On a classification of surjections and embeddings: Surjective characterization of injective classes of compact spaces with applications to compact and locally compact groups

by

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A dissertation submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy

Michaelmas Term 1978

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ABSTRACT. This work has largely grown out of the papers of R. Haydon (Studia Math. 52 (1974), 23-31) and E.V. Shchepin (Russian Math. Surveys 31:5 (1976), 155-191). Among other results affirmative answers to problems 3, 18, 20 and 21 posed by A. Pełczyński in his monograph (Diss. Math. 58, Warszawa 1968) are given.

For pairs (m,n) of extended integers ($-1 \leq m \leq n \leq \infty$) the surjective and injective notions of (m,n)-soft and (m,n)-hard maps respectively are introduced and investigated. It is shown that for a map $X \to Y$ of compact spaces (-1,0)-softness (resp. (0,0)-hardness) entails the existence of a regular averaging (resp. extension) operator and that the converse holds, if X is metrizable (resp. Y is a Dugundji space). (0,0)-hardness, (-1,0)-softness and (0,0)-softness are shown to be local properties and it is deduced that each point of a Dugundji space and of a quotient space of a locally compact group has a neighbourhood base of Dugundji spaces. Moreover, we prove that a space and its cone are Dugundji spaces simultaneously. (∞,∞)-soft group homomorphisms are isomorphisms. This follows from the result that a compact contractible group is trivial.

A compact space is classified as an absolute extensor for m- to n-dimensional spaces or AE(m,n), if the constant map is (m,n)-soft. Equivalently, a compact space is an AE(m,n) if and only if every embedding into a compact space is (m,n)-hard. Haydon's paper entails coincidence of the notions of AE(0,0) and Dugundji space. Here an AE(0,0) (resp. $AE(0,\infty)$) is surjectively characterized as (-1,0)-soft image of a generalized Cantor set (resp. Tychonoff cube). An injective characterization of the class of Peano spaces is given as metrizable AE(0,n), n ($1 \leq n \leq \infty$) being arbitrary.

Using combinatorial techniques we obtain some results concerning the countable chain condition on spaces of probability measures and spaces of $\leq$ n-element subsets.

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Introduction

In [28] Pełczyński introduced the notions of Milutin space and Dugundji space and established some basic results for these spaces. By definition a compact space X is called a Dugundji space (resp. Milutin space) if there exists a continuous embedding of X into a product of closed unit intervals (resp. continuous surjection of the generalized Cantor set onto X) admitting a regular extension operator or r.e.o. (resp. regular averaging operator or r.a.o.)[1]. Hence both Milutin spaces and Dugundji spaces are defined in algebraic terms by the existence of a certain operator acting on the Banach space of continuous real-valued functions. In 1974 R.Haydon [14] succeeded in giving a purely topological characterization of Dugundji spaces as absolute extensors for zero-dimensional spaces or AE(0-dim). At the same time he proved that every Dugundji space is a Milutin space. Before this result was obtained S.Ditor and R.Haydon showed in a joint paper [7] that a Milutin space of topological weight at most ω_1 is necessarily a Dugundji space. Until 1976 it remained an open problem of whether or not coincidence of the notions of Dugundji space and Milutin space holds without any restriction on topological weight. Then in his paper [37] Shchepin produced the first known example of a (zero-dimensional) non-Dugundji Milutin space, of topological weight ω_2. As far as the author is aware no topological characterization of Milutin spaces of topological weight exceeding ω_1 is known at present.

The research for this thesis was triggered off by an effort to find a surjective characterization of Dugundji spaces that

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1) For the terminology see "Preliminaries".

corresponded to the one given for the strictly bigger class of Milutin spaces. That is to say, in the definition of Milutin space we want to strengthen the requirement of existence of a r.a.o. to such an extent that we precisely describe the class of Dugundji spaces. We shall show that the surjective notion of zero-dimensional lifting property or z.d.l.p. provides a positive answer to this problem. However, we are confronted with the curious phenomenon that for a continuous surjection which has the generalized Cantor set of weight ω_1 as domain space the existence of a r.a.o. does not necessarily imply the z.d.l.p.

The new surjective characterization of Dugundji spaces gives access to a range of interesting results on their topological structure and among other things it reveals that a Dugundji space is determined by its local data (and vice versa). This suggests an extension of the notion of Dugundji space to locally compact spaces and it turns out that the important class of locally compact topological groups and their quotient spaces lies within the realm of this extension.

Zero-dimensional Dugundji spaces are retracts of some generalized Cantor set and convex Dugundji spaces (i.e. subspaces of some locally convex topological vector space) are known to be retracts of a product of closed unit intervals. It seems natural to ask for a classification of further subclasses of Dugundji spaces between these two extremes. As a hybrid of the two classes mentioned above one is offered the subclass of Dugundji spaces consisting of those spaces which are obtained as continuous images of a product of closed unit intervals by surjections satisfying the z.d.l.p. As a rather interesting result the restriction of this subclass to metrizable spaces precisely yields the class of Peano spaces. This charac-

terization, we think, might encourage a new approach to a long desired generalization of the celebrated Hahn-Mazurkiewicz theorem stating that Peano spaces are exactly the continuous images of the closed unit interval.

We recall that the injective classification of a compact metrizable space (resp. convex compact metrizable space) as an AE(0-dim) (resp. absolute extensor) is closely related to a selection theorem of E.Michael for lower semicontinuous set-valued mappings. A similar correspondence cannot be expected to hold any more for the injective classification of a Peano space we present in this thesis. The reason for this can be sought in the mixed dimensional restriction of the spaces involved in the extension process.

Every compact topological group is a Dugundji space and obviously there exists a non-trivial topological group which is a Peano space (e.g. the one-dimensional circle group). Pushing the topological restriction on the group a bit further one might ask for a characterization of contractible compact groups. It is shown that there is only one such group, namely the trivial group. In particular there does not exist a non-trivial compact group that is an absolute retract.

Summary of the contents

In the preliminary section we establish the basic notation and terminology of this thesis. Most of the items listed will be familiar. However, to avoid confusion later on we choose to present

them there since they are important for the general understanding of this thesis and will frequently be used in the following sections without any further reference.

Section 1 motivates and introduces the notion of z.d.l.p. For compact metrizable spaces a topological characterization of a continuous surjection allowing a r.a.o. is given in terms of z.d.l.p. This completely solves a problem posed by Pełczyński in his monograph ([28], Problem 3, p.65) and under less restrictive assumptions slight improvements are obtained. However, it is shown that in general this characterization cannot be extended to zero-dimensional compact spaces of uncountable topological weight. On the other hand if a continuous surjection has an extremally disconnected domain or image space, then the existence of a r.a.o. is equivalent to the map being a retraction.

Section 2 is devoted to the study of the zero-dimensional retraction property or z.d.r.p. which can be considered as the injective counterpart of the surjective notion of z.d.l.p. investigated in the preceding section. For an embedding of compact spaces it is shown that the existence of a r.e.o. is always implied by the z.d.r.p. but a converse does not hold in general. However, if one of the spaces involved is a Dugundji space the converse does hold. This induces an embedding characterization of Dugundji spaces in purely topological terms by changing "admitting a r.e.o." into "satisfying the z.d.r.p." in the definition given at the beginning of the introduction. What follows next can be regarded as an extension of this result and at the same time as an effort to develop some sufficient conditions entailing the

z.d.r.p. The notions of fibre-preserving r.e.o. and O-softness are introduced and their equivalence is proved. In the remainder of this section we investigate necessary conditions for embeddings to allow a r.e.o. or to satisfy the z.d.r.p.

In section 3 we introduce the notion of Michael space which is defined as the continuous image of a generalized Cantor set by a map satisfying the z.d.l.p. It turns out that Michael space and Dugundji space are equivalent notions. Utilizing this surjective characterization of Dugundji spaces we prove that a Dugundji space is locally a Dugundji space and hence give an affirmative answer to a problem posed by Pełczyński ([28], Problem 18, p.70). It is shown that pinching a closed G_δ-set of a Michael space to a point yields a Michael space again. This leads to the result that a compact space and its cone are Dugundji spaces simultaneously.

In section 4 we extend the notions of Dugundji space and O-softness to locally compact paracompact spaces and verify that the defining property of a Dugundji space in terms of r.e.o. carries over to this larger class of spaces. It follows from Shchepin's paper [37] that the notions of Dugundji space and O-softness are compatible. We show that this compatibility remains valid for their locally compact extensions and we prove an inverse limit theorem for locally O-soft maps. It entails as a corollary that in a quotient space of a locally compact topological group every point has a neighbourhood base of Dugundji spaces. This result gives a complete answer to problems 20 and 21 of Pełczyński's monograph ([28], p.70).

Section 5 is devoted to the study of a countable partially

ordered spectrum of injective classes of compact spaces. For each pair of integers (m,n) with $0 \leq m \leq n \leq \infty$ the class of absolute extensors for m- to n-dimensional compact spaces or AE(m,n) is defined such that the biggest class, AE(0,0), (resp. smallest class, $AE(\infty,\infty)$,) coincides with the class of Dugundji spaces (resp. absolute retracts). The naturally corresponding notions of (m,n)-soft and (m,n)-hard maps are introduced and investigated. In particular (∞,∞)-soft maps are injectively characterized in terms of fibre-preserving multiplicative r.e.o. We prove that a compact space of type $AE(0,\infty)$ admits a surjective characterization which lies half way between the one for a Dugundji space as Michael space and the one for an absolute extensor as absolute retract. Restricting our attention to compact metrizable spaces we show that for all integers n ($1 \leq n < \infty$) the notions of AE(0,n) and $AE(0,\infty)$ are equivalent and that they exactly coincide with all compact connected locally connected metric spaces or Peano spaces. Turning to compact topological groups we prove that a (∞,∞)-soft group homomorphism is an isomorphism and that a non-trivial compact group is not an absolute retract. These statements are corollaries to the rather surprising result that a compact contractible group is trivial. Several examples of (m,n)-soft maps are exhibited.

In section 6 we slightly digress from the general theme of this thesis and investigate questions concerning the countable chain condition or c.c.c. on compact spaces of the types P(X), $\exp_n(X)$ and exp(X). These spaces deserve our special attention for two main reasons. They are very useful as a source of counterexamples and by Shchepin's functor theorem ([37], p.165) they have the property

that X and P(X) (resp. $\exp_n(X)$, $\exp(X)$) are κ-metrizable simultaneously. An application of two combinatorial lemmas due to Shanin and Ramsey respectively allows us to prove that for a compact space X the product space X^{ω} (resp. X^n) and P(X) (resp. $\exp_n(X)$) satisfy the c.c.c. simultaneously.

Technical Remarks

This thesis is devided into seven sections. Within each section the definitions, propositions, corollaries, theorems, lemmas, remarks and examples are further numbered consecutively so that proposition 4.3 appears as the third item (not necessarily the third proposition) in section 4. The only exception to this rule is section 0, introducing the basic notations, where each number indicates a subsection rather than a specific result. We mention some notational conventions. The end of a proof is denoted by □. In considering statements of the form "A if and only if B" we occasionally use "necessity" to mean that "A implies B" and "sufficiency" to mean the converse, i.e. "B implies A".

Finally let us make a few remarks on the interdependence of the sections and generally on the dependence of this thesis on information from outside sources. For the understanding of sections 1 to 6 the reader is expected to be familiar with the terminology and statements which are gathered in the preliminaries. Each of the sections 1 to 4 relies on results of the preceding one. The general framework of section 5 can probably be comprehended with the knowledge of sections 1 and 2. However, to appreciate it to a larger degree

the background of section 3 is necessary. Section 6 can be followed without having read the preceding sections 1 to 5.

Results from sources outside this thesis are often presented in an appropriate form rather than just being referred to. Occasionally we even furnish them with a proof in cases where a short one is available but not easily accessible or if we find the importance of the result justifies the elaboration. We hope this policy facilitates the understanding of this thesis to such an extent that it can be read without any further study of the references. However, some knowledge of the three major sources of reference [14], [28] and [37] might be rather helpful.

Acknowledgement. I should like to express my gratitude to my supervisor Dr.R.G.Haydon who introduced me to the subject of this piece of research. His guidance and readiness to discuss problems was very stimulating at all times. Furthermore I should like to thank the Michael Foster Foundation and the DAAD for their financial support without which my stay at Oxford University would not have been possible.

§0 Preliminaries

In this section we shall briefly introduce definitions and notations we shall adhere to throughout this thesis. Most of them are standard and more details can be found in at least one of the four references [11], [14], [28] or [37]. Certainly, we shall need further pieces of information in this thesis beyond the ones listed in this section. However, we prefer developing them as we go along, for this will provide us with the opportunity of giving the right kind of motivation for their introduction, which, by its very nature, this section of preliminaries is obviously lacking. In the sections to follow we shall take the results of the preliminaries for granted and frequently use them without any further reference.

The general topological background is presented in the first part. Proofs of the results stated here are contained in [11]. The second part concentrates on some special classes of topological spaces which play a central role in this work and [14], [28] and [37] are the major sources of reference for the proofs of the statements compiled here.

The results and definitions listed below will be used to construct a framework which is sufficient for the arguments of this thesis. However, some of them still hold in a more general setting.

I. Topological spaces and continuous maps

Throughout this thesis the terms "space" and "map" mean "completely regular Hausdorff space" and " continuous map" respectively. An embedding (resp. surjection) is an injective (resp. surjective) map. The terms "projection" and "quotient map" are reserved for special types of surjections. Embeddings (resp. surjections) are often indicated by " $\hookrightarrow$" (resp. " $\twoheadrightarrow$") in diagrams. Note that the notion of "mapping" does not imply continuity and "function" always refers to a real-valued mapping.

1. Space of continuous functions

For a space X we denote by $C^b(X)$ the space of bounded continuous functions on X equipped with the topology induced by the supremum norm. X is called pseudocompact if all continuous functions on X are bounded and in this case we write $C(X)$ instead of $C^b(X)$. If βX denotes the Stone-Čech compactification of the space X the algebras $C(\beta X)$ and $C^b(X)$ are known to be isometrically isomorphic.

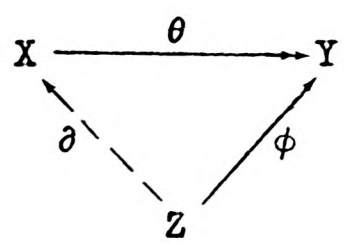
2. Basic topological notations

For any subset Y of a space X we denote by 1_Y the function on X that is 1 on Y and 0 on the complement $X \setminus Y$ of Y in X. Obviously 1_Y is an element of $C^b(X)$ if and only if Y is simultaneously closed and open in X or more conveniently Y is called clopen. Furthermore, for any subset Y of a space X we use $cl_X(Y)$ (resp. $int_X(Y)$) to

indicate the closure (resp. interior) of Y in X and if there is no risk of confusion we sometimes omit the subscript X. The closure of an open set is called a regular closed set. A space X satisfies the Bockstein Separation Property or B.S.P. if every regular closed subset is the intersection of countably many open subsets of X. (We prefer this definition to the equivalent and more familiar one which says that two open disjoint sets can be separated by open F_σ-sets.) Generally a set that is the intersection of countably many open sets is called a G_δ-set.

3. Extremally disconnected spaces

A space with the property that every regular closed subset is open is said to be extremally disconnected. Extremally disconnected compact spaces are precisely characterized as the projective objects in the category of compact spaces and maps, i.e. a compact space Z is extremally disconnected if and only if for any surjection θ: X → Y of compact spaces X, Y and any map ϕ: Z → Y there is a map ∂: Z → X such that the diagram

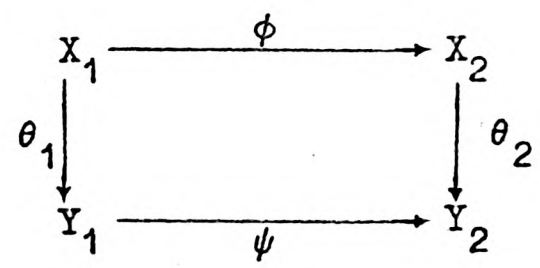


commutes, $\theta \circ \partial = \phi$.

4. Map operations

The identity map on a space X is denoted by id_X. If Y is a sub-

space of a space X and θ is any mapping defined on X we agree to write $\theta|Y$ for the natural restriction. If θ is a map, so is $\theta|Y$. A surjection (resp. embedding) θ: X → Y of spaces X, Y is called a retraction (resp. coretraction) if there exists a map ϕ: Y → X such that $\theta \circ \phi = id_Y$ (resp. $\phi \circ \theta = id_X$). An image of a retraction is a retract. All properties of spaces we are dealing with are invariant under homeomorphisms. Moreover, the same holds for the properties of maps. We recall that for i = 1,2 two maps θ_i: $X_i \to Y_i$ of spaces X_i, Y_i are homeomorphic if and only if there exist homeomorphisms ϕ: $X_1 \to X_2$, ψ: $Y_1 \to Y_2$ such that the diagram



commutes, $\theta_2 \circ \phi = \psi \circ \theta_1$.

Let θ_i: $X_i \to Y_i$ be maps of spaces X_i, Y_i for i = 1,2. The disjoint union of θ_1 and θ_2 is a map θ of the disjoint union of X_1 and X_2 into the disjoint union of Y_1 and Y_2 which is given by $\theta|X_i = \theta_i$ (i = 1,2). Furthermore, if X_1 and X_2 coincide and the space is just denoted by X we define the diagonal product map $\theta_1 \Delta \theta_2$: $X \to Y_1 \times Y_2$ of X into the product space of Y_1 and Y_2 by putting $\theta_1 \Delta \theta_2(x) := (\theta_1(x), \theta_2(x))$ for all $x \in X$.

Let A be any set and $(X_a)_{a \in A}$ a collection of spaces X_a indexed by A. We denote by $\prod \{X_a : a \in A\}$ the product space equipped with the product topology. If $(Y_a)_{a \in A}$ is another collection of spaces and θ_a: $X_a \to Y_a$ is a map for all $a \in A$, we define the product map $(\theta_a)_{a \in A}$ of $\prod \{X_a : a \in A\}$ into $\prod \{Y_a : a \in A\}$ by putting

$$(\theta_a)_{a \in A}((x_a)_{a \in A}) := (\theta_a(x_a))_{a \in A}.$$

5. Ordinals and cardinals

Ordinals are known to be linearly ordered and for two ordinals α, β we write $\alpha < \beta$ (resp. $\alpha \leq \beta$) if α is less than β (resp. α is less than or equal to β). Moreover, we follow the convention of identifying a cardinal with the associated initial ordinal. So e.g. the first three infinite cardinals are indicated by the symbols ω, ω_1 and ω_2.

6. Weight and density

The weight of a space X is the smallest cardinal τ such that there exists a base of open sets for the topology on X of cardinality τ. The density of a space X is the smallest cardinal τ such that there exists a dense subset of X of cardinality τ. A space with countable density is called separable and a compact space with countable weight is known to be metrizable.

7. Some special spaces

Let $\mathbb{R}$ denote the real numbers equipped with the usual topology and let $\mathbb{Q}$ (resp. $\mathbb{N}$) be the subspace of rational numbers (resp. non-negative (finite) integers). Let $\mathbb{I}$ be the closed unit interval of $\mathbb{R}$ and $\mathbb{D}$ the two-element subspace $\{0,1\}$ of $\mathbb{I}$. Sometimes it will be convenient to identify the Cantor set $\mathbb{D}^\omega$ with the closed

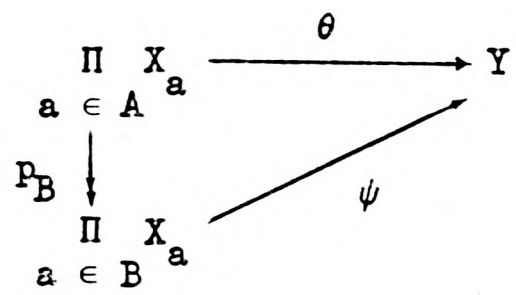
subset of all points of $\mathbb{I}$ which have triadic expansions in which the digit 1 does not occur. A space that is the continuous image of a generalized Cantor set $\mathbb{D}^A$ for some set A is called dyadic. We recall that every compact metrizable space is dyadic. It is known that for any infinite cardinal τ the weight of $\mathbb{D}^\tau$ and $\mathbb{I}^\tau$ is τ and any space of weight τ can be embedded into $\mathbb{I}^\tau$. For $\tau \geq \omega$ the density of $\mathbb{D}^{2^\tau}$ is τ and hence, in particular, $\mathbb{D}^{\omega_1}$ is separable.

8. Dependence of maps

Let $(X_a)_{a \in A}$ be a collection of spaces and X its product space. For any subset B of A let

$$p_B : X \longrightarrow\!\!\!\!\!\rightarrow \prod_{a \in B} X_a$$

denote the canonical projection $(x_a)_{a \in A} \mapsto (x_b)_{b \in B}$. Then a map θ: X → Y of the product space X into a space Y is said to depend on a subset B of A if θ factors through p_B, i.e. there exists a map ψ: $\prod_{a \in B} X_a \to Y$ such that the diagram



commutes. Now suppose the spaces X_a ($a \in A$) are all compact and then so is the product space X. The continuous functions on X depending on only finitely many coordinates form a point-separating subalgebra of C(X) and so by the Stone-Weierstrass theorem it is a dense subalgebra. Hence every element of C(X) is a limit of a

sequence of continuous functions depending on finitely many coordinates. Extending this kind of argument one can verify that for a space Y of infinite weight and compact spaces X_a ($a \in A$) a map of the product space $\Pi \{X_a : a \in A\}$ onto Y depends on a subset B of A where the cardinality of B can be chosen to coincide with the weight of Y.

9. Exponential space and lower semicontinuity

For a normal space X we denote by exp(X) (resp. $\exp_n(X)$ for $n \geq 1$) the space of all non-empty, closed subsets of X (resp. non-empty subsets of X of cardinality $\leq n$) equipped with the finite topology, i.e. a base of open sets is given by sets of the form

$$< U_1,\ldots,U_m > := \{ A \in \exp(X) : A \subset \bigcup_{i=1}^{m} U_i,\ U_i \cap A \neq \emptyset \text{ for } i = 1,\ldots,m \}$$

where m is any finite integer and the sets U_i range over all open subsets of X. $\exp_n(X)$ carries the induced topology. Let Y be a space and $\Phi: Y \to \exp(X)$ a mapping. Then Φ is called lower semicontinuous if for all open sets U of X the set $\{y \in Y: \Phi(y) \cap U \neq \emptyset\}$ is open in Y and a mapping $\partial: Y \to X$ is called a selection for Φ if $\partial(y) \in \Phi(y)$ for all $y \in Y$.

10. Inverse limits

We define an inverse system $(X_\alpha, p_{\alpha\beta})_{\alpha \leq \beta < \tau}$ as a totally ordered system of spaces X_α and surjections $p_{\alpha\beta}: X_\beta \to X_\alpha$ where

$\alpha \leq \beta$ range over all ordinals less a fixed cardinal τ, if it satisfies the following properties for all $\alpha \leq \beta \leq \gamma < \tau$:

$$\text{(i)}\quad p_{\alpha\alpha} = id_{X_\alpha},$$

$$\text{(ii)}\quad p_{\alpha\gamma} = p_{\alpha\beta} \circ p_{\beta\gamma}.$$

The inverse limit space X of the inverse system $(X_\alpha, p_{\alpha\beta})_{\alpha \leq \beta < \tau}$ is defined by

$$X := \{ (x_\alpha)_{\alpha<\tau} \in \prod_{\alpha<\tau} X_\alpha : x_\alpha = p_{\alpha\beta}(x_\beta) \text{ for all } \alpha \leq \beta < \tau \}.$$

The canonical projections $p_\alpha : X \to X_\alpha$, $(x_\alpha)_{\alpha<\tau} \mapsto x_\alpha$, are called limit maps. Maps of the type $p_{\alpha,\alpha+1}$ ($\alpha < \tau$) will be refered to as linking maps.

An inverse limit space X of an inverse system $(X_\alpha, p_{\alpha\beta})_{\alpha \leq \beta < \tau}$ satisfies the following important property. If Y is any space and $\theta_\alpha : Y \to X_\alpha$ ($\alpha < \tau$) a collection of maps such that $p_{\alpha\beta} \circ \theta_\beta = \theta_\alpha$ for all $\alpha \leq \beta < \tau$, then there exists a unique map $\theta : Y \to X$ such that $p_\alpha \circ \theta = \theta_\alpha$ for all $\alpha < \tau$. An inverse system $(X_\alpha, p_{\alpha\beta})_{\alpha \leq \beta < \tau}$ is said to be continuous, if for all limit ordinals $\gamma < \tau$ the natural mapping from X_γ to the inverse limit space of the reduced inverse system $(X_\alpha, p_{\alpha\beta})_{\alpha \leq \beta < \gamma}$ is a homeomorphism.

11. Covering and paracompactness

Let X be a space and $(U_a)_{a \in A}$ a family of subsets of X and let Y be a subspace of X. Then $(U_a)_{a \in A}$ is called a covering of Y or $(U_a)_{a \in A}$ covers Y if the union $\cup \{ U_a : a \in A \}$ contains Y. A covering $(U_a)_{a \in A}$ is called open (resp. closed) if for all $a \in A$ the set U_a is open (resp. closed) in X and it is called finite

if the index set A is finite. Another covering $(V_b)_{b \in B}$ of Y is said to be a refinement of the covering $(U_a)_{a \in A}$ if for each $b \in B$ there is an element $a(b) =: a \in A$ such that V_b is contained in U_a. A covering $(U_a)_{a \in A}$ of a space X is called locally finite, if each point of X has a neighbourhood V such that $V \cap U_a \neq \emptyset$ for at most finitely many indices $a \in A$. We recall that a space X is paracompact, if every open covering of X has a locally finite, open (or, equivalently, closed) refinement. It is well-known that a space X is paracompact if and only if for every open covering $(U_a)_{a \in A}$ of X there exists a partition of unity subordinate to the covering, i.e. there exists a collection $(\lambda_a)_{a \in A}$ of continuous functions of X into the unit interval $\mathbb{I}$ such that

$$\text{(i) } \mathrm{cl}_X(\{ x \in X : \lambda_a(x) \neq 0 \}) \subset U_a \text{ for all } a \in A,$$

$$\text{(ii) } (\mathrm{cl}_X(\{ x \in X : \lambda_a(x) \neq 0 \}))_{a \in A} \text{ is a locally finite covering of X,}$$

$$\text{(iii) } \Sigma \{ \lambda_a : a \in A \} = 1_X.$$

Note that by the local finiteness of the covering the sum in (iii) is well-defined.

Paracompact spaces are normal and a closed subspace of a paracompact space is paracompact too.

12. Dimension

Let $\mathfrak{U} = (U_a)_{a \in A}$ be a finite covering of a space X. We define the order of $\mathfrak{U}$ to be the largest integer n such that $\mathfrak{U}$ contains n+1 sets with non-empty intersection. Let $n \geq -1$ be an integer. Then a normal space X has dimension $\leq n$ or $\dim X \leq n$ if every finite, open

covering of X has a finite, open refinement of order $\leq n$. X has dimension n or dim X = n if dim X $\leq$ n and the inequality dim X $\leq n - 1$ does not hold. X has infinite dimension or dim X = ∞ if the inequality dim X $\leq$ n does not hold for any n. It follows from this definition that dim X = -1 if and only if $X = \phi$. Another characterization of the dimension function is well established. For all integers $n \geq 0$ let S^n denote the n-dimensional sphere $\{ (x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x_i)^2 = 1 \}$. Then a non-empty normal space X has dimension $\leq n$ ($n \geq 0$) if and only if for any closed subset Y of X and any map ϕ: Y $\to S^n$ there exists a continuous extension ψ: X $\to S^n$ such that $\psi|Y = \phi$.

Recall that for each integer $n \geq 1$ the product $\mathbb{I}^n$ has dimension n and for any cardinal $\tau \geq 1$ the space $\mathbb{D}^\tau$ has dimension 0. If Y is any closed subspace of X, then the inequality dim Y $\leq$ dim X holds. For a normal space X the following statements are equivalent.

(i) X is zero-dimensional.

(ii) Every finite, open covering of X has a pairwise disjoint, finite, clopen refinement.

(iii) Each closed subset of X has a neighbourhood base of clopen sets.

II. κ-metrizable spaces, Milutin spaces, Dugundji spaces and absolute extensors

13. κ-metrizability

A space X is called κ-metrizable, if there exists a mapping assigning to each pair (x,C) a non-negative real number d(x,C),

where $x \in X$ and C is a regular closed subset of X, satisfying the following properties:

(i) $d(x,C) = 0$ if and only if $x \in C$.

(ii) If C, D are regular closed subsets of X with $C \subset D$, then $d(x,C) \geq d(x,D)$ for all $x \in X$.

(iii) $d(\cdot,C)$ is a continuous function on X.

(iv) $d(x, \mathrm{cl}(\bigcup_{\alpha<\tau} C_\alpha)) = \inf_{\alpha<\tau} d(x,C_\alpha)$ for any non-decreasing totally ordered system $(C_\alpha)_{\alpha<\tau}$ of regular closed subsets C_α of X and any $x \in X$.

Metrizable spaces and locally compact groups are examples of κ-metrizable spaces. It follows from (i) and (iii) above that κ-metrizable spaces satisfy the B.S.P.

