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Composite supersymmetries in low-dimensional systems

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Abstract. Starting from a $N = 1$ scalar supermultiplet in $2 + 1$ dimensions, we demonstrate explicitly the appearance of induced $N = 1$ vector and scalar supermultiplets of composite operators made out of the fundamental supersymmetric constituents. We discuss an extension to a $N = 2$ superalgebra with central extension, due to the existence of topological currents in $2 + 1$ dimensions. We comment on the relevance of these results for an effective description of the infrared dynamics of planar high-temperature superconducting models with quasiparticle excitations near nodal points of their Fermi surface.

1. Introduction

Supersymmetry is a symmetry that links integer spin excitations (bosons) to half-integer ones (fermions), and, in certain circumstances, provides a satisfactory control of quantum fluctuations, to the extent that many exact analytic results can be obtained on the phase structure of certain relativistic field theories†. From this point of view, supersymmetry is expected mainly to appear, if at all, in theories of the fundamental interactions of Nature, which are by construction relativistic. In fact supersymmetry was a helpful tool for understanding the non-perturbative structure of strongly coupled non-Abelian gauge theories, of relevance to the weak and strong interactions. Seiberg and Witten [1] have managed very effectively to exploit extended $N = 2$ four-dimensional supersymmetry in $SU(2)$ gauge theory so as to obtain complete non-perturbative information on the phase-structure of the theory. This work opened the way to more exact results, some of which pertained to theories in space–time dimensions different from four. There has been considerable interest, for instance, in $N = 2$ supersymmetric gauge theories in

† At present, non-relativistic theories with Galilean invariance are also known to possess some kind of supersymmetry, which, however, is not sufficiently developed or understood to allow exact results in quantum theory.

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three-dimensional space–times [2], where *some* exact non-perturbative information can also be obtained.

At present it is $N = 1$ supersymmetry, which seems to be of phenomenological relevance to Nature in four dimensions, and unfortunately there are no such exact results for it in any dimension. Nevertheless, it may be hoped that, by viewing $N = 1$ supersymmetry as a broken phase of an extended supersymmetry, some exact information on the phase structure can be extracted. Attempts in this direction have already been made in the literature, e.g. four-dimensional softly broken $N = 2$ QCD [3]. Given the necessity of relativistic systems, dynamical supersymmetry was thought to be irrelevant for theories of condensed matter. This is due to the fact that the majority of systems in condensed matter involve non-relativistic excitations around Fermi surfaces of finite-extent without nodes.

High-temperature superconductors, however, are d-wave superconductors, known to have Fermi surfaces and superconducting gaps with nodes, i.e. points in the Fourier space where either the Fermi surface is point-like or the superconducting gap vanishes. The nodal superconducting gap structure occurs in the superconducting phase (optimal doping). On the other hand, for low doping (underdoped cuprates) there is also a nodal structure in the Fermi surface. In this phase a pseudogap region is entered, characterized by gap generation but not phase coherence.

These have prompted the use of relativistic theories for describing the low-lying excitations near the nodes [4]–[6]. As a result, from the perspective of obtaining exact analytic information on the phase structure of such systems it is interesting to examine situations where the effective relativistic theories of such nodal excitations possess, in addition, dynamical supersymmetry.The hope is that such results may also be useful in yielding important information on excitations, even away from the nodes. Such attempts have been initiated in [7, 8] in the context of the so-called spin–charge separation phase of planar antiferromagnets [9], where the fundamental degrees of freedom are not electrons, but rather objects carrying only spin—(spinon) or only electric-charge- (holon) degrees of freedom. In terms of such excitations, a possible continuum low-energy effective field theory describing the dynamics of the antiferromagnets is a CP^1 σ-model coupled to Dirac fermions. The z-magnons of the σ-model represent spinons, while the Dirac fermions represent holon degrees of freedom, which carry electric charge only [5]. The presence of CP^1 field theory necessitates the appearance of non-propagating $U_S(1)$ gauge interactions, which from the effective field theory viewpoint are viewed as interactions with infinitely strong gauge coupling (due to the absence of gauge kinetic (i.e. Maxwell) terms). In some cases [5], the nodal liquid is described in a 'particle–hole' symmetric formalism which allows the existence of *additional* non-Abelian gauge interactions of $SU(2)$ type, representing the redundancy inherent in the ansatz of the spinon and holon degrees of freedom. In [8] it has been asserted that the effective theory of holons can be represented by Dirac fermions near the nodes of a Fermi surface provided the $SU(2)$ gauge interactions are weakly coupled. This will be assumed throughout this work.

The infinitely strong $U_S(1)$ interactions can be integrated out to give [5] effective theories of spinon and holons comprising of *composite objects* made out of these fundamental constituents. These represent the effective physical degrees of freedom of the nodal system. In [8] the conditions for dynamical supersymmetry between spinon and holon constituent degrees of freedom in such effective theories have been analysed. The important point to note is that specific points in the parameter space, depending on particular *doping concentrations*, allow for a scalar $N = 1$ three-dimensional supersymmetry between spinon (z-magnon) and holon (Ψ Dirac fermions) degrees of freedom. We stress that the excitation content of the constituent

$N = 1$ supermultiplet (of spinons and holons) is a direct consequence of spin–charge separation, and hence in this context it *does not* require the introduction of extra degrees of freedom.

In this paper we study the induced *composite supersymmetry* in the effective theory. As we shall show, the existence of a scalar $N = 1$ supermultiplet for spinons and holons suffices to induce $N = 1$ scalar *and* vector supermultiplets at a composite level in the effective theory (obtained from integration of the $U_S(1)$ interactions). It should be stressed that here we *do not* supersymmetrize the $U_S(1)$ interactions. As we shall see, due to the three dimensionality of the theory, it is possible to obtain centrally extended $N = 2$ supersymmetries at both the constituent and the composite level, with the central extension being provided by appropriate topological 'charges' to be defined below [10, 11]. The $N = 2$ supersymmetry couples the two $N = 1$ multiplets of the composite supersymmetry via gauge interactions, leading to interesting models. One possible model is the Abelian–Higgs model; under different circumstances, a broken phase of a non-Abelian $SU(2)$ Georgi–Glashow model may be obtained. These two models can lead to different physics. At present, as we shall discuss in this paper, we are not in position to specify the precise supersymmetric composite model, since this necessitates going beyond composite operators quadratic in the constituent fields.

The structure of this paper is as follows: in the next section we shall introduce notations and conventions pertaining to a scalar $N = 1$ supersymmetry at a constituent spinon–holon level. In section 3 we shall review the appearance of composite operators in these systems, discussed in [5]. In section 4 we shall study an $N = 1$ supersymmetry at a *composite* level, induced by the $N = 1$ scalar supersymmetry at a *constituent* spinon–holon level [7, 8], and in particular we shall define the associated scalar supermultiplet (containing the scalar composites of [5]). In section 5 we shall discuss the induced $N = 1$ vector composite supermulitplet containing the vector composites of [5]. In section 6 we shall discuss the elevation of these supersymmetries in three dimensions into centrally extended ones with the central extension being identified with the topological charge [10, 11]. In section 7 we shall argue that the appropriate effective composite theory is a $N = 2$ supersymmetric Abelian–Higgs-like model, and discuss how this may lead to exact results on superconducting properties of the model at the supersymmetric points, as discussed in section 8. The latter implies the existence of composite complex fermions, which are capable of ensuring the exact masslessness of an appropriate unbroken Abelian subgroup of $SU(2)$ which will be denoted by $U_3(1)$ and will be crucial for superconductivity by a mechanism promulgated before [4]. The superconducting point of the nodal liquid is argued to be a *quantum critical point* in the phase space of the model. A discussion on a pseudogap phase will also be made. Finally, conclusions and outlook are presented in section 9.

