

ON THE ELLIPTIC NONABELIAN FOURIER TRANSFORM FOR UNIPOTENT REPRESENTATIONS OF p -ADIC GROUPS

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To Roger Howe, with admiration

ABSTRACT. In this paper, we consider the relation between two nonabelian Fourier transforms. The first one is defined in terms of the Langlands-Kazhdan-Lusztig parameters for unipotent elliptic representations of a split p -adic group and the second is defined in terms of the pseudocoefficients of these representations and Lusztig's nonabelian Fourier transform for characters of finite groups of Lie type. We exemplify this relation in the case of the p -adic group of type G_2 .

CONTENTS

1.	Introduction	1
2.	Generalities	3
3.	Elliptic representations	4
4.	Unipotent representations	8
	References	16

1. INTRODUCTION

In this paper, we consider the relation between two nonabelian Fourier transforms: the first defined in terms of the Langlands-Kazhdan-Lusztig parameters for unipotent elliptic representations of a split p -adic group and the second, defined in terms of the pseudocoefficients of these representations and Lusztig's nonabelian Fourier transform for characters of finite groups of Lie type. In this introduction, we give a brief outline of the ideas involved, leaving the precise definitions for the main body of the paper.

Let G be a semisimple p -adic group. Lusztig [Lu1] defined the category $\mathcal{C}_u(G)$ of unipotent G -representations. One way to characterize $\mathcal{C}_u(G)$ is that it is the smallest full subcategory of smooth G -representations such that: (a) it contains the irreducible representations with Iwahori-fixed vectors [Bo], and (b) it is closed under partition into L-packets. Let $R_u(G)$ be the complexification of the Grothendieck group of admissible representations in $\mathcal{C}_u(G)$. Let $\mathcal{H}(G)$ be the Hecke algebra of G with respect to a Haar measure on G and let $\overline{\mathcal{H}}(G)$ be the cocenter of $\mathcal{H}(G)$.

If one is interested in the study of characters of admissible representations, the basic case is that of elliptic tempered representations ([Ar, BDK]). Let $\overline{R}_u(G)_{\text{ell}}$ be the elliptic representation space, a certain subspace of $R_u(G)$ isomorphic to the quotient of $R_u(G)$ by the span of all properly parabolically induced characters. To every elliptic character $\pi \in \overline{R}_u(G)_{\text{ell}}$, one attaches a pseudocoefficient $f_\pi \in \overline{\mathcal{H}}(G)$ ([Ar, Ka2, SS]). The functions f_π play an important role in the character formulas for elliptic representations: for example, if π is an irreducible square integrable representation, then $f_\pi(1)$ is the formal degree of π , e.g., [SS, Proposition III.4.4]. Let $\overline{\mathcal{H}}(G)_u^{\text{ell}}$ denote the subspace of $\overline{\mathcal{H}}(G)$ spanned by the pseudocoefficients f_π for $\pi \in \overline{R}_u(G)_{\text{ell}}$. Thus, we have a map, in fact, an isomorphism, $\overline{R}_u(G)_{\text{ell}} \rightarrow \overline{\mathcal{H}}(G)_u^{\text{ell}}$.

The two spaces involved have natural inner products. The elliptic representations space $\overline{R}_u(G)_{\text{ell}}$ carries the Euler-Poincaré pairing EP [Ka2, SS]. With respect to EP, the irreducible square integrable representations are an orthonormal set. On the other hand, the space $\overline{\mathcal{H}}(G)_u^{\text{ell}}$ can be endowed with a character

pairing coming from the ordinary character pairing on the reductive quotients of maximal parahoric subgroups, which are finite groups of Lie type. As noticed in [MW], the pseudocoefficients can be chosen so that the resulting map $\text{res}_{u,\text{ell}} : \bar{R}_u(G)_{\text{ell}} \rightarrow \bar{\mathcal{H}}(G)_u^{\text{ell}}$ is an isometry with respect to the two pairings.

Next, one defines the nonabelian Fourier transform $\mathcal{FT}_{u,\text{ell}} : \bar{\mathcal{H}}(G)_u^{\text{ell}} \rightarrow \bar{\mathcal{H}}(G)_u^{\text{ell}}$ essentially by truncating to the elliptic spaces Lusztig's nonabelian transform [Lu2] on each reductive quotient of a maximal parahoric subgroup.

Now suppose that G is an adjoint simple split group. Let $\text{Inn } G$ denote the class of forms that are inner to G . The irreducible elliptic tempered modules for the groups in $\text{Inn } G$ are parameterized in terms of Kazhdan-Lusztig parameters [Lu1, Op1, Re1, Re2, Wa] in the dual complex group G^\vee . Define a dual elliptic nonabelian Fourier transform

$$\mathcal{FT}_{u,\text{ell}}^\vee : \bigoplus_{G' \in \text{Inn } G} \bar{R}_u(G')_{\text{ell}} \longrightarrow \bigoplus_{G' \in \text{Inn } G} \bar{R}_u(G')_{\text{ell}} \quad (1.1)$$

by the requirement that the diagram

$$\begin{array}{ccc} \bigoplus_{G' \in \text{Inn } G} \bar{R}_u(G')_{\text{ell}} & \xrightarrow{\mathcal{FT}_{u,\text{ell}}^\vee} & \bigoplus_{G' \in \text{Inn } G} \bar{R}_u(G')_{\text{ell}} \\ \text{res}_{u,\text{ell}} \downarrow & & \downarrow \text{res}_{u,\text{ell}} \\ \bigoplus_{G' \in \text{Inn } G} \bar{\mathcal{H}}(G')_u^{\text{ell}} & \xrightarrow{\mathcal{FT}_{u,\text{ell}}} & \bigoplus_{G' \in \text{Inn } G} \bar{\mathcal{H}}(G')_u^{\text{ell}} \end{array} \quad (1.2)$$

be commutative. The expectation is that $\mathcal{FT}_{u,\text{ell}}^\vee$ behaves well with respect to the Kazhdan-Lusztig parameters. For example, if $n \in G^\vee$ is a unipotent element and we write $(\text{Inn } G)_{\text{ell}}^n = \bigoplus_{G' \in \text{Inn } G} \bar{R}_u(G')_{\text{ell}}^n$ for the span of all unipotent representations whose KL parameter has unipotent part conjugate to n , then we expect that $\mathcal{FT}_{u,\text{ell}}^\vee$ is block-diagonal:

$$\mathcal{FT}_{u,\text{ell}}^\vee = \bigoplus_n \mathcal{FT}_{u,\text{ell}}^{\vee,n} \quad (1.3)$$

with respect to the decomposition $\bigoplus_n (\text{Inn } G)_{\text{ell}}^n$. Moreover, we expect that the piece $\mathcal{FT}_{u,\text{ell}}^{\vee,n}$ is an elliptic version of the nonabelian Fourier transform recently defined in [Lu5] in terms of the reductive part of the centralizer $Z_{G^\vee}(n)$. In particular, if n is a distinguished unipotent element, then we believe that $\mathcal{FT}_{u,\text{ell}}^{\vee,n}$ will transform the parameters in the same way as the original nonabelian Fourier transform [Lu2] defined in terms of the group of components of $Z_{G^\vee}(n)$.

One motivation for these expectations is provided by the work of Mœglin and Waldspurger [MW, Wa] where G is the split form of $SO(2n+1)$. In the present paper, we also verify that these conjectures are precisely true when G has type G_2 . Further evidence is provided by [CO], which considered a similar picture for the formal degrees, in other words, the “evaluation at 1” of the diagram (1.2).

In [MW], the commutative diagram (1.2) was used in order to verify the stability of L-packets for odd orthogonal groups. The diagram involving the two nonabelian Fourier transforms should also be related to the recent conjectures of Lusztig regarding unipotent almost characters of semisimple p -adic groups [KmL, Lu4, Lu5]. In fact, we have arrived at this setup from our attempt to express formal degrees of unipotent discrete series representations in terms of certain invariants, which we called elliptic fake degrees in [CO] and which admit a geometric interpretation. It is our hope that this approach will contribute to a better understanding of the relation between the characters of elliptic unipotent representations, in the form of the local character expansion of Howe [Ho] and Harish-Chandra [HC], and the geometry of the affine flag manifold as predicted by the conjectures of Lusztig.

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General notation. If \mathcal{G} is an algebraic group, denote by $Z(\mathcal{G})$ its center. If $x \in \mathcal{G}$, denote by $Z_{\mathcal{G}}(x)$ the centralizer of x in \mathcal{G} and by $Z_{\mathcal{G}}(x)^0$ its identity component. Let $\mathcal{S}_{\mathcal{G}}(x) = Z_{\mathcal{G}}(x)/Z_{\mathcal{G}}(x)^0$ denote the group of components. The same notation applies with a subset $\mathcal{J} \subset \mathcal{G}$ in place of x for the simultaneous centralizer of all elements of \mathcal{J} . Let $N_{\mathcal{G}}(\mathcal{J})$ denote the normalizer of \mathcal{J} in \mathcal{G} .

If \mathcal{G} is a reductive group, we call an element $x \in \mathcal{G}$ *elliptic* if $Z_{\mathcal{G}}(x)$ does not contain a nontrivial split torus.

2. GENERALITIES

Let \mathbf{k} be a nonarchimedean local field of characteristic 0 with ring of integers \mathcal{O} , prime ideal \mathfrak{p} and residue field $\mathcal{O}/\mathfrak{p} = \mathbb{F}_q$. Let G be the \mathbf{k} -points of a connected reductive algebraic group defined over \mathbf{k} . By a G -representation (π, V) of G , we understand a smooth G -representation. We denote by $\mathcal{C}(G)$ the category of smooth G -representations, and by $\mathcal{C}^{\text{adm}}(G)$ the full subcategory of admissible representations. Let $R(G)$ denote the complexification of the Grothendieck group of $\mathcal{C}^{\text{adm}}(G)$. By a parabolic, Levi subgroup, torus of G we will always mean \mathbf{k} -parabolic, \mathbf{k} -Levi, \mathbf{k} -torus, etc. An element of G is called compact if it is contained in a compact subgroup of G .