14. Extension and averaging operators

Let X, Y be any spaces. We agree to call a linear map $u: C^b(X) \to C^b(Y)$ an operator. The operator u is multiplicative if $u(f \cdot g) = u(f) \cdot u(g)$ holds for all $f,g \in C^b(X)$. The operator is regular if it is positive (i.e. $f \geq 0$ implies $u(f) \geq 0$ for all $f \in C^b(X)$) and maps 1_X into 1_Y. For any map $\theta: X \to Y$ we denote by θ° the induced multiplicative operator of $C^b(Y)$ into $C^b(X)$ given by $\theta^\circ(f) := f \circ \theta$ for all $f \in C^b(Y)$.

Let $u: C^b(X) \to C^b(Y)$ be an operator such that $\theta^\circ = \theta^\circ \circ u \circ \theta^\circ$. In the case that θ is an embedding (resp. surjection) u is called an extension operator or e.o. (resp. averaging operator or a.o.) for θ. If, additionally, u is regular, it is called a regular extension operator or r.e.o. (resp. regular averaging

operator or r.a.o.) for θ. We remark that in the standard literature the terms e.o. and a.o. are only defined for compact spaces X, Y.

15. Probability measures

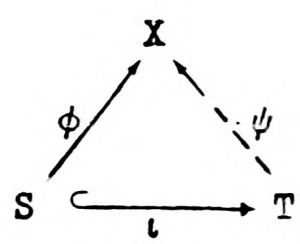
For a compact space X we denote by M(X) the dual of C(X) equipped with the $\sigma(M(X),C(X))$-topology or weak-star topology. M(X) can be identified with the space of all regular signed Borel measures on X. P(X) denotes the compact, convex set of all probability measures on X, $\{ \mu \in M(X) : \mu(X) = 1, \mu \geq 0 \}$, equipped with the induced topology. Note that a base of open sets for P(X) is given by sets of the form $\{ \mu \in P(X) : \mu(U_i) > r_i, i = 1,\ldots,m \}$ where m is any integer, U_i is an open subset of X and r_i is a positive real number ($i = 1,\ldots,m$). If for $x \in X$ the point measure at x is denoted by δ_x, then the mapping of X into P(X) that assigns to each $x \in X$ the probability measure δ_x is an embedding. For each $\mu \in P(X)$ the support of μ or $\mathrm{supp}(\mu)$ is defined to be the smallest closed subset Y of X such that $\mu(Y) = 1$. If Y is a closed subset of a compact space X we identify P(Y) with the set of probability measures on X which are supported by Y (i.e. the support is contained in Y).

Let X, Y be compact spaces and $\theta: X \to Y$ an embedding (resp. surjection). Then θ allows a r.e.o. (resp. r.a.o.) if there exists a map of Y into P(X) taking $y \in Y$ into a probability measure $\mu_y \in P(X)$ such that the support of μ_y is contained in $\theta^{-1}(y)$ for all $y \in \theta(X)$. Conversely, if $u: C(X) \to C(Y)$ is a r.e.o. (resp. r.a.o.) for the embedding (resp. surjection) θ, then the map of Y into P(X) carrying $y \in Y$ into $\delta_y \circ u$ has the property that the support of $\delta_y \circ u$ is contained in $\theta^{-1}(y)$ for all $y \in \theta(X)$. We mention another charac-

terization of maps allowing a r.e.o. (resp. r.a.o.). Let X, Y be compact spaces and θ: X → Y a map. We denote by P(θ)the induced map of P(X) into P(Y) given by $P(\theta)(\mu) := \mu \circ \theta^{\circ}$ for all $\mu \in P(X)$ or equivalently $P(\theta)(\mu)(B) = \mu(\theta^{-1}(B))$ for all Borel sets B in Y. If θ is an embedding (resp. surjection) then θ allows a r.e.o. (resp. r.a.o.) if and only if the induced map P(θ) of probability measures is a coretraction (resp. retraction).

16. Milutin spaces, Dugundji spaces and absolute extensors

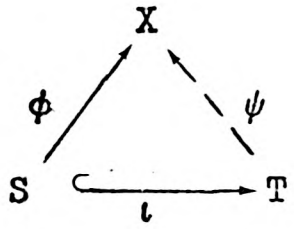
A compact space X is a Milutin space (resp. almost Milutin space) if there exists a surjection $\mathbb{D}^A \twoheadrightarrow X$ allowing a r.a.o. (resp. a.o.). A Dugundji space (resp. almost Dugundji space) is a compact space X such that any embedding of X into a compact space allows a r.e.o. (resp. e.o.). Equivalently, X is a Dugundji space (resp. almost Dugundji space) if and only if there is an embedding of X into some $\mathbb{I}^A$ allowing a r.e.o. (resp. e.o.). For Dugundji spaces the following purely topological characterization is known. A compact space X is a Dugundji space if and only if X is an absolute extensor for zero-dimensional spaces or AE(0-dim),i.e. given any embedding ι: S → T of compact, zero-dimensional spaces S, T and any map ϕ: S → X then there exists a map ψ: T → X such that the diagram



commutes, $\psi \circ \iota = \phi$.

We recall the following inclusions. A Dugundji space is a Milutin space and a Milutin space is κ-metrizable. For compact spaces of weight less than ω_2 the three terms κ-metrizable and compact, Milutin space and Dugundji space coincide. $\exp_2(\mathbb{D}^{\omega_2})$ is a Milutin space, but not a Dugundji space and $\exp(\mathbb{D}^{\omega_2})$ is a compact κ-metrizable space which is not dyadic and thus in particular not a Milutin space. Each of the three classes of compact spaces, namely compact κ-metrizable spaces, Milutin spaces and Dugundji spaces, contains all compact metrizable spaces and is closed under the taking of products and retracts.

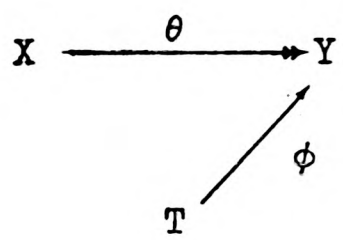
Finally we recall that a compact space X is an absolute extensor or AE if for any embedding ι: S → T of compact spaces S, T and any map ϕ: S → X there exists a map ψ: T → X such that the diagram



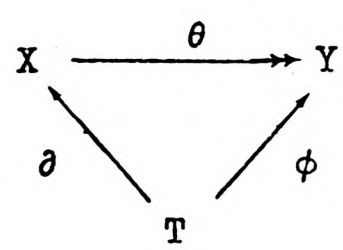
commutes, $\psi \circ \iota = \phi$. X is an AE if and only if X is an absolute retract or AR, i.e. any embedding of X into a compact space is a coretraction. The class of AE's contains $\mathbb{I}$ and is closed under the taking of arbitrary products and retracts.

§1 The zero-dimensional lifting property and the existence of a regular averaging operator

Let us consider the following diagram



where X, Y, T are spaces, θ is a surjection and ϕ any map. A continuous lifting of ϕ with respect to θ (or just a continuous lifting) is a map ∂: T → X which completes the diagram above in a commutative fashion, i.e.



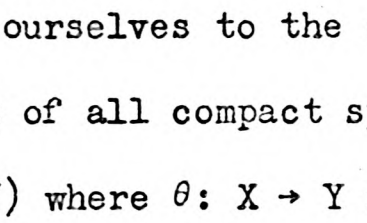
commutes, $\theta \circ \partial = \phi$.

The problem of whether or not there exists such a continuous lifting is in general very difficult to answer. However, because of its importance in various branches of mathematics it has been studied intensively, particularly in algebraic topology where initial interest lay in lifting topological objects like paths and, more generally, simplices.

In order to tackle the general lifting problem one is basically offered two ways of looking at it. Suppose we are given a class $\mathcal{N}$ of triples (X,θ,Y) where X, Y are spaces and θ: X → Y is a surjection. One might ask what class $\mathcal{T}$ of spaces has the property that whenever one is given an element (X,θ,Y) of $\mathcal{N}$, an

element T of $\mathcal{T}$ and a map ϕ: T → Y, then there exists a map ∂: T → X such that $\theta \circ \partial = \phi$ holds. Alternatively we might want to characterize a class $\mathcal{N}$ of triples such that for a given class $\mathcal{T}$ of spaces the lifting problem always has a continuous solution.

We agree to call a pair of classes $(\mathcal{N},\mathcal{T})$ of the above mentioned type a lifting pair if for any choice of elements of $\mathcal{N}$ and $\mathcal{T}$ a continuous lifting always exists, i.e. given a space T of $\mathcal{T}$, a surjection θ: X → Y where (X,θ,Y) belongs to $\mathcal{N}$, and a map ϕ: T → Y, then there exists a map ∂: T → X such that



commutes.

Let us restrict ourselves to the case where all spaces involved belong to the class $\mathcal{C}$ of all compact spaces. Let $\mathcal{M}$ denote the class of all triples (X,θ,Y) where θ: X → Y is a surjection of compact spaces X, Y. Then the biggest subclass $\mathcal{E}$ of $\mathcal{C}$ such that $(\mathcal{M},\mathcal{E})$ is a lifting pair is known to be the class of extremally disconnected compact spaces. Any other class $\mathcal{N}$ of triples is a subclass of $\mathcal{M}$ and so, if we are given a lifting pair $(\mathcal{N},\mathcal{T})$ we may always choose $\mathcal{T}$ to contain the class $\mathcal{E}$ of extremally disconnected compact spaces.

We can easily give a characterization of the other end of the spectrum of lifting pairs $(\mathcal{N},\mathcal{T})$. The biggest sublass $\mathcal{R}$ of $\mathcal{M}$ such that $(\mathcal{R},\mathcal{C})$ is a lifting pair is exactly the class of triples (X,θ,Y) where θ: X → Y is a retraction of compact spaces X, Y. Thus for any lifting pair $(\mathcal{N},\mathcal{T})$ we may always assume that $\mathcal{N}$ contains $\mathcal{R}$.

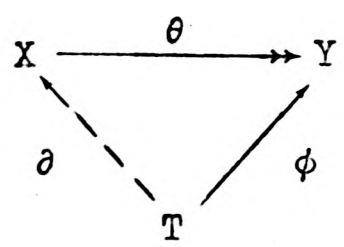
Finally we can mention another non-trivial lifting pair. Let

$\mathcal{O}$ consist of all triples (X, θ, Y) where X, Y are compact metrizable spaces and θ: $X \to Y$ is an open surjection and let $\mathcal{C}_o$ be the class of all zero-dimensional compact spaces. The fact that $(\mathcal{O}, \mathcal{C}_o)$ is a lifting pair follows from a selection theorem of E.Michael (see lemma 1.9) and the observation that θ is an open map if and only if θ^{-1} is lower semicontinuous (see remark 1.12 for more details).

In this section we shall prove that $(\mathcal{A}, \mathcal{C}_o)$ is a lifting pair, where $\mathcal{A}$ denotes the class of all triples (X, θ, Y) such that X, Y are compact metrizable spaces and θ: $X \to Y$ is a surjection allowing a r.a.o. We shall see that the class $\mathcal{O}$ is strictly contained in $\mathcal{A}$ and furthermore that $\mathcal{C}_o$ is indeed the biggest sublass of $\mathcal{C}$ such that $(\mathcal{A}, \mathcal{C}_o)$ is a lifting pair.

Let us start the discussion by giving the obvious definition in this context and making some general observations which are partly valid for any sublass of $\mathcal{C}$ and partly restricted to the class $\mathcal{C}_o$ of zero-dimensional compact spaces.

1.1. DEFINITION. Let X, Y be compact spaces and θ: $X \to Y$ a surjection. θ satisfies the <u>zero-dimensional lifting property</u> or <u>z.d.l.p.</u> if for any zero-dimensional compact space T and any map ϕ: $T \to Y$ there exists a map ∂: $T \to X$ such that the diagram



commutes, i.e. $\theta \circ \partial = \phi$.

Obviously, if θ is a homeomorphism or Y is a finite space,

then θ satisfies the z.d.l.p.

We shall now introduce two notions for surjections which will be used on frequent occasions throughout this thesis.

Let us agree to call a property of a surjection $\theta\colon X \to Y$ of spaces X, Y hereditary if for any closed subset T of Y the restriction map $\theta|\theta^{-1}(T)\colon \theta^{-1}(T) \to T$ has the same property. It is easy to see that the z.d.l.p. is hereditary.

A property of a surjection $\theta\colon X \to Y$ of compact spaces X, Y is called extensive if for any embedding $\varepsilon\colon X \to Z$ of X into a compact space Z and any surjection $\psi\colon Z \to Y$ with $\psi\circ\varepsilon = \theta$ the map ψ has the same property as θ. Evidently the z.d.l.p. is an extensive property.

Let $\theta\colon X \to Y$ and $\psi\colon Y \to Z$ be surjections of compact spaces X, Y, Z. Then if θ and ψ satisfy the z.d.l.p. so does the composition $\psi\circ\theta$. On the other hand, if the composition $\psi\circ\theta$ satisfies the z.d.l.p. so does ψ.

Clearly, any retraction $\theta\colon X \to Y$ of compact spaces X and Y has the z.d.l.p. and hence in particular any projection of a product of compact spaces onto one of its factors has the z.d.l.p.

Lifting maps coordinatewise we easily verify that the z.d.l.p. is stable under arbitrary products. Hence the following holds.

1.2. PROPOSITION. For some index set A let $\theta_a\colon X_a \to Y_a$ ($a \in A$) be a family of surjections satisfying the z.d.l.p. Then the product map

$$(\theta_a)_{a \in A}\colon \prod_{a \in A} X_a \longrightarrow\!\!\!\!\!\rightarrow \prod_{a \in A} Y_a$$

satisfies the z.d.l.p.

Using a fairly straightforward transfinite induction argument

we now prove that the z.d.l.p. is stable under the taking of continuous inverse limits, if we assume the linking maps to satisfy the z.d.l.p.

1.3. PROPOSITION. Let $(X_\alpha, p_{\alpha\beta})_{\alpha\leq\beta<\tau}$ be a continuous inverse system of compact spaces X_α ($\alpha<\tau$) and linking maps $p_{\alpha,\alpha+1}: X_{\alpha+1} \to X_\alpha$ satisfying the z.d.l.p. for all $\alpha < \tau$. Let X denote the inverse limit space of the inverse system and $p_\alpha: X \to X_\alpha$ the limit map for $\alpha < \tau$. Then p_α satisfies the z.d.l.p. for all $\alpha < \tau$.

Proof. For convenience of notation we shall show that p_1 satisfies the z.d.l.p. The proof for any other limit map p_α ($\alpha\geq 1$) is completely analogous. So let T be a zero-dimensional compact space and $\phi: T \to X_1$ a map. By transfinite induction we shall construct a system of maps $\partial_\alpha: T \to X_\alpha$ for all $\alpha < \tau$ such that $p_{\alpha\beta}\circ\partial_\beta = \partial_\alpha$ holds for all $\alpha \leq \beta < \tau$. Then by the property of the inverse limit space there exists a map $\partial: T \to X$ with $p_\alpha\circ\partial = \partial_\alpha$ for all $\alpha<\tau$.

We start the induction by putting $\partial_1 := \phi$. Suppose we had defined maps $\partial_\alpha: T \to X_\alpha$ for all $\alpha<\gamma<\tau$ with $p_{\alpha\beta}\circ\partial_\beta = \partial_\alpha$ for $\alpha\leq\beta<\gamma$. If γ is a limit ordinal then by the continuity of the inverse system there exists a map $\partial_\gamma: T \to X_\gamma$ such that $p_{\alpha\gamma}\circ\partial_\gamma = \partial_\alpha$ for all $\alpha\leq\gamma$. If $\gamma = \alpha+1$ is a non-limit ordinal the z.d.l.p. of $p_{\alpha,\alpha+1}$ assures us the existence of a map $\partial_{\alpha+1}: T \to X_{\alpha+1}$ such that $p_{\alpha,\alpha+1}\circ\partial_{\alpha+1} = \partial_\alpha$ and hence the induction is complete. Since ∂_1 and ϕ coincide we conclude that $p_1\circ\partial = \phi$ holds and the proof is finished. □

1.4. REMARK. We notice that Pełczyński proved a statement analogous to proposition 1.2 for surjections allowing a r.a.o. ([28], Proposition 4.7). Moreover, it is not hard to see that a result corresponding to proposition 1.3 holds for maps allowing

a r.a.o. One has just to observe that for any fixed $\alpha < \tau$ continuous functions of the form $f \circ p_\beta$ with $f \in C(X_\beta)$ and $\alpha \leq \beta < \tau$ establish a dense subalgebra of $C(X)$ and so it is enough to define a r.a.o. for $p_\alpha : X \to X_\alpha$ on this subalgebra.

Pełczyński showed in lemma 3.6 ("Localization Principle") of his monograph [28] that a surjection allows a r.a.o. if it has this property locally. As the following proposition indicates a corresponding result is valid for maps satisfying the z.d.l.p.

1.5. PROPOSITION. Let $\theta : X \to Y$ be a surjection of compact spaces X, Y. Let $(Y_a)_{a \in A}$ be a collection of closed subsets of Y such that the union of the interiors of the sets Y_a ($a \in A$) contains Y. Let $(X_a)_{a \in A}$ be a collection of closed subsets of X with $\theta(X_a) = Y_a$ for all $a \in A$. If $\theta_a : X_a \to Y_a$ denotes the restriction of θ to X_a and θ_a satisfies the z.d.l.p. for all $a \in A$, then θ satisfies the z.d.l.p.

Proof. Let T be a zero-dimensional compact space and $\phi : T \to Y$ a map. We shall construct a map $\partial : T \to X$ such that $\theta \circ \partial = \phi$.

By a compactness argument there exists a finite subset F of A such that $\cup \{ \operatorname{int}(Y_a) : a \in F \} = Y$ and the continuity of ϕ implies that $(\phi^{-1}(\operatorname{int}(Y_a)))_{a \in F}$ is a finite open covering of T. Since T is zero-dimensional there exists a finite refinement $(T_e)_{e \in E}$ of the covering $(\phi^{-1}(\operatorname{int}(Y_a)))_{a \in F}$ consisting of clopen, pairwise disjoint subsets T_e of T ($e \in E$). For each $e \in E$ we pick an element $a(e) \in F$ such that T_e is contained in $\phi^{-1}(\operatorname{int}(Y_{a(e)}))$ and hence the maps $\phi_e : T_e \to Y_{a(e)}$, $\phi_e := \phi | T_e$ are well-defined for all $e \in E$. By hypothesis there exist continuous liftings $\partial_e : T_e \to X_{a(e)}$ such

that $\theta_{a(e)} \circ \partial_e = \phi_e$ for all $e \in E$. We well-define $\partial\colon T \to X$ by setting $\partial|T_e := \partial_e$ ($e \in E$) and notice that the pairwise disjointness of the clopen cover $(T_e)_{e \in E}$ of T yields the continuity of ∂. Obviously $\theta \circ \partial = \phi$ holds by construction. □

As the following example reveals, even in the case of metrizable compact spaces X, Y and a two-element index set A we cannot weaken the covering condition in proposition 1.5 by just requiring $(Y_a)_{a \in A}$ to be a covering of Y.

1.6. EXAMPLE. Let $X = X_1 \cup X_2$ be the disjoint union of two copies of the Alexandroff one-point compactification of the non-negative integers $\mathbb{N}$ with x_1 (resp. x_2) as point at infinity. Let Y be the quotient space of X obtained by identifying x_1 and x_2, $\theta\colon X \to Y$ the canonical quotient map, and $\theta_i\colon X_i \to \theta(X_i) =: Y_i$ ($i = 1,2$) the restriction of θ to X_i. Then θ_i ($i = 1,2$) is the identity map and θ is not a retraction.

Proof. Suppose there exists a continuous map $\sigma\colon Y \to X$ such that $\theta \circ \sigma = id_Y$ holds. Without loss of generality we may assume $\sigma(\theta(x_1)) = x_1$. Let $(y_n)_{n<\omega}$ be a sequence in $X_2 \setminus \{x_2\}$ converging to x_2. Then by continuity of θ and σ we infer that $(\sigma \circ \theta(y_n))_{n<\omega}$ converges to $\sigma \circ \theta(x_2) = \sigma \circ \theta(x_1) = x_1$. But $\sigma \circ \theta(y_n)$ is y_n for all $n < \omega$ and so $(\sigma \circ \theta(y_n))_{n<\omega}$ has x_2 as its unique limit point. This contradiction finishes the proof. □

We notice that the map θ in example 1.6 does not allow a r.a.o. Suppose $u\colon C(X) \to C(Y)$ is a r.a.o. for θ. Then for $y \in Y$ the probability measure $\delta_y \circ u$ is necessarily of the form δ_y if $y \neq \theta(x_1)$ and $t \cdot \delta_{x_1} + (1-t) \cdot \delta_{x_2}$ with some $t \in \mathbb{I}$ if $y = \theta(x_1)$. Letting $(y_n)_{n<\omega}$

be a sequence as in the proof above and $1_{X_2} \in C(X)$ we imply that $1 = 1_{X_2}(y_n) = (\delta_{y_n} \circ u)(1_{X_2})$ converges to $(t\cdot\delta_{x_1} + (1-t)\cdot\delta_{x_2})(1_{X_2})$ $= 1-t$ as n tends to ω and hence we get $t = 0$. Using a corresponding sequence in $X_1 \setminus \{x_1\}$ we infer $t = 1$ and this contradiction completes the argument. Moreover, θ is not an open map, since the image of the open set X_1 is not open in Y.

So far all the properties we mentioned to hold for surjections satisfying the z.d.l.p. are well-known to be also valid for onto maps allowing a r.a.o. Therefore it seems natural to ask whether or not these two notions are equivalent. As a result of this section we shall see that they are indeed equivalent for compact metrizable spaces, but in general this equivalence does not hold any more for compact spaces of uncountable weight. However, as a first step towards a solution we shall prove that a surjection satisfying the z.d.l.p. necessarily allows a r.a.o. As a crucial tool in the verification of this implication we shall need a statement which turns out to be equivalent to Milutin's lemma. Milutin's lemma states that the unit interval is the continuous image of the Cantor set by a map allowing a regular averaging operator. Since there exists a very short and elegant proof of it and one of its equivalent formulations will be referred to on several occasions throughout this thesis, we choose to give a brief outline of the proof for completeness' sake. It relies on an inverse limit argument which is based on an idea of Ditor and can be found in more detail in Bade's book ([3], Theorem 4.13).

1.7. LEMMA. The following equivalent statements hold.

(i) Every compact space is the continuous image of a zero-

dimensional compact space of the same weight by a map allowing a r.a.o.

(ii) Every compact metrizable space is the continuous image of $\mathbb{D}^\omega$ by a map allowing a r.a.o.

(iii) $\mathbb{I}$ is the continuous image of $\mathbb{D}^\omega$ by a map allowing a r.a.o.

Proof. Let us first show the equivalence of the statements (i), (ii) and (iii).

To infer that (i) implies (ii) it is enough to recall that every zero-dimensional compact metrizable space is a retract of $\mathbb{D}^\omega$ (cf. [36], Theorem 8.3.4). Obviously (ii) implies (iii).

Now suppose that (iii) holds; in order to verify (i) we may assume that we are given a closed, infinite subset of $\mathbb{I}^A$ where the cardinality of A coincides with the weight of X. By hypothesis there exists a surjection θ: $\mathbb{D}^\omega \to \mathbb{I}$ allowing a r.a.o. and hence the product map $(\theta_a)_{a \in A}$: $(\mathbb{D}^\omega)^A \to \mathbb{I}^A$ ($\theta_a := \theta$ for all $a \in A$) allows a r.a.o. (see remark 1.4). Letting T be the inverse image of X by this product map and ψ: T → X its restriction map we infer that ψ allows a r.a.o. since admitting a r.a.o. is a hereditary property. Clearly T is zero-dimensional and compact and the weight of T and the cardinality of A coincide.

Finally we give a sketch of the proof of (iii). For each integer $n \geq 0$ and each element a of $\{0,1,2\}^n$ (for convenience of notation we put here $\{0,1,2\}^0 := 0$) we may choose closed subintervals K^n_a of $\mathbb{I}$ such that

(a) $K^0_0 = \mathbb{I}$,

(b) $K^{n+1}_{(a,i)}$ is a closed subinterval of K^n_a for i = 0,1,2 and the union of the interiors (relative to K^n_a) of $K^{n+1}_{(a,i)}$ is a covering of K^n_a,

(c) the length of the interval K_a^n is $\leq 2^{-n}$ for all $a \in \{0,1,2\}^n$.

For $n \geq 0$ let T_n be the disjoint union of the sets K_a^n where a ranges over all elements of $\{0,1,2\}^n$ and we define surjections $\theta_{n,n+1}$: $T_{n+1} \to T_n$ by putting $\theta_{n,n+1}(t) := t$ for all $t \in T_{n+1}$ and for $m \leq n < \omega$ $\theta_{mn} := \theta_{m,m+1} \circ \ldots \circ \theta_{n-1,n}$. It is a consequence of Pełczyński's localization principle ([28], Lemma 3.6) and condition (b) that $\theta_{n,n+1}$ allows a r.a.o. Hence by remark 1.4 we infer the existence of a r.a.o. for the limit map θ_0: $T \to T_0$ where T denotes the inverse limit space of the inverse system $(T_m, \theta_{mn})_{m \leq n < \omega}$. By (a) T_0 coincides with $\mathbb{I}$ and using condition (c) one checks that any two distinct points of the inverse limit space T can be seperated by clopen sets. Therefore T is a zero-dimensional compact metrizable space and any such space is known to be a retract of $\mathbb{D}^\omega$ ([36] Theorem 8.3.4). This completes the proof of (iii). □

Sets K_a^n as required in the proof above can easily be obtained by putting $K_0^0 := [0,1]$, $K_{00}^1 := [0,2^{-1}]$, $K_{01}^1 := [2^{-2}, 3\cdot 2^{-2}]$, $K_{02}^1 := [2^{-1},1]$, $K_{000}^2 := [0,2^{-2}]$, $K_{001}^2 := [2^{-3}, 3\cdot 2^{-3}]$,... etc, where as usual the symbol $[\cdot,\cdot]$ denotes a closed interval of real numbers.

We recall that statement (i) of lemma 1.7 was first mentioned by Ditor in his thesis.([5] Proposition 3.2). However, our proof here is much simpler and independent of his. The equivalence of the statements (i) and (iii) seems to have remained unnoticed.

1.8. PROPOSITION. Let θ: $X \to Y$ be a surjection of compact spaces X, Y satisfying the z.d.l.p. Then θ allows a r.a.o.

Proof. By lemma 1.7 (i) there exist a zero-dimensional compact space T and a surjection ϕ: $T \to Y$ allowing a r.a.o.

$u: C(T) \to C(Y)$. Using the assumption on θ we find a map $\partial: T \to X$ with $\theta \circ \partial = \phi$. An easy verification shows that the linear operator $v: C(X) \to C(Y)$, given by $v(f) := u(f \circ \partial)$ for all $f \in C(X)$, is indeed a r.a.o. for θ. □

In order to discuss conditions under which a converse of proposition 1.8 holds we shall start to consider the case where X (and thus Y) is a metrizable compact space. For the restriction to metrizability allows us to apply a very powerful selection theorem of E.Michael which also plays an important role in R.Haydon's characterization of Dugundji spaces as absolute extensors for zero-dimensional spaces. We shall see later (cf. proposition 1.19) that in essence the metrizable case is the best possible to let the converse of proposition 1.8 hold. For convenience of the reader we shall recall from Michael's paper ([23], Theorem 2) the following lemma.

1.9. LEMMA. Let T be a paracompact, zero-dimensional space and X a complete metric space. Then every lower semicontinuous set-valued mapping from T into the closed, non-empty subsets of X admits a continuous selection.

Finally we shall rely on the following result which is proposition 4.1 of Ditor's thesis [5] and the proof of which we shall give as it is relatively short.

1.10. LEMMA. Let X be a compact space. Then the set-valued mapping from P(X) into exp(X) that takes each probability measure into its support is lower semicontinuous.

Proof. Let us assume the contrary. Then there exist an

open subset U of X and a net (μ_α) of probability measures converging to a probability measure μ such that $\mathrm{supp}(\mu_\alpha) \cap U = \phi$ and $\mathrm{supp}(\mu) \cap U \neq \phi$. Then by regularity of the measure μ we find a compact set K in U with $\mu(K) > 0$. Let $f \in C(X)$ be a function of X into $\mathbb{I}$ such that $f|K = 1$ and $f|X \setminus U = 0$ which exists by normality of X. This yields $\mu(f) \geq \mu(K) > 0$ and $\mu_\alpha(f) = 0$. But this is impossible, since by assumption the net (μ_α) converges to μ, and hence in particular the net of real numbers $(\mu_\alpha(f))$ converges to $\mu(f)$. □

With the preceding two lemmas we have paved the way for a purely topological characterization of surjections of compact metrizable spaces which admit a r.a.o.

1.11. THEOREM. Let θ: X → Y be a continuous surjection of compact metrizable spaces X, Y. Then θ allows a r.a.o. if and only if θ satisfies the z.d.l.p.

