z_i^d \leq 1, \, \forall i = 1, \ldots, n. \tag{4.10}$$

This formulation resembles the one for the LS model in (2.1)-(2.5). Since the lot sizing model is a cost-minimization model, the worst-case is that the production cost function and the demand rise from the nominal value. In this formulation, $z_i^p > 0$ if the production cost

function in period i rises from the nominal value and similarly for $\mathbf{z}^d$. The objective function now includes a maximisation problem over the non-linear terms $\hat{f}_i z_i^p y_i$ and $\hat{p}_i z_i^p x_i$ in order to determine the worst case of setup and unit production cost realisations. Constraints (4.3) take into account of the possible deviation of demand by modeling demand in each period as $\tilde{d}_i + \hat{d}_i z_i^d$, $i = 1, \ldots, n$. Constraints (4.5) and (4.6) impose the upper bounds on total uncertainty. The rest of the constraints are as in the deterministic lot sizing problem. When $Cap = M$ for some large constant M, the production quantity is not restricted, and we denote this by ULSUB. When $Cap = C$, i.e., the production quantity in any period cannot exceed C, we denote this by CCLSUB. Please note that when $\Gamma^p = \Gamma^d = 0$, we have the deterministic lot sizing problem. In the rest of this chapter, we assume that $\Gamma^p > 0$ and $\Gamma^d > 0$ unless otherwise stated. In addition, we call the objective function a minmax cost function and drop the dependency on $\mathbf{x}$ and $\mathbf{I}$ therein.

Let $\mathcal{U}^p \subset \{1, \ldots, n\}$, with $|\mathcal{U}^p| \leq \Gamma^p$, be the set of periods i with $z_i^p > 0$, i.e., the set of periods in which the production cost function rises in order to adversely affect the objective function value of the solution [38]. Similarly, we define $\mathcal{U}^d \subset \{1, \ldots, n\}$, with $|\mathcal{U}^d| \leq \lceil \Gamma^d \rceil$, for the demand. We call such periods $\boldsymbol{p}$-worst-case periods and $\boldsymbol{d}$-worst-case periods respectively.

As in Chapter 3, we denote by $\alpha_{ki} = p_k + h_{k,i-1}$ the cost of delivering a unit of demand in period i from production in period k. We let $\overline{\alpha}_{ki} = \overline{p}_k + h_{k,i-1}$ and $\underline{\alpha}_{ki} = \underline{p}_k + h_{k,i-1}$ respectively. To construct a solution approach to LSUB, we observe the following property:

Lemma 4.1. *Given a setup decision vector* $\mathbf{y}$ *and a realisation of production cost function* $(\mathbf{f}, \mathbf{p})$*, there exists an optimal allocation decisions of* $\boldsymbol{d}$*-worst-case periods,* $(\mathbf{z}^d)^\star$*, satisfying that: there exists a period* k *such that* $(z_k^d)^\star = \Gamma^d - \lfloor \Gamma^d \rfloor$*, and* $(z_i^d)^\star \in \{0, 1\}$ *for* $i \in \{1, \ldots, n\} \setminus \{k\}$.

Proof. By construction of the model, the worst case for demand is that the demand rises. So $\sum_{i=1}^n (z^d)^* = \Gamma^d$. Assume that the statement does not hold. We assume without loss of generality that there exist periods l and k, such that $0 < (z_l^d)^\star < 1$ and $0 < (z_k^d)^\star < 1$, and $(z_i^d)^\star \in \{0, 1\}$ for all other periods. We let m_l and m_l denote the production periods for demand in period l and k respectively. Then if $\alpha_{m_l,l} = \alpha_{m_k,k}$, there exists an alternative optimal solution to $(\mathbf{z}^d)^\star$, $(\mathbf{z}^d)'$, such that $(z_i^d)' = (z_i^d)^*$ for $i \in \{1, \ldots, n\} \setminus \{l, k\}$, $(z_l^d)' = 1$,

and $(z_k^d)' = \Gamma^d - \lfloor \Gamma^d \rfloor$. Then, we can choose $(\mathbf{z}^d)'$ over $(\mathbf{z}^d)^*$ without loss of optimality.

If $\alpha_{m_l,l} > \alpha_{m_k,k}$, the same $(\mathbf{z}^d)'$ leads to an increased objective function value, which contradicts with the fact that $(\mathbf{z}^d)^\star$ is optimal. If $\alpha_{m_l,l} < \alpha_{m_k,k}$, using the same argument we see that modifying $(\mathbf{z}^d)'$, such that $(z_k^d)' = 1$, and $(z_l^d)' = \Gamma^d - \lfloor \Gamma^d \rfloor$, again leads to an increased objective function value. Hence, we arrive again at the contradiction that $(\mathbf{z}^d)^\star$ is optimal. □

The property shows that the model is protected against the most costly case in which up to $\lfloor \Gamma^d \rfloor$ of these uncertain data are allowed to change, i.e., taking the upper bound of the uncertainty interval (and in one period k such that d_k changes by $(\Gamma^d - \lfloor \Gamma^d \rfloor)\hat{d}_k$).

In the following section, we construct a general DP algorithm to solve the proposed robust lot sizing model. We decompose this DP into an outer DP and an inner DP. The outer DP decomposes the planning horizon into regeneration intervals, whilst the inner DP calculates the minmax cost of the regeneration intervals.

4.3 The algorithm

As shown in Chapter 2, the vector $\mathbf{y}$ splits the planning horizon into a sequence of regeneration intervals, where no inventory enters or leaves the intervals and the inventory levels within the interval are positive. In this section we propose a DP algorithm that decomposes the LSUB problem into subplans.

To begin with, we define $\mathbf{1}$ as a vector of the same length of $\mathbf{\Gamma}$, with each element in the vector being 1, and $\mathbf{i}$ as $i \cdot \mathbf{1}$. We construct a network with nodes $(k, \mathbf{\Gamma}_{k-1})$ in which the DP moves from node $(k, \mathbf{\Gamma}_{k-1})$ to $(t, \mathbf{\Gamma}_t)$, defining the subplan $[(k, \mathbf{\Gamma}_{k-1}), (t, \mathbf{\Gamma}_t)]$, for $1 \leq k \leq t \leq n$, $\mathbf{0} \leq \mathbf{\Gamma}_{k-1} \leq \mathbf{\Gamma}_t \leq \mathbf{\Gamma}$. In this subplan, $\mathbf{\Gamma}_i \leq \mathbf{i} - 1$ denotes the uncertainty budget vector in $[1, i-1]$ for $i = 2, \ldots, n+1$ with $\mathbf{\Gamma}_1 = \mathbf{0}$. As incoming inventory into the regeneration interval $[k, t]$ is always zero, none of the production decisions taking in the previous subplan affect the regeneration interval $[k, t]$. Hence given $\mathbf{\Gamma}_{k-1}$ and $\mathbf{\Gamma}_t$ assigning worst-case periods in one subplan is independent from assigning in the others. An illustra-

tion of the subplan $[(k, \mathbf{\Gamma}_{k-1}), (t, \mathbf{\Gamma}_t)]$ is given in Figure 4.1.

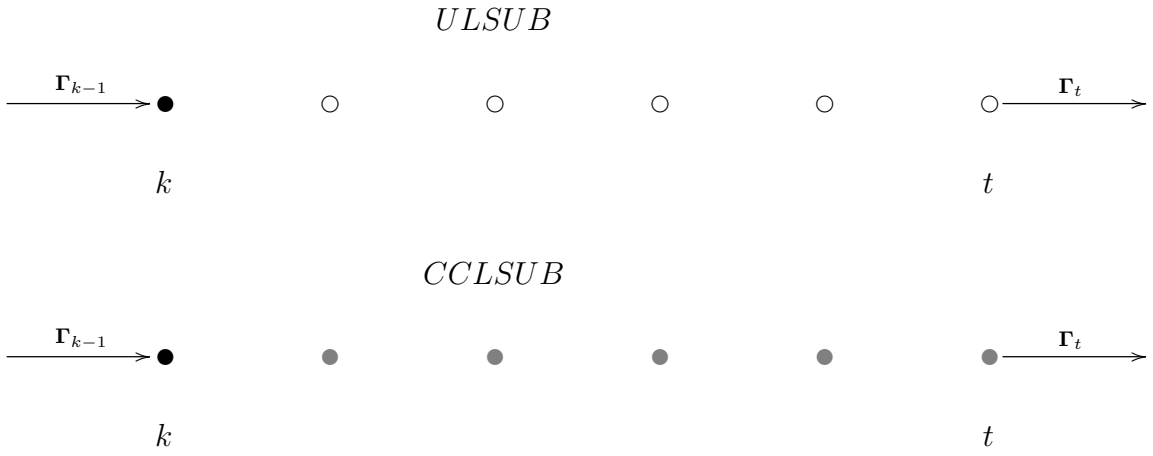


Figure 4.1: An illustration of the subplan $[(k, \mathbf{\Gamma}_{k-1}), (t, \mathbf{\Gamma}_t)]$: ● production period with zero incoming inventory, ○ non-production periods, ● status unknown.

In $[(k, \mathbf{\Gamma}_{k-1}), (t, \mathbf{\Gamma}_t)]$, $\mathbf{\Delta} = \mathbf{\Gamma}_t - \mathbf{\Gamma}_{k-1}$, and denote by $\phi(k, t, \mathbf{\Delta})$ the minmax cost of the subplan with uncertainty budget vector of size $\mathbf{\Delta}$. Let $G(t, \mathbf{\Gamma}_t)$ be the minmax cost of the LSUB model over the first t periods with uncertainty budget of $\mathbf{\Gamma}_t$, where $1 \leq t \leq n$ and $\mathbf{0} \leq \mathbf{\Gamma}_t \leq \mathbf{\Gamma}$, and $S(k, t, \mathbf{\Gamma}_t)$ be the minmax cost over the first t periods with uncertainty budget of $\mathbf{\Gamma}_t$, with the additional information that $I_{k-1} = 0$ and $I_i > 0$ for $k \leq i < t$, i.e., $[k, t]$ is the last regeneration interval in $[1, t]$. We define a sink node at $(1, \mathbf{0})$ with $G(1, \mathbf{0}) = S(1, 1, \mathbf{0}) = 0$. Then by construction, the optimal solution to the LSUB model is equal to $G(n, \mathbf{\Gamma})$. We can derive a recursion for $G(t, \mathbf{\Gamma}_t)$ and $S(k, t, \mathbf{\Gamma}_t)$:

$$G(1, \mathbf{0}) = S(1, 1, \mathbf{0}) = 0 \tag{4.11}$$

$$G(t, \mathbf{\Gamma}_t) = \min_{k:\, 1 \leq k \leq t} S(k, t, \mathbf{\Gamma}_t) \tag{4.12}$$

$$S(k, t, \mathbf{\Gamma}_t) = \max_{\mathbf{\Delta}:\, \mathbf{0} \leq \mathbf{\Delta} \leq \min\{(t-k)\cdot\mathbf{1}, \mathbf{\Gamma} - \mathbf{\Gamma}_{k-1}\}} \{G(k, \mathbf{\Gamma}_t - \mathbf{\Delta}) + \phi(k, t, \mathbf{\Delta})\}. \tag{4.13}$$

For uncertain production cost function with budget Γ^p, it is easy to show that we can assume without loss of optimality that $\Gamma^p_t \in \mathbb{Z}$. Moreover, if $\Gamma^d \in \mathbb{Z}$, a similar assumption can be made on Γ^d_t. Then, for $\mathbf{\Gamma} = (\Gamma^p, \Gamma^d) \in \mathbb{Z} \times \mathbb{Z}$, the LSUB model is also polynomially solvable if $\phi(k, t, \mathbf{\Delta})$ are polynomially solvable for all subplans $[(k, \mathbf{\Gamma}_{k+1}), (t, \mathbf{\Gamma}_t)]$.

In general, if $\Gamma^d \in \mathbb{R}$, Lemma 4.1 indicates that there exists an optimal path from $(1, \mathbf{\Gamma}_1)$ to $(n, \mathbf{\Gamma})$ such that in at most one subplan $[(k, \mathbf{\Gamma}_{k-1}), (t, \mathbf{\Gamma}_t)]$ a $\boldsymbol{d}$-worst-case period, say j, $k \leq j \leq t$, takes $d_j = \tilde{d}_j + (\Gamma^d - \lfloor \Gamma^d \rfloor)\hat{d}_j$, and in all other periods demand takes either upper bound of the uncertainty interval or the nominal value. As a result, we can split Γ^d into two parts. We let $\Gamma^{int} = \lfloor \Gamma^d \rfloor$ and $\Gamma^{frac} = \lceil \Gamma^d \rceil - \lfloor \Gamma^d \rfloor$. Similarly, we define $(\Gamma_t^{int}, \Gamma_t^{frac})$ and $(\Delta^{int}, \Delta^{frac})$. Then by letting $\mathbf{\Gamma} = (\Gamma^p, \Gamma^{int}, \Gamma^{frac})$ and $\mathbf{\Delta} = (\Delta^p, \Delta^{int}, \Delta^{frac})$, the LSUB model is again polynomially solvable if all subplan costs $\phi(k, t, \mathbf{\Delta})$ can be calculated in polynomially time. In the rest of this chapter, we handle the worst general case, i.e., $\Gamma^d \in \mathbb{R}$.

4.4 Minmax cost of a subplan

In this section we calculate $\phi(k, t, \mathbf{\Delta})$. We denote by $\mathcal{P}_{kt}$ with $|\mathcal{P}_{kt}| = P_{kt}$ the set of production periods in $[k, t]$.

It is obvious that a worst-case period only contributes to the objective function if it is also a production period. Hence, within the subplan, if $P_{kt} \leq \Delta^p$, the optimal allocation of worst-case periods must be a subset of the production periods. In addition, if $P_{kt} > \Delta^p$, it is easy to show that $\phi^p(k, t, \mathbf{\Delta}) = \phi^p(k, t, (P_{kt}, \Delta^{int}, \Delta^{frac}))$. For example, consider a 3-period ULSUB^p subplan $[(1, 0, 0, 0), (3, \Gamma_t^p, 0, 0)]$, with $\tilde{f}_i = \tilde{f}$ and $\hat{f}_i = \hat{f}$, $\tilde{p}_i = \tilde{p}$ and $\hat{p}_i = \hat{p}$, and $h_i = 0$ for $i = 1, 2, 3$. Then, for $\Gamma_t^p = 1$, the optimal setup decision vector $\mathbf{y}^\star = (1, 0, 0)$ with $\mathbf{x}^\star = (\tilde{d}_{13}, 0, 0)$ and a total cost of $\phi^p(1, 3, 1) = \overline{f}_1 + \overline{p}_1 \tilde{d}_{13}$. If we increase Γ_{t+1}^p to 2 or 3, the optimal solution remains unchanged, i.e., $\phi^p(1, 3, 1, 0, 0) = \phi^p(1, 3, 2, 0, 0) = \phi^p(1, 3, 3, 0, 0)$. Unlike the production cost uncertainty case, the $\boldsymbol{d}$ worst-case periods are not limited to production periods.

4.4.1 The ULSUB model

As shown in Chapter 2, we can assume without loss of optimality that period k is the only production period within the subplan since the model is uncapacitated. As a result, the total unit cost to deliver one unit in period $i \in [k, t]$ is α_{ki}. We show that by including an $\mathcal{O}(\Gamma^d n)$ time preprocessing procedure, the $\phi(k, t, \mathbf{\Delta})$ can be calculated in constant time.

Indeed, we have that the minmax cost of the subplan is known ahead and equal to

$$\phi(k,t,\mathbf{\Delta}) = \begin{cases} \tilde{f}_k + \sum_{i=k}^{t} \tilde{\alpha}_{ki} d_i + \tilde{E}(k,t,\Delta^{int},\Delta^{frac}) & \text{if } \Delta^p = 0 \\ \overline{f}_k + \sum_{i=k}^{t} \overline{\alpha}_{ki} d_i + \overline{E}(k,t,\Delta^{int},\Delta^{frac}) & \text{if } \Delta^p \geq 1, \end{cases}$$

where

$$\tilde{E}(k,t,\Delta^{int},\Delta^{frac}) := \max_{\substack{\mathbf{0}\leq \mathbf{z}^d \leq \mathbf{1} \\ \sum_{i=k}^{t} z_i^d = \Delta^d}} \sum_{i=k}^{t} (\tilde{\alpha}_{ki}\hat{d}_i) z_i^d,$$

$$\overline{E}(k,t,\Delta^{int},\Delta^{frac}) := \max_{\substack{\mathbf{0}\leq \mathbf{z}^d \leq \mathbf{1} \\ \sum_{i=k}^{t} z_i^d = \Delta^d}} \sum_{i=k}^{t} (\overline{\alpha}_{ki}\hat{d}_i) z_i^d.$$

When $\Delta^{frac} = 0$, function $\tilde{E}(k,t,\Delta^{int},\Delta^{frac})$ gives the maximum cost incurred in $[(k,\Gamma_{k-1}),(t,\Gamma_t)]$ by setting Δ^{int} demands to upper bound. For given values of k, t, and Δ^{int}, calculating $\tilde{E}(k,t,\Delta^{int},0)$ is equivalent to a ranking problem, which can be solved in $\mathcal{O}(n)$ time. By calculating $\tilde{E}(k,n,\Delta^{int},0)$ first and deleting the least value of $\tilde{\alpha}_{ki}\hat{d}_i$, $\tilde{E}(k,t,\Delta^{int},0)$ can be obtained in $\mathcal{O}(n)$ time for all values of t. When $\Delta^{frac} = 1$ solving $\tilde{E}(k,t,\Delta^{int},\Delta^{frac})$ is equivalent to solving $\tilde{E}(k,t,\Delta^{int}+1,0)$, where the least value of $\tilde{\alpha}_{ki}\hat{d}_i$ is modified to $(\Gamma^d - \lfloor\Gamma^d\rfloor)\tilde{\alpha}_{ki}\hat{d}_i$. The same running time can be derived for $\overline{E}(k,t,\Delta^{int},\Delta^{frac})$. Therefore, all values of $\tilde{E}(k,t,\Delta^{int},\Delta^{frac})$ and $\overline{E}(k,t,\Delta^{int},\Delta^{frac})$ can be obtained in an $\mathcal{O}(\Gamma^d n)$ time preprocessing procedure.

4.4.2 The CCLSUB model

We assume that the deviations of demands $\hat{d}_i$ are time-invariant, i.e., $\hat{d}_i = \hat{d}$ for $i = 1,\ldots,n$. Then, with given $(\Delta^{int},\Delta^{frac})$, we known immediately the total demand being produced in $[k,t]$, i.e., $\tilde{d}_{kt} + \Delta^{int}\hat{d} + \Delta^{frac}(\Gamma^d - \lfloor\Gamma^d\rfloor)\hat{d}$. In addition, recall from Chapter 2 that within the regeneration interval $[k,t]$ we can assume there is at most one free production period with production quantity below capacity. Hence, $[k,t]$ consists of $P_{kt} = \lceil \frac{\tilde{d}_{kt}+\Delta^{int}\hat{d}+\Delta^{frac}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}}{C} \rceil$ production periods, in which $\lfloor \frac{\tilde{d}_{kt}+\Delta^{int}\hat{d}+\Delta^{frac}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}}{C} \rfloor$ periods produce at full capacity C, and one period at $\pi = \tilde{d}_{kt}+\Delta^{int}\hat{d}+\Delta^{frac}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d} - \lfloor \frac{\tilde{d}_{kt}+\Delta^{int}\hat{d}+\Delta^{frac}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}}{C} \rfloor C$ if $\pi > 0$. Therefore, if $\Delta^p > \lceil \frac{\tilde{d}_{kt}+\Delta^{int}\hat{d}+\Delta^{frac}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}}{C} \rceil$, $\phi(k,t,\mathbf{\Delta}) = \phi(k,t,\lceil \frac{\tilde{d}_{kt}+\Delta^{int}\hat{d}+\Delta^{frac}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}}{C} \rceil,\Delta^{int},\Delta^{frac})$.

In the following, we propose an inner DP that resembles the one in [75] for the deterministic CCLS model.

We let $\mathbf{\Delta}_{l+1} = (\Delta^p_{l+1}, \Delta^{int}_{l+1}, \Delta^{frac}_{l+1})$ and let $g(l+1, \tau, \eta, \mathbf{\Delta}_{l+1})$ be the minmax cost for $[k, l]$, $l > k$, during which production occurs τ times at full capacity, η times at level π, Δ^p_{l+1} of the production periods have production cost parameters to upper bounds, Δ^{int}_{l+1} of the periods have demand to upper bound, and Δ^{frac}_{l+1} periods have that demand deviates from the nominal value by $(\Gamma^d - \lfloor \Gamma^d \rfloor)\hat{d}$, for $0 \leq \tau \leq \lfloor \frac{\tilde{d}_{kt} + \Delta^{int}\hat{d} + \Delta^{frac}(\Gamma^d - \lfloor \Gamma^d \rfloor)\hat{d}}{C} \rfloor$, $\eta \in \{0, 1\}$, $0 \leq \Delta^p_{l+1} \leq \Delta^p$, $0 \leq \Delta^{int}_{l+1} \leq \Delta^{int}$, and $0 \leq \Delta^{frac}_{l+1} \leq \Delta^{frac}$. Similarly, we define $s(l+1, \tau, \eta, \mathbf{\Delta}_{l+1})$ with the additional constraint that period l is a production period, and therefore $x_l = C$ if $\eta = 0$ and $x_l = \pi$ otherwise. If $\tau C + \eta\pi < \tilde{d}_{kl} + \Delta^{int}_{l+1}\hat{d} + \Delta^{frac}_{l+1}(\Gamma^d - \lfloor \Gamma^d \rfloor)\hat{d}$ or $\tau > l - k + 1$, we let $g(l+1, \tau, \eta, \Delta^p_{l+1}) = \infty$ because the production plan up to period l, where we produce τ times at full capacity C and η times at π, is not feasible. Assuming this is not the case and $\pi > 0$, we have:

$$
\begin{aligned}
& g(k, \tau, \eta, \mathbf{\Delta}_{k+1}) = s(k, k+1, \tau, \eta, \mathbf{\Delta}_{k+1}) \\
= & \left\{ \begin{array}{l}
\left\{ \begin{array}{l}
\left\{ \begin{array}{ll}
\tilde{f}_k + \tilde{p}_k C + h_k(C - \tilde{d}_k) & \text{if } \Delta^{int}_{k+1} = 0 \\
\tilde{f}_k + \tilde{p}_k C + h_k(C - \overline{d}_k) & \text{if } \Delta^{int}_{k+1} = 1 \\
\tilde{f}_k + \tilde{p}_k C + h_k[C - \tilde{d}_k - (\Gamma^d - \lfloor \Gamma^d \rfloor)] & \text{if } \Delta^{frac}_{k+1} = 1
\end{array} \right\} \text{if } \Delta^p_{k+1} = 0 \\
\left\{ \begin{array}{ll}
\overline{f}_k + \overline{p}_k C + h_k(C - \tilde{d}_k) & \text{if } \Delta^{int}_{k+1} = 0 \\
\overline{f}_k + \overline{p}_k C + h_k(C - \overline{d}_k) & \text{if } \Delta^{int}_{k+1} = 1 \\
\overline{f}_k + \overline{p}_k C + h_k[C - \tilde{d}_k - (\Gamma^d - \lfloor \Gamma^d \rfloor)] & \text{if } \Delta^{frac}_{k+1} = 1
\end{array} \right\} \text{if } \Delta^p_{k+1} = 1
\end{array} \right\} \text{if } \eta = 0 \\
\left\{ \begin{array}{l}
\left\{ \begin{array}{ll}
\tilde{f}_k + \tilde{p}_k \pi + h_k(\pi - \tilde{d}_k) & \text{if } \Delta^{int}_{k+1} = 0 \\
\tilde{f}_k + \tilde{p}_k \pi + h_k(\pi - \overline{d}_k) & \text{if } \Delta^{int}_{k+1} = 1 \\
\tilde{f}_k + \tilde{p}_k \pi + h_k[\pi - \tilde{d}_k - (\Gamma^d - \lfloor \Gamma^d \rfloor)] & \text{if } \Delta^{frac}_{k+1} = 1
\end{array} \right\} \text{if } \Delta^p_{k+1} = 0 \\
\left\{ \begin{array}{ll}
\overline{f}_k + \overline{p}_k \pi + h_k(\pi - \tilde{d}_k) & \text{if } \Delta^{int}_{k+1} = 0 \\
\overline{f}_k + \overline{p}_k \pi + h_k(\pi - \overline{d}_k) & \text{if } \Delta^{int}_{k+1} = 1 \\
\overline{f}_k + \overline{p}_k \pi + h_k[\pi - \tilde{d}_k - (\Gamma^d - \lfloor \Gamma^d \rfloor)] & \text{if } \Delta^{frac}_{k+1} = 1
\end{array} \right\} \text{if } \Delta^p_{k+1} = 1
\end{array} \right\} \text{if } \eta = 1
\end{array} \right.
\end{aligned}
$$

$$
\begin{aligned}
& g(l+1, \tau, \eta, \mathbf{\Delta}_{l+1}) \\
= & \left\{ \begin{array}{ll}
\min \left\{ \begin{array}{l}
g(l, \tau, 0, \Delta^p_{l+1}, \Delta^{int}_{l+1}, \Delta^{frac}_{l+1}) + h_l(\tau C - \tilde{d}_{kl} - \Delta^{int}_{l+1}\hat{d} - \Delta^{frac}_{l+1}(\Gamma^d - \lfloor \Gamma^d \rfloor)\hat{d}) \\
s(l+1, \tau, 0, \Delta^p_{l+1}, \Delta^{int}_{l+1}, \Delta^{frac}_{l+1})
\end{array} \right\} & \text{if } \eta = 0 \\
\min \left\{ \begin{array}{l}
g(l, \tau, 1, \mathbf{\Delta}_{l+1}) + h_l(\tau C - \tilde{d}_{kl} - \Delta^{int}_{l+1}\hat{d} - \Delta^{frac}_{l+1}(\Gamma^d - \lfloor \Gamma^d \rfloor)\hat{d}) \\
s(l+1, \tau, 1, \mathbf{\Delta}_{l+1})
\end{array} \right\} & \text{if } \eta = 1
\end{array} \right.
\end{aligned}
$$

$$
\begin{aligned}
& s(l+1,\tau,\eta,\boldsymbol{\Delta}_{l+1}) \\
= & \begin{cases}
\max \begin{cases}
g(l,\tau-1,0,\Delta^p_{l+1},\Delta^{int}_{l+1}-1,\Delta^{frac}_{l+1})+\tilde{f}_l+\tilde{p}_l C+ \\
+h_l(\tau C-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau-1,0,\Delta^p_{l+1},\Delta^{int}_{l+1},\Delta^{frac}_{l+1}-1)+\tilde{f}_l+\tilde{p}_l C+ \\
+h_l(\tau C-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau-1,0,\Delta^p_{l+1},\Delta^{int}_{l+1},\Delta^{frac}_{l+1})+\tilde{f}_l+\tilde{p}_l C+ \\
+h_l(\tau C-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau-1,0,\Delta^p_{l+1}-1,\Delta^{int}_{l+1}-1,\Delta^{frac}_{l+1})+\overline{f}_l+\overline{p}_l C+ \\
+h_l(\tau C-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau-1,0,\Delta^p_{l+1}-1,\Delta^{int}_{l+1},\Delta^{frac}_{l+1}-1)+\overline{f}_l+\overline{p}_l C+ \\
+h_l(\tau C-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau-1,0,\Delta^p_{l+1}-1,\Delta^{int}_{l+1},\Delta^{frac}_{l+1})+\overline{f}_l+\overline{p}_l C+ \\
+h_l(\tau C-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d})
\end{cases} \text{if } \eta=0 \\
\max \begin{cases}
g(l,\tau,0,\Delta^p_{l+1},\Delta^{int}_{l+1}-1,\Delta^{frac}_{l+1})+\tilde{f}_l+\tilde{p}_l \pi+ \\
+h_l(\tau C+\pi-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau,0,\Delta^p_{l+1},\Delta^{int}_{l+1},\Delta^{frac}_{l+1}-1)+\tilde{f}_l+\tilde{p}_l \pi+ \\
+h_l(\tau C+\pi-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau,0,\Delta^p_{l+1},\Delta^{int}_{l+1},\Delta^{frac}_{l+1})+\tilde{f}_l+\tilde{p}_l \pi+ \\
+h_l(\tau C+\pi-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau,0,\Delta^p_{l+1}-1,\Delta^{int}_{l+1}-1,\Delta^{frac}_{l+1})+\overline{f}_l+\overline{p}_l \pi+ \\
+h_l(\tau C+\pi-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau,0,\Delta^p_{l+1}-1,\Delta^{int}_{l+1},\Delta^{frac}_{l+1}-1)+\overline{f}_l+\overline{p}_l \pi+ \\
+h_l(\tau C+\pi-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau,0,\Delta^p_{l+1}-1,\Delta^{int}_{l+1},\Delta^{frac}_{l+1})+\overline{f}_l+\overline{p}_l \pi+ \\
+h_l(\tau C+\pi-d_{kl}-\Delta^{int}_{l+1}\hat{d}-\Delta^{frac}_{l+1}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d})
\end{cases} \text{if } \eta=1
\end{cases}
\end{aligned}
$$

for $l=k,\ldots,t$, $\tau=0,\ldots,\lfloor\frac{\tilde{d}_{kt}+\Delta^{int}\hat{d}+\Delta^{frac}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}}{C}\rfloor$, $\eta=\{0,1\}$, $\Delta^p_{l+1}=0,\ldots,\Delta^p$, $\Delta^{int}_{l+1}=0,\ldots,\Delta^{int}$, and $\Delta^{frac}_{l+1}\in\{0,\Delta^{frac}\}$. If $\pi=0$, it is sufficient to use the same algorithm, with all paths in which $\eta=1$ forbidden.