2.1. The character distribution. Fix a Haar measure μ on G . Let $\mathcal{H}(G)$ denote the Hecke algebra of G , i.e., the associative algebra of functions $f : G \rightarrow \mathbb{C}$ that are compactly supported and locally constant, with the product given by the μ -convolution. The Hecke algebra $\mathcal{H}(G)$ is not unital (unless $G = 1$). As it is well-known, there is an equivalence of categories between $\mathcal{C}(G)$ and the category of nondegenerate $\mathcal{H}(G)$ -modules: if the G -representation is (π, V) , then the action of $\mathcal{H}(G)$ is via

$$\pi(f)v = \int_G f(x)\pi(x)v \, d\mu(x).$$

Since $f \in \mathcal{H}(G)$ is locally constant, there exists a compact open subgroup K such that $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$ and then $\pi(f)V \subset V^K$. If π is admissible, this implies that every $\pi(f)$ has a well-defined trace. Define the distribution

$$\Theta_{\pi}(f) = \text{tr } \pi(f), \quad f \in \mathcal{H}(G). \quad (2.1)$$

Then Θ_{π} is zero on the span $[\mathcal{H}(G), \mathcal{H}(G)]$ of all commutators of $\mathcal{H}(G)$. Thus, we may regard Θ_{π} as a functional on the cocenter $\overline{\mathcal{H}}(G) = \mathcal{H}(G)/[\mathcal{H}(G), \mathcal{H}(G)]$.

Let G_s denote the set of semisimple elements of G . For $x \in G_s$, let $D_G(x)$ denote the Harish-Chandra discriminant, i.e., the coefficient of $t^{\text{rk} G}$ in $\det((t+1)\text{Id} - \text{Ad}(x))$. Let

$$G_{sr} = \{x \in G_s \mid D_G(x) \neq 0\}$$

denote the set of regular semisimple elements of G . For example, when $G = GL(n, \mathbf{k})$, this is the set of all matrices whose eigenvalues are all distinct.

Theorem 2.1 (Harish-Chandra [HC]). *Let π be an admissible G -representation. The character distribution Θ_{π} is represented on G by a locally integrable function Θ_{π} which is locally constant on G_{sr} :*

$$\Theta_{\pi}(f) = \int_G f(x)\Theta_{\pi}(x) \, d\mu(x), \quad f \in \mathcal{H}(G).$$

2.2. Parabolic induction. Let $P = MN$ be a parabolic subgroup of G and (ρ, V_{ρ}) a representation of M . The (normalized) parabolically induced representation is the left G -regular representation on the space

$$\text{Ind}_P^G(\rho) = \{f : G \rightarrow V_{\rho} \mid f(gmn) = \delta_P(mn)^{-1/2} \rho(m)^{-1} f(g), \, g \in G, m \in M, n \in N\},$$

where δ_P is the modulus function of P . If ρ is an admissible M -representation, the image of $\text{Ind}_P^G(\rho)$ in $R(G)$ does not depend on the choice of $P \supset M$, and therefore, we may denote by

$$i_M^G : R(M) \rightarrow R(G),$$

the linear map defined by Ind_P^G . Let A be a maximally split torus of G .

Theorem 2.2 (van Dijk [vD]). *Let $P = MN$ be a parabolic subgroup and let ρ be an admissible M -representation. Let $\pi = \text{Ind}_P^G(\rho)$ be the parabolically (normalized) induced representation. Then:*

$$\Theta_\pi(g) = \begin{cases} \sum_{w \in W(M, G)} \frac{|D_{M^w}(g)|^{1/2}}{|D_G(g)|^{1/2}} \cdot \Theta_{\rho^w}(g), & \text{if } g \in G_{sr} \cap M, \\ 0, & \text{if } g \text{ is not conjugate to an element of } G_{sr} \cap M, \end{cases}$$

where $W(M, G) = N_G(M)/M$ and $M^w = wMw^{-1}$, $\rho^w(m) = \rho(w^{-1}mw)$.

This allows one to compute the character of a parabolically induced representation from the character of the inducing representation of the Levi subgroup. For example, when G is split and $P = B = AN$, if χ is any smooth character of A and $\pi_\chi = \text{Ind}_B^G(\chi)$ is the corresponding minimal principal series, then

$$\Theta_{\pi_\chi}(g) = \begin{cases} |D_G(a)|^{-1/2} \sum_{w \in W} \chi^w(a), & \text{if } g \text{ is conjugate to an element of } A \cap G_{sr}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.2)$$

Here $W = N_G(A)/A$ is the finite Weyl group.

2.3. Jacquet functors. If (π, V) is a G -representation and $P = MN$ a parabolic subgroup, define $V(N) = \{\pi(n)v - v \mid n \in N, v \in V\}$ and $V_N = V/V(N)$. Then V_N is an M -representation. The Jacquet functor

$$r_P : V \mapsto V_N$$

maps admissible G -representations to admissible M -representations by the well-known result of Jacquet. Fix a minimal parabolic subgroup $P_0 = M_0N_0$ of G . By a standard parabolic subgroup, we mean a parabolic subgroup P of G such that $P \supset P_0$. A standard Levi subgroup is a Levi subgroup of a standard parabolic. Let \mathcal{L} denote the set of standard Levi subgroups of G .

For every $M \in \mathcal{L}$, denote the Jacquet functor at the level of the Grothendieck groups by

$$r_G^M : R(G) \rightarrow R(M), \quad V \mapsto r_P(V).$$

For the properties of i_M^G and r_G^M and the combinatorial relations between them, see [BDK, Da].

2.4. Unramified characters. Let G^0 denote the subgroup of G generated by all of the compact open subgroups of G . For example, when $G = GL(n, k)$, $G^0 = \{g \in G \mid \det g \in \mathcal{O}^\times\}$. The subgroup G^0 is normal in G and the quotient G/G^0 is a free abelian group of finite rank. A character $\chi : G \rightarrow \mathbb{C}^\times$ is called *unramified* if $\chi|_{G^0} = 1$. Let

$$\Psi(G) = \text{Hom}_{\mathbb{Z}}(G/G^0, \mathbb{C}^\times) \quad (2.3)$$

denote the torus of unramified characters of G . The same definitions and notations apply to a Levi subgroup M . If M is a Levi subgroup, denote $d(M) = \dim \Psi(M)$. Notice that $d(G) = 0$ if and only if the center of G is anisotropic.

The torus $\Psi(G)$ acts on $R(G)$ by “unramified twists”: $\pi \mapsto \pi\psi$, $\pi \in R(G)$, $\psi \in \Psi(G)$.

2.5. Orbital integrals. Let $\omega \subset G_{sr}$ be a conjugacy class. Let x be a representative of ω and let $\Gamma = Z_G(x)$, a Cartan subgroup of G . For every $f \in \mathcal{H}(G)$, define the orbital integral

$$\Phi(x, f) = \int_{G/\Gamma} f(gxg^{-1}) dg/d\gamma, \quad (2.4)$$

where $d\gamma$ is a Haar measure on Γ . The normalization of Haar measures is important, but we will only be interested in the vanishing of such orbital integrals, and we ignore the normalization issue here. Denote also $\Phi(\omega, f)$ in place of $\Phi(x, f)$.

It is easy to see that if $f \in [\mathcal{H}(G), \mathcal{H}(G)]$, then $\Phi(\omega, f) = 0$ for all $\omega \subset G_{sr}$. Hence, it makes sense to consider $\Phi(\omega, f)$ for $f \in \overline{\mathcal{H}}(G)$.

3. ELLIPTIC REPRESENTATIONS

Define the space of induced representations:

$$R(G)_{\text{ind}} = \sum_{M \in \mathcal{L}, M \neq G} i_M^G(R(M)),$$

and let the space of (virtual) elliptic representations to be the quotient

$$\overline{R}(G)_{\text{ell}} = R(G)/R(G)_{\text{ind}}. \quad (3.1)$$

The space $\overline{R}(G)_{\text{ell}}$ can be identified naturally with a subspace of $R(G)$ as in [BDK, §5.5]. Let

$$A = \frac{1}{p} A_{d(M_0)} \circ A_{d(M_0)-1} \circ \cdots \circ A_{d(G)+1} : R(G) \rightarrow R(G), \quad (3.2)$$

(when G is semisimple, $d(G) = 0$) where

$$A_d = \prod_{M \in \mathcal{L}, d(M)=d} (i_M^G \circ r_G^M - p_M \text{Id}),$$

$p_M = |N_{W_G}(M)/W_M|$, $p = \prod_{M \in \mathcal{L}, M \neq G} (-p_M)$, $W_M = N_M(M_0)/M_0$. Using the combinatorics of the induction and restriction maps [BDK, §5.4] (see also [Da, Proposition 2.5 (i)]), one sees that $\ker A = R(G)_{\text{ind}}$ and therefore

$$\overline{R}(G)_{\text{ell}} \cong A(R(G)). \quad (3.3)$$

The action of the torus $\Psi(G)$ on $R(G)$ preserves $R(G)_{\text{ind}}$ and therefore $\Psi(G)$ acts on $\overline{R}(G)_{\text{ell}}$. For every compact open subgroup K of G , let $R(G)_K$ denote the span of all irreducible admissible G -representations (π, V) such that $V^K \neq 0$. Then $\Psi(G)$ acts on $R(G)_K$. This is because $(\pi\psi)(k)v = \pi(k)\psi(k)v = \pi(k)v$, for all $k \in K$ (since ψ is unramified) and $v \in V$. Let $\overline{R}(G)_{\text{ell}, K}$ denote the image of $R(G)_K$ in $\overline{R}(G)_{\text{ell}}$.

Theorem 3.1 ([BDK, §3.1]). *For every compact open subgroup K , $\overline{R}(G)_{\text{ell}, K}$ is a finitely generated $\Psi(G)$ -module. In particular, if G is semisimple, then $\overline{R}(G)_{\text{ell}, K}$ is finite dimensional.*

Notice that, as a consequence of the Langlands classification, every class in $\overline{R}(G)_{\text{ell}}$ is represented by an essentially tempered module.

3.1. Projective modules. Let $K_0(G)$ denote the complexification of the Grothendieck group of finitely generated projective $\mathcal{H}(G)$ -modules. A typical element in $K_0(G)$ is the compactly induced module:

$$\text{ind}_K^G(\rho) = \mathcal{H}(G) \otimes_{\mathcal{H}(K)} V_\rho,$$

where K is a compact open subgroup and (ρ, V_ρ) is a finite dimensional K -representation. In fact, based on the results of Schneider-Stuhler [SS], Blanc-Brylinski [BB], and Bernstein, one sees ([Da, Corollary 4.22]) that $K_0(G)$ is spanned by the modules $\text{ind}_K^G(\rho)$ as K varies over a base of compact open subgroups of G and ρ varies over the irreducible unitary K -representations.