2. Scalar $N = 1$ supersymmetry of spinon–holon constituents

In [8] a detailed microscopic model was presented, and the associated continuum effective theory of doped antiferromagnets with spin–charge separation was derived. The constituent spinon and holon degrees of freedom formed a scalar $N = 1$ *supermultiplet* at certain points of the parameter space of the microscopic model, depending on the doping concentration.

In this section we set up the notation and conventions in $2 + 1$ dimensions that will be used in this paper and also review the earlier work. The Dirac matrices are 2×2 and are given by $(\gamma^0)^\alpha_\beta = -\mathrm{i}(\sigma^2)_{\alpha\beta}, (\gamma^1)^\alpha_\beta = (\sigma^1)_{\alpha\beta}$ and $(\gamma^2)^\alpha_\beta = (\sigma^3)_{\alpha\beta}$ where σ^a are the Pauli matrices. Hence $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1)$ and $\mu, \nu = 0, 1, 2$. We raise and lower Dirac indices with the real antisymmetric tensor whose components are given by $\epsilon^{\alpha\beta} = (-\gamma^0)^\alpha_\beta = \epsilon_{\alpha\beta}$.

This convention has the property that real spinors remain real under the operation of raising and lowering indices.

It is important to notice that in the model of [8], as a result of the antiferromagnetic structure, there are *two* complex supermultiplets†. Hence starting with four $N = 1$ real scalar supermultiplets, we can construct two $N = 1$ complex scalar superfields Z_a, $a = 1, 2$, made out of complex supermultiplets (z_a, f_a, ψ_a) which transform as

$$\begin{aligned} \delta z_a &= \overline{\varepsilon}\psi_a \\ \delta f_a &= \overline{\varepsilon}\partial\!\!\!/\psi_a \\ \delta\psi_a &= \partial\!\!\!/ z_a\varepsilon + f_a\varepsilon, \end{aligned} \tag{1}$$

where $\overline{\psi} = \psi^\dagger\gamma^0$.

For a product of complex spinors it is useful to note that

$$\begin{aligned} \overline{\psi}_1\psi_2 &= \psi_1^{\star\alpha}\psi_{2\alpha} \\ &= \psi_1^{\star\alpha}\psi_2^\beta\epsilon_{\beta\alpha} \\ &= \psi_2^\beta\psi_1^{\star\alpha}\epsilon_{\alpha\beta} \\ &= (\overline{\psi}_2\psi_1)^\star \end{aligned} \tag{2}$$

and similarly that

$$\overline{\psi}_1\gamma^\mu\psi_2 = -(\overline{\psi}_2\gamma^\mu\psi_1)^\star. \tag{3}$$

We also have the Fierz identity

$$\begin{aligned} (\overline{\chi}_1\gamma_\mu\chi_2)(\overline{\chi}_3\gamma^\mu\chi_4) &= (\chi_1^{\star\alpha}(\gamma_\mu)^\beta{}_\alpha\chi_{2\beta})(\chi_3^{\star\gamma}(\gamma^\mu)^\delta{}_\gamma\chi_{4\delta}) \\ &= -\chi_1^{\star\alpha}\chi_2^\beta\chi_{3\gamma}^\star\chi_{4\delta}(\delta_\beta^\delta\delta_\alpha^\gamma + \delta_\beta^\gamma\delta_\alpha^\delta) \\ &= -(\overline{\chi}_1\chi_4)(\overline{\chi}_3\chi_2) + (\overline{\chi}_1\chi_3^\star)(\overline{\chi}_4\chi_2^\star)^\star, \end{aligned} \tag{4}$$

which is valid for any set of spinors χ_i, $i = 1, 2, 3, 4$. We recall the important property concerning the antisymmetric tensor $\epsilon_{\mu\nu\rho}$:

$$\epsilon_{\mu\nu\rho}\gamma^\rho = [\gamma_\nu, \gamma_\mu] = \gamma_{[\nu}\gamma_{\mu]}. \tag{5}$$

Finally, if we write explicitly the 2×2 matrix $\chi_1\overline{\chi}_2$, for any spinors χ_1 and χ_2, we find the following relations:

$$\begin{aligned} \chi_1\overline{\chi}_2 + (\chi_2\overline{\chi}_1)^\star &= -(\overline{\chi}_2\chi_1)\mathbf{1} \\ \chi_1\overline{\chi}_2 - (\chi_2\overline{\chi}_1)^\star &= -(\overline{\chi}_2\gamma_\nu\chi_1)\gamma^\nu, \end{aligned} \tag{6}$$

where $\mathbf{1}$ is the 2×2 unit matrix in Dirac space. We thus find that for any spinors χ_1 and χ_2

$$\chi_1\overline{\chi}_2 = -\tfrac{1}{2}[(\overline{\chi}_2\chi_1)\mathbf{1} + (\overline{\chi}_{\mathbf{2}}\gamma_\nu\chi_{\mathbf{1}})\gamma^\nu]. \tag{7}$$

The associated (long-wavelength) dynamics is represented by a supersymmetric σ-model which is described by the superfield Lagrangian:

$$\mathcal{L}_\sigma = -\int \mathrm{d}^2\theta\{D^\alpha Z_a^\dagger D_\alpha Z_a + (Z_a^\dagger D^\alpha Z_a)(Z_b^\dagger D_\alpha Z_b)\} \tag{8}$$

† Throughout this work we follow the representation of [4], where spinons are bosonic fields (z magnons), while holons are electrically charged fermions. For alternative approaches, where the holons are represented as bosons and the spinons as fermions see [12].

where $D_\alpha (= \frac{\partial}{\partial \theta^\alpha} + \theta^\beta \not{\partial}_{\alpha\beta})$, with $\not{\partial}_{\alpha\beta} = \gamma^\mu_{\alpha\beta}\partial_\mu$, denotes the supercovariant derivative. The superfield Z satisfies the constraint $Z^\dagger_a Z_a = 1$. In terms of components the constraint becomes

$$\sum_{a=1}^{2} z^\dagger_a z_a = 1, \qquad \sum_{a=1}^{2} z^\dagger_a \psi_a + \text{h.c.} = 0, \qquad \sum_{a=1}^{2} (z^\dagger_a f_a + f^\dagger_a z_a + \psi^\dagger_a \psi_a) = 0. \quad (9)$$

We shall come back to this action in section 6, when we discuss the extension of the $N = 1$ supersymmetry to a centrally extended $N = 2$ supersymmetry, as a result of the special features of gauge theories in $(2 + 1)$ dimensions.

