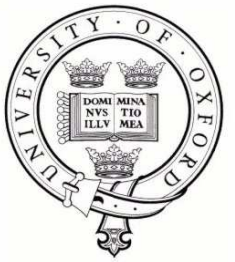
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Dissertation

Investigation of Portfolio Choice that Tracks a Continuously Moving Target

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Abstract

We investigate the problem of tracking a moving target, specifically a given continuously compounded, deterministic growth rate by using linear quadratic control theory. We first present some preliminaries and then derive and solve the corresponding Riccati equations for our problem. We then implement our findings in MATLAB by using one stock from the FTSE100 and a risk-less asset and analyze the results. Furthermore, we focus on situations when the performance of our tracking is not the expected and highlight when this can happen and what the risks involved are. We also present situations when we would need to borrow or short sell, like in the case of aggressively tracking a target. We then briefly present the extension to multi asset portfolios. Finally, we discuss the case of tracking a market index.

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1 Introduction

When managing funds, one might want to measure the performance of his portfolio against certain financial benchmarks that can be either deterministic or stochastic. One way of accomplishing this is by using the theory of Stochastic Control. So it would be useful to be able to form a portfolio in such a way so as to move as close as possible to a given target over time. The target could, for example, be a given deterministic growth rate or be related to financial instruments. We are of course able to rebalance the portfolio a finite number of times during the investment period.

It is important to note that we want our portfolio to track the benchmark and not beat it. Going over the benchmark is equally penalized as going under it. In practice, this can be used by fund managers as a reference tool, in order to compare the portfolio's performance against these benchmarks but not to beat them. This approach is used by tracker funds. The reasoning behind tracker funds is based on the idea that one cannot beat the index, so in the longterm it is better to track it. Whether or not this is true is a controversial question and there is plenty of literature on the subject. Furthermore, we have to note that beating a specific benchmark could some times be taken as a sign of aggressive behavior of the fund manager and therefore is not always optimal. For these reasons it is often that someone would just want to track a benchmark and not beat it. For models with other risk measures see [2].

Note that when tracking certain financial benchmarks, such as a market index, one could always literally track it by holding all the constituents stocks in proportion, and adjusting their weights continuously, but as this is usually impractical, it is important that we focus the problem on portfolios consisting of a small proportion of the stocks available. In the case of tracking a market index this would lead to an incomplete market. We however will be more concerned with tracking a given deterministic growth rate, so the number of stocks in the market is irrelevant from a mathematical point of view insensitive to portfolio selection which is our case. By insensitive to portfolio selection, we mean that we do not want our portfolio performance to depend on the stocks we choose. Of course even in the growth rate case we would want our portfolio to consist of a little number of stocks.

The approach of tracking a given fixed growth rate and a stochastic market index has been investigated in [6] with the use of semidefinite programming (SDP). There, an infinite

time horizon was adopted. This dissertation is mainly based on [6] but we will investigate the problem in a finite time horizon.

We will first give the theory behind the problem of tracking a given deterministic continuously compounded growth rate and derive the optimal portfolio given a number of stocks we can invest in. We formulate our problem as a stochastic linear quadratic control problem and then derive and solve the corresponding Riccati equations. Having their solutions we get the equation for the optimal portfolio. We then implement our findings in MATLAB by using stocks from FTSE 100 as data and discuss the results. Note that we will mainly use a portfolio consisting of only one stock and a risk-less asset, although we will discuss the case of multi asset portfolios later. We also focus on situations when our portfolio did not track the target as expected and discuss the reasons for that. A very important aspect that arises, is the amount we may have to short sell or borrow. This, for example, is the case when we are tracking aggressively, that is tracking a growth rate that is a lot higher than the rate of the risk-less asset. There our portfolio will be more volatile and therefore we may have to borrow or short sell significant amounts. We will discuss these situations and in all cases observe the results by drawing plots.

2 Preliminaries

For clarity we first give some notation used in the dissertation:

A' - Inverse matrix

$S^{n\times n}$ - The set of all $n \times n$ symmetric matrices

$\{\mathcal{F}_t\}_{t\geq 0}$ - Filtration

L^∞ - The set of all bounded measurable functions $f : [0, T] \to \mathbb{R}^n$

$L^p_{\mathcal{G}}(\mathbb{R}^n)$ - The space of all $\mathcal{G}$-measurable random variables $X(\cdot)$ such that $E\,|X|^p < \infty$ with $p \in [1, \infty)$.

$L^p_{\mathcal{F}}(\mathbb{R}^n)$ - The space of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted measurable processes $X(\cdot)$ valued in $\mathbb{R}^n$ that satisfy $\int_0^{+\infty} \|X(t)\|^p < +\infty$.

We will next briefly give some basic notions and definitions on control theory. The chapter is based on [7], [4] and [1] and we refer there for more details.

2.1 Optimal Control

Optimal control problems are often encountered in the real world and the theory behind them is well developed and used in several scientific areas. As stated in [7], although the name of optimal control theory was first used in the late 1950s, similar problems are encountered all the way back to ancient times, when it was discovered that the shortest path between two points is a straight line.

We will now formulate the problem of control theory in a deterministic setting and then we will focus on their stochastic extension. Consider that the dynamics of a system are given by the ordinary differential equation

$$\begin{cases} \dot{x}(t) = b(t, x(t), u(t)), \qquad a.e.\ \ t \in [0, T] \\ x(0) = x_0 \end{cases} \tag{1}$$

where $b : [0, T]\times\mathbb{R}^n\times\Gamma \to \mathbb{R}^n$ is a given map. A measurable map $u(\cdot) : [0, T] \to \Gamma$ is called a *control*. A solution, $x(\cdot)$, of Equation (1), is called the *state trajectory* corresponding to $u(\cdot)$, and x_0 is called the *initial state*. Furthermore, we are given a cost functional

$$J(u(\cdot)) = \int_0^T f(t, x(t), u(t))dt + h(x(T)), \tag{2}$$

for given maps $f : [0, T]\times\mathbb{R}^n\times U \to \mathbb{R}$ and $h : \mathbb{R}^n \to (R)$, that is measuring the performance of the controls. The term $\int_0^T [f(t, x(t), u(t))]dt$ is called the *running cost* and $h(x(T))$ is called the *terminal cost*.

Note that in our problem there may be various constraints involved. For example state constraints $x(t) \in S(t)$ and control constraints $u(t) \in U$ where $S(\cdot) : [0, T] \to 2^{(R)^n}$ and $U(\cdot) : [0, T] \to 2^{\Gamma}$ are given multifunctions. Now define

$$\mathcal{V}[0, T] = \{u : [0, T] \to U |\ u(\cdot) \text{ is measurable}\}.$$

Any control belonging to that set is called a *feasible control*.

Definition: A control $u(\cdot)$ is called *admissible control* and $(u(\cdot), x(\cdot))$ is called an *admissible pair*, if:

(i) $u(\cdot)$ is a feasible control.

(ii) $x(\cdot)$ is the unique solution of the dynamics 1, under $u(\cdot)$.

(iii) the state constraint is satisfied.

(iv) $t \to f(t, x(t), u(t)) \in L^1[0, T]$.

We are only interested in admissible controls and want to minimize the cost functional under the dynamics of the state, by changing the control. The deterministic optimal control problem is now stated as

$$\min J(u(\cdot)) = \int_0^T f(t, x(t), u(t))dt + h(x(T)), \tag{3}$$

over the set of all admissible controls, subject to the dynamics 1 and the constraints. For further details on optimal control theory we refer to [7].

2.2 LQ Control

A particularly important case of optimal control problems is the *linear quadratic (LQ) optimal control problem*. The importance of LQ problems is based on the fact that they can model many real life problems and approximate many non linear ones, and also due to the nice properties that their solutions exhibit. Consider the following problem:

$$\min J(u(\cdot)) = \frac{1}{2}\int_0^T [x(t)'Q(t)x(t) + u(t)'P(t)u(t)]dt + \frac{1}{2}x(T)'Hx(T), \tag{4}$$

subject to

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(0) = x_0, \end{cases} \tag{5}$$

where, if $S^{d\times d}$ the set of all $d \times d$ symmetric matrices:

$Q(\cdot) \in L^\infty(0, T; S^{n\times n}), R(\cdot) \in L^\infty(0, T; S^{m\times m}),$

$A(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times n}), B(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times m}),$

$H \in S^{n\times n}.$

We can get a solution to this problem by introducing the *Riccati equation*

$$\begin{cases} \dot{P}(t) = -P(t)A(t) - A(t)'P(t) - Q(t) + P(t)B(t)R(t)^{-1}B(t)'P(t), \quad t \in [0, T] \\ P(T) = H \in S^{n\times n}, \end{cases} \tag{6}$$

which admits a unique solution under the following assumptions:

$Q(t) \geq 0$ (positive semidefinite)

$R(t) > 0$ (positive definite)

$H \geq 0.$

After finding the solution of the Ricatti equation we can calculate the optimal feedback as

$$u^*(t) = -R(t)^{-1}B(t)'P(t)x^*(t)$$

and the optimal value as

$$J^* = inf J(u(\cdot)) = \frac{1}{2}x_0'P(0)x_0.$$

Note that LQ control can be extended by introducing the non-homogeneous form of Equation (5) as

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t) \\ x(0) = x_0. \end{cases} \tag{7}$$

Then, apart from the Riccati equation we will need to introduce a second equation

$$\begin{cases} \dot{G}(t) + [A(t)' - P(t)B(t)R(t)^{-1}B(t)']G(t) + P(t)f(t) = 0, \quad a.e.\ t \in [0,T] \\ G(T) = 0 \end{cases} \tag{8}$$

and then the optimal feedback control is

$$u(t)^* = -R(t)^{-1}[B(t)'P(t)x(t)^* + B(t)G(t)]$$

and the optimal value is

$$J^* = inf J(u(\cdot)) = \frac{1}{2}x_0'P(0)x_0 + G(0)'x_0 + \frac{1}{2}\int_0^T \left[2G(t)'f(t) - \left|R(t)^{\frac{1}{2}}B'(t)G(t)\right|^2\right]dt.$$

For the proofs and for further details on LQ control we refer to [7].