The algebra $\mathcal{H}(G)$ is not unital, but it is a direct limit of unital associative algebras, $\mathcal{H}(G) = \varinjlim_K \mathcal{H}(G, K)$, where K varies over an appropriate base of compact open subgroups (see [Ka1] for example), and this allows one to introduce for $\mathcal{H}(G)$ -modules standard techniques from homological algebra of associative unital algebras.

Definition 3.2. *If P is a finitely-generated projective $\mathcal{H}(G)$ -module, there exists $n \in \mathbb{N}$ and e_P an idempotent in $M_n(\mathcal{H}(G))$ such that $P = \mathcal{H}(G)^n e_P$. Define the Hattori-Stallings trace map $\tau_{HS} : K_0(G) \rightarrow \overline{\mathcal{H}}(G)$ to be*

$$\tau_{HS}(P) = \text{tr } e_P \text{ mod } [\mathcal{H}(G), \mathcal{H}(G)]. \quad (3.4)$$

This definition is independent of the choice of idempotent e_P .

Define the Euler-Poincaré map $\text{ep} : R(G) \rightarrow K(G)$ by

$$\pi \mapsto \sum_{i=1}^k (-1)^i [P_i], \quad (3.5)$$

where $0 \rightarrow P_k \rightarrow \cdots \rightarrow P_0 \rightarrow \pi \otimes \mathbb{C}[G/G^0] \rightarrow 0$ is a resolution by finitely generated projective modules. It is known that $\mathcal{H}(G)$ has finite cohomological dimension [Be, Theorem 29], [SS, Corollary II.3.3].

Every $f \in \mathcal{H}(K)$, K a compact open subgroup, can be extended by zero to a function in $\mathcal{H}(G)$ which we will denote \tilde{f} . To understand the map τ_{HS} better, notice that if $P = \text{ind}_K^G(\rho) \in K_0(G)$, where ρ is irreducible, then $P = \mathcal{H}(G)\tilde{e}_\rho$, where $e_\rho \in \mathcal{H}(K)$ is the idempotent corresponding to ρ . This means that

$$\tau_{HS}(\text{ind}_K^G(\rho)) = \tilde{e}_\rho \text{ mod } [\mathcal{H}(G), \mathcal{H}(G)].$$

Consider the composition $\tau_{HS} \circ \text{ep} : R(G) \rightarrow \overline{\mathcal{H}}(G)$. By results of [Ka1, SS], the space of induced representations $R(G)_{\text{ind}}$ lies in the kernel of this composition, hence we have a map

$$\tau_{HS} \circ \text{ep} : \overline{R}(G)_{\text{ell}} \longrightarrow \overline{\mathcal{H}}(G). \quad (3.6)$$

To describe the image of this map, we need to define the duals in $\overline{\mathcal{H}}(G)$ of the induction and Jacquet functors. Let $R(G)_{\text{good}}^*$ denote the space of good forms defined in [BDK]. By the Density Theorem of Kazhdan [Ka2, Theorem 0] (see also [Ka1, Theorem B]) and the Trace Paley-Wiener Theorem of Bernstein, Deligne, and Kazhdan [BDK, Theorem 1.2], the trace map

$$\text{tr} : \overline{\mathcal{H}}(G) \rightarrow R(G)_{\text{good}}^*, \quad \text{tr}(f)(\pi) = \Theta_\pi(f) \quad (3.7)$$

is an isomorphism. We can use this isomorphism to define the dual maps using the trace pairing. Namely, for every standard Levi subgroup M , let

$$\begin{aligned} \bar{i}_M^G : \overline{\mathcal{H}}(M) &\rightarrow \overline{\mathcal{H}}(G), \quad \Theta_\pi(\bar{i}_M^G(f)) = \Theta_{r_G^M(\pi)}(f), \quad \pi \in R(G), \quad f \in \mathcal{H}(M), \\ \bar{r}_G^M : \overline{\mathcal{H}}(G) &\rightarrow \overline{\mathcal{H}}(M), \quad \Theta_\pi(\bar{r}_G^M(f)) = \Theta_{i_M^G(\pi)}(f), \quad \pi \in R(M), \quad f \in \mathcal{H}(G). \end{aligned} \quad (3.8)$$

The following results are collected in [Da] and they are based on [BDK, Ka2, SS, BB].

Theorem 3.3. (1) [Da, Theorem 1.6 and Corollary 4.20] *The map $\tau_{HS} : K_0(G) \rightarrow \overline{\mathcal{H}}(G)$ is injective. Its image is the subspace*

$$\overline{\mathcal{H}}_c(G) = \{f \in \overline{\mathcal{H}}(G) \mid \Phi(\omega, f) = 0 \text{ for all conjugacy classes } \omega \subset G_{sr} \setminus G_{src}\}. \quad (3.9)$$

(2) [Da, Theorem 3.3 and Theorem 3.4] *The map $\tau_{HS} \circ \text{ep}$ induces a linear isomorphism of $\overline{R}(G)_{\text{ell}}$ onto the elliptic cocenter*

$$\overline{\mathcal{H}}(G)^{\text{ell}} = \bigcap_{M \in \mathcal{L}, d(M) > d(G)} \ker \bar{r}_G^M. \quad (3.10)$$

Moreover, $\overline{\mathcal{H}}(G)^{\text{ell}} = \{f \in \overline{\mathcal{H}}_c(G) \mid \Phi(\omega, f) = 0 \text{ for all nonelliptic conjugacy classes } \omega \subset G_{sr}\}.$

In light of Theorem 3.3, we make the following definition.

Definition 3.4. Define the pseudo-coefficient map $\text{ps} : \overline{R}(G)_{\text{ell}} \rightarrow \overline{\mathcal{H}}^{\text{ell}}$ to be the isomorphism from Theorem 3.3(2). For every class $[\pi] \in \overline{R}(G)_{\text{ell}}$, let $f_\pi = \text{ps}(\pi)$ and call it the Euler-Poincaré function of π .

It is easy to see from the definitions that

$$\Theta_\pi(f_{\pi'}) = \text{EP}(\pi, \pi'), \quad \text{where } \text{EP}(\pi, \pi') = \sum_{i \geq 0} (-1)^i \dim \text{Ext}_{\mathcal{H}(G)}^i(\pi, \pi'). \quad (3.11)$$

The sum in the right hand side is finite since $\mathcal{H}(G)$ has finite cohomological dimension, as mentioned before.

3.2. Euler-Poincaré functions. We recall an explicit description of f_π from [SS]. For the applications in this paper, it is sufficient to assume that G is semisimple and split, which, for brevity, we assume now. Let A be a maximal split torus and $B = AN$ a Borel subgroup of G . Let R be the set of roots of A in G and let Π be the set of simple roots of A in G with respect to B . Let

$$W_0 = N_G(A)/A \text{ and } W = N_G(A)/A(\mathcal{O}) \quad (3.12)$$

be the finite Weyl group and the Iwahori-Weyl group, respectively. Let $W \rightarrow W_0$ be the natural projection whose kernel is the lattice $X = A/A(\mathcal{O}) \cong \text{Hom}(\mathbf{k}^\times, A)$. Let $Y = \text{Hom}(A, \mathbf{k}^\times)$ be the character lattice. For every $x \in X$, $y \in Y$, the composition $y \circ x : \mathbf{k}^\times \rightarrow \mathbf{k}^\times$ is an algebraic homomorphism, hence it is given by $z \mapsto z^n$ for some $n \in \mathbb{Z}$. Denote $\langle x, y \rangle := n$; this defines a perfect pairing between X and Y . Let $R^\vee \subset X$ be the set of coroots in duality $\alpha \leftrightarrow \alpha^\vee$, $\langle \alpha, \alpha^\vee \rangle = 2$ with the roots R .

Identify $W = W_0 \ltimes X$. A typical element of W is wt_x , where $w \in W_0$ and $x \in X$ and the product is $(w_1 t_{x_1}) \cdot (w_2 t_{x_2}) = w_1 w_2 t_{w_2^{-1}(x_1) + x_2}$. The group W acts naturally on $Y \times \mathbb{Z}$, where $Y = \text{Hom}(A, \mathbf{k}^\times)$, via:

$$wt_x : (y, k) \mapsto (w(y), k - \langle x, y \rangle).$$

Define the positive and negative affine roots:

$$\begin{aligned} R_a^+ &= \{(\alpha, k) \mid \alpha \in R, k > 0\} \cup \{(\alpha, 0) \mid \alpha \in R^+\} \\ R_a^- &= \{(\alpha, k) \mid \alpha \in R, k < 0\} \cup \{(\alpha, 0) \mid \alpha \in R^-\}, \end{aligned}$$

and set $R_a = R_a^+ \cup R_a^- = R \times \mathbb{Z}$. This is a W -stable subset of $Y \times \mathbb{Z}$. The simple affine roots are

$$\Pi_a = \{(\alpha, 0) \mid \alpha \in \Pi\} \cup \{(\alpha, 1) \mid \alpha \in \Pi_m\} \subset R_a^+,$$

where $\Pi_m = \{\beta \in R \mid \beta \text{ minimal with respect to } \leq\}$ where $\beta_1 \leq \beta_2$ if, by definition, $\beta_2 - \beta_1$ is a nonnegative integral combination of roots in Π .

The group W acts on the space $E = \mathbb{R} \otimes_{\mathbb{Z}} X$ via

$$wt_x : e \mapsto w(e + x).$$

Let $W^a = W_0 \ltimes \mathbb{Z}R$ be the affine Weyl group. It is generated by the reflections corresponding to the simple affine roots.

Assume for simplicity that G is simple. Then we may regard E as a simplicial complex with hyperplanes corresponding to the affine reflections, then W^a acts transitively on the open facets of E . If $c \subset E$ is a facet, denote by W_c the stabilizer of c in W . This is a finite group.

There exists a unique open facet c_0 (the fundamental alcove) which contains 0 in its closure and such that all simple roots $\alpha \in \Pi$ take positive values on c_0 . The reflections in the hyperplanes that border c_0 are the simple affine reflections corresponding to the roots in Π_a .