3. Composite meson fields

In the path integral the integration of the strongly-coupled $U_S(1)$ gauge interactions leads to the appearance of composite operators [5], of ‘meson’ and ‘baryon’ type, comprising of holon constituents. In what follows we shall concentrate exclusively on *meson type* operators [5]: if we organize the fermions (holons) into a ‘colour’ doublet $\chi = (\psi_1, \psi_2)$, built out of the two (two-component) fermion species ψ_1 and ψ_2 (corresponding to the two antiferromagnetic sublattices), then the bilinears ($a = 1, 2, 3$)

$$\begin{aligned} \Phi^a &= \overline{\chi}\Sigma^a\chi \\ \mathcal{A}^a_\mu &= \overline{\chi}\mathrm{i}\Sigma^a_\mu\chi \end{aligned} \quad (10)$$

transform as *triplets* under $SU(2)$, where $\Sigma^a = \sigma^a \otimes \mathbf{1}$ and $\Sigma^a_\mu = \sigma^a \otimes \gamma_\mu$. On the other hand, the $SU(2)$ singlets are given by the bilinears

$$\begin{aligned} \mathcal{S} &= \overline{\chi}\chi \\ \mathcal{S}_\mu &= \overline{\chi}\mathrm{i}\Gamma_\mu\chi, \end{aligned} \quad (11)$$

where $\Gamma_\mu = \gamma_\mu \otimes \mathbf{1}$. Note that $\mathcal{S}$ is a parity violating mass term, and $\mathcal{S}_\mu$ a four-component fermion-number current. On the other hand, the parity-invariant fermion mass term transforms as a triplet under the group $SU(2)$. In models with a gauged $SU(2)$ symmetry among spinon and holons, as required by a particle–hole symmetric spin–charge separation ansatz [5], the dynamical generation of a parity-invariant holon mass-gap will induce a dynamical breaking of the $SU(2)$ gauge symmetry down to a $U(1)$ compact subgroup. On ignoring *non-perturbative* effects, such a phase would be superconducting by the anomaly mechanism of [4]. However, due to the compact nature of the $U(1) \subset SU(2)$ subgroup that is left unbroken, there are monopole-instanton effects that in general may be responsible for a small but non-zero mass of the $U(1)$ gauge boson. This will spoil superconductivity, thereby leading to a pseudogap phase for the nodal liquid, as discussed in detail in [5].

Superconductivity, on the other hand, requires an exactly massless $U(1)$ photon. Such a case arises in the Georgi–Glashow supersymmetric model of [13]. In [5] we did not discuss spinon contributions to the composite operators, because we worked in a phase where the spinon gap was much larger than the fermion (holon) dynamically generated mass gap. In the supersymmetric situation [8], any mass gaps of spinons and holons are of equal magnitude. It is natural to consider the effect of spinon composites could affect the mesons (10), (11). This can be answered by invoking supersymmetry at a composite level. Since the mesons are obtained in [5] by integrating out a non-dynamical gauge group, the natural thing to assume is that the supersymmetry at a constituent magnon–holon level will be preserved at a composite level, and this will *define* the appropriate spinon contributions to the composites.

In what follows we shall concentrate on the $SU(2)$ triplet composites, which are the ones that could possibly couple to the gauge $SU(2)$ interactions present in the particle–hole symmetric formalism of spin–charge separation [5]. As we shall demonstrate below, the bilinear composites can lead only to two *decoupled* $N = 1$ supermultiplets at a composite level, a scalar and a vector. Coupling of these two supermultiplets can only be seen if higher order constituent operators appear in the definition of the composite operators (10). As will be discussed in section 7, such a coupling will allow the appearance of a $N = 2$ dynamical supersymmetry, with a central charge that coincides with the topological charge [10, 11] defined by the vector fields in the three-dimensional case at hand.

4. $N = 1$ composite scalar supermultiplet

Let us now consider a scalar composite $\Phi = \overline{\psi}_1\psi_2$. Φ defined in this way is complex but has the generic composite structure that we will study; the real scalar composites will be considered at the end of this section. The supersymmetric transform of Φ induced by the spinons and holons is $\delta\Phi = \overline{\varepsilon}\Psi$ where

$$\Psi = (f_1^\star - \partial\!\!\!/ z_1^\star)\psi_2 + (f_2 - \partial\!\!\!/ z_2)\psi_1^\star. \tag{12}$$

The supersymmetric transform of Ψ is $\delta\Psi = \tilde{F}\varepsilon + M^{(+)}\varepsilon$ where

$$\begin{aligned}\tilde{F} &= 2(f_1^\star f_2 - \partial_\mu z_1^\star \partial^\mu z_2)\\ M^{(+)} &= -\{\gamma^\mu, \psi_2\partial_\mu\overline{\psi}_1\} - \{\gamma^\mu, \psi_1\partial_\mu\overline{\psi}_2\}^\star.\end{aligned} \tag{13}$$

Using equation (7), we can also write

$$\begin{aligned}M^{(+)} &= \tfrac{1}{2}\{\gamma^\mu, \partial_\mu\overline{\psi}_1\psi_2\mathbf{1} + \partial_\mu\overline{\psi}_\mathbf{1}\gamma_\nu\psi_\mathbf{2}\gamma^\nu\} + \tfrac{1}{2}\{\gamma^\mu, \partial_\mu\overline{\psi}_2\psi_1\mathbf{1} + \partial_\mu\overline{\psi}_\mathbf{2}\gamma_\nu\psi_\mathbf{1}\gamma^\nu\}^\star\\ &= \gamma^\mu\partial_\mu(\overline{\psi}_1\psi_2) + \partial_\mu\overline{\psi}_1\gamma^\mu\psi_2 + (\partial_\mu\overline{\psi}_2\gamma^\mu\psi_1)^\star\\ &= \partial\!\!\!/\Phi - \overline{\psi}_1\partial\!\!\!/\psi_2 - (\overline{\psi}_2\partial\!\!\!/\psi_1)^\star,\end{aligned} \tag{14}$$

such that $\delta\Psi = F\varepsilon + \partial\!\!\!/\Phi\varepsilon$ where

$$F = \tilde{F} - \overline{\psi}_1\partial\!\!\!/\psi_2 - (\overline{\psi}_2\partial\!\!\!/\psi_1)^\star. \tag{15}$$

