



Testing Bell inequalities in Higgs boson decays

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ABSTRACT

Higgs boson decays produce pairs of W bosons in a maximally entangled state, the spins of which can be expected to violate Bell inequalities. We show that the spin density matrix of the W^\pm pair may be reconstructed experimentally from the directions of the charged lepton decay products, and from it the expectation values of various Bell operators determined. Numerical simulations of $H \rightarrow WW^*$ decays indicate that violation of a generalised CHSH inequality is unlikely to be measurable, however the CGLMP inequality is near-maximally violated. Experimental Bell tests could be performed at a variety of colliders and in different production channels. If reconstruction effects and backgrounds can be controlled then statistically significant violations could be observable even with datasets comparable to those already collected at the LHC.

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1. Introduction

The predictions of quantum mechanics for entangled particles have long been known to violate inequalities of the type first introduced by Bell [1]. Violations of such inequalities are expected in quantum theory, however they are incompatible with ‘realist’ theories (including classical physics) in which the properties of systems are independent of our observation of them.

Experimental tests showing violation of Bell inequalities have been performed for pairs of two-outcome measurements using photons [2,3], ions [4], superconducting systems [5] and nitrogen vacancy centres [6], and in pairs of three-outcome measurements using photons [7].

Significant advances in testing local realism were made through ‘loophole-free’ Bell tests performed by three different groups in 2015 [8–10]. Proposals have also been made to test Bell inequalities in e^+e^- collisions [11] and positronium decays [12]. Recently it has been proposed to make such tests in entangled $t + \bar{t}$ decays [13,14] and in systems of $B^0 + \bar{B}^0$ mesons [15] at the LHC.¹

The decay $H \rightarrow WW^*$ is that of a scalar to a pair of distinguishable massive spin-one bosons in a maximally entangled state. In the narrow width and non-relativistic approximations the state may be represented in the spin basis as

$$|\psi_s\rangle = \frac{1}{\sqrt{3}}(|+\rangle|-\rangle + |0\rangle|0\rangle + |-\rangle|+\rangle). \quad (1)$$

The subsequent W boson decays maximally violate chirality; a W^+ boson decay preferentially emits a charged lepton along the W^+ spin direction while a W^- boson decay preferentially emits a charged lepton moving against its spin direction. Decaying W bosons are ‘their own polarimeters’ [11], with each decay causing a spin measurement to be made along the axis of the emitted lepton. This results in correlations in the azimuthal directions of the emitted leptons $\ell^\pm \in \{e^\pm, \mu^\pm\}$ which were exploited by the ATLAS and CMS collaborations to separate Higgs decays from W^+W^- backgrounds in their Higgs boson searches [17,18].

Moreover, measurements of the emitted lepton directions over an ensemble of decays allow one to determine the two-particle spin density matrix ρ , and from it the expectation values $\text{tr}(\rho\mathcal{B})$ of various quantum Bell operators \mathcal{B} [19]. One can therefore go further and use these $H \rightarrow WW^*$ boson decays as a laboratory to perform tests of Bell inequalities.

Like similar proposals using the weak decay to analyse spin [11, 13,14] they do not allow the experimentalist to freely choose the W -boson spin measurement directions. Nevertheless, they offer an opportunity to test Bell inequalities in a new regime: at energies of order $m_H \approx 125 \text{ GeV}$ [20], time scales of order $\hbar/\Gamma_W \approx 10^{-25} \text{ s}$ [20] and length scales of order $\hbar c/\Gamma_W \approx 10^{-16} \text{ m}$. These scales are very many orders of magnitude removed from existing experimental results, and offer the prospect of performing Bell tests in a new regime deep within the realm of quantum field theory.

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¹ Following the submission of this Letter a manuscript was submitted to the arXiv exploring sensitivity to entanglement and to Bell inequalities in $t + \bar{t}$ decays at the LHC [16].

2. Testing the generalised CHSH inequality

For a pair of two-outcome experiments, such as measurements of the spins of a pair of spin-half particles, the necessary and sufficient conditions [21] that measurements be compatible with any realist theory is that they satisfy the inequality of Clauser-Horne-Shimony-Holt (CHSH) [22]

$$\mathcal{I}_2 = E(a, b) - E(a, b') + E(a', b) + E(a', b') \leq 2. \quad (2)$$

This equation characterises the results a, b of experiments performed on each of the systems, labelled A and B respectively, where the primes indicate the results obtained from alternative versions of the experiment, for example by changing the axis of a spin measurement.