Let $\Omega = W_{c_0}$. Then

$$W = \Omega \ltimes W^a.$$

For the definition of the building \mathcal{B}_G of G , see [SS] for example. This is a simplicial complex containing E as a subcomplex and such that G acts on \mathcal{B}_G via simplicial maps. Every facet of \mathcal{B}_G can be translated into E by the action of G . Define

$$P_c^+ = \text{stabilizer in } G \text{ of the facet } c \in \mathcal{B}_G. \quad (3.13)$$

This group sits into a short exact sequence

$$1 \longrightarrow U_c \longrightarrow P_c^+ \longrightarrow M_c^+ \longrightarrow 1,$$

where U_c is the prounipotent radical and M_c^+ is the \mathbb{F}_q -points of a possibly disconnected reductive group over k . Let $M_c = M_c^{+,0}$ be the \mathbb{F}_q -points of the identity component of M_c^+ . Define P_c to be the inverse image in P_c^+ of M_c . This is called a parahoric subgroup of G ; it is a compact open subgroup.

Let $I = P_{c_0}$, an Iwahori subgroup. Suppose c is a facet of \mathcal{B}_G such that $c \subset \bar{c}_0$. Then $P_c \supset I$ is a standard parahoric subgroup. We have $P_c^+/P_c \cong \Omega_c$, where Ω_c is the stabilizer of c in Ω . The assignment

$$J \subsetneq \Pi_a \longrightarrow P_J := P_{c_J} \quad (3.14)$$

gives a one-to-one correspondence between Ω -orbits of proper subsets in Π_a and G -conjugacy classes of parahoric subgroups. Here c_J is the facet contained in the closure of c_0 and defined by the hyperplanes corresponding to the roots in J :

$$P_J = \langle I, I s_\alpha I : \alpha \in J \rangle.$$

At one extreme, $P_\emptyset = I$ and at the other, if J is maximal, then P_J is the stabilizer of a vertex of \bar{c}_0 and it is a maximal parahoric subgroup of G .

Using an explicit resolution for admissible representations π in terms of projective modules that are compactly induced from parahoric subgroups, Schneider and Stuhler obtain the following explicit description of the functions f_π .

Theorem 3.5 ([SS, Theorem III.4.20]). *Let (π, V) be an admissible representation of G . Set*

$$F_\pi = \sum_{J \subsetneq \Pi_a/\Omega} (-1)^{|J|} \frac{\tilde{\epsilon}_J \tilde{\chi}_J}{\mu(P_J^+)}, \quad (3.15)$$

where χ_J is the character of the P_J^+ -module V^{U_J} (this is zero or finite dimensional), $\epsilon_J : P_J^+ \rightarrow \{\pm 1\}$ is the orientation character, and $\tilde{\epsilon}_J, \tilde{\chi}_J$ denote the extension by zero. Then $f_\pi = F_\pi \bmod [\mathcal{H}(G), \mathcal{H}(G)]$.

The relevance of the functions F_π is that they give an explicit incarnation of the *pseudo-coefficients* of square integrable representations $\pi \in \overline{R}(G)_{\text{ell}}$.

Theorem 3.6 ([SS, Theorem III.4.6], [Ka2, Corollary p. 29]). *Suppose that π is an irreducible square integrable representation of G and that π' is an irreducible tempered G -representation. Then*

$$\Theta_{\pi'}(f_\pi) = \begin{cases} 1, & \pi' \cong \pi, \\ 0, & \pi' \not\cong \pi. \end{cases}$$

The Euler-Poincaré pairing EP is nondegenerate on $\overline{R}(G)_{\text{ell}}$ and the set of irreducible square integrable representations form an orthonormal set with respect to EP.

In fact, more is true: with the notation as in the previous theorem, $\text{Ext}_{\mathcal{H}(G)}^i(\pi, \pi') = 0$ if $i > 0$ [Me, Theorem 41]. This result has been extended in [OS2, Theorem 3] to a complete determination of $\text{Ext}_{\mathcal{H}(G)}^i(\pi, \pi')$ for all irreducible tempered modules, in terms of the Knapp-Stein theory of standard intertwiners and analytic R-groups.

4. UNIPOTENT REPRESENTATIONS

4.1. Weyl groups. Let \mathcal{W} be a finite Weyl group acting on the n -dimensional reflection representation V . Let $R(\mathcal{W})$ be the complexification of the Grothendieck group of finite dimensional \mathcal{W} -representations. It can be naturally identified with the space of complex class functions on \mathcal{W} . Let $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ be the character pairing of \mathcal{W} on $R(\mathcal{W})$. Let $\text{lrr } \mathcal{W}$ be the set of irreducible characters of \mathcal{W} .

Define $\bigwedge^+ V = \sum_{i=0}^n (-1)^i \bigwedge^i V$, viewed as a virtual \mathcal{W} -representation. Its character is $(\text{tr } \bigwedge^+ V)(w) = \det_V(1 - w)$. Call an element $w \in \mathcal{W}$ elliptic if $\det_V(1 - w) \neq 0$. It is easy to see that w is elliptic if and only if $w \notin \mathcal{W}_J$ for every proper parabolic subgroup $\mathcal{W}_J \subset \mathcal{W}$. Define the hermitian pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{W}}^{\text{ell}} : R(\mathcal{W}) \times R(\mathcal{W}) \rightarrow \mathbb{C}, \quad \langle \sigma, \sigma' \rangle_{\mathcal{W}}^{\text{ell}} = \langle \sigma, \sigma' \otimes \bigwedge^+ V \rangle_{\mathcal{W}}, \quad (4.1)$$

and call it the elliptic pairing of \mathcal{W} . The radical of $\langle \cdot, \cdot \rangle_{\mathcal{W}}^{\text{ell}}$ is $R(\mathcal{W})_{\text{ind}}$ [Re1, Proposition 2.2.2], the span of induced representations from proper parabolic subgroups. Let $\overline{R}(\mathcal{W})_{\text{ell}} = R(\mathcal{W})/R(\mathcal{W})_{\text{ind}}$ be the space of elliptic representation. It may be naturally identified with the space of class functions on \mathcal{W} supported on the elliptic conjugacy classes of \mathcal{W} .

4.2. Finite groups. Let \mathcal{G} be a connected semisimple algebraic $\overline{\mathbb{F}}_q$ -group defined over \mathbb{F}_q with Frobenius morphism $F : \mathcal{G} \rightarrow \mathcal{G}$. Let \mathcal{G}^F be the fixed points of F , a finite group of Lie type. We assume that \mathcal{G} is adjoint and \mathcal{G}^F is split. As before, let $R(\mathcal{G}^F)$ denote the complex class functions on \mathcal{G}^F with $\langle \cdot, \cdot \rangle_{\mathcal{G}^F}$ the character pairing. Let $\text{lrr } \mathcal{G}^F$ denote the set of characters of irreducible representations of \mathcal{G}^F .

To each pair (T, θ) , T an F -stable maximal torus of \mathcal{G} and $\theta : T^F \rightarrow \mathbb{C}^\times$ a character, Deligne and Lusztig [DL] associated a *generalized character* $R_{T, \theta} \in R(\mathcal{G}^F)$.

Definition 4.1. A character $\rho \in \text{lrr } \mathcal{G}^F$ is called *unipotent* if $\langle \rho, R_{T, 1} \rangle_{\mathcal{G}^F} \neq 0$ for some F -stable maximal torus T . Let $\text{lrr}_u \mathcal{G}^F$ denote the set of irreducible unipotent characters and $R_u(\mathcal{G}^F) \subset R(\mathcal{G}^F)$ their span.

A character $\rho \in \text{lrr}_u \mathcal{G}^F$ is called *cuspidal* if $\langle \rho, R_{T, \theta} \rangle_{\mathcal{G}^F} = 0$ for all maximal tori T contained in a proper F -Levi subgroup.

Let T be an F -stable maximal torus of \mathcal{G} and suppose that T is contained in an F -stable parabolic subgroup $P = MU_P$ with M an F -stable Levi subgroup. By [Lu6, Proposition 2.6], if θ is a character of T^F , then

$$R_{T, \theta} = \text{Ind}_{P^F}^{\mathcal{G}^F}(R_{T, \theta}^M), \quad (4.2)$$

where $R_{T, \theta}^M$ is the generalized character of M^F , regarded as a virtual representation of P^F , trivial on U_P^F .

By [Lu6, Corollary 3.2], if $\rho \in \text{Irr}_u \mathcal{G}^F$ and θ' is a nontrivial character of T'^F , where T' is an F -stable maximal torus of \mathcal{G} , then

$$\langle \rho, R_{T', \theta'} \rangle_{\mathcal{G}^F} = 0. \quad (4.3)$$

Combining (4.2) and (4.3), it follows that the functors of parabolic induction and restriction take unipotent representations to unipotent representations.

Fix a maximal F -stable torus T_0 contained in an F -stable Borel subgroup B_0 , and let $\mathcal{W} = N_{\mathcal{G}^F}(T_0)/T_0$ be the finite Weyl group. As it is well known, the irreducible representations occurring in the minimal principal series $\text{Ind}_{B_0^F}^{\mathcal{G}^F}(1)$ (which equals $R_{T_0, 1}$ by (4.2)) are in one-to-one correspondence with the irreducible representations of \mathcal{W} . If $\mu \in \text{Irr } \mathcal{W}$, denote by $\rho_\mu \in \text{Irr}_u \mathcal{G}^F$ the corresponding constituent of the minimal principal series.

Lusztig [Lu2] partitioned $\text{Irr } \mathcal{W}$ into families \mathcal{F} and attached to each family \mathcal{F} a finite group $\Gamma_{\mathcal{F}}$. He defined the sets

$$M(\Gamma_{\mathcal{F}}) = \Gamma_{\mathcal{F}}\text{-orbits on } \{(x, \sigma) \mid x \in \Gamma_{\mathcal{F}}, \sigma \in \text{Irr } Z_{\Gamma_{\mathcal{F}}}(x)\}. \quad (4.4)$$

Definition 4.2. *The nonabelian Fourier transform is the pairing $\{ , \} : M(\Gamma_{\mathcal{F}}) \times M(\Gamma_{\mathcal{F}}) \rightarrow \mathbb{C}$ given by*

$$\{(x, \sigma), (y, \tau)\} = \frac{1}{|Z_{\Gamma_{\mathcal{F}}}(x)||Z_{\Gamma_{\mathcal{F}}}(y)|} \sum_{g \in \Gamma_{\mathcal{F}}, xgyg^{-1} = gyg^{-1}x} \sigma(gyg^{-1}) \overline{\tau(g^{-1}xg)}. \quad (4.5)$$

Set $\mathcal{X}(\mathcal{W}) = \bigsqcup_{\mathcal{F} \subset \text{Irr } \mathcal{W}} M(\Gamma_{\mathcal{F}})$ and extend $\{ , \}$ to $\mathcal{X}(\mathcal{W})$.

