Equivariant scanning and stable splittings of configuration spaces

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For Rozy. You made it possible.

Abstract

We give a definition of the scanning map for configuration spaces that is equivariant under the action of the diffeomorphism group of the underlying manifold. We use this to extend the Bödigheimer-Madsen result for the stable splittings of the Borel constructions of certain mapping spaces from compact Lie group actions to all smooth actions. Moreover, we construct a stable splitting of configuration spaces which is equivariant under smooth group actions, completing a zig-zag of equivariant stable homotopy equivalences between mapping spaces and certain wedge sums of spaces. Finally we generalise these results to configuration spaces with twisted labels (labels in a fibre bundle subject to certain conditions) and extend the Bödigheimer-Madsen result to more mapping spaces.

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Contents

1 Introduction

1.1 Background

Configuration spaces, in various forms, arise naturally as models for a wide range of systems and spaces. From a topological perspective they have been particularly useful for modelling iterated loop spaces and mapping spaces more generally. In the simplest sense a configuration space is the space of all possible configurations of k ordered points (hereafter referred to as particles) on a manifold M, denoted by $\tilde{C}_k(M)$ and topologised so that particles cannot collide. More generally, and with a view to modelling more complicated systems, the configurations can be ordered or unordered, and particles can be allowed to vanish in a submanifold. The highly combinatorial nature of a configuration space often makes it easier to understand than the space that it is modelling.

It is natural to ask what information can be gathered about a given configuration by looking locally at the underlying manifold. The idea is to move a magnifying lens of small radius around the manifold, picking up information about the configuration in the process. When a particle appears in the lens, its position defines a point in a certain fibre bundle associated to the tangent bundle of the manifold. Continuing in this manner, the configuration eventually defines a section of this bundle. This process is known as *scanning*, and the map which sends a configuration to the section it defines is called the *scanning map*.

Maps of this kind were first defined by May [May72] and Segal [Seg73], in the case where the underlying manifold is $\mathbb{R}^n$ and the particles in a configuration have local data in the form of labels in a parameter space X. We denote the space of such configurations by $C(\mathbb{R}^n; X)$. In this case the scanning map is a homotopy equivalence between $C(\mathbb{R}^n; X)$ and the iterated loop space $\Omega^n \Sigma^n X$. McDuff [McD75] studied configuration spaces without labels for more general manifolds and constructed a scanning map into a certain section space. This map is a homotopy equivalence when particles are allowed to vanish in a submanifold of the boundary.

A further application of configuration spaces as combinatorial models has been to obtain stable splittings of spaces. Snaith [Sna74] used May's configuration spaces to prove that iterated loop spaces $\Omega^n\Sigma^n X$ of connected CW-complexes can be split stably into wedges of simpler spaces. Splittings of this kind have been constructed by many others, and all rely on a power set map which (roughly) sends a finite configuration to a collection of it's subconfigurations. Although these are splittings of suspension spectra, they are constructed unstably on the level of spaces. A short and elegant proof of Snaith's splitting using stable arguments on the spectrum level has been given by R. Cohen [Coh80].

Bödigheimer has combined McDuff's scanning map with a stable splitting described by F. Cohen [Coh83] to obtain stable splittings of mapping spcaes and section spaces. As an extension of this Bödigheimer and Madsen have constructed stable splittings of the Borel constructions of mapping spaces with respect to compact Lie group actions [BM88], making use of the equivariance of the scanning map under such actions. Several descriptions of the scanning map have been given, each of which relies on a Riemannian metric on the manifold and disks around the particles for scanning. The purpose of this paper is to give a precise definition of a scanning map which avoids such rigid constructions, allowing equivariance under the action of the diffeomorphism group of the manifold. Equipped with this map we generalise the results of Bödigheimer and Madsen to all smooth actions.

More recently configuration spaces and the scanning map have been viewed as the 0-dimensional case of a more general construction. The space of configurations of k unordered particles on an manifold M is precisely the space of embeddings of k points in the manifold. By removing the ordering we obtain the orbit space under the action of the diffeomorphism group of k points, the symmetric group Σ_k. Now consider the space of embeddings of a manifold X in M and take the orbit space under the action of $\mathrm{Diff}(X)$. We can think of this as the space of configurations of X in M. A scanning map is defined in much the same way; we take a magnifying lens and locally we see small neighbourhoods of X. Such constructions are the subject of the Madsen-Weiss Theorem [MW07]. In particular this theorem is concerned with the spaces of configurations of surfaces $S_{g,1}$ in $\mathbb{R}^\infty$, where $S_{g,1}$ is the compact orientable surface of genus g with one boundary component. The inclusions $S_{g,1} \hookrightarrow S_{g+1,1}$ induce inclusions of the associated configuration spaces and, taking the colimit over g, the scanning map gives an equivalence with the infinite loop space associated to a certain Thom spectrum. This application of configuration spaces is more closely related to the homological stability results of McDuff [McD75] than to stable splitting results. However, the hope is that the scanning map defined in this paper could be adapted to give a more precise description of this generalised scanning map.

1.2 Outline

The layout of this paper is as follows. In section 2 we describe the configuration spaces which are the focus of our study. We look in detail at the current definition of the scanning map and some stable splitting results.

Using fibrewise topology we develop models for configuration spaces in section 3. These models carry more data about the ambient manifold but have the same homotopy type as configuration spaces. We define a smooth version of the *fibrewise mapping space* of James [Jam89] and use this to construct a fibre bundle $E_k(M;X)$ over a given configuration space $C_k(M;X)$. The fibre over a configuration in $C_k(M;X)$ is the space of tubular neighbourhoods of the particles in that configuration. The total space is called the *tubular configuration space*.

We use tubular configuration spaces to define a more flexible scanning map in section 4. As an easy consequence we will see that the scanning map is equivariant under the action of the diffeomorphism group of the underlying manifold. We use Gromov's method as detailed by McDuff [McD77] to show that this new scanning map is a homotopy equivalence for the same choices of manifold, submanifold and labelling space as the original.

In section 5 we extend the Bödigheimer-Madsen result [BM88] for the stable splittings of Borel constructions of mapping spaces for compact Lie group actions to all smooth actions. We also construct a stable splitting of another model for configuration spaces which is equivariant under smooth actions.

The final section is devoted to configuration spaces with twisted coefficients, that is configuration spaces with labels in a fibre bundle over the underlying manifold, where the label of a particle lies in the fibre over that particle. Ordinary configuration spaces are then configuration spaces with labels in a trivial bundle. We extend many of the results previously obtained to these configuration spaces.

We include an appendix for completeness, in which we give a definition of the tubular neighbourhood of a submanifold which differs slightly from the usual definition. We show that with this definition the space of such tubular neighbourhoods is contractible if the submanifold is compact. This is an important result as it ensures our model for the configuration spaces is homotopy equivalent.

2 Configuration spaces

2.1 The configuration space

Definition 2.1. Let M be a smooth n-manifold. The *ordered configuration space* $\tilde{C}_k(M)$ of k particles in M is the space of k-tuples of distinct points in M. More precisely,

$$\tilde{C}_k(M) := M^k \smallsetminus \Delta_k$$

where M^k is the k-fold cartesian product and $\Delta_k \subset M^k$ is the *fat diagonal*,

$$\Delta_k := \{(m_1, \dots, m_k) \subset M^k : m_i = m_j \text{ for some } i \neq j\}.$$

The finite ordered configuration spaces have obvious free actions by the symmetric groups by permuting the particles in a configuration. This leads us to the definition of the *unordered* configuration space of k particles as the orbit space

$$C_k(M) := \tilde{C}_k(M)/\Sigma_k.$$

The configuration space on a manifold M is the disjoint union $C(M) := \coprod_{k \geq 0} C_k(M)$ where the empty configuration space $C_0(M) = *$ serves as the basepoint. We denote an element of the finite and infinite, ordered and unordered configuration space on M by $\boldsymbol{m} = (m_1, \dots, m_k)$. It will be clear from context which is meant.

We may add more structure to a configuration space in a number of ways. Let M_0 be a (possibly empty) compact submanifold of M.

Definition 2.2. The configuration space of particles in M modulo M_0 is the quotient space

$$C(M, M_0) := C(M)/\sim$$

where $(m_1, \dots, m_k) \sim (m_1, \dots, m_{k-1})$ if $m_k \in M_0$. We think of this relation as particles vanishing in M_0.

Remark 2.3. The configuration space in which particles do not vanish in a submanifold is precisely the space $C(M, \emptyset)$ so we need only consider the case with vanishing particles.

The second way to add more structure is to add local data to each particle in the form of a *label*.

Definition 2.4. Let X be a well-pointed space with basepoint x_0. The configuration space of k particles in M *with labels in* X is

$$C_k(M; X) := \tilde{C}_k(M) \times_{\Sigma_k} X^k.$$

The configuration space of particles in M *modulo* M_0 *with labels in* X is the space

$$C(M, M_0; X) := \left(\coprod_{k \geq 0} C_k(M; X) \right) \Big/ \approx$$

where $(m_1, \ldots, m_k; x_1, \ldots, x_k) \approx (m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1})$ if $m_k \in M_0$ or $x_k = x_0$.

We think of this relation as particles in M vanishing in M_0 or if they have label x_0. Points in $C(M, M_0; X)$ are equivalence classes $[(\boldsymbol{m}; \boldsymbol{x})] = [(m_1, \ldots, m_k; x_1, \ldots, x_k)]$ where each particle m_i in a configuration is associated to the label x_i.

Remark 2.5. The unlabelled configuration space is precisely the configuration space with labels in S^0 so we need only consider configuration spaces modulo a submanifold and with labels.

2.2 The scanning map

The scanning map allows us to use configuration spaces as combinatorial models for section spaces. We give the definition due to Bödigheimer & Madsen [BM88]. Let M be a smooth compact manifold and let W be a smooth manifold without boundary containing M as a codimension 0 submanifold, for example let $W = M$ if $\partial M = \emptyset$ and $W = M \cup \partial M \times [0, 1)$ otherwise. Let $\overset{\infty}{T}W$ be the fibrewise one point compactification of TW and let $\overset{\infty}{T}_m W$ denote the compactified fibre over m. The fibres of the one point compactification have an obvious basepoint at ∞. Let $\overset{\infty}{\tau} := \overset{\infty}{T}W \wedge_W (W \times X)$ be the fibrewise smash product. The inclusion of the basepoint $*_m$ into the fibre $\overset{\infty}{T}_m W \wedge X$ defines a section $* : W \to \overset{\infty}{\tau}$. Given $A_0 \subseteq A \subseteq W$ we define $\Gamma(A, A_0; X)$ to be the space of sections of $\overset{\infty}{\tau}$ which are defined in A and agree with $*$ in A_0. The scanning map is a map from the configuration space $C(M, M_0; X)$ to the section space $\Gamma(W \smallsetminus M_0, W \smallsetminus M; X)$.

Before we give the full definition of the map we give an intuitive description. Consider a configuration $[(\boldsymbol{m}; \boldsymbol{x})] \in C(M, M_0; X)$ and a point $z \in W \smallsetminus M_0$. The map is determined by where the section associated to $[(\boldsymbol{m}; \boldsymbol{x})]$ sends z in $\overset{\infty}{\tau}$. Choose a

Riemannian metric d on W and choose any representative $(\boldsymbol{m};\boldsymbol{x}) \in [(\boldsymbol{m};\boldsymbol{x})]$. Around z we consider a little ε-disk which we think of as the lens of a magnifying glass. The idea is to scan around the manifold with our lens and see what information we pick up. The constant $\varepsilon > 0$ is chosen small enough so that we can never see two particles of the configuration with our lens. Now for a chosen z and ε we ask 'what is inside the lens?' If there is no particle of the configuration inside we send z to $*_z$, the basepoint in the fibre over z. If there is a particle m_i with label x_i inside the lens we send z to $\exp_z^{-1}(m_i) \wedge x_i$. Note that we are also assuming that ε is small enough that $\exp_z$ is a diffeomorphism between an ε-disk around z and an ε-disk in T_zW. This description roughly corresponds to the definition we now give.

Definition 2.6. The *scanning map*

$$\gamma : C(M, M_0; X) \longrightarrow \Gamma(W \smallsetminus M_0, W \smallsetminus M; X)$$

is defined as follows [BM88]. For $[(\boldsymbol{m};\boldsymbol{x})] \in C(M, m_0; X)$ and $z \in W \smallsetminus M_0$, $\gamma([(\boldsymbol{m};\boldsymbol{x})])(z)$ is the image of $[(\boldsymbol{m};\boldsymbol{x})]$ under the composition

$$\begin{aligned} C(M, M_0; X) &\longrightarrow C(M, M_0 \cup (M \smallsetminus \mathring{D}(z)); X) \longrightarrow C(D(z), \partial D(z); X) \\ &\longrightarrow C(D_zW, \partial D_zW; X) \longrightarrow (D_zW/\partial D_zW) \wedge X \longrightarrow \overset{\infty}{T}W \wedge_W (W \times X). \end{aligned}$$

Here the first map is the quotient, the second is excision, the third is the inverse of the exponential map ($D(z)$ is a disk around z small enough that exp is a diffeomorphism and D_zW is the corresponding disk in T_zW). The fourth map is a little less straightforward. There is a natural inclusion $(D_zW/\partial D_zW) \wedge X \hookrightarrow C(D_zW, \partial D_zW; X)$ by sending a point $m \wedge x$ to $[(m, x)]$ and $*$ to the empty configuration. The configuration space deformation retracts onto the image of this inclusion by a stretching map which pulls the particles away from the centre of D_zW until there is at most one remaining. The final map is the inclusion of a fibre.

Although it is possible to understand what the scanning map does, it is not a very precise definition. Our aim is to give a description which puts the scanning map in a more natural setting and hence demonstrate equivariance under the action of the diffeomorphism group of M. In order to check this equivariance we need a description of the action on both the configuration space and the section space. We restrict our attention to diffeomorphisms of M which fix the submanifold M_0 and the boundary of M. However, since the section space is defined in terms of a larger manifold W we will instead consider the action of the group $\mathrm{Diff}(W, M_0 \cup (W \setminus M))$. These groups are isomorphic by restricting or extending a diffeomorphism as appropriate.

To define the action on $C(M, M_0; X)$ we first assume X has a (possibly trivial) $\mathrm{Diff}(W, M_0 \cup (W \setminus M))$ action. We require that this action fixes the basepoint x_0. For any $\phi \in \mathrm{Diff}(W, M_0 \cup (W \setminus M))$ the action on $C(M, M_0; X)$ is given by

$$\phi \cdot [(m_1, \ldots, m_k; x_1, \ldots, x_k)] = [(\phi(m_1), \ldots, \phi(m_k); \phi \cdot x_1, \ldots, \phi \cdot x_k)].$$

This the most natural action on the configuration space as it simply maps the manifold under the diffeomorphism and carries the particles along with it.

The action on a section $s \in \Gamma(W \setminus M_0, W \setminus M; X)$ is given by

$$\phi \cdot s = (d\phi \wedge_W (\phi \times \phi)) \circ s \circ \phi^{-1}.$$

The action conjugates the section by the inverse action on the base and the corresponding action on the total space so that the section still maps points in the manifold into the correct fibre.

It is immediate that with the current description the scanning map is not equivariant under this action. It is dependent on a Riemannian metric, disks on the manifold and the exponential map. In particular the fact that an ε-disk on the manifold does not necessarily map to an ε-disk under a diffeomorphism prevents equivariance. It is this particular difficulty that we will overcome by modelling the configuration space with a larger space which defines these disks in a more convenient way. However, the scanning map as it is currently defined is equivariant under that action of compact Lie groups [BM88] since they can be made to act by isometry, preserving the ε-disks.

2.3 Stable splittings

We model mapping and section spaces with configuration spaces because they have particularly nice properties. For example the configuration space $C(M, M_0; X)$ has a filtration by closed sets, determined by the lengths of configurations.

$$C^k(M, M_0; X) := \left(\coprod_{i=0}^{k} \tilde{C}_i(M) \times_{\Sigma_i} X^i \right) \Big/ \approx$$

These filtrations allow us to construct stable splittings of configuration spaces and hence mapping spaces.

Theorem 2.7. *Let $C^k = C^k(M, M_0; X)$ be the filtrations of $C(M, M_0; X)$ and let $D^k = C^k/C^{k-1}$ be quotients of successive filtrations. Then there is a stable homotopy equivalence*

$$C(M, M_0; X) \overset{s}{\simeq} \bigvee_{k \geq 1} D^k$$

for any (M, M_0) and X.

We present the proof given by Bödigheimer [Böd87], following the methods of F. Cohen [Coh83]. We need the following construction.