2.3 Stochastic Control

Stochastic Control is the extension of control theory to systems with uncertainty. It would be convenient if we could always be certain about the outcome of our system. Unfortunately, this is hardly ever the case. Most systems in real life have uncertainty inherent in them and this is also the case for mathematical finance problems. Therefore instead of trying to select the optimal decision in order to get the best result we are usually trying to get the best expected result. These problems are called *stochastic optimal control problems*. In problems of this class, given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ on which an

m-dimensional standard Brownian motion $W(\cdot)$ is defined, the dynamics of the system is given by a stochastic differential equation of the following matrix form:

$$\begin{cases} dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t) \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{9}$$

where, for a fixed $T \in (0, \infty)$, $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times m}$, with U being a given separable metric space. In addition, we are given a cost functional of the form

$$J(u(\cdot)) = E\left(\int_0^T f(t, x(t), u(t))dt + h(x(T))\right). \tag{10}$$

Define the set

$$\mathcal{U}[0, T] = \{u : [0, T] \times \Omega \to U | \; u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t \geq 0} - \text{adapted}\}.$$

Any control, $u(\cdot)$, belonging to that set is called a *feasible control*. Let $S(t)$ be a given multifunction. Then, in our problem we may also have some state constraints given by

$$x(t) \in S(t), \quad \forall t \in [0, T], \quad P - a.s.$$

Other types of state constraints are also possible.

Definition : Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a given probability space and assume $(\Omega, \mathcal{F}, P)$ is complete, $\mathcal{F}_0$ contains all the P-null sets in $\mathcal{F}$ and $\{\mathcal{F}\}_{t \geq 0}$ is right continuous. Furthermore, let $W(t)$ be a given m-dimensional, standard, $\{\mathcal{F}_t\}_{t \geq 0}$ Brownian motion. Then, a control $u(\cdot)$ is called an *admissible control*, and a pair $(x(\cdot), u(\cdot))$ an *admissible pair* if:

(i) $u(\cdot)$ is a feasible control.

(ii) $x(\cdot)$ is the unique solution of the dynamics 9.

(iii) the state constraints are satisfied.

(iv) $f(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$.

So the problem of stochastic optimal control under this formulation can be stated as:

$$\min J(u(.)) = E\left(\int_0^T f(t, x(t), u(t))dt + h(x(T))\right), \tag{11}$$

over the set of all admissible controls, under the dynamics 9. For further details on stochastic control theory see [7].

2.4 Stochastic LQ Control

As in the deterministic case, stochastic linear quadratic control is of particular interest. Consider the following general non-homogeneous problem:

$$\min J(u(.)) = \frac{1}{2}\left\{\int_0^T [x(t)'Q(t)x(t) + u(t)'P(t)u(t)]dt + \frac{1}{2}x(T)'Hx(T)\right\}, \tag{12}$$

subject to

$$\begin{cases} dx(t) = (A(t)x(t) + B(t)u(t) + f(t))dt + \sum_{j=1}^m (C_j(t)x(t) + D_j(t)x(t) + g_j(t))dW_j(t) \\ x(0) = x_0, \end{cases} \tag{13}$$

where A, B, C_j, D_j, b, g_j are deterministic matrix/vector valued functions, $(W^1(t), ..., W^m(t))$ is a m-dimensional Brownian Motion and b, g_j are square integrable. Note that this is not just a routine extension of the deterministic case as the control in the diffusion term makes the problem a lot different (see [7]).

We now introduce the *Stochastic Riccati equation*

$$\begin{cases} \dot{P} = -PA - A'P - \sum_{j=1}^m C_j'PC_j - Q + \\ \quad +[B'P + \sum_{j=1}^m D_jPC_j][R + \sum_{j=1}^m D_j'PD_j]^{-1}[B'P + \sum_{j=1}^m D_jPC_j] \\ P(T) = H \\ K(t) = R(t) + \sum_{j=1}^m D_j(t)'P(t)D_j(t) > 0, \quad \forall t \in [0,T], \end{cases} \tag{14}$$

along with the equation

$$\begin{cases} \dot{G} = -[A - B(R + \sum_{j=1}^m D_j'PD_j)^{-1}(B'P + \sum_{j=1}^m D_jPC_j)]'G - \\ \quad - \sum_{j=1}^m [C_j - D_j(R + \sum_{j=1}^m D_j'PD_j)^{-1}(B'P + \sum_{j=1}^m D_jPC_j)]Pg_j - Pf \\ G(T) = 0, \end{cases} \tag{15}$$

where all the matrices are dependent on t. Note that Equation (14) is an ordinary differential equation. We call it stochastic Ricatti equation because it arises from the stochastic LQ problem. Furthermore, note that we need $R + \sum_{j=1}^m D_j'PD_j > 0$ for the problem to have an optimal control and not R to be positive definite.

Given equaions 14 and 15 we get that the optimal control is

$$\begin{cases} u^*(t) = -(R(t) + \sum_{j=1}^m D_j(t)'P(t)D_j(t))^{-1}[B(t)'P(t) + \sum_{j=1}^m D_j(t)'P(t)C_j(t)]x^*(t) - \\ -(R(t) + \sum_{j=1}^m D_j(t)'P(t)D_j(t))^{-1}[B(t)'G(t) + \sum_{j=1}^m D_j(t)'P(t)g_j] \end{cases} \tag{16}$$

and the optimal value is

$$\begin{cases} J^*(t) = \frac{1}{2}x(0)'P(0)x(0) + G(0)'x(0) + \\ +\frac{1}{2}\int_0^T (-((R(t) + \sum_{j=1}^d D_j(t)P(t)D_j(t))^{-1/2}(B(t)'G(t) + \sum_{j=1}^d D_j(t)P(t)g_j(t)))^2 + \\ +2G'(t)f(t) + \sum_{j=1}^d g_j(t)P(t)g_j(t))dt. \end{cases} \tag{17}$$

For proofs and further details on LQ stochastic control theory we refer to [7].

3 Market Model

We will now build a model for the market. The model is the same as in [6]. We consider m stocks that form a market index. Assume that the price of each stock $S_i(t), i = 1, 2, ...m$ follows a Geometric Brownian Motion:

$$dS_i(t) = b_i S_i(t)dt + \sum_{j=1}^{m} \sigma_{ij} S_i(t)dW_j(t), \quad S_i(0) = S_{i0}, \tag{18}$$

where $W(t) = (W_1(t), ..., W_m(t))'$ is an m-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.
Furthermore, we assume the existence of a risk-less asset with price that is governed by

$$dS_0(t) = rS_0(t)dt. \tag{19}$$

Also, let $\pi_i(t)$, $i = 1, .., n$, be the wealth invested in stock i at time t. So, then $\pi(.) := (\pi_1(t), .., \pi_n(.))'$ is the composition of our portfolio at time t. In our framework, in the next chapters, $\pi(\cdot)$ will be our control. We say that the portfolio $\pi(\cdot)$ is *admissible* if it belongs to $L^2_{\mathcal{F}}(\mathbb{R}^n)$, which is the space of all $\mathcal{F}_t$-adapted measurable processes, valued in $\mathbb{R}^n$ and satisfying $\int_0^{+\infty} ||\pi(t)||^2 < +\infty$.
Denote x(t) the *total wealth* at time t and $N_i(t)$ the number of shares of the i-th stock. Then $x(t) = \sum_{i=0}^{n} \pi_i(t) = \sum_{i=0}^{n} N_i(t)S_i(t)$.
Now, we demand the strategy to be self financing so we can find the wealth process:

$$\begin{aligned}
dx(t) &= \sum_{i=0}^{n} N_i(t)dS_i(t) = N_0(t)dS_0(t) + \sum_{i=1}^{n} N_i(t)dS_i(t) = \\
&= N_0(t)r(t)S_0(t)dt + \sum_{i=1}^{n} N_i(t)S_i(t)(b_i dt + \sum_{j=1}^{m} \sigma_{ij} dW_j(t)) = \\
&= r(t)(x(t) - \sum_{i=1}^{n} \pi_i(t))dt + \sum_{i=1}^{n} b_i(t)\pi_i(t)dt + \sum_{i=1}^{n}\sum_{j=1}^{m} \pi_i(t)\sigma_{ij}(t)dW_j(t) \Rightarrow \\
\Rightarrow dx(t) &= [r(t)x(t) + \sum_{i=1}^{n}(a_i(t) - r(t))\pi_i(t)]dt + \sum_{i=1}^{n}\sum_{j=1}^{m} \sigma_{ij}\pi_i(t)dW_j(t).
\end{aligned} \tag{20}$$

By denoting $b := (b_1 - r, ..., b_n - r)'$, $\sigma := (\sigma_{ij})_{m\times m}$, $\Gamma := \sigma\sigma'$, σ_n the $n \times m$ matrix identical to the matrix consisting of the first n rows of σ , and $\Gamma_n := \sigma_n\sigma_n'$ we get the matrix form

$$\Rightarrow \boxed{dx(t) = [r(t)x(t) + b'\pi(t)]dt + \pi'\sigma_n dW(t)}. \tag{21}$$

These are the dynamics of our portfolio. In the following we will be using a portfolio consisting of n (where $n < m$) stocks. We will assume without loss of generality that these are the first n of the m stocks.