On the other hand, to every $\mu \in \text{Irr } \mathcal{W}$, Lusztig [Lu2] associates an *almost character* R_μ as follows. For every $w \in \mathcal{W}$, choose an F -stable representative $\dot{w} \in \mathcal{G}$ and $g \in \mathcal{G}$ such that $g^{-1}F(g) = \dot{w}$. Define $T_w = gT_0g^{-1}$, an F -stable maximal torus. Define

$$R_\mu = \frac{1}{|W|} \sum_{w \in W} \mu(w) R_{T_w, 1}. \quad (4.6)$$

Theorem 4.3 ([Lu2, Theorem 4.23]). *There is a bijection $\mathcal{X}(\mathcal{W}) \longrightarrow \text{Irr}_u \mathcal{G}^F$, $(x, \sigma) \mapsto \rho_{(x, \sigma)}$ such that, for all $(x, \sigma) \in \mathcal{X}(\mathcal{W})$ and $\mu \in \text{Irr } \mathcal{W}$:*

$$\langle \rho_{(x, \sigma)}, R_\mu \rangle_{\mathcal{G}^F} = \Delta(x, \sigma) \{(x, \sigma), (y, \tau)\},$$

where $\rho_{(y, \tau)} = \rho_\mu$ and $\Delta(x, \sigma) \in \{\pm 1\}$ is defined in [Lu2, §4.21].

If W is an irreducible Weyl group, then $\Delta \equiv 1$ except for the families that contain the representations $512'_a$ in E_7 or $4096'_z$, $4096'_x$ in E_8 . In light of this theorem, Lusztig [Lu2, (4.24.1)] defines the almost character $R_{(y, \tau)}$ for each $(y, \tau) \in \mathcal{X}(\mathcal{W})$:

$$R_{(y, \tau)} = \sum_{\rho_{(x, \sigma)} \in \text{Irr}_u \mathcal{G}^F} \{(x, \sigma), (y, \tau)\} \Delta(x, \sigma) \rho_{(x, \sigma)}. \quad (4.7)$$

When $\rho_{(y, \tau)} = \rho_\mu$ then $R_{(y, \tau)} = R_\mu$. The set $\{R_{(y, \tau)} : (y, \tau) \in \mathcal{X}(\mathcal{W})\}$ is another orthonormal basis of $R_u(\mathcal{G}^F)$ [Lu2, Corollary 4.25] and the change of basis matrix between this and the basis of irreducible characters is $\{ , \}$.

By transporting $\{ , \}$ via the bijection $\mathcal{X}(\mathcal{W}) \longrightarrow \text{Irr}_u \mathcal{G}^F$ and extending it sesquilinearly, we can define the nonabelian Fourier transform, denoted also by $\{ , \}$, on $R_u(\mathcal{G}^F)$. We may also think of it as a linear map:

$$\mathcal{FT} : R_u(\mathcal{G}^F) \rightarrow R_u(\mathcal{G}^F), \quad \rho \mapsto \sum_{\rho' \in \text{Irr}_u \mathcal{G}^F} \{\rho, \rho'\} \rho'. \quad (4.8)$$

In [Lu3], Lusztig introduced another set of class functions on \mathcal{G}^F , the characteristic functions χ_A of F -stable character sheaves A on \mathcal{G} (certain simple perverse sheaves on G) and proved [Lu3, V, Theorem 25.2], under certain restrictions on the characteristic p , that they form an orthonormal basis of the space of class functions on \mathcal{G}^F . In particular, restricting to the space $R_u(\mathcal{G}^F)$, the set of characteristic functions of unipotent character sheaves gives an orthonormal basis. By further work of Lusztig and Shoji [Sh1],

the F -stable character sheaves on \mathcal{G} are parameterized by the same set $\mathcal{X}(\mathcal{W})$ in such a way that if $A_{(y,\tau)}$ is the character sheaf corresponding to $(y, \tau) \in \mathcal{X}(\mathcal{W})$, then

$$\chi_{A_{(y,\tau)}} = \zeta_{(y,\tau)} \cdot R_{(x,\sigma)}, \quad (4.9)$$

for some root of unity $\zeta_{(y,\tau)}$. These roots of unity are explicitly known in most cases, see [Lu7, Sh1, Sh2, Sh3, DLM]. The behavior of the χ_A 's with respect to induction and restriction is studied in [Lu3, III, Section 15], particularly Proposition 15.7 in *loc. cit.*. These results in conjunction with (4.9) allows one to see that, in particular, the nonabelian Fourier transform $\{ , \}$ preserves the subspace of $R_u(\mathcal{G}^F)$ spanned by the proper parabolically induced representations. We will need this fact in subsection 4.4.

4.3. Finite G_2 . Consider the case when G is the group of type G_2 defined over \mathbb{F}_q . Then the finite Weyl group $W(G_2)$ has 6 irreducible representations, four 1-dimensional and two 2-dimensional. In the notation of Carter [Ca], these are labelled $\phi_{(1,0)}$, $\phi'_{(1,3)}$, $\phi''_{(1,3)}$, $\phi_{(1,12)}$, $\phi_{(2,1)}$, and $\phi_{(2,2)}$. Let s_1 denote the long reflection and s_2 the short reflection of $W(G_2)$. The character table is Table 4.3.

TABLE 1. Character table of $W(G_2)$

	1	s_1	s_2	$s_1 s_2$	$(s_1 s_2)^2$	$(s_1 s_2)^3$
$\phi_{(1,1)}$	1	1	1	1	1	1
$\phi'_{(1,3)}$	1	-1	1	-1	1	-1
$\phi''_{(1,3)}$	1	1	-1	-1	1	-1
$\phi_{(1,6)}$	1	-1	-1	1	1	1
$\phi_{(2,1)}$	2	0	0	1	-1	-2
$\phi_{(2,2)}$	2	0	0	-1	-1	2

Denote the Coxeter element $c(G_2) = s_1 s_2$. The elliptic conjugacy classes are $c(G_2)$, $c(G_2)^2$, and $c(G_2)^3$ of sizes 2, 2, 1, respectively. With respect to the *usual character pairing*, on orthonormal basis of $\overline{R}(W(G_2))_{\text{ell}}$ is given by

$$\{\sqrt{6} \cdot 1_{c(G_2)}, \sqrt{6} \cdot 1_{c(G_2)^2}, \sqrt{12} \cdot 1_{c(G_2)^3}\},$$

where

$$\begin{aligned} 1_{c(G_2)} &= \frac{1}{6}(\phi_{(1,1)} - \phi'_{(1,3)} - \phi''_{(1,3)} + \phi_{(1,6)} + \phi_{(2,1)} - \phi_{(2,2)}), \\ 1_{c(G_2)^2} &= \frac{1}{6}(\phi_{(1,1)} + \phi'_{(1,3)} + \phi''_{(1,3)} + \phi_{(1,6)} - \phi_{(2,1)} - \phi_{(2,2)}), \\ 1_{c(G_2)^3} &= \frac{1}{12}(\phi_{(1,1)} - \phi'_{(1,3)} - \phi''_{(1,3)} + \phi_{(1,6)} - 2\phi_{(2,1)} + 2\phi_{(2,2)}). \end{aligned} \quad (4.10)$$

There are three families of Weyl group representations:

$$\mathcal{F}: \quad \{\phi_{(1,0)}\}, \quad \{\phi_{(2,1)}, \phi'_{(1,3)}, \phi''_{(1,3)}, \phi_{(2,2)}\}, \quad \{\phi_{(1,6)}\}.$$

The finite groups $\Gamma_{\mathcal{F}}$ associated to them are:

$$\Gamma_{\mathcal{F}}: \quad 1, \quad S_3, \quad 1.$$

We use the notation of [Lu2] for representatives of the conjugacy classes in S_3 and the representations of the centralizers. Let 1 be the identity in S_3 , g_2 be a transposition and g_3 a 3-cycle. The centralizers are S_3 , $\mathbb{Z}/2\mathbb{Z} = \{1, g_2\}$ and $\mathbb{Z}/3\mathbb{Z} = \{1, g_3, g_3^2\}$, respectively. Let 1 denote the trivial representation for each of these groups. Let r , ϵ denote the standard, respectively sign representation of S_3 . Let ϵ denote the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$ and θ the representation $g_3 \mapsto \theta$ of $\mathbb{Z}/3\mathbb{Z}$, where θ is a nontrivial third root of unity.

Under the parameterization from Theorem 4.3, the unipotent minimal principal series representations attached to irreducible characters of $W(G_2)$ in the middle family correspond to pairs in $M(S_3)$ as follows:

$$\rho_{(2,1)} \rightarrow (1, 1), \quad \rho'_{(1,3)} \rightarrow (1, r), \quad \rho''_{(1,3)} \rightarrow (g_3, 1), \quad \rho_{(2,2)} \rightarrow (g_2, 1). \quad (4.11)$$

The remaining four unipotent representations are all cuspidal, and they correspond to pairs as follows ([Ca, page 478]):

$$G_2[1] \rightarrow (1, \epsilon), \quad G_2[-1] \rightarrow (g_2, \epsilon), \quad G_2[\theta] \rightarrow (g_3, \theta), \quad G_2[\theta^2] \rightarrow (g_3, \theta^2). \quad (4.12)$$

The nonabelian Fourier transform for S_3 in the ordered set

$$(1, 1), (1, r), (1, \epsilon), (g_2, 1), (g_2, \epsilon), (g_3, 1), (g_3, \theta), (g_3, \theta^2)$$

is:

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \quad (4.13)$$

All $\Delta(x, \sigma) = 1$ for G_2 .