Proof. The sufficiency follows from proposition 1.8. To prove the necessity let T be a compact zero-dimensional space and ϕ: T → Y a map. By hypothesis there is a r.a.o. u: C(X) → C(Y) for θ and hence the mapping that assigns to each $t \in T$ the probability measure $\delta_{\phi(t)} \circ u$ is continuous. Moreover, the support of $\delta_{\phi(t)} \circ u$ is contained in the fibre $\theta^{-1}(\phi(t))$ for all $t \in T$. By lemma 1.10 we deduce that the set-valued mapping Φ taking each $t \in T$ into the closed, non-empty subset $\mathrm{supp}(\delta_{\phi(t)} \circ u)$ of X is lower semicontinuous. Lemma 1.9 now guarantees the existence of a continuous selection ∂: T → X for Φ which in fact is a lifting for ϕ with respect to θ, since the inclusion $\partial(t) \in \Phi(t) \subset \theta^{-1}(\phi(t))$ holds for all $t \in T$. □

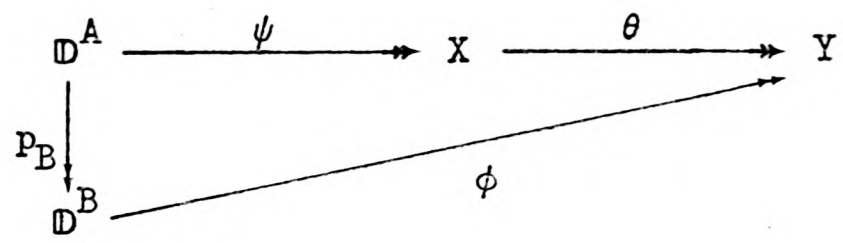
An analysis of the preceding proof exhibits that the existence of a r.a.o. for θ was basically used to define a lower semicontinuous set-valued mapping $Y \to \exp(X)$ assigning to each element y of Y a closed, non-empty subset of X contained in the fibre $\theta^{-1}(y)$. Now, a surjection $\theta: X \to Y$ is open if and only if $\theta^{-1}: Y \to \exp(X)$, $y \mapsto \theta^{-1}(y)$, is a lower semicontinuous mapping and so we can mimic the proof above to show the following.

1.12. REMARK. Let $\theta: X \to Y$ be an open surjection of compact metrizable spaces X, Y. Then θ satisfies the z.d.l.p. (or, equivalently, allows a r.a.o.). Moreover, there exists a non-open surjection of compact metrizable spaces allowing a r.a.o. (e.g. the map in lemma 1.7 (iii) is not open, for otherwise $\mathbb{I}$ would be zero-dimensional).

Theorem 1.11 gives a complete answer to a problem posed by Pełczyński in his monograph ([28], Problem 3, p. 65). It allows, however, a further generalization as indicated by the following corollary.

1.13. COROLLARY. Let X be a Milutin space, Y a compact metrizable space and $\theta: X \to Y$ a continuous surjection. Then θ allows a r.a.o. if and only if θ satisfies the z.d.l.p.

Proof. Suppose θ allows a r.a.o. and let $\psi: \mathbb{D}^A \to X$ be a surjection admitting a r.a.o. The map $\theta \circ \psi$ has metrizable image Y and therefore there exists a countable subset B of A such $\theta \circ \psi$ factors through p_B, where $p_B: \mathbb{D}^A \to \mathbb{D}^B$ denotes the canonical projection. Hence there is a surjection $\phi: \mathbb{D}^B \to Y$ such that the diagram



commutes, $\phi \circ p_B = \theta \circ \psi$. Obviously, ϕ allows a r.a.o. since $\theta \circ \psi$ has this property. $\mathbb{D}^B$ is a compact metrizable space and therefore theorem 1.11 implies that ϕ satisfies the z.d.l.p. and consequently so does $\phi \circ p_B = \theta \circ \psi$. But then θ satisfies the z.d.l.p. too. □

We go on to give another generalization of theorem 1.11 by suppressing the metrizability condition of the spaces and instead requiring that X and Y are close together modulo a compact metrizable space. The appropriate definition in this context was introduced by Haydon in [14].

1.14. DEFINITION. Let θ: X → Y be a map of compact spaces X, Y. θ has a metrizable kernel if there exists a compact metrizable space K and an embedding ι: X → Y × K such that $\theta = p_1 \circ \iota$, where p_1: Y × K → Y is the canonical projection onto the first factor.

1.15. COROLLARY. Let θ: X → Y be a surjection of compact spaces with metrizable kernel. Then θ allows a r.a.o. if and only if θ satisfies the z.d.l.p.

Proof. If K is a metrizable kernel for θ we may without loss of generality assume that X is a closed subspace of Y × K and θ is the restriction of the canonical projection Y × K → Y to the subspace X. Let us suppose that θ allows a r.a.o. u: C(X) → C(Y) and denote by p_2: Y × K → K the projection onto the second factor. Let T be a zero-dimensional compact space and ϕ: T → Y a map.

Analogously to the argument presented in the proof of theorem 1.11 we infer that the mapping Φ that assigns to each $t \in T$ the non-empty closed subset $p_2(\mathrm{supp}(\delta_{\phi(t)} \circ u)) = \Phi(t)$ of K is lower semicontinuous and Φ admits a continuous selection $\psi\colon T \to K$. It is easily verified that the map $\partial\colon T \to X$, $\partial(t) := (\phi(t),\psi(t))$ for all $t \in T$, is a well-defined lifting for ϕ with respect to θ. □

Making use of the whole strength of Michael's selection theorem a proof which mimics arguments of the proofs of lemma 1.7 and theorem 1.11 shows that the following holds.

1.16. REMARK. Every compact space (resp. compact metrizable space) is the continuous image of a zero-dimensional compact space of the same weight (resp. of $\mathbb{D}^\omega$) by a map satisfying the lifting property for zero-dimensional paracompact spaces.

We noticed in remark 1.12 that an open surjection of compact metrizable spaces allows a r.a.o. and that there are non-open surjections of compact metrizable spaces admitting a r.a.o. Hence the proper inclusion $\mathcal{O} \subsetneq \mathcal{A}$ holds. Furthermore, theorem 1.11 exhibits that $(\mathcal{A}, \mathcal{C}_0)$ is a lifting pair. It remains to prove that $\mathcal{C}_0$ is in fact the maximal subclass of the class $\mathcal{C}$ of compact spaces with this property, i.e. we have to verify the following.

1.17. PROPOSITION. Let T be a compact space with the property that for any surjection $\theta\colon X \to Y$ of compact metrizable spaces allowing a r.a.o. and any map $\phi\colon T \to Y$ there exists a continuous lifting $\partial\colon T \to X$ with $\theta \circ \partial = \phi$. Then T is zero-dimensional.

Proof. We may assume T to be a closed subspace of some $\mathbb{I}^A$.

For each $a \in A$ denote by $p_a: \mathbb{I}^A \to \mathbb{I}$ the projection onto the a^{th}-coordinate and let $\phi_a: T \to \mathbb{I}$ be the restriction of p_a to the subspace T of $\mathbb{I}^A$. By Milutin's lemma (proposition 1.7 (iii)) there exists a surjection $\theta: \mathbb{D}^\omega \to \mathbb{I}$ allowing a r.a.o. and the hypothesis of the proposition implies the existence of a collection of maps $\partial_a: T \to \mathbb{D}^\omega$ with $\theta \circ \partial_a = \phi_a$ for all $a \in A$. We claim that the product map $\partial: T \to (\mathbb{D}^\omega)^A$, given by $\partial(t) := (\partial_a(t))_{a \in A}$ for $t \in T$, is a homeomorphic embedding. Let s,t be two distinct points of T. Then there is a coordinate $a \in A$ such that $\phi_a(t) \neq \phi_a(s)$ and consequently $\partial_a(t) \neq \partial_a(s)$ holds. Hence ∂ is an embedding and T is homeomorphic to a closed subspace of $(\mathbb{D}^\omega)^A$. □

Combining the technique of the preceding proof with lemma 1.10 yields the following characterization of zero-dimensional compact spaces.

1.18. PROPOSITION. Let T be a compact space. Then T is zero-dimensional if and only if for all compact metrizable spaces X and lower semicontinuous mappings $\Phi: T \to \exp(X)$ there exists a continuous selection for Φ.

Proof. The necessity is implied by Michael's selection theorem (cf. lemma 1.9).

To prove the sufficiency we mimic the proof of proposition 1.17. The surjection $\theta: \mathbb{D}^\omega \to \mathbb{I}$ allows a r.a.o. $u: C(\mathbb{D}^\omega) \to C(\mathbb{I})$ and hence for all $a \in A$ the mapping $\Phi_a: T \to \exp(\mathbb{D}^\omega)$, given by $\Phi_a(t) := \mathrm{supp}(\delta_{\phi_a(t)} \circ u)$ for all $t \in T$, is lower semicontinuous by lemma 1.10. Our hypothesis yields the existence of a continuous selection ∂_a for Φ_a which is a lifting for ϕ_a with re-

spect to θ, since u is a r.a.o. for θ. To finish the argument we proceed as in the proof above. □

It might be interesting to recall that metrizability of the compact space X in proposition 1.18 is not only sufficient but also necessary for the existence of continuous selections for lower semicontinuous mappings defined on zero-dimensional compact spaces T into exp(X). The necessity was recently shown in [24].

Our next aim is to show that there does not exist an obvious extension of theorem 1.11 to the non-metrizable case.

1.19. PROPOSITION. Let $(K_a)_{a \in A}$ be a collection of non-trivial, compact metrizable spaces. Let K be the product space $\Pi \{ K_a : a \in A \}$ and let θ: $K \times K \to \exp_2(K)$ denote the natural surjection that takes (x,y) into {x,y}. Then the following hold.

(i) θ allows a r.a.o.

(ii) θ satisfies the z.d.l.p. if and only if A is a countable set.

Proof. Ad (i): It is easy to check that the operator u: $C(K \times K) \to C(\exp_2(K))$, given by $uf(\{x,y\}) := \frac{1}{2}(f(x,y) + f(y,x))$ for all $\{x,y\} \in \exp_2(K)$, defines a r.a.o. for θ.

Ad (ii): If A is a countable set then the product space $K \times K$ is metrizable and hence by part (i) and theorem 1.11 θ satisfies the z.d.l.p.

To prove the necessity we assume the contrary. Then it is not difficult to see that we may suppose A to have cardinality ω_1. Since the spaces K_a are non-trivial there exists a copy of $\mathbb{D}$ in

each K_a and so $\mathbb{D}^A$ can be regarded as a subspace of K. Recalling that satisfying the z.d.l.p. is a hereditary property we infer that the natural surjection θ: $\mathbb{D}^A \times \mathbb{D}^A \to \exp_2(\mathbb{D}^A)$, $(x,y) \mapsto \{x,y\}$, satisfies the z.d.l.p. or equivalently, θ is a retraction. Consequently there exists a continuous selection

$$\sigma : \exp_2(\mathbb{D}^A) \longrightarrow \mathbb{D}^A$$

such that $\sigma(\{x,y\}) \in \{x,y\}$ for all $\{x,y\} \in \exp_2(\mathbb{D}^A)$.

For each $x \in \mathbb{D}^A$ we well-define a function

$$\sigma_x : \mathbb{D}^A \setminus \{x\} \longrightarrow \mathbb{D}$$

$$\sigma_x(y) := \begin{cases} 0 & \text{if } \sigma(\{x,y\}) = x \\ 1 & \text{if } \sigma(\{x,y\}) = y \end{cases}$$

Indeed, employing the continuity of σ we deduce that σ_x is a continuous function for all $x \in \mathbb{D}^A$. From Efimov's paper ([10], Theorem 7 and Lemma 3, p.245) we deduce that the Stone-Čech compactification of $\mathbb{D}^A \setminus \{x\}$ coincides with $\mathbb{D}^A$ and hence for each $x \in \mathbb{D}^A$ the map σ_x has a (unique) continuous extension $\beta\sigma_x$: $\mathbb{D}^A \to \mathbb{D}$. For simplicity we shall identify $\beta\sigma_x$ with σ_x. The set A has cardinality ω_1 and so $\mathbb{D}^A$ is a separable space. Let $\{x_n : n < \omega\}$ be a countable dense subset of $\mathbb{D}^A$. Each continuous function σ_{x_n} depends on a countable subset B_n of A and so the union $\cup \{ B_n : n < \omega \} =: B$ is a countable set and each σ_{x_n} ($n < \omega$) depends on coordinates in B. To continue the proof we shall verify the following.

CLAIM. For all $x \in \mathbb{D}^A$ the map σ_x depends on coordinates in B.

Suppose there exists a function σ_x which does not depend on B. Then we can find two distinct points $y, z \in \mathbb{D}^A$ different from x such that $\sigma_x(y) \neq \sigma_x(z)$ and $p_B(y) = p_B(z)$, where p_B: $\mathbb{D}^A \to \mathbb{D}^B$ denotes the canonical projection map.

We notice that for any three pairwise distinct elements $a,b,c \in \mathbb{D}^A$ the following equivalence holds:

(*) $\sigma_a(b) \neq \sigma_a(c)$ if and only if $\sigma_b(a) \neq \sigma_c(a)$.

Hence by (*) and the inequality $\sigma_x(y) \neq \sigma_x(z)$ we conclude $\sigma_y(x) \neq \sigma_z(x)$. The continuity of σ_y and σ_z at the point x and the density of the set $\{ x_n : n < \omega \}$ in $\mathbb{D}^A$ imply the existence of an element x_k ($k < \omega$) such that the inequality $\sigma_y(x_k) \neq \sigma_z(x_k)$ holds. Again applying (*) we obtain $\sigma_{x_k}(y) \neq \sigma_{x_k}(z)$. But this is a contradiction since σ_{x_k} factors through p_B and $p_B(y) = p_B(z)$. So the proof of the claim is complete.

To finish the argument of the main proof we pick any three pairwise distinct elements $x,y,z \in \mathbb{D}^A$ such that $p_B(x) = p_B(y) = p_B(z)$. Then at least two of the three numbers $\sigma_x(x)$, $\sigma_y(y)$, $\sigma_z(z) \in \mathbb{D}$ must coincide. Without loss of generality we may assume $\sigma_x(x) = \sigma_y(y)$. Since by our claim both σ_x and σ_y factor through p_B we infer $\sigma_x(y) = \sigma_x(x) = \sigma_y(y) = \sigma_y(x)$. This is a contradiction, for the equality $\sigma_x(y) = \sigma_y(x)$ implies $x = y$. □

As a consequence of proposition 1.19 we deduce that Michael's selection theorem does not hold even in the case of a two-fold continuous (not just lower semicontinuous) mapping if we suppress metrizability. Just consider the natural surjection $\mathbb{D}^{\omega_1} \times \mathbb{D}^{\omega_1} \to \exp_2(\mathbb{D}^{\omega_1})$ which does not allow a continuous selection. At the same time this surjection furnishes us with an example we alluded to in [18].

In passing we mention that we can exhibit a continuous selection for the natural surjection $\mathbb{D}^\omega \times \mathbb{D}^\omega \to \exp_2(\mathbb{D}^\omega)$ explicitly. We regard $\mathbb{D}^\omega$ as a subset of $\mathbb{I}$ and the map of $\mathbb{D}^\omega \times \mathbb{D}^\omega$ into $\mathbb{D}^\omega$ taking (x,y) into

$\min\{x,y\}$ (resp. into $\max\{x,y\}$) has the required property.

From proposition 1.19 we can deduce another interesting observation. The limit map of a system of surjections each satisfying the z.d.l.p. does not necessarily have this property. This should be contrasted with proposition 1.3.

For each $\alpha < \omega_1$ we denote by $\exp_2(p_\alpha)$ the map which is induced by the canonical projection $p_\alpha: \mathbb{D}^{\omega_1} \to \mathbb{D}^\alpha$, $\exp_2(p_\alpha)(\{x,y\}) := \{p_\alpha(x), p_\alpha(y)\}$ for all $\{x,y\} \in \exp_2(\mathbb{D}^{\omega_1})$. Obviously $\exp_2(p_\alpha)$ is a retraction and so allows a r.a.o. If $\theta: \mathbb{D}^{\omega_1} \times \mathbb{D}^{\omega_1} \to \exp_2(\mathbb{D}^{\omega_1})$ denotes the natural surjection of proposition 1.19, then $\exp_2(p_\alpha)\circ\theta$ allows a r.a.o. and hence by corollary 1.13 satisfies the z.d.l.p. But θ is the limit map of the system $(\exp_2(p_\alpha)\circ\theta)_{\alpha<\omega_1}$ and does not satisfy the z.d.l.p.

Finally we conclude this section with an observation relating the z.d.l.p. and the existence of a r.a.o. for those surjections which have an extremally disconnected compact domain space.

1.20. PROPOSITION. Let $\theta: X \to Y$ be a surjection of compact spaces X, Y. If X is an extremally disconnected space, then the following are equivalent.

(i) Y is extremally disconnected.

(ii) θ is a retraction.

(iii) θ satisfies the z.d.l.p.

(iv) θ allows a r.a.o.

Proof. Obviously, (i) and (ii) are equivalent, (ii) implies (iii) and (iii) implies (iv) by proposition 1.8. To show that (iv)

implies (i) we recall that a compact space Z is extremally disconnected if and only if C(Z) is an injective object in the category of Banach spaces and linear contractions, i.e. C(Z) is a $\mathcal{P}_1$-space (cf. [36], Theorem 24.7.1). Since by (iv) C(Y) is isometric to a norm one retract of the $\mathcal{P}_1$-space C(X) it is a $\mathcal{P}_1$-space too. □

One should see the preceding proposition 1.20 in connection with corollary 9.4 of Pełczyński's monograph [28] which states that a compact space Z is extremally disconnected if and only if every map from an arbitrary compact space onto Z admits a r.a.o.

§2 The zero-dimensional retraction property and the existence of a regular extension operator

In the previous section we were mainly concerned with surjections allowing a r.a.o. (resp. satisfying the z.d.l.p.). Here we shall give a brief introduction into the connections and differences for the corresponding injective notions of r.e.o. (resp. z.d.r.p.) the latter being defined as follows.

2.1. DEFINITION. Let $\varepsilon: X \to Y$ be a homeomorphic embedding of compact spaces X, Y. Then ε satisfies the zero-dimensional retraction property or z.d.r.p. if for any embedding $\iota: S \to T$ of zero-dimensional compact spaces S, T and any maps $\phi: S \to X$, $\psi: T \to Y$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \iota\ } & T \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow[\ \varepsilon\ ]{} & Y \end{array}$$

commutes, i.e. $\varepsilon \circ \phi = \psi \circ \iota$, there exists a map $\partial: T \to X$ with $\partial \circ \iota = \phi$.

It is clear that any coretraction ε satisfies the z.d.r.p. and moreover, if Y is a zero-dimensional space the z.d.r.p. of ε is equivalent to ε being a coretraction.

Before we go on to show that in a certain sense the z.d.r.p. relates to the existence of a r.e.o. as the z.d.l.p. does to the existence of a r.a.o. we follow the pattern of section 1 and verify some elementary observations concerning the z.d.r.p.

2.2. PROPOSITION. Let A be a set and for each $a \in A$ let $\varepsilon_a : X_a \to Y_a$ be an embedding of compact spaces X_a, Y_a satisfying the z.d.r.p. Then the product map

$$(\varepsilon_a)_{a \in A} : \prod_{a \in A} X_a \hookrightarrow \prod_{a \in A} Y_a$$

is an embedding satisfying the z.d.r.p.

Applying the z.d.r.p. coordinatewise yields a straightforward proof of this proposition which we omit to give. Employing the defining property of a zero-dimensional space one can prove that the z.d.r.p. is a local property. Hence the following proposition corresponds to the injective version of Pełczyński's localization principle which states that an embedding allows a r.e.o. if it has this property locally ([28], Lemma 3.6).

2.3. PROPOSITION. Let $\varepsilon : X \to Y$ be an embedding of compact spaces X, Y and let $(Y_a)_{a \in A}$ be a collection of closed subsets of Y such that $\varepsilon X \subset \cup\{\mathrm{int}(Y_a) : a \in A\}$. Let $X_a := \varepsilon^{-1}(Y_a \cap \varepsilon X)$ and $\varepsilon_a : X_a \to Y_a$ be the restriction of ε to X_a for all $a \in A$. If ε_a satisfies the z.d.r.p. for all $a \in A$, then ε satisfies the z.d.r.p.

Proof. Suppose we are given S, T, ι, ϕ and ψ as in definition 2.1. Since εX is a compact space there exists a finite subset F of A such that $\cup\{\mathrm{int}(Y_a) : a \in F\}$ contains εX. Let us put $U_a := \mathrm{int}(Y_a)$ for all $a \in F$ and by normality of Y we can find an open subset U_o of Y such that

$$Y \setminus \cup\{ U_a : a \in F \} \subset U_o \subset Y \setminus \varepsilon X.$$

Setting $F_o := F \cup \{o\}$ the collection $(\psi^{-1}(U_a))_{a \in F_o}$ is a finite

open covering of the zero-dimensional compact space T. Hence there is a finite clopen, pairwise disjoint refinement $(T_e)_{e \in E}$. For each $e \in E$ we pick an element $a(e) \in F_o$ such that $T_e \subset \psi^{-1}U_{a(e)}$. We set $E_1 := \{e \in E : a(e) \in F\}$ and it follows that the clopen set $\cup \{ T_e : e \in E_1 \}$ contains ιS. For each $e \in E_1$ we put $S_e := \iota^{-1}T_e$, take $\iota_e : S_e \to T_e$ as the restriction of ι to S_e and well-define restrictions $\phi_e : S_e \to X_{a(e)}$ (resp. $\psi_e : T_e \to Y_{a(e)}$) of ϕ (resp. ψ). Then the diagram

$$\begin{array}{ccc} S_e & \xrightarrow{\ \iota_e\ } & T_e \\ {\scriptstyle \phi_e}\downarrow & & \downarrow{\scriptstyle \psi_e} \\ X_{a(e)} & \xrightarrow[\ \varepsilon_e\ ]{} & Y_{a(e)} \end{array}$$

commutes for all $e \in E_1$ and by hypothesis there is a map $\partial_e : T_e \to X_{a(e)}$ such that $\partial_e \circ \iota_e = \phi_e$ ($e \in E_1$). If x_o is any point of X we well-define a map $\partial : T \to X$ by putting

$$\partial(t) := \begin{cases} \partial_e(t) & \text{if } t \in T_e \text{ for some } e \in E_1 \\ x_o & \text{otherwise} \end{cases}$$

and it is easily checked that our construction yields $\partial \circ \iota = \phi$. □

We shall now prove an injective counterpart of proposition 1.8 by showing that an embedding satisfying the z.d.r.p. necessarily admits a r.e.o. As in section 1 the crucial tool is an equivalent statement of Milutin's lemma. It allows us to employ the z.d.r.p. on a surjection admitting a r.a.o. which in turn gives rise to a r.e.o.

2.4. PROPOSITION. Let $\varepsilon : X \to Y$ be an embedding of compact spaces X, Y. Then if ε satisfies the z.d.r.p., it admits a r.e.o.

Proof. By lemma 1.7 (i) there exists a zero-dimensional compact space T and a surjection ψ: T → Y allowing a r.a.o. u: C(T) → C(Y). Let us take S := $\psi^{-1}(\varepsilon X)$, ι: S → T as the natural inclusion and ϕ: S → X is given by $\phi(s) := \varepsilon^{-1}(\psi \circ \iota(s))$ for all $s \in S$. Then we can apply the hypothesis on ε to infer the existence of a surjection ∂: T → X with $\partial \circ \iota = \phi$. A fairly elementary calculation shows that the linear operator v: C(X) → C(Y), defined by v(f):= u($f \circ \partial$) for all $f \in C(X)$, is indeed a r.e.o. for ε. □

Because of the difficulties similar to those encountered in section 1 a converse of proposition 2.4 does not hold in general. However, employing Haydon's characterization of Dugundji spaces as absolute extensors for zero-dimensional spaces a straightforward proof shows the validity of the following statement.

2.5. PROPOSITION. Let X, Y be compact spaces and ε: X → Y be an embedding allowing a r.e.o. If X or Y is a Dugundji space, then ε satisfies the z.d.r.p.

As a consequence of this result we deduce an embedding characterization of Dugundji spaces in purely topological terms.

2.6. COROLLARY. Let X be a compact space. Then the following are equivalent.

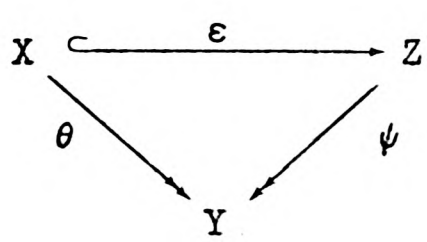
(i) X is a Dugundji space.

(ii) Every embedding of X into a compact space satisfies the z.d.r.p.

(iii) There exists an embedding of X into a Dugundji space satisfying the z.d.r.p.

We should like to point out an interesting difference between the surjective notions of z.d.l.p. and r.a.o. on the one hand and their injective counterparts z.d.r.p. and r.e.o. on the other hand. Recall that the natural surjection $\mathbb{D}^{\omega_1} \times \mathbb{D}^{\omega_1} \to \exp_2(\mathbb{D}^{\omega_1})$, $(x,y) \mapsto \{x,y\}$, does not satisfy the z.d.l.p. but allows a r.a.o. (proposition 1.19) and hence $\exp_2(\mathbb{D}^{\omega_1})$ is a Milutin space of weight ω_1 or, equivalently, a Dugundji space. As proposition 2.5 exhibits,an injective analogue of this phenomenon does not occur, i.e. if an embedding of compact spaces allows a r.e.o. and one of the spaces involved is a Dugundji space, then it always satisfies the z.d.r.p. Generally it appears that the role compact metrizable spaces play for the surjective notions is taken over by Dugundji spaces for the injective counterparts. Just compare the following two results. A surjection allowing a r.a.o. and having metrizable compact domain space satisfies the z.d.l.p.; an embedding of compact spaces allowing a r.e.o. and having a Dugundji space as range space satisfies the z.d.r.p.

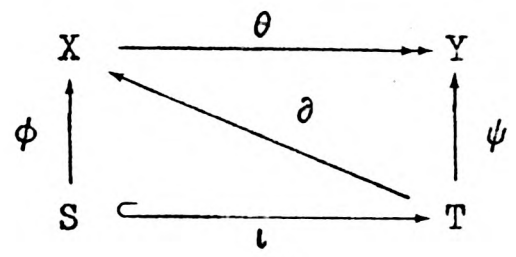
It seems that in general the problem of characterizing embeddings satisfying the z.d.r.p. is a very difficult one. However, in what follows next we shall exhibit some more sufficient conditions for an embedding to satisfy the z.d.r.p. We are interested in embeddings $\varepsilon: X \to Z$ of compact spaces X, Z with the property that there exist a compact space Y and a commutative diagram of the following kind



where θ and ψ are surjections. In particular we shall investigate the case where Z is a product space and ψ a projection map onto a factor space of the product. This will lead to a kind of injective counterpart of corollary 1.15, namely corollary 2.12, where again Dugundji spaces play the role of compact metrizable spaces in section 1.

In order to infer the z.d.r.p. for ε in terms of properties of the surjections ψ and θ in the diagram above we shall recall the notion of O-softness from [37], p.162.

2.7. DEFINITION. Let $\theta: X \to Y$ be a surjection of compact spaces X, Y. θ is called <u>O-soft</u>, if for all embeddings $\iota: S \to T$ of zero-dimensional compact spaces S, T and maps $\phi: S \to X$, $\psi: T \to Y$ with $\theta \circ \phi = \psi \circ \iota$ there exists a map $\partial: T \to X$ such that the diagram



commutes, $\phi = \partial \circ \iota$ and $\theta \circ \partial = \psi$.

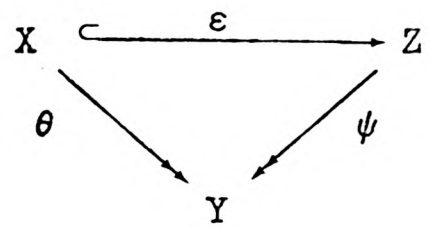
The concept of O-softness was introduced and characterized by Shchepin in his paper [37]. Obviously every O-soft map satisfies the z.d.l.p. but the converse does not even hold in the case of compact metrizable spaces. We recall that by a result of Shchepin ([37], p.163) a surjection of compact metrizable spaces is O-soft if and only if it is open and by theorem 1.11 such a surjection satisfies the z.d.l.p. if and only if it allows a r.a.o.

Remark 1.12 exhibits a non-open surjection of compact metrizable spaces allowing a r.a.o.

It follows immediately from the definition above that 0-softness is a hereditary property. But in contrast to z.d.l.p. we cannot expect it to be an extensive property in general. E.g. let X be a non-empty compact space that is not a Dugundji space or, equivalently, not an AE(0-dim) and let x be any point in X. Then the identity map $\{x\} \to \{x\}$ is 0-soft, but it does not allow a 0-soft extension $X \to \{x\}$, for otherwise X would be an AE(0-dim).

Employing the notion of 0-softness we can state a sufficient condition for certain types of embeddings to satisfy the z.d.r.p. The proof of the following result is straightforward and therefore omitted.

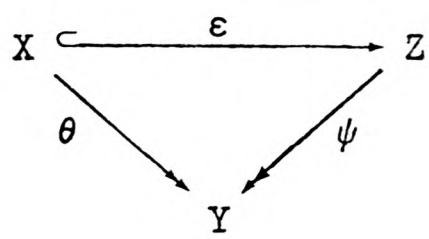
2.8. PROPOSITION. Let X, Y, Z be compact spaces and let



be a commutative diagram, where θ and ψ are surjections and ε is an embedding. If θ is 0-soft, then ε satisfies the z.d.r.p.