For two-outcome measurements such as of the spins of spin-half particles or of photons, the expectation values E are the products of the assigned values, either $+1$ or -1 . If one demands a description of nature which is consistent with realism then the values of \mathcal{I}_2 can be no larger than two, since a larger value would imply negative marginal probabilities.

Theories admitting local realism are based on the further observation that according to special relativity information should not travel faster than the speed of light. In local realistic theories the hypothesised pre-existing values for the experimental outcomes for each system may therefore depend only on events in the past light-cone of that system. This restriction prevents information from the measurement settings or measurements of system A from being transmitted to B, and hence to causally affect the measurement outcome at B (and similarly for signals from B to A).

Quantum mechanics, in conflict both with realism and with local realism, allows values of \mathcal{I}_2 larger than two and indeed up to the Cirel'son bound [23] of $2\sqrt{2}$.

For the case of massive vector bosons we have to assign a result value to each of the three possible outcomes from the measurements, opening up more assignment options. Here we choose to take the eigenvalues $\{1, 0, -1\}$ of the $s = 1$ spin operators meaning that the additional outcome is assigned the value zero [24]. This additional zero outcome dilutes the expectation values and so tends to decrease violation of inequality (2), such that this spin-one generalisation of the spin-half CHSH operator is found in analytical calculations not to violate the Bell inequality (2) either in non-relativistic quantum mechanics or in relativistic quantum mechanics in the narrow-width approximation [24]. Nevertheless, it proves instructive to examine its behaviour in the $H \rightarrow WW^*$ system to allow insight into its construction and behaviour and so that a direct comparison with the spin-half equivalent measurement may be made.

The generalised CHSH operator that we use [24] may be written

$$\mathcal{B}_{\text{CHSH}}^{\text{gen}} = \hat{\mathbf{n}}_1 \cdot \mathbf{S} \otimes (\hat{\mathbf{n}}_2 - \hat{\mathbf{n}}_4) \cdot \mathbf{S} + \hat{\mathbf{n}}_3 \cdot \mathbf{S} \otimes (\hat{\mathbf{n}}_2 + \hat{\mathbf{n}}_4) \cdot \mathbf{S}, \quad (3)$$

in an analogous manner to the spin-half case, where $\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3$ and $\hat{\mathbf{n}}_4$ are unit vectors in \mathbb{R}^3 and $\mathbf{S} \equiv (S_x, S_y, S_z)$ are the dimension-3 Hermitian spin operators. When an explicit matrix representation is required we use the standard representation i.e. with $S_z = \text{diag}(+1, 0, -1)$.

To calculate the expectation value

$$\mathcal{I}_2^{\text{gen}} = \text{tr}(\rho \mathcal{B}_{\text{CHSH}}^{\text{gen}}), \quad (4)$$

we note that the density matrix ρ for a single spin-one particle may be parameterised by

$$\rho_W = \frac{1}{3}I_3 + \sum_{i=1}^3 a_i S_i + \sum_{i,j=1}^3 c_{ij} S_{\{ij\}}, \quad (5)$$

where we denote the anticommutator

$$S_{\{ij\}} \equiv S_i S_j + S_j S_i,$$

and where the parameters a_i form a real vector and c_{ij} a traceless real symmetric matrix.

The two-particle spin density matrix ρ may similarly be parameterised in terms of the S_i and $S_{\{ij\}}$ for each particle. Noting that the S_i are each trace orthogonal with one another, with the identity, and with each of the $S_{\{ij\}}$, and that the spin operators of each particle commute with those of the other, the only terms in ρ contributing to the expectation value (4) are of the form

$$\rho \supset \sum_{i,j=1}^3 \frac{1}{4} d_{ij} S_i \otimes S_j, \quad (6)$$

where d_{ij} are real parameters which contribute to this expectation value through terms of the form

$$\text{tr}(\rho S_i \otimes S_j) = d_{ij}. \quad (7)$$

Our generalised CHSH inequality for a pair of spin-one particles therefore can be reduced to

$$|\hat{\mathbf{n}}_1 \cdot d \cdot (\hat{\mathbf{n}}_2 - \hat{\mathbf{n}}_4) + \hat{\mathbf{n}}_3 \cdot d \cdot (\hat{\mathbf{n}}_2 + \hat{\mathbf{n}}_4)| \leq 2. \quad (8)$$

We next need to determine the elements d_{ij} of the matrix d and to choose four unit vectors that maximise the left hand side of (8). In practice not all choices need be made, since a procedure for testing the inequality in general has been obtained for the spin-half case [25], and remains valid for a pair of spin-one particles.

