Robust Hedging of Variance Swaps: Discrete Sampling & Co-maturing European Options

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Abstract

In the practice of quantitative finance, model risk has raised significant concern and thus model-independent hedging is of particular interest to both academia and industry. In this thesis, we review two methods of constructing robust and model-independent hedging portfolios of variance swaps. One of them assumes a continuum of European options trade but does not require the underlying asset's price path to be continuous. However, the other assumes finite number of options quoted but requires the continuity of underlying asset's price path. We explore numerically the hedging performance as well as upper and lower bounds of several numerical examples by implementing these two methods. Finally, we try to combine these two methods and use an example to show an idea of a possible approach of doing this.

keywords: variance swap, robust and model-independent hedging, Skorokhod embedding problem, linear programming

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Introduction

The core of the modern finance theory is about studying the sophisticated relationship between the return and risk of financial assets. Since the middle of last century, the standard deviation of asset returns (i.e. volatility) has been proposed as the most commonly used measure of risk. In addition to its use in measuring the risk of a portfolio, volatility (variance) also has extensive use in derivative pricing. For example, volatility is a very important input in the Black-Scholes formula for option prices and the effect of volatility smile or volatility skew has been extensively studies (see [**8**] for example). Investors can take exposures to volatility market risk because of the sensitivity of option price with respect to volatility, which is known as the "Vega". However, taking a position in call and put options will not enable investors to bet purely on volatility or variance since dynamic delta-hedging and gamma-hedging is required. According to [**3**], in the 1990s, hedge funds did not even have the necessary infrastructure to delta-hedge a single option. This fact probably motivated the idea that investors could pay banks some fixed amount of money for some payoff depends explicitly on the realised variance of some underlying asset such as equity index, FX rate, and commodity price.

Formally, the payoff of a variance swap is defined as the following,

$$Payoff^{VS} = N^{VS}\left(RV - K\right) \tag{0.0.1}$$

where N^{VS} is the notional amount of the variance swap, which is quite common in other swap contracts, RV is the annualised realised variance of the underlying asset, and K is the variance strike. Note that the payoff of a variance swap looks like the payoff of a forward contract written on the realised variance of the underlying asset. However, one could imagine that the contract of variance swap enforces the two counter-parties to exchange the value of fixed lag $N^{VS}K$ and floating lag $N^{VS}RV$ at maturity, which is quite different from other type of swaps (e.g. interest rate swap, currency swap) in the sense that exchange of cash flow only happens once at the maturity date.

Since there is no cost to enter a variance swap contract (i.e. no exchange of initial cash flow), the pricing problem of variance swap is actually the problem of finding a variance strike K such that the expected value of the payoff is zero at inception, in other words,

$$E^{Q}\left[Payoff^{VS}\right] = E^{Q}\left[N^{VS}\left(RV - K\right)\right] = 0 \tag{0.0.2}$$

hence,

$$K = E^{Q}\left[RV\right] \tag{0.0.3}$$

which means finding the fair value of a variance swap rate is equivalent to finding the expected value of future realised variance under the risk-neutral measure Q.

The traditional approach of pricing over the-counter exotic derivatives such as variance swaps is as the following. A model that specifies the evolution dynamics of underlying asset price is built at the first place. Then the model is calibrated to the market data in the sense that the model parameters is chosen such that the discrepancy between the theoretical prices provided by the model and the prices of liquidly traded financial instruments is minimized. After that, the model along with the parameters can be used to price some exotic derivatives that is not liquidly traded. A simple example of this approach is the use of implied volatility surface in the pricing of exotic options written on a same or similar underlying asset.

However, in practice, as mentioned in [**?**, **?**], different model often provide different prices for the same derivative even they are calibrated to the same set of instruments. This fact raised the concern for model risk among both practitioners and academia. Therefore, a different approach is developed and the approach is robust in the sense that its model-independent. People do not necessarily need to have a probabilistic believe about the dynamics of underlying assets in order to obtain some no-arbitrage bounds of the derivatives prices. This approach will probably be studied and used by more and more people in the derivatives industry.

The overview of this thesis is as follows. In the first chapter, we will introduce some basic concepts about variance swaps as well as robust and model-independent hedging. Examples include different ways of calculating realised variance. In the second chapter, we will follow the procedures in [**10**]. With the assumption that we can trade any co-maturing European options with any positive strikes, we are able to construct sub and super hedges of a variance swap on an asset with non-continuous price paths. We describe the theory behind this method and its relation to the Skorokhod embedding problem in the first part of this chapter and explore numerically some examples in the second part. In the third chapter, we change our view on this whole problem by assuming the continuity of price paths but finite number of options liquidly traded in the market. After introducing the motivation and the method, we use a numerical example to see how the problem is actually solved and discuss some other issues regarding the method. Then we finish this thesis with a conclusion remark. A bibliography is also provided at the end.

CHAPTER 1

Basics about variance swaps and robust hedging

1.1. Different ways of calculating realised variance

First, we need to introduce some different ways of calculating realised variance since the way of doing this has a surprisingly huge impact on the pricing and hedging of variance swaps with a robust approach. We will follow the definitions in [**10**].

DEFINITION 1. A variation swap kernel is a bi-variate function $H : (0,\infty) \times (0,\infty) \mapsto [0,\infty)$ which is continuously differentiable. Also, it satisfies $H(x,x) = H_y(x,x) = 0$.

Based on that, we define a variance swap kernel as the following,

DEFINITION 2. A variance swap kernel is a regular variation swap kernel (i.e. $\in C^2$) and $H_y(x,x) = \frac{1}{x^2}$.

In this paper, we will mainly focus on two variance swap kernel, $H^R(x,y) = \left(\frac{y-x}{x}\right)^2$ and $H^L(x,y) = (\log y - \log x)^2$. Next, we restrict the definition of variation swap payoff to the case of variance swap payoff.

DEFINITION 3. The realised variance of a plain vanilla variance swap with kernel H for a certain underlying price path S and a certain partition P is

$$RV^H(S,P) = \sum_{k=0}^{N} H\left(S_{t_i}, S_{t_{i+1}}\right) \tag{1.1.1}$$

Then it can be easily seen that for H^R and H^L, we have

$$RV^R = \sum_{k=0}^{N} \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2 \tag{1.1.2}$$

$$RV^L = \sum_{k=0}^{N} \left(\log \frac{S_{t_{i+1}}}{S_{t_i}}\right)^2 \tag{1.1.3}$$

For weighted variance swaps, the realised variance is just a simple modification to (1.1). Similarly, for H^R and H^L, we have

$$RV^R = \sum_{k=0}^{N} w\left(S_{t_{i+1}}\right) \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2 \tag{1.1.4}$$

$$RV^L = \sum_{k=0}^{N} w\left(S_{t_{i+1}}\right) \left(\log \frac{S_{t_{i+1}}}{S_{t_i}}\right)^2 \tag{1.1.5}$$

In the practice of the financial industry, although variance swaps are over-the-counter (OTC) derivatives, the market convention is that the realised variance will be calculated by using kernel H^L. Also, the weight function $w(.)$ are often selected as the following, $w(S_t) = 1$ (e.g. plain vanilla variance swap), $w(S_t) = S_t$ (e.g. Gamma variance swap) and $w(S_t) = 1_I(S_t)$ (e.g. corridor variance swap).

Another important feature of a variance swap kernel is its increasing or decreasing property. We say that a variance swap kernel is increasing (decreasing) if

- $\Phi(u,y) = \frac{H_x(u,y)}{u-y}$ is monotone increasing (decreasing) in y
- $H_{yy}(x,y)$ is increasing (decreasing) in its first argument

Based on these two criteria, we could easily verify that H^R is a increasing kernel and H^L is a decreasing kernel, as shown in [**10**].

1.2. Robust hedging and model-independent pricing

In this section, we will continue to follow [**10**] and introduce some basic concepts about robust and model-independent hedging. Actually, throughout the hedging of variance swaps, we are interested in the set of semi-static strategies. We present their definition below,

DEFINITION 4. A hedging strategy is call semi-static if there exists a function pair (ψ, δ) such that the terminal payoff of the strategy is

$$\psi(S_T) + \sum_{i=0}^{N-1} \delta(S_t, 0 \le t \le t_i)\left(S_{t_{i+1}} - S_{t_i}\right) \tag{1.2.1}$$

where δ is the dynamic part of the strategy and ψ is the static part of the strategy, and S_t being a price path of the underlying asset.

Later, also in [**10**], it can be proved that given certain conditions like the differentiability of ψ, a semi-static strategy can be denoted by $\left(\psi, -\psi'\right)$, in which case ψ is called the root of this semi-static strategy.

Followed by the above definition, the definition of sub and super semi-static hedging strategies is just trivial. A semi-static hedging strategy is called a sub (super) hedge if its payoff at maturity is less (greater) than that of the (usually path-dependent) derivative need to be hedged. Using a simple no-arbitrage argument, it is quite obvious that the cost of constructing such a sub (super) hedge is a no-arbitrage lower (upper) bound of the derivative. Also, it is robust and model-independent since we have assumed any probability distribution for the underlying asset price.

We will finish this section by listing some concepts in [**5**], which are of significant importance of understanding model-independent no-arbitrage pricing.

- If we can form a semi-static portfolio with the underlying asset and some options such that the initial portfolio is negative but all cash flows after that are non-negative, then this is a model-independent arbitrage opportunity.
- If there exists a model (i.e. a filtered probability space) $(\Omega, F, (F)_t, P)$ of the underlying asset such that the option prices $C(K)$ are equal to

$E^P\left[(S_T-K)^+\right]$, then we say that the model is consistent with the option prices, and the option prices does not admit a model-dependent arbitrage.