So it remains to characterize the 0-softness of θ in order to find sufficient conditions for ε to satisfy the z.d.r.p. To this end we develop a machinery which allows us to deduce the 0-softness of θ by requiring ψ to be 0-soft and ε to admit a special kind of r.e.o. (We notice that by propositions 2.8 and 2.4 the 0-softness of of θ always implies the existence of a r.e.o. for ε.) What we mean precisely is captured in the following definition.

2.9. DEFINITION. Let X, Y, Z be compact spaces and suppose the diagram



commutes, where ε is an embedding and θ, ψ are surjections. A r.e.o. u: C(X) → C(Z) for ε is called <u>fibre-preserving</u> (for the diagram) if u carries $\theta^{\circ}(C(Y))$ into $\psi^{\circ}(C(Y))$.

Obviously, if Y consists of a single point, then every r.e.o. is fibre-preserving. In a certain sense the terminology is justified as the following characterization shows.

2.10. PROPOSITION. Suppose we are given a diagram as in definition 2.9. Then a r.e.o. u: C(X) → C(Z) for ε is fibre-preserving if and only if the support of the probability measure $\delta_z \circ u$ is contained in the fibre $\theta^{-1}(\psi(z))$ for all $z \in Z$.

Proof. Suppose u is a fibre-preserving r.e.o. and there is an element $z \in Z$ such that the support of $\delta_z \circ u$ is not contained in $\theta^{-1}(\psi(z))$. Then by regularity of the measure there exists a compact subset K in $X \setminus \theta^{-1}(\psi(z))$ with $\delta_z \circ u(K) > 0$. Hence $\theta(K)$ is a compact subset of Y and disjoint from $\psi(z)$ and so there exists a continuous function f: Y → I such that $f(\psi(z)) = 0$, $f|\theta(K) = 1$. Consequently we arrive at the following contradiction:

$$0 < \delta_z \circ u(K) \leq \int f \circ \theta \, d(\delta_z \circ u) = u(\, f \circ \theta \,)(z) = f \circ \psi(z) = 0.$$

This proves the necessity.

To show the converse we have to verify that for $f \in C(Y)$ the

map $u(f\circ\theta)$ is constant on the fibre $\psi^{-1}(\psi(z))$ for all $z \in Z$, i.e. $u(f\circ\theta)(z) = u(f\circ\theta)(z_1)$ for all z_1 in $\psi^{-1}(\psi(z))$. But this is easy to check, since both $\delta_{z_1}\circ u$ and $\delta_z\circ u$ are supported by the same fibre $\theta^{-1}(\psi(z_1)) = \theta^{-1}(\psi(z))$ on which $f\circ\theta$ is constant. □

Suppose we are given a fibre-preserving r.e.o. as in definiton 2.9. Furthermore, we suppose that Z is the product space of Y and some $\mathbb{I}^A$ and ψ the canonical projection $Y \times \mathbb{I}^A \to Y$. Then we claim that θ is O-soft and hence by proposition 2.8 ε satisfies the z.d.r.p. We arrive at this result by the following argument. O-softness of θ in the given situation was characterized by Shchepin [37]. Theorem 7 of [37] states that a surjection $\theta: X \to Y$ is O-soft if and only if there exists a map $\phi: X \to \mathbb{I}^A$ such that the diagonal product map $\theta\Delta\phi: X \to Y \times \mathbb{I}^A$ is an embedding allowing a fibre-preserving r.e.o. Clearly our assumptions imply the existence of such a map ϕ and furthermore ε and $\theta\Delta\phi$ coincide. This verifies the claim. The following generalization of Shchepin's characterization of O-softness shows that we may substitute $\mathbb{I}^A$ by any Dugundji space in the claim above.

2.11. PROPOSITION. Let $\theta: X \to Y$ be a surjection of compact spaces X, Y. Then the following are equivalent.

(i) θ is O-soft.

(ii) There exists a map $\phi: X \to Z$ into some Dugundji space Z such that the diagonal product map $\theta\Delta\phi: X \to Y \times Z$ is an embedding allowing a fibre-preserving r.e.o. $u: C(X) \to C(Y \times Z)$.

Proof. By the before-mentioned it remains to show that (ii) implies (i). Let $\iota: Z \to \mathbb{I}^A$ be any embedding which then allows

a r.e.o. v: $C(Z) \to C(\mathbb{I}^A)$ by hypothesis. From proposition 4.7 of [28] we infer the existence of a r.e.o. w: $C(Y \times Z) \to C(Y \times \mathbb{I}^A)$ for the product map (id_Y, ι): $Y \times Z \to Y \times \mathbb{I}^A$. Investigating the proof of Pełczyński's proposition 4.7 of [28] one finds that the operator w arises from the operators $id_{C(Y)}$ and v in a natural fashion and w is fibre-preserving for the induced diagram

$$\begin{array}{ccc} Y \times Z & \xrightarrow{(id_Y, \iota)} & Y \times \mathbb{I}^A \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

where the surjections are the canonical projections. Since by (ii) u is a fibre-preserving r.e.o. for the diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta \Delta \phi} & Y \times Z \\ & \searrow^{\theta} \quad \swarrow & \\ & Y & \end{array}$$

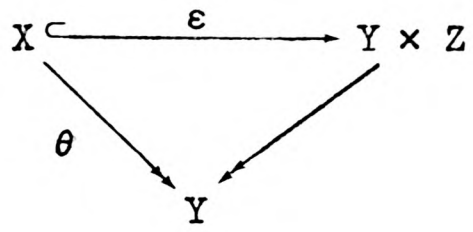
where $Y \times Z \to Y$ is the canonical projection map, the composite operator $w \circ u$: $C(X) \to C(Y \times \mathbb{I}^A)$ is a fibre-preserving r.e.o. for the the diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta \Delta (\iota \circ \phi)} & Y \times \mathbb{I}^A \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

where $Y \times \mathbb{I}^A \to Y$ is the canonical projection. An application of the above mentioned theorem 7 of.[37] completes the proof. □

Combining proposition 2.8 with the preceding one yields the following corollary and we observe that it extends one part of proposition 2.5.

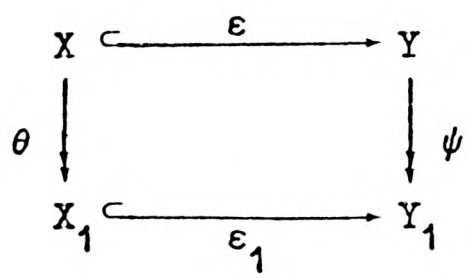
2.12. COROLLARY. Let X, Y be compact spaces, Z a Dugundji space and suppose we are given a commutative diagram



where ε is an embedding, θ a surjection and $Y \times Z \to Y$ the projection onto the first factor. Then if ε allows a fibre-preserving r.e.o., it satisfies the z.d.r.p.

Finally we want to show that in the diagram of definition 2.9 the 0-softness of ψ together with the existence of a fibre-preserving r.e.o. for ε implies the 0-softness of θ and hence by proposition 2.8 the embedding ε satisfies the z.d.r.p. A slightly generalized version of this result is proved in the following proposition.

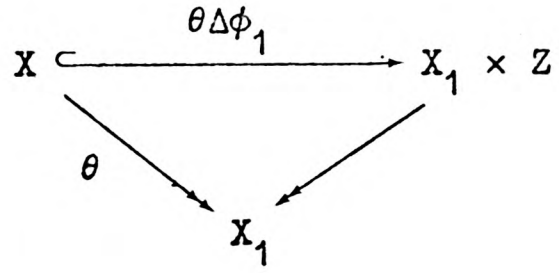
2.13. PROPOSITION. Let $\varepsilon: X \to Y$ (resp. $\varepsilon_1: X_1 \to Y_1$) be an embedding of compact spaces allowing a r.e.o. $u: C(X) \to C(Y)$ (resp. $u_1: C(X_1) \to C(Y_1)$) for ε (resp. ε_1). Let $\theta: X \to X_1$ and $\psi: Y \to Y_1$ be surjections such that the diagram



commutes. Suppose further that $u(f \circ \theta) = u_1(f) \circ \psi$ holds for all $f \in C(X_1)$. Then if ψ is 0-soft, so is θ.

Proof. For simplicity we shall identify X (resp. X_1)

with its image $\varepsilon(X)$ (resp. $\varepsilon_1(X_1)$) and θ with the restriction of ψ to X. By hypothesis and by proposition 2.11 there exist a Dugundji space Z and a map ϕ: $Y \to Z$ such that the diagonal product map $\psi\Delta\phi$: $Y \to Y_1 \times Z$ is an embedding and allows a fibre-preserving r.e.o. v: $C(Y) \to C(Y_1 \times Z)$. Let ϕ_1: $X \to Z$ be the restriction of ϕ to X. Then again by proposition 2.11 it remains to prove that the diagonal product map $\theta\Delta\phi_1$: $X \to X_1 \times Z$ is an embedding and allows a fibre-preserving r.e.o. Obviously $\theta\Delta\phi_1$ is as the restriction of the embedding $\psi\Delta\phi$ of the same type. Let $\iota: X_1 \times Z \to Y_1 \times Z$ denote the natural inclusion map. Then putting $w(f) := \iota \circ v \circ u(f)$ for all $f \in C(X)$ well-defines a r.e.o. for $\theta\Delta\phi_1$. Employing the fibre-preserving property of v and the hypothesis on the interrelationship of u and u_1 shows that w is indeed a fibre-preserving r.e.o. for the diagram



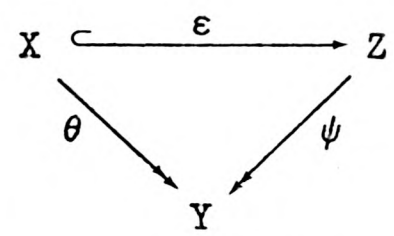
This completes the proof. □

In passing we mention that Haydon showed in lemma 2 of [14] that proposition 2.13 remains valid if we substitute "0-soft" by "open".

Proposition 2.13 yields as a corollary another sufficient condition for certain types of embeddings to satisfy the z.d.r.p. and thus furnishes us with a result we alluded to before.

2.14. COROLLARY. Let X, Y, Z be compact spaces and suppose

we are given a commutative diagram



where ε is an embedding and θ,ψ are surjections. Suppose further that ε allows a fibre-preserving r.e.o. Then if ψ is 0-soft, so is θ and hence in particular ε satisfies the z.d.r.p.

In the final part of this section we shall investigate how certain properties of the space Y are passed down to X where ε: X → Y is an embedding allowing a r.e.o. (resp. satisfying the z.d.r.p.). Pełczyński showed in proposition 5.10 of his monograph [28] that for an embedding ε: X → Y of compact spaces X, Y allowing a r.e.o. the B.S.P. of Y implies the B.S.P. of X. In fact, Pełczyński's proof is based on the following result.

2.15. LEMMA. Let ε: X → Y be an embedding of compact spaces allowing a r.e.o. If (U_o,U_1) is any pair of disjoint open subsets of X, then there exists a pair (V_o,V_1) of disjoint open subsets of Y such that $\varepsilon^{-1}(V_i)$ contains U_i for i = o,1.

2.16. PROPOSITION. Let ε: X → Y be an embedding of compact spaces X, Y allowing a r.e.o. Then if Y is extremally disconnected, so is X.

Proof. The proof is a simple application of lemma 2.15 and the fact that a compact space is extremally disconnected if and only if disjoint open sets have disjoint closures ([36], 24.2.2). □

Comparing this result with its surjective counterpart, proposition 1.20, one might conjecture that the hypotheses of proposition 2.16 are equivalent to ε being a coretraction. However, Amir and Arbel constructed in their joint paper ([1], p.304) an embedding $\varepsilon: X \to \beta\mathbb{N}$ allowing a r.e.o. that is not a coretraction. Furthermore, this shows that the converse of proposition 2.4 does not hold in general. In their example X is the maximal ideal space of the Banach algebra $L^\infty[0,1]$ and so X has uncountable density (cf. [36], 16.5.8) whereas $\beta\mathbb{N}$ is separable, consequently $\varepsilon: X \to \beta\mathbb{N}$ cannot be a coretraction. This kind of argument can be extended to embeddings satisfying the z.d.r.p. and thus furnishes us with a necessary condition for an embedding to satisfy the z.d.r.p.

2.17. PROPOSITION. Let $\varepsilon: X \to Y$ be an embedding of compact spaces X, Y satisfying the z.d.r.p. Then the density of X does not exceed the density of Y.

Proof. This statement follows immediately from the fact that each compact space is the continuous image of a zero-dimensional compact space with the same density. E.g. for each compact space one may choose its Gleason space and Gleason map (cf. [28], Definition 9.2). □

The following shows that embeddings satisfying the z.d.r.p. preserve the surjective notions of dyadic space and Milutin space. For the preservation of κ-metrizability it is enough to require the existence of a r.e.o. for the embedding.

2.18. PROPOSITION. Let $\varepsilon: X \to Y$ be an embedding of compact

spaces X, Y satisfying the z.d.r.p. Then if Y is a dyadic space (resp. Milutin space), so is X.

Proof. By hypothesis there exist a set A and a surjection $\psi\colon \mathbb{D}^A \to X$ (resp. allowing a r.a.o.). Applying definition 2.1 to the situation where one takes $T := \mathbb{D}^A$, $S := \psi^{-1}(\varepsilon X)$, $\iota\colon S \to T$ as the natural inclusion, $\phi\colon S \to X$ given by $\phi(s) := \varepsilon^{-1}(\psi \circ \iota(s))$ for all $s \in S$, we infer the existence of a surjection $\partial\colon \mathbb{D}^A \to X$ by hypothesis. If additionally ψ allows a r.a.o., so does ϕ and then ∂ too, since admitting a r.a.o. is a hereditary and extensive property. □

2.19. PROPOSITION. Let $\varepsilon\colon X \to Y$ be an embedding of compact spaces X, Y allowing a r.e.o. Then if Y is a κ-metrizable space, so is X.

Proof. Since ε allows a r.e.o. the induced map of probability measures $P(\varepsilon)\colon P(X) \to P(Y)$ is a coretraction. For the completion of the proof it remains to recall that a retract of a κ-metrizable space is κ-metrizable and that by Shchepin's functor theorem ([37], p. 165) for any compact space Z the spaces Z and P(Z) are κ-metrizable simultaneously. □

In passing we mention that one can easily prove a surjective counterpart of this proposition by applying an analogous argument. Precisely, if $\theta\colon X \to Y$ is a surjection of compact spaces X, Y allowing a r.a.o. and X is a κ-metrizable space, then so is Y.

§3 Michael spaces and Dugundji spaces

In section 1 we were basically concerned with surjections satisfying the z.d.l.p. (resp. allowing a r.a.o.) in a rather general setting. Corresponding to the definition of Milutin spaces we shall now restrict our attention to those maps with z.d.l.p. which have as domain space some generalized Cantor set. In this context we find it convenient to introduce a notation for the class of spaces which appear as images of these maps.

3.1. DEFINITION. A compact space X is called a Michael space if there exist a set A and a surjection $\mathbb{D}^A \to X$ satisfying the z.d.l.p.

It is evident from Haydon's characterization of a Dugundji space as an AE(0-dim) that for zero-dimensional compact spaces the classes of Dugundji spaces and Michael spaces coincide. It seems natural to ask for a characterization of the biggest subclass of compact spaces to which this coincidence can be extended. The first major aim of this section is to prove that this subclass is as big as we could possibly hope for, i.e. equivalence of the notions of Michael space and Dugundji space holds for all compact spaces.

Let us start our investigation of Michael spaces with some preparatory observations which turn out to be easy consequences of results of section 1. As implications of proposition 1.8, lemma 1.7 (ii) and theorem 1.11 we can mention the following.

3.2. REMARK. Every Michael space is a Milutin space

and every compact metrizable space is a Michael space.

3.3. PROPOSITION. The class of Michael spaces is the smallest class of compact spaces that contains $\mathbb{D}$ and is closed under taking arbitrary products and images by maps satisfying the z.d.l.p.

Proof. That the class of Michael spaces has the above mentioned properties follows from proposition 1.2 and that it is the smallest such class is exactly the definition 3.1. □

In passing we notice that the class of Milutin spaces allows a corresponding characterization. The class of Milutin spaces is the smallest class of compact spaces that contains $\mathbb{D}$ and is closed under taking arbitrary products and images by maps admitting a r.a.o.

3.4. COROLLARY. For any set A the product space $\mathbb{I}^A$ is a Michael space.

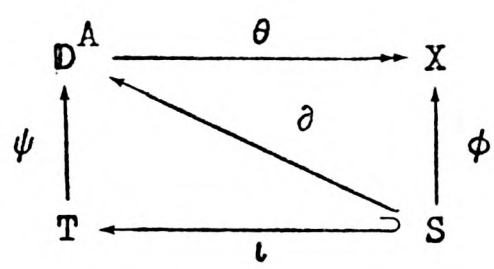
With these considerations we have prepared the ground for a proof of our surjective characterization of Dugundji spaces as Michael spaces.

3.5. THEOREM. Let X be a compact space. Then X is a Michael space if and only if X is a Dugundji space.

Proof. The proof of both implications relies on a theorem of Haydon stating that the class of Dugundji spaces coincides with the class of absolute extensors for zero-dimensional spaces.

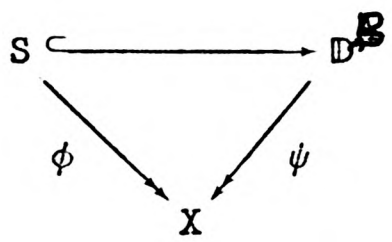
Let X be a Michael space and $\theta\colon \mathbb{D}^A \to X$ be a surjection satisfying the z.d.l.p. Suppose we are given an embedding $\iota\colon S \to T$

of zero-dimensional compact spaces S, T and a map ϕ: S → X. By hypothesis there exists a continuous lifting ∂: S → $\mathbb{D}^A$ with $\theta \circ \partial = \phi$. Since $\mathbb{D}^A$ is an AE(0-dim) we find a continuous extension ψ: T → $\mathbb{D}^A$ of ∂ such that the diagram



commutes, $\psi \circ \iota = \partial$. Thus the map $\theta \circ \psi$ is a continuous extension of ϕ and hence we have proved that X is an AE(0-dim).

To show sufficiency we may suppose that X is a subspace of some product $\mathbb{I}^A$. By corollary 3.4 there exists a set B and a surjection θ: $\mathbb{D}^B$ → $\mathbb{I}^A$ satisfying the z.d.l.p. Let S be the closed subspace $\theta^{-1}(X)$ of $\mathbb{D}^B$ and ϕ: S → X the restriction of θ to S. Since X is an AE(0-dim) by hypothesis, there exists a continuous extension ψ: $\mathbb{D}^B$ → X of ϕ such that the diagram



commutes. Satisfying the z.d.l.p. is a hereditary and extensive property by the results of section 1 and hence with θ the restriction ϕ as well as the extension ψ of ϕ satisfy the z.d.l.p. Consequently X is a Michael space. □

Let us start employing the preceding theorem by exhibiting a rather interesting phenomenon involving surjections which have some generalized Cantor set as domain space and allow a r.a.o.

or satisfy the z.d.l.p.

Let θ: $\mathbb{D}^A \to X$ be a surjection. Without loss of generality we may assume that the cardinality of A coincides with the weight of X. Three cases depending on the cardinality of A are considered.

(a) If A is a countable set, then we deduce from theorem 1.11 that θ admits a r.a.o. if and only if θ satisfies the z.d.l.p.

(b) Now let A have the cardinality ω_1 and assume that θ allows a r.a.o. Then in general θ does not satisfy the z.d.l.p. This statement follows from proposition 1.19 which implies that the natural surjection $\mathbb{D}^{\omega_1} \times \mathbb{D}^{\omega_1} \to \exp_2(\mathbb{D}^{\omega_1})$, $(x,y) \mapsto \{x,y\}$, does not satisfy the z.d.l.p. although it admits a r.a.o. However, the following consideration always holds. Since θ allows a r.a.o. X is a Milutin space of weight ω_1 and hence X is a Dugundji space. Then by theorem 3.5 there exists a surjection ψ: $\mathbb{D}^A \to X$ satisfying the z.d.l.p. and consequently a map ∂: $\mathbb{D}^A \to \mathbb{D}^A$ such that $\psi \circ \partial = \theta$, i.e. the map θ allowing a r.a.o. (but not necessarily satisfying the z.d.l.p.) always factors through a map ψ satisfying the z.d.l.p. and having the same domain space.

(c) Finally, if A has cardinality ω_2 and θ allows a r.a.o., then generally there does not exist a surjection $\mathbb{D}^A \to X$ satisfying the z.d.l.p. as explained under item (b). For instance the natural surjection θ: $\mathbb{D}^{\omega_2} \times \mathbb{D}^{\omega_2} \to \exp_2(\mathbb{D}^{\omega_2})$ allows a r.a.o. by proposition 1.19 (i) but $\exp_2(\mathbb{D}^{\omega_2})$ is not a Dugundji space or, equivalently by theorem 3.5, not a Michael space.

We notice that the pathological counterexamples for the cases

of weight ω_1 and ω_2 already occur within the realm of zero-dimensional compact spaces.

Mimicking the second part of the proof of theorem 3.5 by starting off with a surjection $\theta\colon \mathbb{D}^B \to \mathbb{I}^A$ that satisfies the lifting property for zero-dimensional paracompact spaces one can prove the following.

3.6. REMARK. A compact space X is a Dugundji space if and only if there exist a set A and a surjection of $\mathbb{D}^A$ onto X satisfying the lifting property for zero-dimensional paracompact spaces.

Proof. It remains to prove necessity. We regard X as a subspace of some $\mathbb{I}^A$. For each $a \in A$ there exists a surjection $\theta_a\colon \mathbb{D}^\omega \to \mathbb{I}$ satisfying the lifting property for zero-dimensional paracompact spaces by remark 1.16. Then the product map $\theta := (\theta_a)_{a \in A} : (\mathbb{D}^\omega)^A \to \mathbb{I}^A$ has the same property. Now we continue as indicated above. □

Our surjective characterization of Dugundji spaces as Michael spaces yields several interesting results.

3.7. COROLLARY. Let Y be a closed G_δ-subset of a compact space X. Then if X is a Dugundji space, so is Y. In particular a regular closed subset of a Dugundji space is a Dugundji space.

Proof. Since X is a Michael space there exists a surjection $\theta\colon \mathbb{D}^A \to X$ satisfying the z.d.l.p. The closed G_δ-subset Y can be represented as the zero-set of a continuous function on X, i.e. there is a continuous function $f\colon X \to \mathbb{I}$ such that $f^{-1}(0) = Y$. Then we can find a countable subset B of A such that $f \circ \theta\colon \mathbb{D}^A \to \mathbb{I}$ factors through the canonical projection $p_B\colon \mathbb{D}^A \to \mathbb{D}^B$. Hence

there exists a map $\psi\colon \mathbb{D}^B \to \mathbb{I}$ with $\psi \circ p_B = f \circ \theta$. Consequently we calculate $\theta^{-1}(Y) = \theta^{-1}(f^{-1}(0)) = (f\circ\theta)^{-1}(0) = (\psi\circ p_B)^{-1}(0) = p_B{}^{-1}(\psi^{-1}(0))$ which is homeomorphic to $D^{A\setminus B} \times \psi^{-1}(0)$. Now $\psi^{-1}(0)$ is a compact metrizable space and therefore $\theta^{-1}(Y)$ is, as a product of two Michael spaces of the same type. Since the z.d.l.p. is a hereditary property the restriction map $\theta|\theta^{-1}(Y)\colon \theta^{-1}(Y) \to Y$ satisfies the z.d.l.p. and this shows that Y is a Michael space or, equivalently, a Dugundji space.

For the remaining part we have just to recall that Dugundji spaces satisfy the B.S.P., i.e. regular closed sets are G_δ-sets. □

In passing we mention that a proof corresponding to the one presented above yields that a closed G_δ-subset of a Milutin space is a Milutin space.

To show that the G_δ-condition cannot be weakened in both cases (i.e. for Milutin spaces and Dugundji spaces) we consider the following closed subset Y of $\mathbb{D}^{\omega_1}$ which then certainly is the intersection of a family of open subsets indexed by ω_1. Let Y be any copy of a non-dyadic zero-dimensional compact space of weight ω_1, e.g. the Alexandroff one-point compactification of a discrete space of cardinality ω_1.

Taking from Pełczyński's book ([28], p.36) in a slightly broader fashion the notion of a locally Dugundji space the following proposition gives a characterization of Dugundji spaces in local terms.

3.8. DEFINITION. A space X is called a <u>locally Dugundji</u>

space if each point in X has a compact neighbourhood which is a Dugundji space.

We notice that a locally Dugundji space is a (not necessarily compact) locally compact space and by corollary 3.7 each point has a neighbourhood base of Dugundji spaces.

3.9. PROPOSITION. Let X be a compact space. Then the following are equivalent.

(i) Each point of X has a neighbourhood base of Dugundji spaces.

(ii) X is a Dugundji space.

(iii) X is a locally Dugundji space.

Proof. It remains to show that (iii) implies (ii). By a standard compactness argument there exist finitely many Dugundji spaces $X_1,\ldots,X_n$ in X such that X is covered by the interiors of the subsets X_i ($i = 1,\ldots,n$) of X. We may assume that there exist a set A and surjections $\theta_i: \mathbb{D}^A \to X_i$ satisfying the z.d.l.p. for $i = 1,\ldots,n$. Then the surjection $\theta: \mathbb{D}^A \times \{1,\ldots,n\} \to X$, given by $\theta(x,i) := \theta_i(x)$ for $i = 1,\ldots,n$ and $x \in \mathbb{D}^A$, satisfies the z.d.l.p. by proposition 1.5 and so X is a Dugundji space. □

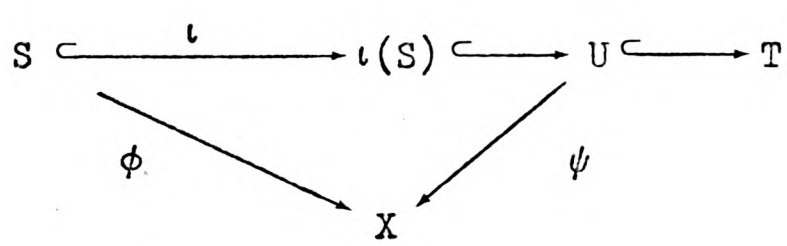
3.10. COROLLARY. An open (resp. clopen) subset of a Dugundji space is a locally Dugundji space (resp. Dugundji space).

The equivalence of (i) and (ii) in proposition 3.9 answers Pełczyński's problem 18 ([28], p.70) for Dugundji spaces in the affirmative. That statement (ii) of proposition 3.9 follows from (iii) was first stated by Pełczyński ([28], Proposition 6.3). However, his proof is not quite correct and in contrast to

Pełczyński's argument we presented here a purely topological proof.

Considering the special role that regular closed sets play in Dugundji spaces an apparently weaker version of Haydon's characterization can be proved. Let us first recall the following conventions.

3.11. DEFINITION. A compact space X is an absolute neighbourhood extensor for zero-dimensional spaces or ANE(0-dim) if for any embedding ι: S → T of zero-dimensional compact spaces S, T and any map ϕ: S → X there exists a neighbourhood U of ι(S) in T and a map ψ: U → X such that the diagram



commutes, $\phi = \psi \circ \iota$.

A compact space X is said to be a locally ANE(0-dim) if each point of X has a compact neighbourhood that is an ANE(0-dim).

3.12. PROPOSITION. Let X be a compact space. Then the following are equivalent.

(i) X is a Dugundji space.

(ii) X is an ANE(0-dim).

(iii) X is a locally ANE(0-dim).

Proof. Because of the results of proposition 3.9 it is enough to prove equivalence of the statements (i) and (ii). Since (i) implies (ii) by Haydon's characterization of Dugundji spaces as AE(0-dim) it remains to verify the converse. Let us assume that

X is a closed subset of some product $\mathbb{I}^A$. By corollary 3.4 there exist a set B and a surjection $\theta\colon \mathbb{D}^B \to \mathbb{I}^A$ satisfying the z.d.l.p. We take $S := \theta^{-1}(X)$, $\iota\colon S \to \mathbb{D}^B$ as the natural inclusion and $\phi\colon S \to X$ as the restriction of θ to S. By hypothesis (ii) there exist a neighbourhood U of S in $\mathbb{D}^B$ and a map $\psi\colon U \to X$ extending ϕ. Since $\mathbb{D}^B$ is a normal space we may choose U to be a regular closed neighbourhood of S in $\mathbb{D}^B$ and hence U is a Dugundji space by corollary 3.7. So it remains to show that $\psi\colon U \to X$ satisfies the z.d.l.p. But this is an easy consequence of the fact that satisfying the z.d.l.p. is a hereditary and extensive property and that θ satisfies the z.d.l.p. by construction. □

As we have seen closed G_δ- subsets of a Dugundji space are fairly interesting objects to study because they are Dugundji spaces too and hence the natural inclusion map satisfies the z.d.r.p. or, equivalently, allows a r.e.o. (cf. corollary 2.6). We shall now proceed and investigate the remainder of the Dugundji space by pinching a closed G_δ-subset to a point.

