Presmooth Geometries

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Abstract

This thesis explores the geometric principles underlying many of the known Trichotomy Theorems. The main aims are to unify the field construction in non-linear o-minimal structures and generalizations of Zariski Geometries as well as to pave the road for completely new results in this direction.

In the first part of this thesis we introduce a new axiomatic framework in which all the relevant structures can be studied uniformly and show that these axioms are preserved under elementary extensions. A particular focus is placed on the study of a smoothness condition which generalizes the presmoothness condition for Zariski Geometries. We also modify Zilber's notion of universal specializations to obtain a suitable notion of infinitesimals. In addition, families of curves and the combinatorial geometry of one-dimensional structures are studied to prove a weak trichotomy theorem based on very weak one-basedness.

It is then shown that under suitable additional conditions groups and group actions can be constructed in canonical ways. This construction is based on a notion of "geometric calculus" and can be seen in close analogy with ordinary differentiation. If all conditions are met, a definable distributive action of one one-dimensional type-definable group on another are obtained.

The main result of this thesis is that both o-minimal structures and generalizations of Zariski Geometries fit into this geometric framework and that the latter always satisfy the conditions required in the group constructions. We also exhibit known methods that allow us to extract fields from this. In addition to unifying the treatment of o-minimal structures and Zariski Geometries, this also gives a direct proof of the Trichotomy Theorem for "type-definable" Zariski Geometries as used, for example, in Hrushovski's proof of the relative Mordell-Lang conjecture.

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Contents

Chapter 1

Introduction

Despite its use throughout mathematics, the word *Trichotomy* has aquired a very specific meaning in Model Theory. Trichotomy Theorems are a family of classification results which give sufficient conditions for an abstract geometry to be algebraic in nature. For example, the Trichotomy Theorem for Zariski Geometries states that a one-dimensional Zariski Geometry is either combinatorially trivial, bi-interpretable with an abelian group with possibly extra linear structure or a finite covering of an algebraically closed field. Since we understand the structure of vector spaces and algebraically closed fields very well, such results have strong implications and can lead to powerful applications. Hrushovski's original solution of the relative version of the Mordell-Lang conjecture [Hru96] relies on a variant of the Trichotomy Theory for Zariski Geometries [HZ96] applicable to minimal sets in separably closed fields. Similarly, his work on the Manin-Mumford conjecture [Hru01] relies on the Trichotomy Theorem for Algebraically Closed Fields with a generic Automorphism (ACFA) [CHP02]. The original Trichotomy Theorem for Zariski Geometries [HZ96] and a version for reducts of fields by E. Rabinovich [Rab93] also inspired some recent work by Yuri Manin [Man10]. In practice, the hardest part of the proof of a Trichotomy Theorem is to recover a field if the geometry is sufficiently rich. In the existing Trichotomy Theorems, this is done using a variety of techniques some of which are quite inaccessible to non-specialists interested in the above applications. In this thesis, I axiomatically expose the underlying principles that are common to many of the contexts in which we already have a Trichotomy Theorem. Using a combination of ideas from Zilber [Zil10] and Starchenko and Peterzil [PS98], we will unify a large proportion of the field construction by carrying out most of the work in this new and more general class of structures. By constructing the algebraic objects based on an intuitive notion of geometric calculus and using abstract analogues of the chain and product rules for usual differentiation, I hope that this will also make these results more readable for the general mathematician. Of course, by giving an axiomatic framework in which this construction works, we should also be able to more easily indentify new classes of geometries for which a Trichotomy Theorem holds. Most of the axioms follow easily form a cell decomposition theorem in the style of that for o-minimal structures and in view of this, both C-minimal structures and p-minimal fields are promising classes to look at in the future.

1.1 Trichotomy Theorems

1.1.1 The Trichotomy Conjecture and Zilber's Principle

To give the reader some context, we start with a brief history leading up to Zilber's Trichotomy Conjecture and a survey of the various existing Trichotomy Theorems. The story begins quite early in Model Theory with the study of categorical structures.

Definition 1.1.1 We say that a structure M is *categorical*, if every elementarily equivalent structure of the same cardinality is isomorphic to M.

Since the Löwenheim Skolem Theorems tell us that a countable theory has models of every cardinality, this is the best kind of uniqueness we could wish for a structure to have. There were known examples of categorical structures from the very early days of Model Theory. Already in 1954, Łoś and Vaught showed that uncountable algebraically closed fields of fixed characteristic are categorical. However, the real breakthrough came some eleven years later when Morley proved the Categoricity Theorem. This states that if a countable theory has one uncountable categorical model, then all uncountable models are categorical. The methods that were developed to prove this theorem can be seen as the beginning of modern model theory and provided a lot of new tools for the study of *uncountably categorical* theories. In 1971 Macintyre gave an interesting converse to the Łoś and Vaught result. He showed that if a field is categorical, then it is algebraically closed, thus giving a complete classification of uncountably categorical fields. This result spawned a large amount of interest in this kind of classification question and in the following decades much work has been done in an attempt to classify categorical groups, rings and other structures. However, success has been limited; even Zilber and Cherlin's Algebraicity Conjecture of 1977, stating that uncountably categorical groups are algebraic groups, is still open.

Not long after this flurry of activity took off, Zilber started thinking in another direction. Łoś, Vaught and Macintyre gave the following equivalence:

An uncountable field K is categorical if and only if it is algebraically closed.

Zilber asked whether we could replace the left-hand side with something completely model-theoretic. In other words, is there a model theoretic property P such that a structure M has P if and only if it is equivalent to an algebraically closed field? To be able to understand the resulting Trichotmoy Conjecture which tried to answer the above question, we need to look at some of the structural results that went into the proof of the Categoricity Theorem. At the core of Morley's theorem is the fact that uncountably categorical theories are in some sense controlled by an underlying combinatorial geometry.

Strongly Minimal Sets and Pregeometries

Definition 1.1.2 Given a structure M, an infinite definable set D is *minimal* if for any other definable set $Y \subseteq D$ either Y or $D \setminus Y$ is finite. Given a theory T, the formula $\phi(x, a)$ is called *strongly minimal* if in any model $\mathrm{M} \models T$ the set $\phi(\mathrm{M}, a)$ is minimal.

Theorem 1.1.3 *Let T be a countable theory with a model which is categorical in an uncountable cardinality. Then*

1. *T has a prime model* M_0*,*

2. *there is a strongly minimal formula* $\phi(x)$ *with parameters from the prime model and*

3. *given any model* $\mathrm{M} \models T$*,* M *is prime over* $\phi(\mathrm{M})$*.*

Classic examples of strongly minimal sets are structures in the language of pure sets (equality), vector spaces and algebraically closed fields. In fact, these are key examples and there is a strong analogy between these and general strongly minimal sets: The aforementioned examples all come with well known geometries, namely algebraic closure in fields, linear span in vector spaces and just the identity map in the pure set. Morley showed that all strongly minimal sets come equipped with a canonical closure operation like this and that this allows us to define a dimension that determines the structure up to isomorphism.

Definition 1.1.4 Given a structure M and a set $A \subseteq \mathrm{M}$, we define the *algebraic closure*

$$\operatorname{acl}(A) = \{b \mid \text{ there is a formula } \phi(x) \text{ with parameters from } A \text{ such that } \mathrm{M} \models \phi(b) \text{ and } \phi(\mathrm{M}) \text{ is finite.}\}$$

Definition 1.1.5 A *pre-geometry* on a set X is a function $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ such that

1. for all $A \subseteq X$, $A \subseteq \mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$,

2. for all $A \subseteq B \subseteq X$, $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ and

3. for all $A \subseteq X$ and $a, b \in X$, if $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$, then $b \in \mathrm{cl}(Aa)$.

Given a set $D \subseteq X$, we define the *localization* of cl at D to be cl_D where $\mathrm{cl}_D(A) = \mathrm{cl}(A \cup D)$.

We can define independence and dimension in analogy with the examples given above.

Definition 1.1.6 Let (X, cl) be a pre-geometry. We say that a set $B \subseteq X$ is *independent* if for all $b \in B$, $b \notin \mathrm{cl}(B \setminus \{b\})$. We define the *combinatorial dimension* $\operatorname{cdim}(A) = \min\{\, |B| \mid B \subseteq A \text{ is independent } \}$.

Theorem 1.1.7 *Given a model* $\mathrm{M} \models T$ *and a strongly minimal formula* $\phi(x)$*,* acl *is a pre-geometry on* $\phi(\mathrm{M})$*. Furthermore, if* N *is another model, then* $\phi(\mathrm{M}) \cong \phi(N)$ *if and only if* $\mathrm{cdim}\,(\phi(\mathrm{M})) = \mathrm{cdim}\,(\phi(N))$*.*

This, together with 1.1.3 tells us that uncountably categorical theories are controlled by the strongly minimal sets they contain. To give a classification of uncountably categorical structures it is therefore enough to classify strongly minimal sets. We can classify pre-geometries using a multitude of dividing lines. The ones that will be important for us right now are the following.

Definition 1.1.8 Let cl be a pre-geometry on X. We say that cl is

trivial if for all $A \subseteq X$, $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$.

modular if the dimension formula holds: For all $A, B \subseteq X$,

$$\mathrm{cdim}\,(A \cup B) = \mathrm{cdim}\,(A) + \mathrm{cdim}\,(B) - \mathrm{cdim}\,(A \cap B)$$

locally modular if there is a tuple a such that the localization cl_a is modular.

Looking back at our examples we see that the pre-geometry given by equality is trivial, span is modular (and therefore locally modular) and algebraic closure in fields is neither. Local modularity of acl has interesting consequences for the complexities of incidence relations that can be interpreted.

Definition 1.1.9 A *pseudo-plane* is a two-sorted structure $\langle P, L, I \rangle$ where $I \subseteq P \times L$ and

1. for all $p \in P$, $\{l \in L | \models I(p,l)\}$ is infinite,
2. for all $l \in L$, $\{p \in P | \models I(p,l)\}$ is infinite,
3. for all $p_1, p_2 \in P$, $\{l \in L | \models I(p_1,l) \wedge I(p_2,l)\}$ is finite and
4. for all $l_1, l_2 \in L$, $\{p \in P | \models I(p,l_1) \wedge I(p,l_2)\}$ is finite.

Even though this definition is symmetric in P and L we should think of elements of P as "points" and elements of L as "lines". In this way a pseudo-plane can then be regarded as a non-linear family of curves. In view of this, we will call strongly minimal sets which interpret a pseudo-plane *non-linear* or *linear* otherwise.

Conjecture and Refutation

By the early 1980s a good deal was known about linear strongly minimal sets and some first classification results started to appear. However, little beyond the one example we have already seen and the interpretability of a pseudo-plane had been established in the non-linear case. In [Zil83] Zilber gives the following weak trichotomy:

Theorem 1.1.10 *Let* M *be a strongly minimal structure. Then exactly one of the following is true:*

1. acl *is trivial,*
2. acl *is non-trivial and locally modular or*
3. M *interprets a pseudo-plane.*

The fact that there were no other examples led Zilber [Zil83] to conjecture that every uncountably categorical pseudo-plane should be bi-interpretable with an algebraically closed field. Using the group configuration theorem (see [Bou89] for a good account) which generalized techniques already known to work in totally categorical theories, Hrushovski finally managed to fully classify linear strongly minimal sets in his PhD thesis [Hru85]. This established what is now commonly known as the Weak Trichotomy Theorem (WTT).

Theorem 1.1.11 *Let* M *be a strongly minimal structure. Then exactly one of the following is true:*

1. acl *is trivial,*

2. acl *is non-trivial, locally modular and* M *is bi-interpretable with an abelian group with possibly extra linear structure or*

3. M *interprets a pseudo-plane.*

It was also Hrushovski who finally gave an answer to Zilber's conjecture, but not in the way one might have hoped for. In [Hru93] he used a new method now known as the Hrushovski Construction to find a strongly minimal set which is not locally modular, but interprets no groups. Despite the failure of Zilber's conjecture for strongly minimal structures which can be regarded as one of the nicest possible contexts from the point of view of Model Theory, there are now a number of contexts where results along these lines do hold. Starchenko and Peterzil [PS98] put it as follows:

Suppose we have class of structures $\mathcal{C}$ with good notions of "trivial", "linear" and "non-linear". We say that $\mathcal{C}$ satisfies the *Zilber Principle* if for any structure $\mathrm{M} \in \mathcal{C}$ either M is trivial, M is linear and all definable sets arise from an interpretable module structure or M is non-linear and M interprets a field.

Typically the non-linear case is the hardest to deal with, but before we look at a strategy to construct a field (or at least a field-like structure), we will give a brief survey of known results.

1.1.2 Known Results

The first successful attempts concerned reducts of algebraically closed fields which still interpret a pseudo-plane. The most general of these is due to Rabinovich [Rab93]:

Theorem 1.1.12 *Let* $\langle K, \Omega\rangle$ *be a non-locally modular reduct of an algebraically closed field* $\langle K, +, \cdot, 0, 1\rangle$*. Then* $\langle K, +, \cdot, 0, 1\rangle$ *is definable in* $\langle K, \Omega\rangle$*.*

This is an improvement on the Łoś, Vaught, Macintyre result, but as we need to know from the outset that the field is present this is by no means the purely model theoretic property P that Zilber envisaged. Some of the methods used are also not likely to generalize to more abstract contexts as the proof relies on keeping track of the difference between the sets definable in $\langle K, \Omega \rangle$ and their Zariski closures. For this reason we make no attempt to include this result in the general framework developed in this text. What we should take away from Rabinovich's work, though, is that a notion of closures of definable sets is key. Indeed, Zilber and Hrushovski noticed that the addition of a definable topology to a general strongly minimal set allowed them to axiomatize a kind of smoothness in form of the dimension theorem. The resulting structures are called Zariski Geometries (see definition 4.1.31). Their main results [HZ96] are

Theorem 1.1.13 *If* M *is a non-locally modular Zariski Geometry, then* M *is a finite covering of an interpretable algebraically closed field* $\langle K, +, \cdot \rangle$ *and furthermore the projections* $\mathrm{M}^n \to K^n$ *are closed maps with respect to the Zariski Topology on* K^n*.*

Theorem 1.1.14 *There is a non-locally modular Zariski Geometry, which is not interpretable in an algebraically closed field.*

A good treatment of Zariski Geometries is given in [Zil10] where the same theorems are proved using new techniques which are one of the main influences on this piece of work. There are also a couple of variants of this theorem which apply to types and will also greatly concern us. There is no proof for the following result in the literature, but it is considered folklore by many.

Theorem 1.1.15 *Let* p *be a non-locally modular minimal type over* A *in a stable theory* T *with quantifier elimination. Let* $\mathrm{M} \models T$ *be an* $|A|$*-saturated model and suppose that the subsets of* $p(\mathrm{M})^n$ *relatively defined by positive quantifier free formulas satisfy the axioms for a Zariski Geometry (with the exception of quantifier elimination). Then* T *interprets an algebraically closed field* K *and* p *is non-orthogonal to the generic type of* K*.*

This theorem played an important part in Hrushovski's proof of the relative version of the Mordell-Lang Conjecture [Hru96]. He showed that certain *thin* types in seperably closed fields do satisfy the required axioms. Delon [Del98] later extended this to all minimal types. However, in [Hru96] the field is obtained by directly showing that these thin types come equipped with a specialization which satisfies similar properties to specializations in Zariski Geometries. In this way a proof of 1.1.15 in full generality is omitted. The trichotomy theorem for ACFA [CHP02] uses another modified version of the original Zariski Geometries Axioms. The main result here is

Theorem 1.1.16 *Let p be a non-locally modular minimal type in ACFA. Then p is almost internal to a fixed field.*

Again, the authors show that a certain "limit structure" almost satisfies the axioms of a Zariski Geometry and state that the proof of Trichotomy Theorem should generalize, but instead they use techniques specific to the context to recover the field.

A further Trichotomy Theorem is that for o-minimal structures which does not fall into the realm of Zariski Geometries at all. O-minimal structures can be seen as a generalization of real analytic geometry. Good introductions to the subject can be found in [vdD98] and [Pet07]. Even though from a model theoretic viewpoint these are fundamentally different from Zariski Geometries, there are some striking similarities. O-minimal structures also come equipped with a definable topology and dimension. acl is still a pre-geometry and they have quantifier elimination in the language of closed sets. We will see later that they also share a suitable smoothness assumption with Zariski Geometries. In [PS98] Starchenko and Peterzil prove

Theorem 1.1.17 *Let M be an o-minimal structure and $a \in \mathrm{M}$. Then exactly one of the following is true.*

1. *The geometry around a is trivial.*
2. *The geometry around a is linear and there is a convex set $V \ni a$ which has the structure of an ordered vector space over an ordered division ring.*
3. *The geometry around a is non-linear and there is an interval $R \ni a$ which has the structure of an o-minimal expansion of real closed field.*

Despite the structural similarities to Zariski Geometries, the proof of this theorem uses entirely different techniques. Unifying this result with that for Zariski Geometries and showing that it is the geometric similarities that underly the Trichotomy Theorems is one of the main goals of this thesis.

1.2 Outline and Statement of Results

In order to make the technical construction in chapters 2–4 a little easier to follow, we will start with an outline of the key concepts and results. To get a general feel for the strategy we are pursuing, let us take a look at how we can recover algebraic structures in a more familiar context. The idea is to first construct a chunk of the multiplicative group and then a chunk of additive group of a field by differentiating suitable definable families of functions. The chain rule turns composition, which is easily definable, into multiplication and the product rule turns multiplication into addition. To illustrate this, let us consider $\mathbb{R}$ and suppose $\mathcal{U}$ is a sufficiently rich and smooth definable family of differentiable functions $\mathbb{R} \to \mathbb{R}$. By this we mean a definable set $\mathcal{U} \subseteq \mathbb{R}^2 \times L$ such that

1. there is a two-dimensional set $S \subseteq \mathbb{R}^2$ such that for all $(a,b) \in S$, the set $\{ l \in L \mid U(a,b,l) \}$ is infinite,
2. for every $l \in L$ the set $\mathcal{U}_l = \{ (x,y) \mid (x,y,l) \in \mathcal{U} \} \subseteq S$ is the graph of a differentiable function $\mathbb{R} \to \mathbb{R}$ and
3. the funcions $l \mapsto \mathcal{U}_l$ and $(a,l) \mapsto \frac{dU_l}{dx}(a)$ are continuous and non-constant.

A good example of such a family would be $I = \{ (x,y,u,v) \mid y = u.x + v \ \& \ (u,v) \in L \}$ where $L \subseteq \mathbb{R}^2$. For convenience, we will write l instead of U_l for the function parametrized by l. We now fix $(a,b) \in S$ and restrict our attention to functions $f \in L_{(a,b)} = \{ f \in L \mid f(a) = b \}$. Consider the set

$$D = \left\{ \frac{d(f^{-1} \circ g)}{dx}(a) \,\middle|\, f,g \in L_{(a,b)} \right\}$$

By assumptions 1 and 3 on $\mathcal{U}$, this set is infinite and contains an interval around 1. Furthermore, by the chain rule

$$\frac{d\left[(f^{-1}\circ g)\circ(i^{-1}\circ h)\right]}{dx}(a)=\frac{d(f^{-1}\circ g)}{dx}(a).\frac{d(i^{-1}\circ h)}{dx}(a)$$

so the trace of composition on D gives us a relation $\otimes=\{(x,y,z)\in D^3 \mid x.y=z\}$. We can proceed similarly if we already have multiplication in place. We consider $L_{(1,c)}$ for some suitable c such that $(1,c)\in S$ and look at the set

$$E=\left\{\frac{d(g/f)}{dx}(a)\;\middle|\; f,g\in L_{(a,1)}\right\}$$

where multiplication and division of g and f are pointwise. Again, this will contain some interval containing 0 and in a similar fashion to the above, the product rule implies that

$$\frac{d\,[(g/f).(h/i)]}{dx}(a)=\frac{d(g/f)}{dx}(a)+\frac{d(h/i)}{dx}(a)$$

thus giving us the relation $\oplus=\{(x,y,z)\in E^3 \mid x+y=z\}$. Of course, neither of these are groups yet, but by restricting to infinitesimal neighbourhoods around 1 and 0 respectively, we obtain two type-definable groups with a definable distributive action of the multiplicative group on the additive group via

$$\frac{d(h/i)}{dx}(a).\frac{d(f^{-1}\circ g)}{dx}(a)=\frac{d((h/i)\circ f^{-1}\circ g)}{dx}(a)$$

Such an object is called a *definable group presentation* in [PS98] (see definition 3.2.24). In both stable theories and o-minimal structuers there are known methods that allow us to construct a field from this. Our strategy will be inspired by this construction and works in a much more abstract context.

In chapter 2 we axiomatically define the class of structures in which we will be working and develop some tools for later use. *Strong Geometries* can be summed up as follows:

A Strong Geometry is a structure M with a distinguished generating class of definable sets called *closed* and a dimension notion dim for existentially definable sets satisfying

1. the dimension is existentially definable and additive (See axioms (DP),(DU),(DD) and lemma 2.1.23),
2. existentially definable sets P have a well defined *closure* $\overline{P}$ and their boundaries are small (See axiom (SB) and lemma 2.1.15),
3. closed sets can be decomposed into finitely many *components* which are *uniform* with respect to the dimension notion (See axiom (DC), definition 2.1.3 and lemma 2.1.13) and
4. uniform sets satisfy a *smoothness* condition (See definition 2.1.17 and axiom (GS)).

The sort of examples we should have in mind are Zariski Geometries, o-minimal and C-minimal structures. However, we should note three key differences that distinguishes general Strong Geometries from the former.

1. The closed sets do not in general form the closed sets of a topology,
2. we do not require quantifier elimination and only work with existentially defined sets and
3. zero-dimensional sets are not required to be finite.

These differences make some arguments and proofs rather more technical than they would otherwise be. The reason we make generalization 1 is that the property that the closed sets form a topology is not generally preserved under elementary extensions. This happens in specific classes like the ones mentioned above, but I see no way of axiomatizing this. We do not ask for quantifier elimination as one of our main aims is to include type-definable Zariski Geometries as in [Hru96] and [CHP02]. Lastly, the reason for generalization 3 is the hope that we might be able to include some Analytic Zariski Geometries which may have infinite zero-dimensional sets. The omission of any structural information about zero- or one-dimensional sets potentially allows many kinds of structures to be considered in the framework. On the other hand, the following examples illustrate that dimension is neither uniquely determined by the structure M, nor does it tell us anything useful without further assumptions:

1. Any structure M where all definable sets are closed and zero-dimensional is a Strong Geometry.

2. Given a Strong Geometry M, M is still a Strong Geometry if we multiply all dimensions by a constant.

3. Suppose we have two Strong Geometries A and B. The structure $\mathrm{M} = A \times B$, where the definable sets in M^n are boolean combinations of sets of the form $C \times D \subseteq A^n \times B^n$ and the dimension $\dim(C \times D) = \dim C + \dim D$ is a Strong Geometry.

We will not have to worry about examples 1 and 2 as we will be working in a one-dimensional structure, but in view of example 3, it should come as no surprise that to carry out the group constructions, we have to introduce some further axioms. However, a surprising amount of theory can be built up without that additional information.

The first ingredient in the outline was the family of functions $\mathcal{U}$. This needed to have a certain degree of complexity so that the sets D and E contained intervals around 1 and 0 respectively. Generally speaking, the complexity of a family of definable sets is a subtle issue, but in section 2.1.3 we develop a meaningful notion of the dimension of families of sets. The key object that will take the place of $\mathcal{U}$ is a *generically normal two-dimensional family of plane curves*. This consists of two uniform one-dimensional sets C_x and C_y, a two-dimensional parameter set L and a uniform $I(x, y, u) \subseteq C_x \times C_y \times L$ such that

1. for all $l \in L$, $\dim I_l = \dim \{ (x, y) \mid (x, y, l) \in I \} = 1$,

2. for all $(x, y) \in C_x \times C_y$, $\dim I_{(x,y)} = \dim \{ l \mid (x, y, l) \in I \} = 1$ and

3. $\dim \left\{ (l_1, l_2) \in L^2 \;\middle|\; \dim \{ (x, y) \mid (x, y) \in I_{l_1} \cap I_{l_2} \} = 1 \right\} = 2$.

It is shown in lemma 2.1.41 that this also implies

$$\dim \left\{ (x_1, y_1, x_2, y_2) \in (C_x \times C_y)^2 \;\middle|\; \dim(\{ l \mid l \in I_{(x_1,y_1)} \cap I_{(x_2,y_2)} \}) = 1 \right\} = 2$$

These four properties are very similar to those of a pseudo-plane. We will see that such an object exists in every sufficiently rich one-dimensional Strong Geometry. We will do so in analogy with strongly minimal sets by studying a suitable pre-geometry on elementary extensions.

The first obvious question we need to address is whether the class of strong geometries is closed under elementary extensions. Not only is this an interesting question in its own right, but it is also an important prerequisite for a well-behaved notion of infinitesimals as well as a weak trichotomy theorem.

Theorem 2.2.2 *If* M *is a Strong Geometry and* $^*\mathrm{M} \succcurlyeq \mathrm{M}$, *then with the induced dimension,* $^*\mathrm{M}$ *is also a Strong Geometry.*

Due to the fact that zero-dimensional sets are not bounded, this may be somewhat surprising to model theorists. However, we have already seen that no meaning is attached to the dimension and if one looks closely at the axioms of Strong Geometries, then they all translate to combinatorial properties of formulas. It is obvious that definable sets in $^*\mathrm{M}$ are of the form $S(a) = \{ x \mid (x, a) \in S \}$ where $S(x, y)$ is an M-definable set and a is generic in the projection of S to the y-coordinate. The smoothness condition ensures that uniformity is preserved by this – if S is uniform, then so is $S(a)$ (see lemma 2.2.5). Since dimension is definable, this more or less turns all the axioms into schemas of first order sentences.

If M is one-dimensional, we can define a combinatorial dimension (or rank) cdim in the usual way and define a pre-grometry cl which takes the place of acl. We obtain a weak trichotomy result similar to that for *geometric structures* by Berenstein, Vasilliev [BV10] and Boxall, et al. [BV10, BBWK$^+$11]:

Theorem 2.2.25. *Let* M *be a one-dimensional Strong Geometry and* $^*\mathrm{M}$ *an elementary extension sufficiently saturated for quantifier-free types. Then exactly one of the following is true:*

1. cl *and all families of plane curves are trivial,*
2. cl *is very weakly one-based and every generically normal family of plane curves is linear,*

3. cl *is not very weakly one-based and there is a two-dimensional generically normal family of plane curves.*

Even though the statement is the same as that for geometric structures given in [BV10], the context and proof are not. In particular, the argument given here avoids the use of *lovely pairs*.

In the last sections of chapter 2 we develop our main tool for the construction of algebraic objects. We use *specializations* (definition 2.3.1) as in [HZ96] or [Zil10] to get a notion of infinitesimals. A specialization $\pi\colon {}^*\mathrm{M} \to \mathrm{M}$ is simply a partial homomorphism with respect to the relations given by closed sets. These should be thought of as generalizations of the standard part map or valuations and in that spirit we call $\mathcal{V}_a = \pi^{-1}(a)$ the *infinitesimal neighbourhood* of a. Of course, the identity map is a specialization so this notion only becomes useful once the sets $\mathcal{V}_a$ are have some non-trivial structure. Using a construction similar to that commonly used to prove the existence of saturated models, Zilber [Zil10] obtained so called *universal specializations* for which every infinitesimal neighbourhood is ω-saturated. However, since we do not require quantifier elimination, universal specializations exhibit some undesirable behaviour. Given an existentially definable set P, we have very little control over its boundary $\overline{P} \setminus P$. In particular, it might happen that the boundatry contains elements a with $\operatorname{cdim}(a) = \dim(P)$. As we indicated in the summary of Strong Geometries, existentially definable sets are meant to have small boundaries so this this has to be avoided. To overcome this problem, we generalize the notion of universal specializations and show in proposition 2.3.15 that for Strong Geometries there are specializations with respect to which all infinitesimal neighbourhoods are ω-saturated for existential formulas, but such that within the domain of the specialization, the boundary of existentially definable sets is the union of closed sets of small dimension. In other words, within $\operatorname{Dom}\pi$, the type of generic boundary elements is omitted. In 2.3.17 we show that this corresponds to a form of local elimination of existential quantifiers. All proofs in chapter 3 could be carried out without referring to specializations, but they allow us to follow the proof strategy given earlier in this section very closely. The reason for this is the Implicit Function Theorem (IFT)

2.3.22. This gives us a much more intuitive characterization of smoothness and allows us to draw many parallels with the strategy outlined in the beginning of this section.

In particular, it allows us to see families of curves as families of functions. Given a family of curves $I(x,y,z,u)$ parametrized by u, we can consider this as a family of local bijections $x \mapsto y$. Given generic $(a,b,c,l) \in I$, we define the *c-branch* of l at (a,b) to be

$$\widetilde{l} = \{\, (a',b') \in \mathcal{V}_{ab} \mid \exists c' \in \mathcal{V}_c \;\&\; I(a',b',c',l)\,\}$$

The Implicit Function Theorem guarantees that if a, l and m are sufficiently generic, this is a bijection $\mathcal{V}_a \mapsto \mathcal{V}_b$. Furthermore, composition and pointwise multiplication behave exactly as we would expect. We discuss branches and their properties in detail in section 2.3.2. The important thing to note is that if $I(x,y,u)$ is a generically normal two-dimensional family of curves, then we can see $I_{(a,b)}$ as a one-dimensional family of bijections $\mathcal{V}_a \mapsto \mathcal{V}_b$ and compose them definably.

With a suitable replacement of $\mathcal{U}$ in place, we turn our attention to the challenge of extracting groups from this. Chapter 3 contains the main constructions. We start by introducing two more assumptions. The first is a generalization of the pre-smoothness axiom for Zariski Geometries. This tells us that after sufficient decomposition, the Implicit Function Theorem holds away from a set of co-dimension 2. The second assumption concerns the structure of one-dimensional sets. We introduce axioms (nT) which limit the number of generic two-types over M that are realized in $\mathcal{V}_{aa}$ for a generic singleton $a \in$ M and thus obtain a notion of *direction* (see definition 3.1.8). For example, in an o-minimal structure there are two such types – the one containing $x < y$ and the one containing $y < x$ corresponding to approaching a from the left or from the right. We will call a structure satisfying these extra axioms a *Presmooth Geometry*. We also introduce the axiom (SL) which makes sure that sets are in some sense locally irreducible. We call Presmooth Geometries satisfying (SL) *Stable-Like*.

We then define a notion of tangency. Differentiating a curve at a point is a matter of finding a straight line which is tangent to the curve and taking as the derivative the gradient of that straight line. In view of this, we will consider ordinary differentiation to be differentiation with respect to the family of straight lines parametrized by gradient. As we do not in general have a notion of straight line, we will just take our family $I_{(a,b)}$ instead and differentiate with respect to that. In that fashion we hope to obtain a form of "Geometric Calculus" in the sense of [PS98]. The *I-derivative* of a curve C should be the tangency equivalence class of curves from I which are tangent to C, so long as this exists. Of course, this only makes sense if the tangency relation is actually an equivalence relation and making sure of this is one of the main challenges in the construction.

Suppose K is a family of c-branches through (a, b). As an example of such a K we should always keep $K = I_{(a,b)}$ in mind in which case we omit the c. We can try do define tangeny in analogy with traditional differentiation. Suppose we have two branches from I parametrized by l and m. We think of the intersection at (a, b) as being tangent if it has multiplicity two or more. Since for all l and m, $\widetilde{l}(a) = \widetilde{m}(a)$, this is equivalent to saying that once we perturb l and m slightly, we get a second point of intersection. Using our infinitesimal toolbox we can say exactly this. We define the tangency relation $\|$ to be

$$l \parallel m \text{ if and only if there exist } (l', m'a') \in \mathcal{V}_{lma} \text{ with } a \neq a' \text{ and } \widetilde{l'}(a') = \widetilde{m'}(a')$$

This is illustrated in the following figure.

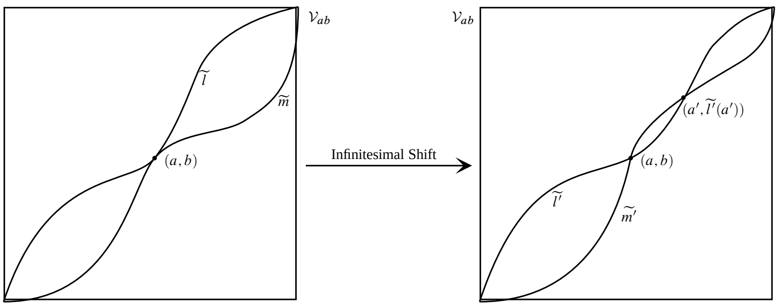

Figure: Tangency for a family of branches through (a, b)

We will see in section 3.2 that given families of branches through (a, b), the relation $\|$ between these two families is given by a closed set. This definition is quite clearly reflexive and symmetric, but in many cases transitivity will fail. This is easily seen in the following examples in $\mathbb{R}$.

Example 1 Take I_1 to be the family of vertical lines, I_2 to be the family of circles of radius 1 with centre on the vertical line $x = -1$ and I_3 the family of circles of radius 1 with centre on $x = 1$. Then the above definition tells us that the circles centred on $(-1,0)$ and $(1,0)$ are each tangent to the line $x = 0$ at $(0,0)$, but it does not detect that they are tangent to each other.

Example 2 Consider the family F where $F(s)$ is the function $y = x.(1 + \frac{sign(s)}{2}.e^{-\frac{x^2}{s^2}})$. Given any differentiable function h with $h(0) = 0$ and gradient between $\frac{1}{2}$ and $\frac{3}{2}$, the above definition will tell us that $F(0) = \mathrm{id}_{\mathrm{R}}$ is tangent to h at 0.

On the other hand, this construction does exactly what we want under some circumstances. Suppose our structure M is an o-minimal extension of a real closed field, C is a curve, (a,b) is generic on C and I is the family of straight lines through (a,b) parametrized by gradient. By general facts about o-minimal structures (see [vdD98]), we know that C is differentiable at (a,b). Suppose that the derivative is d. Then the definition of derivatives in real closed fields tells us that for any open set $U \ni (a,b,d)$ there is a tuple $(a,b,d) \neq (a',b',d') \in U$ such that

$$\frac{b'-b}{a'-a} = d'$$

This implies that there is a triple $(a,b,d) \neq (a',b',d') \in \mathcal{V}_{abd}$ satisfying the same formula. If we write $\mathrm{dx} = a' - a$ and $\mathrm{dy} = b' - b$, this situation gives us the following familiar picture and the formula

$$\pi\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right) = d$$

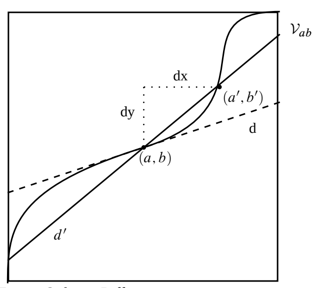

Figure: Ordinary Differentiation.

Our two examples highlight the two main issues that we need to overcome. Our family needs to be sufficiently rich to actually detect tangency and it has to be smooth enough so that $\|$ does not exhibit degenerate behaviour. By picking a generic $(a,b,g) \in I$ and working locally around this, we should be able to make sure of this, but a general proof is missing. What we can show is the following:

Proposition 1.2.1 *Suppose $I(x,y,u)$ is a smooth generically normal family of plane curves and $(a,b,g) \in I$ is generic. Let D be a direction at a such that*

$$K = I^D_{(a,b)} = \{ (x,y,u) \in I \mid x \in D \;\&\; (a,b,u) \in I \}$$

is a one-dimensional family through (a,b). Then $\|$ is generically transitive on the family $K^{-1} \circ K$.

This is proved in corollary 3.1.17. In the group constructions we are interested in tangency around the identity map $g^{-1} \circ g$. Since g is generic, the only non-generic elements in $\mathcal{V}_{gg}$ are of the form (h,h) for $h \in \mathcal{V}_g$. The question whether $\|$ is an equivalence relation on the set $\{ f^{-1} \circ h \mid f,h \in \mathcal{V}_g \}$ therefore boils down to the question whether it is possible to have $n^{-1} \circ m \| o^{-1} \circ o$ for $m \neq n$.

If this is never the case, then $\|$ is an equivalence relation and we will denote the equivalence class of $f^{-1} \circ h$ by $d(f^{-1} \circ h)$. We then wish to define multiplication on

$$H = \{ d(f^{-1} \circ h) \mid f,h \in \mathcal{V}_g \}$$

via

$$d(f^{-1} \circ h).d(u^{-1} \circ v) = d(r^{-1} \circ s) \text{ if and only if } f^{-1} \circ h \circ u^{-1} \circ v \| r^{-1} \circ s$$

Since $h^{-1} \circ f \circ f^{-1} \circ j$ is trivially tangent to $h^{-1} \circ j$, one way of making sure that H is closed under this operation is to make sure that we can match "nominators" with "denominators". There are still a lot of well-definedness issues to overcome, but in fact, a richness condition along these lines suffices.

Proposition 3.2.8. *Suppose that for any family K obtained as above, K satisfies the following two properties:*

(Non-Degeneracy) *Given independent $n, m, o \in \mathcal{V}_g$, $o^{-1} \circ n \not\parallel o^{-1} \circ m$.*

(Richness) *For any generic $(h, i, r) \in \mathcal{V}_{(g,g,g)}$, there exists an $s \in \mathcal{V}_g$ such that $i^{-1} \circ h \parallel s^{-1} \circ r$.*

Then the relation $\otimes(f, h, i) = g^{-1} \circ h \parallel f^{-1} \circ i$ gives a group with identity g on $\mathcal{V}_g$.

Note that $\otimes$ is the lift of the operation . defined above under the map $h \mapsto d(g^{-1} \circ h)$. Once we have done this, we can use standard methods to change the situation to one where we can assume that b is the identity of a type-definable group on $\mathcal{V}_b$. We re-label b to e and now consider $K = I^D_{(a,e)}$ for some generic a. Again, we pick a generic curve g, but this time we look at

$$G = \left\{ \frac{e}{f} \cdot h \;\middle|\; f, h \in \mathcal{V}_g \cap I_{a,b} \right\}$$

where $\frac{e}{f}$ denotes the multiplicative inverse. We do not write f^{-1} to distinguish the multiplicative inverse from the inverse function. Using an identical proof we obtain an analogous proposition.

Proposition 3.2.22. *Suppose that for any family K obtained as above, K satisfies the following two properties:*

(Non-Degeneracy) *Given independent $n, m, o \in \mathcal{V}_g$, $\frac{e}{o} \cdot n \not\parallel \frac{e}{o} \cdot m$.*

(Richness) *For any $h, i, r \in \mathcal{V}_g$, there exists an $s \in \mathcal{V}_g$ such that $\frac{e}{i} \cdot h \parallel \frac{e}{s} \cdot r$.*

Then the relation $\oplus(f, h, i) = \frac{e}{g} \cdot h \parallel \frac{e}{f} \cdot i$ gives a group with identity g on $\mathcal{V}_g$.

In an ideal world the multiplicative group would act on the additive group by composition. If it did, distributivity would follow from lemma 3.2.23. It would be nice if this were possible, but a proof of this is lacking. Instead we construct a second multiplicative group. Instead of letting $\mathcal{V}_e$ act on $\mathcal{V}_g$, we relabel g to 0 and fix an element $1 \in \mathcal{V}_0$. Since

$e, 0$ and 1 are generic independent elements, the family $(e/0 \cdot h)$ for $h \in \mathcal{V}_1$ is generically normal. We can thus apply proposition 3.2.8 to this to obtain another multiplicative group, this time on $\mathcal{V}_1$. This does act distributively on the additive group so that we obtain a group presentation in the following sense.

Definition 3.2.24. A (one-dimensional) *constructible group presentation* consists of curves C and D with elements $0 \in C$, $1 \in D$ and constructible relations $+, \cdot$ and $.$ such that the restrictions of $+$ and $\cdot$ to $\mathcal{V}_0$ and $\mathcal{V}_1$ are the graphs of groups with identities 0 and 1 respectively and such that the restriction of $.$ to $\mathcal{V}_0 \times \mathcal{V}_1 \times \mathcal{V}_0$ is a distributive group action of $\langle \mathcal{V}_1, \cdot \rangle$ on $\langle \mathcal{V}_0, + \rangle$. We say that the presentation is faithful if the action is.

Proposition 3.2.25. *Under the conditions given in 3.2.8 and 3.2.22, if M is a non-linear one-dimensional Presmooth Geometry, then there is a projective group presentation.*

Sadly, a general proof of the non-triviality of this action is missing, but in all the key classes this can be shown to be faithful.

In chapter 4 we look at examples of Strong and Presmooth Geometries. In view of the characterization of Strong Geometries given earlier in this introduction, it is natural to look for examples amongst classes of structures with a good cell decomposition theorem and dimension theory. Such structures include Zariski Geometries, o-minimal structures, C-minimal structures, p-adically closed fields and many others. We will merely show that the first two are Strong Geometries, but the proofs should generalize easily. Whether a structure is a Presmooth Geometry is a much more subtle issue. The only known examples so far are generalizations of Zariski Geometries. The proof of presmoothness for o-minimal relies on a certain strenghthening of the cell-decomposition theorem, which is left open at this point (see property (LC), definition 4.2.14). In order to treat Zariski Geometries and their type-definable derivatives uniformly, we introduce an axiomatization similar to that found in [CHP02].

Definition 4.1.2. An $\mathcal{L}$-structure M with a distinguished generating family of definable sets called *closed* is a *multi-sorted κ-proper Zariski Geometry* if for any tuple of sorts

$X_{i_1}, \ldots, X_{i_n}$, the closed sets form a Noetherian topology on $X_{i_1}(\mathrm{M}) \times \ldots \times X_{i_n}(\mathrm{M})$ and these topologies, together with Krull dimension dim, satisfy

(κMZ1) M is κ-compact and for any closed irreducible set $S \subseteq \mathrm{M}^{n+k}$ and a projection $\mathrm{pr}\colon \mathrm{M}^{n+k} \to \mathrm{M}^n$, $\mathrm{pr}S \supseteq \overline{\mathrm{pr}S} \setminus \bigcup_{j \in J} F$ for proper closed subsets $F_j \subseteq \overline{\mathrm{pr}S}$ and $|J| < \kappa$.

(MZ2) $X_i(\mathrm{M})$ is irreducible and for any closed set $S \subseteq \mathrm{M}^n \times X_i(\mathrm{M})$ there is a natural number m such that for all $a \in \mathrm{M}^n$, either $S(a) = X_i(\mathrm{M})$ or $|S(a)| \leq m$.

(MZ3) For any tuple of sorts $X_{i_1}, \ldots, X_{i_n}$

$$\dim(X_{i_1}(\mathrm{M}) \times \ldots \times X_{i_n}(\mathrm{M})) = n$$

and for any closed irreducible set $S \subseteq \mathrm{M}^n$ and Δ a diagonal of the form $x_i = x_j$ every component of $S \cap \Delta$ has dimension $\geq \dim S - 1$.

If M is one-sorted we will just say *κ-proper Zariski Geometry.*

Proposition 4.1.3. *A multi-sorted κ-proper Zariski Geometry is a Stable-Like Presmooth Geometry.*

Proposition 4.2.2. *If* M *is an o-minimal structure satisfying (LC), then* $\mathrm{M}_{\mathrm{nat}}$ *is a Presmooth Geometry satisfying (2T).*

It then remains to check the conditions of the group constructions in chapter 3.

Theorem 1.2.2 *In both Stable-Like Presmooth Geometries and o-minimal structures the conditions of proposition 3.2.25 are satisfied and furthermore, the action is faithful.*

This follows from propositions 4.1.19 and 4.2.34.

In the case of o-minimal structures, Starchenko and Peterzil [PS98] showed that the existence of a faithful group presentation guarantees the existence of an interpretable field. So if we were able to establish that (LC) holds for o-minimal structures we would obtain a new proof of the non-linear case of theorem 1.1.17. In the stable-like case we show slightly less.

Proposition 4.1.20. *In a non-linear Stable-Like Presmooth Geometry there exists a two-dimensional type-definable group which is solvable and centreless. In particular, this group satisfies the metabelian identity*

$$[a,b].[c,d] = [c,d].[a,b] \text{ where } [a,b] = a.b.a^{-1}.b^{-1}$$

and does not satisfy the class-2 nilpotency equation

$$r.[u,v] = [u,v].r$$

In a stable context there are standard methods for extracting a field from such a group (e.g. Poizat [Poi01]). In particular this proves theorem 1.1.15 and gives a proof of the field construction in theorem 1.1.13.

In chapter 5 we will discuss some open problems and questions for further research.

Chapter 2

Strong Geometries

2.1 Basics

2.1.1 Definitions

We work in a relational language $\mathcal{L}$ with equality. Given an $\mathcal{L}$-structure M we call definable sets $X \subseteq \mathrm{M}^n$ *closed*, *constructible* or *projective* if they are respectively defined using positive quantifier free, quantifier free or existential formulas with parameters from M.

Notation We will denote the classes of closed and projective sets by $\mathcal{C}$ and $\mathcal{P}$, respectively. If there is any ambiguity about the parameters allowed, we will write $\mathcal{C}_A$ and $\mathcal{P}_A$ and say A-closed and A-projective for sets defined over the parameter set A.

Definition 2.1.1 Let $f : \mathrm{M}^n \to \mathrm{M}^m$ be given by

$$\langle x_1, \dots, x_n \rangle \mapsto \langle f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n) \rangle$$

where each f_i is either a constant map or a coordinate map of the form $f_i(x_1, \dots, x_n) = x_j$ for some $j \in \{1, \dots, n\}$. Then f is called a *projection* map.

Notation If $S \subseteq \mathrm{M}^{n+k}$ is definable, we will write S and $S(x, y)$ interchangably for the set itself and the formula defining it. It is implicit that x is a tuple of length n and y is a tuple of length k. If there is ambiguity we write $\#(x)$ for the length of the tuple x. We write pr_y for the projection $(x, y) \mapsto x$ corresponding to the quantifier $\exists y$. If a is a tuple of length k, then S_a and $S(a)$ denote the fibre over a, $S(\mathrm{M}, a) = \mathrm{pr}_y^{-1}(a) \cap S$.

Notation The structures we will consider are not in general of a topological nature, but many concepts still make sense. We will call complements of closed sets *open* and make use of a lot of topological terminology like connectedness, irreducibility, discreteness, etc. When we are working in a projective set X will say that $T \subseteq X$ is *relatively closed (or open)* if $T = X \cap S$ where S is closed (or open). If it is clear which set X we are working in, we will omit the word "relatively". We will use $\subseteq_{\mathrm{cl}}$ and $\subseteq_{\mathrm{op}}$ indicate when subsets are relatively closed or open.

Let dim be an integer valued dimension defined on all projective sets over M. We will exclusively be concerned with the class of projective sets and it will often be implicit that the sets in question are projective.

Definition 2.1.2 We say that a set S has a *component of dimension* k if we can write S as a union $S_1 \cup S_2$ of relatively closed sets with $\dim S_1 = k$ and $S_2 \neq S$. We say that a set S is *pure-dimensional* if S has no component of dimension less than $\dim S$.

Definition 2.1.3 We say that a projective set S is *aligned* with respect to a projection pr if for any $T \subseteq S$

$$\dim T = \dim S \quad \Rightarrow \dim \mathrm{pr} T = \dim \mathrm{pr} S$$

S is *uniform* if it is pure-dimensional and aligned with respect to all projections.

We assume the following properties for the class of projective sets over M with the dimension dim:

(DP) Dimension of **P**oints

Singletons have dimension 0 and $\dim(\emptyset) = -1$.

(DU) Dimension of Unions

If S_1 and S_2 are projective, then $\dim S_1 \cup S_2 = \max(\dim S_1, \dim S_2)$.

(DD) Definability of Dimension

If S is a pure dimensional closed set and pr is a projection, then there is a closed subset $K \subseteq_{\mathrm{cl}} \mathrm{pr}(S)$ with $\dim K < \dim \mathrm{pr}(S)$ such that for all $a \in \mathrm{pr}(S) \setminus K$

$$\dim(S(a)) = \dim S - \dim \mathrm{pr} S$$

(SB) Small Boundaries (of projective sets)

If P is a projective set, then there is a closed set $\widetilde{P} \supseteq P$ of dimension $\dim P$ such that for any other projective set Q,

$$\text{if } \dim(P \cap Q) < \dim(P), \text{ then } \dim(\widetilde{P} \cap Q) < \dim(P)$$

(DC) De-Composition

If S is closed and $\dim S = n$, then there is a closed pure-dimensional subset $T \subseteq_{\mathrm{cl}} S$ such that

$$\dim(S \setminus T) < n$$

(DP), (DU) and (DD) are natural axioms for any geometry. (SB) is a very weak version of properness or quantifier elimination (see 2.3.17) and, as the name suggests, should be interpreted as saying that projective sets have small boundaries or are very closely approximated by closed sets. (DC) gives us a sort of cell decomposition which is made precise in lemma 2.1.13.

Note that these axioms do not imply that zero-dimensional sets are finite. The best we can say is that any projective subset X of a zero-dimensional set is closed. This follows straight from (SB) and the fact that $\dim(\emptyset) = -1$. For suppose $a \in \widetilde{X} \setminus X$. Then $X \cap \{a\} = -1 < 0 = \dim X$, but $\dim(\widetilde{X} \cap \{a\}) = 0 = \dim X$, contradicting (SB).

Definition 2.1.4 Given any natural number n, consider the set

$$\Sigma = \{ (i_1, \ldots, i_k) \mid i_j \in \mathbb{N} \;\&\; 1 \le i_1 < \cdots < i_k \le n \}$$

And order this set by $(i_1, \ldots, i_k) < (j_1, \ldots, j_l)$ iff $k > l$ or $(i_1, \ldots, i_k) <_{\text{lex}} (j_1, \ldots, j_l)$.

We can enumerate this set as $\sigma_1 < \cdots < \sigma_r$ for some r.

Given $\sigma = (i_1, \ldots, i_s) \in \Sigma$, we write pr_σ for the projection

$$(x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_s})$$

Given a set $S \subseteq \mathrm{M}^n$, we call the tuple $\text{alig}(S) = (\dim \text{pr}_{\sigma_1} S, \ldots, \dim \text{pr}_{\sigma_r} S)$ the *alignment* of S. We order alignments using the lexicographic ordering.

Note that a set S is uniform exactly when for all $T \subseteq S$ with $\dim T = \dim S$, T has the same alignment as S.

Lemma 2.1.5 *Any closed set S can be written as a finite union $\bigcup S_i$ of pure-dimensional closed sets with the property that for $i < j$, $\dim(S_i) > \dim(S_j)$ and for all i, $\dim S_i \cap \bigcup_{j \neq i} S_j < \dim S_i$. Furthermore, this decomposition is unique.*

Proof. We argue by induction on $\dim S$. The base case $\dim S = 0$ is trivial since S is already pure-dimensional.

For the inductive step we apply (DC) to S to get a pure-dimensional $S_1 \subseteq_{\text{cl}} S$ such that $\dim(S \setminus S_1) < n$. Let $S' = \widetilde{S \setminus S_1} \cap S$. By (SB), $\dim S' < \dim S$ so that the inductive hypothesis applies. Let $S_2, \ldots, S_k$ be the decomposition of S'. Now by (SB), $S_1, \ldots, S_k$ are as required.

To get uniqueness, suppose there is another such decomposition $S'_1, \ldots, S'_m$ and suppose $n = \min \{ r \le \min(m, k) \mid S_r \neq S'_r \}$ exists.

Without loss of generality, assume that $\dim S_n \ge \dim S'_n$ and that $L = S_n \setminus S'_n \neq \emptyset$. The pure-dimensionality of S_n implies that $\dim L = \dim S_n$. By assumption,

$\dim S_n \cap \bigcup_{j \neq n} S_j < \dim S_n$ so in particular, for all $i \neq n$, $\dim(S_i \cap L) < \dim S_n$. However, now

1. $L \subseteq S = \bigcup S'i$,
2. for $i < n$, $S_i = S'_i$ so that $\dim(S'_i \cap L) < \dim L$
3. $S'_n \cap L = \emptyset$ and
4. for $i > n$, $\dim S'_i < \dim L$.

This contradicts (DU) so no such n can exist.

Now suppose $m \neq k$. Without loss of generality assume that $m < k$. By assumption $S_{m+1} \setminus \bigcup_{i \leq m} S_i \neq \emptyset$, but since for $i \leq m$, $S_i = S'_i$, $\bigcup_{i \leq m} S_i = S$ so this is a contradiction as well. Thus $m = k$. □

Lemma 2.1.6 *Suppose* $S(x, y)$ *is a closed set such that for all* $a \in \mathrm{pr}_y(S)$, $|S(a)| = 1$. *Then* $\dim \mathrm{pr}_y(S) = \dim S$.