4.4.3 The running time

For the ULSUB model, $\phi(k, t, \mathbf{\Delta})$ can be calculated in $\mathcal{O}(\Gamma^d n^2)$ time. For given t and $\mathbf{\Gamma}_t$, calculating $G(t, \mathbf{\Gamma}_t)$ can be done in $\mathcal{O}(n)$ time due to the minimization over k. For each $S(k, t, \mathbf{\Gamma}_t)$, we need to consider the maximisation over Δ^d. In addition we know that $\Delta^p \in \{0, 1\}$. Therefore, all values of $S(k, t, \mathbf{\Gamma}_t)$ and $G(t, \mathbf{\Gamma}_{t+1})$ can be calculated in $\mathcal{O}((\Gamma^d)^2 n^2)$ time, and the DP solves the ULSUB problem in $\mathcal{O}((\Gamma^d)^2 n^2)$ time.

For the CCLSUB model, it is easy to see that all values of $\phi^d(k, t, \mathbf{\Delta})$ can be calculated in $\mathcal{O}(\Gamma^p(\Gamma^d)^2 n^4)$ time. Hence, given t and $\mathbf{\Gamma}$, calculating $G(t, \mathbf{\Gamma}_t)$ can be done in $\mathcal{O}(n)$ time due to the minimization over k. For each $S(k, t, \mathbf{\Gamma})$, we need to consider the maximisation over Δ^p and Δ^d, which takes $\mathcal{O}(\Gamma^p \Gamma^d)$ time. Therefore, all values of $S(k, t, \Gamma_t^{int}, \Gamma_t^{frac})$ and $G(t, \Gamma_t^{int}, \Gamma_t^{frac})$ can be calculated in $\mathcal{O}(\Gamma^p(\Gamma^d)^2 n^4)$ time, and the DP solves the CCLSUB problem with time-invariant $\hat{d}$ in $\mathcal{O}(\Gamma^p(\Gamma^d)^2 n^4)$ time.

4.5 Single type uncertainty

In this section, we consider the case when either the production cost function or the demand is uncertain, but not both. We will show that with slight modifications to the solution approach proposed in Section 4.3 and Section 4.4, we are able to solve the LSUB model with single type uncertainty with reduced running time.

4.5.1 Production cost function uncertainty

The DP now moves from node (k, Γ_{k-1}^p) to (t, Γ_t^p) with $G(t, \Gamma_t^p)$ and $S(k, t, \Gamma_t^p)$ defined as in Section 4.3. Again as there is an optimal solution to the ULSUB model that satisfies the ZIO property, we can assume that $\mathcal{P}_{kt} = \{k\}$ without loss of optimality. Therefore, for the ULSUB model with production cost function uncertainty, it is easy to see that

$$\phi^p(k, t, \Delta^p) = \begin{cases} \tilde{f}_k + \sum_{i=k}^{t} \tilde{\alpha}_{ki} d_i & \text{if } \Delta^p = 0 \\ \overline{f}_k + \sum_{i=k}^{t} \overline{\alpha}_{ki} d_i & \text{if } \Delta^p \geq 1. \end{cases}$$

Hence, all values of $\phi^p(k, t, \Delta^p)$ can be calculated in $\mathcal{O}(n^2)$ time. For given t and Γ_t^p, calculating $G(t, \Gamma_t^p)$ can be done in $\mathcal{O}(n)$ time due to the minimization over k. For each $S(k, t, \Gamma_t^p)$,

we need to consider the maximisation over Δ^p which takes constant time since $\Delta^p \in \{0,1\}$. Therefore, all values of $S(k,t,\Gamma_t^p)$ and $G(t,\Gamma_t^p)$ can be calculated in $\mathcal{O}(\Gamma^p n^2)$ time, and the DP solves the ULSUB problem with production cost function uncertainty in $\mathcal{O}(\Gamma^p n^2)$ time.

For the CCLSUB model with production cost function uncertainty, we let $g(l{+}1,\tau,\eta,\Delta_{l+1}^p)$ be the minmax cost for $[k,l]$, and $s(l{+}1,\tau,\eta,\Delta_{l+1}^p)$ with the additional constraint that period l is a production period as in Section 4.3, we have:

$$g(k,\tau,\eta,\Delta_{k+1}^p) = s(k,k+1,\tau,\eta,\Delta_{k+1}^p) = \begin{cases} \left\{ \begin{array}{ll} \tilde{f}_k + \tilde{p}_k C + h_k(C-d_k) & \text{if } \Delta_{k+1}^p = 0 \\ \overline{f}_k + \overline{p}_k C + h_k(C-d_k) & \text{if } \Delta_{k+1}^p = 1 \end{array} \right\} & \text{if } \eta = 0 \\ \left\{ \begin{array}{ll} \tilde{f}_k + \tilde{p}_k \pi + h_k(C-\pi) & \text{if } \Delta_{k+1}^p = 0 \\ \overline{f}_k + \overline{p}_k \pi + h_k(C-\pi) & \text{if } \Delta_{k+1}^p = 1 \end{array} \right\} & \text{if } \eta = 1 \end{cases}$$

$$g(l+1,\tau,\eta,\Delta_{l+1}^p) = \begin{cases} \min \left\{ \begin{array}{l} g(l,\tau,\eta,\Delta_{l+1}^p) + h_l(\tau C - d_{kl}) \\ s(l+1,\tau,0,\Delta_{l+1}^p) \end{array} \right\} & \text{if } \eta = 0 \\ \min \left\{ \begin{array}{l} g(l,\tau,1,\Delta_{l+1}^p) + h_l(\tau C + \pi - d_{kl}) \\ s(l+1,\tau,1,\Delta_{l+1}^p) \\ s(l+1,\tau,0,\Delta_{l+1}^p) \end{array} \right\} & \text{if } \eta = 1 \end{cases}$$

$$s(l+1,\tau,\eta,\Delta_{l+1}^p) = \begin{cases} \max \left\{ \begin{array}{l} g(l,\tau-1,0,\Delta_{l+1}^p - 1) + \overline{f}_l + \overline{p}_l C + h_l(\tau C - d_{kl}) \\ g(l,\tau-1,0,\Delta_{l+1}^p) + \tilde{f}_l + \tilde{p}_l C + h_l(\tau C - d_{kl}) \end{array} \right\} & \text{if } \eta = 0 \\ \max \left\{ \begin{array}{l} g(l,\tau,0,\Delta_{l+1}^p - 1) + \overline{f}_l + \overline{p}_l \pi + h_l(\tau C + \pi - d_{kl}) \\ g(l,\tau,0,\Delta_{l+1}^p) + \tilde{f}_l + \tilde{p}_l \pi + h_l(\tau C + \pi - d_{kl}) \end{array} \right\} & \text{if } \eta = 1 \end{cases}$$

for $l = k,\ldots,t$, $\tau = 0,\ldots,\lfloor \frac{d_{kt}}{C} \rfloor$, $\eta \in \{0,1\}$, and $\Delta_{l+1}^p = 0,\ldots,\Delta^p$. If $\pi = 0$, it is sufficient to use the same algorithm, with all paths in which $\eta = 1$ forbidden.

Then $\phi^p(k,t,\Delta^p) = g(t,\lfloor \frac{d_{kt}}{C} \rfloor, 1, \Delta^p)$. It is easy to see that all values of $\phi^p(k,t,\Delta^p)$ can be calculated in $\mathcal{O}(\Gamma^p n^4)$ time. Hence given t and Γ_t^p, calculating $G(t,\Gamma_t^p)$ can be done in $\mathcal{O}(n)$ time due to the minimization over k. For each $S(k,t,\Gamma_t)$, we need to consider the maximisation over Δ^p, which takes $\mathcal{O}(\Gamma^p)$ time. Therefore, all values of $S(k,t,\Gamma_t^p)$ and $G(t,\Gamma_t^p)$ can be calculated in $\mathcal{O}((\Gamma^p)^2 n^2)$ time, and the DP solves the CCLSUB problem with uncertain production cost function in $\mathcal{O}((\Gamma^p)^2 n^4)$ time.

4.5.2 Demand uncertainty

The DP now moves from node $(k, \Gamma^{int}_{k-1}, \Gamma^{frac}_{k-1})$ to $(t, \Gamma^{int}_{t}, \Gamma^{frac}_{t})$ with $G(t, \Gamma^{int}_{t}, \Gamma^{frac}_{t})$ and $S(t, \Gamma^{int}_{t}, \Gamma^{frac}_{t})$ defined as in Section 4.3. Then for the ULSUB model, $\phi^d(k, t, \Delta^{int}, \Delta^{frac})$ is very similar to that in Section 4.4, i.e.,

$$\phi(k, t, \Delta^{int}, \Delta^{frac}) = f_k + \sum_{i=k}^{t} \alpha_{ki} d_i + \tilde{E}(k, t, \Delta^{int}, \Delta^{frac}).$$

For given t, Γ^{int}_{t}, and Γ^{frac}_{t}, calculating $G(t, \Gamma^{int}_{t}, \Gamma^{frac}_{t})$ can be done in $\mathcal{O}(n)$ time due to the minimization over k. For each $S(k, t, \Gamma^{int}_{t}, \Gamma^{frac}_{t})$, we need to consider the maximisation over Δ^{int}. Therefore, all values of $S(k, t, \Gamma^{int}_{t}, \Gamma^{frac}_{t})$ and $G(t, \Gamma^{int}_{t}, \Gamma^{frac}_{t})$ can be calculated in $\mathcal{O}((\Gamma^d)^2 n^2)$ time, and the DP solves the ULSUB problem with uncertain demand in $\mathcal{O}((\Gamma^d)^2 n^2)$ time.

For the CCLSUB model, we define $g(l+1, \tau, \eta, \Delta^{int}_{l+1}, \Delta^{frac}_{l+1})$ and $s(l+1, \tau, \eta, \Delta^{int}_{l+1}, \Delta^{frac}_{l+1})$ as in Section 4.4. Then,

$$g(k, \tau, \eta, \Delta^{d}_{k+1}) = s(k, k+1, \tau, \eta, \Delta^{d}_{k+1}) =$$

$$= \begin{cases} \left\{ \begin{array}{ll} f_k + p_k C + h_k(C - \tilde{d}_k) & \text{if } \Delta^{int}_{k+1} = \Delta^{frac}_{k+1} = 0 \\ f_k + p_k C + h_k[C - \tilde{d}_k - (\Gamma^d - \lfloor \Gamma^d \rfloor)\hat{d}] & \text{if } \Delta^{frac}_{k+1} = 1 \\ f_k + p_k C + h_k(C - \overline{d}_k) & \text{if } \Delta^{int}_{k+1} = 1 \end{array} \right\} & \text{if } \eta = 0 \\ \left\{ \begin{array}{ll} f_k + p_k \pi + h_k(C - \tilde{d}_k) & \text{if } \Delta^{int}_{k+1} = \Delta^{frac}_{k+1} = 0 \\ f_k + p_k \pi + h_k[C - \tilde{d}_k - (\Gamma^d - \lfloor \Gamma^d \rfloor)\hat{d}] & \text{if } \Delta^{frac}_{k+1} = 1 \\ f_k + p_k \pi + h_k(C - \overline{d}_k) & \text{if } \Delta^{int}_{k+1} = 1 \end{array} \right\} & \text{if } \eta = 1. \end{cases}$$

$$g(l+1, \tau, \eta, \Delta^{int}_{l+1}, \Delta^{frac}_{l+1}) =$$

$$= \begin{cases} \min \left\{ \begin{array}{l} g(l, \tau, 0, \Delta^{int}_{l+1}, \Delta^{frac}_{l+1}) + h_l(\tau C - d_{kl} - \Delta^{int}_{l+1}\hat{d}) \\ s(l+1, \tau, 0, \Delta^{int}_{l+1}, \Delta^{frac}_{l+1}) \end{array} \right\} & \text{if } \eta = 0 \\ \min \left\{ \begin{array}{l} g(l, \tau, 1, \Delta^{int}_{l+1}, \Delta^{frac}_{l+1}) + \\ + h_l(\tau C + \pi - d_{kl} - \Delta^{int}_{l+1}\hat{d} - \Delta^{frac}_{l+1}(\Gamma^d - \lfloor \Gamma^d \rfloor)\hat{d}) \\ s(l+1, \tau, 1, \Delta^{int}_{l+1}, \Delta^{frac}_{l+1}) \\ s(l+1, \tau, 0, \Delta^{int}_{l+1}, \Delta^{frac}_{l+1}) \end{array} \right\} & \text{if } \eta = 1 \end{cases}$$

$$
\begin{aligned}
&s(l+1,\tau,\eta,\Delta_{l+1}^{int},\Delta_{l+1}^{frac}) = \\
&= \begin{cases}
\max\left\{\begin{array}{l}
g(l,\tau-1,0,\Delta_{l+1}^{int}-1,\Delta_{l+1}^{frac}) + f_l + p_l C + \\
+h_l(\tau C - d_{kl} - \Delta_{l+1}^{int}\hat{d} - \Delta_{l+1}^{frac}(\Gamma^d - \lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau-1,0,\Delta_{l+1}^{int},\Delta_{l+1}^{frac}-1) + f_l + p_l C + \\
+h_l(\tau C - d_{kl} - \Delta_{l+1}^{int}\hat{d} - \Delta_{l+1}^{frac}(\Gamma^d - \lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau-1,0,\Delta_{l+1}^{int},\Delta_{l+1}^{frac}) + f_l + p_l C + \\
+h_l(\tau C - d_{kl} - \Delta_{l+1}^{int}\hat{d} - \Delta_{l+1}^{frac}(\Gamma^d - \lfloor\Gamma^d\rfloor)\hat{d})
\end{array}\right\} \text{if } \eta = 0 \\
\max\left\{\begin{array}{l}
g(l,\tau,0,\Delta_{l+1}^{int}-1,\Delta_{l+1}^{frac}) + f_l + p_l \pi + \\
+h_l(\tau C + \pi - d_{kl} - \Delta_{l}^{int}\hat{d} - \Delta_{l+1}^{frac}(\Gamma^d - \lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau,0,\Delta_{l+1}^{int},\Delta_{l+1}^{frac}-1) + f_l + p_l \pi + \\
+h_l(\tau C + \pi - d_{kl} - \Delta_{l}^{int}\hat{d} - \Delta_{l+1}^{frac}(\Gamma^d - \lfloor\Gamma^d\rfloor)\hat{d}) \\
g(l,\tau,0,\Delta_{l+1}^{int},\Delta_{l+1}^{frac}) + f_l + p_l \pi + \\
+h_l(\tau C + \pi - d_{kl} - \Delta_{l+1}^{int}\hat{d} - \Delta_{l+1}^{frac}(\Gamma^d - \lfloor\Gamma^d\rfloor)\hat{d})
\end{array}\right\} \text{if } \eta = 1
\end{cases}
\end{aligned}
$$

Then $\phi^d(k,t,\Delta_{l+1}^{int},\Delta_{l+1}^{frac}) = g(t, \lfloor\frac{\tilde{d}_{kt}+\Delta^{int}\hat{d}+\Delta^{frac}(\Gamma^d-\lfloor\Gamma^d\rfloor)\hat{d}}{C}\rfloor, 1, \Delta_{l+1}^{int}, \Delta_{l+1}^{frac})$. It is easy to see that all values of $\phi^d(k,t,\Delta_t^{int},\Delta_t^{frac})$ can be calculated in $\mathcal{O}((\Gamma^d)^2 n^4)$ time. Hence, given t and Γ^d, calculating $G(t,\Gamma_t^{int},\Gamma_t^{frac})$ can be done in $\mathcal{O}(n)$ time due to the minimization over k. For each $S(k,t,\Gamma_t^{int},\Gamma_t^{frac})$, we need to consider the maximisation over Δ^d, which takes $\mathcal{O}(\Gamma^d)$ time. Therefore, all values of $S(k,t,\Gamma_t^{int},\Gamma_t^{frac})$ and $G(t,\Gamma_t^{int},\Gamma_t^{frac})$ can be calculated in $\mathcal{O}((\Gamma^d)^2 n^4)$ time, and the DP solves the CCLSUB problem with time-invariant $\hat{d}$ in $\mathcal{O}((\Gamma^d)^2 n^4)$ time.

4.6 Summary

In this chapter, we have formulated the robust lot sizing models using the idea of uncertainty budget, in which both production cost function and demand are considered uncertain. We gave a theoretical analysis to the optimal uncertainty scenarios and proposed an algorithm that solves the problem in $\mathcal{O}((\Gamma^d)^2 n^2)$ time if the model is uncapacitated. Better running times were found for the case in which only the production cost function is uncertain. In particular, an $\mathcal{O}(\Gamma^p n^2)$ time algorithm is proposed. We also studied robust CCLS model. An $\mathcal{O}(\Gamma^p n^4)$ time algorithm is proposed for production cost function uncertainty, an $\mathcal{O}((\Gamma^d)^2 n^4)$ time algorithm for uncertain demand with time-invariant deviation component,

and an $\mathcal{O}(\Gamma^p(\Gamma^d)^2 n^4)$ time algorithm for the combination of both types of uncertainty. In Chapter 5, we propose a robust lot sizing model using a similar methodology but with additional parameters to control the trade-off between robustness and the quality of objective function. Numerical results on the performance of both models are presented.

Chapter 5

The Lot Sizing Model with Uncertainty Range

In this chapter, we study a robust lot sizing model where the robustness is modeled using the absolute criterion [159]. We modify upon the LSUB model proposed in Chapter 4 and show that by introducing two additional parameters to control the range of uncertain parameters and the minimal deviation levels in each period, the proposed model is less sensitive to parameter settings than the LSUB model in general. To show this, comprehensive numerical results are presented. The study in this chapter can be found in [122].

5.1 Introduction

In this study, we focus on the ULS model with uncertain demand, although the proposed model can be generalised to take into account constant capacities and uncertain production cost function as well, as in Chapter 4. We model again d_i for $i \in 1, \ldots, n$ as an uncertain parameter that takes values in the symmetric and bounded interval $[\tilde{d}_i - \hat{d}_i, \tilde{d}_i + \hat{d}_i]$, where the nominal value $\tilde{d}_i$ denotes the mean and $\hat{d}_i$ the deviation from the nominal demand. Again, we denote by Γ^d the uncertainty budget for demand. Recall from Chapter 4, the ULSUB

model with uncertain demand can be formulated as follows,

$$ULSUB: \quad \min_{\mathbf{y},\mathbf{x},\mathbf{I}} \max_{\mathbf{z}} \sum_{i=1}^{n} [f_i y_i + p_i x_i + h_i I_i] \tag{5.1}$$

$$\text{s.t.} \quad I_{i-1} + x_i = I_i + \tilde{d}_i + \hat{d}_i z_i, \; \forall i = 1, \ldots, n \tag{5.2}$$

$$x_i \leq M\, y_i, \; \forall i = 1, \ldots, n \tag{5.3}$$

$$\sum_{i=1}^{n} z_i \leq \Gamma^d \tag{5.4}$$

$$y_i \in \{0, 1\}, \; \forall i = 1, \ldots, n \tag{5.5}$$

$$x_i, I_i \geq 0, \; \forall i = 1, \ldots, n \tag{5.6}$$

$$0 \leq z_i \leq 1, \; \forall i = 1, \ldots, n. \tag{5.7}$$

Here we drop the superscript d from $\mathbf{z}$ for convenience.

The ULSUB model can be understood as a "reasonable worst-case" approach. By adjusting Γ^d, one can control the trade-off between robustness and quality of the objective function value and therefore arrive at a less conservative solution. The ULSUB model chooses to allocate worst-case periods in places that lead to the largest increase in the objective function value. Ideally, we would like to consider a small amount of worst-case periods to ensure a good quality objective function value, and at the same time reach a satisfactory robustness level. Nevertheless, we show in the chapter later that this may not always be the case. Given an uncertainty budget Γ^d, we introduce an additional parameter $\Theta \in \mathbb{Z}$ that restricts the minimum number of periods in which demand is protected against uncertainty. In those periods, we again constrain the level of demand deviation from the nominal value is at least β. We show that in this way, the ULS model with Uncertainty Range (ULSUR) has the chance to improve further the trade-off between robustness and the quality of the objective function value.

In this chapter, after showing that the ULSUR models is polynomially solvable, we provide comprehensive computational experiments on the factors that affect the performance of the proposed models. By benchmarking with the nominal model, a deterministic ULS model in which the demands are equal to the nominal value, and Soyster's absolute robust model, we investigate how cost and demand parameter settings affect the performance of the

proposed models. In addition, we give insights on when controlling the uncertainty budget is sufficient to obtain satisfactory results.

The remainder of the chapter is organised as follows. We formulate the ULSUR model in Section 5.2. To solve the problem, we propose a DP solution approach in Section 5.4. In Section 5.5, we present the numerical results for both the ULSUB model and the ULSUR model. In Section 5.6, a summary of the chapter is given and lines of future research are proposed.

5.2 The formulation

In the following, we propose the ULSUR model. To begin with, we formally introduce some additional notations that will be used throughout this chapter.

- Θ: the minimum number of periods we would like the model to take into account the possibility that the demand may deviate from the nominal values and provide protection against the uncertainty;
- $\mathbf{z}$: $z_i = 1$ if i is a worst-case period, and 0 otherwise;
- $\mathcal{Z}$: the set of worst-case periods, i.e., the set of i in which $z_i = 1$;
- $\mathbf{w}$: the scaled deviation level in each period, $w_i = \frac{d_i - \tilde{d}_i}{\hat{d}_i}$;
- β: the minimum scaled deviation level in each period, i.e., $w_i \geq \beta$.

The uncertainty budget Γ^d controls the "depth" of data uncertainty, i.e., to what extent the total data uncertainty needs to be considered in order to achieve a solution that is robust and gives a less conservative objective function value than Soyster's model. By introducing Θ, one also has the control over the "width" of data uncertainty, i.e., to what extent the Γ^d degree of total data uncertainty should be spread out over the planning horizon. Note that as the ULS is a cost minimization problem, in worst-case periods the demand will go above the nominal value and consequently the total cost rises. In this period, to make the protection

against uncertainty meaningful, we constrain that the scaled deviation of demand, $\frac{d_i - \tilde{d}_i}{\tilde{d}_i}$, is at least $\beta \in (0, 1)$.

Then the ULSUR model can be formulated as follows,

$$ULSUR: \quad \min_{\mathbf{y},\mathbf{x},\mathbf{I}} \max_{\mathbf{z},\mathbf{w}} \sum_{i=1}^{n} [f_i y_i + p_i x_i + h_i I_i] \tag{5.8}$$

$$\text{s.t.} \quad I_{i-1} + x_i = I_i + \tilde{d}_i + w_i \hat{d}_i, \, \forall i = 1, \ldots, n \tag{5.9}$$

$$x_i \leq M \, y_i, \, \forall i = 1, \ldots, n \tag{5.10}$$

$$\sum_{i=1}^{n} z_i \geq \Theta \tag{5.11}$$

$$\sum_{i=1}^{n} w_i \leq \Gamma^d \tag{5.12}$$

$$\beta z_i \leq w_i \leq z_i, \, \forall i = 1, \ldots, n \tag{5.13}$$

$$y_i \in \{0, 1\}, \, \forall i = 1, \ldots, n \tag{5.14}$$

$$x_i, I_i \geq 0, \, \forall i = 1, \ldots, n \tag{5.15}$$

$$z_i \in \{0, 1\}, \, \forall i = 1, \ldots, n. \tag{5.16}$$

In this formulation, constraints (5.9) are similar to their counterpart in the ULS formulation, where we take into account the demand deviation. Constraint (5.11) imposes the minimum number of periods to be protected, whereas constraint (5.12) imposes an upper bound of the total deviation level. Constraints (5.13) ensure that in the worst-case periods the deviation levels are at least β, and in the remaining periods the deviation levels are equal to zero. Constraints (5.16) make sure that $\mathbf{z}$ is well-defined. The remaining constraints, namely (5.10), (5.14) and (5.15), are as in the ULS formulation. In addition, in the rest of the chapter we denote by m_i the production period for demand in period i, and, as in Chapter 3, by $\alpha_{m_i,i} = p_{m_i} + \sum_{k=m_i}^{i-1} h_k$ be the cost of producing and delivering the demand.

Below, we propose a relaxation of the ULSUR model, the ULS model with Uncertainty Budget and β Constraint (ULSUB-β). Unlike the ULSUR model which controls both the "width" of the demand uncertainty via the parameter Θ and the "depth" via the parameter Γ^d, the ULSUB-β model concentrates on the depth by imposing the uncertainty budget only.

Then the two-stage ULSUB-β model is formulated as follows,

$$ULSUB-\beta: \quad \min_{\mathbf{y}} \max_{\mathbf{z},\mathbf{w}} \sum_{i=1}^{n} [f_i y_i + \min_{\mathbf{x},\mathbf{I}} (p_i x_i + h_i I_i)] \tag{5.17}$$

$$\text{s.t.} \quad I_{i-1} + x_i = I_i + \tilde{d}_i + w_i \hat{d}_i, \; \forall i = 1, \ldots, n \tag{5.18}$$

$$x_i \leq M\, y_i, \; \forall i = 1, \ldots, n \tag{5.19}$$

$$\sum_{i=1}^{n} w_i \leq \Gamma^d \tag{5.20}$$

$$\beta z_i \leq w_i \leq z_i, \; \forall i = 1, \ldots, n \tag{5.21}$$

$$y_i \in \{0, 1\}, \; \forall i = 1, \ldots, n \tag{5.22}$$

$$x_i, I_i \geq 0, \; \forall i = 1, \ldots, n. \tag{5.23}$$

The formulation resembles the one of the ULSUR model in (5.8)-(5.16). As the ULSUB-β model no longer models the "width" of demand uncertainty, constraint (5.11) is excluded. With Lemma 4.1 derived in Chapter 4, it can be shown that the ULSUB model is equivalent to the ULSUB-β model when $\Gamma^d \in \mathbb{Z}$ or $\beta \leq \Gamma^d - \lfloor \Gamma^d \rfloor$.