First let us introduce some pieces of notation. For any compact space X and a closed subset Y of X we denote by X/Y the <u>quotient space</u> of X that one gets by identifying the points of Y (cf. [36], 5.2.8) and let $\pi_Y\colon X \to X/Y$ be the induced <u>quotient map</u>. (In general there is no risk of confusing this notation with the algebraic quotient space and quotient map where X and Y are topological groups.)

Our next aim is to show that for any Dugundji space X and any closed G_δ-subset Y of X the quotient space X/Y is a Dugundji space too. The

program for achieving this result is based on the following ideas. Since X is a Dugundji space there is a surjection θ: $\mathbb{D}^A \to X$ satisfying the z.d.l.p. by theorem 3.5. If T denotes the inverse image $\theta^{-1}(Y)$ then the map $\pi_Y \circ \theta$ factors in a natural way through the induced map ψ: $\mathbb{D}^A/T \to X/Y$. Now by proposition 3.3 it remains to show that $\mathbb{D}^A/T$ is a Dugundji space and ψ satisfies the z.d.l.p. The first claim will be verified in a slightly more generalized fashion by using an inverse limit argument. To deduce that ψ satisfies the z.d.l.p. we shall rely heavily on the fact that Y is a closed G_δ-subset of X and that θ satisfies the z.d.l.p.

Before we start to give a proof of the above mentioned result we shall develop a suitable machinery and thus avoid a rather lengthy proof. Some of the tools will be useful to us in sections 4 and 5 too.

From Shchepin's paper ([37], Proposition 6, p.177) we recall the following result which has a straightforward proof by Haydon's characterization of a Dugundji space as an AE(0-dim). Therefore we join Shchepin in omitting to give the verification.

3.13. PROPOSITION. Let θ: $X \to Y$ be a 0-soft map of compact spaces X and Y. Then X and Y are Dugundji spaces simultaneously. Furthermore, X is a Dugundji space if and only if the constant map $X \to \{X\}$ is 0-soft.

Suppose we are given a Dugundji space X and a closed subset Y of X. To infer that X/Y is a Dugundji space too it is enough to show that the quotient map π_Y: $X \to X/Y$ satisfies the z.d.l.p.

However, with certain restrictions put on Y we can do better and show that π_Y is 0-soft.

3.14. LEMMA. Let X be a compact space and Y a clopen subset of X. Then the canonical quotient map π_Y: X → X/Y is 0-soft if and only if Y is a Dugundji space. Moreover, if in addition X is a Dugundji space, so is X/Y.

Proof. The quotient map π_Y can be identified with the disjoint union of the identity map on $X \setminus Y$, $id_{X \setminus Y}$, and the constant map $Y \to \{Y\}$. Obviously, π_Y and the constant map are 0-soft simultaneously and by proposition 3.13 the map $Y \to \{Y\}$ is 0-soft if and only if Y is a Dugundji space.

For the remaining statement we observe that by corollary 3.10 Y is a Dugundji space and so π_Y is 0-soft. Hence X/Y is a Dugundji space. □

3.15. COROLLARY. Let X be a Dugundji space and Y and Z clopen subsets of X such that Y is contained in Z. Then the canonical map $\pi_{Z,Y}$: X/Y → X/Z which is (well-) defined by putting $\pi_{Z,Y}(\pi_Y(x)) := \pi_Z(x)$ for all $x \in X$ is 0-soft.

Proof. We can identify $\pi_{Z,Y}$ with the disjoint union of the identity map $id_{X \setminus Z}$ on $X \setminus Z$ and the constant map $Z/Y \to \{Z/Y\}$. Hence $\pi_{Z,Y}$ is 0-soft if and only if the constant map is 0-soft and by proposition 3.13 this is the case if and only if Z/Y is a Dugundji space. By corollary 3.10 Y is, as a clopen subspace of a Dugundji space of the same type, and lemma 3.14 yields that Z/Y is a Dugundji space. □

The idea now is to extend corollary 3.15 by considering linearly ordered inverse systems where the linking maps are of the type $\pi_{Z,Y}$ as in corollary 3.15 and thus are 0-soft. In analogy to proposition 1.3 one can show that the limit maps induced by a continuous inverse system of compact spaces are 0-soft, if the linking maps have this property. This is proposition 7 of Shchepin's paper [37], p.177. Hence by proposition 3.13 the limit space and the first space of the continuous inverse system are Dugundji spaces simultaneously.

3.16. PROPOSITION. Let X be a Dugundji space and Y a closed subset of X which can be represented as a countable intersection of clopen subsets $Y_1, Y_2, \ldots, Y = \cap \{ Y_n : n \geq 1 \}$. Then the quotient space X/Y is a Dugundji space.

Proof. Without loss of generality we may assume that the clopen sets Y_n ($n \geq 1$) are decreasing and for $1 \leq m \leq n < \omega$ let us set

$$p_{mn} := \pi_{Y_m, Y_n} : X/Y_n \longrightarrow\!\!\!\!\!\rightarrow X/Y_m$$

for the naturally induced map as defined in corollary 3.15. It is well-known that X/Y can be identified with the inverse limit space of the countable inverse system (cf. [11], 3.12.13)

$$(X/Y_m , p_{mn})_{m \leq n < \omega}.$$

Using our hypotheses, lemma 3.14 and corollary 3.15 we infer that X/Y_1 is a Dugundji space and that for all $n < \omega$ the map $p_{n,n+1}$ is 0-soft. Hence the inverse limit map $p_1 : X/Y \to X/Y_1$ is 0-soft ([37], Proposition 7, p. 177) and by proposition 3.13 X/Y is a Dugundji space. □

The following corollary will be useful to us later on.

3.17. COROLLARY. The Alexandroff one-point compactification of the disjoint union of countably many copies of $\mathbb{D}^A$ is a Dugundji space.

Proof. It is not difficult to verify that the one-point compactification of a countable disjoint union of spaces $\mathbb{D}^A$ is homeomorphic to the quotient space $([\omega] \times \mathbb{D}^A)/(\{\omega\} \times \mathbb{D}^A)$, where $[\omega]$ denotes the space of all ordinals $\leq \omega$ equipped with the order topology. $[\omega] \times \mathbb{D}^A$ is, as a product of Dugundji spaces of the same type, and $\{\omega\} \times \mathbb{D}^A$ is representable as a countable intersection of clopen subsets of $[\omega] \times \mathbb{D}^A$. □

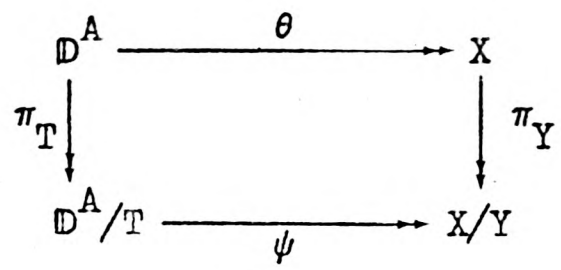
The next theorem shows that the intersection property of the representation of Y in proposition 3.16 can in fact be weakened to just requiring Y to be a closed G_δ-subset of the Dugundji space X. The proof depends heavily on the preceding results of section 3.

3.18. THEOREM. Let X be a Dugundji space and Y a closed G_δ-subset of X. Then the quotient space X/Y is a Dugundji space.

Proof. By hypothesis X is a Dugundji space and so theorem 3.5 guarantees the existence of a set A and a surjection $\theta: \mathbb{D}^A \to X$ satisfying the z.d.l.p. Let T be the inverse image $\theta^{-1}(Y)$ which is a closed G_δ-subset of $\mathbb{D}^A$ by continuity of θ. Since $\mathbb{D}^A$ has a base of clopen sets we can represent T as a countable intersection of clopen sets and so proposition 3.16 implies that $\mathbb{D}^A/T$ is a Dugundji space.

We agree to denote by $\pi_T: \mathbb{D}^A \to \mathbb{D}^A/T$ (resp. $\pi_Y: X \to X/Y$) the canonical quotient map. Then there exists a natural surjection

ψ: $\mathbb{D}^A/T \to X/Y$ such that the diagram



commutes. ψ is well-defined by

$$\psi(\ \pi_T(z)\ )\ :=\ \pi_Y\circ\theta(z)$$

for all $z\in\mathbb{D}^A$. Obviously, the proof of this theorem is complete if we can show that ψ satisfies the z.d.l.p.

Since Y is a closed G_δ-subset of X and X is normal there exists a sequence $G_1, G_2, \ldots$ of open sets of X such that

$$G_n\ \supset\ \mathrm{cl}_X(\ G_{n+1}\ )$$

holds for all $n \geq 1$ and $Y = \cap\{\ G_n : n \geq 1\ \}$. Putting $H_n := \pi_Y(G_n)$ ($n \geq 1$) we get an induced sequence $H_1, H_2, \ldots$ of open subsets of X/Y such that

$$H_n\ \supset\ \mathrm{cl}_{X/Y}(\ H_{n+1}\ )$$

for all $n \geq 1$ and $\pi_Y(Y) = \cap\{\ H_n : n \geq 1\ \}$.

Now suppose we are given a compact zero-dimensional space S and a map ϕ: $S \to X/Y$. We shall construct a continuous lifting ∂: $S \to \mathbb{D}^A/T$ for ϕ with respect to ψ and hence prove that ψ satisfies the z.d.l.p. Let R denote the closed subset $\phi^{-1}(\ \pi_Y(Y)\ )$ of S. Then we imply

$$\cap\{\ \phi^{-1}(\ H_n\ ) : n \geq 1\ \} = R \quad \text{and} \quad \phi^{-1}(\ H_n\ ) \supset \phi^{-1}(\mathrm{cl}_{X/Y}(H_{n+1}))$$

for all $n \geq 1$. Since S is a zero-dimensional compact space we can find clopen sets U_n ($n \geq 1$) such that

$$\phi^{-1}(\ H_n\ )\ \supset\ U_n\ \supset\ \phi^{-1}(\mathrm{cl}_{X/Y}(H_{n+1}))$$

holds for all $n \geq 1$. We define clopen subsets W_n ($n \geq 0$) of S by setting

$$W_0 := S \setminus U_1$$

$$W_n := U_n \setminus U_{n+1} \quad \text{for } n \geq 1.$$

The sets W_n ($n \geq 0$) are pairwise disjoint by definition and for $n \geq 0$ we have $\cup \{ W_i : 0 \leq i \leq n \} = S \setminus U_{n+1}$, consequently

$$\cup \{ W_n : n \geq 0 \} = \cup \{ S \setminus U_{n+1} : n \geq 0 \}$$

$$= S \setminus \cap \{ U_{n+1} : n \geq 0 \}$$

$$= S \setminus R.$$

Let us denote by

$$\phi_n : W_n \longrightarrow X/Y$$

the restriction of ϕ to W_n for $n \geq 0$. Obviously, for $n \geq 0$ the image $\phi_n(W_n)$ is a compact subset of X/Y and disjoint from $\pi_Y(Y)$ and so $(\pi_Y)^{-1} \circ \phi_n$ is a well-defined map. Since by hypothesis θ satisfies the z.d.l.p. $(\pi_Y)^{-1} \circ \phi_n$ allows a continuous lifting

$$\eta_n : W_n \longrightarrow \mathbb{D}^A$$

such that $\theta \circ \eta_n = (\pi_Y)^{-1} \circ \phi_n$. Putting $\partial_n := \pi_T \circ \eta_n$ ($n \geq 0$) we infer

$$\psi \circ \partial_n = (\psi \circ \pi_T) \circ \eta_n = \pi_Y \circ (\theta \circ \eta_n) = \pi_Y \circ (\pi_Y)^{-1} \circ \phi_n = \phi_n$$

for all $n \geq 0$. Hence for each $n \geq 0$ we have constructed a continuous lifting $\partial_n : W_n \to \mathbb{D}^A/T$ such that $\psi \circ \partial_n = \phi_n$. It follows from the pairwise disjointness of the clopen sets W_n that the map

$$\tilde{\partial} : S \setminus R \longrightarrow \mathbb{D}^A/T$$

$$\tilde{\partial} | W_n := \partial_n$$

well-defines a continuous lifting, i.e. $\psi \circ \tilde{\partial} = \phi | S \setminus R$.

Finally let us put

$$\partial(s) := \begin{cases} \tilde{\partial}(s) & \text{if } s \in S \setminus R \\ \pi_T(T) & \text{if } s \in R \end{cases}$$

and we shall prove that ∂ is indeed a continuous lifting of ϕ with respect to ψ. By construction ∂ is a lifting and so it remains to show the continuity of ∂. Clearly ∂ is continuous in each point $s \in S \setminus R$, since $\tilde{\partial}$ has this property. Then in order to verify the continuity of ∂ in some $r \in R$ it is enough to prove that for each closed subset F of $\mathbb{D}^A/T$ which does not contain the point $\pi_T(T)$ the preimage $\partial^{-1}(F)$ is a closed subset of S. Since $\psi^{-1}(\pi_Y(Y))$ coincides with $\pi_T(T)$ we infer that $\pi_T(T)$ is the intersection of the open sets $\psi^{-1}(H_n)$ ($n \geq 1$) and furthermore the inclusion $\psi^{-1}(H_n) \supset \psi^{-1}(cl_{X/Y}(H_{n+1}))$ holds for all $n \geq 1$. A simple compactness argument implies the existence of an integer k such that $F \cap \psi^{-1}(H_n) = \phi$ for all $n \geq k$. Thus F is a subset of the closed set $\mathbb{D}^A/T \setminus \psi^{-1}(H_k) = \psi^{-1}(X/Y \setminus H_k)$. Now consider the commutative diagram

$$\begin{array}{ccc} \psi^{-1}(X/Y \setminus H_k) & \xrightarrow{\bar{\psi}} & X/Y \setminus H_k \\ & \overset{\bar{\partial}}{\nwarrow} \quad \overset{\bar{\phi}}{\nearrow} & \\ & \phi^{-1}(X/Y \setminus H_k) & \end{array}$$

of maps and compact spaces, where $\bar{\psi}$ (resp. $\bar{\phi}$, $\bar{\partial}$) denotes the restriction of ψ (resp. ϕ, ∂ or equivalently $\tilde{\partial}$). Since $\partial^{-1}(F)$ coincides with $\bar{\partial}^{-1}(F)$ and $\bar{\partial}$ is continuous, we deduce that $\partial^{-1}(F)$ is a closed subset of $\phi^{-1}(X/Y \setminus H_k)$ and consequently a closed subset of S. This finishes the verification of the continuity of ∂. Hence ψ satisfies the z.d.l.p. and the proof of the theorem is complete. □

The following example (see [28], Example 5, p. 66) indicates that even in the case of compact zero-dimensional Dugundji spaces

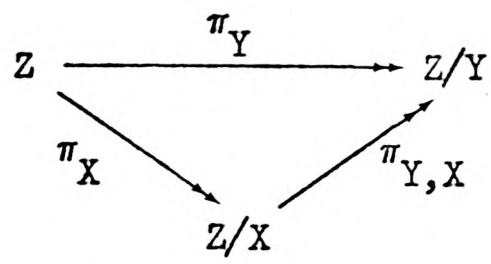
the countability condition in theorem 3.18 (and hence in proposition 3.16) is indeed essential. Note that if X is a compact metrizable space the quotient space X/Y is of the same type for any closed subset Y of X and in particular X/Y is a Dugundji space. Thus any space that might be useful to us as a counterexample has to be of uncountable weight.

3.19. EXAMPLE. Let $X := X_1 \cup X_2$ be the disjoint union of two copies of $\mathbb{D}^{\omega_1}$ and for $i = 1,2$ let x_i be any fixed point of X_i and set $Y := \{x_1, x_2\}$. Then X is a zero-dimensional Dugundji space, Y is a closed subset of X such that there is a family $(U_\alpha)_{\alpha<\omega_1}$ of clopen subsets of X with $Y = \bigcap \{ U_\alpha : \alpha < \omega_1 \}$ and X/Y is not a Dugundji space.

Proof. It remains to show that X/Y is not a Dugundji space. But this is an immediate consequence of the fact that X/Y contains an open set such that its closure is not a G_δ-set and hence X/Y does not satisfy the B.S.P. For $\pi_Y(X_1 \setminus \{x_1\})$ is an open set in X/Y and has $\pi_Y(X_1)$ as its closure. If $\pi_Y(X_1)$ were a G_δ-subset, then $(\pi_Y)^{-1}(\pi_Y(X_1)) = X_1 \cup \{x_2\}$ would be a G_δ-subset of X and consequently x_2 a G_δ-point in X_2. This is a contradiction. □

In §5 of his thesis [5] Ditor proved results concerning almost Dugundji spaces which somehow relate to theorem 3.18. However, in the case of almost Dugundji spaces the situation is much simpler, since by a proposition of Pełczyński ([28], Proposition 2.6) an embedding $\varepsilon: X \to Y$ of compact spaces X, Y has an e.o. if and only if the quotient map $\pi_{\varepsilon X}: Y \to Y/\varepsilon X$ has an **a**.o.

Now, if we are given any three compact spaces X, Y, Z with $X \subset Y \subset Z$ there are natural quotient maps such that the diagram



commutes and hence π_Y allows an a.o. if both π_X and $\pi_{Y,X}$ satisfy this property. Letting Z be some $\mathbb{I}^A$ and using Pełczyński's equivalence ([28], Proposition 2.6) cited above one infers the following.

3.20. REMARK. Let X be a closed subspace of a compact space Y. Then Y is an almost Dugundji space if both X and Y/X are almost Dugundji spaces.

Proof. We embed Y into some $\mathbb{I}^A =: Z$ and notice that the map $\pi_{Y,Z}$ in the diagram above is naturally homeomorphic to the quotient map $\pi_{(Y/X)}: Z/X \to (Z/X)/(Y/X)$. Since Y/X (resp. X) is an almost Dugundji space by hypothesis, the canonical embedding $Y/X \to Z/X$ (resp. $X \to Z$) allows an e.o. or, equivalently, $\pi_{(Y/X)}$ (resp. π_X) allows an a.o. Hence $\pi_Y = \pi_{Y,X} \circ \pi_X$ allows an a.o. That is to say the embedding $Y \to \mathbb{I}^A$ allows an e.o. and so Y is an almost Dugundji space. □

In passing we mention the following corollary. Let Y be the disjoint union of two almost Dugundji spaces X_1 and X_2 and for $i = 1,2$ let x_i be a point in X_i and let $Y := \{x_1, x_2\}$. Then X/Y is an almost Dugundji space. In order to apply remark 3.20 we consider X_1 naturally embedded in X/Y and then $(X/Y)/X_1$ coincides

with X_2. Since X_1 and X_2 are assumed to be almost Dugundji spaces X/Y is an almost Dugundji space by remark 3.20.

In particular this shows that the quotient space in example 3.19 is an almost Dugundji space. Results of this and similar type are discussed in §5 of [5] in a rather more complicated fashion.

For a compact space X the <u>cone</u> of X is defined by $con(X) := (X \times \mathbb{I})/(X \times \{1\})$. Recently Shchepin has shown ([39], Lemma 3) that a compact space X is an absolute neighbourhood retract (ANR) if and only if con(X) is an absolute retract (AR). Theorem 3.18 yields as a corollary a corresponding result for Dugundji spaces or, equivalently, AE(0-dim) bearing in mind that by proposition 3.12 the class of AE(0-dim) and the class of ANE(0-dim) coincide.

3.21. COROLLARY. Let X be a compact space and con(X) the cone of X. Then X is a Dugundji space if and only if con(X) is a Dugundji space.

Proof. The necessity is inferred by theorem 3.18 since $X \times \{1\}$ is a closed G_δ-subset of the Dugundji space $X \times \mathbb{I}$. Conversely, X can be identified with the base space of the cone of X which obviously is a closed G_δ-subset of con(X) and hence corollary 3.7 implies that X is a Dugundji space. □

The final aim of this section is to show that the Alexandroff one-point compactification of the disjoint union of countably many Dugundji spaces is a Dugundji space. To prove this statement we follow a pattern which is akin to the one exhibited in the proof

of theorem 3.18. In order to employ corollary 3.17 we shall first mention a preparatory lemma.

3.22. LEMMA. Let A be an infinite set and let $\theta_a : X_a \to Y_a$ be a surjection of compact spaces X_a and Y_a for all $a \in A$. Let X (resp. Y) denote the Alexandroff one-point compactification of the disjoint union of the spaces X_a (resp. Y_a) ($a \in A$) with x_∞ (resp. y_∞) denoting the point at infinity. Let the mapping $\theta: X \to Y$ be defined by $\theta|X_a := \theta_a$ ($a \in A$) and $\theta(x_\infty) := y_\infty$. Then the following hold.

(i) θ is a continuous surjection.

(ii) If each θ_a ($a \in A$) satisfies the z.d.l.p., then so does θ.

Proof. Ad (i): It remains to show the continuity of θ at x_∞. So let U be an open base neighbourhood of y_∞. Then U is of the form $U = Y \setminus \cup \{ K_a : a \in F \}$ where F is a finite subset of A and K_a is a compact set in Y_a. A simple computation gives $\theta^{-1}(U) = X \setminus \cup \{ \theta_a^{-1}(K_a) : a \in F \}$. Since each $\theta_a^{-1}(K_a)$ ($a \in F$) is a compact subset of X_a and F is finite, $\theta^{-1}(U)$ is an open neighbourhood of x_∞.

Ad (ii): Let T be a zero-dimensional compact space and $\phi: T \to Y$ a map. Putting $R_a := \phi^{-1}(Y_a)$ for $a \in A$ and $R_\infty := \phi^{-1}(y_\infty)$ we define compact subsets of R. Let $\phi_a: R_a \to Y_a$ ($a \in A$) and $\phi_\infty: R_\infty \to \{y_\infty\}$ be the restrictions of the map ϕ. Since by hypothesis each map θ_a ($a \in A$) satisfies the z.d.l.p. there exist maps $\partial_a: R_a \to X_a$ such that $\theta_a \circ \partial_a = \phi_a$ for all $a \in A$. We define a mapping $\partial: R \to X$ by setting $\partial|R_a := \partial_a$ ($a \in A$) and $\partial(R_\infty) := x_\infty$. In fact a calculation similar to the one of part (i) shows that ∂ is continuous and clearly the equation $\theta \circ \partial = \phi$ holds by construction. □

3.23. PROPOSITION. Let $Y_1, Y_2, \ldots$ be a sequence of Dugundji spaces and Y the Alexandroff one-point compactification of the disjoint union of the spaces Y_n ($n \geq 1$). Then Y is a Dugundji space.

Proof. By theorem 3.5 there exist a set A and a countable family of surjections $\theta_n : \mathbb{D}^A =: X_n \to Y_n$ ($n \geq 1$) satisfying the z.d.l.p. An application of lemma 3.22 together with corollary 3.17 finishes the proof. □

§4 Dugundji manifolds, locally compact groups and their quotient spaces

The results of section 3 revealed that for a compact space the property of being a Dugundji space is a local property. This formally suggests an extension of the notion of Dugundji space to locally compact spaces by defining it in local terms. In this way we preserve the coincidence with the already established definition for compact spaces (see proposition 3.9). However, to give this extension a sensible meaning which can be useful to us we naturally want that in analogy to the initial definition of a Dugundji space certain closed embeddings always allow a r.e.o. That is to say we have to ask the following: What subclass $\mathcal{Q}$ of locally compact spaces (containing the compact spaces) has the property that a closed embedding of a locally Dugundji space of $\mathcal{Q}$ into an arbitrary space of $\mathcal{Q}$ allows a r.e.o.

In the sequel it is shown that the class $\mathcal{Q}$ of locally compact paracompact spaces fulfils the desired requirement. Moreover, we produce an example to deduce that an extension of $\mathcal{Q}$ to all locally compact spaces is not feasible. Hence paracompactness is indeed essential. For convenience of notation we introduce the following piece of terminology.

4.1. DEFINITION. A locally Dugundji space is called a Dugundji manifold if it is paracompact.

To verify that the class of locally compact, paracompact spaces has the above mentioned property a generalized version of

Pełczyński's localization principle ([28], Lemma 3.6) is needed. Since his proof for compact spaces depends on a partition of the unity it can be carried over to the case of paracompact spaces. We choose to give the first half of the proof of our generalization in order to show where and how the paracompactness comes into play. Moreover, slight changes of the assumptions are suitable for our purposes.

4.2. LEMMA. Let $\varepsilon: X \to Y$ be a closed embedding of paracompact spaces X and Y. Let $(X_a)_{a \in A}$ (resp. $(Y_a)_{a \in A}$) be a family of closed subsets of X (resp. Y) and let $(U_a)_{a \in A}$ be a family of open subsets of Y such that the following hold.

(1) $\cup \{ U_a : a \in A \} \supset \varepsilon X$,

(2) $U_a \subset Y_a$ for all $a \in A$,

(3) $U_a \cap \varepsilon X \subset \varepsilon X_a \subset Y_a$ for all $a \in A$,

(4) $\varepsilon_a := (\varepsilon | X_a) : X_a \to Y_a$ admits a r.e.o. $u_a: C^b(X_a) \to C^b(Y_a)$ for all $a \in A$.

Then ε admits a r.e.o. $u: C^b(X) \to C^b(Y)$.

Proof. By condition (1) we infer that the union $\cup \{ U_a : a \in A \}$ is an open set containing the closed set εX. Since Y is as a paracompact space in particular normal, there exists an open set U_o in Y such that

$$Y \setminus \cup \{ U_a : a \in A \} \subset U_o \subset Y \setminus \varepsilon X.$$

Hence putting $B := A \cup \{o\}$ the family $(U_a)_{a \in B}$ establishes an open covering of Y. The paracompactness of Y implies the existence of a partition of unity $(\lambda_a)_{a \in B}$ subordinate to the open covering $(U_a)_{a \in B}$, i.e. for all $a \in B$ we have the inclusion

$$cl_Y(\{ y \in Y : \lambda_a(y) \neq 0 \}) \subset U_a .$$

If f is an element of $C^b(X)$ we denote by f_a the restriction of f to X_a and put for all $a \in A$

$$u_a(f)(y) := \begin{cases} u_a(f_a)(y) & \text{if } y \in Y_a \\ 0 & \text{if } y \in Y \setminus Y_a \end{cases}$$

Clearly, $\lambda_a \cdot u_a(f)$ is a bounded continuous function on Y for all $f \in C^b(X)$ and $a \in A$. Let x_o be any fixed point of X. Then we well-define for all $f \in C^b(X)$

$$u(f) := f(x_o)\cdot\lambda_o + \sum_{a \in A} \lambda_a \cdot u_a(f).$$

It follows from an elementary calculation which can be carried over almost literally from Pełczyński's proof of his lemma 3.6 in [28] that u indeed defines a r.e.o. $u: C^b(X) \to C^b(Y)$ for ε. □

4.3. PROPOSITION. Let $\varepsilon: X \to Y$ be a closed embedding of a Dugundji manifold X into a locally compact, paracompact space Y. Then ε allows a r.e.o. $u: C^b(X) \to C^b(Y)$.

Proof. To prove this proposition it obviously remains to construct for each x of X an open subset U_x of Y containing $\varepsilon(x)$ and a compact set X_x (resp. Y_x) of X (resp. Y) satisfying the hypotheses of lemma 4.2. So let x be a given fixed point of X. Since Y is locally compact there exists a compact neighbourhood Y_x of $\varepsilon(x)$ in Y. X is a Dugundji manifold and hence there is a Dugundji space X_x which is contained in the open neighbourhood $\varepsilon^{-1}(\mathrm{int}(Y_x))$ and contains x as an interior point. Now let V_x be any open set in Y such that $\varepsilon^{-1}(V_x)$ coincides with the interior of X_x in X. Such a set exists, since X is equipped with the relative

topology. Letting U_x be the intersection of V_x and the interior of Y_x we easily verify that (1), (2), (3) of the preceding lemma are satisfied and condition (4) holds because X_x is a Dugundji space. □

We notice that it is not true that every locally Dugundji space is paracompact. For instance the space of all countable ordinals is known to be non-paracompact (see e.g. [43], 20.11) and for each countable ordinal α the space of all ordinals $\leq \alpha$ is a metrizable compact neighbourhood of α and hence a Dugundji space containing α as an interior point.

Before we go on to show that the class of Dugundji manifolds encompasses all locally compact groups and their quotient spaces and makes it therefore a rather interesting object to study let us digress for a moment and investigate how restrictive our definition of Dugundji manifold is and how it relates to some other notions. Clearly by Tietze's theorem any closed embedding $\varepsilon: X \to Y$ of normal spaces X, Y is necessarily a C^b-embedding, i.e. the natural restriction algebra homomorphism $C^b(Y) \to C^b(X)$ is surjective.

For a non-compact locally compact space X let us consider the following statements:

(A) X is a Dugundji manifold.

(B) The Stone-Čech compactification βX of X is a Dugundji space.

(C) Whenever X is C^b-embedded as a closed subspace of a locally compact space by an embedding $\varepsilon: X \to Y$, then

ε allows a r.e.o. $C^b(X) \to C^b(Y)$.

We shall now exhibit the interrelationship of the statements (A), (B) and (C).

The discrete space of non-negative integers $\mathbb{N}$ is the simplest example of a non-compact Dugundji manifold. Its Stone-Čech compactification $\beta\mathbb{N}$ is certainly not a Dugundji space, since $\beta\mathbb{N}$ is not dyadic ([10], Remark 3). Hence in general (A) does not imply (B).

