On Galois Correspondences in Formal Logic

Austin Vincent Yim

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Abstract

This thesis examines two approaches to Galois correspondences in formal logic. A standard result of classical first-order model theory is the observation that models of $\mathscr{L}$-theories with a weak form of elimination of imaginaries hold a correspondence between their substructures and automorphism groups defined on them. This work applies the resultant framework to explore the practical consequences of a model-theoretic Galois theory with respect to certain first-order $\mathscr{L}$-theories. The framework is also used to motivate an examination of its underlying model-theoretic foundations.

The model-theoretic Galois theory of pure fields and valued fields is compared to the algebraic Galois theory of pure and valued fields to point out differences that may hold between them. The framework of this logical Galois correspondence is also applied to the theory of pseudoexponentiation to obtain a sketch of the Galois theory of exponential fields, where the fixed substructure of the complex pseudoexponential field $\mathbb{B}$ is an exponential field with the field $\mathbb{Q}^{\mathrm{rab}}$ as its algebraic subfield. This work obtains a partial exponential analogue to the Kronecker-Weber theorem by describing the pure field-theoretic abelian extensions of $\mathbb{Q}^{\mathrm{rab}}$, expanding upon work in the twelfth of Hilbert's problems. This result is then used to determine some of the model-theoretic abelian extensions of the fixed substructure of $\mathbb{B}$.

This work also incorporates the principles required of this model-theoretic framework in order to develop a model theory over substructural logics which is capable of expressing this Galois correspondence. A formal semantics is developed for quantified predicate substructural logics based on algebraic models for their propositional or nonquantified fragments. This semantics is then used to develop substructural forms of standard results in classical first-order model theory. This work then uses this substructural model theory to demonstrate the Galois correspondence that substructural first-order theories can carry in certain situations.

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On Galois Correspondences in Formal Logic, a thesis by Austin Vincent Yim, divided into several sections, to wit

1. a meditation on the general patterns of formal logic that has led to the current state of logic;
2. an introduction to a class of deductive calculi called substructural logics, with semantics defined over them with the express purpose of developing the rudiments of a model theory;
3. an application of this substructural model theory to develop a substructural understanding of Galois correspondences that can occur in certain formal theories satisfying certain properties;
4. some brief applications of this Galois correspondence to various first-order theories over classical first-order predicate logic and how the logical setting may differ from general mathematical practice;
5. a particular application of this Galois correspondence to develop a basic understanding of the Galois theory of exponential fields and elucidation of problems associated with such work;
6. the examination of abelian pure field extensions of the pure field of real abelian numbers and its usefulness in the more general issue of abelian model-theoretic extensions of the fixed exponential field in the Galois theory of exponential fields; and
7. a discourse on continued development of the general patterns constituting this work.

Contents

Chapter 1

Logical Prolegomenon

One of the core motivations for the formal study of logic is the articulation of correct reasoning. Problems associated with this most laudable purpose have been meditated upon since antiquity. The ambiguities in the grammar of the language of Classical China during the Warring States period inspired a bewildering account for why a white horse may not be a horse in [Gongsun Longzi]. The Western tradition of formal logic has flourished since syllogistic principles were laid out in Aristotle's *Organon*, and the pursuit of logic has continued to occupy scholarly interests. The prolific development of Indian logic, which stands comparably with the Western tradition, holds the formalization of Sanskrit grammar as a major source of foundational inspiration.

These early explorations of logic found focus in the language used to carry out reasoning, but mathematics also provided some motivation. In the Western tradition, the interaction between logic and mathematics manifested most notably in the Aristotlian aspiration that syllogistic patterns can capture the mode of Greek mathematical reasoning. A demonstration of the irrationality of the square root of 2 is written in [Aristotle] whereby the assumption of the rationality of that number syllogistically implies that all odd numbers are even numbers. This argument, as noted in [Thomas 2006], may have been articulated by the Pythagoreans, but despite this example, it is cumbersome to formalize general mathematical reasoning into syllogisms. Indeed, one would find the background logic driving the works of Greek mathematicians such as Euclid or Archimedes to employ other patterns of argument than syllogisms.

Consequently, this early foray into mathematical logic found no sustenance. Contrasting with the Peripatetic school of Aristotle, the Stoics developed ideas in logic with resonance to contemporary trends, but their endeavors suffered into obscurity with the close of antiquity. A similar fate much later befell the mathematical inquiries of G.W. Leibniz into formal logic. Unlike his formulation of the differential and integral calculus, Leibnizian logic did not bloom after his time, and the impact of its rediscovery on the advancement of logic in later times has been disputed, as summarized in [Peckhaus 2009]. The long history of formal logic has seen the rise and fall of many attentive interests, but while it has flourished in many disciplines such as law and philosophy, it has too often languished in its interactions with mathematics.

Overcoming the eventual and inevitable oversight of its antecedents, the rise of modern logic in the nineteenth century heralded a particularly mathematical scrutiny into formal logic. Logical discourse and the articulation of correct reasoning became transformed with the ability to manipulate formal logic into a calculus of symbols. The algebraization of logic invited renewed consideration of classic problems in logical discourse and sparked new directions in incorporating mathematical and philosophical concerns about reasoning into this symbolic logic.

The advent of symbolic formal logic brought forth several systems of logical calculi. Two notable proposals for the perceived correct system of reasoning were introduced by G. Frege and the collaboration between A.N. Whitehead and B. Russell. Frege's *Begriffsschrift* laid out a second-order predicate calculus which made use of a novel two-dimensional symbolic notation, a system expanded upon later in his larger *Grundgesetze*. The logicist program of Whitehead and Russell's *Principia Mathematica* which emerged shortly thereafter built upon the technical intricacies of the ramified theory of types at its core.

Later work during this rise of modern symbolic logic coalesced around a common fragment of these pioneering works, the system recognized as the functional calculus. For example, the work of G. Peano on the axioms for arithmetic and the early axiomatizations of set theory could be formalized by readily adopting the functional calculus as the background logic. The functional calculus provided a system sufficiently simple enough to submit to mathematical scrutiny and yet expressively powerful enough to capture a large portion of mathematical reasoning. The proof of completeness by K. Gödel in [Gödel 1930], followed shortly thereafter in [Gödel 1931] by the incompleteness results for Peano arithmetic in functional calculus, eventually established the functional calculus as a standard in subsequent work in formal logic.

The functional calculus is now an archaic designation for the classical first-order predicate logic that is at the center of contemporary practice, but these early days of modern logic also introduced alternative deductive calculi which incorporated principles disputing or extending the functional calculus and competed with others for attention. Among these alternatives to classical logic that emerged from this period is the familiar system of intuitionistic logic. Intuitionistic logic developed as a codification by A. Heyting of the principles of intuitionism, an ideological philosophy of mathematics which emerged from the critiques of L.E.J. Brouwer on the mainstream acceptance of such principles like the Law of Excluded Middle and of Double Negation.

Some logical calculi emerged during this heyday only to fade back into obscurity, but a few of these systems continued to inspire subsequent logical discourse. Among these instances was the logic of intensional relations proposed by E.J. Nelson. This logic of intensional relations did not continue beyond [Nelson 1930], but S. McCall recognized the intuitive principles behind it to be the same as the ones for logical works of antiquity such as the *Organon* as well as his later proposal for connexive logic, a contraclassical formalization developed to witness the idea that antecedents and consequents of implicative statements ought to be connected to each other in some intrinsic way.

Even while mathematical logic developed technical complexity with classical first-order predicate logic at its core, the investigation into other forms of deductive calculi did not cease after the recognition of the versatility of the functional calculus. New philosophical objections to classical logic gave rise to new alternatives, and the tools of formal logic found increased connections to other academic disciplines such as linguistics and computer science. The various traditions of logic that emerged from these connections have enriched the contemporary picture of formal logic so that classical first-order predicate logic is accompanied by diverse arrays of nonclassical logics.

1.1 The Practice and Discipline of Model Theory

In general terms, model theory is the study of the semantics of mathematical logic. Model theory as a discipline in its own right centers upon classical first-order predicate logic. C.C. Chang and J. Keisler's classic [Chang Keisler 1990] gives the equation "Model Theory = Logic + Universal Algebra" to describe the subject matter of their book. Indeed, the semantical definitions readily accepted in model theory borrows liberally the language and style of universal algebra, and many results of the model theory of this book are also foundational results in universal algebra. W. Hodges in [Hodges 1997] in more recent times provided the alternative formulation of "Model Theory = Algebraic Geometry minus Fields," reflecting upon the changes in practice and focus wrought by contemporary practice. A consequence of these changes is the ingratiation of model theory into other areas of mathematics, particularly algebra and number theory. These changes have also motivated expansions of model theory from the traditional confines of classical first-order predicate logic, the very same functional calculus of Frege, Whitehead, Russell, and Gödel, into infinitary systems of logic or even shedding model theory of any pretense of formal mathematical logic.

The formal semantics underpinning the disciplinary practice of model theory was proposed by A. Tarski, who also provided some of the first key results in model theory. It should not be surprising that model theory continues to subscribe to this system of semantics and its logical calculus. The dominant perspective in the philosophy of mathematics of structuralism accedes to the semantics of classical first-order model theory, and key set-theoretic tools mesh well with the classical interpretations of logical connectives. Model-theoretic work based on alternative motivations such as intuitionistic and constructivist principles have become relegated to smaller efforts which concentrate mainly on foundational topics such as arithmetic and elementary analysis, if they manage to proceed beyond settling the semantics of nonclassical propositional systems.

Consequently, the term *model theory* captures two related but distinctly contrasting disciplines. In the realm of mathematical scrutiny, the label describes a highly technical discipline which integrates with various other parts of mathematics. These interactions have also expanded model theory beyond the traditional confines of mathematical logic into taking up patterns from algebra, number theory, and geometry. Part of these changes include shedding the logical artifice in favor of more generalized notions. In the other usage, especially in philosophically motivated studies, the use of model theory is to focus on the details of the semantics of logic, often because the formulation for other deductive calculi is more complicated than the setup used in classical model theory. Except in limited cases, it would be inappropriate to conflate these two pursuits into a common discipline of contemporary formal logic.

1.2 Motivations and Summary of the Work

This work seeks to incorporate the two distinct modes of model theory by exploring consequences of Galois correspondences that hold in the mathematical practice of model theory and using the principles for this framework to motivate a similar technical development using the semantics of a different class of logics. The class of nonclassical systems called substructural logic has been studied in various areas of philosophy, computer science, and mathematics. Substructural logics are characterized by the omission of some combination of the structural rules that are present in classical logic.

This general description can accommodate a range of familiar and relatively unknown systems. The coalescence of substructural logic is of relatively recent origin, but the history of substructural logic, such as one presented by K. Došen in [Došen 1993], recognizes a rich and fascinating history of its constituent branches. Even in current areas of inquiry, substructural logics are usually studied at the propositional level without quantification, so they have not been subject to sustained inquiry for a quantified model theory. This work seeks to rectify this gap by developing a formal semantics and a substructural model theory sophisticated enough to generate Galois correspondences that arise in classical first-order model theory.

First-order theories which hold the property of coding Galois finite sets carry a Galois correspondence between substructures of their models and groups of partial isomorphisms and elementary maps. This Galois correspondence was first observed by B. Poizat for theories having elimination of imaginaries, prompting the idea of an *imaginary Galois theory*. The coding of Galois finite sets, and the uniform variant of coding finite sets, is associated with the behavior of model-theoretic algebraicity in a given first-order theory, and so it is possible to formulate a logical Galois theory that generalizes the presentation of the algebraic Galois theory of fields.

In the language of rings, every theory of fields can code finite sets, so every such theory carries a logical Galois correspondence. This result invites comparisons between algebraic presentations and model-theoretic formulations of Galois theory where both are available. Although this work can be very straightforward, the slight differences between what the logical Galois correspondence can express about pure fields and valued fields compared to the connections that are studied in general mathematical practice are instructive about the ways in which the background logic places certain constraints.

Although Poizat introduced his imaginary Galois theory with an application to differential fields, A. Pillay developed a thorough investigation of differential Galois theory based on a model-theoretic approach. This previous development differs from this work by using the idea of differential algebraicity rather than model-theoretic algebraicity, which in this context of differential fields coincides with the pure field-theoretic understanding of algebraicity. Consequently, the differential Galois theory of this work is much more like the algebraic Galois theory of pure fields than the differential Galois theory of Pillay. This contrast highlights the impact that underlying assumptions made on the general framework of model-theoretic Galois theory can have when applying it to specific first-order theories.

This work devotes considerable attention to understanding a Galois theory of exponential fields by taking up the theory of pseudoexponentiation proposed by B. Zilber. Pseudoexponentiation developed as a model-theoretic response to the difficulties in resolving the algebraic properties of the exponential function. The complex exponential field $\mathbb{C}_{\exp}$ itself sustains mystery due to the unresolved conjecture of S. Schanuel, and a number of key algebraic properties depend on the answer to this conjecture. Indeed, one possibility yet beholden is that an exponential Galois theory may turn out to be a very impoverished system, where the only known automorphisms of the complex exponential field, the identity map and complex conjugation, are the only possible ones. In such a case, the fixed structure is the real exponential field $\mathbb{R}_{\exp}$.

Pseudoexponentiation seeks to address this uncertainty by endowing a field with a group homomorphism that behaves like exponentiation and filling in the missing gaps using geometric methods. Such exponential fields, depending on the literature, are called *pseudoexponential fields*. These pseudoexponential fields accept the problems associated with the model theory of $\mathbb{C}_{\exp}$ wrought by the definability of the integers $\mathbb{Z}$ and the associated consequences of the incompleteness of arithmetic. Despite this wild behavior, the theory of pseudoexponentiation carries a number of desirable properties, such as uncountable categoricity and quasiminimal excellence. A tantalizing conjecture by Zilber is that the complex pseudoexponential field is isomorphic to the true complex exponential field. Schanuel's conjecture, accepted to be a property of pseudoexponentiation, is a major gap in answering this question.

Because pseudoexponentiation is better understood than the theory of the complex exponential field, it is sensible to apply the model-theoretic framework for Galois correspondences to substructures of the complex pseudoexponential field. Despite the possible ambiguity, substructures of this model of the theory with the induced group homomorphism are also called pseudoexponential fields in this work, but this usage is to distinguish the possible difference between the canonical exponential function and the function arising from the model theory. If Zilber's conjecture holds, then the Galois theory of pseudoexponential fields spells out exactly the Galois theory of true exponential fields.

The framework provides only a general setup of the Galois theory, and the crucial granular details can only be gotten through the examination of the structures themselves. Although the theory of pseudoexponentiation carries a number of desirable model-theoretic properties, it continues to lack others, among them model completeness. The lack of quantifier elimination obscures the understanding of model-theoretic algebraicity in the context of pseudoexponentiation. Nevertheless, a result of J. Kirby, A. Macintyre, and A. Onshuus is that the fixed pseudoexponential field of the complex pseudoexponential field is a structure which has the field $\mathbb{Q}^{\mathrm{rab}} = \mathbb{Q}^{\mathrm{tr}} \cap \mathbb{Q}^{\mathrm{ab}}$ as its algebraic subfield. This algebraic field, called the real abelian numbers, is the algebraic core of the closed real abelian pseudoexponential field $\mathbb{Q}^{\mathrm{rab}\,C}$, which is this fixed pseudoexponential field, and concentrating on $\mathbb{Q}^{\mathrm{rab}}$ is a starting point in elaborating upon a Galois theory of exponential fields.

An abelian extension B of a structure A is one such that the Galois group $\mathrm{Gal}(B/A)$ is abelian. In the context of pure fields, the Kronecker-Weber theorem succinctly characterizes abelian extensions of the rational number field so that a field F is an abelian extension of $\mathbb{Q}$ if and only if F is a cyclotomic field or F is contained in a cyclotomic field. This result inspired the twelfth of D. Hilbert's problems which asks for a similar kind of result to the general case of number fields, with partial answers obtained through complex multiplication for totally imaginary quadratic fields and the development of CM fields. This work considers an exponential variant of this result as a source of motivation, the eventual goal being to determine which kinds of extensions of $\mathbb{Q}^{\mathrm{rab}\,C}$ have abelian groups.

As straightforward as it is to posit the goal for a Kronecker-Weber theorem in the context of pseudoexponentiation, tackling the task immediately is rather foolhardy. Determining which extensions of $\mathbb{Q}^{\mathrm{rab}\,C}$ are abelian requires an understanding of which extensions of $\mathbb{Q}^{\mathrm{rab}\,C}$ are model-theoretically algebraic. However, the theory of pseudoexponentiation does not have quantifier elimination, so model-theoretic algebraicity cannot be characterized in a quantifier-free manner as in the case of pure fields. Indeed, examining the functions that can be defined in the language of exponential rings without quantifier elimination gives a sense of the complexity of model-theoretic algebraicity in pseudoexponentiation. Nevertheless, a feasible starting point for laying out an understanding of abelian extensions of $\mathbb{Q}^{\mathrm{rab}\,C}$ is in going back to determining a characterization of pure field-theoretic abelian extensions of the pure field $\mathbb{Q}^{\mathrm{rab}}$, all in the context of pure fields. This task is a nontrivial one since $\mathbb{Q}^{\mathrm{rab}}$ itself is not a number field, and the developments along the lines of Hilbert's twelfth problem do not immediately provide for a solution.

Thus, an analogous result of the Kronecker-Weber theorem is obtained for pure fields which are abelian extensions of the pure field $\mathbb{Q}^{\mathrm{rab}}$ of real abelian numbers. This result is obtained by determining what the Galois group $\mathrm{Gal}((\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{rab}})$ looks like, and it is shown that the group takes the form

$$\mathrm{Gal}((\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{rab}}) = 2^{\omega} \times \hat{\mathbb{Z}}^{\omega}.$$

Model-theoretic algebraicity in the context of pseudoexponentiation is not fully understood, but there is a standard result with respect to models of the theory of pseudoexponentiation constituting a quasiminimal excellent class that can be used to convert certain pure field automorphisms fixing a field F into pseudoexponential field automorphisms which continue to fix F. This result is then used to show that $\mathrm{Gal}((\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{rab}})$ is a particularly distinguished component, specifically a factor group, of the maximal abelian group $\mathrm{Gal}((\mathbb{Q}^{\mathrm{rab}\,C})^{\mathfrak{ab}}/\mathbb{Q}^{\mathrm{rab}\,C})$ of the fixed pseudoexponential field.

The work with exponentiation concludes with a relatively benign exercise discussing exponential fields as two-sorted structures. Two-sorted exponential fields usually contain two maps between sorts, one being identity and the other the exponential function, but removing the identity map creates a situation that yields a particularly well-behaved Galois theory. Originally, it was envisioned that this Galois theory would be a transitional stage between the Galois theory of pure fields and the Galois theory of pseudoexponential fields, but the results of this exercise suggest otherwise.

The model-theoretic Galois theory presented in this work is formulated as a straightforward generalization of the algebraic Galois theory of pure fields in the sense that algebraicity as defined in the classic Galois theory is taken over by model-theoretic algebraicity. Although this framework has its issues with, for example, differential Galois theory, it also provides a relatively straightforward system which can then be used to motivate the development of a substructural model theory.

The semantics of substructural logics is provided here using an approach which seeks to preserve as much of the standard semantics of classical first-order predicate logic as possible. Terms continue to be interpreted by a domain of individuals, but formulae become interpreted as points in another algebraic entity. Truth values in classical logic become *propositions* in substructural logic, and orderings on the algebra of propositions provide a way to designate certain propositions to be true and others to be deemed false.

The fundamental theorems of Soundness and Completeness are provided with respect to this system of semantics. Then, standard results of classical first-order model theory are provided with as much generality as possible. Substructural variants of Compactness and the Löwenheim-Skolem theorems constitute the core of this classic model theory. A brief discussion is provided on substructural variants of reduced products and a question is offered on the substructural variants of Horn sentences.

This development of classic model theory is followed by examinations into the notions of definability and algebraicity in the context of substructural model theory. This second stage covers topics found in contemporary model theory. Useful notions of definability and algebraicity of individuals are the foundational basis for the logical correspondence, and the work introduces substructural understandings of both concepts. In addition, the work provides some remarks on adapting the notion of n-types to substructural logics, and substructural analogues to partial and complete types are provided in the form of weak and structural types.

With the substructural model theory sufficiently explored, a substructural adaptation of model-theoretic Galois theory is provided. The classical source of inspiration is that of a particular presentation by A. Medvedev and R. Takloo-Bighash. The adaptation is for the most part straightforward, especially since the crucial idea of theories coding finite sets, which can be understood in syntactic terms, carries over to the substructural model theory with little modification.

This work concludes with some thoughts and considerations about subsequent work that may be done on the topics that have been covered. Continued investigation into the real abelian numbers and into the theory of pseudoexponentiation can provide more insight into the Galois theory of pseudoexponential fields. The unresolved issues in the substructural model theory provide fertile ground for continued exploration and codification. The difficulties found in the substructural model theory echo similar issues arising from the model-theoretic work on abstract elementary classes. This latter approach can invite continued collaboration between the mathematical practice of model theory and the model theory as based on the semantics of formal logic.

1.3 Fundamentals of Formal Logic

Since this work deals with a number of different deductive calculi, it is sometimes necessary to tinker with the usual conventions of formal logic found in model-theoretic discourse. It is helpful to lay out the fundamental preliminaries of formal logic in order to establish a common foundation for subsequent discussion. The work of formal logic is divided between syntax and semantics. The syntax gives rise to the necessary proof theory which shapes formal logic as a calculus of symbols and as an algebra defined over some language where well-established rules determine the correct manipulation of these symbols. The semantics of logic provides meaning to these symbols and these rules of manipulation. Formally, this meaning is provided by models which map symbols in the syntax to entities which can be understood outside the logical context.

Therefore, in order to set about the syntax and semantics of logic, it is necessary to formally define a language which provides the requisite symbols. In general parlance, a language is defined to be a set $\mathscr{L}$ consisting of logical symbols and nonlogical symbols. Logical symbols are those contained in every language, except when there is a particular purpose in excluding them, and they are intended to always carry a consistent interpretation as determined by the rules and axiom schemata of formal logic. These symbols include the quantifiers $\dot{\forall}$ and $\dot{\exists}$, individual variables $\{\dot{v}_i\}_{i\in\mathbb{N}}$, and the propositional connectives $\dot{\rightarrow}$, $\dot{\wedge}$, $\dot{\vee}$, $\dot{\circ}$, and $\dot{\neg}$. The equality symbol $\doteq$ is regarded as a logical symbol in classical first-order predicate logic, but it is sometimes omitted. Standing in contrast to the logical symbols are the nonlogical ones which are governed by nonlogical axioms, so these symbols thus vary in how they are to be interpreted. The nonlogical symbols are also recognized as constituting a signature.

The logical calculi to be dealt with here are predicate logics, so a language $\mathscr{L}$ contains

1. constant symbols $C_{\mathscr{L}} := \{\dot{c}_i\}_{i\in\mathbb{N}}$,
2. function symbols $F_{\mathscr{L}} := \bigcup_{n\in\mathbb{N}} \{\dot{f}_i^n\}_{i\in\mathbb{N}}$ with n places of arguments for each $n \in \mathbb{N}$, and
3. predicate symbols $R_{\mathscr{L}} := \bigcup_{n\in\mathbb{N}} \{\dot{p}_i^n\}_{i\in\mathbb{N}}$ with n places of arguments for each $n \in \mathbb{N}$.

The collection of terms generated by $\mathscr{L}$ consists of each individual variable; each constant symbol; and where $\dot{t}_1, ..., \dot{t}_n$ are each terms, then each concatenation $\dot{f}_i^n \dot{t}_1 ... \dot{t}_n$ for n-place function symbol $\dot{f}_i^n$. The collection of atomic formulae generated by $\mathscr{L}$ consists of

1. $\doteq \dot{t}_1 \dot{t}_2$, for terms $\dot{t}_1$ and $\dot{t}_2$ and
2. where $\dot{t}_1, ..., \dot{t}_n$ are each terms, then each concatenation $\dot{p}_i^n \dot{t}_1 ... \dot{t}_n$ for n-place predicate symbol $\dot{p}_i^n$.

These concatenations of terms are considered to be well-formed sequences of terms.

Continuing on the construction of symbols, the collection Form($\mathscr{L}$) of well-formed formulae generated by $\mathscr{L}$ consists of

1. every atomic formula generated by $\mathscr{L}$;
2. where $\dot{\alpha}$ and $\dot{\beta}$ are well-formed formulae and $\dot{v}_1$ is an individual variable,
   (a) $\dot{\neg}\dot{\alpha}$
   (b) $\dot{\rightarrow}\dot{\alpha}\dot{\beta}$, $\dot{\wedge}\dot{\alpha}\dot{\beta}$, $\dot{\vee}\dot{\alpha}\dot{\beta}$, $\dot{\circ}\dot{\alpha}\dot{\beta}$, and
   (c) $\dot{\forall}\dot{v}_1\dot{\alpha}$ and $\dot{\exists}\dot{v}_1\dot{\alpha}$.

These concatenations are formally in prefix notation, which has the advantage of requiring no additional punctuation to resolve ambiguities in the syntax. On the other hand, prefix notation is tedious to work with, so the usual practice of infix notation where appropriate is adopted. Each symbol formally has a dot to indicate its status in the syntax; these dots will usually be omitted unless some sort of distinction between syntax and semantics is required. Individual variables generally take up italicized Roman letters at the end of the alphabet, such as x, y, z, u, v, and w. Constant symbols are usually denoted by the symbol usually used in practice, but when generality is called for, Roman letters at the beginning of the alphabet are used, such as a, b, c, and d. Similarly, function symbols will usually be the standard ones used in general practice, but arbitrary symbols may be denoted by Roman letters such as f, g, h, and their neighbors. Logical equality will be denoted by $\doteq$ in infix notation, and those predicate symbols whose interpretations are usually used in such fashion in normal usage follow this same practice in the object language.

Connectives are for the most part written in infix notation, and to avoid ambiguity, parentheses are used where necessary. The connectives have a hierarchy of scope, from the narrowest to widest in the order $\neg$, $\wedge$, $\vee$, $\circ$, and $\to$, to resolve ambiguity without need of parentheses. Well-formed formulae are generally represented by lowercase Greek letters, such as α, β, γ, δ, ϕ, ψ, σ, and τ. A variable x is said to be free in a formula α if it is not in the scope of a quantifier, and the formula may also be written $\alpha(x)$ to highlight the presence of x. If t is some other term, then $\alpha(t)$ denotes the allowable replacement of the free variable x with the term t. Let $\mathrm{Form}(\mathscr{L})[n]$ denote the subset of all $\mathscr{L}$-formulae with n free variables. The cardinality of a language $\mathscr{L}$ is determined by the cardinality of the set $\mathrm{Form}(\mathscr{L})$, so a language is countable if and only if the set of formulae generated by it is countable.

Let $\mathscr{L}$ be a language. A logical calculus **L** is determined by a particular grouping of inference rules and axiom schemata. The relation $\vdash_{\mathbf{L}}$ of formal logical derivability in the logic **L** is explicated as follows. Let $\Sigma \cup \{\tau\}$ be a collection of well-formed formula in some language $\mathscr{L}$. Then, τ is **L**-derivable from Σ, denoted $\Sigma \vdash_{\mathbf{L}} \tau$, if there is a finite sequence of well-formed formulae with the last sequent as τ and each sequent formula being either a formula in Σ, an instance of an axiom schema in **L**, or the application of a rule in **L**. An $\mathscr{L}$-formula α is an **L**-theorem if and only if $\emptyset \vdash_{\mathbf{L}} \alpha$. Let $\mathrm{Thm}_{\mathbf{L}}(\mathscr{L})$ denote the set of **L**-theorems. Then, a logic **L** differs from another logic **M** because of different rules that govern what each logic considers to be the proper manipulation of symbols. More exactly, they distinguish between each other because of the set of theorems that each logic accepts, so that **L** differs from **M** if and only if $\mathrm{Thm}_{\mathbf{L}}(\mathscr{L}) \neq \mathrm{Thm}_{\mathbf{M}}(\mathscr{L})$.

There are different ways of presenting the inferences rules and axiom schemata of a logic **L**. Gentzen-style sequent calculi are usually used in certain branches of substructural logic since they make explicit use of structural rules and can demonstrate how certain substructural logics lack such structural rules. Running counter to this conventional practice, the preference in this work is to develop the presentation using Hilbert-style rules and axiom schemata, an approach often used in developing classical first-order model theory and one which has been used in the relevance branch of substructural logic. Since much of the discussion of substructural logics includes later the consideration of the axiom schemata and inference rules, an enumeration of them is omitted here. In their presentation, α, β, γ, and δ are taken to be arbitrary well-formed formulae, and x is taken as an arbitrary individual variable. Axiom schemata are distinguished by having nothing to the left of $\vdash$, indicating that these formulae stand alone and without the need for additional assumptions; in this way, axioms are taken to be theorems of an especially recognizable kind. Rules have formulae on both sides of $\vdash$, indicating that rules are invoked only when the requisite assumptions to the left of $\vdash$ are met.

With the discussion of different systems of formal logic, it becomes necessary to give them labels. The labels **L** and **LQ** are taken to be of general usage, with the latter especially used when the language includes quantifiers. Specific labels are provided alongside their deductive calculi in this work. Sometimes, the discourse may rely on a language that includes a smaller set of connectives and other logical symbols, and the notation will reflect this change. For a logic **L**, the language without negation is called the positive fragment of $\mathscr{L}$ and denoted $\mathbf{L}_+$. The implicative fragment intuitively contains only $\to$ as a connective, although for practical purposes it is harmless to include its companion connective $\circ$ to form the intensional fragment. The label $\mathbf{L}_\to$ is used to denote the implicative fragment, and $\mathbf{L}_\circ$ is used for the intensional fragment containing $\circ$ and $\to$ when it is necessary to absolutely distinguish this difference.

Let $\mathscr{L}$ be a language. Then, $\mathscr{L}$-theories will usually be denoted using T in general usage, but specific theories will have their own labels. Models of $\mathscr{L}$-theories, and mathematical structures taking up the form determined by the semantics, are generally called $\mathscr{L}$-structures in the context of classical model theory. For substructural logics, where choice of deductive calculus may be needed or where it is useful to distinguish them from the structures of classical semantics, such models may be called $\mathscr{L}$-substructural structures or $\mathscr{L}(\mathbf{LQ})$-models. These semantical objects are usually denoted using Fraktur letters such as $\mathfrak{A}$, $\mathfrak{B}$. Models of an $\mathscr{L}$-theory may also be denoted $\mathfrak{M}$ or $\mathfrak{N}$ to emphasize this relationship.

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are all standard, and they stand for the natural numbers, integers, rational numbers, real numbers, and complex numbers respectively. These number systems may have certain structural features as demanded by context. Contrary to conventional usage in logic, the natural numbers $\mathbb{N}$ in this work usually excludes the number 0, especially with respect to indices which consequently start with 1.

References on occasion are also made to the abstract notion of *Galois connections*. A Galois connection is an entity consisting of two partially ordered objects $(A, <_A)$ and $(B, <_B)$ and two operators $f : (A, <_A) \to (B, <_B)$ and $g : (B, <_B) \to (A, <_A)$ such that for all $a \in A$ and $b \in B$, it is the case that

$$a <_A g(b) \Leftrightarrow b <_B f(a)$$

when f and g are meant to be order-reversing or

$$a <_A g(b) \Leftrightarrow f(a) <_B b$$

when f and g are meant to be order-preserving. Galois connections have been used in the semantics of propositional substructural logics, specifically through systems called *gaggles*; one example of such development may be found in [Bimbó Dunn 2008]. Galois connections have also interacted with the semantics of logic through categorical logic, in which the syntax and semantics of formal logic are themselves regarded as categories. In category theory, Galois connections manifest as a pair of adjoint functors between two categories which themselves arise from partially ordered objects, and taking up a category of all $\mathscr{L}$-theories and a category of all $\mathscr{L}$-structures, for a common language $\mathscr{L}$, yields a Galois connection between syntax and semantics connecting together theories and their models. However, this work primarily exploits the notion of Galois connections through its original source of inspiration, the Galois correspondences of Galois theory. Still, the basic idea behind Galois connections permeates this work in a variety of ways.

Part I

Model Theory over Substructural Logics

Chapter 2

Substructural Logic

2.1 The Syntax of Substructural Logics

The term *substructural logic* was first proposed in 1990 at a meeting held in the *Seminar für natürlich-sprachliche Systeme* of the University of Tübingen. The proceedings of that meeting contributed to the recognition that various traditions of logic held common themes and result in the publication of [Došen Schroeder-Heister 1993]. Substructural logics are systems of formal deductive calculi which reject some or all structural rules of classical logic. The resultant calculi are deductively weaker than the standard system. Structural rules are syntactically most visible with the sequent calculus, where structural rules determine the correct manipulation of sequent structures; substructural logics are hence substructural in the sense that they accept fewer structural rules than classical logic. The conceptualization of structural rules arises in Gentzen-style sequent calculi, and indeed, the adoption of the appellation substructural logic is due to an understanding of logical calculi in terms of sequents and sequent structures. In the familiar Hilbert-style presentation of axiom schemata and inference rules, the structural rules affect the behavior of the $\rightarrow$ connective, the connective symbol for implication or conditional. This presentation is used ere, in light of its use in the branch of substructural logic that developed through the study of relevance logic as well as its common use in the development of classical first-order model theory.

A standard axiomatic presentation of classical propositional logic consists of the following three axiom schemata and one inference rule, where α, β, and γ are taken to be arbitrary well-formed formulae.

1. $\vdash \alpha \rightarrow (\beta \rightarrow \alpha)$

2. $\vdash (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$

3. $\vdash (\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta)$

4. $\alpha \rightarrow \beta, \alpha \vdash \beta$

The language includes two connectives $\neg$ and $\rightarrow$ which are taken to be primitive, with other connectives such as $\wedge$, $\vee$, and $\leftrightarrow$ defined on the primitive ones. Notably, only one axiom schema makes any mention of the $\neg$ connective; everything else is really about the conditional connective $\rightarrow$, and indeed, the first two axiom schemata are powerful enough to exhibit the structural rules carried in classical logic. Hence, most substructural logics reject at least one of these axioms in their own axiomatic presentation. This rejection means that these logics cannot define certain standard connectives like $\wedge$ and $\vee$ in terms of the others, necessitating a larger stock of primitive connectives and axiom schemata to describe them.

2.1.1 The Basics of the Conditional Connective

The distinguishing feature of substructural logics is their adoption of different understandings of the conditional connective $\rightarrow$. In the study of substructural logics, it is necessary and appropriate to establish a minimal intuitive sense of what the $\rightarrow$ connective conveys when used in a formula which serves as a starting point from which different substructural logics can distinguish themselves. For this purpose, let the object language of discourse include only the $\rightarrow$ connective.

A statement of the form $A \rightarrow B$ consists of the antecedent A and consequent B. The arrow of the connective indicates a sense of flow in reasoning from one direction to another; fulfilling the antecedent should lead to fulfilling the consequent, but fulfilling the consequent may say nothing about the antecedent. This reasoning motivates a reading of the inference rule commonly called *modus ponens*. Hence, the inference rule, which is given a different appellation in this work that conforms to general patterns in proof systems, will be a universal feature of the substructural logics.

Another minimal intuitive sense is that a collection of conditional statements can sometimes form a chain of reasoning, with each statement a link in the chain. If $A \rightarrow B$ and $B \rightarrow C$ are accepted to be true, then it makes sense to say that $A \rightarrow C$ is also true. More generally, if one accepts that $A \rightarrow B$ and $C \rightarrow D$ are true, then it is sensible to accept the claim that having $B \rightarrow C$ be true leads to the acceptance of $A \rightarrow D$. Antecedents of antecedents are antecedents of consequents of consequents, and consequents of consequents are consequents of antecedents of antecedents. Along the same track, it seems reasonable to accept that any statement can be the antecedent or consequent of itself, although statements of the form $A \rightarrow A$ are rather lacking in meaning. While there are different logical backgrounds which do reject some of these basic intuitive arguments, the substructural logics of this work always incorporate these principles into their proof theories.

Let $\mathscr{L}$ be a language capable of building up terms, atomic formulae, and well-formed formulae in the usual manner with only $\rightarrow$ as a connective, and let α, β, γ, and δ be arbitrary well-formed formulae. Then, the following inference rules and axiom schema constitute the most basic substructural implicational logic.

1. ($\rightarrow$-Detachment) $\alpha \rightarrow \beta, \alpha \vdash \beta$

2. ($\rightarrow$-Attachment) $\alpha \rightarrow \beta, \gamma \rightarrow \delta \vdash (\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \delta)$

3. ($\rightarrow$-Identity) $\vdash \alpha \rightarrow \alpha$

The rule $\rightarrow$-Detachment is of course the familiar *modus ponens*. The axiom schema $\rightarrow$-Identity encapsulates the reasonable thought that any statement given as an assumption can be in turn provided as a conclusion. As with other axiom schemata in the study of substructural logics, it has an alternative label derived from combinatory logic; it may also be referred to as I.

The rule $\rightarrow$-Attachment formalizes the intuitive sense that the connective $\rightarrow$ allows for (certain) chains of correct reasoning. The rule also highlights the key difference between inference rules and axiom schemata. Inference rules are essentially conditional principles of logical reasoning that depend on specific conditions for them to be invoked. Axioms are inference rules without such conditions; they may be invoked at any time. This view also motivates the presentation of the axiom schemata, where there is nothing before $\vdash$ to signal this liberation. The rule $\rightarrow$-Attachment has axiomatic counterparts that will bolster the idea of chains of correct reasoning as a stronger principle.

Let this system be called the basic implicational logic $\mathbf{B}_{\rightarrow}$, although shortly this calculus will be harmlessly modified to accommodate certain technical issues that arise with respect to substructural logics. It lacks any other connectives, so the presence of the $\rightarrow$ connective in its label is appropriately sensible. It can also be identified by the set of theorems produced by it which are essentially every formula of the schema $\rightarrow$-Identity.

2.1.2 Rudimentary Semantics

Since the overall focus of this work is on semantics, it may be appropriate to sketch out how the changes in syntax affect the behavior of the semantics for the substructural logics. For a language $\mathscr{L}$ there are structures and models that will appropriately interpret the symbols in $\mathscr{L}$ and reflect the principles required for a specific logic $\mathbf{L}$.

An $\mathscr{L}$-substructural structure $\mathfrak{A}$ will consist of four components.

1. The domain $D_{\mathfrak{A}}$ of individuals is the aggregate containing the interpretations of terms. In particular, the interpretations of constant symbols, among others, are individuals in $D_{\mathfrak{A}}$. Function symbols are interpreted to be functions in the domain of individuals, and individual variables are assigned to individuals of this domain. Essentially, this aspect of the semantics changes little from the standard semantics of classical first-order predicate logic.

2. The algebra $K_{\mathfrak{A}}$ of propositions is used to interpret formulae and constitutes the key difference in the semantics of substructural logics from classical logic. A predicate symbol is interpreted to be a function

from the domain of individuals to the algebra of propositions. The collection of propositions constitutes an algebra with the logical connectives interpreted as operators in this algebra. The operators behave differently depending on the logic involved, so different logics will yield different algebraic structures. The connective $\rightarrow$ will be interpreted by an operator denoted $\searrow$. Quantifiers impose additional conditions on the algebra so that universal and existential statements can be modeled successfully.

3. The interpretation function $I_{\mathfrak{A}}$ will be the primary way of discussing how terms and formulae are interpreted. Well-formed formulae are interpreted in a recursive manner connecting the connectives to the operators and the quantifiers to certain conditions on the algebra. Atomic formulae of n free variables are interpreted as functions taking up n individuals as arguments and mapping them to an element in the algebra of propositions. Terms are interpreted as individuals.

4. The collection $\nabla_{\mathfrak{A}}$ of accepted or designated propositions. Since well-formed formulae are not mapped to blatant truth values, it is necessary to come up with some other way of distinguishing those formulae that are regarded as true from those that are not. The most straightforward way of doing so is to designate specific propositions to be accepted, designated, or satisfied; a formula mapped to an accepted proposition would be interpreted to be satisfied. The set $\nabla_{\mathfrak{A}}$ includes the interpretations of every axiom schemata in **L** and is closed with respect to the inference rules of **L**.

For the basic substructural implicational fragment $\mathbf{B}_{\rightarrow}$, the algebra of propositions will be meager and essentially formless. The set of accepted propositions would reflect that only formulae of form $\alpha \rightarrow \alpha$ are theorems. Without more information, the sense of the operator $\searrow$ is not really clear. It is algebraically convenient to introduce another operator such that the operator $\searrow$ would be a residuation operator to this new operator $\otimes$. This semantic relationship would be reflected syntactically between $\rightarrow$ and a new connective $\circ$ commonly called fusion.

To do so, the algebra of propositions is better-defined so that there is a relation $\leq \subseteq K_{\mathfrak{A}} \times K_{\mathfrak{A}}$ and two operators $\otimes$ and $\searrow$ so that, for arbitrary propositions a, b, and c,

$$a \otimes b \leq c \text{ if and only if } a \leq b \searrow c.$$

This property is the residuation condition, demonstrating that $\searrow$ is a (right) residual to $\otimes$. Fusion and the substructural conditional connective constitute the two main *intensional* connectives, in contrast to the main *extensional* connectives of $\wedge$ and $\vee$.

Fusion needs a standard set of rules to correctly align the conditional as its residual. For $\mathbf{B}_{\rightarrow}$, the following rules can be used.

1. (Residuation Rule I) $(\alpha \circ \beta) \to \gamma \vdash \alpha \to (\beta \to \gamma)$
2. (Residuation Rule II) $\alpha \to (\beta \to \gamma) \vdash (\alpha \circ \beta) \to \gamma$

For other substructural logics with stronger structural rules, the following axioms may be used instead.

1. (Residuation Axiom I) $\vdash ((\alpha \circ \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))$
2. (Residuation Axiom II) $\vdash (\alpha \to (\beta \to \gamma)) \to ((\alpha \circ \beta) \to \gamma)$

The only intention for the inclusion of $\circ$ is so that the semantics can express the $\otimes$ operator. Because its syntactic behavior is given through its relationship with $\to$, fusion does not really expand the expressivity of the logic involved. Thus, it is safe to take the pair of connectives and consider them together as one unit, except for the case of intuitionistic and classical logics.

2.1.3 A Spectrum of Intensional Logics

Although substructural logics are recognized in contrast to classical logic, it is also possible to consider classical logic as a maximal substructural logic, a system that cannot accept additional structure without trivializing the entire deductive apparatus. The material conditional of classical logic can be axiomatized as an implicational logic fragment by adding to $\mathbf{B}_{\to}$ the following axioms.

1. ($\to$-Irrelevance) $\vdash \alpha \to (\beta \to \alpha)$
2. ($\to$-Distribution) $\vdash (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
3. ($\to$-Peirce's Law) $\vdash ((\alpha \to \beta) \to \alpha) \to \alpha$

The axioms $\to$-Irrelevance and $\to$-Distribution have the labels K and S, respectively, from combinatory logic. The Heyting implication of intuitionistic logic differs from the material conditional of classical logic in that the former rejects Peirce's Law. Let $\mathbf{H}_{\to}$ denote the intuitionistic implicational fragment and $\mathbf{PC}_{\to}$ denote the classical implicational fragment. In these cases, the residuation condition between $\to$ and $\circ$ would force fusion to behave as the usual notion of conjunction $\wedge$.

In between the basic implicational logic and intuitionistic implicational logic is a spectrum of substructural implicational logics that accept or reject assortments of structural rules. Indeed, it would not be sensible to try to describe every one of them. However, a number of them are notable because they highlight specific combinations of structural rules and have been scrutinized in various areas of philosophy, computer science, linguistics, and other areas of mathematics.

Let $\mathscr{L}$ be a language with connectives $\to$ and $\circ$. The following axiom schemata describe the key structural rules that classical logic holds which are selectively accepted by various substructural logics.

1. I (→-Identity) $\vdash \alpha \to \alpha$

2. B (→-Prefixing) $\vdash (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

3. B' (→-Suffixing) $\vdash (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

4. C (→-Permutation) $\vdash (\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$

5. W (→-Contraction) $\vdash (\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$

6. S (→-Distribution) $\vdash (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

7. K (→-Irrelevance) $\vdash \alpha \to (\beta \to \alpha)$

Now, since $\mathbf{B}_\circ$ accepts I, all of the substructural logics here also accept I. Adding B and B' to $\mathbf{B}_\circ$ yields the implicational logic $\mathbf{TW}_\circ$ of "contractionless ticket entailment," with ticket entailment $\mathbf{T}_\circ$ obtained by also accepting W. The relevance logic $\mathbf{R}_\circ$ accepts B, B', C, W, and S; it too has a contractionless counterpart $\mathbf{RW}_\circ$ that rejects W.

The axiom schema →-Contraction is noteworthy because its rejection is characteristic of linear logic **LL** and other logics of resource management. Without this axiom, the number of occurrences of a formula in the antecedent of a conditional statement has a significant meaning, and these logics can be used to count out and quantify the dependence a particular conclusion may have upon specific statements or resources. Linear logic itself contains additional connectives beyond the standard inventory of mathematical logic, but its implicational fragment is essentially contractionless ticket entailment $\mathbf{TW}_\circ$.

Relevance logic **R** is given the name because it is one of the strongest deductive calculi with the relevance property.

Definition 2.1 (Relevance Property) A logic **L** has the relevance property if every **L**-theorem of the form $\phi \to \psi$ has the feature that both ϕ and ψ have a common subformula χ.

This is part and parcel with the descriptive name for the axiom schema K, →-Irrelevance. This axiom immediately leads to *irrelevance*, permitting theorems that refute the relevance property.

2.1.4 Negation and the Conditional

Although substructural logics distinguish themselves in their treatment of the conditional connective, some of the changes also affect the role of negation with respect to conditional statements. In classical logic, there are three notable properties that highlight the interaction between the two connectives.

1. (Double Negation I) $\vdash \alpha \to \neg\neg\alpha$

2. (Double Negation II) $\vdash \neg\neg\alpha \to \alpha$

3. (Contraposition I) $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$

4. (Contraposition II) $\vdash (\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$

5. (Reductio I) $\vdash (\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$

6. (Reductio II) $\vdash \neg\alpha \to (\alpha \to \beta)$

A notable difference between classical and intuitionistic logic is the latter's rejection of Double Negation II and Contraposition II. With this exception of intuitionistic logic, most discussion of substructural logics accepts the classical principles of double negation and contraposition. The principle of *reductio ad absurdum* is usually derivable in some form or another in the presence of these two principles, although the weaker substructural logics can require them to be included as axiom schemata to be accepted theses.

A notable alternative to these familiar properties of negation is the idea of connexive implication, in which the conditional connective is distinguished through the negation property with one or more of the following properties.

1. (Aristotle Rule) $\alpha \to \alpha \vdash \neg(\alpha \to \neg\alpha)$

2. (Boethius Rule) $\alpha \to \beta \vdash \neg(\alpha \to \neg\beta)$

3. (Aristotle Axiom I) $\vdash (\alpha \to \alpha) \to \neg(\alpha \to \neg\alpha)$

4. (Aristotle Axiom II) $\vdash \neg(\alpha \to \neg\alpha) \to (\alpha \to \alpha)$

5. (Boethius Axiom I) $\vdash (\alpha \to \beta) \to \neg(\alpha \to \neg\beta)$

6. (Boethius Axiom II) $\vdash \neg(\alpha \to \neg\beta) \to (\alpha \to \beta)$

As the invocation of Aristotle and A.M.S. Boethius may suggest, connexive implication is based on principles implied in the *Organon* of Aristotle, specifically the *Prior Analytics* [Aristotle], and the Boethian *De Hypothetico Syllogismo* [Boethius]. Within the context of just the two connectives $\to$ and $\neg$ (and fusion $\circ$), connexive implication holds considerable appeal as a reasonable principle of logic, but it has considerable difficulty reconciling with other reasonable logical principles underlying the extensional connectives. This work will generally avoid confronting these issues by making only an occasional note on this exotic branch of nonclassical logics.

2.1.5 The Extensional Connectives

The discussion of the negation connective $\neg$ cannot be complete without introducing the other major binary connectives of formal symbolic logic and how they fit into the overall picture. The conjunction $\wedge$ and disjunction $\vee$ connectives are collectively referred to as the extensional connectives, in contrast to the intensional connectives of substructural implication and fusion.

The segregation between intensional and extensional connectives is deliberate in part because the changes in the calculi happen almost entirely with the intensional connectives. This is similar to the case of intuitionistic logic, where the extensional connectives $\wedge$ and $\vee$ are left unaltered. Hence, the following theses serve as appropriate axiom schemata for conjunction and disjunction for all of the logics.

1. ($\wedge$-Adjunction) $\alpha, \beta \vdash \alpha \wedge \beta$
2. ($\wedge$-Introduction) $\vdash ((\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma)) \rightarrow (\alpha \rightarrow (\beta \wedge \gamma))$
3. ($\wedge$-Elimination I) $\vdash (\alpha \wedge \beta) \rightarrow \alpha$
4. ($\wedge$-Elimination II) $\vdash (\alpha \wedge \beta) \rightarrow \beta$
5. ($\vee$-Introduction I) $\vdash \alpha \rightarrow (\alpha \vee \beta)$
6. ($\vee$-Introduction II) $\vdash \beta \rightarrow (\alpha \vee \beta)$
7. ($\vee$-Elimination) $\vdash ((\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma)$

In classical logic, the formula $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$ serves as an axiom to introduce the $\wedge$ connective, but substructural logics must reject it. In its place is a cumbersome axiom and the reasonable rule of adjunction.

Missing from the above axiom schemata is how $\wedge$ and $\vee$ interact with each other, namely through some sort of distribution.

1. ($\wedge\vee$-Distribution) $\vdash (\alpha \wedge (\beta \vee \gamma)) \rightarrow ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$

Although there are arguments made against the acceptance of distribution, it will turn out to be necessary for some of the model-theoretic work to flow successfully, so it will be an axiom in all of the logics here. With its inclusion, conjunction and disjunction behave exactly as they would in classical logic.

2.1.6 De Morgan Laws

Continuing the discussion of the negation connective requires outlining its interaction with the extensional connectives. Classical logic accepts the De Morgan laws as another way of showing the relationship between conjunction and disjunction. Intuitionistic logic rejects some aspects of these laws, but the substructural logics in general accept them whole, as follows.

1. ($\wedge\vee$-De Morgan I) $\vdash \neg(\alpha \wedge \beta) \to (\neg\alpha \vee \neg\beta)$
2. ($\wedge\vee$-De Morgan II) $\vdash (\neg\alpha \vee \neg\beta) \to \neg(\alpha \wedge \beta)$
3. ($\vee\wedge$-De Morgan I) $\vdash \neg(\alpha \vee \beta) \to (\neg\alpha \wedge \neg\beta)$
4. ($\vee\wedge$-De Morgan II) $\vdash (\neg\alpha \wedge \neg\beta) \to \neg(\alpha \vee \beta)$

Indeed, just as the material conditional describes implication in classical logic and Heyting implication the conditional for intuitionistic logic, it will be suitable to call the negation of substructural logics to be De Morgan, to distinguish from the more powerful effects of Boolean negation in classical logic. The difference between Boolean and De Morgan negation arises from the differences in the strength of the conditional between substructural and classical logics, however, so it is actually not necessary to include the De Morgan principles as actual axiom schemata.

In contrast to De Morgan negation is an alternative borne out of constructivist principles. Intuitionistic logic has a negation that acts less as a refutation of a statement and more as a denial of that statement's justification. The principle of constructive negation is that a negative statement must be justified in the same manner as a positive statement is justified in intuitionistic logic, and the acceptance of constructive negation requires a slightly different axiomatization of the axioms for $\neg$. Notably, an axiomatization of constructive negation requires the inclusion of the classical principles of double negation, differing from intuitionistic logic, and the De Morgan laws as schemata. A key difference in this regard would be the omission of the contraposition axioms, differing from the Boolean negation of classical logic, De Morgan negation of the conventional substructural logics, and the negation of intuitionistic logic.

2.1.7 Quantifiers

The quantifiers $\forall$ and $\exists$ are often neglected in the discussion of substructural logics due to the particular attention paid upon the propositional connectives. The work here is intended to address the resultant gap. Like the extensional connectives, the universal and particular quantifiers hold little controversy here, and indeed, they essentially act as generalized conjunction and disjunction for their relevant instantiations. Thus, the following axioms will be universally accepted.

1. ($\forall$-Instantiation) $\vdash \forall x\alpha(x) \to \alpha(t)$, where t is a term free for x in α
2. ($\forall$-Distribution) $\vdash \forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)$

3. ($\forall$-Generalization) $\vdash \alpha \to \forall x\alpha$, for x not free in α

4. ($\forall$-Confinement) $\vdash \forall x(\alpha \vee \beta) \to (\alpha \vee \forall x\beta)$

5. ($\exists$-Generalization) $\vdash \alpha(t) \to \exists x\alpha(x)$, where t is a term free for x in α

6. ($\exists$-Distribution) $\vdash \forall x(\alpha \to \beta) \to (\exists x\alpha \to \exists x\beta)$

7. ($\exists$-Instantiation) $\vdash \exists x\alpha \to \alpha$, where x is not free in α

8. ($\exists$-Confinement) $\vdash (\alpha \wedge \exists x\beta) \to \exists x(\alpha \wedge \beta)$, where x is not free in α

Being essentially extensional connectives, it may be of interest to pursue intensional counterparts to $\forall$ and $\exists$ to understand a more intensional variety of quantified model theory. The model theory here, however, focuses exclusively on the extensional notion of quantifiers.

2.1.8 Equality and its Predicate Symbol

In classical logic, the symbol $\doteq$ for equality is usually treated as a logical symbol with a standard set of axioms. Unfortunately, fixing its behavior is rather problematic in the discussion of substructural logics. Like the axiom schemata for the conditional connective, different axiomatizations may be appropriate with different motivations. At minimum, however, the following axiom schema and rules will be accepted as a minimal understanding of equality.

1. (Reflexivity) $\vdash x \doteq x$

2. (Symmetry Rule) $a \doteq b \vdash b \doteq a$

3. (Transitivity Rule) $a \doteq b, b \doteq c \vdash a \doteq c$

4. (Indiscernibility Rule) $a \doteq b, \alpha(a) \vdash \alpha(b)$, where $\alpha(x)$ is atomic and $\alpha(b)$ may have b substituting for a in one or more places

Nevertheless, certain theories may benefit from more concrete formulations of equality. In such cases, it would not be inappropriate to vary the axiomatizations of the symbol to reflect this need.

2.1.9 Relevance Logic R and its Fragments and Extensions

With the publication by Gödel of a completeness result for the functional calculus in [Gödel 1930] and the incompleteness results for first-order arithmetic in [Gödel 1931], first-order predicate logic was shown to be a system that could balance the constraints of a formal calculus with sufficient expressivity to capture large portions of mathematical practice. Tarski exploited the classical principles of the calculus to develop the precursors of model theory, and the first textbook [Hilbert Ackermann] in modern formal logic, Hilbert and W. Ackermann's *Grundzüge der theoretischen Logik*, helped cement this logic become the standard system of choice.

Later in his career with [Ackermann 1956], Ackermann developed a formal system for strong implication which inspired a project by A.R. Anderson and N. Belnap to formalize a logic of entailment through [Entailment I] and [Entailment II]. These systems were motivated by objections to the material conditional of classical logic and the constructive implication of intuitionistic logic accepting thesis with no relevance between antecedents and consequents. The Anderson-Belnap project yielded the system known as relevance logic which became the centerpiece of the Australian school of relevantists. This school, together with other independent and collaborative efforts, played a prominent role in setting up the current understanding of the notion of substructural logics.