The supersymmetric transform of the auxiliary field F is

$$\begin{aligned}\delta F = 2\overline{\varepsilon}[&f_1^\star\partial\!\!\!/\psi_2 + f_2\partial\!\!\!/\psi_1^\star - \partial^\mu\psi_1^\star\partial_\mu z_2 - \partial^\mu z_1^\star\partial_\mu\psi_2]\\ &- \overline{\varepsilon}(f_1^\star - \partial\!\!\!/ z_1^\star)\partial\!\!\!/\psi_2 - \overline{\varepsilon}\partial\!\!\!/(\partial\!\!\!/ z_2 - f_2)\psi_1^\star\\ &- \overline{\varepsilon}(f_2 - \partial\!\!\!/ z_2)\partial\!\!\!/\psi_1^\star - \overline{\varepsilon}\partial\!\!\!/(\partial\!\!\!/ z_1^\star - f_1^\star)\psi_2\\ = &\,\overline{\varepsilon}\gamma^\nu(f_1^\star - \partial\!\!\!/ z_1^\star)\partial_\nu\psi_2 + \overline{\varepsilon}\gamma^\nu(f_2 - \partial\!\!\!/ z_2)\partial_\nu\psi_1^\star\\ &+ \overline{\varepsilon}(\partial\!\!\!/ f_1^\star - \Box z_1^\star)\psi_2 + \overline{\varepsilon}(\partial\!\!\!/ f_2 - \Box z_2)\psi_1^\star\\ = &\,\overline{\varepsilon}\partial\!\!\!/\Psi,\end{aligned} \tag{16}$$

which closes the superalgebra, since we have found the composite superpartners F and Ψ of the field Φ such that

$$\begin{aligned}\delta\Phi &= \overline{\varepsilon}\Psi\\ \delta F &= \overline{\varepsilon}\partial\!\!\!/\Psi\\ \delta\Psi &= \partial\!\!\!/\Phi\varepsilon + F\varepsilon.\end{aligned} \tag{17}$$

Such a closure of the scalar superalgebra is also valid for the $SU(2)$ real triplet $(a = 1, 2, 3)$

$$\Phi^a = \overline{\chi}\Sigma^a\chi, \tag{18}$$

and the $SU(2)$ real singlet

$$\mathcal{S} = \overline{\chi}\chi, \tag{19}$$

with the notations introduced in the previous section.

5. $N = 1$ composite vector supermultiplet

The supersymmetric transformations for a $N = 1$ Abelian vector supermultiplet are in the Wess–Zumino gauge

$$\begin{aligned} \delta a^\mu &= \overline{\varepsilon}\gamma^\mu\lambda \\ \delta\lambda &= -\tfrac{1}{2} f_{\mu\nu}\gamma^\mu\gamma^\nu\varepsilon, \end{aligned} \tag{20}$$

where λ is a real gaugino and $f^{\mu\nu} = \partial^{[\mu}a^{\nu]}$. Let us consider the composite vector field

$$A^\mu = \overline{\psi}_1\gamma^\mu\psi_2 - z_1^\star\partial^\mu z_2 + z_2\partial^\mu z_1^\star \tag{21}$$

and its *Abelian* field strength $F^{\mu\nu}$. The vector field (21) is complex but has the basic composite structure from which we will construct the real gauge fields at the end of this section. We will look for the supersymmetric transform of A^μ, but before that we will derive the following property of $F^{\mu\nu}$ (see equation (24)): using the Fierz identity (4), we can see that for any real Grassmann variables θ_1, θ_2 we have

$$\begin{aligned} \overline{\theta}_1\, \partial_\mu(\overline{\psi}_1\gamma_\nu\psi_2)\gamma^{[\mu}\gamma^{\nu]}\, \theta_2 &= ((\partial_\mu\overline{\psi}_1)\gamma_\nu\psi_2)((\overline{\theta}_1\gamma^\mu)\gamma^\nu\theta_2) + (\overline{\psi}_1\gamma_\nu(\partial_\mu\psi_2))((\overline{\theta}_1\gamma^\mu)\gamma^\nu\theta_2) \\ &\quad - ((\partial_\nu\overline{\psi}_1)\gamma_\mu\psi_2)(\overline{\theta}_1\gamma^\mu(\gamma^\nu\theta_2)) \\ &\quad - (\overline{\psi}_1\gamma_\mu(\partial_\nu\psi_2))(\overline{\theta}_1\gamma^\mu(\gamma^\nu\theta_2)) \\ &= -\,\overline{\theta}_1[\gamma^\mu, \partial_\mu(\psi_2\overline{\psi}_1) - \partial_\mu(\psi_1\overline{\psi}_2)^\star]\theta_2, \end{aligned} \tag{22}$$

and thus

$$\partial_\mu(\overline{\psi}_1\gamma_\nu\psi_2)\gamma^{[\mu}\gamma^{\nu]} = -\partial_\mu[\gamma^\mu, \psi_2\overline{\psi}_1] + \partial_\mu[\gamma^\mu, \psi_1\overline{\psi}_2]^\star. \tag{23}$$

We also have $\partial_\mu(z_a^\star\partial_\nu z_b)\gamma^{[\mu}\gamma^{\nu]} = [\partial\!\!\!/ z_a^\star, \partial\!\!\!/ z_b]$, such that finally

$$F_{\mu\nu}\gamma^\mu\gamma^\nu = -\partial_\mu[\gamma^\mu, \psi_2\overline{\psi}_1] + \partial_\mu[\gamma^\mu, \psi_1\overline{\psi}_2]^\star - 2[\partial\!\!\!/ z_1^\star, \partial\!\!\!/ z_2]. \tag{24}$$

The supersymmetric transform of A^μ is now

$$\delta A^\mu = \overline{\varepsilon}\gamma^\mu\Lambda - \overline{\varepsilon}\,\partial^\mu(z_1^\star\psi_2 - z_2\psi_1^\star), \tag{25}$$

where the gaugino Λ is given by

$$\Lambda = (f_1^\star + \partial\!\!\!/ z_1^\star)\psi_2 - (f_2 + \partial\!\!\!/ z_2)\psi_1^\star, \tag{26}$$

and has the following supersymmetric transform:

$$\delta\Lambda = M^{(-)}\varepsilon + [\partial\!\!\!/ z_1^\star, \partial\!\!\!/ z_2]\varepsilon, \tag{27}$$

with

$$M^{(-)} = [\gamma^\mu, \psi_2\partial_\mu\overline{\psi}_1] - [\gamma^\mu, \psi_1\partial_\mu\overline{\psi}_2]^\star. \tag{28}$$

Then, from the first of equations (6), we can also write that for any spinor χ_1 and χ_2

$$\begin{aligned}\chi_1\overline{\chi}_2 &= -(\overline{\chi}_2\chi_1)\mathbf{1} - (\chi_2\overline{\chi}_1)^\star \\ &= \tfrac{1}{2}(\chi_1\overline{\chi}_2 - (\overline{\chi}_2\chi_1)\mathbf{1} - (\chi_2\overline{\chi}_1)^\star)\end{aligned} \tag{29}$$

such that

$$\begin{aligned}M^{(-)} &= \tfrac{1}{2}[\gamma^\mu, \psi_2\partial_\mu\overline{\psi}_1 - (\partial_\mu\psi_1\overline{\psi}_2)^\star - (\partial_\mu\overline{\psi}_1\psi_2)\mathbf{1}] \\ &\quad - \tfrac{1}{2}[\gamma^\mu, (\psi_1\partial_\mu\overline{\psi}_2)^\star - \partial_\mu\psi_2\overline{\psi}_1 - (\overline{\psi}_1\partial_\mu\psi_2)\mathbf{1}] \\ &= \tfrac{1}{2}[\gamma^\mu, \partial_\mu(\psi_2\overline{\psi}_1) - \partial_\mu(\psi_1\overline{\psi}_2)^\star],\end{aligned} \tag{30}$$

and we finally find with equations (24) and (27)

$$\delta\Lambda = -\tfrac{1}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu\varepsilon. \tag{31}$$