Please note that in the ULSUR and the ULSUB-β model, the worst-case periods are dependent on the setup decision and hence they are considered as a combination of the cost of production and the size of the deviation of demand. This distinguishes our model from other robust models with an uncertainty budget. For instance, a natural way to protect against demand uncertainty in lot sizing problem would be producing more in earlier periods, i.e., setting $z_i = 1$ for $i = 1, \ldots, \Gamma^d$, $\Gamma^d \in \mathbb{Z}$. It is clear that this allocation of Γ^d is a feasible solution to our model, but it may not be the optimal one.

It is obvious that the ULSUB-β model is a special case of the ULSUR model when $\Theta = 0$. In the following we show that the ULSUR model is equivalent to the ULSUB-β model when $\Theta \leq \Gamma^d$. In order to show that, we need the following technical result.

Lemma 5.1. *For a given setup decision vector* $\mathbf{y}$*, there exists an optimal* $(\mathbf{z}^\star, \mathbf{w}^\star)$ *of the ULSUB-*β *model such that* $\lfloor \Gamma^d \rfloor \leq |\mathcal{Z}^\star| \leq \lceil \Gamma^d \rceil$.

Proof. We will first show that $\lfloor \Gamma^d \rfloor \leq |\mathcal{Z}^\star|$. We will prove the statement by contradiction. Suppose that $|\mathcal{Z}^\star| < \lfloor \Gamma^d \rfloor$. We will show that we can construct an alternative $(\mathbf{z}', \mathbf{w}')$ to $(\mathbf{z}^\star, \mathbf{w}^\star)$, which leads to an increased objective function value for $\mathbf{y}$, and therefore we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. We let $\mathcal{Z}' = \mathcal{Z}^\star \cup \mathcal{Z}^{new}$, where $\mathcal{Z}^{new} \subset \{1, \ldots, n\} \backslash \mathcal{Z}^\star$ and $|\mathcal{Z}^{new}| = \lfloor \Gamma^d \rfloor - |\mathcal{Z}^\star|$. Hence we have $|\mathcal{Z}'| = \lfloor \Gamma^d \rfloor$. We let $\mathbf{w}' = \mathbf{z}'$. Then, by construction,

$$\begin{aligned}
&\sum_{i=1}^{n} z_i' = \lfloor \Gamma^d \rfloor \\
&\sum_{i=1}^{n} w_i' = \lfloor \Gamma^d \rfloor \leq \Gamma^d \\
&\beta z_i' \leq w_i' \leq z_i', \, \forall i = 1, \ldots, n.
\end{aligned}$$

Therefore, $(\mathbf{z}', \mathbf{w}')$ is a feasible solution to the ULSUB-β model. Then for $\mathbf{y}$,

$$w_i' = z_i' \geq z_i^\star \geq w_i^*,$$

where this inequality is strict for $i \in \mathcal{Z}' \setminus \mathcal{Z}^\star$, which is not empty. Thus, $(\mathbf{z}', \mathbf{w}')$ leads to a higher production quantity and therefore to a higher objective function value. We arrive at a contradiction with the fact that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal.

We now show that $|\mathcal{Z}^\star| \leq \lceil \Gamma^d \rceil$. Again, we prove the statement by contradiction. Given a setup decision vector $\mathbf{y}$, the production periods for demand in period i, m_i, are known for all $i = 1, \ldots, n$, and hence $\alpha_{m_k,i} \hat{d}_i$ are known. By ordering $\alpha_{m_i,i} \hat{d}_i$ in decreasing order for $i \in \mathcal{Z}^\star$, we define $\mathcal{Z}' = \mathcal{Z}^\star \setminus \{k\}$, where $k \in \arg\min\{\alpha_{m_i,i} \hat{d}_i \, : \, i \in \mathcal{Z}^\star\}$. We construct the corresponding $\mathbf{w}'$, such that

$$\begin{aligned}
&w_i' = 0, \, \forall i \notin \mathcal{Z}' \\
&w_i' \geq w_i^*, \, \forall i \in \mathcal{Z}' \\
&\sum_{i=1}^{n} w_i' = \sum_{i=1}^{n} w_i^\star, \qquad (5.24)
\end{aligned}$$

where (5.24) is possible since $|\mathcal{Z}'| \geq \lceil \Gamma^d \rceil$. Moreover,

$$\sum_{i \in \mathcal{Z}^*} \alpha_{m_i,i} w_i^* \hat{d}_i = \sum_{i \in \mathcal{Z}'} \alpha_{m_i,i} w_i^* \hat{d}_i + \alpha_{m_k,k} w_k^* \hat{d}_k \leq \sum_{i \in \mathcal{Z}'} \alpha_{m_i,i} w_i' \hat{d}_i,$$

i.e., $(\mathbf{z}', \mathbf{w}')$ leads to a greater or equal total cost for $\mathbf{y}$. If the cost is greater, we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. Otherwise, if the cost is equal, we can choose $(\mathbf{z}', \mathbf{w}')$ over $(\mathbf{z}^\star, \mathbf{w}^\star)$. By repeating the same process until $|\mathbf{z}'| = \lceil \Gamma^d \rceil$ we arrive at the desired result. □

Proposition 5.2. *If $\Theta \leq \lfloor \Gamma^d \rfloor$, the ULSUR model is equivalent to the ULSUB-β model.*

Proof. Recall that the ULSUB-β model is equal to the ULSUR model, where constraint (5.11) has been removed. Therefore, to show the equivalence, it is enough to show that if $\Theta \leq \lfloor \Gamma^d \rfloor$, constraint (5.11) is satisfied for optimal solution of the ULSUB-β model. This easily follows from Lemma 5.1. □

In the rest of the chapter, we assume that $\Theta > \lfloor \Gamma^d \rfloor$ when discussing the ULSUR model. The ULSUB-β model can reduce the total cost while maintaining high feasibility. We will illustrate this with the following 4-period example. We assume that inventory holding cost is zero, and let $\mathbf{f} = (f, f, f, f)$ and $\mathbf{p} = (p, p+\epsilon, p+\epsilon, p+\epsilon)$ for some $\epsilon > 0$. We assume that the uncertain demand d_i is a uniform random variable in $[\tilde{d} - \hat{d}, \tilde{d} + \hat{d}]$. For convenience, we let $\Gamma^d \in \mathbb{Z}$, i.e., $\Gamma^d = 0, \ldots, 4$. As $p_i + h_i = p_i \leq p_{i+1}$ for $i = 1, 2, 3$, and there is a setup cost f, the optimal solution of the ULSUB-β model considers the entire planning horizon as a single regeneration interval. It is easy to see that α_{1i} is the unit contribution to the total cost of demand. Hence, with given values for Γ^d and β, and because the uncertain demand intervals are time-invariant, the ULSUB-β allocates the worst-case periods according to the size of α_{1i} for $i = 1, \ldots, 4$, i.e., the Γ^d periods with the highest $\alpha_{1,i}$ are assigned as worst-case periods with demand attaining upper bound.

In the following, we list an optimal worst-case period allocation for each value of Γ^d:

- ULSUB-β model with $\Gamma^d = 0$, which is equivalent to the nominal model, would lead to production in period 1, with $d_i = \tilde{d}$ for $i = 1, \ldots, 4$;
- ULSUB-β model with $\Gamma^d = 1$ would lead to production in period 1, with $d_4 = \tilde{d} + \hat{d}$ while $d_i = \tilde{d}$ for $i = 1, 2, 3$;
- ULSUB-β model with $\Gamma^d = 2$ would lead to production in period 1, with $d_4 = d_3 = \tilde{d} + \hat{d}$ while $d_i = \tilde{d}$ for $i = 1, 2$;

- ULSUB-β model with $\Gamma^d = 3$ would lead to production in period 1, with $d_2 = d_3 = d_4 = \tilde{d} + \hat{d}$ while $d_1 = \tilde{d}$;

- ULSUB-β model with $\Gamma^d = 4$, which is equivalent to Soyster's worst-case model, would lead to production in period 1, with $d_1 = d_2 = d_3 = d_4 = \tilde{d} + \hat{d}$.

Although the solution of the nominal model gives the least total cost, the probability of it being feasible is only 0.5. On the other hand, the Soyster's worst-case model guarantees the solution is feasible for all realizations of demand, but the model also leads to the highest total cost. The ULSUB-β model provides a trade-off between feasibility and total cost. With $\Gamma^d = 2$, the total demand being produced is $4\tilde{d} + 2\hat{d}$. As there is only one production period, using a linear transformation of the Irwin-Hall distribution, which models the sum of i.i.d. uniform random variables, we know that the probability of realized total demand being above $4\tilde{d} + 2\hat{d}$ is less than 5%. That is to say the solution of the ULSUB-β model with $\Gamma^d = 2$ is feasible for more than 95% of the data realizations, whilst reducing the total cost by $2p\hat{d}$ comparing with the Soyster's worst-case model. Letting $\Gamma^d = 3$ leads to an even greater level of feasibility and a $p\hat{d}$ reduction in total cost. Hence in this case, the ULSUB-β model manages to reduce the total cost while maintaining high feasibility.

However, the ULSUB-β model could fail to perform in some situations. Consider again this 4-period with similar data settings. This time we assume that there is no setup cost and $h_i = \epsilon + \eta$ for $i = 1, \ldots, 4$, where η is positive. As $p_i + h_i = p + \epsilon + \eta > p_{i+1}$ for $i = 1, 2, 3$, and there is no setup cost, the optimal solution of the ULSUB-β model considers every period a regeneration interval of its own. In the following, we list an optimal worst-case period allocation for each value of Γ^d:

- ULSUB-β model with $\Gamma^d = 0$ would lead to production in every period, with $d_i = \tilde{d}$ for $i = 1, \ldots, 4$;

- ULSUB-β model with $\Gamma^d = 1$ would lead to production in every period, with $d_2 = \tilde{d} + \hat{d}$ while $d_i = \tilde{d}$ for $i = 1, 3, 4$;

- ULSUB-β model with $\Gamma^d = 2$ would lead to production in every period, with $d_2 = d_3 = \tilde{d} + \hat{d}$ while $d_i = \tilde{d}$ for $i = 1, 4$;

- ULSUB-β model with $\Gamma^d = 3$ would lead to production in every period, with $d_2 = d_3 = d_4 = \tilde{d} + \hat{d}$ while $d_1 = \tilde{d}$;

- ULSUB-β model with $\Gamma^d = 4$, which is equivalent to Soyster's worst-case model, would lead to production in every period, with $d_1 = d_2 = d_3 = d_4 = \tilde{d} + \hat{d}$.

In this example, the first regeneration interval (the first period) is not a worst-case period if $\Gamma^d < 4$, as assigning d_1 to upper bound leads to the smallest increase in total cost compared to other periods. However, with $\Gamma^d < 4$, the solutions are infeasible for any demand realization with $d_1 > \tilde{d}$, which could happen with probability 0.5. As d_1 Hence the probability of the solutions of the ULSUB-β model being feasible in this example is below 0.5. In Section 5.5, we will show that in similar situations, the ULSUR model provides improved results over the ULSUB-β model.

5.3 Properties

In the following, Lemmas 5.3-5.4 are the core in deriving a solution approach for the ULSUR model, and Lemmas 5.5-5.6 are for the ULSUB-β model.

Lemma 5.3. *Given a setup decision vector* $\mathbf{y}$*, there exists an optimal solution* $(\mathbf{z}^\star, \mathbf{w}^\star)$ *of the ULSUR model, such that* $\sum_{i=1}^n z_i^* = \Theta$ *and* $\sum_{i=1}^n w_i^* = \Gamma^d$.

Proof. We will first prove that $\sum_{i=1}^n z_i^* = \Theta$. We prove the statement by contradiction. Suppose that $\sum_{i=1}^n z_i^* > \Theta$. We show that we can construct an alternative $(\mathbf{z}', \mathbf{w}')$ to $(\mathbf{z}^\star, \mathbf{w}^\star)$, with $\sum_{i=1}^n z_i' = \Theta$ and $\sum_{i=1}^n w_i' = \sum_{i=1}^n w_i^*$, which leads to an increased objective function value for $\mathbf{y}$ and we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. The proof resembles that used in Lemma 5.1.

Since $\mathbf{y}$ is given, m_i are known for all $i = 1, \ldots, n$. With an abuse of notation, we define

$\mathbf{z}'$ be the optimal solution of the following optimization problem,

$$\begin{aligned} & \min_{\mathbf{z}} \sum_{i=1}^{n} \alpha_{m_i,i} \hat{d}_i z_i \\ \text{s.t.} \quad & z_i \leq z_i^*, \, \forall i = 1, \ldots, n \\ & \sum_{i=1}^{n} z_i \leq \Theta. \end{aligned}$$

We let $\mathcal{Z}'$ denote the set of i in which $z_i' = 1$. Then $\mathcal{Z}'$ is a subset of $\mathcal{Z}^\star$ consisting of the Θ worst-case periods with the greatest contribution to the objective function value.

Using $\Theta > \lfloor \Gamma^d \rfloor$ and (5.12), we know that

$$\sum_{i=1}^{n} w_i^\star \leq \Gamma^d \ \leq \Theta = \sum_{i=1}^{n} z_i',$$

and therefore, we can construct $\mathbf{w}'$, such that

$$\begin{aligned} & w_i' = 0, \, \forall i \notin \mathcal{Z}' \\ & w_i' \geq w_i^*, \, \forall i \in \mathcal{Z}' \\ & \sum_{i=1}^{n} w_i' = \sum_{i=1}^{n} w_i^\star, \end{aligned} \tag{5.25}$$

where at least one of the inequalities in (5.25) is strict. Since the relocation from $\mathbf{w}^*$ to $\mathbf{w}'$ has been done to periods with at least the same cost, we have

$$\sum_{i \in \mathcal{Z}^*} \alpha_{m_i,i} w_i^* \hat{d}_i \ \leq \ \sum_{i \in \mathcal{Z}'} \alpha_{m_i,i} w_i' \hat{d}_i.$$

If the cost is greater, we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. Otherwise, if the cost is equal, we can choose $(\mathbf{z}', \mathbf{w}')$ over $(\mathbf{z}^\star, \mathbf{w}^\star)$ and we arrive at the desired result.

Note that in a similar way, we can prove that $\sum_{i=1}^{n} w_i^* = \Gamma^d$. Assuming that $\mathbf{z}^*$ satisfies that $\sum_{i=1}^{n} z_i^* = \Theta$, and $\sum_{i=1}^{n} w_i^* < \Gamma^d$, we show that we can construct an alternative $\mathbf{w}'$ to $\mathbf{w}^\star$, with $\sum_{i=1}^{n} w_i' = \Gamma^d > \sum_{i=1}^{n} w_i^*$, which leads to an increased objective function value for $\mathbf{y}$ and we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. Using $\Theta > \Gamma^d$, we can construct

$\mathbf{w}'$, such that

$$\begin{aligned} & w'_i = 0, \forall i \notin \mathcal{Z}^* \\ & w'_i \geq w^*_i, \forall i \in \mathcal{Z}^* \\ & \sum_{i=1}^{n} w'_i = \Gamma^d, \end{aligned} \tag{5.26}$$

where at least one of the inequalities in (5.25) is strict. Thus, it is easy to see that

$$\sum_{i \in \mathcal{Z}^*} \alpha_{m_i,i} w^*_i \hat{d}_i < \sum_{i \in \mathcal{Z}^*} \alpha_{m_i,i} w'_i \hat{d}_i,$$

i.e., $\mathbf{w}'$ leads to an increased total cost, and we arrive at a contradiction that $(\mathbf{z}^\star, \mathbf{w}^\star)$ is optimal. □

Lemma 5.4. *Given a setup decision vector* $\mathbf{y}$*, there exists an optimal solution* $(\mathbf{z}^\star, \mathbf{w}^\star)$ *of the ULSUR model, such that there is at most one period* $r \in \mathcal{Z}^\star$ *in which* $\beta < w^\star_r < 1$*.*

Proof. We will prove the statement by contradiction. Suppose that there exist periods l and r, such that $z^\star_l = 1 = z^*_r$, $\beta < w^\star_l < 1$ and $\beta < w^\star_r < 1$. We will distinguish three cases.

If $\alpha_{m_l,l}\hat{d}_l = \alpha_{m_r,r}\hat{d}_r$,

$$\alpha_{m_l,l}(\tilde{d}_l + w^\star_l \hat{d}_l) + \alpha_{m_r,r}(\tilde{d}_r + w^\star_r \hat{d}_r) = \alpha_{m_l,l}[\tilde{d}_l + (w^\star_l + \epsilon)\hat{d}_l] + \alpha_{m_r,r}[\tilde{d}_r + (w^\star_r - \epsilon)\hat{d}_r],$$

with $\epsilon > 0$ a sufficiently small quantity. Then we can always increase w^*_l while reducing w^*_r until $w^*_l = 1$ or $w^*_r = \beta$ without loss of optimality.

Otherwise, if $\alpha_{m_l,l}\hat{d}_l > \alpha_{m_r,r}\hat{d}_r$,

$$\alpha_{m_l,l}(\tilde{d}_l + w^\star_l \hat{d}_l) + \alpha_{m_r,r}(\tilde{d}_r + w^\star_r \hat{d}_r) < \alpha_{m_l,l}[\tilde{d}_l + (w^\star_l + \epsilon)\hat{d}_l] + \alpha_{m_r,r}[\tilde{d}_r + (w^\star_r - \epsilon)\hat{d}_r].$$

Hence increasing w^*_l while reducing w^*_r leads to an increased total cost, and we arrive at a contradiction.

Finally, if $\alpha_{m_l,l}\hat{d}_l < \alpha_{m_r,r}\hat{d}_r$, increasing w_r^* while reducing w_l^* leads to an increased total cost, and we again arrive at a contradiction. □

Using the same arguments as in the proofs of Lemma 5.3 and Lemma 5.4, we have the following lemmas.

Lemma 5.5. *Given a setup decision vector* $\mathbf{y}$*, there exists an optimal solution* $(\mathbf{z}^\star, \mathbf{w}^\star)$ *of the ULSUB-*β *model, such that*

1. *either* $\sum_{i=1}^n z_i^* = \lceil \Gamma^d \rceil$ *and* $\sum_{i=1}^n w_i^* = \Gamma^d$,
2. *or* $\sum_{i=1}^n z_i^* = \lfloor \Gamma^d \rfloor$ *and* $\sum_{i=1}^n w_i^* = \lfloor \Gamma^d \rfloor$.

Lemma 5.6. *Given a setup decision vector* $\mathbf{y}$*, there exists an optimal solution* $(\mathbf{z}^\star, \mathbf{w}^\star)$ *of the ULSUB-*β *model, such that there is at most one period* $r \in \mathcal{Z}^*$ *in which* $\beta < w_r^* < 1$.

5.4 The solution approach

Recall from Chapter 2 that, using the ZIO property, the vector $\mathbf{y}$ splits the planning horizon into a sequence of regeneration intervals [175], where no inventory enters or leaves the intervals and the inventory levels within the interval are positive. In this section we first propose a DP algorithm for both the ULSUR models that resembles the one proposed in [175]. We then show that with slight modifications, the same DP can also be used to solve the ULSUB-β model.

As shown in Lemma 5.4, in worst-case periods the optimal deviation level of demand, $\mathbf{w}^*$, has three forms. To be more specific, in a worst-case period i, $w_i^* = 1$, $w_i^* = \beta$, or $\beta < w_i^* < 1$. We denote by γ^u the number of worst-case periods in which $w_i^* = 1$, by γ^l the number in which $w_i^* = \beta$, by $\gamma^m \in \{0, 1\}$ whether there exists a period r such that $\beta < w_r^* < 1$. Later in this section, we will count how many of each type are for given Γ^d, Θ, and β.

We denote by $\mathbf{\Gamma} = (\gamma^u, \gamma^m, \gamma^l)$ a vector of dimension three, with the number of each type of worst-case periods. We also denote by $\mathbf{i}$ as a vector of dimension three, with each element in the vector equal to i. We let $\mathbf{\Gamma}_i \leq \mathbf{i}$ denote the vector with the number of each type of worst-case periods in $[1, i]$ for $i = 2, \ldots, n+1$. We define $\mathbf{\Gamma}_1 = \mathbf{0}$. We construct a network with nodes $(k, \mathbf{\Gamma}_{k-1})$ in which the DP moves from node $(k, \mathbf{\Gamma}_{k-1})$ to $(t, \mathbf{\Gamma}_t)$, defining the subplan $[(k, \mathbf{\Gamma}_{k-1}), (t, \mathbf{\Gamma}_t)]$, for $1 \leq k \leq t \leq n$, $\mathbf{0} \leq \mathbf{\Gamma}_{k-1} \leq \mathbf{\Gamma}_t \leq \mathbf{\Gamma}$. As incoming inventory into the regeneration interval $[k, t]$ is always zero, none of the production decisions taken in the previous subplan affect the regeneration interval $[k, t]$. Hence, given $\mathbf{\Gamma}_{k-1}$ and $\mathbf{\Gamma}_t$, assigning worst-case periods in one subplan is independent from assigning in the others.

In $[(k, \mathbf{\Gamma}_{k-1}), (t, \mathbf{\Gamma}_t)]$, we let $\mathbf{\Delta} = (\Delta^u, \Delta^m, \Delta^l) = \mathbf{\Gamma}_t - \mathbf{\Gamma}_k$. We denote by $\phi(k, t, \mathbf{\Delta})$ the minmax cost of the subplan with $\mathbf{\Delta}$ worst-case periods. Let $G(t, \mathbf{\Gamma}_t)$ be the minmax cost over the first t periods with $\mathbf{\Gamma}_t$ worst-case periods, where $1 \leq t \leq n$ and $\mathbf{0} \leq \mathbf{\Gamma}_t \leq \mathbf{\Gamma}$, and $S(k, t, \mathbf{\Gamma}_t)$ be the minmax cost over the first t periods with $\mathbf{\Gamma}_t$ worst-case periods, with the additional information that $I_{k-1} = 0$ and $I_i > 0$ for $k \leq i < t$, i.e., $[k, t]$ is the last regeneration interval in $[1, t]$. As in a ULS model, we minimize the total cost in $[1, t]$ by choosing appropriate setup period. Then, we have

$$G(t, \mathbf{\Gamma}_t) \quad = \quad \min_{k:\, 1 \leq k \leq t} S(k, t, \mathbf{\Gamma}_t).$$

We also need to maximize the total cost in $[1, t]$ by allocating the budget of uncertainty. Hence, we also have

$$S(k, t, \mathbf{\Gamma}_t) \quad = \quad \max_{\mathbf{\Delta}:\, \mathbf{0} \leq \mathbf{\Delta} \leq \min\{(t-k+1)\cdot \mathbf{1}, \mathbf{\Gamma} - \mathbf{\Gamma}_k\}} \{G(k, \mathbf{\Gamma}_t - \mathbf{\Delta}) + \phi(k, t, \mathbf{\Delta})\}.$$

We define $G(1, \mathbf{0}) = S(1, 1, \mathbf{0}) = 0$. Therefore, by construction, the optimal solution to the ULSUR model is equal to $G(n, \mathbf{\Gamma})$.

If all values of $\phi(k, t, \mathbf{\Delta})$ are known, we can show that the DP runs in $\mathcal{O}(\Theta^4 n^2)$ time. We can also show that all values of $\phi(k, t, \mathbf{\Delta})$ can be calculated in $\mathcal{O}(\Theta^2 n^2)$ time. Hence, the DP solves the ULSUR model in $\mathcal{O}(\Theta^4 n^2)$ time.

For given t and $\mathbf{\Gamma}_t = (\gamma^u, \gamma^m, \gamma^l)$, calculating $G(t, \mathbf{\Gamma}_t)$ can be done in $\mathcal{O}(n)$ time due to the minimization over k. For each $S(k, t+1, \mathbf{\Gamma}_t)$, we need to consider the maximization over $\mathbf{\Delta} = (\Delta^u, \Delta^m, \Delta^l)$. As $\Delta^u + \Delta^m + \Delta^l \leq \gamma^u + \gamma^m + \gamma^l = \Theta$, in which Δ^m, $\gamma^m \in \{0, 1\}$, hence solving the maximization takes $\mathcal{O}(\Theta^2)$ time. There are $\mathcal{O}(\Theta^2 n)$ values of $G(t, \mathbf{\Gamma}_t)$ and $\mathcal{O}(\Theta^2 n^2)$ values of $S(k, t, \mathbf{\Gamma}_t)$. Therefore, all values of $S(k, t, \mathbf{\Gamma}_t)$ and $G(t, \mathbf{\Gamma}_t)$ can be calculated in $\mathcal{O}(\Theta^4 n^2)$ time. Next, we will show that all values of $\phi(k, t, \mathbf{\Delta})$ can be obtained in $\mathcal{O}(\Theta^2 n^2)$ time.

As the models are uncapacitated, in the subplan $[(k, \mathbf{\Gamma}_{k-1}), (t, \mathbf{\Gamma}_t)]$, there is only production at period k with the total unit cost to deliver one unit in period i equal to α_{ki}. It remains to allocate $\mathbf{\Delta}$ protected periods in the subplan. The maximum cost of the subplan is equal to

$$\phi(k, t, \mathbf{\Delta}) = f_k + \sum_{i=k}^{t} \alpha_{ki} \tilde{d}_i + E(k, t, \mathbf{\Delta}),$$

where

$$E(k, t, \mathbf{\Delta}) := \max_{\substack{\mathbf{z}^u \in \{0,1\}^{t+1-k}, \\ \mathbf{z}^l \in \{0,1\}^{t+1-k}, \\ \mathbf{z}^m \in \{0,1\}^{t+1-k}, \\ \sum_{i=k}^{t} z_i^u \leq \Delta^u, \\ \sum_{i=k}^{t} z_i^l \leq \Delta^l, \\ \sum_{i=k}^{t} z_i^m \leq \Delta^m, \\ z_i^u + z_i^l + z_i^m \leq 1.}} \sum_{i=k}^{t} \alpha_{ki} \hat{d}_i [z_i^u + \beta z_i^l + w^* z_i^m].$$

The function $E(k, t, \mathbf{\Delta})$ gives the maximum cost incurred in $[(k, \mathbf{\Gamma}_{k-1}), (t, \mathbf{\Gamma}_t)]$ by setting Δ^u demands to upper bound, Δ^l to the minimum protection level, and Δ^m to $\beta < w^* < 1$, where the value of w^* will come from Lemma 5.4. For given values of k, t, and $\mathbf{\Delta}$, calculating $E(k, t, \mathbf{\Delta})$ is equivalent to a unordered partial sorting problem. The optimal solution is obtained in $\mathcal{O}(n)$ time by first assigning the Δ^u periods with largest values of $\alpha_{ki}\hat{d}_i$ with $z_i^u = 1$, followed by Δ^m and then Δ^l periods. By calculating $E(k, n, \mathbf{\Delta})$ first and deleting the lowest value of $\alpha_{ki}\hat{d}_i$, $E(k, t, \mathbf{\Delta})$ can be obtained in $\mathcal{O}(n)$ time for all values of $t \geq k$. Then all values of $E(k, t, \mathbf{\Delta})$ can be calculated in an $\mathcal{O}(\Theta^2 n^2)$ time.

Having calculated all values of $E(k, t, \mathbf{\Delta})$, it is easy to see that all values of $\phi(k, t, \mathbf{\Delta})$ can

also be calculated in $\mathcal{O}(\Theta^2 n^2)$ time. Therefore, the DP solves the ULSUR model in $\mathcal{O}(\Theta^4 n^2)$ time.

Given the DP, it remains to derive $\mathbf{\Gamma}$ and w^* for the ULSUR model.