4 Tracking a Growth Rate

We will use the setting of the previous chapter. Given a portfolio that consists of n stocks ($n < m$), we want to control the investment of an initially valued x_0 wealth among these stocks and the bond, in such a way, so that the portfolio performance will follow as close as possible a given, deterministic, continuously compounded growth rate $x_0e^{\mu t}$ over a finite time horizon. Again we note that we do not seek to beat the target but to track it. Furthermore, from a mathematical point of view, the number of stocks, m, in the index is irrelevant. This means that going over the growth rate is equally penalized with going below it.
We formulate this problem as

$$\min E\left\{\int_0^T (x(t) - x_0e^{\mu t})^2 dt + \left|x(T) - x_0e^{\mu T}\right|^2\right\}, \tag{22}$$

subject to the wealth equation. Note, that with the term $\left|x(T) - x_0e^{\mu T}\right|^2$, we demand the portfolio value to be equal to the target at the end of the investment period. So our problem is

$$\begin{cases} & \min\left\{E\int_0^T (x(t) - x_0e^{\mu t})^2 dt + \left|x(T) - x_0e^{\mu T}\right|^2\right\} \\ s.t. & dx(t) = (rx(t) + \sum_{i=1}^n (b_i - r)\pi_i(t))dt + \sum_{j=1}^m \sum_{i=1}^n \sigma_{ij}\pi_i(t)dW(t) \\ & x(0) = x_0. \end{cases} \tag{23}$$

In order to solve this, let

$$y(t) := x(t) - x_0e^{\mu t}. \tag{24}$$

Then by Ito's formula we get

$$\begin{aligned} dy(t) &= dx(t) - x_0\mu e^{\mu t} = (-x_0\mu e^{\mu t} + rx + b^T\pi))dt + \pi^T\sigma_n dW = \\ &= (-x_0\mu e^{\mu t} + rx + -rx_0e^{\mu t} + rx_0e^{\mu t} + b^T\pi))dt + \pi^T\sigma_n dW. \end{aligned} \tag{25}$$

Thus,

$$dy(t) = (ry(t) + b^T\pi(t) + x_0e^{\mu t}(r-\mu))dt + (\pi^T\sigma_n)dW(t)$$

and hence, our problem becomes

$$\begin{cases} & \min E\left\{\int_0^T y(t)^2dt + y(T)^2\right\} \\ s.t. & dy(t) = (ry(t) + b^T\pi(t) + x_0e^{\mu t}(r-\mu))dt + (\pi^T\sigma_n)dW(t) \\ & y(0) = 0. \end{cases} \tag{26}$$

This is a stochastic LQ problem and can be solved as such. Using control terminology, y is the state and π is the control in our problem. We can proceed in two ways. We can either use the non-homogeneous Riccati equation or transform the problem into a homogeneous one and use the corresponding Riccati equation. The latter approach includes letting $y_0(t) = x_0e^{(\mu-\rho)t}$ so problem 26 becomes

$$\begin{cases} & \min\left\{E\int_0^T y(t)^2dt + y(T)^2\right\} \\ s.t. & dy_0(t) = (\mu-\rho)y_0(t)dt \\ & dy(t) = (ry(t) + b^T\pi(t) + x_0e^{\mu t}(r-\mu))dt + (\pi^T\sigma_n)dW(t) \\ & \begin{bmatrix} y_0(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \end{cases} \tag{27}$$

which is a homogeneous problem and can be solved as such. Note that by using this approach, P will be matrix valued. We will however follow the non-homogeneous approach. Comparing Equation (26) with Equations (12) and (13) we have:

$$Q = 1, \;\; R = 0, \;\; A = r, \;\; B = b^T$$

$$f = x_0e^{\mu t}(r-\mu), \;\; C = 0, \;\; D_j = \sigma_j', \;\; g_j = 0.$$

We also note that the second part, $y(T)^2$, of our objective function, forces our portfolio to be equal with the target at the end of the investment period but we can also proceed without taking this into account.

We then get the following Riccati equation

$$\begin{cases} \dot{P}(t) + 2rP(t) + 1 - P(t)b'(\sigma\sigma')^{-1}b = 0 \\ P(T) = H \\ P(t) > 0, \end{cases} \tag{28}$$

where H can be either 0 or 1 (depending on whether or not we take $y(T)^2$ into account), along with

$$\begin{cases} \dot{G}(t) + \left(r - b'(\sigma\sigma')^{-1}\left(bP(t)\right)\right)' G(t) + P(t)x_0 e^{\mu t}(r-\mu) = 0 \\ G(T) = 0. \end{cases} \tag{29}$$

Equation (28) is an ODE and we can easily solve it:

$$\frac{\dot{P}(t)}{(F-2r)P(t)-1} = 1 \Rightarrow \frac{1}{F-2r} d\left(\ln\left((F-2r)P(t)-1\right)\right) = dt \Rightarrow$$
$$\Rightarrow \ln((F-2r)P(T)-1) - \ln((F-2r)P(t)-1) = (T-t)(F-2r) \Rightarrow$$
$$\Rightarrow \ln((F-2r)P(t)-1) = \ln((F-2r)H-1) - (T-t)(F-2r) \Rightarrow$$

$$\Rightarrow P(t) = \frac{e^{(2r-F)(T-t)}-1}{2r-F} + He^{(2r-F)(T-t)}, \tag{30}$$

where $F = F(t) = b(t)'(\sigma(t)\sigma(t)')^{-1}b(t)$. By plugging this into Equation (29), we can find $G(t)$ by using some numerical software (we used *Mathematica*) as

$$\begin{cases} G(t) = \dfrac{e^{t\mu+(2r-F)(T-t)}((F-2r)FH - (F-\mu-r) + H(F(\mu-r)+2r(\mu+r)))}{(F-2r)(F-\mu-r)}x_0 + \\ + \dfrac{e^{(F-r)t+(\mu+r-F)T}(-(F-2r)(FH+1) - H(F(\mu-r)+2r(\mu+r))) + e^{t\mu}(r-\mu)}{(F-2r)(F-\mu-r)}x_0. \end{cases} \tag{31}$$

Having calculated $P(t)$ and $G(t)$, we can now find the optimal control as

$$\pi^*(t) = -(\sigma(t)\sigma(t)')^{-1}B(t)y^*(t) - (\sigma(t)\sigma(t)')^{-1}B(t)\frac{G(t)}{P(t)} \Rightarrow$$

$$\Rightarrow \boxed{\pi^*(t) = -(\sigma(t)\sigma(t)')^{-1}b(t)'\left(y^*(t) + \frac{G(t)}{P(t)}\right)}. \tag{32}$$

This control is actually the amount of wealth we should invest in each of the stocks of our portfolio in order to track the growth rate as close as possible. We will use this to simulate the problem numerically.

Note that in the numerical simulation we will choose $H = 0$ to simplify the expressions, i.e. $P(T) = 0$. Thus

$$P(t) = \frac{e^{(2r-F)(T-t)} - 1}{2r - F} \tag{33}$$

and

$$G(t) = \left(\frac{e^{t\mu+(2r-F)(T-t)}}{2r - F} + \frac{e^{T\mu+(r-F)(T-t)}}{F - \mu - r} + \frac{e^{t\mu}(r - \mu)}{(F - 2r)(F - \mu - r)} \right) x_0. \tag{34}$$

5 Numerical Simulation of Tracking a Growth Rate

After having addressed the theoretical side of the problem, we will use MATLAB to simulate the results numerically. The program we coded takes as input files that contain the data of stock prices stored in different columns representing date, opening price, closing price etc. It also takes as a parameter the number of times, K, we wish to rebalance our portfolio (if $K = 0$ we only form our portfolio at time 0). Then it finds the mean and volatility of the stock we gave as input (we used the opening price for each stock). Having these, it uses Equation (32) to find the amount we should invest in each of the stocks we have in our portfolio (we could solve the problem numerically but since we have a closed form of the solution this is not necessary). Note that we allow short selling in our problem and that in any case we ignore transaction costs. This is done as many times as we want to rebalance and the value of our portfolio is then plotted over time. The growth rate we want to track is also plotted for comparison, as is the value our portfolio would have if we only invested in the risk-less asset. The exact code that creates the optimal portfolio, when investing in one risky asset and a risk-less one, can be found in the appendix.

5.1 Mean and Volatility

In order to find our optimal control we first need to estimate the drift and the volatility of the stocks from the historical data. In order to do this we use log returns. From historical

data we can estimate the drift d as

$$\hat{d} = \frac{1}{N}\sum_{k=0}^{N-1} \ln\left(\frac{S(k+1)}{S(k)}\right) = \frac{1}{N}\ln\left(\frac{S(N)}{S(0)}\right)$$

and the variance σ^2 by

$$\hat{\sigma}^2 = \frac{1}{N-1}\sum_{k=0}^{N-1}\left[\ln\left(\frac{S(k+1)}{S(k)}\right) - \hat{d}^2\right]^2.$$

Note, that if the data is not from one year, but from a period with period length p part of the year, then we need to transform the drift d_p and the volatility σ_p into annualized values and this is done by

$$d_p = pd, \qquad \sigma_p = \sqrt{p}\sigma.$$

Now, because we are modeling the stocks to follow a Geometric Brownian Motion we have

$$E\left[\ln\left(\frac{S(t_{i+1})}{S(t_i)}\right)\right] = (b - \frac{1}{2}\sigma^2)(t_{i+1} - t_i),$$

so after we estimate d and σ, our real drift estimate will be

$$\hat{b} = \hat{d} + \frac{1}{2}\hat{\sigma}^2.$$

For more details see [3].

Estimating the drift and volatility is of utmost importance in our model and there exist plenty of literature on how someone best can do this. In our code we will use a moving average which means that we recalculate the mean and variance every time we rebalance our portfolio and we calculate their values by using a specific time window that moves along with time.

5.2 Portfolio Simulation by using Vectors

After having estimated the drift and the volatility, we can easily find the optimal control using Equation (32). This will happen every time we form our portfolio, i.e. K+1 times. Our stock data is stored in a vector, and because we rebalance the portfolio, our time horizon is divided into K+1 intervals of length $L = \frac{T}{K+1}$. This fact is what we use in the code to calculate our portfolio value. Every time we rebalance, we take the time from $k \times L$ to $(k+1) \times L$, where $k = 0, ..., K$ and calculate the value of the portfolio value at these times (the last interval could be of smaller length if $L \mod (K+1) \neq 0$) by using vectors.

This approach of vectorizing time instead of using *for* loops, significantly decreases computational cost when using MATLAB. Rebalancing our portfolio can be divided into two parts. One part has to do with the stock in our portfolio and the other with the bond. We allocate funds to the stock according to the formula we derived (we may have to borrow) and whatever is left is invested in the risk-less asset. Of course if the funds needed for the stock are more than the value of our portfolio we need to borrow so we insert a negative amount in the risk-less asset.