Starting from the real matrix d and its transpose d^T one forms the real symmetric positive matrix $M \equiv d^T d$. One orders the three eigenvalues μ_1, μ_2, μ_3 of M such that $\mu_1 \geq \mu_2 \geq \mu_3$. The largest value of our generalised CHSH operator is then, following [25],

$$\mathcal{I}_2^{\text{gen,max}} \equiv \max_{\{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3, \hat{\mathbf{n}}_4\}} (\langle \mathcal{B}_{\text{CHSH}}^{\text{gen}} \rangle) = 2\sqrt{\mu_1 + \mu_2}. \quad (9)$$

This value may be compared against the bound (2) required of a realist theory.

We note that while we choose to make a spin-eigenvalue-based assignment for the outcomes, there are other operators which make different assignments of states to outcomes [26–31], and for which the quantum mechanical expectation values, unlike (3), do violate the inequalities implied by realist theories. We will not investigate all of these possibilities in this Letter, but rather in Section 3 we investigate just the tightest of them.

3. Testing the CGLMP inequality

The optimal [32] Bell inequality for pairs of three-outcome systems is the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality [30,31]. To construct it one again considers two observers A and B, each having two measurement settings, A_1 and A_2 for A, and B_1 and B_2 for B, but with each experiment now having three possible outcomes. One denotes by $P(A_i = B_j + k)$ the probability that the outcomes A_i and B_j differ by k modulo 3. One then constructs the linear function

$$\begin{aligned} \mathcal{I}_3 = & P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) \\ & + P(B_2 = A_1) - P(A_1 = B_1 - 1) - P(B_1 = A_2) \\ & - P(A_2 = B_2 - 1) - P(B_2 = A_1 - 1). \end{aligned} \quad (10)$$

In classical theories, and other theories admitting realism, this function is bounded by [31]

$$\mathcal{I}_3 \leq 2. \quad (11)$$

To test inequality (11) in quantum mechanics we can calculate the expectation value of the Bell operator

$$\mathcal{B}_{\text{CGLMP}}^{\text{xy}} = \frac{2}{\sqrt{3}} (S_x \otimes S_x + S_y \otimes S_y) + \lambda_4 \otimes \lambda_4 + \lambda_5 \otimes \lambda_5, \quad (12)$$

where λ_i is the i th of the eight traceless 3×3 Hermitian Gell-Mann matrices in the standard representation [33]. In this convention the spin operators S_x and S_y are given by

$$S_x = \frac{1}{\sqrt{2}}(\lambda_1 + \lambda_6) \quad \text{and} \quad S_y = \frac{1}{\sqrt{2}}(\lambda_2 + \lambda_7).$$

Our operator (12) is related to the standard CGLMP operator $\mathcal{B}_{\text{CGLMP}}^{\text{xy}}$ [34] through the transformation²

$$\mathcal{B}_{\text{CGLMP}}^{\text{xy}} = (P_{(13)} \otimes I_3) \mathcal{B}_{\text{CGLMP}}^{\text{xy}} (P_{(13)} \otimes I_3), \quad (13)$$

where $P_{(13)}$ is the permutation operator. This procedure has the same effect as mapping our singlet state (1) into the computational basis:

$$|\psi_s\rangle \rightarrow \frac{1}{\sqrt{3}}(|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle).$$

4. Determining expectation values from data

We wish to determine the expectation value $\text{tr}(\rho \mathcal{B})$ of two different Bell operators \mathcal{B} from $H \rightarrow WW^*$ decay data. We may do so by finding the density matrix ρ using as data the directions $\hat{\mathbf{n}}_{\ell^+}$ and $\hat{\mathbf{n}}_{\ell^-}$ of the daughter leptons. Exploiting the trace orthogonality relations

$$\text{tr}(\lambda_i \lambda_j) = 2\delta_{ij}, \quad (14)$$

we now parameterise the density matrix for the W^+W^- spins in the Gell-Mann basis

$$\rho = \frac{1}{9}I_3 \otimes I_3 + \sum_{i=1}^8 f_i \lambda_i \otimes I_3 + \sum_{j=1}^8 g_j I_3 \otimes \lambda_j + \sum_{i,j=1}^8 h_{ij} \lambda_i \otimes \lambda_j, \quad (15)$$

where f_i , g_i and h_{ij} are real coefficients, of which only the h_{ij} contribute to the Bell operators.