Lemma 5.7. *Given Γ^d, Θ, and β, we have that*

$$\begin{aligned}
\gamma^u &= \lfloor \frac{\Gamma^d - \beta\Theta}{1-\beta} \rfloor, \\
\gamma^m &= \begin{cases} 1 & \text{if } \Gamma^d - \gamma^u > \beta \cdot (\Theta - \gamma^u), \\ 0 & \text{otherwise}, \end{cases} \\
\gamma^l &= \Theta - \gamma^u - \gamma^m,
\end{aligned}$$

and in period r, if $\gamma^m = 1$, the level of deviation is

$$w_r^\star = \Gamma^d - \gamma^u - \beta\gamma^l.$$

Proof. By Lemma 5.3, there exists an optimal solution such that

$$\gamma^u + w_r^* \gamma^m + \beta\gamma^l = \Gamma^d, \tag{5.27}$$

$$\gamma^u + \gamma^m + \gamma^l = \Theta. \tag{5.28}$$

Then by (5.27) and (5.28),

$$\begin{aligned}
\Gamma^d - \gamma^u &= \beta\gamma^l + w_r^\star \gamma^m \\
&\geq \beta\gamma^l + \beta\gamma^m \\
&= \beta(\gamma^l + \gamma^m) \\
&= \beta(\Theta - \gamma^u).
\end{aligned}$$

Hence

$$\begin{aligned}(1-\beta)\gamma^u &\leq \Gamma^d - \beta\Theta \\ \gamma^u &\leq \frac{\Gamma^d - \beta\Theta}{1-\beta} \\ \gamma^u &\leq \lceil \frac{\Gamma^d - \beta\Theta}{1-\beta} \rceil.\end{aligned}$$

We now prove that $\gamma^u \geq \lfloor \frac{\Gamma^d - \beta\Theta}{1-\beta} \rfloor$. By Lemma 5.3, we can substitute γ^l with $\Theta - \gamma^u - \gamma^m$ and rewrite (5.27) as follows,

$$\begin{aligned}\gamma^u + w_r^* \gamma^m + \beta\gamma^l &= \Gamma^d \\ \gamma^u + w_r^* \gamma^m + \beta(\Theta - \gamma^u - \gamma^m) &= \Gamma^d \\ (1-\beta)\gamma^u + (w_r^\star - \beta)\gamma^m &= \Gamma^d - \beta\Theta.\end{aligned}$$

By definition, $\beta < w_r^\star < 1$, thus

$$\begin{aligned}(1-\beta) + (1-\beta)\gamma^u &> \Gamma^d - \beta\Theta \\ 1 + \gamma^u &> \frac{\Gamma^d - \beta\Theta}{1-\beta} \\ \gamma^u &> \frac{\Gamma^d - \beta\Theta}{1-\beta} - 1 \\ \gamma^u &\geq \lfloor \frac{\Gamma^d - \beta\Theta}{1-\beta} \rfloor.\end{aligned}$$

The desired result follows using that $\gamma^u \in \mathbb{Z}$.

Given that we have proven the result for γ^u, the results for γ^m and γ^l follow trivially. By Lemma 3.2, we know that in the rest of the worst-case periods, the uncertainty level is exactly β, with the possibility that in one period r, $\beta < w_r^\star < 1$. Then existence of γ^m is determined by the size of $\Gamma^d - \gamma^u - \beta(\Theta - \gamma^u)$, i.e., if $\Gamma^d - \gamma^u - \beta(\Theta - \gamma^u) > 0$, $\gamma^m = 1$ with $w_r^\star = \Gamma^d - \gamma^u - \beta\gamma^l$.

□

Using Lemma 5.5 and Lemma 5.6, we derive a similar result for the ULSUB-β model.

Lemma 5.8. *Given Γ^d, Θ, and β, we have that*

(1) if $\Gamma^d - \lfloor \Gamma^d \rfloor \geq \beta$,

$$\begin{aligned} \gamma^u &= \lfloor \Gamma^d \rfloor, \\ \gamma^m &= 1, \\ \gamma^l &= 0, \end{aligned}$$

and in period r, if $\gamma^m = 1$, the level of deviation is

$$w_r^\star = \Gamma^d - \gamma^u.$$

(2) otherwise, if $\Gamma^d - \lfloor \Gamma^d \rfloor < \beta$,

(a) either

$$\begin{aligned} \gamma^u &= \lfloor \Gamma^d \rfloor, \\ \gamma^m &= 0, \\ \gamma^l &= 0, \end{aligned}$$

(b) or

$$\begin{aligned} \gamma^u &= \lfloor \Gamma^d \rfloor - 1, \\ \gamma^m &= 1, \\ \gamma^l &= 1, \end{aligned}$$

and in period r, the level of deviation is

$$w_r^\star = \Gamma^d - \gamma^u - \beta.$$

Proof. Using Lemma 5.5 and Lemma 5.6 and constructing a similar argument as in the proof of Lemma 5.8, we can show that the statement holds when $\Gamma^d - \lfloor \Gamma^d \rfloor \geq \beta$. While

$\Gamma^d - \lfloor \Gamma^d \rfloor < \beta$, we need to consider two cases. We can either leave $\Gamma^d - \lfloor \Gamma^d \rfloor$ as unused, as in Case (2)(a), or we can reduce γ^u as in Case (2)(b) by one and assign the remaining uncertainty budget of size $1 + \Gamma^d - \lfloor \Gamma^d \rfloor$ so that $\gamma^m = 1 = \gamma^l$.

□

For the ULSUB-β model, if $\Gamma^d - \lfloor \Gamma^d \rfloor \geq \beta$, the proposed DP algorithm coincides with the one proposed in Chapter 4 and hence can be applied directly by using the $\mathbf{\Gamma}$ vector derived in Case (1) in Lemma 5.8. If $\Gamma^d - \lfloor \Gamma^d \rfloor < \beta$, the DP has to run twice for both $\mathbf{\Gamma}$ vectors in Case (2) in Lemma 5.8. The one with the highest cost leads to the optimal solution of the ULSUB-β model. It is easy to see that the ULSUB-β model can be solved in $\mathcal{O}((\Gamma^d)^2 n^2)$ time using Lemma 5.8.

5.5 Numerical results

The purpose of this computational study is to investigate the effectiveness in terms of protection against the demand uncertainty of the proposed models. We will show performance results for the ULSUR and the ULSUB-β models, against the nominal model and Soyster's absolute robust model [159]. In the nominal model, we have a deterministic ULS where demands are equal to $\tilde{d}_i$. Soyster's model is equivalent to the ULSUB-β model with $\Gamma^d = n$. As commonly used in the literature , throughout this section, we assume that the demand in period i, d_i, is uniformly i.i.d., i.e., $d_i \sim U(\tilde{d}_i - \hat{d}_i, \tilde{d}_i + \hat{d}_i)$. We present a comprehensive set of results, where we test different values for

- the cost parameters $\mathbf{f}$, $\mathbf{p}$, and $\mathbf{h}$; and
- the size of the deviation component of the demand $\hat{\mathbf{d}}$.

5.5.1 Experiment setup

Parameter settings

We experiment with $f_i = 50, 100, 200, 300$. For the unit inventory holding costs, we assume that $h_i = \lambda \cdot \sum_{t=1}^{n} p_t / n$, where $\lambda = 0.10, 0.20, 0.30, 0.40$. For the nominal demand, we assume

$\tilde{d}_i = 30$, while for the deviation component, $\hat{d}_i = \delta\tilde{d}_i$, with $\delta = 0.25, 0.5, 0.75, 1.00$. The protection of periods is heavily influenced by $\alpha_{m_i,i}\hat{d}_i$, where recall that $\alpha_{m_i,i} = p_{m_i} + h_{m_i,i-1}$. Therefore, we will distinguish three patterns for $\mathbf{p}$. In Sections 5.5.2 and 5.5.3, we will consider time-invariant unit production costs, followed by non-increasing ones in Section 5.5.4, and non-decreasing ones in Section 5.5.5. In all these experiments, we have $n = 15$.

Both the ULSUR and the ULSUB-β models have parameters. We have tested our models for all possible values of $\Gamma^d \in \mathbb{Z}$ and Θ, while fixing $\beta = 0.2$. We have experimented with other values of β, but the results are similar. Please note that setting $\Gamma^d \in \mathbb{Z}$ makes experiment results also valid for the ULSUB model in Chapter 4.

Performance criteria

The goal of this experiment is to test the protection against demand uncertainty of the robust models using two criteria, namely feasibility and objective function. In order to do so, we generate 5000 realizations of the demand vector and use them as experiment demand instances. For each of the robust models, we report feasibility and average objective function costs among the feasible instances. More precisely, for a given robust model we obtain the optimal solution $(\mathbf{y}^*, \mathbf{x}^*)$. For each of the experiment demand instance, we first check whether $(\mathbf{y}^*, \mathbf{x}^*)$ is feasible, i.e., whether we can find a feasible vector of inventory levels. If that is the case, then we calculate the total ULS costs. In what follows,

- Q denotes the percentage of the feasible experiment demand instances for the robust models, i.e.,

$$Q = \frac{\text{Number of feasible experiment demand instances}}{\text{Total number of experiment demand instances}};$$

- V denotes the average cost for the corresponding robust model, across the feasible experiment demand instances for both the robust models and the nominal model;

- V_n denotes the average cost for the nominal model, across the feasible experiment demand instances for both the robust models and the nominal model;

- R_v denotes the increase (in percentage) in average cost across the feasible experiment

demand instances, compared to nominal model, i.e.,

$$R_v = \frac{V - V_n}{V_n} \cdot 100.$$

For each set of the experiments, we present the results of the "best" parameters of the ULSUR model and the ULSUB-β model. The goodness of the parameters is measured by V, while we impose a minimum feasibility level on Q. We present results for several values of feasibility threshold, namely, 70%, 80%, 90%.

5.5.2 The base experiment

We first present results of the base experiment. Then in the following sections we test our models with various parameter settings, altering one at a time. In the base experiment, the problem has time-invariant setup cost $f_i = 200$, unit production cost $p_i = 3$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n = 0.3$. For the demand, we let $\hat{d}_i = 15$.

We give a visualization for the full results of the base experiment in Figure 5.1. In the figure, colors are given for values of Γ^d and different symbols are for different values of Θ, as indicated by the legend. For each value of Γ^d and Θ, the xy-coordinates of the symbols correspond to the feasibility level and the average cost over those feasible realizations out of the 5000 demand realizations generated. So we have several symbols of the same color for different values of Θ. We have also marked the position of the nominal model and the Soyster's model in the plot. It is clear that the proposed models establish a trade-off curve between the two extremes, which allows more flexibility in the production planning process. We notice in the figure that the beginning of the plot is rather flat and the end is very steep. The flat beginning means that if we tolerate some increase in the average cost, we gain much higher increase in feasibility level. Similarly, the steep end of the figure means that giving up some robustness of the solution would lead to a significant drop in average total cost.

The summarized results can be found in Table 5.1. From the table we can see that the nominal model achieves the lowest average total cost over feasible realizations. However the solution is feasible only for 37% of the realizations. On the other hand, Soyster's model is the most robust in terms of feasibility, at the expense of the highest average cost, i.e., more than

50% higher than the nominal model. There is no big difference between the ULSUR model and the ULSUB-β model, both models showing their effectiveness by reaching a satisfactory feasibility level while maintaining reasonable cost, i.e., increase in average cost is below 20%.

5.5.3 Time-invariant unit production cost

We carry on the experiment by altering the parameter settings of the base experiment, one at a time. To begin with, we experiment with f_i and present the results in Table 5.2, followed by the unit production cost. In addition to $p_i = 3$, we let $p_i = 2$, 4, and 5 and the results are shown in Table 5.3. Then the experiment is repeated with various values of the unit inventory holding cost. The results are presented in Table 5.4. The advantage of the ULSUR model and the ULSUB-β model over the nominal model and Soyster's absolute robust model, as in Section 5.5.2, seems to be insensitive to changes in cost parameters. Taking 90% feasibility level for example, the proposed models lead to a 30%-40% cost reduction compared to the absolute robust model. This reduction rises above 50% if the unit inventory holding cost is increased. Lastly, we vary the size of the deviation component of the demand, $\hat{d}$, and the results are presented in Table 5.5. It can be seen in the table that if the demand deviation is increased, all robust model suffer from relative higher objective function values compared to the nominal model, i.e., the R_n values in Table 5.5 are higher than those in the Tables 5.2-5.4. However, the cost reduction led by the proposed robust models is the most significant. The average cost of Soyster's absolute robust model doubles that of the nominal model, while the proposed models lead to a 30%-40% increase in average cost and reaching at least 90% feasibility level.

5.5.4 Non-increasing unit production cost

In this section, we first let the unit production costs to be a non-increasing sequence with $p_1 = 4 = 200\% \cdot p_{15}$ and $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$, whilst all other parameters are as in the base experiment. The results are shown in Table 5.6. We then experiment with different parameter settings as in Section 5.5.3, and the results are presented in Tables 5.7-5.9 respectively. In addition, we test other three sets of unit production costs, all of which are non-increasing, and the results are shown in Table 5.10.

Feasible level	Model	Γ^d	Θ	V	R_v	Q
70%	ULSUB-β	3	-	2528.8	11.0	71.8
	ULSUR	3	15	2509.3	10.1	78.3
80%	ULSUB-β	4	-	2619.0	14.9	89.7
	ULSUR	4	13	2604.4	14.3	83.8
90%	ULSUB-β	5	-	2703.2	18.6	93.1
	ULSUR	5	12	2688.8	18.0	92.0
	Nominal	-	-	2278.5	-	36.6
	Soyster	-	-	3569.4	56.7	100.0

Table 5.1: The results of the base experiment, with setup cost $f_i = 200$, unit production cost $p_i = 3$, and unit inventory holding cost $h_i = 0.3$. Demand realizations are generated uniformly in $(15, 45)$.

The results coincide with those for the time-invariant unit production cost in Sections 5.5.2 and 5.5.3. The proposed ULSUR and ULSUB-β models show their usefulness, in particular, when unit inventory holding cost or the deviation component of the demand is high. The models achieve average costs that are around 70% less than the Soyster's model, while attaining at least 90% feasibility level in these scenarios. In all cases, the nominal model is only feasible for around 30% of the demand realizations.

5.5.5 Non-decreasing unit production cost

In this section, we let the unit production cost to be a non-decreasing sequence with $p_{15} = 4 = 200\% \cdot p_1$, and let $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$, while keep all other parameters as in the base experiment. The results are shown in Table 5.11. We then experiment with different parameter settings as in Section 5.5.3, and the results are presented in Tables 5.12-5.14 respectively. In addition, we test other three sets of unit production cost, all of which are non-decreasing, and the results are shown in Table 5.15.

While the results in general agree with those in the previous sets of experiments, we see that the ULSUR model in general outperforms the ULSUB-β model, particularly with high setup cost, high unit inventory holding cost, and high demand deviation component. For

instance, the ULSUR model reduces the total cost by 19% if $f_i = 300$ and a minimum of 90% feasibility level is required, by 20% if $h_i = 0.4 \cdot \sum_{i=1}^{n} p_i/n$, and by 16% if $\hat{d}_i = 30$.

5.6 Summary

Using the concept of uncertainty budget, in this chapter we study robust lot sizing models. We assume the uncertainty lies in the demand and propose a polynomial time DP to solve the models. We provide comprehensive computational experiments on the factors that affect the performance of the ULSUR and ULSUB-β models. We see that the proposed models outperform the nominal model and Soyster's absolute robust model in general, being cost-efficient while attaining high feasibility level. The advantage of the proposed robust models is quite insensitive to variations of cost parameters and demand deviations. While the proposed models perform similarly in the first two sets of experiments, the ULSUR shows its effectiveness when unit production cost is non-decreasing, especially when the cost parameters, or the size of demand deviation is large. As allocating worst case periods relies on the size of $\alpha_{m_i,i}\hat{d}_i$, with time-invariant demand intervals in the experiment, non-decreasing unit production cost makes the ULSUB model provide more protection against demand uncertainty to the later periods in the planning horizon. Then the ULSUR model is able to spread this protection more evenly across the planning horizon and hence benefiting from production in earlier cheaper periods. In particular, using the ULSUR leads to a $6 - 20\%$ average objective function value reduction compared to the ULSUB-β model in this case.

$\mathbf{f}$	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
50	70%	ULSUB-β	3	-	2064.0	11.9	75.6
		ULSUR	3	13	2047.5	11.0	72.3
	80%	ULSUB-β	6	-	2281.1	23.6	96.2
		ULSUR	5	14	2178.2	18.1	80.4
	90%	ULSUB-β	6	-	2281.1	23.6	96.2
		ULSUR	6	9	2271.1	23.1	90.5
		Nominal	-	-	1845.0	-	31.3
		Soyster	-	-	3016.9	63.5	100.0
100	70%	ULSUB-β	3	-	2228.8	10.5	71.8
		ULSUR	3	3	2228.8	10.5	72.3
	80%	ULSUB-β	4	-	2319.0	15.0	89.7
		ULSUR	4	13	2304.4	14.2	83.8
	90%	ULSUB-β	8	-	2652.8	31.5	97.8
		ULSUR	8	12	2640.5	30.9	92.0
		Nominal	-	-	2017.1	-	31.3
		Soyster	-	-	3247.9	61.0	100.0
200	70%	ULSUB-β	3	-	2528.8	11.0	71.8
		ULSUR	3	15	2509.3	10.1	78.3
	80%	ULSUB-β	4	-	2619.0	14.9	89.7
		ULSUR	4	13	2604.4	14.3	83.8
	90%	ULSUB-β	5	-	2703.2	18.6	93.1
		ULSUR	5	12	2688.8	18.0	92.0
		Nominal	-	-	2278.5	-	36.6
		Soyster	-	-	3569.4	56.7	100.0
300	70%	ULSUB-β	3	-	2704.3	9.11	72.0
		ULSUR	3	3	2704.3	9.11	72.0
	80%	ULSUB-β	4	-	2795.5	12.8	88.8
		ULSUR	4	13	2792.2	12.7	86.5
	90%	ULSUB-β	5	-	2874.5	16.0	90.1
		ULSUR	5	5	2874.5	16.0	90.5
		Nominal	-	-	2478.5	-	36.6
		Soyster	-	-	3825.9	54.4	100.0

Table 5.2: The results for time-invariant unit production cost, with various settings of $\mathbf{f}$, unit production cost $p_i = 3$, and unit inventory holding cost $h_i = 0.3$. Demand realizations are generated uniformly in $(15, 45)$.

p	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
2	70%	ULSUB-β	3	-	1802.9	9.1	72.0
		ULSUR	3	3	1802.9	9.1	72.0
	80%	ULSUB-β	4	-	1863.7	12.8	88.8
		ULSUR	4	8	1859.3	12.5	84.0
	90%	ULSUB-β	5	-	1916.4	16.0	90.1
		ULSUR	5	5	1916.4	16.0	90.1
		Nominal	-	-	1652.3	-	36.6
		Soyster	-	-	2550.6	54.4	100.0
3	70%	ULSUB-β	3	-	2528.8	11.0	71.8
		ULSUR	3	15	2509.3	10.1	78.3
	80%	ULSUB-β	4	-	2619.0	14.9	89.7
		ULSUR	4	13	2604.4	14.3	83.8
	90%	ULSUB-β	5	-	2703.2	18.6	93.1
		ULSUR	5	12	2688.8	18.0	92.0
		Nominal	-	-	2278.5	-	36.6
		Soyster	-	-	3569.4	56.7	100.0
4	70%	ULSUB-β	4	-	3171.8	9.8	71.8
		ULSUR	4	4	3171.8	9.8	71.8
	80%	ULSUB-β	4	-	3292.0	13.9	89.7
		ULSUR	4	14	3269.4	13.1	81.6
	90%	ULSUB-β	5	-	3404.3	17.8	93.1
		ULSUR	5	12	3385.1	17.2	90.6
		Nominal	-	-	2889.5	-	31.3
		Soyster	-	-	4559.3	57.8	100.0
5	70%	ULSUB-β	3	-	3814.7	10.2	71.8
		ULSUR	3	8	3812.8	10.1	73.4
	80%	ULSUB-β	4	-	3965.0	14.5	89.7
		ULSUR	4	14	3937.1	13.7	81.6
	90%	ULSUB-β	5	-	4105.3	18.6	93.1
		ULSUR	5	12	4081.4	17.9	90.6
		Nominal	-	-	3461.9	-	31.3
		Soyster	-	-	5546.6	60.2	100.0

Table 5.3: The experiments for various settings of the time-invariant unit production cost $\mathbf{p}$, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

$\mathbf{h}$	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
$0.1 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	3	-	2528.8	11.0	71.8
		ULSUR	3	15	2509.3	10.1	78.3
	80%	ULSUB-β	4	-	2619.0	14.9	89.7
		ULSUR	4	13	2604.4	14.3	83.8
	90%	ULSUB-β	5	-	2703.2	18.6	93.1
		ULSUR	5	12	2688.8	18.0	92.0
		Nominal	-	-	2278.5	-	36.6
		Soyster	-	-	3569.4	56.7	100.0
$0.2 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	3	-	2972.6	10.7	71.8
		ULSUR	3	8	2970.0	10.6	72.3
	80%	ULSUB-β	4	-	3108.0	15.8	89.7
		ULSUR	4	14	3074.5	14.5	81.6
	90%	ULSUB-β	8	-	3595.6	33.9	97.8
		ULSUR	5	13	3202.7	19.3	90.6
		Nominal	-	-	2684.3	-	31.3
		Soyster	-	-	4470.9	66.6	100.0
$0.3 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	3	-	3472.1	14.4	75.6
		ULSUR	3	9	3384.0	11.5	70.9
	80%	ULSUB-β	6	-	3853.4	27.0	96.2
		ULSUR	5	14	3688.4	21.5	82.5
	90%	ULSUB-β	6	-	3853.4	27.0	96.2
		ULSUR	6	8	3840.0	26.5	92.9
		Nominal	-	-	3035.0	-	25.4
		Soyster	-	-	5250.8	73.0	100.0
$0.4 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	3	-	3801.1	14.1	75.6
		ULSUR	3	13	3735.1	12.2	72.3
	80%	ULSUB-β	6	-	4264.6	38.1	96.2
		ULSUR	5	8	4025.0	20.9	86.0
	90%	ULSUB-β	6	-	4264.6	28.1	96.2
		ULSUR	6	11	4214.0	26.5	90.5
		Nominal	-	-	3330.2	-	23.9
		Soyster	-	-	5992.8	80.0	100.0

Table 5.4: The experiments for time-invariant unit production cost $\mathbf{p}$, with setup cost $f_i = 200$, and various settings of $\mathbf{h}$. Demand realizations are generated uniformly in $(15, 45)$.

$\hat{d}$	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
7.5	70%	ULSUB-β	3	-	2347.7	3.0	72.0
		ULSUR	3	3	2347.7	3.0	72.0
	80%	ULSUB-β	4	-	2393.3	5.0	88.8
		ULSUR	4	15	2390.0	4.9	84.0
	90%	ULSUB-β	5	-	2432.8	6.8	90.1
		ULSUR	5	5	2432.8	6.8	90.1
		Nominal	-	-	2278.5	-	31.3
		Soyster	-	-	2894.7	27.0	100.0
15	70%	ULSUB-β	3	-	2528.8	11.0	71.8
		ULSUR	3	15	2509.3	10.1	78.3
	80%	ULSUB-β	4	-	2619.0	14.9	89.7
		ULSUR	4	13	2604.4	14.3	83.8
	90%	ULSUB-β	5	-	2703.2	18.6	93.1
		ULSUR	5	12	2688.8	18.0	92.0
		Nominal	-	-	2278.5	-	31.3
		Soyster	-	-	3569.4	56.7	100.0
22.5	70%	ULSUB-β	3	-	2683.2	17.8	71.8
		ULSUR	3	15	2668.4	17.1	78.3
	80%	ULSUB-β	4	-	2818.5	23.7	89.7
		ULSUR	4	14	2793.4	22.6	81.6
	90%	ULSUB-β	5	5	2944.8	29.2	93.1
		ULSUR	5	13	2923.3	28.3	90.6
		Nominal	-	-	2278.5	-	31.3
		Soyster	-	-	4442.2	95.0	100.0
30	70%	ULSUB-β	3	-	2837.6	24.5	71.8
		ULSUR	3	3	2837.6	24.5	71.8
	80%	ULSUB-β	4	-	3018.0	32.5	89.7
		ULSUR	4	14	2984.5	31.0	81.6
	90%	ULSUB-β	5	-	3186.4	39.8	93.1
		ULSUR	5	12	3157.7	38.6	90.6
		Nominal	-	-	2278.5	-	31.3
		Soyster	-	-	4918.9	115.9	100.0

Table 5.5: The experiments for time-invariant unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly using various setting of $\hat{\mathbf{d}}$ as shown in the table.

Feasibility level	Model	Γ^d	Θ	V	R_v	Q
70%	ULSUB-β	2	-	2642.1	7.9	76.0
	ULSUR	2	8	2626.8	7.3	72.3
80%	ULSUB-β	3	-	2753.2	12.4	88.9
	ULSUR	3	12	2702.0	10.3	82.2
90%	ULSUB-β	4	-	2838.2	15.8	95.3
	ULSUR	4	14	2791.0	14.0	91.4
-	Nominal	-	-	2449.2	-	31.3
-	Soyster	-	-	3771.3	54.0	100.0

Table 5.6: The experiments for non-increasing unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

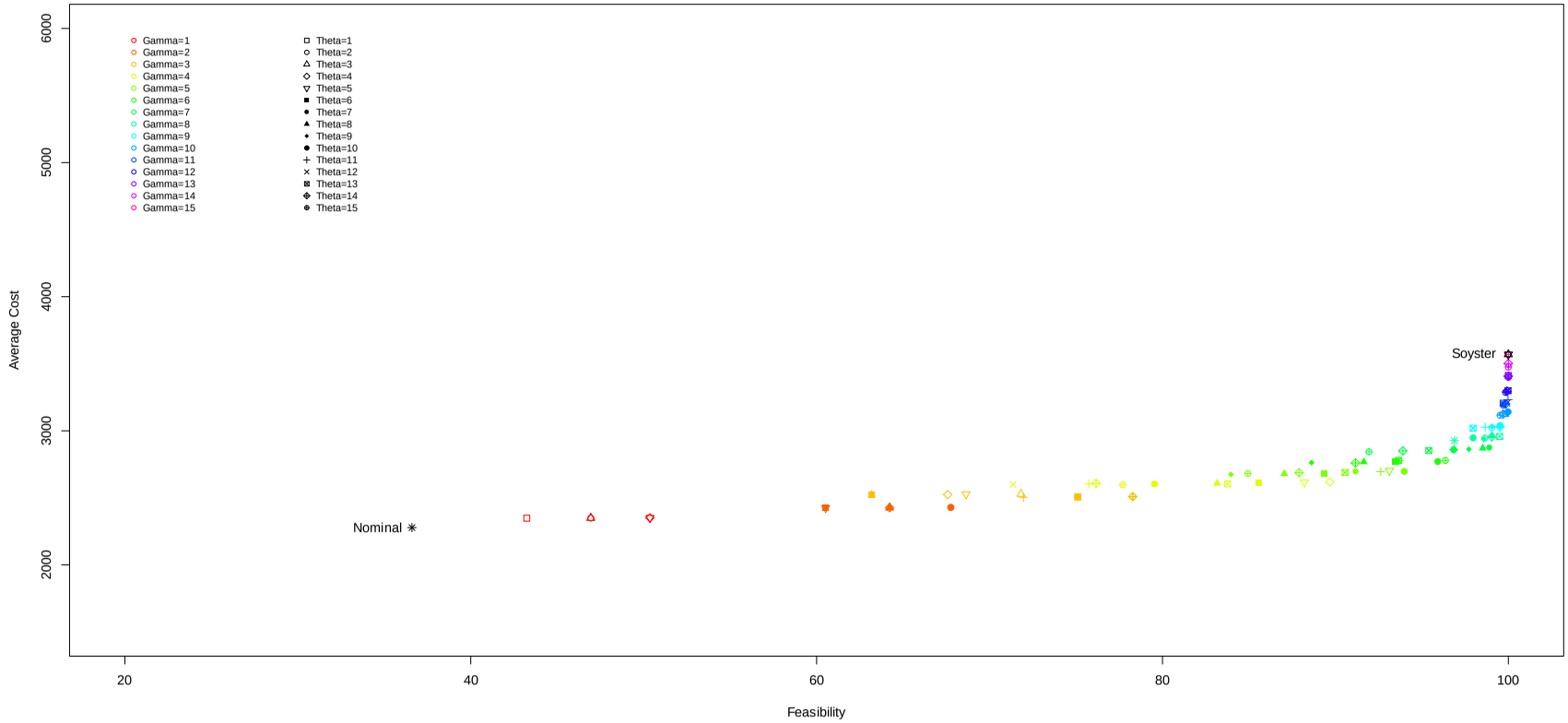


Figure 5.1: The trade-off between the average cost and feasibility for the base experiment. Colors are coded for different values of Γ^d and different symbols are given for different values of Θ. Two special symbols are given for the solutions of the nominal model and Soyster's model.