5.3 Error Measurement

In order to measure how good our tracking is, we need to introduce a measure. We let $Portfolio(i)$ and $GR(i)$ be the value, at day i, of our Portfolio and of the Growth rate respectively, and assume we want to track a target for T days. We are going to measure the performance of our portfolio by

$$E = \sum_{i=1}^{T} (Portfolio(i) - GR(i))^2. \tag{35}$$

This is obviously a discrete version of the function $\int_0^T [x(t) - x_0 e^{\mu t}]^2 dt$ which expected value we wanted to minimize initially (problem (23)). Then, we can evaluate the performance of portfolios by comparing the errors between them, over the same time frame. In addition, we can also observe how the error evolves over time, as there will be time intervals that contribute more to this E than others. By plotting this, we can see where our tracking goes wrong and spot the reasons for that.

5.4 Results

The data of the stock prices were drawn from [8]. We assume that have £1000 at the beginning of the investment period and we will use one stock and a bank account of 4% interest rate per annum, in order to try to track the benchmark, which is a deterministic growth rate of 10% per annum, that is 6% higher than the bank account. The reason we use one stock only, is that it is easier to see the performance of the portfolio compared to the stock and analyse its movement, and this also gives the code a better readability. The extension from one to multiple stocks can be easily implemented and we will discuss this later. Our investment period will be 360 days, which we suppose is a year, and we rebalance

our portfolio 10 times over one year, that is every 36 days. We choose a moving average of 40 days for the estimation of the mean and volatility.

We then run our code, in order to track the target by using some stocks of FTSE 100. First we try to track the growth rate by using Compass Group stocks. Plotting the portfolio value over time we get Figure 1. We can see the portfolio value, the growth rate we wish to track and the value our portfolio would had if we only invested in the risk-less asset. Over the year, we have an error of $E = 1.345 \times 10^4$ and when observing the plot we notice the overall good results we get. Note however, that there are times that our tracking is not that good. This is the case after 4 months for example where our portfolio value falls even under the value of the risk-less asset. After that it recovers and our tracking continues as expected.

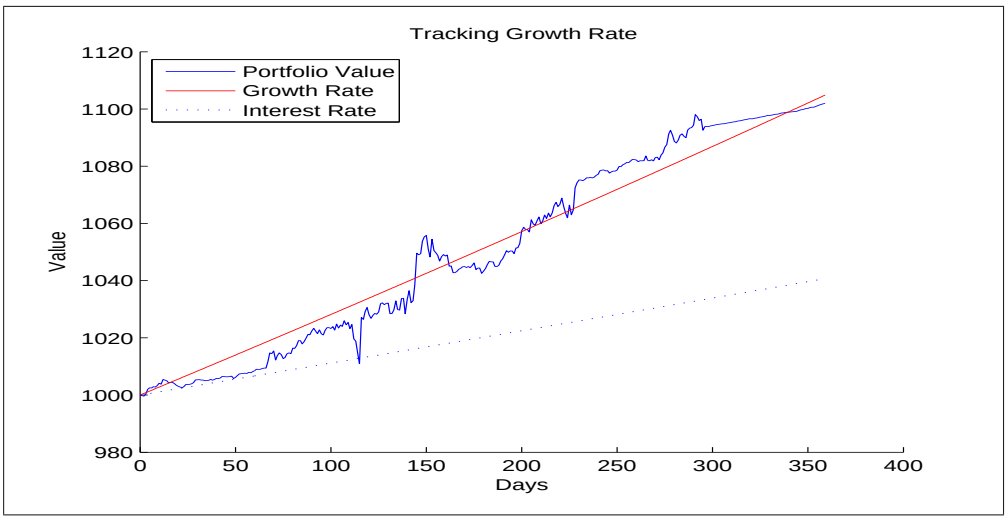


Fig. 1: 1-year tracking using Compass Group stocks

Next, in Figure 2, we also plot the movement of the stock (the stock plot is normalized to start at £1000, i.e. we plot the value of stock worth £1000 at time 0) for comparison and that is how we are going to plot our results from now on. We notice, that after 4 months a sudden jump down of the stock price is what made our portfolio fall as well. We see similar reactions of our portfolio with other jumps of the stock over the investment period.

In Figure 3, we plot the results we get when we are using Cable and Wireless stocks to track our target. Note the jump down the stock realizes after almost 260 days and the reaction of our portfolio. Over the year we get $E = 1.5853 \times 10^4$ which is the same order with the error we got when we are tracking using Compass Group stocks. Both these

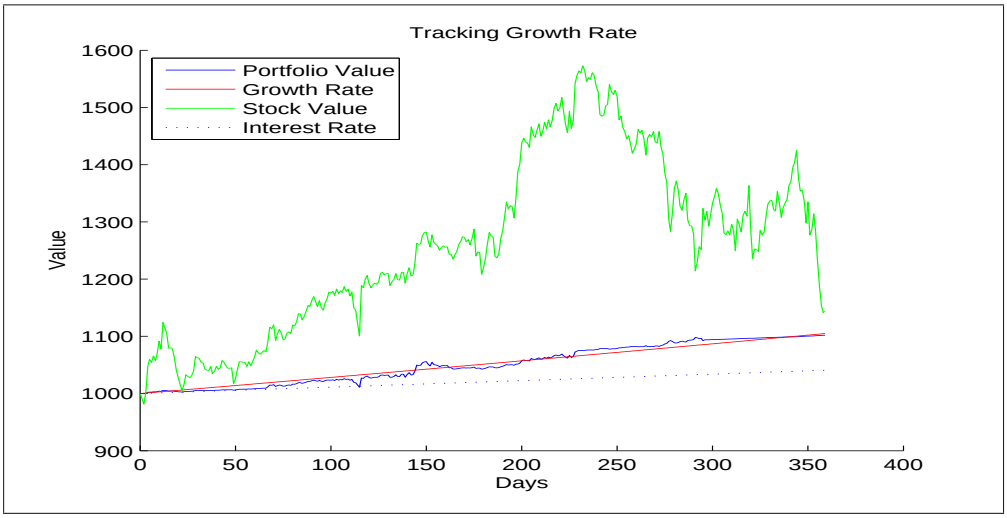


Fig. 2: 1-year tracking using Compass Group stocks

stocks allow us to get good results as they do not deviate a lot from the mean or volatility we predicted. Unfortunately, this is not always the case.

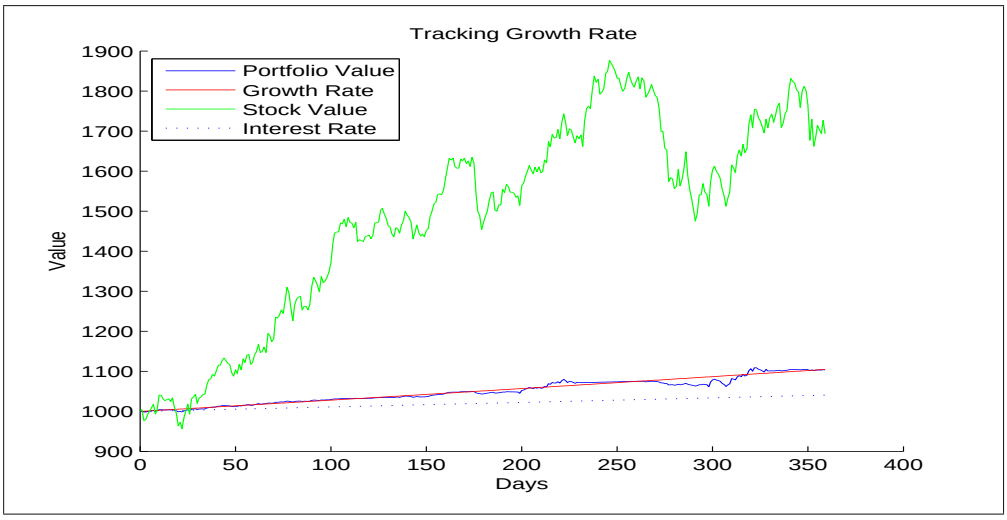


Fig. 3: 1-year tracking using Cable and Wireless stocks

We next turn to use stocks from AstraZeneca. Running our code, we get Figure 4 and a poor error of $E = 2.1985 \times 10^5$ over the year, which is a lot bigger than the ones we got in the previously implementations. We observe that after the second month, the stock started falling rapidly and because of that, the value of our portfolio started falling as well. After that though, the stock continued falling but our moving average 'caught' that movement and therefore our portfolio was able to correctly track our target until the end of the investment period. This example highlights the importance of the moving average

in our implementation. If we used a non-moving average, our simulation would not have taken into account that the stock price had started to fall and we would not get the correct results.

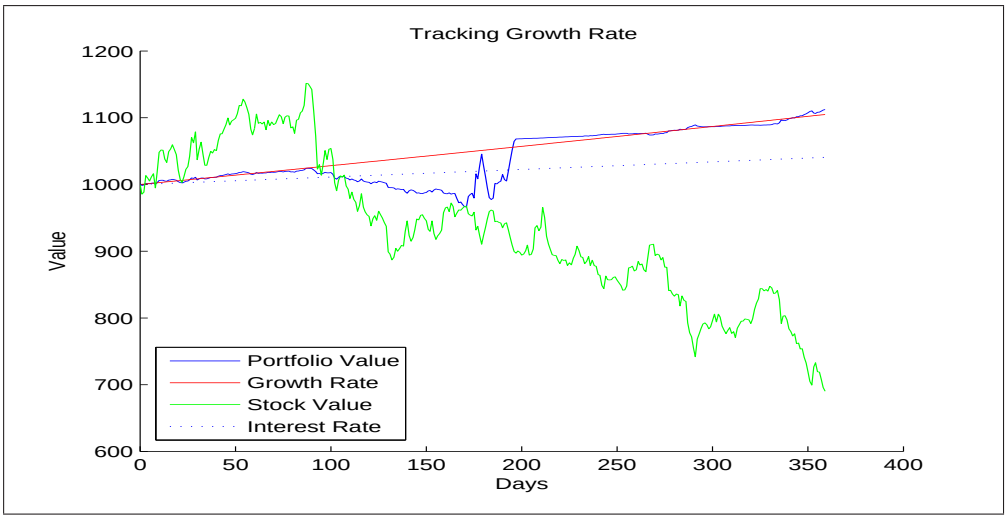


Fig. 4: 1-year tracking using AstraZeneca stocks

We see a similar reaction of our portfolio in Figure 5, when we are using the stocks of BG Group. The stock price jumps down after almost 250 days and because our portfolio does not have the information to react it falls as well, giving an error of $E = 5.4664 \times 10^4$ over the year. In order to better investigate this reaction we plot the error over time in Figure 6. Note that over the first 6 months the error is only $E = 8.5384 \times 10^3$. After the 250 days though, it gets really big in just one month. Then the tracking continues normally and therefore the curve has a smaller slope. Sudden jumps up or down, change the mean or volatility of the stock and make tracking the target hard, as they cannot be predicted.