Definition 2.8. We define the *power set map* $\sigma_k : C(M, M_0; X) \to C(C_k(M); D^k)$ as follows. For convenience denote elements of $C = C(M, M_0; X)$ by formal sums. Choose a configuration $\xi_\alpha = \Sigma_{i \in \alpha} m_i x_i \in C$ and a cardinality k subset $\beta \subset \alpha$ where α is the index set of ξ_α. Let $m_\beta = \Sigma_{i \in \beta} m_i \in C_k(M)$ be an unlabelled subconfiguration of ξ_α and let $\xi_\beta = \Sigma_{i \in \beta} m_i x_i \in C^k$ be the corresponding labelled subconfiguration. Denote by $\bar{\xi}_\beta$ the image of ξ_β under the quotient map $C^k \to D^k$. Then we define the power set map by the formal sum $\sigma_k(\xi_\alpha) = \Sigma_\beta m_\beta \bar{\xi}_\beta$ where β ranges over all cardinality k subsets of α.

Proof of Theorem 2.7. Choose an embedding of $C(M) = \amalg C_k(M)$ into $\mathbb{R}^\infty$. The power set maps together with this embedding and the inclusions $D^k \hookrightarrow V := \bigvee_{k \geq 1} D^k$ define a map $\sigma : C(M, M_0; X) \to C(\mathbb{R}^\infty; V)$. The scanning map is an approximation

$C(\mathbb{R}^n; V) \to \Omega^n\Sigma^n V$ so passing to the colimit over n yields $\gamma : C(\mathbb{R}^\infty; V) \to \Omega^\infty\Sigma^\infty V$. Let $p : \Sigma^\infty C \to \Sigma^\infty V$ be the adjoint of the composition $\gamma \circ \sigma$, then p clearly preserves the filtration of C and the filtration $V^k = \bigvee_{i=1}^k D^i$ of V. We have a commutative digram

$$\begin{tikzcd}
\Sigma^\infty(C^{k-1}) \arrow[r, "p"] \arrow[d] & \Sigma^\infty(V^{k-1}) \arrow[d] \\
\Sigma^\infty(C^k) \arrow[r, "p"] \arrow[d] & \Sigma^\infty(V^k) \arrow[d] \\
\Sigma^\infty(C^k/C^{k-1}) \arrow[r, "p\simeq Id"] & \Sigma^\infty(V^k/V^{k-1})
\end{tikzcd}$$

where the vertical sequences are cofibrations of spectra, and hence fibrations. The lower horizontal arrow is homotopic to the identity. Passing to homotopy groups and noting that $C^1 = V^1$ the result follows by induction on k. $\square$

Since the scanning map is equivariant under the action of a compact Lie group G, there is a version of the theorem for the Borel constructions associated to G.

Theorem 2.9. *Let G be a compact Lie group acting on M by isometry and fixing M_0. Let $C^k = C^k(M, M_0; X)$ be the filtrations of $C = C(M, M_0; X)$ and let $D^k(M, M_0; X) = C^k/C^{k-1}$. Then there is a homotopy equivalence*

$$\Omega^\infty\Sigma^\infty(EG \ltimes_G C(M, M_0; X)) \simeq \prod_{k\geq 1} \Omega^\infty\Sigma^\infty(EG \ltimes_G D^k(M, M_0; X))$$

The proof generalises the proof of the nonequivariant case. Instead of embeddings of $C(M)$ into $\mathbb{R}^\infty$ we embed each $C_k(M)$ G-equivariantly into a G-representation space V_k. Thom spaces of Ex-spaces are used to construct maps

$$\Omega^{n(k,r)}\Sigma^{n(k,r)}(E_rG \ltimes_G C) \longrightarrow \Omega^{n(k,r)}\Sigma^{n(k,r)}(E_rG \ltimes_G D_k)$$

where E_rG is the the preimage under the projection $EG \to BG$ of the r-skeleton of BG and $n(k, r)$ is an integer depending on k and r. By passing to ∞ over r we have maps $q_k : \Omega^\infty\Sigma^\infty(EG \ltimes_G C) \to \Omega^\infty\Sigma^\infty(EG \ltimes_G D^k)$ whose product give the homotopy equivalence in the theorem by induction in a similar manner to the previous proof. For a full proof see Bödigheimer and Madsen [BM88].

3 A model for configuration spaces

We construct a fibre bundle over the ordered configuration space $\tilde{C}_k(M)$, whose fibre over any configuration $\boldsymbol{m} = (m_1, \cdots, m_k)$ is $\mathrm{Emb}^{Id}(\amalg(T_{m_i}M, 0_i); M, \amalg m_i)$. Here the notation of pairs means that each is $0_i \in T_{m_i}M$ mapped to $m_i \in M$ and Emb^{Id} is a subspace of the standard embedding space equipped with the C^∞ topology. The space is further restricted to embeddings f with the the additional property that the derivative df induces the identity on $\amalg T_{m_i}M$ via

$$\amalg T_{m_i}M = T(\amalg T_{m_i}M)|_{\amalg m_i} \xrightarrow{df} TM|_{\amalg m_i} = \amalg T_{m_i}M$$

where each m_i is identified with $0_i \in T_{m_i}M$. We refer to this property as Id.

We define this bundle as the disjoint union of the required fibres, equipped with a topology which restricts to the ordinary C^∞ topology on each fibre. Note that the additional property Id ensures that the embedding space over each configuration is in fact the space of tubular neighbourhoods $\mathrm{Tub}^{Id}(\amalg m_i)$ as defined in Appendix A. Since a collection of points is compact, this space is contractible by Corollary A.7 and the projection of the constructed fibre bundle is a homotopy equivalence.

3.1 Fibrewise topology

Our model for configuration spaces makes use of a fibrewise construction which associates a pair of fibre bundles over the same base to a fibre bundle of maps between their fibres. We begin by describing enough of the theory of fibrewise topology for our purposes. We describe the *fibrewise mapping space* before constructing a smooth version. For a full treatment of the fibrewise theory we refer the reader to Crabb & James [CJ98] and James [Jam89].

Definition 3.1. A *fibrewise topological space* is a topological space X together with a continuous *projection* $p : X \to B$ where B is another topological space, the *base*.

We say X is a fibrewise topological space over B. For every point $b \in B$ the *fibre* over b is the subspace $X_b = p^{-1}(b)$.

Remark 3.2. This is a generalisation of the notion of fibre bundle or fibration. However, there are many less restrictions on the spaces involved. We do not require fibres to homeomorphic as we would in a fibre bundle. We do not even require that the projection is surjective, allowing for the possibility of empty fibres. Obvious examples of fibrewise spaces are a space B over itself with with the identity as the projection and $B \times T$ for some space T together with projection onto the first factor.

Definition 3.3. Let X and Y be fibrewise spaces over B with projections p and q, respectively. A *fibrewise map* $\phi : X \to Y$ is a map such that $q \circ \phi = p$. That is, a map which sends each fibre of X to a subspace of the corresponding fibre of Y.

Remark 3.4. Let $p : X \to B$ be a fibrewise space and let $B' \subset B$ be a subspace. Then we regard $X_{B'} := p^{-1}(B')$ as a fibrewise space over B' with the restriction of p as the projection. Given another fibrewise space $q : Y \to B$ and a fibrewise map $\phi : Y \to X$, the restriction $\phi_{B'} : X_{B'} \to Y_{B'}$ is also a fibrewise map.

Many of the ideas of topology generalise to fibrewise topology.

Definition 3.5. A fibrewise topological space $p : X \to B$ is

1. *fibrewise open* if the projection p is open.
2. *fibrewise closed* if p is closed.
3. *fibrewise compact* if p is proper.

Remark 3.6. Fibrewise compactness is equivalent to compactness in each fibre and a closed projection.

Definition 3.7. Given a set X, a topological space B and a set function $p : X \to B$, a *fibrewise topology* for (X, B, p) is any topology on X which makes the projection p continuous.

For a fibrewise topology there are notions equivalent to base and subbase. By a *fibrewise base* for a fibrewise topology on X we mean a collection $\mathcal{U}$ of subsets of X such that the intersections of members of $\mathcal{U}$ with sets of the form X_W, where $W \subseteq B$ is open, form a base for the topology. A *fibrewise subbase* for a topology is a collection of subsets of X whose finite intersections form a fibrewise base.

3.2 The fibrewise mapping space

Although we have only seen a fragment of the fibrewise theory of topology we have developed enough of the theory to define a fibrewise version of the mapping space.

Definition 3.8. Given two fibrewise topological spaces X and Y over the same base B we define the fibrewise mapping space from X to Y as the fibrewise set

$$\mathrm{Map}_B(X, Y) := \coprod_{b \in B} \mathrm{Map}(X_b, Y_b)$$

over B equipped with an appropriate fibrewise topology.

The topology is a fibrewise generalisation of the compact-open topology. It would be convenient to define this topology in such away that the induced topology on the fibre $\mathrm{Map}(X_b, Y_b)$ over $b \in B$ is the ordinary compact-open topology. The topology we construct is coarser in general, but in certain cases restricts to the compact-open topology in the fibres.

Definition 3.9. Given $W \subseteq B$, $V \subseteq Y_W$ and $K \subseteq X_W$ we denote by $(K, V; W)$ the set of maps $\psi : X_b \to Y_b$, where $b \in W$, for which $\psi(K_b) \subseteq V_b$. If W is open in B, K is fibrewise compact over W and V is open in Y_W we call the subset $(K, V; W)$ of $\mathrm{Map}_B(X, Y)$ *fibrewise compact-open*. These sets form a fibrewise subbase for the *fibrewise compact-open topology*.

Proposition 3.10. *Let* $p : X \to B$ *be a fibre bundle and let* $q : Y \to B$ *be a fibrewise space. Then for any point* $b \in B$ *the topology on the fibre* $\mathrm{Map}(X_b, Y_b)$ *induced by* *the fibrewise compact-open topology on* $\mathrm{Map}_B(X,Y)$ *is the ordinary compact-open topology.*

Proof. In the fibre over $b \in B$, the fibrewise compact-open set $(K,V;W) \subseteq \mathrm{Map}_B(X,Y)$ restricts to the set (K_b, V_b) of maps $\psi : X_b \to Y_b$ such that $\psi(K_b) \subseteq V_b$, where K_b is compact and V_b is open in Y_b. This is precisely a compact-open set in the ordinary mapping space. It suffices to check that every compact-open set is the restriction of some fibrewise compact-open set.

Let W be an open neighbourhood of a point $b \in B$ such that there exists a local trivialisation $\phi : W \times X_b \to X_W$ for b. In particular ϕ is a fibrewise homeomorphism and W is open in B. Let $K \subseteq X_b$ be compact and $V \subseteq Y_b$ be open, then $W \times K$ is fibrewise compact over W and the image $\phi(W \times K) \subseteq X_W$ is fibrewise compact. Let U be an open subset of Y_W such that $U_b = V$, then the fibrewise compact-open set $(\phi(W \times K), U; W)$ restricts to the compact-open set (K,V) in the fibre over b. □

3.3 The fibrewise smooth mapping space

Just as James [Jam89] generalises the compact-open topology for fibrewise spaces, we generalise the C^∞ topology. We begin with a reminder of the definition of the C^r and C^∞ topologies.

Definition 3.11. Let M, N be C^r manifolds (with r finite) and let $C^r(M,N)$ denote the set of C^r maps from M to N. The C^r *topology* on $C^r(M,N)$ is generated by the following sets. Let $f \in C^r(M,N)$, let (ϕ, U) and (ψ, V) be charts for M and N respectively. Let $K \subset U$ be a compact set such that $f(K) \subset V$ and let $0 < \varepsilon \leq \infty$. Define a *subbasic neighbourhood*

$$\mathcal{N}^r(f; (\phi, U), (\psi, V), K, \varepsilon)$$

to be the set of C^r maps $g : M \to N$ such that $g(K) \subset V$ and

$$||D^k(\psi f \phi^{-1})(x) - D^k(\psi g \phi^{-1})(x)|| < \varepsilon$$

for all $x \in \phi(K)$ and $k = 0, \dots, r$ [Hir76].

Definition 3.12. If M and N are C^∞ manifolds, the C^∞ *topology* on $C^\infty(M,N)$ is the union of the topologies induced by the inclusions $C^\infty(M,N) \hookrightarrow C^r(M,N)$.

With this definition in mind we will define a fibrewise topology on the fibrewise C^r mapping space

$$C^r_B(X,Y) := \coprod_{b\in B} C^r(X_b, Y_b)$$

where X and Y are C^r fibre bundles over B. That is, B, X and Y are C^r manifolds and the projections are C^r maps. In this case every $b \in B$ must be a regular value of both projections and the fibres X_b, Y_b are therefore C^r manifolds. We consider only fibre bundles as this is all we need for our purposes.

Definition 3.13. Given B, X and Y as above, the *fibrewise* C^r *topology* on $C^r_B(X,Y)$ is generated by the following sets. Let $W \subset B$ be open, let $f : X_W \to Y_W$ be a C^r bundle map and let (ϕ, U), (ψ, V) be charts on X, Y respectively such that $U \cap X_W$ and $V \cap Y_W$ are nonempty. Let $K \subset U \cap X_W$ be fibrewise compact over W such that $f(K) \subset V \cap Y_W$ and let $\varepsilon : W \to \mathbb{R}_{>0}$ be continuous. Define a *fibrewise subbasic neighbourhood*

$$\mathcal{N}^r(f; (\phi, U), (\psi, V), K, \varepsilon; W)$$

to be the set of C^r maps $g : X_b \to Y_b$, where $b \in W$, such that $g(K_b) \subset V_b$ and

$$||D^k(\psi_b f_b \phi_b^{-1})(x) - D^k(\psi_b g \phi_b^{-1})(x)|| < \varepsilon(b)$$

for all $x \in \phi_b(K_b)$ and $k = 0, \dots, r$. Here (ϕ_b, U_b), (ψ_b, V_b) are charts for X_b and Y_b respectively, obtained by restricting the charts ϕ, ψ to the fibre over b.

Definition 3.14. Let X, Y be C^∞ fibre bundles over B. The *fibrewise* C^∞ or *fibrewise smooth mapping space* is the set

$$C^\infty_B(X,Y) := \coprod_{b\in B} C^\infty(X_b, Y_b)$$

equipped with the *fibrewise* C^∞ *topology*, the union of the topologies induced by the fibrewise inclusions $C^\infty_B(X,Y) \hookrightarrow C^r_B(X,Y)$.

Proposition 3.15. *The fibrewise C^∞ topology is a fibrewise topology.*

Proof. We need to check the obvious projection $q : C^\infty_B(X,Y) \to B$ is continuous. Let $W \subset B$ be open, then $q^{-1}(W) = C^\infty_W(X_W, Y_W)$. Let $f : X_W \to Y_W$ be any fibrewise map, let $(\phi, U), (\psi, V)$ be any charts for X, Y with nonempty intersection with X_W and Y_W respectively. Choose any $\varepsilon > 0$ and any $r \in \mathbb{N}$. The fibrewise subbasic neighbourhood $\mathcal{N}^r(f; (\phi, U), (\psi, V), \emptyset, \varepsilon; W)$ is the set of smooth maps $g : X_b \to Y_b$ since taking the fibrewise compact set to be the empty set removes all the other conditions. This neighbourhood is precisely $q^{-1}(W)$. □

Proposition 3.16. *Let X, Y be C^∞ fibre bundles over B. Then for any point $b \in B$ the topology on the fibre $C^\infty(X_b, Y_b)$ induced by the fibrewise C^∞ topology on $C^\infty_B(X,Y)$ is the ordinary C^∞ topology.*

Proof. It is clear that the fibrewise neighbourhood $\mathcal{N}^r(f; (\phi, U), (\psi, U), K, \varepsilon; W)$ restricts to the neighbourhood $\mathcal{N}^r(f_b; (\phi_b, U_b), (\psi_b, V_b), K_b, \varepsilon(b))$ in the fibre over $b \in B$. We need only check that every subbasic neighbourhood for each of the C^r topologies is the restrictions of some fibrewise subbasic neighbourhood.