The relevance logic **R** is the centerpiece of a sustained program taken up by the Australian school of Relevantists which grew out of the Anderson-Belnap project of Entailment. Its implicational fragment $\mathbf{R}_{\rightarrow}$ differs from the intuitionistic fragment $\mathbf{H}_{\rightarrow}$ only in its rejection of K. A complete axiomatization of the quantifier-free relevance logic **R** includes the extensional connectives $\wedge$ and $\vee$ and the classical principles of negation, and the first-order system **RQ** is obtained by adding the standard rules for quantification.

Relevance logic is primarily characterized as a response to a philosophical objection to the material conditional of classical logic. Classical logic produces as theorems many statements that are intuitively difficult to accept as correct and true statements, particularly when they are conditional statements where the antecedent and consequent have no relationship with each other. The response is thus that a correct logical calculus should at minimum recognize that an acceptable conditional statement should have some relevance between the antecedent and consequent.

The term *relevance logics* also refers to a family of like-minded systems of which **R** is the most conspicuous. Notable ones include the logic **T** of ticket entailment and its contractionless counterpart **TW**. The class of relevance logics also include a similar system that, due to its unusual properties, is called semirelevant, the system **RM**. The logic of **RM** is characterized by adding to **R** the axiom schema $\vdash \alpha \rightarrow (\alpha \rightarrow \alpha)$, which can be called ($\rightarrow$-Self-Reference). It has no associated combinatory-logic-derived label. The logic **RM** is sometimes called semi-relevant due to some of the bizarre theorems it accepts.

2.1.10 Linear Logic LL

Linear logic developed as a response to the need for a logic which can keep track of what statements or resources are needed for a particular objective and how often such resources are invoked. The idea is that a formula of the form $\alpha \to \beta$ states that the resources of α are required to obtain β, and it would be necessary to distinguish between, for example, the formula $\alpha \to (\alpha \to \beta)$ from $\alpha \to \beta$ due to the double presence of α in the first formula. Linear logic is the most well-known logic of resource management. Although inspired by computational needs, the fragment of **LL** called the multiplicative-additive linear logic **MALL** is similar to **RW** by removing S and W as structural rules. Axiom schema S undermines the counting of resources while K muddles the distinction between resources and the end product.

The logic **MALL** differs from **RW**, however, in not allowing extensional conjunction and disjunction to distribute. This prevention yields, in the tradition of linear logic, additive and multiplicative understandings of conjunction and disjunction; in this situation, fusion $\circ$ is regarded as multiplicative conjunction while additive conjunction resembles the extensional $\wedge$. The computational background of linear logic includes the use of different symbols for the syntax and working primarily with Gentzen-style sequent calculi; some of the challenges in translating this approach to Hilbert-style calculi usually used with relevance logics may be found in the work [Avron 1988] of A. Avron.

The full system **LL** includes other connectives and modal operators geared towards fleshing out a system of resource management. For example, the modal operator ! allows its input to be used as a resource without the need to count out how much it is used, allowing a formula of the form $\alpha \to (!\beta \to \alpha)$ to be valid, and it has a dual ?. The full complement of linear logic is outside the scope of this work, and the only aspect of linear logic that may be applicable here would be its fragment **MALL**, or at least an understanding of it in its relation to **RW**.

2.1.11 Substructural Logics with K

Relevance logics are uniformly characterized by their rejection of K due its ability to introduce *irrelevance*, and linear logic also happens to reject K. On the other hand, it is not a necessary feature for a substructural logic to always reject K. Substructural logics accepting K consequently reject S lest the resulting system be intuitionistic. Of such systems, worthy of note is **CK**, the logic obtained by including K and C as primary axioms.

2.1.12 Łukasiewicz and Kleene Logics

The systems devised by J. Łukasiewicz can also be described as substructural logics. However, their syntactic characterization is difficult to pinpoint, and a semantic presentation is more illuminating. Łukasiewicz logics are systems stronger than **CK** that are also weaker than classical logic. The systems named after S.C. Kleene are also similar to Łukasiewicz but with an intuitionistic bent, rejecting the law of excluded middle.

2.1.13 Intuitionistic and Intermediate Logics

Intuitionistic logic **H** can be considered a substructural logic, as can classical logic **PC**, which would be the maximal nontrivial substructural logic. A salient point to consider then is the following property.

Proposition 2.2 There are infinitely many logics between intuitionistic logic and classical logic.

Such systems are called intermediate logics, because the deductive strength of these systems are between that of intuitionistic and classical logic. Consequently, these can also be considered substructural logics for the purposes of this work. Intuitionistic and classical logic differ from the proper substructural logics in that they normally do not include a fusion connective. Indeed, in these and the intermediate logics, fusion and extensional conjunction coincide.

2.1.14 The Material Conditional

The normative view that has been taken to this point is that classical logic is the maximal nontrivial substructural logic and that the material conditional is the most deductively powerful conditional connective possible. So expressive is the material conditional that it can be used to define all of the extensional connectives when paired with the negation connective $\neg$. The set $\{\rightarrow, \neg, \forall\}$ is thus regarded as a set of adequate logical connectives and quantifiers for quantified classical logic.

A contrasting view would be that classical logic actually lacks a true conditional connective. The argument would be that the material conditional is an imposter connective fashioned out of the extensional connectives. Rather than defining, for example, disjunction $\alpha \vee \beta$ as $(\alpha \rightarrow \beta) \rightarrow \beta$, one would instead define the material conditional $\alpha \rightarrow \beta$ using $\neg\alpha \vee \beta$. This alternative view may be interesting to ponder, but its impact here is rather limited. The motivations behind it, to define away the material conditional into a combination of the extensional connectives, though, will be useful when looking at first-order theories of known mathematical structures in a substructural context. Since the conditional connective changes meaning depending on the logic, a safe bet when comparing a particular theory over different background logics would be to avoid the use of the conditional connective to spell out the shape of the underlying mathematical structures.

One of the properties shared by all of the substructural logics being discussed, including intuitionistic logic but notably excluding classical logic, is the idea of conservative extensions and fragments. The discussion of the syntax of substructural logics began by looking only at the conditional connective, then slowly adding in other logical symbols until obtaining the full logical language. For a language $\mathscr{L}$, if $\mathbf{L}_{\rightarrow}$ is the implicative fragment of a logic $\mathbf{L}$, one may wonder what the relationship is between the theorems of $\mathbf{L}_{\rightarrow}$ and the theorems of $\mathbf{L}$ containing only $\rightarrow$.

The answer in all substructural logics but not in classical logic is that language expansions of substructural logics are conservative extensions of their fragments, so that, for example, if α is an **L**-theorem with connectives $\rightarrow$, $\wedge$, and $\vee$, then it is also an $\mathbf{L}_+$-theorem. A glaring counterexample in classical logic is Peirce's law, $\vdash ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$, the distinguishing feature between the material conditional, which satisfies it, and intuitionistic implication, which refutes it.

2.2 The Semantics of Substructural Logics

Having carried a brief discourse on the syntax of substructural logics, it is appropriate to proceed to the semantics. The system of semantics here aims to generalize the semantics of classical first-order predicate logic as much as possible so that those model-theoretic properties can carry through to the current setup without too much complication. Substructural logics are usually studied in their propositional forms, resulting in a relative dearth of sustained work on quantification, but some prior efforts inform the current approach. A previous personal foray proposing a primitive understanding of this approach to the semantics was done for the logic **RQ** in [Yim 2008] which avoided some of the technical issues that arise in modeling quantification, and a previous presentation by G. Restall in [Restall 1994] based on the content semantics of R. Brady as found in [Brady 1988], [Brady 1989], and [Brady 2006] provides important guidance and insight. Other precedents will be discussed and considered following the development of the semantics of this work.

2.2.1 The Basic Picture

Essentially, the semantics is designed so that terms and well-formed formulae of a provided language $\mathscr{L}$ are interpreted by an $\mathscr{L}$-substructural structure. Constant symbols and closed terms are interpreted to be individuals residing in a domain of individuals with function symbols interpreted to be functions in that domain. Predicate symbols are interpreted to be functions that map finite sequences of individuals to a second aggregate, the domain of propositions. An atomic formula with n free variables becomes a function accepting n arguments. Well-formed formulae are interpreted by recursion so that they too essentially act as functions mapping sequences of individuals to propositions. Notably, sentences are interpreted to be propositions.

In philosophical discourse, there is a distinction made between *sentences* and *propositions* which is reflected in how each term is used in logic. A sentence is a syntactic construction of particular well-formed shape, which in this logic is a well-formed formula with no free variables. A proposition is the corresponding meaning of that statement, so a semantic entity, but in the abstract logical context, this idea is not helpful in the formal setting. Treatises may be written about propositions as semantic objects, but it is ideal in this logical work to not read too much into the appellation except as a label with some attached properties.

The propositions have an algebraic structure over them, so that each connective has a corresponding operator. Thus, propositions become objects in an algebra, subject to those operators. In sufficiently strong substructural logics, a partial-order relation arises which also determines the behavior of the extensional connectives. The partial order imposes a lattice structure on the propositions, so that conjunction and disjunction can be interpreted as lattice meet and lattice join, respectively. The quantifiers then act as restricted generalized meets and joins, based on their instantiations with respect to the domain of individuals. Consequently, propositions have a value attached to them allowing for such comparisons.

Propositions as formal objects are implicitly present in classical logic. In particular, one can conceive of the two truth values $\{\text{True}, \text{False}\}$ as propositions, the set forming a two-element Boolean algebra. With substructural logics, and even in general Boolean models of classical logic, propositions become something more than mere truth values, so it becomes necessary to find a replacement for truth values so that an $\mathscr{L}$-substructural structure can speak of true propositions. This role is served by distinguishing a subset of the domain of propositions as the set of *designated*, *accepted*, or *satisfied* propositions.

2.2.2 The Semantics

The basic picture is formalized and made more exact with the following definition of an $\mathscr{L}$-substructural structure.

Definition 2.3 Let $\mathscr{L}$ be a language with V denoting the set of individual variables; connectives $\rightarrow$, $\circ$, $\wedge$, $\vee$, and $\neg$; and quantifiers $\forall$ and $\exists$. Then, an $\mathscr{L}$-substructural structure $\mathfrak{A}$ is an object $\mathfrak{A} := \langle D_{\mathfrak{A}}, K_{\mathfrak{A}}, \nabla_{\mathfrak{A}}, I_{\mathfrak{A}} \rangle$ such that

1. $D_{\mathfrak{A}}$ is the domain of individuals;
2. $K_{\mathfrak{A}}$ is the domain of propositions with transitive binary relation $\leq$ and operators $\searrow$, $\otimes$, $\bar{\wedge}$, $\bar{\vee}$, and $-$ such that
   (a) $\searrow$ is a right residual of the groupoid operator $\otimes$ so that $a \otimes b \leq c \Leftrightarrow a \leq b \searrow c$ for all a, b, c in $K_{\mathfrak{A}}$,
   (b) $\bar{\wedge}$ and $\bar{\vee}$ are lattice meet and join if $\leq$ is a partial order,
   (c) $-$ is a function;
3. $\nabla_{\mathfrak{A}} \subseteq K_{\mathfrak{A}}$ is the subset of designated propositions, a filter with respect to $\leq$, such that for a, b, c, and d in $K_{\mathfrak{A}}$,
   (a) $a \leq b$ and $a \in \nabla_{\mathfrak{A}}$ implies $b \in \nabla_{\mathfrak{A}}$,
   (b) $a \searrow a$ is in $\nabla_{\mathfrak{A}}$,

(c) $a \searrow b$ and $c \searrow d$ in $\nabla_{\mathfrak{A}}$ implies $(b \searrow c) \searrow (a \searrow d)$ in $\nabla_{\mathfrak{A}}$,

(d) $a \searrow b$ and a in $\nabla_{\mathfrak{A}}$ implies $b \in \nabla_{\mathfrak{A}}$, and

(e) a and b in $\nabla_{\mathfrak{A}}$ implies $a \bar{\wedge} b \in \nabla_{\mathfrak{A}}$; and

4. $I_{\mathfrak{A}}$ is the interpretation function defined such that

(a) for every constant symbol c_i in $\mathscr{L}$, there is an individual $c_i^{\mathfrak{A}}$ in $D_{\mathfrak{A}}$;

(b) for every n-place function symbol f_i^n in $\mathscr{L}$, there is an n-place function $f_i^{\mathfrak{A}} : D_{\mathfrak{A}}{}^n \to D_{\mathfrak{A}}$;

(c) for every n-place predicate symbol p_i^n in $\mathscr{L}$, there is an n-place predicate function (or propositional function) $p_i^{\mathfrak{A}} : D_{\mathfrak{A}}{}^n \to K_{\mathfrak{A}}$;

(d) where $\int : V \to D_{\mathfrak{A}}$ is a variable-assignment function, $I_{\mathfrak{A}}(\alpha \to \beta)_{\int} = I_{\mathfrak{A}}(\alpha)_{\int} \searrow I_{\mathfrak{A}}(\beta)_{\int}$,
$I_{\mathfrak{A}}(\alpha \circ \beta)_{\int} = I_{\mathfrak{A}}(\alpha)_{\int} \otimes I_{\mathfrak{A}}(\beta)_{\int}$,
$I_{\mathfrak{A}}(\alpha \wedge \beta)_{\int} = I_{\mathfrak{A}}(\alpha)_{\int} \bar{\wedge} I_{\mathfrak{A}}(\beta)_{\int}$,
$I_{\mathfrak{A}}(\alpha \vee \beta)_{\int} = I_{\mathfrak{A}}(\alpha)_{\int} \bar{\vee} I_{\mathfrak{A}}(\beta)_{\int}$, and
$I_{\mathfrak{A}}(\neg \alpha)_{\int} = -(I_{\mathfrak{A}}(\alpha)_{\int})$; and

(e) where $\int^{x \mapsto d} : V \to D_{\mathfrak{A}}$ is a variable-assignment function coinciding with $\int$ except mapping x to $d \in D_{\mathfrak{A}}$, $I_{\mathfrak{A}}(\forall x \alpha)_{\int} = \bigwedge_{d \in D_{\mathfrak{A}}} (I_{\mathfrak{A}}(\alpha)_{\int^{x \mapsto d}})$ and $I_{\mathfrak{A}}(\exists x \alpha)_{\int} = \bigvee_{d \in D_{\mathfrak{A}}} (I_{\mathfrak{A}}(\alpha)_{\int^{x \mapsto d}})$;

where α and β are arbitrary well-formed formulae in $\mathscr{L}$ and x is an arbitrary individual variable in V, and furthermore, every proposition $\mathfrak{a} \in K_{\mathfrak{A}}$ is syntactically accessible such that there is some formula α and $\int : V \to D_{\mathfrak{A}}$ such that $\mathfrak{a} = I_{\mathfrak{A}}(\alpha)_{\int}$.

This definition attempts to be as comprehensive as possible. Where the language $\mathscr{L}$ is restricted by omitting certain connectives or quantifiers, then the corresponding fragment structures can be defined by striking out the relevant portions of the definition. The definition packages a lot of information which ought to be unpacked into its components and discussed separately.

Comparisons with classical semantics are unavoidable, so some background conventions may be useful. The structures in the standard semantics of classical model theory defined with respect to a language $\mathscr{L}$ are to be denoted $\mathscr{L}$-structures; there will not be confusion between this and the later notion of $\mathscr{L}(\mathbf{LQ})$-structures, which are $\mathscr{L}$-substructural structures for a specific logic **LQ**. An $\mathscr{L}$-structure $\mathfrak{M}$ has a domain of individuals usually also denoted $\mathfrak{M}$, and the function symbols and constant symbols are respectively interpreted as functions and individuals in the domain $\mathfrak{M}$, and predicate symbols are interpreted to be relations on $\mathfrak{M}$.

In the substructural case, the domain of individuals is essentially unchanged from the domain of individuals in an $\mathscr{L}$-structure of classical logic. Thus, the language of rings can have theories whose substructural models have a domain of individuals which is a ring or field. On the level of terms and variables, the situation in substructural logics matches the situation in the standard semantics.

The first deviation is with the modeling of the predicate symbols. In the standard semantics of classical logic, an n-place predicate symbol is mapped to an n-place relation which partitions the n-Cartesian product of the domain of individuals into two disjoint sets, which naturally leads to the realization that there are two truth values at play. Substructural logics may require more than two propositions to be modeled accurately, so it must be possible to partition the domain of individuals into more than two sets. By changing the interpretation of a predicate symbol into a function mapping to propositions, it is possible to fulfill this partition requirement.

The algebra of propositions essentially mirrors the algebra of the logical syntax. Because the behavior of this algebra is key to successfully modeling the various substructural logics, its specific behavior has been left bare. There is one notable exception. The last part of the definition requires that every proposition in an $\mathscr{L}$-substructural structure be accessible in the sense that some formula, with its free variables interpreted by some assignment function, can be mapped to it. This requires that the space of functions generated by the well-formed formulae through the structure map onto the algebra of propositions. This condition requires every proposition to be useful in modeling the syntax, even if it may not be represented by a sentence. A similar situation occurs with the domain of individuals, which may include individuals not named by the language but play a role in the semantics due to the universal and existential quantifiers depending on instantiations that use them. The job of determining truth is left to the set of designated propositions. It will not in general be the case that this set is maximal in the sense that every proposition or its negation is designated.

Definition 2.4 Let $\mathscr{L}$ be a language, and let $\mathfrak{A}$ be an $\mathscr{L}$-substructural structure. Let α be a formula. Then, if $I_{\mathfrak{A}}(\alpha)_{\int} \in \nabla_{\mathfrak{A}}$, then α is said to be *designated*, *accepted*, or *satisfied* by $\mathfrak{A}$ with respect to $_{\int}$, and this property can be notated $\mathfrak{A}_{\int} \models \alpha$.

Definition 2.5 Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Then, an $\mathscr{L}$-substructural structure $\mathfrak{A}$ is appropriate for **LQ**, and so $\mathfrak{A}$ is an $\mathscr{L}$(**LQ**)-structure, if for every axiom schemata of **LQ** is satisfied by $\mathfrak{A}$ and $\nabla_{\mathfrak{A}}$ is closed with respect each inference rule in **LQ**.

As in the syntactic presentation of substructural logics, it may be helpful to look at examples of reducts of $\mathscr{L}$-substructural structures to demonstrate how bare and accommodating the definitions are.

Example 2.6 Let $\mathscr{L}$ be a language with $\circ$, $\rightarrow$, and $\neg$ the only connectives. The $\mathscr{L}$-substructural structure $\mathfrak{A}$ defined such that having $a \searrow b \in \nabla_{\mathfrak{A}}$ implies $-(a \searrow -b) \in \nabla_{\mathfrak{A}}$ for a and b in $K_{\mathfrak{A}}$ is a *weak Boethian $\mathscr{L}$-substructural structure*.

The connexive logic **BB** obtained by adding the Boethius Rule $\alpha \to \beta \vdash \neg(\alpha \to \neg\beta)$ to $\mathbf{B}_\circ$ or $\mathbf{B}_\to$ would thus be modeled by $\mathscr{L}(\mathbf{BB})$-structures. As this example demonstrates, an $\mathscr{L}(\mathbf{L})$-structure $\mathfrak{A}$ must be defined such that the semantic properties corresponding to the inference rules are closed with respect to $\nabla_{\mathfrak{A}}$ and every proposition corresponding to an axiom schema must reside in $\nabla_{\mathfrak{A}}$. The definition for $\mathscr{L}$-substructural structures thus requires them to at least be $\mathscr{L}(\mathbf{B})$-structures.

Now, the listing of axiom schemata forces the propositions to take up their corresponding structural properties which will be reflected in the algebra. Let $\mathfrak{A}$ be an $\mathscr{L}$-substructural structure. The following enumeration outlines some of these potential properties for $K_{\mathfrak{A}}$.

1. B ($\otimes$-Associativity) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$
2. C ($\otimes$-Commutativity) $a \otimes b = b \otimes a$
3. W ($\otimes$-Square-Increasing) $a \leq a \otimes a$
4. K ($\otimes$-Triviality) $a \otimes b \leq a$ and $a \otimes b \leq b$

As the descriptive names and the single-letter labels hint, these properties correspond to the specific structural rules formulated by the corresponding axiom schemata. In stronger logics, the $\otimes$-reduct of the algebra of propositions will be a semi-group.

The interesting deductive calculi are those with the full complement of connectives and quantifiers. In most of the deductive calculi considered here, the extensional connectives will be modeled by a common kind of underlying reduct on the algebra of propositions, a distributive De Morgan lattice.

Definition 2.7 (Distributive De Morgan Lattices) Let $\mathscr{L}$ be a language with extensional connectives, and let $\mathfrak{A}$ be an $\mathscr{L}$-substructural structure. Then, $K_{\mathfrak{A}}$ has a reduct $\langle K_{\mathfrak{A}}, \leq, \bar{\wedge}, \bar{\vee} \rangle$ which is a *lattice* if

1. the relation $\leq$ is a partial order on $K_{\mathfrak{A}}$ such that for all a, b, and c in $K_{\mathfrak{A}}$ it is the case that $a \leq a$, $a \leq b$ and $b \leq a$ implies $a = b$, and $a \leq b$ and $b \leq c$ implies $a \leq c$;
2. the binary operator $\bar{\wedge}$ is lattice meet mapping to the greatest lower bound of its arguments, so that $a \bar{\wedge} b = b \bar{\wedge} a$, $a \bar{\wedge} b \leq a$, $a \bar{\wedge} b \leq b$, and for all c $c \leq a$ and $c \leq b$ implies $c \leq a \bar{\wedge} b$; and
3. the binary operator $\bar{\vee}$ is lattice join and therefore maps to the least upper bound, so that $a \bar{\vee} b = b \bar{\vee} a$, $a \leq a \bar{\vee} b$, $b \leq a \bar{\vee} b$, and for all c $a \leq c$ and $b \leq c$ implies $a \bar{\vee} b \leq c$.

The lattice is *distributive* if $\bar{\wedge}$ and $\bar{\vee}$ distribute over each other such that

$$a \bar{\wedge} (b \bar{\vee} c) = (a \bar{\wedge} b) \bar{\vee} (a \bar{\wedge} c)$$

and

$$a \bar{\vee} (b \bar{\wedge} c) = (a \bar{\vee} b) \bar{\wedge} (a \bar{\vee} c).$$

Furthermore, $K_{\mathfrak{A}}$ has a reduct $\langle K_{\mathfrak{A}}, \leq, \bar{\wedge}, \bar{\vee}, - \rangle$ which is a *distributive De Morgan lattice* if

1. the reduct $\langle K_{\mathfrak{A}}, \leq, \bar{\wedge}, \bar{\vee} \rangle$ is a distributive lattice;
2. the operator $-$ respects the De Morgan laws, so that $-(a \bar{\wedge} b) = -a \bar{\vee} -b$ and $-(a \bar{\vee} b) = -a \bar{\wedge} -b$; and
3. the operator $-$ is idempotent, so that $--a = a$, and has the property that $a \leq b$ if and only if $-b \leq -a$.

A lattice implies a partial order, which is represented by $\leq$. The relation models the syntactical notion $\vdash$ of logical consequence. Thus, the residuation condition binding $\searrow$ and $\otimes$ looks similar to the deduction theorem of classical logic, which is that $\alpha, \beta \vdash \gamma$ if and only if $\alpha \vdash \beta \rightarrow \gamma$. This resemblance is not accidental, and it is one of the reasons why fusion carries the idea of an intensional conjunction. Since $\leq$ models $\vdash$, $\nabla_{\mathfrak{A}}$ must respect the partial order, so if $a \leq b$ and $a \in \nabla_{\mathfrak{A}}$, then it ought to be the case that $b \in \nabla_{\mathfrak{A}}$. Thus, $\nabla_{\mathfrak{A}}$ is a filter on $K_{\mathfrak{A}}$ with respect to $\leq$. The filter becomes an **L**-filter for a logic **L** if every theorem of **L** maps to a proposition in the filter.

Whereas the extensional reduct of the main $\mathscr{L}$-substructural structures is a distributive De Morgan lattice, the intensional reduct will have slightly different names depending on how much structure is present. In any case, the reduct $\langle K_{\mathfrak{A}}, \leq, \otimes, \searrow \rangle$ will at minimum be a *right-residuated partially-ordered groupoid* or *magma*. If $\otimes$ becomes associative, the result is a right-residuated partially-ordered semi-group; commutativity would provide a commutative semi-group. If $K_{\mathfrak{A}}$ happens to have an identity element e for $\otimes$, then the reduct will respectively be a right-residuated partially-ordered loop, monoid, and commutative monoid. A monoid is one step away from being a group, but the structural conditions for becoming a group are not among the axiom schemata primarily discussed here. Having an identity element in the structure can be guaranteed if the language includes a propositional constant t governed by $\vdash t$, $\alpha \vdash t \rightarrow \alpha$, and $t \rightarrow \alpha \vdash \alpha$, the intention being that t is mapped to the identity e. In this situation, it is clear to see that $a \leq b$ if and only if $e \leq a \searrow b$, and since t is always to be a satisfied formula, e must always be in $\nabla_{\mathfrak{A}}$.

The addition of t and its axiomatization is not harmful to a logic **L** in that the augmented logic $\mathbf{L}_t$ will be a conservative extension of **L**, so that the theorems of $\mathbf{L}_t$ without t are already theorems of **L**. The identity element is a helpful luxury, but reliance upon it will be avoided because the presence of the propositional constant t does little beyond provide technical relief to the current setup even with almost no cost.

Reiterating some of the conventions introduced at the start of this work, the following definitions formalize the notion of logical deduction and logical implication, the two relations united by the soundness and completeness results.

Definition 2.8 Let $\mathscr{L}$ be a language, and let $\Sigma \cup \{\tau\}$ be a collection of well-formed formulae. Let **LQ** be a quantified substructural logic. Then, $\Sigma \vdash_{\mathbf{LQ}} \tau$, or Σ logically deduces τ under **LQ**, if there is a formal derivation of τ from Σ using the axioms of **LQ**.

Definition 2.9 Let $\mathscr{L}$ be a language, and let $\Sigma \cup \{\tau\}$ be a collection of well-formed formulae. Let **LQ** be a quantified substructural logic. Then, $\Sigma \models_{\mathbf{LQ}} \tau$, or Σ logically implies τ under **LQ**, if every $\mathscr{L}(\mathbf{LQ})$-structure that satisfies every formula σ in Σ also satisfies τ.

2.2.3 Soundness

Theorem 2.10 Let $\mathscr{L}$ be a language, and let $\Sigma \cup \{\tau\}$ be a collection of $\mathscr{L}$-well-formed formulae. Let **LQ** be a substructural logic. Then, if $\Sigma \vdash_{\mathbf{LQ}} \tau$, then $\Sigma \models_{\mathbf{LQ}} \tau$.

Proof. The proof is essentially the same as in classical logic. Let $\{\tau_1, \tau_2, ..., \tau_n\}$ be a formal proof of τ from Σ in **LQ**, with $\tau_n := \tau$. Let $\mathfrak{A}$ be an $\mathscr{L}(\mathbf{LQ})$-structure that models Σ but not τ. Then, there is some variable assignment function $\int$ and some i with $1 \leq i \leq n$ such that τ_i is the first formula not accepted by $\mathfrak{A}$ with $\int$. Being an $\mathscr{L}(\mathbf{LQ})$-structure, τ_i cannot be an axiom without there being an immediate contradiction. Similarly, since $\mathfrak{A} \models \Sigma$, τ_i cannot be in Σ without contradiction.

Thus, τ_i must be the result of applying an inference rule to formulae earlier in the formal deduction. Depending on the logic, there are two or three rules that need to be checked, excluding the weak connexive logics. In each case, the models are defined to satisfy the rule conditions.

1. ($\to$-Detachment) The formula τ_i is obtained by applying the rule on τ_j and $\tau_k := \tau_j \to \tau_i$ with $j, k < i$. Now, $\mathfrak{A}_{\int} \models_{\mathbf{LQ}} \tau_j$ and $\mathfrak{A}_{\int} \models_{\mathbf{LQ}} \tau_j \to \tau_i$ by assumption. That is, $I_{\mathfrak{A}}(\tau_j)_{\int} \in \nabla_{\mathfrak{A}}$ and $I_{\mathfrak{A}}(\tau_j \to \tau_i)_{\int} \in \nabla_{\mathfrak{A}}$, so $I_{\mathfrak{A}}(\tau_j)_{\int} \searrow I_{\mathfrak{A}}(\tau_j \to \tau_i)_{\int} \in \nabla_{\mathfrak{A}}$. Because $\mathfrak{A}$ is an $\mathscr{L}(\mathbf{LQ})$-structure, $\nabla_{\mathfrak{A}}$ must be closed with respect to $\leq$, so $I_{\mathfrak{A}}(\tau_i)_{\int} \in \nabla_{\mathfrak{A}}$, which contradicts the supposition that τ_i is not accepted by $\mathfrak{A}$ with $\int$.
2. ($\to$-Attachment) The formula τ_i is obtained by applying the rule on $\tau_j := \alpha \to \beta$ and $\tau_k := \gamma \to \delta$ with $j, k < i$, so that τ_i has the form $(\beta \to \gamma) \to (\alpha \to \delta)$. The only way $\mathfrak{A}$ with $\int$ would reject τ_i is if $I_{\mathfrak{A}}((\beta \to \gamma) \to (\alpha \to \delta))_{\int} \notin \nabla_{\mathfrak{A}}$, but this contradicts a primitive notion of $K_{\mathfrak{A}}$ being a residuated magma.
3. ($\wedge$-Adjunction) If τ_i is the result of adjoining τ_j and τ_k with $j, k < i$ so that $\tau_i := \tau_j \wedge \tau_k$, then $I_{\mathfrak{A}}(\tau_j)_{\int} \in \nabla_{\mathfrak{A}}$ and $I_{\mathfrak{A}}(\tau_k)_{\int} \in \nabla_{\mathfrak{A}}$, so $I_{\mathfrak{A}}(\tau_j)_{\int} \bar{\wedge} I_{\mathfrak{A}}(\tau_k)_{\int} \in \nabla_{\mathfrak{A}}$ because of $\nabla_{\mathfrak{A}}$ as a filter, and $I_{\mathfrak{A}}(\tau_j)_{\int} \bar{\wedge} I_{\mathfrak{A}}(\tau_k)_{\int} = I_{\mathfrak{A}}(\tau_j \wedge \tau_k)_{\int}$. A contradiction thus arises.

Clearly, these inference rules are $\nabla_{\mathfrak{A}}$-preserving, so they cannot be the source of rejecting τ_i. Since no possible case of τ_i can be rejected, the original assumption must be erroneous. Consequently, soundness is affirmed. ↻

2.2.4 Completeness

Theorem 2.11 Let $\mathscr{L}$ be a language, and let $\Sigma \cup \{\tau\}$ be a collection of $\mathscr{L}$-well-formed formulae. Let **LQ** be a substructural logic. Then, if $\Sigma \models_{\mathbf{LQ}} \tau$, then $\Sigma \vdash_{\mathbf{LQ}} \tau$.

As in the standard Henkin proof in classical logic, the completeness result is a corollary of the following theorem.

Theorem 2.12 Let $\mathscr{L}$ be a language, and let $\Sigma \cup \{\tau\}$ be a collection of $\mathscr{L}$-well-formed formulae. Let **LQ** be a substructural logic. Then, if $\Sigma \nvdash_{\mathbf{LQ}} \tau$, then $\Sigma \not\models_{\mathbf{LQ}} \tau$.

Proof. The proof procedure is founded on essentially the same idea; one constructs a model of Σ explicitly designed with refuting τ in mind, and this model therefore refutes the idea of Σ logically implying τ. However, the mechanism is more complicated than in classical logic. In classical logic, many properties turn out to coincide with each other, but these coincidences break down in substructural logics, and care must be taken to address the breakdowns of these properties which were relied upon in the classical Henkin construction procedure.

Let $\mathscr{L}'$ be the language $\mathscr{L}$ expanded by adding new constant symbols. For current purposes, suppose $\mathscr{L}$ is countable; the procedure works in uncountable languages if the formulae in such cardinality can be well-ordered. The goal is to expand Σ to a maximal saturated prime theory $\Delta \supseteq \Sigma$ that does not include τ.

Definition 2.13 Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Then, an $\mathscr{L}$(**LQ**)-*theory* T is a collection of $\mathscr{L}$-formulae such that $T \vdash \tau$ if and only if $\tau \in T$. The theory T is *prime* if, for every formula of the form $\alpha \vee \beta$, $\alpha \vee \beta \in T$ implies that at least one of α and β is in T. A theory T is *saturated* if every formula of the form $\exists x \alpha(x)$ in T for some individual variable x implies that $\alpha(c)$ is in T for some constant symbol c.

To obtain the maximal saturated prime theory Δ from Σ, first expand Σ to its deductive closure, the theory Σ', so that $\alpha \in \Sigma' \Leftrightarrow \Sigma \vdash \alpha$. Obviously, $\tau \notin \Sigma'$. Consider the ordered pair $(\Sigma', \{\tau\})$. The next step in the task is essentially to add every single $\mathscr{L}$-formula into one of the two coordinates of the pair. For indexing purposes, it may be convenient to set $T_1 := \Sigma'$ and $F_1 := \{\tau\}$ and index every $\mathscr{L}$-formula as well. Then, given the pair (T_n, F_n) and formula ϕ_n, ϕ_n is added to one of T_n and F_n to obtain T_{n+1} and F_{n+1}, with the consequence that exactly one of $T_{n+1} = T_n$ or $F_{n+1} = F_n$. Eventually, this procedure yields (Δ, Θ) with $\Delta \cup \Theta = \text{Form}(\mathscr{L})$. Then, this Δ is the needed saturated prime theory Δ. The set Θ is the collection of rejected formulae. This overall view guides the following individual steps of the procedure.

Let C denote the set of new constant symbols in $\mathscr{L}'$ not in $\mathscr{L}$. Let (T_1, F_1) be $(\Sigma', \{\tau\})$, and let

$$\{\phi_1, \phi_2, \phi_3, ...\}$$

be an indexing of the well-formed formulae of $\mathscr{L}'$. Now, (T_2, F_2) will be either $(T_1 \cup \{\phi_1\}, F_1)$ or $(T_1, F_1 \cup \{\phi_1\})$. To determine which pair requires a notion of exclusivity; the pair (T_i, F_i) is exclusive if there are no formulae $\alpha_1, ..., \alpha_m \in T_i$ and $\beta_1, ..., \beta_n \in F_i$ such that $\vdash_{\mathbf{LQ}} (\alpha_1 \wedge ... \wedge \alpha_m) \to (\beta_1 \vee ... \vee \beta_n)$. Clearly, (T_1, F_1) is exclusive, and it is evident that if $(T_1 \cup \{\phi_1\}, F_1)$ is not exclusive, then $(T_1, F_1 \cup \{\phi_1\})$ must be exclusive. Let $(T_2, F_2) := (T_1 \cup \{\phi_1\}, F_1)$, unless it is not exclusive, in which case (T_2, F_2) should be set to be $(T_1, F_1 \cup \{\phi_1\})$. This procedure continues for every ϕ_i, with two exceptions.

1. If ϕ_i is of the form $\exists x \psi(x)$ and $(T_i \cup \{\phi_i\}, F_i)$ is exclusive, then add $\psi(c_i)$ for some c_i in C that does not appear in $T_i \cup F_i \cup \{\exists x \psi(x)\}$ to T_i as well, so that

$$(T_{i+1}, F_{i+1}) = (T_i \cup \{\phi_i, \psi(c_i)\}, F_i).$$

2. If ϕ_i is of the form $\forall x \psi(x)$ and $(T_i, F_i \cup \{\phi_i\})$ is exclusive since $(T_i \cup \{\phi_i\}, F_i)$ is not, then also add $\psi(c_i)$ for some c_i in C not appearing in $T_i \cup F_i \cup \{\forall x \psi(x)\}$ to F_i as well. Thus, the result would be that

$$(T_{i+1}, F_{i+1}) = (T_i, F_i \cup \{\phi_i, \psi(c_i)\}).$$

The pair (Δ, Θ) can be obtained by defining

$$\Delta := \bigcup_{n \in \mathbb{N}} T_n \text{ and } \Theta := \bigcup_{n \in \mathbb{N}} F_n.$$

Clearly, (Δ, Θ) is exclusive; otherwise, some (T_n, F_n) will not be exclusive. $\Delta \not\vdash \tau$ still, since $\tau \in \Theta$. Furthermore, Δ is saturated simply through how it was constructed. Also, Δ is prime; wherever $\alpha \vee \beta$ is in Δ, at least one of α and β would have been added to Δ. If both α and β are in Θ, then exclusivity would be lost by taking $\alpha \vee \beta$ in Δ and α and β in Θ; this would yield $\vdash_{\mathbf{LQ}} (\alpha \vee \beta) \to (\alpha \vee \beta)$, contradicting exclusivity since this formula is an axiom in every substructural logic.

The next task is to construct a canonical model denoted $\mathfrak{M}_\Delta$ for Δ and a variable assignment function $\int$ so that $\mathfrak{M}_{\Delta} \int \models \phi$ for every $\phi \in \Delta$. The domain $D_{\mathfrak{M}_\Delta}$ is obtained using the constant symbols of $\mathscr{L}'$; if a predicate for equality is in the language and the intention for that symbol is to treat identity, then a slight modification must be carried out. The algebra $K_{\mathfrak{M}_\Delta}$ is obtained using the formulae themselves.

For the sentences of $\mathscr{L}'$, let $\equiv$ be a relation obtained such that $\phi \equiv \psi$ if and only if $\phi \leftrightarrow \psi \in \Delta$, for sentences ϕ and ψ. Note that $\equiv$ is an equivalence relation, and it will be used to obtain a Lindenbaum-Tarski algebra. Let $[\phi]$ denote the equivalence class of ϕ with respect to $\equiv$. Then, $K_{\mathfrak{M}_\Delta} := \{[\phi] : \phi \text{ is a sentence }\}$. The connectives will define their respective interpretations in this algebra, so that

1. $[\phi] \bar{\wedge} [\psi] := [\phi \wedge \psi]$,

2. $[\phi]\bar{\vee}[\psi] := [\phi \vee \psi]$,

3. $[\phi] \searrow [\psi] := [\phi \to \psi]$,

4. $[\phi] \otimes [\psi] := [\phi \circ \psi]$, and

5. $-[\phi] := [\neg\phi]$.

To define $\leq$ on $K_{\mathfrak{M}_\Delta}$, let $[\phi] \leq [\psi]$ when $[\phi \vee \psi] = [\psi]$. In sufficiently strong substructural logics, namely those with B and B', the ordering relation also yields $[\phi] \leq [\psi]$ if and only if $\phi \to \psi \in \Delta$.

The domain of individuals for $\mathfrak{M}_\Delta$ can simply be the set C of constant symbols in the language $\mathscr{L}'$. If equality is represented in the language with the intention of true equality, then the domain $D_{\mathfrak{M}_\Delta}$ can reflect that. Where $\dot{a}$ and $\dot{b}$ are terms and $\doteq$ is the symbol for equality, let $\approx$ be a relation such that $\dot{a} \approx \dot{b}$ if and only if $\dot{a} \doteq \dot{b} \in \Delta$. Let $[a]$ denote the equivalence class of $\dot{a}$ over $\approx$; then, $D_{\mathfrak{M}_\Delta} := \{[a] : a \in C\}$. With equality, function symbols can be interpreted by functions constructed in the same manner, their graphs determined by the equations in Δ. Predicate symbols can also be interpreted by propositional functions defined so that if $p_i t_1 ... t_n$ is an atomic sentence, then $p_i^{\mathfrak{M}_\Delta}([t_1], ..., [t_n]) = [p_i t_1 ... t_n]$.

The variable assignment function $\int$ can be constructed based on Δ in the presence of the equality symbol. Every variable x will have a constant symbol $\dot{c_x}$ such that $x \doteq \dot{c_x} \in \Delta$ due to the construction procedure for Δ, and let $\int(x) = [c_x]$. This $\int$ can be used along with the structure defined for $\mathfrak{M}_\Delta$ so far to interpret all well-formed formula in $\mathscr{L}'$, and so $I_{\mathfrak{M}_\Delta}$ can therefore be defined.

Finally, $\nabla_{\mathfrak{M}_\Delta}$ can be obtained by defining it so that $\nabla_{\mathfrak{M}_\Delta} := \{[\phi] : \phi \in \Delta\}$. Note every formula satisfied by $\mathfrak{M}_\Delta$ has a corresponding sentential form that will be in Δ.

Through all this work, it becomes evident that $\mathfrak{M}_\Delta$ is a model for Σ that will not accept τ. Consequently, if $\Sigma \nvdash \tau$, then $\Sigma \not\models \tau$. Thus, Completeness follows as a consequence. ↻

2.2.5 The Equality Predicate

In the standard semantics of classical logic, equality is treated as a logical symbol rather than a normal predicate symbol. The consequence of this treatment is the intention that its semantic meaning is fixed; that is, if a structure $\mathfrak{A}$ satisfies the formula $t \doteq u$, then it always to be the case that $t^{\mathfrak{A}} = u^{\mathfrak{A}}$. For the proof theory, the axioms for equality are designed to reflect the principle of indiscernibility, that no predicate or function can distinguish between different terms that are to be interpreted as the same individual. Nevertheless, the axioms cannot be so strong as to distinguish equality from a particularly strong equivalence relation, so by dropping the convention, it is possible to obtain models which treat the symbol as an equivalence relation rather than true equality.

The issue of treating equality is a nontrivial issue in intuitionistic logic. The underlying philosophy of intuitionism can lead to the argument that determining whether two individuals a and b are equal requires a constructive proof of this property; to accept this philosophy would stand at odds with many axiomatic formulations of equality. The semantics must also adjust to such issues.

In the case of substructural logics, the equality axioms of classical logic present issues in their modeling. Predicate symbols are interpreted to be functions from the domain of individuals to the algebra of propositions. Certainly, any atomic formula stating the reflexive principle of equality ought to be accepted and mapped appropriately, so it makes sense to include $x \doteq x$ as an axiom schema. The inference rules that were previously outlined also make intuitive sense; if $a = b$ and $b = c$ are accepted propositions, then so should $b = a$ and $a = c$, since these properties are true of equality in general practice, and a formula of the form $\alpha(a) \to \alpha(b)$ ought to be accepted as well, reflecting the process that the favorable interpretation of $\alpha(a)$ means the same for $\alpha(b)$. On the other hand, certain axiomatizations essentially require the models to accept all true atomic equations to be of the same propositional value, which may not be desirable in, for example, a project on relevance issues.

It is possible to have more than one equality predicate which map pairs of individuals to different propositions, depending on the situation. Restall offers in [Restall 1994] an example of a "Boolean" version. Let $\doteq_{\text{bool}}$ be the symbol for this predicate axiomatized by

1. $\vdash \alpha \to x \doteq_{\text{bool}} x$,
2. $\vdash x \doteq_{\text{bool}} y \to y \doteq_{\text{bool}} x$,
3. $\vdash x \doteq_{\text{bool}} y \to (y \doteq_{\text{bool}} z \to x \doteq_{\text{bool}} z)$, and
4. $\vdash x \doteq_{\text{bool}} y \to (\alpha(x) \to \alpha(y))$ where α is any formula.

This equality relation would essentially be the classical notion, with the aim that all correct equations are given the highest possible propositional value.

With multiple equality relations, there would be a need to synchronize them to "say" the same thing, namely to agree upon which individuals (or their interpretations) should be equal. Thus, if $\doteq$ and $\approx$ are two equality predicates, then the rule $a \doteq b \dashv\vdash a \approx b$ should be introduced to avoid problems.

For current purposes and in developing the model theory, it is sufficient to include one equality predicate and not specify its propositional value.

2.2.6 Examples

Some examples may be helpful in seeing how the semantics work.

Example 2.14 (Classical Structures) Let $\mathscr{L}$ be the language of sets with equality. Consider the structure $\mathfrak{M} := \langle V, \{T, F\}, I_{\mathfrak{M}}, \{T\}\rangle$, where V is a classical model of the ZFC axioms of set theory and $I_{\mathfrak{M}}$ is defined such that $I_{\mathfrak{M}}(a \in b) = T$ if and only if $a^{\mathfrak{M}} \in^{\mathfrak{M}} b^{\mathfrak{M}}$ in the conventional classical sense. $\mathfrak{M}$ is really just a classical two-valued model of ZFC. Indeed, every model in classical logic can be made into a model in substructural logics by formally incorporating the truth values into a two-valued Boolean algebra.

Similarly, Boolean-valued structures are nonstandard models of classical logic. These too can be converted into structures of the kind outlined here for substructural logics.

Example 2.15 (Boolean-Valued Structures) This example is based on work found in [Kutateladze 2007]. The current system of semantics is also designed to accommodate structures that model classical first-order logic without appealing to an underlying two-valued Boolean algebra. Such *Boolean-valued models*, as developed by D. Scott and R. Solovay, have arisen in the exploration of Boolean-valued Analysis proposed by G. Takeuti and others, in which the work of analysis is done in the context of Boolean-valued models of set theory. A naive introduction to such Boolean-valued models is instructive in how it may then be adapted to conform into the shape of an $\mathscr{L}$-substructural model of the current model theory.

Let B be a complete Boolean algebra. By definition, B contains two particular elements which serve as the top and bottom of the lattice structure; let $\top$ and $\bot$ denote the top and bottom elements, respectively, such that $a \bar{\wedge} \top = a$ and $a \bar{\vee} \bot = a$ for all $a \in B$. The algebra is then used to construct B-valued sets, such that $\mathbb{V}(B)(0) := \emptyset$,

$$\mathbb{V}(B)(\alpha+1) := \{x \text{ is a function with } \operatorname{dom}(x) \subseteq \mathbb{V}(B)(\alpha) \text{ and } \operatorname{im}(x) \subseteq B\}$$

as the successor case, and

$$\mathbb{V}(B)(\gamma) := \bigcup_{\beta<\gamma} \mathbb{V}(B)(\beta)$$

for limit ordinals γ, so that the domain of B-valued sets is

$$\mathbb{V}(B) := \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{V}(B)(\alpha)$$

where Ord is an appropriate collection of set-theoretic ordinal numbers.

Boolean-valued sets as defined by B are therefore rigidly defined functions mapping the individuals in their extensions to the Boolean algebra B, with the atomic formula $x \in y$ essentially calculating that value of x in y in the algebra. In a strict sense, the Boolean algebra B need not be complete in order to successfully yield a Boolean-valued model that can be adapted into the semantical setup, but B must be sufficiently rich enough to properly model quantification.

Let $\mathscr{L}$ be the language of sets. Let $\mathfrak{A}^D := \langle \mathbb{V}(B), B, I, \{\top\}\rangle$ be a (draft) $\mathscr{L}$-model with I defined as follows.

1. Where $\int$ is a variable assignment function such that $\int(x) = a$, $\int(y) = b$, and $\int(z) = c$ for individual variables x, y, and z and individuals a, b, and c in $\mathbb{V}(B)$, let

$$I(x\dot{\in}y)_{\int} := \bigvee_{c\in\mathrm{dom}(a)} (a(c) \wedge I(z\dot{=}x)_{\int'})$$

and

$$I(x\dot{=}y)_{\int} := (\bigwedge_{c\in\mathrm{dom}(a)} a(c) \to I(z\dot{\in}y)_{\int'}) \wedge (\bigwedge_{c\in\mathrm{dom}(b)} b(c) \to I(z\dot{\in}x)_{\int'})$$

where $\int'$ differs from $\int$ in how it maps z to individuals in $\mathbb{V}(B)$.

2. Where α and β are well-formed $\mathscr{L}$-formulae and x is an individual variable, let $I(\alpha\wedge\beta)_{\int} = I(\alpha)_{\int}\bar{\wedge}I(\beta)_{\int}$, $I(\alpha \vee \beta)_{\int} = I(\alpha)_{\int}\bar{\vee}I(\beta)_{\int}$, $I(\alpha \to \beta)_{\int} = I(\alpha)_{\int} \searrow I(\beta)_{\int}$, $I(\alpha \circ \beta)_{\int} = I(\alpha \wedge \beta)_{\int}$, $I(\neg\alpha)_{\int} = -I(\alpha)_{\int}$, $I(\forall x\alpha)_{\int} = \wedge I(\alpha)_{\int'}$, and $I(\exists x\alpha)_{\int} = \vee I(\alpha)_{\int'}$.

The draft $\mathfrak{A}^D$ can be refined to a structure $\mathfrak{A} := \langle \mathbb{V}(\bar{B}), B, \bar{I}, \{\top\}\rangle$ with $\mathbb{V}(\bar{B})$ obtained by defining an equivalence relation $\approx$ on $\mathbb{V}(B)$, such that $a \approx b \Leftrightarrow I(x\dot{=}y)_{\int} = \top$ where $\int(x) = a$ and $\int(y) = b$, and taking the quotient

$$\mathbb{V}(\bar{B}) := \mathbb{V}(B)/\approx.$$

This $\mathfrak{A}$ is a Boolean-valued model adapted into the current substructural model theory. By a similar characterization, it is possible to obtain K-valued sets for whatever algebra K that may be used as the algebra of propositions. In this way, one may readily obtain $\mathscr{L}(\mathbf{L})$-models (and $\mathscr{L}(\mathbf{LQ})$-models where appropriate) for the various flavors of logic **L** that are substructural.

Example 2.16 (The Integers) A different way of obtaining models may be by looking at proof-of-concept structures that take a concrete mathematical object as the basis for a substructural structure. For example, take the algebra of propositions to be based on the integers $\mathbb{Z}$. Now, $\mathbb{Z}$ can be regarded as a lattice based on the canonical ordering, and negation arising from additive inverses can be used for the interpretation of the negation connective. Residuation $\searrow$ can be defined conditionally, so that if a and b are integers such that if $a \leq b$, then let $a \searrow b = -a \vee b$, and if $a > b$, then let $a \searrow b = -(a \vee -b)$. Let $\mathscr{L}$ be a language of choice, and let $\mathfrak{A} := \langle D, \mathbb{Z} \cup \{-\infty, \infty\}, I, \{a \in \mathbb{Z} \cup \{-\infty, \infty\} : a \geq 0 \text{ or } a = \infty\}\rangle$ be an $\mathscr{L}$-substructural structure where D sufficiently handles the $\mathscr{L}$-terms and I is defined using the canonical ordering, negation, and residuation as discussed. This model, based on the infinite normal Sugihara matrix, turns out to model the propositional logic **RM** and its quantified variant **RMQ**.

2.2.7 Comparison with Semantic Precedents

The semantics formulated here is based in part on previously defined systems, and it would be helpful to compare the similarities and differences between them. Before even taking quantification into account, there are two general approaches to modeling the propositional connectives of substructural logics which mirror the dual approaches used in intuitionistic and modal logics. The approach taken here is an algebraic one, interpreting formulae as points called propositions and connectives as algebraic operators on them. Certain classes of algebras correspond to specific examples of propositional logics, so that for example the logic **R** is modeled faithfully by the class of De Morgan algebras and classical logic **PC** is modeled by the class of Boolean algebras. Intuitionistic logic **H** has the class of Heyting algebras and modal logics have various flavors of interior algebras.

The other approach in modeling propositional systems is by introducing a frame of points commonly called states, situations, or worlds; these points interpret some formulae in the classical way, but certain connectives are interpreted using relations on these points. In modal and intuitionistic logics, this set is structured by a binary relation, the differences between these logics based on differences held in that binary relation. Substructural logics generally require a ternary relation which is used to model the intensional connectives $\rightarrow$ and $\circ$. The use of relational or frame semantics has significant appeal since these systems lend themselves to philosophical discussions and interpretations about the formal logical calculi.

Because of this appeal, considerable effort was taken in extending the relational semantics to accommodate quantification, but the accomplishment of such a semantics required unexpected complications. In addition to the frame relations modeling the propositional connectives, quantifiers also require their own machinery in order to yield a sound and complete semantics. The prospect of developing a model theory over this setup is not considerably appealing even though it may continue to hold philosophical interest. Indeed, a program of intuitionistic model theory has centered on the use of relational semantics, but the relational semantics of quantified intuitionistic logic is more simplified than the broader case of substructural logics. Consequently, the algebraic approach seems more sensible for this task of creating a substructural model theory.

Systems of algebraic semantics have been defined for quantified classical logic such as through cylindric algebras, and similar efforts have been undertaken for intuitionistic variants, so this direction is not without precedent. Perhaps [Meyer Dunn LeBlanc 1974] is the first example of developing the semantics for quantified substructural logics. A very crude system is sketched out in order to solve the γ problem for quantified relevance logic, which asks whether it follows that $\vdash_{\mathbf{RQ}} \neg\alpha \vee \beta$ and $\vdash_{\mathbf{RQ}} \alpha$ implies $\vdash_{\mathbf{RQ}} \beta$. This sketch nevertheless inspires several subsequent developments of algebraic semantics for quantified substructural logics, including the system developed in this work.

One such development is a hybrid system proposed by Brady. In [Brady 1988] and [Brady 1989], Brady developed a content-based semantics where propositions become collections of content atoms; the connectives are interpreted by a combination of algebraic operators on the atoms and closure conditions imposed upon the propositions. The intuitive appeal of this content semantics is to base partial ordering on content containment and to avoid the use of the fusion connective $\circ$. Content semantics later became the basis for the system in [Brady 2006] which proposes an alternative understanding of the foundations of mathematics by taking up a combination of a relatively weak substructural logic and stronger axioms for set theory than allowed with classical logic.

In contrast to the content semantics of Brady is the project by H. Ono in [Ono 1993] and [Ono 1995] which fully embraces technical questions in modeling quantification. The way quantification is generally handled in classical, intuitionistic, and substructural logics reflects a consensus on the intuitive meaning of universal and existential quantifiers. Universal quantification is modeled as an overall lattice meet of all of the instantiations of the quantified variable in question, and existential quantification is taken as the overall lattice join. An arbitrary lattice will not necessarily be populated with such meets and joins, and one way of avoiding this problem is to make use of complete lattices and develop a way to convert an incomplete lattice into a complete one.

An algebraic model of propositional substructural logic includes other structure besides the lattice, and a key concern is that a completion procedure for the lattice reduct might be incompatible with the residuated groupoid reduct, so that the completed lattice no longer respects the properties required of the groupoid operator and its residual. The work of Ono develops appropriate adaptations of the Dedekind-MacNeille completion of lattices to yield algebraic models for quantified substructural logics. For propositional logics without $\wedge\vee$-Distribution, for example, Dedekind-MacNeille completion can be used on their models without modification to obtain algebraic models with quantification. For logics with $\wedge\vee$-Distribution, however, Dedekind-MacNeille completion of the lattice reduct destroys the modeling ability of the original residual operator, so a new one needs to be developed to restore the interpretation of $\rightarrow$ connective.

One of the key differences between the semantics of this work and the semantics of Ono is the starting point in the development of the semantics. This work is concerned with a semantics useful enough to take up concepts important to model theory, so a key emphasis in the semantics here is in demonstrating how the $\mathscr{L}$-substructural structures handle terms and their roles in affecting the interpretation of formulae. On the other hand, Ono's work is concerned with the question of developing models for a quantified substructural logic based on models of the propositional fragment of that logic. In this work, the semantics eschews a completion procedure for the lattice reduct because the extensional reduct of a $\mathscr{L}$-substructural structure does not have to be complete; the reduct only needs to be populated enough to accommodate all the meets and joins that can be expressed in the language. Indeed, imposing completion requirements on the lattice structure would be detrimental to the development of substructural understandings of n-types and other model-theoretic concepts that are based on the amount of propositions present.

This kind of reasoning also motivates some of the developments in the semantics in [Restall 1994], which is based in part on the content semantics of Brady. Of the various antecedents of this work, the semantics of Restall is of great pertinence since some developments towards a substructural model theory are also presented. Consequently, Restall's system also avoids the completion of lattice reducts for the same reason as this work, but the implementation of the semantics differs from the current presentation in several important ways. The Restall semantics is set up so that there is a *relational structure* built over an underlying *propositional structure*, the latter being an algebra which would model the propositional fragment of the quantified substructural logic. A relational structure $\mathfrak{A}$ has an associated domain of individuals, denoted $D_{\mathfrak{A}}$, and includes functions of that domain for each function symbol in the language. Predicate symbols are interpreted as functions from the domain of individuals to the underlying propositional structure, just as in the semantics for this work. There are also two functions, denoted $\bigcap$ and $\bigcup$, defined on sets of the domain of individuals which are used to interpret the quantifiers $\forall$ and $\exists$.