Now let us return to the transformation (25). The latter is the expected one up to an Abelian gauge transformation (i.e. total derivative): if we define the scalar ρ such that its supersymmetric transform is $\delta\rho = z_1^\star\bar{\varepsilon}\psi_2 - z_2\bar{\varepsilon}\psi_1^\star$, then we can write (25) as

$$\delta(A^\mu + \partial^\mu\rho) = \varepsilon\gamma^\mu\Lambda, \tag{32}$$

such that together with (31), we have defined a $N = 1$ vector composite supermultiplet, up to a gauge transformation. This result can be applied to the $SU(2)$ real triplet $(a = 1, 2, 3)$

$$A_\mu^a = \overline{\chi}\mathrm{i}\Sigma_\mu^a\chi - \mathrm{i}\phi^\dagger\sigma^a\partial_\mu\phi + \mathrm{i}\partial_\mu\phi^\dagger\sigma^a\phi, \tag{33}$$

and the $SU(2)$ real singlet

$$B_\mu = \overline{\chi}\mathrm{i}\Gamma_\mu\chi - \mathrm{i}\phi^\dagger\partial_\mu\phi + \mathrm{i}\partial_\mu\phi^\dagger\phi \tag{34}$$

where $\phi = (z_1, z_2)$ is a two-component scalar field and the other notations were introduced in section 3. We recall that the field strength $F_{\mu\nu}^a$ appearing then in equation (31) is the *Abelian* one for each a (i.e. we obtain three independent Abelian composite supermultiplets).

6. Current supermultiplet and $N = 2$ extended superalgebra

A specific feature of $2 + 1$ dimensions is that we can always construct a topological conserved current J_μ, starting from a vector A_μ. This topological current is

$$J_\mu = \epsilon_{\mu\rho\sigma}\partial^\rho A^\sigma. \tag{35}$$

In general it can be shown [10] that the current J_μ belongs to a new supermultiplet containing A_μ and a spinorial current $\tilde{S}_\mu^\alpha$. As a result, the $N = 1$ supersymmetry is centrally extended, with the central extension being provided by the topological charge associated with the current J_μ [10].

This framework can be applied at the composite level. There is again a current supermultiplet involving the topological current constructed out of a composite vector field A_μ^a, $a = 1, 2, 3$ in (33). Although there are three such currents, our interest is in the effective theory of degrees of freedom which are massless at a perturbative level. As discussed in detail in [5], due to spontaneous symmetry breaking, only one of the vector fields A_μ^a will remain massless in perturbation theory. In the supersymmetric case, we have already seen in section 5 that the vector multiplet is supersymmetric if it is a *gauge* field (which is Abelian in our quadratic composite field order). Hence it is plausible to *conjecture* that the vector composite fields in the supersymmetric case are *gauged*. This stems from the fact that *supersymmetry* cannot be broken by the passage from the constituent theory to the composite one, since the latter is obtained by integrating out a strongly coupled $U_S(1)$ field [5], which does not belong to the supermultiplets under consideration.

Actually, in view of the symmetry breaking patterns of [5], it is only one of the three vector composites that will remain massless. In what follows we shall denote this vector field by the generic symbol A_μ without any component index. Our analysis formally holds for any of the three components of the composite vector field (33). In particular, at a *composite* level we are interested in demonstrating that the current $J_\mu = \epsilon_{\mu\nu\rho}\partial^\nu A^\rho$ belongs to a new supermultiplet containing A_μ and a spinorial composite current $\tilde{S}^\alpha_\mu$. To show this let us return to the supersymmetric transform of A_μ seen in the previous section and define the spinorial current $\tilde{S}^\alpha_\mu$ by

$$\delta A_\mu = \varepsilon^\alpha \tilde{S}_{\mu\alpha} = \bar{\varepsilon}\tilde{S}_\mu, \tag{36}$$

such that $\tilde{S}^\alpha_\mu = \gamma_\mu \Lambda^\alpha$, where Λ denotes the *real* gaugino, obtained from (26) by adding the Hermitian conjugate. Since the translations commute with the supersymmetric transformations, $\tilde{S}_\mu$ is actually a conserved current in the specific gauge $\partial^\mu A_\mu = 0$, in which by definition A_μ is also conserved and we have $\partial\!\!\!/\Lambda = 0$. From the discussion in the previous section it follows that the supersymmetric transform of $\tilde{S}^\alpha_\mu$ is

$$\begin{aligned}\delta\tilde{S}_\mu &= -\tfrac{1}{2}\gamma_\mu F_{\rho\sigma}\gamma^\rho\gamma^\sigma\varepsilon \\ &= -\tfrac{1}{2}\gamma_\mu\partial_\rho A_\sigma\gamma^{[\rho}\gamma^{\sigma]}\varepsilon \\ &= \tfrac{1}{2}\gamma_\mu\partial_\rho A_\sigma\epsilon^{\rho\sigma\lambda}\gamma_\lambda\varepsilon \\ &= \tfrac{1}{2}\gamma_\mu J\!\!\!/\varepsilon.\end{aligned} \tag{37}$$

Let us now look at the supersymmetric transform of J_μ:

$$\begin{aligned}\delta J_\mu &= \epsilon_{\mu\rho\sigma}\partial^\rho\bar{\varepsilon}\gamma^\sigma\Lambda \\ &= \bar{\varepsilon}\gamma_{[\rho}\gamma_{\mu]}\partial^\rho\Lambda \\ &= \bar{\varepsilon}\partial\!\!\!/ S_\mu - \bar{\varepsilon}\gamma_\mu\partial\!\!\!/\Lambda \\ &= \bar{\varepsilon}\partial\!\!\!/\tilde{S}_\mu.\end{aligned} \tag{38}$$

Thus in the gauge $\partial_\mu A^\mu = 0$, we have a conserved current supermultiplet satisfying the supersymmetric transformations

$$\begin{aligned}\delta A_\mu &= \bar{\varepsilon}\tilde{S}_\mu \\ \delta J_\mu &= \bar{\varepsilon}\partial\!\!\!/\tilde{S}_\mu \\ \delta\tilde{S}_\mu &= \tfrac{1}{2}\gamma_\mu J\!\!\!/\varepsilon.\end{aligned} \tag{39}$$

Equations (39) form a closed superalgebra, since two successive supersymmetric transformations applied on any of the currents lead to a translation of this current:

$$\begin{aligned}\delta_1\delta_2 A_\mu &= \tfrac{1}{2}\bar{\varepsilon}_2\gamma_\mu\gamma_\nu\varepsilon_1\epsilon^{\nu\rho\sigma}\,\partial_\rho A_\sigma \\ \delta_1\delta_2 J_\mu &= \tfrac{1}{2}\bar{\varepsilon}_2\gamma^\rho\gamma_\mu\gamma^\sigma\varepsilon_1\,\partial_\rho J_\sigma \\ \delta_1\delta_2\tilde{S}_\mu &= \tfrac{1}{2}\gamma_\mu\gamma^\nu\varepsilon_2\,(\bar{\varepsilon}_1\partial\!\!\!/\tilde{S}_\nu).\end{aligned} \tag{40}$$

It must be stressed that this result is independent of the model that we consider for the composite fields, since the supersymmetric transformations (1) concern the original microscopic fields z_a and ψ_a.