CHAPTER 2

Hedge with a Continuum of European Options

We will first describe the theory behind the method developed in [**10**], The theory relates the problem of finding upper and lower no-arbitrage bounds of a variance swap to the Skorokhod embedding problem without giving a specific model for the price path of the underlying asset of the variance swap. Hence, the hedging strategy is "robust" in the sense that it does not rely on the specification and validity of the model describes the dynamic evolution of the underlying asset.

2.1. Recovering the implied risk-neutral distribution

Suppose that we are given market prices of co-maturing European options with a continuum of strikes, formally, we are given a continuous function $C(K)$ and $P(K)$. Without loss of generality, we only consider the case of $C(K)$, also, we assume that the function is twice differentiable. Recall the risk-neutral pricing formula for a European call option,

$$C(K) = \exp(-rT)\int_{-\infty}^{\infty}(S-K)^{+} f_{S_T}(S)\,dS \tag{2.1.1}$$

$$= \exp(-rT)\int_{K}^{\infty}(S-K) f_{S_T}(S)\,dS \tag{2.1.2}$$

where $f_{S_T}(S)$ is the risk-neutral density function of underlying asset price at time T, taking partial derivatives with respect to strike price K, we obtain

$$\frac{\partial C(K)}{\partial K} = \exp(-rT)\frac{\partial}{\partial K}\left\{\int_{K}^{\infty}(S-K) f_{S_T}(S)\,dS\right\} \tag{2.1.3}$$

$$= \exp(-rT)\left\{\int_{K}^{\infty}\frac{\partial}{\partial K}(S-K) f_{S_T}(S)\,dS - (K-K) f_{S_T}(K)\right\} \tag{2.1.4}$$

$$= \exp(-rT)\int_{K}^{\infty} -f_{S_T}(S)\,dS \tag{2.1.5}$$

$$= -\exp(-rT)\int_{K}^{\infty} f_{S_T}(S)\,dS \tag{2.1.6}$$

taking the second order derivative with respect to strike price K again, we get

$$\frac{\partial^2 C(K)}{\partial K^2} = \exp(-rT) f_{S_T}(K) \tag{2.1.7}$$

Therefore, we can derive the Breeden-Litzenberger formula which calculates the implied risk-neutral distribution

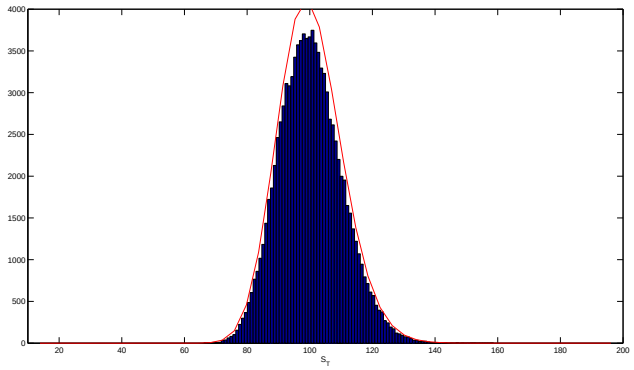


FIGURE 2.1.1. Implied risk-neutral distribution derived from 50 option prices, histogram of a sample from log-normal distribution, sample size 10^5

$$f_{S_T}(K) = \exp(rT)\frac{\partial^2 C(K)}{\partial K^2} \tag{2.1.8}$$

We demonstrate the above equation with a numerical example.

Consider the usual Black-Scholes model with constant risk-free rate and volatility. It is well-known that the model suggests the underlying asset price at maturity S_T follows a log-normal distribution, formally

$$S_T = S_0 \exp\left(\left(r - q - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right) \tag{2.1.9}$$

$$Z \sim N(0,1) \tag{2.1.10}$$

However, consider the equation (2.1.8) and use a second order central difference scheme to approximate the second order derivative $\frac{\partial^2 C(K)}{\partial K^2}$, we get the (discrete) implied risk-neutral distribution of S_T, given by

$$f_{S_T}(K) = \exp(rT)\lim_{dK\to 0}\frac{C(K+dK) + C(K-dK) - 2C(K)}{(dK)^2} \tag{2.1.11}$$

As we said, for simplicity, we assume the validity of Black-Scholes model in this example. Hence, we substitute the call prices with the Black-Scholes prices in equation (2.1.11) and calculate the implied risk-neutral distribution. Also, using Monte Carlo simulation and equation (2.1.9), we can easily get the (approximate) plot of log-normal distribution. We expect these two methods of estimating the probability distribution of S_T will yield two (approximately) same result. As shown below, we present two plots below, the first one uses 50 co-maturing European call options and the second uses 100 options to estimate the implied risk-neutral distribution. See that the second one obtains a better estimation of the risk-neutral density since more options are employed.

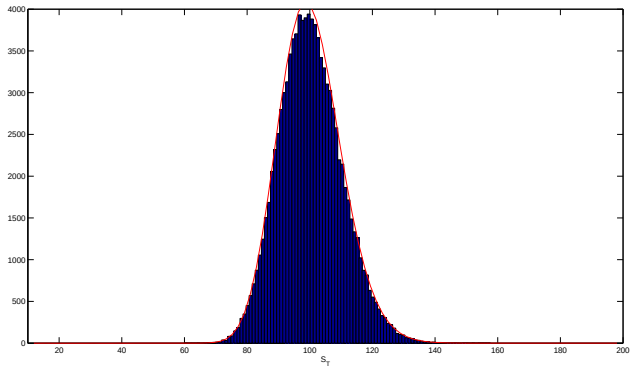


FIGURE 2.1.2. Implied risk-neutral distribution derived from 100 option prices, histogram of a sample from log-normal distribution, sample size 10^5

2.2. Hedging with and without the continuous path assumption

2.2.1. Hedging without jumps. Now we suppose the sample path S_t is continuous. The quadratic variation of path f is given by

$$\langle S \rangle_T = \lim_{n \to \infty} \sum_{\max\{t_i\} \leq T} \left(S_{t_{i+1}} - S_{t_i} \right)^2 \tag{2.2.1}$$

Use Ito's formula on $\log S_t$, we obtain

$$d\left(\log S_t\right) = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} d\left\langle S \right\rangle_t \tag{2.2.2}$$

Integrate both sides from 0 to T yields

$$\log S_T - \log S_0 = \int_0^T \frac{1}{S_t} dS_t - \frac{1}{2} \int_0^T \frac{1}{S_t^2} d\left\langle S \right\rangle_t \tag{2.2.3}$$

Rearranging the above equation gives us

$$\int_0^T \frac{1}{S_t^2} d\left\langle S \right\rangle_t = 2 \int_0^T \frac{1}{S_t} dS_t - 2 \log \frac{S_T}{S_0} \tag{2.2.4}$$

Recall that the realised variance of a (plain vanilla) variance swap is

$$RV^R = \sum_{k=0}^{N} \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \right)^2 \tag{2.2.5}$$

or

$$RV^L = \sum_{k=0}^{N} \left(\log \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2 \tag{2.2.6}$$

in some cases.

Note that if the variance swap is continuously monitored, then the (discretely-sampled) realised variance will have a continuous time limit

$$\int_0^T \frac{1}{S_t^2} d \langle S \rangle_t \tag{2.2.7}$$

or equivalently

$$\int_0^T d \langle \log S \rangle_t \tag{2.2.8}$$

Therefore, the left hand side of equation (2.2.4) is just the realised variance of a continuously monitored of a variance swap on an underlying asset with continuous price path. The right hand side of the equation is a portfolio holding a short position in the European log contract and a dynamically trading position in the underlying asset. [**13**] argued that this is a perfect hedge. To sum up, given the continuity assumption of the underlying asset's price path, we can construct a semi-static hedge for a continuously monitored variance swap. Furthermore, since we have not assumed any model for the underlying asset, the hedge is robust and a model-independent price can be obtained by calculating the cost of constructing such a hedge.

2.2.2. Hedging with jumps. In reality, the price path of the underlying asset may have jumps at some time and our previous hedging argument for continuous path continuously monitored variance swap may become invalid. However, we follow [**10**] and move on to the more general setting based on the results obtained in previous discussions.

Recall that in the with the continuous price path assumption, we have

$$\int_0^T d \langle \log S \rangle_t = \int_0^T \frac{1}{S_t^2} d \langle S \rangle_t = 2 \int_0^T \frac{1}{S_t} dS_t - 2 \log \frac{S_T}{S_0} \tag{2.2.9}$$

Let J be the set of time where the price path jump from S_{t-} to $S_{t-} - \Delta S_t$, using the path-wise Ito's formula on $\log S_T$. Note that the second term on the right hand side of the above equation will not change due to the additive property of the logarithm function. The first term on the right hand side will become $2 \int_0^T \frac{1}{S_t} dS_t - 2 \sum_J \frac{\Delta S_t}{S_{t-}}$. For the left hand side, since

$$\langle S \rangle_t = \langle S \rangle_t^C + \sum_J (\Delta S)^2 \tag{2.2.10}$$

where $\langle S \rangle_t^C$ denotes the quadratic variation of continuous part of price path S_t, we know that

$$d\langle \log S\rangle_t = \frac{d\langle S\rangle_t^C}{S_{t-}^2} + (\Delta \log S_t) \tag{2.2.11}$$

Hence, we have the following equality,

$$\int_0^T \frac{1}{S_{t-}^2} d\langle S\rangle_t^C - \log\left(1+\frac{\Delta S_t}{S_{t-}}\right) = 2\int_0^T \frac{1}{S_{t-1}} dS_t - 2\sum_J \left(\frac{\Delta S_t}{S_{t-}}\right) - 2\log\frac{S_T}{S_0} \tag{2.2.12}$$

Rearranging the above equation and note that $\int_0^T d\langle \log S\rangle_t = \langle \log S\rangle_T = RV^L$, which is the realised variance of a continuously monitored variance swap with kernel H^L. We can calculate the hedging error as the following,

$$\langle \log S\rangle_T - \left(2\int_0^T \frac{1}{S_{t-1}} dS_t - 2\log\frac{S_T}{S_0}\right) = -2\sum_J \left(\frac{\Delta S_t}{S_{t-}}\right) + ... \tag{2.2.13}$$

Similarly, for a continuously monitored variance swap with kernel H^R, we already know that the realised variance is$RV^R = \int_0^T \frac{1}{S_{t-}^2} d\langle S\rangle_t$, so the hedging error is

$$\int_0^T \frac{1}{S_{t-}^2} d\langle S\rangle_t - \left(2\int_0^T \frac{1}{S_{t-1}} dS_t - 2\log\frac{S_T}{S_0}\right) = -2\sum_J \left(\frac{\Delta S_t}{S_{t-}}\right) + ... \tag{2.2.14}$$

The above two equations on hedging errors tell us that one may sub or super hedge his or her position in a variance swap due to the effect of jumps. What's more, the P&L caused by the hedging error may have different size and even different directions for variance swaps with different kernels. Therefore, it is essential to develop a hedging strategy in the presence of jumps.