Similar to the semantics of this work, the well-formed formulae are essentially interpreted as functions from the domain of individuals to some other entity, which is something built upon the propositional structure and the functions $\bigcap$ and $\bigcup$. One difference between the two approaches is that this work interprets a formula with n free variables as a function with n arguments, but the semantics of Restall regards every formula, regardless of the number of free variables, as a function taking each natural number as an argument. What basically happens in this latter case, however, is that an argument at the nth position is ignored if the individual variable x_n is not free in the formula. Nevertheless, the setup provided in this work is arguably more straightforward and less cumbersome to address.

Another distinction arises in that the relational structures avoid spelling out the exact shape of the algebra to which these functions map their arguments. This algebra is the result of modifying the underlying propositional structure with the functions $\bigcap$ and $\bigcup$ as well as other closure conditions needed for an approach based on content semantics; the propositional structure informs the interpretation of the predicate symbols. The $\mathscr{L}$-substructural structure of this work instead lays out an algebra of propositions which is already populated with all of the points needed to interpret every $\mathscr{L}$-formula with parameters from the domain of individuals. Each proposition is essentially accessible using some formula and some finite sequence of individuals; no such requirement is explicitly required in the semantics of Restall.

These different developments of the semantics for quantified substructural logics are motivated by slightly different goals. The content semantics permitted the removal of the fusion connective $\circ$ from the language because its semantic role was no longer needed. The program of Ono continues a line of development that harks back to the work of H. Rasiowa and R. Sikorski in classical logic, focusing on the development of algebraic semantics in a generalized sense. One of the key differences between the semantics of Ono and Restall from the semantics of this work is in the respective inclusion and omission of the propositional constant t, interpreted to be the identity element of the groupoid operator represented by the fusion connective $\circ$. The presence of t simplifies the semantics because designated propositions can be determined using just the identity element and the partial order, so that a proposition a is designated if and only if $e \leq a$ holds. However, the constant t is not necessary for the development of substructural model theory, and since it is possible to develop the semantics without it, this work therefore proceeds by regarding t as an object of convenience rather than one of necessity.

Chapter 3

Elementary Substructural Model Theory

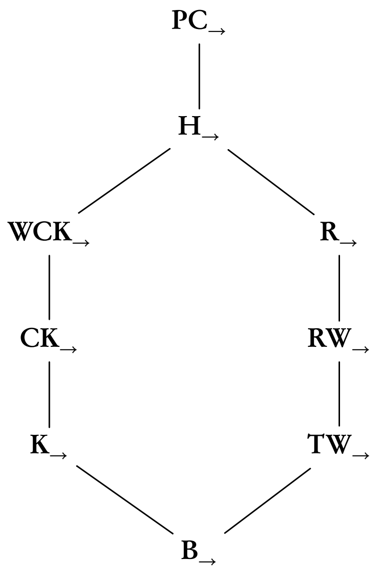
Having set forth a system of syntax and semantics for various substructural logics, the next task is to build up a substructural model theory using these developments. The variety of substructural logics necessitates some restrictions towards obtaining a universal presentation of model theory. Where applicable, the development will attempt to be accommodative as possible, but in some parts, the proposed definitions and results only work when certain conditions are met.

One of the requirements is that conjunction and disjunction must act as classically as possible. Thus, $\wedge$ and $\vee$ need to distribute over each other. Furthermore, it will be necessary to confine the scope of logics to those whose negation connective is a De Morgan one. For example, connexive implication as based on the theses of Aristotle and Boethius is simply too exotic to grasp in the presence of the extensional connectives. Intuitionistic logic has a nonclassical negation connective that refutes certain De Morgan principles, but some of the following developments may apply to it; in such a case, it would be of interest how the model theory here compares to the topological models of intuitionistic analysis or the model theory based over frame-based relational semantics.

3.1 Between Logical Calculi

As noted previously, the theorems of a weaker substructural logic will be theorems in a stronger substructural logic, so any theory built up in a stronger substructural logic will also be a theory in the weaker system. Within the confines of this work, there is partial order on the substructural logics, which can be defined so that $\mathbf{L} < \mathbf{M}$ for logics $\mathbf{L}$ and $\mathbf{M}$ when the theorems of $\mathbf{L}$ are also theorems of $\mathbf{M}$.

An example of this partial order is the following depiction of implicative fragments of substructural logics.



Similarly, where a language $\mathscr{L}$ is set, an $\mathscr{L}(\mathbf{M})$-structure can be construed as an $\mathscr{L}(\mathbf{L})$-structure when $\mathbf{L} < \mathbf{M}$. Thus, the relationship between substructural logics and their models form a Galois connection. Since classical logic is the strongest substructural logic by having all of the structural rules, this situation means that any classical structure, when modified to conform to the shape of substructural structures, can be made into an $\mathscr{L}(\mathbf{L})$-structure for all substructural logics $\mathbf{L}$. In certain situations, it becomes sufficient to show that if something can or cannot happen in classical model theory, then it can or cannot happen in the substructural counterpart.

The tendency in moving from weaker to stronger substructural logics is that the corresponding propositional algebras become simpler in their description. Like the heavens and the earth before the Six Days of Creation, $\mathscr{L}(\mathbf{BQ}_{\circ})$-structures are rather shapeless and without much form, but adding structural rules allow for semigroups, De Morgan algebras, and Boolean algebras to become the associated varieties of algebras. At the top occupied by models for classical logic are Boolean algebras, the most conspicuous of which are the two-valued systems.

Of course, there is one other kind of structure that has even simpler shape than the classical ones. A structure with a singleton as the propositional algebra can indeed be a substructural structure, and this is not a problem. The consequent trivial structure is not nonsensical, and it does mean certain accepted principles of classical logic need to be modified. It no longer makes sense to state that no inconsistent theory over classical logic has a model, for example; rather, this statement means that the only models for inconsistent theories over classical logic are the trivial ones with singleton propositional algebras. The compactness property is another example of where this change is meaningful.

3.2 Compactness

Theorem 3.1 (Compactness I) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\Sigma \cup \{\tau\}$ be a collection of formulae. Then, if $\Sigma \models_{\mathbf{LQ}} \tau$, then there is a finite subset Σ_0 of Σ such that $\Sigma_0 \models_{\mathbf{LQ}} \tau$.

Proof. This is an immediate consequence of soundness and completeness. Formal derivations must be of finite length, which constrains the number of assumptions needed in Σ to obtain τ. ↻

Another form of compactness is also of great use in classical model theory.

Theorem 3.2 (Compactness II) The set Σ of formulae has a model if and only if its finite subsets have models.

Proof. One direction of this statement is readily immediate. Let Σ have a model $\mathfrak{M}$, so that $\mathfrak{M} \models \Sigma$. Then, clearly, $\mathfrak{M} \models \Sigma_0$ for every finite subset Σ_0 of Σ. The converse direction poses some questions on the general usefulness of this result. In classical logic, one would generally proceed by supposing that Σ does not have any models. Then, Σ is inconsistent, so there is some sentence α such that $\Sigma \vdash \alpha$ and $\Sigma \vdash \neg\alpha$. Since α and $\neg\alpha$ can be derived from Σ using finitary formal proofs, only a finite subset Σ_0 of Σ is needed such that $\Sigma_0 \vdash \alpha$ and $\Sigma_0 \vdash \neg\alpha$. This finite subset Σ_0 therefore cannot have models. ↻

On the other hand, this argument does not carry through to substructural logics in general, since inconsistent theories can and do have models. Indeed, trivial models in which the set of designated propositions consists of the entire algebra means that every theory is theoretically satisfiable. Consequently, this form of compactness holds almost trivially. The result is useful if consistency is of relevance. If every finite subset Σ_0 of Σ is consistent, then Σ itself is consistent. However, consistency readily implies satisfiability and existence of nontrivial models by Soundness and Completeness.

Thus, actually, this compactness result therefore becomes less useful in a more wide-ranging substructural model theory. Nevertheless, it may be used in careful contexts to construct new models with nonstandard features as is the case of compactness in classical model theory. Certainly, the usefulness holds when the underlying logic is the same; thus, the versatility of compactness in classical model theory continues when Σ is a classical theory.

3.3 Substructures and Extensions

In classical model theory, an $\mathscr{L}$-structure $\mathfrak{A}$ is a substructure of another $\mathscr{L}$-structure $\mathfrak{B}$ if $\mathfrak{A}$ is a subset of $\mathfrak{B}$ and the $\mathfrak{A}$-restrictions of all of the functions and relations of $\mathfrak{B}$ coincide with the functions and relations of $\mathfrak{A}$. In such a case, $\mathfrak{B}$ would also be an extension of $\mathfrak{A}$. The situation is similar in the current setup, modified to take into account the algebra of propositions.

Definition 3.3 Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Then, an $\mathscr{L}(\mathbf{LQ})$-structure $\mathfrak{A}$ is a substructure of an $\mathscr{L}(\mathbf{LQ})$-structure $\mathfrak{B}$ if $D_{\mathfrak{A}} \subseteq D_{\mathfrak{B}}$, $K_{\mathfrak{A}} \subseteq K_{\mathfrak{B}}$, $\nabla_{\mathfrak{A}} \subseteq \nabla_{\mathfrak{B}}$, and the restriction of $I_{\mathfrak{B}}$ to the domain and algebra of $\mathfrak{A}$ coincides with $I_{\mathfrak{A}}$.

Definition 3.4 Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}(\mathbf{LQ})$-structure. Then, an object $\mathfrak{A} := \langle D_{\mathfrak{A}}, K_{\mathfrak{A}}, I_{\mathfrak{A}}, \nabla_{\mathfrak{A}} \rangle$ is a potential substructure of $\mathfrak{M}$ if $D_{\mathfrak{A}} \subseteq D_{\mathfrak{M}}$, $K_{\mathfrak{A}} \subseteq K_{\mathfrak{M}}$, $I_{\mathfrak{A}}$ is an appropriate restriction of $I_{\mathfrak{M}}$, and $\nabla_{\mathfrak{A}} \subseteq \nabla_{\mathfrak{M}}$ such that every closed term of $\mathscr{L}$ is represented in $D_{\mathfrak{A}}$ and $K_{\mathfrak{A}}$ is closed with respect to the algebraic operators defined on it. The potential structure $\mathfrak{A}$ is also an $\mathscr{L}(\mathbf{LQ})$-structure and therefore a substructure of $\mathfrak{M}$ if $\mathfrak{A}$ is an $\mathscr{L}(\mathbf{LQ})$-structure such that $K_{\mathfrak{A}}$ can properly model quantification and $\nabla_{\mathfrak{A}}$ properly designates the interpretations of all theorems of **LQ**.

The potential problem when considering an arbitrary piece of some true $\mathscr{L}$-substructural structure is the possibility that the slice is too mal-formed to be of use. A similar situation in classical model theory may be the situation where a set of parameters of a provided $\mathscr{L}$-structure is itself not sufficiently constituted to be itself an $\mathscr{L}$-structure. The notion of enrichment would address this issue. An $\mathscr{L}(\mathbf{LQ})$-structure $\mathfrak{M}$ is sufficiently enriched if every potential substructure is indeed a veritable substructure of $\mathfrak{M}$ which is also an $\mathscr{L}(\mathbf{LQ})$-structure. In a way, this concept is akin to sufficient saturation of a theory in which existential statements have witnesses; in the current context, there are sufficiently many propositions to witness each expression of quantification.

Of course, it would be appropriate to compare arbitrary $\mathscr{L}(\mathbf{LQ})$-structures that may not explicitly share common individuals or propositions. It is appropriate to let $\mathfrak{A}$ be a substructure of $\mathfrak{B}$ if there is an embedding map h such that $D_{\mathfrak{A}} \hookrightarrow D_{\mathfrak{B}}$ and $K_{\mathfrak{A}} \hookrightarrow K_{\mathfrak{A}}$ with $\nabla_{\mathfrak{A}} \hookrightarrow \nabla_{\mathfrak{B}}$ which appropriately acts with $I_{\mathfrak{A}}$ and $I_{\mathfrak{B}}$.

Some extensions of a substructure are notable for increasing the domain of individuals while maintaining the algebra of propositions found in the substructure. For these, certain model-theoretic considerations from the classical setting carry through with little need to make constant reference to the propositions. Because of how the structures are generally defined, it is not possible to have an extension that maintains the same domain of individuals while increasing the algebra of propositions.

3.4 Elementary Equivalence and Elementary Substructures

In classical model theory, two $\mathscr{L}$-structures are elementarily equivalent if they have the same theory, so that they satisfy the same set of sentences. The following definition appears to be the most useful formulation of elementary equivalence in substructural model theory.

Definition 3.5 (Elementary Equivalence) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathscr{L}(\mathbf{LQ})$-structures. Then, $\mathfrak{A}$ and $\mathfrak{B}$ are **LQ**-elementarily equivalent, denoted $\mathfrak{A} \equiv_{\mathbf{LQ}} \mathfrak{B}$, if there is a map σ :

$K_{\mathfrak{A}} \to K_{\mathfrak{B}}$ one-to-one and onto in the algebra of propositions that preserves $\nabla_{\mathfrak{A}}$ in $\nabla_{\mathfrak{B}}$ such that, for every sentence τ, $\sigma(I_{\mathfrak{A}}(\tau)) = I_{\mathfrak{B}}(\tau)$, so that $\mathfrak{A} \models \tau$ if and only if $\mathfrak{B} \models \tau$.

A weaker definition of elementary equivalence, one which may seem more intuitively appropriate, may state that $\mathfrak{A} \equiv_{\mathbf{LQ}} \mathfrak{B}$ when $\mathfrak{A} \models \tau$ if and only if $\mathfrak{B} \models \tau$ for every sentence τ. Such a definition would be able to accommodate inconsistent theories and incomplete theories to a certain extent. However, elementary equivalence is also used in classical model theory to establish some deeper results when taking into account parameters and the idea of a diagram of a model. In order to preserve these abilities in substructural model theory, the notion of elementary equivalence must be robust enough to provide information not just about the satisfiability of a particular sentence but also information about the satisfiability of the negation of such a sentence. The definition provided in this work can provide such information by referring to specific propositions whose negations can then be checked through the algebra of propositions, whereas the weaker proposal does not provide such information due to the inexactness of the satisfiability relation.

Another elementary notion is that of a structure $\mathfrak{A}$ being an elementary substructure of another structure $\mathfrak{B}$; in such a case, $\mathfrak{B}$ would be an elementary extension of $\mathfrak{A}$. Such a relation holds if there is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

Definition 3.6 (Elementary Substructures and Extensions) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathscr{L}$(**LQ**)-structures such that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$. Then, $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$, and $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$, if it is also the case that, where i is the embedding of $\mathfrak{A}$ into $\mathfrak{B}$, $i(I_{\mathfrak{A}}(\alpha(\bar{a}))) = I_{\mathfrak{B}}(\alpha(i(\bar{a})))$. Consequently, $I_{\mathfrak{A}}(\alpha)_{\int} \in \nabla_{\mathfrak{A}}$ if and only if $I_{\mathfrak{B}}(\alpha)_{\int} \in \nabla_{\mathfrak{A}}$. In such a case, the relation is notated $\mathfrak{A} \prec_{\mathbf{LQ}} \mathfrak{B}$.

3.5 Downward Löwenheim-Skolem Theorem

Theorem 3.7 (Downward Löwenheim-Skolem) Let $\mathscr{L}$ be a language of cardinality $\varkappa$, and let **LQ** be a logic. Let Σ be a collection of formulae. Then, if Σ has a model, then Σ has a model $\mathfrak{M}$ such that $D_{\mathfrak{M}}$ has cardinality $\varkappa$.

Proof. The theorem is an immediate consequence of the Henkin construction for completeness. ○

A more substantial question about the Löwenheim-Skolem theorem is whether the downward result can be strengthened so that it is possible to obtain not only a model of some $\mathscr{L}$-theory T with the same cardinality as the language, but also whether that such an $\mathscr{L}$-theory T with a model $\mathfrak{M} \models T$ can have a model $\mathfrak{N}$ such that $\mathfrak{N}$ is an elementary substructure of $\mathfrak{M}$. This issue can be resolved in much the same way as in classical model theory, and so what follows is a rough sketch.

Le C be a set of constant symbols not in the language $\mathscr{L}$, with enough constant symbols for every $\mathscr{L}$-formula but not enough to change the cardinality of the augmented language. Since the language $\mathscr{L}$ has cardinality $\varkappa$, there are only such many formulae expressible in the language. For every $\mathscr{L}$-formula $\phi(\bar{x})$ with free variables $\bar{x}$, let the formula $\phi(\bar{x}) \leftrightarrow \phi(\bar{c})$ be built up using constant symbols not in $\mathscr{L}$ for $\bar{c}$. Then, if $\mathfrak{M} \models \phi(\bar{x})$, let $\mathfrak{M} \models \phi(\bar{x}) \leftrightarrow \phi(\bar{c})$ as well. Let Δ denote the $\mathscr{L}$-diagram of $\mathfrak{M}$ adorned with the constant symbols and formulae taken up according to this manner. Then, $\mathfrak{M} \models_{\mathbf{LQ}} \Delta$, so by Theorem 3.7, there is some $\mathfrak{N}$ of cardinality $\varkappa$ such that $\mathfrak{N} \models_{\mathbf{LQ}} \Delta$. Now, since $\Sigma \subseteq \Delta$, it is also the case that $\mathfrak{N} \models_{\mathbf{LQ}} \Sigma$, and $\mathfrak{N}$ could very well have been obtained from the application of Theorem 3.7 to Σ. Nevertheless, $\mathfrak{N}$ has the added benefit that its behavior is strictly outlined according to Δ. An embedding that maps $c^{\mathfrak{N}}$ to $c^{\mathfrak{M}}$ straightforwardly gives an elementary map that witnesses the relation $\mathfrak{N} \prec \mathfrak{M}$.

3.6 Upward Löwenheim-Skolem Theorem

An upward version of the Löwenheim-Skolem result would have the following form.

Theorem 3.8 (Upward Löwenheim-Skolem) Let $\mathscr{L}$ be a language of cardinality $\varkappa$, and let **LQ** be a logic. Let Σ be a collection of formulae. Then, if Σ has a model, then Σ has a model $\mathfrak{M}$ such that $D_{\mathfrak{M}}$ has cardinality λ for every $\lambda > \varkappa$.

Proof. This result can be obtained trivially by expanding the language $\mathscr{L}$ with sufficiently many new constant symbols to then build up a model of the required cardinality. If $\mathscr{L}$ contains an equality predicate $\doteq$, then the preservation of this symbol to the intuitive sense of identity can complicate matters. As observed in [Restall 1994], a more sensible result would preserve whatever notion of equality holds in the original structure or theory rather than demolishing the predicate into a near-meaningless abstraction. To guarantee an upward Löwenheim-Skolem result that preserves some sense of meaningful equality, there are two ways proposed which can readily be adapted in the more general case of the current substructural model theory. One way is to introduce the concept of coherent and compact algebras, such that structures whose algebra of propositions is indeed a coherent compact algebra can be manipulated to provide the desired models of higher cardinality. Another way is to use the notion of Boolean equality discussed previously, either in conjunction with or replacing the original notion of equality, so that a formula expressing $\dot{a} \doteq \dot{b}$ must be interpreted by $\mathfrak{M}$ as $a^{\mathfrak{M}} = b^{\mathfrak{M}}$ and $\neg \dot{a} \doteq \dot{b}$ is to be interpreted as $a^{\mathfrak{M}} \neq b^{\mathfrak{M}}$. ↻

Essentially, these two cases are the only means of obtaining a nontrivial upward result with equality. The issue is not unlike a similar situation that arises in the study of abstract elementary classes. Although abstract elementary classes are beyond the immediate scope of this work, a problem that potentially arises in a given abstract elementary class is that it is not well-formed enough to guarantee structures of high cardinality belonging to that abstract elementary class.

3.7 Isomorphisms and Elementary Maps

In classical model theory, two $\mathscr{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic if there is a map h between them one-to-one and onto such that for every atomic formula $\alpha(\bar{x})$, $\mathfrak{A} \models \alpha(\bar{a})$ if and only if $\mathfrak{B} \models \alpha(h(\bar{a}))$. Two isomorphic structures will look essentially the same model-theoretically.

Similarly, it is possible to define isomorphisms between substructural models.

Definition 3.9 (Isomorphisms Between Models) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathscr{L}(\mathbf{LQ})$-structures. Then, $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ with respect to $\mathscr{L}$ and **LQ**, denoted $\mathfrak{A} \cong_{\mathscr{L}(\mathbf{LQ})} \mathfrak{B}$ (or $\mathfrak{A} \cong \mathfrak{B}$ if the context is clear), if there is a map h between their domains of individuals one-to-one and one such that, for every n-place function symbol f in $\mathscr{L}$ and for every m-place predicate symbol p in $\mathscr{L}$, it is the case that $h(f^{\mathfrak{A}}(t_1^{\mathfrak{A}}, ..., t_n^{\mathfrak{A}})) = f^{\mathfrak{B}}(h(t_1^{\mathfrak{A}}), ..., h(t_n^{\mathfrak{A}}))$ and $h(p^{\mathfrak{A}}(t_1^{\mathfrak{A}}, ..., t_m^{\mathfrak{A}})) = p^{\mathfrak{B}}(h(t_1^{\mathfrak{A}}), ..., h(t_m^{\mathfrak{A}}))$.

The following result holds in substructural model theory just as in classical model theory.

Proposition 3.10 Two $\mathscr{L}(\mathbf{LQ})$-structures $\mathfrak{A}$ and $\mathfrak{B}$ that are isomorphic are also elementarily equivalent. That is, if $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$.

The proof is also essentially the same as in the classical context.

Partial and total elementary maps are functions that demand greater fidelity to the shape of the comparative structures. Of course, a total elementary map is actually an elementary embedding.

Definition 3.11 (Partial Elementary Maps) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathscr{L}(\mathbf{LQ})$-structures. Then, a partial elementary map from $\mathfrak{A}$ to $\mathfrak{B}$ is a map $\mathfrak{i}$ generated by a function $i : X \to D_{\mathfrak{B}}$ with $X \subseteq D_{\mathfrak{A}}$ such that $I_{\mathfrak{A}}(\alpha(\bar{a})) = \mathfrak{a}$ implies $I_{\mathfrak{B}}(\alpha(i(\bar{a}))) = \mathfrak{i}(\mathfrak{a})$, where α is an arbitrary formula, $\bar{a} \in X^n$ is a finite sequence of n individuals, and $\mathfrak{a}$ is a proposition in $K_{\mathfrak{A}}$.

3.8 Model Constructions through Reduced Products

The early development of classical first-order model theory made considerable use of Compactness in order to find nonstandard models of popular mathematical theories. Construction techniques from universal algebra provided concrete examples of these nonstandard models by building new models from existing ones. Key to the contribution of universal algebra is the idea of the ultraproduct, which relies on the notion of an ultrafilter.

Definition 3.12 Let $\mathscr{L}$ be a language, and let I be an index. Let $\mathfrak{M}_i$, $i \in I$, be a collection of $\mathscr{L}$-structures modeling a common $\mathscr{L}$-theory T. The Cartesian product

$$\mathfrak{M} := \prod_{i \in I} \mathfrak{M}_i := \{(c_i)_{i \in I} : c_i \in \mathfrak{M}_i\}$$

defined on I is the *direct product* of the family of $\mathscr{L}$-structures constituting $\mathfrak{M}_i$. Let U be an ultrafilter defined on I, and let $\approx$ be an equivalence relation defined on $\mathfrak{M}$ such that, for a and b in $\mathfrak{M}$, $a \approx b$ when $\{i \in I : a_i = b_i\} \in U$. Then, the *ultraproduct* of $\mathfrak{M}$ over U is the $\mathscr{L}$-structure $\mathfrak{M}^*$ obtained as the quotient structure

$$\mathfrak{M}^* := \mathfrak{M}/U = \{c/\approx : c \in \mathfrak{M} \text{ and } \approx \text{ is the relation defined on } U\}$$

based on this equivalence relation, with all relations and functions of $\mathfrak{M}^*$ built up accordingly in the same manner.

Theorem 3.13 (Łos) Let $\mathscr{L}$ be a language, and let $\phi(\bar{x})$ be a formula with free variables $\bar{x}$. Let $\mathfrak{M}^*$ be an ultraproduct of $\mathfrak{M}$ over U. Then, $\mathfrak{M}^* \models \phi(\bar{c}/\approx)$ if and only if $\{i \in I : \mathfrak{M}_i \models \phi(\bar{c_i})\} \in U$.

Unfortunately, ultraproducts are of little help in the model construction of substructural model theory. The issue is that they unavoidably rely on the Boolean algebraic structure of classical model structures by basing the equivalence relation on an ultrafilter. Indeed, an ultrafilter works by imposing a Boolean structure on its quotient objects. Thus, ultraproducts do not carry through to the current work in substructural model theory and allow for a particular substructural way of forming new models.

Despite their usefulness, ultraproducts do not strictly provide for new objects in classical first-order model theory that could not be obtained by using Compactness. Their usefulness is primarily in providing concrete examples of nonstandard models whose existence is called for by particular applications of Compactness. Thus, the inability to translate ultraproducts to substructural model theory does not mean that substructural models are in a sense more difficult to obtain. Nevertheless, it is rather disappointing that the role of ultraproducts may not extend in its current form into substructural model theory.

Still, it is possible to salvage some aspects of this model construction at a cost. Ultraproducts themselves are examples of *reduced products*. A reduced product is the quotient of a Cartesian product by an equivalence relation based on a filter which need not be an ultrafilter. An ultraproduct is thus a reduced product with an ultrafilter. The ultraproduct is used in classical model theory rather than the more general reduced product because of the theorem of Łos. Theorem 3.13 depends on the ultrafilter when proving the induction case of a well-formed formula $\neg\alpha$ when the case of α is assumed to hold. In order to let reduced products play a similar role with substructural model theory, such objects must overcome this barrier so as to achieve a result similar to Theorem 3.13.

In classical model theory, a more limited statement than Theorem 3.13 holds for reduced products. Specifically, if $\mathfrak{M}_i \models \phi$ for a sentence ϕ, then the reduced product $\mathfrak{M}^*$ has the property that $\mathfrak{M}^* \models \phi$ if and only if ϕ is equivalent to a *Horn sentence*. For a language $\mathscr{L}$, an $\mathscr{L}$-formula ϕ is a *basic Horn formula* if ϕ is a disjunction of literals

$$\phi = \alpha_1 \vee ... \vee \alpha_n$$

where at most one of the α_i $(1 \leq i \leq n)$ is an atomic formula and the others are negations of atomic formulae. The sentence ϕ, if containing more than one literal, would be logically equivalent to a formula ϕ' defined as

$$\phi' = \beta_1 \wedge ... \wedge \beta_{n-1} \rightarrow \alpha_k$$

where α_k is the one atomic formula in ϕ. A well-formed *Horn formula* would be a basic Horn formula or one build up using $\wedge$, $\exists$, and $\forall$, and a Horn formula which is an $\mathscr{L}$-sentence would be a Horn sentence. A sentence logically equivalent to a Horn sentence can be called a *reduced product sentence*.

These problems must also affect the substructural model theory, for if it were possible to directly adapt reduced products into the substructural setting and find that they satisfy the entirety of a substructural version of Theorem 3.13, then the application to classical logic would yield a contradiction. Since model constructions using direct products, reduced products, and ultraproducts do not provide models that cannot be obtained from the invocation of Compactness, these problems will not be elaborated much further, but it may be beneficial to outline what the substructural analogue of reduced products would look like.

Definition 3.14 Let $\mathscr{L}$ be a language, let **LQ** be a logical calculus, and let I be an index with each $\mathfrak{M}_i$, $i \in I$, being an $\mathscr{L}(\mathbf{LQ})$-structure modeling a common theory T. Let U be a filter, and let

$$\approx_D := \{i \in I : a_i = b_i \in D_{\mathfrak{M}_i}\} \in U$$

where a and b are in $\mathfrak{M}$ with a_i and b_i in $\mathfrak{M}_i$ and

$$\approx_K := \{i \in I : \mathfrak{a}_i = \mathfrak{b}_i \in K_{\mathfrak{M}_i}\} \in U$$

with $\mathfrak{a}$ and $\mathfrak{b}$ in $K_{\mathfrak{M}}$ be equivalence relations defined by this filter. Let $*\mathfrak{M} = \langle D_{*\mathfrak{M}}, K_{*\mathfrak{M}}, I_{*\mathfrak{M}}, \nabla_{*\mathfrak{M}} \rangle$ be the object obtained such that

$$D_{*\mathfrak{M}} := \{a/U : a \in \prod_{i \in I} D_{\mathfrak{M}_i}\},$$

$$K_{*\mathfrak{M}} := \{\mathfrak{a}/U : \mathfrak{a} \in \prod_{i \in I} K_{\mathfrak{M}_i}\},$$

and

$$\nabla_{*\mathfrak{M}} \subseteq K_{*\mathfrak{M}} \text{ such that } \mathfrak{a} \in \nabla_{*\mathfrak{M}} \Leftrightarrow \{i \in I : \mathfrak{a}_i \in \nabla_{\mathfrak{M}_i}\} \in U.$$

The interpretation function $I_{*\mathfrak{M}}$ is defined as follows. For a constant symbol c, let $c^{*\mathfrak{M}} := (c^{\mathfrak{M}_i} : i \in I)/\approx_D$, and for each n-place function symbol f, let $f^{*\mathfrak{M}}$ be the quotient function obtained from $f^{\mathfrak{M}_i}$ at each $i \in I$ with respect to $\approx_D$. For an n-place predicate symbol p and with terms $t_1, ..., t_n$, let

$$p^{*\mathfrak{M}}(t_1{}^{*\mathfrak{M}}, ..., t_n{}^{*\mathfrak{M}}) = (p^{\mathfrak{M}_i}(t_1{}^{\mathfrak{M}_i}, ..., t_n{}^{\mathfrak{M}_i}) : i \in I)/\approx_K .$$

Let $I_{*\mathfrak{M}}$ be further defined, where $\int$ is a variable assignment function, so that if α and β are well-formed formulae and x is an individual variable, then

1. $I_{*\mathfrak{M}}(\neg\alpha)_{\int} = -I_{*\mathfrak{M}}(\alpha)_{\int}$;
2. $I_{*\mathfrak{M}}(\alpha \wedge \beta)_{\int} = I_{*\mathfrak{M}}(\alpha)_{\int} \bar{\wedge} I_{*\mathfrak{M}}(\beta)_{\int}$,
   $I_{*\mathfrak{M}}(\alpha \vee \beta)_{\int} = I_{*\mathfrak{M}}(\alpha)_{\int} \bar{\vee} I_{*\mathfrak{M}}(\beta)_{\int}$,
   $I_{*\mathfrak{M}}(\alpha \rightarrow \beta)_{\int} = I_{*\mathfrak{M}}(\alpha)_{\int} \searrow I_{*\mathfrak{M}}(\beta)_{\int}$, and
   $I_{*\mathfrak{M}}(\alpha \circ \beta)_{\int} = I_{*\mathfrak{M}}(\alpha)_{\int} \otimes I_{*\mathfrak{M}}(\beta)_{\int}$; and
3. $I_{*\mathfrak{M}}(\forall x \alpha)_{\int} = \bigwedge I_{*\mathfrak{M}}(\alpha)_{\int}{}'$ and $I_{*\mathfrak{M}}(\exists x \alpha)_{\int} = \bigvee I_{*\mathfrak{M}}(\alpha)_{\int}{}'$.

Essentially, the construction of the substructural reduced product involves taking the reduced products of the domains and algebras of the family of substructural models indexed by some index. These details on the whole are rather too general to be of much further use, and a key problem which remains is to which extent substructural reduced products satisfy the sentences in the $\mathscr{L}$-theory T which are modeled by the family of substructural models.

The topics covered to this point are roughly analogous to what can be considered to be part of the core of the classical development of classical first-order model theory, the field as it stood shortly before more recent developments which fueled new research directions in stability, order-minimality, and geometric model theory. Since the Galois correspondence of model structures and their automorphism groups arose from this latter development, it would be appropriate to continue the current discourse of substructural model theory to address these concerns.

Chapter 4

Definability and Algebraicity

The previous section delved into elementary portions of a substructural model theory. This section will tackle some other concepts that are readily used in the contemporary practice of the subject. As with moving into a new situation, some concepts follow through to the substructural model theory better than other concepts.

For example, consider the notion of an n-type. Let $\mathscr{L}$ be a language, and let $\mathfrak{M}$ be an $\mathscr{L}$-structure. Then, an n-type p (with no parameters) is a collection of $\mathscr{L}$-formulae that is consistent with the theory $\mathrm{Th}(\mathfrak{M})$ of $\mathfrak{M}$, so there is a model $\mathfrak{N}$ of $\mathrm{Th}(\mathfrak{M})$ that realizes p, which is to say that there is some finite sequence $\bar{a}$ of n individuals such that $\mathfrak{N} \models \phi(\bar{a})$ for every $\phi(\bar{x}) \in p$. An n-type is a complete type if at least one of $\phi(\bar{x})$ and $\neg\phi(\bar{x})$ is in it; otherwise, the n-type is a partial type.

In abstract terms, n-types also allow for a way to divide up the (n-Cartesian product of the) domain of an $\mathscr{L}$-structure into identifiably different groupings, and this division of the domain hence inspires the usage of the term. Formally, a finite sequence $\bar{a}$ of n individuals in $\mathfrak{M}$ has a type associated with it, denoted $\mathrm{tp}_{\mathfrak{M}}(\bar{a})$. The sequence $\bar{a}$ will realize its type, satisfying exactly every formula in $\mathrm{tp}_{\mathfrak{M}}(\bar{a})$ and none other, so the type characterizes $\bar{a}$ in a model-theoretic manner. Two sequences which satisfy the same formulae will have the same type, and in the object language, they will be indistinguishable from each other.

In contemporary practice, n-types play an important role in defining additional concepts that are in turn investigated by the latest trends in research. For example, the notion of stability is based on how large the n-type space of a structure of some cardinality is. Relatively small models of theories can omit certain n-types but not others, so there are different kinds of n-types. Saturated models which realize all n-types thus also tend to be large.

Converting this body of results into a substructural model theory presents some difficult challenges, and these issues will be noted later in this work. Other model-theoretic concepts carry through relatively well, and it may be worthwhile to outline these ideas in order to inform which approaches may be appropriate to handling n-types in substructural logics. Specifically, the notions of definability and algebraicity will be addressed.

4.1 Defining Sets and Individuals with Propositions

Let $\mathscr{L}$ be a language, and let the logic be classical. Let $\mathfrak{M}$ be an $\mathscr{L}$-structure. In classical model theory, a set $X \subseteq \mathfrak{M}$ is definable without parameters in $\mathfrak{M}$ if there is some $\mathscr{L}$-formula $\phi(x)$ such that $a \in X$ if and only if $\mathfrak{M} \models \phi(a)$, so that only the individuals in X satisfy $\phi(x)$. This definition follows through when considering arbitrary finite sequences of individuals rather than single ones.

In changing the underlying logic from classical to substructural, the key challenge is addressing how propositions enter the picture. For the moment, let the exact proposition be ignored.

Definition 4.1 (Weak Definability) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}(\mathbf{LQ})$-structure. Then, a set $X \subseteq D_{\mathfrak{M}}$ is *weakly definable* if there is some $\mathscr{L}$-formula $\alpha(x)$ with one free variable such that $\mathfrak{M} \models_{\mathbf{LQ}} \alpha(c)$ if and only if $c \in X$.

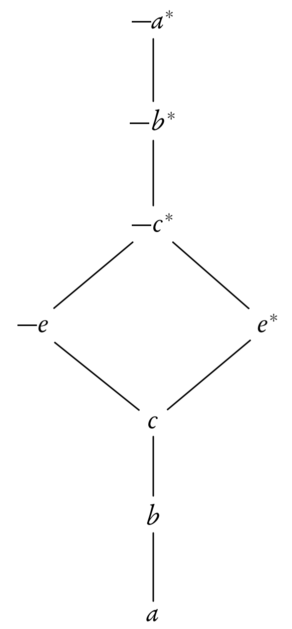
The set $X \subseteq D_{\mathfrak{M}}$ is weakly definable over a set $A \subseteq D_{\mathfrak{M}}$ of parameters if there is some $\mathscr{L}(A)$-formula $\alpha(x)$ with one free variable such that $\mathfrak{M} \models_{\mathbf{LQ}} \alpha(c)$ for every $c \in X$, or equivalently if there is some $\mathscr{L}$-formula $\alpha(x,\bar{y})$ with free variable x and finite sequence $\bar{y}$ of free variables and a finite sequence $\bar{d}$ of individuals in A such that $\mathfrak{M} \models_{\mathbf{LQ}} \alpha(c,\bar{d})$ for every $c \in X$.

Of course, $\mathfrak{M} \models_{\mathbf{LQ}} \alpha(c)$ means $I_{\mathfrak{M}}(\alpha(c)) \in \nabla_{\mathfrak{M}}$. So, X is weakly definable if every individual in X can just manage to map $\alpha(x)$ to an accepted proposition. It may be more desirable to make use of a different notion of definability, one that forces the individuals of a definable set to behave more consistently.

Definition 4.2 (Strong Definability) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}(\mathbf{LQ})$-structure. Let A be a subset of $D_{\mathfrak{M}}$. Let $\mathfrak{a}$ be a proposition in $\nabla_{\mathfrak{M}}$. Then, a set $X \subseteq D_{\mathfrak{M}}$ is *strongly definable* over A at $\mathfrak{a}$, or is $A(\mathfrak{a})$-definable, if there is some $\mathscr{L}(A)$-formula $\alpha(x)$ with one free variable such that $I_{\mathfrak{M}}(\alpha(c)) = \mathfrak{a}$ if and only if $c \in X$.

Weak and strong definability of sets of finite sequences of individuals straightforwardly extends from weak and strong definability of sets of individuals. Clearly, a strongly definable set is also weakly definable. Whether a weakly definable set is also strongly definable seems more difficult to answer in generality. Certainly, it is possible to construct a scenario where a weakly definable set defined by a formula that is not immediately strongly definable can be made strongly definable.

Example 4.3 Let $\mathscr{L}$ be a language with propositional constant t, and let **LQ** be a substructural logic that refutes K. Let $\mathfrak{A}$ be an $\mathscr{L}(\mathbf{LQ})$-structure such that $K_{\mathfrak{A}}$ is defined by the following Hasse diagram and $\otimes$-table.



$\otimes$	a	b	c	$-e$	e	$-c$	$-b$	$-a$
a	a	a	a	a	a	a	a	a
b	a	b	b	$-e$	b	$-e$	$-e$	$-a$
c	a	b	c	$-e$	c	$-e$	$-b$	$-a$
$-e$	a	$-e$	$-e$	$-a$	$-e$	$-a$	$-a$	$-a$
e	a	b	c	$-e$	e	$-c$	$-b$	$-a$
$-c$	a	$-e$	$-e$	$-a$	$-c$	$-a$	$-a$	$-a$
$-b$	a	$-e$	$-b$	$-a$	$-b$	$-a$	$-a$	$-a$
$-a$	a	$-a$	$-a$	$-a$	$-a$	$-a$	$-a$	$-a$

The designated propositions are those with asterisks, and $\otimes$-table is read with the row representing the left argument and the column representing the right argument.

Let $X \subseteq D_{\mathfrak{A}}$ be a set defined by a formula $\phi(x)$ such that $\phi(X) \mapsto \nabla_{\mathfrak{A}}$; that is, every proposition in $\nabla_{\mathfrak{A}}$ is the image of some individual in X. Since t is in the language, t is mapped to the identity element e in $K_{\mathfrak{A}}$ which is also the lowest-valued accepted proposition. Thus, the formula $\phi(x) \wedge t$ is also satisfied by X and it is also the case that X is strongly defined by $\phi(x) \wedge t$ to the proposition e.

However, there is no guarantee in general that every weakly definable set can be made strongly definable. In addition to defining sets of the domain of individuals, it is also possible to define finite sequences of individuals.

Definition 4.4 Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}(\mathbf{LQ})$-structure. Let $\bar{a}$ be a finite sequence of individuals in $D_{\mathfrak{A}}$. Let A be a subset of $D_{\mathfrak{M}}$. Then, $\bar{a}$ is weakly definable over A if the set $\{\bar{a}\}$ is weakly definable over A, and $\bar{a}$ is strongly definable over A at proposition $\mathfrak{a} \in \nabla_{\mathfrak{M}}$ if the singleton $\{\bar{a}\}$ is strongly definable over A.

Here, it is immediately evident that any weakly definable sequence of individuals is strongly definable. In some structures, a sequence of individuals may be strongly definable to several propositions, but such an ability is not a problem. The key matter of importance is that such a sequence verily has the property of strong definability.

4.2 Algebraic Sets

A related concept to definability is the notion of algebraicity. In looking at definable objects, the work proceeded first from weakly and strongly definable sets to definable individuals. An algebraic set is simply something consisting of algebraic individuals, so it is more sensible look at finite sequences of individuals first.

In classical model theory, a finite sequence $\bar{a}$ of individuals in some $\mathscr{L}$-structure $\mathfrak{M}$ is algebraic over a parameter set $A \subseteq \mathfrak{M}$ if there is some $\mathscr{L}(A)$-formula $\phi(\bar{x})$ such that $\mathfrak{M} \models \phi(\bar{a})$ and the solution set of $\phi(\bar{x})$, the set of all finite sequences that satisfy $\phi(\bar{x})$ and denoted $\phi(\mathfrak{M})$, is finite. This formula $\phi(\bar{x})$ is also said to be an algebraic formula. A set A is algebraic when it is defined by an algebraic formula.

Moving to the substructural case, it is immediately clear that the notion of an algebraic set ought not change, although different notions of algebraicity would require different notions of algebraic sets. With respect to individuals and finite sequences of individuals, a straightforward notion of algebraicity readily translates to the substructural case.

Definition 4.5 (Weak Algebraicity) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}(\mathbf{LQ})$-structure. Let $\bar{a}$ be a finite sequence of individuals in $D_{\mathfrak{M}}$. Let A be a subset of $D_{\mathfrak{M}}$. Then, $\bar{a}$ is *weakly algebraic* over A if there is some $\mathscr{L}(A)$-formula $\phi(\bar{x})$ such that $I_{\mathfrak{A}}(\phi(\bar{a})) \in \nabla_{\mathfrak{M}}$, so that $\mathfrak{M} \models_{\mathbf{LQ}} \phi(\bar{a})$, and there are finitely many $\bar{c}$ overall such that $\mathfrak{M} \models_{\mathbf{LQ}} \phi(\bar{c})$.

Weak algebraicity only concerns with whether or not a particular finite sequence of individuals satisfies a formula that is satisfied by finitely many others. A stronger notion of algebraicity would perhaps make some mention of a specific proposition. There are several ways of accomplishing this aim, but the following one seems robust.

Definition 4.6 (Strong Algebraicity) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}(\mathbf{LQ})$-structure. Let $\bar{a}$ be a finite sequence of individuals in $D_{\mathfrak{M}}$. Let A be a subset of $D_{\mathfrak{M}}$, and let $\mathfrak{a}$ be an accepted proposition. Then, $\bar{a}$ is *strongly algebraic* over A to $\mathfrak{a}$, or is $A(\mathfrak{a})$-algebraic, if there is some $\mathscr{L}(A)$-formula $\alpha(\bar{x})$ such that $I_{\mathfrak{M}}(\alpha(\bar{a})) = \mathfrak{a}$, there are finitely many $\bar{b}$ such that $I_{\mathfrak{M}}(\alpha(\bar{b})) = \mathfrak{a}$ also, and there are finitely many $\bar{c}$ overall such that $\mathfrak{M} \models_{\mathbf{LQ}} \alpha(\bar{c})$.

Clearly, any finite sequence of individuals which is strongly algebraic will be weakly algebraic. Similarly, the conditions for weak algebraicity imply strong algebraicity. If $\bar{a}$ is weakly algebraic, then some formula $\phi(\bar{x})$ will map $\bar{a}$ to an accepted proposition $\mathfrak{a}$ while $\phi(\mathfrak{M})$ is a finite solution set. Thus, there can only be finitely many other sequences that also map to $\mathfrak{a}$, and the conditions are satisfied to permit an assertion that $\bar{a}$ is strongly algebraic.

A finite sequence of individuals belongs to a weakly definable set that has finitely many members. A natural question that arises is what is the smallest such definable set possible. If the algebraic sequence is definable, for example, then the smallest definable set is its singleton, and for nondefinable algebraic sequences, this idea can measure how far away the sequence is from becoming definable in a particular sense. To find this smallest set, it is necessary to frame the idea of irreducibility.

Definition 4.7 (Irreducible Formulae) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}(\mathbf{LQ})$-structure. Let $\bar{a}$ be a finite sequence of individuals in $D_{\mathfrak{M}}$, and let $\mathfrak{a}$ be an accepted proposition so that $\bar{a}$ is $\mathfrak{a}$-algebraic. Let A be a subset of $D_{\mathfrak{M}}$. Then, there is an $\mathscr{L}(A)$-formula $\phi(\bar{x})$, called an irreducible formula of $\bar{a}$ over A and denoted $\mathrm{irr}(\bar{a}/A)$, such that $I_{\mathfrak{M}}(\phi(\bar{a})) \in \nabla_{\mathfrak{M}}$ and $\phi(\mathfrak{M})$ is the smallest solution set containing $\bar{a}$. That is, if $\bar{a}$ maps an arbitrary formula $\psi(\bar{x})$ to an accepted proposition, then $\phi(\mathfrak{M}) \subseteq \psi(\mathfrak{M})$.

Irreducible formulae actually lead to another notion of algebraicity, one which imposes more conditions than strong algebraicity.

Definition 4.8 (Normalized Algebraicity) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}(\mathbf{LQ})$-structure. Let $\bar{a}$ be a finite sequence of individuals in $D_{\mathfrak{M}}$, and let $\mathfrak{a}$ be an accepted proposition so that $\bar{a}$ is $\mathfrak{a}$-algebraic. Let A be a subset of $D_{\mathfrak{M}}$. Then, $\bar{a}$ is *normalizably algebraic* over A at $\mathfrak{a}$ if there is an $\mathscr{L}(A)$-formula $\phi(\bar{x})$ such that $I_{\mathfrak{M}}(\phi(\bar{a})) = \mathfrak{a}$ and there are finitely many other $\bar{b}$ such that $I_{\mathfrak{M}}(\phi(\bar{b})) = \mathfrak{b}$ such that there is no other $\bar{c}$ with $I_{\mathfrak{M}}(\phi(\bar{c})) \in \nabla_{\mathfrak{M}}$.

A key result here is that the three kinds of algebraicity coincide in the sense that every finite sequence that is weakly algebraic must also be normalizably algebraic. The key is with the irreducible formula for a finite sequence to an accepted proposition.

Proposition 4.9 Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}(\mathbf{LQ})$-structure with $A \subseteq D_{\mathfrak{M}}$. Let $\bar{a}$ be a finite sequence of individuals in $D_{\mathfrak{M}}$. Then, if $\bar{a}$ is weakly algebraic, then $\bar{a}$ has an irreducible formula.

Proof. If $\bar{a}$ is weakly algebraic, then there is some formula $\phi_1(\bar{x})$ such that $I_{\mathfrak{M}}(\phi_i(\bar{a})) \in \nabla_{\mathfrak{M}}$. If $\phi_1(\bar{x})$ is not irreducible, then there is another formula $\phi_2(\bar{x})$ such that $\bar{a} \in \phi_2(\mathfrak{M})$ and $\phi_2(\mathfrak{M}) \cap \phi_1(\mathfrak{M}) \;\; \phi_1(\mathfrak{M})$. Since $\phi_1(\mathfrak{M})$ is finite, and so indeed is $\phi_2(\mathfrak{M})$, this process of finding successively more irreducible formulae in the sense that their solution sets continue to decrease is a finitary one. Hence, $\bar{a}$ must have an irreducible $\mathscr{L}(A)$-formula. ↻

Proposition 4.10 Let $\bar{a}$ be weakly algebraic over A, and let $\phi(x)$ be an irreducible formula for $\bar{a}$. Then, $\bar{a}$ is normalizably algebraic with respect to $\phi(x)$.

Proof. Suppose that $|\phi(\mathfrak{M})| = n$ for some $n \in \mathbb{N}$. Then, it must be the case that

$$\mathfrak{M} \models_{\mathbf{LQ}} \exists x_1 \dots \exists x_{n-1} (\bigwedge_{1 \leq i \leq n-1} \phi(\bar{x}) \leftrightarrow \phi(x_i))$$

with $\bar{x} \mapsto \bar{a}$. Otherwise, this formula would itself contradict irreducibility by implying the existence of a smaller solution set containing $\bar{a}$. Thus, $\phi(\mathfrak{M})$ has the property that

$$I_{\mathfrak{M}}(\phi(\bar{c})) = I_{\mathfrak{M}}(\phi(\bar{a})) \Leftrightarrow c \in \phi(\mathfrak{M}).$$

Thus, $\bar{a}$ is normalizably algebraic with respect to $\phi(\bar{x})$. ↻

Now, normalizable algebraicity implies strong algebraicity, which in turn implies weak algebraicity. Conversely, weak algebraicity turns out to also imply normalized algebraicity. Thus, all three notions of algebraicity coincide and are extensionally the same.

The equivalence of these intuitively different reactions to algebraicity in the substructural context is a fortuitous result. In this regard, the continued development of this model theory does not depend on a choice made of one particular form of algebraicity to the exclusion of the others. As with the situation of weakly and strongly definable sequences of individuals, the normalized algebraicity of a sequence of individuals to several propositions is not an issue; the greater concern is whether or not a given sequence of individuals is indeed normalizably algebraic.

4.3 Definable and Algebraic Closures

Returning to the setting of classical model theory, let $\mathscr{L}$ be a language, and let $\mathfrak{M}$ be an $\mathscr{L}$-structure with $A \subseteq \mathfrak{M}$. Then, there are two notable supersets of A taking into account the notions of definability and algebraicity. The definable closure dcl(A) is the set of all individuals and finite sequences of individuals definable over A, and the algebraic closure acl(A) is the set of all individuals and finite sequences of individuals algebraic over A. Clearly, it follows that $A \subseteq$ dcl(A) and $A \subseteq$ acl(A). If A is a substructure of $\mathfrak{M}$, then it also happens that dcl(A) and acl(A) are substructures of $\mathfrak{M}$. These properties also hold in the substructural case, and because the various notions of definability and algebraicity of (finite sequences of) individuals coincide, the formulation is readily straightforward.

Definition 4.11 Let $\mathscr{L}$ be a language, and set **LQ** to be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}$(**LQ**)-structure with $A \subseteq D_{\mathfrak{M}}$. The collection of definable finite sequences of individuals over A is denoted dcl(A), and the collection of normalizably algebraic finite sequences of individuals over A is denoted acl(A). The labels dcl(A) and acl(A) may also denote the substructures of $\mathfrak{M}$ generated by the subsets of the domain $D_{\mathfrak{M}}$.

Proposition 4.12 If A generates a substructure of $\mathfrak{M}$, then dcl(A) and acl(A) both generate substructures of $\mathfrak{M}$.

Proof. Suppose A generates a substructure of $\mathfrak{M}$. This essentially means that the propositional functions map A to subsets of $K_{\mathfrak{M}}$ and $\nabla_{\mathfrak{M}}$, which when closed with respect to the algebraic operators of $K_{\mathfrak{M}}$ and the closure conditions necessary for the interpretation of the quantifiers yields respectively a propositional algebra K_A and filter ∇_A that also makes $\langle A, K_A, I_A, \nabla_A \rangle$, where I_A is the suitable restriction of $I_{\mathfrak{M}}$, an $\mathscr{L}$(**LQ**)-structure. This is a construction procedure that also applies to dcl(A) and acl(A). Consequently, dcl(A) and acl(A) also generate substructures. ↻

4.4 Understanding Types in Substructural Model Theory

This section began with a brief discourse on n-types in classical structures, and it would be appropriate to return to that situation. Suppose $\mathscr{L}$ is a language, and let $\mathfrak{M}$ be an $\mathscr{L}$-structure. Let $n = 1$, to simplify the discussion. Then, 1-types partition the domain of the $\mathscr{L}$-structure, although this partition is messy without limiting to complete 1-types and holding that $\mathfrak{M}$ satisfy all such types. Put another way, though, every individual of the domain has an associated complete 1-type and connects this model-theoretic notion to the general logical notion of typing. The domain partitions itself to the realized types, and the partitioning may be modified by taking into account parameter sets.

Let a be an individual in $\mathfrak{M}$, and let $A \subseteq \mathfrak{M}$ be a parameter set. Then, in the classical setting, the 1-type of a over A is the set tp(a/A) of all $\mathscr{L}(A)$-formulae with one free variable that a realizes. If this reasoning moves into the substructural model theory, a corresponding notion of a weak 1-type follows through.

Definition 4.13 (Weak n-types) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}$(**LQ**)-structure. Let $\bar{a}$ be a finite sequence of n individuals in $D_{\mathfrak{M}}$, and let $A \subseteq D_{\mathfrak{M}}$ be a parameter set. Then, $\bar{a}$ has a weak

n-type over A, denoted $\text{wtp}_{\mathfrak{M}}(\bar{a}/A)$, that consists of all $\mathscr{L}(A)$-formulae $\phi(\bar{x})$ with n-free variables such that $\mathfrak{M} \models_{\mathbf{LQ}} \phi(\bar{a})$.

It is also appropriate to expand the idea of weak n-types to also include what would be arbitrary partial n-types in classical model theory, where p is a weak n-type if p is a set of formulae with n free variables such that p can be realized by a model of T. Just as a complete n-type partitions the domain of an $\mathscr{L}$-structure into those that realize the type and those that do not, it certainly also follows that a finite sequence $\bar{a}$ of n individuals partitions the collection of $\mathscr{L}$-formulae with n free variables into two classes, the corresponding n-type and the unsatisfied stock. Suppose this latter collection is the n-antitype $\text{atp}(\bar{a}/A)$. Then, $\bar{a}$ induces a structure on the class $\text{Form}(\mathscr{L})[n]$, namely a two-valued Boolean algebra with $\text{atp}(\bar{a}/A) \leq \text{tp}(\bar{a}/A)$. This formulation can be generalized into the substructural case to the following notion of a structural n-type.

Definition 4.14 (Structural n-types) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let $\mathfrak{M}$ be an $\mathscr{L}(\mathbf{LQ})$-structure. Let $\bar{a}$ be a finite sequence of n individuals in $D_{\mathfrak{M}}$, and let $A \subseteq D_{\mathfrak{M}}$ be a parameter set. Then, $\bar{a}$ has a structural n-type over A, denoted $\text{strtp}_{\mathfrak{M}}(\bar{a}/A)$, that is an algebra induced upon $\text{Form}(\mathscr{L})[n]$ to $K_{\mathfrak{M}}$ such that, if $\phi(\bar{x})$ is an $\mathscr{L}(A)$-formula with n free variables such that $I_{\mathfrak{M}}(\phi(\bar{a})) = \mathfrak{a}$ for some $\mathfrak{a} \in K_{\mathfrak{M}}$, then $\phi(\bar{x}) \in \mathfrak{a}(\text{strtp}_{\mathfrak{M}}(\bar{a}/A))$. The set of all $\mathscr{L}(A)$-formulae that $\bar{a}$ maps to $\mathfrak{a}$ may be called the $\mathfrak{a}(n)$-type of $\bar{a}$ over A.

Essentially, the structural n-type of $\bar{a}$ is the image of $I_{\mathfrak{M}}$ to $K_{\mathfrak{M}}$. As in classical model theory, the reference of the base structure in the notations for types is omitted when the situation is clear. The following result, which is immediately evident, establishes the connection between weak and structural types.