Proof. By the above lemma and (DU) it is enough to show this for pure-dimensional sets so suppose S is pure-dimensional. (DD) implies that there is a closed subset $K(x)$ of dimension $< \dim \mathrm{pr}_y S$ such that for all $a \in \mathrm{pr}_y(S) \setminus K$, $\dim S(a) = \dim S - \dim \mathrm{pr}_y(S)$. However, we know that $\dim S(a) = 0$ so that $\dim \mathrm{pr}_y(S) = \dim S$, as required. □

Lemma 2.1.7 *Dimension of closed sets is preserved under permutation of coordinates.*

Proof. Let $\sigma \in \mathrm{Sym}_n$ be a permulation of n elements and p_σ be the function $(x_1, \ldots, x_n) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Suppose $S \in \mathrm{M}^n$ is a closed set and $T = p_\sigma(S)$. Then the graph of $p_\sigma : S \to T$ is given by the formula

$$S(x_1, \ldots, x_n) \wedge \bigwedge_1^n x_i = x_{\sigma(i)+n}$$

The above formula is quantifier free so that the graph of p_σ is closed. The previous lemma implies that $\dim S = \dim T = \dim \mathrm{graph}(p_\sigma)$. $\Box$

Lemma 2.1.8 *Suppose S is pure-dimensional and closed and* pr *is a projection. Let K be as in the statement of (DD). Define $T' = \mathrm{pr}^{-1}(\mathrm{pr}S \setminus K) \cap S$ and $T_0 = \widetilde{T'} \cap S$. Let T_1 be the pure-dimensional closed set given by applying (DC) to T_0. Then $\dim T_1 = \dim S$ and $\dim \mathrm{pr}(T_1) = \dim \mathrm{pr}(S)$. Furthermore, if $S_1, \ldots, S_k$ is the decomposition of $\widetilde{S \setminus T_1}$ given by lemma 2.1.5, then for all i, either $\dim S_i < \dim T_1$ or $\dim \mathrm{pr}(S_i) < \dim \mathrm{pr}(T_1)$.*

Proof. Let $T_0 = T_1, \ldots, T_k$ be the decomposition of T_0 given by lemma 2.1.5 and suppose for a contradiction that $\dim T_1 < \dim S$ or $\dim \mathrm{pr}(T_1) < \dim \mathrm{pr}(S)$. Let $I = \{ i \mid \dim \mathrm{pr}(T_i) = \dim \mathrm{pr}(S) \}$. For each $i \in I$ we apply (DD) to get a closed set $K_i \subseteq \mathrm{pr}(T_i)$ with $\dim K_i < \dim \mathrm{pr}(T_i) = \dim \mathrm{pr}(S)$ and such that for $a \in \mathrm{pr}(T_i) \setminus K_i$, $\dim T_i(a) = \dim T_i - \dim \mathrm{pr}(T_i) < \dim S - \dim \mathrm{pr}(S)$. Let

$$L = K \cup \bigcup_{i \in I} K_i \cup \bigcup_{i \notin I} \widetilde{\mathrm{pr}(T_i)}$$

By (SB) and (DU), $\dim L < \dim \mathrm{pr}(S)$ so that $\mathrm{pr}(S) \setminus L \neq \emptyset$. However, by construction and (DU), for $a \in \mathrm{pr}(S) \setminus L$, $\dim S(a) = \max_{i \in I}(\dim T_i - \dim \mathrm{pr}(T_i)) < \dim S - \dim \mathrm{pr}(S)$. This contradicts the choice of K.

For the furthermore clause, it is enough to consider S_1 since all other S_i have smaller dimension than T. We will show that either $\dim S_1 < \dim S$ or $\mathrm{pr}(S_1) \subseteq K$.

Suppose that $\mathrm{pr}(S_1) \nsubseteq K$. We can write $S_1 = (S_1 \cap \mathrm{pr}^{-1}(K)) \cup (S_1 \cap T_0)$. By pure-dimensionality of S_1, our assumpion implieas that $\dim(S_1 \cap T_0) = \dim S_1$. By (DU), $\dim(S_1 \cap T_0) = \max_{1<i<k} \dim(S_1 \cap T_i)$. By construction and (SB), $\dim(S_1 \cap T_1) < \dim S$ and for $i > 1$, $\dim T_i < \dim S$ so that $\dim S_1 < \dim S$ as required. $\Box$

Lemma 2.1.9 *The set T_1 in the above lemma is aligned with respect to* pr.

Proof. Suppose not. Then there is a closed subset $R \subseteq_{\rm cl} T_1$ of dimension $\dim R = \dim T_1$ such that $\dim \mathrm{pr}(R) < \dim \mathrm{pr}(T_1)$. Using (DC) we can assume that R is pure-dimensional itself. Apply (DD) to R and pr to get a closed set K_R and define $R' = \mathrm{pr}^{-1}(\mathrm{pr}R \setminus K_R) \cap R$ and $R_0 = \widetilde{R'} \cap R$. Let R_1 be the pure-dimensional closed set given by applying (DC) to R_0. By the previous lemma, $\dim R_1 = \dim R = \dim T_1$.

Since for $a \in \mathrm{pr}(S) \setminus K$ and $b \in \mathrm{pr}(R) \setminus K_R$, $\dim S(a) = \dim T_1 - \dim \mathrm{pr}(T_1) < \dim R - \dim \mathrm{pr}(R) < \dim S(b)$, $(\mathrm{pr}(S) \setminus K) \cap (\mathrm{pr}(R) \setminus K_R) = \emptyset$. By two applications of (SB), this implies that $\dim(R' \cap T') < \dim S = \dim T_1$, but on the other hand, $Q \subseteq R \subseteq T_1 \subseteq T'$. This is clearly a contradiction to (DU). □

Lemma 2.1.10 *Suppose S is a pure-dimensional closed set which is aligned with respect to a projection* pr. *If $T \subseteq_{\rm cl} S$ is of the same dimension, then it is also aligned with respect to* pr.

Proof. Suppose not. Then there exists an $R \subseteq_{\rm cl} T$ of dimension $\dim R = \dim T = \dim S$ such that $\dim \mathrm{pr}(R) < \dim \mathrm{pr}(T) = \dim \mathrm{pr}(S)$. This contradicts the alignment of S. □

Lemma 2.1.11 *Given any pure-dimensional closed set S, we can write S as a union of closed sets $S = \bigcup S_i$ such that S_0 is uniform and for $i > 0$, $\mathrm{alig}(S_i) < \mathrm{alig}(S_0)$.*

Proof. Suppose $S \subseteq \mathrm{M}^n$. Let $\mathrm{pr}_{\sigma_1}, \ldots, \mathrm{pr}_{\sigma_r}$ be the projections from definition 2.1.4. Given any set $L \subseteq \mathrm{M}^n$ we define

$$n_L = r - \max\{m \mid \forall k \leq m \ S \text{ is aligned with respect to } \mathrm{pr}_{\sigma_k}\}$$

We will prove something slightly stronger by induction on n_S:

Given any pure-dimensional closed set S, we can write S as a union of closed sets $S = \bigcup S_i$ such that S_0 is uniform, for $i \leq n_S$, $\dim \mathrm{pr}_{\sigma_i}(S) = \dim \mathrm{pr}_{\sigma_i}(S_0)$ and for $i \geq 1$, $\mathrm{alig}(S_i) < \mathrm{alig}(S_0)$.

In the base case $n_S = 0$, S is aligned with respect to all projections so it is already uniform.

For the inductive step, let $\mathrm{pr} = \mathrm{pr}_{\sigma_{n_S+1}}$ and obtain $T = T_1$ and $S_1, \ldots, S_k$ as in lemma 2.1.8. Since $\dim T = \dim S$, the previous lemma implies that T is aligned with respect to $\mathrm{pr}_{\sigma_1}, \ldots, \mathrm{pr}_{\sigma_n}$ and lemma 2.1.9 implies that it is aligned with respect to $\mathrm{pr}_{\sigma_{n+1}}$. Therefore, the inductive hypothesis applies to T and we can write $T = \bigcup T_i$ such that T_0 is uniform, for $i \leq n_S + 1$, $\dim \mathrm{pr}_{\sigma_i}(T) = \dim \mathrm{pr}_{\sigma_i}(T_0)$ and for $i \geq 1$, $\mathrm{alig}(T_i) < \mathrm{alig}(T_0)$.

Lemma 2.1.8 tells us that $\dim(T_0) = \dim T = \dim S$ and $\dim \mathrm{pr}(T_0) = \dim \mathrm{pr} T = \dim \mathrm{pr} S$. Furthermore, it tells us that and that for all i, $\dim S_i < \dim T$ or $\dim \mathrm{pr}(S_i) < \dim \mathrm{pr}(T)$. Therefore, for all i, $\mathrm{alig}(S_i) < \mathrm{alig}(T_0)$. Relabelling T_0 to S_0 and T_i to S_{k+i} for $i > 0$, we are done. $\square$

Corollary 2.1.12 *Any closed set S can be written as a finite union $\bigcup S_i$ of uniform closed sets.*

Proof. We argue by induction on $\mathrm{alig}(S)$. If $\mathrm{alig}(S) = (0, \ldots, 0)$, then $\dim S = 0$ and then S is already uniform.

We start by applying lemma 2.1.5 to write S as a union $S = S_1 \cup \ldots \cup S_k$ of pure dimensional sets such that for $i < j$, $\dim S_i > \dim S_j$.

Next, we apply lemma 2.1.11 to write S_1 as a union $S_1 = S'_0 \cup \ldots, S'_m$ where S'_0 is uniform and for $i > 0$, $\mathrm{alig}(S'_i) < \mathrm{alig}(S_1)$.

Now we apply the inductive hypothesis to all S_i for $i > 1$ and to all S'_i for $i > 0$. $\square$

Proposition 2.1.13 *Any closed set S can be uniquely written as a finite union $\bigcup S_i$ of uniform closed sets with the property that for $i > j$,* $\mathrm{alig}(S_i) > \mathrm{alig}(S_j)$ *and for all i,* $\dim S_i \cap \bigcup_{j \neq i} S_j < \dim S_i$.

Proof. We prove existence of such a decomposition first. By the above corollary we know that we can write S as the union of uniform sets. By taking unions of sets with identical alignments, we may assume that S is a union of uniform sets $S = T_1 \cup \ldots, T_n$ where $i < j$ implies $\mathrm{alig}(T_i) > \mathrm{alig}(T_j)$.

Now recursively define $S_1 = T_1$, $T'_{i+1} = T_{i+1} \setminus \bigcup_{j=1}^{i} S_j$ and S_{i+1} to be the largest pure-dimensional component of $\widetilde{T'_{i+1}}$. By pure-dimensionality and uniformity of the T_i, each S_i is either empty or has the same alignment as T_i. By construction and (SB), the property that for all i, $\dim S_i \cap \bigcup_{j \neq i} S_j < \dim S_i$ also holds.

For uniqueness we use a similar argument to that employed in lemma 2.1.5. Again , we assume that there is another such decomposition $S = S'_1 \cup \ldots \cup S'_m$ and suppose $n = \min\{\, r \leq \min(m, k) \mid S_r \neq S'_r \,\}$ exists.

Without loss of generality assume that $\mathrm{alig}(S_n) \geq \mathrm{alig}(S'_n)$ and that $L = S_n \setminus S'_n \neq \emptyset$. By the uniformity of S_n, $\mathrm{alig}(L) = \mathrm{alig}(S_n)$. However, now

1. $L \subseteq S = \bigcup S'i$,
2. for $i < n$, $S_i = S'_i$ so that $\dim(S'_i \cap L) < \dim L$
3. $S'_n \cap L = \emptyset$ and
4. for $i > n$, $\mathrm{alig}(S'_n) < \mathrm{alig}(L)$.

This contradicts (DU) so no such n can exist.

A similar argument to that in 2.1.5 shows that $k = m$ so that the two decompositions match up as required. □

Definition 2.1.14 We call the S_i in the above lemma the *uniform components* of S

The above proposition allows us to sharpen up (SB) slightly. The following lemma shows that with respect to projective sets, the closed sets behave a little like the closed sets of a topology.

Lemma 2.1.15 *Given a projective set P, there is a smallest closed set $\overline{P} \supseteq P$.*

Proof. By the previous lemma it is enough to show this for $P = \mathrm{pr}S$ where S is uniform. Let $\widetilde{P}$ be the closed set given by (SB). By (SB) and uniformity of S the set $\widetilde{P}$ can only have one component of dimension $\dim P$, which we will denote $\overline{P}$. Let P' be the union of all other components of $\widetilde{P}$. Then $\dim P' < \dim P$ and by the alignment and pure-dimensionality of S, P' cannot intersect P. So $P \subseteq \overline{P}$.

Now suppose that $P \subseteq T \subseteq_{\mathrm{cl}} \overline{P}$ for some T and define $K = \overline{P} \setminus T$. $P \cap K = \emptyset$ so by (SB), $\dim \widetilde{K} \cap P < \dim P$, but now we can write $\overline{P} = T \cup (\overline{P} \cap \widetilde{K})$, contradicting the pure-dimensionality of $\overline{P}$.□

Definition 2.1.16 Given a projective set $P \subseteq X \subseteq_{\mathrm{cl}} \mathrm{M}^n$, we call $\mathrm{int}_X\, P = P \setminus \overline{(X \setminus \overline{P})}$ the *interior of P in X* and $\mathrm{bd}_X(P) = \overline{P} \setminus \mathrm{int}_X(P)$ the *boundary of P in X*. If it is clear which set X we are working in, we simply write int and bd. If X is not closed, we let $\mathrm{int}_X(P) = X \cap \mathrm{int}_{\overline{X}}(P)$ and $\mathrm{bd}_X(P) = X \cap \mathrm{bd}_{\overline{X}}(P)$.

If the closed sets are those of a topology, then $\mathrm{int}_X\, P$ is the topological interior of $\overline{P}$ intersected with P. This makes sense if we have axiom (κMZ1) for multi-sorted κ-compact Zariski Geometries in mind. We can imagine projective sets as constructible sets with a "small" number of closed sets taken out and with this in mind this definition of interior makes sense.

To be able to do anything interesting with these kinds of structures, we need to understand the structure of sets of the form

$$Q = \{\, (x, y, z) \mid (x, y) \in S \;\&\; (x, z) \in T \,\}$$

Intuitively, if $\mathrm{pr}_y(S)$ and $\mathrm{pr}_z(T)$ are almost the same and and S and T are uniform and "smooth", then Q should turn out to be pure-dimensional itself and its dimension should be determined by (DD). Indeed, if T is M^{n+k} iself, then $Q = S \times \mathrm{M}^k$ so it is pure-dimensional for any S. This motivates the following definition:

Definition 2.1.17 Let $\mathrm{pr}\colon \mathrm{M}^{n+k} \to \mathrm{M}^n$ be a projection map and $S \subseteq \mathrm{M}^{n+k}$ be pure-dimensional and aligned with respect to pr. We say that S is *smooth* with respect to pr if whenever $T \subseteq \mathrm{M}^{n+m}$ is pure-dimensional and aligned with respect to $\mathrm{pr}'\colon \mathrm{M}^{n+m} \to \mathrm{M}^n$ and $\overline{\mathrm{pr}(S)} = \overline{\mathrm{pr}'(T)}$, then

$$\{\, (x, y, z) \in \mathrm{M}^{n+k+m} \mid (x, y) \in S \;\&\; (x, z) \in T \,\}$$

has no components of dimension less than $\dim S + \dim T - \dim \mathrm{pr} S$.

There are numerous ways in which sets can fail to be smooth. We show just a few examples in the context of $\mathbb{R}$ here. The red points are those which would have to be removed to make the sets smooth. It is left as an exercise to the reader to find suitable sets T that witness the non-smoothness of these sets. The word smooth will be justified later when we show that a set is smooth exactly if the Implicit Function Theorem 2.3.22 holds at every point.

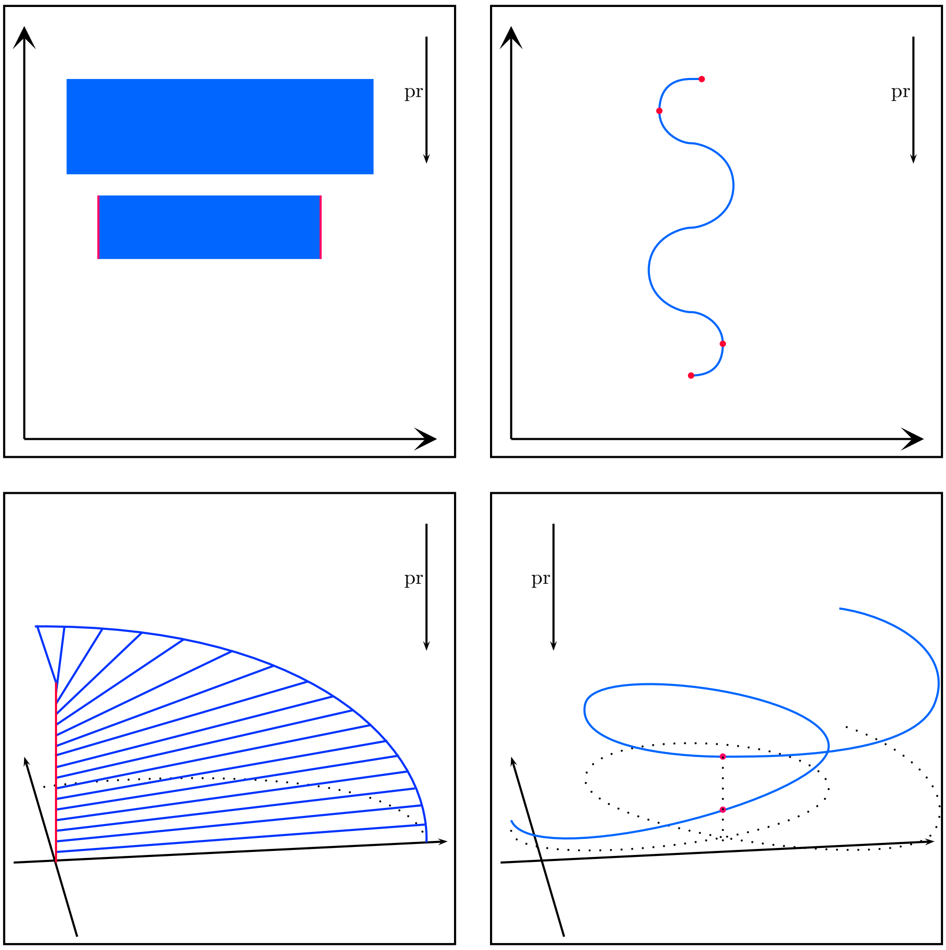

Figure 1: Four different ways in which smoothness can fail with regards to a projection pr.

The first two examples illustrate something rather interesting. The class of smooth sets is not in general closed under unions. As we will see in 4.2.10, smoothness in o-minimal structures is equivalent to the following:

Lemma 4.2.10 *A definable set S is smooth with respect to a projection* pr *if and only if for all $s \in S$ and open sets $U \ni s$, there is an open set $V \ni \mathrm{pr}(s)$ such that* $\mathrm{pr}(S \cap U) \cap V = \mathrm{pr}(S) \cap V$.

The Implicit Function Theorem 2.3.22 will generalize this through infinitesimals and this is what we shoud have in mind when we think about smooth sets: A set $S(x, y)$ is smooth with respect to pr_y if for any point (a, b) the projection locally around (a, b) looks the same as the projection of S around a. In chapter 3 we will make a strong assumption regarding smoothness, but for now we will just require that sets are smooth almost everywhere.

(GS) **G**eneric **S**moothness

Let $S \subseteq_{\mathrm{cl}} \mathrm{M}^{n+k}$ be a closed pure dimensional set and let pr be the projection onto the first n coordinates. Then there is a closed subset $K \subseteq_{\mathrm{cl}} \overline{\mathrm{pr}(S)}$ with $\dim K < \dim \mathrm{pr}(S)$ such that $S \setminus \mathrm{pr}^{-1}(K)$ is smooth with respect to pr.

We will call a structure satisfying all the above axioms a *Strong Geometry.* For the rest of this text we will assume that all geometries M considered are pure-dimensional. In the one-dimensional case this is equivalent to there not being any isolated points.

2.1.2 Some Basic Results

In this section we develop a toolbox of basic lemmas that we will use throughout the rest of this text.

Lemma 2.1.18 *If S is pure-dimensional,* pr *is a projection and $T \subseteq \mathrm{pr}(S)$, then*

$$\dim T = \dim \mathrm{pr}(S) \quad \Rightarrow \dim \mathrm{pr}^{-1} T \cap S = \dim S$$

Proof. Suppose for a contradiction that $\dim \mathrm{pr}^{-1} T \cap S < \dim S$. Then by (SB), $\dim \overline{(\mathrm{pr}^{-1} T \cap S)} < \dim S$. If $S' = S \setminus \overline{(\mathrm{pr}^{-1} T \cap S)}$, then by pure-dimensionality of S, $\overline{S'} = S$ and in particular $\mathrm{pr}^{-1}(T) \cap S \subseteq \overline{S'}$. Now $T \subseteq \overline{\mathrm{pr}(S')}$, but this contradicts (SB) since $T \cap \mathrm{pr}(S') = \emptyset$ and $\dim T = \dim \mathrm{pr}(S')$. $\square$

Corollary 2.1.19 *The projection of a uniform set is uniform.*

Proof. Let C be uniform and pr a projection.

Suppose for a contradiction that $\mathrm{pr}(C)$ is not pure-dimensional. Then $\mathrm{pr}(C) = T_1 \cup T_2$ where T_1 and T_2 are closed sets, $T_1 \neq C$ and $\dim T_2 < \dim C$. Since C is aligned, $\dim \mathrm{pr}^{-1}(T_2) \cap C < \dim C$, but since $C = (\mathrm{pr}^{-1}(T_1) \cap C) \cup (\mathrm{pr}^{-1}(T_2) \cap C)$ and $\mathrm{pr}^{-1}(T_1) \cap C \neq C$, this contradicts the pure-dimensionality of C. Thus, $\mathrm{pr}(C)$ is pure-dimensional.

Let $T \subseteq \mathrm{pr}(C)$ be a subset with $\dim T = \dim \mathrm{pr}(C)$ and pr' a projection. By the previous lemma $\dim(\mathrm{pr}^{-1}(T) \cap C) = \dim C$, so by alignment of C with respect to $\mathrm{pr}' \circ \mathrm{pr}$,

$$\dim \mathrm{pr}' T = \dim \mathrm{pr}'(\mathrm{pr}(\mathrm{pr}^{-1}(T) \cap C)) = \dim \mathrm{pr}'(\mathrm{pr}(C))$$

as required. □

Lemma 2.1.20 *If C is uniform and* pr *is a projection, then*

1. *if $T \subseteq C$ is dense, then $\mathrm{pr}(T) \subseteq \mathrm{pr}(C)$ is dense and*
2. *if $T \subseteq \mathrm{pr} C$ is dense, then $\mathrm{pr}^{-1}(T) \cap C \subseteq C$ is dense.*

Recall that according to the notation on page 27, dense should be interpreted in analogy with topology. $P \subseteq S$ is dense if $S \subseteq \overline{P}$.

Proof.

1. $\overline{T} \subseteq \mathrm{pr}^{-1}(\overline{\mathrm{pr}(T)})$, so if $\mathrm{pr}(T)$ is not dense, then neither is T.
2. Suppose $T' = C \setminus \overline{\mathrm{pr}^{-1}(T)} \neq \emptyset$. Then since C is pure-dimensional, $\dim T' = \dim C$ which means $\dim \mathrm{pr}(T') = \dim \mathrm{pr}(C)$ since C is aligned. However, by definition, $T \cap \mathrm{pr}(T') = \emptyset$ which contradicts the density of T by (SB). □

Lemma 2.1.21 *Given any projective set C, C is uniform if and only if $\overline{C}$ is uniform.*

Proof. Certainly, C is pure-dimensional if and only if $\overline{C}$ is because if one of them were not, then the witnessing sets would also witness non-pure-dimensionality of the other. Since $\dim C = \dim \overline{C}$, the alignment of $\overline{C}$ immediately implies the alignment of C. For the converse we have to show that if $S \subseteq \overline{C}$ is a set of dimension $\dim C$, then $S \cap C$ has dimension $\dim C$, but this follows from density of C and (SB). □

Lemma 2.1.22 *Suppose* $\dim \mathrm{M}^m = s$. *Given a constructible set* $X \subseteq \mathrm{M}^{n+m}$ *and a projection* $\mathrm{pr}\colon \mathrm{M}^{n+m} \to \mathrm{M}^n$, *there are constructible sets* $V_0, \ldots, V_s$ *partitioning* $\mathrm{pr}(X)$ *such that for all* $a \in V_i \cap \mathrm{pr}(X)$, $\dim(X(a)) = i$. *Furthermore, the dimension is additive in the sense that for all* i, *if* $V_i \neq \emptyset$, *then* $\dim(\mathrm{pr}^{-1}(V_i) \cap X) - \dim V_i = i$.

Proof. By definition of constructibility,

$$S = \bigcup T_i^0 \setminus R_i^0$$

for closed sets T_i^0 and R_i^0. By reducing T_i^0 to $T_i = \overline{(T_i^0 \setminus R_i^0)}$ and R_i^0 to $R_i = T_i \cap R_i^0$, we can make sure that $\dim R_i < \dim T_i$. Furthermore, we can split each T_i into finitely many closed uniform sets, so by (DU) it is enough to consider

$$S = T \setminus R$$

where T is closed and uniform, $R \subseteq T$ is closed and $\dim R < \dim T$.

We will now argue by induction on $\dim \mathrm{pr}(S)$.

The base case where $\dim \mathrm{pr}(S) = 0$ is very straightforward. By (SB) and pure- dimensionality of T, every fibre T_a over $\mathrm{pr}(T)$ is pure-dimensional of dimension $\dim T$. So for any $a \in \mathrm{pr}(S)$, either $R(a) = T(a)$ or $\dim(T(a) \setminus R(a)) = \dim S$. Therefore, all fibres over $\mathrm{pr}(S)$ have dimension $\dim S$, as required.

For the induction, suppose we could find a closed subset $L \subseteq \mathrm{pr}(S)$ such that $\dim L < \dim \mathrm{pr}(S)$ and $\dim S(a) = \dim S - \dim \mathrm{pr}(S)$ for $a \in \mathrm{pr}(S) \setminus L$. Then our inductive hypothesis would allow us to partition L into suitable V_i and for $r = \dim S - \dim \mathrm{pr}(S)$, we can replce V_r by $V_r \cup (\overline{\mathrm{pr}(S)} \setminus L)$ to get a suitable partition of $\mathrm{pr}(S)$. To find such an L, we apply (DD) to both T and R. We obtain closed subsets K_T and $K_R \subseteq \mathrm{pr}(S)$

of dimension $< \dim \mathrm{pr}(S)$ such that

$$\text{for all } a \in \mathrm{pr}(S) \setminus K_T \quad \dim T(a) = \dim T - \dim \mathrm{pr}(T) = \dim S - \dim \mathrm{pr}(S) \text{ and}$$

$$\text{for all } a \in \mathrm{pr}(S) \setminus K_R \quad \dim R(a) = \dim R - \dim \mathrm{pr}(R).$$

If $\dim \mathrm{pr}(R) < \dim \mathrm{pr}(S)$, then $L = K_T \cup \overline{\mathrm{pr}(R)}$ is a suitable choice since for any $a \in \mathrm{pr}(S) \setminus L$, $S(a) = T(a)$ and this has dimension $\dim(S) - \dim \mathrm{pr}(S)$.

Otherwise let $L = K_T \cup K_R$. For any $a \in \mathrm{pr}(S) \setminus L$, $\dim R(a) < \dim T(a)$ so that by (DU), $\dim S(a) = \dim T(a) = \dim(S) - \dim \mathrm{pr}(S)$. $\square$

Proposition 2.1.23 *(Definability of Dimension) Suppose* $\dim \mathrm{M}^m = r$*. Given a projective set* $X \subseteq \mathrm{M}^{n+m}$ *and a projection* $\mathrm{pr}\colon \mathrm{M}^{n+m} \to \mathrm{M}^n$*, there are constructible sets* $V_0, \ldots, V_r$ *partitioning* $\mathrm{pr}(X)$ *such that for all* $a \in V_i \cap \mathrm{pr}(X)$*,* $\dim(X(a)) = i$*. Furthermore, the dimension is additive in the sense that for all* i*, if* $V_i \neq \emptyset$*, then* $\dim(\mathrm{pr}^{-1}(V_i) \cap X) - \dim V_i = i$*.*

Proof. By definition, $X = \mathrm{pr}'(S_0)$, where $S_0 \subseteq \mathrm{M}^{n+m+k}$ is constructible. We argue by induction on $\dim S_0$, $\dim X$ and $\dim \mathrm{pr}(X)$.

By definition of constructibility,

$$S_0 = \bigcup T_i^0 \setminus R_i^0$$

for closed sets T_i^0 and R_i^0. By reducing T_i^0 to $T_i = \overline{(T_i^0 \setminus R_i^0)}$ and R_i^0 to $R_i = T_i \cap R_i^0$, we can make sure that $\dim R_i < \dim T_i$. Furthermore, we can split each T_i into finitely many closed uniform sets, so by (DU) it is enough to consider

$$X = \mathrm{pr}'(T \setminus R)$$

where T is closed and uniform, $R \subseteq T$ is closed and $\dim R < \dim T$.

The base case where $\dim \mathrm{pr}(X) = 0$ is similar to the one in the previous lemma. By (SB), $\mathrm{pr}(X)$ is closed and discrete which means that for any $a \in \mathrm{pr}(X)$, $\overline{X}(a) = \overline{X(a)}$. By (SB) this implies that for all $a \in \mathrm{pr}(X)$, $\dim(X(a)) = \dim \overline{X}(a)$. Furthermore, by the previous lemmas, $\overline{X}$ is pure-dimensional so (DD) implies that all fibres over $\mathrm{pr}(X)$ have dimension $\dim(X)$ as required.

For the induction, we proceed as in the previous lemma. If we could find a closed subset $L \subseteq \mathrm{pr}(X)$ such that $\dim L < \dim \mathrm{pr}(X)$ and $\dim(X(a)) = \dim X - \dim \mathrm{pr}(X)$ for $a \in \mathrm{pr}(X) \setminus L$, then we would be done. We will now construct such a set L.

Let $Y = \mathrm{pr}'(T)$. Apply (DD) to T with both pr' and $\mathrm{pr}' \circ \mathrm{pr}$ to get closed sets $K_Y \subseteq_{\mathrm{cl}} Y$ and $K_{\mathrm{pr}(Y)} \subseteq_{\mathrm{cl}} \mathrm{pr}(Y)$ such that

$$\begin{aligned} &\text{for all } (a,b) \in Y \setminus K_Y \quad \dim T(a,b) = \dim T - \dim Y \text{ and} \\ &\text{for all } a \in \mathrm{pr}(Y) \setminus K_{\mathrm{pr}(Y)} \quad \dim T(a) = \dim T - \dim \mathrm{pr}(Y) \end{aligned}$$

T is uniform so $\dim \mathrm{pr}'^{-1}(K_Y) < \dim T$ and $\dim \mathrm{pr}'^{-1}(\mathrm{pr}^{-1}(K_{\mathrm{pr}(Y)})) < \dim T$. By the previous lemma, dimension is definable over K_Y and by hypothesis it is definable over $K_{\mathrm{pr}(Y)}$ and $\mathrm{pr}(K_Y)$. For $j \neq \dim(T) - \dim(Y)$ that allows us to obtain constructible sets $V_{ij} \subseteq \overline{\mathrm{pr}(K_Y)}$ such that for $a \in V_{ij} \cap \mathrm{pr}(Y)$,

$$\dim \{\, b \in K_Y(a) \mid \dim T(a,b) = j \,\} = i$$

Since all the V_{ij} are subsets of $\overline{\mathrm{pr}(K_Y)}$ and the dimension is additive, the previous lemma and the pure-dimensionality of T and Y imply that $\dim V_{ij} \leq \dim(T) - i - j - 1$. Let $I = \{\, (i,j) \mid \dim V_{ij} < \dim \mathrm{pr}(Y) \,\}$ and $K' = K_{\mathrm{pr}(Y)} \cup \bigcup_{(i,j) \in I} V_{ij}$. By (DU), $\dim K' < \dim \mathrm{pr}(Y)$ which means that by (SB) the same is true for $K = \overline{K'}$. We claim that for $a \in \mathrm{pr}(Y) \setminus K$, $\dim \mathrm{pr}'(T)(a) = \dim Y - \dim \mathrm{pr}(Y)$.

Suppose for a contradiction that for some such a this is not the case.

Case 1: $\dim Y(a) < \dim Y - \dim \mathrm{pr}(Y)$

By the previous lemma, dimension is definable and additive on $Y(a)$. We also know that $\dim T(a) = \dim T - \dim \mathrm{pr}(Y)$ so for some i and j with $i + j \leq \dim T - \dim \mathrm{pr}(Y)$ and $i \leq \dim Y(a) < \dim Y - \dim \mathrm{pr}(Y)$, $\dim \{ b \in Y(a) \mid \dim(T(b)) = j \} = i$. These inequalities imply $j > \dim T - \dim Y$ so $a \in V_{ij}$. However, by our choice of a, this means that $\dim V_{ij} = \dim \mathrm{pr}(Y) = \dim T - i - j$ which contradicts our previous observation about $\dim V_{ij}$.

Case 2: $\dim Y(a) > \dim Y - \dim \mathrm{pr}(Y)$
By a similar argument, there are some i and j with $i + j \leq \dim T - \dim \mathrm{pr}(Y)$ and $i = \dim Y(a)$ such that $\dim \{ b \in Y(a) \mid \dim(T(b)) = j \} = i$. Since $i > \dim X - \dim \mathrm{pr}(X)$, $j < \dim T - \dim X$. Now by our choice of a, $\dim V_{ij} = \dim \mathrm{pr}(Y)$ which implies that $\dim \{ b \in Y \mid \dim(T(b)) = j \} > \dim Y$. This is clearly a contradiction.

This proves the claim. Note that by uniformity of T and the fact that $\dim R < \dim T$, $\dim X = \dim Y$ and $\dim \mathrm{pr}(X) = \dim \mathrm{pr}(Y)$. We now apply the previous lemma to R to get the set

$$K_R = \{ (a, b) \in X \mid \dim R(a, b) \geq \dim(T) - \dim(X) \}$$

which has dimension less than $\dim X$. By our hypothesis, the set

$$L' = \{ a \in \mathrm{pr}(X) \mid \dim K_R(a) \geq \dim(X) - \dim \mathrm{pr}(X) \}$$

is projective and must be of dimension less than $\dim \mathrm{pr}(X)$. We define $L = \overline{K \cup L'}$, which also has dimension less than $\dim \mathrm{pr}(X)$ by (SB). For any a, $Y(a) = X(a) \cup K_R(a)$. Since for $a \in \mathrm{pr}(X) \setminus L$, $\dim Y(a) = \dim X - \dim \mathrm{pr}(X)$ and $\dim K_R(a) < \dim X - \dim \mathrm{pr}(X)$, (DU) implies that $\dim X(a) = \dim X - \dim \mathrm{pr}(X)$ as required. $\square$

Definition 2.1.24 We define the set of *regular fibres* of X over $\mathrm{pr}(X)$ to be

$$\mathrm{reg}_{\mathrm{pr}}(X) = \{ a \in \mathrm{pr}(X) \mid \dim X(a) \leq \dim X - \dim \mathrm{pr}(X) \}$$

Other fibres are called *irregular.*

Corollary 2.1.25 *Let C be pure-dimensional and aligned with respect to a projection* pr *and let* $r = \dim C - \dim \mathrm{pr} C$. *Then* V_r *is dense in* $\mathrm{pr}(C)$.

Proof. Suppose otherwise. Then for some $s \neq r$ we have $V_s \setminus \overline{V_r} \neq \emptyset$. Choose s such that

$$\dim V_s = \max\{\dim V_l \mid l \neq r \;\&\; V_s \setminus \overline{V_r} \neq \emptyset\}$$

Let $V = \overline{\bigcup_{l \neq s} V_l}$ so that $\mathrm{pr} C = V \cup V_s$ and by (SB) and choice of s, $V_s \setminus V \neq \emptyset$.

Now (SB) implies that the union $C = (\mathrm{pr}^{-1}(V) \cap C) \cup \overline{\mathrm{pr}^{-1}(V_s) \cap C}$ in non-redundant. C is pure-dimensional so $\dim(\mathrm{pr}^{-1}(V_s) \cap C) = \dim C$. By the alignment of C this means that $\dim V_s = \dim \mathrm{pr} C$, but this implies that C has a component of dimension $\dim \mathrm{pr}(C) + s$ which contradicts the pure-dimensionality of C. $\square$

In particular this proof shows that V_r is the only one of the V_i for with $\dim V_i = \dim \mathrm{pr} C$. Furthermore, the alignment of C implies that if $l > r$, then $\dim V_l \leq \dim \mathrm{pr}(C) - 1 - (l - r)$.

Lemma 2.1.26 *For any pure-dimensional set* $T(x, u, v)$ *which is aligned with respect to* $\mathrm{pr}_u, \mathrm{pr}_v$ *and* pr_{uv}, $\dim \mathrm{pr}_u(T) - \dim \mathrm{pr}_{uv}(T) \geq T - \dim \mathrm{pr}_v(T)$.

Proof. By corollary 2.1.25 we know that

$$V_1 = \{(a, b) \in \mathrm{pr}_v(T) \mid \dim T(a, b) = \dim T - \dim \mathrm{pr}_v(T)\}$$

is dense in $\mathrm{pr}_v(T)$ and that

$$V_2 = \{a \in \mathrm{pr}_{uv}(T) \mid \dim \mathrm{pr}_u(T)(a) = \dim \mathrm{pr}_v(T) - \dim \mathrm{pr}_{uv}(T)\}$$

is dense in $\mathrm{pr}_{uv}(T)$.

In particular this means that there exists an $a \in \mathrm{pr}_u(V_1) \cap V_2$. Let $b \in \mathcal{V}_1(a)$. Now

$$T(a,b) \subseteq \{ c \mid \exists b' \ (a,b',c) \in T \} = \mathrm{pr}_u(T)(a)$$

By (DU) this implies that $\dim T - \dim \mathrm{pr}_v(T) = \dim T(a,b) \leq \mathrm{pr}_u(T)(a) = \dim \mathrm{pr}_u T - \dim \mathrm{pr}_{uv}(T)$ as required. □

Lemma 2.1.27 *If S is pure-dimensional and $V_1, V_2 \subseteq S$ are two dense subsets, then $V_1 \cap V_2$ is dense as well.*

Proof. By (SB) $\dim V_1 = \dim V_2 = \dim S$. As $V_1 \subseteq \overline{V_2}$ and vice versa, another application of (SB) tells us that $\dim(V_1 \cap V_2) = \dim S$. Now let

$$S' = S \setminus \overline{(V_1 \cap V_2)}$$

Since S is pure-dimensional, either $S' = \emptyset$ or S' is also pure-dimensional of dimension $\dim S$. Suppose for a contradiction that $S' \neq \emptyset$. Then by density of V_1 and V_2 in S and (SB) the sets $V_1' = (V_1 \cap S')$ and $V_2' = (V_2 \cap S')$ must be dense in S'. By the above argument, these must then have non-empty intersection, but this contradicts their definition. □

With these tools at our disposal we turn our attention to smoothness. From the examples in Figure 1 we can clearly see that subsets of smooth sets are in general not smooth again, even if they are uniform and of the same dimension. Take for example $[0,2] \times \{0\} \cup [1,2] \times \{1\} \subseteq [0,2] \times \{0,1\} \subseteq \mathbb{R}^2$. Even though the sets are indistinguishable in terms of dimension, the former is not smooth at $(1,1)$ since locally at (1,1), the projection at is contained in $[1,2]$. The characterization of smoothness in terms of local and global projections does, however, suggest that we can restrict a smooth set $S(x,y)$ to $S(x,y) \wedge T(x) \times \mathrm{M}^{\#(y)}$ without affecting smoothness. The following lemma makes this precise.

Lemma 2.1.28 *Let $F \subseteq_{\mathrm{cl}} \mathrm{M}^{n+k}$ be a uniform set and* pr *be the projection $\mathrm{M}^{n+k} \to \mathrm{M}^n$.*

Suppose that F is smooth with respect to pr *and that $T \subseteq \overline{\mathrm{pr}F}$ is pure dimensional of the same dimension as* $\mathrm{pr}F$*. Then $V = F \cap \mathrm{pr}^{-1}(T)$ is also smooth with respect to* pr.

Proof. Let $S \subseteq \mathrm{M}^{n+m}$ be a uniform set such that for $\mathrm{pr}' \colon \mathrm{M}^{n+m} \to \mathrm{M}^n$ $\overline{\mathrm{pr}'(S)} = \overline{T}$.

Let $r = \dim S - \dim T$ and let $Q = \overline{\mathrm{pr}(F)} \times \mathrm{M}^r \times \{c\}$, where c is some tuple of length $m - r$.

Now pick two elements $a, b \in \mathrm{M}$ and let $U = (S \times \{a\}) \cup (Q \times \{b\}) \subseteq_{\mathrm{cl}} \mathrm{M}^{n+k+1}$. This is pure-dimensional of dimension $\dim S$ and aligned with respect to the projection to M^n. Now by (GS) the set $R = \{ (x, y, z) \in \mathrm{M}^{n+k+m+1} \mid (x, y) \in (F \setminus \mathrm{pr}^{-1}(K)) \ \& \ (x, z) \in U \}$ has no components of dimension $< \dim F + r$, but $R \subseteq \mathrm{M}^{n+k+m} \times \{a, b\}$ so in particular, $R(a) = \{ (x, y, z) \in \mathrm{M}^{n+k+k} \mid (x, y) \in (F \setminus \mathrm{pr}^{-1}(K)) \ \& \ (x, z) \in S \}$ has no components of dimension $< \dim F + r = \dim S + \dim F - \dim T$, as required. $\square$

A key consequence of this is that the set T in the definition of smoothness no longer has to be co-dense with S in the projection.

Corollary 2.1.29 *Suppose we have a uniform set $S \subseteq \mathrm{M}^{n+k}$, smooth with respect to a projection $\mathrm{pr} \colon \mathrm{M}^{n+k} \to \mathrm{M}^n$, and a pure-dimensional set $T \subseteq \mathrm{M}^{n+m}$ aligned with respect to $\mathrm{pr}' \colon \mathrm{M}^{n+m} \to \mathrm{M}^n$. If $\mathrm{pr}'(T) \subseteq \overline{\mathrm{pr}(S)}$ is pure-dimensional of the same dimension as $\mathrm{pr}(S)$, then $\{ (x, y, z) \mid (x, y) \in S \ \& \ (x, z) \in T \}$ has no components of dimension $<$ $\dim S + \dim T - \dim \mathrm{pr}S$.*

In particular, by taking $m = 0$, this corollary implies that if S is smooth with respect to pr and $T \subseteq \mathrm{pr}S$ is pure-dimensional of the same dimension as $\mathrm{pr}S$, then $S \cap \mathrm{pr}^{-1}(T)$ is still pure-dimensional. Below Figure 1, we also noted that unions of smooth sets are not generally smooth. However, in view of the characterization in terms of local and global projections, it should not come as a surprise that if two sets overlap in the right fashion, then smoothness is preserved.

Lemma 2.1.30 *Let $S, T \subseteq_{\mathrm{cl}} X \subseteq \mathrm{M}^{n+k}$ have the same alignment and be smooth with respect to $\mathrm{pr} \colon \mathrm{M}^{n+k} \to \mathrm{M}^n$. Let $K = \mathrm{pr}(S \cup T)$ and let $\mathrm{bd}(\mathrm{pr}S)$ and $\mathrm{bd}(\mathrm{pr}T)$ be the* *relative boundaries in K. Then $S \cup T$ is smooth with respect to* pr *if and only if for all* $x \in \mathrm{bd}(\mathrm{pr}S) \cap \overline{\mathrm{pr}T \setminus \overline{\mathrm{pr}S}}$, $S(x) \subseteq T(x)$ *and vice versa.*

Proof. First, suppose $x \in \mathrm{bd}(\mathrm{pr}S) \cap \overline{\mathrm{pr}T \setminus \overline{\mathrm{pr}S}}$, but $(x, y) \in S \setminus T$. Let $W = \overline{\mathrm{pr}T \setminus \overline{\mathrm{pr}S}}$. Let C be the component of $(S \cup T) \cap \mathrm{pr}^{-1} W$ that contains (x, y). As $(x, y) \notin T$ and C is pure-dimensional, the set $C \setminus T$ has the same dimension as C. However, $\mathrm{pr}(C \setminus T)$ is contained entirely in $W \cap \overline{\mathrm{pr}S}$ which has dimension $< \dim S$ by (SB). Since both S and T are aligned with respect to pr, this implies that $\dim C < \dim(S \cup T)$ and therefore $S \cup T$ is not smooth.

For the other direction let $W(x, z) \subseteq \mathrm{M}^{n+m}$ be pure-dimensional and aligned with respect to pr_z such that $\overline{\mathrm{pr}_z(W)} = \overline{\mathrm{pr}_y(S \cup T)}$. Let $K = \mathrm{pr}(S \cup T)$,

$$W_S = \{ (x, z) \mid \exists y \ (x, y) \in S \ \& \ (x, z) \in W \}$$

and

$$W_T = \{ (x, z) \mid \exists y \ (x, y) \in T \ \& \ (x, z) \in W \}$$

Let W_S^0 be the component of W_S of dimension $\dim W$ and let $W_S^1, \ldots, W_S^p$ be the other components. Similarly, let W_T^0 be the component of W_T of dimension $\dim W$ and let $W_T^1, \ldots, W_T^q$ be the other components. Note that $W_S = \mathrm{pr}_z^{-1}(\mathrm{int}(\mathrm{pr}_y(S)))$ so for $i \neq 0$ we may assume $\mathrm{pr}_z(W_S^i) \subseteq \mathrm{bd}(S)$ and similarly for T. Let

$$V_S = \{ (x, y, z) \mid (x, y) \in S \ \& \ (x, z) \in W \}$$

$$V_T = \{ (x, y, z) \mid (x, y) \in T \ \& \ (x, z) \in W \}$$

$$V = \{ (x, y, z) \mid (x, y) \in (S \cup T) \ \& \ (x, z) \in W \}$$

Note that $V = V_S \cup V_T$ and that $\mathrm{pr}_y(V)$ is dense in W. Suppose that V has a component C with $\dim C < \dim(S \cup T) + \dim W - \dim K$. Without loss of generality we may assume that $C \subseteq V_S$. Now if $\mathrm{pr}_y(C) \subseteq \overline{W_S^0}$, then we would get a contradiction to the smoothness of S. However, by the pure-dimensionality of W, $W = \overline{W_S^0} \cup \overline{W_T^0}$ so $\mathrm{pr}_y(C) \subseteq \overline{W_T^0}$. Thus, $\mathrm{pr}_y(C) \subseteq \overline{W_T \setminus \overline{W_S}}$ and $\mathrm{pr}_y(C) \subseteq \overline{T \setminus \overline{S}}$. Now by assumption, for all $x \in \mathrm{pr}_{yz}(C)$, $S(x) = T(x)$ so in particular $C \subseteq V_T$. However, $\mathrm{pr}_y(C) \subseteq \overline{W_T^0}$ so by the smoothness of T, C has dimension at least $\dim(S \cup T) + \dim(W) - \dim K$. This contradicts our choice of C. □

Next we note that irregular fibres, like the one in the fourth example of Figure 1, are usually an indication of non-smooth behaviour. And indeed this is the case:

Lemma 2.1.31 *If $S(x, y)$ is a uniform set and has an irregular fibre at a with respect to pr_y, then S is not smooth with respect to pr_y.*

Proof. Let $T = \{ (x, y) \in S \mid x \in \mathrm{reg}_{\mathrm{pr}_y}(S) \}$, $r = \dim S - \dim \mathrm{pr}_y(S)$ and $s = \dim \mathrm{pr}_y(S)$. Define $S^m = \{ (x, y_1, \ldots, y_m) \mid (x, y_i) \in S \}$ and T^m in a similar fashion. Note that $\dim T^s = s(1+r)$. However, the fibre $S^s(a) = S(a)^s$ has at least the same dimension and therefore cannot be contained in $\overline{T^s}$. Therefore, S^s is not aligned with respect to $\mathrm{pr}_{y_1 \ldots y_s}$. Let $k = \min\{ m \mid S^m$ is not aligned with respect to $\mathrm{pr}_{y_1 \ldots y_m} \}$. Let C be a component of S^k such that $\dim \mathrm{pr}_{y_1 \ldots y_k}(C) < \dim \mathrm{pr}_y(S)$ and pick $(a, b_1, \ldots, b_k) \in C \setminus (\overline{S^k \setminus C})$. Define

$$W = \{ (x, y_1, \ldots, y_{k-1}) \in S^{k-1} \mid x \notin \mathrm{pr}_{y_1 \ldots y_k}(C) \} \cup \{ (a, b_1, \ldots, b_{k-1}) \}$$

Since S^{k-1} is aligned with respect to $\mathrm{pr}_{y_1 \ldots y_{k-1}}$ and $\mathrm{pr}_{y_1 \ldots y_k}(C)$ has small dimension, W is pure-dimensional and aligned with respect to $\mathrm{pr}_{y_1 \ldots y_{k-1}}$. However,

$$\{ (x, y_1, \ldots, y_{k-1}, y_k) \mid (x, y_k) \in S \wedge (x, y_1, \ldots, y_{k-1}) \in W \}$$

has a component contained in $\{(a, b_1, \ldots, b_k)\} \times S(a)$. Since $\dim S(a) < \dim S$, this shows that S is not smooth. □

Smoothness only gives a lower bound on the dimension of components in the intersection. We will sometimes want to know the exact dimensions and alignments of components of the resulting set. The following few lemmas give us a good handle on this.

Lemma 2.1.32 *Let $S(x,y) \subseteq \mathrm{M}^{n+k}$ be a uniform set, which is smooth with respect to* pr_y*. Let $T(x,z) \subseteq \mathrm{M}^{n+m}$ be another uniform set such that $\overline{\mathrm{pr}_y(S)} = \overline{\mathrm{pr}_z(T)}$. Then for every component C of*

$$\{ (x,y,z) \mid (x,y) \in S \;\&\; (x,z) \in T \},$$

$$\dim \{ (x,y) \mid \exists z \; C(x,y,z) \} = \dim S \quad \textit{and} \quad \dim \{ (x,z) \mid \exists y \; C(x,y,z) \} = \dim T$$

Proof. If $\dim \mathrm{pr}_y(S) = 0$, the proof is trivial, so suppose otherwise.

The proof is the same for the two above statements, so we will only prove the first one. Suppose for a contradiction that $\dim \mathrm{pr}_z C < \dim S$. We use 2.1.23 to find a set $V \subseteq \mathrm{pr}_y C$ such that all fibres over V have dimension

$$r = \dim C - \dim \mathrm{pr}_z C > \dim T - \dim \mathrm{pr}_z T$$

If $K \subseteq \mathrm{pr}_z(T)$ is the closed set given to us by (DD), then $\mathrm{pr}_z(V) \subseteq K$. The set $({\mathrm{pr}_y}^{-1}(V)) \cap C$ is dense in C so in particular $C \subseteq {\mathrm{pr}_z}^{-1}(\mathrm{pr}^{-1}(K))$. Now let $(a,b,c) \in C$ be a point not contained in any other component of

$$\{ (x,y,z) \mid (x,y) \in S \;\&\; (x,z) \in T \}$$

Let

$$T' = (T \setminus {\mathrm{pr}_z}^{-1}(K)) \cup \{(a,c)\}$$

We note that T' is dense in T and hence uniform by 2.1.21. Furthermore, by 2.1.20,

$$\overline{\mathrm{pr}_z(T')} = \overline{\mathrm{pr}_z(T)} = \overline{\mathrm{pr}(S)}$$

However, the component C' of $\{ (x,y,z) \mid (x,y) \in S \;\&\; (x,z) \in T' \}$ containing (a,b,c) is contained in $S(a) \times \{c\}$. Since $\dim \mathrm{pr} S \neq 0$ and S is uniform, $\dim S(a) \leq \dim S$, so this component C' contradicts the smoothness of S. $\square$

Lemma 2.1.33 *Let $S(x, y)$ be uniform and smooth with respect to* pr_y *and let $T(x, z)$ be a uniform set such that* $\overline{\mathrm{pr}_y(S)} = \overline{\mathrm{pr}_z(T)}$. *Then every component C of*

$$\{ (x, y, z) \mid (x, y) \in S \ \& \ (x, z) \in T \}$$

has dimension exactly $\dim S + \dim T - \dim \mathrm{pr}_y(S)$

Proof. Suppose not. We know that $\dim \mathrm{pr}_z(C) = \dim S$, so

$$r = \dim C - \dim \mathrm{pr}_z(C) > \dim T - \dim \mathrm{pr}_y(S) = \dim T - \dim \mathrm{pr}_z(T)$$

Let $V_T \subseteq \mathrm{pr}_z(T)$ be the set containing all fibres of dimension r. Then $\mathrm{pr}_y^{-1}(V_T)$ is dense in $\mathrm{pr}_z(C)$, so in particular

$$\dim \mathrm{pr}_{yz}(C) < \dim \mathrm{pr}_z(T) = \dim \mathrm{pr}_y(S)$$

and this contradicts the alignment of S. $\square$

Lemma 2.1.34 *If $S(x, y_1, y_2)$ is uniform and smooth with respect to* $\mathrm{pr}_{y_1 y_2}$, *then* $\mathrm{pr}_{y_2}(S)$ *is smooth with respect to* pr_{y_1}.