As a preliminary, let us consider the spin density matrix

$$\rho_W = \frac{1}{3}I_3 + \sum_{i=1}^8 \Lambda_i \lambda_i, \quad (16)$$

for a single W^+ or W^- boson, where the Λ_i are real coefficients. The probability density function for a W^\pm boson with spin density matrix given by (16) to emit a charged lepton ℓ^\pm into infinitesimal solid angle $d\Omega$ in the direction $\hat{\mathbf{n}}(\theta, \phi)$ is

$$p(\ell_{\hat{\mathbf{n}}}^\pm; \rho_W) = \frac{3}{4\pi} \text{tr}(\rho_W \Pi_{\pm, \hat{\mathbf{n}}}), \quad (17)$$

² A similar permutation operation is also required in principle for the generalised CHSH operator, but has no net effect on the final result after the process of optimisation over measurement directions.

where $\Pi_{\pm, \hat{\mathbf{n}}} \equiv |\pm\rangle_{\hat{\mathbf{n}}} \langle \pm|_{\hat{\mathbf{n}}}$ are projection operators,³ the roles of which are to select negative helicity ℓ^- or positive helicity ℓ^+ in the direction $\hat{\mathbf{n}}$. The normalisation of (17) is such that $\int d\Omega p(\ell_{\hat{\mathbf{n}}}^\pm; \rho_W) = 1$.

Using (17) we may obtain information about the density matrix parameters Λ_i from angular integrals. In particular

$$\begin{aligned} \langle \xi_x^\pm \rangle_{\text{av}} &= \int d\Omega p(\ell_{\hat{\mathbf{n}}}^\pm; \rho_W) \sin \theta \cos \phi \\ &= \pm \frac{1}{\sqrt{2}}(\Lambda_1 + \Lambda_6), \end{aligned} \quad (18)$$

$$\begin{aligned} \langle \xi_y^\pm \rangle_{\text{av}} &= \int d\Omega p(\ell_{\hat{\mathbf{n}}}^\pm; \rho_W) \sin \theta \sin \phi \\ &= \pm \frac{1}{\sqrt{2}}(\Lambda_2 + \Lambda_7), \end{aligned} \quad (19)$$

$$\begin{aligned} \langle \xi_z^\pm \rangle_{\text{av}} &= \int d\Omega p(\ell_{\hat{\mathbf{n}}}^\pm; \rho_W) \cos \theta \\ &= \pm \frac{1}{2}(\Lambda_3 + \sqrt{3}\Lambda_8), \end{aligned} \quad (20)$$

where the direction cosines $\xi_i^+ = \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_{\ell^+}$ and $\xi_i^- = \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_{\ell^-}$ are measured in the rest frames of the W^+ and W^- bosons respectively. Equations (18)–(20) allow us to determine the expectation values

$$\text{tr}(\rho_W S_i) = \pm 2 \langle \xi_i^\pm \rangle_{\text{av}}$$

of the single-particle spin operators from the data.

Extending the calculation to the two-particle density matrix we can calculate the expectation values of the operators required for the generalised CHSH inequality in terms of observables⁴:

$$\text{tr}(\rho S_i \otimes S_j) = -4 \langle \xi_i^- \xi_j^+ \rangle_{\text{av}}. \quad (21)$$

In the absence of experimental cuts, and provided the sample of events is sufficiently large,⁵ the elements of d in (6) are therefore given by

$$d_{ij} = -4 \langle \xi_i^+ \xi_j^- \rangle_{\text{av}}, \quad (22)$$

from which we may calculate the generalised CHSH inequality (8) for any measurement angles.

The CGLMP expectation value in terms of the parameters of ρ is

$$\begin{aligned} \text{tr}(\rho \mathcal{B}_{\text{CGLMP}}^{\text{xy}}) &= 4(h_{44} + h_{55}) + \frac{4}{\sqrt{3}}(h_{11} + h_{16} + h_{61} + h_{66}) \\ &\quad + \frac{4}{\sqrt{3}}(h_{22} + h_{27} + h_{72} + h_{77}). \end{aligned} \quad (23)$$

The terms in the second and third parentheses come from $S_x \otimes S_x$ and $S_y \otimes S_y$ operators respectively so can be determined using (21).