On the other hand, if X is a completely regular space and βX is dyadic, then X is pseudocompact ([10], Corollary 7). Since for regular spaces pseudocompactness and paracompactness together is equivalent to compactness ([11], Theorem 3.10.22) we have proved in particular that the Stone-Čech compactification of a non-compact Dugundji manifold is not a Dugundji space, i.e. (A) implies non-(B).

Now we claim that (C) can be deduced from (B). Let $\varepsilon: X \to Y$ be any C^b-embedding of X into a locally compact space Y such that εX is a closed subspace of Y. So the natural restriction algebra homomorphism $C^b(Y) \to C^b(X)$ is surjective and induces a surjective algebra homomorphism $C(\beta Y) \to C(\beta X)$ which, in a natural way, gives rise to an embedding of βX into βY coinciding with ε on X. By (B) the embedding $\beta X \to \beta Y$ allows a r.e.o. $u: C(\beta X) \to C(\beta Y)$ and this operator can be used to define a r.e.o. $v: C^b(X) \to C^b(Y)$ for $\varepsilon: X \to Y$ by putting $v(f) := u(\beta f)|Y$ for all $f \in C^b(X)$, where βf denotes the unique continuous extension of f to βX. This completes the proof of the claim.

Finally we produce an example of a non-compact Dugundji mani-

fold which does not satisfy (C). Hence (A) does not imply (C) in general. Furthermore, it yields that in proposition 4.3 the paracompactness condition on Y cannot be suppressed.

4.4. EXAMPLE. Let $\beta\mathbb{N}$ be the Stone-Čech compactification of the non-negative integers $\mathbb{N}$. Let us identify the zero-dimensional compact space $\beta\mathbb{N}$ of weight $\tau := 2^{\omega}$ with a subspace of $\mathbb{D}^{\tau}$ and put $X := \mathbb{N}$ and $Y := \mathbb{D}^{\tau} \setminus (\beta\mathbb{N} \setminus \mathbb{N})$. Then X is a non-compact Dugundji manifold which is a closed, C^b-embedded subspace of the locally compact space Y and the natural inclusion map $\varepsilon: X \to Y$ does not admit a r.e.o. $C^b(X) \to C^b(Y)$.

Proof. Since the complement $Y \setminus X = \mathbb{D}^{\tau} \setminus \beta\mathbb{N}$ is open in Y the subspace X is closed in Y. Clearly, Y is, as an open subspace of a Dugundji space, a locally Dugundji space (see corollary 3.10).

It is known that a subspace S of a space T is C^b-embedded in T if and only if βS can be identified with the closure of S in βT (see e.g. [13], 6.9 (a)). By construction βX is the closure of X in $\mathbb{D}^{\tau}$ and hence it is enough to show that βY coincides with $\mathbb{D}^{\tau}$. It follows from lemma 3 and theorem 7 of [10] that $\beta Y = \mathbb{D}^{\tau}$ if and only if the fibres $p_A^{-1}(z)$ have a non-empty intersection with Y for all countable subsets A of τ and $z \in \mathbb{D}^A$, where $p_A: \mathbb{D}^{\tau} \to \mathbb{D}^A$ denotes the canonical projection. Suppose there exists such a fibre $F := p_A^{-1}(z)$ in $\mathbb{D}^{\tau}$ that is disjoint from Y. Then F contains a convergent sequence of distinct points. But this is impossible, since F is contained in $\beta\mathbb{N} \setminus \mathbb{N} \subset \beta\mathbb{N}$ and an extremally disconnected space does not admit such a sequence. Thus we have proved that X is C^b-embedded in Y.

It remains to verify that ε does not allow a r.e.o. Assume the contrary. Any r.e.o. $C^b(X) \to C^b(Y)$ for $\varepsilon: X \to Y$ induces in a natural fashion a r.e.o. $C(\beta X) \to C(\beta Y)$ for $\beta\varepsilon: \beta X \to \beta Y$, $\beta\varepsilon$ being the unique continuous extension of ε. Since βY coincides with the Dugundji space $\mathbb{D}^\tau$, the existence of a r.e.o. for the embedding $\beta X \to \beta Y$ implies that $\beta X = \beta\mathbb{N}$ is a Dugundji space. This contradiction completes the proof. □

In proposition 4.3 we showed that in the class of locally compact, paracompact spaces the Dugundji manifolds play the same role as the Dugundji spaces do in the class of compact spaces, i.e. a closed embedding of a Dugundji manifold (resp. Dugundji space) into a locally compact, paracompact (resp. compact) space allows a r.e.o. Moreover, our preceding considerations seem to suggest that this generalization of Dugundji spaces is the best possible we can hope for.

We shall now continue by proving that for the class of Dugundji manifolds analogues of corollary 3.7 and theorem 3.5 still hold. To this end the following preparatory lemma will be useful to us and furthermore, it yields an extension of the known result that a Dugundji space is κ-metrizable.

4.5. LEMMA. A Dugundji manifold is a κ-metrizable space. In particular, a Dugundji manifold satisfies the B.S.P.

Proof. A Dugundji manifold is locally a Dugundji space and hence locally κ-metrizable. To finish the argument of the proof we recall that by a proposition of Shchepin ([37], Proposition 3, p. 179) a locally κ-metrizable paracompact space is again κ-metrizable

and hence in particular satisfies the B.S.P. □

4.6. PROPOSITION. A closed G_δ-subset of a Dugundji manifold is a Dugundji manifold. In particular a regular closed subset of a Dugundji manifold is a Dugundji manifold.

Proof. Let Y be a closed G_δ-subset of a Dugundji manifold X. Certainly, Y is locally compact and paracompact. Let y be a point in Y and U an open neighbourhood of y in Y. Then there exists an open set V in X with $V \cap Y = U$. By hypothesis we can find a neighbourhood W of y in X which is a Dugundji space and is contained in V and hence $W \cap Y$ is a neighbourhood of y in Y and contained in U. Since Y is a closed G_δ-subset of X we deduce that $W \cap Y$ is a closed G_δ-subset of W and thus $W \cap Y$ is a Dugundji space by corollary 3.7. For the verification of the remaining part it is enough to apply lemma 4.5. □

The surjective characterization of Dugundji spaces as Michael spaces by theorem 3.5 yields a corresponding result for Dugundji manifolds. The proof is straightforward and therefore omitted.

4.7. PROPOSITION. Let X be a locally compact, paracompact space. Then the following are equivalent.

(i) X is a Dugundji manifold.

(ii) There exist discrete spaces A and B and a surjection $\theta: \mathbb{D}^A \times B \to X$ such that for all $b \in B$ the restriction of θ to $\mathbb{D}^A \times \{b\}$ onto its image $\theta(\mathbb{D}^A \times \{b\}) =: K_b$ satisfies the z.d.l.p. and the union of the interiors of the sets K_b ($b \in B$) coincides with X.

It is known that the product of two paracompact spaces is not

necessarily paracompact ([9], Theorem 2.4.(3), p. 165). However, if we additionally assume each factor to be locally compact, paracompactness then follows for the product space. Modulo the compact case this is all we need to know in order to infer that the class of Dugundji manifolds is closed under taking finite products. Locally compact, paracompact spaces are known to admit a representation of the following kind ([9], Theorem 7.3, p. 241).

4.8. LEMMA. A locally compact space is paracompact if and only if it is the disjoint union of a family of σ-compact, locally compact spaces.

4.9. PROPOSITION. If X, Y are locally compact, paracompact spaces, then so is the product space $X \times Y$.

Proof. By lemma 4.8 there are pairwise disjoint σ-compact locally compact subspaces X_a ($a \in A$) (resp. Y_b ($b \in B$)) of X (resp. Y) such that $\cup \{ X_a : a \in A \} = X$ (resp. $\cup \{ Y_b : b \in B \} = Y$). Now $X \times Y$ coincides with the disjoint union of the product spaces $X_a \times Y_b$ where $(a,b) \in A \times B$. Certainly the product of two σ-compact locally compact spaces X_a, Y_b is of the same type and hence lemma 4.8 assures the paracompactness of the product space $X \times Y$, since $X \times Y$ is locally compact. □

Surely this result is well-known, but since we could not find it in the standard literature we chose to give the proof. From proposition 4.9 and the fact that a product of Dugundji spaces is a Dugundji space one immediately deduces the following statement by a simple induction argument.

4.10. PROPOSITION. The product of finitely many Dugundji manifolds is a Dugundji manifold.

In the remaining part of this section we shall develop the tools which allow us to imply as a corollary of our main result (theorem 4.23) that every quotient space of a locally compact group by a closed subgroup is a Dugundji manifold. The general concept for our approach is based on an inverse limit argument which can be regarded as the locally compact extension of the compact case. Before we introduce some new pieces of terminology let us recall for a moment how one goes about proving that a certain inverse limit space is a Dugundji space. We notice that an argument along the following lines was already applied in the proof of proposition 3.16.

4.11. REMARK. Let X be the inverse limit space of the continuous inverse system $(X_\alpha, p_{\alpha\beta})_{\alpha \leq \beta < \tau}$ of compact spaces X_α and surjections $p_{\alpha\beta}: X_\beta \to X_\alpha$, where $p_{\alpha,\alpha+1}$ is a 0-soft map for all $\alpha < \tau$. Using a familiar transfinite induction argument akin to the one produced in the proof of proposition 1.3 one verifies that all maps $p_\alpha: X \to X_\alpha$ ($\alpha < \tau$) are 0-soft.([37], Proposition 7, p. 177). In particular, $p_1: X \to X_1$ is 0-soft and hence by proposition 3.13 the space X is a Dugundji space if and only if X_1 is a Dugundji space.

Led by similar conclusions we shall present here a concept that generalizes the results of remark 4.11 to locally compact spaces. Thus our program is divided into three steps. First we have to find the right kind of definition that extends 0-softness

to locally compact spaces. Then this generalized notion will allow us to prove an analogue of proposition 3.13 and finally we shall show that it is stable under taking continuous inverse limits.

Before introducing a wider definition of O-softness we recall that a surjection is called perfect if it is a closed map and inverse images of points (or, equivalently, compact sets) are compact (see [9], Theorem 5.3(2), p. 236).

4.12. DEFINITION. Let θ: X → Y be a perfect map of spaces X and Y. Then θ is called locally O-soft if for each point y of Y there is a compact neighbourhood K of y such that the induced surjection

$$\theta_K := \theta | \theta^{-1}(K) : \theta^{-1}(K) \longrightarrow\!\!\!\!\!\rightarrow K$$

is O-soft.

Clearly it follows from this definition that the range space Y of a locally O-soft map θ: X → Y is locally compact. Hence by the perfectness of θ the domain space X is locally compact too. Moreover, if Y is paracompact, then so is X. This is a consequence of the fact that paracompactness is preserved under taking inverse images by perfect maps ([9], Theorem 5.3(1), p.236). Finally we notice that local O-softness is a hereditary property, since both O-softness and perfectness satisfy this property.

The next proposition demonstrates that O-softness and local O-softness actually coincide in the realm of compact spaces and hence local O-softness is a sensible generalization of O-softness to locally compact spaces.

4.13. PROPOSITION. Let X, Y be locally compact spaces and θ: X → Y a perfect map. Then θ is locally O-soft if and only if for all compact subsets K of Y the induced map

$$\theta_K := \theta \mid \theta^{-1}(K) \longrightarrow\!\!\!\!\!\rightarrow K$$

is O-soft.

Proof. Let K be a compact subset of Y and assume that θ is locally O-soft. Then by compactness of K and our assumption there exist finitely many compact subsets K_i ($1 \leq i \leq n$) of Y such that the union of the interiors of the sets K_i ($1 \leq i \leq n$) contains K and θ_{K_i} ($1 \leq i \leq n$) is O-soft. Let us set $L_i := K \cap K_i$, $\theta_i := \theta_{L_i}$ for $1 \leq i \leq n$. Then θ_i is a O-soft map by the hereditary property of O-softness. Putting $U_i := K \cap \mathrm{int}_Y(K_i)$ for $1 \leq i \leq n$ we define open subsets U_i of K such that the union of the sets U_i ($1 \leq i \leq n$) coincides with K.

Now suppose we are given an embedding ι: S → T of zero-dimensional compact spaces S, T and maps ϕ: S → $\theta^{-1}(K)$, ψ: T → K such that the diagram

$$\begin{array}{ccc} \theta^{-1}(K) & \xrightarrow{\theta_K} & K \\ \phi \uparrow & & \uparrow \psi \\ S & \xhookrightarrow[\iota]{} & T \end{array}$$

commutes, $\theta_K \circ \phi = \psi \circ \iota$. By continuity of ψ the family $(\psi^{-1}(U_i))_{1 \leq i \leq n}$ establishes an open covering of T and since T is zero-dimensional there exists a finite clopen, pairwise disjoint refinement $(W_j)_{1 \leq j \leq m}$ of $(\psi^{-1}(U_i))_{1 \leq i \leq n}$. For each $j \in \{1,\ldots,m\}$ we can and do pick an element $i(j) \in \{1,\ldots,n\}$ such that $W_j \subset \psi^{-1}(U_{i(j)})$. Since U_i is contained in L_i for all i ($1 \leq i \leq n$) the following induced

commutative diagram is well-defined for all j ($1 \leqslant j \leqslant m$)

$$\begin{array}{ccc} \theta_{i(j)}^{-1}(L_{i(j)}) & \xrightarrow{\theta_{i(j)}} & L_{i(j)} \\ \uparrow \phi_j & & \uparrow \psi_j \\ \iota^{-1}(W_j) & \xrightarrow[\iota_j]{} & W_j \end{array}$$

where ι_j (resp. ϕ_j; ψ_j) denotes the restricition of ι (resp. ϕ; ψ) to $\iota^{-1}(W_j)$ (resp. $\iota^{-1}(W_j)$; W_j). The 0-softness of $\theta_{i(j)}$ implies the existence of a map $\partial_j: W_j \to \theta_{i(j)}^{-1}(L_{i(j)})$ such that $\partial_j \circ \iota_j = \phi_j$ and $\theta_{i(j)} \circ \partial_j = \psi_j$. The pairwise disjointness of the clopen covering $(W_j)_{1 \leqslant j \leqslant m}$ allows us to well-define a map $\partial: T \to \theta^{-1}(K)$ by putting $\partial | W_j := \partial_j$ ($1 \leqslant j \leqslant m$) and obviously $\partial \circ \iota = \phi$ and $\theta_K \circ \partial = \psi$ hold by construction. Hence θ_K is a 0-soft map and the necessity is proved.

The converse is trivial, because Y is locally compact by hypothesis. This ends the proof. □

4.14. COROLLARY. Let $\theta: X \to Y$ be a surjection of compact spaces X and Y. Then θ is 0-soft if and only if θ is locally 0-soft.

It is a consequence of the next result that locally 0-soft maps and Dugundji manifolds are compatible in a sense that generalizes proposition 3.13.

4.15. PROPOSITION. Let $\theta: X \to Y$ be a locally 0-soft map. Then X and Y are Dugundji manifolds simultaneously.

Proof. If Y is a Dugundji manifold, then X is a locally compact, paracompact space by the perfectness of θ. Moreover, the

interiors of compact sets of the form $\theta^{-1}(K)$, where K runs through the Dugundji spaces of Y, cover X. By the local O-softness of θ and proposition 4.13 θ_K: $\theta^{-1}(K) \to K$ is O-soft for all compact subsets K of Y. Hence an application of proposition 3.13 shows that X is a Dugundji manifold.

To prove the converse let y_o be a point of Y. We shall construct a compact neighbourhood of y_o that is a Dugundji space. As remarked earlier, it follows from the local O-softness of θ that Y is a locally compact space. Hence there is a closed G_δ-subset K of Y which is compact and contains y_o as an interior point (e.g. choose a continuous function f: $Y \to \mathbb{I}$ such that $\{ y \in Y : f(y) \geq \frac{1}{2} \}$ is compact and $\{ y \in Y : f(y) > \frac{1}{2} \}$ contains y_o). Certainly, as a consequence of the continuity and perfectness of θ, the inverse image $\theta^{-1}(K)$ is a compact G_δ-subset of X. By hypothesis, X is a Dugundji manifold and so proposition 4.6 immediately implies that $\theta^{-1}(K)$ is a (compact) Dugundji space. Now consider the induced restriction of θ

$$\theta_K: \theta^{-1}(K) \longrightarrow\!\!\!\!\!\rightarrow K.$$

From the local O-softness of θ and proposition 4.13 we deduce that θ_K is a O-soft map. Hence we can utilize proposition 3.13 and infer that $\theta^{-1}(K)$ and K are Dugundji spaces simultaneously. Consequently, K is a Dugundji space and a neighbourhood of y_o by construction. This shows that Y is a locally Dugundji space and it remains to check the paracompactness of Y. But it is well-known that paracompactness is preserved under the taking of images by closed maps ([9], 2.6, p. 165). This finishes the proof. □

To produce a non-trivial example of a locally O-soft map of

non-compact locally compact spaces we state a simple lemma. Its proof is straightforward bearing in mind Haydon's characterization of a Dugundji space as an AE(0-dim).

4.16. LEMMA. Let X, Y be compact spaces and let $p: X \times Y \to X$ be the canonical projection onto the first factor. Then p is 0-soft if and only if Y is a Dugundji space.

For the sake of convenience we introduce the following notation. Let X, Y be spaces and $\theta: X \to Y$ be surjection. θ is called a <u>locally trivial fibration</u> if each point y of Y has a neighbourhood K such that the induced surjection $\theta_K: \theta^{-1}(K) \to K$ is homeomorphic to a projection $K \times L \to K$ onto the first factor for some space L. L is called the <u>fibre</u> of θ (in y). θ is called a <u>trivial fibration</u> if θ is homeomorphic to a projection $Y \times L \to Y$.

4.17. PROPOSITION. Let $\theta: X \to Y$ be a closed surjection of locally compact spaces X and Y. If θ is a locally trivial fibration with a Dugundji space as fibre, then θ is a locally 0-soft map.

Proof. We notice that the inverse image of a point of Y is homeomorphic to the compact fibre. Consequently θ is a perfect map. Since θ is a locally trivial fibration and Y is a locally compact space, there exists for each point y of Y a compact neighbourhood K of y such that θ_K is homeomorphic to the natural projection $K \times F \to K$, where F denotes the fibre. F is a Dugundji space by hypothesis and so an application of lemma 4.16 completes the proof. □

In order to apply the preceding proposition to topological groups we recall the following result ([25], Theorem 2, p. 221),

based on a famous theorem of Gleason (see e.g. [41], p. 60).

4.18. LEMMA. Let G be a locally compact topological group and H a compact Lie subgroup of G. Then the canonical quotient map $G \to G/H$ is a locally trivial fibration.

Clearly for any locally compact group G and compact subgroup H of G the quotient map $G \to G/H$ is perfect. Hence lemma 4.18, proposition 4.17 and the fact that a compact Lie group is metrizable and therefore a Dugundji space yield the promised example of a non-trivial locally O-soft map.

4.19. PROPOSITION. Let G be a locally compact topological group and H a compact Lie subgroup of G. Then the quotient map $G \to G/H$ is locally O-soft. Moreover, if G is a compact group, then the quotient map $G \to G/H$ is O-soft.

We shall now use proposition 4.19 and a well-known inverse limit representation of compact groups to show that a compact group is a Dugundji space. From Pontryagin's book ([29], pp. 326-329) we recall the following.

A compact topological group G is the inverse limit of a continuous inverse system $(G_\alpha, p_{\alpha\beta})_{\alpha \leq \beta < \tau}$ of compact groups G_α and surjective continuous group homomorphisms $p_{\alpha\beta}: G_\beta \to G_\alpha$ ($\alpha \leq \beta < \tau$) such that

(i) G_1 is a compact Lie group,

(ii) $p_{\alpha,\alpha+1}: G_{\alpha+1} \to G_\alpha$ has a compact Lie group as (algebraic) kernel for all $\alpha < \tau$.

From proposition 4.19 we deduce that the maps $p_{\alpha,\alpha+1}$ of (ii)

above are O-soft and hence as a consequence of remark 4.11 and (i) the following holds.

4.20. PROPOSITION. A compact topological group is a Dugundji space.

We notice that this result was first stated by Haydon ([15], Remark 5.8). Since the argument we presented here is slightly different from Haydon's and as far as we know a complete proof does not exist in the literature we chose to give an outline of the proof. Furthermore, recalling from Skljarenko's paper ([41], Theorem 16') that a locally compact group is homeomorphic to a product $A \times \mathbb{D}^B \times \mathbb{R}^n \times G$, where A, B are discrete spaces, $\mathbb{R}$ the real numbers with the usual topology, n a non-negative integer and G a compact connected group, and applying proposition 4.10 we arrive at the following result.

4.21. COROLLARY. A locally compact topological group is a Dugundji manifold.

The remainder of this section is devoted to the problem of proving a natural extension of remark 4.11 to locally compact spaces and then applying this generalization to quotient spaces of certain types of locally compact groups which allow us to carry over Pontryagin's proof of the inverse limit representation of compact groups quoted above. Basically his proof depends on the following result known to hold in compact groups ([29], Theorem 67, p.325). Every neighbourhood of the identity contains a closed normal subgroup such that the factor group is a Lie group. A projectively-

Lie group is defined as a locally compact group in which this statement remains valid. For a detailed study of projectively-Lie groups and their relationship to locally compact groups we refer to Skljarenko's survey paper [41].

Since the bulk of the preparatory work for the following theorem has already been done, we shall be able to give a fairly brief proof of it. However, we prefer to isolate one further result concerning inverse limits of perfect maps. For a proof of the following lemma we mention Puzio's paper ([31], Theorem 7; see also [37], Proposition 7, p. 184).

4.22. LEMMA. Let X be the limit space of the continuous inverse system $(X_\alpha, p_{\alpha\beta})_{\alpha\leq\beta<\tau}$ of spaces X_α and surjections $p_{\alpha\beta}$: $X_\beta \to X_\alpha$ ($\alpha \leq \beta < \tau$). Then if $p_{\alpha,\alpha+1}$ is a perfect map for all $\alpha < \tau$, so is the limit map $p_\alpha: X \to X_\alpha$.

4.23. THEOREM. Let X be the limit space of the continuous inverse system $(X_\alpha, p_{\alpha\beta})_{\alpha\leq\beta<\tau}$ of spaces X_α and surjections $p_{\alpha\beta}$: $X_\beta \to X_\alpha$ ($\alpha \leq \beta < \tau$). Furthermore, suppose that $p_{\alpha,\alpha+1}$ is locally 0-soft for all $\alpha < \tau$. Then $p_\alpha: X \to X_\alpha$ is locally 0-soft for all $\alpha < \tau$. Moreover, X and X_α ($\alpha < \tau$) are Dugundji manifolds simultaneously.

Proof. By lemma 4.22 all maps p_α ($\alpha < \tau$) are perfect. All spaces X_α ($\alpha < \tau$) are locally compact, since X_α is the range space of the locally 0-soft map $p_{\alpha,\alpha+1}$.

Let $\alpha < \tau$ be fixed. To infer the local 0-softness of p_α we shall apply proposition 4.13. Let K_α be any compact subset of X_α and let us put $K_\beta := (p_{\alpha\beta})^{-1}(K_\alpha)$ for $\alpha \leq \beta < \tau$. We consider the

following continuous inverse system of compact spaces and surjections

$$(*) \qquad \left(K_\beta, (p_{\beta\gamma})_{K_\beta} \right)_{\alpha \leqslant \beta \leqslant \gamma < \tau}.$$

It follows from our hypothesis and proposition 4.13 that for all β ($\alpha \leqslant \beta < \tau$) the map $(p_{\beta,\beta+1})_{K_\beta}$ is 0-soft. Then by remark 4.11 the limit map onto the first space K_α of the induced inverse system (*) is 0-soft and this map coincides with $(p_\alpha)_{K_\alpha}$. Hence an application of proposition 4.13 yields the local 0-softness of p_α.

The remaining statement of the theorem can now be deduced from proposition 4.15. □

This theorem yields as a corollary our promised result on quotient spaces of locally compact groups.

4.24. COROLLARY. Let G be a locally compact group and H a closed subgroup of G. Then the quotient space G/H is a Dugundji manifold.

Proof. By theorem 16 of Skljarenko's paper [41] the quotient space G/H is homeomorphic to a product $\mathbb{D}^A \times B \times C$, where A, B are discrete spaces and C is a connected quotient space of some projectively-Lie group. So by proposition 4.10 it is sufficient to consider the case where G is a projectively-Lie group. It follows from theorem 1 of [41] that G/H allows a continuous inverse limit representation of the following kind. G/H is homeomorphic to the inverse limit space of a continuous inverse system $(X_\alpha, p_{\alpha\beta})_{\alpha \leqslant \beta < \tau}$ of spaces X_α and perfect maps $p_{\alpha\beta}: X_\beta \to X_\alpha$ ($\alpha \leqslant \beta < \tau$) such that

(i) X_1 is a quotient space of a Lie group by a closed subgroup,

(ii) $p_{\alpha,\alpha+1}$ is a locally trivial fibration, the fibre being a compact manifold.

Clearly (i) implies that X_1 is a locally compact, metrizable space and hence a Dugundji manifold.
In order to deduce from (ii) that $p_{\alpha,\alpha+1}$ is locally 0-soft we shall apply proposition 4.17. Since the map $p_{1\alpha}$ is perfect by hypothesis, the space X_α is locally compact for all $\alpha < \tau$. Moreover, the fibre of $p_{\alpha,\alpha+1}$ is a Dugundji space by (ii) and hence $p_{\alpha,\alpha+1}$ is locally 0-soft. Finally we apply the preceding theorem to complete the proof. □

We notice that this corollary yields an affirmative answer to problem 21 (and thus to problem 20 too) of Pełczyński's book ([28], p. 70).

Pełczyński showed that for a compact group G and a closed subgroup H of G the canonical quotient map G → G/H allows a r.a.o. ([28], Proposition 7.4). An investigation of the proof of theorem 15 of Skljarenko's paper [41] reveals that for any compact group G and closed subgroups H and K of G with $K \subset H$ the natural quotient map G/K → G/H can be represented as an inverse limit map p_1 of a system of perfect maps $(p_{\alpha\beta})_{\alpha\leq\beta<\tau}$ such that the linking maps $p_{\alpha,\alpha+1}$ are locally trivial fibrations with a compact manifold as fibre. Thus by proposition 4.17 and corollary 4.14 the map $p_{\alpha,\alpha+1}$ ($\alpha < \tau$) is 0-soft and consequently the same holds for the limit map p_1 (see remark 4.11). We have proved the following extension of proposition 4.19.

4.25. REMARK. Let G be a compact topological group and H, K closed subgroups of G with $K \subset H$. Then the natural quotient map G/K → G/H is 0-soft. In particular, the quotient map G → G/H is 0-soft.

From sections 1 and 2 we recall that O-softness is a considerably stronger condition than z.d.l.p. which in turn implies the existence of a r.a.o.

Let G be a compact group and H a closed subgroup. Although the above mentioned linking maps $p_{\alpha,\alpha+1}$ in the inverse limit representation of the quotient map $G \to G/H$ are locally trivial fibrations, it is not true in general that the quotient map itself is a locally trivial fibration. Let us illustrate this phenomenon by describing an example hinted at on page 72 of [41].

4.26. EXAMPLE. Let $\mathbb{T}$ be the multiplicative compact group of complex numbers of modulus 1 and let G be the product group $\mathbb{T}^{\omega}$ obtained by taking countably many copies of the group $\mathbb{T}$. Let H be the solenoidal subgroup of G, i.e. H is the inverse limit group of the countable inverse system $(H_m, p_{mn})_{m \leq n < \omega}$, where H_m is $\mathbb{T}$ for all $m < \omega$ and $p_{m,m+1}: H_{m+1} \to H_m$ is the surjective group homomorphism given by $z \mapsto z^2$ and $p_{mn} := p_{m,m+1} \circ \ldots \circ p_{n-1,n}$ for all $m \leq n < \omega$. Then the quotient map $G \to G/H$ is not a locally trivial fibration.

Proof. Suppose the contrary. Then G is locally homeomorphic to a product of a subset of G/H and H. But it is well-known that H is not locally connected (see e.g. [17], p. 149) and so the product set is not locally connected. But this contradicts the local connectivity of G. □

In this context it seems interesting to recall theorem 8 of Mostert's paper [26]. It states that for a locally compact group G and a closed subgroup H of G the canonical quotient map $G \to G/H$ is

a locally trivial fibration, if G/H is finite dimensional. Note that by lemma 4.18 the restriction on the finiteness of the dimension of G/H can be omitted if H is a compact Lie group.

In his paper [2] Arhangel'skii posed the question whether a ~~closed~~ compact G_δ-subset of a topological group is dyadic. It follows from corollary 4.24 and proposition 4.6 that a compact G_δ-subset of a quotient space of a locally compact group is a Dugundji space and hence in particular a dyadic space.

In [27] Pasynkov introduced and investigated those topological groups which contain a compact G_δ-subset and called them <u>almost metrizable groups</u>. Clearly the class of almost metrizable groups encompasses all locally compact groups and all metrizable groups. In the proof of his theorem 16 Shchepin [37] stated that an almost metrizable group allows an inverse limit representation similar to Pontryagin's representation of compact groups mentioned earlier.