Let $b \in B$ and let $\mathcal{N}^r(f; (\phi, U), (\psi, U), K, \varepsilon; W)$ be a subbasic neighbourhood for $C^\infty(X_b, Y_b)$. We will show it is the restriction of a fibrewise subbasic neighbourhood to the fibre over b. Choose local trivialisations $\tau_1 : W_1 \times X_b \to X_{W_1}$, $\tau_2 : W_2 \times Y_b \to Y_{W_2}$ and a chart (σ, W_3) around b. Let $W = \cap_i W_i$. Now $((\sigma \times \phi) \circ \tau_1^{-1}, \tau_1(W \times U))$ and $((\sigma \times \psi) \circ \tau_2^{-1}, \tau_2(W \times V))$ are charts for X_W and Y_W respectively. Since K is compact, $\tau_1(W \times K) \subset \tau_1(W \times U) \subset X_W$ is fibrewise compact over W. Define a bundle map $\tilde{f} : X_W \to Y_W$ by $x \mapsto (\tau_2 \circ (Id_W \times f) \circ \tau_1^{-1})(x)$, then $\tilde{f}(\tau_1(W \times K)) \subset \tau_2(W \times V)$. Let $\tilde{\varepsilon} : W \to \mathbb{R}_{>0}$ be the constant map with value ε. Then the fibrewise subbasic neighbourhood

$$\mathcal{N}^r(\tilde{f}; ((\sigma \times \phi) \circ \tau_1^{-1}, \tau_1(W \times U)), ((\sigma \times \psi) \circ \tau_2^{-1}, \tau_2(W \times V)), \tau_1(W \times K), \tilde{\varepsilon}; W)$$

restricts to the weak subbasic neighbourhood $\mathcal{N}^r(f; (\phi, U), (\psi, U), K, \varepsilon; W)$. □

3.4 The fibrewise embedding space

The ideas we have just developed can be used to define fibrewise spaces of embeddings which model the configuration spaces. Throughout this section we assume M is a smooth compact n-manifold and W is a smooth manifold without boundary containing M as a codimension 0 submanifold, as in the definition of the scanning map. We will use the notation $T_{m_i}M$ and $T_{m_i}W$ interchangeably to mean the same tangent space when $m_i \in M \subseteq W$.

We restrict our attention to the special case $\tilde{C}_1(M) = M$ at first. Define the fibrewise embedding space

$$\mathrm{Emb}^{Id}_M(TM, M; M \times W, \Delta) := \coprod_{m \in M} \mathrm{Emb}^{Id}(T_m M, 0; \{m\} \times W, \{m\} \times \{m\})$$

as a subspace of the fibrewise smooth mapping space associated to the tangent bundle and the product bundle $\pi_1 : M \times W \to M$. Here $M \subset TM$ is identified with the (image of the) zero section. Since the structure of $M \times W$ is trivial we denote the fibres of the fibrewise embedding space by $\mathrm{Emb}^{Id}(T_m M, 0; M, m)$ for simplicity.

In order to show this is a fibre bundle we must find local trivialisations. Choose $m \in M$ and let $\phi : U \times T_m M \to TM|_U$ be a local trivialisation for TM around m. Then we can attempt to define a local trivialisation for $\mathrm{Emb}^{Id}_M(TM, M; M \times W, \Delta)$ around m by

$$\begin{aligned} \Phi : \coprod_{n \in U} \mathrm{Emb}^{Id}(T_n M, 0; M, n) &\longrightarrow U \times \mathrm{Emb}^{Id}(T_m M, 0; M, m) \\ f &\longmapsto (n, f \circ \phi_n) \end{aligned}$$

where $\phi_n = \phi(n, -) : T_m M = \{n\} \times T_m M \to T_n M$. However, maps in the image do no necessarily map $0 \in T_m M$ to $m \in M$. To solve this problem we will need the following lemma.

Lemma 3.17. *There exists a continuous family of diffeomorphisms of the closed k-ball, parameterised by the open k-ball and fixing the boundary,*

$$\tau_k : D^k \longrightarrow \mathrm{Diff}(\bar{D}^k, \partial D^k)$$

such that $\tau_k(x)$ maps x to the midpoint, 0, and 0 to $-x$.

Proof. For fixed k and $x = (x_1, \dots, x_k) \in D^k$ consider the maps

$$\begin{aligned} \sigma_k(x) : \bar{D}^k &\longrightarrow \bar{D}^k \\ y = (y_1, \dots, y_k) &\longmapsto \frac{y_1 - x_1}{1 - y_1 x_1} \sqrt{1 - y_2^2 - \dots - y_k^2}. \end{aligned}$$

Then we define $\tau_k(x) \coloneqq g_x \circ \sigma_k \circ g_x^{-1}$ where $g_x \in SO_k(\mathbb{R})$ is the rotation which sends $(\|x\|, 0, \dots, 0)$ to x. It is sufficient to check the case $k = 1$, that is that the map $\sigma_1 : (-1, 1) \to \mathrm{Diff}([-1, 1], \{-1, 1\})$ is continuous.

Recall that the subbasic neighbourhoods $\mathcal{N}^r(f; U, K, \varepsilon)$ are the sets of diffeomorphisms $g : [-1, 1] \to [-1, 1]$ such that $g(K) \subset U$ and

$$\|D^n f(y) - D^n g(y)\| < \varepsilon$$

for all $y \in K$, $n = 0, \cdots, r$. It suffices to check that the preimages of such neighbourhoods under σ_1 are open for each pair $U = [-1, a)$, $K = [-1, c]$ or $U = (b, 1]$, $K = [d, 1]$, each $\varepsilon > 0$ and each $r \in \mathbb{N}$. We need only consider such Us and Ks since the intervals given generate the open and compact subsets of $[-1, 1]$. Consider first the compact open sets $(U, K) \coloneqq \{g : g(K) \subset U\}$. Certainly we have $\sigma_1^{-1}([-1, c], [-1, a)) = (-1, \frac{c-a}{1-ca})$ and $\sigma_1^{-1}([d, 1], (b, 1]) = (\frac{d-b}{1-db}, 1)$, and since all the diffeomorphisms fix the boundary $\sigma_1^{-1}([-1, c], (b, 1]) = \emptyset = \sigma_1^{-1}([d, 1], [-1, a)$ for all a, b, c, d. The map is certainly continuous in the compact open topology.

Consider a subbasic neighbourhood $\mathcal{N}^r(f; U, K, \varepsilon)$. We may assume $f = \sigma_1(s)$ for some $s \in (-1, 1)$. For such an s and a fixed n the derivative $D^n\sigma_1(s)(y) - D^n\sigma_1(x)(y)$ is a polynomial fraction and continuous in both x and y. Since K is of the form

$[-1, c]$ or $[d, -1]$ we need only check the condition '$< \varepsilon$' holds for a finite number of $y \in K$. This is an open condition so the preimage of a subbasic neighbourhood is a finite intersection of open sets and hence open. Thus σ_1 and hence all σ_k and τ_k are continuous. □

There exists a tubular neighbourhood $V \subset U$ of m diffeomorphic to an open ball via a diffeomorphism which maps the midpoint to m. By the previous lemma we can find a family of diffeomorphisms $\tau : V \to \mathrm{Diff}(\bar{V}, \partial V)$ such that τ_n maps n to m. Since these diffeomorphisms fix ∂V they extend to a family of diffeomorphisms of W fixing $W \smallsetminus V$, which we also denote by τ. The image of the map $f \mapsto (n, \tau_n \circ f \circ \phi_n)$ now sends $0 \in T_m M$ to $m \in M$ as required.

However, in order to satisfy the property Id we need to ensure the composition

$$T_m M = T_0(T_m M) \xrightarrow{d_0 \phi_n} T_0(T_n M) \xrightarrow{d_0 f} T_n M \xrightarrow{d_n \tau_n} T_m M$$

is the identity on $T_m M$. This is achieved by choosing the trivialisations ϕ carefully. Define $\phi : V \times T_m M \to TM|_V$ by $\phi : (n, v) \mapsto (d_n \tau_n)^{-1}(v)$. Since each τ_n is a diffeomorphism, ϕ restricts to a linear isomorphism on the fibres and the differential on the zero section is given by $d_0 \phi_n = (d_n \tau_n)^{-1}$. The composition now satisfies the required properties and defines a local trivialisation

$$\Phi : \mathrm{Emb}_V^{Id}(TM|_V, V; V \times M, \Delta) \longrightarrow V \times \mathrm{Emb}^{Id}(T_m M, 0; M, m).$$

To see that this is in fact a local trivialisation we construct an explicit inverse

$$\begin{aligned} \Psi : V \times \mathrm{Emb}^{Id}(T_m M, 0; M, m) &\longrightarrow \mathrm{Emb}_V^{Id}(TM|_V, V; V \times M, \Delta) \\ (n, g) &\longmapsto (Id_V \times (\tau_n^{-1} \circ g)) \circ \phi^{-1}. \end{aligned}$$

The construction of the fibrewise embedding space over M generalises to a fibre bundle over $\tilde{C}_k(M)$ for any k. First we need a fibre bundle over $\tilde{C}_k(M)$ whose fibre over $(m_1, \ldots, m_k)$ is the disjoint union $\amalg_k T_{m_i} M$. Recall that $\tilde{C}_k(M)$ is a subspace of the k-fold product M^k obtained by removing the diagonal Δ_k. Let π_i be the projection onto the i-th factor, then the pullback π_i^*TM is a fibre bundle over M^k with fibre $T_{m_i}M$ over $(m_1, \ldots, m_i, \ldots, m_k)$. Let (ϕ_i, U_i) be a local trivialisation for TM around m_i, then (ϕ_i^i, U_i^i), where $\phi_i^i = Id_M \times \cdots \times \phi_i \times \cdots \times Id_M$ and $U_i^i = M \times \cdots \times U_i \times \cdots \times M$ with ϕ_i and U_i in the ith component, is a local trivialisation for π_i^*TM around any point with m_i in the ith component.

Denote by $\dot{\amalg}_k \pi_i^*TM$ the *fibrewise* disjoint union of these pullbacks. This is a fibre bundle over M^k with fibre $\amalg_k T_{m_i}M$ over $(m_1, \ldots, m_k)$ and local trivialisations $(\amalg_k \phi_i^i, \cap_k U_i^i) = (\amalg_k \phi_i^i, \Pi_k U_i)$. Taking the pullback of $\dot{\amalg}_k \pi_i^*TM$ under the inclusion $\tilde{C}_k(M) \hookrightarrow M^k$ gives the required fibre bundle. We assume this pullback has the same local trivialisations and the products $\prod_k U_i$ have empty intersection with Δ_k, else restrict the U_i's.

Definition 3.18. The *ordered tubular configuration space* is the disjoint union

$$\tilde{E}_k(M) := \coprod_{\boldsymbol{m} \in \tilde{C}_k(M)} \mathrm{Emb}^{Id}(\amalg_k(T_{m_i}, 0); \boldsymbol{m} \times W, \boldsymbol{m} \times \amalg_k m_i)$$

topologised as a subspace of the fibrewise smooth mapping space associated to the bundle $\dot{\amalg}_k \pi_i^*TM|_{\tilde{C}_k(M)}$ and the product bundle $\tilde{C}_k(M) \times W \to \tilde{C}_k(M)$.

As above we denote the fibre over $\boldsymbol{m} = (m_1, \ldots, m_k)$ by $\mathrm{Emb}^{Id}(\amalg_k(T_{m_i}, 0); W, \amalg_k m_i)$ for simplicity. An element of $\tilde{E}_k(M)$ is an embedding $f : \amalg_k T_{m_i}M \to W$ for some $\boldsymbol{m}$. Each embedding is defined by a map on each of the components in the domain, denoted by $f_i : T_{m_i}M \to W$. We will use the notation $\amalg_k f_i$ and f interchangeably and sometimes $f_{\boldsymbol{m}}$ to signify that f is in the fibre over $\boldsymbol{m}$. The projection map $\tilde{p}_k : \tilde{E}_k(M) \to \tilde{C}_k(M)$ sends an embedding f to the configuration $(f_1(0), \ldots, f_k(0))$.

To construct local trivialisations for $\tilde{E}_k(M)$ we proceed in a similar manner as before. For each i let $V_i \subset U_i$ be an open neighbourhood of m_i that is diffeomorphic to a disk via a diffeomorphism which maps the midpoint to m_i. Then by Lemma 3.17 there exist continuous families of diffeomorphisms $\tau^i : V_i \to \mathrm{Diff}(\bar{V}_i, \partial V_i)$ such that τ_n^i

maps n to m. Since the product of the U_i's have empty intersection with Δ_k we have $V_i \cap V_j = \emptyset$ for any $i \neq j$. We define a continuous family of diffeomorphisms

$$\begin{aligned} \tau : \prod_k V_i &\longrightarrow \mathrm{Diff}(\amalg_k \bar{V}_i, \amalg_k \partial V_i) \\ \boldsymbol{n} &\longmapsto \tau_{\boldsymbol{n}} := \amalg_k \tau^i_{n_i} \end{aligned}$$

which extend to diffeomorphisms of W fixing $W \smallsetminus \amalg_k V_i$.

Define a local trivialisation for $\tilde{E}_k(M)$ around $\boldsymbol{x}$ by

$$\begin{aligned} \Phi : \tilde{E}_k(M)|_V &\longrightarrow V \times \mathrm{Emb}^{Id}(\amalg_k(T_{x_i}, 0); W, \amalg_k m_i) \\ f_{\boldsymbol{n}} &\longmapsto (\boldsymbol{n}, \tau_{\boldsymbol{n}} \circ f_{\boldsymbol{n}} \circ \phi_{\boldsymbol{n}}) \end{aligned}$$

where $V := \Pi_k V_i$ and $\phi_{\boldsymbol{n}} := \amalg_k \phi^i_i(\boldsymbol{n}, -) : \amalg_k T_{m_i} M \to \amalg_k T_{n_i} M$.

In order to ensure that property Id is satisfied, define each local trivialisation $\phi_i : V_i \times T_{m_i} M \to TM|_{V_i}$ of TM around m_i by $(n_i, v_i) \mapsto (d_{n_i} \tau^i_{n_i})^{-1}(v_i)$.

Just as in the previous section we can define an explicit inverse to Φ. Note that in the case $k = 1$ this tubular configuration space is precisely the fibrewise embeddings space over M as defined previously.

Remark 3.19. The fibre over any configuration $\boldsymbol{m}$ is the space $\mathrm{Tub}^{Id}(\boldsymbol{m})$ of tubular neighbourhoods of the particles in $\boldsymbol{m}$. This space is contractible by Corollary A.7 so the projection $\tilde{p}_k : \tilde{E}_k(M) \to \tilde{C}_k(M)$ is a homotopy equivalence. A homotopy inverse is given by any continuous section $\tilde{s}_k : \tilde{C}_k(M) \to \tilde{E}_k(M)$.

In particular we can define a section using the exponential map. Choose a Riemannian metric d on W. For a configuration $\boldsymbol{m}$ let $\delta_{\boldsymbol{m}} = \min\{d(m_i, m_j) : i \neq j\}$ and for each i let δ_{m_i} be the largest value such that $\exp_{m_i} : B_{\delta_{m_i}}(T_{m_i} M, 0) \to B_{\delta_{m_i}}(W, m_i)$ is a diffeomorphism. Let $\varepsilon_{m_i} = \min\{\delta_{m_i}, \delta_{\boldsymbol{m}}\}$ and for each i define $f_i : T_{m_i} M \to W$ by $v \mapsto \exp_{m_i}(\frac{2\varepsilon_{m_i} v}{\pi |v|} \arctan(|v|))$.

Each of the f_i is an embedding and satisfies the property Id since exp does. Our construction ensures the images of each f_i are distinct so $f = \amalg_k f_i$ is an element of the fibre over $\boldsymbol{m}$. The map $\varepsilon : \tilde{C}_k(M) \to \mathbb{R}^k$, $\boldsymbol{m} \mapsto (\varepsilon_{m_1}, \ldots, \varepsilon_{m_k})$ is continuous so

$$\begin{aligned} \tilde{s}_k : \tilde{C}_k(M) &\longrightarrow \tilde{E}_k(M) \\ \boldsymbol{m} &\longmapsto \amalg_k f_i \end{aligned}$$

is a continuous section and a homotopy inverse to $\tilde{p}_k$.

3.5 Ordering, labels and vanishing particles

So far we have a model for the finite ordered configuration spaces of a manifold. However, the scanning map is defined on the infinite unordered labelled configuration space modulo a submanifold so we have some more work to do.

Ordering We begin by constructing a model for the unordered finite configuration spaces. To motivate our definition recall the definition of the *unordered* configuration space of k particles on M as the orbit space $C_k(M) \coloneqq \tilde{C}_k(M)/\Sigma_k$ where Σ_k is the symmetric group on k elements.

Definition 3.20. The symmetric group acts on the tubular configuration space $\tilde{E}_k(M)$ as follows. Let $f = \amalg_k f_i \in \tilde{E}_k(M)$ be in the fibre over $\boldsymbol{m} = (m_1, \ldots, m_k)$ and define $\sigma \cdot f = \amalg_k f_{\sigma(i)} : (T_{m_{\sigma(i)}}M, 0) \to (W, m_{\sigma(i)})$ in the fibre over $\sigma \cdot \boldsymbol{m}$. Define the *unordered* tubular configuration space to be the quotient space $E_k(M) \coloneqq \tilde{E}_k(M)/\Sigma_k$.