4.4. Finite elliptic Fourier transform. For every F -stable Levi subgroup $\mathcal{L} \supset T_0$, let $i_{\mathcal{L}^F}^{\mathcal{G}^F} : R_u(\mathcal{L}^F) \rightarrow R_u(\mathcal{G}^F)$ and $r_{\mathcal{G}^F}^{\mathcal{L}^F} : R_u(\mathcal{G}^F) \rightarrow R_u(\mathcal{L}^F)$ be the parabolic induction, respectively parabolic restriction maps. (These are well defined since induction/restriction take unipotent representations to unipotent representations.) Define the space of elliptic unipotent representations

$$\overline{R}_u(\mathcal{G}^F)_{\text{ell}} = \bigcap_{\mathcal{L} \neq \mathcal{G}} \ker r_{\mathcal{G}^F}^{\mathcal{L}^F}. \quad (4.14)$$

The space $\overline{R}_u(\mathcal{G}^F)$ can be naturally identified with the space of unipotent class functions of \mathcal{G}^F which vanish on the nonelliptic conjugacy classes of \mathcal{G}^F . This notion of elliptic is compatible with the notion of elliptic representations of the finite Weyl group \mathcal{W} , namely, the injection $R(\mathcal{W}) \hookrightarrow R_u(\mathcal{G}^F)$ induces an injection $\overline{R}(\mathcal{W})_{\text{ell}} \hookrightarrow \overline{R}_u(\mathcal{G}^F)_{\text{ell}}$. This follows at once from the fact that the injections $R(\mathcal{W}) \hookrightarrow R_u(\mathcal{G}^F)$ are compatible with parabolic induction and restriction.

The following fact is extracted from the works of Lusztig [Lu3] and Shoji [Sh1] by Mœglin and Waldspurger [MW, sections 2.7, 2.8, 3.16, 4.3], see our discussion at the end of subsection 4.2.

Proposition 4.4. *The nonabelian Fourier transform preserves the elliptic space, i.e.,*

$$\mathcal{FT}(\overline{R}_u(\mathcal{G}^F)_{\text{ell}}) = \overline{R}_u(\mathcal{G}^F)_{\text{ell}}.$$

In light of this proposition, let $\mathcal{FT}_{\text{ell}}$ denote the restriction of \mathcal{FT} to $\overline{R}_u(\mathcal{G}^F)_{\text{ell}}$.

Example 4.5. *We compute $\mathcal{FT}_{\text{ell}}$ when $\mathcal{G} = G_2$. A basis of $\overline{R}_u(G_2(\mathbb{F}_q))_{\text{ell}}$, orthonormal with respect to the character pairing, is given by:*

$$\mathcal{B}_{\text{ell}}(G_2(\mathbb{F}_q)) = \{\sqrt{6} \cdot \rho_{c(G_2)}, \sqrt{6} \cdot \rho_{c(G_2)^2}, \sqrt{12} \cdot \rho_{c(G_2)^3}, G_2[1], G_2[-1], G_2[\theta], G_2[\theta^2]\}, \quad (4.15)$$

where by $\rho_{c(G_2)^i}$ we denote the class function in $R_u(G_2^F)$ corresponding to $1_{c(G_2)^i}$. In other words, as a virtual representation, $\rho_{c(G_2)} = \frac{1}{6}(\rho_{(1,1)} - \rho'_{(1,3)} - \rho''_{(1,3)} + \rho_{(1,6)} + \rho_{(2,1)} - \rho_{(2,2)})$ etc. In this basis, the elliptic nonabelian Fourier transform is:

$$\mathcal{FT}_{\text{ell}}(G_2(\mathbb{F}_q)) = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} & \frac{\sqrt{2}}{3} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{\sqrt{2}}{3} & 0 & \frac{1}{3} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & \frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & \frac{1}{3} & 0 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}. \quad (4.16)$$

This is an orthogonal, symmetric matrix.

4.5. Simple p -adic groups. We return to the setting of p -adic groups. Since the classification of unipotent representations recalled below has only been obtained for simple (or quasisimple) groups, we will assume that G is a simple split p -adic group.

Definition 4.6. *An irreducible smooth representation (π, V) of G is called unipotent if there exists a parahoric subgroup P_c of G and a cuspidal unipotent representation ρ of M_c such that $\text{Hom}_{M_c}(\rho, V^{U_c}) \neq 0$.*

Let $\text{Irr}_u G$ denote the set of characters of irreducible unipotent G -representations and let $R_u(G)$ be their complex span. Let $\overline{R}_u(G)_{\text{ell}}$ denote the image of $R_u(G)$ in $\overline{R}(G)_{\text{ell}}$.

By [MP, Theorem 6.11], the functors of parabolic induction and the Jacquet functors map unipotent representations to unipotent representations.

Let $\text{Irr}_{u,\text{temp}} G$ denote the set of characters of irreducible tempered unipotent G -representations and define $R_u(G)_{\text{temp}}$ to be their \mathbb{C} -span. As before, we may identify $\overline{R}_u(G)_{\text{ell}}$ with the image of the Bernstein A -operator, and since every class in $\overline{R}(G)_{\text{ell}}$ is represented by a tempered character, we may think of the space of elliptic representations as

$$\overline{R}_u(G)_{\text{ell}} = A(R_u(G)_{\text{temp}}) \subset R_u(G)_{\text{temp}}.$$

Let G^\vee denote the complex simple group dual to G . If $x \in G^\vee$, we will write $x = sn$ for the Jordan decomposition, s semisimple, n unipotent. Denote

$$A_x = Z_{G^\vee}(x)/Z_{G^\vee}(x)^0 Z(G^\vee) = \mathcal{S}_{G^\vee}(x)/Z(G^\vee),$$

where $\mathcal{S}_{G^\vee}(x)$ is the group of components of $Z_{G^\vee}(x)$.

We say that s is compact if s lies in a compact subgroup of G^\vee . Denote by \mathcal{T}_{G^\vee} the set of elements of G^\vee with compact semisimple part. Finally, we say that $x \in G^\vee$ is elliptic if $Z_{G^\vee}(x)$ does not contain any nontrivial torus. The parameterization of irreducible unipotent tempered representations is as follows.

Theorem 4.7 (Langlands-Kazhdan-Lusztig classification). *Suppose that G is simple, split and adjoint. There exists a natural bijection between $\text{Irr}_{u,\text{temp}} G$ and G^\vee -orbits on*

$$\{(x, \phi) \mid x \in \mathcal{T}_{G^\vee}, \phi \in \widehat{A}_x\}.$$

In this bijection, the irreducible square integrable representations correspond to the elliptic elements $x \in G^\vee$.

Decompose

$$R_u(G)_{\text{temp}} = \bigoplus_{x \in \mathcal{T}_{G^\vee}/G^\vee} R_u(G)_{\text{temp}}^x, \quad \overline{R}_u(G)_{\text{ell}} = \bigoplus_{x \in \mathcal{T}_{G^\vee}/G^\vee} \overline{R}_u(G)_{\text{ell}}^x. \quad (4.17)$$

In the Iwahori case, Reeder [Re1, Main Theorem] described the elements x such that $\overline{R}_u(G)_{\text{ell}}^x \neq 0$. This description can be extended to the setting of unipotent representations of a simple, adjoint group G by combining:

- (a) the results of Opdam-Solleveld [OS2] on the elliptic theory of affine Hecke algebras with arbitrary positive parameters, more precisely, the expression of the Euler-Poincaré pairing between two tempered modules in terms of the analytic R-group [OS2, Theorem 6.5];
- (b) the Hecke algebra isomorphisms between the unipotent representations and the categories of modules for affine Hecke algebras with unequal parameters, Lusztig [Lu1, “the arithmetic/geometric correspondence”], see also Opdam [Op2, Theorem 3.4].
- (c) the equality of the analytic R-group with the geometric R-group, as in sections 8 and 9 of [Re1], particularly, [Re1, (9.2.1), (9.2.2), (9.2.3)].

To state the result, we need one more definition.

Definition 4.8. *A semisimple element $s \in G^\vee$ is called isolated if $Z_{G^\vee}(s)$ is semisimple. A unipotent element $n \in G^\vee$ is called quasidistinguished if it is the unipotent Jordan factor of an elliptic element $x \in G^\vee$. An elliptic unipotent element is called distinguished. Denote by $\mathcal{T}_{G^\vee}^0 \subset \mathcal{T}_{G^\vee}$ the set of elements $x = sn \in G^\vee$ such that s is isolated and n is quasidistinguished in $Z_{G^\vee}(s)$.*

Every elliptic element $x \in G^\vee$ has a Jordan decomposition $x = sn$ where s is isolated and n is distinguished in $Z_{G^\vee}(s)$, hence $\mathcal{T}_{G^\vee}^0$ contains all the elliptic elements of G^\vee . Notice however that if $x = n$ is quasidistinguished, but not distinguished in G^\vee , then x is not elliptic, but $x \in \mathcal{T}_{G^\vee}^0$.

Theorem 4.9 ([Re1, Main Theorem]). *The space $\overline{R}_u(G)_{\text{ell}}^x$ is nonzero if and only if $x \in \mathcal{T}_{G^\vee}^0$.*

Let $\mathcal{U}_{G^\vee}^{\text{ell}}$ denote the set of conjugacy classes of elements n such that n is the unipotent Jordan part of an element of $x \in \mathcal{T}_{G^\vee}^0$. For each $n \in \mathcal{U}_{G^\vee}^{\text{ell}}$, denote

$$\overline{R}_u(G)_{\text{ell}}^n = \bigoplus_{x=sn \in \mathcal{T}_{G^\vee}^0} \overline{R}_u(G)_{\text{ell}}^x. \quad (4.18)$$

Example 4.10. (1) *Suppose $G^\vee = Sp(2n, \mathbb{C})$. The set of conjugacy classes of unipotent elements in G^\vee is in one-to-one correspondence, via the Jordan canonical form, with partitions of $2n$ such that the odd parts appear with even multiplicity. The distinguished unipotent classes correspond to partitions where all parts are even and distinct. The quasidistinguished unipotent classes correspond to partitions where all parts are even and the multiplicity of each part is at most 2. On the other hand, the set $\mathcal{U}_{G^\vee}^{\text{ell}}$ is in bijection with partitions of $2n$ consisting of only even parts and where the multiplicity of each part is at most 4.*

(2) *If $G^\vee = G_2$, then $\mathcal{U}_{G^\vee}^{\text{ell}} = \{G_2, G_2(a_1)\}$, where G_2 denotes the regular unipotent class and $G_2(a_1)$ the subregular unipotent class. Both classes are distinguished.*

Let $x = sn \in G^\vee$ be an elliptic element. Choose a Lie homomorphism $\psi : SL(2) \rightarrow Z_{G^\vee}(s)$ such that $\psi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = n$. Let $M = Z_{G^\vee}(\psi)$. Then $\mathcal{S}_{G^\vee}(n) = M/M^0$ as it is well known. Moreover, M^0 is a torus and $Z_M(s)$ is finite [Re2, Lemma 7.1]. In particular, $Z_{G^\vee}(s, \psi)^0 = Z_M(s)^0 = \{1\}$, hence $\mathcal{S}_{G^\vee}(sn) = \mathcal{S}_{G^\vee}(s, \psi) = Z_M(s)$. Again by *loc. cit.*, we have a natural surjective map

$$\mathcal{S}_{G^\vee}(sn) = Z_M(s) \longrightarrow Z_{\mathcal{S}_{G^\vee}(n)}(s) \quad (4.19)$$

whose kernel is $Z_{M^0}(s)$, a finite group. When n is distinguished in G^\vee , equivalently $M^0 = \{1\}$, then

$$\mathcal{S}_{G^\vee}(sn) \cong Z_{\mathcal{S}_{G^\vee}(n)}(s). \quad (4.20)$$

Lemma 4.11. *Suppose n is distinguished in G^\vee . Let Σ_n be the set of $Z_{G^\vee}(n)$ -orbits on*

$$\{(s, \phi) : s \in Z_{G^\vee}(n) \text{ semisimple}, \phi \in \widehat{\mathcal{S}_{G^\vee}(sn)}\}.$$

Then Σ_n can be naturally identified with $M(\mathcal{S}_{G^\vee}(n))$.