2.3. Relation to the Skorokhod embedding problem

In this subsection, we will introduce the connection between the robust hedging of variance swaps and the Skorokhod embedding problem. Let us first introduce the Skorokhod embedding problem.

In the classical analysis of random process, we are usually interested in the marginal distribution of a stochastic process at a specific time, either deterministic or random. However, we change our point of view in some way. Consider if we are given a probability distribution of a stochastic process and we are asked to find a random time such that the random process at the that random time follow the given probability distribution. This problem was proposed and solved by Soviet and Ukrainian mathematician Anatoliy Skorokhod in 1961 and it has stimulated the research in probability theory and stochastic process for over forty years. According to [**14**] and [**15**], the Skorokhod embedding problem can be formulated as the following,

PROBLEM 5. Let $(X_t : t > 0)$ be a given stochastic process and μ a given probability measure, find a minimal stopping time τ such that random variable X_τ has marginal distribution μ, i.e.

$$X_\tau \sim \mu \tag{2.3.1}$$

Later, Skorokhod imposed a condition $E[\tau] < \infty$ on the problem to ensure that the solution is not too "large" and meaningless. The certain conditions under which a solution to the problem could be found are discussed in [**14**].

Consider a continuously monitored variance swap with kernel H^R, we know that it has realised variance

$$RV^R = \int_0^T \frac{d\langle S\rangle_t^C}{(S_{t-})^2} + \sum_J \left(\frac{\Delta S_t}{S_{t-}}\right)^2 \tag{2.3.2}$$

Let S_t^{sup} denote the running maximum process of S_t (i.e. $S_t^{sup} = \sup_{u\le t} S_u$), we have the inequality

$$RV^R = \int_0^T \frac{d\langle S\rangle_t^C}{(S_{t-})^2} + \sum_J \left(\frac{\Delta S_t}{S_{t-}}\right)^2 \tag{2.3.3}$$

$$\geq \int_0^T \frac{d\langle S\rangle_t^C}{\left(S_{t-}^{sup}\right)^2} + \sum_J \left(\frac{\Delta S_t}{S_{t-}^{sup}}\right)^2 \tag{2.3.4}$$

Furthermore, there exists some time-change $t \to A_t$ such that $S_t = B_{A_t}$, where B is a Brownian motion starts at mean m of a probability distribution μ and $X_0 = 0, X_T \sim \mu$. Since S_t is the price path with jumps, A_t is also a stochastic process with jumps. Its continuous part A_t^C satisfies $dA_t^C = \left(dS_t^C\right)^2 = d\left\langle S^C\right\rangle_t$. Hence, inequality (2.1.29) becomes

$$RV^R \geq \int_0^T \frac{d\langle S\rangle_t^C}{\left(S_{t-}^{sup}\right)^2} + \sum_J \left(\frac{\Delta S_t}{S_{t-}^{sup}}\right)^2 \tag{2.3.5}$$

$$\geq \int_0^T \frac{dA_t^C}{\left(B_{A_{t-}}^{sup}\right)^2} + \sum_J \left(\frac{\Delta B_{A_t}}{B_{A_{t-}}^{sup}}\right)^2 \tag{2.3.6}$$

where B_t^{sup} is the running maximum process of Brownian motion B_t (i.e. $B_t^{sup} = \sup_{u\le t} B_u$). Taking expectation on both sides of the inequality above,

$$E\left[\int_0^T \frac{dA_t^C}{\left(B_{A_{t-}}^{sup}\right)^2} + \sum_J \left(\frac{\Delta B_{A_t}}{B_{A_{t-}}^{sup}}\right)^2\right] = E\left[\int_0^T \frac{dA_t^C + \Delta A_t}{\left(B_{A_{t-}}^{sup}\right)^2}\right] \tag{2.3.7}$$

$$= E\left[\int_0^T \frac{dA_t}{\left(B_{A_{t-}}^{sup}\right)^2}\right] \tag{2.3.8}$$

$$\geq E\left[\int_0^{A_T} \frac{du}{(B_t^{sup})^2}\right] \tag{2.3.9}$$

Therefore, what we are actually interested in the following minimisation problem

$$\min_{\tau} E\left[\int_0^{\tau} \frac{du}{(B_u^{sup})^2}\right] \tag{2.3.10}$$

where τ is such a stopping time that $B_\tau \sim \mu$. Recall our description of the Skorokhod embedding problem in the previous part of this paper, it can be seen that what we need is a not only a solution to the Skorokhod embedding problem but also a solution with some certain minimum property that minimizes (2.1.35). In [**11**], they found that the solution is actually the Perkins embedding which can be stated as the following.

PROPOSITION 6. *Let μ be a probability measure on $\mathbb{R}^+$ with mean m, Z a random variable with distribution μ. Define functions $\alpha : (m,\infty) \mapsto [0,m)$ and $\beta : (0,m) \mapsto (m,\infty)$ as below,*

$$\alpha(x) = \arg\min_{y<m} \frac{C(x) - P(y)}{x-y} \tag{2.3.11}$$

$$\beta(x) = \arg\min_{y>m} \frac{P(x) - C(y)}{y-x} \tag{2.3.12}$$

where

$$C(x) = E\left[(Z-x)^+\right] \tag{2.3.13}$$

$$P(y) = E\left[(y-Z)^+\right] \tag{2.3.14}$$

Let B be a Brownian motion starts at m, B^{sup}, B^{inf} be the running maximum and minimum processes. The stopping time

$$\tau = \inf\left\{u > 0 : B_u < \alpha\left(B_u^{sup}\right) or B_u > \beta\left(B_u^{inf}\right)\right\}$$

solves the Skorokhod embedding problem. In addition, this Perkins solution has the property of minimizing $E\left[F\left(B_\tau^{sup}\right)\right]$ over all embeddings τ for increasing function F.

2.3.1. Calculate the hedges step by step. Motivated by the connection between this robust hedging problem and the Perkins solution to the Skorokhod embedding problem, [**10**] calculated such a optimal sub and super hedge for a variance swap on a underlying asset price path which has jumps, and they we also show that there exists a consistent model such that it reproduces the market data (i.e. the hedge is indeed "optimal"). In order to make our paper consistent, we summarize their results and state them as the following without proof.

PROPOSITION 7. *Let's define a class of the roots of semi-static hedges $\left(\psi, -\psi'\right)$ as the following,*

$$\psi_\kappa(x) = \begin{cases} 0 & x = S_0 \\ \int_{S_0}^{x} (x-u)\Phi(u,\kappa(u))du & x > S_0 \\ \psi_\kappa(k(x)) + \psi_\kappa'(k(x))(x-k(x)) + H(k(x),x) & x < S_0 \end{cases} \tag{2.3.15}$$

Algorithm 1 Algorithm of calculating the model-independent robust hedge for a variance swap of certain given kernels using market data of co-maturing European vanilla options

- Revert the implied risk-neutral distribution μ from market prices of co-maturing vanilla European options, use Breeden-Litzenberger formula
- Given the implied risk-neutral distribution μ, calculate the two quantities $\alpha(x), \beta(x)$ defined in the Perkins solution to the Skorokhod embedding problem
- Use Proposition 7, know that the optimal sub and super hedges are associated with functions κ, ι, calculate their inverse function k, l
- Use Proposition 6, calculate the optimal sub and super hedges $\psi_\kappa(x), \psi_\iota(x)$, obtain the root of semi-static hedge $\left(\psi_\kappa, -\psi_\kappa^{'}\right), \left(\psi_\iota, -\psi_\iota^{'}\right)$

where $\kappa \in K$ is the class of monotone decreasing right-continuous function on $[S_0, \infty) \mapsto (0, S_0]$ with $\kappa(S_0) = S_0$. Function k is the inverse of functionκ. Similarly, let's define another class of the roots of semi-static hedges as the following,

$$\psi_\iota(x) = \begin{cases} 0 & x = S_0 \\ \int_x^{S_0} (u - x)\, \Phi(u, \iota(u))\, du & x < S_0 \\ \psi_\iota(l(x)) + \psi_\iota^{'}(l(x))(x - l(x)) + H(l(x), x) & x > S_0 \end{cases} \tag{2.3.16}$$

where $\iota \in L$ is the class of monotone increasing right-continuous function on $(0, S_0) \mapsto (S_0, \infty)$ with $\iota(S_0) = S_0$. Function l is the inverse of function ι. In addition, if $\Phi(u, y) = \frac{H_x(u-y)}{u-y}$ is increasing (decreasing) in y and $H_{yy}(x, y)$ is increasing (decreasing) in x, we say the variance swap kernel H is increasing (decreasing). Furthermore, every root of semi-static hedge in class $\psi_\kappa(x)$ is a sub (super)hedge and every root of semi-static hedge in class $\psi_\iota(x)$ is a super (sub)hedge of the variance swap with kernel H.