Remark 3.21. The fibre over $\boldsymbol{m}$ is homeomorphic to the fibre over $\sigma \cdot \boldsymbol{m}$ by reordering the disjoint union. The action of Σ_k on $\tilde{E}_k(M)$ is free and proper and the projection $\tilde{p}_k$ is clearly equivariant.

Proposition 3.22. *The unordered tubular configuration space $E_k(M)$ is a fibre bundle over the unordered configuration space $C_k(M)$ with projection p_k induced by $\tilde{p}_k$.*

Proof. There is a commutative diagram

$$\begin{tikzcd} \tilde{E}_k(M) \arrow[r, "\tilde{p}_k"] \arrow[d, "q_E"'] & \tilde{C}_k(M) \arrow[d, "q_C"] \\ E_k(M) \arrow[r, "p_k"] & C_k(M) \end{tikzcd}$$

where the vertical maps are covering maps. As such any local trivialisation $U \times \tilde{p}_k^{-1}(\boldsymbol{m}) \to \tilde{p}_k^{-1}(U)$ for $\tilde{E}_k(M)$ around $\boldsymbol{m}$ induces a trivialisation for $E_k(M)$ around $q_C(\boldsymbol{m})$. □

Observing that the fibre over the unordered configuration $\boldsymbol{m} = \{m_1, \dots, m_k\}$ is the space of tubular neighbourhoods $\mathrm{Tub}^{Id}(\{m_1, \dots, m_k\})$ we have the following corollary.

Corollary 3.23. *The projection $p_k : E_k(M) \to C_k(M)$ is a homotopy equivalence. The homotopy inverse is the section s_k induced by $\tilde{s}_k$.*

Labels The unordered configuration space with labels in a well-pointed space X are defined in much the same way as the labelled configuration spaces. There is a fibre bundle $p_k : E_k(M; X) := \tilde{E}_k(M) \times_{\Sigma_k} X^k \to C_k(M; X)$ over the unordered labelled configuration space where the projection is induced by $\tilde{p}_k \times Id_{X^k}$. The fibre over an unordered configuration $\{\boldsymbol{m}; \boldsymbol{x}\}$ is the space $\mathrm{Tub}^{Id}(\{m_1, \dots, m_k\}) \times \{\boldsymbol{x}\}$ which is contractible so p_k is a homotopy equivalence with homotopy inverse s_k induced by $\tilde{s}_k \times Id_{X^k}$.

Remark 3.24. There is another way to add labels to a tubular configuration space which might allow for a little more flexibility. Instead of simply lifting the label from the configuration to the fibre over that configuration, we could associate to each embedding $f : \amalg_k T_{m_i} M \to W$ a map $\amalg_k T_{m_i} M \to X$. The projection is evaluation at 0 in each tangent space and the fibre over the unordered configuration $\{\boldsymbol{m}; \boldsymbol{x}\}$ is $\mathrm{Tub}^{Id}(\{m_1, \dots, m_k\}) \times \mathrm{Map}(\amalg_k (T_{m_i} M, 0_i); X, \amalg_k m_i)$. Although this construction yields a fibre bundle with contractible fibres it adds more complexity than is required for our purposes. However, we will find that such an approach is useful when considering twisted labels in Section 6.

There is a canonical action of $\mathrm{Diff}(W, W \setminus M)$ on $E_k(M; X)$. The idea of the action is to compose each embedding with the diffeomorphism of W, once again yielding an embedding. In order to ensure the image of the zero section is preserved we precompose with the inverse of the derivative of the diffeomorphism. That is, for any $\phi \in \mathrm{Diff}(W, W \setminus M)$ and $f \in E_k(M)$ we define an action by $\phi \cdot f = \phi \circ f \circ d\phi^{-1}$.

Given an action of $\mathrm{Diff}(W, W \setminus M)$ on X fixing the basepoint we have a diagonal action on $E_k(M; X)$ just as for the unordered relative configuration space. It is easy to check this is a well defined action and it takes $f \in p_k^{-1}(\boldsymbol{m}; \boldsymbol{x})$ to $\phi \cdot f \in p_k^{-1}(\phi \cdot \boldsymbol{m}; \phi \cdot \boldsymbol{x})$.

Remark 3.25. The homotopy equivalences p_k between the unordered spaces are equivariant under this action but the sections s_k are not equivariant since they are dependent on structures which are not preserved under diffeomorphism.

Vanishing particles Suppose that M_0 is a (possible empty) compact submanifold of M. Recall that for relative configuration spaces we allowed particles in a configuration on M to vanish in M_0. We would like such a construction for the tubular configuration spaces and we proceed in the same way.

Definition 3.26. Given a (possible empty) compact submanifold $M_0 \subset M$, the *unordered relative tubular configuration space* is

$$E(M, M_0; X) := \coprod_{k \geq 0} E_k(M; X)/\sim$$

where $(f_1, \ldots, f_k; x_1, \ldots, x_k) \sim (f_1, \ldots, f_{k-1}; x_1, \ldots, x_{k-1})$ if $x_k = x_0$ or $f_i(0) \in M_0$.

The idea here is to allow a component f_i of an embedding f to vanish when the underlying particle $m_i = f_i(0)$ does. This is necessary to ensure that these spaces still model the configuration spaces properly.

The lengths of configurations in $C(M, M_0; X)$ and tubular configurations in $E(M, M_0; X)$ induce filtrations

$$E^k(M, M_0; X) = \coprod_{i=0}^{k} E_i(M; X)/\sim .$$

These filtrations have natural inclusions $E^{k-1}(M, M_0; X) \hookrightarrow E^k(M, M_0; X)$. It follows that topologically $E(M, M_0; X)$ is the colimit $\varinjlim E^k(M, M_0; X)$. Similarly $C(M, M_0; X) = \varinjlim C^k(M; M_0; X)$. The projections $p_k : E_k(M; X) \to C_k(M; X)$ induce maps $p^k : E^k(M, M_0; X) \to C^k(M; M_0; X)$. Elements of $C^k(M, M_0; X)$ are equivalence classes $[(\boldsymbol{m}; \boldsymbol{x})]$ and elements of $E^k(M, M_0; X)$ are classes $[(f; \boldsymbol{x})]$. We define p^k by $[(f; \boldsymbol{x})] \mapsto [p_l(f; \boldsymbol{x})]$ where $l = \mathrm{card}(f; \boldsymbol{x})$. The way in which components of embeddings vanish ensures the projection is independent of the choice of representative in each class.

The projection is still a homotopy equivalence. Consider the map

$$\begin{aligned} s^k : C^k(M; M_0; X) &\longrightarrow E^k(M, M_0; X) \\ [(\boldsymbol{m}; \boldsymbol{x})] &\longmapsto [(s_l \times Id)(\boldsymbol{m}; \boldsymbol{x})] \end{aligned}$$

where $l = \mathrm{card}(\boldsymbol{m}; \boldsymbol{x})$ and the s_k's are the sections defined previously. Then $p^k \circ s^k = Id$ and the homotopy $s^k \circ p^k \simeq Id$ is given by the contraction of each fibre, $\mathrm{Tub}^{Id}(\amalg_l m_i) \times \{\boldsymbol{x}\}$. With a little more work we can show the homotopy equivalences on the filtrations carry over to $E(M, M_0; X)$.

Proposition 3.27. *If X has the homotopy type of a CW-complex then the spaces $C_k(M, X)$, $C^k(M, M_0; X)$, $C(M, M_0; X)$, $E_k(M, X)$, $E^k(M, M_0; X)$ and $E(M, M_0; X)$ do too.*

Proof. We generalise a method of McDuff [McD75]. Each $C_k(M)$ is a manifold and hence has the homotopy type of a CW-complex. If X has the homotopy type of a CW-complex then so does each $C_k(M; X)$. Each $C_k(M; X)$ has a subcomplex $B_k \coloneqq \{(\boldsymbol{m}; \boldsymbol{x}) : \exists i \text{ s.t. } m_i \in M_0 \text{ or } x_i = x_0\}$. Now $C^k(M, M_0; X)$ is obtained from

$C^{k-1}(M, M_0; X)$ by attaching $C_k(M; X)$ along the obvious map $B_k \to C^{k-1}(M, M_0; X)$. Therefore each $C^k(M, M_0; X)$ has the homotopy type of a CW-complex and so does the colimit $C(M, M_0; X) = \varinjlim C^k(M, M_0; X)$.

The same argument shows that $E_k(M, X)$, $E^k(M, M_0; X)$ and $E(M, M_0; X)$ have the homotopy types of CW-complexes. □

Corollary 3.28. *The canonical projection $p : E(M, M_0; X) \to C(M, M_0; X)$ defined by the p^k's is a weak homotopy equivalence for any X and a homotopy equivalence provided X has the homotopy type of a CW-complex.*

Proof. It follows immediately from the construction of $E(M, M_0; X)$ and $C(M, M_0; X)$ as colimits that p is a weak homotopy equivalence for any X. If X has the homotopy type of a CW-complex then by Whitehead's theorem p is a homotopy equivalence. The homotopy inverse s is obtained from the s^k's in the same way. □

4 The Scanning Map

With our model for configuration spaces we can define a scanning map which is equivariant under the action of the diffeomorphism group of the underlying manifold. The idea is to use the embeddings to define the section rather than the exponential map, avoiding the use of a metric and ε-disks. In this section we will define a scanning map $E(M, M_0; X) \to \Gamma(W \setminus M_0, W \setminus M; X)$ and prove that it is equivariant and a nonequivariant homotopy equivalence in certain cases.

4.1 Definition

We construct the scanning map using a sequence of maps

$$\gamma_k : E_k(M; X) \longrightarrow \Gamma(W \setminus M_0, W \setminus M; X)$$

in the same way as we construct the projection p. The idea of the map is to define the section of $\overset{\infty}{T}W \wedge_W (W \times X)$ associated to an embedding $f : \amalg_k T_{m_i}M \to W$ by f^{-1}. That is, given a point $z \in W \smallsetminus M_0$ we want $\gamma_k(f;\boldsymbol{x})(z)$ to be $f_i^{-1}(z) \wedge x_i$ if z is in the image of f_i and agree with the canonical section $*$ otherwise.

Remark 4.1. Throughout this section we will not distinguish between maps between tangent spaces and maps between their one point compactifications as it will be clear from context which is meant.

Simply using f^{-1} is not a section since $f_i^{-1}(z) \in \overset{\infty}{T}_{m_i}M$, not in $\overset{\infty}{T}_zW$. In order to correct this we need to move $f_i^{-1}(z)$ to the fibre over z. Although it might be nicer to use flow on the manifold to move the fibre, the main objective is to construct a diffeomorphism equivariant map. As such we prefer an explicit map and so we compose with a map $\overset{\infty}{T}_{m_i}M \to \overset{\infty}{T}_zW$. To find such a map we make use of the following lemma.

Lemma 4.2. *Let V be a finite dimensional vector space. Then given a vector $x \in V$, there is a canonical linear isomorphism $V \to T_xV$. Moreover, for any finite dimensional vector space W and any linear map $L : V \to W$ the following diagram commutes.*

$$\begin{tikzcd} V \arrow[r, "\cong"] \arrow[d, "L"'] & T_xV \arrow[d, "d_xL"] \\ W \arrow[r, "\cong"'] & T_{Lx}W \end{tikzcd}$$

Proof. Given a vector $v \in V$ we define a derivation

$$\begin{aligned} D_v|_x : C^\infty(V) &\longrightarrow V \\ f &\longmapsto \left.\frac{d}{dt}\right|_{t=0} f(x+vt). \end{aligned}$$

Now the map $v \mapsto D_v|_x$ is the required isomorphism. Given a linear map $L : V \to W$ we have $(d_xLD_v|_x)f = D_v|_x(L \circ f) = D_{Lv}|_{Lx}f$ by the definition of the derivation and the linearity of L [Lee03]. $\square$

By Lemma 4.2 there is a canonical linear isomorphism $T_{f_i^{-1}(z)}(T_{m_i}M) \cong T_{m_i}M$ so the composition of $d_{f_i^{-1}(z)}f_i$ with this isomorphism is the required map. The isomorphism is certainly equivariant under the action of the diffeomorphism group so for simplicity we omit it from our notation. Our section is now given by

$$\begin{aligned}\gamma_k(f;\boldsymbol{x}) : W \smallsetminus M_0 &\longrightarrow \overset{\infty}{T}W \wedge_W (W \times X)\\ z &\longmapsto \begin{cases} *_z & \text{if } z \notin Im(f)\\ (d_{f_i^{-1}(z)}f_i \circ f_i^{-1}(z)) \wedge x_i & \text{if } z \in Im(f_i).\end{cases}\end{aligned}$$

Although this is a well defined section there are still more problems to overcome. Consider the case where $Im(f_i) \cap (W \smallsetminus M) \neq \emptyset$. If $z \in Im(f_i) \cap (W \smallsetminus M)$ the image of z, $(d_{f_i^{-1}(z)}f_i \circ f_i^{-1}(z)) \wedge x_i$, is not necessarily the basepoint in the fibre over z. As such our section does not agree with $*$ in $W \smallsetminus M$ and so is not in $\Gamma(W \smallsetminus M_0, W \smallsetminus M; X)$. This problem can be resolved but requires a norm on the tangent bundle and as such, a metric on the manifold. This appears at first to be a problem for equivariance but we will see that it is not.

The idea is to stretch the map f_i^{-1} so $Im(f_i) \cap (W \smallsetminus M)$ maps to $*$. Such a stretching map $r_{f_i} : \overset{\infty}{T}_{m_i}M \to \overset{\infty}{T}_{m_i}M$ is defined as follows. If $Im(f_i) \cap (W \smallsetminus M) = \emptyset$ then $r_{f_i} = Id$. Else let $t_{f_i} = \inf\{|v| : v \in f_i^{-1}(W \smallsetminus M)\}$ and define

$$r_{f_i}(u) := \begin{cases} \infty & \text{if } |u| \geq t_{f_i}\\ u \tan\left(\frac{\pi |u|}{2t_{f_i}}\right) & \text{otherwise.}\end{cases}$$

Now for fixed $(f;\boldsymbol{x})$ the section is defined to be

$$z \longmapsto \begin{cases} *_z & \text{if } z \notin Im(f)\\ (d_{f_i^{-1}(z)}f_i \circ r_{f_i} \circ f_i^{-1}(z)) \wedge x_i & \text{if } z \in Im(f_i).\end{cases}$$

In order to define the scanning map $\gamma_\infty : E(M, M_0; X) \to \Gamma(W \smallsetminus M_0, W \smallsetminus M; X)$ we must take into consideration what happens to the section when components of embeddings vanish in M_0. Recall that a component vanishes when it's midpoint is in M_0. If we leave the section unaltered when a component f_i vanishes, some of $Im(f_i)$ is still contained in $W \smallsetminus M_0$ and so the section is not well defined. In order to overcome this problem we make a small adjustment to the maps r_{f_i}. If $Im(f_i)$ has empty intersection with $(M_0 \cup (W \smallsetminus M)) = \emptyset$ let $r_{f_i} = Id$. Else let $t_{f_i} = \inf\{|v| : v \in f_i^{-1}(M_0 \cup (W \smallsetminus M))\}$ and define r_{f_i} as before. Since M_0 is compact it is closed and we do not require the section to be defined on M_0 so it is easy to see that the section remains continuous when components of embeddings vanish.

Remark 4.3. The section will not be continuous if we allow particles to move into $\partial M \smallsetminus (M_0)$ since the stretching map in the fibre over that particle will contract a sphere to a point.

Definition 4.4. The *interior tubular configuration space* $\mathring{E}(M, M_0; X) \subseteq E(M, M_0; X)$ is the subspace of tubular configurations whose midpoints are in $M_0 \cup (M \smallsetminus \partial M)$

Proposition 4.5. *The inclusion* $\mathring{E}(M, M_0; X) \hookrightarrow E(M, M_0; X)$ *is a homotopy equivalence.*

Proof. Choose a closed collar neighbourhood $\partial M \times [-1, 0] \subset M$ where $\partial M \times \{0\}$ is identified with the boundary. The notation $[-1, 0]$ is chosen to be compatible with the notation $\partial M \times [0, 1)$ for the collar attached to define W. We use this collar to define an isotopy inverse to the inclusion $i : M_0 \cup (M \smallsetminus \partial M) \hookrightarrow M$ as a map $\beta : M \to M_0 \cup (M \smallsetminus \partial M)$. Outside of the collar neighbourhood let β be the identity. There are three possibilities for each component of the collar neighbourhood. Let A be a component of ∂M.