Proof. Let $T(x, z)$ be uniform with $\overline{\mathrm{pr}_z(T)} = \overline{\mathrm{pr}_{y_1 y_2}(S)}$. For each component C of

$$\{ (x, y_1, y_2, z) \mid (x, y_1, y_2) \in S \ \& \ (x, z) \in T \}$$

$\dim \mathrm{pr}_z(C) = \dim S$ and $\dim \mathrm{pr}_{y_1 y_2}(C) = \dim T$ by the previous lemma. We have to show that

$$\dim \mathrm{pr}_{y_2}(C) \geq \dim T + \dim \mathrm{pr}_{y_2}(S) - \dim \mathrm{pr}_{y_1 y_2}(S)$$

For a contradiction suppose not. Then since by smoothness of S, $\dim C \geq \dim S + \dim T - \dim \mathrm{pr}_{y_1 y_2}(S)$ this would mean

$$r = \dim C - \dim \mathrm{pr}_{y_2}(C) > \dim S - \mathrm{pr}_{y_2}(S) = l$$

Let $V_C \subseteq \mathrm{pr}_{y_2}(C)$ be the set containing fibres of dimension r, $V_S \subseteq \mathrm{pr}_{y_2}(S)$ the set containing fibres of dimension r and $V_l \subseteq \mathrm{pr}_{y_2}(S)$ be the set containing fibres of dimension r_S. Clearly, $\mathrm{pr}_z^{-1}(V_r^S) \supseteq V_r^C$, so in particular

$$\dim \mathrm{pr}_{y_2 z}(C) < \dim S$$

However, this contradicts 2.1.32 and the uniformity of S. □

Lemma 2.1.35 *Let* $S(x, y_1, y_2)$ *and* $T(x, z_1, z_2)$ *be uniform and suppose* S *is smooth with respect to* $\mathrm{pr}_{y_1 y_2}$*. Then for every component* C *of*

$$\{ (x, y_1, y_2, z_1, z_2) \mid (x, y_1, y_2) \in S \& (x, z_1, z_2) \in T \}$$

$$\begin{aligned} \dim \mathrm{pr}_{y_2 z_2}(C) &= \dim \{ (x, y_1, z_1) \mid (x, y_1) \in \mathrm{pr}_{y_2}(S) \ \& \ (x, z_1) \in \mathrm{pr}_{z_2}(T) \} \\ &= \dim \mathrm{pr}_{y_2}(S) + \dim \mathrm{pr}_{z_2}(T) - \dim \mathrm{pr}_{y_1, y_2}(S) \end{aligned}$$

Proof. First of all, note that by 2.1.34 the set pr_{y_2} is uniform and smooth with respect to pr_{y_1}, so by 2.1.33

$$\begin{aligned} l &= \dim \{ (x, y_1, z_1) \mid (x, y_1) \in \mathrm{pr}_{y_2}(S) \ \& \ (x, z_1) \in \mathrm{pr}_{z_2}(T) \} \\ &= \dim \mathrm{pr}_{y_2}(S) + \dim \mathrm{pr}_{z_2}(T) - \dim \mathrm{pr}_{y_1, y_2}(S) \end{aligned}$$

Suppose for a contradiction that $\dim \mathrm{pr}_{y_2 z_2}(C) < l$. By 2.1.32, $\dim \mathrm{pr}_{y_2 z_1 z_2}(C) = \dim \mathrm{pr}_{y_2}(S)$ and $\dim \mathrm{pr}_{y_1 y_2 z_2}(C) = \dim \mathrm{pr}_{z_2}(T)$. Let $V_S \subseteq \mathrm{pr}_{y_2}(S)$ be the set of points with fibres of dimension $\dim S - \dim \mathrm{pr}_{y_2}(S)$ and let $V_T \subseteq \mathrm{pr}_{z_2}(T)$ be the set of points with fibres of dimension $\dim T - \dim \mathrm{pr}_{z_2}(T)$. These are dense in $\mathrm{pr}_{y_2}(S)$ and $\mathrm{pr}_{z_2}(T)$, respectively. By 2.1.20 this implies that

$$\{ (x, y_1, z_1) \mid (x, y_1) \in V_S \ \& \ (x, z_1) \in V_T \}$$

is dense in $\mathrm{pr}_{y_2 z_2}(C)$ and this, in turn, means that

$$\begin{aligned}\dim C &= \dim \mathrm{pr}_{y_2 z_2}(C) + \dim S - \dim \mathrm{pr}_{y_2}(S) + \dim T - \dim \mathrm{pr}_{z_2}(T) \\ &< \dim l + \dim S - \dim \mathrm{pr}_{y_2}(S) + \dim T - \dim \mathrm{pr}_{z_2}(T) \\ &= \dim S + \dim T - \dim \mathrm{pr}_{y_1, y_2}(S)\end{aligned}$$

This contradicts smoothness. □

Lemma 2.1.36 *Let* $S(x, y_1, y_2)$ *be uniform and smooth with respect to* pr_{y_1}. *Then* $\mathrm{pr}_{y_2}(S)$ *is also smooth with respect to* pr_{y_1}.

Proof. Let $T(x, z)$ be uniform with $\overline{\mathrm{pr}_z(T)} = \overline{\mathrm{pr}_{y_1 y_2}(S)}$. Let C be a component of

$$\{ (x, y_1, y_2, z) \mid (x, y_1, y_2) \in S \ \& \ (x, z) \in T \}$$

Then by 2.1.35,

$$\dim \mathrm{pr}_{y_2}(C) = \dim \mathrm{pr}_{y_2}(S) + \dim T - \dim \mathrm{pr}_{y_1 y_2}(S)$$

as required. □

Lemma 2.1.37 *Let* $S(x, y, z)$ *be uniform and smooth with respect to* pr_z *and assume that* $\mathrm{pr}_z(S)$ *is smooth with respect to* pr_y. *Then* S *is smooth with respect to* pr_{yz}.

Proof. Let $T(x, w)$ be uniform and such that $\overline{\mathrm{pr}_{yz}(S)} = \overline{\mathrm{pr}_w(T)}$.

$$T_1 = \{ (x, y, w) \mid (x, y) \in \mathrm{pr}_z(S) \ \& \ (x, w) \in T \}$$

has no components of small dimension and for all comoponents C of T_1, $\dim \mathrm{pr}_w C = \dim \mathrm{pr}_z(S)$. The result follows by smoothness of S and lemma 2.1.29. □

2.1.3 Families of Sets and their Dimension

When we talk about a family of sets in Model Theory, we usually mean a definable family given by pair of formulas $\phi(x, y)$ and $\psi(y)$ which represent the family

$$\{ U_l \mid \models \psi(l) \;\&\; U_l = \phi(\mathrm{M}, l) \}$$

In a context with some notion of dimension it makes sense to ask for the dimension of a family of sets. In general, this is a delicate issue since there may be redundancy in the parameter set $\psi(\mathrm{M})$. In many geometric contexts this problem can be solved using *canonical bases*. In the spirit of this project we try to obtain a purely geometric definition of the dimension of a family of sets.

Since we are working in a context with a definable dimension we can make a few assumptions on our families.

Definition 2.1.38 A *family of r-dimensional sets* is a set $I(x, y) \subseteq \mathrm{M}^{n+k}$ such that for any $a \in \mathrm{pr}_x(I)$

$$\dim \{ x \mid \mathrm{M} \models I(x, a) \} = r$$

We call

I the *incidence relation*,

$L = \{ y \mid \exists x\; I(x, y) \}$ the *parameter space* of the family and

$S = \{ x \mid \exists y\; I(x, y) \}$ the *domain* of the family.

We say that I is *smooth* if it is uniform and smooth with respect to the projections pr_x and pr_y and such that all fibres over $\mathrm{pr}_x(I)$ and $\mathrm{pr}_y(I)$ are regular.

By lemma 2.1.13 and (GS) is clear that for any family I of sets of dimension r, there is an open set U such that $I \cap U$ is a smooth family. We will restrict our attention to these. A good notion of dimension should measure the degrees of freedom in choosing a set from the family. We do, however, have to be very careful in distinguishing the local and global behaviour of families. To illustrate this point consider the family

$$\mathcal{F} = \{ (x, y, s, t) \mid y = s.x \vee y = t.x \} \subseteq \mathbb{R}^4$$

where the (x, y)-coordinates represent the domain and the (s, t)-coordinates represent the parameter set. This is clearly a two-dimensional family, since we have two degrees of freedom using parameters s and t. However, if we choose a generic $(a, b, u, v) \in \mathcal{F}$, the situation locally around this point is rather different. Since $u \neq v$, the point (a, b) can only lie on one of the straight lines, say $y = u.x$. A small change in v does not alter the set around (a, b), so locally we only have one degree of freedom. In this sense this family is locally one-dimensional.

Given a smooth family of sets $I(s, l)$, we define

$$I_r^k(s_1, \ldots, s_r, l_1, \ldots, l_k) = \bigwedge_{i=1}^{r} \bigwedge_{j}^{k} I(s_i, l_j)$$

and write I_r for I_r^1 and I^k for I_1^k. Since I is smooth, lemma 2.1.33 implies that the sets I_r and I^k are pure-dimensional and of dimensions $r.\dim I - (r-1).\dim L$ and $k.\dim I - (k-1).\dim S$ respectively. Given $(a_1, \ldots, a_{r+1}) \in \mathrm{pr}_l(I_{r+1})$, it follows from the definitions that

$$I_{r+1}(a_1, \ldots, a_{r+1}) \subseteq I_r(a_1, \ldots, a_r)$$

So the sequence $t_r = \dim(I_r) - \dim(\mathrm{pr}_l(I_r))$ of the dimension of a regular fibre over $\mathrm{pr}_l(I_r)$ is a decreasing sequence. We can do even better than that.

Lemma 2.1.39 *For the sequence t_r above, $t_{r-1} - t_r \leq t_r - t_{r+1}$.*

Proof. First we note that $\mathrm{pr}_{s_n}(I_n) = I_{n-1}$. Therefore, $\dim I_n - \dim \mathrm{pr}_{s_n}(I_n) = \dim I_n -$

$\dim I_{n-1} = \dim I - \dim L$. Thus,

$$\begin{aligned} t_{r-1} - t_r &= \dim \mathrm{pr}_{s_n}(I_n) - \dim \mathrm{pr}_{ls_n}(I_n) - (\dim I_n - \dim \mathrm{pr}_l I_n) \\ &= \dim \mathrm{pr}_l I_n - \dim \mathrm{pr}_{ls_n}(I_n) - (\dim I - \dim L) \\ &= \dim \mathrm{pr}_{ls_{n+1}} I_{n+1} - \dim \mathrm{pr}_{ls_n s_{n+1}}(I_{n+1}) - (\dim I - \dim L) \end{aligned}$$

Since I_{n+1} is symmetric in s_n and s_{n+1}, $\dim \mathrm{pr}_{ls_{n+1}} I_{n+1} = \dim \mathrm{pr}_{ls_n} I_{n+1}$. Furthermore, since I is uniform, lemma 2.1.35 implies that lemma 2.1.26 applies to $\mathrm{pr}_l I_{n+1}$ so that

$$\dim \mathrm{pr}_{ls_n} I_{n+1} - \dim \mathrm{pr}_{ls_n s_{n+1}}(I_{n+1}) \geq \dim \mathrm{pr}_l I_{n+1} - \dim \mathrm{pr}_{ls_{n+1}}(I_{n+1})$$

Therefore,

$$t_{r-1} - t_r \geq \dim \mathrm{pr}_l I_{n+1} - \dim \mathrm{pr}_{ls_{n+1}}(I_{n+1}) - (\dim I - \dim L) = t_r - t_{r+1}$$

□

Since $t_1 \leq \dim L$, the above lemma implies that the decreasing sequence t_i must stabilize in at most $\dim L$ steps. This allows us to calculate the dimension of the family I using the set $I_{\dim L}$. Essentially, the "redundancy" in the parameter set is measured by a regular fibre over $\mathrm{pr}_l(I_{\dim L})$. Since $I_{\dim L}$ has dimension $\dim L . \dim I - (\dim L - 1) . \dim L$, the dimension of such a fibre is

$$\dim L . \dim I - (\dim L - 1) \dim L - \dim(\mathrm{pr}_l(I_{\dim L}))$$

The dimension of the family should be $\dim L$ minus the redundancy, so we define the following.

Definition 2.1.40 Given a family of sets I the *dimension* of the family is

$$\begin{aligned}&\dim L - (\dim L . \dim I - (\dim L - 1) \dim L - \dim(\mathrm{pr}_l(I_{\dim L})))\\&= \dim(\mathrm{pr}_l(I_{\dim L})) - \dim L(\dim I - \dim L)\end{aligned}$$

and given a point $(a, b) \in I$, we say that I has got *local dimension* $\geq k$ at (a, b) if there is a component C of $I_{\dim L}$ such that

$$\dim(\mathrm{pr}_l(C)) - \dim L(\dim I - \dim L) \geq k$$

and $(a, \ldots, a, b) \in C$.

We say that I has a *k-dimensional component* if there is a component C of $I_{\dim L}$ such that

$$\dim(\mathrm{pr}_l(C)) - \dim L(\dim I - \dim L) = k$$

We say that I has a *locally k-dimensional component* if there is a component C of $I_{\dim L}$ which intersects the diagonal $s_1 = \cdots = s_{\dim L}$ such that

$$\dim(\mathrm{pr}_l(C)) - \dim L(\dim I - \dim L) = k$$

When $k = \dim S + \dim L - \dim I$, we call C a *linear component.*

By definition, local dimension is smaller than global dimension. Our example suggests that the converse is not in general true, but at least we could hope that if a family has a component of low dimension, then it has low local dimension somewhere. It would be nice to get a general result along these lines, but here we will only prove this for linear components. We start by giving a number of characterizations of linearity.

Lemma 2.1.41 *If I is a smooth family, then the following are equivalent:*

1. *I has a linear component.*

2. $\dim(I_2^2) = \dim I + (\dim I - \dim L) + (\dim I - \dim S)$.

3. *I_2 has a component C such that* $\dim \mathrm{pr}_l(C) = \dim S + (\dim I - \dim L)$.

4. I^2 *has a component* C *such that* $\dim \mathrm{pr}_s(C) = \dim L + (\dim I - \dim S)$.

Proof. (1) $\Rightarrow$ (3) If I has a linear component, then $I_{\dim L}$ has a component C such that $\dim \mathrm{pr}_l(C) = \dim S + (\dim L - 1)(\dim I - \dim L)$. Furthermore, $\dim \mathrm{pr}_l(C) - \dim \mathrm{pr}_{l s_3 \ldots s_{\dim L}}(C) \geq \dim C - \dim \mathrm{pr}_{s_3 \ldots s_{\dim L}}(C) = (\dim L - 2)(\dim I - \dim L)$. Let $D = \mathrm{pr}_{s_3 \ldots s_{\dim L}}(C)$. This is part of a uniform component of I_2 and by the above inequalities $\dim \mathrm{pr}_l(D) \leq \dim S + (\dim I - \dim L)$. However, since $\dim \mathrm{pr}_l(D) \geq \dim S - \dim I + \dim D = \dim S + (\dim I - \dim L)$, we must have equality and thus D is as required.

(3) $\Rightarrow$ (1) If D is a component of I_2 such that $\dim \mathrm{pr}_l(D) = \dim S + (\dim I - \dim L)$, then after removing any irregular fibres from D, consider the set $D_{\dim L}$ defined by

$$\bigwedge_{i=2}^{\dim L} D(s_1, s_i, l)$$

This has the same dimension as $I_{\dim L}$ and is a subset thereof so there is a component C of $I_{\dim L}$ with the same alignment as a component of $D_{\dim L}$. A quick dimensional calculation shows

$$\dim \mathrm{pr}_l(D_{\dim L}) = \dim S + (\dim L - 1)(\dim I - \dim L)$$

so the same must be true of C.

(2) $\Rightarrow$ (3) Suppose $\dim(I_2^2) = \dim I + (\dim I - \dim L) + (\dim I - \dim S)$ and let C be a component of this dimension. Then $\dim \mathrm{pr}_{l_2}(C) \leq \dim I_2 = \dim I - \dim(I - \dim L)$. However, $\dim C - \dim \mathrm{pr}_{l_2}(C) \leq \dim I - \dim S$, so we must have equality and there must be a component D of I_2 with the same alignment as $\mathrm{pr}_{l_2}(C)$. Now without loss of generality we may assume that $\dim C - \dim \mathrm{pr}_{l_2}(C) \leq \dim C - \dim \mathrm{pr}_{l_1}(C)$ and deduce

that

$$\begin{aligned}
\dim \mathrm{pr}_{l_1}(D) &= \dim \mathrm{pr}_{l_1 l_2}(C) \\
&= \dim \mathrm{pr}_{l_1 l_2}(C) - \dim \mathrm{pr}_{l_2}(C) + \dim \mathrm{pr}_{l_2}(C) \\
&\leq \dim \mathrm{pr}_{l_1}(C) - \dim(C) + \dim \mathrm{pr}_{l_2}(C) \\
&\leq \dim(I_2) - (\dim I - \dim S) \\
&= \dim S + (\dim I - \dim L)
\end{aligned}$$

as required.

(3) $\Rightarrow$ (2) Let C be a component of I_2 such that $\dim \mathrm{pr}_l(C) = \dim S + (\dim I - \dim L)$. Then $\dim C - \dim \mathrm{pr}_l(C) = \dim I - \dim S$ so the set defined by $C(s_1, s_2, l_1) \wedge C(s_1, s_2, l_2)$ is a subset of I_2^2 and has dimension at least

$\dim S + (\dim I - \dim L) + 2(\dim I - \dim S) = \dim I + (\dim I - \dim L) + (\dim I - \dim S)$

(2) $\Leftrightarrow$ (4) follows by symmetry. □

Lemma 2.1.42 *If I has a linear component, then I has a locally linear component.*

Proof. By the previous lemma, we know that there is a component C of I_2 such that $\dim \mathrm{pr}_l(C) = \dim S + \dim I - \dim L$. We may assume that C is smooth and has no irregular fibres with respect to any projection. Therefore the set $D = C(s_1, s_2, l) \wedge C(s_1, s_3, l)$ is of dimension $\dim I + 2(\dim I - \dim L)$ and $E = \mathrm{pr}_{s_1}(D)$ is of dimension $\dim I + (\dim I - \dim L)$. Thus D is a component of I_3 and E is a component of I_2.

Furthermore, $\dim \mathrm{pr}_l(D) = \dim S + 2(\dim I - \dim L)$ so $\dim \mathrm{pr}_l(E) = \dim \mathrm{pr}_{l s_1}(D) \leq \mathrm{pr}(E) - \mathrm{pr}(D) + \mathrm{pr}_l(D) = \dim S + (\dim I - \dim L)$. However, this is the minimum for any component of I_2 so we must have equality. Clearly D intersects the diagonal $s_2 = s_3$ so E is part of a linear component of I_2 intersecting the diagonal. □

Corollary 2.1.43 *If a family I has no locally linear component at a point (a, b), then there is an open subset $U \subseteq I$ such that $J = U \cap I$ has no linear component.*

Proof. Let C be the union of all those components of I_2 such that $\dim \mathrm{pr}_l(C) = \dim S + \dim I - \dim L$. If I has no locally linear component at a point (a, b), then $(a, a, b) \notin C$. Let U be defined by $\neg C(s, s, l)$. Now the components of J_2 are exactly the components of I_2 intersected with $U(s_1, l) \wedge U(s_2, l)$. By the construction of U, no linear component of J_2 can intersect the diagonal so by the previous lemma J can not have a linear component. $\square$

Definition 2.1.44 We call a family I *non-linear* if it has no linear component. We call M *non-linear* if there is a non-linear family I.

Definition 2.1.45 We will call a family of sets I *generically normal* if

$$\dim \{ (l_1, l_2) \in L^2 \mid \dim \{ s \mid I^2(s, l_1, l_2) \} = \dim I - \dim L \} = \dim L$$

Definition 2.1.46 A *family of curves* is a family of one-dimensional sets and a *family of plane curves* is a family of curves in a two-dimensional domain $C \times D$ where C and D are uniform one-dimensional sets.

A smooth family of plane curves $I(x, y, l) \subseteq \mathrm{M}^{2+n}$ is *trivial* if one of the following is true:

1. I is zero-dimensional,
2. $\dim \mathrm{pr}_x(I) < \dim I$ or
3. $\dim \mathrm{pr}_y(I) < \dim I$.

The last two conditions of this definition are symmetric so there are really two ways in which a family of plane curves can be trivial.

Example 1: The first way a family can be trivial is if it is zero-dimensional. This should be the case for constant families so let's consider the family $I = \{ (x, y, l) \in \mathbb{C}^3 \mid x = y \}$. Here $\dim I = 2$ and $L = \{ l \in \mathbb{C} \}$ so that $\dim L = 1$. $S = \{ (x, y) \in \mathbb{C}^2 \mid x = y \}$ so $\dim S = 1$. So the dimension of our family is

$$\dim(\mathrm{pr}_l(I_{\dim L})) - \dim L(\dim I - \dim L) = \dim S - \dim L(\dim I - \dim 0) = 1 - 1.(2-1) = 0$$

Therefore, this family is trivial.

Example 2: The second way a family can be trivial is if the curves from the family are unions of sets of the form $K \times \{ a \}$ (or $\{ a \} \times K$) where $\dim K = 1$. An example of such a family would be $I = \{ (x, y, l) \in \mathbb{C}^3 \mid y = a \}$. Here $\dim I = 2$, but $\dim \mathrm{pr}_x(I) = 1$ so the family is indeed trivial.

Ideally, we would like to go farther and show that we can decompose a family of sets I into "pure" families $I_1, \ldots, I_n$. By "pure" we mean that $I_j(s_1, l) \wedge I_j(s_2, l)$ is uniform. I do not know whether this is possible in general, but we can show that this is always the case for two-dimensional generically normal non-linear families of plane curves.

Lemma 2.1.47 *If I is a two-dimensional generically normal familiy of plane curves in a one-dimensional strong geometry* M*, then I is non-linear,* $\dim L = 2$ *and I_2 is uniform.*

Proof. Note that since I is two-dimensional we must have $\dim \mathrm{pr}_l(I_2) = 4$ which forces $\dim I_2 \geq 4$ and $\dim L \geq 2$. Now if I were not non-linear, then there would be a component C of I^2 with $\dim \mathrm{pr}_s(C) = 2\dim L - 1$, which means

$$\dim \{ (l_1, l_2) \in L^2 \mid \dim \{ s \mid I^2(s, l_1, l_2) \} = 1 \} = 2\dim L - 1 \geq \dim L + 1$$

contradicting generic normality.

From non-linearity we get that $\dim I_2^2 = 2\dim L$ and that I_3^2 has a component of dimension $2\dim L + 1$. So I_2^2 must have a component C with $\dim \mathrm{pr}_{s_1 s_2}(C)$ of dimension $2\dim L - 2$ and $\dim \mathrm{pr}_{s_2}(C) = 2\dim L - 1$. If $\dim L > 2$, then this would contradict generic normality. Hence, $\dim L = 2$.

Let C be a component of I_2. By 2.1.35 and the fact that $\dim \mathrm{pr}_l(C) = 4$ we can explicitly compute the dimensions of projectons of C and conclude that I_2 is uniform. $\square$

We will see that in a one-dimensional Strong Geometry, we can always reduce a non-linear family of sets to a two-dimensional generically normal non-linear familiy of plane curves with $S, L \subseteq \mathrm{M}^2$, but the proof of this fact is very cumbersome to write in this geometric language. We will prove it after the next sectoion, once we have a pre-geometry and an independence relation at our disposal.

2.2 Elementary Extensions

2.2.1 Preservation of the Axioms

Let M be a Strong Geometry and let *M be an elementary extension of M.

Using the projective definition of dimension given in 2.1.23, we can naturally define a dimension dim on *M. This is well-defined, for if $P(x, y)$ and $Q(x, z)$ are projective (over M), then

$$\mathrm{M} \models \forall a \forall b (\forall x (P(x, a) \leftrightarrow Q(x, b)) \rightarrow \dim P(\mathrm{M}, a) = \dim Q(\mathrm{M}, b))$$

The last part of this formula is first order by 2.1.23. Since *M is an elementary extension, the same formula holds in the larger model.

In addition to the dimension for projective sets, we can define a rank for tuples of *M. We keep the notation from [Zil10] and call this the *combinatorial dimension* cdim .

Definition 2.2.1 Given any $a \in {}^*\mathrm{M}^n$ we define

$$\mathrm{cdim}\,(a) = \min \{ \dim S \mid S \text{ is projective } \&^*\mathrm{M} \models a \in S \}$$

Given another tuple $b \in {}^*\mathrm{M}^k$,

$$\mathrm{cdim}\,(a/b) = \mathrm{cdim}\,(ab) - \mathrm{cdim}\,(b)$$

and given any set $A \subseteq {}^*\mathrm{M}$,

$$\mathrm{cdim}\,(a/A) = \min\{\,\mathrm{cdim}\,(a/b) \mid b \in A^k\,\}$$

If $\mathrm{cdim}\,(a/A) = \mathrm{cdim}\,(a)$, we say that a is *independent* of A. A itself is called independent if for any $a \in A$, a is independent of $A \setminus \{a\}$.

If $b \in {}^*\mathrm{M}^k$ and S is M-closed, we say that $a \in S(b)$ is *generic* (in $S(b)$) if $\mathrm{cdim}\,(a/b) = \dim S(b)$.

Given an element $a \in {}^*\mathrm{M}^n$, a *locus* of a is a uniform closed set $a \in S \subseteq_{\mathrm{cl}} \mathrm{M}^n$ such that $\dim S = \mathrm{cdim}\,(a)$.

Note that in the definition of cdim, it does not matter whether we consider projective sets or just closed sets. By (SB) the two definitions are equivalent. Note also that unlike in Zariski Geometries, the locus of a point is not unique since there may not be a minimal closed set containing it.

Theorem 2.2.2 *The structure* $^*\mathrm{M}$ *with the dimension defined above is a Strong Geometry.*

We will prove this theorem in a series of lemmas.

Lemma 2.2.3 *(DP) and (DU) hold.*

Proof. Easy applications of 2.1.23. □

Lemma 2.2.4 *Every closed set in* $^*\mathrm{M}^n$ *is of the form* $S({}^*\mathrm{M}, a)$*, where*

1. $S(x, y) \subseteq \mathrm{M}^{n+m}$ *is closed,*

2. a *is generic in the projection of* S*,*

3. *for every uniform component* C *of* S*,* $\mathrm{pr}_x(C) \ni a$ *and*

4. *for all* a' *generic in the projection of* S*,* $\dim S(a) = \dim S(a')$*.*

Proof. By definition, every closed set is defined by a positive quantifier-free formula $\phi(x, a, b)$, where $b \in \mathrm{M}^m$ and $a \in {}^*\mathrm{M}^k$. $\phi(x, y, b)$ defines a closed set $R \subseteq_{\mathrm{cl}} \mathrm{M}^{n+k}$ and hence every closed set is of the form $R(a)$, where $a \in {}^*\mathrm{M}^k$ and R is a closed set in M. Let L be a locus of a and let $Q = \{ (x, y) \in R \mid y \in L \}$. Note that $Q(a) = R(a)$ and a is now generic in the projection of Q. By 2.1.23, we can find a constructible set V such that $a \in V$ and for all $a' \in V$,

$$\dim Q(a) = \dim Q(a')$$

By (SB), the intersection of $\overline{V}$ with a set V_i containing those points with fibres of dimension $i \neq \dim(Q(a))$ is small so for all generic $a' \in \overline{V}$,

$$\dim Q(a) = \dim Q(a')$$

Now let $S' = \{ (x, y) \in Q \mid x \in \overline{V} \}$ and S be the union of all those components C of S' such that $\mathrm{pr}_x(C) \ni a$. Then clearly $S(a) = Q(a) = R(a)$ and S has properties 1–4. $\square$

The same argument holds for projective sets, except that the S would also be projective. We call such a representation of a closed (or projective) set a *canonical representation.*

By the definition of canonical representations and dimension in ${}^*\mathrm{M}$ it is clear that for a canonical representation $S(a)$, $\dim S(a) = \dim S - \mathrm{cdim}\,(a)$.

Lemma 2.2.5 *If* $S \subseteq_{\mathrm{cl}} \mathrm{M}^{n+m}$ *is pure-dimensional and* $a \in {}^*\mathrm{M}^m$ *is generic in the projection of* S*, then* $S({}^*\mathrm{M}, a)$ *is pure-dimensional.*

Proof. Since S is a finite union of closed uniform sets it is enough to consider the case where S itself is uniform.

Now suppose $S(a)$ is not pure dimensional. Then there are closed sets $T \subseteq_{\mathrm{cl}} \mathrm{M}^{n+k}$ and $T_2 \subseteq_{\mathrm{cl}} \mathrm{M}^{n+k_2}$ and tuples $c \in {}^*\mathrm{M}^k$ and $c_2 \in {}^*\mathrm{M}^{k_2}$ such that $T(c)$ and $T_2(c_2)$ are canonical representations of closed sets, $S(a) = T(c) \cup T_2(c_2)$, $T(c) \neq S(a)$ and $\dim T_2(c_2) < \dim S(a)$. This means $\dim(S(a) \setminus T(c)) < \dim S(a)$ and $S(a) \setminus T(c) \neq \emptyset$.

Let $\mathrm{pr}\colon \mathrm{M}^{n+m} \to \mathrm{M}^m$ and $\mathrm{pr}'\colon \mathrm{M}^{m+k} \to \mathrm{M}^m$ be projections. We can find a closed set $V \subseteq_{\mathrm{cl}} \overline{\mathrm{pr}(S)} \times \mathrm{M}^k$ in which (a, c) is generic and such that for all generic $(a', c') \in V$ we have $S(a') \setminus T(c') \neq \emptyset$ and

$$\dim(S(a') \setminus T(c')) = \dim(S(a) \setminus T(c)) < \dim(S(a'))$$

Now we apply (GS) to S and pr to find a closed set $K \subseteq_{\mathrm{cl}} \mathrm{M}^m$ with $\dim K < \dim \mathrm{pr}(S)$ such that $S \setminus \mathrm{pr}^{-1}(K)$ is smooth with respect to pr. Since a is generic in $\mathrm{pr}(S)$, $a \notin K$. By 2.1.29,

$$U = \{ (x, y, z) \mid (x, y) \in S \setminus \mathrm{pr}^{-1}(K) \ \& \ (y, z) \in V \}$$

has no components of dimension less than $\dim S + \dim V - \dim L = \dim V + \dim S(a)$. Let $R = \{ (x, y, z) \mid (x, z) \in T \}$. Since $T(c) \neq S(a)$,

$$U \setminus R = \{ (x, y, z) \mid (x, y) \in S \setminus \mathrm{pr}^{-1}(K) \ \& \ (x, z) \notin T \ \& \ (y, z) \in V \} \neq \emptyset$$

By pure-dimensionality of components, $U \setminus R$ still does not have a component of dimension less than $\dim V + \dim S(a)$ and $U \setminus R$ covers (a, c). However, this is a contradiction because by definition of V, a generic fibre over V has dimension smaller than $\dim S(a)$. $\square$

Corollary 2.2.6 ${}^*\mathrm{M}$ *satisfies (DC).*

Proof. Let our closed set be given by a canonical representation $S(a)$. Apply (DC) to S to get $S = \bigcup S_i$ where each S_i is pure-dimensional of dimension i. Without loss of generality we may assume that $a \in \mathrm{pr}(S_i)$ for all i. Then a is generic in $\mathrm{pr}(S_i)$ for each i and so by the previous lemma $S(a)$ is a union of pure dimensional sets $S_i(a)$ of dimensions $i - \mathrm{cdim}\,(a)$, respectively. By (DU)

$$\dim(S(a) \setminus S_{\dim(S)}(a)) \leq \dim \bigcup_{i<\dim S} S_i(a) < \dim S(a)$$

□

Lemma 2.2.7 *M *satisfies (DD).*

Proof. Again let $S(a)$ be a canonical representation where $S(x,y,z) \subseteq_{\mathrm{cl}} \mathrm{M}^{n+m+k}$ is pure-dimensional and $a \in {}^*\mathrm{M}^k$ is generic in $\mathrm{pr}_{xy} S$. We can use (DD) to find $K \subseteq_{\mathrm{cl}} \mathrm{M}^{m+k}$ with $\dim K < \dim \mathrm{pr}_x(S)$ such that

$$\text{for all } (b,a') \in \mathrm{pr}_x(S) \setminus K\ ,\ \dim(S(b,a')) = \dim S - \dim \mathrm{pr}_x(S)$$

Since a is generic in $\mathrm{pr}_{xy}(S)$ and $\dim K < \dim S$, $\dim K(a) < \dim \mathrm{pr}_x(S) - \mathrm{cdim}\,(a)$. We have to show that

$$\text{if } b \in \mathrm{pr}_x(S(a) \setminus K(a)), \text{ then } \dim(S(b,a)) = \dim S(a) - \dim \mathrm{pr}_x(S(a))$$

However, by definition of dimension and the genericity of a, $\dim S(a) = \dim S - \mathrm{cdim}\,(a)$ and $\dim \mathrm{pr}_x(S(a)) = \dim \mathrm{pr}_x(S) - \mathrm{cdim}\,(a)$ and if $b \in \mathrm{pr}_x(S(a) \setminus K(a))$, then $(b,a) \in \mathrm{pr}_x(S) \setminus K$, so

$$\begin{aligned} \dim(S(b,a)) &= \dim S - \dim \mathrm{pr}_x(S) \\ &= \dim S - \mathrm{cdim}\,(a) + \mathrm{cdim}\,(a) - \dim \mathrm{pr}_(S) \\ &= \dim(S(a)) - \dim \mathrm{pr}_x(S(a)) \end{aligned}$$

as required. □

Lemma 2.2.8 *M *satisfies (SB).*

Proof. We consider two projective sets $S(c_1)$ and $T(c_2)$. Without loss of generality, we may assume that c_1 and c_2 are generic in the projections of S and T respectively. By choosing a locus V of (c_1, c_2) and defining

$$T' = \{ (x, z_1, z_2) \mid (x, z_2) \in T \;\&\; (z_1, z_2) \in V \} \text{ and}$$
$$S' = \{ (x, z_1, z_2) \mid (x, z_1) \in S \;\&\; (z_1, z_2) \in V \}$$

we may assume that $c = c_1 = c_2$. Now $S(c) \cap T(c)$ and $\overline{S}(c) \cap T(c)$ are also such that c is generic in their projections. By (SB),

$$\begin{aligned} \dim(S(c) \cap T(c)) < \dim S(c) \;&\Rightarrow\; \dim(S \cap T) < \dim S \\ &\Rightarrow\; \dim(\overline{S} \cap T) < \dim S \\ &\Rightarrow\; \dim(\overline{S}(c) \cap T(c)) < \dim S(c) \end{aligned}$$

and so $\overline{S}(^*\mathrm{M}, c)$ works as a closure of $S(^*\mathrm{M}, c)$. □

Lemma 2.2.9 *If $S(x, y, z)$ is smooth with respect to pr_y and a is generic in $\mathrm{pr}_{xy}(S)$, then $S(a)$ is also smooth with respect to pr_y.*

Proof. Let $T(x, u, b)$ be the canonical representation of a uniform set such that

$$\overline{\mathrm{pr}_u(T(b))} = \overline{\mathrm{pr}_y(S(a))}$$

Let U be a locus of (a, b) and V be a locus of b such that both $\mathrm{pr}_{xu}(T)$ and $\mathrm{pr}_z(U)$ are dense in V. Without loss of generality we may assume that $U = U \cap \mathrm{pr}_z^{-1}(V)$ and $T = T \cap \mathrm{pr}_{xu}^{-1}(V)$.

Apply (GS) to U and pr_z to get a closed set $K \subseteq V$ with $\dim K < \dim V$ and such that $U' = U \setminus \mathrm{pr}_z^{-1}(K)$ is smooth with respect to pr_z. Since $\dim K < \dim V = \mathrm{cdim}\,(b)$, U'

must still contain (a, b). Using (DD) we can truncate U' further and assume that for all $a' \in \mathrm{pr}_v(U')$, $\dim(U'(a')) = \mathrm{cdim}\,(b/a)$. Now by 2.1.33,

$$T' = \{\, (x, u, z, v) \mid (x, u, v) \in T \;\&\; (z, v) \in U' \,\}$$

has no components of dimension

$$\neq \dim T + \dim U - \dim V = \dim T + \mathrm{cdim}\,(ab) - \mathrm{cdim}\,(b) = \mathrm{cdim}\,(ab) + \dim T(b)$$

Let D be a component of $T' \cap \mathrm{pr}_{uv}^{-1}(\overline{\mathrm{pr}_y(S)})$ with $(a, b) \in \mathrm{pr}_{xu}(D)$ and suppose for a contradiction that $\dim(\mathrm{pr}_{uv}(D)) < \dim(\mathrm{pr}_y(S))$. By 2.1.35 and our assumptions on U', T' is aligned with respect to pr_{uv} so this implies that $\dim D < \dim T'$. Since T' is pure-dimensional, this implies that there is a uniform set $D \subseteq C \subseteq T'$ of dimension $\mathrm{cdim}\,(ab) + \dim T(b)$ and such that $C \cap \mathrm{pr}_{uv}^{-1}(\overline{\mathrm{pr}_y(S)}) = D$. Since C is part of a component of T', 2.1.35 implies $\dim \mathrm{pr}_u(C) = \dim C - \dim T + \dim \mathrm{pr}_u(T)$ and by our last assumption on U' we get

$$\begin{aligned}
\dim(\mathrm{pr}_u(C)(a, b)) &= \dim \mathrm{pr}_u(C) - \mathrm{cdim}\,(ab) \\
&= \dim C - \dim T + \dim \mathrm{pr}_u(T) - \mathrm{cdim}\,(ab) \\
&= \mathrm{cdim}\,(ab) + \dim T(b) - \dim T + \dim \mathrm{pr}_u(T) - \mathrm{cdim}\,(ab) \\
&= \dim \mathrm{pr}_u(T) - \mathrm{cdim}\,(b) \\
&= \dim \mathrm{pr}_u(T(b)) \\
&= \mathrm{pr}_y(S(a))
\end{aligned}$$

By assumption, $\mathrm{pr}_u(C)(a, b) \subseteq \mathrm{pr}_u(T)(b) \subseteq \overline{\mathrm{pr}_y(S(a))}$ so in particular

$$\dim(\mathrm{pr}_u(C)(a, b) \cap \mathrm{pr}_y(S)(a)) = \dim(\mathrm{pr}_u(C)(a, b)) = \dim(\mathrm{pr}_y(S)(a))$$

By our assumptions on U',

$$\begin{aligned}
\dim(\mathrm{pr}_{uv}(C)\cap \mathrm{pr}_y(S)) &\geq \dim(\mathrm{pr}_u(C)\cap \mathrm{pr}_v^{-1}\mathrm{pr}_y(S)) - \mathrm{cdim}\,(b/a)\\
&= \dim(\mathrm{pr}_u(C)(a,b)\cap \mathrm{pr}_y(S)(a)) + \mathrm{cdim}\,(a)\\
&= \dim(\mathrm{pr}_y(S)(a)) + \mathrm{cdim}\,(a)\\
&= \dim(\mathrm{pr}_y(S))
\end{aligned}$$

But this last statement contradicts our choice of C.

It now follows from smoothness of S and 2.1.29 that the set

$$V(x,y,z,u,v) = \{\,(x,y,z,u,v) \mid (x,y,z)\in S \;\&\; (x,u,v)\in T \;\&\; (z,v)\in U'\,\}$$

has no components of dimension

$$\begin{aligned}
&< \mathrm{cdim}\,(ab) + \dim T(b) + \dim S - \dim \mathrm{pr}_y(S)\\
&= \mathrm{cdim}\,(ab) + \dim T(b) + \dim S(a) - \dim \mathrm{pr}_y(S)(a)
\end{aligned}$$

Since (a,b) is generic in its projection, by 2.2.5 every component of $V(a,b)$ has at least dimension $\dim S(a) + T(b) - \dim(\mathrm{pr}_y(S(a)))$ as required. $\square$

Corollary 2.2.10 *M *satisfies (GS).*

Proof. Suppose $S(x,y,a)$ is a canonical representation of a closed pure-dimensional set. Then by (GS), there is a closed set $K\subseteq \overline{\mathrm{pr}_y(S)}$ such that $\dim K < \dim \mathrm{pr}_y S$ and $S' = S\setminus \mathrm{pr}_y^{-1}(K)$ is smooth with respect to pr_y. In order to verify (GS), we will show that

1. $\dim K(a) < \dim \mathrm{pr}_y(S(a))$ and
2. $S(a)\setminus \mathrm{pr}_y^{-1}(K(a))$ is smooth with respect to pr_y.

Note that $S'(a) = S(a)\setminus \mathrm{pr}_y^{-1}(K)(a) = S(a)\setminus \mathrm{pr}_y^{-1}(K(a))$ and that since a is generic,

$\dim(K(a)) < \dim(\mathrm{pr}_y(S)(a))$.

The fact that $S'(a)$ is smooth follows from the previous lemma. □

This concludes the proof of theorem 2.2.2.

2.2.2 The Pre-Geometry cl

From now on, we will always assume that we are working with a one-dimensional strong geometry M.

We start by recalling some basic definitions.

Definition 2.2.11 Let M be a set and cl be a function $\mathcal{P}(\mathrm{M}) \to \mathcal{P}(\mathrm{M})$. Then $(\mathrm{M}, \mathrm{cl})$ is called a *pre-geometry* if cl satisfies

1. $A \subseteq B \subseteq \mathrm{M} \quad \Rightarrow \quad \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$
2. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ for all $A \subseteq \mathrm{M}$
3. $a \in \mathrm{cl}(B \cup \{b\}) \setminus \mathrm{cl}(B) \quad \Rightarrow \quad b \in \mathrm{cl}(B \cup \{a\})$

Definition 2.2.12 $B \subseteq \mathrm{M}$ is *independent* over A if for all $b \in B$

$$b \notin \mathrm{cl}((B \setminus \{b\}) \cup A)$$

In the context of Strong Geometries there is an obvious candidate for a pre-geometry. We define

$$\mathrm{cl}(A) = \{ a \mid \mathrm{cdim}\,(a/A) = 0 \}$$

Proposition 2.2.13 *If* M *is a one-dimensional Strong Geometry, then* cl *is a pre-geometry on any elementary extencion* $^*\mathrm{M}$.

Proof. We check the three properties of cl:

1. $A \subseteq B \Rightarrow A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ is trivial.

2. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$

   The right to left inclusion follows from 1. For the other direction, suppose there is an element $a \in \mathrm{cl}(\mathrm{cl}(A))$. Then there are tuples $c \in A^n$ and $b \in A^k$ such that $\mathrm{cdim}\,(b/c) = \mathrm{cdim}\,(a/b) = 0$. Now

$$\begin{aligned}\mathrm{cdim}\,(a/c) =& \mathrm{cdim}\,(ac) - \mathrm{cdim}\,(c)\\ =& \mathrm{cdim}\,(ac) - \mathrm{cdim}\,(bc) + \mathrm{cdim}\,(bc) - \mathrm{cdim}\,(c)\\ =& \mathrm{cdim}\,(ac) - \mathrm{cdim}\,(abc)\\ &+ \mathrm{cdim}\,(abc) - \mathrm{cdim}\,(bc)\\ &+ \mathrm{cdim}\,(bc) - \mathrm{cdim}\,(c)\\ =& \mathrm{cdim}\,(a/bc) + \mathrm{cdim}\,(b/c) - \mathrm{cdim}\,(b/ac)\end{aligned}$$

   From the definitions, it is easy to see that these are all zero, so $\mathrm{cdim}\,(a/c) = 0$ and $a \in \mathrm{cl}(A)$.

3. $a \in \mathrm{cl}(A \cup \{b\}) \setminus \mathrm{cl}(A) \Rightarrow b \in \mathrm{cl}(A \cup \{a\})$

   There is a $c \in A^k$ such that $\mathrm{cdim}\,(a/bc) = 0$ and $\mathrm{cdim}\,(a/c) = 1$. Now by the above,

$$0 \leq \mathrm{cdim}\,(b/ac) = \mathrm{cdim}\,(a/bc) + \mathrm{cdim}\,(b/c) - \mathrm{cdim}\,(a/c) = \mathrm{cdim}\,(b/c) - 1$$

   Since $\mathrm{cdim}\,(b/c) \leq \mathrm{cdim}\,(b) = 1$, we must have equality. $\square$

Definition 2.2.14 We say that B *is independent of* A *over* C if for any $B_0 \subseteq B$, if B_0 is independent over C, then it is also independent over $A \cup C$. We write

$$B \underset{C}{\perp} A$$

to denote this.

For finite $b = B$ and $C = \emptyset$, this is equivalent to saying $\operatorname{cdim}(b/A) = \operatorname{cdim}(b)$ and thus matches the notion of independence introduced earlier.

Independence relations are studied a lot in Model Theory. We refer the reader to [KP97] or [Adl09] for further reading. We follow the axiomatization of independence relations given in [Adl09] since this is broken down into more and simpler properties. We will list the axioms and prove that the above relation $\downarrow$ satisfies all of them. Note that the axioms here differ from Adler's in two respects. First, we replace "elementarily equivalent" by "existentially equivalent" and second, the acl in anti-reflexivity is replaced by cl. Neither of these changes makes a big difference in the applications we have in mind.

In the following, $\operatorname{tp}_{\mathcal{P}}(a/B) = \operatorname{tp}_{\exists}(a/B)$ will denote the projective or existential type of a over B.

$$\operatorname{tp}_{\mathcal{P}}(a/B) = \left\{ P(x,b) \,\middle|\, P \text{ is projective } \& \, b \in B^k \,\&\, {}^*\mathrm{M} \models P(a,b) \right\}$$

Definition 2.2.15 We define the following axioms for a ternary relation $\downarrow$.

(Invariance) If $A \underset{C}{\downarrow} B$ and $\operatorname{tp}_{\mathcal{P}}(A', B', C') = \operatorname{tp}_{\mathcal{P}}(A, B, C)$, then $A' \underset{C'}{\downarrow} B'$.

(Monotonicity) If $A \underset{C}{\downarrow} B$, $A' \subseteq A$ and $B' \subseteq B$, then $A' \underset{C}{\downarrow} B'$.

(Base Monotonicity) If $D \subseteq C \subseteq B$ and $A \underset{D}{\downarrow} B$, then $A \underset{C}{\downarrow} B$.

(Transitivity) If $D \subseteq C \subseteq B$, $B \underset{C}{\downarrow} A$ and $C \underset{D}{\downarrow} A$, then $B \underset{D}{\downarrow} A$.

(Normality) If $A \underset{C}{\downarrow} B$, then $AC \underset{C}{\downarrow} B$.

(Extension) If $A \underset{C}{\downarrow} B$ and $B \subseteq B'$, then there is an A' with $\operatorname{tp}_{\mathcal{P}}(A/BC) = \operatorname{tp}_{\mathcal{P}}(A'/BC)$ such that $A' \underset{C}{\downarrow} B'$.

(Finite Character) $A \underset{C}{\downarrow} B$ if and only if $A_0 \underset{C}{\downarrow} B$ for all finite $A_0 \subseteq A$.

(Local Character) For all A, there is a cardinal $\kappa(A)$ such that for every B there is some $C \subseteq B$ with $|C| < \kappa(A)$ and $A \mathop{\smile\!\!\!\!\!\mid}_{C} B$.

(Anti-Reflexivity) If $a \mathop{\smile\!\!\!\!\!\mid}_{C} a$, then $a \in \mathrm{cl}(C)$.

(Full Existence) For all A, B, C, there is an A' with $\mathrm{tp}_{\mathcal{P}}(A/C) = \mathrm{tp}_{\mathcal{P}}(A'/C)$ such that $A' \mathop{\smile\!\!\!\!\!\mid}_{C} B$.

(Symmetry) $A \mathop{\smile\!\!\!\!\!\mid}_{C} B$ if and only if $B \mathop{\smile\!\!\!\!\!\mid}_{C} A$.

Proposition 2.2.16 $\smile\!\!\!\!\!\mid$ *in 2.2.14 satisfies all of the axioms in 2.2.15.*

Before we start to prove this, we give a useful alternative characterization of $\smile\!\!\!\!\!\mid$.

Lemma 2.2.17 *$A \mathop{\not\smile\!\!\!\!\!\mid}_{C} B$ if and only if there are tuples $a \in A^m$, $b \in B^n$ and $c \in C^k$ such that*

$$\mathrm{cdim}\,(a/bc) < \mathrm{cdim}\,(a/C)$$

Proof. This is just a matter of playing with the definitions. $A \mathop{\not\smile\!\!\!\!\!\mid}_{C} B$ if and only if there is a subset A_0, independent over C, but not independent over BC. By definition this means that there is an $a_0 \in A_0$ such that

$$a_0 \in \mathrm{cl}((A_0 \setminus \{a_0\})BC) \setminus \mathrm{cl}((A_0 \setminus \{a_0\})C)$$

which in turn means that there are tuples a_1, b and c such that

$$a_0 \in \mathrm{cl}(a_1, b, c) \setminus \mathrm{cl}(a_1, C)$$

Let $a = (a_0, a_1)$ and let d be a tuple extending c such that $\mathrm{cdim}\,(a/d) = \mathrm{cdim}\,(a/C)$. Now $\mathrm{cdim}\,(a/bd) < \mathrm{cdim}\,(a/C)$, as required.

For the converse, suppose there are a, b and c such that $\mathrm{cdim}\,(a/bc) < \mathrm{cdim}\,(a/C)$. Let $a = (a_0, \ldots, a_n)$. Let $i = \min\{\, j \mid a_j \in \mathrm{cl}(a_0, \ldots, a_{j-1}, b, c) \setminus \mathrm{cl}(a_0, \ldots, a_{i-j}, C)\,\}$. Then $(a_0, \ldots, a_i)$ is independent over C, but not over BC. $\square$

We will now begin the proof of proposition 2.2.15. We will check the axioms one by one.

(Invariance) Since dimension is existentially definable, this is an easy consequence of the above characterization of independence.

(Monotonicity) Again, this follows from the above characterization. A witness to $A' \underset{C}{\not\perp} B'$ would also be a witness for $A \underset{C}{\not\perp} B$

(Base Monotonicity) If $A \underset{C}{\not\perp} B$, then there is some tuple a from A such that

$$\operatorname{cdim}(a/B) < \operatorname{cdim}(a/C) \leq \operatorname{cdim}(a/D)$$

which by the above characterization implies $A \underset{D}{\not\perp} B$

(Transitivity) Assume $B \underset{C}{\perp} A$ and $B \underset{D}{\not\perp} A$. Let $b_1, \ldots, b_n$ be a tuple of minimal length from B such that

$$\operatorname{cdim}(b_1, \ldots, b_n/D) > \operatorname{cdim}(b_1, \ldots, b_n/DA)$$

Furthermore, let $c_1, \ldots, c_m$ be a tuple of minimal length from C such that

$$\operatorname{cdim}(b_1, \ldots, b_n/c_1, \ldots, c_m, D) = \operatorname{cdim}(b_1, \ldots, b_n/CD) = n - r$$

Then $b_1, \ldots, b_n$ and $c_1, \ldots, c_m$ are independent over D. Without loss of generality assume

$$b_1, \ldots b_r \in \operatorname{cl}(b_{r+1}, \ldots, b_n, c_1, \ldots, c_m, D)$$

so that by exchange

$$c_1, \ldots c_r \in \operatorname{cl}(b_1, \ldots, b_n, c_{r+1}, \ldots, c_m, D)$$

Now $b_1 \in \operatorname{cl}(b_2, \ldots, b_n, D, A)$, so

$$c_1, \ldots c_r \in \operatorname{cl}(b_2, \ldots, b_n, c_{r+1}, \ldots, c_m, D, A)$$

which means that $\operatorname{cdim}(c_1, \ldots, c_m/D) = m > m-1 \geq \operatorname{cdim}(c_1, \ldots, c_m/DA)$. By the previous characterization,

$$C \underset{D}{\not\downarrow} A$$

(Normality) For all tuples a from A and c from C,

$$\operatorname{cdim}(ac/BC) = \operatorname{cdim}(a/BC) = \operatorname{cdim}(a/C) = \operatorname{cdim}(ac/C)$$

(Extension) Enumerate $A = (a_\lambda)_{\lambda<\kappa}$. Let

$$\begin{aligned} p(x_\lambda)_{\lambda<\kappa} = \{\neg K(x_{\lambda_1}, \ldots, x_{\lambda_n}, \bar{b}, \bar{c}) \mid K \text{ is closed },\\ \dim K(\bar{b}, \bar{c}) < \operatorname{cdim}(a_{\lambda_1}, \ldots, a_{\lambda_n}/C),\\ \bar{b} \in \widetilde{B}^k, \bar{c} \in C^m\} \end{aligned}$$

This expresses exactly that for all $\lambda_1, \ldots, \lambda_n$ we have

$$\operatorname{cdim}(x_{\lambda_1}, \ldots, x_{\lambda_n}/\widetilde{B}C) \geq \operatorname{cdim}(a_{\lambda_1}, \ldots, a_{\lambda_n}/C)$$

We will now show that $q = p \cup \operatorname{tp}_{\mathcal{P}}(A/BC)$ is consistent. Then any realisation of q would be a suitable $\widetilde{A}$.

$\operatorname{tp}_{\mathcal{P}}(A/BC)$ is closed under intersections so if q is inconsistent, there is a

$$P(x_{\lambda_1}, \ldots, x_{\lambda_n}, \bar{b}_1, \bar{c}_1) \in \operatorname{tp}_{\mathcal{P}}(A/BC)$$

and a

$$\neg K(x_{\lambda_1}, \ldots, x_{\lambda_n}, \bar{b}_2, \bar{c}_2) \in p$$

such that

$$P(\bar{b}_1, \bar{c}_1) \subseteq K(\bar{b}_2, \bar{c}_2)$$

However, by definition of p and the fact that $A \underset{C}{\downarrow} B$,

$$\dim P(\bar{b}_1, \bar{c}_1) \geq \operatorname{cdim}(a_{\lambda_1}, \ldots, a_{\lambda_n}/C) > \dim K(\bar{b}_2, \bar{c}_2)$$

This is a contradiction to (DU) so q is consistent as required.

(Finite Character) Follows since dependence is witnessed by tuples, which are finite.

(Local Character) For every finite subset $F \subseteq A$, let f be F as a tuple. Now there is a tuple b_f such that

$$\operatorname{cdim}(f/B) = \operatorname{cdim}(f/b_f)$$

Let B_F be the set of elements of b_f and let

$$C = \bigcup \{ B_F \mid F \subseteq A \text{ is finite } \}$$

Then $|C| \leq \max(2^{|A|}, \aleph_0)$ and $A \underset{C}{\downarrow} B$.

(Anti-Reflexivity) If $a \underset{C}{\downarrow} a$, then $\operatorname{cdim}(a/C) = \operatorname{cdim}(a/aC) = 0$, so $a \in \operatorname{cl}(C)$

(Full Existence) The proof is very similar to Extension, but by [Adl09] is also follows from the other axioms.

(Symmetry) Suppose $A \underset{C}{\not\downarrow} B$. Let a be a minimal tuple from A such that

$$\operatorname{cdim}(a/BC) < \operatorname{cdim}(a/C)$$

and b a minimal tuple from B such that

$$\operatorname{cdim}(a/bC) = \operatorname{cdim}(a/BC)$$

Then a and b are independent (as sets) over C, but then by an easy application of exchange,

$$\operatorname{cdim}(b/AC) \leq \operatorname{cdim}(b/aC) < \operatorname{cdim}(b/C)$$

so $B \underset{C}{\not\downarrow} A$.