This effect of the mean and the volatility values changing from the ones we predicted and its consequences on our portfolio can also be observed in Figure 7, where we plot the portfolio value we get when we are using stocks of AVIVA to track the target. Although the error over the year is $E = 3.9878 \times 10^4$, which is smaller than some of the previous portfolios, we notice that the volatile movement of the stock price constantly changes the mean from the one we are estimating as our moving average, making it hard to track our target, not just over a small time interval, but during the whole tracking period.

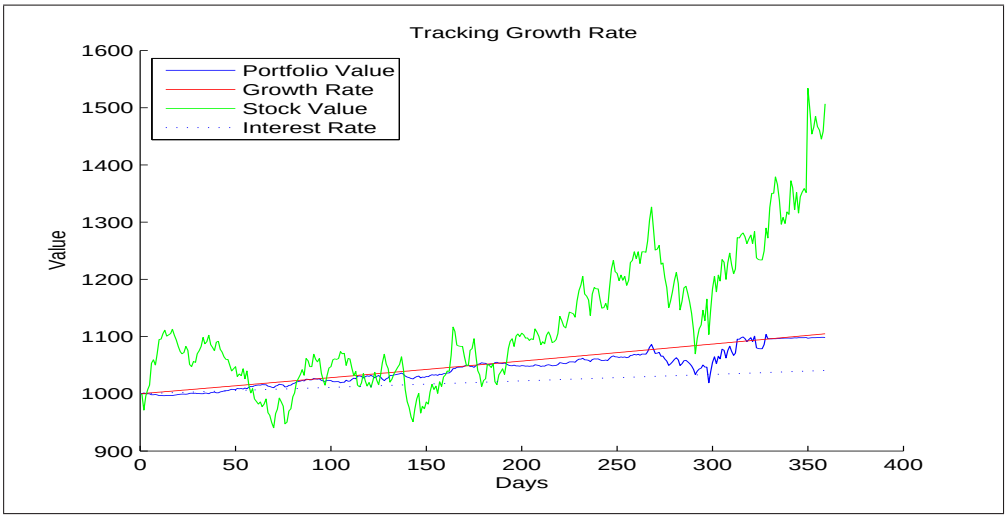


Fig. 5: 1-year tracking using BG Group stocks

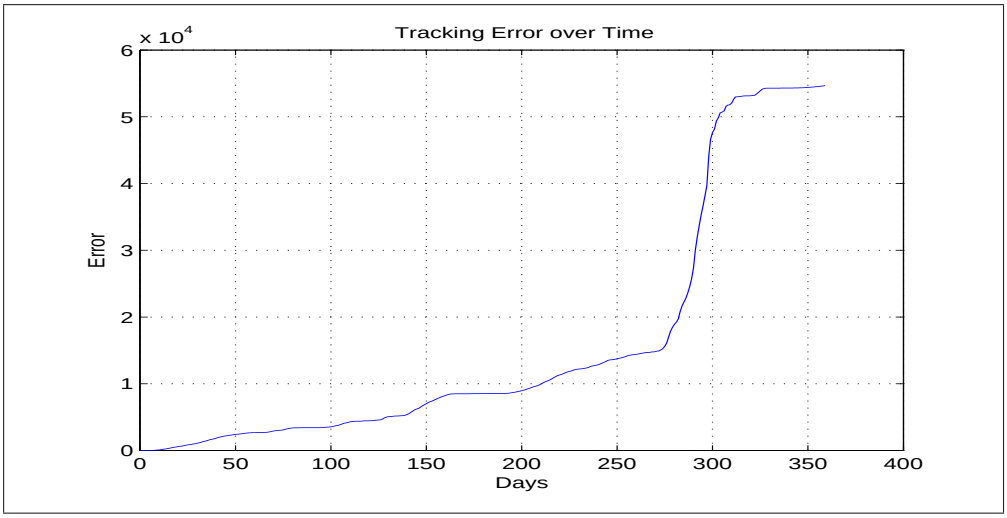


Fig. 6: Error in tracking using BG Group stocks over time

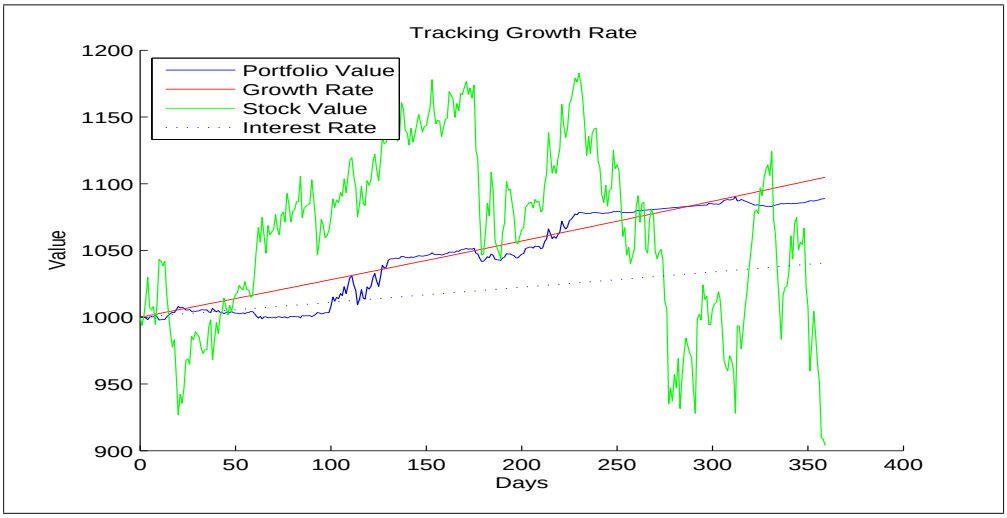


Fig. 7: 1-year tracking using AVIVA stocks

5.5 Borrowing and Short Selling

Another issue that may arise when dealing with volatile stocks, is the amount of money that we may have to borrow and the amount of short selling we may have to do, in order to be able to track the target. In our code, we assume that the money we can actually borrow and the stocks we can short sell are infinite, but this of course is not true in the real world. We modified our code so it can keep track of the amount of long selling and short selling and we tried to track the growth rate by using stocks of All and Leics getting Figure 8. We see that after 60 days the stock is not behaving as predicted and our portfolio is unable to track the target, leading to an error $E = 1.7857 \times 10^6$. In order to recover, at some point we need to short sell 0.5071×10^3. Our portfolio does eventually recover after a while, but only if we are able to short sell this amount.

In order to better observe this behavior, we draw a new plot. In Figure 9, the red line is the value of the portfolio, while the blue stars represent the amount of stock we need to buy or sell every time we rebalance the portfolio, in order to track our target. If some star is above the red line, we need to borrow money. If, on the other hand, some star is on the negative domain, we would need to short sell this amount, which is the case when we use stocks of All and LEICS. At some of the rebalances, like the 6^{th}, or even worse the 10^{th}, we need to short sell a significant amount of stocks and that may not be possible.

Finally, we turn to look at a more extreme situation. If we try to track the target with stocks of BP, we get Figure 10. Our portfolio is moving up and down, because it cannot

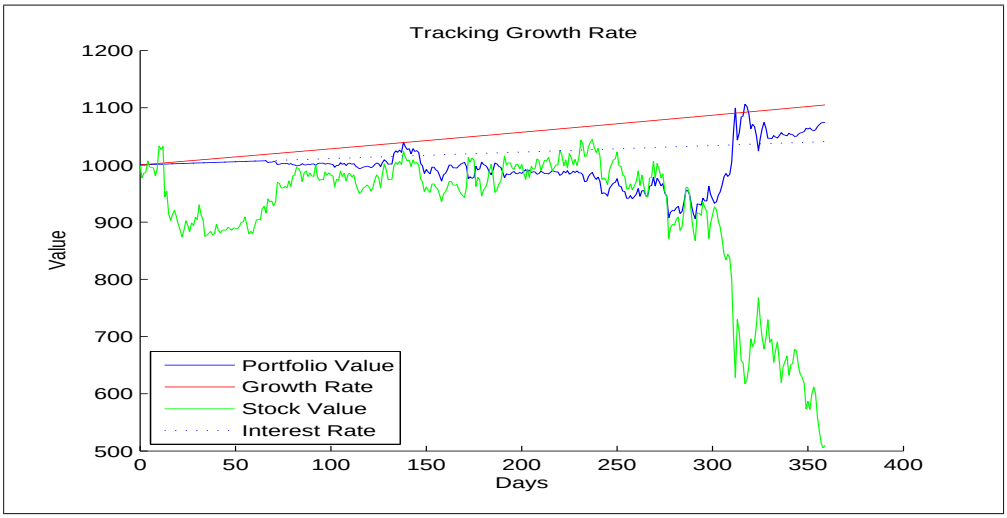


Fig. 8: 1-year tracking using All and LEICS stocks

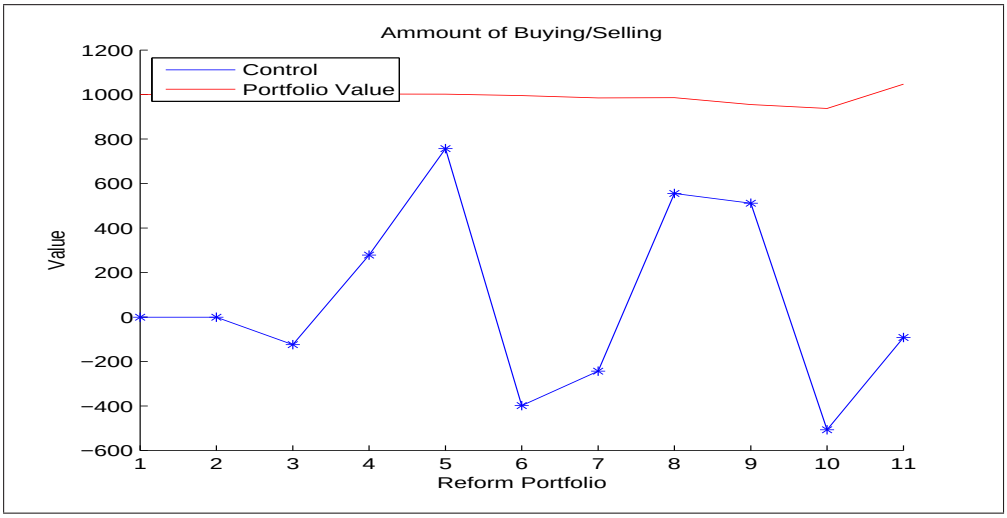


Fig. 9: 1-year tracking using All and LEICS stocks (risk of borrowing or short selling)

follow the target properly. After that, it tracks correctly, but only if we are in position to borrow money (Fig. 11). Note, that the sum we have to borrow at the 10^{th} rebalance of our portfolio is 4.4468×10^3. Compared to our initial £1000 this is a big sum. If we are unable to borrow, we cannot keep on tracking the target and we may be stuck far away from it.