1. If $A \subseteq M_0$ let β be the identity on $A \times [-1, 0]$.

2. If $A \cap M_0 = \emptyset$ let β be a contracting map $A \times [-1, 0] \to A \times [-1, -1/2]$.

3. If $A \cap M_0 \neq \emptyset$ but $A \nsubseteq M_0$ let $B \subsetneq A$ be an open neighbourhood of $A \cap M_0$ in A. Let β be a contracting map which fixes $(A \cap M_0) \times [-1, 0]$, contracts $(A \smallsetminus B) \times [-1, 0]$ to $(A \smallsetminus B) \times [-1, -1/2]$ and contracts $B \smallsetminus (A \cap M_0) \times [-1, 0]$ so that β is smooth.

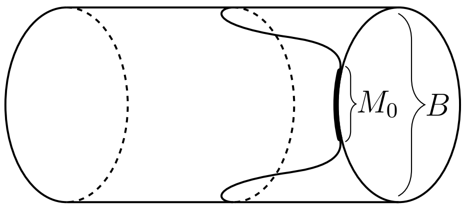

Figure 4.1: A contraction of a collar $\partial M \times I$ fixing $M_0 \times I$.

We choose all the components of β so that the whole map in an embedding isotopic to the identity on M. The embedding extends to an embedding $\bar{\beta} : W \to W$ which is isotopic to the identity on W by appropriate stretching maps on the collar $\partial M \times [0,1) \to \partial M \times [-1/2, 1)$.

A homotopy inverse to the inclusion $\mathring{E}(M, M_0; X) \hookrightarrow E(M, M_0; X)$ is given by

$$\begin{aligned} \alpha : E(M, M_0; X) &\longrightarrow \mathring{E}(M, M_0; X) \\ f &\longmapsto \beta \circ f \circ (d\beta)^{-1}. \end{aligned}$$ □

Remark 4.6. Although the proof assumes W is M with a collar attached to the boundary, the theorem still holds if W is any manifold without boundary containing M as a codimension 0 submanifold. The proof is easily adapted as there is a collar $\partial M \times [-1, 1] \subset W$ where $\partial M \times \{0\}$ is identified with the boundary.

Remark 4.7. The map α is equivariant under the action of $\mathrm{Diff}(W, M_0 \cup (W \smallsetminus M))$. Any diffeomorphism fixing $W \smallsetminus M$ must also fix ∂M so we may assume it fixes a closed collar. Taking this as the collar defining β we find that any such diffeomorphism commutes with β.

The homotopy equivalence α preserves the filtrations and each finite tubular configuration space E_k. The scanning map is now defined by factoring through the interior tubular configuration space.

$$\begin{aligned} \gamma_\infty : E(M, M_0; X) &\longrightarrow \Gamma(W \smallsetminus M_0, W \smallsetminus M; X) \\ [(f; \boldsymbol{x})] &\longmapsto \gamma_k \circ \alpha(f; \boldsymbol{x}) \end{aligned}$$

where $(f;\boldsymbol{x}) \in E_k(M;X)$ is any representative of the class $[(f;\boldsymbol{x})]$. From now on we will omit the composition by α to simplify notation since it does not change the group action.

4.2 Diffeomorphism equivariance

It is clear from our definition of the scanning map that, apart from perhaps the maps r_{f_i} which depend on a metric, it should be equivariant under the action of the $\mathrm{Diff}(W, M_0 \cup (W \smallsetminus M)) \cong \mathrm{Diff}(M, M_0 \cup \partial M)$. However, the maps r_{f_i} have been carefully chosen so that the whole scanning map is equivariant. We begin by establishing the result for the γ_k.

Proposition 4.8. *For each k the map γ_k is equivariant under the action of the diffeomorphism group* $\mathrm{Diff}(W, M_0 \cup (W \smallsetminus M))$.

Proof. We need to check that for any $(f;\boldsymbol{x}) \in E_k(M;X)$, any $z \in W \smallsetminus M_0$ and any $\phi \in \mathrm{Diff}(W, M_0 \cup (W \smallsetminus M))$ we have $(\phi \cdot \gamma_k(f;\boldsymbol{x}))(z) = \gamma_k(\phi \cdot (f;\boldsymbol{x}))(z)$.

For the left hand side, let $\phi^{-1}(z) \in Im(f_i)$, then

$$\begin{aligned}
(\phi \cdot \gamma_k(f;\boldsymbol{x}))(z) &= \left((d\phi \wedge_W (\phi \times \phi)) \circ \gamma_k(f;\boldsymbol{x}) \circ \phi^{-1}\right)(z) \\
&= (d\phi \wedge_W (\phi \times \phi)) \left(d_{f_i^{-1} \circ \phi^{-1}(z)} f_i \circ r_{f_i} \circ f_i^{-1} \circ \phi^{-1}(z) \wedge x_i\right) \\
&= d_{\phi^{-1}(z)}\phi \circ d_{f_i^{-1} \circ \phi^{-1}(z)} f_i \circ r_{f_i} \circ f_i^{-1} \circ \phi^{-1}(z) \wedge \phi \cdot x_i.
\end{aligned}$$

To calculate the $\gamma_k(\phi \cdot (f;\boldsymbol{x}))(z)$ note that $(\phi \cdot f_i)^{-1} = d_{m_i}\phi \circ f_i^{-1} \circ \phi^{-1}$ and $d(\phi \cdot f_i) = d\phi \circ df_i \circ d(d_{\phi(m_i)}(\phi^{-1})) = d\phi \circ df_i \circ (d_{m_i}\phi)^{-1}$ by Lemma 4.2. We also need to understand how the diffeomorphisms act on r_{f_i}, that is how $r_{\phi \cdot f_i}$ is related to r_{f_i}. Let $u \in \overset{\infty}{T}_{\phi(m_i)}M$.

$$r_{\phi \cdot f_i}(u) = \begin{cases} \infty & \text{if } |u| \geq t_{\phi \cdot f_i} \\ u \tan\left(\frac{\pi |u|}{2 t_{\phi \cdot f_i}}\right) & \text{otherwise.} \end{cases}$$

Since ϕ fixes M_0 and $W \setminus M$, their intersection with the image of f_i is invariant under the action. Note that $t_{\phi \cdot f_i} = \det(d_{m_i}\phi)t_{f_i}$, it follows that $\frac{\pi|u|}{2t_{\phi \cdot f_i}} = \frac{\pi|(d_{m_i}\phi)^{-1}(u)|}{2t_{f_i}}$. Putting these identities into

$$d_{m_i}\phi \circ r_{f_i} \circ (d_{m_i}\phi)^{-1}(u) = \begin{cases} \infty & \text{if } |(d_{m_i}\phi)^{-1}(u)| \geq t_{f_i} \\ d_{m_i}\phi \circ \Big((d_{m_i}\phi)^{-1}(u)\tan\Big(\frac{\pi|(d_{m_i}\phi)^{-1}(u)|}{2t_{f_i}}\Big)\Big) & \text{otherwise} \end{cases}$$

we see that $r_{\phi \cdot f_i} = d_{m_i}\phi \circ r_{f_i} \circ (d_{m_i}\phi)^{-1}$.

Now for the right hand side, let $z \in Im(\phi \cdot f_i)$, then

$$\begin{aligned} \gamma_k(\phi \cdot (f; \boldsymbol{x}))(z) &= d_{(\phi \cdot f_i)^{-1}(z)}(\phi \cdot f_i) \circ r_{\phi \cdot f_i} \circ (\phi \cdot f_i)^{-1}(z) \wedge \phi \cdot x_i \\ &= d_{\phi^{-1}(z)}\phi \circ d_{f_i^{-1} \circ \phi^{-1}(z)} f_i \circ (d_{m_i}\phi)^{-1} \circ d_{m_i}\phi \circ r_{f_i} \\ &\quad \circ (d_{m_i}\phi)^{-1} \circ d_{m_i}\phi \circ f_i^{-1} \circ \phi^{-1}(z) \wedge \phi \cdot x_i \\ &= d_{\phi^{-1}(z)}\phi \circ d_{f_i^{-1} \circ \phi^{-1}(z)} f_i \circ r_{f_i} \circ f_i^{-1} \circ \phi^{-1}(z) \wedge \phi \cdot x_i. \end{aligned}$$

The result follows by observing that $z \in Im(\phi \cdot f_i) \iff \phi^{-1}(z) \in Im(f_i)$. $\square$

Corollary 4.9. *The scanning map $\gamma_\infty : E(M, M_0; X) \to \Gamma(W \setminus M_0, W \setminus M; X)$ is equivariant under the action of* $\mathrm{Diff}(W, M_0 \cup (W \setminus M))$.

Proof. Recall that $\gamma_\infty[(f; \boldsymbol{x})]$ is precisely $\gamma_k(f; \boldsymbol{x})$ for any representative of the class. Since any diffeomorphism in $\mathrm{Diff}(W, M_0 \cup (W \setminus M))$ fixes $M_0 \subseteq M$ and $x_0 \in X$ the equivalence classes are invariant under the action. As such it is enough that each γ_k is equivariant. $\square$

4.3 Homotopy equivalence

Theorem 4.10. *Let M be a compact manifold. The new scanning map*

$$\gamma_\infty : E(M, M_0; X) \to \Gamma(W \setminus M_0, W \setminus M; X)$$

is a weak homotopy equivalence provided (M, M_0) or X is 0-connected. Moreover, γ_∞ is a homotopy equivalence if X has the homotopy type of a CW-complex.

Throughout the rest of this paper the term *(weak) homotopy equivalence* will be used when a map is a homotopy equivalence if the labelling space of the configuration space involved has the homotopy type of a CW-complex and a weak homotopy equivalence otherwise. The proof of this theorem makes use of several lemmas.

Lemma 4.11. *The construction C satisfies the following properties [Böd87]:*

1. *A homotopy functor in X, that is a pointed homotopy equivalence $X \xrightarrow{\simeq} Y$ induces a homotopy equivalence $C(M, M_0; X) \xrightarrow{\simeq} C(M, M_0; Y)$.*
2. *An isotopy functor in (M, M_0), that is an isotopy equivalence $(M, M_0) \xrightarrow{\cong} (N, N_0)$ induces a homotopy equivalence $C(M, M_0; X) \xrightarrow{\simeq} C(N, N_0; X)$.*
3. *The excision property $C(M, M_0; X) \cong C(M \setminus U, M \setminus U; X)$ for $U \subset M_0$ and open in M.*
4. *The product property $C(M, M_0; X) \cong C(N_1, N_1 \cap M_0; X) \times C(N_2, N_2 \cap M_0; X)$ for $M = N_1 \cup N_2$ and $N_1 \cap N_2 \subset M_0$.*

It is immediate that the construction E satisfies properties 1 and 2 above and satisfies properties 3 and 4 up to homotopy equivalence.

Lemma 4.12. *Let $H \subset M$ be a compact codimension 0 submanifold. Then the isotopy cofibration*

$$(H, H \cap M_0) \longrightarrow (M, M_0) \xrightarrow{q} (M, H \cup M_0)$$

induces a quasifibration

$$C(H, H \cap M_0; X) \longrightarrow C(M, M_0; X) \xrightarrow{Q} C(M, H \cup M_0; X)$$

provided $(H, H \cap M_0)$ or X is 0-connected, that is provided X is path connected or every path component of H has non-empty intersection with M_0 [Böd87].

Using the properties in Lemma 4.11 this can be reformulated as follows.

Corollary 4.13. *Let $H \subset M$ be a compact codimension 0 submanifold. Then the sequence*

$$C(M \smallsetminus \mathring{H}, (M \smallsetminus \mathring{H}) \cap M_0; X) \longrightarrow C(M, M_0; X) \longrightarrow C(H, ((M \smallsetminus \mathring{H}) \cup M_0) \smallsetminus \overbrace{(M \smallsetminus \mathring{H})}^{\circ}; X)$$

is a quasifibration provided $(M \smallsetminus \mathring{H}, (M \smallsetminus \mathring{H}) \cap M_0)$ or X is 0-connected. Moreover, if $M_0 = \partial M$ the quasifibration can be simplified to

$$C(M \smallsetminus \mathring{H}, (M \smallsetminus \mathring{H}) \cap \partial M; X) \longrightarrow C(M, \partial M; X) \xrightarrow{r_C} C(H, \partial H; X).$$

The map r_C is the obvious restriction map, $[(\boldsymbol{m}; \boldsymbol{x})] \mapsto [(\boldsymbol{m} \cap H; \boldsymbol{x})]$, where the labels are removed with their corresponding particles. With this corollary in mind we make the following definition.

Definition 4.14. A submanifold $H \subset M$ is called *nice* if it is compact, of codimension 0 and $(M \smallsetminus \mathring{H}, (M \smallsetminus \mathring{H}) \cap \partial M)$ is 0-connected, that is if every path component of $M \smallsetminus \mathring{H}$ has non-empty intersection with ∂M.

We will use Gromov's method to prove Theorem 4.10, as detailed by McDuff [McD77]. The idea is to prove that the theorem holds for disks modulo their boundary and prove that if a manifold can be decomposed into two submanifolds in a sufficiently nice way and the theorem holds each of the pieces and their intersection, then it also holds for the whole manifold. This will allow us to build up our manifolds using handles. We adapt the procedure of McDuff [McD75], and the generalisation due to Bödigheimer [Böd87].

Lemma 4.15. *Theorem 4.10 holds for a disk modulo it's boundary (D^n, S^{n-1}) and any labelling space X.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} E(D^n, S^{n-1}; X) & \longrightarrow & \Gamma(\mathring{D}^n, S^{n-1} \times [0,1); X) \\ \big\downarrow & & \big\downarrow \\ E^1(D^n, S^{n-1}; X) & \longrightarrow & S^n X \end{array}$$

where the horizontal maps are induced by γ_∞, the right hand vertical map is given by evaluation at the origin. The left hand vertical map is induced by radial expansion of the disk so that at most one component of an embedding remains, the radial expansions can be chosen so as to vary continuously in $E(D, S^{n-1}; X)$. The left, bottom and right hand maps are homotopy equivalences so the top map is too. $\square$

Lemma 4.16. *Suppose M is the union of two compact codimension 0 submanifolds M_1, M_2 whose intersection $M_{12} = M_1 \cap M_2$ is a compact codimension 0 submanifold of M_1 and M_2. Suppose Theorem 4.10 holds for $(M_{12}, \partial M_{12})$, $(M_1, \partial M_1)$ and $(M_2, \partial M_2)$. Then it also holds for $(M, \partial M)$ if M_{12} is nice in both M_1 and M_2 and each of M_1 and M_2 are nice in M or if X is path connected.*

Proof. First note that the map $q : (M, M_0) \to (M, H \cup M_0)$ of Lemma 4.12 also induces a map $E(M, M_0; X) \to E(M, H \cup M_0; X)$. Then by excision and isotopy invariance we have a well defined restriction map $r_E : E(M, \partial M; X) \to E(H, \partial H; X)$ which removes any component of a tubular neighbourhood whose midpoint lies outside of H. Here we use the same manifold W containing each of M, M_1, M_2 and M_{12} as codimension 0 submanifolds. The following diagram commutes.

$$\begin{tikzcd}
& C(M_1, \partial M_1; X) \arrow[dd] & & E(M_1, \partial M_1; X) \arrow[ll, "p", "\simeq"'] \arrow[dd] \\
C(M, \partial M; X) \arrow[ur] \arrow[dd] & & E(M, \partial M; X) \arrow[ll, "p", "\simeq"'] \arrow[ur] \arrow[dd] & \\
& C(M_{12}, \partial M_{12}; X) & & E(M_{12}, \partial M_{12}; X) \arrow[ll, "p", "\simeq"'] \\
C(M_2, \partial M_2; X) \arrow[ur] & & E(M_2, \partial M_2; X) \arrow[ll, "p", "\simeq"'] \arrow[ur] &
\end{tikzcd}$$

All the horizontal maps to the left are projections and all the other maps are the restrictions r_C and r_E. The left hand face is homotopy cartesian since it is a cartesian square and the vertical maps are quasifibrations by the assumptions. All the projections are (weak) homotopy equivalences so the right hand face is also homotopy cartesian [Mat76, Lemma 6].