This concludes the proof of 2.2.15. $\square$

Definition 2.2.18 An independence relation $\mathop{\smile\!\!\!\!\!\!|}$ is said to be

Trivial if for all A, B, C and $C \subseteq C'$, $A \underset{C}{\mathop{\smile\!\!\!\!\!\!|}} B \Rightarrow A \underset{C'}{\mathop{\smile\!\!\!\!\!\!|}} B$,

Modular if $A \underset{cl(A) \cap cl(B)}{\mathop{\smile\!\!\!\!\!\!|}} B$ for all $A, B \subseteq \mathrm{M}$,

Weakly Locally Modular if for all $A, B \subseteq \mathrm{M}$ there exists a $C \subseteq \mathrm{M}$ such that $C \mathop{\smile\!\!\!\!\!\!|} AB$ and $A \underset{cl(AC) \cap cl(BC)}{\mathop{\smile\!\!\!\!\!\!|}} B$,

Very Weakly Locally Modular if for all tuples a and $B \subseteq \mathrm{M}$ there exists a $C \subseteq \mathrm{M}$ such that $C \underset{a}{\mathop{\smile\!\!\!\!\!\!|}} B$, $C \underset{B}{\mathop{\smile\!\!\!\!\!\!|}} a$ and $a \underset{cl(aC) \cap cl(BC)}{\mathop{\smile\!\!\!\!\!\!|}} B$,

Weakly One-Based if for all tuples a and $B \subseteq M$ there is a $B \subseteq C \subseteq M$ such that $a \underset{B}{\mathop{\smile\!\!\!\!\!\!|}} C$ and for all a' realizing $\mathrm{tp}_{\mathcal{P}}(a/C)$, $a \underset{C}{\mathop{\smile\!\!\!\!\!\!|}} a'$ implies $a \underset{a'}{\mathop{\smile\!\!\!\!\!\!|}} C$ and

Very Weakly One-Based if for all tuples a and $B \subseteq M$ there is an a' realizing $\mathrm{tp}_{\mathcal{P}}(a/B)$ such that $a \underset{B}{\mathop{\smile\!\!\!\!\!\!|}} a'$ and $a \underset{a'}{\mathop{\smile\!\!\!\!\!\!|}} B$.

Lemma 2.2.19 *The independence coming from* cl *is trivial if and only if* cl *is trivial in the sense that for all* A, $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$.

Proof. For the left to right implication suppose that there is a set A such that $\mathrm{cl}(A) \neq \bigcup_{a \in A} \mathrm{cl}(a)$. Let $b \in \mathrm{cl}(A) \setminus \bigcup_{a \in A} \mathrm{cl}(a)$ and let $a = (a_0, \ldots, a_n)$ be a minimal tuple from A such that $b \in \mathrm{cl}(a)$. Then $b \underset{\emptyset}{\mathop{\smile\!\!\!\!\!\!|}} (a_0, \ldots, a_{n-1})$, but $b \underset{a_n}{\not\mathop{\smile\!\!\!\!\!\!|}} (a_0, \ldots, a_{n-1})$ so $\mathop{\smile\!\!\!\!\!\!|}$ is not trivial.

For the other implication, suppose that for all A, $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$ and pick A, B, C and $C' \supseteq C$. $A \underset{C'}{\not\mathop{\smile\!\!\!\!\!\!|}} B$ if and only if there is a subset $A_0 \subseteq A$ such that A_0 is independent over C' but not over BC'. This means there is an $a \in A$ such that $a \in \mathrm{cl}((A \setminus \{a\})BC') \setminus \mathrm{cl}((A \setminus \{a\})C')$. By our assumption on cl this implies that there is a $b \in B$ such that $a \in \mathrm{cl}(b)$, but then $a \in \mathrm{cl}((A \setminus \{a\})BC) \setminus \mathrm{cl}((A \setminus \{a\})C)$ so that $A \underset{C}{\not\perp} B$. □

Lemma 2.2.20 *Let $a \in {}^*\mathrm{M}^n$ and $B \subseteq {}^*\mathrm{M}$. Suppose a' is another realization of* $\mathrm{tp_{qf}}(a/B)$*. If $a \underset{a'}{\perp} B$, then $a' \underset{a}{\perp} B$.*

Proof. By symmetry $a' \underset{a}{\perp} B$ if and only if $B \underset{a}{\perp} a'$. This fails if there is a tuple b from B such that

$$\mathrm{cdim}\,(b/aa') < \mathrm{cdim}\,(b/a)$$

but all information about $\mathrm{cdim}\,(b/a)$ is contained in $\mathrm{tp_{qf}}(a/B)$, so

$$\mathrm{cdim}\,(b/a') = \mathrm{cdim}\,(b/a)$$

and $B \underset{a'}{\not\perp} a$. □

Proposition 2.2.21 *([BBWK+11]) The following are equivalent:*

1. *$(M, \perp)$ is very weakly locally modular.*
2. *$(M, \perp)$ is weakly one based.*
3. *$(M, \perp)$ is very weakly one based.*

The proof of this proposition is a combination of results from [BBWK+11] and [BV10]. We include it for completeness.

Proof. We individually prove equivalence of (1) and (2) to (3).

(1) $\Rightarrow$ (3) Let $a \in {}^*\mathrm{M}^n$ and $B \subseteq {}^*\mathrm{M}$. From our assumption we can find $C \subseteq {}^*\mathrm{M}$ such that $C \underset{B}{\perp} a$, $C \underset{a}{\perp} B$ and $a \underset{cl(aC)\cap cl(BC)}{\perp} B$. By full existence, we can find some a' with $\mathrm{tp}_{\mathcal{P}}(a/\mathrm{cl}(BC)) = \mathrm{tp}_{\mathcal{P}}(a'/\mathrm{cl}(BC))$ and $a \underset{cl(BC)}{\perp} a'$. Now $C \underset{a'}{\perp} B$ implies $\mathrm{cl}(a'C) \cap \mathrm{cl}(BC) \underset{a'}{\perp} B$. Since a and a' have the same type over $\mathrm{cl}(BC)$, $\mathrm{cl}(a'C) \cap \mathrm{cl}(BC) = \mathrm{cl}(aC) \cap \mathrm{cl}(BC)$ and $\mathrm{cl}(aC) \cap \mathrm{cl}(BC) \underset{a'}{\perp} B$.

(3) $\Rightarrow$ (1) Let $a \in {}^*\mathrm{M}^n$ and $B \subseteq {}^*\mathrm{M}$. By assumption and 2.2.20 there exists some a' such that $\mathrm{tp}_{\mathcal{P}}(a/B) = \mathrm{tp}_{\mathcal{P}}(a'/B)$, $a \mathop{\downarrow}\limits_{B} a'$, $a \mathop{\downarrow}\limits_{a'} B$ and $a' \mathop{\downarrow}\limits_{a} B$. Let C be the set containing the elements of a'. Then by base monotinicity $a \mathop{\downarrow}\limits_{cl(aC)\cap cl(BC)} B$.

(2) $\Rightarrow$ (3) Let $a \in {}^*\mathrm{M}^n$ and $B \subseteq {}^*\mathrm{M}$. Let $B \subseteq C \subseteq {}^*\mathbb{N}$ be such that $a \mathop{\downarrow}\limits_{B} C$ and for all a' realizing $\mathrm{tp}_{\mathcal{P}}(a/C)$ if $a \mathop{\downarrow}\limits_{C} a'$ then $a \mathop{\downarrow}\limits_{a'} C$. By full existence, we can find an a' realizing $\mathrm{tp}_{\mathcal{P}}(a/C)$ such that $a \mathop{\downarrow}\limits_{C} a'$. By transitivity we have $a \mathop{\downarrow}\limits_{B} a'$ and by monotonicity we have $a \mathop{\downarrow}\limits_{a'} B$.

(3) $\Rightarrow$ (2) Let $a \in {}^*\mathrm{M}^n$ and $B \subseteq {}^*\mathrm{M}$. By assumption and 2.2.20 there exists some a'' such that $\mathrm{tp}_{\mathcal{P}}(a/B) = \mathrm{tp}_{\mathcal{P}}(a''/B)$, $a \mathop{\downarrow}\limits_{B} a''$, $a \mathop{\downarrow}\limits_{a''} B$ and $a'' \mathop{\downarrow}\limits_{a} B$. Let $C = a''B$ and use full existence to find a' such that $\mathrm{tp}_{\mathcal{P}}(a/C) = \mathrm{tp}_{\mathcal{P}}(a'/C)$ and $a \mathop{\downarrow}\limits_{C} a'$. Then certainly $a'' \mathop{\downarrow}\limits_{a'} B$, but from $a \mathop{\downarrow}\limits_{a''} B$ and $a \mathop{\downarrow}\limits_{C} a'$ we can also derive $a \mathop{\downarrow}\limits_{a'a''} B$, so by transitivity $a \mathop{\downarrow}\limits_{a'} B$. Another quick independence calculation shows $a \mathop{\downarrow}\limits_{B} Ca'$, so by another application of transitivity we get $a \mathop{\downarrow}\limits_{a'} C$ as required. $\square$

The chain of implications

$$\text{Modular} \Rightarrow \text{ Weakly Locally Modular } \Rightarrow \text{ Very Weakly Locally Modular}$$

is easy to see. In o-minimal structure the reverse of the first implication does not hold so we can not hope for this in Strong Geometries, but it might be interesting to investigate whether the reverse of the second implication holds.

The property we will use in the weak trichotomy is very weak one-basedness so it will be useful to have a slightly more precise characterizations of its converse. We start by making sure that we can choose B finite.

Lemma 2.2.22 *(Due to Gareth Boxall) Suppose a, b is not a counter-example to very weak one-basedness so that there is a tuple a' with the same projective type as a such that $a \mathop{\downarrow}\limits_{b} a'$ and $a \mathop{\downarrow}\limits_{a'} b$. Whenever a'' has the same type as a over $a'b$ and $a \mathop{\downarrow}\limits_{a'b} a''$, then $a \mathop{\downarrow}\limits_{a''} a'b$.*

Proof. $\operatorname{cdim}(a'/a) = \operatorname{cdim}(a/a')$ and $\operatorname{cdim}(a'/ab) = \operatorname{cdim}(a'/b) = \operatorname{cdim}(a/b) = \operatorname{cdim}(a/a'b)$. So from $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{a'} b$ we can deduce $a' \mathop{\smile\hspace{-0.9em}\mid}\limits_{a} b$. By choice of a'', this also means $a' \mathop{\smile\hspace{-0.9em}\mid}\limits_{a''} b$. On the other hand, from $b \mathop{\smile\hspace{-0.9em}\mid}\limits_{a'} a$ and $a'' \mathop{\smile\hspace{-0.9em}\mid}\limits_{ba'} a$ we can deduce that $b \mathop{\smile\hspace{-0.9em}\mid}\limits_{a'a''} a$. By several applications of transitivity $a'a \mathop{\smile\hspace{-0.9em}\mid}\limits_{a''} b$, $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{b} a'a''$ and $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{ba'} a''$. $\square$

Lemma 2.2.23 *(Due to Gareth Boxall) Suppose a and B form a counter-example to very weak one-basedness. Let b be a tuple from B such that $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{b} B$. Then a and b form a counter-example to very weak one-basedness.*

Proof. Suppose not. Then there exists a tuple a' with the same projective type as a over b such that $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{b} a'$ and $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{a'} b$. By full existence, we may choose such an a' with $a' \mathop{\smile\hspace{-0.9em}\mid}\limits_{ab} B$. By transitivity this gives $a' \mathop{\smile\hspace{-0.9em}\mid}\limits_{b} aB$ and by base monotonicity and monotonicity $a' \mathop{\smile\hspace{-0.9em}\mid}\limits_{B} a$.

Now let a'' have the same projective type as a over $a'B$ such that $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{a'B} a''$. Then $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{b} a'a''B$ and hence $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{ba'} a''$. Now by the previous lemma $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{a''} a'b$. We may conclude that $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{B} a''$ and $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{a''} B$. This is a contradiction, since a and B are assumed to be a counter-example to very weak one-basedness. $\square$

So if cl is not very weakly one-based, then there are tuples a and b such that whenever a' has the same projective type as a over b and $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{b} a'$, then $a \mathop{\not\smile\hspace{-0.9em}\mid}\limits_{a'} b$. If our elementary extension *M is sufficiently saturated we can strengthen this condition slightly so that we only have to consider quantifier free types.

Lemma 2.2.24 *Suppose *M is ω-saturated for projective types. If cl is not very weakly one-based, then there are tuples a and b such that whenever a' has the same quantifier-free type as a over b and $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{b} a'$, then $a \mathop{\not\smile\hspace{-0.9em}\mid}\limits_{a'} b$.*

Proof. Let a and b be such that whenever a' has the same projective-free type as a over b and $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{b} a'$, then $a \mathop{\not\smile\hspace{-0.9em}\mid}\limits_{a'} b$. Suppose $a' \models \operatorname{tp}_{\mathrm{qf}}(a/b)$ is such that $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{b} a'$ and $a \mathop{\smile\hspace{-0.9em}\mid}\limits_{a'} b$.

Let $S(x, x', y)$ be a locus of (a, a', b). Since S is uniform, the dimensions of projections of S correspond to the combinatorial dimensions of sub-tuples of (a, a', b). The above two independence relations correspond to saying

$$\dim S - \dim \mathrm{pr}_x(S) = \dim \mathrm{pr}_{x'}(S) - \dim \mathrm{pr}_{xx'}(S)$$

and

$$\dim S - \dim \mathrm{pr}_x(S) = \dim \mathrm{pr}_y(S) - \dim \mathrm{pr}_{xy}(S)$$

Now if we could show that

$$q(x') = \mathrm{tp}_{\mathcal{P}}(a/b)(x') \cup \{ S(a, x', b) \} \cup \{ \neg K(a, x', b) \mid K \text{ is closed and } \dim K < \dim S \}$$

is consistent, then the saturation would make sure that there is a realization of q. However, any such realization would contradict our choice of a and b.

Suppose it is not consistent. Then there is a projective set $P(x', b)$ with $a \in P(b)$ and a closed set K with $\dim K < \dim S$ such that $P({}^*\mathrm{M}, b) \cap S(a, {}^*\mathrm{M}, b) \subseteq K(a{}^*\mathrm{M}, b)$.

Without loss of generality we may assume that $\dim P = \mathrm{cdim}\,(a, b)$. The above and (SB) imply that $R = \overline{P}({}^*\mathrm{M}, b) \wedge S(a, {}^*\mathrm{M}, b)$ has dimension $< \dim S(a, {}^*\mathrm{M}, b)$. However, since $(a, b) \in \overline{P}$, $(a', b) \in \overline{P}$. Therefore, $a' \in R$. This contradicts the genericity of $a' \in S(a, {}^*\mathrm{M}, b)$.

2.2.3 The Weak Trichotomy

In this section we will prove the following weak trichotomy for Strong Geometries:

Proposition 2.2.25 *Let* M *be a Strong Geometry and* *M *an elementary extension sufficiently saturated for projective types. Exactly one of the following is true:*

1. cl *and all families of plane curves are trivial,*

2. cl *is very weakly one-based, non-trivial and every generically normal family of plane curves has dimension at most* 1 *or*

3. cl *is not very weakly one-based and there is a two-dimensional generically normal family of plane curves.*

Dealing with the first case is fairly standard.

Lemma 2.2.26 cl *is trivial if and only if all familes of plane curves are trivial.*

Proof. Suppose every family of plane curves is trivial and suppose $a \in \mathrm{cl}(B)$. If $a \notin \mathrm{cl}(\emptyset)$ we can find a minimal tuple $b_1, \ldots, b_n \in B$ such that $a \in \mathrm{cl}(b_1, \ldots, b_n)$. Suppose for a contradiction that $n \neq 1$. By minimality, $a, b_1 \notin \mathrm{cl}(b_2, \ldots, b_n)$ and so by definition of cl, there is a closed set $I \subseteq \mathrm{M}^{n+1}$ such that

$$\dim I(a, b_2, \ldots, b_n) = \dim I(b_1, \ldots, b_n) = 0 \text{ and } \dim I(b_2, \ldots, b_n) = 1$$

Thus $I \in \mathrm{M}^{2+(n-1)}$ is a non-trivial family of curves, a contradiction. So $n = 1$ and $\mathrm{cl}(B) = \bigcup_{b \in B} \mathrm{cl}(b)$.

For the converse, suppose $I \in \mathrm{M}^{2+n}$ is a non-trivial family of plane curves. Since the family has dimension at least 1, then the domain S has dimension 2. Pick a generic $(a, b) \in S$ and $l_1, \ldots, l_n$ generic in $I_(a, b)$. By definition of a family of curves, $a \in \mathrm{cl}(b, l_1, \ldots, l_n)$, but by non-triviality $a \notin \mathrm{cl}(l_1, \ldots, l_n)$ and by choice of (a, b), $a \notin \mathrm{cl}(b)$. Hence, if there is a non-trivial family, then cl is not trivial. □

The other two cases follow from the following lemma and proposition.

Lemma 2.2.27 cl *is not very weakly one-based if and only if there is a non-linear family of sets.*

Proof. Suppose I is a non-linear family of sets. Let $(a, b) \in I$ be generic and suppose $\mathrm{qftp}(a'/b) = \mathrm{qftp}(a/b)$ and $a' \underset{b}{\perp} a$. Then $(a', b) \in I$, so that $(a, a', b) \in I_2$. Since a and

a' are independent we have $\operatorname{cdim}(aa'b) = \dim I + \dim I - \dim L$ and can conclude that (a, a', b) is generic in I_2. Let C be the component of I_2 containing (a, a', b). Then (a, a') is generic in $\mathrm{pr}_I(C)$ and hence by non-linearity and 2.1.41, $\operatorname{cdim}(aa') = \dim \mathrm{pr}_I(C) > \dim S + \dim I - \dim L$. a' is generic in S so

$$\operatorname{cdim}(a/a') > \dim I - \dim L = \operatorname{cdim}(a/a'b)$$

By definition, this means $a \not\mathop{\smile\!\!\!\!\!\!|}_{a'} b$ so cl is not very weakly one-based.

Suppose on the other hand that cl is not very weakly one-based. Then by the last results of the previous section, there are tuples a and b such that whenever $a' \models \mathrm{qftp}(a/b)$ and $a' \mathop{\smile\!\!\!\!\!\!|}_{b} a$, then $a \not\mathop{\smile\!\!\!\!\!\!|}_{a'} b$. Given a small enough locus I of (a, b), I is a family of $\operatorname{cdim}(a/b)$-dimensional sets. Now suppose any such I has a locally linear component at (a, b). Pick an I and note that since I has only got finitely many components and $J \subseteq I$ implies $J_2 \subseteq I_2$, there is a component C of I_2 such that $(a, a, b) \in C$, $\dim \mathrm{pr}_I(C) = \dim S + \dim I - \dim L = \operatorname{cdim}(a) + \operatorname{cdim}(a/b)$ and for any other locus $J \subseteq I$ of (a, b), $\dim(C \cap J_2) = \dim C$. On the other hand, any element of $\mathrm{qftp}(a/b)$ contains a locus of (a, b) so for any $J \in \mathrm{qftp}(a/b)$, the set $C(a, b) \cap J(b) \subseteq I(b)$ has dimension $\operatorname{cdim}(a/b)$. That means we can find an $a' \models \mathrm{qftp}(a/b)$ such that $(a, a', b) \in C$. Since C is a linear component, this contradicts our choice of (a, b). $\square$

Proposition 2.2.28 *There is a two-dimensional generically normal family of plane curves if and only if there is a non-linear family of sets.*

We prove this proposition in two lemmas, but before we do so we want to describe the strategy and problems by looking at some examples. In the proof of the last lemma, we saw that the locus of a pair of tuples a and b witnessing non very weak one-basedness is the incidence relation of a non-linear family of curves. If a is a two-tuple, then this family would be a family of plane curves. If b is also a two-tuple, then the family is generically normal so to prove the above proposition we need to show that we can reduce any witnessing pair (a, b) to a pair of two-tuples. By 2.1.41, the situation is symmetric, so in fact it is sufficient to show that we can reduce a to a two-tuple while keeping b the same. The strategy for doing so should be fairly clear. We can reduce $\operatorname{cdim}(a/b)$ by naming elements of a until we get to $\operatorname{cdim}(a/b) = 1$. Then we project to the plane. However, as the following two examples show, we cannot just name any elements of a, nor can we project to the plane in any direction we like. If we do, we may turn a non-linear family into a linear one.

Example 1. We work in $\mathbb{C}$. Let $\phi_n(x, y, z_0, \ldots, z_{n-1})$ be the formula $y = x^n + z_{n-1}x^{n-1} + \cdots + z_0$. We define a family of hyperplanes with the formula

$$\psi(x_1, \ldots, x_m, z_1, \ldots z_n) = \phi_n(x_1, x_2, z_1, \ldots, z_n)$$

If we let $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_n)$ be generic in $\psi(\mathbb{C})$ and choose a' to be another point on $\psi(b)$ independent of a, then $\operatorname{cdim}(aa') = 2m$, so this is a witness of non very weak one-basedness. However, if we try to fix a_1 or a_2 the result will be completely trivial because the only relation b induced on a was between a_1 and a_2.

Example 2. Similarly, take the n-dimensional family of curves defined by

$$\psi(x_1, \ldots, x_m, y_1, \ldots, y_n) = \phi_n(x_1, x_2, y_1, \ldots, y_n) \wedge \bigwedge_3^n \phi_1(x_1, x_i, y_1)$$

If we project to any plane other than (x_1, x_2) we get the linear family $\phi_1(x, y, z)$ so in each of the two stages of our reduction (fixing coordinates and projecting to the plane) we must make sure that there is a choice that preserves non-linearity.

Lemma 2.2.29 *Let $a = (a_1, \ldots, a_n)$ and b be tuples such that whenever $a' \models \operatorname{qftp}(a/b)$ and $a' \mathop{\smile\!\!\!\!\!\!\,|}_b a$, then $a' \not\mathop{\smile\!\!\!\!\!\!\,|}_a b$. Then there is a sub-tuple c of a such that $\operatorname{cdim}(a/bc) = 1$ and whenever $a' \models \operatorname{qftp}(a/bc)$ and $a' \mathop{\smile\!\!\!\!\!\!\,|}_{bc} a$, then $a' \not\mathop{\smile\!\!\!\!\!\!\,|}_{ac} b$.*

Proof. Without loss of generality we may assume that $a_1, \ldots a_m$ are a basis of a over b. Let $c_k = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_m)$ and suppose for a contradiction that for all k there exists some $a'_k \models \operatorname{qftp}(a/bc_k)$ with $a'_k \mathop{\smile\!\!\!\!\!\!\,|}_{bc_k} a$ and $a'_k \mathop{\smile\!\!\!\!\!\!\,|}_{ac_k} b$.

We will construct a sequence $a^0, \dots, a^m$ of tuples realizing $\mathrm{qftp}(a/b)$ such that for all i

1. $\mathrm{cdim}\,(a_1^i, \dots, a_i^i/ab) = i$
2. $\mathrm{cdim}\,(a^i/a) \leq i$

The element a^m will then be such that $a^m \underset{b}{\downarrow} a$ and $a^m \underset{a}{\downarrow} b$ and so we have a contradiction.

Let $a^0 = a$. This trivially has the required properties.

Suppose we have constructed a^r and let $c = (a_1^r \dots, a_r^r, a_{r+2}^r, \dots, a_m^r)$. Since $a^r \models \mathrm{qftp}(a/b)$ there is a $a' \models \mathrm{qftp}(a^r/bc)$ such that $a' \underset{bc}{\downarrow} a^r$ and $a' \underset{a^r c}{\downarrow} b$.

Let $a^{r+1} \models \mathrm{qftp}(a'/ba^r)$ such that $a^{r+1} \underset{ba^r}{\downarrow} a$. Now

$$\begin{aligned}
\mathrm{cdim}\,(a_1^{r+1}, \dots, a_{r+1}^{r+1}/ab) &= \mathrm{cdim}\,(a_1^{r+1}, \dots, a_r^{r+1}/ab) + \mathrm{cdim}\,(a_{r+1}^{r+1}/aba_1^{r+1}, \dots, a_r^{r+1}) \\
&= \mathrm{cdim}\,(a_1^r, \dots, a_r^r/ab) + \mathrm{cdim}\,(a_{r+1}^{r+1}/aba_1^r, \dots, a_r^r) \\
&= r + \mathrm{cdim}\,(a_{r+1}^{r+1}/aba_1^r, \dots, a_r^r) \\
&\geq r + \mathrm{cdim}\,(a_{r+1}^{r+1}/aba^r) \\
&= r + \mathrm{cdim}\,(a_{r+1}^{r+1}/ba^r) \\
&= r + \mathrm{cdim}\,(a_{r+1}^{r+1}/bc) \\
&= r + \mathrm{cdim}\,(a_{r+1}^r/bc) \\
&= r + 1
\end{aligned}$$

It follows that $\mathrm{cdim}\,(a^{r+1}/a) \leq \mathrm{cdim}\,(a^r/a) + \mathrm{cdim}\,(a^{r+1}/a^r) \leq r + 1$.□

Lemma 2.2.30 *Let* $a = (a_1, \dots, a_n)$ *and* b *be tuples such that* $\mathrm{cdim}\,(a/b) = 1$ *and whenever* $a' \models \mathrm{qftp}(a/b)$ *and* $a' \underset{b}{\downarrow} a$*, then* $a' \underset{a}{\not\downarrow} b$*. Then there exists a sub-tuple* $c = (c_1, c_2)$ *of* a *such that whenever* $c' \models \mathrm{qftp}(c/b)$ *and* $c' \underset{b}{\downarrow} c$*, then* $c' \underset{c}{\not\downarrow} b$*.*

Proof. We may assume that a_1 is a basis for a over b. Now suppose for $i = 2, \ldots, n$ that there are $(a_1^i, a_i^i) \models \mathrm{qftp}(a_1, a_i/b)$ such that $a_1^i a_i^i \underset{b}{\downarrow} a_1 a_i$ and $a_1^i a_i^i \underset{a_1 a_i}{\downarrow} b$. By 2.2.22 we may replace b with $ba_1^2 \ldots a_1^n a_2^2 \ldots a_n^n$ and assume for all i and for all $(a_1^i, a_i^i) \models \mathrm{qftp}(a_1, a_i/b)$ with $a_1^i a_i^i \underset{b}{\downarrow} a_1 a_i$ that we have $a_1^i a_i^i \underset{a_1 a_i}{\downarrow} b$.

Now if we pick $a' \models \mathrm{qftp}(a/b)$ with $a' \underset{b}{\downarrow} a$, then $a_i' \in \mathrm{cl}(a, a_1')$ and hence $\mathrm{cdim}\,(a'/a) = 1$. This contradicts $a' \underset{a}{\not\downarrow} b$. □

Note that in the above lemma $\mathrm{cdim}\,(c/b) = 1$, for if $cdim(c/b) = 0$, then for $c' = c$, the conclusion would not hold.

This concludes the proof of the proposition and the Weak Trichotomy Theorem.

2.3 Specializations

In keeping with [Zil10], we will use specializations to develop a theory of infinitesimals. This notion becomes useful when the infinitesimal neighbourhoods have a certain degree of saturation which allows us to find points by checking the consistency of types. Zilber's proof of the existence of *universal specializations* goes through without problems, but as we have seen in the introduction and will discuss in some more detail here, they can exhibit some bad behaviour on the boundaries of projective sets. To get around this problem, we generalize the notion of universality and show that it is possible to consruct specializations which are sufficiently saturated for projective sets while omitting infinitesimal points of large combinatorial dimension in the boundaries of projective sets.

Definition 2.3.1 Let ${}^*\mathrm{M} \succcurlyeq \mathrm{M}$ be an elementary extension of M and let $A \subseteq {}^*\mathrm{M}$. A map $\pi\colon A \to \mathrm{M}$ is a *(partial) specialization* if for all closed sets $P \subseteq_{\mathrm{cl}} \mathrm{M}^n$ and $a \in A^n$,

$$^*\mathrm{M} \models P(a) \Rightarrow \mathrm{M} \models P(\pi(a))$$

where π acts coordinate-wise on the tuple a. In other words, π is a partial $\mathcal{L}$- homomorphism.

If we have $\mathrm{M}' \succcurlyeq {}^*\mathrm{M} \succcurlyeq \mathrm{M}$, $A \subseteq \mathrm{M}'$, $B \subseteq {}^*\mathrm{M}$ and specializations $\pi_1 \colon A \to {}^*\mathrm{M}$ and $\pi_2 \colon B \to \mathrm{M}$, then we write $\pi_1 \circ \pi_2$ for the map $\pi_1 \circ \pi_2 \colon A \cap \pi_1^{-1}(B) \to \mathrm{M}$.

Lemma 2.3.2 *Compositions of specializations as above are again specializations.*

Proof. If $a \in (A \cap \pi_1^{-1}(B))^n$ and S is closed, then by definition

$$\mathrm{M}' \models S(a) \Rightarrow {}^*\mathrm{M} \models S(\pi_1(a)) \Rightarrow \mathrm{M} \models S(\pi_2(\pi_1(a)))$$

□

Definition 2.3.3 Given a (partial) specialization $\pi: {}^*\mathrm{M} \to \mathrm{M}$, we will call a definable set $P \subseteq \mathrm{M}^n$ *π-closed* if for every $a \in \mathrm{Dom}\,\pi$, ${}^*\mathrm{M} \models P(a) \implies \mathrm{M} \models P(\pi(a))$. We denote the family of π-closed sets by $\mathcal{C}_\pi$.

Definition 2.3.4 Given a family of definable sets $\mathcal{F} \supseteq \mathcal{C}$, a tuple a from ${}^*\mathrm{M}$ and $A \subseteq {}^*\mathrm{M}$, we define the *$\mathcal{F}$-type* of a over A to be

$$\mathrm{tp}_{\mathcal{F}}(a/A) = \{\, F(y,b) \mid F \in \mathcal{F} \ \ \& \ \ b \in A^n \ \ \& \ \ {}^*\mathrm{M} \models F(a,b)\,\}$$

A map $\alpha \colon A \to B$ is said to be an $\mathcal{F}$-embedding over C if for any $a \in A^n$

$$\mathrm{tp}_{\mathcal{F}}(a/C) \subseteq \mathrm{tp}_{\mathcal{F}}(\alpha(a)/C)$$

Definition 2.3.5 Let $\mathcal{F}$ be a family of definable sets extending $\mathcal{C}$, ${}^*\mathrm{M} \succcurlyeq \mathrm{M}$ and $\pi \colon {}^*\mathrm{M} \to \mathrm{M}$ a specialization. We say that π is *$\mathcal{F}$-universal* if for any $\mathrm{M}' \succcurlyeq {}^*\mathrm{M}$, finite $A \subseteq \mathrm{M}'$ and specialization $\pi' \colon (\mathrm{Dom}\,\pi \cup A) \to \mathrm{M}$ extending π, there is an $\mathcal{F}$-embedding $\alpha \colon A \to \mathrm{Dom}\,\pi$ over $(A \cap {}^*\mathrm{M}) \cup \mathrm{M}$ such that $\pi \circ \alpha = \pi'$.

If $\mathcal{F}$ is the family of all definable sets, we omit its mention and just say *universal*.

Proposition 2.3.6 *(Zilber) For any relational structure* M *there are* ${}^*\mathrm{M} \succcurlyeq \mathrm{M}$ *and a universal specialization* $\pi \colon {}^*\mathrm{M} \to \mathrm{M}$.

For completeness we include the proof:

Proof. We will construct a chain of elementary extensions

$$\mathrm{M} = \mathrm{M}_0 \preccurlyeq \mathrm{M}_1 \preccurlyeq \cdots \preccurlyeq \mathrm{M}_i \preccurlyeq \ldots$$

and specializations $\pi_i \subseteq \pi_{i+1}$, $\pi_i : \mathrm{M}_i \to \mathrm{M}$ with the following property:

Let $\langle A, \bar{a}, p(\bar{x})\rangle$ be a triple, where $A \subseteq \mathrm{M}_i$ is finite, p is a n-type over $A \cup \mathrm{M}$ and $a \in \mathrm{M}^n$. Suppose that there exists a specialization $\pi : \mathrm{M}' \to \mathrm{M}$ extending π_i and $a' \in \mathrm{Dom}(\pi)$ with $\models p(a')$ and $\pi(a') = a$. Then p is also realized in $\pi_{i+1}^{-1}(a)$.

It then easily follows that the pair $^*\mathrm{M} = \bigcup \mathrm{M}_i$, $\pi = \bigcup \pi_i$ is universal.

To construct the M_i and π_i we start with any specialization $\pi_1 : \mathrm{M}_1 \to \mathrm{M}$ – for example, the identity map on M. Given M_i and π_i, we enumerate all triples

$$\{\langle A_\alpha, a_\alpha, p_\alpha\rangle, \alpha < \kappa\}$$

where as above $A_\alpha \subseteq \mathrm{M}_i$ is finite, $a_\alpha \in \mathrm{M}^n$ and p_α is an n-type over $A_\alpha \cup \mathrm{M}$. We recursively construct $N_{i,\alpha}$ and $\pi_{i,\alpha}$. Let $N_{i,0} = \mathrm{M}_i$, $\pi_{i,0} = \pi_i$. If α is a limit we take unions. At successor steps we consider two cases:

If there exists a specialization $\pi : \mathrm{M}' \to \mathrm{M}$ extending $\pi_{i,\alpha}$ and $a' \in \mathrm{Dom}(\pi)$ with $\models p(a')$ and $\pi(a') = a$, then let $N_{i,\alpha^+} = \mathrm{M}'$ and $\pi_{i,\alpha^+} = \pi$.

Otherwise, let $N_{i,\alpha^+} = N_{i,\alpha}$, $\pi_{i,\alpha^+} = \pi_{i,\alpha}$.

$\mathrm{M}_{i+1} = N_{i,\kappa}$ and $\pi_{i+1} = \pi_{i,\kappa}$ have the desired properties by construction. □

Corollary 2.3.7 *Given any family of sets $\mathcal{F}$, there is a $\mathcal{F}$-universal specialization $\pi : {}^*\mathrm{M} \to \mathrm{M}$.*

Proof. We know that universal specializations exist and a universal specialization is $\mathcal{F}$-universal for any $\mathcal{F}$. □

Definition 2.3.8 Let $\pi\colon {}^*\mathrm{M} \to \mathrm{M}$ be a specialization and $a \in \mathrm{M}^n$. Then we define

$$\mathcal{V}_a = \pi^{-1}(a)$$

$$\mathrm{Nbd}_a(y) = \{\neg Q(y, c') \mid Q \in \mathcal{C} \ \& \ \mathrm{M} \models \neg Q(a, c) \ \& \ c' \in \mathcal{V}_c \ \& \ c \in \mathrm{M}^k \}$$

Since π acts coordinate-wise on tuples, it is clear that $\mathcal{V}_{ab} = \mathcal{V}_a \times \mathcal{V}_b$.

If there is any ambiguity about which specialization we are talking, we write $\mathcal{V}_a^\pi$ and $\mathrm{Nbd}_a^\pi(y)$. If there are specializations $\pi_1, \pi_2, \ldots$, we also write $\mathcal{V}_a^1$, $\mathrm{Nbd}_a^1(y)$ etc.

Lemma 2.3.9 *If $\pi\colon {}^*\mathrm{M} \to \mathrm{M}$ is a specialization and ${}^*\mathrm{M} \models \mathrm{Nbd}_a(a')$, then $\pi \cup \{\langle a', a\rangle\}$ is also a specialization. (If a is a tuple, $\langle a', a\rangle$ is to be read coordinate-wise).*

Proof. From definition of specializations.

Lemma 2.3.10 *Let $\pi\colon {}^*\mathrm{M} \to \mathrm{M}$ be an $\mathcal{F}$-universal specialization. Then*

1. *Given $a' \in {}^*\mathrm{M}^n$ and an $\mathcal{F}$-type $p(a', y)$, if $\mathrm{Nbd}_b(y) \cup \{p(a', y)\}$ is consistent, then there exists $b' \in \mathcal{V}_b$ satisfying $p(a', b')$ and*
2. *$\mathcal{V}_b = \mathrm{Nbd}_b({}^*\mathrm{M})$*

Proof.

1. Suppose the type is consistent. Then we can find a model $\mathrm{M}' \succcurlyeq {}^*\mathrm{M}$ and e realizing $p(a', y) \cup \mathrm{Nbd}_b(y)$. Now the map $\pi' = \pi \cup \{\langle e, b\rangle\}$ is a specialization by the preceding lemma. Since π is $\mathcal{F}$-universal, we can find an $\mathcal{F}$-embedding $\alpha\colon e \to \mathrm{Dom}\,\pi$ such that $\pi' = \pi \circ \alpha$. Let $b' = \alpha(e)$. Then $\pi(b') = b$ and $tp_{\mathcal{F}}(e/\mathrm{M}a') \subseteq tp_{\mathcal{F}}(b/\mathrm{M}a')$. In particular ${}^*\mathrm{M} \models p(a', b')$.
2. Note that by definition of a specialization, if $a' \in \mathcal{V}_a$, then $\models \mathrm{Nbd}_a(a')$.

   To get the reverse inclusion we note that equality is closed and hence in $\mathcal{F}$. If

$\models \mathrm{Nbd}_a(a')$, then clearly the type

$$\mathrm{Nbd}_a(y) \cup \{y = a'\}$$

is consistent, so by the previous part we must have $a' \in \mathcal{V}_a$.

□

Lemma 2.3.11 *Let* $\mathrm{M}_1 \succcurlyeq \mathrm{M}$ *and* $\mathrm{M}_2 \succcurlyeq \mathrm{M}$ *be elementary extensions with* $\mathcal{F}$*-universal specializations* $\pi_1 : \mathrm{M}_1 \to \mathrm{M}$ *and* $\pi_2 : \mathrm{M}_2 \to \mathrm{M}$. *Then given an* $\mathcal{F}$*-type* $p(y)$ *over* M, p *is realized in* $\mathcal{V}_a^{\pi_1}$ *if and only if it is realized in* $\mathcal{V}_a^{\pi_2}$.

Proof. By 2.3.10 it is enough to show that if $\mathrm{Nbd}_a^{\pi_1} \cup p$ is inconsistent, then so is $\mathrm{Nbd}_a^{\pi_2} \cup p$. If the former is the case, then there is a $\neg Q(y, c') \in \mathrm{Nbd}_a^{\pi_1}(y)$ with $c' \in \mathcal{V}_c^{\pi_1}$ and $\mathrm{M} \models \neg Q(a, c)$ and a $F \in p$ such that

$$\mathrm{M}_1 \models F(y) \to Q(y, c')$$

Take any $f \in F(\mathrm{M})$. Then $\mathrm{M}_1 \models Q(f, c')$ and since π_1 is a specialization $\mathrm{M} \models Q(f, c)$. In other words, $\mathrm{M} \models F(y) \subseteq Q(y, c)$. But $\neg Q(y, c) \in \mathrm{Nbd}_a^{\pi_2}$, so $\mathrm{Nbd}_a^{\pi_2} \cup p$ is also inconsistent. □

Lemma 2.3.12 *For an* $\mathcal{F}$*-universal specialization* π *and* $F \in \mathcal{F}$,

$$F \in \mathcal{C}_\pi \quad \Leftrightarrow \quad F = \bigcap_{S \in \mathcal{C} \ \&\ F \subseteq S} S$$

Proof. The right to left inclusion follows from definitions. $\mathcal{C} \subseteq \mathcal{C}_\pi$ and clearly $\mathcal{C}_\pi$ is closed under intersections.

Now suppose for a contradiction $F \in \mathcal{F}$ is π-closed and

$$a \in \Big(\bigcap_{S \in \mathcal{C} \ \& \ F \subseteq S} S \Big) \setminus F$$

Claim: The type $\mathrm{Nbd}_a(y) \cup \{F(y)\}$ is consistent.

Proof: Suppose not. Then $^*\mathrm{M} \models F(y) \to Q(y, c')$ for some $Q \in \mathcal{C}$ and $c' \in \mathcal{V}_c$ such that $\mathrm{M} \models \neg Q(a, c)$. Now take any $f \in F$, then $^*\mathrm{M} \models Q(f, c')$ and since π is a specialization $\mathrm{M} \models Q(f, c)$. Hence $F \subseteq Q(\mathrm{M}, c)$ and this contradicts $\neg Q(a, c)$, proving the claim.

Now by 2.3.10 we can find $a' \in F \cap \mathcal{V}_a$. Since F π-closed, this implies that $a \in F$, resulting in a contradiction. $\square$

From now on we will look at specializations in Strong Geometries so in all the following lemmas the underlying structure M is assumed to be one.

Corollary 2.3.13 *If π is $\mathcal{P}$-universal, then projective sets are closed if and only if they are π-closed.*

Proof. This follows from 2.1.15, since for a projective set P

$$\Big(\bigcap_{S \in \mathcal{C} \ \& \ P \subseteq S} S \Big) = \overline{P}$$

is closed. $\square$

In view of axiom (SB) we should expect points in the boundary of a projective set X to have small combinatorial dimension. Unfortunately, unless $\overline{X} \setminus X$ is actually locally projective at $a \in \overline{X} \setminus X$, the type

$$\mathrm{Nbd}_a(y) \cup \{ \overline{X} \setminus X(y) \} \cup \{ y \notin K \mid \dim K < \dim X \}$$

is consistent so if π is universal, it is realized by some element a' which would be a boundary element of X with $\mathrm{cdim}\,(a') = \dim X$. However, this type is only projective if

$\overline{X} \setminus X$ is. This means that if we only require the specialization to be $\mathcal{P}$-universal, then we do not necessarily have to realize this type. The best possible property we could hope for is the following.

Definition 2.3.14 Given a specialization $\pi\colon {}^*M \to \mathrm{M}$, we say that π *omits generic boundary points* over a set A, if for any projective set X defined over A and any $a \in \overline{X} \cap \operatorname{Dom}\pi$, if $\operatorname{cdim}(a/A) = \dim X$, then $a \in X$.

Proposition 2.3.15 *There exists an elementary extension* ${}^*\mathrm{M}$ *and a* $\mathcal{P}$*-universal specialization* $\pi\colon {}^*\mathrm{M} \to \mathrm{M}$ *which omits generic boundary points over* M.

Proof. A direct proof in the style of Zilber's proof of the existence of universal specializations is possible. This would involve picking up points with the desired type one by one while making sure at each step that we do not include any generic boundary points. We take an approach which makes the essential use of (SB) more apparent. Given a universal specialization, we move all offending points $a \in (\overline{X} \setminus X) \cap \operatorname{Dom}(\pi)$ into X. In doing so we destroy some of the universality, but retain enough to build a chain of these and get full $\mathcal{P}$-universality in the limit.

We define $\mathrm{M}_0 = \mathrm{M}$ and $\pi_0 = \mathrm{id}_{\mathrm{M}}$ and construct M_n and π_n recursively as follows:

Given M_n and π_n, let $N \succcurlyeq \mathrm{M}_n$ and let π_N be a universal specialization $N \to \mathrm{M}_n$.

We now recursively construct sets $Q_0 \subseteq \ldots \subseteq Q_n$.

Let $Q_0 \supseteq \operatorname{tp}_{\mathcal{P}}(\operatorname{Dom}\pi_N/\mathrm{M}_0)$ be a maximal set of projective sets over M_0 under the condition of being finitely satisfiable in N.

Given Q_i, let $Q_{i+1} \supseteq Q_i$ be a maximal set of projective sets over M_{i+1} under the condition of being finitely satisfiable in N.

Finally, let $Q = Q_n$. Let $\mathrm{M}_{n+1} \succcurlyeq N$ model Q.

In effect we have introduced constants c_a for each $a \in \operatorname{Dom}\pi_N$ so we can define $\pi'_{n+1}\colon \mathrm{M}_{n+1} \to \mathrm{M}_n$ by

$$\pi'_{n+1}(b') = b$$

where b' is the interpretation of c_a in M_{n+1} and $\pi_N(a) = b$. Then define $\pi_{n+1} = \pi_n \circ \pi'_{n+1}$.

Since M_n and π_n are ascending chains we can define the limits

$$^*\mathrm{M} = \bigcup \mathrm{M}_n \text{ and } \pi = \bigcup \pi_n$$

Claims:

1. π_{n+1} and π are specializations.
2. π_{n+1} and π omit generic boundary points over M_i for all $i \leq n$.
3. Given a $\mathcal{P}$-type $p(y)$ over a finite subset $A \subseteq \mathrm{M}_n$ which is consistent with $\mathrm{Nbd}_a^{\pi'_{n+1}}(y)$ there is an $a' \in \mathcal{V}_a^{\pi'_{n+1}}$ such that $\mathrm{M}_{n+1} \models p(a')$.
4. π is $\mathcal{P}$-universal.

Proof of claim 1:

The fact that $\pi_n \circ \pi_N$ is a specialization is encoded in $\mathrm{Diag}(\mathrm{Dom}\,\pi_N/\mathrm{M}_0)$. By construction, this implies that $\pi_{n+1} = \pi'_{n+1} \circ \pi_n$ is also a specialization.

Now the π_n form an increasing sequence of maps. Since the definition of specializations only concerns finite tuples, this implies that $\pi = \bigcup \pi_i$ must also be a specialization.

Proof of claim 2:

Suppose $i \leq n$ and X is a projective set defined over M_i. By definition of Q we only have to show that if $a \in (\mathrm{Dom}\,\pi_n)^k$ is a generic boundary point of X over M_i, then $Q_i \cup \{X(a)\}$ is finitely satisfiable in N.

Suppose not. Then there exists a $P(a,b) \in Q_i$ such that

$$N \models \nexists (a,b) \ \ P(a,b) \wedge X(a)$$

Let R be the set defined by $\exists y \ \ P(x,y)$, let V be a locus of a and let $S = \overline{R \cap V}$. Since a is generic in $\overline{X}$, we must have $\dim(R \cap V) = \dim X$. However, $R \cap X = \emptyset$, so by two application of (SB),

$$\dim(S \cap X) < \dim X \text{ and } \dim(S \cap \overline{X}) < \dim X$$

This contradicts the genericity of a in $\overline{X}$. If π had a generic boundary point in its domain this would have to appear at some finite stage in the construction, say at M_n. However, this is impossible since we have shown above that for every $k > n$, π_k omits generic boundary points over M_n.

Proof of claim 3:
Since $\mathrm{Nbd}_a^{\pi'_{n+1}}(y) = \mathrm{Nbd}_a^{\pi_N}(y)$, we can find such an a' in N by universality of π_N. Let b' be the interpretation of $c_{a'}$ in M_{n+1}. Then by construction of π_{n+1},

$$\mathrm{tp}_{\mathcal{P}}(a'/\mathrm{Dom}\,(\pi_N)) \subseteq \mathrm{tp}_{\mathcal{P}}(b'/\mathrm{Dom}\,(\pi_{n+1}))$$

Hence, b' also realizes the required type in M_{n+1}.

Proof of claim 4:
Suppose π' is a finite extension of π by some set A. Let $A_0 = A \cap {}^*\mathrm{M}$ and a be a tuple made up of the elements of $A \setminus A_0$. Now by construction $A_0 \subseteq \mathrm{M}_n$ for some n. By claim 3 we can find an $b \in \pi_{n+1}^{-1}(a)$ such that

$$\mathrm{tp}_{\mathcal{P}}(b/A_0 \cup \mathrm{M}) \supseteq \mathrm{tp}_{\mathcal{P}}(a'/A_0 \cup \mathrm{M})$$

So that $a' \mapsto b$ is the $\mathcal{P}$-embedding required for $\mathcal{P}$-universality. □

Corollary 2.3.16 *There are elementary extensions* $\mathrm{M}' \succcurlyeq {}^*\mathrm{M} \succcurlyeq \mathrm{M}$ *and* $\mathcal{P}$*-universal specializations* $\pi\colon \mathrm{M}' \to {}^*\mathrm{M}$ *and* $\pi^*\colon {}^*\mathrm{M} \to \mathrm{M}$ *which omit generic boundary points over* ${}^*\mathrm{M}$ *and* M *respectively, such that* $\pi \circ \pi^*$ *is also* $\mathcal{P}$*-universal and omits generic boundary points over* M.

Proof. Essentially, we repeat the construction in 2.3.15 twice. We obtain $^*\mathrm{M}$ from the first applicaton. For the second application, we write $\mathrm{M}_0 = \mathrm{M}$, $\mathrm{M}_1 = {}^*\mathrm{M}$ and $\pi^* = \pi_1$. Now we use the construction in 2.3.15 to build a chain

$$\ldots \mathrm{M}_4 \succcurlyeq \mathrm{M}_3 \succcurlyeq \mathrm{M}_2 \succcurlyeq \mathrm{M}_1 \succcurlyeq \mathrm{M}_0$$

with a $\mathcal{P}$-universal specialization $\pi'_n\colon \mathrm{M}_n \to \mathrm{M}_{n-1}$ omitting generic boundary points over all smaller models in the chain. Let $\pi_2 = \pi'_2$ and $\pi_n = \pi_{n-1} \circ \pi'_n$ for $n > 2$. We define

$$\mathrm{M}' = \bigcup \mathrm{M}_i \text{ and } \pi = \bigcup_2^\infty \pi_i$$

Then by exactly the same arguments as claims 1 – 4 of 2.3.15, π and $\pi_1 \circ \pi$ are as required. □

From now on, we will not explicitly mention specializations much. When we write $\mathcal{V}_a$ it is implicit that $\mathcal{V}_a = \pi^{-1}(a)$ for some $\mathcal{P}$-universal specialization omitting generic boundary points. We will also often make use of two specializations as in the above corollary. When we do so, we will assume that there are specializations π_1 and π_2 with $\pi = \pi_1 \circ \pi_2$ such that all three are $\mathcal{P}$-universal and omit generic boundary points. In this case we write $\mathcal{V}^1_a = \pi_1^{-1}(a)$, $\mathcal{V}^2_a = \pi_2^{-1}(a)$ and $\mathcal{V}_a = \pi^{-1}(a)$. Note that $\mathcal{V}_a = \bigcup_{b \in \mathcal{V}^1_a} \mathcal{V}^2_b$. By lemma 2.3.11, all results that we obtain for $\mathcal{V}^1$ or $\mathcal{V}^2$ also hold for $\mathcal{V}$.

When we say something along the lines of "let $\mathcal{V}^1$ and $\mathcal{V}^2$ be the infinitesimal neighbourhoods of two levels of specializations", it is implicit that there are specializations $\pi_1\colon \mathrm{M}_1 \to \mathrm{M}$ and $\pi_2\colon \mathrm{M}_2 \to \mathrm{M}_1$ omitting generic boundary points over M and M_1 respectively and that $\pi = \pi_1 \circ \pi_2$ is also $\mathcal{P}$-universal and omits generic boundary points over M.

Lemma 2.3.17 *Let $a_1, a_2 \in \mathcal{V}_a$. Then a_1 and a_2 have the same projective type over* M *if and only if they have the same quantifier-free type.*

Proof. Suppose a_1 and a_2 have the same quantifier-free type. If P is a projective set such that $a_1 \in P$ and $a_2 \notin P$, let S be a locus of a_1. Then $a_1 \in S \cap \overline{P}$ and $S \cap \overline{P}$ is expressed by a quantifier-free formula, so $a_2 \in S \cap \overline{P}$. This, however, makes a_2 a generic boundary point contradicting our choice of π. $\square$

Lemma 2.3.18 *Let P be a projective set over* M *and $a \in \operatorname{Dom} \pi$ such that $a \notin P$. Then there is a locus L of a such that* $\dim(L \cap P) < \dim L$.

Proof. Let L' be any locus of a. Then

$$\dim(P \cap L') \leq \dim L' = \operatorname{cdim} a$$

Since π omits generic boundary points and $a \notin P$, this means that $a \notin \overline{(P \cap L')}$. Let

$$L = \overline{(L' \setminus \overline{(P \cap L')})} \subseteq L'$$

Then $a \in L$, L is a locus of a and $\dim(L \cap P) < \dim L$ by (SB). $\square$

Lemma 2.3.19 *Given a projective set P,*

$$\dim P = \max\{\operatorname{cdim}(b) \mid b \in P \;\&\; b \in \operatorname{Dom} \pi\}$$

Proof. Using 2.1.13 we may assume that $\overline{P}$ is uniform. Let $a \in P(\mathrm{M})$. Then 2.3.15 tells us that finding $b \in \mathcal{V}_a \cap P$ with $\operatorname{cdim}(b) = \dim P$ is a matter of showing that

$$\mathrm{Nbd}_a(x) \cup \{P(x)\} \cup \{\neg K(x) \mid K \text{ is closed and } \dim(K \cap P) < \dim P\}$$

is consistent. However, if $\neg Q(x, c) \in \mathrm{Nbd}_a(x)$ and K is as above, then by pure dimensionality of $\overline{P}$

$${}^*\mathrm{M} \models \forall y(\forall x(P(x) \to (K(x) \vee Q(x, y)) \leftrightarrow \forall x(\overline{P}(x) \to Q(x, y))$$

Since ${}^*\mathrm{M} \models \neg Q(a,c) \cap \overline{P}(a)$, we can conclude that $P(x) \wedge \neg K(x) \wedge \neg Q(x,c)$ is non-empty. □

Lemma 2.3.20 *Suppose $F(x,y) \subseteq \mathrm{M}^{n+k}$ is a uniform projective set and $(a,b) \in F$ are such that for all generic $a' \in \mathrm{pr}_y(F) \cap \mathcal{V}_a$ there is some $b' \in \mathcal{V}_b$ such that ${}^*\mathrm{M} \models F(a',b')$. Then for all $a' \in \mathrm{pr}_y(F) \cap \mathcal{V}_a$, there is a $b' \in \mathcal{V}_b$ such that ${}^*\mathrm{M} \models F(a',b')$.*

Proof. We will prove this by making use of two-levels of infinitesimals as explained after corollary 2.3.16. Instead of showing the result for π itself, we will show it for π_1.

Let $a' \in \mathrm{pr}_y(F) \cap \mathcal{V}^1_a$. Let $a'' \in \mathcal{V}^2_{a'} \cap \mathrm{pr}_y F$ be generic. Then $a'' \in \mathcal{V}_a \cap \mathrm{pr}_y F$ so by our assumption, 2.3.10 and 2.3.11, this means we can find $b'' \in \mathcal{V}_b$ such that

$$\mathrm{M}' \models F(a'', b'')$$

but now by applying π_2 we get that

$${}^*\mathrm{M} \models F(a', \pi_2(b''))$$

Let $b' = \pi_2(b'')$. Since $\pi_2 \colon \mathcal{V}_b \to \mathcal{V}^1_b$, we are done. □

2.3.1 The Implicit Function Theorem

We could ask whether a result similar to lemma 2.3.19 holds for closed sets definable using parameters from $\mathrm{Dom}\,\pi$:

Given F and $a \in \mathrm{pr}F \cap \mathrm{Dom}\,\pi$, is there some $b \in F(a) \cap \mathrm{Dom}\,\pi$ such that $\dim F(a) = \mathrm{cdim}\,(b/a)$?