$\mathbf{f}$	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
50	70%	ULSUB-β	2	-	2095.2	9.6	73.3
		ULSUR	2	6	2082.3	8.9	70.6
	80%	ULSUB-β	3	-	2200.9	15.1	87.7
		ULSUR	3	11	2153.4	12.6	80.2
	90%	ULSUB-β	4	-	2278.9	19.2	94.5
		ULSUR	4	11	2261.2	18.3	91.0
		Nominal	-	-	1911.7	-	23.9
		Soyster	-	-	3100.1	62.2	100.0
100	70%	ULSUB-β	2	-	2314.4	8.4	74.1
		ULSUR	2	7	2307.5	8.1	70.6
	80%	ULSUB-β	3	-	2409.3	12.8	86.6
		ULSUR	3	11	2388.1	11.8	82.4
	90%	ULSUB-β	4	-	2506.8	17.4	93.7
		ULSUR	4	14	2462.4	15.3	90.6
		Nominal	-	-	2135.3	-	27.7
		Soyster	-	-	3370.4	57.8	100.0
200	70%	ULSUB-β	2	-	2642.1	7.9	76.0
		ULSUR	2	8	2626.8	7.3	72.3
	80%	ULSUB-β	3	-	2753.2	12.4	88.9
		ULSUR	3	12	2702.0	10.3	82.2
	90%	ULSUB-β	4	-	2838.2	15.9	95.3
		ULSUR	4	14	2791.0	14.0	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	3771.3	54.0	100.0
300	70%	ULSUB-β	2	-	2893.5	7.4	77.7
		ULSUR	2	10	2865.6	6.4	70.4
	80%	ULSUB-β	3	-	3005.4	11.6	89.8
		ULSUR	3	14	2952.0	9.6	81.2
	90%	ULSUB-β	4	-	3079.9	14.3	94.7
		ULSUR	4	15	3056.5	13.5	91.8
		Nominal	-	-	2693.6	-	36.6
		Soyster	-	-	4071.3	51.1	100.0

Table 5.7: The experiments for non-increasing unit production cost, with various settings of $\mathbf{f}$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

$\mathbf{h}$	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
$0.1 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	2	-	2642.1	7.9	76.0
		ULSUR	2	8	2626.8	7.3	72.3
	80%	ULSUB-β	3	-	2753.2	12.4	88.9
		ULSUR	3	12	2702.0	10.3	82.2
	90%	ULSUB-β	4	-	2838.2	15.9	95.3
		ULSUR	4	14	2791.0	14.0	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	3771.3	54.0	100.0
$0.2 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	3	-	3207.3	14.7	86.6
		ULSUR	2	6	3061.9	9.5	70.6
	80%	ULSUB-β	3	-	3207.3	14.7	86.6
		ULSUR	3	11	3157.9	12.9	81.5
	90%	ULSUB-β	4	-	3299.4	18.0	91.6
		ULSUR	4	11	3293.7	17.8	90.2
		Nominal	-	-	2796.5	-	27.7
		Soyster	-	-	4619.4	65.2	100.0
$0.3 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	3	-	3571.5	18.3	75.6
		ULSUR	3	13	3511.2	16.3	72.3
	80%	ULSUB-β	4	-	3771.3	24.9	93.4
		ULSUR	3	9	3557.6	17.8	80.9
	90%	ULSUB-β	4	-	3771.3	24.9	93.4
		ULSUR	4	6	3744.2	24.0	90.3
		Nominal	-	-	3019.8	-	27.7
		Soyster	-	-	5368.4	77.8	100.0
$0.4 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	3	-	3903.6	21.9	75.6
		ULSUR	3	13	3826.7	19.5	72.3
	80%	ULSUB-β	4	-	4150.1	29.6	93.4
		ULSUR	3	9	3885.2	21.4	80.9
	90%	ULSUB-β	4	-	4150.1	29.6	93.4
		ULSUR	4	6	4115.6	28.6	90.3
		Nominal	-	-	3201.3	-	23.9
		Soyster	-	-	6117.4	91.1	100.0

Table 5.8: The experiments for non-increasing unit production cost, with setup cost $f_i = 200$, and various settings of $\mathbf{h}$. Demand realizations are generated uniformly in $(15, 45)$.

$\hat{d}$	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
7.5	70%	ULSUB-β	2	-	2496.6	1.9	76.0
		ULSUR	2	8	2489.0	1.6	72.3
	80%	ULSUB-β	3	-	2552.2	4.2	88.9
		ULSUR	3	12	2526.6	3.2	82.2
	90%	ULSUB-β	4	-	2594.7	5.9	95.3
		ULSUR	4	14	2571.1	5.0	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	3061.2	25.0	100.0
15	70%	ULSUB-β	2	-	2642.1	7.9	76.0
		ULSUR	2	8	2626.8	7.3	72.3
	80%	ULSUB-β	3	-	2753.2	12.4	88.9
		ULSUR	3	12	2702.0	10.3	82.2
	90%	ULSUB-β	4	-	2838.2	15.9	95.3
		ULSUR	4	14	2791.0	14.0	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	3771.3	54.0	100.0
22.5	70%	ULSUB-β	2	-	2787.6	13.8	76.0
		ULSUR	2	8	2764.6	12.9	72.3
	80%	ULSUB-β	3	-	2954.2	20.6	88.9
		ULSUR	3	12	2877.5	17.5	82.2
	90%	ULSUB-β	4	-	3069.7	25.3	93.1
		ULSUR	4	14	3010.9	22.9	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	4470.8	82.5	100.0
30	70%	ULSUB-β	3	-	3118.8	27.3	87.4
		ULSUR	2	8	2902.4	18.5	72.3
	80%	ULSUB-β	3	-	3118.8	27.3	87.4
		ULSUR	3	12	3053.0	24.7	82.2
	90%	ULSUB-β	4	-	3304.8	24.6	93.1
		ULSUR	4	14	3230.8	34.9	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	5150.8	110.3	100.0

Table 5.9: The experiments for non-increasing unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly using various setting of $\hat{\mathbf{d}}$ as shown in the table.

p	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
$p_1 = 4.25$ and $p_{15} = 1.75$	70%	ULSUB-β	2	-	2682.8	6.8	76.0
		ULSUR	2	8	2665.8	6.1	72.3
	80%	ULSUB-β	3	-	2797.7	11.4	88.9
		ULSUR	3	12	2741.7	9.1	82.2
	90%	ULSUB-β	4	-	2884.0	14.8	95.3
		ULSUR	4	14	2832.4	12.7	91.4
		Nominal	-	-	2512.5	-	31.3
		Soyster	-	-	3821.8	52.1	100.0
$p_1 = 4$ and $p_{15} = 2$	70%	ULSUB-β	2	-	2642.1	7.9	76.0
		ULSUR	2	8	2626.8	7.3	72.3
	80%	ULSUB-β	3	-	2753.2	12.4	88.9
		ULSUR	3	12	2702.0	10.3	82.2
	90%	ULSUB-β	4	-	2838.2	15.9	95.3
		ULSUR	4	14	2791.0	14.0	91.4
		Nominal	-	-	2449.2	-	31.3
		Soyster	-	-	3771.3	54.0	100.0
$p_1 = 3.75$ and $p_{15} = 2.25$	70%	ULSUB-β	2	-	2601.4	5.9	76.0
		ULSUR	2	7	2583.2	5.2	71.2
	80%	ULSUB-β	3	-	2681.3	9.2	86.4
		ULSUR	3	12	2657.4	8.2	81.4
	90%	ULSUB-β	4	-	2792.5	13.7	95.3
		ULSUR	4	14	2749.6	12.0	91.4
		Nominal	-	-	2455.9	-	31.3
		Soyster	-	-	3720.8	51.5	100.0
$p_1 = 3.5$ and $p_{15} = 2.5$	70%	ULSUB-β	2	-	2560.7	5.5	76.0
		ULSUR	2	7	2544.7	4.8	71.2
	80%	ULSUB-β	3	-	2639.5	8.7	86.4
		ULSUR	3	12	2618.3	7.9	81.4
	90%	ULSUB-β	4	-	2746.8	13.1	95.3
		ULSUR	4	14	2708.3	11.6	91.4
		Nominal	-	-	2427.7	-	31.3
		Soyster	-	-	3670.4	51.2	100.0

Table 5.10: The experiments for various settings of the non-increasing unit production cost $\mathbf{p}$, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

Feasibility level	Model	Γ^d	Θ	V	R_v	Q
70%	ULSUB-β	4	-	2347.3	13.8	72.0
	ULSUR	3	14	2270.4	10.1	74.6
80%	ULSUB-β	6	-	2501.1	21.3	88.6
	ULSUR	4	15	2345.9	13.8	82.2
90%	ULSUB-β	8	-	2664.9	29.2	96.6
	ULSUR	6	15	2501.5	21.3	91.2
	Nominal	-	-	2061.9	-	37.9
	Soyster	-	-	3299.5	60.0	100.0

Table 5.11: The experiments for non-decreasing unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

$\mathbf{f}$	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
50	70%	ULSUB-β	10	-	2442.0	37.3	78.0
		ULSUR	7	15	2210.8	24.3	73.7
	80%	ULSUB-β	11	-	2525.8	42.0	80.2
		ULSUR	11	15	2519.6	41.7	91.0
	90%	ULSUB-β	12	12	2604.0	46.4	96.2
		ULSUR	11	15	2519.6	41.7	91.0
		Nominal	-	-	1778.2	-	31.6
		Soyster	-	-	2879.4	61.9	100.0
100	70%	ULSUB-β	6	-	2280.4	19.8	74.9
		ULSUR	3	15	2069.1	8.7	72.0
	80%	ULSUB-β	9	-	2508.5	31.8	91.7
		ULSUR	7	11	2355.7	23.8	83.4
	90%	ULSUB-β	9	-	2508.5	31.8	91.7
		ULSUR	8	15	2436.6	28.0	91.7
		Nominal	-	-	1903.2	-	31.6
		Soyster	-	-	3048.8	60.2	100.0
200	70%	ULSUB-β	4	-	2347.3	13.8	72.0
		ULSUR	3	14	2270.4	10.1	74.6
	80%	ULSUB-β	6	-	2501.1	21.3	88.6
		ULSUR	4	15	2345.9	13.8	82.2
	90%	ULSUB-β	8	-	2664.9	29.2	96.6
		ULSUR	6	15	2501.5	21.3	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	3299.5	60.0	100.0
300	70%	ULSUB-β	2	-	2355.7	7.1	80.1
		ULSUR	2	2	2355.7	7.1	80.1
	80%	ULSUB-β	2	-	2355.7	7.1	80.1
		ULSUR	2	2	2355.7	7.1	80.1
	90%	ULSUB-β	8	-	2864.9	30.3	96.6
		ULSUR	3	11	2440.0	11.0	90.9
		Nominal	-	-	2199.0	-	50.1
		Soyster	-	-	3499.5	59.1	100.0

Table 5.12: The experiments for non-decreasing unit production cost, with various settings of $\mathbf{f}$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

$\mathbf{h}$	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
$0.1 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	4	-	2347.3	13.8	72.0
		ULSUR	3	14	2270.4	10.1	74.6
	80%	ULSUB-β	6	-	2501.1	21.3	88.6
		ULSUR	4	15	2345.9	13.8	82.2
	90%	ULSUB-β	8	-	2664.9	29.2	96.6
		ULSUR	6	15	2501.5	21.3	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	3299.5	60.0	100.0
$0.2 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	3	-	2807.7	16.3	70.5
		ULSUR	3	3	2807.7	16.3	70.5
	80%	ULSUB-β	6	-	3127.2	29.5	91.2
		ULSUR	5	11	3014.7	24.84	83.1
	90%	ULSUB-β	6	-	3127.2	29.5	91.2
		ULSUR	6	6	3127.2	29.5	91.2
		Nominal	-	-	2414.9	-	31.6
		Soyster	-	-	4294.5	77.8	100.0
$0.3 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	5	-	3466.4	32.4	75.8
		ULSUR	5	5	3466.4	32.4	75.8
	80%	ULSUB-β	9	-	4050.6	54.8	96
		ULSUR	6	10	3608.7	37.9	82.8
	90%	ULSUB-β	9	-	4050.6	54.8	96
		ULSUR	7	15	3758.8	43.6	90.6
		Nominal	-	-	2617.4	-	27.4
		Soyster	-	-	5109.6	95.2	100.0
$0.4 \cdot \sum_{t=1}^{n} p_t/n$	70%	ULSUB-β	7	-	4198.8	51.7	80.2
		ULSUR	7	7	4198.8	51.7	80.2
	80%	ULSUB-β	7	-	4198.8	51.7	80.2
		ULSUR	7	7	4198.8	51.7	80.2
	90%	ULSUB-β	12	-	5127.9	85.2	98.1
		ULSUR	9	15	4578.9	65.4	91.4
		Nominal	-	-	2768.6	-	25.4
		Soyster	-	-	5868.1	112.0	100.0

Table 5.13: The experiments for non-decreasing unit production cost, with setup cost $f_i = 200$, and various settings of $\mathbf{h}$. Demand realizations are generated uniformly in $(15, 45)$.

$\hat{d}$	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
7.5	70%	ULSUB-β	4	-	2162.0	4.9	72.0
		ULSUR	3	13	2123.3	3.0	74.6
	80%	ULSUB-β	6	-	2239.0	8.6	88.6
		ULSUR	4	15	2161.3	4.8	82.2
	90%	ULSUB-β	8	-	2320.9	12.6	96.6
		ULSUR	6	15	2239.2	8.6	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	2638.2	27.9	100.0
15	70%	ULSUB-β	4	-	2347.3	13.8	72.0
		ULSUR	3	14	2270.4	10.1	74.6
	80%	ULSUB-β	6	-	2501.1	21.3	88.6
		ULSUR	4	15	2345.9	13.8	82.2
	90%	ULSUB-β	8	-	2664.9	29.2	96.6
		ULSUR	6	15	2501.5	21.3	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	3299.5	60.0	100.0
22.5	70%	ULSUB-β	4	-	2532.5	22.8	72.0
		ULSUR	3	13	2416.3	17.2	74.6
	80%	ULSUB-β	6	-	2763.3	34.0	88.6
		ULSUR	4	15	2530.4	22.7	82.2
	90%	ULSUB-β	8	-	3009.0	45.9	96.6
		ULSUR	6	15	2763.9	34.0	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	3960.9	92.1	100.0
30	70%	ULSUB-β	4	-	2717.7	31.8	72.0
		ULSUR	3	13	2562.8	24.3	74.6
	80%	ULSUB-β	6	-	3025.4	46.7	88.6
		ULSUR	4	15	2715.0	31.7	82.2
	90%	ULSUB-β	8	-	3353.1	63.1	96.6
		ULSUR	6	15	3026.3	46.8	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	4621.3	124.1	100.0

Table 5.14: The experiments for non-decreasing unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly using various setting of $\hat{\mathbf{d}}$ as shown in the table.

p	Feasibility level	Model	Γ^d	Θ	V	R_v	Q
$p_1 = 1.75$ and $p_{15} = 4.25$	70%	ULSUB-β	5	-	2369.3	19.4	77.2
		ULSUR	3	14	2212.1	11.5	72.3
	80%	ULSUB-β	7	-	2520.3	27.0	88.7
		ULSUR	4	15	2287.1	15.3	82.2
	90%	ULSUB-β	9	-	2680.7	35.1	96.6
		ULSUR	7	14	2519.8	27.0	91.2
		Nominal	-	-	1984.1	-	50.1
		Soyster	-	-	3217.9	62.2	100.0
$p_1 = 2$ and $p_{15} = 4$	70%	ULSUB-β	4	-	2347.3	13.8	72.0
		ULSUR	3	14	2270.4	10.1	74.6
	80%	ULSUB-β	6	-	2501.1	21.3	88.6
		ULSUR	4	15	2345.9	13.8	82.2
	90%	ULSUB-β	8	-	2664.9	29.2	96.6
		ULSUR	6	15	2501.5	21.3	91.2
		Nominal	-	-	2061.9	-	37.9
		Soyster	-	-	3299.5	60.0	100.0
$p_1 = 2.25$ and $p_{15} = 3.75$	70%	ULSUB-β	3	-	2327.6	11.9	74.6
		ULSUR	3	13	2327.3	11.9	70.7
	80%	ULSUB-β	5	-	2483.2	19.4	88.4
		ULSUR	4	15	2404.6	15.6	82.2
	90%	ULSUB-β	7	-	2650.1	27.4	96.6
		ULSUR	6	15	2563.8	23.3	91.2
		Nominal	-	-	2080.1	-	36.6
		Soyster	-	-	3381.1	62.5	100.0
$p_1 = 2.5$ and $p_{15} = 3.5$	70%	ULSUB-β	3	-	2384.2	12.5	70.7
		ULSUR	3	-	2384.2	12.5	70.7
	80%	ULSUB-β	4	-	2467.4	16.4	87.4
		ULSUR	4	7	2463.4	16.2	82.2
	90%	ULSUB-β	6	-	2636.3	24.4	96.6
		ULSUR	5	14	2545.4	20.1	90.9
		Nominal	-	-	2119.4	-	36.6
		Soyster	-	-	3462.7	63.4	100.0

Table 5.15: The experiments for various settings of the non-decreasing unit production cost $\mathbf{p}$, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $(15, 45)$.

Chapter 6

The *bw*-Robust Uncapacitated Lot Sizing Model

In this chapter, we study a robust lot sizing model where the robustness is modeled using the *bw*-robustness criterion [151]. In particular, we model the demand uncertainty using both scenarios and uncertainty intervals. As stated in Chapter 1, the model aims to identify a solution that guarantees an objective function value of at most w in all circumstances, and maximizes the number of demand scenarios, alternatively, the proportion of demand realizations in the interval case, in which the objective function is below b. As far as the author is aware, this is the first *bw*-robustness study for the lot sizing model and the first study concerning uncertain parameter that helps define the feasible region. The study in this chapter can be found in [121].

6.1 Introduction

Although originally proposed to deal with uncertainty in the coefficients of the objective function, the *bw*-robustness criterion also has its relevance when applied to uncertain demand, which is a parameter that defines the feasible region. Uncertainty in demand has its impact in two ways. First of all, it harms the feasibility of solutions, making it risky when deciding a production plan. Secondly, it also harms the quality of the objective function

value. While the main priority of most classic robust optimization approaches is to deal with feasibility, the *bw*-robustness criterion focuses on the quality of the objective function value, which is equally important to the decision maker. As far as the author is aware, this is the first study in which the *bw*-robustness criterion is used to deal with an uncertain parameter, in this study demand, that defines the feasible region.

In the robust optimization literature, uncertain parameters are usually modeled in two ways, namely the discrete set of scenarios [117] and the uncertainty interval [33, 36]. In this chapter, we model the uncertain demand using both methodologies. For the former, we assume that the final realized demand is one of the S demand scenarios, i.e., $\mathbf{d} \in \{\mathbf{d}^1, \ldots, \mathbf{d}^S\}$ where $\mathbf{d}^s$ denotes the s-th demand scenario, with $s = 1, \ldots, S$. For the latter, like in the previous chapters, we again model d_i, for $i = 1, \ldots, n$, as an uncertain parameter that takes values in the symmetric and bounded interval $[\tilde{d}_i - \hat{d}_i, \tilde{d}_i + \hat{d}_i]$, where the nominal value $\tilde{d}_i$ denotes the nominal demand and $\hat{d}_i$ the deviation from the nominal demand.

Studies of robust optimization models with interval uncertainty have been reviewed extensively in the previous chapters. With scenario-represented uncertainty, Kouvelis and Yu [117] propose a framework for discrete robust optimization problems. Under their approach, the robust formulation of many polynomially solvable discrete optimization problems becomes $\mathcal{NP}$-hard. Discrete robust optimization models with scenario-represented uncertainty have been applied to many problems. Studies include, but are not limited to, the shortest path problem [77, 117, 140], the spanning tree [117], the 1-median problem [107, 117], and the assignment problem [8, 61].

The remainder of the chapter is organized as follows. In Section 6.2, we formally introduce the *bw*-robustness criterion for demand uncertainty. We formulate the scenario model as a Mixed Integer Linear Programming (MILP) problem. For the interval case, we propose a Mixed Integer Nonlinear Programming (MINLP) formulation, which we subsequently formulate as an MILP. In Section 6.3 we present the computational experiment where we benchmark the performance of the proposed models against the corresponding benchmark models. In Section 6.4, a summary of the chapter is given and lines of future research are proposed.

6.2 The formulation

Given a demand realization vector $\mathbf{d}$ and a setup decision $\mathbf{y}$, we first define by $F(\mathbf{d}, \mathbf{y}) = \sum_{i=1}^{n}(f_i y_i + p_i x_i + h_i I_i)$ the objective function value of the ULS problem formulated in (2.11)-(2.15) in Chapter 2. Please note that the corresponding production quantity $\mathbf{x}$ and the inventory $\mathbf{I}$ are determined by using the ZIO property. Applying the bw-robustness criterion to the ULS problem with uncertain demand amounts to determining a setup decision $\mathbf{y}$ that maximizes the number of scenarios, alternatively, the proportion of demand realizations in the interval case, for which $F(\mathbf{d}, \mathbf{y}) \leq b$, while $F(\mathbf{d}, \mathbf{y}) \leq w$ for all possible demand instances $\mathbf{d}$. Therefore, w can be interpreted as the maximum value allowed in any situation and b, with $b \leq w$, the desired value to be achieved. In the rest of this section, we formulate the bw-robust ULS model with uncertain demand. We begin with the case in which the uncertainty demand is represented by scenarios, yielding an MILP formulation. Then, for the interval based uncertain demand, we formulate the model as an MINLP problem and show that by employing linearization techniques the problem can be transformed to an MILP one.

6.2.1 The bw-robust ULS with scenario-represented uncertain demand

To formulate the bw-ULS problem with demand scenarios, we introduce additional variables. With an abuse of notation, we define $\mathbf{z} = (z^s)$, $s = 1, \ldots, S$, in order to count the number of scenarios in which the objective function is less or equal to b. Moreover, for each demand scenario $\mathbf{d}^s$, we define $\mathbf{x}^s$ and $\mathbf{I}^s$ as the production quantity decision vector and the inventory decision vector. Then, we have

$$bw - ULS - sce: \quad \max_{\mathbf{y},\mathbf{x}^s,\mathbf{I}^s,\mathbf{z}} \sum_{s=1}^{S} z^s \tag{6.1}$$

$$\text{s.t.} \quad I_{i-1}^s + x_i^s = d_i^s + I_i^s, \; \forall s = 1, \ldots, S, \; i = 1, \ldots, n \tag{6.2}$$

$$F(\mathbf{d}^s, \mathbf{y}) \leq w(1 - z^s) + b z^s, \; \forall s = 1, \ldots, S \tag{6.3}$$

$$x_i^s \leq M \, y_i, \; \forall s = 1, \ldots, S, \; i = 1, \ldots, n \tag{6.4}$$

$$y_i \in \{0, 1\}, \; \forall i = 1, \ldots, n \tag{6.5}$$

$$z^s \in \{0, 1\}, \; \forall s = 1, \ldots, S \tag{6.6}$$

$$x_i^s, I_i^s \geq 0, \; \forall s = 1, \ldots, S, \; i = 1, \ldots, n. \tag{6.7}$$

Constraints (6.2) are the flow conservation constraints, as in the ULS problem, for each demand scenario. If $z^s = 0$, constraint (6.3) becomes $F(\mathbf{d}^s, \mathbf{y}) \leq w$, otherwise, $F(\mathbf{d}^s, \mathbf{y}) \leq b$. Therefore, $\mathbf{z}$ is well-defined. All other constraints are similar to those in the ULS formulation (2.11)-(2.15) with the demand scenario $\mathbf{d}^s$.

6.2.2 The bw-robust ULS with interval uncertain demand

As in Chapters 3-5, we let $d_i \in [\underline{d}_i, \overline{d}_i]$, where $\underline{d}_i = \tilde{d}_i - \hat{d}_i$ is the lower bound of the realized demand d_i and $\overline{d}_i = \tilde{d}_i + \hat{d}_i$ is the upper bound. The total cost associated with the setup decision $\mathbf{y}$ varies in the range $[F(\underline{\mathbf{d}}, \mathbf{y}), F(\overline{\mathbf{d}}, \mathbf{y})]$. So we know that the probability of the lot sizing cost being smaller than $F(\underline{\mathbf{d}}, \mathbf{y})$, or being greater than $F(\overline{\mathbf{d}}, \mathbf{y})$, is zero. In Figure 6.1, we give an illustration of the empirical cumulative distribution of the lot sizing cost for a given setup decision $\mathbf{y}$, with randomly generated parameters. With $d_i \in [\underline{d}_i, \overline{d}_i]$, the cumulative distribution of the lot sizing cost can be estimated quite accurately using a normal distribution. Indeed, the shape of the empirical cumulative distribution in Figure 6.1 is very similar to a normal cumulative distribution, for which the middle section does not bend much. Inspired by those observations, we give a reasonable linear approximation to this middle section of the cumulative distribution.

We introduce $\mathbf{d}^{+c}$. For each period i, we let $d_i^{+c} = \tilde{d}_i + c\hat{d}_i$ for $0 < c \leq 1$. Similarly, we introduce $\mathbf{d}^{-c}$ with $d_i^{-c} = \tilde{d}_i - c\hat{d}_i$. For all $d_i \in [d_i^{-c}, d_i^{+c}]$, we define by $P_b(\mathbf{y})$ as follows,

$$P_b(\mathbf{y}) = \begin{cases} 0 & \text{if } b \leq F(\mathbf{d}^{-c}, \mathbf{y}) \\ \frac{b - F(\mathbf{d}^{-c}, \mathbf{y})}{F(\mathbf{d}^{+c}, \mathbf{y}) - F(\mathbf{d}^{-c}, \mathbf{y})} & \text{if } F(\mathbf{d}^{-c}, \mathbf{y}) \leq b \leq F(\mathbf{d}^{+c}, \mathbf{y}) \\ 1 & \text{if } F(\mathbf{d}^{+c}, \mathbf{y}) \leq b. \end{cases}$$

The idea is that if $F(\mathbf{d}^{-c}, \mathbf{y}) \leq b \leq F(\mathbf{d}^{+c}, \mathbf{y})$, $F(\mathbf{d}^{+c}, \mathbf{y}) - F(\mathbf{d}^{-c}, \mathbf{y})$ is the range of all possible objective function values, and $b - F(\mathbf{d}^{-c}, \mathbf{y})$ is the range of objective function values below b. Hence $P_b(\mathbf{y}) = \frac{b - F(\mathbf{d}^{-c}, \mathbf{y})}{F(\mathbf{d}^{+c}, \mathbf{y}) - F(\mathbf{d}^{-c}, \mathbf{y})}$ is an approximation to the proportion of demand realizations, defined by uncertain intervals $[d_i^{-c}, d_i^{+c}]$, with a lot sizing objective function value being less or equal to b. Please note that $P_b(\mathbf{y})$ is defined if and only if $F(\mathbf{d}^{+c}, \mathbf{y}) - F(\mathbf{d}^{-c}, \mathbf{y}) > 0$. This is satisfied when $\hat{d}_i > 0$ for at least one period $i = 1, \ldots, n$.

Using $P_b(\mathbf{y})$, a piecewise linear function of b, for a given $\mathbf{y}$, we aim to estimate the middle range of the corresponding cumulative distribution of the lot sizing cost, as shown in Figure 6.2. The left tail of the empirical cumulative distribution is taken care of by choosing an appropriate value of b. That is, unless b is chosen too small, for which results in $P_b(\mathbf{y}) = 0$, the approximation of the proportion of demand realizations with an objective function value being less or equal to b for $d_i \in [d_i^{-c}, d_i^{+c}]$ is a good indicator for the proportion of demand realizations with an objective function value being less or equal to b for all $[\underline{d}_i, \overline{d}_i]$. Similarly, the right tail of the empirical cumulative distribution is taken care of by choosing a value of b that is not too large, otherwise $P_b(\mathbf{y}) = 1$ which may be too optimistic.