Proof. By (4.20), $\mathcal{S}_{G^\vee}(sn) \cong Z_{\mathcal{S}_{G^\vee}(n)}(s)$. Recall that the elements of $M(\mathcal{S}_{G^\vee}(n))$ are $\mathcal{S}_{G^\vee}(n)$ -orbits of pairs (y, τ) , where y is an element of $\mathcal{S}_{G^\vee}(n)$ and τ is an irreducible representation of $Z_{\mathcal{S}_{G^\vee}(n)}(y)$. The claim is immediate. \square

Example 4.12. *In the case $G = G_2$, the spaces $\overline{R}_u(G)_{\text{ell}}^n$, $n \in \{G_2, G_2(a_1)\}$ admit bases consisting of square integrable representations. The space $\overline{R}_u(G)_{\text{ell}}^{G_2}$ is one-dimensional, spanned by v_1 , the Steinberg representation. The space $\overline{R}_u(G)_{\text{ell}}^{G_2(a_1)}$ is 8-dimensional. The eight square integrable are as follows: 4 have Iwahori-fixed vectors and 4 are supercuspidal. The 4 supercuspidal representations are compactly induced $c\text{-ind}_{K_0}^G(\tilde{\rho})$, where K_0 is the maximal hyperspecial compact open subgroup $K_0 = G(\mathcal{O})$ and $\tilde{\rho}$ is the pull-back to K_0 one of the 4 cuspidal unipotent representations of $G_2(\mathbb{F}_q)$.*

The component group is $\mathcal{S}_{G^\vee}(n) = S_3$. The set $M(S_3)$ has precisely cardinality 8 same as the number of irreducible unipotent square integrable representations with unipotent part n . This is not a coincidence, since $\Sigma_n \cong M(\mathcal{S}_{G^\vee}(n))$ by Lemma 4.11.

There are 3 isolated semisimple conjugacy classes in G_2 , whose representatives we denote by $s_0 = 1$, s_1 , and s_2 . The corresponding centralizers in $G^\vee = G_2$ have types G_2 , $A_1 \times \bar{A}_1$, and A_2 , respectively. The parameterizations and the division of the 8 unipotent discrete series representations into L -packets are in Table 2. We label the representations v_2, \dots, v_9 for later reference.

4.6. The elliptic restriction map. Suppose (π, V) is a unipotent representation of G . For every standard parahoric subgroup P_J , $J \subsetneq \Pi_a$, we define the *restriction* of π to be:

$$\text{res}_{P_J}(\pi) = \text{the character of the } M_J\text{-representation on } V^{U_J}. \quad (4.21)$$

We recall an important result of Moy and Prasad.

TABLE 2. Unipotent parameters for $n = G_2(a_1)$, $A_n = S_3$, in G_2

$Z_{G^\vee}(s)$	A_{su}	$\phi \in \widehat{A_{su}}$	$(y, \tau) \in M(A_u)$	Label; Representation
G_2	S_3	1	$(1, 1)$	v_2 ; Iwahori, generic, dual of the affine reflection repn.
		refl	$(1, r)$	v_3 ; Iwahori, nongeneric, short reflection sign repn.
		sgn	$(1, \epsilon)$	v_6 ; supercuspidal $G_2[1]$
$A_1 + \tilde{A}_1$	$\mathbb{Z}/2\mathbb{Z}$	1	$(g_2, 1)$	v_5 ; Iwahori, endoscopic $A_1 \times \tilde{A}_1$
		sgn	(g_2, ϵ)	v_7 ; supercuspidal $G_2[-1]$
A_2	$\mathbb{Z}/3\mathbb{Z}$	1	$(g_3, 1)$	v_4 ; Iwahori, endoscopic A_2
		ζ	(g_3, θ)	v_8 ; supercuspidal $G_2[\theta]$
		ζ^2	(g_3, θ^2)	v_9 ; supercuspidal $G_2[\theta^2]$

Lemma 4.13 (see [MP, Theorem 3.5]). *Let G be a simple p -adic group and (π, V) an admissible G -representation. Suppose that $J \subsetneq \Pi_a$ and that V^{U_J} contains the cuspidal unipotent representation μ of M_J . If $J' \subsetneq \Pi_a$ is such that $V^{U_{J'}} \neq 0$ and J' is minimal with respect to this property, then $J' = \omega J \omega^{-1}$ for some $\omega \in \Omega$ and $V^{U_{J'}}$ is a direct sum of copies of the twist μ^ω .*

This means that the image of the restriction lands in the unipotent M_J -characters, i.e., it defines a map $\text{res}_{P_J} : R_u(G) \rightarrow R_u(M_J)$. Let

$$\text{proj}_{\text{ell}}^{M_J} : R_u(M_J) \rightarrow \bar{R}_u(M_J)_{\text{ell}} \quad (4.22)$$

be the projection with respect to the ordinary character pairing. This is the same as defining $\text{proj}_{\text{ell}}^{M_J}(\sigma)(\mathcal{C}) = \text{tr } \sigma(\mathcal{C})$ if $\mathcal{C} \subset M_J$ is an elliptic conjugacy class and $\text{proj}_{\text{ell}}^{M_J}(\sigma)(\mathcal{C}) = 0$ otherwise. For the computations later on, if \mathcal{B}_{ell} is an orthonormal (with respect to the character pairing) basis of $\bar{R}_u(M_J)_{\text{ell}}$ (viewed as a subspace of $R_u(M_J)$), then $\text{proj}_{\text{ell}}(\sigma) = \sum_{\rho \in \mathcal{B}_{\text{ell}}} \langle \sigma, \rho \rangle_{M_J} \cdot \rho$.

Definition 4.14. *Suppose that G is simple and simply-connected. Let $\Pi_{a, \max}$ denote the set of maximal subsets $J \subsetneq \Pi_a$. Define the unipotent elliptic restriction map:*

$$\begin{aligned} \text{res}_{u, \text{ell}} : \bar{R}_u(G)_{\text{ell}} = A(R_u(G)_{\text{temp}}) &\longrightarrow \bigoplus_{J \in \Pi_{a, \max}} \bar{R}_u(M_J)_{\text{ell}}, \\ \pi &\longmapsto \sum_{J \in \Pi_{a, \max}} (\text{proj}_{\text{ell}}^{M_J} \circ \text{res}_{P_J})(\pi). \end{aligned} \quad (4.23)$$

Define $\bar{\mathcal{H}}(G)_u^{\text{ell}}$ to be the image of $\bar{R}(G)_u^{\text{ell}}$ under the homomorphism $\tau_{HS} \circ \text{ep}$. In other words, $\bar{\mathcal{H}}(G)_u^{\text{ell}}$ is the span in $\bar{\mathcal{H}}(G)$ of all the pseudocoefficients f_π of elliptic unipotent representations π of G .

On the other hand, clearly extension by zero outside P_J realizes $\bar{R}_u(M_J)_{\text{ell}}$ as a subspace of $\bar{\mathcal{H}}_c(G)$. Let $\iota : \bigoplus_{J \in \Pi_{a, \max}} \bar{R}_u(M_J)_{\text{ell}} \rightarrow \bar{\mathcal{H}}_c(G)$ be the linear map defined by $\sum_J f_J \mapsto \tilde{f}_J$. The results in the following proposition are proved in [MW, Theorem 1.9].

Proposition 4.15 ([MW]). *Suppose G is simple and simply-connected. Then:*

- (1) ι is injective and the image is $\bar{\mathcal{H}}(G)_u^{\text{ell}}$. In particular, we may identify $\bar{\mathcal{H}}(G)_u^{\text{ell}} = \bigoplus_{J \in \Pi_{a, \max}} \bar{R}_u(M_J)_{\text{ell}}$.
- (2) $f_\pi \equiv \text{res}_{u, \text{ell}}(\pi)$ in $\bar{\mathcal{H}}(G)_u^{\text{ell}}$ for every $\pi \in \bar{R}_u(G)_{\text{ell}}$;
- (3) The map $\text{res}_{u, \text{ell}} : \bar{R}_u(G)_{\text{ell}} \rightarrow \bar{\mathcal{H}}(G)_u^{\text{ell}}$ is an isometry with respect to the Euler-Poincaré pairing EP for $\bar{R}_u(G)_{\text{ell}}$ and the ordinary character pairing for $\bar{\mathcal{H}}(G)_u^{\text{ell}} = \bigoplus_{J \in \Pi_{a, \max}} \bar{R}_u(M_J)_{\text{ell}}$.

An analogous result was proved in [OS1, Section 3] in the setting of the affine Hecke algebra \mathcal{H} of an affine Weyl group W and with arbitrary positive parameters. In that case, one has an isometric isomorphism $\bar{R}(\mathcal{H})_{\text{ell}} \rightarrow \bar{R}(W)_{\text{ell}}$ with respect to the Euler-Poincaré pairings on both spaces. Moreover, $\bar{R}(W)_{\text{ell}}$ is naturally isometrically isomorphic with the direct sum

$$\bar{R}(W)_{\text{ell}} \cong \bigoplus_v \bar{R}(W_v)_{\text{ell}} \quad (4.24)$$

over the vertices v of the fundamental alcove c_0 . Here W_v is the finite Weyl group which centralizes in W the vertex v , but the pairing for $\bar{R}(W_v)_{\text{ell}}$ needed for the isometry in (4.24) is the ordinary character pairing of the finite Weyl group.