Unfortunately, this is not true in general. For example, let us consider the plane $F \subseteq K^3$ defined by $x = y.z$ in some algebraically closed field K. Let ${}^*K$ be an elementary extension with specialization π and let pr be the projection to the x, y plane. This is a one-to-one covering except at the point $(0,0)$. Now given $(a,b) \neq (0,0)$ in the infinitesimal neighbourhood of $(0,0)$, there is a unique c such that $a = b.c$, but if we chose $b = a^2$, then c is not in any infinitesimal neighbourhood. Lemma 2.1.31 indicates that this kind of phenomenon is likely to occur around all irregular fibres, but we will now see that smoothness prevents this.

Proposition 2.3.21 *Given a uniform set $V(x,y) \subseteq \mathrm{M}^{n+k}$ which is smooth with respect to pr_y, a point $a \in \mathrm{pr}_y V$, an arbitrary $b \in V(a)$ and a generic $a' \in \mathcal{V}_a \cap \mathrm{pr}_y V$, then for any finite set $a \subseteq A \subseteq {}^*\mathrm{M}$, there is a point $b' \in \mathcal{V}_b \cap V(a')$ with* $\mathrm{cdim}\,(b'/A) = \dim V(a')$.

Proof. By lemma 2.1.31, we may assume that all fibres over $\mathrm{pr}_y V$ are of dimension $\leq \dim V - \dim \mathrm{pr}_y V$. By $\mathcal{P}$-universality of the specialization, it is enough to show that the following type is consistent:

$$q(y) = \mathrm{Nbd}_b(y) \cup \{\, V(a',y) \,\} \cup \{\, \neg K(d,y) \mid K \text{ is closed } \ \& \ \dim K < \dim V \ \& \ d \in A^l \,\}$$

It is enough consider a single $\neg Q(y,c') \in \mathrm{Nbd}_b$ and a single $K(d,y)$ since these sets are closed under unions and intersections. Define

$$P(x,y,z) = V(x,y) \wedge \neg Q(y,z) \wedge \neg K(d,y)$$

We need to show that $(a',c') \in \mathrm{pr}_y P$. Suppose not. Then by 2.3.18 we can find a locus L of (a',c') such that $\dim L > \dim(L \cap \mathrm{pr}_y P)$.

Both V and L have a' in their projection to M^n so that $\dim(\mathrm{pr}_y(V) \cap \mathrm{pr}_z(L)) = \dim \mathrm{pr}_y(V)$. After truncating L sufficiently, 2.1.29 tells us that

$$Z(x,y,z) = V(x,y) \wedge L(x,z)$$

has no components of dimension less than $\dim L + \dim V(a')$ and is non-empty since $(a,b,c) \in Z$. Consequently the set

$$P(x,y,z) \wedge L(x,z) = (V(x,y) \cap L(x,z)) \cap \neg Q(y,z) \cap \neg K(x,y)$$

also does not have any small components. By the additivity of dimension and our assumption on the dimension of fibres over $\mathrm{pr}_y V$, we get $\dim\{\,(x,z) \mid \exists y P(x,y,z) \wedge L(x,z)\,\} = \dim L$ which contradicts the assumption about the small dimension of $L \cap \mathrm{pr}_y P$. $\square$

Corollary 2.3.22 *(The Implicit Function Theorem) Let $V(x,y) \subseteq \mathrm{M}^{n+k}$ be a pure-dimensional set which is smooth with respect to pr_y and let $a \in \mathrm{pr}_y V$. Then for any $b \in V(a)$ and any $a' \in \mathcal{V}_a \cap \mathrm{pr}_y V$, there is a point $b' \in \mathcal{V}_b \cap V(a')$.*

Proof. This follows from the preceding proposition and 2.3.20 $\square$

On the surface, this version of the Implicit Function Theorem is not really about functions at all. To see why this is indeed a kind of Implicit Function Theorem we need to restrict our attention to discrete coverings.

Definition 2.3.23 Let $F(x,y) \subseteq \mathrm{M}^{n+k}$ be pure-dimensional and suppose that for all $a \in \mathrm{pr}_y(F)$, $\dim(F(a)) = 0$. Define

$$F^n = \left\{ (x, y_1, \ldots, y_n) \;\middle|\; \bigwedge F(x, y_i) \wedge \bigwedge y_i \neq y_j \right\}$$

We say that (a,b) has multiplicity $\geq n$ over a if

$$(a, b, \ldots, b) \in \overline{F^n}$$

and define

$$\mathrm{mult}_{\mathrm{pr}_y}(a,b) = \max\{\, n \mid (a,b) \text{ has multiplicity } \geq n \text{ over } a \,\}$$

if this exists or ∞ otherwise.

Note that in the above definition $\dim F^n = \dim F$ and

$$F^n \cap \{\,(x, y_1, \ldots, y_n) \mid \bigwedge (y_i = y_j)\,\} = \emptyset$$

By (SB) this means that $\dim(\overline{F^n} \cap \{ (x, y_1, \ldots, y_n) \mid \bigwedge(y_i = y_j) \}) < \dim F$. This tells us that the points of multiplicity greater than one are contained in a set of small dimension.

Lemma 2.3.24 *Given a pure-dimensional discrete covering F as in the previous definition,* $\mathrm{mult}_{\mathrm{pr}_y}(a, b) \geq n$ *if and only if there is a $a' \in \mathrm{pr}(F) \cap \mathcal{V}_a$ and distinct $b_1, \ldots, b_n \in \mathcal{V}_b \cap F(a')$.*

Proof. First assume that $\mathrm{mult}_{\mathrm{pr}_y}(a, b) \geq n$. Then by definition

$$(a, b, \ldots, b) \in \overline{F^n}$$

This implies that there is a tuple $(a', b_1, \ldots, b_n) \in F^n \cap \mathcal{V}_{ab\ldots b}$. By definition of F, we can immediately conclude that all of the b_i are distinct.

For the converse, suppose we have $a' \in \mathrm{pr}(F) \cap \mathcal{V}_a$ and distinct $b_1, \ldots, b_n \in \mathcal{V}_b \cap F(a')$. Then $(a', b_1, \ldots, b_n) \in F^n$ and we get $(a, b, \ldots, b) \in \overline{F^n}$ by applying the specialization. $\square$

For the second part of the lemma, it was not necessary that a' be generic, so the following corollary follows.

Corollary 2.3.25 *If* $\mathrm{mult}_{\mathrm{pr}_y}(a, b) = 1$, *then for any* $a' \in \mathcal{V}_a \cap \mathrm{pr}(F)$ $|F(a') \cap \mathcal{V}_b| \leq 1$.

This corollary together with 2.3.22 lets us prove a proposition which looks much more like a form of the Implicit Function Theorem.

Proposition 2.3.26 *Let $F(x, y) \subseteq \mathrm{M}^{n+k}$ be smooth with respect to pr_y and suppose that for all $a \in \mathrm{pr}_y(F)$, $\dim(F(a)) = 0$. If $(a, b) \in F$ has multiplicity 1 over a, then $F \cap \mathcal{V}_{ab}$ is the graph of a function $\mathcal{V}_a \to \mathcal{V}_b$.*

Proof. By the previous lemma, for any $a' \in \mathcal{V}_a \cap \mathrm{pr}_y(F)$ we have $|F(a') \cap \mathcal{V}_b| \leq 1$, but by the Implicit Function Theorem 2.3.22 there is a $b' \in F(a' \cap \mathcal{V}_b)$. Therefore,

$|F(a') \cap \mathcal{V}_b| = 1$. □

In 2.1.31 we saw that irregular fibres are a clear indicator of non-smoothness. We are now in a position to show that such "singularities" always look very similar to the fourth example in Figure 1.

Lemma 2.3.27 *Let $F(x, y) \subseteq \mathrm{M}^{n+k}$ be uniform. If F has an irregular fibre at a and $b \in F(a)$ is generic, then*

$$\{x \in \mathcal{V}_a^2 \mid \exists y \in \mathcal{V}_b \; F(x, y)\} \supsetneq \{x \in \mathcal{V}_a^2 \mid \exists y \in \mathcal{V}_b^2 \; F(x, y)\}$$

Proof. We use a similar idea to the proof of 2.1.31 and will use the same notation. In this lemma, given any set $V(x, y)$ we will write $V^k(x, y_1, \ldots, y_k) = V(x, y_1) \wedge \ldots \wedge V(x, y_k)$.

Let $T = \{(x, y) \in F \mid x \in \mathrm{reg}_{\mathrm{pr}}(F)\}$ and $k = \min\{m \mid F^m(a) \cap \mathcal{V}_{b\ldots b}^1 \nsubseteq \overline{T^m}(a)\}$. Since b is chosen generic there are some $b_i \in \mathcal{V}_b^1$ such that $(a, b_2, \ldots, b_k) \in \overline{T^{k-1}}$, but $(a, b, b_2, \ldots, b_k) \notin F^k \setminus \overline{T^k}$. Consequently, there exists a tuple $(a', b_2', \ldots, b_k') \in \mathcal{V}_{ab_2\ldots b_k}^2$ such that $(a', b_2', \ldots, b_k') \in T^{k-1}$.

However, if $a' \in \{x \mid \exists y \in \mathcal{V}_b^2 \; T(x, y)\}$, then there exists a $b' \in \mathcal{V}_b^2$ such that $(a', b', b_2', \ldots, b_k') \in T^k$. Applying the specialization π_2 would give $(a, b, b_2, \ldots, b_k) \notin \overline{T^k}$ which is a contradiction. Since $b_2' \in \mathcal{V}_b$, we have $a' \in \{x \mid \exists y \in \mathcal{V}_b \; T(x, y)\}$ so

$$a' \in \{x \in \mathcal{V}_a^2 \mid \exists y \in \mathcal{V}_b \; F(x, y)\} \setminus \{x \in \mathcal{V}_a^2 \mid \exists y \in \mathcal{V}_b^2 \; F(x, y)\}$$

□

Lemma 2.3.28 *Suppose $F(x, y) \subseteq \mathrm{M}^{n+k}$ is closed and smooth with respect to pr_y. Then $\mathrm{pr}_y(F)$ is locally closed in the sense that for all $a \in \mathrm{pr}_y(F)$, $\mathcal{V}_a \cap \mathrm{pr}_y(F) = \mathcal{V}_a \cap \overline{\mathrm{pr}(F)}$.*

Proof. Let $a \in \mathrm{pr}_y(F)$ and suppose $(a, b) \in F$ and $c \in \mathcal{V}_a^1 \cap \overline{\mathrm{pr}_y(F)}$. Then there exists a $c' \in \mathcal{V}_c^2 \cap \mathrm{pr}_y(F)$. Since $\mathcal{V}_c^2 \subseteq \mathcal{V}_a$, the Implicit Function Theorem gives us a $b' \in \mathcal{V}_b$

such that $(c', b') \in F$. Thus, $(c, \pi_2(b')) \in F$ and $c \in \mathrm{pr}_y(F)$ as required. $\square$

2.3.2 Families of Branches

We are now in a position to introduce the key objects of study for chapter 3: Families of branches of curves.

First of all, a quick re-cap what the general strategy in chapter 3 will be. The implicit function theorem 2.3.22 implies that if $C \subseteq \mathrm{M}^2$ is a smooth plane curve and $(a, b) \in C$ is a point of multiplicity 1 with respect to both projections, then $\widetilde{C}_{ab} = C \cap \mathcal{V}_{ab}$ is either of the form $\{ a \} \times \mathcal{V}_b$ or $\mathcal{V}_a \times \{ b \}$, or it is a bijection $\mathcal{V}_a \to \mathcal{V}_b$. If C is uniform, then $\widetilde{C}_{ab}$ falls into the last case for either no $(a, b) \in C$ or for almost all $(a, b) \in C$.

To obtain algebraic objects, we will want to compose, invert and "differentiate" these local bijections. Suppose $(a, b) \in C \subseteq \mathrm{M}^2$ and $(b, c) \in D \subseteq \mathrm{M}^2$ are a smooth plane curves such that $\widetilde{C}_{ab}$ and $\widetilde{D}_{bc}$ are bijections. How can we invert and compose these?

Define $C^{-1}(x, y) = C(y, x)$. Then $\widetilde{C^{-1}}_{ba}$ is exactly the inverse function of $\widetilde{C}_{ab}$. The composition would be given by

$$\widetilde{D}_{bc} \circ \widetilde{C}_{ab} = \{ (x, z) \in \mathcal{V}_{ac} \mid \exists y \in \mathcal{V}_b \ (x, y) \in C \ \& \ (y, z) \in D \}$$

Here we have a problem. To compose these functions, we have to quantify over $\mathcal{V}_b$, which takes us out of a first order context. One might hope that instead we could just replace the $\mathcal{V}_b$ with M so that for $\{ (x, z) \in \mathrm{M}^2 \mid \exists y \in \mathrm{M} \ (x, y) \in C \ \& \ (y, z) \in D \}$, $\widetilde{E}_{ac} = \widetilde{D}_{ac} \circ \widetilde{C}_{ab}$, but this is not the case in general. The problem is multiplicity.

Example: Consider the curves

$$C = \{ (x, t) \in \mathbb{C}^2 \mid x = t^2 - 1 \}$$

and

$$D = \{ (t, y) \in \mathbb{C}^2 \mid y = t(t^2 - 1) \}$$

Both of these are smooth and of multiplicity 1 at $(x,t) = (0,1)$ and $(t,y) = (1,0)$. However, composing these naively as described above gives

$$E = \{\, (t^2 - 1, t(t^2 - 1)) \in \mathbb{C}^2 \mid t \in \mathbb{C} \,\}$$

This curve has a singularity at $(0,0)$ as this point is approached from different directions as $t \to 1$ and $t \to -1$.

To overcome this problem, we switch from using curves to using branches of plane curves. All that really means is that we keep the y (or t) variable around as a "dummy" and simply remember that for all intents and purposes it should have been projected out. For curves like the above, we define

$$F(x,z;y) = (D \circ C)(x,y;z) = \{\, (x,z;y) \mid (x,y) \in C \ \&\ (y,z) \in D \,\}$$

and

$$\widetilde{F}_{ac;b} = \{\, (x,z) \in \mathcal{V}_{ac} \mid \exists y \in \mathcal{V}_b \ (x,y) \in C \ \&\ (y,z) \in D \,\}$$

In the case of our example that would give us

$$\widetilde{F}_{00;1} = \{\, (t^2 - 1, t(t^2 - 1)) \in \mathcal{V}_{00} \in C^2 \mid t \in \mathcal{V}_1 \,\}$$

as we would have hoped. This idea can be generalised to families.

Definition 2.3.29 A family of branches of curves is a tuple $\langle S; C; L, I \rangle$ such that $\langle S \times C, L, I \rangle$ is a family of curves. The set C here is optional in the sense that we allow the coordinate tuple of C to have length 0. That allows us to consider families of curves as families of branches of curves directly.

For any $(s;c;l) \in I$ we define

$$\widetilde{I(l)}_{s;c} = \{\, x \in \mathcal{V}_s \mid \exists y \in \mathcal{V}_c \ \&\ (x;y;l) \in I \,\}$$

We will call this the *the c-branch of l at s*. Given $(s;c)$, if for all $l \in L$, $(s;c;l) \in I$, we say that I is *a family of c-branches through s*.

Often the family I will be implicit so that we will write $\widetilde{l}_{s;c}$ instead of $\widetilde{I(l)}_{s;c}$. In the case of families of c-branches through a point s, the parameters s and c will also be implicit so that we will reduce this further to just writing l or $\widetilde{l}$.

Note that we use semi-colons to separate domain, branching and parameter variables here. By re-ordering and splitting the coordinates of $S \times C$ up at different points, a single family of curves can give rise to many different families of branches of curves in different planes. Thus, using semi-colons helps avoid confustion as to which coordinates are to be considered part of the domain and which ones are merely "dummy" variables.

Even though in general $\widetilde{I(l)}_{s;c}$ is not be the restriction of a definable relation to $\mathcal{V}_s$, using the Implicit Function Theorem we can still study branches. Just as we did for families of curves, we could define notions of linarity and dimension for families of branches. However, we will not need this so it is left as an exercise for the reader.

We will usually consider *smooth families of branches of plane curves*. By this we mean a family of branches $\langle C_x \times C_y; Z; L, I\rangle$ where

1. C_x and C_y are one-dimensional sets,
2. I is aligned with respect to its projections to $C_x, C_y, C_x \times C_y, L, C_x \times L$ and $C_y \times L$,
3. I is smooth with respect to its projections to $C_x, C_y, L, C_x \times L$ and $C_y \times L$ and
4. $I \cap \mathrm{pr}_{zl}^{-1}\mathrm{reg}_{\mathrm{pr}_{zl}(I)}$ is smooth with respect to pr_{zl}.

The reason for treating the projection to $C_x \times C_y$ specially in the above definition is that we will want to consider families of c-branches through a point (a, b). Since I has an irregular fibre over (a, b), 2.1.31 implies that the above condition 3 is the best sort of smoothness we could hope for in this case.

As was the case for the curve C in the beginning of this section, given $l \in L$ and $(a, b; c) \in I(l)$ with $\mathrm{mult}_{\mathrm{pr}_{yz}}(I(l)) = \mathrm{mult}_{\mathrm{pr}_{xz}}(I(l)) = 1$, the Implicit Function Theorem implies that the c-branch of l at (a, b),

$$\widetilde{l} = \{ (x, y) \in \mathcal{V}_a \times \mathcal{V}_b \mid \exists c' \in \mathcal{V}_c \ \&\ (x, y; c') \in I(l') \}$$

is either of the form $\{ a \} \times \mathcal{V}_b$, $\mathcal{V}_a \times \{ b \}$ or a bijection $\mathcal{V}_a \to \mathcal{V}_b$. For generic (a, b, l) the case depends on the alignment of I. We are generally interested in families of the third kind so that we can see I as a family of local functions.

Definition 2.3.30 Given two smooth families of branches of plane curves $\langle C_x \times C_y; V; F, I \rangle$ and $\langle C_y \times C_z; W; G, J \rangle$ as above, we define

1. the inverse family I^{-1} by $\langle C_y \times C_x; V; F, I^{-1} \rangle$ where $I^{-1}(y, x; v, f) = I(x, y; v, f)$ and
2. the composition family $J \circ I$ by $\langle C_x \times C_z; V \times C_y \times W; F \times G, J \circ I \rangle$ where

$$J \circ I(x, z; v, y, w; f, g) = I(x, y; v; f) \wedge J(y, z; w; g)$$

Note that in the latter, C_y has become part of the branching coordinates.

When we are talking about branches from I^{-1}, will write f^{-1} for the parameter f to make it clear which family we are parametrizing. Similarly, we will write $g \circ f$ instead of (f, g) to indicate that we are parametrizing a branch in the family $J \circ I$. This is more than just notational convenience. If we take $(a, b; v; f) \in I$ and $(b, c; w; g) \in J$ and pick $f' \in \mathcal{V}_f$ and $g' \in \mathcal{V}_g$, then $(\widetilde{f'}_{ab;v})^{-1} = \widetilde{(f'^{-1})}_{ba;v}$ and $\widetilde{(g' \circ f')}_{ac;vbw} = \widetilde{g'}_{bc;w} \circ \widetilde{f'}_{ab;v}$.

If I is a family of v-branches through (a, b) and J is a family of w-branches through (b, c), it is implied that I^{-1} is a family of v-branches through (b, a) and $J \circ I$ is a family of (v, b, w)-branches through (a, c). In other words, if $f \in F$ and $g \in G$, we write f for $\widetilde{I(f)}_{ab;v}$, g for $\widetilde{J(g)}_{bc;w}$, f^{-1} for $\widetilde{I^{-1}(f)}_{ba;v}$ and $g \circ f$ for $\widetilde{J \circ I}_{ac;vbw}$.

We can play similar games if there is a group structure present around b on C_y.

Definition 2.3.31 Suppose we have two smooth families of branches of plane curves $\langle C_x \times C_y; V; F, I\rangle$ and $\langle C_x \times C_y; W; G, J\rangle$ as above. Furthermore, assume that $\cdot \subseteq C_y^3$ is a relation such that for all $y_1, y_2 \in C_y$, $\dim \cdot(y_1, y_2, C_y) = 0$ and let $e \in C_y$. We define

1. the e-inverse family e/I by $\langle C_x \times C_y; V \times C_y; F, e/I\rangle$ where $e/I(x, y; v, y'; f) = I(x; y', v; f) \wedge \cdot(y, y', e)$ and
2. the $\cdot$-composition family $J \cdot I$ by $\langle C_x \times C_y; V \times C_y^2 \times W; F \times G, J \cdot I\rangle$ where $J \cdot I(x, y; v, y_1, y_2, w; f, g) = I(x, y_1; v; f) \wedge J(x, y_2; w; g) \wedge \cdot(y_1, y_2, y)$.

Again, we will write e/f and $f \cdot g$ for the parameters of these families and omit tildas and subscripts when we are dealing with families of braches through a fixed point. In general we should not expect this to behave too nicely, but let us suppose that $\cdot \cap \mathcal{V}_e^3$ is the graph of a group with identity e. If we fix $(a, e, v) \in I$ and $(a, e, w) \in J$ and work around these points, then $f' \mapsto e/f'$ is just pointwise inversion and $(f', g') \mapsto f' \cdot g'$ is pointwise multiplication by $\cdot$. Such "type-definable" groups will be central in chapter 3.

2.4 Smooth Type-Definable Groups

Definition 2.4.1 Suppose we have a pure-dimensional constructible relation $F \subseteq X^3$ on a pure-dimensional projective set X with a distinguished element $e \in X$. We will say that $\langle X, F\rangle$ is a *group chunk* if the following holds:

1. for all $x \in X$, $(e, x, e), (x, e, e) \in F$,
2. for all $x \in X$ there exists an $x^{-1} \in X$ with $(x, x^{-1}, e), (x^{-1}, x, e) \in F$,
3. for all pure-dimensional closed sets $R \subseteq X^2$ with $(e, e) \in R$,

$$\dim \{ (x, y) \in R \mid \exists z \in X \;\; (x, y, z) \in X \} = \dim R$$

4. there is an open set $W \ni (e,e,e,e,e,e,e) \subseteq X^7$ such that

$$\{ (x,y,z,u,v,w_1,w_2) \in W \mid (x,y,u),(y,z,v),(x,v,w_1),(u,z,w_2) \in F \}$$
$$\subseteq \{ x,y,z,u,v,w_1,w_2 \mid w_1 = w_2 \}$$

5. for any projection $F \to X^2$, all fibres are discrete and (e,e,e) has multiplicity 1.

We say that F is *abelian* if it is symmetric in x and y and say it is *smooth* if the F is smooth with respect to any projection $F \to X^2$.

Definition 2.4.2 Given a pure-dimensional constructible relation $F \subseteq X^3$ on a pure-dimensional projective set X with a distinguished element $e \in X$, we say that $\langle \mathcal{V}_e \cap X, F \rangle$ is a *type-definable group* if $F \cap (\mathcal{V}_e \cap X)^3$ is the graph of a group operation. We say that the group is smooth if F is smooth with respect to all projections to X^2.

Lemma 2.4.3 *Suppose $e \in X$. $\langle \mathcal{V}_e, F \rangle$ is a smooth type-definable group if and only if there is a projective set $Y \subseteq X$ with $Y \cap \mathcal{V}_e = X \cap V_e$ and an open set $U \ni (e,e,e)$ such that for $E = F \cap U$, $\langle E, Y \rangle$ is a smooth group chunk.*

Proof. First suppose that $\langle \mathcal{V}_e, F \rangle$ is a smooth type-definable group. Let $U \ni (e,e,e)$ be an open set such that $F \cap U$ has no irregular fibres or multiplicities with respect to any projection $X^3 \to X^2$. That such a set U exists follows from 2.3.24 and the fact that there are no irregular fibres over $\mathcal{V}_{ee}$. Let

$$Y = \{ x \in X \mid \exists z \in Z \ \ (x,e,x),(e,x,e),(x,z,e),(z,x,e) \in F \cap U \}$$

We claim that $\langle Y, E \rangle$ is indeed a group chunk. It is importamt to note here that $F \cap \mathcal{V}_{eee} = E \cap \mathcal{V}_{eee}$ so that $\langle \mathcal{V}_e \cap Y, E \rangle$ is still a type-definable group. Conditions 1 and 2 and 5 follow from definition of Y. If condition 3 were false, then there would be a pure-dimensional set $R \subseteq Y^2$ with $(e,e) \in R$ and such that $\dim(R \cap \mathrm{pr}_z E) < \dim R$. Existential saturation of $\mathcal{V}_{ee}$ would allow us to find a generic $(a,b) \in R \cap \mathcal{V}_{ee}$. By (SB) we would have $(a,b) \notin \mathrm{pr}_z F$ so that ab is undefined, contradicting the fact that

$\langle \mathcal{V}_e, E\rangle$ is a group. Condition 4 also follows from the existential saturation of infinitesimal neighbourhoods. If it were false, we could find $(a,b,c,r,s,d_1,d_2) \in V_{eeeeeee}$ such that $ab = r, bc = s, as = d_1, rc = d_2$ and $d_1 \neq d_2$. This would contradict the associativity of the multiplication on $\mathcal{V}_e$. The smoothness of the group chunk follows straight from the smoothness of the type-definable group.

Now suppose F is a smooth group chunk on X. Condition 3 ensures that $\mathcal{V}_{ee} \cap Y^2 \subseteq \mathrm{pr}_z F$. Smoothness of F and the IFT 2.3.22 thus ensure that for all $(a,b) \in \mathcal{V}_{ee}$ there exists a $c \in \mathcal{V}_e$ such that $(a,b,c) \in F$. The fact that (e,e,e) has multiplicity one with respect to any projection $X^3 \to X^2$ and 2.3.24 imply that this c is unique so that $F^3 \cap \mathcal{V}_{eee}$ is the graph of a binary operation. Condition 1,2 and 4 translate to the identity, inverse and associativity axioms so that we do indeed get a group. □

We call the group chunk $\langle Y, E\rangle$ constructed in the previous lemma the *group chunk associated with* $\langle \mathcal{V}_e, F\rangle$.

Lemma 2.4.4 *Suppose $\langle X, F\rangle$ is a smooth group chunk and $h \in X$. Let h^{-1} be as in condition 2. Then for all $e' \in \mathcal{V}_e$ there is a $h' \in \mathcal{V}_h$ such that $(h', h^{-1}, e') \in F$ and for all $h' \in \mathcal{V}_h$, $g \in \mathcal{V}_{h^{-1}}$, there is an $e' \in \mathcal{V}_e$ with $(h', g, e') \in F$.*

Proof. This follows straight from the IFT and the smoothness of F. □

Suppose we have two levels of infinitesimals $\mathcal{V}^1$ and $\mathcal{V}^2$. The way to interpret the above lemma is that for $h \in \mathcal{V}^1_e$, $h^{-1}\mathcal{V}^2_h = \mathcal{V}^2_{h^{-1}}\mathcal{V}^2_h = \mathcal{V}_e$.

Lemma 2.4.5 *Let F be a smooth group chunk on a one-dimensional set X and $C = \{(x,y,z) \in X^3 \mid (x,y,z),(y,x,z) \in F\}$. Let D be the two-dimensional part of C given by (DC). If $(e,e,e) \in \overline{D}$, then the type-definable group $\langle \mathcal{V}_e, F\rangle$ is abelian.*

Proof. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two levels of infinitesimals. Under the given conditions there are generic $(r,s) \in D \cap \mathcal{V}^1_{ee}$.

By the genericity of (r,s), $\mathcal{V}_{rs}^2 \subseteq D$ so that for all $(r',s') \in \mathcal{V}_{rs}$, $r's' = s'r'$.

Now let $f, g \in \mathcal{V}_e$. By the previous lemma there are $s' \in \mathcal{V}_s$ and $r' \in \mathcal{V}_r$ such that $r^{-1}r' = f$ and $s^{-1}s' = g$ and therefore we get $fg = (r^{-1}r')(s^{-1}s') = (s^{-1}s')(r^{-1}r') = gf$.

This proves that $\mathcal{V}_e^2$ is abelian, but by existential saturation of infinitesimal neighbourhoods, if there was a witness to the non-abelianity of $\mathcal{V}_e$, then there would also be a witness for the non-abeliantity of $\mathcal{V}_e^2$ so that we can conclude that $\mathcal{V}_e$ is abelian. □

Lemma 2.4.6 *A one-dimensional smooth type-definable group $\langle \mathcal{V}_e, \cdot \rangle$ is either centreless or abelian.*

Proof. Suppose $a \in \mathcal{V}_e \setminus \{e\}$ is in the center. Let $b \in \mathcal{V}_e$ be independent of a so that $\operatorname{cdim}(ab) = 2$. Since $(a,b) \in \mathcal{V}_{ee}$, there is a $c \in \mathcal{V}_e$ such that $(a,b,c), (b,a,c) \in F$. In other words, $(a,b,c) \in D$ where D is the one defined in the previous lemma so that after specializing, $(e,e,e) \in \overline{D}$. Now abelianity follows from the previous lemma. □

If M is a geometric structure in the sense of [BV10] so that cl = acl and zero-dimensional sets are finite, then this leads to a standard theorem:

Proposition 2.4.7 *If* M *is a geometric structure, then one-dimensional smooth type-definable groups of unbounded exponent are abelian.*

Proof. Let $\langle \mathcal{V}_e, \cdot \rangle$ be a smooth one-dimensional type-definable group of unbounded exponent. For $n \in \mathbb{N}$, the formulas $x^n \neq e$ are projective and $\mathrm{Nbd}_e(x) \cup \{x^n \neq e \mid n \in \mathbb{N}\}$ is consistent so by 2.3.11 there exists an $x \in \mathcal{V}_e$ of infinite order. Now all elements from the inifinite set $\{x^n \mid n \in \mathbb{N}\}$ commute with each other. Since M is a geometric structure this forces $\{(x,y) \mid xy = yx\}$ to be two-dimensional, which by 2.4.5 implies that $\mathcal{V}_e$ is abelian. □

This is as far as we can go in this general setting. In Zariski Geometries and o-minimal structures we will show that all smooth one-dimensional type-definable groups are abelian. However, in C-minimal structures, no such result is known so it is not surprising that we have to stop at this point.

Definition 2.4.8 Given a type-definable group $\langle \mathcal{V}_e, \cdot \rangle$ we say that an automorphism σ of $\mathcal{V}_e$ is definable if there is a constructible set $S \subseteq X^2$ such that $S \cap \mathcal{V}_e^2 = \mathrm{graph}(\sigma)$. We say that σ is smooth if S is smooth with respect either projection to X.

Definition 2.4.9 Given a smooth group chunk F on X, we say that Y is a definable sub group chunk if Y is a projective set and $F \cap Y^3$ is a group chunk on Y. We say that Y is *proper* if X and Y are not codense at e: $\overline{X} \cap \mathcal{V}_e \neq \overline{Y} \cap \mathcal{V}_e$.

Suppose we have a smooth group chunk F on X. By 2.4.4, it is clear that a smooth sub group chunk Y of the same dimension defines the same type-definable group and is therefore not proper. However, later on we will be interested in subgroups $Y = \{ x \in \mathcal{V}_e \mid \sigma(x) = \tau(x) \}$ where σ and τ are definable automorphisms defined using parameters from $\mathcal{V}_e^1$. The argument derived from 2.4.4 only tells us that $Y \cap \mathcal{V}_{ee} = X \cap V_e^2$. In order to show that $Y \cap V_e = X \cap V_e$, we use the following stronger result.

Lemma 2.4.10 *Given a smooth group chunk F on X and a smooth sub group chunk $Y \subseteq X$ of the same dimension, there is an open set $W \ni e$ such that $Y \cap W$ is clopen in $X \cap W$.*

Proof. Let U and V be the open sets from the definition of a group chunk (w.l.o.g. for both X and Y). Let $W = U(e) \cap V \cap \mathrm{int}\{ x \mid (x, x^{-1}) \in U \}$. If $y \in Y \cap W$, then by 2.4.4, $\mathcal{V}_y = y\mathcal{V}_e \subseteq X$ and thus $Y \cap W$ is open in $X \cap W$. Now suppose $y \in \overline{Y} \cap W$. Then there exists a $y' \in Y \cap \mathcal{V}_y$. By 2.4.4, $y^{-1}y' = e' \in \mathcal{V}_e \subseteq Y$ and thus $y = y'e'^{-1} \in Y$. Hence $Y \cap W$ is closed in $X \cap W$.$\square$

Chapter 3

Presmooth Geometries

3.1 Presmoothness and One-Dimensionality Axioms

In this section we introduce two axioms which will allow us to define a notion of one-sided tangency between branches of curves and to show that this notion is generically transitive. We start by generalizing the presmoothness axiom for Zariski Geometries.

Definition 3.1.1 We call a set $S \subseteq \mathrm{M}^n$ *presmooth* if whenever $T(x, y) \subseteq S \times \mathrm{M}^k$ is a pure-dimensional closed set which is aligned with respect to pr_y and such that $\dim \mathrm{pr}(T) = \dim S$, then we can decompose T into closed sets T_i and find closed sets $K_i \subseteq S$ of dimension at most $\dim S - 2$ such that $T_i \setminus (K_i \times \mathrm{M}^k)$ is smooth with respect to pr_y. Furthermore, if $\dim T = \dim S$, we require $K_i \subseteq \{ a \in S \mid \dim T_i(a) > 0 \}$.

Note that by 2.3.27 we already have $K_i \supseteq \{a \in S | \dim T_i(a) > 0\}$ and so in the case $\dim T = \dim S$, we have equality.

To see an example of a set which is not presmooth, let us go back to the third set of Figure 1:

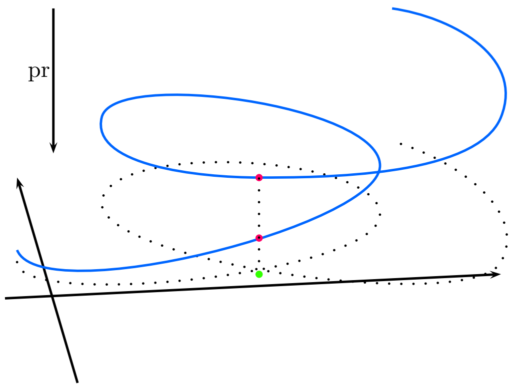

Figure: A set, which is not presmooth.

The points which need to be removed to make this set smooth are indicated with red dots. As indicared with the green dot, they have co-dimension 1 in the projection and thus, the projection is not presmooth. However, this is clearly a singularity in the projection and not in the curve itself. Once we remove the offending point from the projection it is locally isomorphic to $\mathbb{R}$ which is presmooth as we will see in chapter 4. The situation in a general Strong Geometry is not so different. The Implicit Function Theorem and (GS) tell us that all sets are almost everywhere locally isomorphic to some power of M so as long as M^n is presmooth, we should expect all sets to be presmooth almost everywhere. We prove this in the following lemmas.

Lemma 3.1.2 *If D is presmooth and $D' \subseteq D$ is of the same dimension, then D' is presmooth.*

Proof. Any witness to non-presmoothness of D' would also witness non-presmoothness of D. $\square$

Lemma 3.1.3 *Let $D \subseteq \mathrm{M}^n$ and $F(x, y) \subseteq D \times \mathrm{M}^k$ be a discrete covering of D. Let K be the set of points of multiplicity greater than 1 in F over D. Suppose $S \subseteq F \times \mathrm{M}^m$ is smooth with respect to the projection to D. Then $S \setminus (K \times \mathrm{M}^m)$ is smooth with respect to the projection to F.*

Proof. Suppose not. Then this failure is witnessed by some pure-dimensional aligned set $T \subseteq F \times \mathrm{M}^r$. Now if T is aligned with respect to $\mathrm{M}^{n+k+r} \to F$, then it is also aligned with respect to $\mathrm{M}^{n+k+r} \to D$ so by smoothness of S, the set

$$W = \{ (x, y_1, y_2, u, v) \mid (x, y_1, u) \in S \ \& \ (x, y_2, v) \in T \}$$

has no components of dimension less than $\dim S + \dim T - \dim D$. The set

$$Q = \{ (x, y, u, v) \mid (x, y, u) \in S \ \& \ (x, y, v) \in T \}$$

is homeomorphic to $W' = \{ (x, y_1, y_2, u, v) \in W \mid y_1 = y_2 \}$. Suppose Q has a component of small dimension. Then so does W'. Let $C \subseteq W'$ be a component of dimension less than $\dim S + \dim T - \dim D$ and $(a, b, b, c, d) \in C \setminus \overline{(W' \setminus C)}$. Now $(a, b, b, c, d) \in \overline{(W \setminus W')}$, but by the definition of multiplicity, this implies $(a, b) \in F$ has multiplicity at least 2 over D. Thus, non-smoothness only occurs above points with multiplicity greater than 1, as required. □

Corollary 3.1.4 *Let $D \subseteq \mathrm{M}^n$ be presmooth, let $F \subseteq D \times \mathrm{M}^k$ be a discrete covering of D and let $K \subseteq F$ be the set of points of multiplicity greater than* 1*. Then $F \setminus K$ is presmooth.*

Proof. By the previous lemma, any decomposition into smooth sets over D also gives a decomposition into smooth sets over $F \setminus K$. □

Corollary 3.1.5 *If M^n is presmooth for all n and S is uniform, then there is a closed subset $K_S \subseteq S$ of dimension at most* $\dim S - 1$ *such that $S \setminus K_S$ is presmooth. Furthermore, for T another uniform set and K_T constructed in the same way, the set $(S \setminus K_S) \times (T \setminus K_T)$ is presmooth.*

Proof. Let $k = \dim S$. Then we may assume that $S(x, y) \subseteq \mathrm{M}^{k+r}$ is such that $\dim \mathrm{pr}_y S = \dim S = k$. By uniformity, S is aligned with respect to pr_y so by (GS) we can remove a closed set K' of dimension at most $\dim S - 1$, which leaves $S \setminus K'$

smooth with respect to pr_y. By 2.3.27, the set K contains all large fibres over M^k, so we are left with a discrete covering. Now by the previous lemmas and presmoothness of M^k, it is enough to remove points of multiplicity greater than 1 which are also contained in a set of dimension at most $\dim S - 1$.

By the Implicit Function Theorem, the set $S \setminus K_S$ obtained in this way is locally isomorphic to its projection to M^k. For T and K_T obtained in the same way, $T \setminus K_T$ would be locally isomorphic to its projection to M^m for some m. Now $(S \setminus K_S) \times (T \setminus K_T)$ would be locally isomorphic to its projection to M^{k+m} and the projection would have dimension $k+m$. Presmoothness of $(S \setminus K_S) \times (T \setminus K_T)$ follows from 3.1.2 and 3.1.4. $\square$

With these results in mind, it is enough to give a presmoothness axiom very similar to (sPS) in [Zil10]:

(PS) M^n is presmooth for all n.

From now on, when we say "S is presmooth", we will mean that S of the form $S \setminus K_S$ given in the above lemma so that we can assume that products of presmooth sets are presmooth.

The next issue we need to address is our concept of one-dimensionality. We will now introduce an assumption that locally limits the the number of generic 2-types and allows us to obtain a good notion of direction.

(nT) At most **n** **T**ypes

Let $D \subseteq \mathrm{M}^2$ be the diagonal $x_1 = x_2$. Then for any pure-dimensional sets $S_1, \ldots, S_{n+1} \subseteq \mathrm{M}^2$ of dimension 2 such that for all i, j, $\dim S_i \cap S_j \cap D = 1$,

$$\dim(S_r \cap S_s) = 2 \text{ for some } r, s$$

(SL) **S**table - **L**ike

Whenever $S, T \in \mathrm{M}^n$ are pure-dimensional n-dimensional sets, either $\dim(S \cap T) = n$ or $S \cap T = \emptyset$.

We will call a one-dimensional Strong Geometry satisfying (PS) and (nT) for some n a *Presmooth Geometry* and a Presmooth Geometry satisfying (SL) a *Stable-Like Presmooth Geometry*.

Lemma 3.1.6 *Suppose* M *satisfies (nT) and we have a generic singleton* $a \in {}^*\mathrm{M} \succcurlyeq \mathrm{M}$. *Then there are* M*-closed sets* $S_1, \dots S_r \subseteq \mathrm{M}^2$ *for some* $r \leq n$ *such that*

1. *for all* i, S_i *is pure-dimensional of dimension* 2,
2. *for all* i, $(a,a) \in S_i$,
3. $(a,a) \in \operatorname{int}(\bigcup S_i)$,
4. *for* $i \neq j$ $\dim(S_i \cap S_j) < 2$,
5. *for all* 2*-dimensional pure-dimensional* M*-closed* $S \subseteq \mathrm{M}^2$ *with* $(a,a) \in S$, *there is an open set* $U \ni (a,a)$ *such that for all* i,

$$\dim(S \cap U \cap S_i) = 2 \Rightarrow S_i \cap U \subseteq S \cap U$$

Proof. Suppose not. We are going to recursively construct M-closed sets $S_1^i, \dots S_i^i$ satisfying properties 1 - 4. $S_1^{n+1}, \dots, S_{n+1}^{n+1}$ will then contradict (nT).

We start with $S_1^1 = \mathrm{M}^2$. Now suppose we have $S_1^k, \dots, S_k^k$ satisfying 1–4. By assumption, if $k \leq n$, then they do not satisfy 5, so there is a closed set $S \subseteq \mathrm{M}^2$ pure-dimensional of dimension 2 such that $(a,a) \in S$ and for some i and all open sets $U \ni (a,a)$

$$\dim(S \cap U \cap S_i^k) = 2, \text{ but } S_i^k \cap U \not\subseteq S \cap U$$

In particular, we can find a component C of $S \cap S_i^k$ which is pure-dimensional of dimension 2 and containds (a,a). We let $S_j^{k+1} = S_j^k$ for $j \notin \{i, k+1\}$, $S_i^{k+1} = C$ and $S_{k+1}^{k+1} = \overline{S_i^k \setminus C}$. □

Lemma 3.1.7 *For the S_i and a from the previous lemma, property 5 holds in the following stronger form:*

For all 2-dimensional pure-dimensional M*-closed $S \subseteq \mathrm{M}^{2+m}$ and $l \in {}^*\mathrm{M}^m$ independent of a, if $(a,a) \in S(l)$, then there is a an* M*-open set $U \ni (a,a,l)$ such that for all i*

$$\dim(S(l) \cap U(l) \cap S_i) = 2 \Rightarrow S_i \cap U(l) \subseteq S(l) \cap U(l)$$

Proof. If not, then we could use exactly the same construction as in the previous lemma. The existence of an l contradicting this lemma can be expressed in a first-order sentence, so if it exists in *M, it would also exist in M and we would get a contradiction to (nT). It is easy to check that these sets satisfy properties 1–4.□

It is unclear whether there are any interesting geometries not satisfying (1T) or (2T) but satisfying (nT) for some higher n. In some sense 3.1.6 gives us a notion of direction. It should be fairly clear that o-minimal structures satisfy axiom (2T). There are two types around the diagonal at a point (a,a) which correspond to the two directions we can approach a from: $x < a$ and $x > a$. In the same way, we can consider the n types around (a,a) given by (nT) as the different directions that we can approach the point a from. This idea will help us make sense of a "one-sided" derivative for Presmooth Geometries, just like differentiating from one or the other direction in $\mathbb{R}$ would give us the left or right derivative.

Given that $\mathbb{R}$ satisfies (2T), the axiom (nT) is hard to illustrate, but we can imagine that every point in a Presmooth Geometry satisfying (3T) would look like a node in a graph with three edges at every node.

Definition 3.1.8 Given a generic point $a \in \mathrm{M}$, we call the sets $D_i(x) = S_i(a,x)$ the *directions* at a.

Note that this is only well-defined locally. There is a choice of sets S_i here and we can only guarantee that they match up in a sufficiently small neighbourhood around a. On the other hand, the local nature of directions allows us to lift them. We know that a one-dimensional curve C is generically locally isomorphic to M. So given a generic $a \in C$, we can lift the directions via a suitable projection to M and thus extend the concept of directions to all one-dimensional sets.

(SL) says that there is locally only one generic (projective) type in M^n, which is what we would expect to happen in a stable structure. This clearly holds in Zariski Geometries since M^n is irreducible.

Lemma 3.1.9 *Let $a_1, \ldots, a_n \in \mathrm{M}$ and suppose* M *satisfies (SL). Then all generic elements in $\mathcal{V}_{a_1 \ldots a_n}$ have the same projective type over* M.

Proof. If not, then there would have to be two disjoint projective sets P and Q, both of dimension n, with $(a_1, \ldots, a_n) \in \overline{P} \cap \overline{Q}$. Then $\overline{P}$ and $\overline{Q}$ satisfy the assumption of (SL), so they have an n-dimensional intersection. However, this contradicts (SB). □

Indeed, since all sets are locally homeomorphic to M^n at generic points, this is true for all sets. If $a_1 \in S_1, \ldots, a_n \in S_n$ are (individually) generic, then all elements in $S_1 \times \ldots \times S_n \cap \mathcal{V}_{a_1 \ldots a_n}$ have the same projective type.

Presmoothness (PS) is preserved under elementary extensions, but the same may not hold for the axioms (SL) or (nT). To get around that problem, we introduce some slightly stronger axioms:

(UnT) **U**niform (nT)

> There are sets $T_1, \ldots, T_n \subseteq \mathrm{M}^2$ such that for all $a \in \mathrm{M}$, $T_1(a), \ldots, T_n(a)$ are the directions at a.

(USL) **U**niform (SL)

> M is irreducible in the sense that M is the only one-dimensional closed subset of M.

Proposition 3.1.10 *(PS) is preserved under elementary extensions.*

Proof. Let $^*\mathrm{M} \succcurlyeq \mathrm{M}$. We showed in section 2.2.1 that if $S(a)$ is a canonical presentation of a uniform set and that if S is smooth with respect to a projection pr, then so is $S(a)$. Since all projective sets in $^*\mathrm{M}$ have a canonical projection, this immediately implies that $^*\mathrm{M}^n$ is presmooth for all n. $\square$

The key observation is that we don't lose anything by assuming (USL) or (UnT) instead of (SL) or (nT).

Proposition 3.1.11 *If* M *is a non-linear Presmooth Geometry satisfying (SL) or (nT), then there is a non-linear Presmooth Geometry satisfying (USL) or (UnT) definable in* M.

Proof. Let $I \in \mathrm{M}^4$ be the incidence relation of a non-linear family of plane curves. Let $K = \mathrm{pr}_y(I^{-1} \circ I)$ be defined by

$$K(x_1, x_2, l_1, m_1, l_2, m_2) = \exists y \; I(x_1, y, l_1, m_1) \wedge I(x_2, y, l_2, m_2)$$

After possibly removing a set of small dimension this is again a smooth non-linear family of plane curves. Let J be defined by

$$\begin{aligned} J(x, y, u_1, v_1, u_2, v_2) = \exists(w_1, z_1, w_2, z_2) \; & K(x, y, w_1, z_1, w_2, z_2) \\ & \wedge K(u_1, v_1, w_1, z_1, w_2, z_2) \\ & \wedge K(u_2, v_2, w_1, z_1, w_2, z_2) \end{aligned}$$

J expresses that $(x_1, x_2), (u_1, v_1)$ and (u_2, v_2) are collinear. It is easy to check that after removing a set of small dimension, J is also a non-linear family of plane curves, now parametrized by (u_1, v_1, u_2, v_2).

Note that J is very symmetric and that if

$$D = \{ (x_1, x_2, u_1, v_1, u_2, v_2) \mid (x_1 = x_2 = u_1 = v_1 = u_2 = v_2 \}$$

then $\dim(\overline{J} \cap D) = 1$. Let R' be the projection of $\overline{J} \cap D$ to M and $R = \operatorname{int} R'$.

In the case that M satisfies (nT), let $a \in R$ and let $S_1, \ldots S_n$ be the locally irreducible sets around (a, a) given by 3.1.6. Let E' be the projection of $S_1 \cap \ldots \cap S_n$ to M and $E = \operatorname{int} E$. Let $N = R \cap E$.

In the case that M satisfies (SL) simply let $N = R$.

Now the claim is that N with the induced structure is a non-linear Presmooth Geometry satisfying (UnT) or (USL) respectively. The axioms of a presmooth geometry are easily checked. N is non-linear since we have a non-linear family of plane curves given by the restriction of J. The fact that the uniform axioms hold follows from the construction. $\square$

In light of this proposition, we may as well assume that M satisfies the uniform versions of the respecive axioms so for the rest of this chapter we will assume that M is a Presmooth Geometry satisfying (UnT) or (USL). The advantage of this is that the uniform versions are preserved under elementary extensions so that we can also assume that all elementary extensions are Presmooth Geometries.

3.1.1 Type-Definable Groups in Presmooth Geometries

Lemma 3.1.12 *Suppose C is presmooth and $F \subseteq C \times D$ is a closed set such that $F \cap \mathcal{V}_{ab}$ defines a function $\mathcal{V}_a \to \mathcal{V}_b$ for some $(a, b) \in C \times D$. Then there is an open set $U \ni (a, b)$ such that $F \cap U$ is smooth with respect to the projection $F \to C$.*

Proof. By definability of dimension, we can assume that all fibres over C are are discrete so presmoothness guarantees that we can decompose F into smooth components $F_1, \ldots, F_n$. By discarding all other components, we can assume that $(a, b) \in F_i$ for all i. We now truncate the F_i by taking

$$E_i = F_i \setminus \{ (x, y) \mid x \in \operatorname{pr}(F_j) \text{ and } (x, y) \notin F_j \text{ for some } j \neq i \}$$

Since $F \cap \mathcal{V}_{ab}$ is a function, the components match up nicely in $\mathcal{V}_{ab}$ and so $E_i \cap \mathcal{V}_{ab} = F_i \cap \mathcal{V}_{ab}$. In particular, there is an open set U such that for $E = \bigcup E_i$, $E \cap U = F \cap U$. By 2.1.30 and construction, E is smooth, as required. □

Corollary 3.1.13 *If $\langle \mathcal{V}_e, \cdot \rangle$ is a type-definable group on presmooth set and $\cdot$ is given by a closed relation, then $\mathcal{V}_e$ is smooth.*

Corollary 3.1.14 *If a definable automorphism σ of type-definable group on a presmooth set is given by a closed relation, then it is smooth.*

3.1.2 Tangency

We are now in a position to define a notion of one-sided tangency for branches of curves through a fixed point and will make use of a construction very similar to Zilber's in [Zil10].

Definition 3.1.15 Let $\langle S; C_1; L_1, I_1 \rangle$ be a smooth family of c_1-branches through s and let $\langle S; C_2; L_2, I_2 \rangle$ be a smooth family of c_2-branches through s. Define

$$J' = \{ (x, z_1, z_2, u_1, u_2) \mid (x; z_1; u_1) \in I_1 \ \& \ (x; z_2; u_2) \in I_2 \ \& \ x \neq s \}$$

We say that l_1 is tangent to l_2 and simply write $l_1 \parallel l_2$ if and only if

$$(s; c_1, c_2; l_1, l_2) \in J = \overline{J'}$$

Note that J is pure-dimensional of dimension $\dim I_1 + \dim I_2 - \dim S$ and by (SB), the relation $\parallel$ is at most $\dim I_1 + \dim I_2 - \dim S - 1$ dimensional. This definition illustrates nicely that the relation $\parallel$ is definable. For any other purpose we should think of it slightly differently. Remember that for $l_1 \in L_1$ and $l_2 \in L_2$, l_1 represents the branch $\widetilde{I_1(l_1)}_{s;c_1}$ and l_2 represents $\widetilde{I_2(l_2)}_{s;c_2}$. The above definition is equivalent to saying that l_1 and l_2

are tangent if there are $l_1' \in \mathcal{V}_{l_1}$ and $l_2' \in \mathcal{V}_{l_2}$ such that the branches represented by l_1' and l_2' have a second intersection point $s' \in \mathcal{V}_s$.

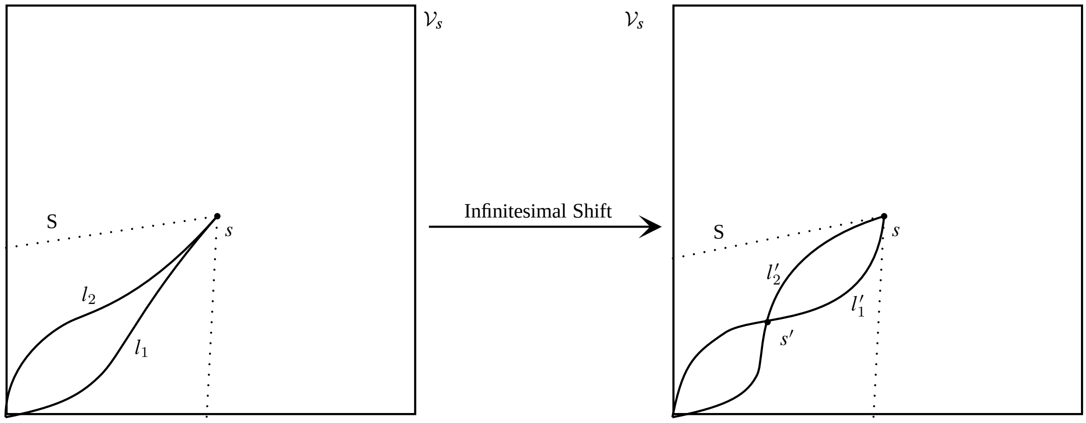

Figure: Witnessing tangency for a family through a fixed s

The above picture illustrates this definition in terms of infinitesumals. Note in particular how the point s is shown to lie on the boundary of S. Symmetry and reflexivity of $\parallel$ follow easily from the definition. Transitivity, however, is tricky and in order to get anywhere, we have to restrict S to one direction. From now on we will assume that our family is non-trivial and each curve is locally parametrized by the first coordinate of s. We will assume $S(x,y) = C_x(x) \times T(y)$ where C_x is a one-dimensional presmooth curve and write $s = (a,b) \subseteq S(x,y)$.

Lemma 3.1.16 *For $i = 1,2$, let $\langle S; C_i; L_i, I_i \rangle$ be smooth families of c_i-branches through (a,b), where L_i is presmooth. Let D be one of the directions at a and suppose $\mathrm{pr}_y(S) \subseteq D$. Then for generic $l_i \in L_i$ the following are equivalent:*

1. $l_1 \parallel l_2$

2. *For all $l_1' \in \mathcal{V}_{l_1} \cap L_1$ and $a_1' \in \mathcal{V}_a \cap D$ there are $(l_2', c_1', c_2', b') \in \mathcal{V}_{l_2 c_1 c_2 b}$ such that*

$$(a', b'; c_1', c_2'; l_1', l_2') \in J$$

3. *For all $l_2' \in \mathcal{V}_{l_2} \cap L_2$ and $a_1' \in \mathcal{V}_a \cap D$ there are $(l_1', c_1', c_2', b') \in \mathcal{V}_{l_1 c_1 c_2 b}$ such that*

$$(a', b'; c_1', c_2'; l_1', l_2') \in J$$

Proof. The directions (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are very straightforward from the definition of specializations and the fact that J is a closed set. Conditions (2) and (3) are strictly stronger than the characterization of tangency given above the diagram as they allow us to pick a' and one of the l_i'.