Proposition 4.15 Let $\phi(\bar{x})$ be an $\mathscr{L}(A)$-formula with n free variables. Then, the formula $\phi(\bar{x})$ is in the weak n-type $\text{wtp}(\bar{a}/A)$ if and only if the proposition which $\phi(\bar{x})$ is mapped to in the structural n-type $\text{strtp}(\bar{a}/A)$ is in $\nabla_{\mathfrak{M}}$.

Let $\mathscr{L}$ be a language, and let $\mathfrak{M}$ be an $\mathscr{L}(\mathbf{LQ})$-model. Let $\bar{a}$ and $\bar{b}$ be finite sequences of individuals in $D_{\mathfrak{M}}$ with the same length, A a set of parameters, and $\mathfrak{a}$ a proposition in $K_{\mathfrak{M}}$. Then, $\bar{a}$ and $\bar{b}$ have the same weak n-type over A if they satisfy the same formulae with n free variables, so that $\mathfrak{M} \models_{\mathbf{LQ}} \phi(\bar{a}) \Leftrightarrow \mathfrak{M} \models_{\mathbf{LQ}} \phi(\bar{b})$. This is the same definition as in classical first-order model theory. Similarly, $\bar{a}$ and $\bar{b}$ have the same structural type if $I_{\mathfrak{M}}(\phi(\bar{a})) = I_{\mathfrak{M}}(\phi(\bar{b}))$. Thus, two sequences having the same structural type implies having the same weak type.

It may be appropriate to compare how types in classical model theory compare with types in this substructural picture. Let $\mathscr{L}$ be a language, T an $\mathscr{L}$-theory, $\mathfrak{M} \models T$, $A \subseteq \mathfrak{M}$, and $\bar{a} \in \mathfrak{M}^n$. Then, the n-type $\text{tp}(\bar{a}/A)$ is the set of all $\mathscr{L}(A)$-formulae $\phi(\bar{x})$ with n free variables such that $\mathfrak{M} \models \phi(\bar{a})$. This type is a complete type, so every $\mathscr{L}(A)$-formula with n free variables or its negation belongs to $\text{tp}(\bar{a}/A)$. Now, since formulae are the same syntactic objects in classical or substructural logic, $\text{tp}(\bar{a}/A)$, as a set of $\mathscr{L}(A)$-formulae, is also a weak type in the lines of substructural model theory just as it is a complete type in the classical context. Indeed, all partial and complete n-types are weak types in this manner.

It is also possible to represent structural types, to an extent, as particular forms of weak types. Informally, a structural type essentially encodes how a sequence of individuals behaves between formulae and propositions. So, if $\mathscr{L}$ is a language, **LQ** is a logic, T is an $\mathscr{L}(\mathbf{LQ})$-theory, $\mathfrak{M} \models_{\mathbf{LQ}} T$ is a $\mathscr{L}(\mathbf{LQ})$-model, $A \subseteq D_{\mathfrak{M}}$ is a set of parameters, and $\phi(\bar{x})$ and $\psi(\bar{x})$ are $\mathscr{L}(\mathbf{LQ})$-formulae, then if $I_{\mathfrak{M}}(\phi(\bar{a})) = I_{\mathfrak{M}}(\psi(\bar{a}))$, then $\mathfrak{M} \models \phi \leftrightarrow \psi(\bar{a})$ as well, so a structural type $\text{strtp}(\bar{a}/A)$ can be represented by a weak type $\text{wtp}'(\bar{a}/A)$ such that

1. $\text{wtp}'(\bar{a}/A) \supseteq \{\phi(\bar{x}) : \mathfrak{M} \models_{\mathbf{LQ}} \phi(\bar{a})\}$;
2. for every formulae $\alpha(\bar{x})$ and $\beta(\bar{x})$, if $I_{\mathfrak{M}}(\alpha(\bar{a})) = I_{\mathfrak{M}}(\beta(\bar{a}))$, then $\alpha \leftrightarrow \beta \in \text{wtp}'(\bar{a}/A)$ and $\neg\alpha \leftrightarrow \neg\beta \in \text{wtp}'(\bar{a}/A)$; and
3. if $I_{\mathfrak{M}}(\alpha(\bar{a})) \leq I_{\mathfrak{M}}(\beta(\bar{a}))$, then $\alpha \to \beta \in \text{wtp}'(\bar{a}/A)$ and $\neg\beta \to \neg\alpha \in \text{wtp}'(\bar{a}/A)$.

In this way, the idea is that this weak type encodes the algebra that shapes the structural type, and so it can act in the stead of the structural type. Clearly, complete types can also be regarded a structural type since a complete type will satisfy these conditions.

Before proceeding with the development of the substructural model theory necessary for a Galois correspondence, it may be appropriate to pause and consider a substructural version of the classical Omitting Types Theorem.

Theorem 4.16 (Omitting Types) Let $\mathscr{L}$ be a countable language, and let T be an $\mathscr{L}$-theory. Let p be an n-type with no parameters such that there is no formula $\phi(\bar{x})$ such that $T \models \forall\bar{x}(\phi(\bar{x}) \to \psi(\bar{x}))$ for every $\psi(\bar{x}) \in p$. That is, suppose that p is not an isolated type. Then, there is a countable model $\mathfrak{M}$ that omits p.

The standard proof for the theorem uses a Henkin construction, the key induction step which is added being the omission of the type p. This proof carries over to the substructural context with some limitations. The notion of an isolated type carries over to an isolated weak type and similarly with nonisolated types as well. The Henkin construction for the classical case exploits the classical equivalence of $\mathfrak{M} \models \neg\phi \Leftrightarrow \mathfrak{M} \not\models \phi$, a property that does not hold with substructural logics in general. A complete and consistent weak type must be a classical complete type; for proper substructural cases, to allow this reasoning to also hold, the weak type must be consistent, so it cannot be complete. In this way, a substructural version of Theorem 4.16 can be obtained.

Theorem 4.17 (Omitting Weak Types) Let $\mathscr{L}$ be a countable language, and let **LQ** be a substructural logic which is not classical. Let T be a consistent $\mathscr{L}(\mathbf{LQ})$-theory. Let p be a consistent weak type with no parameters which is not isolated. Then, there is a countable $\mathscr{L}(\mathbf{LQ})$-model $\mathfrak{M}$ such that $\mathfrak{M}$ omits p.

At this point, with a reliance on some external structure imposed on the formulae to get an adequate notion of structural type, it may be more appropriate to discard the parlance of formulae and consider n-types in purely semantic terms. Such a view of n-types motivates the type theory of abstract elementary classes, and as in those cases, the notion of a Galois n-type can emerge from the semantics.

Definition 4.18 (Galois n-types) A *Galois n-type* $\text{gtp}(\bar{a}/A)$ of $\bar{a}$ over A is the portion of the domain $D_{\mathfrak{M}}$ that is fixed by every partial elementary map or automorphism of $\mathfrak{M}$ fixing A pointwise that also fixes $\bar{a}$. The Galois n-type $\text{gtp}(\bar{a}/A)[\mathfrak{a}]$ of $\bar{a}$ over A with respect to the proposition $\mathfrak{a}$ is the partition of $D_{\mathfrak{M}}$ that is fixed by any partial elementary map or automorphism fixing A pointwise that also fixes a with respect to $\mathfrak{a}$.

For Galois types, $\bar{a}$ and $\bar{b}$ have the same such n-type when every partial elementary map or automorphism which fixes A pointwise either fixes both $\bar{a}$ and $\bar{b}$ or otherwise permutes both $\bar{a}$ and $\bar{b}$. Between weak types, structural types, and Galois types, it is perhaps unappealing to pick one of these proposals and assert that the chosen concept best embodies the most versatile definition with which the work of classical model theory on n-types can be adapted to the current substructural setting.

Admittedly, this discourse into n-types is not considerably developed, and a careful look into the theory of types in substructural model theory would likely resolve the issues that have been raised so far. Although the Galois correspondence in classical model theory can be formulated in terms of n-types, they are not strictly necessary to demonstrate some sort of correspondence framework, so the work can proceed to do so without settling these current problems. Indeed, it may be that the substructural Galois correspondence to be developed can provide insight into resolving the notions of weak types, structural types, Galois types, and other possible notions which have not been proposed in this work.

Chapter 5

Galois Correspondences in Substructural Model Theory

The previous developments in substructural model theory are enough to formalize and demonstrate the following result.

Theorem 5.1 (Fundamental Theorem) Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let T be a maximally consistent first-order **LQ**-theory which codes finite sets, and let $\mathfrak{M} := \langle D_{\mathfrak{M}}, K_{\mathfrak{M}}, I_{\mathfrak{M}}, \nabla_{\mathfrak{M}} \rangle$ be a model of T. Then, there is a Galois correspondence between substructures $\mathfrak{A}$ of $\mathfrak{M}$ and groups $\mathscr{G}$ of strong partial elementary maps of $\mathfrak{M}$ given by the two morphisms Fix and Stab, which are defined such that

1. Fix is an operator from subgroups to substructures that maps a subgroup H of the group of elementary maps of $\mathfrak{M}$ to the structure Fix(H) that is the substructural substructure obtained from the individuals of the domain of $\mathfrak{M}$ fixed pointwise by the maps in H and

2. Stab is an operator from substructures to subgroups that maps a substructure $\mathfrak{B}$ of $\mathfrak{A}$ to a group Stab($\mathfrak{B}$) consisting of all partial elementary maps defined on $\mathfrak{A}$ that fix the individuals in the domain of $\mathfrak{B}$.

5.1 Conventions

It was the usual practice to denote single instances of variables, constant symbols, and individuals by lowercase letters while designating finite sequences of the same with a bar above the letter. For example, a would denote an individual or the constant symbol for such an individual while $\bar{a}$ would indicate a finite sequence of such items. There is no need to distinguish between the two situations in generality, so the bar is dropped in the following discourse. Single instances are then singleton variants of the finite sequences.

Let $\mathscr{L}$ be a language, and let **LQ** be a logic. Let T denote a collection of $\mathscr{L}$-formulae. Depending on what is in T, its **LQ**-deductive closure may differ from the choice of the logic **LQ**, so if there is concern about the output differences, then the variants can be recognized notationally. Therefore, let the $\mathscr{L}$(**LQ**)-deductive closure of T

be denoted $\mathrm{Th}_{\mathbf{LQ}}(T)$. Clearly, if $\mathbf{LQ} < \mathbf{MQ}$, then $\mathrm{Th}_{\mathbf{LQ}}(T) \subseteq \mathrm{Th}_{\mathbf{MQ}}(T)$ as sets of $\mathscr{L}$-formulae, and the classical theory $\mathrm{Th}_{\mathbf{PC}}(T)$ would then be the strongest possible nontrivial theory. If T is consistent, then each of its logic-dependent theories is consistent; if inconsistent, then each would be inconsistent along with the consequences of inconsistency.

In classical model theory, it is possible to obtain a large model of a theory that realizes all n-types. This saturated model can act as a container within which all model-theoretic work is actually done. This property is slightly restricted in the general case of substructural logics. Saturated models which are elementary extensions of a canonical structure would be guaranteed in cases where the upward Löwenheim-Skolem theorem holds, though they may also be found independent of this result. A goal for full generality would have need to address these limitations. In the current work, it is implicitly assumed that the underlying universal model is saturated, so the results would not immediately follow for theories and logics that prevent such models. However, these examples tend to be highly exotic, and the usual examples of nontrivial models do satisfy the requirements for the theorem.

With this limiting assumption in place, let $\mathscr{L}$ be a language, and let $\mathbf{LQ}$ be a logic. Let T be an $\mathscr{L}(\mathbf{LQ})$-theory, so that $T \vdash_{\mathbf{LQ}} \sigma$ implies $\sigma \in T$. Let $\mathfrak{M}$ be a large model that satisfies $\mathbf{LQ}$-saturation, realizing all n-types in the manner described previously. Furthermore, let the algebra of propositions in $\mathfrak{M}$ be sufficiently enriched such that its substructures are properly $\mathscr{L}(\mathbf{LQ})$-structures. This model $\mathfrak{M}$ becomes the container in which all subsequent work is to be done. Where $A \subseteq D_{\mathfrak{M}}$ is an arbitrary subset of the domain, it is possible to obtain a substructure $\mathfrak{A}$ generated by A such that $D_{\mathfrak{A}} = A$, $K_{\mathfrak{A}}$ and $\nabla_{\mathfrak{A}}$ are subsets respectively of $K_{\mathfrak{M}}$ and $\nabla_{\mathfrak{M}}$ generated by A, and $I_{\mathfrak{A}}$ is the appropriate restriction of $I_{\mathfrak{M}}$. To simplify notation, let this structure $\langle A, K_{\mathfrak{A}}, I_{\mathfrak{A}}, \nabla_{\mathfrak{A}} \rangle$ be simply denoted $A \subseteq \mathfrak{M}$; the potential ambiguity can be resolved by the local context.

Let A and B be substructures of $\mathfrak{M}$ generated in such a manner. It would be useful and interesting to determine how they can be compared to each other. If their domains are set up so that $A \subseteq B \subseteq \mathfrak{M}$, then it follows that $K_{\mathfrak{A}} \subseteq K_{\mathfrak{B}}$ and $\nabla_{\mathfrak{A}} \subseteq \nabla_{\mathfrak{B}}$. Thus, the notation $A \subseteq B$ is not subject to ambiguity and can be well-defined. This situation becomes even more evident if it turns out that the algebra of propositions and the set of designated propositions coincide, so that $K_{\mathfrak{A}} = K_{\mathfrak{B}}$ and $\nabla_{\mathfrak{A}} = \nabla_{\mathfrak{B}}$. The notation of $\mathscr{L}(A)$-formulae continues to mean $\mathscr{L}$-formulae with parameters from A.

Let $\phi(x)$ be an $\mathscr{L}(A)$-formula with x denoting a sequence of free variables, and let a be a sequence of individuals. It is generally assumed that a and x are of the same length if discussion of $\phi(a)$ arises. In general, $\phi(\mathfrak{M})$ is to denote the solution set of $\phi(x)$ in $\mathfrak{M}$ which is the set of all individuals of $\mathfrak{M}$ that satisfy $\phi(x)$. A sequence a is a solution for $\phi(x)$ if $\mathfrak{M} \models_{\mathbf{LQ}} \phi(a)$, and it is an $\mathfrak{a}$-solution if $I_{\mathfrak{M}}(\phi(a)) = \mathfrak{a}$ for a designated proposition $\mathfrak{a}$. Likewise, $\phi(\mathfrak{M}|\mathfrak{a})$ can denote the $\mathfrak{a}$-solution set of $\phi(x)$, although this notion is not used.

Since the Galois correspondence makes comparisons among the substructures of the universal $\mathfrak{M}$, the setup becomes vastly simplified if the substructures under consideration share a common domain of propositions. So, let $A \subseteq B$ be two substructures of $\mathfrak{M}$ with common set of propositions. In such a case, the maps acting upon the substructures can be fully characterized looking at how they affect the domains of individuals.

The definition of partial elementary maps were defined previously, but it is provided again here.

Definition 5.2 (Partial Elementary Maps) Let $\mathscr{L}$ be a logic, and let **LQ** be a logic. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathscr{L}(\mathbf{LQ})$-structures. Then, a partial elementary map from $\mathfrak{A}$ to $\mathfrak{B}$ is a map $\mathfrak{i}$ generated by a function $i : X \to D_{\mathfrak{B}}$ with $X \subseteq D_{\mathfrak{A}}$ such that $I_{\mathfrak{A}}(\alpha(\bar{a})) = \mathfrak{a}$ implies $I_{\mathfrak{B}}(\alpha(i(\bar{a}))) = \mathfrak{i}(\mathfrak{a})$, where α is an arbitrary formula, $\bar{a} \in X^n$ is a finite sequence of n individuals, and $\mathfrak{a}$ is a proposition in $K_{\mathfrak{A}}$.

In the current context of all structures being contained in $\mathfrak{M}$, the definition would apply where $\mathfrak{A}$ and $\mathfrak{B}$ are themselves substructures of $\mathfrak{M}$. Consequently, these partial elementary maps are partial automorphisms of $\mathfrak{M}$. Let G be a set of such partial maps such that they all have a common domain for whose individuals the maps can have defined outputs. Let that domain be $\mathfrak{B}$. Let $\mathfrak{A}$ denote the subset of that domain which is fixed pointwise by every map in G. Then, G becomes an automorphism group of $\mathfrak{B}$ fixing $\mathfrak{A}$. Since it is possible to have maps fully defined on $\mathfrak{M}$, there can be set of totally defined maps that fix whichever substructures may be needed. For a structure generated by A, let $\mathrm{Gal}(\mathfrak{M}/A)$ denote the group of maps defined on all of $\mathfrak{M}$ where every such map fixes the individuals of A. Such collections of maps form groups with the same reason why automorphisms of classical model theory also form groups; the identity map is always available, and the maps being one-to-one guarantees that their inverses are also maps.

The development cleaves to the presentation provided in [Medvedev Takloo-Bighash 2009]. Elementary maps and other structural maps must respect the underlying structure of $\mathfrak{M}$, so in particular they must respect all instances of algebraicity and definability. Thus, every such map must fix definable individuals, and permutation of algebraic individuals is constrained by their irreducible formulae. If a is an individual in $D_{\mathfrak{M}}$ and A is a set of parameters, let $\mathrm{Orb}(a/A) := \{b \in D_{\mathfrak{M}} : i(a) = b \text{ for some } i \in \mathrm{Gal}(\mathfrak{M}/A)\}$ denote the orbit of a over A.

Since irreducibility is a well-defined notion with respect to normalized algebraicity, it is possible to attach a quantitative measurement of the cardinality of orbits. Let $\deg(a/A) := |\mathrm{Orb}(a/A)|$ denote the degree of a over A. The different proposed forms of algebraicity coincide, so it is appropriate to refer to algebraicity unadorned without fear of ambiguity.

Proposition 5.3 Let a be algebraic over A. Then, $\deg(a/A)$ is finite.

Proposition 5.4 If c is definable over Ab and b is definable over Ac, then $\deg(c/A) = \deg(b/A)$.

Proof. The first statement immediately holds since algebraic sequences have finitely many possible permutations governed by the irreducible solution set. The second statement relies on a straightforward adaptation of the same result in classical model theory. ↻

5.2 Algebraic Extensions

Let $A \subseteq B \subseteq \mathfrak{M}$ be substructures of $\mathfrak{M}$. Then, when certain conditions are met, one can regard B as a finite algebraic extension of A.

Definition 5.5 (Finite Algebraic Extensions) Let $A \subseteq B \subseteq \mathfrak{M}$. Then, B is a finite algebraic extension of A if there is some b in B that is algebraic over A such that every individual in B is definable over Ab.

The extension B is finite in the sense that its individuals are generated by a finite addition to A, and it is algebraic since the definition implies that every individual in B is algebraic over A. The ideal situation occurs when A is itself definably closed and B is itself the definable closure of Ab; there is then no possible ambiguity arising between one finite algebraic extension C and another D both over A satisfying the definition using the same b over A. A finite algebraic extension can be measured with respect to its degree of extension.

Definition 5.6 Let $A \subseteq B \subseteq \mathfrak{M}$, such that B is a finite extension of A generated by some sequence b. Then, the degree of the extension, denoted $\deg(B/A)$, is defined such that $\deg(B/A) = \deg(b/A)$.

In a more general context, it is possible to denote any substructure B between A and $\mathrm{acl}(A)$ to be an algebraic extension of A. As in the finitary case, the situation is best manageable when B is itself definably closed over definably closed A.

5.3 Normal Extensions

Definition 5.7 Let $A \subseteq B$ be substructures of $\mathfrak{M}$ such that B is a finite algebraic extension of A. Then, B is a *normal extension* of A if $\mathrm{Orb}(c/A) \subseteq D_B$ for every sequence c of individuals in B.

Perhaps at first glance, the situation with orbits in substructural model theory might seem as if it ought to be more complicated than in the case of classical model theory. However, because all orbits are fixed upon a particular proposition, so that if a is normalizably algebraic to a proposition $\mathfrak{a}$, then all sequences in the orbit of a also map to $\mathfrak{a}$ with the appropriate irreducible formula. Thus, the following proposition is an immediate consequence of the previous exploration of the substructural notion of normalized algebraicity.

Proposition 5.8 Let $\dot{\mathrm{irr}}(b/A)$ be an irreducible formula for b over A. Then, $\dot{\mathrm{irr}}(b/A)(\mathfrak{M}) = \mathrm{Orb}(b/A)$.

The idea of splitting extensions carries through to the logical framework, although some modifications make the notion well-grounded model-theoretically.

Definition 5.9 Let d be a sequence of individuals in B, and let B be a finite algebraic extension of A. Let $\mathrm{Orb}(d/A)$ be a subset of B and B a subset of $\mathrm{dcl}(A \cup \mathrm{Orb}(d/a))$. Then, B is a *splitting extension* of d over A.

Although it may be more appropriate to assert that B is a splitting extension for the irreducible formula of d over A, since the irreducible formula itself is not unique and reference of irreducibility ultimately points to the size of the orbit of d with respect to A, the current language is not ambiguous and hence is well-defined. Splitting extensions in the general substructural model-theoretic sense can also be normal extensions when they satisfy some additional conditions.

Definition 5.10 Let B be a splitting extension of some sequence d in B over A. Then, B is a normal extension of A if B is also definably closed so that $B = \mathrm{dcl}(B)$.

5.4 Towers of Extensions

The definitions presented to this point are now sufficient to lay out some of the results that the fundamental theorem depends upon. In some respects, the results represent substructural model-theoretic analogues to general results in the basic theory of fields. Among such field-theoretic results is the Tower Law, in which the logical version has the following form.

Definition 5.11 (Towers) Let A, B, and C be substructures of $\mathfrak{M}$. Then, they constitute a tower if it is the case that $A \subseteq B \subseteq C \subseteq \mathfrak{M}$, that B is a finite algebraic extension of A, and that C is a finite algebraic extension of B.

Proposition 5.12 (Tower Law) Let $A \subseteq B \subseteq C \subseteq \mathfrak{M}$ be a tower of structures. Then,

$$\deg(C/A) = \deg(C/B)\deg(B/A).$$

Proof. Let b and c be generators of B and C over A and B, respectively. The first goal is to show that bc then generates C over A, from which the result follows. Let d be an arbitrary sequence of C. By definition, d is definable over Bc, so there is an $\mathscr{L}(Bc)$-formula $\phi(x)$ that defines d. Let $b_1, ..., b_n$ be parameters in B actually used to define d in $\phi(x)$. Each b_i $(1 \leq i \leq n)$ is itself definable over Ab, so for each parameter there is an $\mathscr{L}(Ab)$-formula $\beta_i(x)$ that defines each b_i. Then, the $\mathscr{L}(Abc)$-formula which essentially asserts

$$\exists x_1 ... \exists x_n (\bigwedge_{i=1}^{n} \beta_1(x_i) \wedge \phi(x)),$$

where $\phi(x)$ is modified so that each occurrence of b_i is replaced with an occurrence of x_n, defines d. Since c is algebraic over B and b is algebraic over A, it is clear that bc is algebraic over A. Therefore, bc generates C over A as a finite algebraic extension.

From the definition of extension degrees, it is the case that $\deg(C/B) = \deg(c/B)$ and $\deg(B/A) = \deg(b/A)$, so $\deg(C/B) = |\mathrm{Orb}(c/B)|$ and $\deg(B/A) = |\mathrm{Orb}(b/A)|$. Hence, the second goal is to show that

$$|\mathrm{Orb}(bc/A)| = |\mathrm{Orb}(c/B)||\mathrm{Orb}(b/A)|.$$

The argument is to show that $\mathrm{Orb}(bc/A)$ can basically be split into $|\mathrm{Orb}(b/A)|$-many pairwise disjoint sets that each have $|\mathrm{Orb}(c/B)|$-many items. Take an arbitrary $e \in \mathrm{Orb}(b/A)$. Let

$$P_e := \{ef : f = j(c) \text{ for a map } j \text{ such that } j(b) = e\}$$

be a subset of $\mathrm{Orb}(bc/A)$ where b is always mapped to e. For each such $e \in \mathrm{Orb}(b/A)$, P_e is disjoint from the others. In each such P_e, the f portion of the sequences ef in P_e must be in $\mathrm{Orb}(c/B)$. Any map which permutes c to the sequences in its orbit can be used to show that f can then be an arbitrary permutation of c, so there are as many f-variants as the size of $\mathrm{Orb}(c/B)$. Thus, $\mathrm{Orb}(bc/A)$ consists of $|\mathrm{Orb}(c/B)|$ disjoint sets distinguished by which permutation of b over A it fixes with the size of each such disjoint set determined by the number of permutations of c over B. Therefore, it is clear that $|\mathrm{Orb}(bc/A)| = |\mathrm{Orb}(c/B)||\mathrm{Orb}(b/A)|$. ↻

Proposition 5.13 Let $A \subseteq B \subseteq \mathfrak{M}$ such that B is a finite algebraic extension of A generated by b. Let $B = \mathrm{dcl}(Ab)$, and take $B_0 := B \cup \mathrm{Orb}(b/A)$. Then, $|\mathrm{Gal}(B/A)| = |B_0|$.

Proof. Clearly, both $\mathrm{Gal}(B/A)$ and B_0 have finite cardinality, so constructing a one-to-one map from $\mathrm{Gal}(B/A)$ onto B_0 yields the required result. Let θ be defined as such a map with $\theta(i) = i(b)$, where i is a map in $\mathrm{Gal}(B/A)$. Suppose i and j are two maps such that $i(b) = j(b)$. Then, $i \circ j^{-1} = \mathrm{id}_B$ since $B = \mathrm{dcl}(Ab)$, so θ is one-to-one. For an arbitrary $c \in B_0$, let i be a map so that $i(b) = c$. Then, $i(B) = B$, so $\theta^{-1}(c) = B$ and therefore θ is onto. ↻

Proposition 5.14 (Degrees of Normal Extensions) Let $A \subseteq B \subseteq \mathfrak{M}$ such that B is a finite extension of A generated by b and $B = \mathrm{dcl}(B)$. Then, $\deg(B/A) = |\mathrm{Gal}(B/A)|$ if and only if B is a normal extension of A.

Proof. Since $B = \mathrm{dcl}(Ab)$ is definably closed, having $\deg(B/A) = |\mathrm{Gal}(B/A)|$ implies that B splits b over A. A definably closed splitting extension is a normal extension. Conversely, if B is normal, then it is immediately evident that $|\mathrm{Orb}(b/A)| = |\mathrm{Gal}(B/A)|$ and that B splits every c in B over A. ↻

Proposition 5.15 (Towers of Normal Extensions) Let $A \subseteq B \subseteq C \subseteq \mathfrak{M}$ be a tower of structures. Then, if C is a finite normal extension of A, then C is also a finite normal extension of B.

Proof. Since C is normal over A, the orbit of every c in C over A is in C. The orbit of every c in C over B is a subset of such orbits over A, so C must be normal over B. ↻

Corollary 5.16 For normal extensions B and C over A and B respectively,

$$|\mathrm{Gal}(C/A)| = |\mathrm{Gal}(C/B)||\mathrm{Gal}(B/A)|.$$

Recall that a normal subgroup H of a group G is one such that ghg^{-1} is in H for every h in H and g in G.

Proposition 5.17 Let $A \subseteq B \subseteq C \subseteq \mathfrak{M}$ be a tower of structures such that C and B are both finite normal extensions of A. Then, Gal(C/B) is a normal subgroup of Gal(C/A).

Proof. Let i be a map in Gal(C/B). Then, since i is fully defined on C and $A \subseteq B$, it is also the case that i is a map in Gal(C/A), so Gal(C/B) is a subgroup of Gal(C/A). For an arbitrary map j in Gal(C/A), it follows that $jij^{-1} = i$ due to the requirement that maps in Gal(C/A) be one-to-one and onto, so Gal(C/B) is a normal subgroup of Gal(C/A). ↻

Consequently, for any structure A of $\mathfrak{M}$ and a normal extension B of A in $\mathfrak{M}$, there is a correspondence between normal intermediate extensions and normal subgroups. This special case of the Fundamental Theorem holds without the requirement that the models and theories under question hold some special property such as the ability to code finite sets. Still, to formally express the limited Galois connection requires the development of the operators that carry through the correspondence.

5.5 Operators on the Galois Connection

A Galois connection depends upon two maps that move between two ordered collections of objects which respects that ordering. In the current case, the two maps are the Fix and Stab operators. At this point, substructures will generally be considered to have strong definable closure since the correspondence directly connects them to subgroups of partial elementary maps.

Definition 5.18 (The Fix **and** Stab **operators)** Let $A \subseteq C \subseteq \mathfrak{M}$ be two definably closed substructures. Let Gal(C/A) denote the group consisting of the elementary maps defined on C that fix every individual in D_A. If H is a subgroup of Gal(C/A), then denote by Fix(H) the structure obtained such that the domain of individuals is $\{c \in D_C : h(c) = c \forall h \in H\}$.

Similarly, let $A \subseteq B \subseteq C \subseteq \mathfrak{M}$ be a tower of definably closed substructures such that C is a finite algebraic extension of A. Then, denote by $\mathrm{Stab}(B) := \{h \in \mathrm{Gal}(C/A) : h(b) = b \forall b \in D_B\}$ the subgroup of Gal(C/A) where every elementary map fixes the individuals of B pointwise.

The result that demonstrates the connection between towers of normal extensions normal subgroups then leads to the following statement.

Corollary 5.19 (Normal Correspondence) Let $A \subseteq B \subseteq C \subseteq \mathfrak{M}$, each with strong definable closure. Let C be a normal finite extension over A. Then, $\mathrm{Gal}(C/B)$ is a normal subgroup of $\mathrm{Gal}(C/A)$ if and only if B is a normal finite extension of A.

5.6 The Fundamental Theorem

In order to obtain the full correspondence between arbitrary subgroups instead of normal subgroups, the underlying large model and its theory must have additional structure imposed by finitary coding of finite sets.

Definition 5.20 Let T be an $\mathscr{L}(\mathbf{LQ})$-theory, and let $\mathfrak{M} \models T$ be a large model. Then, $\mathfrak{M}$ codes finite sets of sequences of individuals if, for every number $n \in \mathbb{N}$ and for every finite set $F \subseteq D^n_{\mathfrak{M}}$, there is some sequence b of individuals such that $i(b) = b$ if and only if $i(F) = F$ setwise for any elementary map i of $\mathfrak{M}$. This b is then the finitary code of F. If every model $\mathfrak{M}$ of T can code finite sets, then T is said to be able to code finite sets.

The previous results and the properties associated with coding can then provide a foundation for the Fundamental Theorem.

Theorem 5.21 (Fundamental Theorem) Let T be an $\mathscr{L}(\mathbf{LQ})$-theory, and let $\mathfrak{M}$ be a large model which can code finite sets. Let $A \subseteq C \subseteq \mathfrak{M}$ be substructures definably closed such that C is a normal finite extension of A. Then, there is a Galois correspondence between subgroups of $G := \mathrm{Gal}(C/A)$ and intermediate extensions definably closed between A and C provided by the operators Fix and Stab.

Proof. The operator Fix already provides a mapping of subgroups to intermediate extensions, and the operator Stab connects intermediate extensions to subgroups. Thus, the unresolved question is in whether applying the two operators on an arbitrary subgroup or intermediate extension, one after the other, yields the original input. The result would then be obtained if

$$\mathrm{Stab}(\mathrm{Fix}(H)) = H$$

and

$$\mathrm{Fix}(\mathrm{Stab}(B)) = B$$

for subgroups H and intermediate extensions B. So, $\mathrm{Fix}(H)$ is the substructure of $\mathfrak{M}$ generated by those individuals fixed by H, and $\mathrm{Stab}(\mathrm{Fix}(H))$ is the subgroup of G determined by those that do not affect the individuals in $\text{Fix}(H)$. Clearly, H will be a subgroup of $\text{Stab}(\text{Fix}(H))$ since every map in H will by definition not move the individuals it fixes. Since C is a normal finite extension of A, there is some algebraic sequence c of C that generates C over A so that $C = \text{dcl}(Ac)$. Being algebraic, c will have a finite orbit over A. Let $O(H) := \{h(c) : h \in H\}$ be the subset of $\text{Orb}(c/A)$ determined by the actions of H. It is evident that $O(H) = \text{Orb}(c/\text{Fix}(H))$. Since $\text{Orb}(c/A)$ is finite, $O(H)$ is finite, so $O(H)$ has a code f such that any map which fixes f pointwise will also fix $O(H)$ setwise and any map fixing $O(H)$ setwise will fix f pointwise. Certainly, every map in H will fix $O(H)$ setwise, so every such map in H will fix f pointwise. Therefore, the sequence f must be in $\text{Fix}(H)$. Consequently, any map in $\text{Stab}(\text{Fix}(H))$ fixes f pointwise, which means that any such map fixes $O(H)$ setwise. Therefore, every map in $\text{Stab}(\text{Fix}(H))$ is in H, so the two subgroups of G are identical.

For the other direction, $\text{Stab}(B)$ is the subgroup of G generated by those maps which stabilize to B, and $\text{Fix}(\text{Stab}(B))$ is the intermediate extension between A and C which is fixed by the maps in the stabilizer of B. Clearly, B itself is contained in $\text{Fix}(\text{Stab}(B))$. If d is a finite sequence not in B, then there is a map i in $\text{Stab}(B)$ which does not fix d pointwise. Consequently, d is not a fixed point which will be in $\text{Fix}(\text{Stab}(B))$. So, $\text{Fix}(\text{Stab}(B))$ is exactly B. ↻

5.7 Branches

The Fundamental Theorem thus holds that substructural model theory can articulate a well-formed Galois correspondence between definably closed substructures and subgroups of partial elementary maps situated within a large model when certain conditions are met. The substructural semantics as developed to this point was motivated particularly so that this result of classical model theory can be expressed in this altered logical setting. Continued ponderance of substructural model theory and the Galois correspondence thereof can be possible along some readily apparent directions, but there may also be unexpected connections with other areas of logic, model theory, and mathematics that may motivate other paths.

The Fundamental Theorem is not the most general statement of classical first-order model-theoretic Galois theory. As demonstrated in [Casanovas Farré 2004], a theory with Galois finite coding can have large models which can host Galois connections between its substructures and subgroups. The current state of the semantics for substructural logic is also able to accommodate a substructural version of Galois finite coding and therefore lead to a substructural version of the wider correspondence. Galois finite coding, which is defined subsequently, changes the idea of coding finite sets to be dependent on that choice of the parameter structure. In order to sketch such an accommodation, the key is that all algebraic extensions are inhabited by the algebraic closure of the base parameter structure, so fixing the parameter set fixes the codes necessary for the Galois correspondence.

During the course of developing the basis for the Fundamental Theorem, some intuitive appeals were made to the behavior of the large substructural model and its various components. Conspicuously absent have been concrete examples of large substructural models that are significantly different from classical examples. Obtaining such models and explicating a Galois correspondence would therefore be a reasonable direction for a subsequent effort in the work of substructural model theory.

Substructural logics, being motivated mainly through philosophical and informatical questions, remain relatively isolated from the wider concerns of model theory as a mathematical craft. The current effort served as an endeavor to bridge this gap by developing the semantics to be able to handle the technical complexity required for the classical model-theoretic result for Galois theory. Still, model theory itself is moving in new directions with some efforts shedding the confines of first-order logic. One such direction involves the exploration of model theory over abstract elementary classes.

Part II

Galois Theory of First-Order Theories

Chapter 6

The Galois Correspondence in Classical First-Order Logic

The previous development of the model theory over substructural logics was motivated by the goal of obtaining a substructural variant of the following result in classical model theory.

Theorem 6.1 (Galois Correspondences) Let $\mathscr{L}$ be a language, and let T be an $\mathscr{L}$-theory that can code finite sets. Let $\mathfrak{M} \models T$ be a large and saturated model. Let A be a substructure of $\mathfrak{M}$. Then, there is a Galois correspondence between substructures of acl(A) that are extensions of A and the groups of partial elementary maps defined on acl(A), such that every closed subgroup H of Aut(acl(A)/A) has a corresponding definably closed substructure Fix(H) with $A \subseteq \text{Fix}(H) \subseteq \text{acl}(A)$ and every substructure B with $A \subseteq B \subseteq \text{acl}(A)$ has a corresponding subgroup Stab(B) of Aut(acl(A)/A).

There is a slight gap between what was achieved with Theorem 5.1 in the substructural setting and Theorem 6.1. The preceding development when done in classical model theory leads to the following result.

Theorem 6.2 (Fundamental Theorem of Galois Theory) Let $\mathscr{L}$ be a language, and let T be an $\mathscr{L}$-theory that can code finite sets. Let $\mathfrak{M} \models T$ be a large and saturated model. Let A be a substructure of $\mathfrak{M}$ definably closed, and let $C \supseteq A$ be a normal algebraic extension of A that is also definably closed. Then, there is a correspondence between intermediate algebraic extensions of A contained in C and the subgroups of Gal(C/A).

Since the extensions discussed in this theorem are algebraic and hence finite, the Galois groups themselves are also finite. This finitary result naturally leads to a profinite formulation which develops straightforwardly from the established situation. Let $C \supseteq A$ be called a *profinite* extension if its Galois group Gal(C/A) is a profinite group, obtained as the inverse limit of a directed system of Galois groups of finite algebraic extensions of A. Then, Theorem 6.2 quickly leads to the following result.

Theorem 6.3 (Profinite Fundamental Theorem) Let $\mathscr{L}$ be a language, and let T be an $\mathscr{L}$-theory that can code finite sets. Let $\mathfrak{M} \models T$ be a large and saturated model. Let A be a definably closed substructure of $\mathfrak{M}$. Let $C \supseteq A$ be a normal profinite extension of A that is also definably closed. Then, there is a correspondence between intermediate profinite extensions of A contained in C and the closed subgroups of $\mathrm{Gal}(\mathrm{dcl}(C)/\mathrm{dcl}(A))$.

Proof. The proof follows through in the same manner in which the profinite formulation of the Fundamental Theorem of Galois theory for pure fields is a consequence of the finitary formulation of the Fundamental Theorem of Galois theory for pure fields. The key is that the finite groups can be used to form an inverse system so that profinite groups are obtained as inverse limits of these finite groups, and the correspondence then holds with respect to closed subgroups of these profinite groups. ↻

If C is a normal extension of A, then the group $\mathrm{Gal}(C/A)$ is profinite. Notably, $\mathrm{acl}(A)$ is always normal with respect to A, so $\mathrm{Gal}(\mathrm{acl}(A)/A) := \mathrm{Aut}(\mathrm{acl}(A)/A)$ is profinite. Hence, Theorem 6.3 leads to Theorem 6.1.

As demonstrated in [Casanovas Farré 2004], an $\mathscr{L}$-theory needs only a slightly weaker property than coding finite sets to exhibit the correspondence.

Definition 6.4 (Coding Galois Finite Sets) Let $\mathscr{L}$ be a language, and let T be an $\mathscr{L}$-theory with $\mathfrak{M} \models T$ a large and saturated model. Then, T codes Galois finite sets if, given an arbitrary parameter set $A \subseteq \mathfrak{M}$, for a finite set X of finite sequences of n individuals, it is the case that there is some finite sequence a of m individuals such that every map that fixes A pointwise will fix X setwise if and only if the map fixes a pointwise.

Clearly, the ability to code finite sets implies the ability to code Galois finite sets. Coding finite sets means that the finite sequence a does not depend on the choice of the parameter set A. With the slightly weaker notion of Galois coding, it is also possible to get a farther-reaching correspondence result.

Theorem 6.5 (Interdefinable Fundamental Theorem) Let $\mathscr{L}$ be a language, and let T be an $\mathscr{L}$-theory that can code Galois finite sets. Let $\mathfrak{M} \models T$ be a large and saturated model. Let A be a definably closed substructure of $\mathfrak{M}$. Let $C \supseteq A$ be an interdefinably closed extension of A in $\mathrm{acl}(A)$. Then, there is a correspondence between intermediate interdefinable extensions of A contained in C and the closed subgroups of $\mathrm{Gal}(\mathrm{dcl}(C)/\mathrm{dcl}(A))$.

Definition 6.6 (Interdefinability) Let $\mathscr{L}$ be a language, T an $\mathscr{L}$-theory, and $\mathfrak{M} \models T$ a large and saturated model. Let a and b be sequences of $\mathfrak{M}$. Then, a and b are *interdefinable* if $a \in \mathrm{dcl}(b)$ and $b \in \mathrm{dcl}(a)$. Let A and B be substructures of $\mathfrak{M}$. Then, A and B are interdefinable if $a \in \mathrm{dcl}(B)$ and $b \in \mathrm{dcl}(A)$ for every $a \in A$ and $b \in B$, and a and b are interdefinable over A if $a \in \mathrm{dcl}(Ab)$ and $b \in \mathrm{dcl}(Aa)$. In addition, B is *interdefinably closed* over A if every individual in $\mathrm{dcl}(B)$ is interdefinable over A with some individual in B. That is, B is A-interdefinably closed if every $d \in \mathrm{dcl}(B)$ has some $b \in B$ such that $d \in \mathrm{dcl}(Ab)$ and $b \in \mathrm{dcl}(Ad)$.

For the most part, the subsequent discussion will rely on theories that are able to code finite sets rather than with theories that code Galois finite sets, and the correspondence to look at is between definably closed substructures; this slightly weaker formulation will then be exploited to develop results about a Galois theory of pseudoexponential fields.

Observations of Galois correspondences in model theory began with work by Poizat in [Poizat 1983], where the fundamental theorem is explicated concerning intermediate extensions between a set A and its algebraic closure $\mathrm{acl}(A)$. This result is then used to study strong types.

Definition 6.7 (Strong n-Types) Let $\mathscr{L}$ be a language, and let T be an $\mathscr{L}$-theory with model $\mathfrak{M} \models T$. Let a and b be finite sequences of length n in $\mathfrak{M}$, and let $A \subseteq \mathfrak{M}$ be a set of parameters. Then, a and b have the same strong n-type over A if they are in the same equivalence class for every equivalence relation with finitely many classes definable with parameters from A. The strong type of a over A is denoted $\mathrm{stp}(a/A)$.

The original result claimed to rely on the property of eliminating imaginaries, but it became apparent that the correspondence relies on a finitary notion, which is indeed the ability to code finite sets. The subsequent discussion will try to minimize explicit usage of types and instead rely on generators of extensions and implied reliance of isolated types. This approach is not necessarily problematic in the applications within the scope of this work. It essentially simulates the approach taken up in the Galois theory for pure fields, where one often looks at generators of field extensions and the polynomials that describe them. Likewise, isolated types allow for a concise and finitary way of describing generators of model-theoretic extensions. Most applications are with fields, so this approach is rather sensible.

An occasional way in which a theory or its model may manifest the Galois correspondence determined by Theorem 6.1, 6.2, or 6.3 is through a rather empty result. A Galois correspondence is *trivial* if a substructure cannot have proper extensions contained in its algebraic closure. A clear way of this state happening is when a substructure is itself algebraically closed. Another way in which this situation may occur is if the notions of definability and algebraicity coincide. A Galois correspondence is nontrivial otherwise.

Chapter 7

Some Applications to First-Order Theories

The following work consists of case studies of what information may be gleaned from the application of the model-theoretic principles of the Galois correspondence to specific mathematical structures and their theories. The case studies have a generally common shape and direction which clarify that the model-theoretic Galois correspondence within some structure is really about its substructures. It is sensible to work with large models as used in the proof for the Fundamental Theorem. An appropriate place to start is to determine the prime substructure fixed by all structure-preserving maps defined over the large model. Any Galois correspondence developed within the large model, with respect to its structure-preserving maps, would be with substructures which all contain this prime structure.

The substructures themselves will also be definably closed, as required by the Fundamental Theorem. Working with structures which are not definably closed would resolve into working with their definable closures, for an elementary map defined over a substructure not definably closed can be extended uniquely to the definable closure. Because two different substructures may turn out to have the same definable closure, it is simply more prudent to work only with definably closed structures. Most of the case studies involve fields with additional structure, and these additional properties affect which fields turn out to be definably closed. Some of these examples have Galois-theoretic formulations from general mathematical practice which may differ with the model-theoretic realization, and often the differences between them manifest due to definable closure.

7.1 Recovering the Galois Theory of Fields

Galois connections originate from the Galois theory of fields, so an obvious case to consider is to look at recovering this original system from the model-theoretic setting. Fields are best described in the language of rings or fields; the preference here is for the language of rings, with subtraction and multiplicative inverse defined operations. The theory T_f of fields in the language $\langle +, \times, 0, 1\rangle$ can be axiomatized by the following list.

1. $\forall x \forall y \forall z(x+(y+z) \doteq (x+y)+z)$

2. $\forall x \forall y \forall z(x \times (y \times z) \doteq (x \times y) \times z)$

3. $\forall x \forall y(x+y \doteq y+x)$

4. $\forall x \forall y(x \times y \doteq y \times x)$

5. $\forall x(x+0 \doteq x)$

6. $\forall x(x \times 1 \doteq x)$

7. $\forall x \exists y(x \doteq 0 \vee x \times y \doteq 1)$

8. $\forall x \exists y(x+y \doteq 0)$

9. $\forall x \forall y \forall z(x \times (y+z) \doteq x \times y + x \times z)$

10. $\neg 0 \doteq 1$

Any field F models T_f, so unfortunately, this theory is useless without more information. The theory must describe the container in which the Galois correspondence is to be carried out; substructures of the model need not be models themselves, unless certain conditions prevail. In the algebraic understanding of the Galois theory of fields, the Galois correspondence explicated over some base field F is done within an algebraic closure of F, so that all of the field extensions of F are algebraic over F and all automorphisms fixing F are taken to be in the absolute Galois group of F. Let F^{alg} denote the field-theoretic algebraic closure, to distinguish from the related notion of model-theoretic algebraic closure, which in this case would be denoted $\text{acl}(F)$.

A field K is (field-theoretically) algebraically closed if it has no proper algebraic extensions, so the theory of fields with the axiom schemata which states that every polynomial of degree n has a root, formally

$$\vdash \forall x_0 \forall x_1 \forall x_2 ... \forall x_{n-1} \exists y(x_0 + x_1 y + x_2 y^2 + ... + y^n = 0), \tag{7.1}$$

for every natural number n, axiomatizes the theory of algebraically closed fields. This theory is denoted ACF.

Fields can be compared against each other, in the sense that it is possible to determine whether one is an extension of the other or what their common subfields and compositum look like, only when they share the same characteristic. Adding the axiom for field characteristic p or 0 to the theory ACF yields the complete theory ACF_p or ACF_0.

In the field characteristic of 0, the canonical model of ACF_0 is the complex number field $\mathbb{C}$. The prime field of characteristic 0 is of course the rational number field $\mathbb{Q}$, and its field-theoretic algebraic closure $\mathbb{Q}^{\text{alg}}$ is also a model of ACF_0. The theories ACF_0 and ACF_p, where p is prime, are complete and have elimination of quantifiers. Quantifier elimination is a very useful property which has the consequence that these theories are also model-complete, so every submodel of a model is an elementary substructure. In this example, $\mathbb{Q}^{\mathrm{alg}}$ is an elementary substructure of $\mathbb{C}$ (and indeed of any model of ACF_0). Model completeness makes harmless the move into an excessively larger container, a large and saturated model of ACF_0, which is guaranteed to host a plethora of automorphisms.

Much more can be noted of the theory of algebraically closed fields of a fixed characteristic and its models, but such observations are standard in model theory. For the current context, the main observations of relevance include that the rational number field, the prime field of characteristic 0, is a definably closed substructure of every algebraically closed field of zero characteristic, along with the following result.

Proposition 7.1 Let $\mathscr{L}$ be the language of rings, and let T be the theory ACF_0 of algebraically closed fields with characteristic 0. Let $\mathfrak{M}$ be a model of ACF_0. Then, the definably closed substructures of $\mathfrak{M}$ are those substructures A that are fields, and every field F is definably closed, so that $F = \mathrm{dcl}(F)$.

In characteristic 0, all fields are perfect, so perfection is trivial. If $A \subseteq \mathfrak{M}$ is a substructure, then obviously it is the case that $A \subseteq \mathrm{dcl}(A)$. The substructure A needs to be closed with respect to addition and multiplication in order to be definably closed. Furthermore, additive and multiplicative inverses are unique and hence definable for each individual in A. Therefore, if $\mathrm{dcl}(A) = A$, then A must be a field.

Suppose a is algebraic over A. Then, because of quantifier elimination, the algebraicity of a is witnessed by some quantifier-free $\mathscr{L}(A)$-formula ϕ, which really then represents a system of polynomials, so a is field-theoretically an algebraic number over A. Field-theoretic and model-theoretic algebraic closure therefore coincide in ACF_0, so $\mathrm{acl}(\mathbb{Q}) = \mathbb{Q}^{\mathrm{alg}}$. In the parlance of [Junker Koenigsmann 2010], algebraically closed fields of characteristic 0 are slim fields, which are those fields wherein the model-theoretic meaning of algebraicity coincides with the pure field-theoretic meaning of algebraicity. Algebraic extensions of $\mathbb{Q}$ in the logical sense will be algebraic extensions of $\mathbb{Q}$ in the algebraic sense. Indeed, these mirroring results hold for any field of zero characteristic. All in all, then, the Galois theory of pure fields of characteristic 0 is recovered from the model-theoretic application to the theory ACF_0.

In fields of positive characteristic, the situation becomes slightly more complicated. Let p be a prime number, and let the fields subsequent be of that prime characteristic. In each characteristic there is a prime field, denoted $\mathbb{F}_p$, and it is clearly definably closed, being finite. Less immediately evident is what shape the definably closed substructures of an algebraically closed field of positive characteristic may be.

Proposition 7.2 Let $\mathscr{L}$ be the language of rings, let the theory T be the theory ACF_p for some positive prime number $p \in \mathbb{N}$, and let $\mathfrak{M} \models \mathrm{ACF}_p$ an algebraically closed field of characteristic p. Then, the definably closed substructures of $\mathfrak{M}$ are fields that are perfect and contained in $\mathfrak{M}$. If F and K are two fields such that $F \subseteq K$, then K is a model-theoretically algebraic extension of F if K is algebraically a separable extension of F.

Proof. Since ACF_p has quantifier elimination, the same reasoning as in the case of ACF_0 applies with respect to model-theoretic algebraicity coinciding with field-theoretic algebraicity. Thus, algebraically closed fields of prime characteristic are also slim. Perfection of fields is necessary; if a field F is imperfect, then its Frobenius image F^p will be properly smaller, and it would be possible to define individuals not in F but in its perfect closure. Otherwise, the same arguments in the zero characteristic situation also hold here. If $K \supseteq F$ are two fields, separability is now an issue. If K is finitely algebraic over F but not separable, then K is not perfect, so K is not definably closed. ↻

The task of characterizing definably closed substructures of algebraically closed fields succeeds without great difficulty in no small part to the coincidence between model-theoretic algebraicity and field-theoretic algebraicity. The need to categorize fields with this property as slim fields accompanies the observation that some theories of fields do not conflate these two understandings of algebraicity. In such cases, model-theoretic algebraicity, which always includes field-theoretic algebraicity, may be used to define field-theoretically transcendental numbers, a phenomenon that has been explored in [Koenigsmann 2002].

So, the Galois theory of fields as described model-theoretically in the theory of algebraically closed fields of some fixed characteristic captures many of the algebraic Galois theory of fields in a straightforward manner. Some aspects are lost, however; one cannot speak of algebraic but inseparable extensions, for instance, and consequently, the discourse cannot recognize imperfect fields.

7.2 Regarding Imperfect Fields

Background information for this section of the work may be found in standard references such as [Delon 1998]. An imperfect field F is one not equal to its Frobenius image $F^p := \{x^p \in F\}$. The Frobenius image itself is not perfect but is a subfield of F. Since field extensions can be regarded as linear spaces over an underlying subfield, there is a linear basis spanning F over F^p. A p-basis of F is a set $B \subseteq F$ which has the property that each element $b \in B$ is p-free over F^p (so that each b is not in $F^p(B \setminus \{b\})$) and that $F = F^p(B)$; this p-basis B generates a linear basis spanning F over F^p consisting of the monomials of B with the form

$$\prod_{i \in I} b_i^{j(i)}$$

where $I = |B|$ and $j : I \to \{0, 1, 2, ..., p-1\}$ does not map all of I to 0.

The cardinality of the p-basis determines the degree of imperfection of F; this work assumes that the p-basis is finite for model-theoretic purposes, but it is also possible to have a p-basis be infinite. In such a case, F has the property $[F : F^p] = p^n$, with n then being the degree of imperfection. The degree of imperfection essentially describes how far away F is from being a perfect field. Each imperfect field F has a perfect closure, denoted $F^{p^{-\infty}}$, obtained by adjoining all p^nth roots of every individual in F for every $n \in \mathbb{N}$. Notably, F is a purely inseparable extension F^p, and $F^{p^{-\infty}}$ is a purely inseparable extension of F.

One way of speaking about imperfect fields in a model-theoretic environment is to change the background theory from ACF to the theory SCF of separably closed fields. The theory SCF of separably closed fields can be obtained from the theory ACF of algebraically closed fields by changing the axiom schemata of 7.1 for each $n \in \mathbb{N}$ to one that effectively states that an arbitrary polynomial of degree n has a root or otherwise is inseparable. Inseparability of a polynomial can be expressed in a first-order fashion by using the discriminant of the polynomial, with an inseparable polynomial having discriminant value of 0. Equivalently, inseparability may be expressed using formal derivatives since an inseparable polynomial has a formal derivative value of 0. Like ACF, SCF is an incomplete theory which lacks some essential information. One of these is the field characteristic. Another, which is indeed a key ingredient, is the information about the cardinality of the p-basis of the container field. The following work relies on the degree of imperfection to be finite, since the following techniques do not follow through to fields with infinite degree of imperfection.

Let K be an arbitrary imperfect field of characteristic $p \neq 0$. Then, K has a p-basis obtained over $\mathbb{F}_p$. Suppose that K is imperfect with a finite degree of imperfection. The theory of the separable closure of K can then be obtained by completing the theory SCF with the axiom for field characteristic p and by including constant symbols to be interpreted as a p-basis for K. In the case of algebraically closed fields, ACF_p denotes the completion of ACF with the inclusion of the field characteristic. In the separable case, let $\mathrm{SCF}_p(K)$ denote the completion of SCF based on the information provided by the imperfect field K of characteristic p, or alternatively if there is no desire to make explicit reference to a field one can denote by $\mathrm{SCF}_{p,B}$ the completion by the inclusion of the field characteristic and n new constant symbols to describe the p-basis.

This setup permits the discussion of imperfect fields as definably closed structures. Specifically, let $\mathscr{L}$ be the language of rings, plus the new constant symbols for a p-basis. Take the complete theory $\mathrm{SCF}_{p,B}$ for that language, by describing how the constant symbols actually act as a p-basis. Let $\mathfrak{M} \models \mathrm{SCF}_{p,B}$ be a model. The following result makes the choice of $\mathfrak{M}$ not a crucial issue in setting up the resultant Galois theory.

Proposition 7.3 The theory $\mathrm{SCF}_{p,B}$ in the language of rings adjoined with constant symbols determining the p-basis B is model complete.