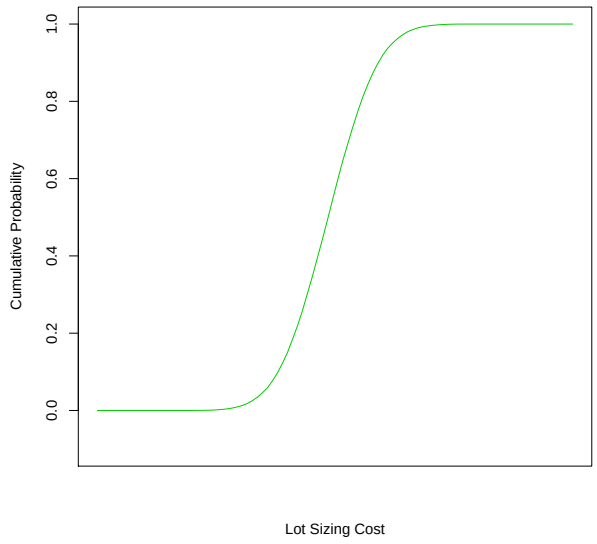


Figure 6.1: An illustration of the empirical cumulative distribution of the lot sizing cost for a given setup decision $\mathbf{y}$.

Then using the *bw*-robustness criterion, our goal is to maximize $P_b(\mathbf{y})$. This leads to the

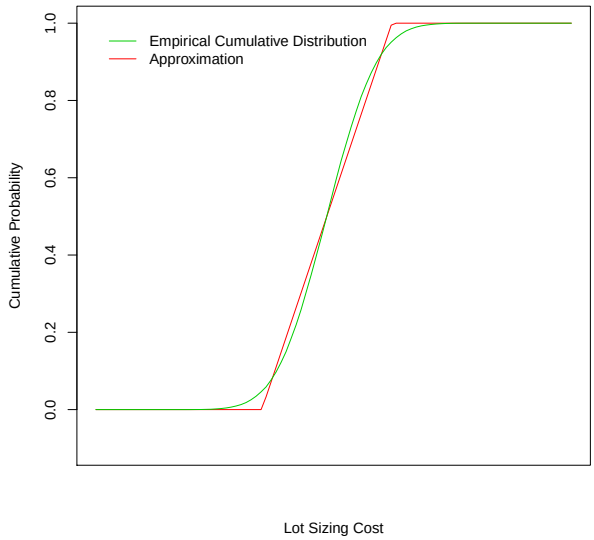


Figure 6.2: An illustration of using $P_b(\mathbf{y})$ with $c = 0.25$ to approximate the empirical cumulative distribution of the lot sizing cost for a given setup decision $\mathbf{y}$. $P_b(\mathbf{y})$ is plotted for continuous values of b.

following MINLP problem:

$$bw - ULS - int - NL: \quad \max_{\mathbf{y},\overline{\mathbf{x}},\overline{\mathbf{I}},\underline{\mathbf{x}},\underline{\mathbf{I}}} \min\left\{\frac{b - F(\underline{\mathbf{d}},\mathbf{y})}{F(\overline{\mathbf{d}},\mathbf{y}) - F(\underline{\mathbf{d}},\mathbf{y})};\, 1\right\} \tag{6.8}$$

$$\text{s.t.} \quad \overline{I}_{i-1} + \overline{x}_i - \overline{I}_i - \overline{d}_i = 0 \tag{6.9}$$

$$\underline{I}_{i-1} + \underline{x}_i - \underline{I}_i - \underline{d}_i = 0 \tag{6.10}$$

$$F(\mathbf{d}^{+c},\mathbf{y}) \leq w \tag{6.11}$$

$$F(\mathbf{d}^{-c},\mathbf{y}) \leq b \tag{6.12}$$

$$\underline{x}_i \leq M\, y_i,\ \forall i = 1,\ldots,n \tag{6.13}$$

$$\overline{x}_i \leq M\, y_i,\ \forall i = 1,\ldots,n \tag{6.14}$$

$$y_i \in \{0,1\},\ \forall i = 1,\ldots,n \tag{6.15}$$

$$\underline{x}_i, \underline{I}_i, \overline{x}_i, \overline{I}_i \geq 0,\ \forall i = 1,\ldots,n. \tag{6.16}$$

In this formulation, $\overline{\mathbf{x}}$ and $\overline{\mathbf{I}}$ are the production and inventory quantities of $F(\mathbf{d}^{+c},\mathbf{y})$, determined by the ZIO property, and similarly for $\underline{\mathbf{x}}$ and $\underline{\mathbf{I}}$.

In the following, we linearize the bw-ULS-int-NL problem in (6.8)-(6.16) using standard techniques in the literature. We first define the following variables,

$$\begin{aligned}
\varphi &= \frac{1}{\sum_{i=1}^{n}[p_i(\overline{x}_i - \underline{x}_i) + h_i(\overline{I}_i - \underline{I}_i)]} \\
\psi_i^{\mathbf{y}} &= \frac{y_i}{\sum_{i=1}^{n}[p_i(\overline{x}_i - \underline{x}_i) + h_i(\overline{I}_i - \underline{I}_i)]} = y_i\varphi \\
\psi_i^{\overline{\mathbf{x}}} &= \frac{\overline{x}_i}{\sum_{i=1}^{n}[p_i(\overline{x}_i - \underline{x}_i) + h_i(\overline{I}_i - \underline{I}_i)]} = \overline{x}_i\varphi \\
\psi_i^{\overline{\mathbf{I}}} &= \frac{\overline{I}_i}{\sum_{i=1}^{n}[p_i(\overline{x}_i - \underline{x}_i) + h_i(\overline{I}_i - \underline{I}_i)]} = \overline{I}_i\varphi \\
\psi_i^{\underline{\mathbf{x}}} &= \frac{\underline{x}_i}{\sum_{i=1}^{n}[p_i(\overline{x}_i - \underline{x}_i) + h_i(\overline{I}_i - \underline{I}_i)]} = \underline{x}_i\varphi \\
\psi_i^{\underline{\mathbf{I}}} &= \frac{\underline{I}_i}{\sum_{i=1}^{n}[p_i(\overline{x}_i - \underline{x}_i) + h_i(\overline{I}_i - \underline{I}_i)]} = \underline{I}_i\varphi.
\end{aligned}$$

We notice that

$$\begin{aligned}
\sum_{i=1}^{n}[p_i(\psi_i^{\overline{\mathbf{x}}} - \psi_i^{\underline{\mathbf{x}}}) + h_i(\psi_i^{\overline{\mathbf{I}}} - \psi_i^{\underline{\mathbf{I}}})] &= \varphi\sum_{i=1}^{n}[p_i(\overline{x}_i - \underline{x}_i) + h_i(\overline{I}_i - \underline{I}_i)] \\
&= \varphi\frac{1}{\varphi} \\
&= 1.
\end{aligned}$$

Therefore, we need to add the constraint $\sum_{i=1}^{n}[p_i(\psi_i^{\overline{\mathbf{x}}} - \psi_i^{\underline{\mathbf{x}}}) + h_i(\psi_i^{\overline{\mathbf{I}}} - \psi_i^{\underline{\mathbf{I}}})] = 1$ to the feasible region.

We now rewrite the objective function (6.8). We have

$$\begin{aligned}
\max_{\mathbf{y}} \min \left\{ \frac{b - F(\underline{\mathbf{d}}, \mathbf{y})}{F(\overline{\mathbf{d}}, \mathbf{y}) - F(\underline{\mathbf{d}}, \mathbf{y})}; 1 \right\} &= \min_{\mathbf{y}} \max \left\{ 1 - \frac{b - F(\underline{\mathbf{d}}, \mathbf{y})}{F(\overline{\mathbf{d}}, \mathbf{y}) - F(\underline{\mathbf{d}}, \mathbf{y})}; 1 - 1 \right\} \\
&= \min_{\mathbf{y}} \max \left\{ \frac{F(\overline{\mathbf{d}}, \mathbf{y}) - b}{F(\overline{\mathbf{d}}, \mathbf{y}) - F(\underline{\mathbf{d}}, \mathbf{y})}; 0 \right\} \\
&= \min_{\mathbf{y}} \max \left\{ \frac{\sum_{i=1}^{n} [f_i y_i + p_i \overline{x}_i + h_i \overline{I}_i] - b}{\sum_{i=1}^{n} [f_i (y_i - y_i) + p_i (\overline{x}_i - \underline{x}_i) + h_i (\overline{I}_i - \underline{I}_i)]}; 0 \right\} \\
&= \min_{\mathbf{y}} \max \left\{ \frac{\sum_{i=1}^{n} [f_i y_i + p_i \overline{x}_i + h_i \overline{I}_i] - b}{\sum_{i=1}^{n} [p_i (\overline{x}_i - \underline{x}_i) + h_i (\overline{I}_i - \underline{I}_i)]}; 0 \right\} \\
&= \min_{\mathbf{y}} \max \left\{ \sum_{i=1}^{n} [f_i \psi_i^{\mathbf{y}} + p_i \psi_i^{\overline{\mathbf{x}}} + h_i \psi_i^{\overline{\mathbf{I}}}] - b\varphi; 0 \right\}.
\end{aligned}$$

We introduce a variable q which represents $\max\{\sum_{i=1}^{n} [f_i \psi_i^{y} + p_i \psi_i^{\overline{\mathbf{x}}} + h_i \psi_i^{\overline{\mathbf{I}}}] - b\varphi; 0\}$. Hence the objective function can be rewritten as

$$\min q.$$

In addition, we know that

$$q = \begin{cases} 0 & \text{if } \sum_{i=1}^{n} [f_i \psi_i^{y} + p_i \psi_i^{\overline{\mathbf{x}}} + h_i \psi_i^{\overline{\mathbf{I}}}] - b\varphi \leq 0 \\ \sum_{i=1}^{n} [f_i \psi_i^{\mathbf{y}} + p_i \psi_i^{\overline{\mathbf{x}}} + h_i \psi_i^{\overline{\mathbf{I}}}] - b\varphi & \text{else.} \end{cases}$$

Therefore, we need to add the following constraints to the feasible region

$$\begin{aligned}
q &\geq 0 \\
q &\geq \sum_{i=1}^{n} [f_i \psi_i^{y} + p_i \psi_i^{\overline{\mathbf{x}}} + h_i \psi_i^{\overline{\mathbf{I}}}] - b\varphi.
\end{aligned}$$

In the following, we handle the constraints of the bw-ULS-int-NL problem. We have two flow balance constraints (6.9) and (6.10), which can be written as:

$$\begin{aligned}
\psi_{i-1}^{\overline{\mathbf{I}}} + \psi_i^{\overline{\mathbf{x}}} - \psi_i^{\overline{\mathbf{I}}} - \overline{d}_i \varphi &= 0 \\
\psi_{i-1}^{\underline{\mathbf{I}}} + \psi_i^{\underline{\mathbf{x}}} - \psi_i^{\underline{\mathbf{I}}} - \underline{d}_i \varphi &= 0.
\end{aligned}$$

Constraints (6.13) and (6.14) can be rewritten as

$$\psi_i^{\overline{\mathbf{x}}} - M\psi_i^{\mathbf{y}} \leq 0$$
$$\psi_i^{\underline{\mathbf{x}}} - M\psi_i^{\mathbf{y}} \leq 0.$$

For constraint (6.11), we have that

$$\begin{aligned}
F(\overline{\mathbf{d}}, \mathbf{y}) \leq w \quad &\Leftrightarrow \quad F(\overline{\mathbf{d}}, \mathbf{y}) - w \leq 0 \\
&\Leftrightarrow \quad \sum_{i=1}^{n} [f_i y_i + p_i \overline{x}_i + h_i \overline{I}_i] - w \leq 0 \\
&\Leftrightarrow \quad \sum_{i=1}^{n} [f_i \frac{\psi_i^{\mathbf{y}}}{\varphi} + p_i \frac{\psi_i^{\overline{\mathbf{x}}}}{\varphi} + h_i \frac{\psi_i^{\overline{\mathbf{I}}}}{\varphi}] - w \leq 0 \\
&\Leftrightarrow \quad \sum_{i=1}^{n} [f_i \psi_i^{\mathbf{y}} + p_i \psi_i^{\overline{\mathbf{x}}} + h_i \psi_i^{\overline{\mathbf{I}}}] - w\varphi \leq 0.
\end{aligned}$$

Since $\varphi = \frac{1}{\sum_{i=1}^{n} [p_i(\overline{x}_i - \underline{x}_i) + h_i(\overline{I}_i - \underline{I}_i)]} = \frac{1}{F(\underline{\mathbf{d}}, \mathbf{y}) - F(\overline{\mathbf{d}}, \mathbf{y})}$, we have $F(\underline{\mathbf{d}}, \mathbf{y}) = F(\overline{\mathbf{d}}, \mathbf{y}) - \frac{1}{\varphi}$. Hence constraint (6.12) can be reformulated as

$$\begin{aligned}
F(\underline{\mathbf{d}}, \mathbf{y}) \leq b \quad &\Leftrightarrow \quad F(\overline{\mathbf{d}}, \mathbf{y}) - \frac{1}{\varphi} \leq b \\
&\Leftrightarrow \quad F(\overline{\mathbf{d}}, \mathbf{y}) - \frac{1}{\varphi} - b \leq 0 \\
&\Leftrightarrow \quad \sum_{i=1}^{n} [f_i \frac{\psi_i^{\mathbf{y}}}{\varphi} + p_i \frac{\psi_i^{\overline{\mathbf{x}}}}{\varphi} + h_i \frac{\psi_i^{\overline{\mathbf{I}}}}{\varphi}] - \frac{1}{\varphi} - b \leq 0 \\
&\Leftrightarrow \quad \sum_{i=1}^{n} [f_i \frac{\psi_i^{\mathbf{y}}}{\varphi} + p_i \frac{\psi_i^{\overline{\mathbf{x}}}}{\varphi} + h_i \frac{\psi_i^{\overline{\mathbf{I}}}}{\varphi}] - \frac{1}{\varphi} - b \leq 0 \\
&\Leftrightarrow \quad \sum_{i=1}^{n} [f_i \frac{\psi_i^{\mathbf{y}}}{\varphi} + p_i \frac{\psi_i^{\overline{\mathbf{x}}}}{\varphi} + h_i \frac{\psi_i^{\overline{\mathbf{I}}}}{\varphi}] - b \leq \frac{1}{\varphi} \\
&\Leftrightarrow \quad \sum_{i=1}^{n} [f_i \psi_i^{\mathbf{y}} + p_i \psi_i^{\overline{\mathbf{x}}} + h_i \psi_i^{\overline{\mathbf{I}}}] - b\varphi \leq 1.
\end{aligned}$$

Recall that $\psi_i^{\mathbf{y}}$ is the product of two variables, namely $\psi_i^{\mathbf{y}} = y_i\varphi$. To ensure that $\psi_i^{\mathbf{y}}$ is

well-defined, we need to add the following constraints:

$$\begin{aligned}
\psi_i^{\mathbf{y}} &\leq M' y_i \\
\psi_i^{\mathbf{y}} &\leq \varphi \\
\psi_i^{\mathbf{y}} &\geq \varphi - (1 - y_i) M' \\
\psi_i^{\mathbf{y}} &\geq 0,
\end{aligned}$$

where M' is a sufficiently large number.

Therefore, we can transform the original MINLP model (6.8)-(6.16), to an MILP formulation, as follows

$$\begin{aligned}
bw - ULS - int: \quad & \min q \\
\text{s.t.} \quad & \psi_{i-1}^{\bar{\mathbf{I}}} + \psi_i^{\bar{\mathbf{x}}} - \psi_i^{\bar{\mathbf{I}}} - \overline{d}_i \varphi = 0, \, \forall i = 1, \ldots, n \\
& \psi_{i-1}^{\underline{\mathbf{I}}} + \psi_i^{\underline{\mathbf{x}}} - \psi_i^{\underline{\mathbf{I}}} - \underline{d}_i \varphi = 0, \, \forall i = 1, \ldots, n \\
& q \geq \sum_{i=1}^{n} [f_i \psi_i^{y} + p_i \psi_i^{\bar{\mathbf{x}}} + h_i \psi_i^{\bar{\mathbf{I}}}] - b\varphi \\
& \sum_{i=1}^{n} [f_i \psi_i^{y} + p_i \psi_i^{\bar{\mathbf{x}}} + h_i \psi_i^{\bar{\mathbf{I}}}] - w\varphi \leq 0 \\
& \sum_{i=1}^{n} [f_i \psi_i^{y} + p_i \psi_i^{\bar{\mathbf{x}}} + h_i \psi_i^{\bar{\mathbf{I}}}] - b\varphi \leq 1 \\
& \psi_i^{\bar{\mathbf{x}}} - M \psi_i^{\mathbf{y}} \leq 0, \, \forall i = 1, \ldots, n \\
& \psi_i^{\underline{\mathbf{x}}} - M \psi_i^{\mathbf{y}} \leq 0, \, \forall i = 1, \ldots, n \\
& \sum_{i=1}^{n} [p_i (\psi_i^{\bar{\mathbf{x}}} - \psi_i^{\underline{\mathbf{x}}}) + h_i (\psi_i^{\bar{\mathbf{I}}} - \psi_i^{\underline{\mathbf{I}}})] = 1 \\
& \psi_i^{\mathbf{y}} \leq M' y_i, \, \forall i = 1, \ldots, n \\
& \psi_i^{\mathbf{y}} \leq \varphi, \, \forall i = 1, \ldots, n \\
& \psi_i^{\mathbf{y}} \geq \varphi - (1 - y_i) M', \, \forall i = 1, \ldots, n \\
& y_i \in \{0, 1\}, \forall i = 1, \ldots, n \\
& q, \, \varphi, \, \psi_i^{\mathbf{y}}, \, \psi_i^{\bar{\mathbf{x}}}, \, \psi_i^{\bar{\mathbf{I}}}, \, \psi_i^{\underline{\mathbf{x}}}, \, \psi_i^{\underline{\mathbf{I}}} \geq 0, \, \forall i = 1, \ldots, n.
\end{aligned}$$

6.3 Numerical results

In this section, we show some numerical results for the proposed robust models, namely the bw-ULS-sce model and the bw-ULS-int model. The purpose of this computational study is to investigate the effectiveness of these two models using two criteria, the level of feasibility and the quality of the objective function. The experiments are carried out with various parameter settings. For each parameter setting, we will show performance results for the bw-ULS-sce model and the bw-ULS-int model, against the corresponding benchmark model. In the rest of this section, we first present the experiment setup in Section 6.3.1 and then present the experiment results in Section 6.3.2 for the scenario case and in Section 6.3.3 for the interval case.

6.3.1 Experiment setup

Parameter settings

In all experiments, we have $n = 15$. We experiment with setup costs equal to $f_i = 200$. For the unit inventory holding costs, we assume that $h_i = \lambda \cdot \sum_{t=1}^{n} p_t/n$, where $\lambda = 0.1, 0.2, 0.3, 0.4$.

We distinguish three patterns for $\mathbf{p}$. We consider a sequence of randomly generated unit production costs according to a uniform distribution between 2.5 and 3.5, $p_i \sim U(2.5, 3.5)$ ($\mathbf{p}^R$), followed by increasing and equally spaced production costs with $p_1 = 2.5$ and $p_{15} = 3.5$ ($\mathbf{p}^I$), and decreasing and equally spaced ones with $p_1 = 3.5$ and $p_{15} = 2.5$ ($\mathbf{p}^D$).

We also experiment with various settings of the uncertain demand, i.e., the scenarios in the bw-ULS-sce model and the uncertainty intervals in the bw-ULS-int model. We let the nominal demand $\tilde{d}_i = 45$, while for the deviation component, we consider $\hat{d}_i = \delta \cdot \tilde{d}_i$, with $\delta = 0.25, 0.50, 0.75, 1.00$. Then, for the bw-ULS-sce model, each demand scenario d_i^s is randomly generated according to a uniform distribution between $\tilde{d}_i + \hat{d}_i$ and $\tilde{d}_i - \hat{d}_i$, i.e., $d_i^s \sim U(\tilde{d}_i + \hat{d}_i, \tilde{d}_i - \hat{d}_i)$. For the bw-ULS-int model, we assume that the uncertainty interval is equal to $[\tilde{d}_i + \hat{d}_i, \tilde{d}_i - \hat{d}_i]$.

We have tested the models against other parameter settings such as the size of setup cost and the size of the nominal demand $\tilde{d}_i$. However, the effect of those mimic the ones of unit inventory holding cost and the demand deviation $\hat{d}_i$. Therefore the results are not included.

We need to define an appropriate choice of b and w. We define

$$F(\mathbf{d}^{bes}, \mathbf{y}^{bes}) = \min_{\mathbf{d}} \min_{\mathbf{y}} F(\mathbf{d}, \mathbf{y})$$
$$F(\mathbf{d}^{wor}, \mathbf{y}^{wor}) = \max_{\mathbf{d}} \min_{\mathbf{y}} F(\mathbf{d}, \mathbf{y}),$$

where $\mathbf{d} \in \{\mathbf{d}^1, \ldots, \mathbf{d}^S\}$ for the scenario case, and $d_i \in [\underline{d}_i, \overline{d}_i]$ for the interval case. We let $w = 1.1 \cdot F(\mathbf{d}^{wor}, \mathbf{y}^{wor})$, i.e., the proposed bw-robust models would tolerate a 10% increase in the objective function value in the worst case. We consider several values of b, namely, $b = F(\mathbf{d}^{bes}, \mathbf{y}^{bes}) + \theta \cdot \{F(\mathbf{d}^{wor}, \mathbf{y}^{wor}) - F(\mathbf{d}^{bes}, \mathbf{y}^{bes})\}$, where $\theta = 0.1$, 0.2, 0.3, 0.4, 0.5.

For the interval case, we let $c = 0.25$.

Benchmarking

A natural and intuitive way to overcome demand uncertainty in practice is to use the deterministic ULS model formulated in (2.11)-(2.15), with some statistical description of the uncertain demand, $\mathbf{d}'$. In this way, $\mathbf{d} = \mathbf{d}'$ in the flow conservation constraints (2.12). In this section, we use this benchmark to assess the performance of the proposed bw-robust models. The choice of $\mathbf{d}'$ depends on the modeling of the uncertainty. For the scenario case, we let d'_i be the γ-th percentile of $\{d_i^1, \ldots, d_i^S\}$. For the interval case, a sequence of 100 demand instances are drawn uniformly from $[\underline{d}_i, \overline{d}_i]$ and d'_i is defined similarly as for the scenario case. In our experiments, we choose $\gamma = 60$ and 70.

We will use the same $\mathbf{d}'$ in the bw-robust models to make the comparison fair. Recall that bw-robust model returns a setup vector, namely $\mathbf{y}^*$. For each of them, we define the

corresponding production amounts using the vector $\mathbf{d}'$. Given $\mathbf{y}^*$ and $\mathbf{d}'$, $\mathbf{x}^*(\mathbf{y}^*, \mathbf{d}')$ is determined using the ZIO property.

In reality, the choice of $\mathbf{d}'$ may come from a more accurate statistical description of the uncertain demand. Alternatively, one can obtain $\mathbf{d}'$ by using other robust models to ensure extra robustness. For instance, one may let $\mathbf{d}'$ be the worst-case demand defined by Soyster's absolute robust model, in which case $\mathbf{d}' = \overline{\mathbf{d}}$. The interpretation is that, across models producing the same demand, $\mathbf{y}^*$ ensures the largest the number of scenarios, alternatively, the largest proportion of demand realizations in the interval case, for which the lot sizing cost is smaller or equal to the desire value b.

Performance criteria

To assess the performance of the proposed models, we present average results on feasibility level and quality of the objective function across a collection of experiment demand instances. The experiment demand instances are defined as follows

- bw-ULS-sce model: The total number of demand scenarios is equal to $S = 100$. The experiment demand instances are the demand scenarios.

- bw-ULS-int model: The experiment demand instances are 5000 demand realizations generated uniformly from $[\underline{d}_i, \overline{d}_i]$.

We report average results across the experiment demand instances for the feasibility level and the quality of the objective function. In order to do so, for each experiment demand instance, we first check whether $(\mathbf{y}^*, \mathbf{x}^*(\mathbf{y}^*, \mathbf{d}'))$ is feasible, i.e., whether we can find a feasible vector of inventory levels. If that is the case, then we calculate the total ULS costs. A similar procedure is followed for the optimal solution of the benchmark model.

To present the results, for given $(\mathbf{y}^*, \mathbf{x}^*(\mathbf{y}^*, \mathbf{d}'))$ and $(\mathbf{y}^n, \mathbf{x}^n(\mathbf{y}^n, \mathbf{d}'))$, we define

- $Q(\mathbf{d}')$ as the percentage of the feasible experiment demand instances for the *bw*-robust model, i.e.,

$$Q(\mathbf{d}') = \frac{\text{Number of feasible experiment demand instances}}{\text{Total number of experiment demand instances}};$$

- $Q_n(\mathbf{d}')$ as the percentage of the feasible experiment demand instances for the benchmark model, defined similarly as $Q(\mathbf{d}')$;

- $V(\mathbf{d}')$ as the average cost for the *bw*-robust model, across the feasible experiment demand instances for both the *bw*-robust model and the benchmark model;

- $V_n(\mathbf{d}')$ as the average cost for the benchmark model, across the feasible experiment demand instances for both the *bw*-robust model and the benchmark model;

- $R_q(\mathbf{d}')$ as the increase (in percentage) in the number of feasible experiment demand instances compared to the corresponding benchmark model, i.e.,

$$R_q(\mathbf{d}') = \frac{Q(\mathbf{d}') - Q_n(\mathbf{d}')}{Q_n(\mathbf{d}')} \cdot 100;$$

- $R_v(\mathbf{d}')$ the increase (in percentage) in average cost across the feasible experiment demand instances, compared to the corresponding benchmark model, i.e.,

$$R_v(\mathbf{d}') = \frac{V(\mathbf{d}') - V_n(\mathbf{d}')}{V_n(\mathbf{d}')} \cdot 100.$$

In the rest of this section, we report Q, V, R_q, R_v for each of the experiment parameter settings.

6.3.2 The scenario case

This section is devoted to the scenario model, namely, the bw-ULS-sce model. We first present experiment results for the three types of unit production costs, $\mathbf{p}^R$, $\mathbf{p}^I$, and $\mathbf{p}^D$ with $\delta = 0.5$. Then using the experiment for unit production cost as a base experiment, we test the effect of the size of unit inventory holding cost and the size of demand deviation, altering one at a time.

For p

The results can be found in Table 6.1. At a first glance, we notice that the differences in average total cost of the proposed model and the corresponding benchmark model are negligible, all being less then 1%. The bw-ULS-sce model gains its advantage with $\mathbf{p}^I$, improving feasibility by 6% to 8% compared to the corresponding benchmark model. On the other hand, with $\mathbf{p}^R$ and $\mathbf{p}^D$, there is not a significant difference between the *bw*-robust model and the benchmark model in most circumstances. One possible reason for this is that the optimal setup decision of the bw-ULS-sce model considers not only high demand scenarios but also those with low demand. With $\mathbf{p}^I$, the bw-ULS-sce model gives a setup decision with longer regeneration intervals in order for those scenarios with low demand to benefit more from the low production cost at earlier periods, and hence the number of scenarios in which the total cost is smaller or equal to b is maximized. In lot sizing problems, longer regeneration interval are more robust as demand uncertainty may be overcome by holding inventory.