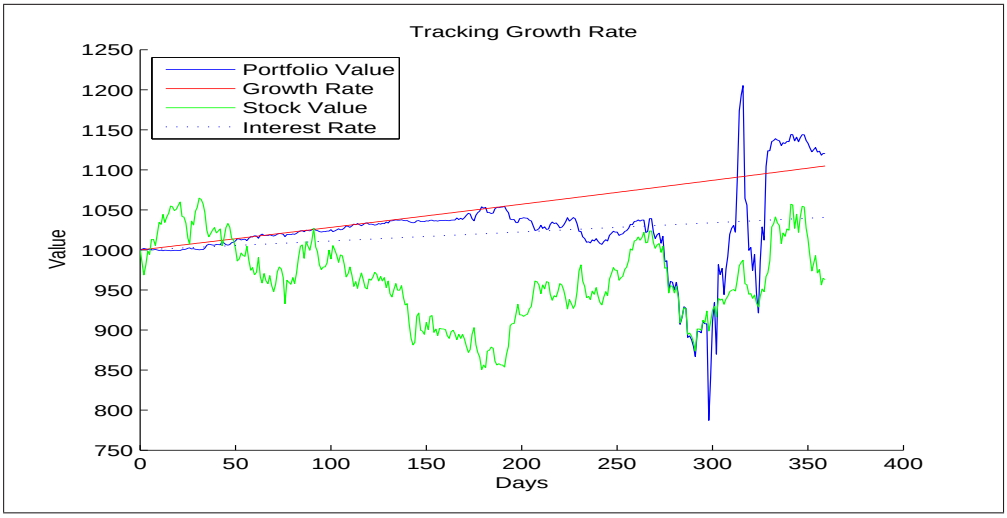


Fig. 10: 1-year tracking using BP stocks

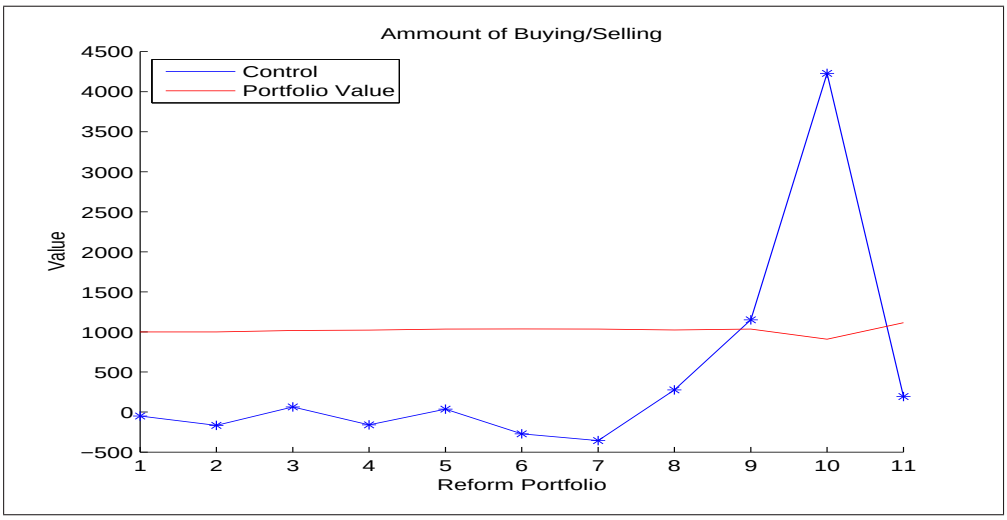


Fig. 11: 1-year tracking using BP stocks (risk of borrowing or short selling)

The ideal case is when we do not need to borrow money or short sell huge amount of stocks. This is the case when using stocks of Compass Group for example (see Fig. 1). As we see in Figure 12 there is no need to borrow money and the amount of short selling is kept within reasonable levels.

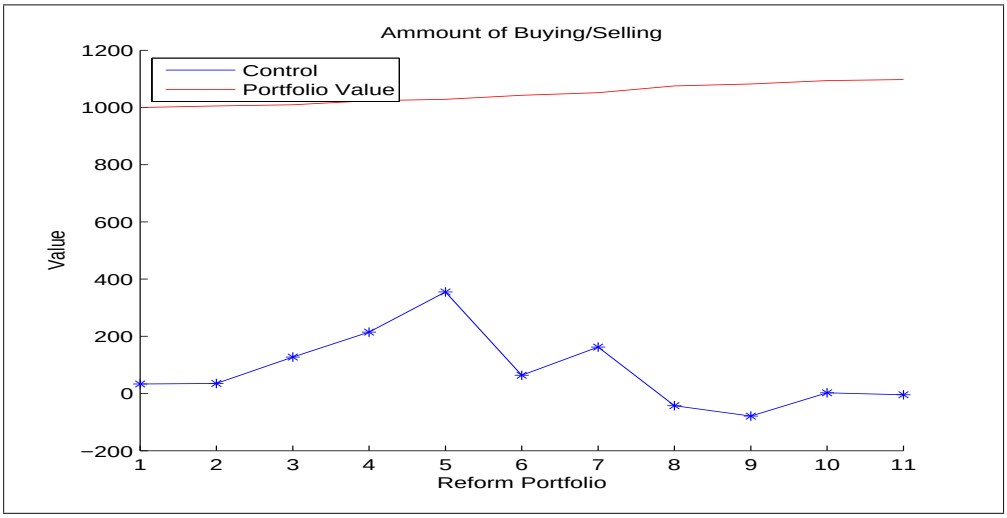


Fig. 12: 1-year tracking using Compass Group stocks (risk of borrowing or short selling)

Let us examine what would happen if we were not able to buy or sell the amount of stocks that the theory demanded in order to track the target. For this purpose, we add the following two lines in our code:

```
if (u<Umin) u=Umin; end
if (u>Umax) u=Umax; end
```

These will not allow us to spend more than Umax to buy stocks, or sell stocks with value more than Umin. Now we can observe the reaction of our portfolio to these constraints. We are focusing on the case where we are using stocks of AstraZeneca again, but this time we are not allowed to buy or sell more than £1000. Running our code, we plot the results in Figure 13. Comparing with Figure 4, we see that our portfolio is not able to recover as fast this time, so although it is able to track again, it gives an error of $E = 2.3128 \times 10^5$ compared to 2.1985×10^5 when we was tracking without constraints. This is an example where our portfolio was able to recover and these constraints did not have a big impact on the tracking, but of course, in other situations, that might not be the case.

5.6 Tracking more aggressively

One of the most important parameters in our model is the growth rate we wish to track. Assume now that we want to track a higher growth rate. Then, we would need to buy more risky assets and our portfolio would consequently be more volatile. This is of utmost

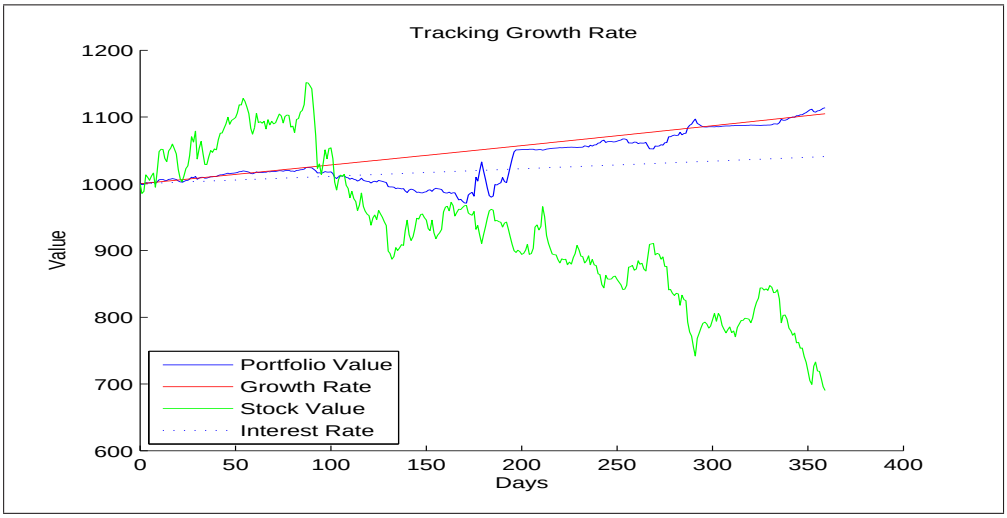


Fig. 13: 1-year tracking under constraints using AstraZeneca stocks

importance, not only because the error of our tracking would increase, but also because of the amount of borrowing and short selling we would need to do in practice, as we previously discussed. So in real life situations tracking a lower target means less risk and we always need to take this into consideration. If we want to track a growth rate equal to the risk-free asset we will not be exposed to any risk, as we will put all our money in the bank, and therefore the portfolio will not be volatile at all. The higher the growth rate we want to track the more volatile our portfolio will be and we will be exposed to more risk of having to borrow or short sell. We will now illustrate this, by focusing again on the results we got when we used stocks of Cable and Wireless. In Figure 3, we were tracking a growth rate of 10% per annum. If we plot the value of our portfolio when tracking a higher growth rate, we will observe this more volatile movement of our portfolio. In Figure 14, we plot the same portfolio, together with the one we get from tracking a growth rate of 30% per annum. Indeed, we observe how much more volatile our portfolio is in the second case. The error this time is $E = 3.766410^5$, compared to $E = 1.5853 \times 10^4$ we got initially. This can also be translated into money we need to buy, or stocks we need to short sell. In Figure 15, we can see the change in the amount of stock we need to buy. When tracking a 30% growth rate, we actually need to have leverage to achieve that. This means that we might be forced to borrow a lot of money in order to continue tracking a high target and this should be taken into account in practice, as it might not always be possible or acceptable.