Proof. This is a standard result, so a brief sketch may be sufficient for current purposes. Since B is a p-basis, any model $\mathfrak{M}$ of $\mathrm{SCF}_{p,B}$ must have B as its p-basis, so every such separably closed field will be separable over the prime $\mathbb{F}_p(B)$, even if (especially so) such a field is transcendental over this base field. If $\mathfrak{M} \subseteq \mathfrak{N}$ such that $\mathfrak{N} \models \mathrm{SCF}_{p,B}$ as well, then $\mathfrak{M}$ is relatively algebraically closed in $\mathfrak{N}$, since the algebraic closure of $\mathfrak{M}$, if not $\mathfrak{M}$ itself, is purely inseparable. Furthermore, $\mathfrak{M}^{\mathrm{alg}}$ and $\mathfrak{N}$ are linearly disjoint over $\mathfrak{M}$. For every $a \in \mathfrak{N}^n$ and quantifier-free $\mathscr{L}(\mathfrak{M})$-formula $\phi(x)$ with $\mathfrak{N} \models \exists x \phi(X)$ so that $\mathfrak{N} \models \phi(a)$, the sequence a generates a prime ideal $\mathfrak{p}$ of $\mathfrak{M}\left[X^{n+1}\right]$. Consequently, $\mathfrak{M}$ is a field of definition for $\mathfrak{p}$, so a corresponding variety V exists in $\mathfrak{M}$. The polynomials of $\mathfrak{p}$ may not all be separable, but the separable ones in $\mathfrak{p}$ are such that their solutions in $\mathfrak{M}$ are dense in V. The separably closed field $\mathfrak{M}$ therefore contains these separable points, and they in turn can be used to find some $b \in \mathfrak{M}$ such that $\mathfrak{M} \models \phi(b)$. Consequently, the theory $\mathrm{SCF}_{p,B}$ is a model complete theory. ⭮

The role that the constants for the p-basis play is key, since it forces all models of $\mathrm{SCF}_{p,B}$ to have the same p-basis. Crucially, model completeness also provides the following observation which is key for formulating the Galois theory of this context.

Corollary 7.4 Let $\mathrm{SCF}_{p,B}$ be formulated in an appropriate language such that the finite set B constitutes the p-basis described in the theory. Then, the field $(\mathbb{F}_p(B))^{\mathrm{sep}}$, the separable closure of the field $\mathbb{F}_p(B)$, is a model of $\mathrm{SCF}_{p,n}$ and an elementary substructure of any such model $\mathfrak{M} \models \mathrm{SCF}_{p,B}$. The field $\mathbb{F}_p(B)$ is a definably closed and prime substructure of any model of $\mathrm{SCF}_{p,B}$.

Proposition 7.5 Let $\mathfrak{M} \models \mathrm{SCF}_{p,B}$ be a separably closed field. Then, the definably closed substructures of $\mathfrak{M}$ are exactly those subfields of $\mathfrak{M}$ which have B as their p-basis.

Proof. Once it becomes evident that $(\mathbb{F}_p(B))^{\mathrm{sep}}$ is a model of the model-complete theory $\mathrm{SCF}_{p,B}$, the prime structure must be a substructure of this prime separably closed field. With the addition of constant symbols for B, it is not possible to definably close the prime field $\mathbb{F}_p$ of characteristic p, which is finite and therefore perfect. On the other hand, the field generated by the inclusion of the p-basis, which has been denoted $\mathbb{F}_p(B)$, incorporates the imperfection caused by B. Clearly, it is prime. Definable closure arises basically by definition; the field is generated by the p-basis B, so the only way $\mathbb{F}_p(B)$ would not be definably closed is if it is not a field or is not generated by B in the conventional sense of generating fields through their generators. ⭮

The question of which substructures are definably closed is related to the question of what the model-theoretic algebraic extensions look like in the language of field theory. The case of the prime imperfect field $\mathbb{F}_p(B)$ and its separable closure $(\mathbb{F}_p(B))^{\mathrm{sep}}$ is instructive. Being separably closed, any intermediate extension K between $\mathbb{F}_p(B)$ and $(\mathbb{F}_p(B))^{\mathrm{sep}}$ must be separable. Consequently, K must have B as its p-basis. With model completeness, it follows that, for an arbitrary $\mathfrak{M} \models \mathrm{SCF}_{p,B}$, all intermediate fields between $\mathbb{F}_p(B)$ and $\mathfrak{M}$ are separable.

Theorem 7.6 (Application to Separably Closed Fields) Let the theory $\mathrm{SCF}_{p,B}$ be formulated in an appropriate language of rings augmented by a finite number of additional constant symbols. Let $\mathfrak{M}$ be a model. Let $A \subseteq \mathfrak{M}$ be a substructure. Then, A is definably closed if and only if it is an imperfect field containing $\mathbb{F}_p(B)$ with B as its p-basis. Let F and K be definably closed substructures of $\mathfrak{M}$ with $F \subseteq K$. Then, K is model-theoretically a finite algebraic extension of F if K is field-theoretically a finite algebraic separable extension of F.

7.3 Differential Fields and Differentially Closed Fields

When Poizat developed his *théorie de Galois imaginaire* in [Poizat 1983], the work laid out two worked examples of the Galois correspondence. The first example explicated a Galois theory of strong types. The second developed a model-theoretic formulation of differential Galois theory. Through a comprehensive effort, Pillay clarified and refined this latter application to develop a largely comprehensive corpus of model-theoretic differential Galois theory in [Pillay 1998], [Pillay 1997], and [Marker Pillay 1997]. A complete summary of the local application of model-theoretic notions of Galois theory to differential fields would be redundant. However, it would be productive to compare the similarities between the work developed by Pillay and the consequences of the general framework here to the appropriate theory.

A field F becomes a differential field (F,d) when paired with a differential operator or derivative function $d : F \to F$ that follows the Leibniz product rule and linearity.

1. (Product Rule) $\forall x \forall y (d(xy) = x dy + y dx)$
2. (Linearity) $\forall x \forall y (d(x+y) = dx + dy)$

When (F,d) is a differential field, it has a subfield called the field of constants which has the property that $dx = 0$ for all x in the field of constants. A differential field has trivial differentiation when it coincides with its field of constants. A differential field (F,d) gives rise to differential polynomial rings, which are polynomial rings of F that are closed with respect to derivation and denoted $F\{x\}$ in the case of one variable. Just as a polynomial has a degree taken as the largest sum of the exponents of the variables of the constituent monomials differential polynomials also have an order, determined by the differential monomial with the most iterations of the differential operator. A differential field also has a differential closure, defined as follows.

Definition 7.7 Let $\mathscr{L}_{dr} := \langle +, \times, 0, 1, d \rangle$ be the language of differential rings. Then, the theory DCF_0 of differentially closed fields (of characteristic 0) consists of the axioms for algebraically closed fields of characteristic 0, axioms describing d as a differential operator, and axioms that assert differential closure, specifically that if $f(x)$ and $g(x)$ are differential polynomials with order of f greater than the order of g, then there is a solution to the system asserting $f(x) = 0$ and $g(x) \neq 0$.

Natural examples of differentially closed fields are not readily obtained. The difficulty in obtaining natural examples of differentially closed fields may be exemplified by the case of $\mathbb{C}(t)$. The field $\mathbb{C}(t)$ has the usual derivative function that maps $at^n \mapsto ant^{n-1}$ but is not differentially closed.

Proposition 7.8 Let $\mathfrak{M} \models \mathrm{DCF}_0$ be a model in the language of differential rings. Then, a substructure $A \subseteq \mathfrak{M}$ is definably closed if and only if A paired with the derivation induced by $\mathfrak{M}$ is a differential field.

Proposition 7.9 The notion of model-theoretic algebraic closure coincides with the field-theoretic notion of algebraic closure in the sense that $a \in \mathrm{acl}(A)$ if and only if $a \in A^{\mathrm{alg}}$.

Proof. That model-theoretic algebraicity coincides with field-theoretic algebraicity in the theory of differentially closed fields is remarked upon in [Pillay 1997]. Since DCF has quantifier elimination, the same reasoning as with pure fields can be used to obtain that differential subfields of a differentially closed field $\mathfrak{M}$ are exactly the definably closed substructures of $\mathfrak{M}$. ↻

A possible source of confusion may be had with the notion of *differential algebraicity*. In general, a sequence of individuals in a differentially closed field being differentially algebraic over a differential field does not imply that it is model-theoretically algebraic over that same differential field; rather a sequence is differentially algebraic if it satisfies a finite system of differential equations. Differential closure then is based on this idea of differential algebraicity. The current framework relies on model-theoretic algebraicity rather than differential algebraicity, so the differential Galois theory of this work says little beyond the Galois theory of pure fields. The differential Galois theory of Pillay, however, is based on differential algebraic groups inaccessible from model-theoretic algebraicity, and so that approach yields a much richer understanding of differential Galois theory.

7.4 Unstable Theories

The first-order theories that have been considered so far share a number of common model-theoretic properties. When provided a small amount of additional information, they each become complete, and an understandably enriched language enables them to eliminate quantifiers, straightforwardly leading to model-completeness as well. In addition, the theories are all stable, though to varying degrees.

This consistency should not be surprising. The theories ACF_0 and ACF_p for prime p are among the first-studied theories in model theory, and they capture in a logical context the results of classical algebraic geometry. The separable variants $\mathrm{SCF}_p(K)$ or $\mathrm{SCF}_{p,B}$ do not majorly affect the properties that arise from the algebraic origins. The theory of differentially closed fields is an extension of the algebraically closed one, and the addition of a derivation further mollifies its model-theoretic properties.

The model-theoretic Galois correspondence can be demonstrated abstractly without the need for stability, but the framework developed from notions of stability theory, so it is not surprising that these stable theories can accommodate nontrivial systems of Galois theory. Since stable theories are those which basically have relatively few n-types, moving to theories which are not stable may provide insight into the role that the number of n-types may play in affecting the development of Galois connections. The following examples show how their instability can induce significant hurdles that prevent nontrivial correspondences to develop.

7.4.1 Real Closed Fields

The collection of stable theories has a classic exemplar manifested in the theories ACF_0 and ACF_p. Another division of first-order theories is the collection of order-minimal theories, of which the theory RCF of real closed fields is the classic example. The canonical model of ACF_0 is the complex number field $\mathbb{C}$; the counterpart for RCF is the real number field $\mathbb{R}$, which is also axiomatized as RCOF, the theory of real closed ordered fields, in the first-order language of ordered rings $\mathscr{L}_{r<} := \langle +, \times, 0, 1, < \rangle$ by the standard field axioms, the assertion that the field characteristic is 0, that every polynomial of odd degree has a solution, and that $<$ describes a total linear ordering with $0 < c$ implying c is a square which respects addition and multiplication so that, where a, b, and c are individuals, $a < b$ implies $a + c < b + c$ and, where $c > 0$, $ac < bc$, with $0 < c$ implying c is a square.

Being a theory of fields, RCF can code finite sets, and it does so in the usual way as in other encountered field theories. The theory RCF is complete, and its almost-trivial extension RCOF eliminates quantifiers. The only barrier to quantifier elimination for RCF is the ordering relation which is covered by RCOF, so these two theories are essentially interchangeable.

Let $\mathfrak{M} \models \mathrm{RCOF}$ be a large and saturated real closed ordered field. An obvious question to consider is what substructures of $\mathfrak{M}$ are definably closed. Recall that in the algebraic case, the definably closed substructures of an algebraically closed field are perfect subfields. The same broadness does not apply here.

Proposition 7.10 Let $\mathfrak{M} \models \mathrm{RCOF}$ be a real closed ordered field in the language of ordered rings. Then, a substructure $A \subseteq \mathfrak{M}$ is definably closed if and only if A is also a model of RCOF. Consequently, A must be an elementary substructure of $\mathfrak{M}$.

Proof. Consider, for example, the prime pure field $\mathbb{Q}$. Definably closed in the language of rings with the theory ACF_0, it turns out to fail definable closure when an ordering is present. For example, the standard irreducible formula for $\sqrt{2}$ in ACF_0, which would be the polynomial equation $x^2 - 2 = 0$ with solution set $\{\sqrt{2}, -\sqrt{2}\}$, can be augmented in the language of ordered rings so that $\sqrt{2}$ can be separated from $-\sqrt{2}$, so that the formula $x^2 - 2 = 0 \wedge 0 < x$ defines $\sqrt{2}$ as an individual. Consequently, the rational number field is not definably closed, and since this same process of reasoning can be used for arbitrary algebraic numbers over $\mathbb{Q}$, the definable closure of $\mathbb{Q}$ must be relatively algebraically closed in $\mathfrak{M}$. The only relatively algebraically closed real fields are real closed fields, so these are the only substructures of $\mathfrak{M}$ that are definably closed. Model completeness then means that such a real closed subfield A would be an elementary substructure of $\mathfrak{M}$. ⟲

Theorem 7.11 The Galois correspondence between substructures of $\mathfrak{M} \models$ RCOF is trivial.

Another way of determining this result is looking at what the model-theoretic notion of algebraicity means in the context of real closed fields. Because of the total linear ordering relation, any individual in the large real closed field that would be algebraic in the corresponding field-theoretic algebraic closure which models ACF_0 with the language of rings can now be distinguished from its algebraic conjugates.

Theorem 7.12 In the theory RCF in the language of rings and the theory RCOF in the language of ordered rings, a finite sequence a of individuals in a model $\mathfrak{M}$ is algebraic over a parameter set A if and only if it is definable over A as well. Consequently, model-theoretic algebraic closure coincides with model-theoretic definable closure.

The key culprit is the linear order. Since the order is characteristic of all order-minimal structures, the trivial Galois correspondence is characteristic of all such structures.

Theorem 7.13 If $\mathfrak{M}$ is an arbitrary o-minimal structure, then $\mathfrak{M}$ will only have trivial Galois connections between its substructures and their corresponding groups of partial elementary maps.

7.4.2 Arithmetic

The problem with ordering also affects sufficiently expressive systems of arithmetic. For example, Peano arithmetic, over its arithmetical languages $\mathscr{L}_{\mathrm{arith}\,1} := \langle s, +, \times, 1\rangle$ or $\mathscr{L}_{\mathrm{arith}\,0} := \langle s, +, \times, 0\rangle$, can code finite sets. Indeed, Peano arithmetic is expressive enough to eliminate imaginaries uniformly, a task set out as an exercise in [Hodges 1997]. The elimination of imaginaries in a theory is uniform if the procedure by which imaginaries are encoded in the home sort of the structure does not depend on the use of parameters. Because ordering can be defined in arithmetic, any sequence of individuals that is algebraic can be defined, so again, algebraicity collapses into definability.

Proposition 7.14 The arithmetic over natural numbers $\mathbb{N}$ and any nonstandard model of Peano arithmetic have a trivial Galois correspondence.

Trivial as such a Galois theory may be, this application is an example of how seemingly-unlikely theories can still possess a theoretic Galois correspondence. Here, the problem would be as in the real closed case. The only definably closed substructures of a model of arithmetic would be elementary substructures of that model.

Chapter 8

Fields with Valuations

The case studies involving order-minimal structures and arithmetic may suggest that only stable theories have interesting Galois correspondences. The following applications involve fields with valuations, one of which refutes this suggestion while the other may reinforce the underlying impression that unstable theories with nontrivial correspondences are relatively rare. The first valued field application is looking at valued fields that inhabit an algebraically closed valued field. The second case study looks into p-adically closed fields $\mathbb{Q}_p$ and their corresponding first-order theories.

8.1 Valued Fields

A valued field (F, v) consists of a pure field F and a valuation function v on F. Associated with a valuation is an ordered abelian group Γ such that v maps individuals in F to elements in $\Gamma \cup \{\infty\}$; only 0 maps to ∞, which has the usual properties of infinity such as $\infty + \infty = \infty \times \infty = \infty + \gamma = \gamma + \infty = \infty$ for all γ in Γ. This group Γ is the value group of (F, v). For current purposes, it is always assumed that the valuation maps onto its value group. A formal definition summarizes this presentation.

Definition 8.1 (Valued Rings and Fields) Let R be an integral domain. Then, a valuation on R is a function $v : R \twoheadrightarrow \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group and the following properties hold.

1. For all x and y in R, $v(x) \neq \infty$ if and only if $x \neq 0$.
2. Furthermore, $v(xy) = v(x) + v(y)$.
3. Where $<$ is the inherent ordering in Γ, $v(x + y) \geq \min(\{v(x), v(y)\})$.

A valued ring (R, v) is a valued field if R is a field.

Every field can be matched with the trivial valuation with $\Gamma := \{0\}$. Furthermore, any finite field will only have trivial valuations. Some basic properties of valued fields follow.

Proposition 8.2 (Basic Properties) Let (F, v) be a valued field.

1. It is always the case that $v(1) = 0$.
2. For all $x \in F$, $v(x) = v(-x)$, where $-x$ is the additive inverse of x.
3. For all $x \in F^\times$, $v(x^{-1}) = -v(x)$, where x^{-1} is the multiplicative inverse of x.
4. For all x and y in F, if $v(x) < v(y)$, then $v(x + y) = v(x)$.

Let (F, v) be a valued field. In addition to its value group Γ (Γ_v if ambiguity may otherwise arise), the valued field also induces a valuation ring $\mathcal{O}_v := \{x \in F : 0 \leq v(x)\}$ of individuals in the field. The value group can be obtained using the valuation ring, such that $\Gamma \cong F^\times / \mathcal{O}_v^\times$. The valuation ring has a maximal ideal $\mathcal{M}_v := \{x \in F : 0 < v(x)\}$. The valuation ring and its maximal ideal generate together the residue class field $k_v := \mathcal{O}_v / \mathcal{M}_v$; the residue field may also be denoted F_v when different valued fields are being discussed.

Much of the standard results of this section are based on [Engler Prestel 2005]. While valuation rings may carry more than one exact valuation, the ring has the form that at least every nonzero field element or its multiplicative inverse is in it. Conversely, if F is a field and $R \subseteq F$ is a valuation ring, then there is a valuation induced by that ring. Therefore, any such ring may be called a valuation ring. Two arbitrary valuations v and w on a field F can be called equivalent if their valuation rings are the same, so that $\mathcal{O}_v = \mathcal{O}_w$.

8.2 Valuation Theory

Let (F, v) and (K, w) be valued fields such that $F \subseteq K$. Since F and K may have more than one valuation, it is not necessarily the case that the restriction of w to F will be v. However, when this situation does occur, so that v agrees with w on F, then it is appropriate to say that (K, w) is a valued field extension of (F, v). The valuation w is then a *prolongation* of the valuation v, and v is a restriction of w to F.

If K/F is a field extension in the sense of pure fields, then the following theorem guarantees that (F, v) as a valued field has an extension to some valued field (K, w) with w a prolongation of v.

Theorem 8.3 (Chevalley) Let (F, v) be a valued field, with Γ the induced value group. Let K/F be a field extension. Then, there is a value group Δ such that $\Gamma \leq \Delta$ and a valuation $w : K \to \Delta$ extending v.

If K is a pure field extension of F, then the degree of the extension is a measuring device that can be used to compare different pure fields. In the context of valued fields (K, w) and (F, v) with (K, w) an extension of (F, v), there are two additional ways of measuring extensions by referring to the induced valuation structures. The first method is with the ramification index, based on the value groups Γ_w and Γ_v.

Definition 8.4 (Ramification Index) Let (K, w) be a valued field extension of (F, v), with Γ_w and Γ_v their respective value groups. Then, the *ramification index* of (K, w) over (F, v), denoted $e((K, w)/(F, v))$ is determined by their valued groups such that $e((K, w)/(F, v)) := [\Gamma_w : \Gamma_v]$, the index of Γ_w over Γ_v.

The other method is the *residue degree* of (K, w) over (F, v) which is determined by the extension degree of the residue field K_w over the residue field F_v. The ramification index and residue degree can be used to outline various kinds of valued field extensions.

Definition 8.5 (Immediate Extensions) Let (K, w) be a valued field extension of (F, v), such that Γ is the induced value group of (F, v) and Δ the value group of (K, w). Then, (K, w) is an immediate extension of (F, v) if $\Gamma = \Delta$ and $k_v = k_w$; that is, if the induced value groups and the residue class fields are the same. Consequently, the ramification index and residue degree of (K, w) and (F, v) both take on the value 1.

Definition 8.6 (Maximally Complete Extensions) A valued field (F, v) is maximally complete if it has no proper immediate extension. That is, (F, v) is maximally complete if any valued field extension (K, w) if either not immediate or is (F, v) itself.

Proposition 8.7 Every valued field (F, v) has a maximally complete extension (K, w) that is immediate.

Let (F, v) be a valued field, and consider its unique separable closure F^{sep}. There is the possibility that there are infinitely many different prolongations of v to F^{sep}, but henselian fields do not have this possibility.

Definition 8.8 (Henselian Fields) Let (F, v) be a valued field. Then, (F, v) is a henselian field if the valuation v extends uniquely to the separable closure F^{sep} of F.

Henselian fields are so-called because they satisfy Hensel's Lemma as available in several forms.

Proposition 8.9 (Hensel's Lemma I) Let (F, v) be a henselian valued field. Let $f \in \mathcal{O}_v[X]$ and let $c \in \mathcal{O}_v$ so that $2v(f'(c)) < v(f(c))$. Then, there is some $d \in \mathcal{O}_v$ such that $f(d) = 0$ and $v(f'(c)) < v(c - d)$.

Proposition 8.10 (Hensel's Lemma II) Let (F, v) be a henselian valued field. Let $f \in \mathcal{O}_v[X]$ and let $c \in \mathcal{O}_v$ so that their residue images $\bar{f}$ and $\bar{c}$ have the property that $\bar{f}(\bar{c}) = 0$ and $\bar{f}'(\bar{c}) \neq 0$. Then, there is some $d \in \mathcal{O}_v$ such that $f(d) = 0$ and $\bar{d} = \bar{c}$.

Just as a field F may have a separable closure, algebraic closure, and the like, a valued field will also have a henselian closure, obtained by its henselization.

Definition 8.11 (Henselizations) Let (F, v) be a valued field. Then, (F, v) has a valued field extension (F^h, v^h), known as a henselization of (F, v), such that

1. F^h is an algebraic field extension of F,

2. (F^h, v^h) is henselian,

3. for any other valued field extension (F', v') with the property that F' is an algebraic field extension of F and (F', v') is henselian, there is a unique embedding $\varphi : F^h \to F'$ fixing F pointwise such that $v' \circ \varphi = v^h$.

While a valued field may have more than one henselization, it is also the case that each such henselization will be isomorphic to the others. Furthermore, every valued field (F, v) has a henselization (F^h, v^h), which can be obtained in the following manner.

Obviously, a henselian valued field (F, v) has itself as its henselization, so suppose that (F, v) is not henselian. It turns out then that F cannot be separably closed. So, take the separable closure of F, denoted F^{sep}, along with an appropriate extension v^{sep} of the valuation. Take the subgroup G of the permutation group $\text{Aut}(F^{\text{sep}}/F)$ defined such that $G := \{\sigma \in \text{Aut}(F^{\text{sep}}/F) : v^{\text{sep}} \circ \sigma = v^{\text{sep}}\}$. The field generated by the fixed elements of G, denoted $\text{Fix}(G)$, is the intended field reduct of the henselization. Take $(\text{Fix}(G), v^{\text{sep}}|_{\text{Fix}(G)})$ as the henselization.

The property of being henselian is monotonic with respect to valued field extensions, so that the following result holds.

Proposition 8.12 Let (F, v) be a henselian field. Then, any algebraic valued field extension of (F, v) is itself henselian.

Since the first application is looking at algebraically closed valued fields, it is sensible to crystallize this notion and outline some of such fields' properties.

Definition 8.13 (Algebraically Closed Valued Fields) Let (F, v) be a valued field. Then, if F is algebraically closed, then (F, v) is also said to be an algebraically closed valued field.

Proposition 8.14 (Properties) Let (F, v) be an algebraically closed valued field. Then, the induced value group Γ is a divisible group and the residue class field k_v is itself also algebraically closed.

The characteristics of an algebraically closed valued field and its residue class field can either be $(0, 0)$, $(0, p)$, or (p, p) for a prime number p. Just as ACF_0 was the suitable theory for looking at the Galois theory of pure fields of characteristic 0 and DCF_0 for differential Galois theory, an appropriately formulated theory of algebraically closed valued fields is the host for a Galois theory of valued fields. Before delving into model-theoretic studies, some review of the algebraic Galois theory of valued fields may be appropriate.

8.3 The Structure Theory of Valued Fields

Let (K,w) be a valued field extension of (F,v). Then, (K,w) is a Galois extension if K is a Galois field extension of F and w is a prolongation of v. Between K and F, structure theory outlines intermediate valued field extensions that demonstrate through the associated induced value groups, rings, and residue fields the transition from (F,v) to (K,w). An overall diagram may be helpful to observe first before getting into specific details.

Automorphism Group	Field	Value Group	Residue Field
1	K	Γ_w	K_w
R	K^{ram}	$\frac{1}{p^{\prime\infty}}\Gamma_v \cap \Gamma_w$	$K_w \cap F_v{}^{\mathrm{sep}}$
I	K^{inert}	Γ_v	$K_w \cap F_v{}^{\mathrm{sep}}$
D	K^{decomp}	Γ_v	F_v
G	F	Γ_v	F_v

Thus, between a particular valued field and a Galois extension of it, there are certain distinguished intermediate extensions that are recognized by the structure theory of valued fields. Some quick comments can provide some orientation. The group denoted 1 is the trivial automorphism group of (K,w) over itself. The base field (F,v) and its induced structures constitute the bottom row of the figure, the Galois extension (K,w) and its induced structures occupying the top row. The intermediate extensions are motivated by certain subgroups of the Galois group $G := \mathrm{Gal}((K,w)/(F,v))$.

The first intermediate extension of (F,v) is the field obtained by the decomposition subgroup D of G.

Definition 8.15 (Decomposition Group) Let (K,w) be a Galois extension of (F,v), and let G be the corresponding Galois group $\mathrm{Gal}((K,w)/(F,v))$. Then, the *decomposition group* D of G is defined to be the subgroup

$$D := \{\sigma \in G : \forall x \in K v(\sigma(x)) = v(x)\}$$

which fixes the valuation.

The corresponding field is the *decomposition* extension of (F,v), which will be denoted $(K,w)^{\mathrm{decomp}}$ and obtained by $(K^{\mathrm{decomp}}, w|_{K^{\mathrm{decomp}}})$. The decomposition extension must be an immediate extension since the value group and residue field of $(K,w)^{\text{decomp}}$ are the same as of (F,v). The valued field $(K,w)^{\text{decomp}}$ is also henselian when $K = F^{\text{sep}}$. If $K \neq F^{\text{sep}}$, then $(K,w)^{\text{decomp}}$ is relatively henselian with respect to (K,w).

Following the decomposition extension is the *purely inert* extension of (F,v) under (K,w), characterized by its automorphism group, the inertia subgroup I of G.

Definition 8.16 (Inertia Group) Let (K,w) be a Galois extension of (F,v), and let $G := \text{Gal}((K,w)/(F,v))$ be the corresponding Galois group. Then, the *inertia group I* of G is the subgroup which fixes the intersection of the residue field of K and the separable closure of the residue field of F. Specifically, I is the group defined by

$$I := \{\sigma \in \text{Gal}((K,w)/(F,v)) : (\sigma(x) - x \in \mathcal{M}_v)\forall x \in \mathcal{O}_v\}.$$

The definition of the inertia group immediately outlines the shape of the purely inert extension's residue field; it will be precisely the intersection of the residue field of (K,w) and the separable closure of the residue field of (F,v). Since the residue field changes, the purely inert extension is not immediate over the decomposition extension. The value group remains unchanged, though. Following on up, there is the tamely ramified extension corresponding to the ramification subgroup R of G.

Definition 8.17 (Ramification Group) Let (K,w) be a Galois extension of (F,v), and let

$$G := \text{Gal}((K,w)/(F,v))$$

be the corresponding Galois group. Then, the *ramification group R* of G is the subgroup such that, if σ is in R, then for all nonzero x in K, $\frac{\sigma(x)}{x} - 1$ is in the maximal ideal $\mathcal{M}_w$. Put another way, R can be defined so that

$$R := \{\sigma \in \text{Gal}((K,w)/(F,v)) : (\frac{\sigma(x)}{x} - 1 \in \mathcal{M}_v)\forall x \in K\}.$$

The value group has now entered a transitional state from that of (F,v) to (K,w). It then happens that (K,w) is a wildly ramified extension of this tamely ramified extension.

8.4 Logical Perspectives on the Structure Theory

Like pure fields embedding into algebraically closed fields and differential fields absorbed into differentially closed fields, valued fields are resident substructures of algebraically closed valued fields. The consequences of the model-theoretic Fundamental Theorem 6.2 for the Galois theory over algebraically closed valued fields lead to a simplified array between a valued field (F,v) and its Galois extension (K,w).

There are various ways of presenting the theory ACVF of algebraically closed valued fields and its permissible completions in the style of logic. A given valued field (F,v) has several induced structures associated with it, and there is a choice to be made on which of these additional objects should be explicitly denoted in the syntax. The consequence is that the structures involved may include a large number of sorts.

The minimalist approach, one which diverges little from the language of rings employed by pure fields, is extending the language of rings with a two-place valued order relation Div such that the formula $\mathrm{Div}(x, y)$ would be interpreted as $v(x) \leq v(y)$. The behavior of the valuation function would be encoded into this predicate, which would also be used to extrapolate the value group, valuation ring, and residue field.

Another approach would be to look at a three-sorted language, so that a valued field (F, v) would be represented by the home sort F, a sort for the valuation ring $\mathcal{O}_v$, and a sort for the residue field k_v. A two-sorted alternative is to have the home sort F and the value group Γ, with the valuation v being a map between the sorts. The additive identity of F would be conveniently elided to avoid having to include infinity to Γ.

For particular personal reasons, the valued fields here will be formally treated in the one-sorted language with the ordering predicate. Let $\mathscr{L}_{vr} := \langle +, \times, 0, 1, \mathrm{Div} \rangle$ denote this language of valued rings with Div being a two-place predicate symbol whose interpretation is to denote the order given by the value group.

In essence, this language is similar to the language of ordered rings, with the difference being that the order is based on the value group rather than on the field itself. Being a one-sorted structure, some of the induced structures cannot be modeled directly as objects of their own right, but rather through representations embodied within the home field itself. In any case, with these issues in mind, the work may proceed on the model theory of algebraically closed valued fields.

Like ACF, ACVF is not yet complete, since it is missing information about field characteristic. Since there are two fields to consider, it is necessary to include information about the characteristic of both fields. The characteristic of the residue field depends upon the characteristic of the home field. If the characteristic of the home field is 0, then the residue field may be of characteristic 0 too or of p for some prime number p. If the characteristic of the home field is p for some prime p, then the characteristic of the residue field must also be p. Let $\mathrm{ACVF}_{(p,q)}$ denote such a completion of ACVF, subject to the constraints about which field characteristics are actually permissible.

In the one-sorted language, the information about p is axiomatized in the usual manner, by explicitly stating the field characteristic in the language of rings. To encode the information about q, though, the divisibility predicate needs to be used carefully.

Theorem 8.18 (Elimination of Quantifiers) The theory $\mathrm{ACVF}_{(p,q)}$ of algebraically closed valued fields with characteristic (p, q) has elimination of quantifiers in the language $\mathscr{L}_{vr}$ of valued fields.

Proof. The result is a consequence of work by A. Robinson in [Robinson 1956] about the model completeness of the theory in this language; further discussion about the connection between quantifier elimination and valued fields in this language may be found in [Macintyre McKenna van den Dries 1983]. ↻

Thus, the theory is also model complete. Since the language of rings can encode finite sets for pure fields, it can code finite sets for valued fields. However, elimination of imaginaries is a different situation. A key result by H.D. Macpherson, D. Haskell, and E. Hrushovski in [Haskell Hrushovski Macpherson 2007] is that elimination of imaginaries for algebraically closed valued fields requires a language with a countably infinite number of sorts.

Certainly, the more sorts accommodated by the language, the easier it would be to express certain structural properties of the valued fields. Thus, the challenge of using a one-sorted language is to determine how much one can get away with it in describing the various induced structures of valued fields without resorting to additional symbols.

So, let $\mathscr{L}_{vr}$ be the language of choice, and let $(\mathfrak{M}, \mathfrak{v}) \models \mathrm{ACVF}_{(p,q)}$ be an algebraically closed valued field. The value group, valuation ring, and the residue field of $\mathfrak{M}$ can be regarded as equivalence classes of the home field $\mathfrak{M}$, which is to say that these structures can be interpreted over $\mathfrak{M}$.

Since the value group is exactly the image of the valuation map, an individual $c \in \mathfrak{M}$ can represent an element of the value group by its valuation. Then, the equivalence relation $\equiv_{\Gamma_{\mathfrak{v}}}$ defined by $a \equiv_{\Gamma_{\mathfrak{v}}} b \Leftrightarrow \mathfrak{v}(a) = \mathfrak{v}(b)$ can be represented by the formula $\mathrm{Div}(x, y) \wedge \mathrm{Div}(y, x)$, with $(a, b) \mapsto (x, y)$. The group operator $+_{\Gamma_{\mathfrak{v}}}$ is defined on multiplication in the home field, so its interpretation carries through to the language in a straightforward manner. Group inverses can be obtained by exploiting the property that $\mathfrak{v}(1) = 0_{\Gamma_{\mathfrak{v}}}$; thus, for every nonzero $c \in \mathfrak{M}$, $c \times c^{-1} = 1$ implies $\mathfrak{v}(cc^{-1}) = \mathfrak{v}(1) \Rightarrow \mathfrak{v}(c) + \mathfrak{v}(c^{-1}) = 0_{\Gamma_{\mathfrak{v}}} \Rightarrow \mathfrak{v}(c^{-1}) = -\mathfrak{v}(c)$.

Also, since $\mathfrak{v}(1) = 0_{\Gamma_{\mathfrak{v}}}$, it is relatively straightforward to interpret the domain of the induced valuation ring $\mathscr{O}_{\mathfrak{v}}$ so that an individual $c \in \mathfrak{M}$ is in $\mathscr{O}_{\mathfrak{v}}$ if and only if it realizes $\mathrm{Div}(1, c)$. The maximal ideal would be obtained by requiring that $c \in \mathscr{O}_{\mathfrak{v}}$ refutes $\mathrm{Div}(c, 1)$. The residue field is the quotient structure of $\mathscr{O}_{\mathfrak{v}}$ over $\mathscr{M}_{\mathfrak{v}}$. Thus, the domain of the residue field $k_{\mathfrak{v}}$ can be represented using the equivalence relation $\equiv_{k_{\mathfrak{v}}}$ so that $a \equiv_{k_{\mathfrak{v}}} b$ through the formula $\mathrm{Div}(1, x) \wedge \mathrm{Div}(x, 1) \wedge \mathrm{Div}(1, y) \wedge \mathrm{Div}(y, 1) \wedge \mathrm{Div}(1, x - y) \wedge \neg \mathrm{Div}(x - y, 1)$ with $(a, b) \mapsto (x, y)$.

Thus, the one-sorted language $\mathscr{L}_{vr}$ is expressive enough to recognize the various structures associated with the algebraically closed valued field $(\mathfrak{M}, \mathfrak{v})$. Furthermore, the model-theoretic understanding of algebraicity continues to be the same as the field-theoretic notion of algebraicity. Although the new predicate increases the expressivity of the language, the definable sets involving the predicate for valuation are always infinite and therefore do not affect algebraicity.

A substructure of $(\mathfrak{M}, \mathfrak{v})$ will carry the valuation induced by the model itself. In addition, a definably closed substructure of the algebraically closed valued field $(\mathfrak{M}, \mathfrak{v})$ must, at minimum, be a perfect subfield of $\mathfrak{M}$. The additional structure brought on by the valuation does affect the situation, though, and so there is a notable difference here than in the case of pure fields.

Proposition 8.19 Let $(\mathfrak{M}, \mathfrak{v}) \models \mathrm{ACVF}_{(p,q)}$ be a large and saturated algebraically closed valued field in the language $\mathscr{L}_{vr}$. Let $A \subseteq \mathfrak{M}$ be a substructure of $\mathfrak{M}$. Then, A is definably closed, so that $A = \mathrm{dcl}(A)$, if and only if A is a perfect subfield of $\mathfrak{M}$, and the induced valuation v_A on A is a henselian valuation, so that (A, v_A) is a perfect henselian valued subfield of $(\mathfrak{M}, \mathfrak{v})$.

Proof. The key new part of this result is the claim that the definably closed structures must be henselian. Let (F, v) be a perfect henselian field. By the arguments that are used in the case of perfect fields as the definably closed substructures of algebraically closed fields in the language of rings, a definably closed field must be perfect. Suppose (F, v) is not definably closed; then the reason for failure would have to be due to henselianity. Let $x \in \mathrm{dcl}((F, v)) \setminus (F, v)$. If $x \in \mathrm{acl}((F, v))$, then take an automorphism σ in the decomposition subgroup of the absolute Galois group of F such that $\sigma(x) \neq x$. This is clearly a contradiction. If x is transcendental over (F, v) but definable over (F, v), then x is model-theoretically algebraic over (F, v) in $(\mathfrak{M}, \mathfrak{v})$. However, since both understandings of algebraicity coincide with algebraically closed valued fields, x being model-theoretically algebraic over (F, v) implies x being field-theoretically algebraic over (F, v), contradicting the assumption that x is transcendental over (F, v). Thus, every perfect henselian field must be definably closed. That every definably closed valued subfield of an algebraically closed valued field is perfect henselian takes some further consideration.

Let (F, v) be a valued subfield of the algebraically closed valued field $(\mathfrak{M}, \mathfrak{v})$, with $(F^{\mathrm{sep}}, v^{\mathrm{sep}})$ the separable closure of F and v and v^{sep} the valuations that are obtained from restricting $\mathfrak{v}$ to F and F^{sep}, respectively. Suppose (F, v) is definably closed. Let f be a polynomial in $\mathscr{O}_v[X]$, and let $a \in \mathscr{O}_v$ be an individual such that $f(a) \in \mathscr{M}_v$ and $f'(a) \notin \mathscr{M}_v$. Note then that $\mathscr{M}_v \subseteq \mathscr{M}_{v^{\mathrm{sep}}}$, so $f(a) \in \mathscr{M}_{v^{\mathrm{sep}}}$, and $f'(a) \notin \mathscr{M}_{v^{\mathrm{sep}}}$ since $f'(a) \in \mathscr{M}_{v^{\mathrm{sep}}}$ implies $f'(a) \in \mathscr{M}_v$.

Notably, $(F^{\mathrm{sep}}, v^{\mathrm{sep}})$ is henselian, so $(F^{\mathrm{sep}}, v^{\mathrm{sep}})$ satisfies Hensel's lemma. So, there exists a unique $d \in \mathscr{O}_{v^{\mathrm{sep}}}$ such that $f(d) = 0$ and $d - a \in \mathscr{M}_{v^{\mathrm{sep}}}$. The formula $f(x) \doteq 0 \wedge \mathrm{Div}(1, x-a) \wedge \neg \mathrm{Div}(x-a, 1)$ sufficiently describes this d and indeed defines d, so d is definable over (F, v). The valued field (F, v) being definably closed then must have d already in F, so (F, v) must satisfy Hensel's lemma. Therefore, (F, v) is a henselian valued field. ↻

If (F, v) is not definably closed, then there are three situations where the failure may arise. If F is not a field, then it is not closed with respect to addition, multiplication, subtraction, and the existence of multiplicative inverses for all nonzero elements. If F is not perfect, then it is possible to specify an individual that is not in the Frobenius image of F. If F is not henselian, then it fails Hensel's lemma, so the reasoning used to show that a definably closed valued field is henselian can be used to show that there is some polynomial over F satisfying the conditions for Hensel's lemma which does not have a solution in F that this polynomial defines in the separable closure of F.

Consequently, the model-theoretic Galois correspondence simplifies the structure theory into the following picture.

Automorphism Group	Field	Value Group	Residue Field
1	K	Γ_w	K_w
R	K^{ram}	$\frac{1}{p^{/\infty}}\Gamma_v \cap \Gamma_w$	$K_w \cap F_v{}^{\text{sep}}$
I	K^{inert}	Γ_v	$K_w \cap F_v{}^{\text{sep}}$
$D = G$	$K^{\text{decomp}} = F$	Γ_v	F_v

Specifically, if (F, v) and (K, w) are definably closed substructures of an algebraically closed valued field $(\mathfrak{M}, \mathfrak{v})$ and K/F is Galois, then (F, v) must be its own decomposition extension with the Galois group $\text{Gal}(K/F)$ coinciding with its decomposition subgroup. The other intermediate extensions themselves are henselian, so they are included in the stock of definably closed substructures of the overall algebraically closed valued field.

This result is not entirely surprising, but it is interesting to note that the relatively weak language used here nevertheless forces model-theoretic valued field automorphisms to always respect the valuation function. Recall that in this language, the valuation function can only be indirectly included by relying on the ordering of the value group. Furthermore, the language can only make general references to the other induced structures of a valued field, including the residue class field.

8.5 The p-adic Valuation

The completion of the rational numbers $\mathbb{Q}$ to the real numbers $\mathbb{R}$ can be modified to define a process of building up p-adic numbers, where p is a prime number. The p-adic fields may be among the most prominent number systems considered beyond the standard ones such as the rationals, reals, and complex numbers. One way of considering p-adic numbers is through a valuation that is originally defined on the rational number field $\mathbb{Q}$ which then is used to complete $\mathbb{Q}$ into the field $\mathbb{Q}_p$ of p-adic numbers.

Definition 8.20 (p-adic Valuations) Let $\mathbb{Q}$ be the field of rational numbers. Let p be a prime number in $\mathbb{N}$. Then, a p-adic valuation on $\mathbb{Q}$ is a valuation function $v_p : \mathbb{Q} \to \mathbb{Z}$ such that $v_p(q) = v_p(p^n \frac{a}{b}) = n$, q is any rational number, which can consequently be written in the form $p^n \frac{a}{b}$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{N}$ are pairwise prime and prime to p.

So, let $q \in \mathbb{Q}$ be a rational number such that $q = \frac{a}{b}$, where a and b are natural numbers and q is written as a fraction in simplified form. Since v_p is a valuation, $v_p(q) = v_p(\frac{a}{b}) = v_p(a) - v_p(b)$ by the basic properties of any valuation. The values of a and b are then determined by the power of the prime that divides a and b respectively, so if p^α divides a and $p^{\alpha+1}$ does not, then $v_p(a) = \alpha$, and similarly if p^β divides b but not $p^{\beta+1}$, then $v_p(b) = \beta$.

Thus, let p be a prime number. Then, $v_p(p) = 1$, $v_p(1) = 0$, and $v_p(2p) = 1$ when p is not 2. True to the basic properties of valuations, $v_p(\frac{1}{p}) = -1$, and since 0 is divided by everything, it is appropriate that $v_p(0)$ be assigned ∞.

Since p-adic fields arose out of tinkering the completion procedures of $\mathbb{Q}$ into $\mathbb{R}$, it is sensible that there is a notion of a field being closed with respect to the p-adic valuation.

Definition 8.21 (p-adically Closed Fields) Let (F, v) be a valued field. Then, (F, v) is *p-adically closed* if

1. F is of characteristic 0 and so $\mathbb{Q} \subseteq F$;
2. (F, v) is a henselian valued field;
3. the residue class field F_v is the prime field $\mathbb{F}_p$ of characteristic p;
4. the value group Γ_v has a minimal positive element, namely $v(p)$;
5. for every $m \geq 2$ and for every nonzero $c \in F$, there is some n such that $0 \leq n \leq m-1$ and that $v(cp^{-n})$ is divisible by m in Γ_v.

The canonical p-adically closed field of $\mathbb{Q}$ is the field $(\mathbb{Q}_p, v_p)$ of p-adic numbers. Clearly, $\mathbb{Q} \subseteq \mathbb{Q}_p$ and $\mathbb{Q}$ is dense in $\mathbb{Q}_p$. The p-adic valuation gives rise to a p-adic absolute value, such that $|q|_p := p^{-v_p(q)}$. The set of p-adic integers $\mathbb{Z}_p$ is then the subring $\mathbb{Z}_p := \{q \in \mathbb{Q}_p : |q|_p \leq 1\} = \mathcal{O}_{v_p}$. The requirement that the residue field of a p-adically closed field be the prime field of characteristic p implies that a p-adically closed field is not algebraically closed. This property further reinforces the connection that p-adic closure has with real closure, rather than algebraic closure.

Whereas the structure theory of valued fields provides a nuanced view of the Galois theory of valued fields inhabiting an algebraically closed valued field, a counterpart discussion of p-adic fields is not available. This absence can be explained by the following result of the Fundamental Theorem.

Theorem 8.22 Let $\mathscr{L}$ be the language of valued rings, and let T be the theory of p-adically closed fields, for some prime number p. Let $\mathfrak{M} \models T$ be a p-adically closed field. Then, the Galois correspondence between substructures of $\mathfrak{M}$ and automorphism groups of $\mathfrak{M}$ is trivial, in that the only definably closed substructures of $\mathfrak{M}$ are elementary substructures $\mathfrak{A}$ of $\mathfrak{M}$, so that $\mathfrak{A}$ is also a p-adically closed field.

Proof. So, let $(\mathfrak{M}, \mathfrak{v})$ be a p-adically closed field with the relevant valuation. It is a standard result that the theory of p-adically closed fields is model-complete, so if preferred, the following work may be done at a closer level than the possibly very large $\mathfrak{M}$. In any case, the residue class field $\mathfrak{M}_{\mathfrak{v}}$ is the prime field $\mathbb{F}_p$ of characteristic p. Its value group $\Gamma_{\mathfrak{v}}$ is discrete with $\mathfrak{v}(p) = 1$ the minimal nonzero element in the positive cone of the ordered group. Obviously, $p \in \mathbb{Q}$, so $p \in F$ for any substructure of $\mathfrak{M}$.

Let A be a substructure of $\mathfrak{M}$ which is definably closed. Then, for the same reasons as in the case of definably closed substructures of algebraically closed fields, A must be a perfect field. Similarly, for the same reasoning used in the case of definably closed substructures of algebraically closed valued fields, A must be henselian as well. So, A, if definably closed, is already a perfect henselian field. ↻

The valuation induced on A is the p-adic valuation carried up from the rational number field $\mathbb{Q}$ and sent down from the p-adically closed field $\mathfrak{M}$. Let v denote the induced valuation at A. Recall that if (F, v_F) and (K, v_K) are valued fields with $(F, v_F) \subseteq (K, v_K)$, then the residue degree is the degree of the extensions of the respective residue class fields, which is $\left[K_{v_K} : F_{v_F}\right]$. In comparing $(A, v) \subseteq (\mathfrak{M}, \mathfrak{v})$, the residue degree is thus $[\mathfrak{M}_{\mathfrak{v}} : A_v]$. Since $\mathfrak{M}_{\mathfrak{v}} = \mathbb{F}_p$ is prime, it must be the case that $A_v = \mathbb{F}_p$ as well. A quicker way of arriving at this conclusion is the result that A, as a substructure of a p-adically closed field, is a formally p-adic field. The following definition is relevant as is the accompanying theorem, provided without proof.

Definition 8.23 A valued field (F, v) is *finitely ramified* if the residue class field F_v is of characteristic 0 or, if not of characteristic 0, the residue class field has characteristic p and there are finitely many elements in the value group Γ_v between 0 and $v(p)$.

Theorem 8.24 A valued field (F, v) which is finitely ramified is henselian if and only if it is algebraically maximal.

Now, A must be finitely ramified since the residue class field $A_v = \mathbb{F}_p$ is of characteristic p and $v(p) = 1$ is the minimal positive element in the value group. So, A is a finitely ramified henselian valued field with a p-adic valuation. Consequently, A must be algebraically maximal, which in this case means that it cannot admit proper algebraic extensions with the same valuation induced by $\mathfrak{M}$. This condition is equivalent to being p-adically closed. Therefore, (A, v) is a p-adically closed subfield of $(\mathfrak{M}, \mathfrak{v})$.

The theory being model complete, A is not only a model of the theory of p-adically closed fields but is also an elementary substructure of $\mathfrak{M}$. The Galois correspondence is trivial since no algebraic extension of A has a p-adic valuation and is in $\mathfrak{M}$. The parallels between p-adically closed fields and real closed fields are hence rather considerable. Just as a real closed field connects with a p-adically closed field, the idea of formally real fields has a counterpart in formally p-adic fields, the definably closed substructures of real closed fields are real closed subfields while the definably closed substructures of p-adically closed fields are p-adically closed subfields, and both theories admit trivial Galois correspondences.

Chapter 9

Fields with Exponentiation

The application of the Fundamental Theorem 6.2 has to this point been rather straightforward. The next direction seeks to elucidate a Galois theory where a prior system has not yet been established. Consider the complex number field $\mathbb{C}$. The language of rings handicaps the formalization of general mathematical practice by only describing algebraic functions to the exclusion of the more intricate machinery of analysis, but this exclusion is not permanent. The inability to express a particular function can be remedied by expanding the language to accommodate it.

Thus, if there is a desire to make use of differentiation, the language can be expanded to that of differential rings, and the derivative becomes a plaything of the language. Valued fields can be accommodated by expanding the language in a wide assortment of ways. Similarly, the language can be expanded to include analytic functions that naturally arise in mathematical practice but are not special kinds of polynomial functions.

One such analytic function in the complex numbers is the exponential function that maps an individual $c \in \mathbb{C}$ to the number $e^c \in \mathbb{C}$. Let $\mathscr{L}_{er}$ denote the language of exponential rings, with symbols for addition $+$, multiplication $\times$, and the exponential function exp. The canonical interpretation of exp when considered in the field $\mathbb{C}$ is that $\exp(x)$ is interpreted to be e^x, where e is a transcendental number that has certain interesting properties.

Let $\mathbb{C}_{\exp}$ denote the complex exponential field described by the language of exponential rings. In general mathematical parlance, much can be said of the exponential function; for example, it is the function which coincides with its derivative, so that $\frac{de^x}{dx} = e^x$. The image e^1 maps to the transcendental number e, a process that cannot occur with polynomials, where coefficients algebraic over the rationals have solutions by definition algebraic over the rationals. When considered together with the square root i of -1 and the number π, the exponential function is one which is very different than the functions built from polynomials and from the language of rings. Indeed, questions about the algebraic properties of the complex exponential field are not yet resolved, despite the importance of exponentiation in general mathematical practice. Chief among them is the following conjecture by Schanuel.

Conjecture 9.1 (Schanuel) Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be numbers linearly independent over $\mathbb{Q}$. Then, the transcendence degree of

$$\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n)$$

over $\mathbb{Q}$ is at least n. That is,

$$\mathrm{ldim}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n) \leq \mathrm{tr.deg}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n)).$$

Schanuel's conjecture is a generalization of the following known result.

Theorem 9.2 (Lindemann-Weierstrass) Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers in $\mathbb{C}$ linearly independent over $\mathbb{Q}$. Then, the transcendence degree of

$$\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n)$$

over $\mathbb{Q}$ is at least n. That is,

$$\mathrm{ldim}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n) \leq \mathrm{tr.deg}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n)).$$

Thus, Schanuel's conjecture is the assertion that the algebraicity requirement in the Lindemann-Weierstrass theorem can be omitted. A definitive answer to Schanuel's conjecture has remained elusive since its proposal, but a recent result by Kirby is that, if Schanuel's conjecture does not hold in the complex exponential field $\mathbb{C}_{\exp}$, there are essentially countably many counterexamples refuting it. The conjecture's unresolved status has been a reason why the model theory of the complex exponential field has not been extensively compiled. Thus, definitive statements about $\mathrm{Th}(\mathbb{C}_{\exp})$ are not as readily available.

A major reason for this problem, besides the presence of Schanuel's conjecture, is the curious ability to define the set $\mathbb{Z}$ of integers using the exponential function. The integers, as a set, can be obtained through the kernel of the exponential map, since any number of the form $2\pi i z$, $z \in \mathbb{Z}$, has the property $e^{2\pi i z} = 1$. The definability of the set of integers implies the definability of the sets of rational numbers and the natural numbers, and so Peano arithmetic becomes a part of the model theory of the complex exponential field. Consequently, all of the issues associated with Incompleteness affect this model-theoretic situation.

The definability of the integers as a set means that the model theory of the complex exponential field is no longer minimal, since now there are definable sets that are infinite and coinfinite. Stability is also out due to arithmetic. Model completeness and quantifier elimination are not possible, and it is not immediately apparent how to obtain a model completion.

Proposition 9.3 The theory $\mathrm{Th}(\mathbb{C}_{\exp})$ in the language of exponential rings is not model complete.

Proof. This result was proved by D. Marker in [Marker 2006] essentially in the following manner. Model completeness implies that definable sets are projections of closed sets on the induced model-theoretic topology. Because the set $\mathbb{Z}$ of integers is definable from the kernel of the exponential function, the rationals $\mathbb{Q}$ are also definable as a set. Consequently, every definable set is the countable union of closed sets and also the countable intersection of open sets. The rationals as a set, however, are not a countable intersection of open sets. Such a conclusion contradicts the Baire category theorem which applies to $\mathbb{Q}$. Hence, model completeness fails. ↻

9.1 Pseudoexponentiation

Thus, the model theory of the complex exponential field, a natural extension of the canonical study of the complex number field, is a complicated endeavor. In a bid to overcome this uncertainty, Zilber proposed in [Zilber 2005] a possible alternative way of capturing the model-theoretic properties of exponentiation by using Hrushovski amalgamation methods to introduce the idea of *pseudoexponentiation*. Pseudoexponentiation accepts all of the known problems associated with true exponentiation, such as the relevance of the Gödel incompleteness result, and limits their detrimental effects as much as possible to yield a theory which possesses potential for model-theoretic analysis currently unavailable to the model theory of true complex exponentiation.

The language of pseudoexponentiation is the language of exponential rings, and the theory PSE of pseudo-exponential fields is axiomatized by the following axiom schemata, some of which will be elaborated upon in the continuing discourse.

1. The axioms of ACF_0, the algebraically closed fields of characteristic 0.
2. The function symbol exp is interpreted so that exp is a group homomorphism that maps addition onto multiplication, so that $\exp(0) = 1$ and $\forall x \forall y \exp(x+y) = \exp(x)\exp(y)$. Furthermore, $\forall x \exists y (x \neq 0 \rightarrow \exp(y) = x)$ holds as well.
3. The kernel of the pseudoexponential exp is $\tau\mathbb{Z}$, where the transcendental number τ is the generator of the kernel.
4. Schanuel's conjecture is satisfied as a property, so that if $\alpha_1, \ldots, \alpha_n$ are linearly independent numbers over $\mathbb{Q}$, then $\mathrm{tr.deg}_{\mathbb{Q}}(\alpha_1, \ldots., \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n)) \geq n$.
5. The pseudoexponential field has the *countable closure property*, that the exponential-algebraic closure of any finite set X of the field is countable.

6. The pseudoexponential field is also *strongly exponentially and algebraically closed*, so that a free and rotund variety V of the product of the additive and multiplicative groups of the field with dimension n has, given an arbitrary finite sequence $\bar{a}$ of individuals in the field, a finite sequence $\bar{b}$ such that $(\bar{b}, \exp(\bar{b}))$ is in V and is generic over $\bar{a}$.

The notions of exponential-algebraic closure and the concept of free and rotund varieties are left undefined for the moment, but the standard references for the work in pseudoexponentiation such as [Kirby 2010 FPEF] provide a comprehensive introduction to these definitions. Thus, pseudoexponential fields preemptively settle the open question of Schanuel's conjecture, answering in the affirmative, and the theory makes additional assumptions about the geometric structure that the complex exponential field may have. Notably, the theory as formulated would fail to be a first-order theory, and the original axiomatization was in an infinitary logic with a quantifier specifying uncountable quantities, specifically $\mathscr{L}_{\omega_1,\omega}(Q)$. Recent developments include [Kirby Zilber 2011] on the possible first-order axiomatization, or rather an axiomatization to the extent possible within a first-order language, contingent upon the resolution of the Conjecture on the Intersection of Tori (CIT). The following summary of the model theory of PSE is background information which may be found in standard references for pseudoexponentiation.

Proposition 9.4 Let $\mathscr{L}_{er}$ be the language of exponential rings, and let PSE be the theory of pseudoexponentiation. Then, the theory is complete, quasiminimal, and uncountably categorical, but the theory fails to be model complete and is undecidable.