For h

The experiment results for various settings of unit inventory holding cost are presented in Tables 6.2-6.4. With $\mathbf{p}^R$, we notice that the performance of the bw-ULS-sce model starts to surpass that of the benchmark model at $\lambda = 0.3$ with $\mathbf{d}'$ being the 70-percentile. When the unit inventory holding cost is further increased by setting $\lambda = 0.4$, the bw-ULS-sce model manages to improve the feasibility level by 5% to 16% compared to the corresponding benchmark model, while the increase in total cost is negligible. On the contrary, with $\mathbf{p}^I$ the bw-ULS-sce model loses its advantage compared to the benchmark model as λ grows. With $\lambda = 0.4$, this advantage disappears completely with $\mathbf{d}'$ being the 60-percentile. With $\mathbf{p}^D$, both the bw-ULS-sce model and the benchmark model behave very consistently in response to the change in unit inventory holding cost. The difference in performance remains the same across all values of λ, with the benchmark model reaching a 4% higher feasibility level when $\mathbf{d}'$ is the 60-th percentile.

For $\mathbf{d}^s$

From Tables 6.5-6.7, we see that as the deviation of demand used to generate the scenarios increases, both the bw-ULS-sce model and the benchmark model suffer by showing a

sharp reduction in feasibility level. However, the bw-ULS-sce model suffers marginally less by showing an improvement in R_q. In particular, for $\mathbf{p}^I$, the bw-ULS-sce model consistently outperforms the benchmark model when δ is at least 0.50, improving the feasibility level by around 5% to 9%. In addition, we see that with $\mathbf{p}^R$ and $\mathbf{p}^D$, the performance of the bw-ULS-sce model starts to surpass that of the corresponding benchmark model with $\delta = 0.75$ and 1.00.

6.3.3 The interval case

For the interval case, we again use experiment results for the three types of unit production costs, $\mathbf{p}^R$, $\mathbf{p}^I$, and $\mathbf{p}^D$, and $\delta = 0.50$ as the base experiment. As before, we test our models with various parameter settings, namely the size of unit inventory holding cost and the size of demand deviation, altering one at a time.

For p

The results can be found in Table 6.8. When producing at 60-th percentile, the bw-ULS-int model gains its advantage with $\mathbf{p}^I$, improving the feasibility level by 9%-16% while reaching 3%-5% reduction in average total cost. The bw-ULS-int model show signs of better performance with $\mathbf{p}^D$ as well when $\theta = 0.5$. We see a 13.8% improvement in the feasibility level while reaching similar average total cost as the benchmark model. With $\mathbf{p}^R$, the bw-ULS-int model shows a very similar performance as the benchmark model in general. Noticeable difference occurs when $\theta = 0.5$, the bw-ULS-int model improves the feasibility level by 2.7% at the expense of a 4.1% increase in average total cost.

When producing at 70-th percentile, we notice the differences in the performances shrink. With $\mathbf{p}^I$, the improvements of the bw-ULS-int model in feasibility level ranges from 4.4%-6.4% while obtaining similar average total cost as the benchmark model. We still see the bw-ULS-int model reaching higher feasibility level with $\mathbf{p}^D$ when $b = 3870.4$. However the difference is 5.9% comparing with 13.8% when producing at 60-th percentile. Again the bw-ULS-int model performs similarly with the benchmark model in terms of feasibility level. The bw-ULS-int model leads to 5.6% higher average total cost when $b = 3604.7$.

For h

The experiment results for various settings of unit inventory holding cost are presented in Tables 6.9-6.11.

With $\mathbf{p}^D$, the relative performance of the bw-ULS-int model compared to the benchmark model fluctuates as λ increases. We see a significant increase in the relative performance of the bw-ULS-int model when $\lambda = 0.2$, increasing the feasibility level acrose all values of b and reducing the average total cost in most cases. In particular, when $b > 3117.8$, the increase in the feasibility level ranges from 9.1%-12.3% when producing at 60-th percentile, and this increase ranges from 5.4%-6.2% when producing at 60-th percentile. However, when λ is further increased to 0.3, the improvements in the feasibility level drops and the bw-ULS-int model gives a similar performance as in the base experiment. Finally when $\lambda = 0.4$, the bw-ULS-int model increases the feasibility level by around 8% while reducing the average total cost by around 6% when producing at 70-th percentile.

Similar behaviour is noticed for $\mathbf{p}^R$. We see a slight improvement in the relative performance of the bw-ULS-int model compared to the benchmark model when λ is increased to 0.2, and a drop in the relative performance when λ is increased to 0.3. The bw-ULS-int model outperforms the benchmark model with $\lambda = 0.4$. For $b = 3655.5$ and 4206.3, the bw-ULS-int model leads feasibility levels that are 9%-18% higher while reducing the average total cost by 5%-10%.

With $\mathbf{p}^I$, the relative performance of the bw-ULS-int model compared to the benchmark model behaves in a different manner. As λ is increased to 0.2, the performance of the benchmark model surpasses that of the bw-ULS-int model, by giving 2%-7% higher feasibility levels and 2%-7% average cost reductions for different values of b. When $\lambda = 0.3$, the bw-ULS-int model outperforms the benchmark model again. Especially for $b = 4351.5$, the bw-ULS-int model leads to an 18.3% increase in the feasibility level and a 6.8% average cost reduction when producing at 60-th percentile, and to a 10.7% increase in the feasibility level

and a 3.7% average cost reduction when producing at 70-th percentile. When λ is further increased to 0.4, the two models give the same optimal solution for the first four values of b. For $b = 4582.8$, The bw-ULS-int model fails to perform by giving feasibility levels that are around 27% lower compared to the benchmark model, for both the 60-th and the 70-th percentile. The average total costs are about 10% higher for the bw-ULS-int model as well.

For $\hat{\mathbf{d}}$

From Tables 6.12-6.14, we notice that in general the bw-ULS-int model performs better as demand deviation gets larger for $\mathbf{p}^R$ and $\mathbf{p}^D$. This trend is not very clear for $\mathbf{p}^I$.

With $\mathbf{p}^R$, we see that in majority of the parameter settings, the bw-ULS-int model gives higher feasibility levels, ranging from 3.5% to 16.3%. The corresponding average total costs are similar for the bw-ULS-int model and the benchmark model. The most noticeable difference in average total cost lies in $\delta = 1$ and $b = 4116.7$. The average total cost of the bw-ULS-int model is 4.4% and 7.7% higher for the 60-th and 70-th percentile respectively. However in these two cases the corresponding feasibility levels are 16.3% and 9.9% higher for the bw-ULS-int model as well. Similarly, for $\mathbf{p}^D$, the bw-ULS-int model outperforms the benchmark model in terms of feasibility in general, especially for those with larger values of b. For $\delta = 0.75$ and 1.00, the increase in the feasibility level led by the bw-ULS-int model is accompanied by the increase in the average total cost.

With $\mathbf{p}^I$, the results are quite consistent in terms of the feasibility level, across values of δ. With most parameter settings, the bw-ULS-int model reports an increased feasibility level compared with the benchmark model. As δ increases, there is a trend that the bw-ULS-int model leads to slightly higher average total cost in some cases, but not as much as with the other two unit production cost patterns.

6.4 Summary

Using the concept of *bw*-robustness, in this chapter we have studied robust lot sizing models. The uncertain demand is modeled by means of scenarios as well as uncertainty intervals. We have formulated the scenario model as an MILP problem. For the interval case, we propose an MINLP formulation, which we subsequently formulate as an MILP. We have provided a comprehensive computational experiment on the factors that affect the performance of the models. We see that the proposed scenario model outperforms the benchmark model in many circumstances especially when unit inventory holding cost or the deviation of demand is high. For the interval case, the proposed model leads to higher feasibility levels in general.

$\mathbf{p}$	Percentile	b	Q	V	R_q	R_v
$\mathbf{p}^R$	60	2581.9	65.0%	3389.2	-10.8%	1.3%
		2663.9	72.0%	3351.5	0.0%	0.2%
		2746.0	72.0%	3344.9	0.0%	0.0%
		2828.1	72.0%	3344.9	0.0%	0.0%
		2910.2	72.0%	3351.5	0.0%	0.2%
		Benchmark	72.0%	3344.9	0.0%	0.0%
	70	2581.9	94.0%	3673.0	0.0%	0.0%
		2663.9	97.0%	3677.9	3.1%	0.1%
		2746.0	94.0%	3673.0	0.0%	0.0%
		2828.1	94.0%	3673.0	0.0%	0.0%
		2910.2	97.0%	3677.9	3.1%	0.1%
		Benchmark	94.0%	3673.0	0.0%	0.0%
$\mathbf{p}^I$	60	2584.0	78.0%	3391.6	2.6%	0.9%
		2664.9	81.0%	3366.8	6.2%	0.2%
		2745.8	83.0%	3368.6	8.4%	0.2%
		2826.7	83.0%	3368.6	8.4%	0.2%
		2907.6	81.0%	3366.8	6.2%	0.2%
		Benchmark	76.0%	3360.7	0.0%	0.0%
	70	2584.0	99.0%	3699.1	6.1%	0.6%
		2664.9	99.0%	3699.1	6.1%	0.6%
		2745.8	99.0%	3704.9	6.1%	0.8%
		2826.7	99.0%	3704.9	6.1%	0.8%
		2907.6	99.0%	3699.1	6.1%	0.6%
		Benchmark	93.0%	3676.7	0.0%	0.0%
$\mathbf{p}^D$	60	2774.7	75.0%	3577.7	-1.3%	0.3%
		2862.6	73.0%	3583.4	-4.1%	0.4%
		2950.4	73.0%	3583.4	-4.1%	0.4%
		3038.2	76.0%	3567.7	0.0%	0.0%
		3126.0	76.0%	3567.7	0.0%	0.0%
		Benchmark	76.0%	3567.7	0.0%	0.0%
	70	2774.7	93.0%	3926.2	0.0%	0.6%
		2862.6	93.0%	3911.5	0.0%	0.2%
		2950.4	97.0%	3910.7	4.1%	0.2%
		3038.2	93.0%	3902.9	0.0%	0.0%
		3126.0	93.0%	3902.9	0.0%	0.0%
		Benchmark	93.0%	3902.9	0.0%	0.0%

Table 6.1: Scenario case. The experiment results for the three patterns of unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$.

λ	Percentile	b	Q	V	R_q	R_v
0.1	60	2581.9	65.0%	3389.2	-10.8%	1.3%
		2663.9	72.0%	3351.5	0.0%	0.2%
		2746.0	72.0%	3344.9	0.0%	0.0%
		2828.1	72.0%	3344.9	0.0%	0.0%
		2910.2	72.0%	3351.5	0.0%	0.2%
		Benchmark	72.0%	3344.9	0.0%	0.0%
	70	2581.9	94.0%	3673.0	0.0%	0.0%
		2663.9	97.0%	3677.9	3.1%	0.1%
		2746.0	94.0%	3673.0	0.0%	0.0%
		2828.1	94.0%	3673.0	0.0%	0.0%
		2910.2	97.0%	3677.9	3.1%	0.1%
		Benchmark	94.0%	3673.0	0.0%	0.0%
0.2	60	2581.9	59.0%	3980.4	-6.3%	1.3%
		2663.9	63.0%	3937.2	0.0%	0.3%
		2746.0	63.0%	3925.4	0.0%	0.0%
		2828.1	63.0%	3925.4	0.0%	0.0%
		2910.2	63.0%	3937.2	0.0%	0.3%
		Benchmark	63.0%	3925.4	0.0%	0.0%
	70	2581.9	91.0%	4370.6	0.0%	0.0%
		2663.9	94.0%	4374.4	3.3%	0.1%
		2746.0	91.0%	4370.6	0.0%	0.0%
		2828.1	91.0%	4370.6	0.0%	0.0%
		2910.2	94.0%	4374.4	3.3%	0.1%
		Benchmark	91.0%	4370.6	0.0%	0.0%
0.3	60	2581.9	59.0%	4519.9	-6.3%	1.3%
		2663.9	63.0%	4471.2	0.0%	0.2%
		2746.0	63.0%	4462.3	0.0%	0.0%
		2828.1	63.0%	4462.3	0.0%	0.0%
		2910.2	63.0%	4471.2	0.0%	0.2%
		Benchmark	63.0%	4462.3	0.0%	0.0%
	70	2581.9	91.0%	5035.4	0.0%	0.0%
		2663.9	94.0%	5040.8	6.6%	0.1%
		2746.0	91.0%	5035.4	0.0%	0.0%
		2828.1	91.0%	5035.4	0.0%	0.0%
		2910.2	94.0%	5040.8	3.3%	0.1%
		Benchmark	91.0%	5035.4	0.0%	0.0%
0.4	60	2581.9	59.0%	4987.4	-1.7%	0.8%
		2663.9	63.0%	4957.7	5.0%	0.2%
		2746.0	63.0%	4957.7	5.0%	0.2%
		2828.1	63.0%	4952.7	5.0%	0.1%
		2910.2	63.0%	4926.6	5.0%	0.3%
		Benchmark	60.0%	4947.8	0.0%	0.0%
	70	2581.9	91.0%	5635.5	12.3%	0.1%
		2663.9	94.0%	5687.3	16.0%	0.1%
		2746.0	91.0%	5625.4	12.3%	-0.1%
		2828.1	91.0%	5625.4	12.3%	-0.1%
		2910.2	94.0%	5636.7	16.0%	0.1%
		Benchmark	81.0%	5631.0	0.0%	0.0%

Table 6.2: Scenario case. The experiments for $\mathbf{p}^R$, with setup cost $f_i = 200$, and $h_i = \lambda \cdot \sum_{t=1}^{n} p_t/n$.

λ	Percentile	b	Q	V	R_q	R_v
0.1	60	2584.0	78.0%	3391.6	2.6%	0.9%
		2664.9	81.0%	3366.8	6.2%	0.2%
		2745.8	83.0%	3368.6	8.4%	0.2%
		2826.7	83.0%	3368.6	8.4%	0.2%
		2907.6	81.0%	3366.8	6.2%	0.2%
		Benchmark	76.0%	3360.7	0.0%	0.0%
	70	2584.0	99.0%	3699.1	6.1%	0.6%
		2664.9	99.0%	3699.1	6.1%	0.6%
		2745.8	99.0%	3704.9	6.1%	0.8%
		2826.7	99.0%	3704.9	6.1%	0.8%
		2907.6	99.0%	3699.1	6.1%	0.6%
		Benchmark	93.0%	3676.7	0.0%	0.0%
0.2	60	2584.0	65.0%	4045.7	1.6%	0.7%
		2664.9	67.0%	4033.8	4.7%	0.4%
		2745.8	69.0%	4021.6	7.8%	0.1%
		2826.7	69.0%	4021.6	7.8%	0.1%
		2907.6	67.0%	4033.8	4.7%	0.4%
		Benchmark	64.0%	4017.6	0.0%	0.0%
	70	2584.0	95.0%	4484.9	5.7%	1.1%
		2664.9	95.0%	4484.9	5.7%	1.1%
		2745.8	95.0%	4471.6	5.7%	0.8%
		2826.7	95.0%	4471.6	5.7%	0.8%
		2907.6	95.0%	4484.9	5.7%	1.1%
		Benchmark	90.0%	4436.1	0.0%	0.0%
0.3	60	2584.0	63.0%	4577.1	0.0%	1.4%
		2664.9	63.0%	4532.0	0.0%	0.4%
		2745.8	65.0%	4541.0	3.2%	0.6%
		2826.7	65.0%	4541.0	3.2%	0.6%
		2907.6	63.0%	4532.0	0.0%	0.4%
		Benchmark	63.0%	4513.9	0.0%	0.0%
	70	2584.0	91.0%	5079.4	4.6%	0.6%
		2664.9	91.0%	5079.4	4.6%	0.6%
		2745.8	91.0%	5059.2	4.6%	0.2%
		2826.7	91.0%	5059.2	4.6%	0.2%
		2907.6	91.0%	5079.4	4.6%	0.6%
		Benchmark	87.0%	5049.1	0.0%	0.0%
0.4	60	2584.0	59.0%	5071.1	-6.3%	1.5%
		2664.9	60.0%	5016.2	-5.0%	0.4%
		2745.8	61.0%	5031.2	-3.2%	0.7%
		2826.7	61.0%	5031.2	-3.2%	0.7%
		2907.6	60.0%	5016.2	-5.0%	0.4%
		Benchmark	63.0%	4996.2	0.0%	0.0%
	70	2584.0	89.0%	5712.5	2.3%	1.6%
		2664.9	89.0%	5656.2	2.3%	0.6%
		2745.8	89.0%	5650.6	2.3%	0.5%
		2826.7	89.0%	5650.6	2.3%	0.5%
		2907.6	89.0%	5656.2	2.3%	0.6%
		Benchmark	87.0%	5622.5	0.0%	0.0%

Table 6.3: Scenario case. The experiments for $\mathbf{p}^I$, with setup cost $f_i = 200$, and $h_i = \lambda \cdot \sum_{t=1}^{n} p_t/n$.

λ	Percentile	b	Q	V	R_q	R_v
0.1	60	2774.7	75.0%	3577.7	-1.3%	0.3%
		2862.6	73.0%	3583.4	-4.1%	0.4%
		2950.4	73.0%	3583.4	-4.1%	0.4%
		3038.2	76.0%	3567.7	0.0%	0.0%
		3126.0	76.0%	3567.7	0.0%	0.0%
		Benchmark	76.0%	3567.7	0.0%	0.0%
	70	2774.7	93.0%	3926.2	0.0%	0.6%
		2862.6	93.0%	3911.5	0.0%	0.2%
		2950.4	97.0%	3910.7	4.1%	0.2%
		3038.2	93.0%	3902.9	0.0%	0.0%
		3126.0	93.0%	3902.9	0.0%	0.0%
		Benchmark	93.0%	3902.9	0.0%	0.0%
0.2	60	2774.7	75.0%	3577.7	-1.3%	0.3%
		2862.6	64.0%	3583.4	-4.1%	0.4%
		2950.4	63.0%	3583.4	-4.1%	0.4%
		3038.2	63.0%	3567.7	0.0%	0.0%
		3126.0	63.0%	3567.7	0.0%	0.0%
		Benchmark	63.0%	3925.4	0.0%	0.0%
	70	2774.7	93.0%	3926.2	0.0%	0.6%
		2862.6	93.0%	3911.5	0.0%	0.2%
		2950.4	97.0%	3910.7	4.1%	0.2%
		3038.2	93.0%	3902.9	0.0%	0.0%
		3126.0	93.0%	3902.9	0.0%	0.0%
		Benchmark	87.0%	4589.2	0.0%	0.0%
0.3	60	2774.7	75.0%	3577.7	-1.3%	0.3%
		2862.6	73.0%	3583.4	-4.1%	0.4%
		2950.4	73.0%	3583.4	-4.1%	0.4%
		3038.2	76.0%	3567.7	0.0%	0.0%
		3126.0	76.0%	3567.7	0.0%	0.0%
		Benchmark	63.0%	4615.9	0.0%	0.0%
	70	2774.7	93.0%	3926.2	0.0%	0.6%
		2862.6	93.0%	3911.5	0.0%	0.2%
		2950.4	97.0%	3910.7	4.1%	0.2%
		3038.2	93.0%	3902.9	0.0%	0.0%
		3126.0	93.0%	3902.9	0.0%	0.0%
		Benchmark	87.0%	5162.6	0.0%	0.0%
0.4	60	2774.7	75.0%	3577.7	-1.3%	0.3%
		2862.6	73.0%	3583.4	-4.1%	0.4%
		2950.4	73.0%	3583.4	-4.1%	0.4%
		3038.2	76.0%	3567.7	0.0%	0.0%
		3126.0	76.0%	3567.7	0.0%	0.0%
		Benchmark	63.0%	5068.2	0.0%	0.0%
	70	2774.7	93.0%	3926.2	0.0%	0.6%
		2862.6	93.0%	3911.5	0.0%	0.2%
		2950.4	97.0%	3910.7	4.1%	0.2%
		3038.2	93.0%	3902.9	0.0%	0.0%
		3126.0	93.0%	3902.9	0.0%	0.0%
		Benchmark	83.0%	5752.4	0.0%	0.0%

Table 6.4: Scenario case. The experiments for $\mathbf{p}^D$, with setup cost $f_i = 200$, and $h_i = \lambda \cdot \sum_{t=1}^{n} p_t/n$.

δ	Percentile	b	Q	V	R_q	R_v
0.25	60	2581.9	80.0%	3289.2	-9.1%	0.3%
		2663.9	88.0%	3287.3	0.0%	0.2%
		2746.0	88.0%	3284.4	0.0%	0.0%
		2828.1	88.0%	3284.4	0.0%	0.0%
		2910.2	88.0%	3277.8	0.0%	-0.2%
		Benchmark	88.0%	3284.4	0.0%	0.0%
	70	2581.9	100.0%	3643.0	0.0%	0.0%
		2663.9	100.0%	3647.4	3.1%	0.1%
		2746.0	100.0%	3643.0	0.0%	0.0%
		2828.1	100.0%	3643.0	0.0%	0.0%
		2910.2	100.0%	3647.4	3.1%	0.1%
		Benchmark	100.0%	3643.0	0.0%	0.0%
0.50	60	2581.9	65.0%	3389.2	-10.8%	1.3%
		2663.9	72.0%	3351.5	0.0%	0.2%
		2746.0	72.0%	3344.9	0.0%	0.0%
		2828.1	72.0%	3344.9	0.0%	0.0%
		2910.2	72.0%	3351.5	0.0%	0.2%
		Benchmark	72.0%	3344.9	0.0%	0.0%
	70	2581.9	94.0%	3673.0	0.0%	0.0%
		2663.9	97.0%	3677.9	3.1%	0.1%
		2746.0	94.0%	3673.0	0.0%	0.0%
		2828.1	94.0%	3673.0	0.0%	0.0%
		2910.2	97.0%	3677.9	3.1%	0.1%
		Benchmark	94.0%	3673.0	0.0%	0.0%
0.75	60	2581.9	59.0%	3448.0	-6.3%	1.5%
		2663.9	64.0%	3408.6	1.6%	0.3%
		2746.0	63.0%	3398.7	0.0%	0.0%
		2828.1	63.0%	3398.7	0.0%	0.0%
		2910.2	64.0%	3408.6	1.6%	0.3%
		Benchmark	63.0%	3398.7	0.0%	0.0%
	70	2581.9	81.0%	3725.6	0.0%	0.0%
		2663.9	84.0%	3735.3	3.7%	0.3%
		2746.0	81.0%	3725.6	0.0%	0.0%
		2828.1	81.0%	3725.6	0.0%	0.0%
		2910.2	84.0%	3735.3	3.7%	0.3%
		Benchmark	81.0%	3725.6	0.0%	0.0%
1.00	60	2581.9	55.0%	3523.9	-3.5%	1.5%
		2663.9	57.0%	3790.5	0.0%	0.5%
		2746.0	57.0%	3473.5	0.0%	0.0%
		2828.1	57.0%	3473.5	0.0%	0.0%
		2910.2	57.0%	3790.5	0.0%	0.5%
		Benchmark	57.0%	3473.5	0.0%	0.0%
	70	2581.9	76.0%	3780.1	0.0%	0.0%
		2663.9	79.0%	3801.3	3.9%	0.5%
		2746.0	76.0%	3780.1	0.0%	0.0%
		2828.1	76.0%	3780.1	0.0%	0.0%
		2910.2	79.0%	3801.3	3.9%	0.5%
		Benchmark	76.0%	3780.1	0.0%	0.0%

Table 6.5: Scenario case. The experiments for $\mathbf{p}^R$, with setup cost $f_i = 200$, and $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$ with various settings of $\mathbf{d}^s$.

δ	Percentile	b	Q	V	R_q	R_v
0.25	60	2584.0	97.0%	3312.5	0.0%	0.2%
		2664.9	97.0%	3312.5	0.0%	0.2%
		2745.8	97.0%	3312.5	0.0%	0.2%
		2826.7	97.0%	3306.2	0.0%	0.0%
		2907.6	97.0%	3306.2	0.0%	0.0%
		Benchmark	97.0%	3306.2	0.0%	0.0%
	70	2584.0	100.0%	3668.7	0.0%	0.6%
		2664.9	100.0%	3668.7	0.0%	0.6%
		2745.8	100.0%	3676.4	0.0%	0.8%
		2826.7	100.0%	3676.4	0.0%	0.8%
		2907.6	100.0%	3668.7	0.0%	0.6%
		Benchmark	100.0%	3647.9	0.0%	0.0%
0.50	60	2584.0	78.0%	3391.6	2.6%	0.9%
		2664.9	81.0%	3366.8	6.2%	0.2%
		2745.8	83.0%	3368.6	8.4%	0.2%
		2826.7	83.0%	3368.6	8.4%	0.2%
		2907.6	81.0%	3366.8	6.2%	0.2%
		Benchmark	76.0%	3360.7	0.0%	0.0%
	70	2584.0	99.0%	3699.1	6.1%	0.6%
		2664.9	99.0%	3699.1	6.1%	0.6%
		2745.8	99.0%	3704.9	6.1%	0.8%
		2826.7	99.0%	3704.9	6.1%	0.8%
		2907.6	99.0%	3699.1	6.1%	0.6%
		Benchmark	93.0%	3676.7	0.0%	0.0%
0.75	60	2584.0	65.0%	3644.0	3.2%	0.7%
		2664.9	68.0%	3705.8	7.9%	0.2%
		2745.8	69.0%	3678.3	9.5%	0.3%
		2826.7	69.0%	3678.3	9.5%	0.3%
		2907.6	68.0%	3306.9	7.9%	0.2%
		Benchmark	63.0%	3573.4	0.0%	0.0%
	70	2584.0	98.0%	4081.4	5.4%	0.6%
		2664.9	98.0%	4081.4	6.4%	0.6%
		2745.8	98.0%	4088.2	5.4%	0.8%
		2826.7	98.0%	4088.2	5,4%	0.8%
		2907.6	98.0%	4078.5	5.4%	0.6%
		Benchmark	93.0%	4055.8	0.0%	0.0%
1.00	60	2584.0	58.0%	3669.7	3.6%	0.7%
		2664.9	59.0%	3652.7	5.4%	0.2%
		2745.8	61.0%	3668.8	8.9%	0.4%
		2826.7	61.0%	3668.8	8.9%	0.4%
		2907.6	60.0%	3656.8	7.1%	0.3%
		Benchmark	56.0%	3646.0	0.0%	0.0%
	70	2584.0	92.0%	4520.1	5.7%	0.8%
		2664.9	92.0%	4520.1	5.7%	0.8%
		2745.8	93.0%	4493.7	6.9%	1.2%
		2826.7	93.0%	4493.7	6.9%	1.2%
		2907.6	93.0%	4493.7	6.9%	1.2%
		Benchmark	87.0%	4486.5	0.0%	0.0%

Table 6.6: Scenario case. The experiments for $\mathbf{p}^I$, with setup cost $f_i = 200$, and $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$ with various settings of $\mathbf{d}^s$.