Moreover, the result about the optimality of the semi-static sub and superhedges can be stated as below,

PROPOSITION 8. *For variance swaps with increasing kernel H, its most expensive semi-static sub-hedge in class ψ_κ is given by $\kappa = \alpha$, its cheapest semi-static super-hedge in class ψ_ι is given by $\iota = \beta$, where α and β are two quantities involved in the Perkins solution to the Skorokhod embedding problem. For variance swaps with decreasing kernels, a similar results can be obtained.*

Note that in order to find a model that makes the inequalities (2.1.31) and (2.1.34) becomes equalities (i.e. find a model such that the optimal sub and superhedge is attainable), the process must jump at its current maximum, that's probably the reason why this problem is related to the Perkins solution to the Skorokhod embedding problem. So far, we have finished describing the theory behind the method proposed in [**10**]. We summarize the steps as the following algorithm,

2.4. Numerical Implementation

2.4.1. Case I: uniform distribution. Let's consider a simple case first. Suppose we are considering a variance swap with increasing variance kernel

$$H(x,y) = H^R(x,y) = \left(\frac{y-x}{x}\right)^2 \tag{2.4.1}$$

Also, assume that the final distribution of the underlying asset follows a uniform distribution, formally,

$$Z \sim U[0,2] \tag{2.4.2}$$

Recall the Perkins solution to the Skorokhod embedding problem stated in the previous section.

Let ν be a probability measure with support on the half real line R^+. Let Z denote a random variable with law ν and mean m. Define

$$C(z) = E\left[(Z-z)^+\right] \tag{2.4.3}$$

$$P(z) = E\left[(z-Z)^+\right] \tag{2.4.4}$$

then we have

$$\alpha(z) = \arg\min_{y<m} \frac{C(z)-P(y)}{z-y} \tag{2.4.5}$$

$$\beta(z) = \arg\min_{y>m} \frac{P(z)-C(y)}{y-z} \tag{2.4.6}$$

In our case, we can derive the expression (2.4.5) and (2.4.6) explicitly as the following,

$$\frac{C(z)-P(y)}{z-y} = \frac{E\left[(Z-z)^+\right]-E\left[(y-Z)^+\right]}{z-y} \tag{2.4.7}$$

$$= \frac{\int_0^2 (Z-z)^+ f_\nu(Z)dZ - \int_0^2 (y-Z)^+ f_\nu(Z)dZ}{z-y} \tag{2.4.8}$$

$$= \frac{\frac{1}{2}\int_z^2 (Z-z)dZ - \frac{1}{2}\int_0^y (y-Z)dZ}{z-y} \tag{2.4.9}$$

$$= \frac{1-z+\frac{1}{4}z^2-\frac{1}{4}y^2}{z-y} \tag{2.4.10}$$

Similarly,

$$\frac{P(z)-C(y)}{y-z} = \frac{E\left[(z-Z)^+\right]-E\left[(Z-y)^+\right]}{y-z} \tag{2.4.11}$$

$$= \frac{\int_0^2 (z-Z)^+ f_\nu(Z)dZ - \int_0^2 (Z-y)^+ f_\nu(Z)dZ}{y-z} \tag{2.4.12}$$

$$= \frac{\frac{1}{2}\int_0^z (z-Z)dZ - \frac{1}{2}\int_y^2 (Z-y)dZ}{y-z} \tag{2.4.13}$$

$$= \frac{-1+y+\frac{1}{4}y^2+\frac{1}{4}z^2}{z-y} \tag{2.4.14}$$

In order to minimize (2.4.10) and (2.4.14), check the first order condition and obtain

$$\frac{d}{dy}(\frac{C(z)-P(y)}{z-y}) = \frac{d}{dy}(\frac{1-z+\frac{1}{4}z^2-\frac{1}{4}y^2}{z-y}) \tag{2.4.15}$$

$$= \frac{1-z+\frac{1}{4}z^2+\frac{1}{4}y^2-\frac{1}{2}yz}{(z-y)^2} \tag{2.4.16}$$

Similarly,

$$\frac{d}{dy}(\frac{P(z)-C(y)}{y-z}) = \frac{d}{dy}(\frac{-1+y+\frac{1}{4}y^2+\frac{1}{4}z^2}{z-y}) \tag{2.4.17}$$

$$= \frac{-1+z+\frac{1}{4}z^2-\frac{1}{4}y^2+\frac{1}{2}yz}{(z-y)^2} \tag{2.4.18}$$

Set (2.4.16) and (2.4.18) to zero and get

$$\alpha(z) = \arg\min_{y<m} \frac{C(z)-P(y)}{z-y} \tag{2.4.19}$$

$$= z-2\sqrt{z-1}, z>1 \tag{2.4.20}$$

$$\beta(z) = \arg\min_{y>m} \frac{P(z)-C(y)}{y-z} \tag{2.4.21}$$

$$= z+2\sqrt{1-z}, 0<z<1 \tag{2.4.22}$$

Next, recall the important fact about the most expensive sub-hedge stated in Proposition 5.2 in [**10**]. It says

Let H be an increasing variance swap kernel, then the function κ of class K that maximize $\int_0^\infty \psi_\kappa(x)\mu(dx)$ is

$$\kappa = \alpha \tag{2.4.23}$$

where α is the same one as in (2.4.5), which is related to the definition of Perkins solution to the Skorokhod embedding problem.

Therefore, in our case, after knowing κ, we can evaluate ψ_κ based on Definition 4.2 in [**10**]. The definition of ψ_κ is as the following

Suppose function κ of class K has inverse k, we can define a function $\psi_\kappa : (0,\infty) \mapsto \mathbb{R}^+$ as the following

$$\psi_\kappa(x) = \begin{cases} 0 & x = f(0) \\ \int_{f(0)}^x (x-u)\Phi(u,\kappa(u))du & x > f(0) \\ \psi_\kappa(k(x)) + \psi_\kappa'(k(x))(x-k(x)) + H(k(x),x) & x < f(0) \end{cases} \tag{2.4.24}$$

Similarly, we can describe another

$$\psi_\iota(x) = \begin{cases} 0 & x = f(0) \\ \int_x^{f(0)} (u-x)\Phi(u,\iota(u))du & x < f(0) \\ \psi_\iota(l(x)) + \psi_\iota'(l(x))(x-l(x)) + H(l(x),x) & x > f(0) \end{cases} \tag{2.4.25}$$

where

$$\Phi(u,y) = \frac{H_x(u,y)}{u-y} \tag{2.4.26}$$

In our case, since we are considering the variance swap kernel $H^R(x,y) = (\frac{y-x}{x})^2$, we can obtain the explicit formula for $\Phi(u,y)$, which is

$$\Phi(u,y) = \frac{2y}{u^3} \tag{2.4.27}$$

Also, since $k(z)$ is nothing but the inverse function of $\kappa(z)$, its expression can be easily derived in this case, which is

$$k(z) = z + 2\sqrt{z} + 2 \tag{2.4.28}$$

In addition, also note that for our simple case, when $x > f(0)$, the expression for ψ_κ can be explicitly written as

$$\psi_\kappa(x) = \int_{f(0)}^{x} (x-u)\Phi(u,\kappa(u))du \tag{2.4.29}$$

$$= \int_{f(0)}^{x} (x-u)\frac{2\kappa(u)}{u^3}du \tag{2.4.30}$$

$$= \int_{f(0)}^{x} (x-u)\frac{2(u-2\sqrt{u-1})}{u^3}du \tag{2.4.31}$$

Similarly, when $x < f(0)$ the expression for $\psi_\iota(x)$ can be written as

$$\psi_\iota(x) = \int_{x}^{f(0)} (u-x)\Phi(u,\iota(u))du \tag{2.4.32}$$

$$= \int_{x}^{f(0)} (u-x)\frac{2\iota(u)}{u^3}du \tag{2.4.33}$$

$$= \int_{x}^{f(0)} (u-x)\frac{2(u+2\sqrt{1-u})}{u^3}du \tag{2.4.34}$$

Use Matlab, we obtain

(2.4.35)

$$\psi_\kappa(x) = -2(\frac{\sqrt{u-1}(ux+4u-2x)}{2u^2} + \frac{x}{u} + \frac{1}{2}(x-4)\tan^{-1}(\sqrt{u-1}) + \log u)|_{f(0)}^{x}, x > f(0)$$

Therefore, up to now, we have derived the explicit expressions for $\psi_\kappa(x)$ and $\psi_\iota(x)$. However, since the expression contains some rather complicated integration and differentiation, we consider using numerical techniques to do the implementation. The graphs of $\psi_\kappa(x)$ and $\psi_\iota(x)$ are given below.