Quasiminimality is the countable analogue for minimality, so every definable set would be countable or cocountable in a quasiminimal structure. Model completeness fails, so quantifier elimination is not available. Failure of quantifier elimination will turn out to be a considerable problem in the following work. Undecidability is an immediate consequence of arithmetic being encoded into the theory. Uncountable categoricity means that, combined with the fact that the theory ACF_0 of algebraically closed fields of characteristic 0 is also uncountably categorical, there is only one model of PSE of cardinality continuum, up to isomorphism. Let $\mathbb{B}$ denote this structure, the complex pseudoexponential field. Its field reduct is clearly the complex number field $\mathbb{C}$. Thus, the following conjecture is a natural one to consider.

Conjecture 9.5 (Zilber) The complex pseudoexponential field $\mathbb{B}$ is isomorphic to the complex exponential field $\mathbb{C}_{\exp}$.

A consequence of this conjecture is that the transcendental generator τ of the kernel would be one of $2\pi i$ or $-2\pi i$. It has been demonstrated in [Zilber 2005] that the complex exponential field $\mathbb{C}_{\exp}$ has the countable closure property, so the conjecture is equivalent to the following proposition.

Proposition 9.6 The complex pseudoexponential field $\mathbb{B}$ is isomorphic to the complex exponential field $\mathbb{C}_{\text{exp}}$ if and only if Schanuel's conjecture holds in the complex exponential field and if the complex exponential field has the property of strong exponential and algebraic closure.

The following work lays out the foundations for a Galois theory of exponential fields by working primarily in the context of pseudoexponential fields. The previous applications of the Galois correspondence led to mainly straightforward results that did not differ too significantly from standard algebraic results where such efforts are possible, but the logical properties that enabled this similarity are missing in the case of exponentiation and pseudoexponentiation. Thus, the work here must be done with some care. Specifically, the biggest obstacle here is the failure of model completeness which precludes the possibility of quantifier elimination. Without quantifier elimination, definability of individuals may be subject to formulae that cannot be reduced to polynomials and exponential polynomials.

9.2 A Sketch of the Galois Theory of Exponential Fields

As with the Galois theory of pure fields and of structures in general, the Galois correspondence between exponential fields and their automorphism groups takes place above some prime structure and below some maximal structure. In the current example, all work is assumed to take place in the complex pseudoexponential field $\mathbb{B}$, which has as its field reduct the complex number field $\mathbb{C}$, and there is an underlying motivation that the complex pseudoexponential field is essentially the same structure as the canonical complex exponential field. The previous applications worked in saturated models which guaranteed rich spaces of automorphisms, but the previous theories all have model completeness which guarantees that all submodels of the saturated model are elementary substructures. The failure of model completeness in pseudoexponentiation makes the current approach more prudent.

Recall that the Fundamental Theorem 6.1 asserts specifically a Galois correspondence between definably closed substructures containing a base structure A and contained within a definably closed substructure B of some large model $\mathfrak{M}$ and the closed subgroups of the group $\text{Aut}(B/A)$. In the current application, it is intended that B and $\mathfrak{M}$ coincide by being the complex pseudoexponential field $\mathbb{B}$ whose field reduct is the complex number field $\mathbb{C}$. The substructure A, at the smallest possibility, must be the field fixed by all automorphisms and maps acting on the complex pseudoexponential field or complex exponential field.

The appellation of *exponential fields* and *pseudoexponential fields* are mainly used for those structures which are closed with respect to the field operations and the exponential or pseudoexponential function. The terms may be used interchangeably since the relevant operator is always assumed to be the appropriate restriction of the pseudoexponential function of $\mathbb{B}$. Definably closed substructures will be called *closed exponential fields*, to distinguish from arbitrary exponential fields or pseudoexponential fields. If F is an arbitrary field or even a set, let F^E denote the exponential closure of F, which is simply the smallest exponential field containing F, and let $F^C = \text{dcl}(F)$ be the definable closure of F, which is the smallest closed exponential field containing F.

The distinction is nontrivial since the lack of quantifier elimination means that definable closure cannot be determined by appeal to the language of exponential rings. The prime structure for the Galois theory of exponential fields is not actually the prime exponential field. The prime exponential field is obtained in the same manner as the prime pure field is obtained in the language of rings and the theory of algebraically closed fields of a fixed characteristic. In the current case of characteristic 0, the prime field is the rational number field $\mathbb{Q}$. When closed with respect to exponentiation, the resulting prime exponential field is the rational exponential field $\mathbb{Q}^E$, which will be elaborated upon in subsequent discourse. Obviously, the rational number field would not be the prime structure for pseudoexponential fields since it is not closed with respect to exponentiation, but even its exponential closure is not definably closed and so cannot be the prime structure for pseudoexponential fields because the loss of quantifier elimination allows for the definability of other numbers outside exponential closure.

A result by Kirby, Macintyre, and Onshuus sheds light on the composition of the prime structure. Specifically, [Kirby Macintyre Onshuus 2011] describes the algebraic numbers which can be defined without parameters and hence in the exponential field $\text{dcl}(\emptyset)$. Significantly, a consequence of this result is that the Galois theory of pseudoexponential fields is not the simple picture that would result if complex conjugation is the only true exponential field automorphism of the complex exponential field. A summary of their work proceeds in the subsequent section.

Let A and B with $A \subseteq B$ be closed exponential subfields of the complex pseudoexponential field. Then, the Galois correspondence is pertinent when B is model-theoretically algebraic over A. Whereas in the case of valued fields the notion of model-theoretic algebraicity continued to coincide with the field-theoretic notion of algebraicity, the loss of quantifier elimination and the increased expressivity of the language does mean that the model-theoretic notion of algebraicity for pseudoexponential fields is significantly different than the usual algebraicity of pure fields. This difference, which is also explored in subsequent work, complicates the Galois theory of pseudoexponential fields because an extension B of A may be a transcendental extension when they are considered as pure fields but be an algebraic extension when regarded as pseudoexponential fields in a subtle and not immediately evident way.

9.3 Algebraic Numbers in the Fixed Pseudoexponential Field

The language of rings, with the constant symbols $\dot{0}$ and $\dot{1}$, is expressive enough to label every integer and identify every rational number; fractions would be obtained as the only solutions to formulae of the form $\dot{b}x \dot{=} \dot{a}$ where $\dot{a}$ and $\dot{b}$ are syntactic representations of the integers a and b. The exponential function enables the language of exponential rings to also individually denote transcendental numbers inaccessible from the pure ring language. Obviously, every number of the form $\exp(z)$, $z \in \mathbb{Z}$, can be syntactically represented as such a term, and arbitrary numbers of the form $\exp(q)$, $q \in \mathbb{Q}$, are readily pointwise definable.

The collection of these individually definable numbers constitute the smallest exponential field, which is denoted the rational exponential field $\mathbb{Q}^E$. It is possible to construct what $\mathbb{Q}^E$ looks like. Let $E_0 := \mathbb{Q}$ denote the set of definable individuals without any usage of the exponential function. Let $E_1 := E_0(\{\exp(x) : x \in E_0\}) = \mathbb{Q}(\{\exp(q) : q \in \mathbb{Q}\})$ be the field generated by the set of terms obtained with one application of the exponential function. Then, the sets E_i, $i \in \mathbb{N}$, can be defined and obtained by induction so that $E_{i+1} = E_i(\{\exp(x) : x \in E_i\})$, and let

$$E_\infty := \bigcup_{i \in \mathbb{N}} E_i$$

then be the field generated by the set of iterated applications of the exponential function. Clearly, with the exception of E_0, every number in $E_\infty \setminus E_0$ is transcendental. In the context of pseudoexponential fields, the Schanuel property allows then that sums of numbers in E_∞ are also transcendental, and the definition of exponentiation limits algebraic numbers in E_∞ to the rationals.

Proposition 9.7 The rational exponential field $\mathbb{Q}^E$ is the field generated by E_∞ so that $\mathbb{Q}^E = E_\infty$.

Proof. This result is clearly straightforward. Since E_∞ is defined as the field generated by adjoining all numbers in E_i, $i \in \mathbb{N}$, to $\mathbb{Q}$, E_∞ is a field, so it is closed with respect to addition, subtraction, multiplication, and division. That E_∞ is exponentially closed follows from the recursive construction of its constituents. ↻

Notably absent in the collection of definable numbers in $\mathbb{Q}^E$ is thus the imaginary number $i = \sqrt{-1}$, the principal square root of -1. Indeed, as before in the context of pure fields, $\{i, -i\}$ is the irreducible set for i defined by $x^2 + 1 = 0$. Hence, in this tiny respect the Galois theory of pseudoexponential fields conforms to the known aspect of the Galois theory of the true complex exponential field. Without quantifier elimination, the language can be used to define functions and relations which can be used to define new numbers. In particular, it is possible to define trigonometric functions that can in turn be used to define certain irrational algebraic numbers so that $\mathrm{dcl}(\emptyset) \cap \mathbb{Q}^{\mathrm{alg}} \neq \mathbb{Q}$.

The two principal trigonometric functions are the sine and cosine functions, and these are indeed definable in the language of exponential rings using existential quantifiers. In particular, the formula $\exists z(z^2+1=0 \wedge y = \frac{1}{2z}(\exp(zx)-\exp(-zx)))$ defines the sine function such that $y=\sin(x)$, and the formula $\exists z(z^2+1=0 \wedge y = \frac{1}{2}(\exp(zx)+\exp(-zx)))$ defines the cosine function such that $y=\cos(x)$. In both the sine and the cosine functions, the roles that i and $-i$ each play are not actually dependent on distinguishing between the two; substituting $-i$ for i and i for $-i$ results in the same equation. These functions appear to be examples of how the failure of quantifier elimination for the theory of pseudoexponentiation in the language of exponential rings can lead to unexpected definability results.

Consequently, the sine and cosine functions can be used to define new numbers that are not in E_∞. For example, for every number $c \in E_\infty$, the numbers $\sin(c)$ and $\cos(c)$ are newly definable numbers not in E_∞. Note that these numbers must be new since otherwise their membership in E_∞ implies that the trigonometric functions are quantifier-free definable. By themselves, the trigonometric functions cannot give rise to new algebraic numbers, since the trigonometric image of algebraic numbers are transcendental. It is crucial to be able to define π which can be used to derive the new algebraic numbers. Fortunately, π, or something that looks like it in the context of pseudoexponentiation, is indeed definable.

The axioms for the pseudoexponential field require that the kernel of the pseudoexponential function be the standard one of the true exponential function in the sense that there is some transcendental number τ that generates the kernel such that $\ker(\exp)=\tau\mathbb{Z}$. Thus, the formula $\forall x(\exp(x)=1 \rightarrow \exp(xy)=1)$ defines $\mathbb{Z}$ as a set. The generator τ is intended to be the number $2\pi i$ or its additive inverse $-2\pi i$. Let $\mathbb{Z}(\cdot)$ and $\mathbb{Q}(\cdot)$ denote the definable predicates for membership in the integers and rationals, respectively; note that definability of the rational numbers as a set follows readily from the definability of the integers as a set. Being in the kernel of the exponential function is a definable property, so let $\mathrm{Ker}(\cdot)$ be a predicate for membership in the kernel.

Now, τ and its additive inverse $-\tau$ are definable as a set. The set $\{\tau,-\tau\}$ is defined by the formula $\mathrm{Ker}(x) \wedge (\forall y(\mathrm{Ker}(y) \rightarrow \exists z(\mathbb{Z}(z) \wedge zx=y)))$. With the knowledge that τ is to be $2\pi i$ or $-2\pi i$, the set $\{\pi,-\pi\}$ then becomes definable, simply by dividing out 2 and the indeterminate square root of -1.

The definable trigonometric functions then can be used to distinguish between π and $-\pi$. Specifically, using the sine and cosine functions, $\sin(\frac{\pi}{2})=1$ and $\cos(\frac{\pi}{2})=0$, whereas $\sin(\frac{-\pi}{2})=-1$ and $\cos(\frac{-\pi}{2})=0$. Of course, $\sin(\frac{\pi}{2}+2\pi n)=1$ and $\cos(\frac{\pi}{2}+2\pi n)=0$ for $n \in \mathbb{Z}$, but the periodicity of the sine and cosine functions are not problematic. The number π can be distinguished from its periodic counterparts since π is a component of the generator of the kernel. Consequently, only one number will satisfy the formula $\sin(\frac{x}{2})=1 \wedge \cos(\frac{x}{2})=0 \wedge \exists w(w^2=-1 \wedge (\mathrm{Ker}(2xw) \wedge (\forall y(\mathrm{Ker}(y) \rightarrow \exists z(\mathbb{Z}(z) \wedge z2xw=y)))))$.

The definability of π means that the sine and cosine functions can be used to define the familiar algebraic values associated with trigonometry. Indeed, with π defined, the sine and cosine functions are interdefinable, since $\cos(x) = \sin(x + \frac{\pi}{2})$. Hence, one can make do with just the cosine function and π to churn out newly definable algebraic numbers.

Example 9.8 The number $\sqrt{2}$, the principal square root of 2, is the unique individual that satisfies the formula stating $\exists y(x = 2\cos(\frac{y}{4}) \wedge \sin(\frac{y}{2}) = 1)$.

Although the formula $x^2 - 2 = 0$ defines the set $\{\sqrt{2}, -\sqrt{2}\}$, previously the irreducible set of either number in the theory ACF_0, $\sqrt{2}$ can be defined without even making reference to a polynomial equation. The following fact is from elementary mathematics.

Proposition 9.9 Let q be a rational number. Then, $\cos(q\pi)$ is algebraic.

The algebraic numbers defined using the trigonometric functions and π are not random or haphazardly found. They are exactly those numbers that are the real coefficients of the numbers found in the maximal abelian field of the rationals, the field denoted $\mathbb{Q}^{\mathrm{ab}}$. By the Kronecker-Weber theorem, $\mathbb{Q}^{\mathrm{ab}}$ is also the maximal cyclotomic field $\mathbb{Q}(\{\zeta_n : n \in \mathbb{N}\})$, so the algebraic numbers defined are generated by the real coefficients of the primitive roots of unity. For convenience, let a number a in $\mathbb{Q}^{\mathrm{ab}}$ be called an abelian number.

Proposition 9.10 Let $x \in \mathbb{Q}^{\mathrm{ab}}$ be an abelian number. Then, x can be written as a finite sum such that

$$x = \sum_{j=1}^{n} z_j \cos(q_j i \pi)$$

for some $n \in \mathbb{N}$ and for each z_j and q_j in $\mathbb{Q}$.

Theorem 9.11 Let $x \in \mathbb{Q}^{\mathrm{ab}}$ be an abelian number, and let $x = a + bi$ with $a, b \in \mathbb{R}$. Then, a and b are definable, such that a is of the form

$$a = \sum_{i=1}^{n} z_i \cos(q_i \pi)$$

for rationals z_i and q_i $(i \in \mathbb{N})$ and b is of the form

$$b = \sum_{j=1}^{m} z_j \sin(q_j \pi)$$

with rationals z_j and q_j $(j \in \mathbb{N})$.

Corollary 9.12 The real and imaginary components of an abelian number can be written with the form

$$\sum_{i=1}^{n} z_i \cos(q_i \pi)$$

with natural number $n \in \mathbb{N}$ and the rationals z_i and q_i in $\mathbb{Q}$.

These results, when combined with a particular property of pseudoexponentiation, then lead to the following statement which summarizes the algebraic numbers definable in the pseudoexponential field.

Theorem 9.13 (Kirby, Macintyre, Onshuus) Let $\mathscr{L}$ be the language of exponential rings, and let PSE be the theory of pseudoexponentiation, and let $\mathbb{B} \models \mathrm{PSE}$ be the complex pseudoexponential field. The algebraic numbers individually definable with no parameters are those numbers that are totally real and abelian in the set $\mathbb{Q}^{\mathrm{rab}} = \mathbb{Q}^{\mathrm{tr}} \cap \mathbb{Q}^{\mathrm{ab}}$. Let such a number be called a real abelian number.

Proposition 9.14 The set of individuals which can be written in the form of

$$\sum_{i=1}^{n} z_i \cos(q_i \pi),$$

where z_i and q_i are in $\mathbb{Q}$ for each $i \in \mathbb{N}$, is the set $\mathbb{Q}^{\mathrm{rab}}$ of real abelian numbers.

The field $\mathbb{Q}^{\mathrm{rab}}$ is formally real and is also the maximal real subfield of $\mathbb{Q}^{\mathrm{ab}}$. Furthermore, $\mathbb{Q}^{\mathrm{rab}}$ is also the intersection of the abelian numbers and the totally real numbers, hence $\mathbb{Q}^{\mathrm{rab}} = \mathbb{Q}^{\mathrm{ab}} \cap \mathbb{Q}^{\mathrm{tr}}$. Also, $\mathbb{Q}^{\mathrm{rab}}$ can also be regarded as the field $\mathbb{Q}(\{\zeta_n + \zeta_n^{-1} : n \in \mathbb{N}\})$ since $\mathbb{Q}^{\mathrm{ab}}$ is the maximal cyclotomic field.

The work to this point only shows one direction of Theorem 9.13, that real abelian numbers are individually definable using the cosine function and π. Indeed, this work can be done entirely in the model theory of the complex exponential field to this point. To have the other direction in pseudoexponentiation, that the real abelian numbers are the *only* algebraic numbers individually definable, one can construct an automorphism of the pseudoexponential field that fixes on $\mathbb{Q}^{\mathrm{alg}}$ only real abelian numbers while moving all other algebraic numbers.

Thus, the prime structure that serves as the floor for a Galois theory of pseudoexponential fields is significantly larger than the rational number field for the Galois theory of pure fields, but it is also far away from the possible scenario in the complex exponential field that the fixed substructure is the real exponential field $\mathbb{R}_{\exp}$, where the only nontrivial true exponential field automorphism would be complex conjugation. Clearly, the real abelian numbers $\mathbb{Q}^{\mathrm{rab}}$ constitute a proper subfield of the reals $\mathbb{R}$.

9.4 Galois Theory over the Real Abelian Numbers

Since every exponential field automorphism and pseudoexponential field automorphism is a pure field automorphism, working with the algebraic Galois theory of pure fields can elucidate the investigation into the Galois theory of pseudoexponential fields. If the pure field K is a Galois extension of the pure field F, then the Galois group $\mathrm{Gal}(K/F)$ will in a certain sense constrain the behavior of the pseudoexponential definable closures dcl(K) and dcl(F) of K and F and their pseudoexponential Galois group Gal(dcl(K)/ dcl(F)). This constraint would affect the field-theoretic algebraic part of these Galois groups.

With respect to the complex pseudoexponential field $\mathbb{B}$, the fixed algebraic numbers constitute the real abelian numbers $\mathbb{Q}^{\text{rab}}$, so it is inappropriate to look at the Galois theory over the rationals $\mathbb{Q}$. Nevertheless, known results about the rational number field and its pure field extensions can serve as motivating goals for pseudoexponential analogues with the real abelian field and its pseudoexponential field extensions. For example, one might at a glance see how the inverse Galois problem may fare when considering $\mathbb{Q}^{\text{rab}}$ instead of $\mathbb{Q}$. Recall that the inverse Galois problem is the question of whether or not every finite group can be realized as a Galois group over $\mathbb{Q}$; thus, one can ask which finite groups are realizable as Galois groups over $\mathbb{Q}^{\text{rab}}$. This work takes a known result about abelian extensions of the rational field $\mathbb{Q}$ as the source of motivation for working with the real abelian field $\mathbb{Q}^{\text{rab}}$.

Theorem 9.15 (Kronecker-Weber) A field extension K of $\mathbb{Q}$ is abelian if and only if it is a cyclotomic extension $\mathbb{Q}(\zeta_i : i \in I)$ ($I \subseteq \mathbb{N}$) generated by roots ζ_i of unity or if it is a subfield of such a cyclotomic extension. The maximal abelian extension $\mathbb{Q}^{\text{ab}}$ of $\mathbb{Q}$ is obtained by adjoining all roots of unity to $\mathbb{Q}$, resulting in the maximal cyclotomic field.

The maximal abelian extension of the rationals looks familiar to the current situation since $\mathbb{Q}^{\text{ab}} = \mathbb{Q}^{\text{rab}}(i)$. The real abelian numbers are indeed the maximal formally real subfield of the abelian numbers; moreover, $\mathbb{Q}^{\text{rab}}$ is "cyclotomically explosive" in the sense that adjoining any primitive nth root of unity with $n > 2$ results in obtaining all roots of unity that inhabit $\mathbb{Q}^{\text{ab}}$.

In any case, the Kronecker-Weber theorem characterizes abelian extensions of the rational number field as subfields of cyclotomic extensions. To find a pseudoexponential analogue, it is therefore necessary, though not sufficient, to find which pure field Galois extensions of the real abelian numbers are abelian extensions. Since pure field-theoretic algebraicity is a case of model-theoretic algebraicity in the context of pseudoexponentiation, these abelian extensions in the context of pure fields give rise to abelian extensions in the context of pseudoexponentiation. Although full elucidation of model-theoretic algebraicity is unavailable, it is a feasible task to explore the real abelian number field $\mathbb{Q}^{\text{rab}}$ and its pure field-theoretic abelian extensions as a task in its own right and use this information to shed light on model-theoretic abelian extensions of the exponential field $\mathbb{Q}^{\text{rab}\,C} = \text{dcl}(\emptyset)$.

Determining which model-theoretically algebraic extensions yield abelian groups first requires knowing what such algebraic extensions look like, and that task requires a better understanding of model-theoretic exponential algebraicity. The lack of model completeness is a serious hindrance to this elucidation. However, a feasible task is to explore the real abelian number field $\mathbb{Q}^{\text{rab}}$ as an object of study in its own right. In particular, determining what its abelian extensions look like would yield insight into the much more complicated case of the prime structure dcl(∅), which here also hold the appellation the closed real abelian pseudoexponential field $\mathbb{Q}^{\mathrm{rab}\,C}$.

9.5 Abelian Extensions of the Field $\mathbb{Q}^{\mathrm{rab}}$ of Real Abelian Numbers

Since $\mathbb{Q}^{\mathrm{rab}}$ is the maximal real subfield of the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ of the rational numbers, $\mathbb{Q}^{\mathrm{rab}}$ is not an algebraic number field of finite dimension. Consequently, the standard results of algebraic number theory in general and of class field theory in particular that rely on the local-global principle are of little help in shaping up an understanding of the abelian extensions of $\mathbb{Q}^{\mathrm{rab}}$. Abelian extensions can be obtained by understanding cyclic extensions since cyclic groups are the building blocks of abelian groups. Clearly, $\mathbb{Q}^{\mathrm{rab}}$ has cyclic extensions at least of degree 2.

One possible thought about the properties of $\mathbb{Q}^{\mathrm{rab}}$ is that the field $\mathbb{Q}^{\mathrm{rab}}$ is Pythagorean and admits no cyclic extensions of odd prime degree. Consequently, $\mathbb{Q}^{\mathrm{rab}}$ would have as its maximal abelian extension the field $\mathbb{Q}^{\mathrm{rab}}(\{\sqrt{x} : x \in \mathbb{Q}^{\mathrm{rab}}\})$.

Definition 9.16 (Pythagorean Fields) A field F is *Pythagorean* if, for every a and b in F, there is some c in F such that $a^2 + b^2 = c^2$. That is, Pythagorean fields admit the Pythagorean property.

Proposition 9.17 (Whaples) A field F is Pythagorean if and only if F admits no cyclic extensions of degree 4.

Proposition 9.18 (Koenigsmann) A field F which is Pythagorean and admits no cyclic extensions of odd prime degree has no projective extensions.

The consequence of this possibility follows since the only cyclic extensions allowed are of degree 2, so abelian extensions are multiquadratic ones. Since $\mathbb{Q}^{\mathrm{rab}}$ is totally real, it is clearly not quadratically closed or Euclidean, so there are infinitely many nonsquares inhabiting the field. If $\mathbb{Q}^{\mathrm{rab}}$ is Pythagorean and admits no cyclic extensions of odd prime degree, then it would host a smaller variety of Galois extensions than the field $\mathbb{Q}$ of rational numbers because of the dearth of extensions with projective Galois groups. A necessary condition for the real abelian numbers to be Pythagorean is for $\mathbb{Q}^{\mathrm{rab}}$ to contain the unique Pythagorean closure $\mathbb{Q}^{\mathrm{py}}$ of the rational numbers.

Proposition 9.19 (Hilbert) The Pythagorean closure $\mathbb{Q}^{\mathrm{py}}$ of the rational field $\mathbb{Q}$ is the field obtained by taking the maximal totally real subfield of the field of constructible numbers. Consequently,

$$\mathbb{Q}^{\mathrm{py}} = \mathbb{Q}^{\mathrm{tr}} \cap \mathbb{Q}(2),$$

where $\mathbb{Q}^{\text{tr}}$ is the totally real closure of $\mathbb{Q}$ and $\mathbb{Q}(2)$ is the maximal pro-2-extension of the rational field, which is also its quadratic closure.

Corollary 9.20 The Galois group $\text{Gal}(\mathbb{Q}^{\text{py}}/\mathbb{Q})$ is not abelian. Therefore, $\mathbb{Q}^{\text{py}}$ is not a subfield of $\mathbb{Q}^{\text{ab}}$.

Since the Galois group is not abelian, the rational Pythagorean closure, despite being totally real, is not a subfield of the maximal abelian extension of the rational numbers. Thus, there are two numbers a and b in $\mathbb{Q}^{\text{rab}} \cap \mathbb{Q}^{\text{py}}$ such that there is no c in $\mathbb{Q}^{\text{rab}}$ with the property $a^2 + b^2 = c^2$. Therefore, $\mathbb{Q}^{\text{rab}}$ is not Pythagorean.

As a consequence, $\mathbb{Q}^{\text{rab}}$ must have cyclic extensions of degree 4. On the second claim of the original motivational possibility, that $\mathbb{Q}^{\text{rab}}$ admits no cyclic extensions of odd prime degree, it is also possible to demonstrate examples for each prime p that refute such a claim. To do so, let p be an odd prime. Let D_{2p} denote the dihedral group of order $2p$, and let F be a totally real D_{2p}-extension over $\mathbb{Q}$. Such extensions over $\mathbb{Q}$ can be obtained since finite dihedral groups are solvable and hence realizable over $\mathbb{Q}$ by [Fried Jarden 2008], and by a result of J.-P. Serre in [Klüners Malle 2001], finite groups realizable over $\mathbb{Q}$ can be realized by totally real extensions. Since D_{2p} is not abelian (unless $p = 1$ and $p = 2$), F is not an abelian extension of $\mathbb{Q}$ and hence not a subfield of the maximal $\mathbb{Q}^{\text{ab}}$. However, for each such F, there is an intermediate extension K such that K is normal over $\mathbb{Q}$ of degree 2 and F is cyclic over K with degree p. The field K is abelian over $\mathbb{Q}$ and hence in $\mathbb{Q}^{\text{ab}}$. Since F is totally real, K must be totally real as well, so K will also be in $\mathbb{Q}^{\text{rab}}$. Then, F will be a cyclic extension of $\mathbb{Q}^{\text{rab}}$ with degree p.

The conjecture therefore is completely rejected, since $\mathbb{Q}^{\text{rab}}$ is neither Pythagorean nor deprived of cyclic extensions of odd prime. The consequent characterization of abelian extensions as multiquadratic extensions is therefore misleadingly wrong. Although the Kronecker-Weber theorem characterizes abelian extensions of the rational numbers as cyclotomic fields and complex multiplication describes the maximal abelian extension of CM-fields using Weber functions on torsions points of elliptic curves, no uniform characterization is available for totally real fields, whether of finite or infinite degree over $\mathbb{Q}$. Thus, a brute-force and piecemeal approach must be taken to obtain the maximal abelian extension $(\mathbb{Q}^{\text{rab}})^{\text{ab}}$. It is possible to describe what the Galois group $\text{Gal}((\mathbb{Q}^{\text{rab}})^{\text{ab}}/\mathbb{Q}^{\text{rab}})$ looks like. In order to describe the group, it is helpful to state a theorem of G. Whaples, provided with notation original to the work [Whaples 1957].

Theorem 9.21 (Whaples) If p is an odd prime and k has a cyclic extension of degree p, then it has a cyclic extension of degree p^∞. If k has a cyclic extension of degree 4, then it has a cyclic extension of degree 2^∞. A field has a cyclic extension of degree 2 and none of degree 4 if and only if it is an ordered field in which every sum of squares is a square.

The last portion of the theorem is of course a restatement of the Theorem 9.17. If F is a field with K a cyclic extension of degree p^∞, then the Galois group $\mathrm{Gal}(K/F)$ is isomorphic to the p-adic integers $\mathbb{Z}_p$. Intermediate extensions between F and K include cyclic extensions for each power n of p, $n \in \mathbb{N}$.

The proof presented in [Whaples 1957] for the theorem contains five cases depending on the cyclotomic properties of the field. For example, $\mathbb{Q}^{\mathrm{ab}}$ is covered by the first case, while the third case provides the arguments that apply to $\mathbb{Q}$. The real abelian number field $\mathbb{Q}^{\mathrm{rab}}$ is covered by the fourth and fifth cases, which respectively address cyclic extensions of odd prime and of degree 4, and these cases turn out to be relevant to the current situation.

Let p be an odd prime number. The fourth case concerns a field F which does not have primitive pth roots of unity, but the field $F(\zeta_p)$ obtained by adjoining such a pth root of unity ζ_p to F is one containing p^nth powers of unity for every $n \in \mathbb{N}$. Recall that adjoining any nonreal root of unity to $\mathbb{Q}^{\mathrm{rab}}$ yields $\mathbb{Q}^{\mathrm{ab}}$, which has all primitive roots of unity. In this case, once it is established that F has a cyclic extension of degree p^i, every such cyclic extension can be embedded into a cyclic extension of degree p^j, for natural numbers $i < j$.

The real abelian field $\mathbb{Q}^{\mathrm{rab}}$ thus not only has cyclic extensions of every odd prime p and their powers p^n, $n \in \mathbb{N}$, but for every such case, it is possible to build a tower

$$\mathbb{Q}^{\mathrm{rab}} \hookrightarrow F_p \hookrightarrow F_{p^2} \hookrightarrow F_{p^3} \hookrightarrow \dots .$$

where F_{p^n} is a cyclic extension of $\mathbb{Q}^{\mathrm{rab}}$ of degree p^n. Conversely, every cyclic extension of degree n has intermediate extensions which are cyclic of degree dividing n. Therefore, for every odd prime p, a cyclic extension of degree p yields a profinite extension of $\mathbb{Q}^{\mathrm{rab}}$ which has Galois group isomorphic to the p-adic integers $\mathbb{Z}_p$.

The remaining prime number to consider is the case when $p = 2$. Clearly, $\mathbb{Q}^{\mathrm{rab}}$ has $\mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}^{\mathrm{rab}}(i)$ as a quadratic extension, so the field of real abelian numbers is not quadratically closed. Indeed, $\mathbb{Q}^{\mathrm{rab}}$ has infinitely many quadratic extensions. Since it is also not Pythagorean, the field contains cyclic extensions of degree 4 as well. However, even though every $\mathbb{Z}/4\mathbb{Z}$-extension has a quadratic intermediate extension, it is not necessarily the case that every quadratic extension can be embedded into a cyclic extension of degree 4.

Theorem 9.22 Let F be a field with $a \in F$ not a square. Then, the embedding problem $\mathbb{Z}/4\mathbb{Z} \to \mathrm{Gal}(F(\sqrt{a})/F)$ is solvable if and only if there are x and y in F such that $a = x^2 + y^2$.

Therefore, a quadratic extension $\mathbb{Q}^{\mathrm{rab}}(\sqrt{a})$ can be embedded into a cyclic extension of degree 4 if and only if a is the sum of two squares of $\mathbb{Q}^{\mathrm{rab}}$. This embeddability is a necessary condition for a quadratic extension to be liftable to extensions of higher degree, including having procyclic Galois group $\mathbb{Z}_2$. Theorem 9.21 provides that $\mathbb{Q}^{\mathrm{rab}}$ has at least one such extension.

Unfortunately, the proof in [Whaples 1957] relevant to the current case, which constitutes the fifth case of the argument, turns out to be erroneous. The argument builds purported cyclic extensions of degree 2^n over a field F if $F(i)$ is not quadratically closed and F has another quadratic extension $F(\sqrt{a})$ with $F(\sqrt{a}) \neq F(i)$. Unfortunately, this argument assumes that the candidate field extension has two different quadratic extensions, namely $F(i)$ and $F(\sqrt{a})$, but a cyclic extension of degree 2^n for any $n \in \mathbb{N}$ can only have one quadratic extension.

Nevertheless, the following result does hold for the real abelian numbers.

Theorem 9.23 Let F be a field extension of the real abelian numbers $\mathbb{Q}^{\text{rab}}$ with $\text{Gal}(F/\mathbb{Q}^{\text{rab}}) \cong \mathbb{Z}/4\mathbb{Z}$. Then, F can be embedded into a field K such that K is a Galois extension of $\mathbb{Q}^{\text{rab}}$ and the corresponding Galois group $\text{Gal}(K/\mathbb{Q}^{\text{rab}})$ is $\mathbb{Z}_2$.

Proof. Of relevance is the following result of A. V. Yakovlev in [Yakovlev 2002], with language adapted to the current relevant context.

Theorem 9.24 (Yakovlev) Let F be a field, let $a \in F$ be nonsquare, and let ζ be a 2^nth root of unity for some $n \in \mathbb{N}$ greater than 1. Suppose that $F(\zeta)$ is of degree 2 over F with $\tau(\zeta) = \zeta^{-1}$ for the automorphism τ that generates $\text{Gal}(F(\zeta)/F)$. Then, the quadratic extension $F(\sqrt{a})$ over F can be embedded into a cyclic extension of degree 2^n if and only if there is some number z in $F(\zeta)$ such that a is the norm of z with respect to the field, so that

$$a = \text{Nm}_{F(\zeta)/F}\, z.$$

The theorem immediately leads to the following result.

Corollary 9.25 Let F be a quadratic extension of the field $\mathbb{Q}^{\text{rab}}$. Then, if F can be lifted up to a $\mathbb{Z}/4\mathbb{Z}$-extension, then F can be lifted up to any cyclic extension of powers of 2.

However, even though a quadratic extension F embeddable into a cyclic extension of degree 4 can be embedded into a cyclic extension of degree 2^n for arbitrary $n \in \mathbb{N}$, it is not immediately clear that F can be embedded into a procyclic extension. An example presented in [Fried Jarden 2008] but originally in [Geyer Jensen 1996] is the field $\mathbb{Q}(\sqrt{-17})$ and the quadratic extension $\mathbb{Q}(\sqrt{-17})(\sqrt{-1})$, which can be embedded into cyclic extensions of degree 2^n for every $n \in \mathbb{N}$ but cannot be embedded into an extension of $\mathbb{Q}(\sqrt{-17})$ with Galois group $\mathbb{Z}_2$. In this case, the reason is because it is possible to demonstrate that $i = \sqrt{-1}$ cannot be present in the maximal $\mathbb{Z}_2$-extension $\mathbb{Q}(\sqrt{-17})^{(2)}$, so the quadratic extension therefore is prevented from lifting up to a procyclic extension.

The conditions that permit the imaginary quadratic field $\mathbb{Q}(\sqrt{-17})$ to exhibit this behavior are not present for $\mathbb{Q}^{\text{rab}}$, however. Of important relevance in the case of the real abelian numbers is that the field $\mathbb{Q}^{\text{rab}}(i)$ is the maximal cyclotomic field which contains all roots of unity, which also means that $\mathbb{Q}^{\mathrm{rab}}(\zeta_n) = \mathbb{Q}^{\mathrm{rab}}(i) = \mathbb{Q}^{\mathrm{ab}}$ for any root of unity ζ_n. The cyclotomic explosiveness means that Theorem 9.24 has further applicability to the current situation.

Theorem 9.26 Let a be a number in $\mathbb{Q}^{\mathrm{rab}}$ which is not a square, and let $\mathbb{Q}^{\mathrm{rab}}(\sqrt{a})$ be a quadratic extension of the real abelian numbers. Then, if $\mathbb{Q}^{\mathrm{rab}}(\sqrt{a})$ can be embedded into a cyclic extension of degree 4, then it can be embedded into a procyclic extension F_a with Galois group $\mathrm{Gal}(F_a/\mathbb{Q}^{\mathrm{rab}}) \cong \mathbb{Z}_2$.

Theorem 9.24, when applied to the case of $2^n = m = 4$, yields the situation of looking at the field $F(i)$ over the base given field F. In this case, the norm of a complex number, when regarding these field elements as complex numbers, is the square of the modulus, so

$$\mathrm{Nm}_{F(i)/F}\, z = \mathrm{Nm}_{F(i)/F}\, a + bi = a^2 + b^2$$

for the complex number $z = a + bi$ in $F(i)$ with a and b in F. The theorem therefore has as a special case the conditions for Theorem 9.22. Furthermore, since the choice of root of unity for adjoining to $\mathbb{Q}^{\mathrm{rab}}$ makes no difference to the resultant field $\mathbb{Q}^{\mathrm{ab}}$, the condition for embeddability into a $\mathbb{Z}/2^n\mathbb{Z}$-extension for any $n \in \mathbb{N}$ is exactly the same as for embeddability into a $\mathbb{Z}/4\mathbb{Z}$-extension. The norm operator Nm does not change with the choice of the root of unity since the field extension never changes, so there is no need to make reference to the underlying field extension.

So, let a be a real abelian number which is the sum of two squares of real abelian numbers. Then, $\mathbb{Q}^{\mathrm{rab}}(\sqrt{a})$ is a quadratic extension embeddable into any cyclic extension with degree power of 2. It is possible to formulate what each such cyclic extension looks like, and the characterization consequently yields the procyclic extension. It is to be shown that if F is a cyclic extension of degree 2^n containing $\mathbb{Q}^{\mathrm{rab}}(\sqrt{a})$ and K is a cyclic extension of degree 2^{n+1} containing $\mathbb{Q}^{\mathrm{rab}}(\sqrt{a})$, then F is contained in K such that K is a quadratic extension of F.

Let v be the number which satisfies $\mathrm{Nm}\, v = a$, hence encoding the fact that a is the sum of two squares. By the proof of Theorem 9.24 in [Yakovlev 2002], the fields F and K can be obtained as the fixed fields respectively of $\mathbb{Q}^{\mathrm{ab}}(\sqrt[2^n]{\frac{v}{\tau(v)}})$ and of $\mathbb{Q}^{\mathrm{ab}}(\sqrt[2^{n+1}]{\frac{v}{\tau(v)}})$ with respect to the nontrivial automorphism τ of complex conjugation in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{rab}})$. The terms $\sqrt[2^n]{\frac{v}{\tau(v)}}$ and $\sqrt[2^{n+1}]{\frac{v}{\tau(v)}}$ are not well-defined, but let them denote respectively two individuals β and γ such that $\beta^{2^n} = \frac{v}{\tau(v)}$ and $\gamma^{2^{n+1}} = \frac{v}{\tau v}$. The fields $\mathbb{Q}^{\mathrm{ab}}(\beta) = \mathbb{Q}^{\mathrm{ab}}(\sqrt[2^n]{\frac{v}{\tau(v)}})$ and $\mathbb{Q}^{\mathrm{ab}}(\gamma) = \mathbb{Q}^{\mathrm{ab}}(\sqrt[2^{n+1}]{\frac{v}{\tau(v)}})$ are then cyclic extensions of $\mathbb{Q}^{\mathrm{ab}}$ of degree 2^n and 2^{n+1}, respectively. Hence, $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}(\beta)/\mathbb{Q}^{\mathrm{ab}}) \cong \mathbb{Z}/2^n\mathbb{Z}$ and $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}(\gamma)/\mathbb{Q}^{\mathrm{ab}}) \cong \mathbb{Z}/2^{n+1}\mathbb{Z}$, but with some inspection it is also possible to show that $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}(\beta)/\mathbb{Q}^{\mathrm{rab}}) \cong \mathbb{Z}/2^n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}(\gamma)/\mathbb{Q}^{\mathrm{rab}}) \cong \mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

This inspection can be done by defining the group action of $\mathbb{Z}/2^n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2^{n+1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Q}^{\mathrm{ab}}(\beta)$ and $\mathbb{Q}^{\mathrm{ab}}(\gamma)$ respectively with $\tau(\beta) = \beta^{-1}$ and $\tau(\gamma) = \gamma^{-1}$ for τ generating $\mathbb{Z}/2\mathbb{Z}$ in each instance and $\xi_{2^n}(\beta) =$

$\beta\zeta_{2^n}$ and $\xi_{2^{n+1}}(\gamma) = \gamma\zeta_{2^{n+1}}$ for ξ_{2^n} generating $\mathbb{Z}/2^n\mathbb{Z}$ and $\xi_{2^{n+1}}$ generating $\mathbb{Z}/2^{n+1}\mathbb{Z}$. In both instances, it is evident that $\mathbb{Q}^{\mathrm{ab}}(\beta)$ and $\mathbb{Q}^{\mathrm{ab}}(\gamma)$ are normal extensions of $\mathbb{Q}^{\mathrm{rab}}$ with Galois groups as characterized. Furthermore, because of Kummer theory, it follows that $\mathbb{Q}^{\mathrm{ab}}(\beta)$ is a subfield of $\mathbb{Q}^{\mathrm{ab}}(\gamma)$.

Then, the cyclic extensions F and K of respective degree 2^n and 2^{n+1} over $\mathbb{Q}^{\mathrm{rab}}$ can be obtained by taking $F := \mathrm{Fix}(\mathbb{Z}/2\mathbb{Z})$ of $\mathbb{Q}^{\mathrm{ab}}(\beta)$ and $K := \mathrm{Fix}(\mathbb{Z}/2\mathbb{Z})$ of $\mathbb{Q}^{\mathrm{ab}}(\gamma)$. The fields F and K are normal because $\mathbb{Q}^{\mathrm{ab}}(\beta)$ and $\mathbb{Q}^{\mathrm{ab}}(\gamma)$ are normal, and their Galois groups over the real abelian numbers are the cyclic ones of degree 2^n and 2^{n+1} respectively.

Furthermore, it is possible to recover $\sqrt{a}$ in both F and K. In F, take the number $\beta^{2^{n-1}}\tau(v)$, and in K, the number $\gamma^{2^n}\tau(v)$. Then, let $\alpha = \beta^{2^{n-1}}\tau(v) = \gamma^{2^n}\tau(v)$ with $\sqrt{a} = \pm\alpha$ as appropriate. Recall that β is a number such that it is a 2^nth root of $\frac{v}{\tau(v)}$ and γ is a 2^{n+1}th root, so α can be rewritten so that

$$\alpha = \sqrt{\frac{v}{\tau(v)}}\tau(v).$$

This result is sensible since a would then be the square of α, so $a = (\sqrt{\frac{v}{\tau(v)}}\tau(v))^2 = \frac{v}{\tau(v)}\tau(v)\tau(v) = v\tau(v) = \mathrm{Nm}\, v$. Thus, the cyclic extensions both contain $\mathbb{Q}^{\mathrm{rab}}(\sqrt{a})$ as their intermediate quadratic extension.

Suppose x is in F. Then, x will also be in $\mathbb{Q}^{\mathrm{ab}}(\beta)$. The extension $\mathbb{Q}^{\mathrm{ab}}(\gamma)$ is Galois over $\mathbb{Q}^{\mathrm{ab}}(\beta)$, so x is in $\mathbb{Q}^{\mathrm{ab}}(\gamma)$. Since x is invariant with respect to complex conjugation, x will be in the fixed field with respect to complex conjugation, and this fixed field is by definition the field K. Therefore, F is a subfield of K. The field K which is cyclic over $\mathbb{Q}^{\mathrm{rab}}$ of degree 2^{n+1} contains exactly one field which is cyclic over $\mathbb{Q}^{\mathrm{rab}}$ with degree 2^n, so F is exactly the intermediate cyclic extension of such degree.

Consequently, F is now a field extension of $\mathbb{Q}^{\mathrm{rab}}$ such that K is an extension of it by degree 2. Therefore, K is a quadratic extension of F, such that

$$K = F(\sqrt{b})$$

for some b in F. Thus, every number in K has the form $f + g\sqrt{b}$ for numbers f and g in F.

Since the choice of $n \in \mathbb{N}$ has no impact on this form of argument, the relationship between F and K holds for arbitrary cyclic extensions of degrees powers of 2 sharing a common quadratic extension. By induction, it follows then that if $\mathbb{Q}^{\mathrm{rab}}(\sqrt{a})$ is a quadratic extension of $\mathbb{Q}^{\mathrm{rab}}$ embeddable into a cyclic extension of degree 4, then there is a tower

$$\mathbb{Q}^{\mathrm{rab}} \hookrightarrow \mathbb{Q}^{\mathrm{rab}}(\sqrt{a}) \hookrightarrow F_4 \hookrightarrow F_8 \hookrightarrow \ldots \hookrightarrow F_{2^n} \hookrightarrow \ldots$$

for every $n \in \mathbb{N}$ such that $F_{2^{n+1}}$ is a quadratic extension of F_{2^n}. Let $\mathbb{Q}^{\mathrm{rab}}(\mathbb{Z}_2(a))$ be the field defined as the compositum of the fields in the tower; it can be characterized as

$$\mathbb{Q}^{\text{rab}}(\mathbb{Z}_2(a)) := \bigcup_{n\in\mathbb{N}} F_{2^n} \supseteq \mathbb{Q}^{\text{rab}}.$$

It is clear that this process yields an inverse system, and the corresponding cyclic Galois groups $\text{Gal}(F_{2^n}/\mathbb{Q}^{\text{rab}})$ can be directed to form a procyclic group $\text{Gal}(\mathbb{Z}_2(a)/\mathbb{Q}^{\text{rab}}) \cong \mathbb{Z}_2 = \lim_{\leftarrow} \mathbb{Z}/2^n\mathbb{Z}$. ↻

Consequently, the choice of a was only dependent on being the sum of two squares. Thus, any quadratic extension which can be embeddable into a cyclic extension of degree 4 can be admitted into cyclic extension of any degree power of 2, and this embeddability is such that every such quadratic extension can be lifted up into a profinite field extension which is normal over $\mathbb{Q}^{\text{rab}}$ and has the group $\mathbb{Z}_2$. Let U be a subset of $\mathbb{Q}^{\text{rab}}$ defined such that

$$U := \{x \in \dot{\mathbb{Q}}^{\text{rab}} \setminus \dot{\mathbb{Q}}^{\text{rab}^2} : \text{there is a } \mathbb{Z}/4\mathbb{Z}\text{-extension that contains } \mathbb{Q}^{\text{rab}}(\sqrt{x})\ \}.$$

Note that $\dot{\mathbb{Q}}^{\text{rab}} := \mathbb{Q}^{\text{rab}\times} = \mathbb{Q}^{\text{rab}} \setminus \{0\}$ is a convention which will be convenient in subsequent work. Because $\mathbb{Q}^{\text{rab}}$ is not Pythagorean, it has cyclic extensions of degree 4, so U must be nonempty. Indeed, there are infinitely many such individuals in U. Each such x in U consequently corresponds to a separate field $\mathbb{Q}^{\text{rab}}(\mathbb{Z}_2(x))$ with Galois group

$$\text{Gal}(\mathbb{Q}^{\text{rab}}(\mathbb{Z}_2(x))/\mathbb{Q}^{\text{rab}}) = \mathbb{Z}_2.$$

Therefore, the compositum of all such $\mathbb{Z}_2$-extensions, which can be denoted $\mathbb{Q}^{\text{rab}}(U^\infty)$, has Galois group isomorphic to the direct product of infinitely many copies of $\mathbb{Z}_2$. The copies of $\mathbb{Z}_2$ form a direct product since each of the individuals in U are linearly independent, such that if a and b are in U, then $\mathbb{Q}^{\text{rab}}(a) \cap \mathbb{Q}^{\text{rab}}(b) = \mathbb{Q}^{\text{rab}}$.

Although $\mathbb{Q}^{\text{rab}}$ is not Pythagorean, it is closer to being Pythagorean in a measurable manner than the rational number field $\mathbb{Q}$ is. For an arbitrary field F and some number $x \in F$, a question that may be asked of x is if it can be written as the sum of squares in F, and if so, what is the minimum number of such squares to yield a sum of x. Let this minimal number be the length of x, and let $\Sigma(F) \subseteq F$ denote the set of such individuals in F which can be written as sums of squares. Then, the *Pythagoras number* of the field F, denoted $P(F)$, is the maximal number of the length of the numbers in $\Sigma(F)$. As an example, the theorem of Lagrange that demonstrates that every rational number which can be written as the sum of squares needs at most four squares means that $P(\mathbb{Q}) = 4$. For the real and complex number fields, where every positive number in the former and every number in the latter is a square, it is the case that $P(\mathbb{R}) = P(\mathbb{C}) = 1$.

Proposition 9.27 Let F be a field. Then, F is a Pythagorean field if and only if the Pythagoras number $P(F)$ of F is 1.

Theorem 9.28 The Pythagoras number of the real abelian numbers $\mathbb{Q}^{\mathrm{rab}}$ is 2.

Proof. If $P(F)=2$ for some field F, then every sum of squares in F is in fact the sum of two squares. In such a case, one can divide the nonsquares of F into two disjoint sets, one consisting of sums of squares and the other consisting of those numbers which cannot be written as sums of squares. In the case of $\mathbb{Q}^{\mathrm{rab}}$, $P(\mathbb{Q}^{\mathrm{rab}})=2$ would mean that $U=\Sigma(\mathbb{Q}^{\mathrm{rab}})$ and that every quadratic extension which cannot be embedded into cyclic extensions of higher degree is generated by the square root of a number which cannot be the sum of squares.

The task of determining the Pythagoras number of a field F is simplified when F carries certain properties, and [Pfister 1995] includes such results. Among these results which will be useful in the current case is the following theorem.

Theorem 9.29 (Pfister) Let F be a nonarchimedean local field or a nonreal number field. Then, the Pythagoras number $P(F)$ will the minimum of $s(F)+1$ and 4, where $s(F)$ denotes the level of F.

Definition 9.30 The level of a field F, denoted $s(F)$, is the minimal number of individuals $a_1,...,a_n\in F$ needed such that $-1=a_1^2+...+a_n^2$. If -1 cannot be written as the sum of squares in F, then $s(F)=\infty$.

The determination of the Pythagoras number for $\mathbb{Q}^{\mathrm{rab}}$ is not as straightforward as the examples provided in [Pfister 1995] are of classes of fields that are more frequently encountered. It is nevertheless possible to obtain the Pythagoras number by appealing where necessary to the global fields contained in $\mathbb{Q}^{\mathrm{rab}}$ which can be elucidated by the local-global principle with respect to quadratic forms.

The relevant quadratic form is ϕ defined such that

$$\phi := x^2+y^2.$$

Let $\phi(F)$ denote the set of numbers represented by the quadratic form ϕ in the field F. Then, the local-global principle can be expressed by the equivalence

$$\forall a\in F(a\in\phi(F)\Leftrightarrow\forall p\in\mathbb{P}_F(a\in\phi(F_p)))$$

where $\mathbb{P}_F$ denotes the set of primes in F, including the archimedean ones.

When a global field has the Pythagoras number of every local completion be 2, then its own Pythagoras number is 2. This in general will not happen when the global field is formally real, since by [Pfister 1995] real number fields of finite dimension will have Pythagoras number 3 or 4, but this is not problematic in the current situation. The basic argument for showing that $\mathbb{Q}^{\mathrm{rab}}$ has Pythagoras number 2 is that every real abelian number inhabits a global subfield of $\mathbb{Q}^{\mathrm{rab}}$ wherein it is the sum of two squares.

Let a be a real abelian number. Obviously, if a cannot be written as the sum of squares, then it cannot be written as the sum of two squares, so suppose a is the sum of an arbitrary number of squares of real abelian numbers. By [Pfister 1995], the Pythagoras number of any real number field is at most 4, so this limit also holds for $\mathbb{Q}^{\text{rab}}$. Suppose not and let a be the sum of at least 5 squares with $a = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2$. Since a is algebraic and each a_i is also algebraic, the field $\mathbb{Q}(a, a_1, a_2, a_3, a_4, a_5)$ is a real number field. So, $P(\mathbb{Q}(a, a_1, a_2, a_3, a_4, a_5)) \leq 4$, so a, which can be written as the sum of five squares in $\mathbb{Q}^{\text{rab}}$ and therefore in $\mathbb{Q}(a, a_1, a_2, a_3, a_4, a_5)$, can actually be written as the sum of at most four squares. Since $\mathbb{Q}(a, a_1, a_2, a_3, a_4, a_5) \subseteq \mathbb{Q}^{\text{rab}}$, those four squares must also be in $\mathbb{Q}^{\text{rab}}$, so a can be written as the sum of four squares in $\mathbb{Q}^{\text{rab}}$, contradicting the original motivating assumption.

Thus, $P(\mathbb{Q}^{\text{rab}})$ is at most 4 by this initial reasoning. Let the real abelian number a be the sum of four squares. Take $L := \mathbb{Q}(a)$ as a global field. Then, for all but finitely many primes $\mathfrak{p}$, a is represented by the quadratic form ϕ in the local completion $L_{\mathfrak{p}}$, including all of the infinite primes. So, for sufficiently large rational primes and other primes $\mathfrak{p}$, a can be written as the sum of two squares in the completions $L_{\mathfrak{p}}$. The finitely many primes with which a is not represented in the corresponding local completions are then above finitely many rational primes; let $p_1, ..., p_n$ denote these rational primes. Define the field $M := L(\sqrt{3}, \sqrt{5}, \sqrt{p_1 - 1}, ..., \sqrt{p_n - 1})$. Now, M is a subfield of $\mathbb{Q}^{\text{rab}}$ as well because the square roots of every natural number are real abelian. For any prime $\mathfrak{p}$ of M above a prime of L which denies representation of a by the quadratic form, $\sqrt{-1}$ is then in the local completion $M_{\mathfrak{p}}$; this property is clear if $\mathfrak{p}$ is above a rational prime p_i $(1 \leq i \leq n)$ which is odd, and for when $p_i = 2$, $\sqrt{-1}$ is in $\mathbb{Q}_2(\sqrt{3}, \sqrt{5})$ since $3 \times 5 = 15 \equiv -1 \mod 8$. Consequently, all of the local completions $M_{\mathfrak{p}}$ of M contain $\sqrt{-1}$. By Theorem 9.29, the Pythagoras number for each $M_{\mathfrak{p}}$ is $P(M_{\mathfrak{p}}) = 2$. Hence, a is represented by ϕ in each local completion of M, so by the local-global principle, a is represented by ϕ in M itself. Since M is a subfield of $\mathbb{Q}^{\text{rab}}$, it follows that $a \in \phi(\mathbb{Q}^{\text{rab}})$, so a can be written as the sum of two squares in $\mathbb{Q}^{\text{rab}}$. Since a can be an arbitrary sum of squares, every sum of squares in $\mathbb{Q}^{\text{rab}}$ can be written as the sum of two squares. Thus, $P(\mathbb{Q}^{\text{rab}}) = 2$. ↻

Having $P(\mathbb{Q}^{\text{rab}}) = 2$ means that the class of nonsquare real abelian numbers can be divided into those which are sums of two squares and those which are not sums of squares at all. The former collection has previously been denoted U, so the current result clarifies U as not only the set of all sums of two squares but also the set of all sums of arbitrary number of squares. Let the latter collection of nonsquare numbers which are not sums of squares be denoted E.

Quadratic extensions of $\mathbb{Q}^{\text{rab}}$ generated by square roots of numbers in U can be extended to procyclic extensions each with Galois group $\mathbb{Z}_2$. Since $\mathbb{Q}^{\text{rab}}$ is a totally real field and a number in U must be the sum of two squares, every number in U is positive and in fact is totally positive. Indeed, since U is exactly the set of all sums of squares of $\mathbb{Q}^{\text{rab}}$, it is possible to conclude that $\mathbb{Q}^{\text{rab}}(U^{\infty})$ is the relative totally real closure of $\mathbb{Q}^{\text{rab}}$ in the 2-part of the maximal abelian extension $(\mathbb{Q}^{\text{rab}})^{\text{ab}}$.