An almost metrizable group G is the inverse limit of a continuous inverse system of paracompact topological groups G_α ($\alpha < \tau$) and surjective group homomorphisms $p_{\alpha\beta}: G_\beta \to G_\alpha$ ($\alpha \leq \beta < \tau$) such that

(i) G_1 is a metrizable group,

(ii) $p_{\alpha,\alpha+1}$ has a compact Lie group as algebraic kernel for all $\alpha < \tau$.

In particular the maps $p_{\alpha,\alpha+1}$ are perfect and open. One infers from Gleason's well-known theorem (see e.g. [41], p. 60) that $p_{\alpha,\alpha+1}$ is a locally trivial fibration for all $\alpha < \tau$. Shchepin used

these facts and an inverse limit theorem for κ-metrizable spaces ([37], Theorem 14) to conclude that an almost metrizable group is a κ-metrizable space. This result appears as theorem 16 of Shchepin' s paper [37]. Hence for locally compact groups and their quotient spaces corollary 4.24 establishes an improvement on Shchepin's theorem (see lemma 4.5).

On the other hand in general corollary 4.21 (and hence corollary 4.24) does not allow an extension to almost metrizable groups. For any normed vector space is a metrizable group and so in particular an almost metrizable group. But a normed vector space is locally compact if and only if it is finite dimensional.

§5 A classification of Dugundji spaces, Peano spaces and applications to compact groups

In this section we shall mainly be interested in compact spaces. In the category of compact spaces and continuous maps Dugundji spaces are classified as absolute extensors for zero-dimensional spaces or, equivalently, they are precisely those spaces which can be represented as an inverse limit of a continuous inverse system starting with a metrizable space and having 0-soft linking maps. This is basically Haydon's result [14]. More details can be found in Shchepin's work [37]. Inside the class of Dugundji spaces there is the fairly small class of absolute extensors or, equivalently, absolute retracts. Absolute extensors are characterized by Shchepin [37] as those compact spaces that can be represented as an inverse limit of a continuous inverse system starting with a metrizable AE and having soft linking maps (i.e. the dimension restriction in definition 2.7 is suppressed). Here we shall present a countable spectrum of classes of compact spaces which can be located between the Dugundji spaces on one end and the absolute extensors on the other end; moreover, the spectrum admits a partial ordering.

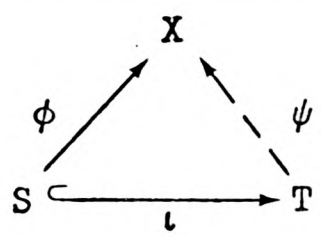
Each class of our spectrum is related to a surjective and an injective class of maps in a similar way as the class of AE(0-dim) is compatible with 0-soft maps (see proposition 3.13) and embeddings satisfying the z.d.r.p. (see corollary 2.6). In particular, each class of spaces is closed under taking continuous inverse limits where the linking maps of the inverse system belong to the corresponding surjective class of maps. For the case of AE(0-dim) this was pointed out in remark 4.11.

Restricting ourselves to compact metrizable spaces our classification yields a good characterization of Peano spaces in the following sense. We can classify Peano spaces in a category-theoretical way naturally corresponding to the ones known to hold for Dugundji spaces as AE(0-dim) and for absolute retracts as AE. Hence it seems sensible to extend this characterization to introduce the notion of generalized Peano spaces of arbitrary weight. We describe them in a surjective fashion too.

Within the realm of compact topological groups (resp. compact convex metrizable spaces) we shall investigate certain surjections of the classes of maps mentioned above. As a key result in this context we prove the triviality of compact contractible groups.

Defining each class of compact spaces of the countable spectrum in a fashion corresponding to the notion of AE(0-dim) and AE we require that certain maps into the given space always admit continuous extensions to specified bigger spaces. What we mean exactly is indicated by the following definition. For convenience of notation we shall regard ∞ as an integer which is bigger than all finite integers.

5.1. DEFINITION. Let X be a compact space and (m,n) a pair of integers with $0 \leq m \leq n \leq \infty$. Then X is an <u>absolute extensor for m- to n-dimensional spaces</u> or <u>AE(m,n)</u> if for all embeddings $\iota: S \to T$ of compact spaces S, T with $\dim S \leq m$, $\dim T \leq n$ and all maps $\phi: S \to X$ there is a map $\psi: T \to X$ such that the diagram



commutes, i.e. $\psi \circ \iota = \phi$.

Obviously X is a Dugundji space or, equivalently, an AE(0-dim) (resp. an AE or, equivalently, an AR) if and only if X is an AE(0,0) (resp. AE(∞,∞)). We shall start our investigation of spaces of the type AE(m,n) by making some general observations.

5.2. PROPOSITION. Let $0 \leq m \leq n \leq \infty$. Then the class of AE(m,n) is closed under taking arbitrary products.

The next result yields the promised partial ordering of the spectrum of classes of AE(m,n). The straightforward proof is omitted.

5.3. PROPOSITION. Let m, n, k, l be integers with $0 \leq m \leq k \leq \infty$, $0 \leq n \leq l \leq \infty$ and $m \leq n$, $k \leq l$. Then the class of AE(k,l) is contained in the class of AE(m,n).

We can check that the class of AE(m,m) coincides with the class AE(n,n) if and only if m = n. For the following is valid.

5.4. PROPOSITION. For all n ($0 \leq n < \infty$) S^n is an AE(n,n) but not an AE(n,n+1).

Proof. The first statement is an immediate consequence of the known characterization of the dimension function dim. Moreover, if S^n were an AE(n,n+1) it would be a retract of the (n+1)-dimensional closed unit ball of $\mathbb{R}^{n+1}$ and thus contractible. That is a contradiction. □

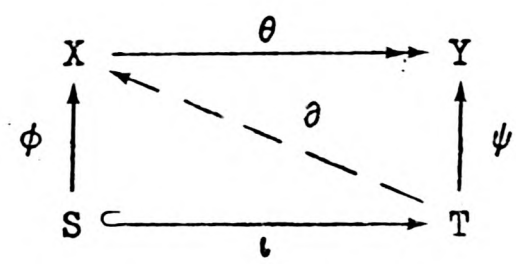
5.5. PROPOSITION. Let $0 \leq m \leq n \leq \infty$ and $n \geq 1$. If X is

an AE(m,n), then X is pathwise connected.

Proof. By proposition 5.3 it is enough to show that AE(0,1) is pathwise connected. Let x_o, x_1 be two points of X and define a map ϕ: $\mathbb{D} \to X$ by $\phi(i) := x_i$ for i=o,1. This map allows a continuous extension ψ: $\mathbb{I} \to X$ by hypothesis and ψ is a path connecting x_o and x_1. □

In section 2 we used the notion of z.d.r.p. to characterize AE(0,0) (see corollary 2.6) and in section 3 an AE(0,0) is described by considering maps with z.d.l.p. Furthermore, Shchepin employed 0-soft (resp. soft) maps to classify AE(0,0) (resp. AE). All these terms fit into a more generalized pattern in which spaces of type AE(m,n) are naturally tied up with (m,n)-soft and (m,n)-hard maps. These notions are defined as follows.

5.6. DEFINITION. Let θ: $X \to Y$ be a surjection of compact spaces X, Y and let (m,n) be a pair of integers with $-1 \leq m \leq n \leq \infty$, $n \geq 0$. Then θ is <u>(m,n)-soft</u>, if for all embeddings ι: $S \to T$ of compact spaces S, T with dim $S \leq m$, dim $T \leq n$ and for all maps ϕ: $S \to X$ and ψ: $T \to Y$ such that $\psi \circ \iota = \theta \circ \phi$ there exists a map ∂: $T \to X$ such that the diagram

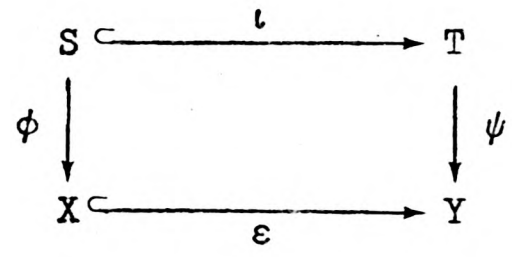


commutes, i.e. $\theta \circ \partial = \psi$ and $\phi = \partial \circ \iota$.

Moreover, for convenience of notation, for $0 \leq n < \infty$ (n,n)-soft maps (resp. (∞,∞)-soft maps) are frequently called <u>n-soft</u> (resp. <u>soft</u>).

Note that this notation coincides with the one given in definition 2.7. Furthermore dim S = -1 means S = ϕ and so θ is (-1,0)-soft if and only if θ satisfies the z.d.l.p. Clearly, θ is a retraction if and only if θ is (-1, dim Y)-soft.

5.7. DEFINITION. Let ε: X → Y be an embedding of compact spaces X, Y and let (m,n) be a pair of integers with $0 \leq m \leq n \leq \infty$. Then ε is <u>(m,n)-hard</u> if for all embeddings ι: S → T of compact spaces S, T with dim S ≤ m, dim T ≤ n and for all maps ϕ: S → X and ψ: T → Y such that the diagram



commutes, i.e. $\varepsilon \circ \phi = \psi \circ \iota$, there exists a map ∂: T → X with $\partial \circ \iota = \phi$.

Obviously, ε is a coretraction if and only if ε is (dim X,dim Y)-hard and ε satisfies the z.d.r.p. if and only if ε is (0,0)-hard.

We continue with some elementary results which we list in the following propositions. Employing a by now familiar method the proofs of the statements can be verified easily and therefore they are omitted.

5.8. PROPOSITION. Let θ: X → Y be a surjection of compact spaces X, Y and let $0 \leq k \leq m \leq \infty$, $0 \leq l \leq n \leq \infty$ and $k \leq l$, $m \leq n$. Then the following hold.

(i) If θ is (m,n)-soft, then θ is (k,l)-soft.

(ii) If θ is (m,n)-soft, then X is an AE(k,l) if and only if Y is an AE(k,l).

(iii) If θ is a trivial fibration with fibre Z, then θ is (m,n)-

soft if and only if Z is an AE(m,n).

(iii) If $n = m+1$ and θ is (m,n)-soft, then θ has the homotopy lifting property for compact spaces of dimension $\leq m$.

(iv) If X is an AE(m,n) and θ is (-1,m)-soft, then Y is an AE(m,n).

A standard transfinite induction argument yields the next result.

5.9. PROPOSITION. Let X be the inverse limit space of the continuous inverse system $(X_\alpha, p_{\alpha\beta})_{\alpha\leq\beta<\tau}$ of compact spaces X_α and surjections $p_{\alpha\beta}: X_\beta \to X_\alpha$ for $\alpha \leq \beta < \tau$. Let m, n be integers with $-1 \leq m \leq n \leq \infty$, $n \geq 0$. Furthermore, suppose that for all $\alpha < \tau$ the linking maps $p_{\alpha,\alpha+1}: X_{\alpha+1} \to X_\alpha$ are (m,n)-soft. Then $p_\alpha: X \to X_\alpha$ is a (m,n)-soft map for all $\alpha < \tau$. Moreover, if $m \geq 0$ then the spaces X and X_α are AE(m,n) simultaneously for all $\alpha < \tau$.

In passing we mention that (m,n)-soft maps ($0 \leq m \leq n \leq \infty$) are open, since by proposition 5.8 (i) they are (0,0)-soft and consequently open (see remarks following definition 2.7). We now state the injective counterpart of proposition 5.8.

5.10. PROPOSITION. Let $\varepsilon: X \to Y$ be an embedding of compact spaces X, Y and let $0 \leq k \leq m \leq \infty$, $0 \leq l \leq n \leq \infty$ and $k \leq l$, $m \leq n$. Then the following hold.

(i) If ε is (m,n) hard, then ε is (k,l)-hard.

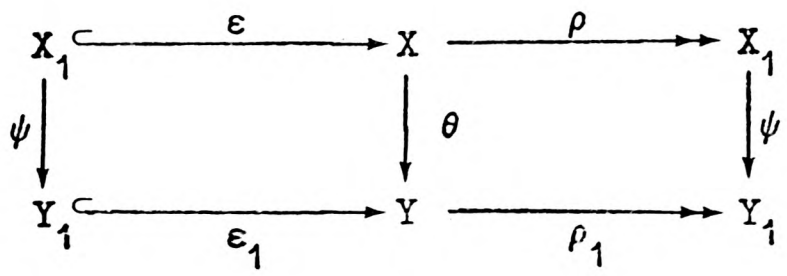
(ii) If Y is an AE(m,n), then ε is (k,l)-hard if and only if X is an AE(k,l).

(iii) Let Y be the product space $Y_1 \times Y_2$ of two compact spaces Y_1, Y_2 and denote by $p: Y_1 \times Y_2 \to Y_1$ the canonical projection. If $p\circ\varepsilon: X \to Y_1$ is surjective and (m,n)-soft, then ε is (m,n)-hard.

In section 1 we noticed that z.d.l.p. is an extensive and hereditary property. Generally we cannot expect (m,n)-softness to be an extensive property for $m \geq 0$, since by a result following definition 2.7 (0,0)-softness does not satisfy this property. However, it is an immediate consequence of definition 5.6 that (m,n)-softness is a hereditary property.

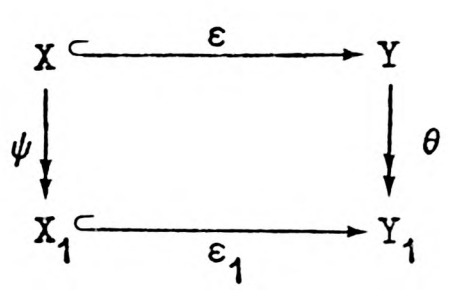
Extending the techniques developed in section 2 we shall now proceed and give a sufficient condition for a restriction of a (m,n)-soft map to be of the same type. In contrast to the definition of the hereditary property we do not require the domain space of the restriction to be an inverse image by the original map. The key result is provided by the following lemma. We choose to omit its straightforward proof.

5.11. LEMMA. Suppose we are given a commutative diagram



of compact spaces and maps such that $\rho \circ \varepsilon = id_{X_1}$ and $\rho_1 \circ \varepsilon_1 = id_{Y_1}$. Then if θ is (m,n)-soft, so is ψ.

5.12. PROPOSITION. Let $\varepsilon: X \to Y$ (resp. $\varepsilon_1: X_1 \to Y_1$) be an embedding of compact spaces admitting a multiplicative r.e.o. $u: C(X) \to C(Y)$ (resp. $u_1: C(X_1) \to C(Y_1)$). Let θ, ψ be surjections such that the diagram



commutes and suppose $u(f\circ\psi) = u_1(f)\circ\theta$ holds for all $f \in C(X_1)$. Then if θ is (m,n)-soft, so is ψ.

Proof. It follows from the proof of proposition 3.3 of Pełczyński's book [28] that there exists a retraction $\rho\colon Y \to X$ (resp. $\rho_1\colon Y_1 \to X_1$) for ε (resp. ε_1) with $u = \rho^\circ$ (resp. $u_1 = (\rho_1)^\circ$). In order to apply lemma 5.11 it remains to verify the equation $\psi\circ\rho = \rho_1\circ\theta$.

Let y be a point of Y and f an element of $C(X_1)$. The additional hypothesis on the interrelationship of u and u_1 yields

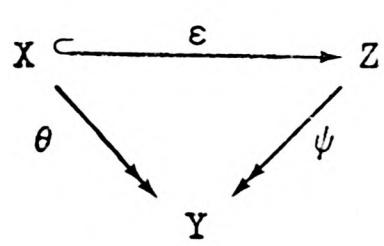
$$f(\psi\circ\rho(y)) = \rho^\circ(f\circ\psi)(y) = u(f\circ\psi)(y) = u_1(f)\circ\theta(y) = (\rho_1)^\circ(f)(\theta(y))$$
$$= f(\rho_1\circ\theta(y)).$$

Since the elements of $C(X_1)$ separate the points of X_1 we infer $\psi\circ\rho = \rho_1\circ\theta$. □

If we are just interested in 0-softness, we can suppress the multiplicativity in the preceding proposition. This result was precisely proved in proposition 2.13. Multiplicativity is a rather strong condition and in fact it is suitable for the smallest class of (m,n)-soft maps (i.e. the class of soft maps) for which proposition 5.12 still applies. This follows from the injective characterization of softness we shall give in the sequel.

In analogy to the notion of fibre-preserving r.e.o. we investigated in section 2 the following piece of terminology is introduced to suit our setting here.

5.13. DEFINITION. Suppose we are given a commutative diagram



of compact spaces X, Y, Z, an embedding ε and surjections ψ, θ. A retraction $\rho\colon Z \to X$ for ε is called fibre-preserving (for the diagram) if $\theta \circ \rho = \psi$.

Copying the method we used in the proof of proposition 5.12 we can easily verify the following.

5.14. PROPOSITION. Suppose we are given a commutative diagram as in definition 5.13. Then the following are equivalent.

(i) ε allows a fibre-preserving multiplicative r.e.o.

(ii) There exists a fibre-preserving retraction for ε.

Comparing this result with proposition 2.10 reveals that the additional assumption of the multiplicativity assures that the probability measures considered there are in fact point measures (and vice versa). The tools we now have at our disposal allow us to give a brief proof of a soft analogue of proposition 2.11.

5.15. PROPOSITION. Let $\theta\colon X \to Y$ be a surjection of compact spaces X, Y. Then the following are equivalent.

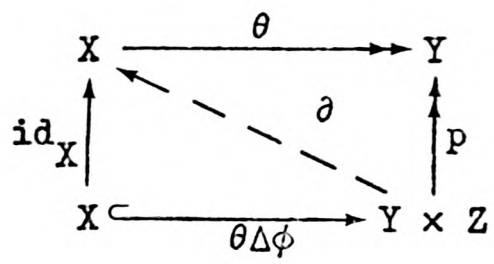
(i) θ is soft.

(ii) Whenever $\phi\colon X \to Z$ is a map into a compact space Z such that the diagonal product map $\theta\Delta\phi\colon X \to Y \times Z$ is an embedding, then $\theta\Delta\phi$ allows a fibre-preserving multiplicative r.e.o.

(iii) There exist an absolute extensor Z and a map $\phi\colon X \to Z$ such that the diagonal product map $\theta\Delta\phi\colon X \to Y \times Z$ is an embedding allowing a fibre-preserving retraction.

Proof. To prove that (i) implies (ii) let us denote by $p\colon Y \times Z \to Y$ the canonical projection. Using the hypotheses we in-

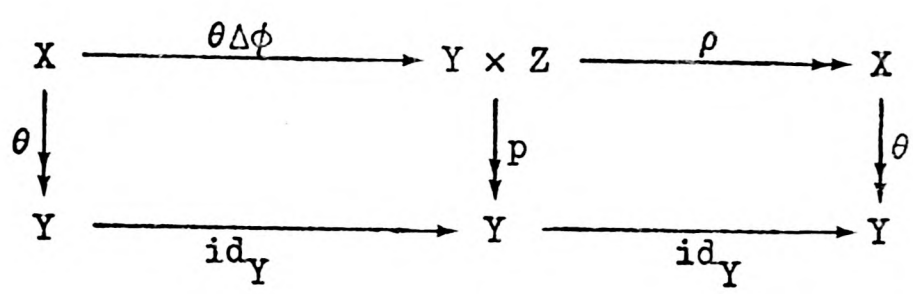
fer the existence of a map ∂: $Y \times Z \to X$ such that the diagram



commutes. But this exactly means that ∂ is a fibre-preserving retraction for $\theta\Delta\phi$ and proposition 5.14 completes this implication.

In order to deduce (iii) from (ii) we let ϕ be any embedding of X into some I^A and apply proposition 5.14 to finish the argument.

Now let us assume (iii) holds. Let p: $Y \times Z \to Y$ be the canonical projection and ρ: $Y \times Z \to X$ the fibre-preserving retraction for $\theta\Delta\phi$. Then the diagram



commutes and satisfies the assumptions of lemma 5.11. Hence to infer the softness of θ it suffices to argue that p is soft. But by proposition 5.8 (iii) p is soft if and only if Z is an AE. □

In particular this result shows that in the case of soft maps multiplicativity in proposition 5.12 is essential.

So far all results in this section on (m,n)-softness have been of a rather general nature. In order to get a better idea of what it really means for a map to be (m,n)-soft let us consider the case where we deal with compact metrizable spaces. As we have already noticed in section 1 even the (-1,0)-soft maps are difficult to describe for the non-metrizable case (for the corresponding diffi-

culties for (0,0)-hard maps see section 2).

5.16. THEOREM. Let be a compact metrizable space. Then the following are equivalent.

(i) Every embedding of X into a compact space is $(0,\infty)$-hard.

(ii) X is an AE$(0,\infty)$.

(iii) X is an AE(0,n) where n ($1 \leq n < \infty$) is arbitrary.

(iv) X is a continuous image of $\mathbb{I}$.

(v) X is locally connected and connected, i.e. X is a Peano space.

(vi) X is a continuous image of $\mathbb{I}$ by a map allowing a r.a.o.

(vii) X is a continuous image of $\mathbb{I}$ by a (-1,0)-soft map.

Proof. Certainly, by proposition 5.10 (ii) the statements (i) and (ii) are equivalent, since X is homeomorphic to a closed subspace of $\mathbb{I}^\omega$ and $\mathbb{I}^\omega$ is an AE.

Without restriction to metrizability (ii) implies (iii) by proposition 5.3.

We prove that (iv) follows from (iii). X is as a compact metrizable space the continuous image of the Cantor set $\mathbb{D}^\omega$ by a map ϕ: $\mathbb{D}^\omega \to X$. Now we regard $\mathbb{D}^\omega$ as a closed subset of the one-dimensional unit interval $\mathbb{I}$ and,if X is an AE(0,1), then the map ϕ allows a continuous extension ψ: $\mathbb{I} \to X$.

The equivalence of the statements (iv) and (v) is the well-known Hahn-Mazurkiewicz theorem (see e.g. [43], Theorem 31.5).

To verify that (v) implies (vi) we investigate the standard technique that is employed to deduce (iv) from (v).One starts with any surjection $\mathbb{D}^\omega \to X$ and uses the local connectedness and connectedness of the metrizable space X to continuously extend this map to a surjection $\mathbb{I} \to X$ (see e.g. [43], Theorem 31.5 or, alternatively,

[17], Theorem 3.30). Now, instead of starting with an arbitrary surjection $\mathbb{D}^\omega \to X$ we may choose a surjection $\mathbb{D}^\omega \to X$ allowing a r.a.o. by proposition 1.7 (ii). Since the existence of a r.a.o. is an extensive property the continuous extension $\mathbb{I} \to X$, constructed as indicated above, allows a r.a.o. too.

The equivalence of the statements (vi) and (vii) is a consequence of theorem 1.11.

It remains to show that (vii) implies (i) or, equivalently, (ii). But this follows immediately from proposition 5.8 (v), since $\mathbb{I}$ certainly is an AE$(0,\infty)$; in fact $\mathbb{I}$ is an AE by Tietze's theorem. □

In his monograph Pełczyński showed that the surjection $\theta: \mathbb{I} \to S^1$, given by $\theta(t) := e^{2\pi it}$ for all $t \in \mathbb{I}$, does not allow a r.a.o. ([28], Example 2, p. 65). Statement (vi) of the preceding theorem however implies that for all n ($1 \leq n < \infty$) we can always find a surjection $\mathbb{I} \to S^n$ allowing a r.a.o.

Since a one-point space is an AE we infer from proposition 5.8 (ii) and the preceding theorem the following equivalence.

5.17. COROLLARY. Let X be a compact metrizable space. Then the following are equivalent.

(i) The constant map $X \to \{X\}$ is $(0,n)$-soft for an arbitrary fixed integer n ($1 \leq n \leq \infty$).

(ii) X is a Peano space.

We shall continue our investigation of Peano spaces by verifying a property that is shared by all spaces of type AE$(0,\infty)$. As a corollary it generalizes a result known to hold for Peano

spaces; namely,if X and Y are non-trivial Peano spaces, then there exists a surjection of X onto Y ([43], Problem 31 A, p. 222).

5.18. PROPOSITION. Let Y be an AE(0,∞) of infinite weight τ and let X be a compact space containing a copy of $\mathbb{D}^{\tau}$. Then there is a (-1,0)-soft map of X onto Y.

Proof. By remark 1.16 there exists a zero-dimensional compact space S of weight τ and a (-1,0)-soft map ϕ: S → **Y**. Our hypotheses imply the existence of an embedding ι: S → X and of a continuous extension ψ: X → Y of ϕ, i.e. $\psi \circ \iota = \phi$. (-1,0)-softness is an extensive property and hence with ϕ the map ψ is (-1,0)-soft too. □

In passing we mention that by proposition 8.10 of Pełczyński's book [28] every compact group of infinite weight τ contains a copy of $\mathbb{D}^{\tau}$.

5.19. COROLLARY. Let X, Y be non-trivial Peano spaces. Then there exists a surjection of X onto Y satisfying the z.d.l.p.

Proof. Since X is non-trivial there is a copy of $\mathbb{I}$ in X and hence $\mathbb{D}^{\omega}$ embeds into X. □

Proposition 5.18 immediately yields the following surjective characterization of the class of AE(0,∞).

5.20. PROPOSITION. A compact space is an AE(0,∞) if and only if it is the continuous image of some $\mathbb{I}^{A}$ by a (-1,0)-soft map.

Proof. Obviously, if the compact space is finite either

hypothesis implies that it consists of a single point and so, in this case, there is nothing to prove.

Let us now assume that we are given a compact space Y of infinite weight τ. The necessity follows from the preceding proposition by setting $X := I^{\tau}$. To show sufficiency we apply statement (v) of proposition 5.8 with $m = 0$ and $n = \infty$. □

Although there have been made several attempts in different directions (see [22] and [43], Notes on section 31, p. 315 for further references) - as far as the author knows - there does not exist a good generalization of the notion of Peano space for the non-metrizable case. It seems to us that theorem 5.16 as well as propositions 5.18 and 5.20 all suggest defining the class of $AE(0,\infty)$ as the class of <u>generalized Peano spaces</u> or, equivalently, the generalized Peano spaces as precisely those compact spaces which are continuous images of some Tychonoff cube I^{A} by a map satisfying the z.d.l.p. That is proposition 5.20. This definition gives us a good characterization of generalized Peano spaces in complete analogy to the one of Michael spaces (definition 3.1). An elementary verification shows that proposition 3.3 admits the following counterpart.

5.21. PROPOSITION. The class of generalized Peano spaces is the smallest class of compact spaces that contains I and is closed under taking arbitrary products and continuous images by maps satisfying the z.d.l.p.

This again shows that the notion of generalized Peano space can be regarded as a hybrid of the notions of $AE(0,0)$ (or, equiv-

alently, Michael space) and AE(∞,∞) (or, equivalently, absolute retract).

In passing we notice that there exists a locally connected, connected compact abelian group (of necessarily uncountable weight) that is not path-connected ([8], Théoreme 4 (iii)). Hence generally a locally connected, connected Dugundji space (cf. proposition 4.20) is not a generalized Peano space, in fact not even an AE(0,1).

Proceeding with our investigation of (m,n)-soft maps we shall now exhibit some interesting examples. First let us illustrate a non-trivial (0,0)-soft map that is not (0,1)-soft. Note that constant maps (resp. projections) of this type are easy to construct. Let X be any compact metrizable space which is not a Peano space. Then $X \to \{X\}$ (resp. $X \times Y \to Y$, where Y is any compact space) has the desired properties by corollary 5.17 (resp. proposition 5.8 (iii)).

5.22. EXAMPLE. Let $\mathbb{T}$ be the multiplicative compact group of complex numbers of modulus 1. Then the surjective continuous group homomorphism $\theta\colon \mathbb{T} \to \mathbb{T}$, $t \mapsto t^2$, is not (-1,1)-soft (and hence not (0,1)-soft) but (0,0)-soft.

Proof. Obviously, θ is homeomorphic to the canonical quotient map $\mathbb{T} \to \mathbb{T}/H$ where H denotes the kernel of θ and so remark 4.25 implies the 0-softness of θ. Since $\mathbb{T}$ is one-dimensional, θ is (-1,1)-soft if and only if θ is a retraction which certainly isn't the case. □

The following example exhibits a (1,1)-soft map of compact groups which is not (1,2)-soft.

5.23. EXAMPLE. Let G be a compact, connected abelian topological group and H a closed subgroup of G which is isomorphic (as a topological group) to a product of one-dimensional tori $\mathbb{T}$. Then the canonical quotient homomorphism θ: G → G/H is (1,1)-soft but not (1,2)-soft.

Proof. Since H is isomorphic to some $\mathbb{T}^A$ we infer that H is a direct factor of G ([16], 25.31) and the quotient homomorphism is homeomorphic to a trivial fibration with fibre $\mathbb{T}^A$. It follows from propositions 5.4 and 5.2 that $\mathbb{T}^A$ is an AE(1,1) but not an AE(1,2). Consequently, the trivial fibration is (1,1)-soft but not (1,2)-soft by proposition 5.8 (iii) and then the same holds for the quotient homomorphism θ. □

Before we continue to prove some more, rather interesting, results about compact topological groups we shall mention a family of non-trivial soft maps. A type of selection theorem of E.Michael ([24], Theorem 1.2), different from the one considered in section 1, is again the key tool for us.