To determine the remaining terms we return to the single-particle density matrix (16), and note that the angular integrals

³ The positive helicity ℓ^+ travelling in direction $\hat{\mathbf{n}}$ in the W^+ rest frame is accompanied by a negative helicity ν_ℓ travelling in the opposite direction $-\hat{\mathbf{n}}$, and similarly the negative helicity ℓ^- with a positive helicity ν_ℓ in the W^- frame, so these are spin-one projections.

⁴ We note that the factor of 4 in (21) differs from a factor of 9 which would be obtained in the spin-half case.

⁵ Care is necessary in evaluating $\mathcal{I}_2^{\text{gen, max}}$ when event samples become very small, due to the procedure of maximising over the choice of axes. In the limit where only a single event satisfies the selection requirements there exists a choice of axes for which $\xi_i^+ = \xi_j^- = 1$, hence $\mu_1 + \mu_2 = 16$ and so the inferred value of $\mathcal{I}_2^{\text{gen, max}}$ would be $2\sqrt{16} = 8$.

$$\begin{aligned} \langle (\xi_x^\pm)^2 - (\xi_y^\pm)^2 \rangle_{\text{av}} &= \int d\Omega p(\ell_{\mathbf{n}}^\pm; \rho_W) \sin^2 \theta \cos(2\phi) \\ &= \frac{2}{5} \Lambda_4 \end{aligned} \quad (24)$$

and

$$\begin{aligned} 2 \langle \xi_x^\pm \xi_y^\pm \rangle_{\text{av}} &= \int d\Omega p(\ell_{\mathbf{n}}^\pm; \rho_W) \sin^2 \theta \sin(2\phi) \\ &= \frac{2}{5} \Lambda_5, \end{aligned} \quad (25)$$

extract the parameters of interest, so that the expectation values are

$$\begin{aligned} \text{tr}(\rho_W \lambda_4) &= 5 \langle (\xi_x^\pm)^2 - (\xi_y^\pm)^2 \rangle_{\text{av}} \\ \text{tr}(\rho_W \lambda_5) &= 10 \langle \xi_x^\pm \xi_y^\pm \rangle_{\text{av}}. \end{aligned} \quad (26)$$

Extending (26) to the two-particle density matrix, the CGLMP expectation value can be expressed

$$\begin{aligned} \text{tr}(\rho \mathcal{B}_{\text{CGLMP}}^{xy}) &= \frac{8}{\sqrt{3}} \langle \xi_x^+ \xi_x^- + \xi_y^+ \xi_y^- \rangle_{\text{av}} \\ &\quad + 25 \left(\langle (\xi_x^+)^2 - (\xi_y^+)^2 \rangle_{\text{av}} \langle (\xi_x^-)^2 - (\xi_y^-)^2 \rangle_{\text{av}} \right. \\ &\quad \left. + 100 \langle \xi_x^+ \xi_y^+ \xi_x^- \xi_y^- \rangle_{\text{av}} \right), \end{aligned} \quad (27)$$

in terms of the x - and y -direction cosines of the lepton emission directions in the respective W^\pm boson rest frames. This is our main result, and provides an experimental observable that can be used to test the CGLMP Bell operator experimentally against the classical bound in any process in which Higgs bosons are produced and subsequently decay to WW^* .

The expectation value in (27) is calculated using only x and y axis direction cosines. Corresponding operators and expectation values could also be constructed for other pairs of mutually orthogonal axes, and tested against the CGLMP inequality. Each such rotated operator will have its own particular dependence on the two-particle density matrix parameters. The expectation value of the CGLMP operator for the new axes can be calculated from a generalisation of Eqn. (27) but now using the corresponding direction cosines. In the case of $H \rightarrow WW^*$ decays the ensemble of decays has rotational symmetry in the Higgs boson rest frame around the direction of the W boson momenta. Hence in this Letter rather than testing every possible pair of axes⁶ we choose a set of Cartesian coordinates in which one axis is aligned with this privileged direction, construct expectation values for each of the (x, y) , (y, z) and (z, x) pairs of axes, and compare the largest of them

$$\mathcal{I}_3^{xyz} = \max(\langle \mathcal{B}_{\text{CGLMP}}^{xy} \rangle, \langle \mathcal{B}_{\text{CGLMP}}^{yz} \rangle, \langle \mathcal{B}_{\text{CGLMP}}^{zx} \rangle) \quad (28)$$

to the classical bound (11).