Thus, the numbers constituting E are those nonsquares which are exactly not totally positive. Numbers which are not totally positive, when in a formally real field, can be either positive or negative depending on the choice of the field ordering. The choice of ordering is what gives rise to involutions, and it is clear that there are infinitely many possible orderings. Because U is not just the set of all sums of two squares but the set of all totally positive nonsquares of $\mathbb{Q}^{\text{rab}}$, no torsion can arise from quadratic extensions generated by square roots of totally positive real abelian numbers.

This partition that E and U participate in with the nonsquares of $\mathbb{Q}^{\text{rab}}$ therefore makes for a straightforward characterization of the 2-part of $\text{Gal}((\mathbb{Q}^{\text{rab}})^{\text{ab}}/\mathbb{Q}^{\text{rab}})$. The sets U and $E = (\dot{\mathbb{Q}}^{\text{rab}} \setminus \dot{\mathbb{Q}}^{\text{rab}^2}) \setminus U$ then have the following properties.

$$\mathbb{Q}^{\text{rab}}(\{\sqrt{e} : e \in E\}) =: \mathbb{Q}^{\text{rab}}(\sqrt{E}) = \text{Fix(torsion-free part of the 2-part of } \text{Gal}((\mathbb{Q}^{\text{rab}})^{\text{ab}}/\mathbb{Q}^{\text{rab}})(2))$$

$$\mathbb{Q}^{\text{rab}}(U^\infty) = \text{Fix(torsion part of the 2-part of } \text{Gal}((\mathbb{Q}^{\text{rab}})^{\text{ab}}/\mathbb{Q}^{\text{rab}})(2))$$

The 2-part of $\text{Gal}((\mathbb{Q}^{\text{rab}})^{\text{ab}}/\mathbb{Q}^{\text{rab}})$ is therefore exactly

$$\prod_{\mathbb{N}} \mathbb{Z}_2 \times \prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$$

since each of the infinitely many involutions yields $\mathbb{Z}/2\mathbb{Z}$ independently of each other and each quadratic extension of $\mathbb{Q}^{\text{rab}}$ liftable to a procyclic extension is independent from each other. For each odd prime p, the p-part of $\text{Gal}((\mathbb{Q}^{\text{rab}})^{\text{ab}}/\mathbb{Q}^{\text{rab}})$ can be characterized as $\prod_{\mathbb{N}} \mathbb{Z}_p$. Since

$$\hat{\mathbb{Z}} \cong \prod_{p \in \mathbb{N}(\mathbb{P})} \mathbb{Z}_p,$$

where $\mathbb{N}(\mathbb{P})$ denotes the prime numbers, it is possible to simplify the characterization of the maximal abelian group into the following result.

Theorem 9.31 Let $\mathbb{Q}^{\text{rab}}$ be the field of totally real abelian numbers. Then, the maximal abelian extension $(\mathbb{Q}^{\text{rab}})^{\text{ab}}$ of $\mathbb{Q}^{\text{rab}}$ has Galois group

$$\text{Gal}((\mathbb{Q}^{\text{rab}})^{\text{ab}}/\mathbb{Q}^{\text{rab}}) = \prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} \times \prod_{\mathbb{N}} \hat{\mathbb{Z}},$$

which can alternatively be written as $2^\omega \times \hat{\mathbb{Z}}^\omega$.

In contrast to the piecemeal approach needed for the maximal abelian extension of $\mathbb{Q}^{\text{rab}}$, the maximal abelian extension of $\mathbb{Q}^{\text{ab}}$ has a straightforward characterization since $\mathbb{Q}^{\text{ab}}$ is a cyclotomic field which can take full advantage of the results of Kummer theory.

Proposition 9.32 The maximal abelian extension of $\mathbb{Q}^{\mathrm{ab}}$ is the field $(\mathbb{Q}^{\mathrm{ab}})^{\mathrm{ab}}$ such that

$$(\mathbb{Q}^{\mathrm{ab}})^{\mathrm{ab}} = \mathbb{Q}^{\mathrm{ab}}(\{\sqrt[n]{q} : q \in \mathbb{Q}^{\mathrm{ab}}, n \in \mathbb{N}\}).$$

The group $\mathrm{Gal}((\mathbb{Q}^{\mathrm{ab}})^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{ab}})$ is then the group

$$\prod_{\mathbb{N}} \hat{\mathbb{Z}}.$$

As tempting as it may be to have $(\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}} = (\mathbb{Q}^{\mathrm{ab}})^{\mathrm{ab}}$, such an outcome is not possible because there are fields which are abelian extensions of $\mathbb{Q}^{\mathrm{ab}}$ but not abelian over $\mathbb{Q}^{\mathrm{rab}}$. Furthermore, the Galois groups do not match; the difference between $\mathbb{Q}^{\mathrm{rab}}$ and $\mathbb{Q}^{\mathrm{ab}}$ can be captured by one instance of $\mathbb{Z}/2\mathbb{Z}$ as the corresponding Galois group between them, yet there are infinitely many involutions that would need to be addressed. Furthermore, it is not correct to assume that the essential change from $\mathbb{Q}^{\mathrm{rab}}$ to $\mathbb{Q}^{\mathrm{ab}}$ is that the involutions get resolved while leaving everything else intact.

A straightforward example of such a field that is abelian over $\mathbb{Q}^{\mathrm{ab}}$ but not over $\mathbb{Q}^{\mathrm{rab}}$ is a cyclic extension F of degree 4 over $\mathbb{Q}^{\mathrm{ab}}$ taken by having F generated by the fourth root of some positive nonsquare number a in $\mathbb{Q}^{\mathrm{rab}}$. Then, $F = \mathbb{Q}^{\mathrm{ab}}(\sqrt[4]{a})$ is cyclic with degree 4 over $\mathbb{Q}^{\mathrm{ab}}$, but because $\mathbb{Q}^{\mathrm{rab}}(\sqrt[4]{a})$ is not normal over $\mathbb{Q}^{\mathrm{rab}}$, the Galois group of F over $\mathbb{Q}^{\mathrm{rab}}$ is the dihedral group $D_8 = \mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ and not the cyclic product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

The following proposition is an immediate result that follows readily from the previous discussion.

Proposition 9.33 The maximal abelian extension $(\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}$ of the real abelian numbers constituting $\mathbb{Q}^{\mathrm{rab}}$ is contained in the field

$$\mathbb{Q}^{\mathrm{ab}}(\{\sqrt[n]{a+bi} : a, b \in \mathbb{Q}^{\mathrm{rab}}, b \neq 0, n > 2\} \cup \{\sqrt{q} : q \in \mathbb{Q}^{\mathrm{rab}}\})$$

which is the subfield of $(\mathbb{Q}^{\mathrm{ab}})^{\mathrm{ab}}$ generated by roots of nonreal abelian numbers.

A more refined result can be gotten if the component-wise reasoning used for what $\mathrm{Gal}((\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{rab}})$ looks like is used to determine $(\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}$ itself. For example, it is certainly the case that every quadratic extension of $\mathbb{Q}^{\mathrm{rab}}$ is abelian, so $\mathbb{Q}^{\mathrm{rab}}(\sqrt{\mathbb{Q}^{\mathrm{rab}}}) := \mathbb{Q}^{\mathrm{rab}}(\{\sqrt{q} : q \in \mathbb{Q}^{\mathrm{rab}}\})$ is a subfield of the maximal abelian extension. Furthermore, the elaboration of the 2-part of the maximal abelian group demonstrates that cyclic extensions of degree 2^n for every $n \in \mathbb{N}$ over $\mathbb{Q}^{\mathrm{rab}}$ have a uniform characterization when composed with $\mathbb{Q}^{\mathrm{ab}}$. This part can be described as

$$\mathbb{Q}^{\mathrm{ab}}(\{\sqrt[2^n]{\frac{a+bi}{a-bi}} : n \in \mathbb{N}; a, b \in \mathbb{Q}^{\mathrm{rab}} \setminus \{0\}\})$$

since each such $a+bi$ with a and b nonzero represents the cyclic extensions that contain the quadratic extension $\mathbb{Q}^{\mathrm{rab}}(\sqrt{a^2+b^2})$.

The scrutinous investigation of the 2-part of $\mathrm{Gal}((\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{rab}})$ leads to this characterization of the 2-part of $(\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}$. For the odd prime p, the argument constituting the fourth case of the proof of Theorem 9.21 in [Whaples 1957] can be used to make a not-quite-clean characterization. Suppose that a is a number in $\dot{\mathbb{Q}}^{\mathrm{rab}} \setminus \dot{\mathbb{Q}}^{\mathrm{rab}^p}$, so that a is not the pth power of some $b \in \mathbb{Q}^{\mathrm{rab}}$, such that $\frac{\bar{a}}{a}$ is nevertheless the p^nth power of some $c \in \mathbb{Q}^{\mathrm{rab}}$. Then, the lemma in this argument of Whaples implies that the field $\mathbb{Q}^{\mathrm{ab}}(\sqrt[p^n]{a})$ for some $n \in \mathbb{N}$ is cyclic over $\mathbb{Q}^{\mathrm{rab}}$ of degree $f p^n$ for some $f \in \mathbb{N}$. Although this argument does not directly characterize what a cyclic extension over $\mathbb{Q}^{\mathrm{rab}}$ of degree p^n looks like, only guaranteeing the existence of such an extension as an intermediate Galois extension, it is clear that this characterization is nonetheless suitable for the current purposes of trying to develop a concise but comprehensive description of $(\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}$. Thus, it is possible to put forth the following statement.

Theorem 9.34 Let $\mathbb{Q}^{\mathrm{rab}}$ be the field of real abelian numbers. Then, the maximal abelian extension $(\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}$ of the real abelian field can be characterized as the field compositum

$$\mathbb{Q}^{\mathrm{rab}}(\sqrt{\mathbb{Q}^{\mathrm{rab}}})\mathbb{Q}^{\mathrm{ab}}(\{\sqrt[2^n]{\frac{a+bi}{a-bi}} : n \in \mathbb{N}, a, b \in \dot{\mathbb{Q}}^{\mathrm{rab}}\} \cup \{\sqrt[p^n]{c} : p \text{ odd prime}, n \in \mathbb{N}, c \in \dot{\mathbb{Q}}^{\mathrm{ab}} \setminus \dot{\mathbb{Q}}^{\mathrm{ab}^p}, \frac{\bar{c}}{c} \in \dot{\mathbb{Q}}^{\mathrm{ab}^{p^n}}\})$$

which can also be characterized as

$$\mathbb{Q}^{\mathrm{ab}}(\{\sqrt[2^n]{\frac{a+bi}{a-bi}} : n \in \mathbb{N} \text{ and } a, b \in \dot{\mathbb{Q}}^{\mathrm{rab}}\} \cup \{\sqrt[p^n]{c} : p \text{ odd prime }, n \in \mathbb{N}, c \in \dot{\mathbb{Q}}^{\mathrm{ab}} \setminus \dot{\mathbb{Q}}^{\mathrm{ab}^p}, \frac{\bar{c}}{c} \in \dot{\mathbb{Q}}^{\mathrm{ab}^{p^n}}\} \cup \sqrt{\mathbb{Q}^{\mathrm{rab}}}).$$

Had $\mathbb{Q}^{\mathrm{rab}}$ been a Pythagorean field admitting no cyclic extensions of odd prime degree, then its maximal abelian extension would have been easily describable with a simple description of the group $\mathrm{Gal}((\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{rab}})$. Nevertheless, the outcome of this study yields a fairly neat characterization of what abelian extensions of the real abelian field look like.

9.6 The Closed Real Abelian Pseudoexponential Field $\mathbb{Q}^{\mathrm{rab}\,C}$

The previous attention paid upon the real abelian number field $\mathbb{Q}^{\mathrm{rab}}$ is due to it being the algebraic subfield contained in the parameter-free definably closed prime structure of the complex pseudoexponential field in the language of exponential rings. Let this prime structure be called the closed real abelian pseudoexponential field or exponential field and denoted $\mathbb{Q}^{\mathrm{rab}\,C}$. Notably, this entity contains the quantifier-free prime exponential field $\mathbb{Q}^E$, and the closed real abelian exponential field properly extends $\mathbb{Q}^E$ on both its algebraic and transcendental numbers.

Since the relationship between $\mathbb{Q}$ and $\mathbb{Q}^{\mathrm{rab}}$ is known, such that $\mathbb{Q}^{\mathrm{rab}}$ is obtained by taking the maximal real subfield of the maximal abelian extension of $\mathbb{Q}$, a closer look at the transcendental portion of $\mathbb{Q}^{\mathrm{rab}\,C}$ may now be warranted. It is somewhat inappropriate to use the rational exponential field $\mathbb{Q}^E$ as a reference when it is possible to use the real abelian exponential field $\mathbb{Q}^{\text{rab}E}$ instead. To do so, the notion of exponential closure is formally defined.

Definition 9.35 Let F be a subfield of $\mathbb{C}$. Then, the exponential closure F^E of the F is the smallest field containing F that is closed with respect to exponentiation; that is, if $x \in F^E$, then $\exp(x) \in F^E$.

The process of constructing F^E from F is the same as the process used to obtain $\mathbb{Q}^E$ from $\mathbb{Q}$. Specifically, let $E(F)_0 := F$, $E(F)_{i+1} = E(F)_i(\{\exp(x) : x \in E(F)_i\})$ for each $i \in \mathbb{N}$, and $F^E := \bigcup_{i\in\mathbb{N}} E(F)_i$. Checking this recursive definition yields a way to regard the exponential closure of a field.

Proposition 9.36 Let $\mathbb{Q}^{\text{rab}}$ denote the field of real abelian numbers. Then, its exponential closure $\mathbb{Q}^{\text{rab}E}$ is the field obtained by adjoining to $\mathbb{Q}^{\text{rab}}$ every individual in $E(\mathbb{Q}^{\text{rab}})_\infty$, so that

$$\mathbb{Q}^{\text{rab}E} = \bigcup_{i\in\mathbb{N}} E(\mathbb{Q}^{\text{rab}})_i.$$

Proof. The same argument used to check $\mathbb{Q}^E = E_\infty$ as previously described can be used to verify that $\mathbb{Q}^{\text{rab}E}$ as described is indeed the exponential closure of $\mathbb{Q}^{\text{rab}}$. ↻

Needless to say, a definably closed pseudoexponential field will be exponentially closed. An exponentially closed field will not usually be definably closed in the language of exponential rings, however. The chief culprit for this break between definable and exponential closure is the lack of a quantifier elimination result for the theory of pseudoexponentiation.

Thus, there are two closure operators that may be considered for a provided field F. With respect to $\mathbb{Q}^{\text{rab}}$, the exponential closure $\mathbb{Q}^{\text{rab}E}$ is obtained by closing the real abelian numbers with respect to exponentiation in addition to addition, subtraction, multiplication, and division. The definable closure $\mathbb{Q}^{\text{rab}C}$ is one obtained by closing the real abelian numbers with respect to all parameter-free definable functions in the language of rings with respect to the complex pseudoexponential field. A brief foray into these functions may elucidate the considerable distance that can separate the exponential closure of a given field and the definable closure of that same field.

The most obvious example is the previous study on the rational numbers. The field $\mathbb{Q}$ is clearly not exponentially or definably closed. The exponentially closed field $\mathbb{Q}^E$ clearly cannot be the definable closure $\mathbb{Q}^C$ since $\mathbb{Q}^{\text{rab}}$ is not contained in it; exponential closure does not provide the enrichment of definable functions necessary to define the trigonometric functions and therefore the irrational real abelian numbers. The definable closure $\mathbb{Q}^C$ is of course the closed real abelian pseudoexponential field $\mathbb{Q}^{\text{rab}C}$.

Because model completeness is not available for the theory of pseudoexponentiation, it is devoid of quantifier elimination, and it becomes clear that considerable expansion of the language is required to achieve such a result. Although the rationals can be individually defined in a quantifier-free manner, the dependence upon the trigonometric functions for the other algebraic numbers in $\mathbb{Q}^{\mathrm{rab}\,C}$ means that they cannot be defined in a similar manner. Similarly, whereas the transcendental numbers that reside in $\mathbb{Q}^E$ may be pointwise definable without the use of quantifiers, the same cannot be said of the remaining transcendental numbers in $\mathbb{Q}^{\mathrm{rab}\,C}$.

In contrast to these algebraically-motivated definitions, the definably closed structures of the complex pseudoexponential field are denoted closed exponential fields when ambiguity is not an issue.

Definition 9.37 (Closed Exponential Fields) Let F be a field. Then, F^E is the exponential closure of F, obtained by closing F with respect to the exponential or pseudoexponential function, depending on context. Let F^C denote the exponential field obtained by taking the definable closure of F with respect to the language of exponential rings in the theory of pseudoexponentiation. That is, $F^C := \mathrm{dcl}(F)$ in such a setting.

If F is a field, let F^C be denoted the closed (pseudoexponential) field of F. Hence, since the algebraic numbers that are individually definable without parameters are precisely the real abelian numbers in $\mathbb{Q}^{\mathrm{rab}}$, it is appropriate to then call $\mathrm{dcl}(\emptyset)$ as the closed real abelian exponential field $\mathbb{Q}^{\mathrm{rab}\,C}$.

It should be immediately evident that $\mathbb{Q}^{\mathrm{rab}\,C}$ is properly larger than $\mathbb{Q}^{\mathrm{rab}\,E}$. The sine and cosine functions, being definable without parameters and used to obtain the irrational real abelian numbers, can be used to obtain transcendental numbers that cannot be obtained directly from addition, multiplication, and exponentiation. Thus, cos(2), which can be rewritten as

$$\frac{e^{2i}+e^{-2i}}{2},$$

is defined by the formula

$$\exists j(j^2+1\doteq 0 \wedge 2y\doteq \exp(2j)+\exp(-2j))$$

and cannot be represented without getting rid of the square roots of -1. With the trigonometric functions being definable, the following motivational conjecture may seem reasonable to consider.

Motivation 9.38 For the complex pseudoexponential field $\mathbb{B}$ in the theory of pseudoexponentiation with the language of rings, every individual in $\mathrm{dcl}(\emptyset) \subseteq \mathbb{B}$ can be expressed without quantifiers in terms of the rational numbers $\mathbb{Q}$, the transcendental number π, addition $+$, multiplication $\times$, pseudoexponentiation exp, and the trigonometric functions sin and cos.

Unfortunately, this thought can be refuted by coming up with other definable functions that resemble the trigonometric functions to come up with more numbers not previously obtained with addition, multiplication, exponentiation, and the sine and cosine functions. The circular trigonometric functions of sine and cosine have hyperbolic counterparts denoted sinh and cosh, with hyperbolic sine defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

and hyperbolic cosine defined as

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

The hyperbolic cosine function, by itself, does not yield new numbers from the exponential closure of a field, but it can be manipulated to do so. Since hyperbolic cosine is related to circular cosine by the observation that $\cosh(x) = \cos(ix)$ (or $\cos(x) = \cosh(ix)$), the manipulation of the $\mathscr{L}_{er}$-formula defining $\cosh(x)$ can yield new numbers just as the circular cosine function has. An example of this also shows how some numbers may become definable without their conventional building blocks being also definable.

A basic trigonometric property is that $\cos(x) = \cos(-x)$ and $\sin(x) = -\sin(-x)$. Since square roots of real numbers are themselves real with one being the additive inverse of the other, it is possible, for example, to define the number $\cos(\sqrt[4]{2})$, where $\sqrt[4]{2}$ is taken to be the principal square root of the principal square root of 2. Now, $\sqrt{2} \in \mathbb{Q}^{\mathrm{rab}}$ is definable and its principal square root $\sqrt[4]{2}$ is not. The set $\{\sqrt[4]{2}, -\sqrt[4]{2}\}$ is irreducible, but the number $\cos(\sqrt[4]{2})$ becomes definable without parameters by using the formula stating

$$\exists x \exists j (j^2 + 1 \doteq 0 \wedge x^2 - \sqrt{2} \doteq 0 \wedge y \doteq \frac{\exp(jx) + \exp(-jx)}{2}).$$

Also, since $\sin(x) = \cos(\frac{\pi}{2} - x)$, $\sin(\sqrt[4]{2})$ is also definable. This result can therefore be generalized as follows.

Proposition 9.39 Let $\mathbb{B}$ be the complex pseudoexponential field described using the language of exponential rings. Let $c \in \mathbb{B}$ be a number pointwise definable without parameters, so that $c \in \mathrm{dcl}(\emptyset) = \mathbb{Q}^{\mathrm{rab}\,C}$. Then, the numbers $\cos(\sqrt{c})$ and $\sin(\sqrt{c})$ are both definable without parameters and hence in $\mathbb{Q}^{\mathrm{rab}\,C}$.

Corollary 9.40 If $c \in \mathbb{B}$ is a number in $\mathbb{Q}^{\mathrm{rab}\,C}$, then $\cosh(c)$ is also in $\mathbb{Q}^{\mathrm{rab}\,C}$.

A similar extension to the hyperbolic sine function is not straightforwardly possible, although a result that fuses the circular and hyperbolic sine functions can be obtained.

Proposition 9.41 If $c \in \mathbb{B}$ is a nonzero number in $\mathbb{Q}^{\mathrm{rab}\,C}$, then the number

$$\frac{\exp(\sqrt{c}) - \exp(-\sqrt{c})}{2\sqrt{c}}$$

is definable and hence in $\mathbb{Q}^{\mathrm{rab}\,C}$.

The kind of numbers definable using the circular and hyperbolic trigonometric functions can be generalized as follows.

Theorem 9.42 Let $c \in \mathbb{B}$ be a number in $\mathbb{Q}^{\mathrm{rab}\,C}$. Let $n \in \mathbb{N}$ be an arbitrary prime number. Let $\sqrt[n]{c}_1, ..., \sqrt[n]{c}_n$ denote the system of nth roots of c. Then, the number

$$\exp(\sqrt[n]{c}_1) + \exp(\sqrt[n]{c}_2) + ... + \exp(\sqrt[n]{c}_n)$$

is definable.

Proof. The key is that, although the individual nth roots of the number c cannot distinguished from each other, such pointwise definability is not necessary. The only requirement needed is that the nth roots as provided are exhaustive and nonrepeating, so that there are exactly n nth roots of c. Taking the pseudoexponential image of each nth root and adding them up is therefore a definable operation. ○

Multiplying the pseudoexponential images of course just yields the number 1. Although the constituent images may be complex numbers, the sum is always a real number, so $\mathbb{Q}^{\mathrm{rab}\,C}$ is a real field.

In any case, whatever shape $\mathbb{Q}^{\mathrm{rab}\,C}$ may be, it is obvious that, without the language of exponentiation, the structures under discussion are very transcendental in the sense that the transcendental bases over $\mathbb{Q}^{\mathrm{rab}}$ contain multitudes of items that field-theoretically have no connection to the original real abelian number field. The exponential closure $\mathbb{Q}^{\mathrm{rab}\,E}$ has a straightforward transcendental basis over $\mathbb{Q}^{\mathrm{rab}}$ whose elements are obtained from iterations of the exponential function as shown in the construction of $\mathbb{Q}^{\mathrm{rab}\,E}$, but the wild definability of other kinds of transcendental numbers in $\mathbb{Q}^{\mathrm{rab}\,C}$ makes the description of its transcendental basis out of reach, at least for now. A full characterization of the basis may indeed be intractable, which perhaps makes the concise description of the algebraic numbers pointwise definable as being exactly the real abelian numbers an all-the-more remarkable result. In any case, it is clear that the change from pure fields to pseudoexponential fields is a significant one.

9.7 Abelian Extensions of the Closed Real Abelian Exponential Field

With the great expressivity of the language of exponential rings and the previous results about the pure field-theoretic abelian extensions of the real abelian number field $\mathbb{Q}^{\mathrm{rab}}$ in mind, some work on finding the abelian extensions of $\mathbb{Q}^{\mathrm{rab}\,C}$ may be appropriate. Clearly, a full analogue of the Kronecker-Weber Theorem is not in reach until more is known about the pseudoexponential understanding of algebraicity. It is possible to divide the simple algebraic extensions of the closed real abelian exponential field $\mathbb{Q}^{\mathrm{rab}\,C}$ into three general classes.

1. Those extensions generated by an individual field-theoretically algebraic over $\mathbb{Q}^{\mathrm{rab}}$

2. Those extensions generated by an individual field-theoretically algebraic over a transcendental number in $\mathbb{Q}^{\mathrm{rab}\,C} \setminus \mathbb{Q}^{\mathrm{rab}}$

3. Those extensions generated by an individual model-theoretically algebraic over $\mathbb{Q}^{\mathrm{rab}\,C}$ which is however field-theoretically transcendental over the same field

Each class of extensions pose certain issues that need to be addressed in order to coherently understand algebraic and abelian extensions of $\mathbb{Q}^{\mathrm{rab}\,C}$. Provisionally, some way of capturing these individuals which are model-theoretically algebraic but field-theoretically transcendental is needed.

Definition 9.43 (n-Irreducible Numbers) Let n be a natural number. Then, a number $c \in \mathbb{B}$ is n-irreducible without parameters if the irreducible formula for c yields a solution set with n individuals. Let A be a set of parameters. Then, $c \in \mathbb{B}$ is n-irreducible over A if the irreducible formula for c using parameters in A yields a solution set with n individuals.

The idea of n-irreducible numbers is intended to generalize the idea of algebraic numbers of degree n. The slight change is needed also since the host of new definable functions in the language of exponential rings can affect algebraic degree. If a is a number algebraic over $\mathbb{Q}^{\mathrm{rab}}$ of degree n even in the context of exponential fields, then a is an n-irreducible number; however, the idea of n-irreducible numbers will primarily be used to look at those individuals that would not be algebraic in the field-theoretic sense.

Let A and B be closed exponential fields contained in $\mathbb{B}$ such that $A \subseteq B$. If $B = \mathrm{dcl}(Ab)$ for some $b \in \mathrm{acl}(A)$, then it is properly the case that B is a finite algebraic extension of A and that b is an n-irreducible number for some n. Although B is definably closed, it is not necessarily the case that B is a normal extension of A. However, if b is 2-irreducible with $\{b, b'\}$ the relevant irreducible set, then having b as a parameter means that b' is also definable. Consequently, B would be a normal extension of A.

Proposition 9.44 Let $A = \mathrm{dcl}(A)$ and $B = \mathrm{dcl}(B)$ be closed exponential fields in $\mathbb{B}$ such that $A \subseteq B$ and B is an algebraic extension of A generated by b. Then, if b is 2-irreducible over A, then B is a normal extension of A of degree 2. Furthermore, $\mathrm{Gal}(B/A)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is therefore abelian.

Proof. The results are immediate from the provided conditions. The individual b being 2-irreducible means its conjugate b' is also definable using b as a parameter and hence $b' \in B$. The permutations allowed by the irreducible set would strictly be moving b to b' and b' to b, which is essentially like the case of square roots in the context of pure fields. ↻

Indeed, 2-irreducible numbers generate cyclic extensions of degree 2 which may not be cyclic extensions in the field-theoretic algebraic sense. Unfortunately, a similar conclusion cannot be made with other n-irreducible numbers, just as similar conclusions cannot be made with algebraic numbers of degree greater than 2. For those 2-irreducible numbers which are field-theoretically transcendental, these individuals generate abelian extensions which are properly not accessible in the context of pure fields.

Such an example includes using the hyperbolic sine function. Let $\{\sqrt[4]{2}, -\sqrt[4]{2}\}$ form the irreducible set for the square roots of $\sqrt{2}$. Then, $\{\sinh(\sqrt[4]{2}), \sinh(-\sqrt[4]{2})\}$ also forms an irreducible set over $\mathbb{Q}^{\text{rab}\,C}$. These latter numbers are clearly transcendental with respect to pure fields, yet with the inclusion of exponentiation, they become more accessible than many other field-theoretically algebraic numbers in the sense that their irreducible sets are smaller than the irreducible sets of such algebraic numbers.

Another example of 2-irreducible numbers is the following one which does not rely on square roots.

Example 9.45 Let a be an individual in $\mathbb{Q}^{\text{rab}\,C}$, and let b be the principal quartic root of a. Then, the numbers $\cos(b)$ and $\cosh(b)$ form an irreducible solution set to the formula

$$\phi(y) = \exists x(x^4 - a \doteq 0 \wedge y \doteq \frac{e^x + e^{-x}}{2})$$

and can be construed as 2-irreducible numbers.

Little else may be concluded about n-irreducible numbers and the algebraic extensions they generate. A more descriptive understanding of these extensions requires greater understanding of the expressive power of the language of exponential rings for the theory of pseudoexponentiation. In a way, the theory is very far away from model completeness in the sense that the only way to reach effective model completeness by quantifier elimination is to add function symbols for every existentially-defined function. Similarly, since the transcendental component of $\mathbb{Q}^{\text{rab}\,C}$ is largely unknown, it is not readily possible to examine algebraic extensions which are field-theoretically generated by individuals that are algebraic over $\mathbb{Q}^{\text{rab}\,C} \setminus \mathbb{Q}^{\text{rab}}$.

An important issue to address is also how exponentiation affects field-theoretic algebraicity. Clearly, this language change has a nontrivial impact. All real numbers algebraic over $\mathbb{Q}$ with degree 2, for example, are now definable, and accordingly, the relation between individuals algebraic over $\mathbb{Q}$ and the rational number field changes significantly. Since the prime structure has $\mathbb{Q}^{\text{rab}}$ as its algebraic subfield, it is more appropriate to frame the question by wondering what happens to numbers which are field-theoretically algebraic over $\mathbb{Q}^{\text{rab}}$ in the context of pseudoexponentiation.

Clearly, every number field-theoretically algebraic over $\mathbb{Q}^{\text{rab}}$ remains model-theoretically algebraic over $\mathbb{Q}^{\text{rab}}$. No such algebraic number not already in $\mathbb{Q}^{\text{rab}}$ becomes pointwise definable; otherwise, such a number would contradict the result that the algebraic subfield of the prime structure is exactly $\mathbb{Q}^{\text{rab}}$. Consequently, those numbers field-theoretically algebraic over $\mathbb{Q}^{\text{rab}}$ of degree 2 remain 2-irreducible.

Suppose a is field-theoretically algebraic over $\mathbb{Q}^{\mathrm{rab}}$ of degree 3 but is 2-irreducible. Let $\{a, a', a''\}$ be the field-theoretical irreducible set for a. If a is 2-irreducible, then one of a' or a'' must in fact be definable to obtain an model-theoretically irreducible set for a with two individuals. Such an outcome contradicts the conclusion reached about algebraic numbers not in $\mathbb{Q}^{\mathrm{rab}}$ being pointwise definable. Therefore, algebraic numbers of degree 3 over $\mathbb{Q}^{\mathrm{rab}}$ are also 3-irreducible.

Tackling if algebraic numbers field-theoretically of higher degree n must be n-irreducible cannot be directly addressed in the same manner as with algebraic numbers of degree 2 and 3. Nevertheless, the following conjecture holds considerable appeal.

Conjecture 9.46 Let a be a number in $\mathbb{Q}^{\mathrm{alg}}$ which is algebraic over $\mathbb{Q}^{\mathrm{rab}}$ of degree n. Then, a is an n-irreducible number over $\mathbb{Q}^{\mathrm{rab}\,C}$.

Partial support for this conjecture may be obtained later when some examination is done of partial exponential fields. If this conjecture holds, then the following proposition would also hold.

Proposition 9.47 Let $\mathbb{Q}^{\mathrm{rab}\,C}$ denote the closed real abelian pseudoexponential field, and let $(\mathbb{Q}^{\mathrm{rab}\,C})^{\mathfrak{ab}}$ denote the model-theoretically maximal abelian extension of $\mathbb{Q}^{\mathrm{rab}\,C}$ in $\mathbb{B}$. Suppose Conjecture 9.46 holds. Then, the logical Galois group

$$\mathrm{Gal}((\mathbb{Q}^{\mathrm{rab}\,C})^{\mathfrak{ab}}/\mathbb{Q}^{\mathrm{rab}\,C})$$

has the field-theoretic Galois group

$$\mathrm{Gal}((\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{rab}}) = 2^{\omega} \times \hat{\mathbb{Z}}^{\omega}$$

as a quotient group if every individual field-theoretically algebraic over $\mathbb{Q}^{\mathrm{rab}}$ with degree n is also n-irreducible over $\mathbb{Q}^{\mathrm{rab}\,C}$.

Unfortunately, without more elucidation about algebraicity, this result may be the only feasible one to obtain.

9.8 Partial Exponential Fields

The preceding work made a jump from pure fields to closed exponential fields without perhaps taking an appropriate pause to consider intermediate structures that rest between fields without any exponentiation and fields with full exponentiation. These structures are partial exponential fields, where the exponential function is defined on a portion of the underlying field. The use of partial exponential fields has been integral to previous work that has been done with respect to pseudoexponentiation, and it would be negligent to ignore them in this work.

Definition 9.48 (Partial Exponential Fields) Let F be a field of characteristic 0. Then, a partial exponential field F^{PE} over F is a two-sorted structure $\langle D(F), F, \mathrm{id}, \exp\rangle$ such that $D(F) \subseteq F$ is a $\mathbb{Q}$-linear subspace of F, $\mathrm{id} : D(F) \to F$ is the identity map such that $x \in D(F) \mapsto x \in F$, and $\exp : D(F) \to F$ is a function such that $\exp(0) = 1$ and $\exp(x+y) = \exp(x)\exp(y)$ for any x and y in $D(F)$.

Definition 9.49 (Partial Pseudoexponential Fields) A partial exponential field $\langle D(F), F, \mathrm{id}, \exp\rangle$ is a partial pseudoexponential field if the function exp is the one induced by the pseudoexponential function of $\mathbb{B}$.

The following gloss permits the use of partial exponential fields to formally construct exponential closures where the exponential map is the one induced by the complex pseudoexponential field. Let F be an arbitrary field resident in the complex number field $\mathbb{C}$. Let $E(F)$ denote the set $\{\exp(a) : a \in F\}$. The partial exponential field $F^{PE_1} := \langle F; F(E(F)); \mathrm{id}; \exp\rangle$ is generated by taking up F as the domain of the partial pseudoexponential map. Take then $F^{PE_2} := \langle F(E(F)); F(E(F))(E(F(E(F)))); \mathrm{id}; \exp\rangle$, and define F^{PE_i} for each $i \in \mathbb{N}$ in the same manner. Then, the set $\{F^{PE_i} : i \in \mathbb{N}\}$ forms a chain wherein the field sort of each such partial pseudoexponential field embeds into the field sort of its successor, with

$$F \hookrightarrow F^{PE_1} \hookrightarrow F^{PE_2} \hookrightarrow F^{PE_3} \hookrightarrow \ldots.$$

where each F^{PE_i} ($i \in \mathbb{N}$) here refers to the field sort of the respective partial pseudoexponential field. The union

$$\bigcup_{i \in \mathbb{N}} F^{PE_i}$$

then forms the pseudoexponential-based exponential closure of F, which is denoted F^E.

So, the pseudoexponential field $\mathbb{Q}^{\mathrm{rab}\,E}$ is closed with respect to addition, multiplication, and exponentiation. Each operator has a dual that is also familiar in general mathematical usage, namely subtraction, division, and logarithms, and these counterparts lose some of the properties held by the original operations. For example, whereas addition and multiplication are commutative, subtraction and division are clearly not. Since subtraction and division are based on the inverses obtained from addition and multiplication through their respective identity elements, it follows that groups are structures that are closed with respect to addition and subtraction or multiplication and division, as opposed to monoids that are closed only with respect to addition or multiplication, and fields are structures closed with respect to addition, multiplication, subtraction, and division, in contrast to unital commutative rings which are not closed with respect to division.

In like manner, the pseudoexponential field $\mathbb{Q}^{\mathrm{rab}\,E}$ is closed with respect to exponentiation but not with respect to logarithms. When considered with definable individuals, though, this lack of closure is not a problem, since the principal logarithm function is not definable, and the best that can be achieved in finding logarithms is in defining the set of all branches of the logarithm. Thus, $\ln(2)$ is not definable over $\mathbb{Q}^{\mathrm{rab}\,E}$, but the set $\ln(2)+$

$2\pi i\mathbb{Z}$ is definable. Consequently, it is not necessary to look into logarithmic closures of fields, which would be called L-fields, or EL-fields that are closed with respect to both exponentiation and logarithms. For the sake of completeness, the term ELA-field is also used when the underlying EL-field is also algebraically closed.

9.9 Working with Partial Exponential Fields

One of the reasons that the general model-theoretic Galois correspondence of structures is developed over a large and saturated model of some theory is that such an entity is guaranteed to be full of the necessary automorphisms and partial elementary maps needed in order to populate the Galois groups that form the correspondence with that model's substructures. A theory which is model-complete has the added benefit that the choice of the large model does not affect what happens in submodels which may be more familiar in general mathematical practice. Although the theory of pseudoexponentiation does not have model completeness, it turns out that partial exponential fields can be used to obtain all of the necessary automorphisms needed to properly speak of a Galois theory of pseudoexponential fields.

The complex pseudoexponential field in the language of exponential rings is altered into a two-sorted structure $\langle \mathbb{C}^+; \mathbb{C}; \mathrm{id}; \exp \rangle$ consisting of two copies of the complex number field $\mathbb{C}$, the first sort having the additive group structure in the language of torsion-free divisible abelian groups as $\mathbb{Q}$-linear spaces and the second sort having the full field structure in the language of rings, and two homomorphisms between the two. The first morphism is the pseudoexponential function exp induced by the complex pseudoexponential field $\mathbb{B}$; the second is the identity map that identifies individuals in the first sort with themselves in the second sort.

Pseudoexponential field automorphisms can be constructed, or at least their existence can be demonstrated, by defining fragments that are defined on partial pseudoexponential fields which are then extended full pseudoexponential fields. This procedure will be outlined shortly, but it may be beneficial to mull over the substructures of the two-sorted complex pseudoexponential field. Let $\mathbb{B}_2$ denote the structure $\langle \mathbb{C}^+; \mathbb{C}; \mathrm{id}; \exp \rangle$. Substructures of $\mathbb{B}_2$ will then be partial pseudoexponential fields.

Definably closed substructures of $\mathbb{B}_2$ are straightforwardly obtained from definably closed substructures of $\mathbb{B}$.

Proposition 9.50 Let $\mathfrak{A}$ be a substructure of $\mathbb{B}_2$. Then, $\mathfrak{A}$ is definably closed, so that $\mathfrak{A} = \mathrm{dcl}(\mathfrak{A})$, if and only if $\mathfrak{A}$ is of the form $\langle F^+; F; \mathrm{id}; \exp \rangle$ where F^+ is the additive group reduct of F and F when extended to accommodate the pseudoexponential function exp is a definably closed substructure of $\mathbb{B}$.

Proof. If $\mathfrak{A} := \langle D(A); A; \mathrm{id}; \exp \rangle$ is a substructure definably closed, then its constituent sorts ought to be definably closed. Consequently, A must be definably closed as a field and the additive group $D(A) \subseteq A$ ought to be closed as a $\mathbb{Q}$-linear space. The identity map between the sorts and the pseudoexponential function affect definable closure; the identity map can be used to uniquely define individuals in the second sort into the first sort, so to respect this ability, $D(A)$ must be the additive group reduct of A itself, so $D(A) = A^+$. The pseudoexponential function then demands that A be exponentially closed, since every individual in A is in $D(A)$. Any formula in the language of exponential rings can be converted into a formula in the two-sorted language, so the trigonometric functions and π are in particular definable, and so would any other object definable in the one-sorted language be definable in the two-sorted language. Thus, A, when considered as a pseudoexponential field, needs to be a definably closed substructure of the one-sorted complex pseudoexponential field.

The converse claim is straightforward. If F is a definably closed substructure of the complex pseudoexponential field, then it is exponentially closed, so whatever $D(F)$ might be, its exponential image will be in F. So, if $D(F)$ is the additive group reduct F^+, F will nonetheless have every individual that exp maps to from $D(F)$. ⟲

If F is an arbitrary field, let F^{PE} denote a partial pseudoexponential field over F such that F is the second sort. This F^{PE} is clearly not a well-defined notion, since every nontrivial field F will have many partial pseudoexponential fields. Rather, this convention will be a device of convenience when considering different partial pseudoexponential fields that can be distinguished by not sharing the same second field sort.

Corollary 9.51 The smallest definably closed substructure $\mathrm{dcl}(\emptyset)$ of the two-sorted complex pseudoexponential field $\mathbb{B}_2$ is the structure obtained from the definably closed substructure of the one-sorted pseudoexponential field $\mathbb{B}$ such that $\mathrm{dcl}(\emptyset) = \langle \mathbb{Q}^{\mathrm{rab}\,C\,+}; \mathbb{Q}^{\mathrm{rab}\,C}; \mathrm{id}; \exp\rangle$.

In addition, it is also convenient to outline the idea of generating a partial pseudoexponential field. A set A generates a partial pseudoexponential field F^{PE} if $A \cap D(F)$ is a linear span of $D(F)$ and F is generated as a field by $A \cup \exp(D(F))$.

As far as the model-theoretic framework of Galois theory is concerned, there is no major difference between working in the one-sorted complex pseudoexponential field or its two-sorted counterpart. The partial pseudoexponential fields provide a fertile ground for producing automorphisms which would otherwise be difficult to create in the one-sorted case. Although the model-theoretic framework of Galois theory justifies a Galois correspondence between definably closed substructures, it is possible to look at partial pseudoexponential fields outside of this slight limitation and consider such fields which have properly partial pseudoexponential functions.

Definition 9.52 Let F^{PE} and K^{PE} be two partial pseudoexponential fields with $F \subseteq K$ and $D(F) \subseteq D(K)$. Then, K^{PE} is a finite algebraic extension of F^{PE} if K is a finite algebraic extension of F as pure fields.

Proposition 9.53 Let K^{PE} be a finite algebraic extension of the partial pseudoexponential field F^{PE}. Then, the partial pseudoexponential field-theoretic Galois group $\text{Gal}(K^{PE}/F^{PE})$ is characterized by the pure field-theoretic Galois group $\text{Gal}(K/F)$ such that $\text{Gal}(K^{PE}/F^{PE}) \cong \text{Gal}(K/F)$.

Proof. The key to this result is the identity map that identifies individuals in the first sort into the second sort. Thus, even though the first sort is only a $\mathbb{Q}$-linear space, with the associated constrained language, the identity map allows for it to be described using the full language of rings. This impact is greater than that provided by the exponential map, which also maps individuals in the first sort into the second sort. The identity map imposes two conditions upon the first sort that the exponential map cannot.

1. The identity map embeds the first sort into the second sort, so that the first sort $D(F)$ must be a subset of the second sort F.
2. The identity map is an additive homomorphism, mapping vector-addition operator in the first sort to the addition operator of the second sort.
3. Whether or not the identity map is onto, it is certainly one-to-one, so no ambiguity arises about the individuals involved with the identity map.

The exponential map also maps individuals from the first sort into the second sort, but it maps the vector-addition operator of the first sort to the multiplication operator of the second sort and maps 0 to 1. In essence, the exponential map places a less-faithful copy of the first sort into the second sort in the examples that have been considered. Also, the exponential map in the full structure $\langle \mathbb{C}^+, \mathbb{C}, \text{id}, \exp \rangle$ is clearly not one-to-one, since $\exp(2\pi i \mathbb{Z}) = 1$. So long as the identity map is present, however, the stock of potential automorphisms must be limited, fully respecting the connection maintained by that map between the individuals of the first sort with the individuals of the second sort. Thus, the situation concerning K^{PE} and F^{PE} yields to the situation concerning K and F. ↻

Eventually, it may be worthwhile to look at these two-sorted structures where the only function permitted between the first sort and the second sort is the exponential map. This view is the subject of a subsequent section of this work. Continuing on with the current context, though, partial exponential fields and partial pseudoexponential fields facilitate the ability to construct exponential or pseudoexponential field automorphisms from pure field automorphisms by gradually building them from subfields and controlling the behavior of the exponential function. An example of such action can be found in proving the converse claim of Theorem 9.13; a summary of the work in [Kirby Macintyre Onshuus 2011] follows.

Two partial exponential and pseudoexponential fields of note are the rational $\mathbb{Q}_0 := \langle \{0\}; \mathbb{Q}; \text{id}; \exp \rangle$ and the standard kernel $SK := \langle 2\pi i \mathbb{Q}; \mathbb{Q}^{ab}(2\pi i); \text{id}; \exp \rangle$ partial exponential fields. Given a partial exponential field F^{PE}

and an extension K^{PE}, an embedding of F^{PE} into K^{PE} is a map $\phi : F^{PE} \to K^{PE}$ such that $\exp_F(a) = b$ implies $\exp_K(\phi(a)) = \phi(b)$. A *strong embedding* is one where for every finite set $X \subseteq K$ the property

$$\mathrm{ldim}_{\mathbb{Q}}(X/D(F)) \leq \mathrm{tr.deg}(X, \exp(X)/F)$$

is affirmed; in that case, the relation between F^{PE} and K^{PE} can be denoted $F^{PE} \lhd K^{PE}$. The Schanuel Property of the complex pseudoexponential field is then equivalent to having $\mathbb{Q}_0 \lhd \mathbb{B}_2$, which also leads to $SK \lhd \mathbb{B}_2$.

Let F^{PE} be a partial exponential field with $X \subseteq F$ a finite set thereof. Then, X is said to *finitely generate* F^{PE}, and F^{PE} is a *finitely generated* partial exponential field, if $X \cap D(F)$ spans $D(F)$, where X is identified with its preimage of the identity map, and F is generated as a field by $X \cup \exp(D(F))$. As an example, the set $X = \{2\pi i\}$ finitely generates the partial exponential field SK. The complex pseudoexponential field also satisfies the following property due to the model-theoretic properties of pseudoexponentiation, particularly with respect to quasiminimal excellence.

Proposition 9.54 (Kirby) Let $\mathfrak{B}$ be a model of the theory of pseudoexponentiation. Let F^{PE} be a finitely generated partial exponential subfield containing SK which is finitely generated such that $F^{PE} \lhd \mathfrak{B}$. Then, any automorphism of F^{PE} can be extended to an automorphism of $\mathfrak{B}$.

Corollary 9.55 Any automorphism of such F^{PE} can therefore be extended to a pseudoexponential field automorphism on the complex pseudoexponential field $\mathbb{B}$.

The automorphism σ_1 of complex conjugation for $\mathbb{Q}^{\mathrm{ab}}$ defined by $\sigma_1(\zeta_i) = \zeta_i^{-1}$ for every non-real root of unity ζ_i can be expanded to the automorphism $\sigma_2(2\pi i) = -2\pi i$. This automorphism σ_2 is an automorphism of SK, so it can be extended to an automorphism of any pseudoexponential field. Any nonreal abelian number in $\mathbb{Q}^{\mathrm{ab}}$ is permuted by σ_0, so every nonreal abelian number is also permuted by σ.

The extension of σ_2 to the complex pseudoexponential field can be mediated to show that any algebraic number which is not a real abelian number can be permuted. Let a be a number algebraic in $\mathbb{Q}^{\mathrm{alg}} \setminus \mathbb{Q}^{\mathrm{ab}}$. Then, the partial exponential field $SK(a)$ defined as

$$SK(a) := \langle 2\pi i\mathbb{Q}; \mathbb{Q}^{\mathrm{ab}}(2\pi i)(a); \mathrm{id}; \exp\rangle$$

leaves the domain of exponentiation untouched but extends the field sort so that $\mathbb{Q}^{\mathrm{ab}}(2\pi i)(a)$ is algebraic over $\mathbb{Q}^{\mathrm{ab}}(2\pi i)$. Since $\mathbb{Q}^{\mathrm{rab}}(a)$ has a field automorphism over $\mathbb{Q}^{\mathrm{rab}}$ permuting a, it is also the case that there is an automorphism of $\mathbb{Q}^{\mathrm{ab}}(2\pi i)(a)$ permuting a. This automorphism σ_3 can be defined such that it extends σ_2 by continuing to fix $\mathbb{Q}^{\mathrm{rab}}$. The partial exponential field $SK(a)$ is finitely generated by the set $\{2\pi i, a\}$; the domain of exponentiation is clearly spanned by $2\pi i$ while a plays no role in that sort, whereas the field sort is clearly generated in the field-theoretic sense according to the definition of being finitely generated.

Furthermore, the conditions for Proposition 9.54 can be satisfied by the observation that $SK \triangleleft \mathbb{B}$ implies $SK(a) \triangleleft \mathbb{B}$. Suppose not, and let $X \cup \{y\}$ be such a set such that

$$\mathrm{ldim}_{\mathbb{Q}}(X \cup \{y\}/D(SK(a))) > \mathrm{tr.deg}(X \cup \{y\}, \exp(X \cup \{y\})/SK(a)) \tag{9.1}$$

and

$$\mathrm{ldim}_{\mathbb{Q}}(X/D(SK(a))) \leq \mathrm{tr.deg}(X, \exp(X)/SK(a)).$$

For any possibility of this situation to occur, the linear dimension of X over $D(SK(a))$ must be equal to the transcendence degree. Then, y must increase the linear dimension without affecting transcendence degree, which implies that y and $\exp(y)$ are both algebraic over $SK(a)$ adjoined with the individuals in X and $\exp(X)$. Since there are finitely many of these components that are involved over $SK(a)$, and $\mathbb{Q}^{\mathrm{ab}}(2\pi i)(a)$ itself is finitely algebraic over $\mathbb{Q}^{\mathrm{ab}}(2\pi i)$, the condition 9.1 implies refutation of the Schanuel Property for the pseudoexponential function exp. Thus, 9.1 cannot occur in this situation.

Thus, the automorphism σ_3 which is an extension of σ_2 based on σ_1 can then be extended to an automorphism σ of the complex pseudoexponential field. It permutes any algebraic number which is not a real abelian number. Thus, $\mathbb{Q}^{\mathrm{rab}}$ constitutes the algebraic subfield of the fixed pseudoexponential field, demonstrating that all of Theorem 9.13 holds.

Now, this construction can be used to partially address the question of whether field-theoretic algebraicity of degree n implies being n-irreducible in the model-theoretic sense, the subject of Conjecture 9.46. The algebraic number a in the preceding proof can be arbitrary, so let a indeed be a number such that $\mathbb{Q}^{\mathrm{rab}}(a)$ is a cyclic extension of degree n. Let ξ denote a generating automorphism of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{rab}}(a)/\mathbb{Q}^{\mathrm{rab}})$. The cyclic extension $\mathbb{Q}^{\mathrm{rab}}(a)$ can be extended to $\mathbb{Q}^{\mathrm{ab}}(a)$ by adjoining the square root of -1, and if τ is the automorphism of complex conjugation for $\mathbb{Q}^{\mathrm{ab}}$ over $\mathbb{Q}^{\mathrm{rab}}$, then ξ and τ generate together the group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}(a)/\mathbb{Q}^{\mathrm{rab}})$. Each automorphism in this group can be extended to the whole complex pseudoexponential field; in particular, it preserves each automorphism in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}(a)/\mathbb{Q}^{\mathrm{rab}})$ from each other. Consequently, each automorphism generated by ξ alone can be extended to the complex pseudoexponential field in distinction to the other automorphisms in the cyclic group. Therefore, a can be permuted to n possible places in the pseudoexponential context in the same way in can be permuted to n possible places in the pure field-theoretic sense. So, a must be n-irreducible, resolving the Conjecture 9.46 with respective to algebraic numbers which generate cyclic extensions in the affirmative.

The strength of Proposition 9.54 means that it can be used to construct pseudoexponential field automorphisms recognized sufficiently enough to address this question about algebraicity and n-irreducibility with respect to abelian extensions. The automorphism which extends, for example, the generator ξ of the cyclic group to the entire complex pseudoexponential field must necessarily define itself completely over $\mathbb{B}$; in particular, it defines its behavior upon the complex numbers as the domain of the pseudoexponential function. Since this automorphism is now not just a field automorphism but also a pseudoexponential field automorphism, it must respect all functions definable in the language of exponential rings. Thus, this automorphism fixes all definable numbers in the closed real abelian pseudoexponential field and can be restricted so that its domain is strictly the definable closure of the partial exponential field involved. The definable closure of a partial exponential field is itself a fully defined exponential field and can be regarded in the single-sorted language of exponential rings. Thus, these automorphisms can be used to find the shape of pseudoexponential field automorphisms needed for the Galois theory of pseudoexponential fields.

Proposition 9.56 Let a be a number which is algebraic over the field $\mathbb{Q}^{\mathrm{rab}}$ of real abelian numbers with degree n such that $\mathbb{Q}^{\mathrm{rab}}(a)$ is a cyclic extension of $\mathbb{Q}^{\mathrm{rab}}$. Then, a is an n-irreducible number over the closed real abelian pseudoexponential field $\mathbb{Q}^{\mathrm{rab}\,C}$.

Corollary 9.57 Let F be an abelian extension of $\mathbb{Q}^{\mathrm{rab}}$. Then, the closed exponential field $\mathrm{dcl}(F)$ is an abelian extension of $\mathbb{Q}^{\mathrm{rab}\,C}$. Furthermore, $\mathrm{dcl}((\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}})$ is contained in the maximal abelian extension $(\mathbb{Q}^{\mathrm{rab}\,C})^{\mathfrak{ab}}$ in the model-theoretic pseudoexponential sense of $\mathbb{Q}^{\mathrm{rab}\,C}$.

Corollary 9.58 The field-theoretic group $\mathrm{Gal}((\mathbb{Q}^{\mathrm{rab}})^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{rab}})$ is a quotient group of the group

$$\mathrm{Gal}((\mathbb{Q}^{\mathrm{rab}\,C})^{\mathfrak{ab}}/\mathbb{Q}^{\mathrm{rab}\,C})$$

as understood in the sense of exponential fields.

9.10 A Relatively Benign Exercise

Partial exponential fields have been very useful in working with exponential fields that may not be amenable to other model-theoretic tools. Now, the multisorted approach taken here need not be the only way to present partial exponential fields. An alternative method would be to retain one field sort which is augmented by the addition of a relation symbol R such that $R(a,b)$ means $\exp(a)=b$ for the concept of partial exponentiation.

The multisorted approach having been taken, the identity map between the sorts is the glue which binds them together which highlights then the particular properties of exponentiation as a group homomorphism. Both maps play key parts in this concept of partial exponential fields. Without the exponential function, the purpose of the first sort would be lost. Without the identity map, the identifying connection between the two sorts becomes lost, with the only way of communicating between the two structures being through the exponential map.

Nevertheless, consider the structure which is the reduct of a partial exponential field where the identity map between sorts is removed. Such structures have been discussed in previous literature, among them [Kirby 2007] and [Zilber 2000]. In both works, two-sorted exponentiation is not discussed with great detail, and the only discourse involving this kind of structure is with the correct formulation of Schanuel's conjecture in such a setting.

Conjecture 9.59 (Schanuel) Let $\langle \mathbb{C}^+; \mathbb{C}; \exp \rangle$ be a two-sorted structure where $\mathbb{C}^+$ denotes the additive group of the complex number field $\mathbb{C}$ and exp is a group homomorphism from the first sort onto the second induced by the canonical exponential map of the complex exponential field $\mathbb{C}_{\exp}$. Let $\alpha_1, \ldots, \alpha_n$ be numbers in $\mathbb{C}^+$ linearly independent over $\mathbb{Q}$. Then,

$$\operatorname{tr.deg}(\alpha_1, \ldots, \alpha_n / \mathbb{Q}^+) + \operatorname{tr.deg}(\exp(\alpha_1), \ldots, \exp(\alpha_n) / \mathbb{Q}) \geq n.$$

As a relatively benign exercise, it may be interesting to look at $\langle \mathbb{C}^+; \mathbb{C}; \exp \rangle$ and consider a Galois theory of its substructures. This enterprise turns out to be surprisingly straightforward with little hint of the complexities that do arise in the language of full exponentiation that has plagued the previous work.