Consider the following commutative diagram.

$$\begin{tikzcd}[row sep=small, column sep=small]
& E(M_1, \partial M_1; X) \arrow[rr, "\gamma_\infty", "\simeq"'] \arrow[dd] & & \Gamma(W \setminus \partial M_1, W \setminus M_1; X) \arrow[dd] \\
E(M, \partial M; X) \arrow[ur] \arrow[rr, "\gamma_\infty" near start] \arrow[dd] & & \Gamma(W \setminus \partial M, W \setminus M; X) \arrow[ur] \arrow[dd] & \\
& E(M_{12}, \partial M_{12}; X) \arrow[rr, "\gamma_\infty" near start, "\simeq"' near start] & & \Gamma(W \setminus \partial M_{12}, W \setminus M_{12}; X) \\
E(M_2, \partial M_2; X) \arrow[ur] \arrow[rr, "\gamma_\infty", "\simeq"'] & & \Gamma(W \setminus \partial M_2, W \setminus M_2; X) \arrow[ur] &
\end{tikzcd}$$

The right hand face a cartesian square and therefore homotopy cartesian since the right most vertical map is a fibration [Böd87]. The horizontal maps to the right are all γ_∞ and by our assumptions they are all (weak) homotopy equivalences except possibly for the top front map $\gamma_\infty : E(M, \partial M; X) \to \Gamma(W \setminus \partial M, W \setminus M; X)$. As we have seen the left square is also homotopy cartesian so all four maps are (weak) homotopy equivalences [Mat76, Corollary 7]. $\square$

Lemma 4.17. *Theorem 4.10 holds for $(S^k \times D^{n-k}, S^k \times S^{n-k-1})$ when $0 \leq k < n$, and additionally when $k = n$ if X is path connected. Here S^{-1} is the empty set.*

Proof. We proceed by induction on k, following McDuff [McD75]. We have already shown the theorem holds for $S^0 \times D^n$. Suppose it holds for $k - 1 < n$ and let $M = S^k \times D^{n-k}$. Since S^k can be decomposed as two copies of D^k with intersection $S^{k-1} \times I$ we can choose $M_1 \cong M_2 \cong D^k \times D^{n-k}$ so that $M_1 \cap M_2 = S^{k-1} \times I \times D^{n-k}$. Then if $k < n$ or X is path connected, the conditions of the previous lemma hold and the result follows. $\square$

Lemma 4.18. *Suppose Theorem 4.10 is true for $(M, \partial M)$ and let M' be M with a handle of index k attached. Then the theorem holds for $(M', \partial M')$ when $k < n$, and additionally when $k = n$ if X is path connected.*

Proof. A handle $D^k \times D^{n-k}$ is attached to M by an embedding $\psi : S^{k-1} \times D^{n-k} \to \partial M$. There exists a collar neighbourhood $\partial M \times I \subset M$ so ψ may be extended to an embedding $\tilde{\psi} : S^{k-1} \times I \times D^{n-k} \to \partial M \times I$. Now $M' = M \cup_{\tilde{\psi}} (D^k \times D^{n-k})$ and $M \cap (D^k \times D^{n-k}) = S^{k-1} \times I \times D^{n-k}$. This decomposition satisfies the conditions of the previous lemma if $k < n$, and additionally for $k = n$ if X is path connected. Since Theorem 4.10 holds for each of (D^n, S^{n-1}), $(S^{k-1} \times D^{n-k+1}, S^{k-1} \times S^{n-k})$ and $(M, \partial M)$ it also holds for $(M', \partial M')$. $\square$

To prove that Theorem 4.10 holds for $(M, \partial M)$ for any compact n-manifold M choose a handle decomposition of M. If $(M, \partial M)$ is 0-connected it is a finite union of compact connected manifolds with nonempty boundary so we can choose a decomposition without handles of index n [Kos93] and the result follows from Lemma 4.18 by induction on the number of handles. If $(M, \partial M)$ is not 0-connected the decomposition contains handles of index n. Provided X is path connected we can attach such a handle by Lemma 4.18 and the result follows.

Lemma 4.19. *Suppose $M_0 \subsetneq \partial M$, then Theorem 4.10 holds for (M, M_0).*

Proof. Choose a complementary submanifold $L \in \partial M$ so that $L \cup M_0 = \partial M$ and $L \cap M_0 = \partial L = \partial M_0$. We may attach a closed collar to L and obtain a manifold (with corners) $N = M \cup_L (L \times I)$. If (M, M_0) is 0-connected then $L \times I$ is a nice submanifold of N otherwise let X be path connected. We have a quasifibration $C(M, M_0; X) \to C(N, \partial N; X) \to C(L \times I, \partial(L \times I); X)$. Consider the following commutative diagram.

$$\begin{tikzcd}
 & C(N,\partial N;X) \arrow[dd] & & E(N,\partial N;X) \arrow[ll, "p", "\simeq"'] \arrow[dd] \\
C(M,M_0;X) \arrow[ur] \arrow[dd] & & E(M,M_0;X) \arrow[ll, "p", "\simeq"'] \arrow[ur] \arrow[dd] & \\
 & C(L\times I,\partial(L\times I);X) & & E(L\times I,\partial(L\times I);X) \arrow[ll, "p", "\simeq"'] \\
[(\boldsymbol{m};\boldsymbol{x})] \arrow[ur, hook] & & [(f;\boldsymbol{x})] \arrow[ll, "p", "\simeq"'] \arrow[ur, hook] &
\end{tikzcd}$$

All the horizontal maps are projections and (weak) homotopy equivalences, we choose $[(\boldsymbol{m};\boldsymbol{x})] \in C(L \times I, \partial(L \times I); X)$ so that $C(M, M_0; X) = r_C^{-1}([(\boldsymbol{m};\boldsymbol{x})])$, and

$[(f;\boldsymbol{x})] \in p^{-1}([(\boldsymbol{m};\boldsymbol{x})])$ so that $E(M,M_0;X) = r_E^{-1}([(f;\boldsymbol{x})])$. The front vertical arrows are constant maps the rear vertical arrows are the restrictions r_C and r_E. The left hand face is a homotopy cartesian square and therefore so is the right hand face. We also have a fibration in the section spaces so that the diagram

$$\begin{tikzcd}
 & E(N,\partial N;X) \arrow[rr, "\gamma_\infty", "\simeq"'] \arrow[dd] & & \Gamma(W\smallsetminus\partial N, W\smallsetminus N;X) \arrow[dd] \\
E(M,M_0;X) \arrow[ur] \arrow[rr, "\gamma_\infty"] \arrow[dd] & & \Gamma(W\smallsetminus M_0, W\smallsetminus M;X) \arrow[ur] \arrow[dd] & \\
 & E(L\times I,\partial(L\times I);X) \arrow[rr, "\gamma_\infty", "\simeq"'] & & \Gamma(W\smallsetminus\partial(L\times I), W\smallsetminus(L\times I);X) \\
{[(f;\boldsymbol{x})]} \arrow[ur, hook] \arrow[rr, "\gamma_\infty", "\simeq"'] & & \gamma_\infty([(f;\boldsymbol{x})]) \arrow[ur, hook] &
\end{tikzcd}$$

commutes. Here the manifold W has been chosen so that it contains N and therefore M and $L \times I$ as codimension 0 submanifolds. Both the left and right faces are homotopy cartesian squares and three of the maps are (weak) homotopy equivalences, hence so is the fourth. $\square$

The theorem now follows when $M_0 \subset M$ is any compact submanifold by replacing M_0 with a closed tubular neighbourhood (that is, the image of the unit disk subbundle in a tubular neighbourhood) and removing the interior of this neighbourhood. By the isotopy invariance and excision properties the homotopy type of $E(M,M_0;X)$ is unaltered and we are now in the case where $M_0 \subseteq \partial M$, for which Theorem 4.10 is already proved.

4.4 The pointed Borel construction

Lemma 4.20. *Let G be a topological group and let X and Y be G-spaces. Let $f : X \to Y$ be a G-equivariant map and a nonequivariant homotopy equivalence. Then f induces a homotopy equivalence*

$$Id \times_G f : EG \times_G X \xrightarrow{\simeq} EG \times_G Y$$

on the Borel constructions of X and Y with respect to G provided X and Y are connected and each has the homotopy type of a CW-complex. The induced map is a weak homotopy equivalence otherwise.

Proof. Consider the principal G-bundle, $G \to EG \to BG$. The Borel constructions on X and Y are the total spaces of the associated X and Y-bundles over BG. Since f is G-equivariant $Id \times_G f$ is a well defined map between the total spaces and the diagram following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
EG \times_G X \arrow[r, "Id\times_G f"] \arrow[d] & EG \times_G Y \arrow[d] \\
BG \arrow[r, "Id"] & BG
\end{tikzcd}$$

Here the vertical sequences are fibre bundles, the upper horizontal map is a homotopy equivalence by assumption and the lower horizontal map is the identity. Passing to homotopy groups it is immediate that we have a weak homotopy equivalence. The result for connected CW-complexes follows by Whitehead's theorem. □

Lemma 4.21. *If X and Y are well pointed G-spaces, then f induces a homotopy equivalence*

$$Id \ltimes_G f : EG \ltimes_G X \xrightarrow{\simeq} EG \ltimes_G Y$$

on pointed Borel constructions if each of X and Y has the homotopy type of a CW-complexes, and a weak homotopy equivalence otherwise.

Proof. The pointed Borel construction is the pushout of the diragram

$$* \longleftarrow BG \longrightarrow EG \times_G X$$

where the right arrow is the canonical inclusion $EG \times_G * \hookrightarrow EG \times_G X$. Since X is well-pointed this inclusion is a cofibration so the pushout square is a homotopy pushout square. The map $Id \times_G f : EG{\times_G} \to EG \times_G Y$ induces a map of homotopy pushout squares

$$\begin{tikzcd}
& EG \times_G X \arrow[rr, "Id\times_G f"] \arrow[dd] & & EG \times_G Y \arrow[dd] \\
BG \arrow[ur] \arrow[rr, "Id_{BG}"] \arrow[dd] & & BG \arrow[ur] \arrow[dd] & \\
& EG \ltimes_G X \arrow[rr, "Id\ltimes_G f"] & & EG \ltimes_G Y \\
* \arrow[ur] \arrow[rr, "Id_*"] & & * \arrow[ur] &
\end{tikzcd}$$

in which the upper rear horizontal map is a weak homotopy equivalence by the previous lemma. Applying Corollary 9 of [Mat76] we have a weak homotopy equivalence between the pointed Borel constructions. □

Proposition 4.22. *Let M be a compact manifold and let $M_0 \subseteq M$ be a compact submanifold. Let G be a topological group which acts smoothly on M, fixing M_0 and ∂M. That is, G acts through a homomorphism $\psi : G \to \mathrm{Diff}(M, \partial M \cup M_0)$. Let X be a well pointed G-space. Then*

(i) the scanning map $\gamma_\infty : E(M, M_0; X) \to \Gamma(W \setminus M_0; W \setminus M; X)$ induces a (weak) homotopy equivalence of pointed Borel constructions

$$EG \ltimes_G E(M, M_0; X) \xrightarrow{\simeq} EG \ltimes_G \Gamma(W \setminus M_0; W \setminus M; X)$$

provided (M, M_0) or X is 0-connected;

(ii) the projection $p : E(M, M_0; X) \to C(M, M_0; X)$ induces a (weak) homotopy equivalence on pointed Borel constructions

$$EG \ltimes_G E(M, M_0; X) \xrightarrow{\simeq} EG \ltimes_G C(M, M_0; X)$$

for any (M, M_0) and X.

Proof. The results follow immediately from Corollary 4.9, Theorem 4.10 and Lemma 4.21. □

Corollary 4.23. *There is a (weak) homotopy equivalence of the pointed Borel constructions*

$$EG \ltimes_G C(M, M_0; X) \simeq EG \ltimes_G \Gamma(W \smallsetminus M_0; W \smallsetminus M; X)$$

provided (M, M_0) *or* X *is* 0*-connected.*

5 Stable splittings

The aim of this section is to generalise the results of Bödigheimer and Madsen [BM88] to obtain stable splittings of the Borel constructions of mapping spaces for any smooth group action. Moreover, we obtain an equivariant stable splittings of a generalisation of configuration spaces, completing a zigzag of equivariant stable maps which are nonequivariant stable homotopy equivalences between mapping spaces and a wedge of simpler spaces.

Throughout this section we make use of the following assumptions and notation. As usual M is a compact manifold, $M_0 \subseteq M$ is a compact submanifold and X is a well-pointed space with basepoint x_0. Let $\mathrm{Diff} \coloneqq \mathrm{Diff}(M, \partial M \cup M_0)$ be a subgroup of the diffeomorphism group of M and assume it acts on X fixing the basepoint. Let $C^k \coloneqq C^k(M, M_0; X)$ be the filtrations of the configuration space $C \coloneqq C(M, M_0; X)$ and let $D^k = D^k(M, M_0; X) \coloneqq C^k/C^{k-1}$ be the quotients of successive filtrations.

5.1 Equivariant splittings in homology

In section 5.3 we obtain an equivariant version of Theorem 2.7. However, our proof takes a more complicated form so we include the following weaker theorem in virtue of it's simplicity.

Theorem 5.1. *Suppose the pair (M, M_0) or the space X is 0-connected and X has the homotopy type of a CW-complex. Then there is an isomorphism*

$$\tilde{H}_*(C(M, M_0; X)) \cong \bigoplus_{k \geq 1} \tilde{H}_*(D^k(M, M_0; X))$$

induced by a Diff*-equivariant map.*

The proof makes use of the infinite symmetric product, SP. We recall the definition.

Definition 5.2. Given a pointed space A, we define the *n-fold symmetric product* $SP_n(X)$ as the quotient of the n-fold product A^n by the action of the symmetric group Σ_n which permutes the coordinates. The inclusion $A^n \hookrightarrow A^{n+1}$ which adds a the basepoint in the last coordinate induces an inclusion $SP_n(A) \hookrightarrow SP_{n+1}(A)$. We define the *infinite symmetric product* $SP(X)$ as the colimit of this sequence of finite symmetric products.

Remark 5.3. The elements of an n-fold symmetric product $SP_n(A)$ are formal sums $\sum k_i a_i$ where $a_i \in A$ and $\sum k_i = n$. Similarly the elements of the infinite symmetric product are finite formal sums.

Proof. The idea of this proof is to alter the power set map to give a map between infinite symmetric products and apply the Dold-Thom theorem [DT58], which asserts that $\pi_* SP(A) = \tilde{H}_*(A)$ for any connected CW-complex A, to obtain a homology equivalence.

Let $V^k := \bigvee_{i=1}^k D^i$ and $V := \bigvee_{i \geq 1} D^i$ be wedges of the filtration quotientss. Note that the assumption that (M, M_0) or X is 0-connected ensures that C and the C^k's are path connected, hence so is each D^k, V^k and V. We could define D^0 as the quotient of C^0 by the empty set and include it in the wedges V^k and V, C^0 is a point so $D^0 = */\emptyset = S^0$ and it's inclusion would add a disjoint point. However, we rely on the connectivity of V to use the Dold-Thom theorem. Restricting to a labelling space with the homotopy type of a CW-complex ensures that the configuration spaces and related spaces each have the homotopy types of CW-complexes.

Choose a configuration $\xi_\alpha = \sum_{i \in \alpha} m_i x_i \in C$ where α is the index set. Given a subset $\beta \subseteq \alpha$ of size k we have a subconfiguration $\xi_\beta := \sum_{i \in \beta} m_i x_i$. Let $\bar{\xi}_\beta$ denote it's image under the composition $C^k \to D^k \hookrightarrow V$. We define a map

$$\begin{aligned} \sigma : C &\longrightarrow SP(V) \\ \xi_\alpha &\longmapsto \sum_{\beta \subseteq \alpha} \bar{\xi}_\beta \end{aligned}$$

which extends to a map $SP(C) \to SP(V)$ by $\sum k_\alpha \xi_\alpha \mapsto \sum k_\alpha \sigma(\xi_\alpha)$. This map respects the filtrations, restricting to maps $SP(C^k) \to SP(V^k)$. Moreover, the map is clearly equivariant under the action of the diffeomorphism group of M. The infinite symmetric product functor takes cofibration sequences to quasifibrations so we have a commutative diagram

$$\begin{tikzcd} SP(C^{k-1}) \arrow[r, "\sigma"] \arrow[d] & SP(V^{k-1}) \arrow[d] \\ SP(C^k) \arrow[r, "\sigma"] \arrow[d] & SP(V^k) \arrow[d] \\ SP(C^k/C^{k-1}) \arrow[r, "\sigma = Id"] & SP(V^k/V^{k-1}) \end{tikzcd}$$

in which the vertical sequences are quasifibrations and all horizontal maps are restrictions of σ. Note that $\sigma : SP(D^k) \to SP(D^k)$ is the identity. Now starting with $C^1 = V^1$ we proceed by induction on k to see that σ induces isomorphisms

$$\pi_* SP(C(M, M_0; X)) \xrightarrow{\cong} \pi_* SP(\bigvee_{k \geq 1} D^k).$$

The result follows by the Dold-Thom theorem. $\square$

Remark 5.4. The connected component of the identity, $\mathrm{Diff}_0(M, \partial M \cup M_0) \subseteq \mathrm{Diff}$ acts trivially on homology since all the diffeomorphisms are isotopic and hence homotopic to the identity. It is immediate that the isomorphism

$$\tilde{H}_*(C(M, M_0; X)) \xrightarrow{\cong} \bigoplus_{k \geq 1} \tilde{H}_*(D^k(M, M_0; X))$$

is equivariant under the action of the mapping class group $\pi_0 \mathrm{Diff}(M, \partial M \cup M_0)$.