If we change the variance swap kernel to $H^L(x,y) = (log(y) - log(x))^2$, we can follow the same procedures described above to derive all the expressions. The plot is given below. Note that they are essentially the same, the only difference is that

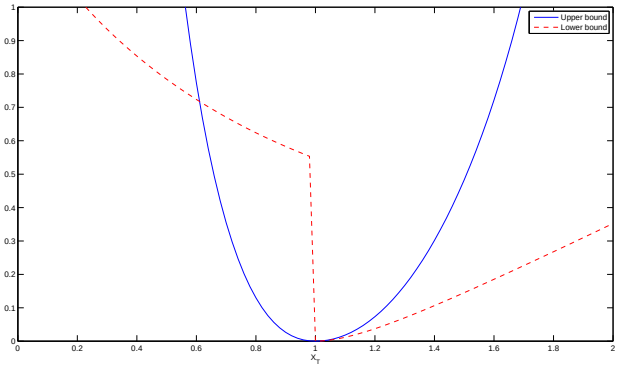


FIGURE 2.4.1. Model-independent bounds of a variance swap with kernel H^R, bounds are calculated relative to the cost of -2 log contracts

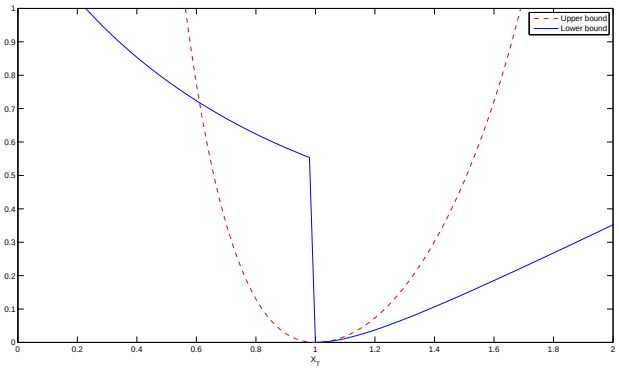


FIGURE 2.4.2. Model-independent bounds of a variance swap with kernel H^L, bounds are calculated relative to the cost of -2 log contracts

for kernel H^R, ψ_κ is associated with a lower bound and ψ_ι is associated with a upper bound, however, for kernel H^L, the bounds are reversed.

2.4.2. Case II: Log-normal distribution and Black-Scholes model. In this subsection, we will investigate another numerical example that is more realistic than the previous one. Let's consider the classical Black-Scholes world with zero interest rate, zero dividend yield and constant volatility. It is well known that the underlying price at maturity is believed to have a log-normal distribution, given by

$$S_T = S_0 exp\left(\left(r - q - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right) \tag{2.4.36}$$

and hence

$$S_T = S_0 exp\left(-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}Z\right) \tag{2.4.37}$$

is a martingale under risk-neutral measure $\mathbb{Q}$, where Z is a standard normal distribution. As described in the previous algorithm, we will start with the market data of co-maturing European option prices. However, since it is assumed that we are living in a Black-Scholes world, the option prices are given by usual Black-Scholes option pricing formula, which make it no need to revert the market data to obtain the implied risk-neutral distribution.

So we proceed to the second step of the algorithm, given the implied risk-neutral distribution μ, which can be explicitly written down as in (2.4.37), we want to calculate the two quantities α and β first. Recall that,

$$\alpha(z) = \arg\min_{y<m} \frac{C(z) - P(y)}{z - y} \tag{2.4.38}$$

$$\beta(z) = \arg\min_{y>m} \frac{P(z) - C(y)}{y - z} \tag{2.4.39}$$

where

$$C(x) = E\left[(Z - x)^+\right] \tag{2.4.40}$$

$$P(y) = E\left[(y - Z)^+\right] \tag{2.4.41}$$

It can be easily seen that this is equivalent to

$$\alpha(z) = \arg\min_{y<m} \frac{BS^{Call}(z) - BS^{Put}(y)}{z - y} \tag{2.4.42}$$

$$\beta(z) = \arg\min_{y>m} \frac{BS^{Put}(z) - BS^{Call}(y)}{y - z} \tag{2.4.43}$$

where $BS^{Call}(z)$ and $BS^{Put}(y)$ denote the Black-Scholes call and put prices with zero interest rate and strike z and y respectively. This observation will simplify our search for the minimizing argument y a lot since we no longer need to calculate the expectation $E\left[(Z - x)^+\right]$ and $E\left[(y - Z)^+\right]$ numerically using Monte Carlo simulation.

After doing this, we carry on to calculate k, l , which are the inverse of functions κ, ι. Although different from the previous uniform distribution case in the sense that the expressions cannot be explicitly written down, we can evaluate the functions numerically. That is to say, given an array of two-dimensional points $(x_1, y_1), ..., (x_n, y_n)$ such that $y_i = \kappa(x_i), \forall i$, the inverse function $k = \kappa^{-1}$ will pass points $(y_1, x_1), ..., (y_n, x_n)$, we will use this fact to evaluate the inverse function numerically. However, this approach does have some drawbacks, which will be discussed in the next subsection.

Next, we are well prepared to calculate the function pair $\left(\psi_\kappa, -\psi_\kappa^{'}\right)$ and $\left(\psi_\iota, -\psi_\iota^{'}\right)$, which is the root of semi-static hedging portfolios. Recall that,

$$\psi_\kappa(x) = \begin{cases} 0 & x = f(0) \\ \int_{f(0)}^{x}(x-u)\Phi(u,\kappa(u))du & x > f(0) \\ \psi_\kappa(k(x)) + \psi_\kappa^{'}(k(x))(x-k(x)) + H(k(x),x) & x < f(0) \end{cases} \tag{2.4.44}$$

note that in this case, we have $f(0) = S_0 = E[S_T]$ since we are working in a zero interest rate environment and the underlying asset price is a martingale. Hence, for $x < f(0)$, we have $k(x) > 0$. Therefore,

$$\psi_\kappa^{'}(k(x)) = \frac{d\psi_\kappa(z)}{dz}|_{z=k(x)} \tag{2.4.45}$$

$$= \frac{d}{dz}\left\{\int_{f(0)}^{z}(z-u)\Phi(u,\kappa(u))du\right\}|_{z=k(x)} \tag{2.4.46}$$

$$= \left\{\int_{f(0)}^{z}\Phi(u,\kappa(u))du + 0\right\}|_{z=k(x)} \tag{2.4.47}$$

$$= \int_{f(0)}^{k(x)}\Phi(u,\kappa(u))du \tag{2.4.48}$$

The reason of doing this equivalent transform is that in terms of computational simplicity, doing numerical integration is much more easier than doing numerical differentiation since the latter may bring problems such as numerically unstable. In this concrete case, since $\Phi(u,\kappa(u))$ is continuous and bounded on $[f(0), k(x)]$, doing numerical integration is not so difficult compared to doing numerical differentiation.

Finally, we can calculate the root of semi-static hedging portfolios and compare the expectation under the log-normal distribution with that of the -2 log contract for different level of volatility parameters in the log-normal distribution. Note that the expectation of the payoff of log contract is

$$E\left[-2\log\frac{S_T}{S_0}\right] = E\left[-2\left(\left(r-q-\frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right)\right] \tag{2.4.49}$$

$$= -2\left(r-q-\frac{1}{2}\sigma^2\right)T \tag{2.4.50}$$

$$= \sigma^2 T \tag{2.4.51}$$

the final equality is true since we have assumed zero interest rate and zero dividend yield.

The plot is provided below. We observe that for higher value of volatility parameter, the underlying asset price at maturity will tend to move far away from its mean value and the bounds of the variance swap will be wider, which means the model-independent hedge is less efficient.

2.4.3. Issues in the numerical implementation. During the process of implementing the methods in [**10**], we have found some practical issues that have some effect on the performance of our numerical computing.

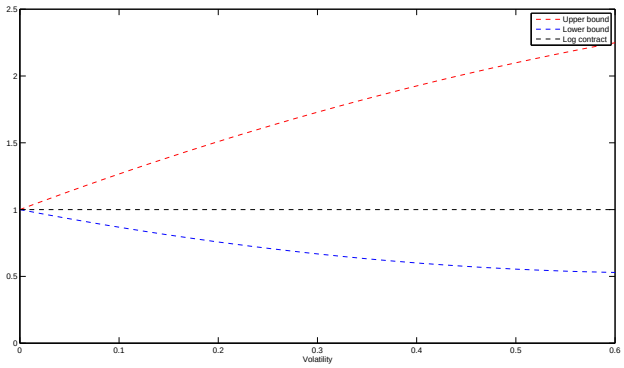


FIGURE 2.4.3. Upper and lower bounds relative to -2 log contracts vs. volatility parameter in the log-normal distribution

First, when calculating the two functions α and β in the Perkins solution to the Skorokhod embedding problem, we are actually solving an optimization problem with constraint $y < m$. For some problems with an objective function that can be explicitly written down, this process is not so difficult since we can apply the usual optimization techniques such as first order condition, etc. However, when the objective function is not known explicitly (e.g. Heston's stochastic volatility model), solving this minimisation problem is difficult and the solution may not be stable.

Second, even if we can solve the minimisation problem and evaluate function α and β numerically, we still need to know their inverse function k and l. This task is not so difficult if α, β are monotone functions on their respective domain. However, if they are not, the task of evaluating their numerical inverse function is not trivial and might need some further investigation.

Third, in the process of calculating functions ψ_κ and ψ_ι, we need to evaluate their derivatives $\psi_\kappa^{'}$ and $\psi_\iota^{'}$ sometime. It seems that this derivative may not be accurately approximated by using central difference scheme sometime. For example, consider the following common approximation

$$\psi_\kappa^{'}(x) \approx \frac{\psi_\kappa(x+h) + \psi_\kappa(x-h) - 2\psi_\kappa(x)}{2h} \tag{2.4.52}$$

In this scheme, the choice of h is of vital importance because ψ_κ is defined on three different domains, which means $x+h$ and $x-h$ may be in different domains and hence have completely different definitions. This problem will make the solution unstable sometime as shown in our uniform distribution example. However, in some cases, the use of numerical differentiation could be avoided and we can use numerical integration instead. The log-normal distribution example demonstrated this point.