δ	Percentile	b	Q	V	R_q	R_v
0.25	60	2774.7	90.0%	3510.7	-2.2%	0.0%
		2862.6	86.0%	3513.4	-6.5%	0.0%
		2950.4	86.0%	3513.4	-6.5%	0.0%
		3038.2	92.0%	3512.7	0.0%	0.0%
		3126.0	92.0%	3512.7	0.0%	0.0%
		Benchmark	92.0%	3513.2	0.0%	0.0%
	70	2774.7	99.0%	3877.2	-1.0%	0.1%
		2862.6	100.0%	3875.7	0.0%	0.0%
		2950.4	100.0%	3875.7	0.0%	0.0%
		3038.2	100.0%	3876.9	0.0%	0.0%
		3126.0	100.0%	3876.9	0.0%	0.0%
		Benchmark	100.0%	3874.1	0.0%	0.0%
0.50	60	2774.7	75.0%	3577.7	-1.3%	0.3%
		2862.6	73.0%	3583.4	-4.1%	0.4%
		2950.4	73.0%	3583.4	-4.1%	0.4%
		3038.2	76.0%	3567.7	0.0%	0.0%
		3126.0	76.0%	3567.7	0.0%	0.0%
		Benchmark	76.0%	3567.7	0.0%	0.0%
	70	2774.7	93.0%	3926.2	0.0%	0.6%
		2862.6	93.0%	3911.5	0.0%	0.2%
		2950.4	97.0%	3910.7	4.1%	0.2%
		3038.2	93.0%	3902.9	0.0%	0.0%
		3126.0	93.0%	3902.9	0.0%	0.0%
		Benchmark	93.0%	3902.9	0.0%	0.0%
0.75	60	2774.7	63.0%	3635.5	0.0%	0.0%
		2862.6	63.0%	3648.2	0.0%	0.4%
		2950.4	63.0%	3648.2	0.0%	0.4%
		3038.2	66.0%	3638.8	4.7%	0.1%
		3126.0	66.0%	3638.8	4.7%	0.1%
		Benchmark	63.0%	3635.5	0.0%	0.0%
	70	2774.7	84.0%	3960.2	2.4%	0.2%
		2862.6	84.0%	3963.0	2.4%	0.2%
		2950.4	87.0%	3960.7	6.1%	0.2%
		3038.2	84.0%	3961.8	2.4%	0.2%
		3126.0	84.0%	3961.8	2.4%	0.2%
		Benchmark	82.0%	3953.5	0.0%	0.0%
1.00	60	2774.7	57.0%	3716.9	1.7%	0.3%
		2862.6	56.0%	3729.8	3.4%	0.6%
		2950.4	56.0%	3729.8	3.4%	0.6%
		3038.2	58.0%	3718.4	0.0%	0.3%
		3126.0	58.0%	3718.4	0.0%	0.3%
		Benchmark	56.0%	3707.6	0.0%	0.0%
	70	2774.7	77.0%	4035.0	1.3%	0.6%
		2862.6	77.0%	4022.0	1.3%	0.2%
		2950.4	79.0%	4020.7	3.9%	0.2%
		3038.2	77.0%	4024.1	1.3%	0.3%
		3126.0	77.0%	4024.1	1.3%	0.3%
		Benchmark	76.0%	4012.5	0.0%	0.0%

Table 6.7: Scenario case. The experiments for $\mathbf{p}^D$, with setup cost $f_i = 200$, and $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$ with various settings of $\mathbf{d}^s$

$\mathbf{p}$	Percentile	b	Q	V	R_q	R_v
$\mathbf{p}^D$	60%	2830.4	70.8%	4073.5	0.0%	0.0%
		3090.4	70.8%	4073.5	0.0%	0.0%
		3350.4	70.8%	4073.5	0.0%	0.0%
		3610.4	73.2%	4087.5	3.3%	0.3%
		3870.4	82.1%	4067.2	13.8%	-0.2%
		Benchmark	70.8%	4073.5	0.0%	0.0%
	70%	2830.4	92.1%	4114.2	0.0%	0.0%
		3090.4	92.1%	4114.2	0.0%	0.0%
		3350.4	92.1%	4114.2	0.0%	0.0%
		3610.4	94.2%	4136.2	2.2%	0.5%
		3870.4	97.9%	4231.4	5.9%	2.8%
		Benchmark	92.1%	4114.2	0.0%	0.0%
$\mathbf{p}^I$	60%	2881.3	77.8%	3755.3	9.0%	-2.8%
		3130.3	70.8%	3860.1	0.0%	0.0%
		3379.3	84.1%	3716.3	15.8%	-3.9%
		3628.3	78.9%	3695.8	11.5%	-4.8%
		Benchmark	70.8%	3860.1	0.0%	0.0%
	70%	2632.4	97.2%	3824.1	5.3%	-1.5%
		2881.3	96.3%	3840.8	4.4%	-1.1%
		3130.3	92.1%	3882.2	0.0%	0.0%
		3379.3	98.4%	3877.6	6.4%	-0.1%
		3628.3	96.8%	3858.9	5.0%	-0.6%
		Benchmark	92.1%	3882.2	0.0%	0.0%
$\mathbf{p}^R$	60%	2646.9	71.3%	3856.1	0.0%	0.0%
		2886.4	71.3%	3856.1	0.0%	0.0%
		3125.8	71.3%	3856.1	0.0%	0.0%
		3365.3	70.0%	3929.5	-1.9%	1.9%
		3604.7	73.3%	4021.1	2.7%	4.1%
		Benchmark	71.3%	3856.1	0.0%	0.0%
	70%	2646.9	92.2%	3886.3	0.0%	0.0%
		2886.4	92.2%	3886.3	0.0%	0.0%
		3125.8	92.2%	3886.3	0.0%	0.0%
		3365.3	91.8%	3949.3	-0.4%	1.6%
		3604.7	92.5%	4117.9	0.4%	5.6%
		Benchmark	92.2%	3886.3	0.0%	0.0%

Table 6.8: Interval case. The experiment results for the three patterns of unit production cost, with setup cost $f_i = 200$, and unit inventory holding cost $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $[22.5, 67.5]$.

λ	Percentile	b	Q	V	R_q	R_v
0.1	60%	2830.4	70.8%	4073.5	0.0%	0.0%
		3090.4	70.8%	4073.5	0.0%	0.0%
		3350.4	70.8%	4073.5	0.0%	0.0%
		3610.4	73.2%	4087.5	3.3%	0.3%
		3870.4	82.1%	4067.2	13.8%	-0.2%
		Benchmark	70.8%	4073.5	0.0%	0.0%
	70%	2830.4	92.1%	4114.2	0.0%	0.0%
		3090.4	92.1%	4114.2	0.0%	0.0%
		3350.4	92.1%	4114.2	0.0%	0.0%
		3610.4	94.2%	4136.2	2.2%	0.5%
		3870.4	97.9%	4231.4	5.9%	2.8%
		Benchmark	92.1%	4114.2	0.0%	0.0%
0.2	60%	3117.8	61.8%	5400.5	2.5%	-1.0%
		3389.0	66.4%	5250.4	9.3%	-3.9%
		3660.2	66.3%	5254.6	9.1%	-3.8%
		3931.3	66.3%	5254.6	9.1%	-3.8%
		4202.5	68.7%	5374.3	12.3%	-1.5%
		Benchmark	60.2%	5452.6	0.0%	0.0%
	70%	3117.8	84.9%	5197.7	0.9%	-0.2%
		3389.0	89.0%	5055.7	5.5%	-3.0%
		3660.2	88.9%	5057.7	5.4%	-3.0%
		3931.3	88.9%	5057.7	5.4%	-3.0%
		4202.5	89.7%	5267.6	6.2%	1.1%
		Benchmark	84.1%	5207.0	0.0%	0.0%
0.3	60%	3359.0	60.2%	6535.2	0.0%	0.0%
		3641.0	60.2%	6535.2	0.0%	0.0%
		3923.0	60.2%	6535.2	0.0%	0.0%
		4204.9	61.2%	6624.7	1.5%	1.4%
		4486.9	67.7%	6767.3	11.2%	3.4%
		Benchmark	60.2%	6535.2	0.0%	0.0%
	70%	3359.0	84.1%	6062.8	0.0%	0.0%
		3641.0	84.1%	6062.8	0.0%	0.0%
		3923.0	84.1%	6062.8	0.0%	0.0%
		4204.9	84.2%	6226.7	0.1%	2.6%
		4486.9	85.8%	6281.1	2.0%	3.6%
		Benchmark	84.1%	6062.8	0.0%	0.0%
0.4	60%	3521.7	60.2%	7617.7	0.0%	0.0%
		3807.8	60.2%	7617.7	0.0%	0.0%
		4093.8	60.2%	7617.7	0.0%	0.0%
		4379.9	60.1%	7692.3	-0.3%	1.0%
		4666.0	60.6%	7964.1	0.5%	4.3%
		Benchmark	60.2%	7617.7	0.0%	0.0%
	70%	3521.7	84.1%	6918.6	8.4%	-6.7%
		3807.8	84.1%	6918.6	8.4%	-6.7%
		4093.8	84.1%	6918.6	8.4%	-6.7%
		4379.9	83.9%	7000.2	8.2%	-5.5%
		4666.0	84.0%	7351.7	8.3%	-0.4%
		Benchmark	77.0%	7383.1	0.0%	0.0%

Table 6.9: Interval case. The experiments for $\mathbf{p}^D$, with setup cost $f_i = 200$, and $h_i = \lambda \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $[22.5, 67.5]$.

λ	Percentile	b	Q	V	R_q	R_v
0.1	60%	2881.3	77.8%	3755.3	9.0%	-2.8%
		3130.3	70.8%	3860.1	0.0%	0.0%
		3379.3	84.1%	3716.3	15.8%	-3.9%
		3628.3	78.9%	3695.8	11.5%	-4.8%
		Benchmark	70.8%	3860.1	0.0%	0.0%
	70%	2632.4	97.2%	3824.1	5.3%	-1.5%
		2881.3	96.3%	3840.8	4.4%	-1.1%
		3130.3	92.1%	3882.2	0.0%	0.0%
		3379.3	98.4%	3877.6	6.4%	-0.1%
		3628.3	96.8%	3858.9	5.0%	-0.6%
		Benchmark	92.1%	3882.2	0.0%	0.0%
0.2	60%	3003.0	70.8%	4992.8	6.1%	-2.2%
		3269.0	66.4%	5101.0	0.0%	0.0%
		3535.0	61.8%	5251.0	-7.5%	2.9%
		3801.1	64.0%	5288.4	-3.7%	3.5%
		4067.1	64.9%	5366.1	-2.3%	4.9%
		Benchmark	66.4%	5101.0	0.0%	0.0%
	70%	3003.0	92.1%	4842.8	3.4%	-1.1%
		3269.0	89.0%	4893.8	0.0%	0.0%
		3535.0	84.9%	5035.5	-4.8%	2.8%
		3801.1	85.6%	5151.3	-3.9%	5.0%
		4067.1	85.9%	5256.6	-3.6%	6.9%
		Benchmark	89.0%	4893.8	0.0%	0.0%
0.3	60%	3244.2	61.8%	6396.9	2.5%	-0.5%
		3521.0	60.2%	6428.4	0.0%	0.0%
		3797.8	60.2%	6428.4	0.0%	0.0%
		4074.7	65.1%	6318.1	7.5%	-1.7%
		4351.5	73.7%	6016.8	18.3%	-6.8%
		Benchmark	60.2%	6428.4	0.0%	0.0%
	70%	3244.2	84.9%	5975.0	0.9%	0.5%
		3521.0	84.1%	5947.2	0.0%	0.0%
		3797.8	84.1%	5947.2	0.0%	0.0%
		4074.7	88.1%	5902.8	4.5%	-0.8%
		4351.5	94.2%	5735.0	10.7%	-3.7%
		Benchmark	84.1%	5947.2	0.0%	0.0%
0.4	60%	3447.1	60.2%	7511.0	0.0%	0.0%
		3731.0	60.2%	7511.0	0.0%	0.0%
		4015.0	60.2%	7511.0	0.0%	0.0%
		4298.9	60.2%	7511.0	0.0%	0.0%
		4582.8	47.5%	8472.2	-26.8%	11.3%
		Benchmark	60.2%	7511.0	0.0%	0.0%
	70%	3447.1	84.1%	6803.0	0.0%	0.0%
		3731.0	84.1%	6803.0	0.0%	0.0%
		4015.0	84.1%	6803.0	0.0%	0.0%
		4298.9	84.1%	6803.0	0.0%	0.0%
		4582.8	66.5%	8135.3	-26.5%	16.4%
		Benchmark	84.1%	6803.0	0.0%	0.0%

Table 6.10: Interval case. The experiments for $\mathbf{p}^I$, with setup cost $f_i = 200$, and $h_i = \lambda \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $[22.5, 67.5]$.

λ	Percentile	b	Q	V	R_q	R_v
0.1	60%	2646.9	71.3%	3856.1	0.0%	0.0%
		2886.4	71.3%	3856.1	0.0%	0.0%
		3125.8	71.3%	3856.1	0.0%	0.0%
		3365.3	70.0%	3929.5	-1.9%	1.9%
		3604.7	73.3%	4021.1	2.7%	4.1%
		Benchmark	71.3%	3856.1	0.0%	0.0%
	70%	2646.9	92.2%	3886.3	0.0%	0.0%
		2886.4	92.2%	3886.3	0.0%	0.0%
		3125.8	92.2%	3886.3	0.0%	0.0%
		3365.3	91.8%	3949.3	-0.4%	1.6%
		3604.7	92.5%	4117.9	0.4%	5.6%
		Benchmark	92.2%	3886.3	0.0%	0.0%
0.2	60%	2973.6	66.3%	5031.1	0.0%	0.0%
		3230.0	66.3%	5031.1	0.0%	0.0%
		3486.5	66.3%	5031.1	0.0%	0.0%
		3742.9	71.3%	4994.3	7.1%	-0.7%
		3999.3	67.3%	5176.0	1.5%	2.8%
		Benchmark	66.3%	5031.1	0.0%	0.0%
	70%	2973.6	88.9%	4814.1	0.0%	0.0%
		3230.0	88.9%	4814.1	0.0%	0.0%
		3486.5	88.9%	4814.1	0.0%	0.0%
		3742.9	92.2%	4861.9	3.5%	1.0%
		3999.3	89.1%	5022.8	0.2%	4.2%
		Benchmark	88.9%	4814.1	0.0%	0.0%
0.3	60%	3223.4	66.3%	6104.5	0.0%	0.0%
		3494.3	66.3%	6104.5	0.0%	0.0%
		3765.2	66.3%	6104.5	0.0%	0.0%
		4036.1	59.8%	6540.5	-10.9%	6.7%
		4307.1	61.2%	6663.8	-8.4%	8.4%
		Benchmark	66.3%	6104.5	0.0%	0.0%
	70%	3223.4	88.9%	5684.2	1.4%	-1.1%
		3494.3	88.9%	5684.2	1.4%	-1.1%
		3765.2	88.9%	5684.2	1.4%	-1.1%
		4036.1	83.8%	6056.9	-4.6%	5.1%
		4307.1	84.2%	6270.1	-4.1%	8.4%
		Benchmark	87.7%	5745.2	0.0%	0.0%
0.4	60%	3380.0	0.0%	0.0	0.0%	0.0%
		3655.5	59.3%	7584.6	11.5%	-5.4%
		3930.9	52.4%	7995.2	0.0%	0.0%
		4206.3	64.2%	7315.7	18.3%	-9.3%
		4481.8	55.1%	8123.6	4.8%	1.6%
		Benchmark	52.4%	7995.2	0.0%	0.0%
	70%	3380.0	0.0%	0.0	0.0%	0.0%
		3655.5	83.6%	6845.8	9.4%	-6.7%
		3930.9	75.8%	7305.9	0.0%	0.0%
		4206.3	87.7%	6619.2	13.6%	-10.4%
		4481.8	77.3%	7567.7	1.9%	3.5%
		Benchmark	75.8%	7305.9	0.0%	0.0%

Table 6.11: Interval case. The experiments for $\mathbf{p}^R$, with setup cost $f_i = 200$, and $h_i = \lambda \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $[22.5, 67.5]$.

δ	Percentile	b	Q	V	R_q	R_v
0.25	60%	2821.7	71.3%	3619.7	0.0%	0.0%
		2935.4	71.3%	3619.7	0.0%	0.0%
		3049.2	71.3%	3619.7	0.0%	0.0%
		3162.9	67.3%	3704.2	-6.0%	2.3%
		3276.6	80.0%	3606.1	10.8%	-0.4%
		Benchmark	71.3%	3619.7	0.0%	0.0%
	70%	2821.7	92.2%	3457.7	0.0%	0.0%
		2935.4	92.2%	3457.7	0.0%	0.0%
		3049.2	92.2%	3457.7	0.0%	0.0%
		3162.9	89.1%	3528.8	-3.5%	2.0%
		3276.6	97.2%	3516.9	5.2%	1.7%
		Benchmark	92.2%	3457.7	0.0%	0.0%
0.50	60%	2881.3	77.8%	3755.3	9.0%	-2.8%
		3130.3	70.8%	3860.1	0.0%	0.0%
		3379.3	84.1%	3716.3	15.8%	-3.9%
		3628.3	78.9%	3695.8	11.5%	-4.8%
		Benchmark	70.8%	3860.1	0.0%	0.0%
	70%	2632.4	97.2%	3824.1	5.3%	-1.5%
		2881.3	96.3%	3840.8	4.4%	-1.1%
		3130.3	92.1%	3882.2	0.0%	0.0%
		3379.3	98.4%	3877.6	6.4%	-0.1%
		3628.3	96.8%	3858.9	5.0%	-0.6%
		Benchmark	92.1%	3882.2	0.0%	0.0%
0.75	60%	2472.7	0.0%	0.0	0.0%	0.0%
		2829.7	71.3%	4072.9	0.0%	0.0%
		3186.7	71.3%	4072.9	0.0%	0.0%
		3543.7	80.0%	4083.9	10.8%	0.3%
		3900.7	80.0%	4083.9	10.8%	0.3%
		Benchmark	71.3%	4072.9	0.0%	0.0%
	70%	2472.7	0.0%	0.0	0.0%	0.0%
		2829.7	92.2%	4279.2	3.5%	-1.0%
		3186.7	92.2%	4279.2	3.5%	-1.0%
		3543.7	97.2%	4392.4	8.5%	1.6%
		3900.7	97.2%	4392.4	8.5%	1.6%
		Benchmark	88.9%	4322.9	0.0%	0.0%
1.00	60%	2058.4	0.0%	0.0	0.0%	0.0%
		2572.9	71.3%	4289.6	0.0%	0.0%
		3087.5	71.3%	4289.6	0.0%	0.0%
		3602.1	67.3%	4377.3	-6.0%	2.0%
		4116.7	85.2%	4485.2	16.3%	4.4%
		Benchmark	71.3%	4289.6	0.0%	0.0%
	70%	2058.4	0.0%	0.0	0.0%	0.0%
		2572.9	92.2%	4672.1	3.5%	-0.6%
		3087.5	92.2%	4672.1	3.5%	-0.6%
		3602.1	89.1%	4753.8	0.2%	1.1%
		4116.7	98.7%	5092.8	9.9%	7.7%
		Benchmark	88.9%	4701.6	0.0%	0.0%

Table 6.12: Interval case. The experiments for $\mathbf{p}^R$, with setup cost $f_i = 200$, and $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $[\tilde{d}_i - \delta\tilde{d}_i, \tilde{d}_i + \delta\tilde{d}_i]$.

δ	Percentile	b	Q	V	R_q	R_v
0.25	60%	2788.1	80.0%	3475.2	11.6%	-4.3%
		2907.4	77.8%	3512.0	9.0%	-3.2%
		3026.7	70.8%	3623.8	0.0%	0.0%
		3146.0	77.8%	3512.0	9.0%	-3.2%
		3265.4	85.2%	3517.8	17.0%	-3.0%
		Benchmark	70.8%	3623.8	0.0%	0.0%
	70%	2788.1	97.2%	3380.4	5.3%	-2.2%
		2907.4	96.3%	3396.0	4.4%	-1.7%
		3026.7	92.1%	3453.9	0.0%	0.0%
		3146.0	96.3%	3396.0	4.4%	-1.7%
		3265.4	98.7%	3491.7	6.6%	1.1%
		Benchmark	92.1%	3453.9	0.0%	0.0%
0.50	60%	2881.3	77.8%	3755.3	9.0%	-2.8%
		3130.3	70.8%	3860.1	0.0%	0.0%
		3379.3	84.1%	3716.3	15.8%	-3.9%
		3628.3	78.9%	3695.8	11.5%	-4.8%
		Benchmark	70.8%	3860.1	0.0%	0.0%
	70%	2632.4	97.2%	3824.1	5.3%	-1.5%
		2881.3	96.3%	3840.8	4.4%	-1.1%
		3130.3	92.1%	3882.2	0.0%	0.0%
		3379.3	98.4%	3877.6	6.4%	-0.1%
		3628.3	96.8%	3858.9	5.0%	-0.6%
		Benchmark	92.1%	3882.2	0.0%	0.0%
0.75	60%	2454.1	0.0%	0.0	0.0%	0.0%
		2827.9	77.8%	3978.3	9.0%	-2.5%
		3201.6	70.8%	4076.6	0.0%	0.0%
		3575.4	70.9%	4235.5	0.1%	3.8%
		3949.2	91.0%	4199.1	22.2%	2.9%
		Benchmark	70.8%	4076.6	0.0%	0.0%
	70%	2454.1	0.0%	0.0	0.0%	0.0%
		2827.9	96.3%	4248.5	4.4%	-0.6%
		3201.6	92.1%	4274.8	0.0%	0.0%
		3575.4	90.0%	4526.7	-2.4%	5.6%
		3949.2	90.0%	4526.7	-2.4%	5.6%
		Benchmark	92.1%	4274.8	0.0%	0.0%
1	60%	2106.9	0.0%	0.0	0.0%	0.0%
		2633.6	77.8%	4201.3	9.0%	-2.2%
		3160.3	70.8%	4293.2	0.0%	0.0%
		3687.0	77.8%	4201.3	9.0%	-2.2%
		4213.7	91.0%	4441.1	22.2%	3.3%
		Benchmark	70.8%	4293.2	0.0%	0.0%
	70%	2106.9	0.0%	0.0	0.0%	0.0%
		2633.6	96.3%	4656.2	4.4%	-0.2%
		3160.3	92.1%	4667.5	0.0%	0.0%
		3687.0	96.3%	4656.2	4.4%	-0.2%
		4213.7	94.8%	4764.1	2.6%	2.0%
		Benchmark	92.1%	4667.5	0.0%	0.0%

Table 6.13: Interval case. The experiments for $\mathbf{p}^I$, with setup cost $f_i = 200$, and $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $[\tilde{d}_i - \delta\tilde{d}_i, \tilde{d}_i + \delta\tilde{d}_i]$.

δ	Percentile	b	Q	V	R_q	R_v
0.25	60%	3002.9	70.8%	3826.5	0.0%	0.0%
		3126.4	70.8%	3826.5	0.0%	0.0%
		3249.9	70.8%	3826.5	0.0%	0.0%
		3373.5	74.0%	3790.5	4.4%	-0.9%
		3497.0	77.8%	3842.6	9.0%	0.4%
		Benchmark	70.8%	3826.5	0.0%	0.0%
	70%	3002.9	92.1%	3665.4	0.0%	0.0%
		3126.4	92.1%	3665.4	0.0%	0.0%
		3249.9	92.1%	3665.4	0.0%	0.0%
		3373.5	94.2%	3647.4	2.2%	-0.5%
		3497.0	96.3%	3741.4	4.4%	2.0%
		Benchmark	92.1%	3665.4	0.0%	0.0%
0.50	60%	2830.4	70.8%	4073.5	0.0%	0.0%
		3090.4	70.8%	4073.5	0.0%	0.0%
		3350.4	70.8%	4073.5	0.0%	0.0%
		3610.4	73.2%	4087.5	3.3%	0.3%
		3870.4	82.1%	4067.2	13.8%	-0.2%
		Benchmark	70.8%	4073.5	0.0%	0.0%
	70%	2830.4	92.1%	4114.2	0.0%	0.0%
		3090.4	92.1%	4114.2	0.0%	0.0%
		3350.4	92.1%	4114.2	0.0%	0.0%
		3610.4	94.2%	4136.2	2.2%	0.5%
		3870.4	97.9%	4231.4	5.9%	2.8%
		Benchmark	92.1%	4114.2	0.0%	0.0%
0.75	60%	2645.6	0.0%	0.0	0.0%	0.0%
		3033.2	70.8%	4299.9	0.0%	0.0%
		3420.8	70.8%	4299.9	0.0%	0.0%
		3808.4	77.8%	4341.5	9.0%	1.0%
		4196.0	86.3%	4577.7	18.0%	6.1%
		Benchmark	70.8%	4299.9	0.0%	0.0%
	70%	2645.6	0.0%	0.0	0.0%	0.0%
		3033.2	92.1%	4525.6	0.0%	0.0%
		3420.8	92.1%	4525.6	0.0%	0.0%
		3808.4	96.3%	4657.5	4.4%	2.8%
		4196.0	98.9%	5042.2	6.9%	10.2%
		Benchmark	92.1%	4525.6	0.0%	0.0%
1.00	60%	2220.6	0.0%	0.0	0.0%	0.0%
		2775.7	70.8%	4526.3	0.0%	0.0%
		3330.9	70.8%	4526.3	0.0%	0.0%
		3886.0	74.0%	4489.7	4.4%	-0.8%
		4441.1	85.2%	4690.5	17.0%	3.5%
		Benchmark	70.8%	4526.3	0.0%	0.0%
	70%	2220.6	0.0%	0.0	0.0%	0.0%
		2775.7	92.1%	4936.9	3.4%	-0.9%
		3330.9	92.1%	4936.9	3.4%	-0.9%
		3886.0	94.2%	4917.7	5.6%	-1.3%
		4441.1	98.7%	5331.2	9.8%	6.5%
		Benchmark	89.0%	4983.5	0.0%	0.0%

Table 6.14: Interval case. The experiments for $\mathbf{p}^D$, with setup cost $f_i = 200$, and $h_i = 0.1 \cdot \sum_{t=1}^{n} p_t/n$. Demand realizations are generated uniformly in $[\tilde{d}_i - \delta\tilde{d}_i, \tilde{d}_i + \delta\tilde{d}_i]$.

Chapter 7

Conclusions and Future Research Directions

In this thesis, we considered robust versions of the lot sizing problem. We followed four robust optimization methodologies in formulating robust lot sizing problems. Existing robust lot sizing models rewrite the feasible region of the lot sizing model to account for possible deviations of the demand, thus focusing on the feasibility level of the problem instances. The robust models proposed in this thesis are more cost driven.

The minimax regret (robust deviation) criterion is by definition cost-concerned itself. It aims to minimize the worst surplus cost over all realisations of the input data. To be more precise, the criterion finds a solution that exhibits the smallest worst case deviation from optimality over all potential realisations. We study a minimax regret ULS model with uncertainty in both the coefficients of the objective function and the demand. After decomposing the problem into subplans, all possible regret situations are discussed and a polynomial time Dynamic Programming (DP) algorithm is proposed to solve the problem. When only the demand is uncertain, the proposed DP algorithm reduces the running time of the algorithm proposed in [184]. Following the line of research in this thesis, an interesting direction for future research is to integrating uncertainty budget with the minimax regret criterion. In this case the budget allowance is assigned to periods with the largest regret. The complexity of each subplan remains a challenge.

With the appealing idea of uncertainty budget, we study robust lot sizing models with the worst-case periods defined as the periods in which uncertain parameters harm the quality of the objective function value the most. By adjusting the size of the uncertainty budget, one can control the trade-off between robustness, i.e., to what extend the solution is feasible, and quality of the objective function value, and therefore arrive at a less conservative solution. In addition, we introduced a range parameter Θ, and proposed a robust lot sizing model with both the uncertainty budget and the uncertainty range. The aim of this is to further enhance the trade-off between robustness and quality of the objective function value. Our numerical results show that this is indeed the case. In most experiment instances we are able to achieve high feasibility of the solution and a satisfying objective function value. Linking uncertainty budget with decisions is an interesting future research direction. Poss [144] studies the so-called variable budgeted uncertainty in combinatorial optimization model, and shows that the model is less conservative for decision vectors with few non-zero components. In our case, it is also interesting to see whether making Γ^d (and Θ) dependent on the setup decision further improves the performance of the proposed models.

The *bw*-robustness criterion is another one that focuses on the quality of the objective function value. With the ULS problem, we use the bw-robustness criterion to deal with a parameter that defines the feasible region. The numerical results suggest that, besides reaching the goals of the objective function value set by b and w, the solution of the *bw*-robust ULS model achieves satisfying feasibility levels. An interesting future research direction is the combination of the *bw*-robustness criterion with other classic robust criteria. For example, the *bw*-deviation, based on the same principles as the *bw*-robustness, generalizes the minimax regret criterion.

Another interesting line of future research is to bridge robust optimization with stochastic programming [44]. For instance, the two frameworks can be adapted to address the same uncertainty in the model as alternative approaches to each other, e.g., quantifying the performance of the robust optimization solution in the stochastic programming problem. In addition, robust optimization can be considered in the context of uncertain probability distributions, i.e., in the stochastic programming framework itself, in which the worst case

is computed over a set of probability distributions of the uncertain parameters.

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