Let us suppose that we have a Lagrangian for the composite vector field A_μ. We can then define the corresponding spinorial Noether's current S_μ^α and the associated supercharge $Q^\alpha = \int \mathrm{d}^2x\, S_0^\alpha$. The new spinorial current $\tilde{S}_\mu^\alpha$ enables us to define a new supercharge $\tilde{Q}^\alpha = \int \mathrm{d}^2x\, \tilde{S}_0^\alpha$. We will show that the anticommutator $\{Q^\alpha, \tilde{Q}^\beta\}$ contains an antisymmetric part, proportional to the topological charge, which thus defines a $N = 2$ supersymmetric structure. From the infinitesimal transformation (37), we know that

$$\{\overline{Q}\varepsilon, \tilde{S}_\mu\} = \tfrac{1}{2}\gamma_\mu \not{J}\varepsilon, \tag{41}$$

and thus

$$\{Q^\alpha, \tilde{S}_\mu^\beta\}\varepsilon^\gamma\epsilon_{\gamma\alpha} = \tfrac{1}{2}(\gamma_\mu \not{J})_\gamma^\beta\varepsilon^\gamma. \tag{42}$$

Since the inverse of $\epsilon_{\gamma\alpha}$ is $\epsilon^{\alpha\gamma} = (-\gamma^0)_\gamma^\alpha$, we then have

$$\{Q^\alpha, \tilde{S}_\mu^\beta\} = -\tfrac{1}{2}(\gamma_\mu \not{J})_\gamma^\beta\epsilon^{\gamma\alpha}. \tag{43}$$

The anticommutator between the two supergenerators is therefore

$$\{Q^\alpha, \tilde{Q}^\beta\} = \epsilon^{\alpha\beta}\frac{T}{2} + (\gamma^k)^{\alpha\beta}\frac{1}{2}\int \mathrm{d}^2x\, J_k \tag{44}$$

where $k = 1, 2$ is a spatial index, and $T = \int \mathrm{d}^2x\, J_0$ is the topological charge [10, 11]. Since the matrices γ^k are symmetric, (44) also implies

$$\{Q^\alpha, \tilde{Q}_\alpha\} = T, \tag{45}$$

which indicates a $N = 2$ superalgebra structure. However, it must be stressed that the mere appearance of a current supermultiplet *does not* necessarily imply a truly $N = 2$ extended dynamical supersymmetry in the physical spectrum of the model. This can only happen if the topological current is an independent quantity. In the case of the composite model discussed here, the topological current is constructed out of the vector composites, and hence, in order to promote the $N = 2$ structures to a true dynamical supersymmetry of the spectrum we must discuss in detail the dynamics of the composite model, in terms of the form of the associated Lagrangian. This will be the topic of the next section, where we shall see that the existence of a true $N = 2$ dynamical supersymmetry implies additional constraints for the coupling constants of the model [11].

Before closing this section we would like to stress that an extension of the $N = 1$ supersymmetry to a centrally extended $N = 2$ as above *also* characterizes the *constituent supersymmetric theory* (8) of spinon and holons, studied briefly in section 2. Details have been presented in [10], and shall not be repeated here.

We only mention briefly that, in this case, the topological current belongs to a current supermultiplet $\mathcal{J}$, which is defined in terms of the Z-scalar-superfields incorporating the magnon z and spinon ψ degrees of freedom:

$$\mathcal{J}_\alpha = \frac{1}{4\pi}D^\beta D_\alpha Z^\dagger D_\beta Z, \tag{46}$$

where α, β are spinorial indices and D denotes the chiral superspace derivative, introduced previously. Due to the basic identity

$$D^\alpha D^\beta D_\alpha = 0 \tag{47}$$

one can see that this current supermultiplet is identically conserved, i.e. $D^\alpha \mathcal{J}_\alpha = 0$, without the use of the equations of motion.

In this way, the passage from the constituent theory to the effective composite model, by integrating out strongly coupled $U_S(1)$ interactions as in [5], preserves the order and nature of supersymmetry algebra.

7. The $N = 2$ supersymmetric Abelian–Higgs composite model

As we have seen, the existence of constituent supersymmetries implies, via the appropriate transformation laws, the existence of *fermionic* composite excitations, which belong to $N = 1$ supersymmetric composite multiplets of scalar and vector type. At a bilinear composite field level, to which we restricted our attention for the purposes of the current work, we are unable to couple these two types of supermultiplets. Such a coupling is provided by the gauge fields themselves. We have already argued in the previous section that, in view of supersymmetry, the latter can be identified with the vector composites, which is not the case for the non-supersymmetric model of [5]. Indeed the viewing of the vector composites as gauge fields was essential for the closure of the $N = 1$ vector supermultiplets in section 5 since the closure of the vector supermultiplet under the constituent supersymmetry transformations occurs only *up to Abelian gauge transformations*. Moreover, since supersymmetry cannot be broken in passing from the constituent to the composite theory (by the integration of strongly coupled $U_S(1)$ interactions that do not belong to the supermultiplets), one is led logically to the above-mentioned identification. We should stress here that the supersymmetric model requires the coupling of the $U_S(1)$ gauge interactions to both z and ψ constituent fields, in which case it expresses fractional statistics of these excitations in $(2+1)$ dimensions, as discussed in [5]. This should be contrasted with the situation presented in [14], where the $U_S(1)$ coupled only to the fermions (holons) ψ, and in that case it represented spin frustrations.

Such a gauge coupling between the $N = 1$ supermultiplets would necessitate higher-than-bilinear constituent-field interactions among spinons and holons in the definition of the various composite fields (10), (11), (33), (34). This would imply sixth-order terms of constituent field operators in the constituent Lagrangian, which are irrelevant operators at low energies (i.e. in the infrared), at least from a naive renormalization group point of view. One might then hope that, as far as the underlying constituent theory of holons and spinons is concerned, the infrared fixed point universality class will not be affected.

The analysis of [5] in the non-supersymmetric model shows that there is dynamical gauge symmetry breaking in the composite Lagrangian in the phase where a parity-invariant mass term appears, or equivalently in the phase where the scalar composite field Φ_3 (10) acquires a non-trivial vacuum expectation value (v.e.v.), $\langle\Phi_3\rangle = u \neq 0$. Since u is a fermion (holon) condensate which is generated dynamically by means of the strongly coupled $U_S(1)$ interactions, whose coupling is formally infinite (of the order of the ultraviolet cut-off of the effective theory), quantum fluctuations of the condensate Φ_3 *will be completely suppressed*. Two of the $SU(2)$ gauge bosons, $A_\mu^{1,2}$ acquire masses in that case, proportional to the above vacuum expectation value, $\kappa^2u^2 \to \infty$, and, hence, they will decouple from the effective theory of light degrees of freedom, while the gauge field A_μ^3 remains perturbatively massless. The effective continuum theory of the light composite degrees of freedom is an Abelian–Higgs model [5, 14].