5.24. LEMMA. Let T be a paracompact space and X a metrizable subset of a locally convex topological vector space E. Let Φ: T → exp(X) be a lower semicontinuous set-valued mapping and suppose for some metric on X the sets $\Phi(t)$ are complete for all $t \in T$. Then there exists a continuous map ∂: T → E such that $\partial(t)$ is contained in the closed convex hull of $\Phi(t)$ for all $t \in T$.

5.25. PROPOSITION. Let X, Y be convex compact metrizable subsets of some locally convex topological vector spaces and let θ: X → Y be an affine continuous surjection. Then θ is soft if and

only if θ is open.

Proof. Since for compact metrizable spaces 0-softness of θ is equivalent to openness of θ (cf. [37], p. 163) it remains to show the sufficiency. Suppose we are given an embedding $\iota\colon S \to T$ of compact spaces S and T and maps $\phi\colon S \to X$, $\psi\colon T \to Y$ such that $\psi \circ \iota = \theta \circ \phi$ holds. Then the set-valued mapping $\Phi\colon T \to \exp(X)$ defined by

$$\Phi(t) := \begin{cases} \phi(t) & \text{if } t = \iota(s) \text{ for some } s \in S \\ \theta^{-1}(\psi(t)) & \text{if } t \in T \setminus \iota S \end{cases}$$

is lower semicontinuous and satisfies the hypotheses of the preceding lemma. Since $\Phi(t)$ is compact and convex for all $t \in T$ there exists a continuous selection $\partial\colon T \to X$ for Φ and it yields $\theta \circ \partial = \psi$, $\partial \circ \iota = \phi$ by construction. □

As the following example shows this proposition gives rise to a range of soft maps.

5.26. EXAMPLE. Let X, Y be compact metrizable spaces and $\theta\colon X \to Y$ an open surjection. Then the induced map of probability spaces $P(\theta)\colon P(X) \to P(Y)$ is soft.

Proof. By a result of Ditor and Eifler [6] the map $P(\theta)$ is open if and only if θ is open (and this statement holds without any restriction to metrizability). Hence $P(\theta)$ satisfies the assumptions of the preceding proposition. □

We notice that the metrizability in proposition 5.25 is indeed essential. For theorem 1 (resp. theorem 3) of [7] states that $P(\mathbb{D}^{\omega_1})$ is (resp. $P(\mathbb{D}^{\omega_2})$ is not) an AE. Let $\theta\colon \mathbb{D}^{\omega_1} \times \mathbb{D}^{\omega_1} \to \mathbb{D}^{\omega_1}$ denote the projection onto the first factor. Then it is a con-

sequence of proposition 5.9 that the softness of the induced map $P(\theta)$: $P(\mathbb{D}^{\omega_1} \times \mathbb{D}^{\omega_1}) \to P(\mathbb{D}^{\omega_1})$ implies that $P(\mathbb{D}^{\omega_2})$ is an AE. This is a contradiction and so $P(\theta)$ is not soft but open by the result of Ditor and Eifler quoted above.

Finally we shall characterize soft homomorphisms θ: $G \to H$ of compact topological groups G and H. Obviously θ is homeomorphic to the canonical quotient homomorphism $G \to G/K$ where K denotes the kernel of θ. As we have noticed in remark 4.25 a quotient map of compact groups is always 0-soft and then so is θ. But in contrast to proposition 5.25 we shall show that in the realm of compact groups a surjective homomorphism is soft if and only if it is an isomorphism. This result turns out to be a corollary of the following theorem which we prove through a series of lemmas.

We recall that a topological space is called <u>contractible</u> if the identity map and any constant map are homotopic.

5.27. THEOREM. A compact contractible topological group is trivial.

Before we go on to give a proof of this result let us consider two special types of compact contractible topological groups, namely abelian and Lie groups.

5.28. LEMMA. Let A be a compact contractible abelian topological group. Then A is trivial.

Proof. Suppose $A \neq 0$. Since the characters of A separate the points of A we infer that the character group of A is non-trivial. On the other hand by theorem 1.4 of [20] the character

group of a compact connected abelian group is isomorphic to its one-dimensional Čech cohomology group over the integers which is trivial in our case, since A is contractible. This contradiction finishes the proof. □

5.29. LEMMA. Let L be a compact contractible Lie group. Then L is trivial.

Proof. Suppose L has dimension $n \geq 1$. Then the Poincaré duality theorem (see e.g. [42], 5.18) implies that the n-dimensional singular cohomology group of L over the integers is isomorphic to the 0-dimensional singular homology group of L over the integers. This is a contradiction, since by the contractibility of L the first one is trivial whereas the second one coincides with the integers. Thus L is a 0-dimensional manifold, i.e. L = 0. □

We notice that the statement of lemma 5.29 is a consequence of the corollary of Scheerer's paper [35]. However, our proof is quite independent from his and very brief; so we gave it.

Finally to reduce the proof of theorem 5.27 to topological groups of the kinds mentioned in lemmas 5.28 and 5.29 we shall need to lift contractibility of a quotient group to the group itself.

5.30. LEMMA. Let G be a connected compact group and H a closed zero-dimensional subgroup of G. If the quotient space G/H is contractible, then G is contractible.

Proof. By theorem 15 of Skljarenko's paper [41] the natural quotient map θ: G → G/H has the homotopy lifting property for arbitrary spaces. Let e denote the identity element of the group G.

Since G/H is contractible, the quotient map θ: G → G/H and the constant map ϕ: G → G/H taking each g ∈ G into θ(e) are homotopic. Now θ has the identity map on G as a continuous lifting and by the homotopy lifting property of θ the map ϕ allows a continuous lifting ψ: G → G which belongs to the homotopy class of the identity map. The image ψ(G) of G is contained in the zero-dimensional subgroup H and ψ(G) is connected, since by hypothesis G has this property and ψ is continuous. Hence ψ is a constant map which is homotopic to the identity map of G, i.e. G is contractible. □

With these three lemmas we have developed all the tools we need to give a brief proof of theorem 5.27.

Proof of theorem 5.27. Let G be a compact contractible topological group. In particular, G is connected and hence G is known to be isomorphic to a quotient group of the type

$$(A \times \prod_{k \in K} L_k)/H$$

where A is a connected compact abelian group, L_k ($k \in K$) is a connected compact Lie group and H is a closed zero-dimensional central subgroup of the product. This characterization can be found in Price's book ([30], 6.5.6). By lemma 5.30 the contractibility of G implies the contractibility of the product group $A \times \prod_{k \in K} L_k$, consequently each factor of the product is contractible and so an application of lemmas 5.28 and 5.29 concludes the proof. □

5.31. COROLLARY. Let G be a contractible locally compact topological group. Then G is homeomorphic to a finite product of real lines.

Proof. By theorem 16' of [41] G is homeomorphic to a product of a compact group H and a finite product of real lines. The contractibility of G implies the contractibility of H and so H is trivial. □

5.32. COROLLARY. Let G and H be compact topological groups and θ: G → H a surjective homomorphism. Then θ is soft if and only if θ is an isomorphism.

Proof. Let K denote the kernel of θ. Clearly θ is homeomorphic to the canonical quotient map G → G/K . Then if θ is soft, so is the quotient map and hence it is homeomorphic to a trivial fibration with fibre K. By statement (iii) of proposition 5.8 the set K is an AE and thus, in particular, K is a compact contractible group. This implies the triviality of K or, equivalently, that θ is an isomorphism. The sufficiency is evident and the proof is complete. □

§6 The countable chain condition in P(X), $\exp_n(X)$ and $\exp(X)$

Let us recall that a topological space is said to satisfy the countable chain condition or c.c.c. if every family of pairwise disjoint, non-empty, open sets is countable. Closely related to c.c.c. is Shanin's condition or S.c. which requires that every pointwise countable family of non-empty, open sets is countable. It is not hard to see that S.c. is equivalent to the following: Every uncountable family of open, non-empty sets has an uncountable subfamily with non-empty intersection. Trivially, S.c. implies c.c.c. It is well-known that all dyadic spaces satisfy S.c. and on page 165 of [37] Shchepin states that all compact κ-metrizable spaces satisfy S.c. Hence for any κ-metrizable compact space X the space P(X) (resp. exp(X), $\exp_n(X)$) satisfies S.c. For it immediately follows from Shchepin's functor theorem ([37], p. 165) that a compact space Y is κ-metrizable if and only if P(Y) (resp. exp(Y), $\exp_n(Y)$) is κ-metrizable.

In this section we shall be concerned with a wider question: Does it hold in the class of compact spaces that X and P(X) (resp. exp(X), $\exp_n(X)$) satisfy c.c.c. simultaneously? Assuming Martin's Axiom we shall show that a compact space X satisfies c.c.c. if and only if P(X) (resp. $\exp_n(X)$) does. However, we shall see that under the assumption of the Continuum Hypothesis or supposing the existence of a Souslin continuum this equivalence does not hold in general.

Our interest in spaces of the type P(X) was triggered off by the question whether or not P(X) is dyadic for a dyadic space X. Obviously, if one could prove that $P(\mathbb{D}^A)$ is dyadic for all sets A

we could give an affirmative answer to this question. As far as the author is aware the following results are known at present. P(X) is an AE if and only if X is a Dugundji space of weight not exceeding ω_1 ([7], Theorem 4). If X has countable weight we can announce an improvement of this statement: P(X) is homeomorphic to $\mathbb{I}^\omega$ for every infinite compact metrizable space X. This result can be deduced from theorem 2.1 and theorem 3.1 of [4] stated on page 98 and 100 respectively. By Shchepin's result quoted above P(X) is κ-metrizable if and only if X is κ-metrizable, in particular $P(\mathbb{D}^A)$ satisfies S.c. and hence c.c.c. for all sets A. In this section we shall extend this result and show that for any compact space X the spaces P(X) and X^ω satisfy c.c.c. simultaneously.

Spaces of the type exp(X) and $\exp_n(X)$ supply us with a rich source of counterexamples and therefore they are of special interest to us. We recall from proposition 1.19 that the natural surjection $\mathbb{D}^{\omega_1} \times \mathbb{D}^{\omega_1} \to \exp_2(\mathbb{D}^{\omega_1})$, $(x,y) \to \{x,y\}$, was used as an example of a surjection allowing a r.a.o. but not satisfying the z.d.l.p. The space $\exp_2(\mathbb{D}^{\omega_2})$ was exhibited by Shchepin ([37], p. 163) as the first example of a non-Dugundji Milutin space. However, we notice that both $\exp_n(\mathbb{D}^{\omega_1})$ for $n \geq 1$ and $\exp(\mathbb{D}^{\omega_1})$ are known to be homeomorphic to $\mathbb{D}^{\omega_1}$. This follows from Sirota's proof of his theorem 5 of [40]; alternatively see theorem 9 of [37] which states that a zero-dimensional Dugundji space of infinite weight τ that is homogeneous with respect to weight is homeomorphic to $\mathbb{D}^\tau$. The compact space $\exp(\mathbb{D}^{\omega_2})$ is an example of a non-dyadic κ-metrizable space (cf. [33], Theorem and [37], p. 165). In theorem 4 of [34] Šapiro announced the somewhat surprising result that $\exp_n(\mathbb{I}^{\omega_1})$

($n \geq 2$) is not an AR. Furthermore, he proved that $\exp(\mathbb{I}^{\omega_1})$ is not a continuous image of $\mathbb{I}^{\omega_1}$ ([34], Corollary 4). Hence $\exp(\mathbb{I}^{\omega_1})$ is a pathwise connected, locally pathwise connected Dugundji space which is not a generalized Peano space. It would be interesting to know whether or not $\exp_2(\mathbb{I}^{\omega_1})$ is a generalized Peano space. That is to say: Does there exist a surjection $\mathbb{I}^{\omega_1} \to \exp_2(\mathbb{I}^{\omega_1})$ satisfying the z.d.l.p. In passing we mention that by proposition 1.19 the natural map $\mathbb{I}^{\omega_1} \times \mathbb{I}^{\omega_1} \to \exp_2(\mathbb{I}^{\omega_1})$ allows a r.a.o. but does not satisfy the z.d.l.p.

Returning to our problem whether $\exp_n(X)$ (resp. $\exp(X)$) and X satisfy c.c.c. simultaneously we have seen above that the answer is in the affirmative if X is a compact κ-metrizable space. Here we shall generalize this result by proving that $\exp_n(X)$ and X^n satisfy c.c.c. simultaneously. Moreover, if X^ω satisfies c.c.c., so does $\exp(X)$.

Let us begin the investigation concerning the c.c.c. on spaces of the above mentioned types by indicating the crucial difference between the notions of c.c.c. and S.c.

6.1. PROPOSITION. Let $(X_n)_{1 \leq n < \omega}$ be a countable family of spaces and let X be the product space. If each X_n ($1 \leq n < \omega$) satisfies S.c., then so does X.

Proof. Suppose we are given any uncountable family $(W_a)_{a \in A}$ of non-empty, open subsets of X. Denoting by $p_n : X \to X_n$ the canonical projection for each $a \in A$ we can find a finite subset F_a of ω and open subsets U_n^a ($n \in F_a$) of X_n such that W_a contains the intersection

$V_a := \cap \{ p_n^{-1}(U_n^a) : n \in F_a \}$. Hence there exist an uncountable subset B of A and an element $k < \omega$ such that $F_a \subset \{1,2,\ldots,k\}$ for all $a \in B$. By hypothesis X_k satisfies S.c. and so there is an element x_k and an uncountable subset B_k of B such that for all $a \in B_k$ the point x_k is an element of the open set $p_k(V_a)$ which coincides with U_k^a or X_k. By a simple induction argument we successively construct uncountable sets $B_k \supset B_{k-1} \supset \ldots \supset B_1$ and elements $x_k, x_{k-1},\ldots,x_1$ such that $p_i(V_a) \ni x_i$ for all $a \in B_i$ and $1 \leq i \leq k$. If x is any element of X with $p_i(x) = x_i$ for $1 \leq i \leq k$, it follows that x is contained in the uncountable intersection $\cap \{ V_a : a \in B_1 \}$ which is a subset of $\cap \{ W_a : a \in B_1 \}$. This finishes the proof. □

Regarding c.c.c. we are faced with a much more delicate situation. As far as the author knows there are three rather fundamental results which show that under certain set-theoretical hypotheses in addition to the standard ones c.c.c. is stable or unstable under taking finite products. This should be contrasted with proposition 6.1. In item 5.3 of [19] it is proved that under the assumption of Martin's Axiom c.c.c. implies S.c. Hence the following remark is a consequence of the preceding proposition.

6.2. REMARK. Assuming Martin's Axiom c.c.c. is preserved under taking countable products.

A Souslin continuum is a compact connected linearly ordered space of uncountable density satisfying the c.c.c. The existence of a Souslin continuum is known to be independent of the usual axioms of set theory. Furthermore, the following result is well-established (see [11], 2.7.10 (a)).

6.3. REMARK. Suppose there exists a Souslin continuum X. Then $X \times X$ does not satisfy c.c.c.

In a recent preprint [12] Galvin summarized and generalized unpublished work of R.Laver. In particular, he states and proves the following result ([12], Corollary 4.9).

6.4. REMARK. Assuming the Continuum Hypothesis,for every integer n ($1 \leq n < \omega$) there is an extremally disconnected compact space X such that X^n satisfies c.c.c. but X^{n+1} does not.

In order to show that for an arbitrary product space all the information concerning c.c.c. is already contained in its finite subproducts we shall need the following combinatorial lemma which is originally due to Shanin. For a short proof we refer the reader to appendix 2.2 of [19].

6.5. LEMMA. Let A be a set of cardinality ω_1 and $(F_a)_{a \in A}$ a family of finite sets. Then there exist a subset B of A of cardinality ω_1 and a fixed set F such that $F_a \cap F_b = F$ for all distinct elements a, b of B.

6.6. PROPOSITION. Let $(X_a)_{a \in A}$ be a family of spaces. Then the product space $\Pi \{ X_a : a \in A \}$ satisfies c.c.c. if and only if for every finite subset F of A the subproduct $\Pi \{ X_a : a \in F \}$ satisfies c.c.c.

Proof. The necessity is obvious, since certainly c.c.c. is stable under taking images by continuous maps. To prove the converse let us suppose there exists a family $(U_b)_{b \in B}$ of pairwise disjoint, non-empty,open subsets of $\Pi \{ X_a : a \in A \}$ indexed by a set B of

cardinality ω_1. For any subsets C, D of A with $C \subset D$ we denote by

$$p_{C,D}: \Pi \{ X_a : a \in D \} \longrightarrow\!\!\!\!\!\rightarrow \Pi \{ X_a : a \in C \}$$

$$p_D: \Pi \{ X_a : a \in A \} \longrightarrow\!\!\!\!\!\rightarrow \Pi \{ X_a : a \in D \}$$

the canonical projections. By definition of the product topology for each $b \in B$ there exist a finite subset F_b of A and a non-empty, open subset V_b of $\Pi \{ X_a : a \in F_b \}$ such that $(p_{F_b})^{-1}(V_b)$ is contained in U_b. Then the pairwise disjointness implies that for any two distinct b,c of B the intersection $F_b \cap F_c$ is non-empty. Hence by lemma 6.5 there exist a subset C of B of cardinality ω_1 and a non-empty set F such that $F_b \cap F_c = F$ for all distinct elements b,c of C. It follows from the hypothesis that the uncountable family $(p_{F,F_c}(V_c))_{c \in C}$ of non-empty, open subsets of $\Pi \{ X_a : a \in F \}$ is not pairwise disjoint. So there exist two distinct elements b,c of C and elements $y \in V_b$, $z \in V_c$ such that $p_{F,F_b}(y) = p_{F,F_c}(z)$. This equation together with $F_b \cap F_c = F$ implies the existence of an element $x \in \Pi \{ X_a : a \in A \}$ such that $p_{F_b}(x) = y$, $p_{F_c}(x) = z$. But this contradicts the disjointness of U_b and U_c, since we have $x \in (p_{F_b})^{-1}(V_b) \cap (p_{F_c})^{-1}(V_c) \subset U_b \cap U_c$. □

6.7. COROLLARY. Let X be a space. Then X^n satisfies c.c.c. for all $n < \omega$ if and only if X^ω does.

In the case where the index set A in proposition 6.6 is uncountable and the spaces X_a are non-trivial and compact we can mention some further equivalences which were recently proved by Shchepin ([38], Theorem 2). The product space satisfies c.c.c. if and only if for every regular closed subset F of the product space there exists a countable subset B of A such that $p_B^{-1} p_B(F) = F$ or, equiv-

alently, each continuous real-valued function on an open set of the product depends on countably many coordinates.

Before stating and proving our main results characterizing c.c.c. on $P(X)$ and $exp_n(X)$ in terms of a condition on X we list some preparatory lemmas. The first two of them are of topological nature whereas the other two are basically combinatorial lemmas.

6.8. LEMMA. Let X be a space and Y a dense subspace of X. Then X satisfies c.c.c. if and only if Y does.

Proof. To prove sufficiency let us assume the contrary, i.e. there exists an uncountable family $(W_a)_{a \in A}$ of pairwise disjoint, non-empty, open subsets of X. Obviously, by the density of Y in X the family $(W_a \cap Y)_{a \in A}$ consists of non-empty subsets of Y and the sets retain their openness and pairwise disjointness. This is a contradiction.

Conversely, if $(V_a)_{a \in A}$ is an uncountable family of non-empty, open subsets of Y there exist open sets W_a ($a \in A$) of X such that $W_a \cap Y = V_a$. By hypothesis X satisfies c.c.c. and so there exist distinct elements a, b of A such that $W_a \cap W_b \neq \emptyset$. Hence $W_a \cap W_b$ is a non-empty, open subset of X and by the density of Y in X we deduce $W_a \cap W_b \cap Y \neq \emptyset$ or, equivalently, $V_a \cap V_b \neq \emptyset$. Thus we showed that Y satisfies c.c.c. □

6.9. LEMMA. Let X, Y be spaces such that X satisfies c.c.c. and Y is separable. Then $X \times Y$ satisfies c.c.c.

Proof. Suppose the contrary. Without loss of generality we may assume there exists an uncountable family $(U_a)_{a \in A}$ (resp.

$(V_a)_{a \in A}$) of open, non-empty subsets of X (resp. Y) such that the sets $U_a \times V_a$ ($a \in A$) are pairwise disjoint. If D denotes a countable dense subset of Y we deduce that $D \cap V_a$ is non-empty for all $a \in A$ and hence by countability of D there exist an uncountable subset B of A and an element $d \in D$ such that $d \in V_a$ for all $a \in B$. For any two distinct elements a, b of B the sets $U_a \times V_a$ and $U_b \times V_b$ are disjoint if and only if U_a and U_b are disjoint, since both V_a and V_b contain d. Hence $(U_a)_{a \in B}$ is an uncountable family of pairwise disjoint, non-empty, open subsets of X. This contradicts the c.c.c. of X. □

The following combinatorial lemma ([19], Appendix, Theorem 4.6) will be used to select "good" subsequences to provoke a contradiction and hence plays a similar role as lemma 6.5 did in the proof of proposition 6.6. For any set A and any integer n ($1 \leq n < \omega$) we shall agree to denote by $[A]^n$ the set of n-element subsets of A.

6.10. LEMMA. (Ramsey's Theorem). Let A be a countably infinite set and m, n two integers and let $(\mathcal{A}_i)_{1 \leq i \leq m}$ be subsets of $[A]^n$ such that $\cup \{ \mathcal{A}_i : 1 \leq i \leq m \} = [A]^n$ holds. Then there are a countably infinite subset B of A and a fixed index $i \in \{1,\ldots,m\}$ such that $[B]^n$ is contained in $\mathcal{A}_i$.

Finally we shall need the following result which is lemma 4.2 of Rosenthal's paper [32]. His proof uses basically combinatorial methods.

6.11. LEMMA. Let X be a compact space satisfying c.c.c. and let $(U_a)_{a \in A}$ be an uncountable family of non-empty, open subsets of X. Then there exists a countably infinite subset B of A such that the

intersection $\cap \{ U_a : a \in B \}$ is non-empty.

We have now introduced all the tools which we shall need to give a proof of the main result of this section.

6.12. THEOREM. Let X be a compact space. Then the space of probability measures P(X) satisfies c.c.c. if and only if X^ω does.

Proof. We start by verifying the sufficiency. Recall that P(X) is the closed convex hull of the set of point measures $\{ \delta_x : x \in X \}$ and so by an easy density argument and lemma 6.8 it is enough to prove that

$$co_{\mathbb{Q}}(X) := \{ \sum_{i=1}^{n} a_i \cdot \delta_{x_i} : 0 \leqslant a_i \in \mathbb{Q}, \sum_{i=1}^{n} a_i = 1, 1 \leqslant i \leqslant n, n < \omega \}$$

satisfies c.c.c. To this end let us define a countable subset Y of $\mathbb{Q}^\omega$ by putting

$$Y := \{ (a_n)_{n<\omega} \in \mathbb{Q}^\omega : a_n \geqslant 0, a_n > 0 \text{ for only finitely many } n < \omega, \sum_{n<\omega} a_n = 1 \}$$

We endow Y with the discrete topology and well-define a continuous surjection

$$X^\omega \times Y \longrightarrow\!\!\!\!\!\rightarrow co_{\mathbb{Q}}(X)$$

$$((x_n)_{n<\omega}, (a_n)_{n<\omega}) \longmapsto \sum_{n<\omega} a_n \cdot \delta_{x_n}$$

By hypothesis and lemma 6.9 the space $X^\omega \times Y$ satisfies c.c.c. and so does the continuous image $co_{\mathbb{Q}}(X)$. This completes the easier part of the proof.

To show necessity let us assume there exists an element $n < \omega$ such that X^n does not satisfy c.c.c. (see corollary 6.7). Hence

there is an uncountable family $(V_a)_{a \in A}$ of pairwise disjoint, non-empty, open sets in X^n. Without loss of generality we may assume the sets V_a to be of the form $V_a = U_a^1 \times \ldots \times U_a^n$ where U_a^i is a non-empty, open subset of X for all $1 \leq i \leq n$, $a \in A$. Let r be a real number such that $0 < r < n^{-1}$ holds and define open, non-empty subsets W_a of P(X) by putting for all $a \in A$

$$W_a := \{ \mu \in P(X) : \mu(U_a^i) > r, \; i = 1,\ldots,n \}.$$

Since P(X) satisfies c.c.c. by hypothesis, we can apply lemma 6.11 to infer the existence of a countably infinite subset B of A and of a probability measure $\mu \in P(X)$ such that $\mu \in W_a$ for all $a \in B$. For each element $\{a,b\} \in [B]^2$ the intersection $V_a \cap V_b$ is empty and so we can pick a coordinate $i := i(\{a,b\}) \in \{1,\ldots,n\}$ with $U_a^i \cap U_b^i = \emptyset$. Let us set for all $j \in \{1,\ldots,n\}$

$$\mathcal{B}_j := \{ \{a,b\} \in [B]^2 : U_a^j \cap U_b^j = \emptyset \}.$$

We just showed that the finite family $(\mathcal{B}_j)_{1 \leq j \leq n}$ covers $[B]^2$ and so by lemma 6.10 there exist a countably infinite subset C of B and a fixed index $i \in \{1,\ldots,n\}$ such that $[C]^2$ is contained in $\mathcal{B}_i$. But this means that $(U_a^i)_{a \in C}$ is an infinite family of pairwise disjoint, non-empty, open sets and hence the countable additivity of the probability measure μ yields the following contradiction:

$$1 = \mu(X) \geq \mu(\cup \{ U_a^i : a \in C \}) = \sum_{a \in C} \mu(U_a^i) \geq k \cdot r$$

for all $k < \omega$, since $\mu \in W_a$ implies $\mu(U_a^i) > r$. □

With respect to c.c.c. the spaces $\exp_n(X)$ behave similarly to P(X). The proof of the following proposition is based on ideas analogous to the ones exhibited in the proof above.

6.13. PROPOSITION. Let X be a compact space and n ($1 \leq n < \omega$) an integer. Then $\exp_n(X)$ satisfies c.c.c. if and only if X^n does.

Proof. Let X^n satisfy the c.c.c. Since $\exp_n(X)$ is the continuous image of the natural surjection $X^n \to \exp_n(X)$, $(x_1,\ldots,x_n) \mapsto \{x_1,\ldots,x_n\}$, we infer that $\exp_n(X)$ satisfies the c.c.c. too.

Conversely, let $(V_a)_{a \in A}$ be an uncountable family of pairwise disjoint, non-empty, open subsets of X^n. We may assume V_a to be a basic open set of the form $V_a := U_a^1 \times \ldots \times U_a^n$ where U_a^i is a non-empty, open subset of X for all $a \in A$ and $1 \leq i \leq n$. Putting

$$W_a := \langle U_a^1,\ldots,U_a^n \rangle$$

for all $a \in A$ we define non-empty, open subsets of $\exp_n(X)$. By hypothesis $\exp_n(X)$ satisfies the c.c.c. and hence by lemma 6.10 there exist a countably infinite subset B of A and an element D of $\exp_n(X)$ such that $D \in W_a$ for all $a \in B$. Using a similar argument as in the proof of theorem 6.12 we find a countably infinite subset C of B and a coordinate $i \in \{1,\ldots,n\}$ such that for any two distinct a, b of C the intersection $U_a^i \cap U_b^i$ is empty. But this is a contradiction, since $D \cap U_a^i$ is non-empty for $a \in C$ and D is finite. □

Combining theorem 6.12 (resp. proposition 6.13) with the remarks at the beginning of this section yields the following.

6.14. COROLLARY. The following statements hold.

(i) Assuming Martin's Axiom the spaces X and P(X) satisfy the c.c.c. simultaneously for all compact spaces X.

(ii) Assuming the existence of a Souslin continuum X the space P(X) (resp. $\exp_2(X)$) does not satisfy the c.c.c. although X does.

(iii) Assuming the Continuum Hypothesis there exists a compact space X such that X satisfies the c.c.c. but P(X) (resp. $exp_n(X)$ ($n \geq 2$)) does not.

6.15. COROLLARY. Let X be a compact space. If X^{ω} satisfies the c.c.c. then so does exp(X).

Proof. By corollary 6.7 X^n satisfies the c.c.c. for all $n < \omega$ and then so does $exp_n(X)$ by the preceding proposition. Consequently the countable union $\cup \{ exp_n(X) : n < \omega \}$ satisfies the c.c.c. and furthermore it is dense in exp(X). Hence an application of lemma 6.8 finishes the proof. □

Obviously, it holds for any space X that the c.c.c. for exp(X) implies the c.c.c. for X. We notice that an uncountable family $(U_a)_{a \in A}$ of pairwise disjoint, non-empty , open subsets of X induces a family $(< U_a >)_{a \in A}$ of subsets of exp(X) with the same property.

Finally we shall make an observation on the calibre of X and P(X). Recall that a space has the regular uncountable cardinal τ as <u>calibre</u> if for any family of non-empty, open subsets of X of cardinality τ there exists a subfamily of cardinality τ with non-empty intersection. Hence ω_1 is a calibre of X if and only if X satisfies S.c.

Mimicking the first part of the proof of theorem 6.12 and using proposition 6.1 one can show the following.

6.16. REMARK. Let X be a subspace of a locally convex topological vector space and let τ be a calibre of X. Then τ is a calibre of the closed convex hull of X.

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