5. Numerical simulations

In the non-relativistic and narrow-width limits, a measurement of the CGLMP operator for the spin-singlet state of a pair of W bosons from a Higgs boson decay is expected to violate the corresponding Bell inequality (11), since the problem reduces to the non-relativistic quantum mechanical calculation [31]. To investigate the impact of relativistic and finite-width effects a numerical

Table 1

'Truth-level' numerical calculations of the generalised CHSH and the CGLMP expectation values in $H \rightarrow WW^*$ decays, as a function of the minimum invariant mass of either of the W bosons.

$m_W^<$ [GeV]	0	20	30	40	50
$\mathcal{I}_2^{\text{gen,max}}$	1.78	1.91	1.96	1.94	1.95
\mathcal{I}_3^{xyz}	2.62	2.76	2.81	2.82	2.77

calculation is employed. These numerical simulations will also allow us to perform a first investigation of the impact of experimental resolutions and event selections, in particular for the existing general-purpose detectors at the LHC.

We performed Monte Carlo simulations of $gg \rightarrow H \rightarrow \ell^+ \nu_\ell \ell^- \bar{\nu}_\ell$ events, where $\ell \in \{e, \mu\}$, using the Madgraph v2.9.2 software [35] which includes full spin correlation, relativistic and Breit-Wigner effects. A sample of 10^6 events was generated at leading order at a proton-proton centre-of-mass energy of 13 TeV, using the Higgs effective-field theory model. Higher order corrections to the shapes of the normalised angular distributions, which for Higgs boson decays to four leptons are typically at the $\lesssim 5\%$ level [36–38], are neglected in this initial study. The LHC Higgs cross-section working group has calculated the $gg \rightarrow H$ cross section for a 125 GeV Higgs boson to N³LO in the effective theory to be 48.6 pb [39]. The branching ratio $H \rightarrow \ell^+ \nu_\ell \ell'^- \bar{\nu}_{\ell'}$ for $\ell, \ell' \in \{e, \mu\}$ has been also calculated to be 1.055×10^{-2} [39]. Thus our sample of 10^6 events corresponds to an integrated luminosity of 1950 fb^{-1} . Using these values, the simulations were scaled to the target integrated luminosity of 140 fb^{-1} , approximately that recorded by each of the ATLAS and CMS experiments during the period 2015–2018 [40]. Events containing an e^+e^- or $\mu^+\mu^-$ pair were rejected in order to remove $H \rightarrow ZZ^*$ contributions.

Our choice of orthonormal basis for the matrix d is a modification of that proposed for measuring spin correlation in top quarks [41]. In the W^+W^- centre-of-mass frame the direction of the W^+ is denoted $\hat{\mathbf{k}}$. The direction $\hat{\mathbf{p}}$ of one of the beams in that frame is determined, and a mutually orthogonal basis constructed from them:

$$\hat{\mathbf{k}}, \quad \hat{\mathbf{r}} = \frac{1}{r}(\hat{\mathbf{p}} - y\hat{\mathbf{k}}), \quad \hat{\mathbf{n}} = \frac{1}{r}(\hat{\mathbf{p}} \times \hat{\mathbf{k}}),$$

where $y = \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}$ and $r = \sqrt{1 - y^2}$. This provides a right-handed orthonormal basis $\{\hat{\mathbf{n}}, \hat{\mathbf{r}}, \hat{\mathbf{k}}\}$ defined in the Higgs boson rest frame. Boosts are then performed into each of the W^\pm rest frames, and a new basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\} = \{\hat{\mathbf{n}}, \hat{\mathbf{r}}, \hat{\mathbf{k}}\}$ defined in each such that $\hat{\mathbf{n}}$ and $\hat{\mathbf{r}}$ are unmodified, while each $\hat{\mathbf{k}}'$ is parallel to $\hat{\mathbf{k}}$ but has been unit-normalised after the corresponding boost. The correlation matrix d is then constructed according to (22), and the CGLMP expectation values according to (27) and (28).

Since $m_H < 2m_W$, at least one of the W bosons must be off its mass-shell, and therefore can be expected to have some scalar component. This component will behave like noise, reducing the observed correlations, so we might expect the degree of the observed correlations in the simulation to depend on the range of the W boson masses accepted. Changing the selected range of W^* boson masses can also be expected to modify the impact of relativistic and finite-width corrections. The values of $\mathcal{I}_2^{\text{gen,max}}$ and \mathcal{I}_3^{xyz} were therefore determined for several different selections, each being defined by the veto on same-flavour leptons and a lower bound $m_W^<$ on the smaller of the masses of the two reconstructed W bosons.