(1) $\Rightarrow$ (2) Let C be the component of J witnessing the tangency. By 2.1.35 we know that the projection of C to $C_x \times L$ is pure-dimensional of dimension $\dim L + 1$. It also contains (a, l_1) and is contained in $D \times L_1$ so by 3.1.7 it must locally be equal to $D \times L_1$. Furthermore, (a, l_1) has co-dimension 1 in $D \times L_1$ so by presmoothness of C_x and L_1, we can assume that C is smooth at (a, l_1) and find the required b', c_1', c_2', l_2' using the Implicit Function Theorem 2.3.22.

(1) $\Rightarrow$ (3) follows the same way. $\square$

Corollary 3.1.17 *For families as in the preceeding lemma the tangency relation $\|$ is transitive on generics.*

Proof. Suppose $f \| g$ and $g \| h$ where g is generic. Pick $a' \in \mathcal{V}_a$. By the above lemma, there exist $f' \in \mathcal{V}_f$ and $h' \in \mathcal{V}_h$ such that $f'(a') = g(a') = h'(a')$ and therefore f and h are tangent. $\square$

3.2 Trichotomy Considerations

3.2.1 Actions

In the following sections we will try to use the tangency relation we just defined to recover groups and group actions. In order to give the reader some idea how this is done and in order to break the proofs up into managable pieces, we will first investigate the properties $\|$ and relations of the kind $F(f, g, h) = f \circ g \| h$ need to have in order to define algebraic structures.

Definition 3.2.1 Suppose we have sets Y and X, equivalence relations E_X and E_Y on X and Y respectively and a relation $F \subseteq X \times Y \times X$. We define the following properties for F:

(Closure) For all $x \in X$ and $y \in Y$ there exists an $x' \in X$ such that $(x, y, x') \in F$.

(Congruence) For all $x, x', z, z' \in X$ and all $y, y' \in Y$, if $yE_Y y'$, $xE_X x'$ and (x, y, z) and $(x', y', z') \in F$, then $zE_X z'$.

(Faithfulness) For all $x, x', z, z' \in X$ and all $y, y' \in Y$, if $(x, y, z), (x, y', z') \in F$ and $zE_X z'$, then $yE_Y y'$ and if $(x, y, z), (x', y, z') \in F$ and $zE_X z'$, then $xE_X x'$.

If F satisfies the closure and congruence conditions, we say that F is an *action of Y on X with respect to E_X and E_Y*. We say that the action is definable (or projective or constructible), if F is the restriction of such a relation to $X \times Y \times X$). If F satisfies faithfulness, we call F a *faithful action*.

We will write $\widetilde{X}$ for X/E_X, $\widetilde{x}$ for x/E_X, etc. It should be clear that closure and congruence conditions ensure that F induces a map $\widetilde{F}\colon \widetilde{X} \times \widetilde{Y} \to \widetilde{X}$. For fixed $\widetilde{x}$ and $\widetilde{y}$ we write $\widetilde{F}_{\widetilde{x}}$ for the map $\widetilde{F}_{\widetilde{x}}(\widetilde{y}) = \widetilde{F}(\widetilde{x}, \widetilde{y})$ and $\widetilde{F}_{\widetilde{y}}$ for the map $\widetilde{F}_{\widetilde{y}}(\widetilde{x}) = \widetilde{F}(\widetilde{x}, \widetilde{y})$. If F is faithful, then for any $\widetilde{x}$ and $\widetilde{y}$, the maps $\widetilde{F}_{\widetilde{x}}$ and $\widetilde{F}_{\widetilde{y}}$ are injections.

We say that the action is *surjective* or *transitive* if $\widetilde{F}$ has these properties.

Definition 3.2.2 Suppose we have sets X, Y and Z with equivalence relations E_X, E_Y and E_Z and actions F of Y on X and G of Z on Y with respect to the corresponding equivalence relations. We say that F and G are *composable* if for all $x, x' \in X$, all $y, y' \in Y$ and all $z \in Z$,

$$\widetilde{F}(\widetilde{x}, \widetilde{y}) = \widetilde{F}(\widetilde{x}', \widetilde{y}') \Rightarrow \widetilde{F}(\widetilde{x}, \widetilde{G}(\widetilde{y}, \widetilde{z})) = F(\widetilde{x}', \widetilde{G}(\widetilde{y}', \widetilde{z}))$$

Lemma 3.2.3 *In the setup of the above definition, if F and G are composable and F is surjective, then there is a unique action H of Z on X satisfying that for all $x \in X$, $y \in Y$ and $z \in Z$, $\widetilde{F}(\widetilde{x}, \widetilde{G}(\widetilde{y}, \widetilde{z})) = \widetilde{H}(\widetilde{F}(\widetilde{x}, \widetilde{y}), \widetilde{z})$. If F and G are faithful, then so is H. If F and G are transitive, then so is H.*

Proof. We define the map $\widetilde{H}: \widetilde{X} \times \widetilde{Z} \to \widetilde{X}$. To obtain H, we simply take the pre-image of $\widetilde{H}$ under the quotient maps. Define $\widetilde{H}(\widetilde{x}, \widetilde{z}) = \widetilde{F}(x', \widetilde{G}(\widetilde{y}, \widetilde{z}))$ where $y \in Y$ and $\widetilde{x} = \widetilde{F}(\widetilde{x}', \widetilde{y})$. By surjectivity of $\widetilde{F}$, we can find such x' and y. Compatibility of F and G says exactly that this is well-defined. Also by definition of H,

$$\widetilde{H}(\widetilde{F}(\widetilde{x}, \widetilde{y}), \widetilde{z}) = \widetilde{F}(\widetilde{x}, \widetilde{G}(\widetilde{y}, \widetilde{z}))$$

which gives us the required property. Uniqueness follows from construction. If both F and G are faithful, then injectivity of $\widetilde{H}_{\widetilde{x}}$ and $\widetilde{H}_{\widetilde{z}}$ follow straight from the definitions and similarly for transitivity. □

Definition 3.2.4 If $Y = Z$ in the above lemma and $G = H$, then we say that F and G are *associative* since for all x, y_1, y_2, $\widetilde{F}(\widetilde{x}, \widetilde{G}(\widetilde{y_1}, \widetilde{y_2})) = \widetilde{F}(\widetilde{F}(\widetilde{x}, \widetilde{y_1}), \widetilde{y_2})$. If $X = Y = Z$ and $F = G = H$, we simply say that F is *associative*.

Example 3.2.5 *Let us consider some algebraic binary relations in $\mathbb{Z}$.*

Addition and multiplication are composable and associative for obvious reasons.

Subtraction is composable with itself, but not associative. If F and G are subtraction, then $H(x, z) = F(x', G(y, z)) = x' - (y - z) = x + z$.

Exponentiation is not composable with itself. $x^y = u^v$ does not entail $x^{(y^z)} = u^{(v^z)}$.

Lemma 3.2.6 *Suppose we have a set X with an equivalence relation E_X and an action $\circ$ of X on X with respect to E_X. If $\circ$ is composable and associative with itself, then $\cdot = \widetilde{\circ}$ gives semigroup structure to $\widetilde{X}$. If $\circ$ is faithful, then this semigroup is cancellative. Furthermore, if $\circ$ is transitive, then given a fixed a, the map $a \cdot x = y$ is a bijection on $\widetilde{X}$. If in addition $\widetilde{X}$ has an identity, then $\langle \widetilde{X}, \cdot \rangle$ is a group.*

Proof. $\cdot$ is a binary operation by surjectivity and definition of an action. Associativity follows from 3.2.3 and the cancellation property follows from injectivity of the maps $a \cdot x \mapsto x$ and $x \cdot a \mapsto x$. Transitivity means that for all a, b there is a c such that

$a \cdot c = b$ as required. If there is an identity e, then taking $b = e$ ensures the existence of inverses. $\square$

Corollary 3.2.7 *Suppose X, E_X and $\circ$ are as in the previous lemma so that $\langle \widetilde{X}, \cdot \rangle$ is a group and G is an action on Y with respect to E_X and E_Y. If F and G are composable and associative, then $\widetilde{G}$ is a group action of $\widetilde{X}$ on $\widetilde{Y}$.*

Using the ideas developed for lemma 3.2.6 and corollary 3.2.7, we will now construct groups and group actions. By studying the tangency relation $\|$ for particular families of curves, we will obtain associative actions and ultimately group actions.

3.2.2 Constructing Multiplicative Groups

Suppose we have curves C_x and C_y and a generically normal two-dimensional family of c-branches of plane curves I in $C_x \times C_y$ parametrized by the set L. We fix a generic $(a, b, c, g) \in I$ as well as a direction D at a. We will be working locally around (a, b, c, g) so we may assume that C_x, C_y, L and $L_{a,b} = \{ u \in L \mid (a, b, c, u) \in I \}$ are presmooth. By (PS) and the definition of a generically normal family of curves, we can find a three dimensional K of $I^D_{(a,b)} = \{ (x, y, z, u) \in I \mid x \in D \;\&\; u \in L_{(a,b)} \}$ which satisfies the following:

1. $(a, b, c, g) \in K$,
2. K is smooth away from a subset of co-dimension 2 with respect to any projection,
3. $G \subseteq L_{(a,b)}$, the parameter set of K, is presmooth,
4. $\mathrm{pr}_{zu}(K)$ is pure-dimensional of dimension 2.

From now on, since g lives on G, when we write $\mathcal{V}_g$, we will mean $\mathcal{V}_g \cap G$. Similarly, when we write $\mathcal{V}_a$ we implicitly mean $\mathcal{V}_a \cap D$.

Recall from 2.3.30 that when we are dealing with families $\langle S; C; L, I \rangle$ of c-branches through s, the parameter $l \in L$ implicitly represents the branch $\widetilde{I(l)}_{s;c}$ and that we write things like l^{-1} and $l \circ m$ to indicate which family of branches the parameters refer to. In particular, given our family K above and given $g, h, u, v \in G$,

1. g stands both for an element of G and for $\widetilde{K(g)}_{ab;c}$
2. g^{-1} also stands for $g \in G$, but for the branch $\widetilde{K^{-1}(g)}_{ba;c}$
3. $g^{-1} \circ h$ stands for $(h, g) \in G^2$ and for $\widetilde{(K^{-1} \circ K)(h, g)}_{aa;cbc}$
4. $g \circ h^{-1}$ stands for $(h, g) \in G^2$ and for $\widetilde{(K \circ K^{-1})(h, g)}_{bb;cac}$
5. $(g^{-1} \circ h) \circ (u^{-1} \circ v)$ stands for $(v, u, h, g) \in G^4$ and for
$$\widetilde{((K^{-1} \circ K) \circ (K^{-1} \circ K))(v, u, h, g)}_{aa;cbcacbc}$$
6. $(g \circ h^{-1}) \circ (u \circ v^{-1})$ stands for $(v, u, h, g) \in G^4$ and for
$$\widetilde{((K \circ K^{-1}) \circ (K \circ K^{-1}))(v, u, h, g)}_{bb;cacbcac}$$

A few more compositions of K with itself appear, but their correspondence to branches can easily be deduced using 2.3.30.

In the remainder of this secion we will prove the following key proposition:

Proposition 3.2.8 *Suppose that for any family K obtained as above, K satisfies the following two properties:*

(Non-Degeneracy) *Given independent $n, m, o \in \mathcal{V}_g$, $o^{-1} \circ n \nparallel o^{-1} \circ m$.*

(Richness) *For any generic $(h, i, r) \in \mathcal{V}_{ggg}$, there exists an $s \in \mathcal{V}_g$ such that $i^{-1} \circ h \parallel s^{-1} \circ r$.*

Then the relation $\otimes(f, h, i) = g^{-1} \circ h \parallel f^{-1} \circ i$ gives a group with identity g on $\mathcal{V}_g$.

Note that since K^{-1} is obtained from I^{-1} the same way K is obtained from I, richness and Non-Degeneracy are also assumed for K^{-1}. Furthermore, by 2.3.20, we can easily remove the genericity condition in richness.

Suppose that tangency is an equivalence relation on $\mathcal{V}_{g^{-1}\circ g}$. We will write $d(f^{-1}\circ h)$ for the tangency equivalence class of $f^{-1}\circ h$ and define the relation $F(g_1^{-1}\circ h_1, g_2^{-1}\circ h_2, g_3^{-1}\circ h_3)$ on $\mathcal{V}_{g^{-1}\circ g}$ by $g_3^{-1}\circ h_3 \parallel g_1^{-1}\circ h_1\circ g_2^{-1}\circ h_2$.

Our two assumptions will allow us to show that this is a faithful, transitive, associative action of $\mathcal{V}_{g^{-1}\circ g}$ on itself so that the trace of composition $\circ$ on $\mathcal{V}_{d(g^{-1}\circ g)}$ gives the binary operation of a group via

$$d(g_1^{-1}\circ h_1)\cdot d(g_2^{-1}\circ h_2) = d(g_1^{-1}\circ h_1\cdot g_2^{-1}\circ h_2) = d(g_3^{-1}\circ h_3)$$

To lift this group to $\mathcal{V}_g$, we note that faithfulness and richness ensure that the map $h\mapsto d(g^{-1}\circ h)$ is a bijection. $\otimes$ is the lift of $\widetilde{F}$ under this map.

Another way to see the above proposition is to consider multiplication on $\mathcal{V}_{g^{-1}\circ g}/\parallel$ like multiplication in $\mathbb{Q}$. By the trivial tangency $u^{-1}\circ h\circ h^{-1}\circ v \parallel u^{-1}\circ v$, F contains the relation defined by

$$\exists s \;\; g_1^{-1}\circ h_1 \parallel g_3^{-1}\circ s \;\wedge g_2^{-1}\circ h_2 \parallel s^{-1}\circ h_3$$

Since $\widetilde{F}$ is a map, richness ensures that F is exactly this relation.

Before we prove that $\parallel$ is indeed an equivalence relation, we look at some useful symmetries.

Lemma 3.2.9 *Suppose I and K are two families of branches through (a,b) and (b,c) respectively. Let g and g' be branches from I and h and h' be branches from K respectively. Then $h^{-1}\circ g \parallel h'^{-1}\circ g'$ if and only if $h\circ h'^{-1} \parallel g\circ g'^{-1}$.*

Proof. By definition, $h^{-1}\circ g \parallel h'^{-1}\circ g'$ means that there is some $a'\in\mathcal{V}_a$ and g_1, h_1, g_1', h_1' in $\mathcal{V}_g$, $\mathcal{V}_h$, $\mathcal{V}_{g'}$ and $\mathcal{V}_{h'}$ respectively such that $h_1^{-1}(g_1(a')) = {h_1'}^{-1}({g_1'}^{-1}(a'))$. Let $b' = {g_1'}^{-1}(a')$. Then $h_1^{-1}(h_1'(b')) = {g_1}^{-1}({g_1'}^{-1}(b'))$ which by definition of $\parallel$ means $h\circ h'^{-1}\parallel g\circ g'^{-1}$. $\square$

In our particular situation this leads to a very nice corollary.

Corollary 3.2.10 *For $g_1, g_2, g_3, g_4 \in \mathcal{V}_g$, the following are equivalent:*

1. $g_2^{-1} \circ g_1 \parallel g_4^{-1} \circ g_3$
2. $g_4 \circ g_2^{-1} \parallel g_3 \circ g_1^{-1}$
3. $g_1^{-1} \circ g_2 \parallel g_3^{-1} \circ g_4$
4. $g_2 \circ g_4^{-1} \parallel g_1 \circ g_3^{-1}$

With this in mind, we can show that $\parallel$ is an equivalence relation. Remember that $\parallel$ is reflexive, symmetric and generically transitive. Since g is generic, the only non-generic elements in $\mathcal{V}_{g^{-1} \circ g}$ are of the form $h^{-1} \circ h$. If we could show that $v^{-1} \circ u \parallel h^{-1} \circ h$ implies $v = u$, then we would be done. However, by the above symmetry, this is equivalent to showing that $v \circ h^{-1} \parallel u \circ h^{-1}$ implies $u = v$.

Lemma 3.2.11 *Given non-degeneracy and richness for both K and K^{-1}, $v \circ h^{-1} \parallel u \circ h^{-1}$ implies $u = v$.*

Proof. If $u = v = h$, we are done. So without loss of generality $u \neq h$ and $u \circ h^{-1}$ is a generic element. Let f be generic over u, g. By richness there exists a u' such that $u' \circ f^{-1} \parallel u \circ h^{-1}$. There are now several cases to consider.

First, suppose $u' = f$. Then by the above symmetries we have $f^{-1} \circ u \parallel f^{-1} \circ h$, which contradicts the non-degeneracy of K. Hence, we must have $u' \neq f$.

Next, assume that $\mathrm{cdim}\,(u', h, f) = 2$. Since $u \neq h$ and f was chosen generic over these this means that $\mathrm{cdim}\,(u/u', h, f) = 1$ and therefore that there is a u'' generic over u, u', f, h such that $u' \circ f^{-1} \parallel u'' \circ h^{-1}$. However, $\parallel$ is generically transitive so $u'' \circ h^{-1} \parallel u \circ h^{-1}$. Since we chose u'' generic over u, h, this contradicts non-degeneracy of K^{-1}.

Lastly, assume that $\mathrm{cdim}\,(u', h, f) = 3$. By generic transitivity of $\parallel$ we have both $u' \circ f^{-1} \parallel u \circ h^{-1}$ and $u' \circ f^{-1} \parallel v \circ h^{-1}$ which by symmetry means $u' \circ f^{-1} \circ h \parallel u$

and $u' \circ f^{-1} \circ h \parallel v$. But (u', f, h) is generic as well so by another application of generic transitivity we get $u \parallel v$. K is a one-dimensional family so $\parallel$ is at most a one-dimensional relation between branches from K. Since g was chosen generic, $\parallel$ can only be equality on $\mathcal{V}_{(g,g)}$ and thus $u = v$ as required. $\square$

We now turn our attention to F.

Lemma 3.2.12 *F satisfies the closure condition.*

Proof. Suppose $u, v, r, s \in \mathcal{V}_g$. We want to show that there are $m, n \in \mathcal{V}_g$ such that $v^{-1} \circ u \circ s^{-1} \circ r \parallel n^{-1} \circ m$.

If $r = s$, then $v^{-1} \circ u \parallel v^{-1} \circ u \circ s^{-1} \circ r$ for trivial reasons so there is nothing to show. Otherwise, richness allows us to find an r' such that $u^{-1} \circ r' \parallel s^{-1} \circ r$. By the characterization of tangency in 3.1.16, there exists an $a' \in \mathcal{V}_a$ and $s', r'' \in \mathcal{V}_{s,r}$ such that

$$v^{-1} \circ r'(a') = v^{-1} \circ u \circ u^{-1} \circ r'(a') = v^{-1} \circ u \circ s'^{-1} \circ r''(a')$$

By definition this gives $v^{-1} \circ r' \parallel v^{-1} \circ u \circ s^{-1} \circ r$, as required. $\square$

Congruence is a bit more tricky. We start with an easy observation.

Lemma 3.2.13 *Suppose I and K are two families of branches through $(a, b) \in C_x \times C_y$ and $(b, c) \in C_y \times C_z$ respectively. Suppose that C_y is presmooth and that D is a direction at b such that the projections of I and K to C_y are contained in D. Let g and g' be branches from I and h and h' be branches from K. If $g \parallel g'$ and $h \parallel h'$, then $h \circ g \parallel h' \circ g'$.*

Proof. By the symmetry lemma 3.2.10, $g \parallel g'$ and $h \parallel h'$ are equivalent to $g^{-1} \circ g' \parallel \mathrm{id}$ and $h^{-1} \circ h' \parallel \mathrm{id}$ where id is the identity function through (b, b). We can consider id to be part of the trivial zero-dimensional family of curves given by the graph of equality. The generic transitivity of $\parallel$ still applies so $g^{-1} \circ g' \parallel h^{-1} \circ h'$. A further application of the symmetry lemma gives the required result. □

This lemma could be read to mean that $\parallel$ satisfies the congruence condition, but it is not quite enough. Suppose $v^{-1} \circ u \parallel v'^{-1} \circ u'$. The above lemma tells us that for any $s^{-1} \circ r$,

$$(v^{-1} \circ u) \circ (s^{-1} \circ r) \parallel (v'^{-1} \circ u') \circ (s^{-1} \circ r)$$

but $(v^{-1} \circ u) \circ (s^{-1} \circ r)$ or $(v'^{-1} \circ u') \circ (s^{-1} \circ r)$ may not be generic in $(K^{-1} \circ K) \circ (K^{-1} \circ K)$ so that generic transitivity does not apply. If $h^{-1} \circ f \parallel (v^{-1} \circ u) \circ (s^{-1} \circ r)$ and $h'^{-1} \circ f' \parallel (v'^{-1} \circ u') \circ (s^{-1} \circ r)$ it does therefore not follow that $h'^{-1} \circ f' \parallel h^{-1} \circ f$. To overcome this, we have to show that $\parallel$ is an equivalence relation on all of $\left[(K^{-1} \circ K) \circ (K^{-1} \circ K)\right] \cup (K^{-1} \circ K)$.

Lemma 3.2.14 *Suppose r, s and u are independent and $r^{-1} \circ s \parallel u^{-1} \circ v$. Then for all $(r', s', u') \in \mathcal{V}_{rsu}$ and all $a' \in \mathcal{V}_a$, there exists a $v' \in \mathcal{V}_v$ such that $r'^{-1} \circ s'(a') = u'^{-1} \circ v'(a')$.*

Proof. This works exactly the same way as the proof of 3.1.16. □

Indeed, this lemma is a slight improvement of 3.1.16, but in order for it to be really useful, we need to get rid of the independence condition on r, s and u.

Lemma 3.2.15 *If $r \neq s$ and $r_1^{-1} \circ s_1 \in \mathcal{V}_{r^{-1} \circ s}$, then for any $r_2 \in \mathcal{V}_r$ and $a' \in \mathcal{V}_a$, there is an $s_2 \in \mathcal{V}_s$ such that $r_1^{-1} \circ s_1(a') = r_2^{-1} \circ s_2(a')$.*

Proof. By richness we can find $u^{-1} \circ v$ independent of r such that $r^{-1} \circ s \parallel u^{-1} \circ v$. By 3.1.16 there exist u_1, v_1 such that $r_1^{-1} \circ s_1(a') = u_1^{-1} \circ v_1(a')$ and by the above lemma there exists a s_2 such that $u_1^{-1} \circ v_1(a') = r_2^{-1} \circ s_2(a')$. □

By symmetry we also get that there is an $s_2' \in \mathcal{V}_s$ such that $r_1^{-1} \circ r_2(a') = s_1^{-1} \circ s_2'(a')$. Thus, we can get rid of the $r \neq s$ condition in the above lemma:

Corollary 3.2.16 *If $r_1^{-1} \circ s_1 \in \mathcal{V}_{r^{-1}\circ s}$, then for any $r_2 \in \mathcal{V}_r$ and $a' \in \mathcal{V}_a$, there is an $s_2 \in \mathcal{V}_s$ such that $r_1^{-1} \circ s_1(a') = r_2^{-1} \circ s_2(a')$.*

Proof. We only have the case $r = s$ left to prove so write $s_1 = r_3$. Now pick $u \neq r$ and apply the above remark twice. The first application gives us a u_1 such that $r_1^{-1} \circ r_3(a') = u^{-1} \circ u_1(a')$. The second application gives r_4 such that $u^{-1} \circ u_1(a') = r_2^{-1} \circ r_4(a')$ as required. □

We are now in a position to prove a version of 3.2.14 without the independence condition and thereby get a much stronger result than lemma 3.1.16 for these particular families.

Corollary 3.2.17 *Given $r^{-1} \circ s \parallel u^{-1} \circ v$, $r' \in \mathcal{V}_r$, $s' \in \mathcal{V}_s$, $u' \in \mathcal{V}_u$ and $a' \in \mathcal{V}_a$, there exists a $v' \in \mathcal{V}_v$ such that $r'^{-1} \circ s'(a') = u'^{-1} \circ v'(a')$*

Proof. Suppose $(r, s, t) \in \mathcal{V}^1_{(g,g,g)}$ and let J be the relation used to define tangency (see definition 3.1.15). By applying the above corollary to the tangency $g^{-1} \circ g \parallel g^{-1} \circ g$, we see that the projection of J to $\mathcal{V}_{aggg}$ is surjective. $\mathcal{V}^2_{arst} \subseteq \mathcal{V}_{aggg}$ and lemma 3.2.11 guarantees that the fibre over (a, g, g, g) is zero-dimensional. Thus, by presmoothness of M and $L_{(a,b)}$, the implicit function theorem applies to J at (a, r, s, t) and for any $r' \in \mathcal{V}^2_r$, $s' \in \mathcal{V}^2_s$, $u' \in \mathcal{V}^2_u$ and $a' \in \mathcal{V}^2_a$, there exists a $v' \in \mathcal{V}^2_v$ such that $r'^{-1} \circ s'(a') = u'^{-1} \circ v'(a')$. □

This extends nicely to curves from $(K^{-1} \circ K) \circ (K^{-1} \circ K)$ and allows us to prove the necessary transitivity lemma to get congruence. For later use, we will prove something slightly stronger.

Lemma 3.2.18 *If $(r^{-1} \circ s) \circ (u^{-1} \circ v) \parallel (m^{-1} \circ n)$, $r' \in \mathcal{V}_r$, $s' \in \mathcal{V}_s$, $u' \in \mathcal{V}_u$, $v' \in V_v$, $m' \in \mathcal{V}_m$ and $a' \in \mathcal{V}_a$, then there exists an $n' \in \mathcal{V}_n$ such that*

$$(r'^{-1} \circ s') \circ (u'^{-1} \circ v')(a') = (m'^{-1} \circ n')(a')$$

Proof. By definition of tangency there exist r_1, s_1, u_1, v_1, m_1 and n_1 in the respective neighbourhoods such that

$$(r_1{}^{-1} \circ s_1) \circ (u_1{}^{-1} \circ v_1)(a') = (m_1{}^{-1} \circ n_1)(a')$$

Let r_u be such that $r_u^{-1} \circ u \parallel r^{-1} \circ s$ and let $a'' = (u_1{}^{-1} \circ v_1)(a')$. By the above lemma there exists an r'_u such that

$$r'_u{}^{-1} \circ u_1(a'') = r_1^{-1} \circ s_1(a'')$$

Hence $r'_u{}^{-1} \circ v_1(a') = (m_1{}^{-1} \circ n_1)(a')$ and therefore $r_u^{-1} \circ v \parallel m \circ n$. By a similar argument, there exists a r''_u such that

$$(r'^{-1} \circ s') \circ (u'^{-1} \circ v')(a') = (r''^{-1}_u \circ v')(a')$$

but now the previous corollary gives us n' such that

$$m^{-1} \circ n'(a') = (r''^{-1}_u \circ v')(a')$$

as required. □

We varied r, s, u, v and m and to found an n, but by symmetry we can vary any five parameters and find the sixth.

Corollary 3.2.19 *$\parallel$ is an equivalence relation on* $\left[(K^{-1} \circ K) \circ (K^{-1} \circ K)\right] \cup (K^{-1} \circ K)$.

Proof. Similar to 3.1.17. □

In particular, F satisfies congruence.

Lemma 3.2.20 *F is faithful.*

Proof. Suppose that $(v^{-1} \circ u) \circ (d^{-1} \circ c) \parallel (s^{-1} \circ u)$ and $(v^{-1} \circ u) \circ (d'^{-1} \circ c') \parallel (s^{-1} \circ u)$. By the symmetry lemma 3.2.10 and the fact that $\parallel$ is an equivalence relation on $\left[(K^{-1} \circ K) \circ (K^{-1} \circ K)\right] \cup (K^{-1} \circ K)$, it immediately follows that $d^{-1} \circ c \parallel d'^{-1} \circ c'$, as required. □

Lastly we note that composability and associativity of F follows straight from properties of composition of functions. Therefore, $\widetilde{F}$ is the graph of a group. This concludes the proof of the proposition.

Remark 3.2.21 So far, elements h of $\mathcal{V}_g$ represent $d(g^{-1} \circ h)$ and the group is defined by $h \cdot i = j$ if $(g^{-1} \circ h) \circ (g^{-1} \circ i) \parallel (g^{-1} \circ j)$ where the curves are considered part of the family $K^{-1} \circ K$. This will become problematic later as we will consider further compositions and products of these curves. Since g appears as a parameter in both $(g^{-1} \circ h)$ and $(g^{-1} \circ i)$, the situation becomes non-generic and we cannot apply smoothness results like 3.1.16 as easily. However, 3.2.18 shows that it is in fact enough to consider the family $g^{-1} \circ K$.

3.2.3 Constructing Additive Groups

Let us suppose now that we have found a presmooth curve H such that for some $e \in H$ and some definable relation $\cdot \subseteq H^3$, $\cdot \cap \mathcal{V}_e^3$ is the graph of a group with identity e. Suppose furthermore that we have a generically normal two-dimensional family of c-branches of plane curves I in $C_x \times H$ parametrized by the set L. We can now proceed as in the previous section, but with $\cdot$ in place of $\circ$.

We fix a generic $(a, e, c, g) \in I$ as well as a direction D at a. We will be working locally around (a, e, c, g) so we may assume that C_x, C, L and $L_{a,e} = \{ u \in L \mid (a, e, c, u) \in I \}$ are presmooth. By (PS) and the definition of a generically normal family of curves, we can find a three dimensional component K of $I^D_{(a,e)} = \{ (x, y, z, u) \in I \mid x \in D \;\&\; u \in L_{(a,e)} \}$ which satisfies the following:

1. $(a, e, c, g) \in K$,

2. K is smooth away from a subset of co-dimension 2 with respect to any projection,

3. $G \subseteq L_{(a,b)}$, the parameter set of K is presmooth,

4. $\mathrm{pr}_{zu}(K)$ is pure-dimensional of dimension 2.

As we did in the multiplicative case, we will use parameters and the branches they represent interchangably as was outlined in 2.3.31. In addition to the notation already used in section 3.2.2, given $g, h, u, v \in G$,

1. e/g^{-1} stands for $g \in G$ and the branch $\widetilde{e/K(g)}_{ae;ce}$
2. $e/g \cdot h$ stands for $(h, g) \in G^2$ and for $\widetilde{(e/K \cdot K)(h, g)}_{ae;ceec}$
3. $g \cdot e/h$ stands for $(h, g) \in G^2$ and for $\widetilde{(K \cdot e/K)(h, g)}_{bb;ceec}$
4. $(e/g \cdot h) \cdot (e/u \cdot v)$ stands for $(v, u, h, g) \in G^4$ and for

$$\widetilde{((e/K \cdot K) \cdot (e/K \cdot K))(v, u, h, g)}_{ae;ceeceeceec}$$

5. $(g \cdot e/h) \cdot (u \cdot e/v)$ stands for $(v, u, h, g) \in G^4$ and for

$$\widetilde{((K \cdot e/K) \cdot (K \cdot e/K))(v, u, h, g)}_{ae;ceeceeceec}$$

Combinations of both multiplication and composition will also be used in both this and the next section, but there is little to be gained from writing out exactly which branches all of these refer to. The reader should refer to 2.3.30 and 2.3.31 for the general rules to work these out. Just as an example, if J is a family of d-branches of curves through (p, a) and r is a curve from J, then $(e/g \cdot h) \circ r$ stands for

$$\widetilde{((e/K \cdot K) \circ J)(r, h, g)}_{pe;daceec}$$

Proposition 3.2.22 *Suppose that for any family K obtained as above, K satisfies the following two properties:*

(Non-Degeneracy) *Given independent* $n, m, o \in \mathcal{V}_g$*,* $e/o \cdot n \not\parallel e/o \cdot m$*.*

(Richness) *For any* $h, i, r \in \mathcal{V}_g$*, there exists an* $s \in \mathcal{V}_g$ *such that* $e/i \cdot h \parallel e/s \cdot r$*.*

Then the relation $\oplus(f, h, i) = e/g \cdot h \parallel e/f \cdot i$ *gives a group with identity* g *on* $\mathcal{V}_g$*.*

The proof of this theorem is entirely analogous to the proof of the corresponding proposition 3.2.8 for the composition case. The only additional result here is the key lemma for distributivity.

Lemma 3.2.23 *Suppose* J *is a family of branches of curves through* (c, a)*. Let* h *and* f *be a curves from* K *and* r *be a curve from* J*. Then*

$$(h \cdot f) \circ r \parallel (h \circ r) \cdot (f \circ r)$$

Proof. This is trivial since the two represent the same branch. □

Note that unlike b in the previous section, e is not generic. In the previous section this was important for the symmetry since K^{-1} is a family through (b, a). In contrast, the family e/K is still a family through (a, e). Because we always parametrize curves by the first coordinate, it is sufficient that a is chosen generic in this case.

3.2.4 Constructing a definable group presentation

Definition 3.2.24 A (one-dimensional) *constructible group presentation* consists of curves C and D with elements $0 \in C$, $1 \in D$ and constructible relations $+, \cdot$ and $.$ such that the restrictions of $+$ and $\cdot$ to $\mathcal{V}_0$ and $\mathcal{V}_1$ are the graphs of groups with identities 0 and 1 respectively and such that the restriction of $.$ to $\mathcal{V}_0 \times \mathcal{V}_1 \times \mathcal{V}_0$ is a distributive group action of $\langle \mathcal{V}_1, \cdot \rangle$ on $\langle \mathcal{V}_0, + \rangle$. We say that the presentation is faithful if the action is.

In order to obtain a projective group presentation we assume that the hypotheses of propositions 3.2.8 and 3.2.22 hold. To recap: If C_x and C_y are presmooth curves and I

is a two-dimensional generically normal family of (branches of) curves in $C_x \times C_y$, then we require the following:

1. If (a, b, g) is generic, D is a direction at a, and K is the family obtained by taking the component of $I^D_{(a,b)}$ that containins (a, b, g), then K satisfies the richness and non-degeneracy conditions of 3.2.8.
2. Suppose there is a generic element $1 \in C_y$, a direction D at a and a relation $\cdot \subseteq C_y^3$. Let a and g be generic and K be the family obtained by taking the component of $I^D_{(a,1)}$ that contains $(a, 1, g)$. Then K satisfies the richness and non-degeneracy conditions of 3.2.22.

Proposition 3.2.25 *Under the above conditions, if M is non-linear than there is a projective group presentation.*

Proof. By non-linearity, there exists a generically normal two-dimensional family of plane curves with incidence relation $I \subseteq \mathrm{M}^4$.

We first construct a multiplicative group using 3.2.8. We pick generic $(r, s, t, b) \in I$ and let $e = (t, b)$. We obtain a group $\cdot$ on $\mathcal{V}_e \cap C_e$ where C_e is a smooth and presmooth curve contained in $\{ (u, v) \mid (r, s, u, v) \in I \}$.

Next, we lift the family I to $\mathrm{M} \times C_e \times \mathrm{M}^2$ using the projection $\mathrm{pr}_u : C_e \to \mathrm{M}$ onto the second (or v-) coordinate of C_e. Via this projection, C_e is locally isomorphic to M so there is an open set such that the family $J = \left\{ (w, (u, v), x, y) \in \mathrm{M} \times C_e \times \mathrm{M}^2 \;\middle|\; (x, y, w, v) \in I \right\}$ is a generically normal two-dimensional family of curves. Note in particular that $(t, e, r, s) \in J$. Since e was generic and the fibre over e in J is non-empty, it is two-dimensional. So we can find a generic $z = (x_z, y_z)$ and a such that $(a, e, z) \in J$. Now if we let K be the family obtained by picking that component of $J_{(a,e)}$ which contains (a, e, z) then by our assumptions, proposition 3.2.22 applies and we can define a relation $\oplus \subseteq C_z^3$ such that the restriction of $\oplus$ to $\mathcal{V}_z$ is the graph of a group.

The elements g of the additive group represent curves of the form $(e/z \cdot g)$. z was chosen generic so the family $K = e/z \cdot J$ is still generically normal away from z. Furthermore, by an analogous result to 3.2.18, $K_{(a,e)}$ witnesses all necessary tangencies and the action of $K_{(a,e)}$ on itself via multiplication gives us the same additive group as above. We will relabel z to 0 so that 0 represents $(e/z \cdot z)$, the constant map e.

We will now make use of a second level of infinitesimals. $\mathcal{V}'$ will denote smaller infinitesimal neighbourhoods than $\mathcal{V}$.

If we fix a generic $1 \in \mathcal{V}_0$, 3.2.8 applies to $K_{(a,e)}$ on $\mathcal{V}'_1$ so we get a multiplicative group $\times$ on $\mathcal{V}'_1$. One dimensional type-definable groups in Presmooth Geometries are smooth so $\mathcal{V}'_0 = \mathcal{V}'_1 - \mathcal{V}'_1$. We can thus try to induce an action of $\langle \mathcal{V}'_1, \times \rangle$ on $\langle \mathcal{V}'_0, + \rangle$. In terms of $\times$ and $+$ this should look as follows. If $g, f \in \mathcal{V}'_0$ and $h \in \mathcal{V}_1$, then $g.h = f$ if and only if there are h_1, h_2, h_3 and $h_4 \in \mathcal{V}'_1$ such that

$$g.h = (h_1 - h_2).h = h_1 \times h - h_2 \times h = h_3 - h_4 = f$$

To show that this is a well behaved distributive group action we will show that this is just the action induced by composition. Remember, elements h of $\mathcal{V}'_1$ represent the curve $(e/z \cdot 1)^{-1} \circ (e/z \cdot h)$ and elements g of $\mathcal{V}'_0$ represent curves of the form $(e/z \cdot g)$. We will show that for any $g' \in \mathcal{V}'_0$, $h' \in \mathcal{V}'_1$ and $a' \in \mathcal{V}'_a$, there is an $f' \in \mathcal{V}'_0$ such that

$$\left[(e/z \cdot g') \circ (e/z \cdot 1)^{-1} \circ (e/z \cdot h')\right](a') = (e/z \cdot f')(a')$$

This will have two major consequences: Firstly it shows that the operation . is indeed just the one induced by composition and therefore that it is distributive by 3.2.23. Secondly, it gives congruence as the above shows that if $\left[(e/z \cdot g) \circ (e/z \cdot 1)^{-1} \circ (e/z \cdot h)\right] \parallel (e/z \cdot s)$, then $(e/z \cdot s) \parallel (e/z \cdot f)$ and thus $f = s$.

So let $g', h', a' \in \mathcal{V}_{ghe}$ and let $a_1 = (e/z \cdot 1)^{-1} \circ (e/z \cdot h')(a')$. By the group version of 3.2.18, there are h'_1, h'_2 such that

$$(e/z \cdot g')(a_1) = [(e/z \cdot h'_1) \cdot e/(e/z \cdot h'_2)](a_1) = [(e/z \cdot h'_1)(a_1) \cdot e/(e/z \cdot h'_2)(a_1)]$$

By 3.2.18 there is an h_3' such that

$$(e/z \cdot h_1')(a_1) = [(e/z \cdot h_1') \circ (e/z \cdot 1)^{-1} \circ (e/z \cdot h)](a') = (e/z \cdot h_3')(a')$$

and similarly for h_2' and h_4'. Hence

$$(e/z \cdot g')(a_1) = (e/z \cdot h_3')(a') \cdot e/(e/z \cdot h_4')(a') = [(e/z \cdot h_3') \cdot e/(e/z \cdot h_4')](a')$$

By another application of the group version of 3.2.18 there is an f' such that

$$(e/z \cdot g')(a_1) = [(e/z \cdot h_3') \cdot e/(e/z \cdot h_4')](a') = (e/z \cdot f')(a')$$

as required.

Thus, $\langle \mathcal{V}_1', \times \rangle$ acts distributively on $\langle \mathcal{V}_0', + \rangle$ and we have obtained a constructible group presentation.□

The remaining question is whether the action of $\times$ on $+$ is faithful or even non-trivial. Since elements of $\mathcal{V}_0'$ represent curves of the form $e/z \cdot g$ and elements of $\mathcal{V}_1'$ represent $(e/z \cdot 1)^{-1} \circ (e/z \cdot h)$, it is tempting to say that the non-degeneracy condition for composition should ensure that $(e/z \cdot g)^{-1} \circ (e/z \cdot g) \not\parallel (e/z \cdot 1)^{-1} \circ (e/z \cdot h)$ for $g \neq h$. However, g and h come from different neighbourhoods here and we can not extend the picture to all of $\mathcal{V}_0$ as things go wrong for $g = 0$. $(e/z \cdot 0)$ is the constant function e which is not invertible. Because of this, we actually have to view $(e/z \cdot h)$ and $(e/z \cdot g)$ as belonging to different families.

As with richness and non-degeneracy, a uniform proof of the faithfulness of this action is missing so we defer it to each individual example.

Chapter 4

Examples

In this chapter we look at several examples of Strong and Presmooth Geometries. The two main examples are generalizations of Zariski Geometries and o-minimal structures. We will show that both of these are Strong Geometries satisfying (SL) and (2T) respectively. We will also show that the former are Presmooth Geometries and give a sufficient condition for the same to be true of o-minimal structures. We will also show that in both these cases the conditions of proposition 3.2.25 are always met, that the group presentation thus obtained is faithful and how we can construct a field from such an object.

4.1 Generalizations of Zariski Geometries

Most of the time, generalizations of the Trichotomy Theorem for Zariski Geometries go via an axiomatization similar to that given in [HZ96]. In this text, however, we have kept the axioms for Strong and Presmooth Geometries in the style of [Zil10] in order to keep the presentation geometric rather than model-theoretic. We will now give an axiomatization in the style of [HZ96] ourselves and show that this is indeed a Stable-Like Presmooth Geometry. Even though the conditions of proposition 3.2.25 are met in all one-dimensional Stable-Like Presmooth Geometries, we need slightly stronger assumptions to extract a field from a group presentation. We will therefore postpone the verification of the conditions of 3.2.25 until we have introduced this new context.

We work in a multi-sorted relational language $\mathcal{L}$ with sorts X_i. As before, we call the sets defined by positive quantifier free formulas (with parameters) closed. We write M for $\bigcup X_i(\mathrm{M})$. When we take a definable $S \subseteq \mathrm{M}^n$, it is implicit that $S \subseteq X_{i_1}(\mathrm{M}) \times \ldots \times X_{i_n}(\mathrm{M})$ for some sorts X_{i_j}.

Definition 4.1.1 We call M κ-compact if whenever $\{ A_i \mid i \in I \}$ is a consistent family of constructible sets with $|I| < \kappa$, then $\bigcap_i A_i \neq \emptyset$.

It is easy to check that κ-saturation implies κ-compactness.

Definition 4.1.2 An $\mathcal{L}$-structure M is a *multi-sorted κ-proper Zariski Geometry* if for any tuple of sorts $X_{i_1}, \ldots, X_{i_n}$, the closed sets form a Noetherian topology on $X_{i_1}(\mathrm{M}) \times \ldots \times X_{i_n}(\mathrm{M})$ and these topologies, together with Krull dimension dim, satisfy

(κMZ1) M is κ-compact and for any closed irreducible set $S \subseteq \mathrm{M}^{n+k}$ and a projection $\mathrm{pr}\colon \mathrm{M}^{n+k} \to \mathrm{M}^n$, $\mathrm{pr}S \subseteq \overline{\mathrm{pr}S} \setminus \bigcup_{j \in J} F$ for proper closed subsets $F_j \subseteq \overline{\mathrm{pr}S}$ and $|J| < \kappa$.

(MZ2) $X_i(\mathrm{M})$ is irreducible and for any closed set $S \subseteq \mathrm{M}^n \times X_i(\mathrm{M})$ there is a natural number m such that for all $a \in \mathrm{M}^n$, either $S(a) = X_i(\mathrm{M})$ or $|S(a)| \leq m$.

(MZ3) For any tuple of sorts $X_{i_1}, \ldots, X_{i_n}$

$$\dim(X_{i_1}(\mathrm{M}) \times \ldots \times X_{i_n}(\mathrm{M})) = n$$

and for any closed irreducible set $S \subseteq \mathrm{M}^n$ and Δ a diagonal of the form $x_i = x_j$ every component of $S \cap \Delta$ has dimension $\geq \dim S - 1$.

If M is one-sorted we will just say *κ-proper Zariski Geometry*

In the case of $\kappa = \omega$ these are exactly the axioms that are given in [CHP02] and from which the authors construct a field there. We will look at that in more detail in section 4.1.4.

Proposition 4.1.3 *A multi-sorted κ-proper Zariski Geometry is a Stable-Like Presmooth Geometry.*

Proving this takes a bit of work. The main difficulty is to prove that dimension is definable. This is reasonably straightforward if we have quantifier elimination, but here we have to be very careful to keep track of the "holes" in projective sets. It is interesting to note the way in which we use (MZ3) in the proof of axiom (DD). It is unclear to me whether this is essential.

Lemma 4.1.4 *If $S(x) \subseteq \mathrm{M}^n$ and $T(y) \subseteq \mathrm{M}^m$ are closed and irreducible, then so is $S \times T$.*

Proof. Suppose $S \times T = X \cup Y$. By irreducibility of T, for every $s \in S$, either $X(s) = T$ or $Y(s) = T$. Let $K = \mathrm{pr}_y(X \setminus Y)$, $L = \mathrm{pr}_y(Y \setminus X)$ and $R = S \setminus (\overline{K} \cup \overline{L})$. The irreducibility of S implies that one of K, L or R is dense in S. However, for every $t \in T$, $K \subseteq X(t)$, $L \subseteq Y(t)$ and $R \subseteq X(t) \cap Y(t)$. Thus, either X or Y are dense in $S \times T$. □

We should note that the topology on M^{n+m} is not just the product topology of $\mathrm{M}^n \times \mathrm{M}^m$. Otherwise the above lemma, as well as many of the following results, would just be a standard facts about Noetherian topologies

Lemma 4.1.5 *A multi-sorted κ-proper Zariski Geometry with the above dimension satisfies (DP), (DU), (DC), (SB) and (SL).*

Proof. Since the dimension of a projective set is Krull dimension, axioms (DP) and (DU) follow immediately. (DC) is also trivial since irreducible closed sets are pure-dimensional.

For (SB) suppose P and Q are projective. Clearly it is enough to consider the case where P is irreducible. Suppose $\dim(\overline{P} \cap Q) = \dim(P)$. Then by irreducibility of $\overline{P}$ we have $\overline{\overline{P} \cap Q} = \overline{P}$ and hence by (κMZ1)

$$P \supseteq \overline{P} \setminus \bigcup S_i \text{ and } \overline{P} \cap Q \supseteq \overline{P} \setminus \bigcup T_j$$

where $S_i \subseteq \overline{P}$ and $T_j \subseteq \overline{P}$ for $i, j < \kappa$ are closed sets of smaller dimension. Then

$$P \cap Q \supseteq \overline{P} \setminus \left(\bigcup S_i \cup \bigcup T_j \right)$$

κ-compactness tells us that the right hand side is dense in $\overline{P}$ so we have $\overline{P \cap Q} = \overline{P}$ and $\dim(P \cap Q) = \dim P$, as required.

(SL) is a direct consequence of the first parts of (MZ2) and (MZ3) together with the previous lemma which tells us that any $X_{i_1}(\mathrm{M}) \times \ldots \times X_{i_n}(\mathrm{M})$ is irreducible of dimension n. $\square$

Lemma 4.1.6 *Suppose R is irreducible and $S(x, y) \subseteq \mathrm{M}^n \times R$ is a closed irreducible set. Then the set*

$$K = \{ a \in \mathrm{pr}_y(S) \mid S(a) = R \}$$

is closed.

Proof. By the definition of K, $\mathrm{pr}^{-1}(K) \cap S = K \times R$. Let $T = \overline{K \times X_i(\mathrm{M})} \subseteq S$ and suppose $(a, b) \in (\overline{K} \times R) \setminus T$. Then $K \subseteq T(b) \subsetneq \overline{K}$, which is a contradiction since $T(b)$ is closed. Therefore $T = \overline{K} \times R \subseteq S$, which means $K = \overline{K}$ by definition of K. $\square$

Corollary 4.1.7 *Suppose $S(x, y) \subseteq \mathrm{M}^n \times X_i(\mathrm{M})$ is a closed irreducible set. Then the set*

$$K = \{ a \in \mathrm{pr}_y(S) \mid \dim S(a) = 1 \}$$

is closed.

Proof. Just note that $\dim S(a) = 1$ if and only if $S(a) = X_i(\mathrm{M})$. $\square$

Lemma 4.1.8 *Suppose $S(x, y) \subseteq \mathrm{M}^n \times X_i(\mathrm{M})$ is a closed irreducible set and let K be as in the previous lemma. Then exactly one of the following is true:*

1. $S = \mathrm{pr}_y(S) \times X_i(\mathrm{M})$ *and* $\dim(S) = \dim \mathrm{pr}_y(S) + 1$.

2. $\dim K \leq \dim \mathrm{pr}_y(S) - 2$ *and* $\dim(S) = \dim \mathrm{pr}_y(S)$

Proof. We proceed by induction on $\dim(S)$. The base case $\dim(S) = 0$ is trivial, since in this case S is a singleton.

We start by noting that for any chain of irreducibles in $\mathrm{pr}_y(S)$ gives rise to a chain of irreducibles in S, so that $\dim \mathrm{pr}_y(S) \leq \dim(S)$.

For the first case, suppose $S = \mathrm{pr}_y(S) \times X_i(\mathrm{M})$. $S(b) \times \{ b \}$ is a proper closed subset of S and is homeomorphic to $S(b) = \mathrm{pr}_y(S)$ so $\dim S > \dim \mathrm{pr}_y(S)$. Let $T \subseteq S$ be an irreducible subset of S of dimension $\dim S - 1$. By hypothesis, either $T = \mathrm{pr}_y(T) \times X_i(\mathrm{M})$ or $\dim T = \dim \mathrm{pr}_y(T)$. In the first case, $\mathrm{pr}_y(T)$ is a proper closed irreducible subset of $\mathrm{pr}_y(S)$ of dimension $\dim(S) - 2$, so $\dim \mathrm{pr}_y(S) > \dim(S) - 2$. In the second case $\mathrm{pr}_y(T)$ is an irreducible closed subset of $\mathrm{pr}_y(S)$ of dimension $\dim S - 1$. Either way, $\dim \mathrm{pr}_y(S) \geq \dim(S) - 1$ and we must have equality.

In the second case $\mathrm{pr}_y^{-1}(K) \cap S$ is a proper closed subset, so it must have dimension $\leq \dim S - 1$. By hypothesis and definition of K this means that $\dim K \leq \dim(S) - 2$. Let m be the constant given by (MZ2) and let $T_1, \ldots, T_{m+1}$ be distinct proper closed irreducible subsets of S of dimension $\dim(T_i) = \dim S - 1$ and such that $\mathrm{pr}_y(T_i) \nsubseteq K$. We claim that for some k the set $\mathrm{pr}_y(T_k)$ is not dense in $\mathrm{pr}_y(S)$. If the claim is true, then by hypothesis, $\overline{\mathrm{pr}_y(T_k)}$ is a proper closed subset of $\mathrm{pr}_y(S)$ of dimension $\dim S - 1$ of $\mathrm{pr}_y(S)$, which would mean that $\dim \mathrm{pr}_y(S) \geq \dim S$.

To prove the claim, we suppose that $\mathrm{pr}_y(T_k)$ is dense in $\mathrm{pr}_y(S)$ for all k. Let $T = \bigcup_{i \neq j} T_i \cap T_j$. Note that $\dim T \leq \dim S - 2$ which implies that $\dim \mathrm{pr}_y(T) \leq \dim S - 2$ by hypothesis and the dimension of K. Now by (κMZ1), the set

$$\left(\bigcap \mathrm{pr}_y(T_i) \right) \setminus (K \cup \mathrm{pr}_y(T))$$

is non-empty. But a fibre over this set would have to be finite of size $> m$ which contradicts (MZ2). □

Lemma 4.1.9 *For all projective sets* $P(x,y) \subseteq \mathrm{M}^{n+1}$, $\overline{\mathrm{pr}_y(P)} = \overline{\mathrm{pr}_y(\overline{P})}$.

Proof. By (κMZ1) we have $P \supseteq S \setminus \bigcup S_i$, where $S = \overline{P}$ and S_i are closed subsets of S of dimension $\leq \dim S - 1$. Let $P' = S \setminus \bigcup \overline{S_i}$. It is clearly enough to consider the case where S is irreducible and to show that $\mathrm{pr}_y(P')$ is dense in $\mathrm{pr}_y(S)$. We use (κMZ1) again to write $\mathrm{pr}_y(S) = T \setminus \bigcup T_i$.

We split into two cases:

Case 1: $\dim S = \dim T$.
In this case $\dim \mathrm{pr}_y(S_i) < \dim T$ for all i, so by κ-compactness $\overline{\mathrm{pr}_y(P')} = T$, as required.

Case 2: $\dim S = \dim T + 1$.
In this case we have seen that $S = T \times X_i(\mathrm{M})$ for some sort X_i. For each S_i let $K_i = \{ x \in T \mid \overline{S}_i(x) = X_i(\mathrm{M}) \}$. Then

$$\mathrm{pr}_y(P') \supseteq T \setminus \bigcup K_i$$

and all K_i have smaller dimension than T so by κ-compactness, $\overline{\mathrm{pr}_y(P')} = T$ as required.□

Corollary 4.1.10 *For all projective sets* $P(x,y)$, $\overline{\mathrm{pr}_y(P)} = \overline{\mathrm{pr}_y(\overline{P})}$.

Proof. Apply the previous lemma one coordinate at a time. □

Corollary 4.1.11 *If* $S(x,y) \subseteq \mathrm{M}^{n+m}$ *is closed and irreducible and* $\dim(S) = \dim \mathrm{pr}_y(S) + k$, *then at least* $m-k$ *of the* m *possible projections* $\mathrm{pr}\colon \mathrm{M}^{n+m} \to \mathrm{M}^{n+m-1}$ *are such that* $\dim S = \dim \mathrm{pr}(S)$.