Finally, since we are sampling from the implied risk-neutral distribution to calculate the hedges, Monte Carlo simulation is used. Hence, the total computational complexity should be taken into consideration if a large sample is needed.

CHAPTER 3

Hedge with Finite Number of European Options

3.1. Motivation and method

Up to this moment, we have assumed that in order to hedge our position in a variance swap, we can traded co-maturing European options with infinite many strikes. In other words, we are assuming a continuum of European option strikes. However, in practice, only a finite number of European options are quoted. This difference brings us a question. How do we hedge variance swaps using finite number of co-maturing European options? As proposed by [**?, ?**], linear programming techniques can be used to search for the most expensive sub-hedge and cheapest super-hedge of the variance swap. In this section, we will explore numerically the bounds of variance swaps in [**?, ?**].

Let's first assume that underlying asset of the variance swap has a continuous price path, formally, the evolution of the asset's price process is diffusive, as described as the following,

$$dS_t = S_t\mu(t,...)dt + S_t\sigma(t,...)dW_t \tag{3.1.1}$$

Applying Ito's formula to $\log S_T$ yields,

$$d(\log S_t) = \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}d\langle S\rangle_t \tag{3.1.2}$$

integrating on both sides yields,

$$\log S_T - \log S_0 = \int_0^T \frac{1}{S_t}dS_t - \frac{1}{2}\langle \log S\rangle_T \tag{3.1.3}$$

rearranging the above yields

$$\langle \log S\rangle_T = 2\int_0^T \frac{1}{S_t}dS_t - 2\log(\frac{S_T}{S_0}) \tag{3.1.4}$$

By [**17**], it has been proved that the realised variance converges to some quantity almost surely, formally,

$$RV_T \longrightarrow \langle \log S\rangle_T, a.s. \tag{3.1.5}$$

Therefore, (3.1.4) means that in order to capture the realised variance, we need to hold a dynamic position in the underlying asset and a static position in the log contract. Since there are actually no liquid market for log-contract, we would like to use a static portfolio of co-maturing European options to replicate the payoff of the log-contract. Also, we need the options we used in our hedging portfolio to be

liquidly traded. This motivates us to use a combination of out-of-money call options (with high strikes) and out-of-money put options (with low strikes). Therefore, we need to introduce some cutoff level S^* such that we will use put options for strikes below this cut off and call options for strikes above this cutoff. Following [**?**, **?**, **?**], the mathematical derivation are as the following,

$$\log\frac{S_T}{S_0} = \log\frac{S_T}{S^*} + \log\frac{S^*}{S_0} \tag{3.1.6}$$

$$= \frac{S_T - S^*}{S^*} - \int_0^{S^*} \frac{1}{K^2}(K - S_T)^+ dK - \int_{S^*}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK + \log\frac{S^*}{S_0} \tag{3.1.7}$$

the detailed proof can be found in [**?**, **?**, **?**]. Therefore, we consider hedging the log contract using the underlying asset, the zero-coupon bond and some co-maturing European options with a finite set of strikes.

Our next effort will be made so as to make our hedge robust with respect to the model of underlying asset prices (e.g. log-normal distribution assumption in the Black-Scholes model). Therefore, we consider using any European call or put prices that do not admit arbitrage but not necessarily consistent with the Black-Scholes model. We refer to the results in [**5**] on the range of traded co-maturing European options with different strikes in the absence of arbitrage. We state a simplified version of their results for co-maturing options without proof.

PROPOSITION 9. *Let $R(.)$ denote the support function of the set of normalized call strikes and prices$\{(k_0, r_0), (k_1, r_1), ..., (k_n, r_n)\}$, where (k_i, r_i) are defined by*

$$r_i = \frac{P_i}{DF} \tag{3.1.8}$$

$$= \frac{E\left[(S_T - K_i)^+\right]}{F} \tag{3.1.9}$$

$$= E\left[(M_T - k_i)^+\right] \tag{3.1.10}$$

$$k_i = \frac{K_i}{F} \tag{3.1.11}$$

where F is the forward price and M_T is a Q-martingale. The prices are consistent with absence of arbitrage if and only if $R(.)$ is a strictly decreasing function on $[0, k_{n_0 \wedge n}]$, $\frac{dR}{dk}|_{k=0+} \geq -1$ and $R(k_i) = r_i$. If r_n is positive and the latter two conditions are satisfied but $R(.)$ is not strictly decreasing on $[0, k_n]$, then there is a weak arbitrage opportunity. Otherwise there is a model-independent arbitrage.

There is another similar proposition with regard to the put options stated in [**?**, **?**], also, the formal definition of weak arbitrage and model-independent arbitrage can be found in [**5**].

As long as we are given co-maturing European call and put prices that do not admit arbitrage (we can verify the conditions stated in the proposition above), we can find the most expensive sub replicating and the cheapest super replicating portfolio of the log contract by solving linear programming problems numerically. As a result, the the the upper and lower bounds of the log contract can be obtained. Note that this hedging strategy is model-independent since we are only assuming the European options prices do not admit arbitrage instead of using a specific model to price those European options. Now we are left with the final step of moving from

robust hedge of the log contract to robust hedge of the (weighted) variance swaps. Actually, based on the following formula derived above,

$$RV_T \to \langle \log S \rangle_T = 2 \int_0^T \frac{1}{S_t} dS_t - 2 \log(\frac{S_T}{S_0}) \tag{3.1.12}$$

we can show that hedging a variance swap is nothing but constructing a semi-static portfolio with dynamically traded underlying asset and a static portfolio of co-maturing European options replicating the European contract with a convex payoff (e.g. a log contract in our case). Furthermore, the dynamic trading part of the hedging portfolio is model-independent since our trading decision is only based on the observation of the market price of the underlying asset (i.e. $\frac{1}{S_t}$) and our money invested in this position is a constant. Hence, the difference between a variance swap and a log contract is a model-independent position of underlying asset, which means these two derivatives are essentially "equivalent" in a model-independent way. The mathematical proof of no-arbitrage conditions of weighted variance swaps are given in [**?**, **?**].

Based on the definition of sub and super replicating portfolio as well as a simple no-arbitrage argument, we can see that the lower (upper) bound of the contract should be the same as the cost of constructing the most expensive sub-replicating portfolio (cheapest super-replicating portfolio) problem, which is associated with the following two linear programming problems.

$$\sup_{X \epsilon \mathbb{R}^{\mathbb{M}}} \quad f^T X \tag{3.1.13}$$

$$s.t. \quad A(S_T)X \leq B(S_T), \forall S_T \tag{3.1.14}$$

$$\inf_{X \epsilon \mathbb{R}^{\mathbb{M}}} \quad f^T X \tag{3.1.15}$$

$$s.t. \quad A(S_T)X \geq B(S_T), \forall S_T \tag{3.1.16}$$

where vector $X_{(N+2)\times 1}$ denote the positions in the replicating portfolio, $f_{(N+2)\times 1}$ denote the initial cost of these hedging instruments, $B(S_T)_{1\times 1}$ denote the payoff of the log contract when the underlying price is S_T, and $A(S_T)_{1\times(N+2)}$ denote the payoff of the hedging instruments. In more details,

$$A_{:,1}(S_T) = 1 \tag{3.1.17}$$

$$A_{:,2}(S_T) = S_T \tag{3.1.18}$$

$$A_{:,3,4,\ldots N}(S_T) = (S_T - K)^+ or (K - S_T)^+ \tag{3.1.19}$$

3.2. Numerical implementation

In this section, we will implement the linear programming technique described in [**?**, **?**] and apply the general method for hedging European options with convex payoff to the specific case of hedging the log contract (and hence the variance swap). After that, we will also investigate several issues that arises in the process of numerical implementation. For example, in practice, we are faced up with the problem of selecting the range of the strikes and the number of the options included in our hedging portfolio. Therefore, the effect of the choice of these two quantities on the hedging performance should also be studied. In addition, as we use more and

Parameter	Input value
Underlying price	100
Time to maturity	3M
Risk free rate	1%
Continuous dividend yield	0%
Volatility	20%

TABLE 1. List of the input values of the Black-Scholes model parameters

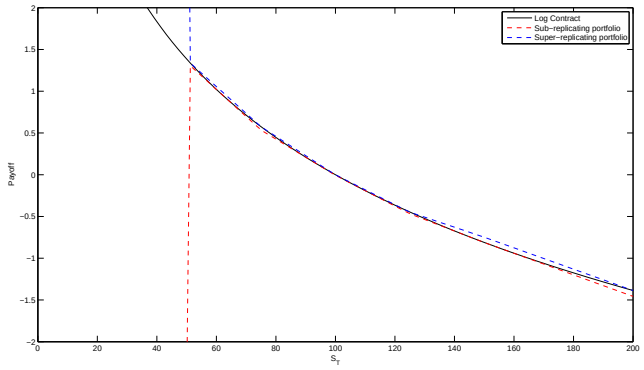


FIGURE 3.2.1. Payoff of the -2log contract and its sub and super replicating portfolio under different underlying asset prices at maturity, using $[0.5S_0, 1.5S_0]$ as the strike range

more options in the hedging portfolio, whether the value will converge to the value of the hedging portfolio in the continuous strikes case is of our particular interest. Finally, since the hedging strategy is believed to be model-independent and robust with respect to its inputs, we will try to change the inputs a little bit (e.g. assuming the volatility skew is linear in strike) and see how the hedging strategy performs. Without loss of generality, we only calculate the case of replicating only use put options, the case of using a combination of put and call options is just the same since we have the put-call parity.