The numerical results in Table 1 show that as $m_W^<$ is increased, and the W bosons approach their mass shell, the value of $\mathcal{I}_2^{\text{gen,max}}$ approaches two. However, that classical limit is not exceeded so no experimental Bell-inequality violation would be expected if using

⁶ This procedure does not formally exclude the possibility that there exists another pair of orthogonal axes which would show larger violation in the general case, but it is sufficient for our purposes.

Table 2

Sensitivity of the CGLMP expectation value to experimental selection and resolution for three different sets of experimental assumptions: ‘Truth’, ‘A’ and ‘B’. Rows 2–4 show the experimental cuts applied respectively to the lepton transverse momentum, and their pseudorapidity, and the smearing parameter for the reconstructed $W^{(*)}$ boson rest frames. Rows 5–8 show for an integrated pp luminosity of 140 fb^{-1} : the number and the fraction of the $H \rightarrow \ell^+ \nu \ell^- \bar{\nu}$ events passing the selection, including the same-family lepton veto; the value of the measured CGLMP expectation value; and finally, for the same integrated luminosity, the statistical significance by which the CGLMP expectation value exceeds the classical limit of 2.

Expt. Assumptions	Truth	‘A’	‘B’	‘C’
Min $p_T(\ell)$ [GeV]	0	5	20	20
Max $ \eta(\ell) $	—	2.5	2.5	2.5
σ_{smear} [GeV]	0	5	5	10
Number of events	34.3k	19.7k	6.5k	5.4k
Fraction of events	0.48	0.27	0.090	0.075
$\mathcal{I}_3^{\text{xyz}}$	2.62	2.40	2.75	2.16
Signif. ($\mathcal{I}_3^{\text{xyz}} - 2$)	11.7 σ	5.2 σ	5.3 σ	1.0 σ

the generalised CHSH operator (3). These findings are consistent with analytical and numerical results for this operator performed previously for entangled states of non-relativistic spin-one systems and of relativistic spin-one bosons in the narrow width approximation [24].

The results for the CGLMP inequality are also shown in Table 1. In this case the expectation values for all values of m_W^- are well in excess of the classical limit of 2, and as large as 2.82. This is close to the largest possible value in non-relativistic quantum mechanics for a maximally entangled state⁷ which is $4/(6\sqrt{3} - 9) \approx 2.8729$. This result confirms that near-maximal violation of the CGLMP inequality is achieved despite relativistic and finite-width corrections.

In any real experiment corrections will be needed to account for detector acceptance and efficiency effects, for backgrounds, and for indeterminacies in the reconstructed W bosons’ rest frames. The appropriate selections and corrections will vary from experiment to experiment, and so only some initial estimates of their approximate magnitudes are considered here.

At a pp collider such as the LHC one might employ kinematic methods similar to those of the ‘ M_{T2} -assisted on-shell’ method [42] to estimate the unobserved neutrino four-momenta. Here we provide a first estimate the effect of such reconstruction effects by independently⁸ smearing each of the three spatial components of the momentum of each of the two W bosons with a Gaussian with two different values of the width parameter chosen to illustrate the size of the effect on the measurement.

In Table 2 we show results for three different experimental scenarios. The first ‘Truth’ scenario shows the idealised results without modelling any experimental acceptance requirements or resolution effects. Scenario ‘A’ includes a resolution on the $W^{(*)}$ bosons’ reconstructed momenta, modelled by a Gaussian of width 5 GeV in each Cartesian direction, and also lepton acceptance requirements similar to those required employed by the LHC general-purpose detectors. Scenario ‘B’ has tighter lower bound of 20 GeV on the leptons’ transverse momenta, similar to that required by the leptonic triggers of the LHC detectors. This reduces the number of events, but also has the effect of increasing the expectation value $\mathcal{I}_3^{\text{xyz}}$ as calculated from those remaining events. In a real experiment any bias introduced by the selection could be corrected by applying a similar selection requirement to the angular integrals ((18)–(20) and (24)–(25)) from which the density matrix coeffi-

cients are extracted, but that correction is not applied here so that the magnitude of the uncorrected effect can be seen. Scenario ‘C’ is like ‘B’ but with an increased smearing of the W boson momentum to 10 GeV in each component. In each case all values of the $W^{(*)}$ bosons’ masses are permitted, and the experimental backgrounds are neglected.