4.7. The elliptic nonabelian Fourier transform for G_2 . In this subsection, we investigate the behavior of the nonabelian Fourier transform with respect to the restriction map just defined.

Definition 4.16. *In light of Proposition 4.15, define the elliptic unipotent nonabelian Fourier transform of G to be*

$$\mathcal{FT}_{u,\text{ell}} : \bar{\mathcal{H}}(G)_u^{\text{ell}} \longrightarrow \bar{\mathcal{H}}(G)_u^{\text{ell}}, \quad \mathcal{FT}_{u,\text{ell}} = \bigoplus_{J \in \Pi_{a,\text{max}}} (\mathcal{FT}_{\text{ell}} : \bar{R}_u(M_J)_{\text{ell}} \rightarrow \bar{R}_u(M_J)_{\text{ell}}). \quad (4.25)$$

Now suppose that $n \in \mathcal{U}_{G^\vee}^{\text{ell}}$ is distinguished. As explained in Example 4.12, we may identify the parameterizing set of $\text{Irr}_u(G)_{\text{ell}}^n$ with $M(A_n)$. For a square integrable representation $\pi \in \text{Irr}_u(G)_{\text{ell}}^n$, let $\zeta_\pi \in M(A_n)$ denote the resulting parameter. We can define a *dual Fourier transform* on $\bar{R}_u(G)_{\text{ell}}^n$ via:

$$\mathcal{FT}_{u,\text{ell}}^{\vee,n} : \bar{R}_u(G)_{\text{ell}}^n \rightarrow \bar{R}_u(G)_{\text{ell}}^n, \quad \pi \mapsto \sum_{\pi' \in \text{Irr}_u(G)_{\text{ell}}^n} \{\zeta_\pi, \zeta_{\pi'}\}_{A_n} \cdot \pi', \quad (4.26)$$

where $\{\cdot, \cdot\}_{A_n}$ denotes the nonabelian Fourier transform for the group A_n .

Definition 4.17. *Suppose $G = G_2$. By Theorem 4.9 and (4.18), $\bar{R}_u(G)_{\text{ell}} = \bigoplus_{n \in \mathcal{U}_{G^\vee}^{\text{ell}}} \bar{R}_u(G)_{\text{ell}}^n$. By Example 4.12, we may consider Lusztig's nonabelian Fourier transform, which we denote $\mathcal{FT}_{u,\text{ell}}^{\vee,n}$, with respect to the parameter space $\Sigma_n = M(\mathcal{S}_{G^\vee}(n))$ for $n \in \mathcal{U}_{G^\vee}^{\text{ell}}$. Define the dual elliptic unipotent nonabelian Fourier transform to be*

$$\mathcal{FT}_{u,\text{ell}}^\vee : \bar{R}_u(G)_{\text{ell}} \rightarrow \bar{R}_u(G)_{\text{ell}}, \quad \mathcal{FT}_{u,\text{ell}}^\vee = \bigoplus_{n \in \mathcal{U}_{G^\vee}^{\text{ell}}} \mathcal{FT}_{u,\text{ell}}^{\vee,n}. \quad (4.27)$$

We remark that, in the natural bases described above, the map $\mathcal{FT}_{u,\text{ell}}^\vee$ is block diagonal

$$\mathcal{FT}_{u,\text{ell}}^\vee = \begin{pmatrix} \{\cdot, \cdot\}_1 & 0 \\ 0 & \{\cdot, \cdot\}_{S_3} \end{pmatrix}$$

with blocks of sizes 1 and 8, while $\mathcal{FT}_{u,\text{ell}}$ is block diagonal

$$\mathcal{FT}_{u,\text{ell}} = \begin{pmatrix} \mathcal{FT}_{\text{ell}}(G_2(\mathbb{F}_q)) & 0 & 0 \\ 0 & \mathcal{FT}_{\text{ell}}((A_1 + \tilde{A}_1)(\mathbb{F}_q)) & 0 \\ 0 & 0 & \mathcal{FT}_{\text{ell}}(A_2(\mathbb{F}_q)) \end{pmatrix}$$

with blocks of sizes 7, 1, 1.

Theorem 4.18. *Suppose $G = G_2$. The diagram*

$$\begin{array}{ccc} \bar{R}_u(G)_{\text{ell}} & \xrightarrow{\mathcal{FT}_{u,\text{ell}}^\vee} & \bar{R}_u(G)_{\text{ell}} \\ \text{res}_{u,\text{ell}} \downarrow & & \downarrow \text{res}_{u,\text{ell}} \\ \bar{\mathcal{H}}(G)_u^{\text{ell}} & \xrightarrow{\mathcal{FT}_{u,\text{ell}}} & \bar{\mathcal{H}}(G)_u^{\text{ell}} \end{array} \quad (4.28)$$

is commutative.

Proof. The proof is based on a direct computation. The restrictions $\text{res}_{P_J}(v_i)$ are given in Table 3. This has been computed using the result of Moy-Prasad, see Lemma 4.13, and the reduction to Iwahori-Hecke algebras. In particular, if we apply Lemma 4.13 to the supercuspidal representations $v_i = \text{c-ind}_{P_{J_0}}^{G_2}(\mu_i)$, $6 \leq i \leq 9$, where μ_i is a cuspidal unipotent representation of $G_2(\mathbb{F}_q)$ as in the table, then we see that $\text{res}_{P_{J_j}}(v_i) = \delta_{i,j} \mu_i$.

The representations $v_1 - v_5$ are all Iwahori-spherical. Let $\mathcal{H}(G, I)$ denote the Iwahori-Hecke algebra of compactly supported, smooth, I -biinvariant functions [Bo, IM]. This is isomorphic to the affine Hecke algebra of type G_2 with equal parameters. Let $\mathcal{H}(P_J, I)$ denote the subalgebra of $\mathcal{H}(G, I)$ of functions

whose support is in P_J . This is isomorphic to the finite Hecke subalgebra of type W_J . The subspace of Iwahori-fixed vectors v_i^I is naturally an $\mathcal{H}(G, I)$ -module and therefore, we may consider the restriction of v_i^I to $\mathcal{H}(P_J, I)$.

The restriction $\text{res}_{P_J}(v_i)$ can be computed at the level of the Iwahori-Hecke algebra. More precisely, suppose that μ is an M_J -type such that μ appears in $\text{res}_{P_J}(v_i)$, $1 \leq i \leq 5$. Let $\tilde{\mu}$ denote the pullback of μ to P_J . Then, by Lemma 4.13, $\tilde{\mu}^I \neq 0$. Moreover:

$$\dim \text{Hom}_{M_J}(\mu, v_i^{U_J}) = \dim \text{Hom}_{\mathcal{H}(P_J, I)}(\tilde{\mu}^I, v_i^I). \quad (4.29)$$

The structure of v_i^I as Hecke algebra modules is well known, and this is how we compute these restrictions.

TABLE 3. Restrictions of unipotent elliptic G_2 -representations

$\text{Irr}_u(G_2)_{\text{ell}}$	$J_0 = \{\alpha_1, \alpha_2\} = G_2$	$J_1 = \{\alpha_0, \alpha_2\} = A_1 + \tilde{A}_1$	$J_2 = \{\alpha_0, \alpha_1\} = A_2$
v_1	$\phi_{(1,6)}$	$\text{sgn}_0 \otimes \text{sgn}_2$	sgn
v_2	$\phi_{(1,6)} + \phi_{(2,1)}$	$\text{sgn}_0 \otimes \text{sgn}_2 + \text{sgn}_0 + \text{triv}_2 + \text{triv}_0 \otimes \text{sgn}_2$	$\text{sgn} + \text{refl}$
v_3	$\phi_{(1,3)}''$	$\text{triv}_0 \otimes \text{sgn}_2$	triv
v_4	$\phi_{(1,0)} + \phi_{(1,3)}'$	$\text{sgn}_0 \otimes \text{triv}_2 + \text{triv}_0 \otimes \text{triv}_2$	refl
v_5	$\phi_{(1,6)} + \phi_{(2,2)}$	$\text{sgn}_0 \otimes \text{sgn}_2 + \text{sgn}_0 \otimes \text{triv}_2 + \text{triv}_0 \otimes \text{sgn}_2$	$\text{sgn} + \text{refl}$
v_6	$G_2[1]$	0	0
v_7	$G_2[-1]$	0	0
v_8	$G_2[\theta]$	0	0
v_9	$G_2[\theta^2]$	0	0

Next, once we have $\text{res}_{P_J}(v_i)$, we need to compute $\text{res}_{u, \text{ell}}(v_i)$ in terms of the orthonormal bases $\mathcal{B}_{\text{ell}}(G_2(\mathbb{F}_q))$ and

$$\mathcal{B}_{\text{ell}}((A_1 + \tilde{A}_1)(\mathbb{F}_q)) = \left\{ \frac{1}{2}(\text{triv}_0 - \text{sgn}_0) \otimes (\text{triv}_2 - \text{sgn}_2) \right\} \text{ and } \mathcal{B}_{\text{ell}}(A_2(\mathbb{F}_q)) = \left\{ \frac{1}{\sqrt{3}}(\text{triv} - \text{refl} + \text{sgn}) \right\}. \quad (4.30)$$

This is done by projecting $\sum_{J \text{ max}} \text{res}_{P_J}(v_i)$ onto the elliptic space. In terms of the ordered bases listed before, the matrix of $\text{res}_{u, \text{ell}}$ is the orthogonal matrix:

$$[\text{res}_{u, \text{ell}}] = \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/\sqrt{6} & 0 & 1/\sqrt{6} & 2/\sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ 1/2\sqrt{3} & -1/2\sqrt{3} & -1/2\sqrt{3} & 0 & \sqrt{3}/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1/2 & -1/2 & -1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 \\ 1/\sqrt{3} & 0 & 1/\sqrt{3} & -1/\sqrt{3} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.31)$$

The claim now amounts to the direct verification that

$$[\text{res}_{u, \text{ell}}] \cdot [\mathcal{T}_{u, \text{ell}}^\vee] \cdot [\text{res}_{u, \text{ell}}]^T = [\mathcal{F}\mathcal{T}_{u, \text{ell}}^\vee].$$

□

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