3.2.1. Choice of the range and number of strikes. Let's first examine the robustness of this model-independent hedging strategy. For simplicity, we will use the option price produced by the Black-Scholes formula as an example since this will obviously satisfy the no-arbitrage condition proposed in [**5**]. The parameters of the Black-Scholes model for comparison is listed below,

We use five put options with equally spaced strikes to construct the semi-static hedge of the log contract. However, the range of put strikes in our hedging portfolio could vary from $[0.5S_0, 1.5S_0]$ to $[0, 2S_0]$, where S_0 is the at-the-money option strike. The linear programming gives us the following results,

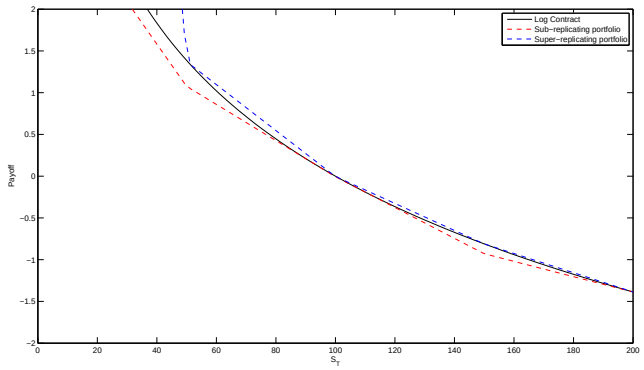


FIGURE 3.2.2. Payoff of the -2log contract and its sub and super replicating portfolio under different underlying asset prices at maturity, using $[0, 2S_0]$ as the strike range

See that there is some sort of trade-off between these two choice of range of strike given that they use the same number of options in the replicating portfolio. Within the neighbor centered at the ATM option strike, the portfolio which chooses $[0.5S_0, 1.5S_0]$ outperforms that of the portfolio which chooses $[0, 2S_0]$ as its strike range. However, outside the neighbor centered at the ATM option strike, especially for those values far away, the situation reverses, hedging with a wide strike range will outperform hedging with a narrow range. In our point of view, the choice of the strike range totally depends on one's view of the possible range of underlying asset prices at maturity. Given same number of options available, one that believe the underlying asset price will not move too far from its current level will have better hedging performance within the range of his choice, however, it the underlying moves out of the range, his hedge will no longer work well. That's the trade-off between the in and out of strike range hedging performance.

Next, we consider the effect of the number of options included on the performance of the hedging portfolio. We choose $[0.5S_0, 1.5S_0]$ as the given range of strikes and we compare the hedging performance between using four options and nine options with equally spaced strikes. We can see that the one with more option strikes (small ΔK will generally outperform the other one with bigger ΔK). This is consistent with our intuition.

See that the bounds are tighter given more options in the replicating portfolio, which is consistent with our intuition.

3.2.2. Convergence analysis. In order to analyze the convergence to the actual payoff of the log contract with more and more number of options included in the replicating portfolio, we first define some sort of mean squared error of the sub and super replication. It is as the following,

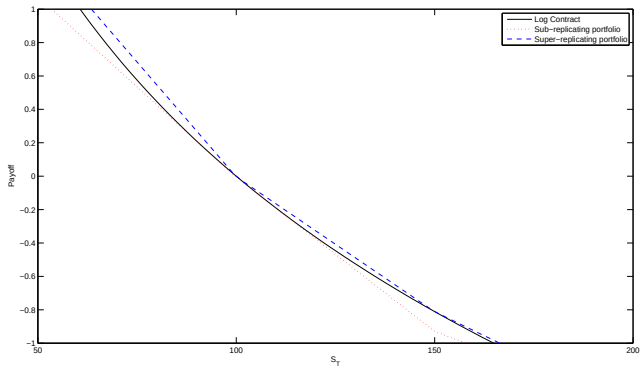


FIGURE 3.2.3. Payoff of the -2log contract and its sub and super replicating portfolio under different underlying asset prices at maturity, using 4 put options with equally spaced strikes

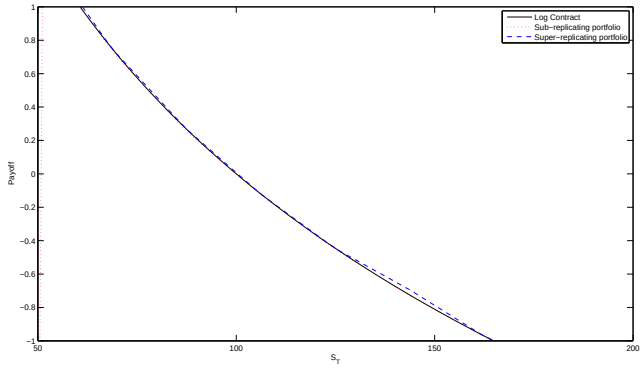


FIGURE 3.2.4. Payoff of the -2log contract and its sub and super replicating portfolio under different underlying asset prices at maturity, using 9 put options with equally spaced strikes

$$MSE^{sub} = E\left[\left(V^{log} - V^{sub}\right)^2\right] \tag{3.2.1}$$

$$MSE^{super} = E\left[\left(V^{log} - V^{super}\right)^2\right] \tag{3.2.2}$$

Since the expectation is taken over all possible outcomes of the underlying asset price at maturity, the above two expressions are just arithmetic averages if the underlying asset price at maturity is believed to be uniformly distributed. We plot the sum of mean squared errors of sub and super replicating with respect to the number of options included in the following graph. It can be seen that as the number of options included in the replicating portfolio increase, the sum of mean squared errors decrease to zero, which means it do converges to the perfect hedge of log contract.

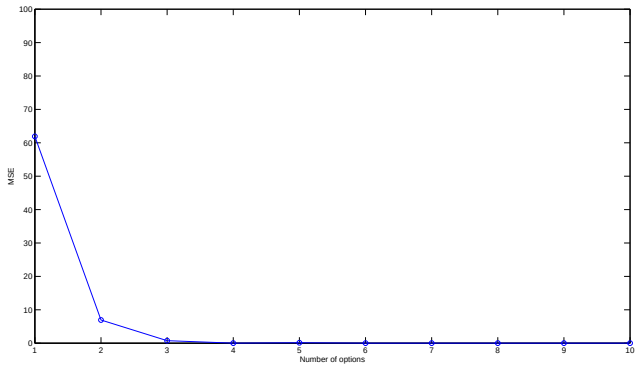


FIGURE 3.2.5. Sum of mean squared error of sub and super replicating vs. number of options included in the hedging portfolio

3.3. Extensions of the method

Motivated by both [**10**] and [**?, ?**], we are quite interested in the possibility of combining these two methods. In [**10**], the continuous price path condition for the underlying asset is not required but a continuum of co-maturing European options strikes is assumed. However, in [**?, ?**], the number of options strikes is assumed to be finite, which is good since it models the real world more accurately, but the continuous price path of the underlying asset is required. There seems to exist some sort of trade-off between the continuous assumption with regard to the price path and the continuous assumption with regard to the options strikes. Therefore, in order to handle the problem of using only a finite number of options to hedge a variance swap written on an underlying asset that has a price path with jumps, we need to combine the two methods such that both of their advantages can be used.

Since we want to stick to the assumption that only a finite number of options can be traded, we will finally use linear programming techniques as it is used in [**?, ?**]. Before this final step, since we want to make use of the advantages of method in [**10**], we need to feed the algorithm with some "continuous curve" of price/strike plot from which the implied risk-neutral distribution can be obtained. As long as this can be done, we can follow the procedures given in [**10**] and get the root of semi-static sub and super hedge $\left(\psi_\kappa, -\psi_\kappa^{'}\right)$ and $\left(\psi_\iota, -\psi_\iota^{'}\right)$. Finally, as said before, we want to stick to the assumption that only a finite number of options can be traded, so we will use the linear programming technique in [**?, ?**] to replicate the sub and super hedge obtained. To sum up, we list the procedure below,

As can be seen above, the difficulty of combing the two methods comes from interpolating the scatter plot of option prices vs. strike to obtain a continuous curve for all options strikes. It is difficult because (1) we need to ensure the no-arbitrage condition in [**5**] is always satisfied, otherwise our work will be meaningless, (2) it is hard to find the "optimal" way of interpolating these points. By "optimal", we mean that after interpolating these points, the implied risk-neutral distribution obtained

Algorithm 2 Algorithm of calculating the model-independent robust hedge for a variance swap without the assumption of a continuum of European option trades

- Use the market data to obtain the scatter plot of option prices vs. strikes
- Interpolate the points to get a continuous curve for options of all strikes
- Follow Algorithm 1 to calculate the root of semi-static sub and super hedge
- Use linear programming to sub and super replicate the hedging portfolio above with finite number of options

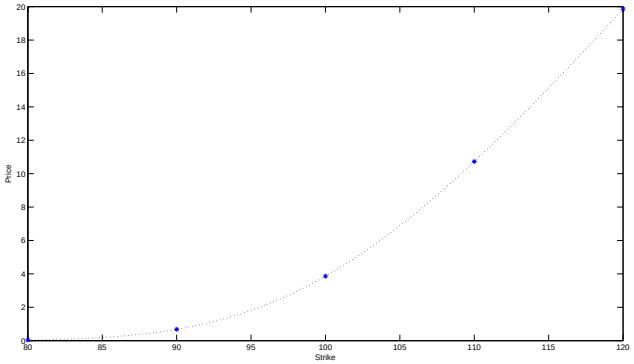


FIGURE 3.3.1. Blue points are the put prices and strikes traded in the market, the black dashed line is the Black-Scholes put price vs. all strikes plot

from reverting the continuous curve coincides with the implied distribution resulting from the linear programming.