These symmetry-breaking patterns can be carried over to the supersymmetric cases discussed here. The supersymmetry transformations derived in sections 4 and 5 imply that the effective composite action in the naive continuum limit will be given by the supersymmetric

Abelian–Higgs model [11]:

$$I = \int \mathrm{d}^3x\,\{-\tfrac{1}{4}F_{\mu\nu}^2 + \tfrac{1}{2}(\partial_\mu \mathcal{S})^2 + \tfrac{1}{2}(D_\mu(A^3)\phi)^*(D^\mu(A^3)\phi) + \frac{\mathrm{i}}{2}\overline{\Lambda}\partial\!\!\!/\Lambda + \frac{\mathrm{i}}{2}\overline{\chi}\partial\!\!\!/\chi + \frac{\mathrm{i}}{2}\overline{\psi}D\!\!\!\!/(A^3)\psi\} \tag{48}$$

where χ is the real superpartner of the scalar singlet $\mathcal{S}$, and $\phi = \Phi_1 + \mathrm{i}\Phi_2$ is a complex scalar, with ψ its Dirac (complex) spinor superpartner, constructed out of the two real superpartners of the scalar composites $\Phi_i, i = 1, 2$ of section 4. The Abelian gauge field A_μ^3, whose field strength is denoted by $F_{\mu\nu}$, is the unbroken subgroup of the original $SU(2)$ gauge group. Notice that, despite the parity violating character of the composite excitations in the singlet supermultiplet, the corresponding terms in the action preserve parity†.

The action (48) is invariant under the following $N = 1$ supersymmetry:

$$\delta\Lambda = -\tfrac{1}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho\varepsilon, \qquad \delta A_\mu^3 = \overline{\varepsilon}\gamma_\mu\Lambda,$$
$$\delta\psi = \gamma^\mu\varepsilon D_\mu(A^3)\phi, \qquad \delta\mathcal{S} = \overline{\varepsilon}\chi, \qquad \delta\phi = \overline{\varepsilon}\psi. \tag{49}$$

We observe that a major part of these transformations has already been derived in sections 4 and 5 by demanding a scalar $N = 1$ supersymmetry among the constituent spinon and holons.

Unfortunately, however, at the level of bilinear composites, discussed in section 3, one cannot see the gauge-potential-dependent parts in the corresponding gauge covariant derivatives in the transformation for $\delta\psi$ in (49). The latter part involves four fermions and hence it can only be derived from composite fields which contain higher-order products of constituent fields *in addition to* the quadratic bilinear contributions studied here. As already mentioned, at a constituent level such coupling terms, which couple the vector and scalar $N = 1$ supermultiplets, appear to be irrelevant operators in a naive renormalization group sense. We hope to come to a more systematic study of such higher-order terms in the composite operators in a future work.

Assuming for the purposes of this work that the $N = 1$ supersymmetric action of the composite fields has the form (48), one may make a comparison with the corresponding action appearing in [11]. In this case we observe that the action does not have potential terms, in contrast to [11], which implies that it constitutes a specific case of the actions of [11] corresponding to zero couplings $e = \lambda = 0$ in the notation of that work, where λ is a coupling constant in front of a Higgs-type potential for the doublets ϕ, and e is a coupling for interacting terms of the form $\overline{\psi}\Lambda\phi - \overline{\Lambda}\psi\phi^*$.

It is also important to notice that in our case with the trivial condition on the couplings $e = \lambda = 0$ the $N = 1$ supersymmetry can be *elevated* trivially to a $N = 2$ supersymmetry in our case. This is because the two real fermions Λ and χ can be *assembled* in a single but complex Dirac fermion

$$\Sigma \equiv \chi - \mathrm{i}\Lambda \tag{50}$$

with the corresponding kinetic term in the effective Lagrangian:

$$\frac{\mathrm{i}}{2}\overline{\Sigma}\partial\!\!\!/\Sigma. \tag{51}$$

† For this reason we have not written explicitly the vector singlet terms $\mathcal{S}_\mu$ and their superpartners. Due to their parity violating nature, such terms should form a mutliplet by themselves, and should not interact with the rest of the terms. To understand this, one should notice that, as a result of the gauge nature of these excitations, if such interactions existed they should have the form $[(\partial_\mu - \mathcal{S}_\mu)\phi]^2$, which would violate parity, and as such should not appear in our parity-conserving composite effective action [5].

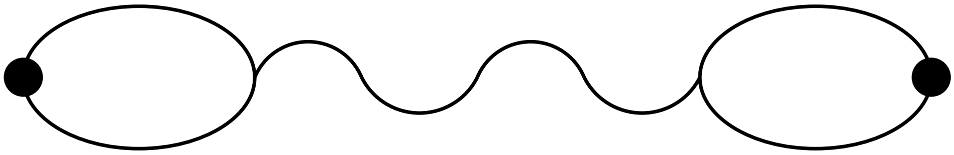

Figure 1. Contribution to electric current–current correlator in the phase where the holon (fermion) fields (solid curves) acquire a mass gap in the models of [4, 5]. The wavy line represents a A^3_μ gauge field. The blob indicates an insertion of the electric current (fermion number) operator J_μ.

The action (48) then has an $N = 2$ supersymmetry invariance, as shown in [11]. The corresponding infinitesimal transformations are characterized by a complex parameter $\eta \equiv \varepsilon e^{i\alpha}$, and they are equivalent to (49) with real parameter ε, followed by a phase transformation for the fermions $\Sigma \to e^{i\alpha}\Sigma$ and $\psi \to e^{i\alpha}\psi$. In the more general case of [11], where $e, \lambda \neq 0$ one has that the $N = 2$ supersymmetry transformations imply a *condition* on the couplings $e^2 = 8\lambda$ in the normalization of [11].

In our case, the absence of potential terms is compatible with the fact that the Abelian–Higgs model is obtained here from a spontaneous breakdown of the Georgi–Glashow model. Indeed in that model $N = 2$ supersymmetry implies a vanishing superpotential [13].

Notice that the analysis of [11], shows that the Noether supersymmetry currents corresponding to the model (48) are such that the pertinent supercharges satisfy a $N = 2$ algebra with central charge given by the topological charge of the Abelian–Higgs model. This is in accordance with the general arguments of [10], discussed in some detail in section 6. For completeness we will explicitly give the spinor charges

$$Q = \int d^2x\, [(-\tfrac{1}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho + i\not{\partial}\mathcal{S})\gamma^0\Sigma + i(\not{D}(A^3)\phi)^*\gamma^0\psi] \tag{52}$$

and correspondingly for $\overline{Q}$. The resulting centrally extended superalgebra is [11]

$$\{Q_\alpha, \tilde{Q}^\beta\} = 2(\gamma_0)^\beta_\alpha P^0 + \delta^\beta_\alpha T \tag{53}$$

where $P^0 = \int d^2x\, (\frac{1}{4}F^2_{ij} + \frac{1}{2}|D_j(A^3)\phi|^2)$, with $i, j = 1, 2$, is the total energy and the central charge T is the topological charge of the model, as discussed in the previous section.