The statistical significances⁹ by which the simulated values of $\mathcal{I}_3^{\text{xyz}}$ exceed the classical bound were calculated for pp integrated luminosity of 140 fb^{-1} , and are also shown in Table 2. A significance of about 12σ is found in the idealised ‘Truth’ simulation, falling to about 5σ for experimental scenarios ‘A’ and ‘B’ and to approximately 1σ for scenario ‘C’. The large significance for the idealised simulation demonstrates that the LHC has already produced sufficient numbers of Higgs bosons to perform the measurements if experimental considerations could be neglected. The wide range of significances for different scenarios shows that making this measurement at the LHC is likely to be experimentally demanding, and to require a careful study of – and optimisation of – these sorts of experimental effects.

6. Discussion

The near-maximal violation of the CGLMP inequality in numerical simulations motivates the more detailed study of Bell violation in Higgs boson decays at the LHC and also at future colliders. We note that unlike in the case of e.g. $t\bar{t}$ production, $H \rightarrow WW^*$ involves an intermediate state comprising only a single narrow-width scalar – the Higgs boson. This means that the spin density matrix ρ of the W boson pair does not depend on how the Higgs bosons have been produced. The expectation values of the Bell observables and the overall method are therefore the same for any Higgs boson whether produced in e.g. gluon-gluon fusion, in Higgsstrahlung, or vector-boson fusion.

The overall method will translate directly to Higgs bosons produced at other high-energy accelerators, regardless of the types of particles (p, e, μ, \dots) that are collided. At a future e^+e^- collider of particular interest would be the $Z + H$ Higgsstrahlung production process, for which the Higgs signal events could be selected using the invariant mass of the object(s) recoiling against the Z boson. Such a selection would cleanly remove backgrounds such as those from $t\bar{t}$ and non-resonant WW^* that would need to be accounted for at a pp collider such as the LHC.

An experimental observation of CGLMP violation in $H \rightarrow WW^*$ decays would provide a striking conflict with realism deep within the regime in which field theory is expected to reign. Furthermore, since the $W^{(*)}$ bosons from the $H \rightarrow WW^*$ decay separate at relativistic speeds they have a mixture of space-like and time-like separations at decay, so observation of Bell violation in this system might have implications for tests of causality and local realism – again at these extreme length, time and energy scales. If, by contrast, experiments are found to be unable to measure a Bell inequality violation where one is expected, then this would be an even more surprising and consequential result.

7. Conclusion

We have outlined methods by which two Bell inequalities – a generalised CHSH inequality and the CGLMP inequality – may be tested experimentally in $H \rightarrow WW^*$ decays, using the spin-analysing nature of the weak decays. Numerical simulations, agreeing with previous analytical and numerical work for pairs of spin-one bosons, suggest that one cannot expect to observe violation

⁷ See Ref. [31]. A slightly larger extremal value of $1 + \sqrt{11/3} \approx 2.9149$ can be achieved for other states that are not maximally entangled [34].

⁸ We neglect the kinematic constraints from the known masses and from the measured missing transverse momentum which could be used to constrain the overall set of momenta.

⁹ The significance is here defined to be $(\mathcal{I}_3^{\text{xyz}} - 2)/\sigma_{\mathcal{I}}$, where $\sigma_{\mathcal{I}}$ is the standard error of the mean of $\mathcal{I}_3^{\text{xyz}}$.

of the generalised CHSH inequality in this process. By contrast the CGLMP inequality, the tightest inequality for pairs of three-state systems, is expected to be near-maximally violated in $H \rightarrow WW^*$ decays.

The method described offers the opportunity to test Bell inequalities in the quantum field theory regime, providing prospects for experimental tests far removed from the length-scales, time-scales and energies of existing measurements. The many orders of magnitude difference in such scales provides ample scope for unexpected experimental results.

Similar test can be performed across a range of production channels (gg fusion, VBF, Higgsstrahlung, ...) and at any type of collider (pp , ee , ep , $\mu\mu$, ...) that produces Higgs bosons, since the narrow scalar H does not retain information about its production mechanism. The experimental challenges in each collider and in each production mechanism will differ, motivating dedicated studies of each.

Numerical simulations suggest that, provided that experimental resolutions and selection effects can be controlled, then statistically significant violations of the CGLMP inequality might be observable by the LHC experiments using datasets comparable to those already collected.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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