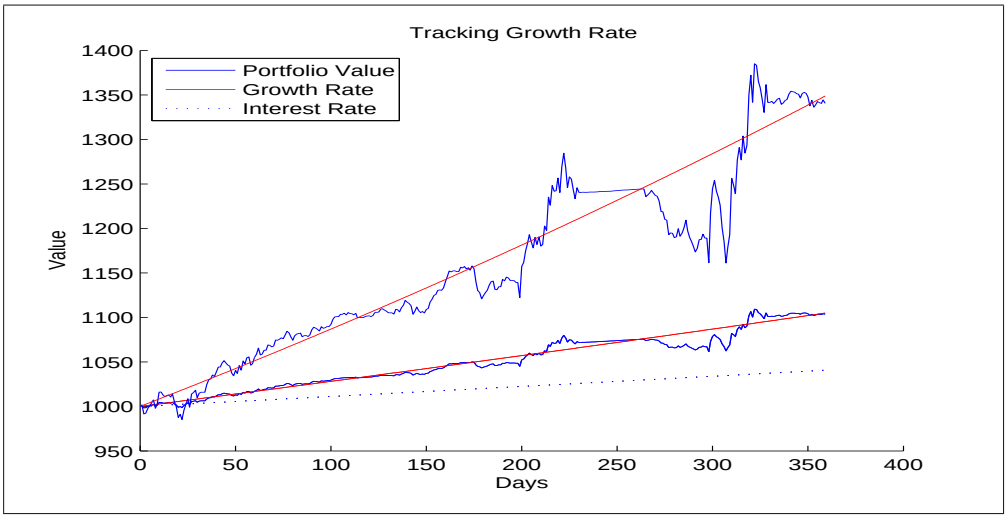


Fig. 14: 1-year tracking (10% and 30% growth rates) using Cable and Wireless stocks

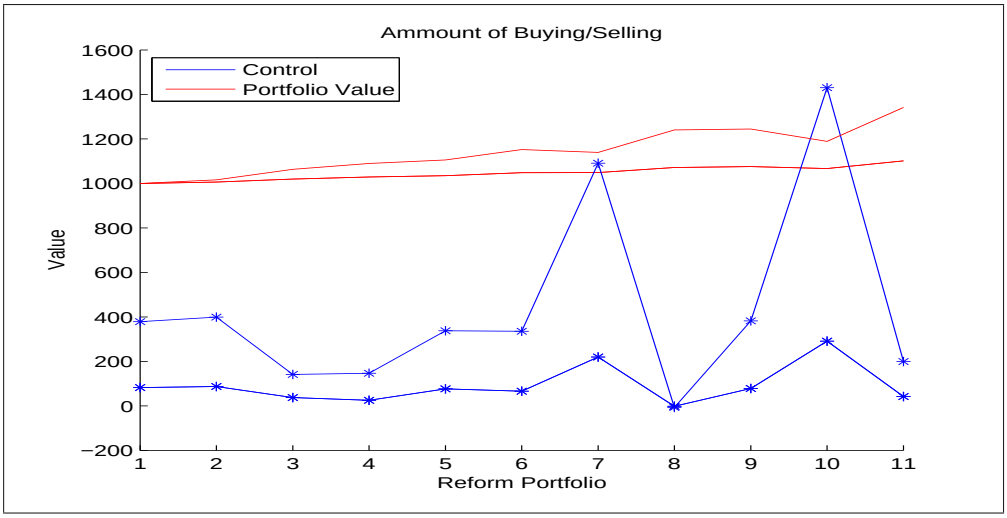


Fig. 15: 1-year tracking(10% and 30% growth rate) using Cable and Wireless stocks (risk of borrowing or short selling)

5.7 Multiple Stocks

For the sake of completeness, we will discuss the case where our portfolio will consist of multiple stocks. In the dissertation, until now, we only presented data from tracking a growth rate, by using only one stock and a risk-less asset. This made the results more easy to analyze and the code in MATLAB easier to read. We can, of course, modify the code, in order to form portfolios consisting of more than one stocks and analyze the results. This would be of some importance, as it would be the case in a real life scenario, because diversification would make our portfolio less risky. For that, we would need to change the code in order to work with matrices as in the theory. That is trivial, but note that now we would not calculate a variance but a covariance matrix, because as we have the stocks prices modeled as

$$dS_i(t) = b_i S_i(t)dt + \sum_{j=1}^{m} \sigma_{ij} S_i(t) dW_j(t), \quad S_i(0) = S_{i0},$$

by Ito's formula, for the time interval (t_l, t_{l+1}) we get that

$$E\left[ln\left(\frac{S_i(t_{l+1})}{S_i(t_l)}\right)\right] = \left(b_i - 0.5\sum_{j=1}^{m}\sigma_{ij}^2\right)(t_{l+1} - t_l), \quad S_i(0) = S_{i0},$$

so we need to modify the code accordingly. What we can do, is first calculate the covariance matrix, Σ, of the returns and then use the *cholesky factorisation* in order to calculate the volatilities, i.e. find a matrix L such that $L'L = \Sigma$. After that we get the optimal control in matrix form and we can perform the same analysis we did for one stock. We will not be focusing on this case any deeper, but in Figure 16 we see the performance of a portfolio when we are using risky assets both from AstraZeneca and BG Group to track the target. Note, that the error we get is $E = 3.9726 \times 10^4$. Using even more stocks, will help diversify our portfolio even more, although there are practical limits as to how many stocks should form a single portfolio.

6 Tracking a market Index

To be thorough, we make a discussion on the case that we would want to track a market index. There we would have to find the new optimal control. In practice, this case may

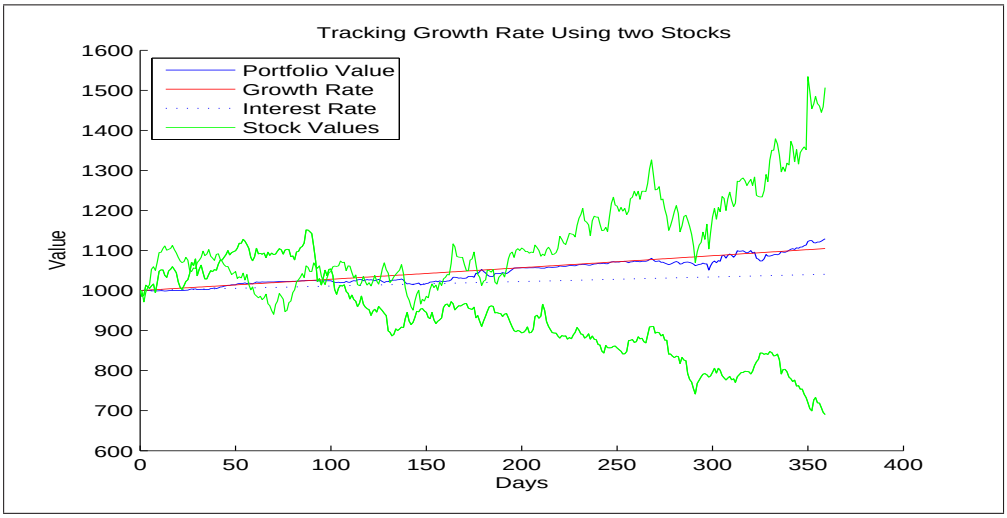


Fig. 16: 1-year tracking using AstraZeneca and BG Group stocks

more important for tracker funds, as usually one would like to track an index and not a deterministic growth. The market index can be represented as:

$$I(t) = \sum_{j=1}^{m} a_j S_j(t), \quad I(0) = I_0, \tag{36}$$

where a_j are the weight shares of each stock in the index. We want to track index $I(t)$ using a portfolio of n stocks. Following [6], our problem is formulated as

$$\min E \int_0^T [x(t) - I(t)]^2 dt, \tag{37}$$

subject to the wealth equation and the dynamics of I and S. So our problem is

$$\begin{cases} \min E \int_0^T [x(t) - I(t)]^2 dt \\ dx(t) = (rx(t) + \sum_{i=1}^{n} (b_i - r)\pi_i(t))dt + \sum_{j=1}^{m} \sum_{i=1}^{n} \sigma_{ij}\pi_i(t)dW(t) \\ dI(t) = \sum_{i=1}^{m} a_i b_i S_i(t)dt + \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \sigma_{ij} S_i(t)dW_j(t) \\ dS_i(t) = b_i S_i(t)dt + \sum_{j=1}^{m} \sigma_{ij} S_i(t)dW_j(t), i = 1, ..., m \\ (x(0), I(0), S_i(0)) = (x_0, I_0, S_{i0}). \end{cases} \tag{38}$$

We then set

$$y(t) := x(t) - I(t) \tag{39}$$

and denote

$$L(t) := [a_1 S_1(t), ..., a_m S_m(t)]'. \tag{40}$$

Then by using Ito's formula, we get

$$dy(t) = dx(t) - dI(t) = (rx + b'\pi(t))dt + \pi'\sigma_n dW(t) - \sum_{j=1}^{m} a_j dS_j(t) =$$

$$= (rx(t) + b'\pi(t))dt + \pi'\sigma_n dW(t) - \sum_{j=1}^{m} a_j(b_j S_j dt + \sum_{i=1}^{m} \sigma_{ij} S_j dW_j(t)) =$$

$$= (rx(t) + b'\pi(t) - \sum_{j=1}^{m} a_j b_j S_j(t))dt + (\pi'\sigma_n + L(t)^t\sigma)dW(t) \Rightarrow$$

$$\Rightarrow dy(t) = (ry(t) + b'\pi(t) + rI(t) - \sum_{j=1}^{m} a_j b_j S_j(t))dt + (\pi'\sigma_n + L(t)^t\sigma)dW(t).$$

Hence, we have the problem

$$\begin{cases} \min E\int_0^T y(t)^2 dt \\ dy(t) = (ry(t) + b'\pi(t) + rI(t) - \sum_{j=1}^m a_j b_j S_j(t))dt + (\pi'\sigma_n + L(t)^t\sigma)dW(t) \\ dI(t) = \sum_{i=1}^m a_i b_i S_i(t)dt + \sum_{i=1}^m \sum_{j=1}^m a_i \sigma_{ij} S_i(t)dW_j(t) \\ dS_i(t) = b_i S_i(t)dt + \sum_{j=1}^m \sigma_{ij} S_i(t)dW_j(t), i = 1, ..., m \\ (y(0), I(0), S_i(0)) = (0, I_0, S_{i0}). \end{cases} \tag{41}$$

This problem can be analyzed as well as the growth rate but this is out of the scope of this dissertation.