Proof. Suppose not. By reordering coordinates, suppose that whenever pr is a projection projecting out one of the last $k+1$ coordinates, we have $\dim S = \dim \mathrm{pr}(S) + 1$.

Let $X_{i_1}, \ldots, X_{i_{k+1}}$ be the sorts corresponding to these coordinates. Let pr be the projection projecting out the last $k+1$ coordinates. Then by repeated applications of 4.1.8, $S = \mathrm{pr}(S) \times X_{i_1}(\mathrm{M}) \times \ldots \times X_{i_{k+1}}(\mathrm{M})$ and thus $\dim S = \dim \mathrm{pr}(S) + k + 1 \geq \mathrm{pr}_y(S) + k + 1$. This is clearly a contradiction to $\dim S = \mathrm{pr}_y(S) + k$. □

Lemma 4.1.12 *For all closed sets S*

$$\dim S = \max\{\, m \mid \mathrm{pr}(S) \textit{ is dense in } X_{i_1} \times \ldots \times X_{i_m} \textit{ for some projection } \mathrm{pr}\,\}$$

Proof. $\dim S \geq \max\{\, m \mid \mathrm{pr}(S)$ is dense in $X_{i_1} \times \ldots \times X_{i_m}$ for some projection $\mathrm{pr}\,\}$ is trivial. The other direction follows from 4.1.11 and 4.1.10 and the irreducibility of $X_{i_1} \times \ldots \times X_{i_m}$. □

Lemma 4.1.13 *For closed sets S and T*

$$\dim(S \times T) = \dim S + \dim T$$

Proof. This is immediate from the characterization of dimension in the previous lemma and the fact that for $P \subseteq \mathrm{M}^n$ and $Q \subseteq \mathrm{M}^k$, $P \times Q$ is dense in M^{n+k} if and only if P is dense in M^n and Q is dense in M^k. □

Lemma 4.1.14 *Let $S(x, y) \subseteq \mathrm{M}^{n+k}$ be closed and irreducible and suppose that for some open set $U \subseteq \mathrm{pr}_y(S)$ and all $a \in U$ that we have $\dim S(a) \leq r$. Then $\dim S \leq \dim \mathrm{pr}_y(S) + r$.*

Proof. Suppose $\dim S = \dim \mathrm{pr}_y(S) + m$. By 4.1.11 there must be projections $\mathrm{pr}_1\colon \mathrm{M}^{n+k} \to \mathrm{M}^{n+m}$ and $\mathrm{pr}_2\colon \mathrm{M}^{n+m} \to \mathrm{M}^n$ such that $\mathrm{pr}_y = \mathrm{pr}_2 \circ \mathrm{pr}_1$ and $\dim \mathrm{pr}_1(S) = \dim S$. This means that $\mathrm{pr}_1(S)$ must be dense in $\mathrm{pr}_y(S) \times X_1(\mathrm{M}) \times \ldots X_m(\mathrm{M})$, where X_1, X_m are the sorts of the $n+1$st to $n+m$th coordinates. Now by (κMZ1)

$$\mathrm{pr}_1(S) \supseteq (\mathrm{pr}_y(S) \times X_1(\mathrm{M}) \times \dots X_m(\mathrm{M})) \setminus \bigcup S_i$$

where the S_i are closed sets of smaller dimension. For each i let

$$K_i = \{ a \in \mathrm{pr}_y(S) \mid S_i(a) = X_1(\mathrm{M}) \times \dots X_m(\mathrm{M}) \}$$

Each K_i has dimension $\leq \mathrm{pr}_y(S) - 1$ so by κ-compactness we can find $a \in U \setminus \bigcup K_i$. Now $\dim S_i(a) < m$ for all i, so $\mathrm{pr}_1(S)(a)$ is dense in $\{a\} \times X_1(\mathrm{M}) \times \dots X_m(\mathrm{M})$, which means that $\dim S(a) \geq m$. By definition of U this means $m \leq r$, as required. $\square$

Lemma 4.1.15 *Given a closed irreducible set* $S(x,y) \subseteq \mathrm{M}^{n+k}$*, there is a proper closed subset* $K \subseteq \mathrm{pr}_y(S)$ *such that* $\dim S(a) \leq \dim S - \dim \mathrm{pr}_y(S)$ *for all* $a \in \mathrm{pr}_y(S) \setminus K$*. Furthermore,* K *can be chosen so that* $\dim K \leq \dim S - 2$*.*

Proof. Suppose $\dim S = \dim \mathrm{pr}_y(S) + r$. By reordering coordinates we can assume that $y = (u, v)$ where $\#(u) = r$ and $\dim \mathrm{pr}_v(S) = \dim S$. Let $v = (v_1, \dots, v_{k-r})$. Let pr_i be the projection $(x, u, v) \mapsto (x, u, v_i)$ and $S_i = \overline{\mathrm{pr}_i(S)}$.

Let $a \in \mathrm{pr}_y(S) = \mathrm{pr}_{uv_i}(S_i)$. If $\dim S(a) > r$, then for some i, $S_i(a)$ has dimension $r+1$. However, by lemma 4.1.6

$$K_i = \{ a \in \mathrm{pr}_{uv_i}(S_i) \mid \dim S_i(a) = r+1 \}$$

is a closed subset of $\mathrm{pr}_{uv_i}(S_i)$. Since $\dim S_i = \dim \mathrm{pr}_{uv_i}(S_i) + r$, $\mathrm{pr}_{uv_i}^{-1}(K_i)$ is a proper closed subset of $\mathrm{pr}_{uv_i}(S_i)$ and by irreducibility of S_i, this implies $\dim K < \dim S_i - (r+1) = \dim \mathrm{pr}_y(S) - 1$. Now the set $K = \bigcup K_i$ is as required. $\square$

Lemma 4.1.16 *Let* $S, T \subseteq \mathrm{M}^n$ *be closed and irreducible. Then all components of* $S \cap T$ *have dimension at least* $\dim S + \dim T - n$*.*

Proof. We note that $S \cap T$ is homeomorphic to $\{ (x, y) \in S \times T \mid x = y \}$. $S \times T$ has dimension $\dim S + \dim T$ and the result follows from n applications of (MZ3).

Lemma 4.1.17 *A multi-sorted κ-proper Zariski Geometry satisfies (DD).*

Proof. Let $S(x, y, z) \subseteq \mathrm{M}^{n+k+m}$ be closed and irreducible, such that $\dim \mathrm{pr}_z(S) = \dim \mathrm{pr}_{yz}(S) = n$. By 4.1.15 and 4.1.10 there is a proper closed subset $K_1 \subseteq \mathrm{pr}_{yz}(S)$ such that for all $a \subseteq \mathrm{pr}_{yz}(S) \setminus K_1$, the fibre $\mathrm{pr}_z(S)(a)$ is finite. By irreducibility of $\mathrm{pr}_z(S)$, ${\mathrm{pr}_y}^{-1}(K_1) \cap \mathrm{pr}_z(S)$ has dimension at most $\dim \mathrm{pr}_z(S) - 1$. Again by 4.1.15, there is a proper closed subset $K_2 \subseteq \mathrm{pr}_z(S)$ such that for all $(a, b) \subseteq \mathrm{pr}_z(S) \setminus K_2$, $\dim S(a, b) \leq \dim S - \dim \mathrm{pr}_z(S)$. Let $K = K_2 \cup \mathrm{pr}_y^{-1}(K_1) \cap \mathrm{pr}_z(S)$. $\dim K \leq \dim \mathrm{pr} S - 1$ so K is a proper closed subset of $\mathrm{pr}_z(S)$.

If $a \in \mathrm{pr}_{yz}(S)$, then $S(a) = S \cap (\{ a \} \times \mathrm{M}^{m+k})$. By 4.1.16, all components of this have dimension at least $\dim S - n = \dim S - \dim \mathrm{pr}_z(S)$. If $(a, b) \in \mathrm{pr}_z(S) \setminus K$, then since $\mathrm{pr}_z(S)(a)$ is finite, $S(a, b)$ is one of the finitely many components of $S(a)$ and therefore must have dimension at least $\dim S - \dim \mathrm{pr}(S)$. By definition of K_2, this means that we must have equality. $\square$

Corollary 4.1.18 *A multi-sorted κ-proper Zariski Geometry satisfies (GS) and (PS).*

Proof. Note that (GS) is a consequence of (PS) so we will just prove the latter. Let $S(x, y) \subseteq \mathrm{M}^{n+k}$ be closed and irreducible such that $\dim \mathrm{pr}_y(S) = n$. There is a proper closed subset $K \subseteq \mathrm{pr}_y(S)$ of co-dimension 2 such that all fibres over $\mathrm{pr}_y(S) \setminus K$ have dimension exactly $\dim S - \dim \mathrm{pr}(S)$. Now let $T(x, z) \subseteq \mathrm{M}^{n+m}$ be irreducible with $\dim \mathrm{pr}_z(T) = n$. Note, this is equivalent to saying $\overline{\mathrm{pr}_z(T)} = \overline{\mathrm{pr}_y(S)}$. Consider

$$W = \{ (x, y, z) \in \mathrm{M}^{n+k+m} \mid (x, y) \in S \ \& \ (x, z) \in \overline{T} \ \& \ x \notin K \}$$

By lemma 4.1.16, all components of this have dimension at least $\dim S + \dim T - n$. Now if V is a component of W, then every fibre over $\mathrm{pr}_y(V)$ has dimension at least $\dim T$ so its projection to $\overline{T}$ must be dense. This means $\mathrm{pr}_y^{-1}(T) \cap V$ is dense in V and therefore

$$Z = \{ (x, y, z) \in \mathrm{M}^{n+k+m} \mid (x, y) \in S \ \&\ (x, z) \in T \ \&\ x \notin K \}$$

has no components of dimension $\leq \dim S + \dim T - n$. $\square$

This concludes the proof of the proposition.

4.1.1 Completing the Trichotomy Results

In this section we will prove three things:

Proposition 4.1.19 *In a Stable-Like Presmooth Geometry the non-degeneracy and richness conditions for propositions 3.2.8 and 3.2.22 are always given and the group action obtained in proposition 3.2.25 is faithful.*

Proposition 4.1.20 *In a non-linear one-dimensional Stable-Like Presmooth Geometry there exists a two-dimensional type-definable group which is solvable and centreless. In particular, this group satisfies the metabelian identity*

$$[a, b].[c, d] = [c, d].[a, b] \ \textit{where}\ [a, b] = a.b.a^{-1}.b^{-1}$$

and does not satisfy the class-2 nilpotency equation

$$r.[u, v] = [u, v].r$$

Theorem 4.1.21 *If* M *is a non-linear multi-sorted* κ*-compact Zariski Geometry then* M *interprets a field.*

Throughout this section $K \subseteq C_x \times C_y \times G$ is the family from 3.2.8 and J is the relation used to define tangency (see 3.1.15). We will only show that Non-Degeneracy and Richness hold for the compositional case. The multiplicative case follows by analogy.

Non-Degeneracy

Lemma 4.1.22 *In a Presmooth Geometry satisfying (1T), non-degeneracy in the sense of 3.2.8 holds.*

Proof. This is exactly lemma 4.1.6 of [Zil10]. Suppose for some generic o, m, n, $o^{-1} \circ n \parallel o^{-1} \circ m$ so that the set $\{ r \mid o^{-1} \circ r \parallel o^{-1} \circ m \}$ is one-dimensional. As a consequence of (1T), (o, m) and (m, o) have the same projective type so the set $R = \{ r \mid r^{-1} \circ m \parallel o^{-1} \circ m \}$ is also one-dimensional. Therefore we can find a $q \in R$ with $\mathrm{cdim}\,(q/mno) = 1$ so that $\mathrm{cdim}\,(mnoq) = 4$. However, generic transitivity now gives $q^{-1} \circ m \parallel o^{-1} \circ n$ which imples that the relation $\parallel$ is four-dimensional. This is a contradiction. $\square$

Remark 4.1.23 By simple dimensional considerations the set

$$\{ (x, u, v, r, s) \mid v^{-1} \circ u(x) = s^{-1} \circ r(x) \}$$

is pure-dimensional of dimension 4. Furthermore, if C is a component not contained in $\{ c \} \times G^4$ for some c, then its projection to $C_x \times G^3$ must still be four. For suppose this is not the case. Then there is a generic $a' \neq a$ and f, g, h such that the set $\{ s \mid g^{-1} \circ f(a') = h^{-1} \circ s(a') \}$ is one-dimensional. But then all curves from G pass through $(a', h \circ g^{-1} \circ f(a'))$. However, a' was chosen generic and the set

$$\{ (x, y) \mid \forall g \in G \; g(x) = y \}$$

is zero-dimensional so this is a contradiction. This essentially establishes the "Moreover" clause of 4.1.7 in [Zil10].

Richness

Lemma 4.1.24 *Suppose* M *satisfies (SL). Then for all independent* m, n, o *there exists a* p *such that* $n^{-1} \circ m \parallel p^{-1} \circ o$*. Furthermore, if* $o \in \mathcal{V}_m$*, then* $p \in \mathcal{V}_n$*.*

Proof. This is 4.1.7 of [Zil10], but this proof is slightly different. Let C be an irreducible component of J which witnesses $n^{-1} \circ m \parallel n^{-1} \circ m$. By non-degeneracy, the fibre $C(a, m, n, m)$ is zero-dimensional so that we can assume C to be smooth with respect to the projection to $C_x \times G^3$. Furthermore, by the above remark, this projection is four-dimensional. Choose $m' \in \mathcal{V}_m^1$ and $a' \in \mathcal{V}_a^2 \setminus \{a\}$. By the Implicit Function Theorem 2.3.22, there exists an $n'' \in \mathcal{V}_n$ such that $n^{-1} \circ m(a') = n''^{-1} \circ m'(a')$. If we specialize with respect to π_2, we obtain an $n' \in \mathcal{V}_n^1$ with $n^{-1} \circ m \parallel n'^{-1} \circ m'$. This establishes the "furthermore clause", but also shows that $\{(m, n, o) \mid \exists p\ n^{-1} \circ m \parallel p^{-1} \circ o\}$ is three-dimensional. (SL) implies that there is only one generic type consistent with $\mathcal{V}_{ggg}$ so we are done. □

Remark 4.1.25 This lemma is sufficient to ensure that the proof of 3.2.11 goes through. In 3.2.11 we never use full richness. We just use the fact that given $u \neq h$ we can find a generic f over u, g such that there exists a u' with $u' \circ f^{-1} \parallel u \circ h^{-1}$.

Corollary 4.1.26 $\parallel$ *satisfies richness.*

Proof. By presmoothness and 3.2.11 we may assume that $\parallel$ is smooth with respect to any projection to G^3. We know that $(g, g, g, g) \in \parallel$ and by the above $\parallel$ projects to a dense subset of G^3. Since $\parallel$ is closed, this together with 2.3.28 gives richness. □

Faithfulness

Lemma 4.1.27 *If* M *satisfies (SL), then the group action obtained in 3.2.25 is faithful.*

For this we have to go back to the remark after proposition 3.2.25. We wrote 0 for z and took a generic $1 \in \mathcal{V}_0$. Elements of $\mathcal{V}_0^2$ represent curves of the form $(e/z \cdot g)$ and elements of $\mathcal{V}_1^2$ represent curves of the form $(e/z \cdot 1)^{-1} \circ (e/z \cdot h)$. If $f.h = f$ for some generic $f \in \mathcal{V}_0^2$ and $h \in \mathcal{V}_1^2$, then this is true for all generic $(1, f, h) \in \mathcal{V}_{000}$. However, we can then use the same argument as we did in 4.1.22 to show that the tangency relation

$$(e/z \cdot r)^{-1} \circ (e/z \cdot s) \parallel (e/z \cdot u)^{-1} \circ (e/z \cdot v)$$

is four-dimensional. This is a contradiction and therefore the action is faithful. □

This concludes the proof of proposition 4.1.19.

Recovering a Field

Lemma 4.1.28 *(Type-Definable Reinecke's Theorem) Suppose* M *satisfies (SL) and* $\langle \mathcal{V}_e, . \rangle$ *is a smooth type-definable group on an irreducible one-dimensional curve* $G \ni e$. *Then* $\mathcal{V}_e$ *is abelian.*

Proof. Consider the set $G_n = \{ g \in G \mid g^n = e \}$. Since . is given by a relation, G_n is a projective subset of G. By (SL) all non-identity elements of $\mathcal{V}_e$ have the same projective type so in particular they have the same order. Suppose $\mathcal{V}_e$ is not abelian. Then lemma 2.4.7 implies that this order is finite and therefore prime. Hence, we may assume that $G = G_p$ for some prime p. If $p = 2$ then $\mathcal{V}_e$ is ablelian by a standard group theoretic argument.

Since we assumed G is not to be abelian, the set $\{ (x, y) \in G \mid \exists z \in G \; x^z = y \}$ is two-dimensional and therefore contains all generic elements of $\mathcal{V}_{ee}$. In particular this means that for any two independent $f, g \in \mathcal{V}_e$ there exists a $h \in G$ such that $f^h = g$ and thus for any two $f, g \in \mathcal{V}_e$ there are $i, j \in G$ such that $f^{i^j} = g$. Pick a generic $f \in \mathcal{V}_e$ and i, j such that $f^{i^j} = f^2$. $2^p \equiv 2 (\text{mod } p)$ so by associativity

$$f = f^e = f^{(ij)^p} = f^{2^p} = f^2$$

This is clearly a contradiction. □

Lemma 4.1.29 *The multiplicative group of the group presentation does not have exponent* 2.

Proof. Suppose it did. Then for all $g \in \mathcal{V}_1$ and all $h \in \mathcal{V}_0$, $h + h.g$ is a fixed point of g. However, we know that the only fixed point of g is 0 so this would imply that for all g and h, $h.g = -h$. If we pick another element $f \in \mathcal{V}_1 \setminus \{g, g^{-1}, 1\}$, then $h.(fg) = --h = h$, but this constradicts the previous assertion. □

We are now in a position to prove 4.1.20.

Consider the the set $\mathcal{V}_1 \times \mathcal{V}_0$ with the operation $(u, v).(r, s) = (u.r, s.u + v.r^{-1})$. This corresponds to thinking of elements $(u, v) \in \mathcal{V}_0 \times \mathcal{V}_1$ as matrices

$$(u, v) = \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix}$$

and doing matrix multiplication. Since $\mathcal{V}_1.\mathcal{V}_1 = \mathcal{V}_1$ and $\mathcal{V}_0.\mathcal{V}_1 \subseteq \mathcal{V}_0$, this is well defined. Since both groups are abelian, we have inverses $(u, v)^{-1} = (u^{-1}, -v)$ and this defines a group. It is easy to check that this group satisfies the required identities. It is also centreless since for any u, v and r, (u, v) commutes with $(r, 0)$ if and only if $v.r^{-1} = v.r$. The multiplicative group does not have exponent 2 and the action is faithful so this is only the case if $v = 0$. □

We will now show that in the context of multi-sorted κ-compact Zariski Geometries this gives rise to a field.

Proposition 4.1.30 *(Group Chunk Theorem) If* M *is a multi-sorted* κ*-compact Zariski Geometry,* $H \subseteq \mathrm{M}^n$ *is irreducible and* $\langle H, \cdot \rangle$ *is a smooth group chunk with identity* f*, then there exists an open subset* $f \in U \subseteq H$ *and a closed equivalence relation* $\sim$ *on*

$U \times U$ and a closed relation $\otimes \subseteq U^4$ such that the trace . of $\otimes$ on $G = U^2/\sim$ is a group. Furthermore, the map $\phi: u \mapsto (u, f)/\sim$ is an embedding of $\langle \mathcal{V}_f, \cdot \rangle$ into G.

Proof. We define the equivalence relation $\sim$ on $H \times H$ by

$$(h_1, h_2) \sim (g_1, g_2) \text{ iff } \dim \{ u \in H \mid h_1 \cdot (h_2 \cdot u) = g_1 \cdot (g_2 \cdot u) \} = \dim H$$

Since H is irreducible, this is the same as saying that $\{ u \in H \mid h_1 \cdot (h_2 \cdot u) = g_1 \cdot (g_2 \cdot u) \}$ is dense in H and therefore $\sim$ is an equivalence relation. By definability of dimension the set $\sim$ is projective. Suppose that $(a, b, c, d) \in \overline{\sim}$, the closure of the relation $\sim$. Let $u \in H$ be independent of a, b, c, d. Let $(a', b', c', d') \in \mathcal{V}_{abcd}$ be such that $(a', b') \sim (c', d')$. Then by definition of $\sim$, there is a $u' \in \mathcal{V}_u$ such that $a' \cdot (b' \cdot u') = c' \cdot (d' \cdot u')$. By smoothness of the group-chunk, all elements involved in this calculation are in the domain of the specialization so we can specialize to get $a \cdot (b \cdot u) = c \cdot (d \cdot u)$. However, we chose u independent of a, b, c, d so we must have $\dim \{ u \in H \mid a \cdot (b \cdot u) = c \cdot (d \cdot u) \} \geq \operatorname{cdim}(u/abcd) = \dim H$. Therefore $(a, b) \sim (c, d)$.

We define multiplication on G by $(a_1, a_2)/\sim . (b_1, b_2)/\sim = (c_1, c_2)/\sim$ iff

$$\dim \{ u \in H \mid a_1 \cdot (a_2 \cdot (b_1 \cdot (b_2 \cdot u))) = c_1 \cdot (c_2 \cdot u) \} = \dim H$$

It is easy to check that this is a well-defined relation and using the same argument as above we can show that this is the trace of a closed relation on H^4.

To show that this defines a binary relation we have to check the closure property. For this it is enough to check that for all $a, b, c \in H$ there are $d, e \in H$ such that $\dim \{ u \in H \mid a \cdot (b \cdot (c \cdot u)) = d \cdot (e \cdot u) \} = \dim H$. We see this by writing $b = b_1 \cdot b_2$ where b_1 and b_2 are generic over a and c. Then $d = a \cdot b_1$ and $e = b_2 \cdot c$ exist and for u generic over a, b, c, b_1, b_2 we get the required equality.

The last thing to show is that G is indeed a group. Clearly $(f, f)/\sim$ is an identity element and . is associative since $\cdot$ is associative. To see that G has inverses, consider $(a, b)/\sim$. We write $a = a_1 \cdot a_2$ and $b = b_1 \cdot b_2$ for generic a_1, a_2, b_1, b_2. Since H is irreducible, all generic elements have inverses so we see that $(b_2^{-1}, b_1^{-1})/\sim .(a_2^{-1}, a_1^{-1})/\sim$ is an inverse for $(a,b)/\sim$.

The fact that the map ϕ is an embedding of $\mathcal{V}_f, \cdot$ into G follows from construction. □

It follows from the previous two propositions that M quantifier-free interprets a two-dimensional type-definable group which is solvable and has finite centre. A multi-sorted κ-compact Zariski Geometry is quantifier-free ω-stable and it is stated in [CHP02] that classical results about the construction of a field from a two-dimensional group as above apply in such a context. We refer the reader to [Poi01], [Che79] or [Zil10]. This proves 4.1.21.□

We will now look at some specific examples classes of structures to which these results apply.

4.1.2 Zariski Geometries

The easiest examples of multi-sorted κ-proper Zariski Geometries are of course one-dimensional Noetherian Zariski Geometries. As mentioned before, there are two quite distinct definitions in the literature.

First we give the axiomatization from [HZ96]:

Definition 4.1.31 A *one-dimensional Zariski Geometry* is a structure M in which the positive quantifier free definable sets form the closed sets of a Noetherian Topology on M^n for each n and which satisfies the following three axioms:

(Z1) For any closed irreducible set $S \subseteq \mathrm{M}^{n+k}$ and a projection $\mathrm{pr}\colon \mathrm{M}^{n+k} \to \mathrm{M}^n$, $\mathrm{pr}S \subseteq \overline{\mathrm{pr}S} \setminus F$ for some proper closed subset $F \subseteq \overline{\mathrm{pr}S}$.

(Z2) For any closed set $S \subseteq \mathrm{M}^{n+1}$, there is some m such that for all $a \in \mathrm{M}^n$, either $S(a,\mathrm{M}) = \mathrm{M}$ or $|S(a,\mathrm{M})| \leq m$.

(Z3) $\dim \mathrm{M}^n = n$ and for $S \subseteq \mathrm{M}^n$ closed and irreducible, every component of the diagonal $S \cap \{x_i = x_j\}$ is of dimension $\geq \dim S - 1$.

We also give the axiomatization from [Zil10] which inspired our axiomatization of Presmooth Geometries.

Definition 4.1.32 Let M be a structure in a relational language $\mathcal{C}$. We say that M is a *topological structure if the closed (positive quantifier-free* M*-definable) subsets of* M^n *form a topology for every* n*.*

Definition 4.1.33 Suppose there is a natural number $\dim S$ attached to evrey constructible set in a topological structure M. We say that M has *good dimension if it satisfies*

(DP) *(**D**imension of a **P**oint) The dimension of a point is* 0.

(DU) *(**D**imension of **U**nions) Given projective sets* $S_1, S_2 \subseteq \mathrm{M}^n$,

$$\dim(S_1 \cup S_2) = \max\{\dim(S_1), \dim(S_2)\}$$

(SI) *(**S**trong **I**rreducibility) If* $S \subseteq_{\mathrm{cl}} U \subseteq_{\mathrm{op}} \mathrm{M}^n$ *is irreducible, then* S *is strongly irreducible:*

$$T \subsetneq_{\mathrm{cl}} S \Rightarrow \dim T < \dim S$$

(AF) *(**A**ddition **F**ormula) If* $S \subseteq_{\mathrm{cl}} U \subseteq_{\mathrm{op}} \mathrm{M}^n$ *is irreducible and* pr *is a projection, then*

$$\dim S = \dim\ \mathrm{pr}S + \min_{a \in \mathrm{pr}S} \dim(\mathrm{pr}^{-1}(a) \cap S)$$

(FC) *(**F**ibre **C**ondition) If* $S \subseteq_{\mathrm{cl}} U \subseteq_{\mathrm{op}} \mathrm{M}^n$ *is irreducible, then there is a* $V \subseteq_{\mathrm{op}} \mathrm{pr}S$ *such that*

$$\forall v \in V \quad \dim(\mathrm{pr}^{-1}(v) \cap S) = \min_{a \in \mathrm{pr}S} (\dim(\mathrm{pr}^{-1}(a) \cap S))$$

Definition 4.1.34 A *Noetherian Zariski Structure is a Noetherian topological structure* M *satisfying*

(SP) *(**S**emi- **P**roperness) For $S \subseteq_{\text{cl}} \mathrm{M}^{n+k}$ and $\mathrm{pr}\colon \mathrm{M}^{n+k} \to \mathrm{M}^n$ a projection, $\mathrm{pr}S \supseteq \overline{\mathrm{pr}S} \setminus F$ for some proper closed subset $F \subseteq_{\text{cl}} \overline{\mathrm{pr}S}$.*

Definition 4.1.35 A *Noetherian Zariski Geometry is a Noetherian Zariski Structure* M *satisfying*

(PS) *(**P**re**S**moothness)If $S_1, S_2 \subseteq_{\text{cl}} \mathrm{M}^n$ are irreducible, then any component of $S_1 \cap S_2$ has dimension at least $\dim S_1 + \dim S_2 - \dim \mathrm{M}^n$.*

(EU) *(**E**ssential **U**ncountability) If $S \subseteq_{\text{cl}} \mathrm{M}^n$ is the union of countably many closed subsets, then there are finitely many of these, the union of which is S.*

Note that (EU) is equivalent to ω_1-compactness (see 3.2.6 of [Zil10]) which in turn is a consequence of ω_1-saturation. We get the following equivalence, answering a question left open in [Zil10] (see the beginning of section 3.3).

Proposition 4.1.36 *An ω_1-saturated structure is a Zariski Geometry in the sense of 4.1.31 if and only if it is an irreducible one-dimensional Noetherian Zariski Geometry in the sense of 4.1.35.*

Proof. Our first task is to show that the dimension in definition 4.1.35 is Krull dimension which we shall denote by kdim. We will show by induction on n that for all $S \subseteq \mathrm{M}^n$, $\dim(S) = \mathrm{kdim}(S)$. By (DU) it is enough to consider irreducible S and (SI) clearly implies that $\dim(S) > \mathrm{kdim}(S)$. For the base case, suppose $S \subseteq \mathrm{M}$ is irreducible. $\dim \mathrm{M} = 1$ so by (DU), M is infinite and $\mathrm{kdim}(\mathrm{M}) = 1$. By Theorem 3.2.1 of [Zil10], M has quantifier elimination so S is constructible and therefore finite or co-finite. Either way, (DU) implies $\dim(S) = \mathrm{kdim}(S)$. Now suppose dim and kdim are the same on M^k for $k < n$ and let $S(x, y) \subseteq \mathrm{M}^n$ where $\#(x) = n - 1$ and $\#(y) = 1$. By (AF), either $\dim(S) = \dim \mathrm{pr}_y(S)$ of $\dim(S) = \dim \mathrm{pr}_y(S) + 1$. In the former case, any descending chain of irreducibles in $\mathrm{pr}_y(S)$ gives rise to a chain in S so $\mathrm{kdim}(S) \geq \mathrm{kdim}\ \mathrm{pr}_y(S) = \dim \mathrm{pr}_y(S) = \dim(S)$. In the latter case, $\dim \mathrm{pr}_x(S) = 1$ so that by (AF), for some $a \in \mathrm{M}$, $\dim(S(a)) = n - 1$. Now $S(a) \times \{a\}$ is homeomorphic to $\mathrm{pr}_y(S)$ and a proper closed subset of S so $\mathrm{kdim}(S) \geq \mathrm{kdim}\ \mathrm{pr}_y(S) + 1 = \dim \mathrm{pr}_y(S) + 1 = \dim(S)$.

With the equivalence of the two dimension notions established, we just have to check the axioms:

(Z1) is just (SP).

(Z3) is an immediate consequence of (PS).

(Z2) follows from (EU) by lemma 3.2.8 of [Zil10].

For the other direction we use the work that we have done to show that a multi-sorted κ-proper Zariski Geometry is a Presmooth Geometry. That means (DP), (DU) are immediate. (SI) is a straightforward consequence of the definition of Krull dimension. (SP) is the same as (Z1) and (AF) and (FC) follow from 4.1.17. Presmoothness is proved in 4.1.16. □

Details of many examples of Zariski Gemetries can be found in [Zil10]. We merely give a list with references where appropriate:

1. An uncountable set with equality as the only relation,
2. Vector Spaces,
3. Algebraic Varieties over algebraically closed fields (see [Zil10] for references to relevant literature on algebraic geometry),
4. Orbifolds (quotients of algebraic varieties by groups of regular automorphisms) [Zil10],
5. Compact Complex Manifolds [Zil10],
6. Proper Rigid Analytic Varieties [Zil10],
7. Sets of finite Morley rank and degree 1 definable in Differentially Closed Fields (after removing a set of smaller rank) [Pil02],
8. Finite covers of algebratically closed fields. Section 10 of [HZ96] or 5.1 and 5.2 of [Zil10],

9. Quantum Harmonic Oscillator [SZ],
10. Compact Analytic Zariski Geometries [PZ04] and [Zil10].

4.1.3 Type Definable Zariski Geometries

In this section we work in M, a κ-saturated structure with quantifier elimination in a language $\mathcal{L}$ of size less than κ.

Definition 4.1.37 Suppose that $p(x)$ is a type defined over some set A of size less than κ and let $P = p(\mathrm{M})$. We say that a set $S \subseteq P^n$ is *relatively definable* over M if there is a formula $\phi(x, y)$ and a tuple $a \in \mathrm{M}^k$ such that $S = \phi(\mathrm{M}, a) \cap P^n$.

We define the structure $\mathfrak{X}(P)$ to be the set P with relations for all sets relatively defined by positive quantifier free formulas over M.

$\mathfrak{X}(P)$ is a *one-dimensional type-definable Zariski Geometry*, if the closed sets form a Noetherian topology on P^n for each n and if these together with Krull Dimension satisfy (Z2) and (Z3).

The major difference to normal one-dimensional Zariski Geometries is the missing axiom (Z1). This has been replaced by quantifier elimination of the ambient structure M. We will show that thanks to the saturation of M, the structure $\mathfrak{X}(P)$ satisfies (κMZ1) so that we get the following proposition.

Proposition 4.1.38 *A type-definable Zariski Geometry in a κ-saturated stucture* M *is a κ-proper Zariski Geometry.*

To prove that (κMZ1) holds we need to relate "internal" and "external" definability.

Definition 4.1.39 Given types $p(x), r(r)$ and $q(x, y)$ over A, we write

$$
\begin{aligned}
p^{(n)}(x_1, \ldots, x_n) &= \operatorname{ded}\{\, \phi(x_i) \mid \phi(x) \in p \,\} \\
(\exists y) q(x, y) &= \operatorname{ded}\{\, \exists y \phi(x, y) \mid \phi \in q \,\} \\
(\exists^{\geq n} y) q(x, y) &= \operatorname{ded}\{\, \exists^{\geq n} y \phi(x, y) \mid \phi \in q \,\} \\
(\exists^{\infty} y) q(x, y) &= \operatorname{ded}\{\, \exists^{\geq n} y \phi(x, y) \mid \phi \in q \;\&\; n \in \omega \,\} \\
p \wedge r &= p \cup r \\
p \vee r &= \operatorname{ded}\{\, \phi \vee \psi \mid \phi \in p \& \psi \in r \,\}
\end{aligned}
$$

where ded stands for the deductive closure of a set of formulas.

Lemma 4.1.40 *Let $P = p(\mathrm{M})$, $R = r(\mathrm{M})$ and $Q = q(\mathrm{M})$ for types p, q, r defined over set of parameters of size $< \kappa$. Then*

1. $p^{(n)}(\mathrm{M}) = P^n$
2. $p \wedge r(\mathrm{M}) = P \cap R$
3. $p \vee r(\mathrm{M}) = P \cup R$
4. $(\exists y) q(\mathrm{M}) = \mathrm{pr}_y(Q)$
5. $(\exists^{\geq n} y) q(\mathrm{M}) = \{ a \in \mathrm{pr}_y(Q) \mid \; |Q(a)| \geq n \}$
6. $(\exists^{\infty} y) q(\mathrm{M}) = \{ a \in \mathrm{pr}_y(Q) \mid \; Q(a) \text{ is infinite} \}$

Proof. 1, 2 and 3 follow straight from definitions. 4, 5 and 6 follow from an easy compactness argument using the saturation of M. □

Lemma 4.1.41 *P is κ-compact and for any closed irreducible set $S \subseteq P^{n+k}$ and a projection $\mathrm{pr}\colon P^{n+k} \to P^n$, $\mathrm{pr} S \subseteq \overline{\mathrm{pr} S} \setminus \bigcup F_i$ for fewer than κ proper closed subsets $F_i \subseteq \overline{\mathrm{pr} S}$.*

Proof. κ-compactness is an easy consequence of κ-saturation of M. By the previous lemma, if Q is a projective set, then Q is type-definable in M by a type q over a set of parameters of size at most $|A|$. If Q_i is a consistent family of projective sets, then $\bigcap_{i\in I} Q_i$ is defined in M by a consistent type over a set of parameters of size at most $\max(|I|, |A|)$. By saturation of M this type has a realization so the set is non-empty.

For the second part of the lemma we note that in M, the set S is defined by a type $p^{(n+k)} \cup \{\phi(x, y)\}$ for some positive quantifier free formula ϕ. So by the previous lemma the set $\mathrm{pr}(S)$ is defined by the type $s = (\exists y)(p^{(n+k)}(x, y) \cup \{\phi(x, y)\})$. This contains at most $|p|$-many formulas and by quantifier elimination, we can assume all of these to be quantifier-free. Furthermore $s(\mathrm{M}) \subseteq P^n$, so $\mathrm{pr}(S) = s(\mathrm{M})$ is the intersection of $|p|$ many constructible sets in P^n. Since P^n is Noetherian, this proves the lemma. $\square$

The key examples of these appear in Hrushovski's proof of the relative version of the Mordell-Lang conjecture in [Hru96]. He proves the following:

Proposition 4.1.42 *(Hrushovski) Let T be the theory of differentially closed fields (or separably closed fields of charactersistic p and dimension p^ν over K^p). Suppose that $\mathrm{dcl}(aB)$ is a field extension of finite transcendence degree over $\mathrm{dcl}(B)$. Then $p = \mathrm{tp}(a/B)$ with the induced structure is a type-definable Zariski Geometry.*

4.1.4 ACFA

Definition 4.1.43 A *difference field* is a structure $\langle K, ., +, \sigma\rangle$ where $\langle K, ., +\rangle$ is a field and $\sigma: K \to K$ is a field automorphism.

In the 90s Macintyre isolated axioms for ACFA, the model compantion of the theory of difference fields. The following version is from [CH99]:

Definition 4.1.44 A structure $\langle K, ., +, \sigma\rangle$ is a model of ACFA if

1. σ is an automorphism of K,

2. K is an algebraically closed field and

3. for any absolutely irreducible algebraic varieties U and $V \subseteq U \times \sigma(U)$ projecting generically onto U and $\sigma(U)$ and every algebraic set W properly contained in V, there is $a \in U(K)$ such that $(a, \sigma(a)) \in V \setminus W$.

Models of ACFA are often called algebraically closed fields with a *generic automorphism*. The genericity of σ is encapsulated in axiom schema 3 above. Just like algebraically closed fields are the natural domain for the study of polynomial equations, models of ACFA are commonly used to study the algebra and model theory of difference fields. Completions of ACFA are simple, but not stable and do not in general have quantifier elimination. To overcome the associated problems we have to carry out the field construction in a *limit structure* instead of a model of ACFA. The limit structure is obtained by considering successively smoother reducts of ACFA and turns out to be a multi-sorted ω-proper Zariski Geometry. See [CHP02] for a detailed account of this construction and the associated proofs. We merely give an outline.

We consider a countable algebraically closed difference field k_0 embedded in a saturated model of ACFA $K \prec \Omega$ for some universal domain Ω.

The place of polynomials is taken by difference equations so given a difference field E and $X = (X_1, \ldots, X_n)$, we define $E[X]_\sigma$ to be the (inversive) difference polynomial ring in variables $X_1, \ldots, X_n$ over E. This can be seen as the polynomial ring in vatiables $\sigma^j(X_i)$ with $j \in \mathbb{Z}$ and the obvious action of σ. Given a difference ring R, we say that $I \subseteq R$ is a σ-ideal if I is an ideal in R and $\sigma(I) = I$. I is a prime σ-ideal if it is prime as an ideal in R. I is *perfect* if for all $i_1, \ldots, i_n \in \mathbb{N}$ and a difference equation f, $\sigma^{i_1}(f) \ldots \sigma^{i_n}(f) \in I$ implies $f \in I$.

Given a prime σ-ideal $P \subseteq E[X]_\sigma$, we get a sigma domain $E[x]_\sigma = E[X]_\sigma / P$ and a domain $E[x] = E[X]/P \cap E[X]$. We write $E(x)_\sigma$ and $E(x)$ for the fields of fraction. Given a multiplicative subset S of $k_0[x]$ not containing 0, we define $k_0\{x\} = k_0[x, 1/s \mid s \in S]$. For $m \geq 1$, $k_0\{x\}_{\sigma^m}$ denotes the σ^m-difference subring of $k_0(x)$ generated by $k_0\{x\}$. If $k_1 \supseteq k_0$, we define $k_1\{x\} = k_1 \otimes_{k_0} k_0\{x\}$ and $k_1\{x\}_{\sigma^m} = k_1 \otimes_{k_0} k_0\{x\}_{\sigma^m}$.

A *pair of coordinate rings* (R, R_σ) is of the form $(K\{x\}, K\{x\}_\sigma)$ as defined above where $I = P$ is a prime σ-ideal and S is some multiplicative subset. We write R_{σ^m} for $K\{x\}_{\sigma^m}$. Given such a pair, we say that $I \subseteq R_\sigma$ is a *virtual ideal* if $I \cap R_{\sigma^m}$ is a perfect σ^m-ideal of R_{σ^m} for some $m \geq 1$. We say that I is a *virtual prime ideal* if I is prime as an ideal. Given a virtual ideal I, we define $V(I)$ to be the set of tuples of K such that for some m there is a σ^m-K-homomorphism $K\{x\}_{\sigma^m}/(I \cap K\{x\}_{\sigma^m}) \to K$ which sends x to a. We can define an equivalence relation $\sim$ on virtual ideals by $I \sim J$ if for some $m \geq 1$, $I \cap R_{\sigma^m} = J \cap R_{\sigma^m}$. Clearly if $I \sim J$, then $V(I) = V(J)$.

If $\mathcal{L} = \{.,+,0,1,\sigma\}$ is the language of difference fields, we consider the reducts $\mathcal{L}_m = \{.,+,0,1,\sigma^m\}$. For any type p over k_0 we can consider $p[m]$, the restriction of p to $\mathcal{L}_m$. We define the equivalence relation $\sim$ on quantifier free types over k_0 by $p \sim q$ if and only if $p[m] = q[m]$ for some m and write $[p]$ for the $\sim$-equivalence class of p and call this a *virtual type*. As we remarked above, models of ACFA are simple so each type $p[m]$ has SU-rank which we will denote by $\mathrm{SU}(p)[m]$.

Given a tuple a and a difference-subfield $E \subseteq K$, we write $E(a)_\sigma$ for the difference field generated by a over E. We we define $\deg_\sigma(a/E) = \mathrm{tr.deg}(E(a)_\sigma/E)$.

Suppose that $E \subseteq K$ with $E^{\mathrm{alg}} = \sigma^n(E)^{\mathrm{alg}}$ for some n and a is a tuple such that $\deg_\sigma(a/E) < \infty$. We define the *eventual* SU-*rank* of a over E by

$$\mathrm{evSU}(a/E) = \lim_{m \geq n}(\mathrm{SU}(\mathrm{qftp}(a/E)[m!]))$$

If $p = \mathrm{qftp}(a/E)$ we define $\mathrm{evSU}(p) = \mathrm{evSU}(a/E)$.

It is shown in 1.13 of [CHP02] that this is well defined. It should be clear that if $p \sim q$, then $\mathrm{evSU}(p) = \mathrm{evSU}(q)$. We say that a quantifier free type p has property (ALGm) if every realization a of p satisfies $\sigma^m(a) \in k_0(a)^{\mathrm{alg}}$. We define $\mathcal{Y}_1(k_0)$ to be the set of all complete quantifier free types over k_0 which satisfy (ALGm) for some $m \geq 1$ and which have eventual SU-rank 1. We call these *basic types* and $\sim$-equivalence classes of basic types *basic virtual types*. We say that q is a *semi-basic type* if for every realization a of q there are independent realizations $a_1, \ldots, a_k \in k_0(a)^{\mathrm{alg}}$ of basic types $p_1, \ldots, p_k$ such that $a \in k_0(a_1, \ldots, a_k)^{\mathrm{alg}}$. We write $\mathcal{Y}(k_0)$ for the set of semi-basic types over k_0.

To each semi-basic type p we can associate a pair of coordinate rings $(R_p, R_{p,\sigma})$ as follows. We let

$$P = I(p) = \{ f \in k_0[X]_\sigma \mid f(x) = 0 \in p \},$$

$S = \emptyset$, $R_p = K\{x\}$ and $R_{p,\sigma} = K\{x\}_\sigma$. If $p = (p_1, \ldots, p_n)$ is a tuple of semi-algebraic types we let $R_p = R_{p_1} \otimes_K \ldots \otimes_K R_{p_n}$ and $R_{p,\sigma} = R_{p_1,\sigma} \otimes_K \ldots \otimes_K R_{p_n,\sigma}$ and call this the *pair of coordinate rings associated with the tuple* $(p_1, \ldots, p_n)$.

If p is an n-type, we write $\mathfrak{X}_p(K) = \{ a \in K^n \mid K \models p[m](a) \text{ for some } m \}$.

We now define languages $\mathcal{L}^*$ and $\mathcal{L}^*_K$ for structures on the set

$$\mathfrak{X}(K) = \bigcup_{p \in \mathcal{Y}(k_0)} \mathfrak{X}_p(K)$$

In each case the sorts are indexed by $\sim$-equivalence classes of types from $\mathcal{Y}(k_0)$ and we let $\mathfrak{X}_p(K)$ be the set of elements of sort $[p]$. The atomic formulas of $\mathcal{L}^*_K$ are of the form $(x_1, \ldots, x_n) \in V(P)$ where P is a prime virtual ideal in the pair of coordinate rings associated with $(p_1, \ldots, p_n)$ where x_i is of sort $[p_i]$. This is well-defined and the interpretation is the obvious one. $\mathcal{L}^*$ is the reduct of $\mathcal{L}^*_K$ consisting of formulas $(x_1, \ldots, x_n) \in V(P)$ where P is generated by its restriction to $k_0\{x_1, \ldots, x_n\}_\sigma$.

The $\mathcal{L}^*$-positive quantifier free definable sets in $\mathfrak{X}(K)$ form a Noetherian topology on each $\mathfrak{X}_p(K)$ where p is a tuple of semi-basic types. Furthermore, every closed set in $\mathfrak{X}_p(K)$ is definable over $\mathfrak{X}_p(K)$ (Propositions 4.2 and 4.7 of [CHP02]). Therefore, the same is true for the substructure

$$\mathfrak{Y}(K) = \bigcup_{p \text{ basic}} \mathfrak{X}_p(K)$$

The key result for our purposes is

Theorem 4.1.45 *(4.16 of [CHP02]) The $\mathcal{L}^*$-structure $\mathfrak{Y}(K)$ is an ω-proper multi-sorted Zariski Geometry.*

They give an argument that $\mathfrak{Y}(K)$ interprets an algebraically closed field and infer the trichotomy theorem 1.1.16 from that. That field construction can now be seen in the context of Premooth Geometries.

4.2 O-Minimal Structures

Definition 4.2.1 A structure $< \mathrm{M}, <, \cdots >$ is called o-minimal if all definable sets $X \subseteq \mathrm{M}$ are finite unions of points and intervals.

Since there is an order, we have a good notion of closed sets: the closed sets in the order topology. In the context of Strong Geometries we want the closed sets to correspond to positive quantifier free definable fomrulas so we need to change our language accordingly. Let $\mathcal{L}_{\mathrm{nat}}$ be the language containing relations for all M-definable sets which are closed in the order topology. Clearly the resulting structure $\mathrm{M}_{\mathrm{nat}}$ is a reduct of M, but it follows from standard facts about o-minimal structures that this is inter-definable with M. We will see this in the course of this section.

Proposition 4.2.2 *If* M *is o-minimal, then* $\mathrm{M}_{\mathrm{nat}}$ *is a Presmooth Geometry satisfying (2T).*

The main ingredient for both the interdefinability of M and $\mathrm{M}_{\mathrm{nat}}$ and this proposition is the Cell Decomposition Theorem. I recommend [vdD98] or [Pet07] for a detailed treatment of o-minimal structures.

Definition 4.2.3 We say that a function $f : \mathrm{M}^n \to \mathrm{M}^m$ is definable if the graph of f is definable.

Definition 4.2.4 (n-Cells) We define a 1-cell to be a point or an open interval in M. If $Y \subseteq \mathrm{M}^n$ is an n-cell, then we define an $(n+1)$-cell to be either of the form

$$\{ (y, z) \mid y \in Y \ \& \ z = f(y) \}$$

where f is a definable continuous function $Y \to \mathrm{M}$ or

$$\{ (y, z) \mid y \in Y \ \& \ z \in (f(y), g(y)) \}$$

where f and g are definable continuous functions $Y \to \mathrm{M}$ with $g(y) > f(y)$.

Theorem 4.2.5 *(Cell Decomposition) Every definable set $X \subseteq \mathrm{M}^n$ is a finite union of disjoint n-cells.*

With this theorem we can easily define a dimension of a set using the recursive definition of a cell.

Definition 4.2.6 (Dimension) We define the dimension of an n-cell recursively. A 1-cell has dimension 0 if it is a point and 1 if it is an interval. Given an $(n+1)$-cell $Z \subseteq \mathrm{M}^{n+1}$ with projection Y in M^n, we define $\dim Z = \dim Y$ if Z is obtained from Y using the first construction rule for cells and $\dim Z = \dim Y + 1$ if it is obtained using the second.

For a definable set X we define

$$\dim(X) = \max\{ \dim(C) \mid C \subseteq X \ \& \ C \text{ is a cell } \}$$

Cell decompositions are not unique, but it should be clear that given any cell decomposition of a definable set X, there is always at least one cell C in the decomposition such that $\dim C = \dim X$.

The order of the coordinates is firmly embedded in the definition of cells so it is not obvious that the dimension notion we defined above is invariant under permutation of coordinates. The following result gives an alternative characterization of dimension which makes this fact obvious.

Proposition 4.2.7 *Given a definable set $X \subseteq \mathrm{M}^n$, $\dim(X)$ is the largest natural number m such that there is a projection $\mathrm{pr}\colon \mathrm{M}^n \to \mathrm{M}^m$ such that $\mathrm{pr}(X)$ has interior.*

From this proposition it is also easy to see that the dimension is definable and that for any definable set X, the dimension of the boundary is small: $\dim \overline{X} \setminus X < \dim X$. Furthermore, for a any cell C and any open set U, either $C \cap U = \emptyset$ or $\dim(C \cap U) = \dim C$. This means that cells and in particular the topological closures of cells are pure-dimensional.

We can now show that M and $\mathrm{M}_{\mathrm{nat}}$ are inter-definable. Suppose not. Let X be a set of minimal dimension under the condition that it is definable in M, but not in $\mathrm{M}_{\mathrm{nat}}$. $\overline{X}$ is closed and definable in M so it is also definable in $\mathrm{M}_{\mathrm{nat}}$. However, $\dim \overline{X} \setminus X < \dim X$ so by the minimality of the dimension of X, $\overline{X} \setminus X$ is also definable in $\mathrm{M}_{\mathrm{nat}}$. This is a contradiction since X is now definable in $\mathrm{M}_{\mathrm{nat}}$. From now on we will assume that all o-minimal structures are given in $\mathcal{L}_{\mathrm{nat}}$ so that we can omit the subscript. From the definitions and results above it is clear that M satisfies axioms (DP),(DU),(DD),(SB) and (DC), which leaves (2T) and (PS) to complete the proof of proposition 4.2.2.

Lemma 4.2.8 *Let $S(x, y) \subseteq \mathrm{M}^2$ be pure-dimensional of dimension 2 and $D(x, y)$ be the diagonal $x = y$. Define*

$$S^{\geq} = \{ a \mid (a, a) \in S \cap D \text{ and for some } b > a,\ [a, b] \subseteq S_1(a) \} \text{ and}$$
$$S^{\leq} = \{ a \mid (a, a) \in S \cap D \text{ and for some } b < a,\ [b, a] \subseteq S_1(a) \}$$

If $\dim(S \cap D) = 1$, then $S^{\geq} \cup S^{\leq} \subseteq \mathrm{pr}_y(S \cap D)$ is dense and open.

Proof. Let $C_1, \ldots, C_k$ be the 2-dimensional cells in a cell-decomposition of S. By pure-dimensionality of S, for all $(a, a) \in S \cap D$ and all open sets $U \ni (a, a)$, $U \cap C_i \neq \emptyset$ for some i. In particular, $\bigcup \mathrm{pr}_y(C_i)$ is dense in $\mathrm{pr}_y(S \cap D)$ so the set $U = \bigcup \mathrm{pr}_y(C_i)$ is open by the definition of 2-dimensional 2-cells. If $a \in U \cap \mathrm{pr}_y(S \cap D)$ then $(a, a) \in \overline{C_i}$ for some i. By definition, there are continuous definable functions f and g such that $C_i = \{ (x, y) \mid x \in \mathrm{pr}_y(C_i) \ \&\ y \in (f(x), g(x)) \}$. If $f(a) > a$ or $g(a) < a$, then by continuity $(a, a) \notin \overline{C_i}$. Therefore, there is a $b \neq a$ such that $[a, b] \in S(a)$ or $[b, a] \in S(a)$ as required. $\square$

Corollary 4.2.9 M *satisfies (2T).*

Proof. Let $D(x, y) \subseteq \mathrm{M}^2$ be the diagonal $x = y$ and $S_1(x, y), S_2(x, y), S_3(x, y) \subseteq \mathrm{M}^2$ be pure-dimensional of dimension 2 such that $\dim(S_1 \cap S_2 \cap S_3 \cap D) = 1$. By the previsous lemma. The eight sets $T_\sigma = S_1^{\sigma(1)} \cap S_2^{\sigma(2)} \cap S_3^{\sigma(3)}$ where $\sigma\colon \{1, 2, 3\} \to \{\geq, \leq\}$ cover a dense subset of $\mathrm{pr}_y(S_1 \cap S_2 \cap S_3 \cap D)$. So for some σ, $\dim T_\sigma = 1$. However, for some $i \neq j$, $\sigma(i) = \sigma(j)$ so by additivity of dimension, $\dim(S_i \cap S_j) = 2$. $\square$

Presmoothness is a more subtle issue. We start by giving a different characterizatrion of smoothness by showing that a set is smooth if it satisfies a version of the Implicit Function Theorem. Thanks to the definable nature of the topology, we do not need to refer to specializations to express this.