We will use a concrete and simple example to demonstrate how this interpolation can be done. Same as above, for the reason of simplicity, we use Black-Scholes prices of vanilla European options and only consider put options without loss of generality. We first present the scatter plot of put price vs. strike as well as the continuous curve below. Note that in practice we only have the scatter plot of available put prices and strikes, the continuous curve is the curve we want our interpolation to be as close to as possible. In this specific example, since we are using the Black-Scholes price for puts, we can plot it. In the more general case, this curve cannot be obtained accurately and it corresponds to the implied risk-neutral distribution. Our aim is to interpolate the points and to approximate the continuous curve.

For super hedges, it is quite obvious to see that piecewise linear interpolation of the five points will always be above the convex continuous curve and will not admit arbitrage since it satisfies conditions in [**5**]. Hence, such a interpolated curve will produce a super hedge. We present the plot here.

However, this piecewise interpolation method cannot to be applied directly to obtain a sub hedge (i.e. an interpolated continuous curve that is below the actual

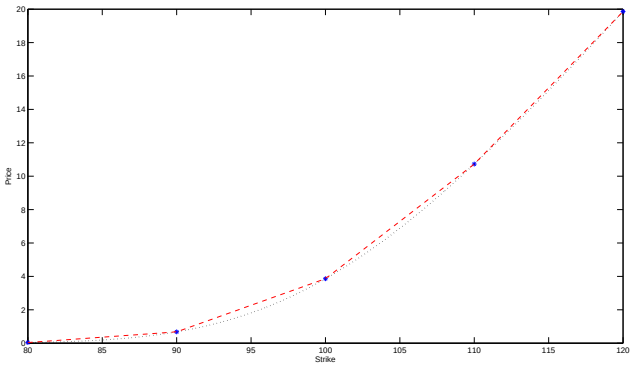


FIGURE 3.3.2. Piecewise linear interpolation of put prices and strikes

curve). However, we may want to try the following method. Thinking about adding one kink in each interval between two points of put prices and strikes. Recall the Proposition 2.1 in [**?**, **?**], which extended the no-arbitrage condition for call options to the case of put options, state that the interpolation need to satisfy

- non-negative, convex, increasing function in option strike
- $P(0) = 0, P(K) \geq K - 1, \frac{dP}{dK}|_{K^*} < 1$, where K^* is the point the intersects with line $K - 1$ and the derivative is actually a left derivative

After adding the kinks, we connect those kinks to form a piecewise linear function that passes all the original data points we have and stays below the continuous curve we want to approximate. The following plot will illustrate the idea. See that we added several kinks at the mid point between two neighboring points (i.e. strikes 85, 95, 105, 115) and connect them to construct a piecewise linear interpolation. This interpolated curve passes all five original data points and lies below the continuous curve (black dashed line) due to the convexity of the curve. Also, it can be easily verified that it satisfies all the no-arbitrage conditions above. Hence, it can be used to create a sub hedge using the algorithm described previously.

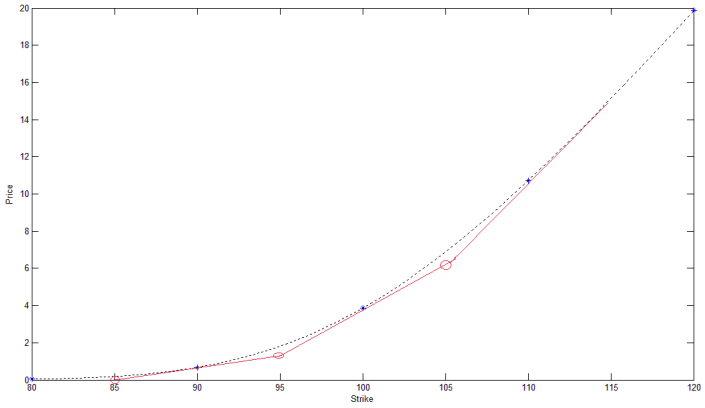


FIGURE 3.3.3. Piecewise interpolation of the data points, can be used to construct a sub hedge

Concluding Remarks

In this thesis, we studied the problem of hedging variance swaps using co-maturing European options without any prior probability settings. Such kind of hedge is quite different from the traditional hedge (e.g. delta hedge in the Black-Scholes model) in the sense that it does not rely crucially on the property of price paths and hence is more robust. We review two paper, [**10**] and [**?, ?**] and implement their different methods developed to solve the problem. During the process of implementation, we explore numerically the hedging performance as well as the no-arbitrage bounds of the variance swaps and found several interesting issues regarding the numerical approximation. We found that in practice, the method in [**10**] may raise some concerns for practitioners since the numerical solution is not always stable. However, for the method in [**?, ?**], the hedging performance of the strategy solved from linear programming seems to be good, at least within the range of chosen options strikes. Also, it has some sort of convergence property to the continuous strikes case. In order to know more about the performance of these two methods, probably more cases need to be studied. For example, one could use Heston's stochastic volatility model to generate price paths and explore the performance of this strategy. Also, the effect of volatility skew on this robust hedging strategy could be studied.

In addition, we tried to combine the two methods of constructing model-independent robust hedging strategies for variance swap so that one could use finite number of co-maturing European options to hedge his or her position in variance swaps no matter whether the underlying asset's price path has jumps or not. We proposed an idea that could be applied to solve this problem. However, the difficult part is not obtaining a hedge but obtaining the optimal hedge. This problem is left open for future investigation and could probably be solved either via numerical experiments or mathematical proofs.

Appendix: MATLAB Code

```
% Implied RN distribution
S0 = 100;
T = 0.25;
r = 0.01;
q = 0.00;
v = 0.2;

S_min = 10;
S_max = 200;

N = 100;
K = linspace(S_min,S_max,N);
dK = (S_max - S_min) / N;

C = zeros(1,N);
for i = 1:1:N
        C(i) = EuropeanVanillaBS(S0,K(i),T,r,q,v,'Call');
end

f_ST = zeros(1,N-2);

for i = 1:1:N-2
        f_ST(i) = (C(i) + C(i+2) - 2 * C(i+1)) / (dK^2) * exp(r * T);
end

M = 100000;
ST = S0 * exp((r-q-0.5*v*v)*T + v*sqrt(T)*randn(1,M));

figure;
hist(ST,100);
axis([10,200,0,4000]);
hold on;
plot(K(2:N-1),f_ST* M,'r-');
xlabel('S_{T}');

% Semi-static hedge of the log contract
function Return = LinProg(N)
S0 = 100;
T = 0.25;
r = 0.01;
q = 0.00;
v = 0.2;

N = 5;
N2 = 9;

K = linspace(0*S0,2*S0,N);
K2 = linspace(0*S0,2*S0,N);

Put = zeros(N,1);
Put2 = zeros(N,1);

for i=1:1:N
        Put(i) = EuropeanVanillaBS(S0, K(i), T, r, q, v, 'Put');
end

for i=1:1:N
        Put2(i) = EuropeanVanillaBS(S0, K2(i), T, r, q, v, 'Put');
```

```
end

figure;
plot(K,Put,'k:o');
xlabel('Put Strike');
ylabel('Put Price');

M = 100;
ST_range = linspace(1,2*S0,M);
LogPayoff_range = -2 * log(ST_range / S0);

f = zeros(N+2,1);
f(1) = exp(-r*T);
f(2) = S0;
f(3:N+2) = Put(1:N);

f2 = zeros(N+2,1);
f2(1) = exp(-r*T);
f2(2) = S0;
f2(3:N+2) = Put2(1:N);

B = LogPayoff_range';

A = zeros(M,N+2);

for i=1:1:M
        for j=1:1:N+2
                if (j == 1)
                        A(i,j) = 1;
                end
                if (j == 2)
                        A(i,j) = ST_range(i);
                end
                if (j >= 3 )
                        A(i,j) = max(K(j-2) - ST_range(i),0);
                end
        end
end

A2 = zeros(M,N+2);

for i=1:1:M
        for j=1:1:N+2
                if (j == 1)
                        A2(i,j) = 1;
                end
                if (j == 2)
                        A2(i,j) = ST_range(i);
                end
                if (j >= 3 )
                        A2(i,j) = max(K2(j-2) - ST_range(i),0);
                end
        end
end

SubHedge = linprog(-f,A,B);
LowerBound = zeros(M,1);
SubCost = f' * SubHedge;

SubHedge2 = linprog(-f2,A2,B);
LowerBound2 = zeros(M,1);

for i=1:1:M
        LowerBound(i) = A(i,:) * SubHedge;
end

for i=1:1:M
        LowerBound2(i) = A2(i,:) * SubHedge2;
end

SuperHedge = -linprog(-f,A,-B);
UpperBound = zeros(M,1);
```

```
SuperCost = f' * SuperHedge;

SuperHedge2 = -linprog(-f2,A2,-B);
UpperBound2 = zeros(M,1);

for i = 1:1:M
        UpperBound(i) = A(i,:) * SuperHedge;
end

for i = 1:1:M
        UpperBound2(i) = A2(i,:) * SuperHedge2;
end

% mean square error
MSE = mean((LowerBound-LogPayoff_range').^2+(LogPayoff_range'-UpperBound).^2);

figure;
subplot(1,2,1);
plot(ST_range,LogPayoff_range','k-',ST_range,LowerBound,'r--',ST_range,UpperBound
    ,'b--');
xlabel('S_{T}');
ylabel('Payoff');
legend('Log Contract','Sub-replicating portfolio','Super-replicating portfolio');
axis([0*S0,2*S0,-2,2]);
subplot(1,2,2);
plot(ST_range,LogPayoff_range','k-',ST_range,LowerBound2,'r--',ST_range,
    UpperBound2,'b--');
xlabel('S_{T}');
ylabel('Payoff');
legend('Log Contract','Sub-replicating portfolio','Super-replicating portfolio');
axis([0*S0,2*S0,-2,2]);

Return = MSE;

end
```

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