We also notice that the existence of an $N = 2$ dynamical supersymmetry at a composite level is compatible with the elevation of the $N = 1$ constituent supersymmetry of the CP^1 σ-model to a $N = 2$ supersymmetry, due to the existence of topological currents [10], as explained in section 6.

8. $N = 2$ supersymmetry implies superconductivity

The precise form of the effective theory is crucial for an understanding of the phase structure of the nodal liquid. As already mentioned, at a perturbative level, the gauge field A^3_μ is massless, and in fact the theory is superconducting [5]. This can be understood by looking at the electric current–current correlator in Fourier space of the constituent theory of holons [4]: $\langle J_\mu(-k)J_\nu(k)\rangle$, where $J_\mu \sim \overline{\Psi}\gamma_\mu\Psi$, with Ψ a Dirac spinor denoting a nodal holon field. The relevant Feynman diagram is given in figure 1.

In the models of [4, 5] the loop corrections of this diagram are non-vanishing only in the phase where there is a mass gap for the electrically charged holon (fermion) fields. This feature is specific to $(2+1)$ dimensions. According to Landau, a theory is superconducting if there is a massless gauge-field pole in this diagram. In the model of [5] what is exchanged (the wavy line) is the gauge field A^3_μ. If this is perturbatively massless, then the theory is superconducting.

However this may not be in general true when non-perturbative effects are taken into account, such as monopoles, which are instantons in the $(2+1)$-dimensional theory [15]. The monopole is a Euclidean configuration which behaves asymptotically as

$$\hat{\Phi}^a = \hat{r}^a, \qquad \tilde{F}^a_\mu(x) = \frac{1}{g}\frac{\hat{r}^a\hat{r}_\mu}{r^2} \tag{54}$$

where the caret indicates a unit vector and the tilde indicates the dual field tensor.

In the $N = 2$ supersymmetric Georgi–Glashow model, or its $N = 2$ Abelian counterpart discussed here, the photon A^3_μ remains exactly massless, even at a non-perturbative level, due to the presence of Dirac (complex) fermions. This has been discussed in [13], and we shall not repeat the discussion here. Thus the nodal liquid at the supersymmetric point leads to the superconductivity mechanism proposed in [4]. It is worth noticing that such a masslessness is a property of the existence of complex fermions rather than composite supersymmetry. Simply, in our $N = 2$ supersymmetric case the existence of complex composite fermions is a necessary consequence of the extended supersymmetry, and in this sense it is the $N = 2$ *constituent supersymmetry* of spinon and holons rather than the composite one which *guarantees superconductivity*. Nevertheless, the existence of a $N = 2$ dynamical supersymmetric effective composite theory is important on its own in yielding exact non-perturbative results on the phase structure, in the way explained in [2].

The important point is that such a supersymmetry-argument based mechanism will work at strictly zero temperature, where supersymmetry is unbroken, and hence the above considerations may point towards a *quantum critical (superconducting) point* of the nodal liquid at the supersymmetric point of the parameter space of the microscopic model [8]†.

An interesting question concerns a transition from the superconducting to the pseudogap phase of the nodal liquid in our case, where indeed the supersymmetric point necessarily implies superconductivity of the nodal liquid of excitations. This may be provided by a simple variation of the doping concentration, which could take one away from the constituent supersymmetric point. In that case there is an explicit breaking of supersymmetry, and the masses of the spinon and holons are unequal. One may then arrive at the situation of [5], where the spinons are very massive and hence should be integrated out. In that case one is left with composites made only of holons, the light degrees of freedom in the problem. In such a case the effective theory is just the Georgi–Glashow model [14] without composite fermions (which may be thought of as being 'very massive and thus decoupled' like the spinons). In such theories, according to the standard analysis [15], non-perturbative effects yield the photon A^3_μ a small mass, thereby leading to a pseudogap phase. If the above scenario is true, it would imply that there is a quantum critical

† On the other hand, in the case of the supersymmetric Abelian–Higgs model with *a non-trivial Higgs potential for the scalars* of [11], which is *not our case here*, there is a Higgs phase in which the photon becomes massive. Such a phase would not be superconducting according to the mechanism of [4], since it would lead to a massive photon exchange in the respective current–current correlator, and hence the Landau criterion for superconductivity would be violated. But given the holon mass-gap generation, such a phase would have been simply a pseudogap phase.

point of the nodal liquid where the onset of superconductivity is identified with the onset of centrally extended $N = 2$ supersymmetries at both constituent and composite levels.

9. Conclusions

In this work we have presented novel results (in sections 4, 5) concerning some non-trivial properties of supersymmetric theories in specific models in $(2+1)$ dimensions. Specifically, we have shown how underlying constituent supersymmetries, which as we have stressed may occur naturally in the context of spin–charge separation, carry over to a composite theory, obtained after integrating out strongly coupled gauge fields in certain $(2 + 1)$-dimensional models of relevance to high-temperature superconductivity.

Moreover, in this context, we have reproduced (section 6) a generally expected result in theories with topologically conserved currents, namely the extension of $N = 1$ supersymmetries into centrally extended ones, of $N = 2$ type. In $(2 + 1)$-dimensions, any vector field leads to topological currents, conserved without the use of equations of motion.

Finally, in section 7, we have associated the existence of $N = 2$ supersymmetries to superconducting properties. These results may therefore have some relevance to efforts in understanding as accurately as possible the phase structure of high-temperature superconducting cuprates. The relativistic nature of our continuum effective theories is a result of looking at quasiparticle excitations near nodes in the Fermi surface of these materials.

At present we have only been able to demonstrate these results within the context of composite operators which are quadratic in the constituent fields. This unfortunately yields only decoupled $N = 1$ vector and scalar composite multiplets, and prevents one from seeing explicitly the coupling of these two multiplets by means of gauge fields. Work is currently in progress towards the inclusion of quartic order terms in the composite operators.

Therefore, much more work is needed before one arrives at a complete understanding of the underlying dynamics of the nodal liquids. In this work we have discussed a possible role of supersymmetry in yielding some exact results concerning the passage to a superconducting phase. Nevertheless, the exciting features on the existence of extended supersymmetries, presented here, which could allow some exact results on the phase structure of the nodal liquids to be obtained, already open up interesting directions for theoretical research in such systems. We hope to come back to such studies in the near future.

Before closing we consider it as useful to remark that our considerations pertain to continuum limits of effective theories. It is at such limits that the emergence of supersymmetries has been discussed. It should be stressed that one hopes to extend these results to lattice models, up to operators *irrelevant* in a renormalization group sense, so that the low-energy (infrared) universality class of the model is not affected. This has not been done here, and constitutes an interesting avenue of research to be explored in the future.

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