5.2 Stable splittings of homotopy quotients

Another reformulation of the power set map yields a generalisation of Theorem 2.9. The proof of Theorem 2.9 involves embedding the configuration space $C(M)$ into representation spaces, and while this can be done G-equivariantly when G is a compact Lie group [BM88], it cannot in general. As such we take a somewhat different approach.

Theorem 5.5. *There is a homotopy equivalence of suspension spectra*

$$\Sigma^\infty \left(E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} C(M, M_0; X)\right) \simeq \Sigma^\infty \bigvee_{k \geq 1} E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} D^k(M, M_0; X).$$

Proof. The space of embeddings of M into $\mathbb{R}^\infty$ is contractible and admits a free action by the diffeomorphism group of M. In particular we take $\mathrm{Emb}(M, \mathbb{R}^\infty)$ as our model for $E\,\mathrm{Diff}$. Given an embedding $h : M \hookrightarrow \mathbb{R}^\infty$ we define an associated embedding $\widehat{h} : \amalg_k C_k(M) \to \mathbb{R}^\infty$. There is a canonical embedding

$$\begin{aligned} \mu_k : (\mathbb{R}^\infty)^k &\longrightarrow \mathbb{R}^\infty \\ ((x_{11}, x_{12}, \ldots); \ldots; (x_{k1}, x_{k2}, \ldots)) &\longmapsto (x_{11}, x_{21}, \ldots, x_{k1}, x_{12}, \ldots) \end{aligned}$$

by reordering coordinates. We extended this to an embedding $\amalg_k (\mathbb{R}^\infty)^k \to \mathbb{R}^\infty$ by further reordering. This defines an embedding $\amalg C_k(\mathbb{R}^\infty) \to \mathbb{R}^\infty$ as follows. The k-th *ordered* configuration space is a subspace of the k-fold product, $\tilde{C}_k(\mathbb{R}^\infty) = (\mathbb{R}^\infty)^k \smallsetminus \Delta_k$, so each k we have an embedding $\tilde{C}_k(\mathbb{R}^\infty) \to \mathbb{R}^\infty$. Now to obtain an embedding of the *unordered* configuration spaces we take the orbit space of the Σ_k action. Since Σ_k is finite we can average over the group to get an embedding

$$\begin{aligned} C_k(\mathbb{R}^\infty) &\longrightarrow \mathbb{R}^\infty \\ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) &\longmapsto \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \mu_k \left(\boldsymbol{x}_{\sigma(1)}, \ldots, \boldsymbol{x}_{\sigma(k)}\right) \end{aligned}$$

where each $\boldsymbol{x}_i$ is an element of $\mathbb{R}^\infty$. By reordering coordinates this extends to an embedding $\mu : \amalg_k C_k(\mathbb{R}^\infty) \to \mathbb{R}^\infty$.

The embedding $h : M \hookrightarrow \mathbb{R}^\infty$ defines a canonical embedding $\amalg_k C_k(M) \to \amalg_k C_k(\mathbb{R}^\infty)$. The associated embedding $\widehat{h} : \amalg C_k(M) \to \mathbb{R}^\infty$ is defined by composition and we have the following commutative diagram.

$$\begin{tikzcd} \coprod_k C_k(\mathbb{R}^\infty) \arrow[r, "\mu"] & \mathbb{R}^\infty \\ \coprod_k C_k(M) \arrow[u, "h"] \arrow[ur, "\widehat{h}"'] & \end{tikzcd}$$

Choose a configuration $\xi_\alpha = \sum_{i\in\alpha} m_i x_i \in C$ where α is the index set. Given a subset $\beta \subseteq \alpha$ of size k we have a subconfiguration $\xi_\beta := \sum_{i\in\beta} m_i x_i$ and an unlabelled subconfiguration $m_\beta := \sum_{i\in\beta} m_i \in C_k(M)$. Let $\bar{\xi}_\beta$ denote the image of ξ_β under the composition $C^k \to D^k \hookrightarrow V$. Here $V^k := \bigvee_{i=1}^k D^i$ and $V := \bigvee_{i\geq 1} D^i$ are wedges of the quotients of the filtrations. A power set map is defined by

$$\begin{aligned} \sigma : \mathrm{Emb}(M, \mathbb{R}^\infty) \ltimes_{\mathrm{Diff}} C(M, M_0; X) &\longrightarrow C(\mathbb{R}^\infty; \mathrm{Emb}(M, \mathbb{R}^\infty) \ltimes_{\mathrm{Diff}} V) \\ (h, \xi_\alpha) &\longmapsto \sum_{\beta\subseteq\alpha} \widehat{h}(m_\beta)(h, \bar{\xi}_\beta). \end{aligned}$$

This certainly gives a well defined map from the product $\mathrm{Emb}(M, \mathbb{R}^\infty) \times C$ to the configuration space with labels in the half smash product $\mathrm{Emb}(M, \mathbb{R}^\infty) \times V$. However to see that it is well defined on the Borel construction we must check the map is well defined on the half smash product and on the orbits of the Diff-action.

To see that σ is well defined on the half smash product, consider the restriction

$$\mathrm{Emb}(M, \mathbb{R}^\infty) \times C^0 \longrightarrow C(\mathbb{R}^\infty; \mathrm{Emb}(M, \mathbb{R}^\infty) \ltimes V).$$

Observe that the filtration $C^0 \subset C$ consists of a single point, the empty configuration $\xi_\emptyset$. Since $\xi_\emptyset$ has no subconfigurations, for each embedding h, σ maps $(h, \xi_\emptyset)$ to the empty configuration in $C(\mathbb{R}^\infty; \mathrm{Emb}(M; \mathbb{R}^\infty) \ltimes V)$. It is immediate that σ is well defined on the half smash product.

Let $\phi \in \mathrm{Diff}$ be a diffeomorphism. To see that σ is well defined with respect to Diff-orbits it is enough to check that $\sigma(\phi \cdot (h, \xi_\alpha)) = \phi \cdot \sigma(h, \xi_\alpha)$ for each pair

$(h, \xi_\alpha) \in \mathrm{Emb}(M, \mathbb{R}^\infty) \ltimes C$. The action on $\mathrm{Emb}(M, \mathbb{R}^\infty)$ is by composition with the inverse, $\phi \cdot h = h \circ \phi^{-1}$, the action on the $C(M, M_0; X)$ is $\phi \cdot \xi_\alpha = \sum_{i \in \alpha} \phi(m_i)(\phi \cdot x_i)$, and the action on $C(\mathbb{R}^\infty; \mathrm{Emb}(M; \mathbb{R}^\infty) \ltimes V)$ is trivial on $\mathbb{R}^\infty$ and the diagonal action on the labels. Now σ maps $\phi \cdot (h, \xi_\alpha)$ to $\sum_{\beta \subseteq \alpha} \widehat{h \circ \phi^{-1}} \left(\sum_{i \in \beta} \phi(m_i)\right) (h \circ \phi^{-1}, \overline{\phi \cdot \xi_\beta})$ whereas acting on $\sigma(h, \xi_\alpha)$ with ϕ gives $\sum_{\beta \subseteq \alpha} \widehat{h}(m_\beta)(h \circ \phi^{-1}, \phi \cdot \bar{\xi}_\beta)$. By definition $\phi \cdot \bar{\xi}_\beta = \overline{\phi \cdot \xi_\beta}$ and

$$\widehat{h \circ \phi^{-1}} \left(\sum_{i \in \beta} \phi(m_i) \right) = \mu \left(\sum_{i \in \beta} h \circ \phi^{-1} \circ \phi(m_i) \right) = \widehat{h}(m_\beta).$$

We conclude that σ is well defined. We have avoided the need to embed the configuration spaces equivariantly into $\mathbb{R}^\infty$ by building homotopy quotients into our power set map.

Let p be the adjoint of the composition

$$E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} C \xrightarrow{\sigma} C(\mathbb{R}^\infty; E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} V) \longrightarrow \Omega^\infty \Sigma^\infty \left(E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} V\right),$$

of the power set map with the scanning map. This map respects the filtrations of C and V since the components do. The cofibration sequences $C^{k-1} \hookrightarrow C^k \to C^k/C^{k-1}$ induce fibration sequences of suspension spectra. We have a commutative diagram

$$\begin{array}{ccc}
\Sigma^\infty(E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} C^{k-1}) & \xrightarrow{\quad p \quad} & \Sigma^\infty(E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} V^{k-1}) \\
\downarrow & & \downarrow \\
\Sigma^\infty(E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} C^{k}) & \xrightarrow{\quad p \quad} & \Sigma^\infty(E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} V^{k}) \\
\downarrow & & \downarrow \\
\Sigma^\infty(E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} C^k/C^{k-1}) & \xrightarrow{p \simeq Id} & \Sigma^\infty(E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} V^k/V^{k-1})
\end{array}$$

in which the vertical sequences are fibrations and the lower horizontal map is homotopic to the identity. Observing that $C^1 = V^1$ we proceed by induction on k to get an equivalence

$$\Sigma^\infty \left(E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} C(M, M_0; X)\right) \simeq \Sigma^\infty E\,\mathrm{Diff} \ltimes_{\mathrm{Diff}} \bigvee_{k \geq 1} D^k(M, M_0; X).$$

The result follows by the following lemma. $\square$

Lemma 5.6. *Let G be a topological group and let $A_1, A_2, \dots$ be a sequence of well-pointed G-spaces with basepoints $a_1, a_2, \dots$ respectively. Then the pointed Borel construction on the wedge $\bigvee_{k\geq 1} A_k$ is the wedge of pointed Borel constructions on the A_k's.*

$$EG \ltimes_G \bigvee_{k\geq 1} A_k = \bigvee_{k\geq 1} EG \ltimes_G A_k$$

Proof. Given a pointed G-space X, as well as being the orbit space of the half-smash product $EG \ltimes X$, the pointed Borel construction is also the quotient of the Borel construction on X by BG, the Borel construction on the basepoint $*$.

$$EG \ltimes_G X = EG \times_G X / EG \times_G * = EG \times_G X / BG$$

The result is now established through the following chain of equalities.

$$\begin{aligned} EG \ltimes_G \bigvee_{k\geq 1} A_k &= EG \times_G \coprod_{k\geq 1} A_k \Big/ EG \times_G \coprod_{k\geq 1} a_k \\ &= \coprod_{k\geq 1} (EG \times_G A_k) \Big/ \coprod_{k\geq 1} (EG \times_G a_k) = \bigvee_{k\geq 1} EG \ltimes_G A_k \end{aligned}$$

□

Corollary 5.7. *Let G be a topological group which acts smoothly on M, fixing M_0 and ∂M. That is, G acts through a homomorphism $\psi : G \to \mathrm{Diff}$. Suppose also that X is a well pointed G-space. Then there is a homotopy equivalence of suspension spectra*

$$\Sigma^\infty \left(EG \ltimes_G C(M, M_0; X) \right) \simeq \Sigma^\infty \bigvee_{k\geq 1} EG \ltimes_G D^k(M, M_0; X).$$

Proof. Assume first that G acts *faithfully* on M, that is $\psi : G \to \mathrm{Diff}$ is a *monomorphism.* Then G acts freely on $\mathrm{Emb}(M; \mathbb{R}^\infty)$ by $g \cdot h = h \circ \psi(g)^{-1}$ so we choose this as a model for EG. The proof of Theorem 5.5 now gives the result.

If G does not act faithfully we must be a little more careful since the action on $\mathrm{Emb}(M; \mathbb{R}^\infty)$ is not necessarily free. Given universal spaces EG for G and $E\,\mathrm{Diff}$ for Diff, we form the product $EG \times E\,\mathrm{Diff}$ and equip it with the diagonal G and

Diff-actions. Here Diff is assumed to act trivially on EG. This product is then universal for both G and Diff. Replacing $\mathrm{Emb}(M, \mathbb{R}^\infty)$ by $EG \times \mathrm{Emb}(M, \mathbb{R}^\infty)$ in the proof of Theorem 5.5 and simply including elements of EG at each stage gives the splitting. □

We now generalise the results of Bödigheimer and Madsen [BM88].

Corollary 5.8. *Let G be a group which acts on a closed n-manifold M by diffeomorphism and on a well-pointed, connected space X. Suppose M is parallelisable, then there are G-spaces $D^n(M; X)$ such that*

$$\Omega^\infty \Sigma^\infty \left(EG \ltimes_G \mathrm{Map}(M; \Sigma^n X) \right) \simeq \prod_{n \geq 1} \Omega^\infty \Sigma^\infty EG \ltimes_G D^n(M; X)$$

Proof. Let us begin in more generality. Let M be a parallelisable, compact n-manifold on which G acts by diffeomorphism, fixing the boundary. Let M_0 be a G-invariant compact submanifold. Consider the pair $(M \setminus M_0, \partial M \setminus M_0)$, understanding the second space to be $\partial M \setminus (\partial M \cap M_0)$ if $M_0 \nsubseteq \partial M$. The scanning map gives a homotopy equivalence

$$EG \ltimes_G C(M \setminus M_0, \partial M \setminus M_0; X) \simeq EG \ltimes_G \mathrm{Map}(M, M_0; \Sigma^n X, *)$$

via the following chain of equivalences, each of which is induced by a G-equivariant map.

$$\begin{aligned} C(M \setminus M_0, \partial M \setminus M_0; X) &\simeq \Gamma(W \setminus (\partial M \setminus M_0), W \setminus (M \setminus M_0); X) && \text{(i)} \\ &= \Gamma((W \setminus \partial M) \cup M_0), (W \setminus M) \cup M_0); X) \\ &\cong \Gamma(M \setminus \partial M, M_0 \setminus \partial M; X) && \text{(ii)} \\ &\simeq \Gamma(M, M_0; X) && \text{(iii)} \\ &\cong \mathrm{Map}(M, M_0; \Sigma^n X; *) && \text{(iv)} \end{aligned}$$

(i) Scanning. The new scanning map from the tubular configuration space to the section space together with the projection from the tubular configuration space onto the ordinary configuration space gives this equivalence.

(ii) Excision of $W \smallsetminus M$. The section space is defined and agrees with the canonical section on all of $W \smallsetminus M$. Since this subset is open in $(W \smallsetminus M) \cup M_0$ we can define a homeomorphism by restricting the sections. An inverse is given by extending sections over the excised subset by the canonical section.

(iii) Extension over the boundary. As in Proposition 4.5 an embedding of pairs $\beta : (M, M_0) \to (M \smallsetminus \partial M, M_0 \smallsetminus \partial M)$ is constructed as an isotopy inverse to the inclusion. The isotopy equivalence induces a homotopy equivalence between the section spaces.

(iv) Parallelisability. Since M is parallelisable, $\overset{\infty}{T}M \wedge_M (M \times X)$ is a trivial $\Sigma^n X$ bundle over M. Sections of this bundle are precisely graphs of maps $M \to \Sigma^n X$.

Putting this homotopy equivalence together with Corollary 5.7 we have a (weak) homotopy equivalence of suspension spectra

$$\Sigma^\infty(EG \ltimes_G \mathrm{Map}(M, M_0; \Sigma^n X, *)) \simeq \Sigma^\infty \left(\bigvee_{k \geq 1} EG \ltimes_G D^k(M \smallsetminus M_0, \partial M \smallsetminus M_0; X) \right).$$

Apply the infinite loop space functor and observe that the free infinite loop space of a wedge of spaces is (weakly) homotopy equivalent to the product of free infinite loop spaces of the wedge summands. We have proved

$$\Omega^\infty\Sigma^\infty(EG \ltimes_G \mathrm{Map}(M, M_0; \Sigma^n X, *)) \simeq \prod_{k \geq 1} \Omega^\infty\Sigma^\infty\big(EG \ltimes_G D^k(M \smallsetminus M_0, \partial M \smallsetminus M_0; X)\big).$$

The corollary is the case where M is a closed manifold and $M_0 = \emptyset$. $\square$

5.3 Equivariant stable splittings

Whenever we have attempted to stably split a space we have made use of the configuration space $C(\mathbb{R}^\infty; A)$ as a model for the free infinite loop space $Q(A) := \varinjlim \Omega^n \Sigma^n A$. This is a problem when trying to construct stable splittings equivariantly since we cannot in embed configuration spaces equivariantly into $\mathbb{R}^\infty$, in general. By replacing our model for Q with the free monoid construction Γ^+ of Barratt and Eccles [BE74] we can avoid such issues. Recall that $\Gamma^+(A)$ is a model for $Q(A)$ when A is a connected, pointed space of the homotopy type of a CW-complex. Our method makes use of a slightly larger configuration space $\mathscr{C}$, from which we construct an equivariant map into $\Gamma^+(V)$ where $V = \bigvee_{i\geq 1} D^i$. We extend this to an equivariant map of infinite loop spaces $\Gamma^+(\mathscr{C}) \to \Gamma^+(V)$ to obtain an equivariant splitting.