Perhaps the most significant difference in this picture is the inability to identify individuals in one sort to individuals in the other sort. A major consequence is also the inability to nest the exponential function into iterated applications. Without being able to identify individuals between the two sorts, for example, it is not possible to define the circular trigonometric functions, and the expressivity of the hyperbolic trigonometric functions become severely limited. The loss of iterated exponential terms also limits the stock of transcendental numbers that populate the exponential closure of a given field.

Part of the descriptive loss is in part due to the limited language of the first sort, which is only given the structure of a torsion-free divisible abelian group. Overall, the language of this restricted partial exponentiation, which will be denominated as such and denoted $\mathscr{L}_{rper}$, accommodates a modest expansion from the language of rings. In addition to polynomial equations which can be expressed through the field sort, it is also possible to express sums and products in that sort which contain exponential outputs of linear combinations obtained from the group sort.

To be comprehensive, the language $\mathscr{L}_{rper}$ of restricted partial exponentiation consists of addition $+_1$ in the first sort with constant symbol for 0_1, addition $+_2$ and multiplication $\times_2$ with 0_2 and 1_2 for the second sort, and a function symbol exp mapping individuals of the first sort to individuals in the second sort. The theory of $\langle \mathbb{C}^+; \mathbb{C}; \exp \rangle$ as described with the language $\mathscr{L}_{rper}$ then describes $\mathbb{C}^+$ to be a torsion-free divisible abelian group, $\mathbb{C}$ as an algebraically closed field of characteristic 0, and exp a group homomorphism that maps addition of the first sort onto multiplication of the second sort such that $\exp(0) = 1$.

Essentially, $\mathrm{Th}(\langle \mathbb{C}^+; \mathbb{C}; \exp\rangle)$ is described as a combination of two well-behaved first-order theories, and a concise description of the theory as

$$\mathrm{Th}(\langle \mathbb{C}^+; \mathbb{C}; \exp\rangle) = (\mathrm{TFDAG}; \mathrm{ACF}_0; \exp(0) = 1, \forall x \forall y \exp(x+y) = \exp(x) \times \exp(y), \forall y \exists x \exp(x) = y)$$

lays out how the structure behaves. The structure, while restricted significantly from the complex exponential field or the complex pseudoexponential field, arises also as exponentiation acting as the universal covering map for the multiplicative group reduct of the complex number field.

Now, both $\mathrm{Th}(\mathbb{C}_{\exp})$ and $\mathrm{PSE} = \mathrm{Th}(\mathbb{B})$ lose many properties held by ACF_0. In contrast, the respective sorts of $\langle \mathbb{C}^+; \mathbb{C}; \exp\rangle$ are both stable and indeed strongly minimal, and the exponential map is rather benign such that the overall theory $\mathrm{Th}(\langle \mathbb{C}^+; \mathbb{C}; \exp\rangle)$ is stable and indeed superstable. Quantifier elimination is obtained if the rational scalars are added into the language, so the theory is also model complete, a key property missing in the full exponential theories.

Since both TFDAG and ACF_0 are uncountably categorical, it is arguably the case that $\langle \mathbb{C}^+; \mathbb{C}; \exp\rangle$ is the only such structure of its cardinality. Nonstandard kernels are still possible, so the same cannot be exactly said of the combined theory. Although the kernel of the exponential map is clearly a definable set (using the formula $\exp(x) \doteq 1$), the set of integers by themselves is still not quite recoverable. The damage of Incompleteness cannot happen since at best the first sort can only yield Presburger arithmetic. Adding an $\mathscr{L}_{\omega_1,\omega}$-formula specifying the exact size of the kernel prevents nonstandard models based on nonstandard kernels, and the theory augmented by such an axiom would make the combined theory uncountably categorical. Elimination of imaginaries or the coding of finite sets is not possible in TFDAG, so a Galois theory of such groups is not possible. Nevertheless, every individual of the first sort maps to an individual of the second sort, and the second sort is capable of coding finite sets, the same issue does not arise in the restricted partial exponential case.

Although TFDAG and ACF_0 are both strongly minimal, the same is not true for $\mathrm{Th}(\langle \mathbb{C}^+; \mathbb{C}; \exp\rangle)$ since the kernel is an infinite coinfinite definable set. The theory, however, is clearly quasiminimal, and the only time minimality is violated must be through the use of the exponential map.

Proposition 9.60 Let X be a set in one sort defined by some $\mathscr{L}_{rper}$-formula $\phi(x)$ in $\langle \mathbb{C}^+; \mathbb{C}; \exp\rangle$. Then, if X is not finite and it is not cofinite, then X is a countably infinite set, or the set defined by $\neg\phi(x)$ is countably infinite. Furthermore, if X is countably infinite or its relative complement is countably infinite, then the function symbol exp occurs in $\phi(x)$.

Proof. If $\phi(x)$ is a formula that does not include an occurrence of exp, then each atomic formula in $\phi(x)$ handles only one sort, so the minimality of each sort limits the conditions on the cardinality of the corresponding definable set. Quantifier elimination means that the definable set of $\phi(x)$ is the Boolean combination of the definable sets of quantifier-free formulae, so $\phi(x)$ can only define a finite or a cofinite set. If $\phi(x)$ contains occurrences of exp, then at worst each occurrence might help define a countably infinite set with a very straightforward correspondence to the set of integers. Since $\phi(x)$ can only have finitely many occurrences of exp, $\phi(x)$ can only be at worst a finitary combination of countable and cocountable sets, so the set X defined by $\phi(x)$ is either countable or cocountable. ↻

Substructures of $\langle \mathbb{C}^+; \mathbb{C}; \exp \rangle$ are altered variants of partial exponential fields. Because the identity map no longer embeds individuals of the additive group into the field, the two sorts are more detached from each other. Definably closed substructures in this context hence look very different from the previous two-sorted case, wherein such definably closed substructures were exactly closed pseudoexponential fields which are definably closed substructures of the complex pseudoexponential field.

For example, the structure $\mathbb{Q}_0 = \langle \{0\}; \mathbb{Q}; \exp \rangle$ is definably closed. Without the identity map, it is not possible to define every individual of the rational field into the domain of the exponential function, and the principal logarithm is not a definable function. Since 0 maps to 1, the group sort does not have to affect the field sort, so the rationals do not need expansion as in the previous case. Indeed, each pure field F gives rise to a definably closed substructure.

Proposition 9.61 Let F be a field of characteristic 0 contained in the complex number field $\mathbb{C}$. Then, the structure $\langle \{0\}; F; \exp \rangle$ is definably closed in the context of $\langle \mathbb{C}^+; \mathbb{C}; \exp \rangle$.

Formulae of the second sort are exactly the formulae obtained from the language of rings concerning fields, so if A and B with $A \subseteq B$ are substructures of $\langle \mathbb{C}^+; \mathbb{C}; \exp \rangle$ with trivial group sorts, then B is an algebraic extension of A if and only if the field sort of B is an algebraic extension of the field sort of A. In this case, the Galois theory would become exactly like the Galois theory of pure fields; groups of automorphisms are determined exactly by the behavior of the field sorts, since nothing is happening literally in the group sorts.

The kernel $2\pi i\mathbb{Z}$ of the exponential map would in itself not affect the size of the field sort, since by definition they all map to 1 through the exponential map. However, since the rational scalars are definable functions, if the group sort contains any nontrivial individual in the kernel, then the set $2\pi i\mathbb{Q}$ must also be included, so in such a case the field must include $\exp(2\pi i\mathbb{Q})$, which is the set of all roots of unity. Consequently, the field sort for structures with nontrivial kernels must contain the field $\mathbb{Q}^{\mathrm{ab}}$ of abelian numbers. The structure $\langle 2\pi i\mathbb{Q}; \mathbb{Q}^{\mathrm{ab}}; \exp \rangle$ then becomes a prime structure for such structures.

Some examples of the kinds of formulae expressible in this language setup may be helpful. The base substructure is $\mathbb{Q}_0 = \langle \{0\}; \mathbb{Q}; \exp \rangle$. The individual variables x and y will usually vary over the field sort, while the individual variable t will vary over the group sort, although the exact usage of the variables should be evident in the actual formulae. The sorts of named individuals are indicated by the subscripted index.

Example 9.62 Suppose $\phi(x)$ is the formula $x^2 \doteq_2 \dot{2}_2$. Then, $\phi(x)$ clearly expresses the polynomial equation $x^2 - 2 = 0$ in the second sort, so $\phi(\langle\mathbb{C}^+;\mathbb{C};\exp\rangle) = \left\{\sqrt{2}_2, -\sqrt{2}_2\right\}$. Then, the substructure

$$\mathfrak{B} := \langle\{0\};\mathbb{Q}(\sqrt{2});\exp\rangle$$

is an extension of $\mathbb{Q}_0$ generated by the finite sequence $b := (\sqrt{2}_2)$. The formula $\phi(x)$ is algebraic, so $\mathfrak{B}$ is a finite algebraic extension of $\mathbb{Q}_0$ in the logical sense. The group $\mathrm{Gal}(\mathfrak{B}/\mathbb{Q}_0)$ is then the same as the pure field-theoretic Galois group of $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$.

Example 9.63 Suppose $\phi(x)$ is the formula $t + 0_1 \doteq_1 t$. Then, $\phi(x)$ is merely claiming a property true in any monoid, and in fact $\phi(\langle\mathbb{C}^+;\mathbb{C};\exp\rangle) = \mathbb{C}^+$. Clearly, $\phi(x)$ is not algebraic, so $\phi(x)$ cannot generate any finite algebraic extension over $\mathbb{Q}_0$.

Example 9.64 Suppose $\phi(x)$ is the formula $\exp(t) \doteq_2 \dot{2}_2$. Then, the solution for $\phi(x)$ is a logarithm of 2, so $\phi(\langle\mathbb{C}^+;\mathbb{C};\exp\rangle) = \{\ln(2) \pm 2\pi i\mathbb{Z}\} \subseteq \mathbb{C}^+$. So, a substructure $\mathfrak{B}$ generated by $b := (\ln(2)_1)$ in some fashion is an extension of $\mathbb{Q}_0$. Since the solution set of $\phi(x)$ is not finite, the formula is not algebraic, so $\mathfrak{B}$ is not strictly a finite algebraic extension. However, if the first sort were to divide out the kernel $(2\pi i)\mathbb{Z}$ of the exponential map, then the solution would in fact be unique.

This last example is vague about the composition of $\mathfrak{B}$. A construction procedure for $\mathfrak{B}$ can resolve this problem. Clearly, $\mathfrak{B}$ must solve $\phi(x)$ in its first sort; let this solution be $\ln(2)$, the principal logarithm of 2. For $\mathfrak{B}$ to be a substructure of any coherent use it needs to be definably closed over itself. Multiplication by rational scalars is definable in the first sort, so the group sort must take the form $\ln(2)\mathbb{Q}$. This expansion of the first sort must be accompanied by a similar expansion in the second sort. For individuals contained in $\ln(2)\mathbb{Z}$, it is clear that the exponential function maps them to rational numbers, so no expansion is needed for these items. The other rational products do force a change. Take the number $\frac{1}{2}\ln(2)$ as a problematic example. It is necessary to include $\exp(\frac{1}{2}\ln(2))$ in the second sort; $\exp(\frac{1}{2}\ln(2)) = 2^{\frac{1}{2}} = \sqrt{2}$ is clearly not rational. Indeed, for every $n \in \mathbb{N}$, the second sort needs to contain some nth root of 2. Therefore, the second sort must be the field obtained by adjoining to $\mathbb{Q}$ an nth root of 2 for every $n \in \mathbb{N}$. Let this field be denoted $\mathbb{Q}(\sqrt[n]{2} : n \in \mathbb{N})$. The exponential map as the third component of the substructure also needs to grow to accommodate all of this expansion, but this step is done invisibly.

Therefore, the substructure $\mathfrak{B}$ that is generated in some way by $\ln(2)$ over $\mathbb{Q}_0$ is $\langle\ln(2)\mathbb{Q};\mathbb{Q}(\sqrt[n]{2} : n \in \mathbb{N});\exp\rangle$. A choice was made to adjoin each new element to $\mathbb{Q}$, and these choices for the nth roots of 2 are governed by their pure minimal polynomial equations, which constitute a coherent system of such roots. Thus, $\mathfrak{B}$ is generated over $\mathfrak{A}$ by an infinite sequence of ln(2) and the various nth roots of 2. However, to simplify matters, it is convenient and harmless to assert that ln(2) is the (canonical) generator of $\mathfrak{B}$ over $\mathfrak{A}$. In these cases, it is also convenient to use the principal roots as the appropriately needed nth roots.

With $\mathfrak{B}$ settled, it is now possible to examine its Galois-theoretic relationship with $\mathbb{Q}_0$. It turns out that the group $\mathrm{Gal}(\mathfrak{B}/\mathfrak{A})$ is trivial; for one thing, the first sort contains only one solution to the question of $\exp(t) = 2$, omitting all others. Thus, the orbit of ln(2) is not contained in $\mathfrak{B}$. Furthermore, it is already the case in the classical theory that $\mathbb{Q}(\sqrt[n]{2})$ is not a normal extension of $\mathbb{Q}$. For these reasons, $\mathfrak{B}$ is not normal over $\mathbb{Q}_0$.

Even if all of the $2\pi i\mathbb{Z}$-variants of ln(2) were to be removed by quotients, $\mathfrak{B}$ would not be a normal extension since the second sort is field-theoretically not a normal extension in the appropriate reduct. Indeed, unless the first sort contains the full standard kernel $2\pi i\mathbb{Z}$, no extension that adds on the first sort can be a normal extension. Therefore, with respect to $\mathbb{Q}_0$, the only normal extensions are those that fix the group sort and whose field sort reducts are normal extensions in the classical theory.

While $\mathbb{Q}_0$ may be the definable closure of the empty set in $\langle\mathbb{C}^+;\mathbb{C};\exp\rangle$, it is in many respects the inappropriate foundational structure to work with. The only Galois extensions are those whose field reducts would already have been Galois extensions in the classical theory. Any candidate pseudoexponential field automorphism in the full exponential language will fix the closed real abelian pseudoexponential field, so if the goal of looking at the Galois theory of $\langle\mathbb{C}^+;\mathbb{C};\exp\rangle$ is to elucidate the Galois theory of the full pseudoexponential field, then it is not helpful to use $\mathbb{Q}_0$ as the main fixed field.

Furthermore, it is clearly the case that any structure $\mathfrak{A}$ with the second sort being $\mathbb{Q}^{\mathrm{rab}}$ cannot have the full standard kernel $2\pi i\mathbb{Q}$ in the first sort, so such a structure would still not have normal extensions that properly expands upon the first sort. Therefore, it seems that a better base structure to use here is the substructure SK, even though it will not capture some of the exponential field automorphisms that might permute roots of unity over the real abelian numbers, particularly the complex conjugate map.

Since the full kernel is included in SK, any expansion of the first sort by an exponential polynomial will include the $2\pi i\mathbb{Z}$-variants, so normal extensions that extend the first sort are now possible. Since exponential polynomials with roots have infinitely many of them, the notion of algebraicity is useless here. However, the quasiminimality of pseudoexponentiation and of the complex exponential field means that a countably infinite version of algebraicity is an appropriate analogue here. It may be helpful to return to the previous example that demonstrated the problem with using $\mathbb{Q}_0$, now using SK as the base structure.

Example 9.65 Suppose $\phi(x)$ is the formula $\exp(t) \doteq_2 \dot{2}_2$, and take the solution $\ln(2)_1$. Then, the structure $\mathfrak{B} := \langle 2\pi i\mathbb{Q} \oplus \ln(2)\mathbb{Q}; \mathbb{Q}^{\mathrm{ab}}(\sqrt[n]{2} : n \in \mathbb{N}); \exp\rangle$ generated by $b := (\ln(2)_1)$ is an extension of SK. Although not a finite algebraic extension in the original sense, since ln(2) is not algebraic, this extension is some sort of a quasifinite-algebraic extension of $\mathfrak{A}$. For every individual in $\mathfrak{B}$, its orbit with respect to SK is contained in $\mathfrak{B}$, so $\mathfrak{B}$ is a normal extension of SK.

Indeed, it is intriguing to examine what the corresponding Galois group $\mathrm{Gal}(\mathfrak{B}/SK)$ looks like. On the first sort, a candidate partial elementary map should be able to permute ln(2) to any other individual of the form $\ln(2) \pm 2\pi i z$ with $z \in \mathbb{Z}$ while nevertheless fixing every vector in $2\pi i\mathbb{Q}$. These permutation maps stay within $2\pi i\mathbb{Q} \oplus \ln(2)\mathbb{Q}$, so normality is preserved here. The field-sort groups can be determined by direct appeal to classical results. The group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}(\sqrt[n]{2})/\mathbb{Q}^{\mathrm{ab}})$ is the profinite group $\hat{\mathbb{Z}}$.

Because the maps of the first sort permute ln(2) along the integers generated by $2\pi i$, the assertion

$$\mathrm{Gal}((2\pi i\mathbb{Q} \oplus \ln(2)\mathbb{Q})/(2\pi i\mathbb{Q})) \cong \mathbb{Z}$$

is clear. Of course, the aim here is to describe the overall group $\mathrm{Gal}(\mathfrak{B}/SK)$. A key observation to note is that the maps of each sort work independently of each other. Permutation of individuals in the first sort has no impact on the individuals of the second sort. Similarly, permuting the individuals of the second sort only affects those in the first sort after the exponential function is applied to them. Thus, the group is clearly $\mathrm{Gal}(\mathfrak{B}/SK) = \mathbb{Z} \times \hat{\mathbb{Z}}$.

Clearly, the preceding construction and observation holds for any integer $z \in \mathbb{Z}$ replacing the role 2 had played. Furthermore, the previous work does not actually required the fact that 2 is an integer. Rather, the same argument works for any rational number $q \in \mathbb{Q}$. Hence, these construction methods establish what simple extensions generated by the logarithm of rational numbers look like, along with their corresponding Galois groups.

9.11 Lessons

The model theory of the complex exponential field $\mathbb{C}_{\exp}$ is a disappointing endeavor compared to the model theory of the complex number field $\mathbb{C}$ particularly because the problems associated with the Gödel Incompleteness results and the unresolved status of Schanuel's Conjecture break down much of the properties that characterize the latter system. The formulation of pseudoexponentiation attempts to contain these problems and restore some sense of direction in the realm of exponentiation. The uncertainty is cleared enough to obtain known results such as that $\mathrm{dcl}(\emptyset) \cap \mathbb{Q}^{\mathrm{alg}}$ is the real abelian number field $\mathbb{Q}^{\mathrm{rab}}$, but the failure of model completeness and relative difficulty in dealing with the language of exponential rings nevertheless means that navigating the complex pseudoexponential field is still a fraught task.

Because of the insight obtained from work on partial exponential fields, it is clear that field-theoretic algebraicity over the real abelian field $\mathbb{Q}^{\mathrm{rab}}$ can shed light on model-theoretic algebraicity over the closed real abelian exponential field $\mathbb{Q}^{\mathrm{rab}\,C}$. Thus, if a is field-theoretically algebraic over $\mathbb{Q}^{\mathrm{rab}}$ of degree n, then a must also be model-theoretically algebraic over $\mathbb{Q}^{\text{rab}\,C}$ of degree n. Consequently, concentration on $\mathbb{Q}^{\text{rab}}$ as a pure field should provide insight to $\mathbb{Q}^{\text{rab}\,C}$ as an exponential field structure.

Returning to the question about model-theoretic abelian extensions of $\mathbb{Q}^{\text{rab}\,C}$, it was observed earlier that such extensions would belong to one of three possible classes. Two of these can be addressed by appealing to $\mathbb{Q}^{\text{rab}\,C}$ as a pure field. The first case concerning abelian extensions generated by numbers algebraic over $\mathbb{Q}^{\text{rab}}$ has already been addressed. The second case can also be addressed. Suppose a is a number transcendental over $\mathbb{Q}^{\text{rab}}$ but algebraic over $\mathbb{Q}^{\text{rab}\,C}$ with degree n. Then, a must be algebraic over some transcendental extension F residing between $\mathbb{Q}^{\text{rab}}$ and $\mathbb{Q}^{\text{rab}\,C}$ such that F has finite transcendence degree over $\mathbb{Q}^{\text{rab}}$. The field $K = F(a)$ can be transformed into a finitely generated partial exponential field and thus subject to Proposition 9.54 provided the condition for strong embedding can be met. If so, then an automorphism of K fixing F can be extended into an automorphism of $\mathbb{B}$ fixing $\operatorname{dcl}(F) = \mathbb{Q}^{\text{rab}\,C}$. Such a result would then allow the conclusion that a becomes model-theoretically algebraic with degree n and that $\operatorname{dcl}(\mathbb{Q}^{\text{rab}\,C})(a)$ is a model-theoretically abelian extension of $\mathbb{Q}^{\text{rab}\,C}$. However, the strong embedding condition for Proposition 9.54 seems rather difficult to fulfill in this situation.

Suppose by some other means of obtaining such a result that $\operatorname{Gal}((\mathbb{Q}^{\text{rab}\,C})^{\text{ab}}/\mathbb{Q}^{\text{rab}\,C})$ is a quotient group of

$$\operatorname{Gal}((\mathbb{Q}^{\text{rab}\,C})^{\mathfrak{ab}}/\mathbb{Q}^{\text{rab}\,C}),$$

so that the field-theoretic maximal abelian group fits into the model-theoretic maximal abelian group. A question to be addressed is what $\operatorname{Gal}((\mathbb{Q}^{\text{rab}\,C})^{\text{ab}}/\mathbb{Q}^{\text{rab}\,C})$ looks like. Fortuitously, it turns out that most of the work on $\operatorname{Gal}((\mathbb{Q}^{\text{rab}})^{\text{ab}}/\mathbb{Q}^{\text{rab}})$ carries through, so the p-part of $\operatorname{Gal}((\mathbb{Q}^{\text{rab}\,C})^{\text{ab}}/\mathbb{Q}^{\text{rab}\,C})$ for p odd prime is the same. Similarly, the torsion-free 2-part also carries through, and all of the torsion points in $\operatorname{Gal}((\mathbb{Q}^{\text{rab}})^{\text{ab}}/\mathbb{Q}^{\text{rab}})$ are also such in $\operatorname{Gal}((\mathbb{Q}^{\text{rab}\,C})^{\text{ab}}/\mathbb{Q}^{\text{rab}\,C})$. However, the Pythagoras number of $\mathbb{Q}^{\text{rab}\,C}$ clearly cannot be 2, and indeed, it must be infinite, so that $P(\mathbb{Q}^{\text{rab}\,C}) = \infty$; being finite would contradict Schanuel's conjecture. Thus, although it would be possible to characterize the group such that

$$\operatorname{Gal}((\mathbb{Q}^{\text{rab}\,C})^{\text{ab}}/\mathbb{Q}^{\text{rab}\,C}) = 2^{\omega} \times \hat{\mathbb{Z}}^{\omega},$$

it would not be possible to obtain a complete characterization of what $(\mathbb{Q}^{\text{rab}\,C})^{\text{ab}}$ without appealing to some other technique that may or may not be available. Thus, even as this work recognizes the difficulty in working with field structures with exponential functions, it is also evident that difficult work may also be found in contexts that do not require the complications of exponentiation.

Chapter 10

Continuance

This work explores many wide-ranging topics collated from the presence of Galois connections in model theory. The observation that classical model theory can produce a Galois correspondence between structures and their automorphisms or partial elementary maps is used to motivate the development of a substructural model theory that can carry out this same observation in a nonclassical setting. The general correspondence makes appropriate the idea of Galois theory for various structures and helps motivate a closer examination of a Galois theory of exponential field. This model-theoretic Galois theory also signals another means in which mathematical logic may continue to ingratiate itself other areas of mathematics. It is hoped that the ideas in this work may provide new lines of research and meditation while also giving insights into existing efforts. It is appropriate to conclude this work by providing some thoughts on what these lines and insights may prompt beyond the current confines.

10.1 Alternative Means of Presentation

The Galois correspondence carried in model theory may be useful for didactic presentations of this area of mathematical logic. The continued integration of model theory into other areas of mathematics accompanies an increasing technical complexity of model theory. As model theory continues to develop beyond its primary first-order context, standard ways of introducing model theory risk may benefit from other pedagogical efforts. The presentation of model-theoretic Galois theory as a generalization of algebraic Galois theory offers up one such effort.

10.2 The Recognition of Previously Obscure Structures

The logical correspondence as formalized in Theorem 6.1 developed as a consequence of the work in model theory on the elimination of imaginaries. There have been efforts to utilize this understanding to extract a particular Galois theory of some class of structures, such as the work of Poizat and Pillay on differential fields, but the observation of Galois correspondences usually arises as tangential remarks in model-theoretic discourse.

Subsequent endeavors concentrating along the lines of this work may be feasible, and one consequence of such work may be in the recognition of certain objects that would remain overlooked otherwise.

The realization that the real abelian numbers $\mathbb{Q}^{\text{rab}}$ are the only fixed algebraic numbers of the complex pseudoexponential field answers a question about the existence of exponential field automorphisms besides complex conjugation with respect to the theory of pseudoexponentiation. The real abelian field, it is dare said, is only seen as the maximal real subfield of the maximal cyclotomic and abelian extension of the rational numbers $\mathbb{Q}$. The work of classic and contemporary field theory does not readily provide insight into the properties of $\mathbb{Q}^{\text{rab}}$. This work, in its small way, rectifies this oversight in exploring what the abelian extensions of $\mathbb{Q}^{\text{rab}}$ look like, but there are of course other properties of $\mathbb{Q}^{\text{rab}}$ that subsequent work can explore. In addition to providing greater insight to this field for its own sake, such work can also yield more understanding of the role that the closed real abelian exponential field $\mathbb{Q}^{\text{rab}\,C}$ plays in the Galois theory of exponential fields.

Applications of this model-theoretic Galois correspondence to other first-order theories whose models do not already have an established Galois theory may similarly promote similar situations whereby a relatively obscure structure may hold some significance to such a project. For example, the model theory of pseudoexponentiation provides a possible source for the model-theoretic development of the Weierstrass $\wp$ function, and the work here on the Galois theory of exponential fields may turn out to guide a Galois theory of fields with the $\wp$ function. Projects motivated in this way can turn up structures which can be studied for their own sake as well as to understand their role in these Galois-theoretic efforts. These potential efforts would likely fill in gaps to well-established areas of study rather than lead to new frontiers of current mathematical understanding, but answering these unresolved questions may yet let to other unexpected directions.

10.3 Logical Galois Connections and the Classification of Theories

Hardly any mention has been made about the relationship between the logical Galois correspondence and the work in classification theory. First-order classification theory, largely built upon the prolific school of S. Shelah, seeks to categorize theories based upon certain model-theoretic properties that they may hold. Clearly, it is pertinent to see if there may therefore be some sort of relationship between the robustness of a theory's Galois theory and its model-theoretic properties.

Most of the specific theories that have been considered in this work are various extensions of the basic theory of fields. This tendency is reinforced by two main factors. One is that any theory of fields satisfies the ability to code finite sets, rendering it capable of the requisite Galois correspondence. Another reason is the recognition that model theory itself has developed as a mathematical discipline by taking different kinds of fields as key sources of inspiration. Contemporary applied model theory may be divided into at least three main factions consisting of work in stable theories, order-minimal structures, and a general model theory of fields, and in each of these groupings fields of historical interest play an inspirational role.

Stable theories are intuitively ones which have a relatively small number of types or means of expressing different statements. Clearly, such a situation provides a favorable environment for a rich Galois theory, or at least one which is not doomed to triviality. Indeed, with the exception of the algebraically closed valued fields, each of the theories considered in this work which carries a nontrivial Galois correspondence is stable.

What has not been scrutinized though is how different kinds of stable theories may provide different flavors of Galois correspondences. It is not apparent that the framework established for this work as based on model-theoretic algebraicity can detect the differences which may exist. For example, model-theoretic algebraicity coincides with pure field-theoretic algebraicity in the context of differential fields inhabiting a differentially closed field, yet the substantial work into differential Galois theory demonstrates that it is considerably different from the usual algebraic Galois theory of pure fields. This limitation would need to be rectified if one would seek to pursue this effort. However, other tasks must also be accomplished in aid of this direction. It would be necessary to establish exactly what it would mean to have different flavors of Galois correspondences and how such differences may be measured or distinguished.

Not every unstable theory induces a trivial Galois correspondence, so it would not be beneficial to limit this approach to those which are stable. Clearly however, it is apparent that many swaths of unstable theories can only carry trivial Galois correspondences. Ordered structures can exploit the ordering relation to distinguish between individuals or sequences of individuals which may otherwise be algebraic and not definable. The case of p-adically closed fields shows another way in which a trivial Galois correspondence can be carried without dependence upon an ordering relation.

Classification theory has of course moved beyond the classification of stable theories, having introduced other classes of theories based on various new notions of independence and other such properties. These divisions may or may not guide the relationship that may hold between robustness of a Galois correspondence and the tameness of model-theoretic behavior. The understanding of certain theories to contain a stable part and an unstable part, as formulated in the work on stable domination, may turn out to be useful in this kind of work.

10.4 The Model Theory of Substructural Logics

A considerable portion of this work developed the formal semantics of substructural logic necessary to host the model-theoretic principles that underpin the Galois correspondence of first-order theories. The system of semantics itself is not of great originality; it is beholden to antecedents which have similarly developed as adaptations of the algebraic semantics for propositional substructural logics. Still, with the model theory of nonclassical logics mainly relegated to propositional systems, this foray into quantified predicate calculi is an initial effort that can promote continued examination of the first-order model theory for a broad class of nonclassical logics.

The primary goal of the substructural development in this work was to gain the requisite machinery necessary to carry over the logical Galois correspondence from classical first-order model theory. There are thus inevitably some gaps in this substructural model theory left behind in this pursuit of a substructural Galois correspondence. The priority given to those aspects of the model theory which are necessary for the final result means that other parts have been given less attention or they may have been overlooked.

A sensible next task may be to address these gaps and pursue a more complete picture of substructural model theory as based on the current system of semantics. There are many specific projects that would be geared towards this overall goal. For example, it is evident that some substructural logics, particularly the stronger ones closer to classical logic than to the basic substructural logic **BQ**, are better-shaped to handle certain model-theoretic properties. The upward Löwenheim-Skolem theorem as presented here, as based on similar work in [Restall 1994], requires the background logic to induce some desirable properties on the corresponding models. However, it may be possible to obtain farther-reaching results using other alternative techniques.

The substructural model theory of reduced products seems ripe for continued work. In classical model theory, the theorem of Łos, as formalized in Theorem 3.13, provides that ultraproducts of models of a theory T are themselves models of T. Ultraproducts themselves cannot carry over to the substructural case since they essentially rely on the two-valued semantics of classical model theory; this reliance is captured by the ultrafilter itself which imputes a two-valued Boolean structure. The more general class of reduced products enjoys a limited result than that of Theorem 3.13 in that reduced products of models of a theory T preserve the Horn sentences with respect to T. Reduced products and the substructural analogues of Horn sentences may be particularly sensitive to the choice of underlying substructural logic.

The current work only provides one possible working definition for a substructural understanding of reduced products. It is possible that there is a more appropriate definition of these reduced products available. In either case, a key task would be to find out which class of sentences plays the same role that Horn sentences play in classical reduced products. Such a result may also provide insight into the feasibility of a substructural analogue to ultraproducts and Theorem 3.13.

The theory of types is another area of potential continued research. These types are not strictly necessary for the substructural Galois correspondence, so the discussion of weak, structural, and Galois types as provided in this work is not particularly extensive. Weak and structural n-types are proposals for a substructural understanding of partial and complete n-types. Galois types are roughly naturally defined based on their namesake source of inspiration. A thorough understanding of the correct notion of types in this substructural model theory is a possible project. A major difficulty in obtaining a substructural theory of types is that the connection between formulae and partitions on the domain of individuals is complicated by the role of propositions and the weakened expressive power of formulae. A choice may have to be made between regarding types as collections of formulae and as ways of dividing up the individuals in a structure.

The substructural model theory presented in this work is based on an algebraic semantics for predicate substructural logics. At the propositional level, there are two main approaches to the semantics, one relying on an algebraic understanding of truth values and the other taking up a relational view based on frames. For those substructural systems motivated by philosophical reasons, the latter approach of the relational semantics holds considerable appeal. The relational semantics provides a formal machinery which can be used to carry certain philosophical arguments about logical reasoning, whereas the algebraic semantics seems more like a technical contrivance.

Now, this work provides a quantified variant of the algebraic semantics. The move from the propositional to the quantified case is a rather straightforward one. The relational semantics for propositional substructural logics, on the other hand, cannot be extended to the quantified case so easily. A reasonable quantified version of the relational semantics eluded a completeness result ever since the relational semantics for relevance logics was proposed by R. Routley and R. Meyer, and the reason became clear when L. Fine showed its incompleteness; a quantified variant of the relational semantics will always admit certain formulae as valid which are not deducible from the axioms of a substructural logic which contains B and B' unless a very nonstandard understanding of quantification is used. A hybrid system of relational semantics which also incorporates some algebraic aspects was later formulated by R. Goldblatt and E. Mares. In both cases, little work has been done beyond proving the completeness of these semantics. Certainly, a possible subsequent project may involve addressing this gap and formulating a substructural model theory over relational semantics.

Despite this gap, some work has been done in working with relational models. The work in [Restall 2008] by Restall is one such example, and it highlights some of the difficulty in working with the relational semantics in order to achieve model-theoretic results. The models for Peano arithmetic in substructural logics as defined in [Restall 2008] are highly idiomatic in the sense that they rely on arithmetical properties to avoid becoming classical models of Peano arithmetic. It is difficult to see how such properties may exist for other theories in devising substructural models for other mathematical theories.

Meanwhile, the current semantics may also provide other directions for subsequent research. The key innovation is the reliance on a domain of individuals for interpreting terms and on an algebra of propositions for interpreting well-formed formulae. It happens that in the current situation that the algebras are those conform to the behavior patterns of substructural logics, but it is not inconceivable that they may be replaced with other kinds of algebras which may allow this system of semantics to model other kinds of nonclassical logic. A systematic study of comparative semantics may be appropriate to see the feasibility of using the current system for other such deductive calculi. It may be the case that, for example, the model theory of continuous logic and metric structures, as explicated in [Metric], can be adapted into the current shape carried for substructural logics in this work.

Another direction in continuing this look into substructural model theory may be in the viability of an intensional understanding of quantification. Since substructural logic focuses on one propositional connective, the quantifiers take lower priority. The quantifiers in this work act as generalized versions of extensional conjunction and disjunction. The fusion connective $\circ$ has also been described as intensional conjunction since it holds a few properties held by conjunction $\wedge$ in classical logic that no longer applies in substructural ones, and it is not difficult to define an intensional disjunction $+$ using De Morgan principles especially in the stronger substructural logics. Consequently, a substructural logic can have extensional and intensional varieties of conjunction and disjunction. Intensional quantification, perhaps defined like extensional quantification as a generalized conceptualization of intensional conjunction and disjunction or even in some unforeseen manner, offers a way to enrich substructural model theory in a way not possible in classical model theory.

10.5 On Abstract Elementary Classes

One of the concepts of classical model theory that has not been discussed in the explication of substructural model theory is that of elementary classes. For a language $\mathscr{L}$, an elementary class is a class of $\mathscr{L}$-structures which are models of some $\mathscr{L}$-sentence σ or more widely of some $\mathscr{L}$-theory T. This definition can carry over to the substructural case in a straightforward manner, but there some issues that do arise in adopting such a definition.

Let ϕ be an $\mathscr{L}$-sentence, and let EC_ϕ be the collection of $\mathscr{L}$-substructural structures which satisfy ϕ. Let $\mathfrak{A}$ and $\mathfrak{B}$ be in EC_ϕ. Unfortunately, it is possible that there is no great way to compare the differences between these two structures $\mathfrak{A}$ and $\mathfrak{B}$. If $\mathfrak{A}$ is an $\mathscr{L}(\mathbf{LQ})$-model and $\mathfrak{B}$ an $\mathscr{L}(\mathbf{MQ})$-model with $\mathbf{LQ} < \mathbf{MQ}$, then one can conclude that $\mathfrak{B}$ is also an $\mathscr{L}(\mathbf{LQ})$-model and that ϕ is compatible, in a certain sense, with both $\mathbf{LQ}$ and $\mathbf{MQ}$, but it does not seem readily apparent, for example, how to compare $D_{\mathfrak{A}}$ and $D_{\mathfrak{B}}$ or even $K_{\mathfrak{A}}$ and $K_{\mathfrak{B}}$.

This inability to compare different objects in EC_ϕ renders the idea of substructural elementary class less helpful than it would ideally be. One of the properties held by classical elementary classes is that models belonging to a common elementary class can be compared to each other and manipulated to form other models. Indeed, elementary classes are closed under ultraproducts by Theorem 3.13, the downward Löwenheim-Skolem theorem, and the upward Löwenheim-Skolem theorem, and elementary classes are therefore compatible with notions of category theory. Because of the lack of a substructural version of Theorem 3.13 and the limited availability of the upward Löwenheim-Skolem theorem, substructural elementary classes are less coherent as collections of objects.

In the expansion of model theory beyond first-order constraints, the definition of abstract elementary classes arose as a generalization of classical elementary classes.

Definition 10.1 Let $\mathscr{L}$ be a language. Then, a collection $(X, \prec)$ of $\mathscr{L}$-structures is an *abstract elementary class* if it satisfies the following properties.

1. X is a collection of $\mathscr{L}$-structures and $\prec \subseteq X \times X$ is a binary relation such that $\prec$ is a partial order on X.

2. For $\mathfrak{M}$ and $\mathfrak{N}$ in X, if $\mathfrak{M} \prec \mathfrak{N}$, then $\mathfrak{M} \subseteq \mathfrak{N}$.

3. Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be $\mathscr{L}$-structures in X. If $\mathfrak{A} \prec \mathfrak{C}$, $\mathfrak{B} \prec \mathfrak{C}$, and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$.

4. Let a chain $(\mathfrak{A}_i : i < \mu)$ of $\mathscr{L}$-structures be a subset of X. Then, the object $\mathfrak{M} := \bigcup_{i<\mu} \mathfrak{A}_i$ is also an $\mathscr{L}$-structure in X such that it is the case that $\mathfrak{A}_i \prec \mathfrak{M}$ for every $i < \mu$. Furthermore, if there is some $\mathfrak{N}$ in X with $\mathfrak{A}_i \prec \mathfrak{N}$ for every $i < \mu$, then $\mathfrak{M} \prec \mathfrak{N}$ as well.

5. There is a Löwenheim-Skolem number $\text{LS}(X)$ for X as follows. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be two $\mathscr{L}$-structures in X. Then, there is some $\mathscr{L}$-structure $\mathfrak{A}'$ in X with $\mathfrak{A} \subseteq \mathfrak{A}'$ and $\mathfrak{A}' \prec \mathfrak{B}$ such that $|\mathfrak{A}'| < \text{LK}(X) + |\mathfrak{A}|$.

An elementary class is an abstract elementary class in which the relation $\mathfrak{M} \prec \mathfrak{N}$ is taken to be that $\mathfrak{M}$ is an elementary substructure of $\mathfrak{N}$. To the definition of abstract elementary classes, other properties can be added to yield other kinds of classes, such as excellent classes. The class of fields with pseudoexponentiation is itself an excellent class.

Since abstract elementary classes generalize the elementary classes of classical first-order model theory, the work of classification theory in abstract elementary classes loses some of the constraining conveniences associated with working in first-order logic. Among these is that one cannot appeal to the notion of n-types using references to formulae and other syntactical objects. The succeeding notion of Galois types relies on references to orbits induced by the presence of automorphisms and other maps. This approach informs the idea of Galois types previously defined in the work of substructural model theory.

Indeed, some of the problems associated the substructural mode theory of this work mirror some of the issues that arise in working with abstract elementary classes. For example, although abstract elementary classes satisfy a downward Löwenheim-Skolem property, one built right into the definition, they do not necessarily satisfy an upward Löwenheim-Skolem property. Furthermore, a major conjecture by Shelah asks whether abstract elementary classes have a categoricity in power result similar to the theorem of M. Morley, with one of the problems being of the existence of models of higher power. This worry echoes the limitations on the upward Löwenheim-Skolem theorem to models which have propositional algebras that are relatively well-behaved.

A major project may thus include a comparative examination of substructural elementary classes with the general abstract elementary classes in classification theory. It is likely that the success of such work would depend on finding a suitable notion of a *substructural elementary class*; the straightforward adaptation of a substructural elementary class as merely a classical elementary class but with $\mathscr{L}$-substructural structures replacing $\mathscr{L}$-structures appears to invite too much variety into a given substructural elementary class. A potential restriction of some sensibility may be in requiring substructural elementary classes to impose some structure in the same that structural types were defined, so that a model must not only satisfy some $\mathscr{L}$-theory but that $\mathscr{L}$-theory must be particularly well-described or that the models in such a substructural elementary class must satisfy additional properties not immediately capture by the syntactic requirements. For example, a substructural elementary class may be required to include only models of a particular substructural logic to the exclusion of others, preventing the scenario described earlier.

It is perhaps appropriate to note that $\mathscr{L}$-substructural structures are themselves objects which can be studied with classical techniques. Indeed, one of the great ironies in the semantics of nonclassical logics is that, while the object language provides discourse in nonclassical deductive calculi, the language of discourse at the metamathematical level remains much the same as in the standard semantics of classical logic. This is clearly evident in the sometimes-complicated mechanisms involved in the substructural models of this work. A development which proceeds with the assumption that a particular substructural logic must be the one true logic, then it would be interesting to see how the techniques in the metalanguage can reflect this thesis.

10.6 The Galois Correspondence over Infinitary Logic

The move into abstract elementary classes is a natural progression for continued work into a model-theoretic understanding of Galois theory. This move is readily apparent in the use of Galois types, but since there are difficulties in working with Galois types with full generality, the progression out of first-order constraints can be done step-by-step. A possible next step would be in working in the infinitary logic $\mathscr{L}_{\omega_1,\omega}$. Since there is still a syntax accompanying the semantics, the work in $\mathscr{L}_{\omega_1,\omega}$ can be a transitional effort moving from the tight harmony of syntax and semantics in first-order model theory to the full expressive possibilities of abstract elementary classes.

The model-theoretic Galois theory examined in this work is not concerned specifically on the $\mathscr{L}$-structures that would make up these abstract elementary classes but rather with their substructures. The key in first-order model theory that allows for these Galois correspondences is the ability to code finite sets, to various degrees. It would be interesting to find what the infinitary analogue of coding finite sets or eliminating imaginaries may be in working out an infinitary model-theoretic Galois theory.

In one strict sense, the theory of pseudoexponentiation is itself an $\mathscr{L}_{\omega_1,\omega}$-theory, but it is treated in this work essentially as a first-order theory. The commitment to first-order model-theoretic algebraicity in this work reinforces this approach. The increased expressivity of infinitary formulae would clearly affect algebraicity. A proper treatment of an infinitary Galois theory of pseudoexponentiation would likely need to take this increased expressivity into account. Galois theory over infinitary logic would therefore be a considerable project to take up as a subsequent undertaking.

10.7 Summary Resolution

No work of sustained discourse can stand outside its foundational context; the ideas and arguments of this work build upon those found in previous endeavors in the field of model theory and formal logic. Since the rise of modern logic in the late nineteenth century, the study of formal logic has enriched the study of mathematics and invigorated philosophical meditations while growing into a discipline with many specialties. The diversity of contemporary logic is great evidence for its vitality, but such diversity also leads to great divergence between its branches.

It is hoped that this work successfully overcomes one such division by incorporating systems of formal logic holding unusual theses and nonstandard principles with ideas of classic and contemporary first-order model theory, however small this effort may be. Indeed, this work also seeks to highlight interactions between different aspects of mathematics by taking up the common theme binding the major components of this work to be the Galois correspondence that emerges from the presence of certain model-theoretic properties. This logical phenomenon becomes the inspiration for the whole of this work.

This work is an attempt to fuse components of model theory as understood in its broadest sense as the study of the semantics of formal logic. Classical first-order model theory has developed into a highly technical discipline of pure mathematics whereas philosophical and informatical concerns about the semantics of deductive calculi rely on other technical work. The research efforts that may take this work as its starting point and its foundational context will likely need to be in one of these directions to the exclusion of the other. Nevertheless, it is hoped that this work demonstrates how the diversity of contemporary logic nevertheless continues to be based on a unifying and uniting goal of systematically understanding and articulating correct reasoning.

Bibliography

[Ackermann 1956] Wilhelm Ackermann, "Begründung einer strengen Implikation," *The Journal of Symbolic Logic*, vol. 21, no. 2 (June 1956), 113-128.

[Entailment I] Alan R. Anderson and Nuel D. Belnap, Jr., *Entailment: The Logic of Relevance and Necessity*, vol. 1, Princeton University Press, 1975.

[Entailment II] Alan R. Anderson, Nuel D. Belnap, Jr., and J. Michael Dunn, *Entailment: The Logic of Relevance and Necessity*, vol. 2, Princeton University Press, 1992.

[Aristotle] Aristotle, *Analytica Priora*.

[Avron 1988] Arnon Avron, "The Semantics and Proof Theory of Linear Logic," *Theoretical Computer Science*, vol. 57, no. 2-3 (May 1988), 161-184.

[Metric] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model Theory for Metric Structures*, Manuscript, 2007.

[Bimbó Dunn 2008] Katalin Bimbó and J. Michael Dunn, *Generalized Galois Logics: Relational Semantics of Nonclassical Logical Calculi*, CSLI Lecture Notes Number 188, CSLI Publications, 2008.

[Boethius] Anicius Manlius Severinus Boethius, *De Syllogismo Hypothetico*.

[Brady 1988] Ross T. Brady, "A Content Semantics for Quantified Relevant Logics I," *Studia Logica*, vol. 47, no. 2 (1988), 111-127.

[Brady 1989] Ross T. Brady, "A Content Semantics for Quantified Relevant Logics II," *Studia Logica*, vol. 48, no. 2 (1989), 243-257.

[Brady 2006] Ross T. Brady, *Universal Logic*, CSLI Lecture Notes Number 109, CSLI Publications, 2006.

[Casanovas Farré 2004] Enrique Casanovas and Rafel Farré, "Weak Forms of Elimination of Imaginaries," *Mathematical Logic Quarterly*, vol. 50, no. 2 (2004), 126-140.

[Chang Keisler 1990] Chen Chung Chang and H. Jerome Keisler, *Model Theory*, Third Edition, Studies in Logic and the Foundations of Mathematics Volume 73, Elsevier Science Publishers, 1990.

[Delon 1998] Françoise Delon, "Separably Closed Fields," *Model Theory and Algebraic Geometry: An Introduction to E. Hrushovski's Proof of the Geometric Mordell-Lang Conjecture*, Lecture Notes in Mathematics 1696, Springer-Verlag, 1998.

[Diller Dress 1965] Justus Diller and Andreas Dress, "Zur Galoistheorie Pythagoreischer Körper," *Archive of Mathematics*, vol. 16 (1965), 148-152.

[Došen 1993] Kosta Došen, "A Historical Introduction to Substructural Logics," in Kosta Došen and Peter Schroeder-Heister (editors), *Substructural Logics*, Studies in Logic and Computation 2, Oxford University Press, 1993.

[Došen Schroeder-Heister 1993] Kosta Došen and Peter Schroeder-Heister (editors), *Substructural Logics*, Studies in Logic and Computation 2, Oxford University Press, 1993.

[Engler Prestel 2005] Antonio J. Engler and Alexander Prestel, *Valued Fields*, Springer Monographs in Mathematics, Spring-Verlag, 2005.

[Frege 1879] Gottlob Frege, *Begriffsschrift*, in Jean van Heijenoort (editor), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Source Books in the History of the Sciences, Harvard University Press, 2002.

[Fried Jarden 2008] Michael D. Fried and Moshe Jarden, *Field Arithmetic*, Third Edition revised by Moshe Jarden, Springer-Verlag, 2008.

[Geyer Jensen 1996] Wulf-Dieter Geyer and Christian Ulrik Jensen, "Prodihedral Groups as Galois Groups over Number Fields," *Journal of Number Theory*, vol. 60, no. 2 (1996), 332-372.

[Gödel 1930] Kurt Gödel, "The Completeness of the Axioms of the Functional Calculus of Logic," in Jean van Heijenoort (editor), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Source Books in the History of the Sciences, Harvard University Press, 2002.

[Gödel 1931] Kurt Gödel, "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems I," in Jean van Heijenoort (editor), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Source Books in the History of the Sciences, Harvard University Press, 2002.

[Gongsun Longzi] Gongsun Long (Traditional Attribution), *Gongsun Longzi*.

[Haskell Hrushovski Macpherson 2007] Deirdre Haskell, Ehud Hrushovski, and H. Dugald Macpherson, *Stable Domination and Independence in Algebraically Closed Valued Fields*, Lecture Notes in Logic 30, Cambridge University Press, 2007.

[Hilbert Ackermann] David Hilbert and Wilhelm Ackermann, *Grundzüge der theoretischen Logik*.

[Hinman 2005] Peter G. Hinman, *Fundamentals of Mathematical Logic*, A.K. Peters, Ltd., 2005.

[Hodges 1997] Wilfrid Hodges, *A Shorter Model Theory*, Cambridge University Press, 1997.

[Junker Koenigsmann 2010] Markus Junker and Jochen Koenigsmann, "Schlanke Körper (Slim Fields)," *Journal of Symbolic Logic*, vol. 75, no. 2 (June 2010), 481-500.

[Kirby 2007] Jonathan Kirby, "Variants of Schanuel's Conjecture," Preprint version 1.1, September 27, 2007.

[Kirby 2010 QEC] Jonathan Kirby, "On Quasiminimal Excellent Classes," *The Journal of Symbolic Logic*, vol. 75, no. 2 (June 2010), 551-564.

[Kirby 2010 FPEF] Jonathan Kirby, "Finitely Presented Exponential Fields," Preprint arXiv:0912.4019v2 [math.LO], June 4, 2010.

[Kirby Macintyre Onshuus 2011] Jonathan Kirby, Angus Macintyre, and Alf Onshuus, "The Algebraic Numbers Definable in Various Exponential Fields," Preprint arXiv:1101.4224v1 [math.LO], January 21, 2011.

[Kirby Zilber 2011] Jonathan Kirby and Boris Zilber, "Exponential Fields and Atypical Intersections," Preprint arXiv:1108.1075v1 [math.LO], August 4, 2011.

[Klüners Malle 2001] Jürgen Klüners and Gunter Malle, "A Database for Field Extensions of the Rationals," *LMS Journal of Computation and Mathematics*, vol. 4 (January 2001), 182-196.

[Koenigsmann 2002] Jochen Koenigsmann, "Defining Transcendentals in Function Fields," *Journal of Symbolic Logic*, vol. 67, no. 3 (September 2002), 947-956.

[Koenigsmann 2006] Jochen Koenigsmann, "Projective Extensions of Fields," *Journal of the London Mathematical Society*, Second Series vol. 73 no. 3 (2006), 639-656.

[Kutateladze 2007] S.S. Kutateladze, "What is Boolean Valued Analysis?," *Siberian Advances in Mathematics*, vol. 17, no. 2 (2007), 91-111.

[Lang 1993] Serge Lang, *Algebra*, Addison-Wesley, Inc., 1993.

[Macintyre McKenna van den Dries 1983] Angus Macintyre, Kenneth McKenna, and Lou van den Dries, "Elimination of Quantifiers in Algebraic Structures," *Advances in Mathematics*, vol. 47, no. 1 (January 1983), 74-87.

[Marker 1996] David Marker, "Model Theory of Differential Fields," *Model Theory of Fields*, Lecture Notes in Logic 5, Springer-Verlag, 1996.

[Marker 2002] David Marker, *Model Theory: An Introduction*, Graduate Texts in Mathematics 217, Springer-Verlag, 2002.

[Marker 2006] David Marker, "A Remark on Zilber's Pseudoexponentiation," *Journal of Symbolic Logic*, vol. 71, no. 3 (September 2006), 791-798.

[Marker Pillay 1997] David Marker and Anand Pillay, "Differential Galois Theory III: Some Inverse Problems," *Illinois Journal of Mathematics*, vol. 41, no. 3 (Fall 1997), 453-461.

[Medvedev Takloo-Bighash 2009] Alice Medvedev and Ramin Takloo-Bighash, "An Invitation to Model-Theoretic Galois Theory," Preprint arXiv:0909.4340v1 [math.LO], September 2009.

[Meyer Dunn LeBlanc 1974] Robert K. Meyer, J. Michael Dunn, and Hugues LeBlanc, "Completeness of Relevant Quantification Theories," *Notre Dame Journal of Formal Logic*, vol. 25, no. 1 (1974), 97-121.

[Nelson 1930] Everett J. Nelson, "Intensional Relations," *Mind*, New Series, vol. 39, no. 156 (October 1930), 440-453.

[Ono 1993] Hiroakira Ono, "Semantics for Substructural Logics," in Kosta Došen and Peter Schroeder-Heister (editors), *Substructural Logics*, Studies in Logic and Computation 2, Oxford University Press, 1993.

[Ono 1995] Hiroakira Ono, "Algebraic Semantics for Predicate Logics and Their Completeness," *RIMS Kôkyûroku*, no. 927, 88-103.

[Peckhaus 2009] Volker Peckhaus, "Leibniz's Influence on 19th Century Logic," *The Stanford Encyclopedia of Philosophy*, Fall 2009 Edition, edited by Edward N. Zalta.

[Pfister 1995] Albrecht Pfister, *Quadratic Forms with Applications to Algebraic Geometry and Topology*, London Mathematical Society Lecture Notes Series 217, Cambridge University Press, 1995.

[Pillay 1997] Anand Pillay, "Differential Galois Theory II," *Annals of Pure and Applied Logic*, vol. 88 (1997), 181-191.

[Pillay 1998] Anand Pillay, "Differential Galois Theory I," *Illinois Journal of Mathematics*, vol. 42, no. 4 (Winter 1998), 678-699.

[Poizat 1983] Bruno Poizat, "Une Théorie de Galois imaginaire," *The Journal of Symbolic Logic*, vol. 48, no. 4 (December 1983), 1151-1170.

[Poizat 2000] Bruno Poizat, *A Course in Model Theory*, Universitext, Springer-Verlag, 2000.

[Restall 1994] Greg Restall, *On Logics without Contraction*, Thesis for the Degree of Doctor of Philosophy, University of Queensland, 1994.

[Restall 2008] Greg Restall, "Models for Substructural Arithmetics," *Miscellanea Logica*, vol. 7 (2009), 1-20.

[Robinson 1956] Abraham Robinson, *Complete Theories*, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, 1956.

[Thomas 2006] Ivor Bulmer-Thomas, *Selections Illustrating the History of Greek Mathematics, with an English Translation by Ivor Thomas I: From Thales to Euclid*, Loeb Classical Library 335, Harvard University Press, 2006. First published 1939, with additions and revisions 1980 and 1991.

[Whaples 1957] George Whaples, "Algebraic Extensions of Arbitrary Fields," *Duke Mathematics Journal*, vol. 24 (1957), 201-204.

[Yakovlev 2002] Anatoly V. Yakovlev, "Embedding of a Cyclic Extension into a Cyclic Extension," *Journal of Mathematical Sciences*, vol. 112, no. 4 (2002), 4410-4413. A translation by Anatoly V. Yakovlev from *Zapinski Nauchnykh Seminarov POMI*, vol. 265 (1999), 306-313.

[Yim 2008] Austin Vincent Yim, *Relevance Logic and Relevant Arithmetic*, Thesis in Partial Fulfillment for the Degree of Bachelor of Arts with Honors, Amherst College, 2008.

[Zilber 2000] Boris Zilber, "Fields with Pseudo-exponentiation," Preprint arXiv:math/0012023v1 [math.LO], December 2000.

[Zilber 2005] Boris Zilber, "Pseudo-exponentiation on Algebraically Closed Fields of Characteristic Zero," *Annals of Pure and Applied Logic*, vol. 132 (2005), 67-95.

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