Lemma 4.2.10 *Suppose $S(x, y) \subseteq \mathrm{M}^{n+k}$ is definable and aligned with respect to pr_y. Furthermore, assume that for all points $(a, b) \in S$ and for all open sets $V \ni b$ there is an open set $U \ni a$ such that $\mathrm{pr}_y(S \cap (U \times V)) = \mathrm{pr}_y(S) \cap U$. Then S is smooth with respect to pr_y.*

Proof. Let $T(x, y) \subseteq \mathrm{M}^{n+m}$ be aligned with respect to pr_z and such that $\overline{\mathrm{pr}_y(S)} = \overline{\mathrm{pr}_z(T)}$. Let

$$(a, b, c) \in R = \{(x, y, z) \mid (x, y) \in S \;\&\; (x, z) \in T\}$$

We need to show that for any open set $Z \ni (a, b, c)$ the set $R \cap Z$ has dimension $\dim S + \dim T - \dim \mathrm{pr}_y(S)$. Since we are working in the product topology we may assume that $Z = U \times V \times W$ for $a \in U$, $b \in V$ and $c \in W$. By our assumption on S, we may also shrink U to make sure that $\mathrm{pr}_y(S \cap (U \times V)) = \mathrm{pr}_y(S) \cap U$. By the alignment of T we know that $\dim \mathrm{pr}_z(T \cap (U \times W)) = \dim \mathrm{pr}_z(T)$. So since $\overline{\mathrm{pr}_y(S)} = \overline{\mathrm{pr}_z(T)}$, we must have

$$\dim(\mathrm{pr}_z(T \cap (U \times W)) \cap \mathrm{pr}_y(S \cap (U \times V))) = \dim \mathrm{pr}_y S$$

This, together with the alignments of S and T means that there must be cells $C_S \subseteq S \cap (U \times V)$ and $C_T \subseteq T \cap (U \times W)$ with $\dim(C_S) = \dim S$ and $\dim(C_T) = \dim(T)$ such that $\dim(\mathrm{pr}_y(C_S) \cap \mathrm{pr}_z(C_T) \cap U) = \dim \mathrm{pr}_y(S)$. Let $C \subseteq \mathrm{pr}_y(C_S) \cap \mathrm{pr}_z(C_T) \cap U$ be a cell of dimension $\dim \mathrm{pr}_y(S)$. Then by definition of cells and their dimension, the set

$$\{ (x, y, z) \mid x \in C \ \&\ (x, y) \in C_S \ \&\ (x, z) \in C_T \}$$

is contained in $R \cap (U \times V \times W)$ and is a cell of dimension

$$\dim(S) + \dim(T) + \dim \mathrm{pr}_y(S)$$

as required. □

Note that a cell $C(x, y) \subseteq \mathrm{M}^{n+k}$ is always aligned with respect to pr_y and satisfies the assumption of the previous lemma. However, unless $\dim \mathrm{pr}_y(C) = \dim(C)$, $\dim \mathrm{pr}_y(\overline{C} \setminus C) = \dim \mathrm{pr}(C)$ so even to verify (GS), we need to analyze smoothness on the boundary of a cell.

Lemma 4.2.11 *Let $C(x, y) \subseteq \mathrm{M}^{n+k}$ be a cell and $D = \overline{C} \cap (\mathrm{pr}_y(C) \times \mathrm{M}^k)$. Then for all $(a, b) \in D$ and all open $V \ni b$, there is an open $U \ni a$ such that* $\mathrm{pr}_y(C \cap (U \times V)) = \mathrm{pr}_y(D) \cap U$.

Proof. We argue by induction on k. So let $C(x, y, z) \subseteq \mathrm{M}^{n+k+1}$. Let $(a, b, c) \in D$ and let $Z \ni (b, c)$ be an open set. We may assume that $Z = V \times W$, where $b \in V$ and $c \in W$.

$W \subseteq \mathrm{M}$ is open so we may assume that it is an interval (q, r) for some $q < c < r$. C is obtained from $\mathrm{pr}_z(C)$ using one of the two construction rules for cells. Either way, since the functions involved are continuous and $C(a, b) \cap (q, r) \neq \emptyset$, there is an open set $U' \ni (a, b)$ such that for all $(a', b') \in U' \cap \mathrm{pr}_z(C)$, there is a $c' \in C(a', b') \cap (p, q)$. By shrinking V, we may assume that $U' = X \times V$ for some $X \ni a$.

By hypothesis, there is another open set $Y \ni a$ such that $\mathrm{pr}_y(\mathrm{pr}_z(C) \cap (Y \times V)) = \mathrm{pr}_{yz}(D) \cap Y$. Let $U = X \cap Y$. Then for any $a' \in U$ there is a b' such that $(a', b') \in \mathrm{pr}_z(C) \cap U \subseteq \mathrm{pr}_z(C) \cap U'$. By our choice of U', there is a $c' \in C(a', b') \cap W$. Hence, $\mathrm{pr}_{yz}(C \cap (U \times W)) = \mathrm{pr}_{yz}(D) \cap U$ as required. □

Corollary 4.2.12 *If $C(x,y) \subseteq \mathrm{M}^{n+k}$ is a cell and $D = \overline{C} \cap (\mathrm{pr}_y(C) \times \mathrm{M}^k)$, then any definable set E with $C \subseteq E \subseteq D$ is smooth.*

Proof. follows from the previous lemma and the characterization of smoothness in 4.2.10.□

Corollary 4.2.13 M *satisfies (GS).*

Proof. Let $S(x,y) \subseteq \mathrm{M}^{n+k}$ be aligned with respect to pr_y. Let $C_1, \ldots, C_k$ be the $\dim S$-dimensional cells of some cell-decomposition of S. We may choose a cell-decomposition such that for all i,j either $\mathrm{pr}_y(C_i) = \mathrm{pr}_y(C_j)$ or $\mathrm{pr}_y(C_i) \cap \mathrm{pr}_y(C_j) = \emptyset$. By the alignment of S, $T = \bigcup \mathrm{pr}_y(C_i)$ is dense in $\mathrm{pr}_y(S)$. Let $K = \overline{\mathrm{pr}_y(S) \setminus T}$. Then $\dim K < \dim \mathrm{pr}_y(S)$ and by the previous corollary and lemma 2.1.30, $S \setminus (K \times \mathrm{M}^k)$ is smooth. □

To get (PS), we would need to do slightly better. In a sense, most of the above K is only needed to deal with cells overlapping in bad ways. If we could show that for a single cell $C(x,y) \subseteq \mathrm{M}^{n+k}$, K can be chosed with co-dimension 2, then we would obtain (PS). Sadly, this is not the case as the following example shows:

Example: Consider the cell

$$C = \{ (w,x,y,z) \in \mathbb{R}^4 \mid w > 0 \;\&\; y = (x/w)^2 \;\&\; z = x/w \}$$

Let $S = \overline{C}$. For $a^2 \neq 0$, the fibre of S over $(w,x,y) = (0,0,a^2)$ is $\{a, -a\}$. The local projections around $(0,0,a^2,a)$ and $(0,0,a^2,-a)$ are contained in $x \geq 0$ and $x \leq 0$ respectively S is not smooth at either point. Thus we would have to remove a one-dimensional set from S to make is smooth with respect to pr_z.

The silver lining is that this is not a counterexample to the presmoothness of $\mathbb{R}$ as a field. We simply have to decompose C into two cells with $x > 0$ and $x < 0$ respectively. The question is whether this is always possible.

Definition 4.2.14 We will say that an o-minimal structure M has the property (LC) if for every definable set $X \subseteq \mathrm{M}^n$, there is a cell decomposition $C_1, \ldots, C_k$ of X such that for all i, $\overline{C_i}$ is locally connected.

From now on, we will work under the assumption that our o-minimal structure has the property (LC) and we will implicitly assume that closures of cells are locally connected.

Since cells are defined in terms of continuous functions, we start by analyzing the possible limiting behaviours of continuous functions $\mathrm{M}^n \to \mathrm{M}$.

Lemma 4.2.15 *Let C be a cell and F be the graph of a definable continuous function $f : C \to \mathrm{M}$. Then for any $a \in \overline{C}$, one of the following is true:*

1. *$f(x) \to \pm\infty$ as $x \to a$*
2. *f has a limit $f(a)$ at a*
3. *$\overline{F}(a)$ is an interval.*

Proof. For any open set $U \ni a$ the set $C \cap U$ is connected so the image $f(C \cap U)$ is a connected subset of M and hence it must be an interval. Furthermore,

$$\overline{F}(a) = \bigcap_{U \ni a} \overline{f(C \cap U)}$$

so we can do a case distinction as follows:

Case 1: For all U $f(C \cap U) = (-\infty, \infty)$.
In this case clearly $\overline{F}(a)$ is the interval $(-\infty, \infty)$.

Case 2: There is a U_0 such that for all $U \subseteq U_0$ $f(C \cap U)$ is of the form (d_U, ∞).
In this case either d_U is bounded above by some d in which case $(d, \infty) \subseteq \overline{F}(a)$ or the d_U are not bounded above, in which case $f(x) \to \infty$ as $x \to a$.

Case 3: There is a U_0 such that for all $U \subseteq U_0$ $f(C \cap U)$ is of the form $(-\infty, d_U)$.

This is like case 2.

Case 4: There is a U_0 such that for all $U \subseteq U_0$ $f(C \cap U)$ is of the form (c_U, d_U).

In this case $\overline{F}(a) = \bigcap_{U \ni a} \overline{f(C \cap U)}$ is either a point $f(a)$ or a closed interval, as required.

$\square$

Proposition 4.2.16 *Let $D(x,y) \subseteq \mathrm{M}^{n+1}$ be a cell and $C = \mathrm{pr}_y(D)$. Then there is a set $K \subseteq \overline{C}$ of dimension at most $\dim C - 2$ such that for all $(a,b) \in \overline{D} \setminus (K \times M)$ and all open $V \ni b$ there is an open $U \ni a$ such that $\mathrm{pr}_y(\overline{D} \cap (U \times V)) = \mathrm{pr}_y(\overline{D} \cap U)$. Furthermore, if $\dim(D) = \dim(C)$, then $K = \{ a \in C \mid \dim \overline{D}(a) = 1 \}$.*

Proof. We have already dealt with all (a,b) where $a \in C$ so we may assume that $a \in \overline{C} \setminus C$. Let F (and possibly G) be the graphs of the continuous functions used to obtain D from C. Let $K' = \{ x \mid \dim \overline{F}(x) = 1 \;\text{ or }\; \dim \overline{G}(x) = 1 \}$ and $K = \overline{K}$. F and G are cells and hence pure-dimensional and aligned with respect to pr_y so $\dim K \leq \dim C - 2$. We will prove that for $a \in \overline{C} \setminus K$ and any $b \in \overline{D}(a)$ and open $V \ni b$ we can find a suitable open set $U \ni a$. Without loss of generality we may assume that $V = (c,d)$ for some $c < b < d$.

By the previous lemma, for all $a' \in \overline{C} \setminus K$, the functions f and g approach limits in $\mathrm{M} \cup \{ \pm\infty \}$.

Suppose first that $D = \{ (x,y) \mid x \in C \;\&\; y = f(x) \}$. Then f must continuously approach b at a. So we can find a U such that for all $a' \in U$ $f(a') \in (c,d)$ as required.

Now suppose $D = \{ (x,y) \mid x \in C \;\&\; y = (f(x), g(x)) \}$. If $b \in \overline{F}(a)$ or $b \in \overline{G}(a)$, then we can proceed as above. Otherwise $b \in (f(a), g(a))$, where $f(a)$ and $g(a)$ could be $\pm\infty$. By possibly making (c,d) smaller we may assume $f(a) < c < b < d < g(a)$. Then there is an open set $U \ni a$ such that for all $a' \in U \cap C$ $f(a') < c < d < g(a')$. This means that $(U \cap C) \times (c,d) \subseteq D$ and hance $(U \cap \overline{C}) \times [c,d] \subseteq \overline{D}$. This certainly implies $\mathrm{pr}_y(\overline{D} \cap (U \times (c,d))) = \mathrm{pr}_y(\overline{D}) \cap U$. $\square$

Proposition 4.2.17 M *satisfies (PS).*

Proof. Let $S(x,y)\subseteq \mathrm{M}^{n+k}$ be aligned with respect to pr_y and such that $\mathrm{pr}_y(S)$ has dimension M^n. Write $y=(y_1,\ldots,y_{k+1})$ and let $z_i=y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_{k+1}$ so that for all i, $\mathrm{pr}_y=\mathrm{pr}_{y_i z_i}$. By possibly reordering and relabelling coordinates, we may assume that $\dim(S)=n$ or $\dim \mathrm{pr}_{z_1}(S)=n+1$. Let $S_1,\ldots,S_m$ be a cell-decomposition of S where $S_1,\ldots,S_p$ are the cells of dimension $\dim S$. We will show that for all $i\leq p$, there exists a subset $K_r\subseteq \mathrm{pr}_y(\overline{S_r})$ of co-dimension 2 such that $\overline{S_r}\setminus(K_r\times \mathrm{M}^k)$ is smooth with respect to pr_y.

Let D be one of the S_i such cell. We will argue by induction on k. The base case is given by the previous proposition. For the inductive step, suppose $D(x,y)\subseteq \mathrm{M}^{n+(k+1)}$. Define $E=\mathrm{pr}_{z_1}(D)$ and $C=\mathrm{pr}_y(D)$. Since D is a cell with respect the ordering of coordinates xy_1z_1, C and E are also a cells. In particular, C is open in M^n. We can apply the hypothesis to obtain $K_E\subseteq\overline{E}$ and $K_C\subseteq\overline{C}$ such that $\overline{D}\setminus \mathrm{pr}_{z_1}^{-1}(K_E)$ is smooth with respect to pr_{z_1} and $\overline{E}\setminus \mathrm{pr}_{y_1}^{-1}(K_C)$ is smooth with respect to pr_{y_1}.

First we consider the case where $\dim(E)=\dim(C)$. Let $K=\overline{\mathrm{pr}_{y_1}(K_E)\cup K_C}$. Let $a\in\overline{C}\setminus K$, $(b,c)\in\{(y_1,z_1)\mid(a,y_1,z_1)\in\overline{D}\}$ and $V\ni(b,c)$ open. We may assume that $V=X\times Y$ for $b\in X$ and $c\in Y$. Since $(a,b)\notin K_E$, there is an open set $Z\ni(a,b)$ such that $\mathrm{pr}_{z_1}(\overline{D})\cap Z=\mathrm{pr}_{z_1}(\overline{D}\cap(Z\times Y))$. We also may assume that $Z=A\times B$. Now since $a\notin K_C$, there exists an open set $U\ni a$ such that $\mathrm{pr}_{y_1}(\mathrm{pr}_{z_1}(\overline{D}))\cap U=\mathrm{pr}_{y_1}(\mathrm{pr}_{y_1}(\overline{D})\cap(U\times B))$. After restricting U to $U\cap A$ this implies $\mathrm{pr}_{y_1}(\mathrm{pr}_{z_1}(\overline{D}))\cap U=\mathrm{pr}_{y_1}(\mathrm{pr}_{y_1}(\overline{D})\cap(U\times B))=\mathrm{pr}_y(\overline{D}\cap(U\times B\times Y))$.

Finding a suitable K in the case $\dim(E)=\dim(C)+1$ is rather more tricky. For each i, let $D^i_1,\ldots,D^i_{s_i}$ be a cell-decomposition of D with respect ot the ordering xy_iz_i and with the property that for all f and g, $\mathrm{pr}_y(D^i_f)\cap\mathrm{pr}_y(D^i_g)=\mathrm{pr}_y(D^i_g)$ or $\emptyset$. We may assume that for all i, $D^i_1\ldots,D^i_{r_i}$ are the cells of dimension $\dim(D)$. Let $C^i_1,\ldots,C^i_{t_i}$ be the distinct projections $\mathrm{pr}_y(D^i_j)$. Let $B=\mathrm{bd}(C)=\overline{C}\setminus C$ and $B^i_j=\mathrm{bd}_B(\overline{C^i_j}\cap B)=(\overline{C^i_j}\cap B)\setminus\mathrm{int}(\overline{C^i_j}\cap B)$. $\dim(B^i_j)<\dim(B)<\dim(C)$ so all B^j_i have co-dimension 2 in $\overline{C}$.

Let $E^i_j = \mathrm{pr}_{z_i}(D^i_j)$ and define $K_{E^i_j}$ and K^i_C in analogy with K_E and K_C. For every (i,j) the sets $K'_{ij} = \{a \in \overline{C} \mid \dim(K_{E^i_j}(a)) = 1\}$ and $K_{ij} = \overline{K'_{ij}}$ also have co-dimension 2 in $\overline{C}$ so

$$K = K_C \cup K_0 \cup \bigcup_{i=1}^{k+1} \left(K^i_C \cup K_{ij} \cup \bigcup_{j=1}^{t_i} B^i_j \right)$$

does, too. Let $(a,b) \in \overline{D} \setminus \mathrm{pr}_y^{-1}(K)$ and $V \ni b$ be open. We will prove that this K is suitable in three claims:

Claim 1: If $\dim(\mathrm{pr}_{z_1}(\overline{D})(a)) = 1$, then there exists an open $U \ni a$ such that

$$\mathrm{pr}_y \overline{D} \cap (U \times V) = \mathrm{pr}_y(\overline{C}) \cap U$$

Claim 2: If $\dim(\overline{D}(a)) = 0$, then there exists an open $U \ni a$ such that $\mathrm{pr}_y \overline{D} \cap (U \times V) = \mathrm{pr}_y(\overline{C}) \cap U$.

Claim 3: If a is not as in claims 1 or 2, then there is a pair (i,j) and an open set $W \ni a$ such that $(a,b) \in \overline{D^i_j}$, $\dim(\mathrm{pr}_{z_i}(\overline{D^i_j})(a)) = 1$ and $\mathrm{pr}_y(D) \cap W = \mathrm{pr}_y(D^i_j) \cap U$.

If a is as in claim 3, then the result of claim 1 applies with D^i_j in place of D and pr_{z_i} in place of pr_{z_1} so that we also get an open $U \ni a$ with $\mathrm{pr}_y \overline{D} \cap (U \times V) = \mathrm{pr}_y(\overline{C}) \cap U$. We will now prove the claims.

Proof of claim 1:

Write $b = (b_1, \ldots, b_k)$ and $e = b_1$, $d = (b_1, \ldots, b_k)$. We may assume that $V = Y \times Z$ where $e \in Y$ and $d \in Z$. Since $(\mathrm{pr}_{z_1}(\overline{D})(a)) = 1$ and $\dim(K_e(a)) = 0$, we can find $(e', d') \in Y \times Z$ such that $(a, e', d') \in \overline{D}$ and $e' \notin K_E(a)$. Now $a \notin K_C$ and $(a, e') \notin K_E$ so we can repeat the argument we used in the case where $\dim(E) = \dim(C)$.

Proof of claim 2:

We do an induction on k again. The base case is given by the previous proposition. Again, we write $b = (b_1, \ldots, b_k)$ and $e = b_1$, $d = (b_1, \ldots, b_k)$. $\dim(\overline{D}(a,e)) = 0$ so by hypothesis we can assume that $(a,e) \notin K_E$. $a \notin K_C$ so again we can repeat the argument from the case where $\dim(E) = \dim(C)$.

Proof of claim 3:

Since $\dim(E) = \dim(C)+1$, we must have $a \in B$ and $\dim(\overline{D}(a)) > 0$. The latter implies that for i, there is a cell D^i_j such that $\dim(\mathrm{pr}_{z_i}(\overline{D^i_j})(a)) = 1$. Let $C^i_q = \mathrm{pr}_y(D^i_j)$. $a \notin B^i_j$ so a is in $\mathrm{int}_{\overline{C}}(\overline{\mathrm{pr}_y(D^i_q)})$. Thus, there is an open set $W \ni a$ such that $\overline{C^i_q} \cap W = \overline{C} \cap W$.

□

This concludes the proof of proposition 4.2.2.

Corollary 4.2.18 *Let $F(x, y) \subseteq X \times Y$ be the graph of a definable function f. Then f is continuous on a subset $S \subseteq X$ if and only if $F \cap (S \times Y)$ is smooth with respect to the projection* $\mathrm{pr}\colon S \mapsto X$.

Proof. If f is continuous on S, then 4.2.10 implies that F is smooth. If F is smooth, and $f(a) = b$, then the implicit function theorem 2.3.22 implies that for all open $V \ni b$, there is an open $U \ni a$ such that for all $a' \in U$, $f(a') \in V$ and thus, f is continuous. □

Again we give some examples of o-minimal structures with references:

1. Dense linear orders (trivially)
2. Ordered divisible abelian groups [Mar02]
3. Real closed fields [Mar02]
4. The real field with exponentiation and restricted analytic functions [Wil99]

4.2.1 Another Tangency Notion

In order to show that our tangency relation satisfies the non-degeneracy condition we will borrow heavily from Peterzil and Starchenko [PS98]. We will introduce their tangency notion and show that for the particular families we are considering, this is highly related to the one defined in 3.1.15. We start with a few theorems about definable functions in o-minimal structures.

Theorem 4.2.19 *(Monotonicity Theorem). If $f:(a,b) \to \mathrm{M}$ is a definable function, then there are $a = a_0 < a_1 < \cdots < a_n = b$ such that on any interval (a_i, a_{i+1}), f is continuous and either strictly increasing, strictly decreasing or constant.*

Theorem 4.2.20 *(Inverse Function Theorem). If $f:(a,b) \to \mathrm{M}$ is continuous and locally strictly increasing or decreasing at a point c, then f has a continuous inverse around $(c, f(c))$.*

Theorem 4.2.21 *(Intermediate Value Theorem). If $f:[a,b] \to \mathrm{M}$ is continuous, then f attains every value between $f(a)$ and $f(b)$.*

Proofs can be found in most standard texts on o-minimality.

Definition 4.2.22 Given two definable functions f and g with $f(x) = g(x)$, we write

- $f \prec_x^+ g$ if there is an $x' > x$ such that for all $a \in (x, x')$, $f(a) < g(a)$,
- $f \succ_x^+ g$ if there is an $x' > x$ such that for all $a \in (x, x')$, $f(a) > g(a)$,
- $f \approx_x^+ g$ if there is an $x' > x$ such that for all $a \in (x, x')$, $f(a) = g(a)$.

We write $\preccurlyeq_x^+ = \prec_x^+ \vee \approx_x^+$ and $\succcurlyeq_x^+ = \succ_x^+ \vee \approx_x^+$.

If f and g are continuous and $f \prec_x^+ g$, then we should think of g as steeper than f. By o-minimality, exactly one of these three is true for any two definable functions f and g with $f(x) = g(x)$. The construction in [PS98] is quite subtle, but the general idea of defining a tangency relation is easily seen using these definitions. Suppose that $\langle C_x \times C_y, L, I \rangle$ is a generically normal two- dimensional family of plane curves. We define

$$A_{(a,b)} = \{ (f,h) \in L^2 \mid f(a) = h(a) = b \;\&\; f \preccurlyeq_x^+ h \}$$

$$B_{(a,b)} = \{ (f,h) \in L^2 \mid f(a) = h(a) = b \;\&\; f \succcurlyeq_x^+ h \}$$

Now suppose $(f,h) \in T = \overline{A_{(a,b)}} \cap \overline{B_{(a,b)}}$. Then there are $(f_1,h_1),(f_2,h_2) \in \mathcal{V}_{fh}$ such that $f_1 \succcurlyeq_a^+ h_1$ and $f_2 \preccurlyeq_a^+ h_2$. In other words, by perturbing f and h infinitesimally we can make one or the other steeper. This is something we would only expect to happen for curves that are tangent. Before we show how this is related to our tangency notion $\|$, we first have to make some assumptions on our families of curves.

Lemma 4.2.23 *Suppose $I \subseteq \mathrm{M}^4$ is the incidence relation of a generically normal family of plane curves and $(a,b,l,m) \in I$ is generic. Then for $g = (l,m)$, there is an open interval $R \ni a$ and an open set $V \ni g$ such that for every $h \in V$, $I(h)$ is the graph of a continuous increasing or increasing function on R.*

Proof. This is en easy application of the monotonicity theorem. □

We can thus restrict I to $I \cap (R \times \mathrm{M} \times V)$ and assume I to be a family of decreasing functions.

Definition 4.2.24 Given a family of functions as obtained by the lemma, we say that it is *increasing p-nice* if

1. for every $h \in V$, the $I(h)$ is the graph of a continuous increasing function $R \to \mathrm{M}$,
2. for all $h_1, h_2 \in V$ there is a unique $x \in R$ such that $h_1(x) = h_2(x)$,
3. for every $(x,y) \in \mathrm{M}^2$, either $V_{xy} = \{h \in U \mid h(x) = y\}$ is empty or it homeomorphically projects to an open interval in M via the projection $\mathrm{pr}\colon (x,y) \mapsto x$ and
4. for the h_1, h_2, x from 2, $\mathrm{pr}(h_1) < \mathrm{pr}(h_2)$ if and ony if $h_1 \prec_x^+ h_2$.

We can define an analogous notion of an *increasing n-nice* family by replacing $\prec_x^+$ by $\succ_x^+$ in 4. Equally we could define *decreasing p-nice* and *decreasing n-nice* by changing item 1 accordingly. We call all such families *nice*.

Note that by item 3, we can consider V_{xy} as an ordered set. Instead of writing $\mathrm{pr}(h_1) < \mathrm{pr}(h_2)$ as we did in item 4, we will just write $h_1 < h_2$.

Proposition 4.2.25 *Suppose $I \subseteq \mathrm{M}^4$ is the incidence relation of a generically normal family of plane curves and $(a,b,l,m) \in I$ is generic. Then there is an open interval $R \ni a$ and an open set $V \ni (l,m)$ such that $I \cap (R \times \mathrm{M} \times V)$ is nice.*

Proof. This is the content of Theorem 4.3 of [PS98]. □

In this text we mostly work locally in the sense of infinitesimals so it makes sense to define a local version.

Definition 4.2.26 Suppose that $\langle C_x \times C_y, D, L, I\rangle$ is a generically normal two- dimensional family of branches of plane curves and that $(a,b,c,g) \in I$ is generic. Let $\mathrm{pr}\colon L \to \mathrm{M}$ be the projection on the first coordinate of L. We say that I is *locally increasing p-nice* at (a,b,c,g) if

1. $\dim \mathrm{pr}(L) = 1$,
2. for every $g' \in \mathcal{V}_g$, the branch g' is increasing,
3. for all $g_1, g_2 \in \mathcal{V}_g$ there is a unique $x \in \mathcal{V}_a$ such that $g_1(x) = g_2(x)$ and
4. for the x from 2, $\mathrm{pr}(g_1) < \mathrm{pr}(g_2)$ if and ony if $g_1 \prec_x^+ g_2$.

Analogously we obtain local versions of all the niceness notions.

Given any set S and a generic $s \in S$, the cell-decomposition theorem tells us that there is a projection $\mathrm{pr}\colon S \to \mathrm{M}^{\dim(S)}$ such that locally around s, pr is a homeomorphism. We wish to chose one such canonically and will do so by choosing $\mathrm{pr}_S^s\colon (x_1,\dots x_n) \mapsto (x_{i_1},\dots,x_{i_{\dim S}})$ to be such a projection where $(i_1,\dots i_{\dim S})$ is least in the lexicographic order. If it is clear which neighbourhood or which set we are talking about we will just write pr_S or pr^s. Given a one-dimensional set $C \subseteq \mathrm{M}^n$, and a generic point $a \in C$, we consider $\mathcal{V}_a$ as an ordered set via pr_C^a.

Proposition 4.2.27 *A generically normal two-dimensional family of branches of plane curves $\langle C_x \times C_y, D, L, I\rangle$ is locally increasing p-nice at a generic point (a,b,c,g) if and only if there is an open set $U \ni (a,b,c,g)$ such that $I \cap U$ is homeomorphic to an increasing p-nice family of curves via the projection* $\mathrm{pr}\colon (x,y,z,u) \mapsto (\mathrm{pr}^a_{C_x}(x), \mathrm{pr}^b_{C_y}(y), \mathrm{pr}^g_L(u))$.

Proof. The right to left direction is trivial. For the other direction we note that locally around (a,b,c,g), I is homeomorphic to $\mathrm{pr}(I)$. We can then apply the previous proposition and obtain a suitable U by lifting the open set $R \times \mathrm{M} \times V$ via pr^{-1}. $\square$

We chose to state the proposition for increasing p-nice families, but it holds for all notions of niceness. In view of these propositions we can assume that the generically normal families of plane curves used in the group constructions are nice (and therefore locally nice) families of functions. Without loss of generality we will always assume that they are increasing p-nice, but it should be clear that the same arguments would work in all four cases.

We now go back to the constructions used in the proof of proposition 3.2.8. In obtaining the family $\langle C_x \times C_y, G, K\rangle$ of curves through (a,b) we had to choose a direction at a. Without loss of generality we will assume that we chose $x > a$.

To sum up we have shown that we can assume

1. K is a family of increasing continuous functions $(a)^+ \to (b)^+$,
2. C_x, C_y and G are ordered and
3. for $x \in (a)^+$, $h(x)$ is continuous and strictly increasing in h.

Lemma 4.2.28 *Suppose that for independent $f, h, u \in \mathcal{V}_g$ and some (possibly dependent) $v \in \mathcal{V}_g$ we have $h^{-1} \circ f \parallel v^{-1} \circ u$ in the sense of 3.1.15 and $h^{-1} \circ f \prec^+_a v^{-1} \circ u$. Then for all $v' \in (v)^+$, $h^{-1} \circ f \succ^+_a v'^{-1} \circ u$.*

Proof. By tangency and the implicit function theorem, for any $a' \in \mathcal{V}_a$ there exists a $v' \in \mathcal{V}_v$ such that $h^{-1} \circ f(a') = v'^{-1} \circ u(a')$. Since $v^{-1} \circ u(a')$ is decreasing in v and

$h^{-1} \circ f(a') < v^{-1} \circ u(a')$, this implies $v' > v$. Assume now for a contradiction that for all $v' \in (v)^+$, $h^{-1} \circ f \prec_a^+ v'^{-1} \circ u$. Then since this is also evidently true for $v' \in (v)^-$, this would mean $(a, v) \in \operatorname{int}\{\, (a', v') \mid h^{-1} \circ f(a') < v'^{-1} \circ u(a') \,\}$ and therefore this is would be true for all $a', v' \in \mathcal{V}_{av}$. This is clearly a contradiction since we already found a' and v' for which the opposite holds. Therefore, for all $v' \in (v)^+$, $h^{-1} \circ f \succ_a^+ v'^{-1} \circ u$ as required. □

Obviously an equivalent lemma holds for $h^{-1} \circ f \succ_a^+ v^{-1} \circ u$. This means that at least for generic f, h, u and some v, tangency in the sense of 3.1.15 implies tangency in the sense of [PS98].

Remark 4.2.29 The proof of this lemma relies on being able to use the implicit function theorem which is why we had to assume that f, h, u are independent. However, once we prove that the non-degeneracy and richness assumptions hold, we can drop that assumption.

Corollary 4.2.30 *If non-degeneracy and richness hold and $h^{-1} \circ f \parallel v^{-1} \circ u$, then $h < f$ if and only if $v < u$*

Proof. Suppose for a contradiction $h > f$ and $v < u$. Then for all $v' \in \mathcal{V}_v$, $h^{-1} \circ f \prec_a^+ v'^{-1} \circ u$. This contradicts the above lemma. □

Corollary 4.2.31 *If non-degeneracy and richness hold, then the group obtained by 3.2.8 is an ordered group.*

Proof. Suppose $e < f$ and u are elements of $\mathcal{V}_g$. Let h be such that $g^{-1} \circ u \parallel h^{-1} \circ g$ so that for any $r \in \mathcal{V}_g$, $(g^{-1} \circ u) \circ (g^{-1} \circ r) \parallel h^{-1} \circ r$. If $e' = u.e$ and $f' = u.f$, then $g^{-1} \circ e' \parallel h^{-1} \circ e$ and $g^{-1} \circ f' \parallel h^{-1} \circ f$. By symmetry lemma 3.2.10 and transitivity of $\parallel$, this implies $f^{-1} \circ e \parallel f'^{-1} \circ e'$ and thus $f' < e'$ by the previous corollary. □

Remark 4.2.32 By exactly the same argument, the group action obtained in 3.2.24 is order-preserving. If $e \in \mathcal{V}_0$ and $u < v$ are in V_1, then $e.u < f.u$.

Corollary 4.2.33 *If non-degeneracy and richness hold, then the groups obtained by 3.2.8 and 3.2.22 are abelian.*

Proof. Ordered groups always have unbounded exponent so this follows from 2.4.7. □

4.2.2 Completing the Trichotomy Results

As we did in the chapter for Zariski Geometries, we will now prove two key results.

Proposition 4.2.34 *In o-minimal structures the non-degeneracy and richness conditions for propositions 3.2.8 and 3.2.22 are always given and the group action obtained in proposition 3.2.25 is faithful.*

Theorem 4.2.35 *If* M *is a non-linear o-minimal structure then there is a field definable in* M.

Again, we will only prove non-degeneracy and richness for the compositional case 3.2.8 here. The group case is analogous.

Thanks to our work in the previous section, we can borrow non-degeneracy from [PS98]. Lemma 5.2 of [PS98] can be rewritten to say the following.

Lemma 4.2.36 *Let F and G be nice families of functions, let (a,b,c) be generic and let K_F and K_G be families through (a,b) and (b,c) obtained from F and G as K was obtained from I. Then for every $g,h \in K_G$ and $e \in K_F$ independent of (a,b,c),*

1. *if $g < h$, then there exists an $f > e$ such that $g \circ f \prec_a^+ h \circ e$ and*
2. *if $g < h$, then there exists an $f < e$ such that $g \circ e \prec_a^+ h \circ f$.*

If $F = I$ and $G = I^{-1}$, this, together with 4.2.28, almost gives us the non-degeneracy we need. However, we want to have $a = c$ so we need an extra argument to weaken the genericity condition on (a, b, c). To do this, we introduce some new notation. Given a nice family of functions $\mathbb{R} \to \mathbb{R}$, we let $f[a, b, l]$ be the unique function from F such that $f[a, b, l](a) = b$ and such that the first coordinate of the parameter of $f[a, b, u]$ is equal to l. For example, if $g = (l, m)$ parametrizes a curve through (a, b), then $f[a, b, l]$ is just g and $f[a, c, l]$ is parametrized by some $h = (l, n)$. In [PS98], the following corollary never appears explicitly, but the idea of the proof is there in several places.

Corollary 4.2.37 *Let F and G be nice families of functions, let both (a, b) and (b, c) be generic and let K_F and K_G be families through (a, b) and (b, c) obtained from F and G as K was obtained from I. Then for every $u, v \in K_G$ and $e \in K_F$ independent of (a, b, c),*

1. *if $g < h$, then there exists an $f > e$ such that $g \circ f \prec_a^+ h \circ e$ and*
2. *if $g < h$, then there exists an $f < e$ such that $g \circ e \prec_a^+ h \circ f$.*

Proof. We assume that both families are increasing p-nice, but all other cases are similar. Suppose for a contradiction that 1 fails. Then there are $u < v$ and r such that for all $s > r$,

$$g[b, c, v] \circ f[a, b, r] \prec_a^+ g[b, c, u] \circ f[a, b, s]$$

By making u a little bit bigger and v a little bit smaller, we can assume that

$$\operatorname{cdim}(a, b, u, v, r) = 5$$

Since all functions are increasing, we can rearrange this to get that for all $s > r$,

$$g[b, c, u]^{-1} \circ g[b, c, v] \prec_a^+ \circ f[a, b, s] \circ f[a, b, r]^{-1}$$

Consider the relation

$$R(i,j) = g[b,i,u]^{-1} \circ g[b,i,v] \prec_a^+ g[b,j,u]^{-1} \circ g[b,j,v]$$

Since c is independent of b, u, v, it is easy to check that locally around c, this is just the order $i < j$ or $j < i$. Without loss of generality we will assume that it is $i < j$. We now pick $d < c$ independent of (a, b, r, u, v). Then for all $s > r$

$$g[b,d,u]^{-1} \circ g[b,d,v] \prec_a^+ g[b,c,u]^{-1} \circ g[b,c,v] \prec_a^+ \circ f[a,b,s] \circ f[a,b,r]^{-1}$$

and after rearranging again we get that for all $s > r$

$$g[b,d,v] \circ f[a,b,r] \prec_a^+ g[b,d,u] \circ f[a,b,s]$$

but this contradicts the previous lemma. □

In particular this implies Lemma 5.5 of [PS98] which states the following for our family K:

Lemma 4.2.38 *For every f, g, u from K with f generic over a,*

1. *if $f > g$, then there exists $v < u$ such that $g^{-1} \circ f \succ_a^+ v^{-1} \circ u$,*
2. *if $f < g$, then there exists $v > u$ such that $g^{-1} \circ f \prec_a^+ v^{-1} \circ u$,*
3. *if $f > g$, then there exists $v > u$ such that $g \circ f^{-1} \succ_a^+ v \circ u^{-1}$,*
4. *if $f < g$, then there exists $v < u$ such that $g \circ f^{-1} \succ_a^+ v \circ u^{-1}$,*

Non-Degeneracy

Corollary 4.2.39 *K satisfies the non-degeneracy condition of 3.2.8.*

Proof. We assume that I was an increasing p-nice family of functions so that we can make the same assumptions on K as we did for 4.2.28. Suppose non-degeneracy fails. Then for some generic $m, n, o \in \mathcal{V}_g$ we have $n^{-1} \circ m \parallel n^{-1} \circ o$ or equivalently $n^{-1} \circ n \parallel o^{-1} \circ m$. Without loss of generality assume $m > o$ so that $n^{-1} \circ n \prec_s^+ o^{-1} \circ m$. By 4.2.28 this implies that for $n' \in (n)^-$ we get $n'^{-1} \circ n \succ_a^+ o^{-1} \circ m$. However, this contradicts item 1 of the above lemma. □

Richness

Proposition 4.2.40 *K satisfies the richness condition of 3.2.8.*

Proof. Again we will assume that our family I was an increasing p-nice family of functions so that we can make the same assumptions on K as we did for 4.2.28. Before proving richness, we first prove the following weaker lemma:

Lemma 4.2.41 *Given generic $m < n$, for every $o > n$ there exists an m' such that $o^{-1} \circ m' \parallel n^{-1} \circ m$ and for any $l < m$ there exists an n' with $n'^{-1} \circ o \parallel n^{-1} \circ m$. Furthermore, the sets $\{ o \mid \exists m' \ o^{-1} \circ m' \parallel n^{-1} \circ m \}$ and $\{ l \mid \exists n' \ n'^{-1} \circ o \parallel n^{-1} \circ m \}$ are convex.*

Proof. The proof of these two statements is very similar so we will only prove the first.

By assumption 3 on K, given a fixed a', the function $\phi_{a'} \colon f \mapsto f(a')$ is continuous and strictly increasing and therefore, by the intermediate value theorem, has convex image.

Let $o > n$ and $a' > a$. Then $a'' = n^{-1} \circ m(a') < a'$ so $m(a') < o(a'') < o(a')$ and $o(a'') \in \mathrm{Im}(\phi_{a'})$. By the intermediate value theorem this implies that there exists an $m < m' < o$ such that $m'(a') = o(a'')$ or in other words $n^{-1} \circ m(a') = o^{-1} \circ m'(a')$. If we choose $a' \in \mathcal{V}_a$ this tells us that $n^{-1} \circ m \parallel o^{-1} \circ \pi(m')$. □

We now finish the proof of the proposition. As was the case for Zariski Geometries (remark 4.1.25), this lemma is sufficient to ensure that the proof of 3.2.11 goes through so that $\parallel$ is an equivalence relation. Thus, in analogy with 4.1.26, it is enough to show that for generic $m, n, o \in \mathcal{V}_g$ there exists a p such that $n^{-1} \circ m \parallel o^{-1} \circ p$. The fact that $p \in \mathcal{V}_g$ then follows from the Implicit Function Theorem. Assume that such a p does not exist for some $m, n, o \in \mathcal{V}_g^2$. Without loss of generality we can assume assume that $n > m$. Then $o < n$ for otherwise such a p would be given by the previous lemma. Now pick $l \in \mathcal{V}_g$ such that $l < \mathcal{V}_g^2$. In particular $l < m$ so the previous lemma gives a $q \in \mathcal{V}_g$ such that $n^{-1} \circ m \parallel q^{-1} \circ l$. We must have $q > o$ for otherwise by the previous lemma and transitivity of $\parallel$, $o \in \{r \mid \exists m' \; r^{-1} \circ m' \parallel n^{-1} \circ m\}$. Now $\mathcal{V}_g^2$ is convex and $n > q > o$ so $q \in \mathcal{V}_g^2$. We now have $m, n, q \in V_g^2$ which implies $l \in \mathcal{V}_l^2$ by the Implicit Function Theorem. This is a contradiction to our choice of l. □

Faithfulness

Corollary 4.2.42 *The action obtained in 3.2.25 is faithful.*

Proof. By an analogous argument to that used in 4.2.39 we can use 4.2.37 to show that for any $h_1, h_2 \in \mathcal{V}_1$ and any $r \in \mathcal{V}_0$ generic over a, $r.h_1 \neq r.h_2$. Of course, all $r \neq 0$ are generic so this suffices. □

This concludes the proof of proposition 4.2.34.

Non-degeneracy implied the smoothness of the set J from 3.1.15 and thus the smoothness of $\parallel$ with respect to any projction to three coordinates. In the same way, faithfullness implies that the graph of the group action is smooth with respect to the projection $\mathcal{V}_0 \times \mathcal{V}_1 \times \mathcal{V}_0 \to \mathcal{V}_0 \times \mathcal{V}_1$. By 4.2.18 this implies that the action is continuous.

Recovering a Field

We now have all the ingredients in place. We have two smooth abelian convex type-definable ordered groups $\langle \mathcal{V}_0, <, + \rangle$ and $\langle \mathcal{V}_1, <, . \rangle$ with an order-preserving continuous definable group action . of $\mathcal{V}_1$ on $\mathcal{V}_0$. Propositions 7.12 and 7.11 of [PS98] now complete the field construction. For completeness, we give an outline of their argument.

Define Λ to be the ring of definable automorphisms of $\mathcal{V}_0$. It follows easily from 2.4.10 that a one-dimensional torsion-free type-definable group has no definable subgroups. Thus, an automorphism σ is uniquely determined by its action on a single element $a \in \mathcal{V}_0 \setminus \{0\}$. So by fixing an element $\mathbb{1} \in \mathcal{V}_0 \setminus \{0\}$ we can embed $\mathcal{V}_1$ into Λ by $h \mapsto \sigma$ where $\sigma(\mathbb{1}) = \mathbb{1}.h - \mathbb{1}$. A straightforward argument using the intermediate value theorem shows that via this embedding $\mathcal{V}_1$ is isomorphic to an ordered commutative subring of Λ.

To get a field from a convex type-definable ordered commutative ring R, we note that we can identify the fraction field F with any interval $(-\beta, \beta) \subseteq R$ via the map $a \mapsto a/(-|a|+\beta)$. Thus we obtain a field which proves theorem 4.2.35.

Chapter 5

Outlook

Further Examples

Of course, one of the big questions given a new framework like the one developed in this text is whether there are any interesting examples. The key examples I had in mind developing Strong and Presmooth Geometries were Zariski Geometries and o-minimal structures.

The proof that all o-minimal structures are indeed presmooth is still incomplete, but I am optimistic that this can be done. Whether this is by proving property (LC) or otherwise remains to be shown.

Furthermore, given how different Zariski Geometries and o-minimal structures are in nature, I would be surprised if the were no other examples.

Recent work carried out by Kovacsisc [Kov13] indicates that the cell-decomposition theorem for C-minimal structures may not be as nice as claimed in [HM94], but they would be an obvious next class of structures to study. If they did turn out to be Strong Geometries, it is not too hard to verify that C-minimal structures would satisfy axiom (1T) as well. In view of this, we would expect them to be easier to deal with than o-minimal structures, but the rather more complicated cell-decomposition theorem for C-minimal structures makes the verification of axiom (PS) considerably more difficult. Even if we consider a pre-smooth C-minimal structure, there are open questions. The non-degeneracy proof for Stable-Like Presmooth Geometries goes through without modification, but neither the richness proof from the Stable-Like context, nor the one from the o-minimal context can be adapted easily. Haskell and Macpherson have proved an analogue of the monotonicity theorem for o-minimal structures in [HM94].

Theorem 5.0.43 *([HM94]) Let $f : \mathrm{M} \to Z$ be a definable partial function on* M *where Z is one of* M*,T,$\mathcal{C}$ and $\mathcal{L}_n$. Then the domain of f can be decomposed into finitely many cells on each of which f is continuous and either a local isomorphism or locally constant.*

Maybe this could be used to prove richness in a similar way we did in o-minimal structures. However, from an informal standpoint this seems unlikely. The issue is the total disconnectedness of the topology in C-minimal structures. We essentially defined tangency as a limit as one point approaches another and there is no reason to believe that such limits should exist in the context of a totally disconnected structure.

Delon and Maalouf have made considerable progress on a Trichotomy Theorem for C-minimal structures, but they also need an extra condition that ensures that certain limits exist.

Definition 5.0.44 We say that a C-minimal structure M is *definably maximal* if every uniformly definable family of cones which is completely ordered by inclusion has non-empty intersection

I see this property as promisig for both the issue of presmoothness and richness. It might allow us to analyse the bahaviour of cells on their boundaries in a similar way we did for o-minimal structures and thus allow us to follow the same path to presmoothness. For richness, we could try the following.

Let us consider the relation $\|$ from 3.2.8. The C-relation gives us a uniformly definable family of cones around a. Let $f^{-1} \circ h$ be a generic curve from $K^{-1} \circ K$ and m be another curve from K. By the monotonicity theorem, for any sufficiently small cone $C \ni a$, the set $\{n \mid \exists a' \in C \;\; n^{-1} \circ m(a') = g^{-1} \circ f(a')\}$ should be a cone. As we shrink C to the singleton a, a condition along the lines definable maximality may guarantee the existance of an n such that $g^{-1} \circ f \parallel n^{-1} \circ m$.

One consequence of the cell-decomposition theorem that is worth mentioning here is that the set of cones at a node is a strongly minimal set and every maximal branch comes endowed with an o-minimal structure. We can view C-minimal structures as a mixture of stable and o-minimal in this way. This is best illustrated with an example. ACVF is the theory of algebraically closed valued fields. It is well known that with the relation $C(x, y, z) = v(x - y) < v(x - z)$ models of ACVF are C-minimal structures. It is shown in [HHM08] that all the non-stability comes from the value group (which has the same order-type as every maximal branch). All "other" sets are stable and stably embedded. Given that Presmooth Geometries are a suitable context to prove the Trichotomy Theorems for both o-minimal structures and strongly minimal structures with a topology, it stands to reason that a Trichotomy Theorem for C-minimal structures should also fit into this framework.

The other examples that I had wished to investigate, but never found the time for were universal covers of abelian varieties. A prime example of such a structure is the universal cover of $\mathbb{C}^{\times}$. We consider a two sorted structure with sorts $V = \langle \mathbb{C}, 0, 1, + \rangle$ and $\langle \mathbb{C}^{\times}, 1, ., \mathcal{C} \rangle$ where $\mathcal{C}$ contains relations for all Zariski closed subsets of $(\mathbb{C}^{\times})^n$. We connect these two sorts with the map $\exp\colon \mathbb{C} \to \mathbb{C}^{\times}$. It was shown in [Bur08] that this is an analytic Zariski Geometry with quantifier elimination which satisfies the dimenstion theorem. It is not too hard to see that such structures could be treated in the context of Stable-Like Presmooth Geometries.

Different Generalizations of Zariski Geometries

In chapter 4 we introduced multi-sorted κ-compact Zariski Geometries as a transitional context from Stable-Like Presmooth Geometries to the world of Zariski Geometries. We did so to keep the definition easy to compare to existing ones in the literature and to highlight how the field construction can be carried out using tools from stability theory. However, I do wonder whether this is the slickest approach. For example, one might instead try to define a "generalized Zariski Geometry" as a Stable-Like one-dimensional Presmooth Geometry in which the closed sets form the closed sets of Noetherian Topologies on powers of M. Further conditions are probably needed to ensure that Noetherianity is preserved under elementary extensions, but the existence of the matrix group constructed in 4.1.20 should guarantee the exsitence of a suitable group chunk in M and Noetherianity might be enough to construct the field in M. Maybe it would even suffice to look at quantifier-free stable Stable-Like Presmooth Geometries. In any case, it would be worthwhile to investigate exactly what is needed on top of axiom (SL) to be able to carry out the field construction.

Unifying the remaining proofs

One of the main results of this thesis is that the construction of the group presentation from 3.2.25 can be carried out uniformy between Zariski Geometries and o-minimal structures. However, the fact that the proofs of non-degeneracy, richness and faithfulness as well as the field construction are not uniform is slightly unsatisfactory. As we already discussed in the context of C-minimal structures, richness might depend on additional assumptions that guarantee the existence of certain limits, but non-degeneracy and faithfulness look very natural. It is somewhat surprising that the proofs are almost trivial with axiom (1T), but very hard in o-minimal structures. Using the symmetry argument used in 4.1.22 one can quickly check that if non-degeneracy fails in an o-minimal structure, then the tangency becomes

$$h^{-1} \circ f \parallel n^{-1} \circ m \text{ iff } h = f \;\&\; m, n \geq h$$

up to symmetry (or the same where $\geq$ is replaced by $\leq$). My view is that this is such unnatural behaviour that there must be a simple argument to exclude it, but I have not been able to come up with a better argument than the one used in [PS98]. If no better argument can be found, it might still be possible to put the argument from [PS98] it into a language where it applies to Presmooth Geometries in general. Again, I have been unable to do so as of yet.

The purity of the field

One of the key results in many of the Zariski-type Trichotomy Theorems is that the field obtained is “pure”, which means that the structure M induces no structure beyond the algebraic one on the field. This is evidently false in o-minimal structures, where o-minimal expansions of real closed fields are very commonly encountered. In C-minimal structures we would expect the field to be a valued field, but we could still ask the question whether there can be any structure beyond that.

Possible fields

So far we only know of two possible types of fields that can be the result of a Trichotomy Theorem: Real Closed and Algebraically Closed Fields. However, I do not see sny reason why there should not be any others. p-adically closed fields, for example, have a well known cell-decomposition theorem and an Implicit Function Theorem so they might also fit into the context of Presmooth Geometries. If they do, this would suggest the possibility of p-adically closed fields. We then have to ask: How can we distinguish the resulting field from the geometric side? One might conjecture that structure satisfying (2T) should result in ordered fields and that stable-like structures should give algebraically closed fields, but further dividing lines might be needed in between.

Bibliography

[Adl09] Hans Adler. A geometric introduction to forking and thorn-forking. *Journal of Mathematical Logic*, 9(01):1–, 2009.

[BBWK+11] Gareth Boxall, David Bradley Williams, Charlotte Kestner, Alexandra Omar Aziz, and Davide Penazzi. Weak one-basedness. 2011.

[Bou89] Elizabeth Bouscaren. Hrushovski's group configuration. *The Model Theory of Groups (A. Nesin and A. Pillay, eds.), Notre Dame Math. Lectures*, 11, 1989.

[Bur08] Lucy Burton. *Toric geometry.* Phd thesis, 2008.

[BV10] Alexander Berenstein and Evgueni Vassiliev. Weakly one-based geometric theories. 2010.

[CH99] Zoe Chatzidakis and Ehud Hrushovski. The model theory of difference fields. *Transactions of the American Mathematical Sociey*, 351:2997–3071, 1999.

[Che79] Gregory Cherlin. Groups of small morley rank. *Annals of Mathematical Logic*, 17:1–28, 1979.

[CHP02] Zoe Chatzidakis, Ehud Hrushovski, and Ya'acov Peterzil. Model theory of difference fields, II: Periodic ideals and the trichotomy in all characteristics. *Proceedings of the London Mathematical Society*, 85(2):257–311, 2002.

[Del98] Françoise Delon. Separably closed fields. In *Model theory and algebraic* *geometry*, volume 1696 of *Lecture Notes in Math.*, pages 143–176. Springer, Berlin, 1998.

[HHM08] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson. *Stable domination and independence in algebraically closed valued fields.* Lecture notes in logic. Cambridge University Press Chicago (Ill.), Cambridge, New York, Melbourne, 2008. Table des matires http://www.loc.gov/catdir/enhancements/fy0805/2007036704-t.html.

[HM94] Deirdre Haskell and Dugald Macpherson. Cell decompositions of c-minimal structures. *Annals of Pure and Applied Logic*, 66(2):113 – 162, 1994.

[Hru85] E. Hrushovski. *Contributions to stable model theory.* Phd thesis, 1985.

[Hru93] Ehud Hrushovski. A new strongly minimal set. *Annals of Pure and Applied Logic*, 62(2):147 – 166, 1993.

[Hru96] Ehud Hrushovski. The Mordell-Lang conjecture for function fields. *Journal of the American Mathematical Society*, 9:667–690, 1996.

[Hru01] Ehud Hrushovski. The manin-mumford conjecture and the model theory of difference fields. *Annals of Pure and Applied Logic 112, no. 1, 43?115.*, 2001.

[HZ96] Ehud Hrushovski and Boris Zilber. Zariski Geometries. *Journal of the American Mathematical Society*, 9:1–56, 1996.

[Kov13] Pablo Cubides Kovacsisc. *Definissabilite dans les structures C-minimales.* Phd thesis, 2013.

[KP97] Byunghan Kim and Anand Pillay. Simple theories. *Annals of Pure and Applied Logic*, 88(23):149 – 164, 1997. ¡ce:title¿Joint AILA-KGS Model Theory Meeting¡/ce:title¿.

[Man10] Y. I. Manin. Combinatorial cubic surfaces and reconstruction theorems. *ArXiv e-prints*, January 2010.

[Mar02] David Marker. *Model Theory: An Introduction.* Springer, Graduate Texts in Mathematics No. 217, 2002.

[Pet07] Y. Peterzil. A self-guide to o-minimality, 2007.

[Pil02] Anand Pillay. Two remarks on differential fields, 2002.

[Poi01] Bruno Poizat. *Stable Groups.* American Mathematical Society, 2001.

[PS98] Ya'acov Peterzil and Sergei Starchenko. A trichotomy theorem for o-minimal structures. *Proc London Math Soc 3*, 77(3):481–523, 1998.

[PZ04] Nicholas Peatfield and Boris Zilber. Analytic zariski structures and the hrushovski construction. *Annals of Pure and Applied Logic*, 132(2-3):127–180, 2004.

[Rab93] Evgenia Rabinovich. Definability of a field in sufficiently rich incidence systems, 1993.

[SZ] Vinesh Solanki and Boris Zilber. Quantum harmonic oscillator as a zariski geometry.

[vdD98] L. P. D. van den Dries. *Tame Topology and O-minimal Structures.* Cambridge University Press, 1998.

[Wil99] A. J. Wilkie. A theorem of the complement and some new o-minimal structures. *Selecta Mathematica, New Series*, 5:397–421, 1999. 10.1007/s000290050052.

[Zil83] Boris Zilber. The structure of models of uncountably categorical theories. *Proceedings of the International Congress of Mathematicians*, 1983.

[Zil10] Boris Zilber. *Zariski Geometries.* London Mathematical Society Lecture Notes Series No. 360, 2010.