7 Discussion

Rebalancing a portfolio in order to make it have a specific behavior is one of the most crucial problems of finance. There are a lot of different kind of reactions we might want our portfolio to have. In practice, one may want to compare his portfolio performance to some financial benchmark. It is therefore often desirable to track a certain target. This target could be deterministic or stochastic and by rebalancing our portfolio a finite number of times during the investment period, we want to move as close as possible to it.

In this dissertation, we investigated the problem tracking of such a moving target, specifically a deterministic growth rate over a given investment period. We formulated the problem and found the optimal portfolio by using linear quadratic control theory. We were able to do this after solving the corresponding Riccati equations. Then, we implemented our findings numerically in MATLAB, used our code for portfolios consisting of a

risk-less asset and one stock from the FTSE 100 and presented the results. In addition, we focused on specific situations when our tracking did not give the results we anticipated and situations when our portfolio might not be feasible in practice. Furthermore, we discussed how these states can occur when we are trying to track higher growth rates. Tracking a higher growth rate leads to a more volatile portfolio and this means that we may have to borrow a lot of money or short sell a significant amount of stocks in order to continue to track our target and this is not always possible. Although it made our analysis more clear and gave the code a certain readability, in a real life scenario we would not invest in only one stock in order to track a target and therefore we also showed how these findings can be extended to multi stock portfolios and we also gave some results. For the sake of completeness, we also presented the problem of tracking a market index but without deriving the Riccati equations or implementing the problem numerically.

Although we found that the method we used is indeed giving correct results, there are some very interesting extensions that can be investigated. First, it would be interesting to compare our findings against portfolios that are tracking other benchmarks, like various market indexes. This could highlight the differences that would occur in our tracking dynamics because of the differences between those benchmarks. Furthermore, it would be useful to quantify and further investigate the risks involved when tracking a target. The situations when our tracking goes wrong could have a huge impact in our financial planing, so such an analysis would be of great importance. In addition, it would be interesting to use different methods of estimating the drift and volatilities in order to improved the results we get in some certain situations, like some of those we encountered in this dissertation. Estimating these parameters is of crucial importance to our model, as they represent our forecasts about the future. The better we can estimate them, the better results our model will give. There exist a lot of different methods in the literature in order to do this and it would be interesting to investigate how our tracking performance change with each of those methods. It would also be interesting to impose more constraints on our problem and then further analyze the reaction of our portfolio. This could also have practical importance, because as we mentioned, in practice we cannot borrow or short sell any amount. Finally, there is lot more analysis to be done concerning the parameters that infiltrate the problem, like the times we rebalance our portfolio, the time window of the moving average and the number of stocks constituting our portfolio, as this could give useful results and insight

concerning the choices one should make when tracking a target.

8 Appendix

The code we used to get the numerical results in MATLAB is the following:

```
%The programme takes a growth rate, and stock values as input
%and outputs the portfolio that tracks the rate..
clear all
close all
%VARIABLES_____________________________________________

T=360;              %Time steps

tt=0:1:T-1;
dt=1/T;

K=10;                   %Times the portfolio will be reformed
x0=1000;                %Starting cash
r=0.04;                 %Interest rate
mu=0.1;                 %Growth rate
GR = x0*exp(mu*tt*dt); %Growth rate we wish to track
BA=x0*exp(r*tt*dt);    %Bank account

Past = 720;          %how far in the past will the data go.
Av=40;               %moving average
FileInput = 's3.TXT';
l1= 2; %column of stock prices..

 UMemory=zeros(2,K+1); %Keep track of the positions
%FILE INPUT-----------------------------------------


  Temp1 = MIFileManipulation(FileInput,Past,T); %Data input

    %The next two lines give an indication of how we will aproach more than
    %one stocks.
     t1=strcat('Temp',num2str(1));
     t1=eval(t1);

     b=MeanS(t1(:,l1),Av);
     sigma=VolatilityS(t1(:,l1),Av); %Calculate Variance
     b=b+0.5*(sigma)^2;      %Calculate mean

B=b-r;
F=B*inv(sigma*sigma)*B;
L=-inv(sigma*sigma)*B;

%_____________________SIMULATION_____________________
%______________________________________________________
Temps1 = MIFileManipulation(FileInput,T,T);  %Data input

Portfolio = zeros(1,T); %Portfolio initialization
money=x0;               %Value of the portfolio at the begining
```

```
step = ceil(T/(K+1)); %When the reforming of the portfolio will happen.
SP=zeros(1,T);        %StockPrice



for j=1:K+1  %Loop over the reformations

%FIND OPTIMAL PORTFOLIO-----------------------

     ti=(j-1)*step;


     if ti==0  Y0=0;  end
     if ti~=0  Y0=Portfolio(ti)-x0*exp(dt*mu*(ti));  end


     PR=(exp((2*r-F)*(T-ti)*dt)-1)/(2*r-F);
     GR=x0*exp(mu*ti*dt+(2*r-F)*(T-ti)*dt)/(2*r-F)+...
     ...x0*exp(mu*T*dt+(r-F)*(T-ti)*dt)/(F-mu-r)+...
     ...x0*exp(mu*ti*dt)*(-mu+r)/((F-2*r)*(F-mu-r));
     PG=GR/PR;
     u=L*(Y0 +PG);  %Optimal Portfolio

     %Add the following two lines if there are constraints
     %if (u<Umin) u=Umin; end
     %if (u>Umax) u=Umax; end

    UMemory(1,j)=u;
    UMemory(2,j)=Y0+x0*exp(dt*mu*(ti));

 %DEAL WITH STOCK-----------------------------
     ts=Temps1; %get data
     Nn=u/ts(ti+1,l1); %Find the number of the ith stocks we will keep

     %take the interval we are working with
     ts=[zeros((j-1)*step,1) ; ts(((j-1)*step+1):min(j*step,T) ,l1);...
     ... zeros(T-j*step,1)];

     SP=SP+ts'; %Store StockPrice Values

     ts=ts'.*Nn;
     Portfolio=Portfolio+ts; %Find portfolio Value

%DEAL WITH BANK ACCOUNT-----------------------
    BankMoney = money-sum(u); %Money for the bank

    for i=ti+1:min(j*step,T)
      Portfolio(1,i)=Portfolio(1,i)+ (BankMoney)*exp(dt*r*(i-1-ti));
    end

    money = Portfolio(min(j*step,T));   %Value of the portfolio


 %MOVING AVERAGE------------------------------
```

```
     t1 = MIFileManipulation(FileInput,Past,T-(j-1)*step);

     b=MeanS(t1(:,l1),Av);
     sigma=VolatilityS(t1(:,l1),Av);
     b=b+0.5*(sigma)^2;

     B=b-r;
     F=B*inv(sigma*sigma)*B;
     L=-inv(sigma*sigma)*B;

end

%RESULTS_____________________________________

UMemory  %Return positions

stop=T;
sum(GR(1:stop)-Portfolio(1:stop));
sum((GR(1:stop)-Portfolio(1:stop)).^2)
Error1 = cumsum((GR(1:stop)-Portfolio(1:stop)).^2);
Error2=(GR(1:stop)-Portfolio(1:stop)).^2;

%Plots---------------------------------
figure(1)
hold on
plot(tt,Portfolio)
plot(tt,GR,'r')
Pl2=(x0/SP(1))*SP;
plot(tt,Pl2,'g')
plot(tt,BA,':')
title('Tracking Growth Rate');
xlabel('Days');
ylabel('Value');
legend('Portfolio','Growth Rate','Interest Rate',2);
hold off

figure(2)
plot(tt,Error1)
title('Tracking Error over Time');
xlabel('Days');
ylabel('Error');
grid on


figure(3)
plot(tt,Error2)
title('Tracking Error over Time');
xlabel('Days');
ylabel('Error');
grid on

figure(4)
hold on
plot(UMemory(1,:))
```

```
plot(UMemory(2,:),'r')
title('Ammount of Buying/Selling');
xlabel('Reform Portfolio');
ylabel('Value');
legend('Control','Portfolio Value',2);
plot(UMemory(1,:),'*')
hold off


-------------------------------------------------------
```

together with the functions:

```
function fil = FileIn(y,P,PStart)

fid = fopen(y,'rt');
if (fid < 0)
   error('could not open file "x.txt"');
end;
tab = fscanf(fid,'%f',[6,inf]);

fclose(fid);
temp = tab';

if (P>length(temp(:,1)))||(PStart>length(temp(:,1)))
   error('The array/matrix is not that big"');
end;
if (P>PStart)
 %return array after P until PStart
    fil = temp((length(temp(:,1))-P):length(temp(:,1))-PStart,:);
end
if (P<=PStart)
%returns array after P until end
    fil = temp(length(temp(:,1))-P:length(temp(:,1)),:);
end


function m=MeanS(x,Av)

%input as column vectors

x=x(length(x)-Av:end,1);
mperiod = (log(x(end))-log(x(1)) )/Av;
m=mperiod*360;


function vol=VolatilityS(x,Av)
%input as column vectors

 x=x(length(x)-Av:end,1);
for i=1:length(x)-1
    x(i)=x(i+1)/x(i);
end
```

```
x=x(1:end-1,1);
 x=log(x);

 volperiod=var(x);
 vol=sqrt(volperiod)*sqrt(360);
```

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