In order to define $\mathscr{C}$, Γ^+ and the maps between them we will need to understand the homogeneous bar construction for the symmetric groups.

Definition 5.9. Let G be a discrete group. We define a simplicial group WG by $(WG)_n = G^{n+1}$, the direct sum of $n+1$ copies of G, with face and degeneracy maps $\partial_i\langle g_0, \dots, g_n\rangle = \langle g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_n\rangle$ and $s_i\langle g_0, \dots, g_n\rangle = \langle g_0, \dots, g_i, g_i, \dots, g_n\rangle$. This is a simplicial group with the usual product in G^{n+1}. There is a free action of G on WG by $\langle g_0, \dots, g_n\rangle \cdot g = \langle g_0 \cdot g, \dots, g_n \cdot g\rangle$.

The universal space of G is then the geometric realisation of the above simplicial group. That is, $EG \coloneqq |WG|$. The identity $\langle 1\rangle \in (WG)_0$ serves as basepoint for WG and defines a basepoint in EG.

Throughout this section we will not distinguish between the simplicial group and it's geometric realisation. Note that this definition means that any group homomorphism $G \to H$ induces a map $EG \to EH$. In particular the canonical inclusion $\Sigma_{k-1} \hookrightarrow \Sigma_k$ induces an inclusion $E\Sigma_{k-1} \hookrightarrow E\Sigma_k$. We can define a reduction homomorphism $R : \Sigma_k \to \Sigma_{k-1}$ by

$$R(\sigma)(i) \coloneqq \begin{cases} \sigma(i) & \text{if } \sigma(i) < \sigma(k) \\ \sigma(i) - 1 & \text{if } \sigma(i) > \sigma(k) \end{cases}$$

where $\sigma \in \Sigma_k$ is acting on $\{1, \dots, k\}$. The reduction homomorphism is a right inverse to the inclusion and induces a reduction map $E\Sigma_k \to E\Sigma_{k-1}$.

Definition 5.10. Let M be a compact manifold and let M_0 be a compact submanifold. Let X be a well-pointed space with basepoint x_0. The *Borel configuration space* of k particles in M with labels in X is

$$\mathscr{C}_k(M;X) := E\Sigma_k \times_{\Sigma_k} \tilde{C}_k(M;X)$$

where $\tilde{C}_k(M;X)$ is the usual unordered configuration space. The Borel configuration space of particles in M vanishing in M_0 with labels in X is

$$\mathscr{C}(M,M_0;X) := \coprod_{k\geq 0} \mathscr{C}_k(M;X) \Big/ \sim$$

where $(w;m_1,\ldots,m_k;x_1\ldots,x_k) \sim (R(w);m_1,\ldots,m_{k-1};x_1\ldots,x_{k-1})$ if $x_k = x_0$ or $m_k \in M_0$. Here $w \in E\Sigma_k$ and $R : E\Sigma_k \to E\Sigma_{k-1}$ is the reduction map.

Remark 5.11. The Borel configuration spaces admit filtrations $\mathscr{C}^k := \coprod_{i=0}^k \mathscr{C}_i \big/ \sim$ analogous to those for ordinary configuration spaces. We will make use of the filtration quotients $\mathscr{D}^k := \mathscr{C}^k/\mathscr{C}^{k-1}$ which are closely related to ordinary configuration spaces. Consider a sequence of spaces $\tilde{D}^k = \tilde{C}_k(M;X)/\approx$ where $(m_1,\ldots,m_k;x_1,\ldots,x_k) \approx *$ if $x_i = x_0$ or $m_i \in M_0$ for any i. These spaces are essentially filtration quotients of an ordered configuration space with vanishing particles. The filtration quotients of the Borel configuration spaces are then $\mathscr{D}^k = E\Sigma_k \ltimes_{\Sigma_k} \tilde{D}^k$ since collapsing $\mathscr{C}^{k-1} \subset \mathscr{C}^k$ to a point is the same as taking the quotient of $\mathscr{C}_k$ under the relation $(w;m_1,\ldots,m_k;x_1,\ldots,x_k) \approx *$, which is precisely $E\Sigma_k \ltimes_{\Sigma_k} \tilde{D}^k$.

Proposition 5.12. *Suppose the pair (M,M_0) or X is 0-connected. Then the* $\mathrm{Diff}(M;M_0 \cup \partial M)$*-equivariant projection $\mathscr{C}(M,M_0;X) \to C(M,M_0;X)$ is a nonequivariant (weak) homotopy equivalence. Moreover, the restriction to the filtration quotients $\mathscr{D}^k \to D^k$ is also equivariant and a nonequivariant homotopy equivalence.*

Proof. The projection $\pi : E\Sigma_k \times \tilde{C}_k(M;X) \to \tilde{C}_k(M;X)$ is a homotopy equivalence since $E\Sigma_k$ is contractible. In fact, the contraction $E\Sigma_k \to *$ is a Σ_k-equivariant homotopy equivalence, and hence so is π. The projection then induces a homotopy equivalence on the orbit spaces $\mathscr{C}_k(M;X) \to C_k(M;X)$. These homotopy equivalences give a well defined map $\mathscr{C}(M,M_0;X) \to C(M,M_0;X)$ since they behave with respect to the equivalence relations on each side. Equivariance of this map is routine to check. This map respects the filtrations and the restriction to the filtration quotients $\mathscr{D}^k \to D^k$ is well defined.

Recall from Proposition 3.27 that each configuration space $C_k(M;X)$ contains a subspace $B_k := \{(\boldsymbol{m};\boldsymbol{x}) : \exists i \text{ s.t. } m_i \in M_0 \text{ or } x_i = x_0)\}$ and each filtration C^k is obtained from C^{k-1} by attaching $C_k(M;X)$ along the obvious map $B_k \to C^{k-1}$. Similarly $\mathscr{C}^k$ is obtained from $\mathscr{C}^{k-1}$ by attaching $\mathscr{C}_k(M;X)$ along a subspace $\mathscr{B}_k \subseteq \mathscr{C}_k(M;X)$, defined as the Borel construction on a subset $\tilde{B}_k$ of $\tilde{C}_k(M;X)$. Form the following commutative diagram in which the horizontal maps from left to right are all induced by the map $\mathscr{C}(M,M_0;X) \to C(M,M_0;X)$.

$$\begin{tikzcd}
& \mathscr{C}_k(M;X) \arrow[rr, "\simeq"] \arrow[dd] & & C_k(M;X) \arrow[dd] \\
\mathscr{B}_k \arrow[ur] \arrow[rr, "\simeq"] \arrow[dd] & & B_k \arrow[ur] \arrow[dd] & \\
& \mathscr{C}^k \arrow[rr] & & C^k \\
\mathscr{C}^{k-1} \arrow[ur] \arrow[rr] & & C^{k-1} \arrow[ur] &
\end{tikzcd}$$

The left and right faces are pushout squares. By our assumptions all the maps from front to back are cofibrations so the left and right faces are in fact homotopy pushout squares. Then if the first three maps are (weak) homotopy equivalences, so is the last [Mat76, Corollary 9]. Starting with $k = 1$ and noting that $\mathscr{C}^0 = * = C^0$ we proceed by induction to obtain (weak) homotopy equivalences between filtrations. Taking the colimit over k gives the result.

To see that the restriction to the filtrations quotients is a (weak) homotopy equivalence we note that D^k is the pushout of the diagram $* \leftarrow C^{k-1} \hookrightarrow C^k$ and $\mathscr{D}^k$ is the pushout of the analogous diagram for Borel configuration spaces. A similar argument to the above completes the proof. $\square$

The reason for introducing the Borel configuration spaces will become clear once we have defined Γ^+. The extra structure is required to define maps between them.

Definition 5.13. Let A be a well-pointed space with basepoint $*$. We define the *free monoid on A* by

$$\Gamma^+(A) := \coprod_{k\geq 0} E\Sigma_k \times_{\Sigma_k} A^k \Big/ \sim$$

where $(w; a_1, \ldots, a_k) \sim (R(w); a_1, \ldots, a_{k-1})$ if $a_k = *$.

Our objective is now to define a map $\mathscr{C}(M; M_0; X) \to \Gamma^+(V)$. We begin with a collection of maps which generalise the power set map. For each k and each $0 \leq i \leq k$, every ordered configuration in $\tilde{C}_k$ has $\binom{k}{i}$ *unordered* subconfigurations of length i. We take the images of these subconfigurations under the composition $C^i \to D^i \hookrightarrow V$. The collection of $\binom{k}{i}$ points in V determines a point in $V^{\binom{k}{i}}$, using a lexicographic ordering based on the ordering of the parent configuration. More precisely, we define maps

$$\begin{aligned} f_i^k : \tilde{C}_k(M; X) &\longrightarrow V^{\binom{k}{i}} \\ (m_1, \ldots, m_k) &\longmapsto (\{m_1, \ldots, m_i\}; \{m_1, \ldots, m_{i-1}, m_{i+1}\}; \ldots; \{m_{k-i+1}, \ldots, m_k\}) \,. \end{aligned}$$

For simplicity we have not included the labels in the definition. Accompanying each such map is a homomorphism $\phi_i^k : \Sigma_k \to \Sigma_{\binom{k}{i}}$ which permutes the subconfigurations in such a way as to make the map above Σ_k equivariant. For example the permutation $(1,2)$ is mapped under ϕ_i^k to the permutation in $\Sigma_{\binom{k}{i}}$ which interchanges any unordered subconfiguration containing the particle m_1, but not the particle m_2, with the corresponding subconfiguration containing m_2 and the same remaining $i-1$ particles. That is, the permutation interchanges sets of the form $\{m_1, m_{j_1}, \ldots, m_{j_{i-1}}\}$ with $\{m_2, m_{j_1}, \ldots, m_{j_{i-1}}\}$, where $m_1 \neq m_{j_s} \neq m_2$ for all s.

For each k the collection of homomorphisms ϕ_i^k give a homomorphism

$$\begin{aligned} \phi^k : \Sigma_k &\longrightarrow \Sigma_{\binom{k}{0}} \oplus \Sigma_{\binom{k}{1}} \oplus \cdots \oplus \Sigma_{\binom{k}{k}} \subset \Sigma_{2^k} \\ \sigma &\longmapsto \left(\phi_0^k(\sigma),\, \phi_1^k(\sigma),\, \ldots\, ,\phi_k^k(\sigma)\right) \end{aligned}$$

which in turn defines a map $\phi^k : E\Sigma_k \to E\Sigma_{2^k}$. Similarly the f_i^k's induce a map $f^k : \tilde{C}_k(M;X) \to V^{2^k}$. Putting these maps together and considering the action of Σ_k on each side we have a well defined map

$$f_k : \mathscr{C}_k(M;X) \longrightarrow E\Sigma_{2^k} \times_{\Sigma_{2^k}} V^{2^k}$$

Proposition 5.14. *The maps f_k give a well defined map $f : \mathscr{C}(M, M_0; X) \to \Gamma^+(V)$ which is equivariant under the action of* $\mathrm{Diff}(M; M_0 \cup \partial M)$.

Proof. The f_k's certainly define a map $\coprod_{k\geq 0} E\Sigma_k \times_{\Sigma_k} \mathscr{C}_k(M;X) \to \coprod_{k\geq 0} E\Sigma_k \times_{\Sigma_k} V^k$. We need to check it behaves well with respect to the equivalence relations on each side. Choose two equivalent Borel configurations. We may assume they are of the form $(w; m_1, \ldots, m_k; x_1, \ldots, x_k)$ and it's reduction $(R(w); m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1})$. Now we check that their images under f are equivalent. Since $x_k = x_0$ or $m_k \in M_0$, any subconfiguration containing $(m_k; x_k)$ will be mapped to the basepoint in the appropriate copy of V. Recall that the equivalence relation in Γ^+ identifies a point in V^n with a point in V^{n-1} if a coordinate lies at the basepoint of V. Precisely 2^{k-1} subconfigurations contain the particle $(m_k; x_k)$ so iterating the relation 2^{k-1} times we have a relation between $f_k(m_1, \ldots, m_k; x_1, \ldots, x_k) \in V^{2^k}$ and $f_{k-1}(m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1}) \in V^{2^{k-1}}$.

To see that the map behaves on the $E\Sigma_k$'s it is enough to consider the homomorphisms ϕ^k and R on the symmetric group. From the definitions of the homomorphisms it follows that the square

$$\begin{array}{ccc} \Sigma_k & \xrightarrow{\phi^k} & \Sigma_{2^k} \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R^{2^{k-1}}} \\ \Sigma_{k-1} & \xrightarrow[\phi^{k-1}]{} & \Sigma_{2^{k-1}} \end{array}$$

commutes up to an isomorphism $(- \circ \tau_k) : \Sigma_{2^k} \to \Sigma_{2^k}$ where $\tau_k \in \Sigma_{2^k}$ is a permutation which ensures the reduction maps on the right remove the correct permutations. As a consequence the same square with the groups replaced by their universal spaces also commutes. This descends to the required equivalence in Γ^+,

$f_k(w; m_1, \ldots, m_k; x_1, \ldots, x_k) \sim f_{k-1}(R(w); m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1})$, and f is a well defined map. It is immediate from the definition that f is $\mathrm{Diff}(M; M_0 \cup \partial M)$-equivariant, assuming a trivial action on $E\Sigma_k$. □

Remark 5.15. By construction f respects the filtrations of the configuration spaces. That is, let $\mathscr{C}^k := \mathscr{C}^k(M, M_0; X)$ be the space of Borel configurations of length $\leq k$ and let $V_k = \bigvee_{i=0}^k D^i$, then f restricts to an equivariant map $\mathscr{C}^k \to \Gamma^+(V_k)$.

Definition 5.16. A Γ^+-*structure* on a pointed space A is a pointed map $\alpha : \Gamma^+(A) \to A$ making the following diagrams commute, where $h_A : \Gamma^+\Gamma^+(A) \to \Gamma^+(A)$ is the natural map defined in [BE74].

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "Id_A"'] & \Gamma^+(A) \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} \Gamma^+\Gamma^+(A) \arrow[r, "h_A"] \arrow[d, "\Gamma^+\alpha"'] & \Gamma^+(A) \arrow[d, "\alpha"] \\ \Gamma^+(A) \arrow[r, "\alpha"'] & A \end{tikzcd}$$

Proposition 5.17. *If a well-pointed space has a Γ^+ structure, then it is a homotopy abelian monoid with basepoint as unit [BE74].*

Remark 5.18. The map h_A is a Γ^+ structure on $\Gamma^+(A)$, for any A. In particular this makes $\Gamma^+(A)$ a homotopy abelian monoid. In fact Γ^+ is a homotopy functor from well-pointed spaces to topological monoids, taking pointed maps $g : A \to B$ to Γ^+ maps $\Gamma^+g : \Gamma^+(A) \to \Gamma^+(B)$ by $\Gamma^+f([w; a_1, \ldots, a_k]) = [w; f(a_1), \ldots, f(a_k)]$.

Definition 5.19. Given spaces A and B with Γ^+-structures $\alpha : \Gamma^+(A) \to A$ and $\beta : \Gamma^+(B) \to B$, a Γ^+ map $g : A \to B$ is a continuous monoid homomorphism making the following square commute.

$$\begin{tikzcd} \Gamma^+(A) \arrow[r, "\Gamma^+g"] \arrow[d, "\alpha"'] & \Gamma^+(B) \arrow[d, "\beta"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

Proposition 5.20. *The map f extends to a* $\mathrm{Diff}(M; M_0 \cup \partial M)$*-equivariant Γ^+ map*

$$\bar{f} : \Gamma^+(\mathscr{C}(M, M_0; X)) \longrightarrow \Gamma^+(V).$$

Proof. We define the extension of f as the composition

$$\Gamma^+(\mathscr{C}(M, M_0; X)) \xrightarrow{\Gamma^+ f} \Gamma^+\Gamma^+(V) \xrightarrow{h_V} \Gamma^+(V).$$

The first map is a Γ^+ map by definition and the second is a Γ^+ map by [BE74, Proposition 3.6]. It is easily checked that each map is $\mathrm{Diff}(M, M_0 \cup \partial M)$-equivariant. □

Theorem 5.21. *Let G be a group acting on a compact manifold M by diffeomorphism and let $M_0 \subseteq M$ be a G-invariant compact submanifold. Let X be a well-pointed G space. Then there is a stable homotopy equivalence*

$$\mathscr{C}(M, M_0; X) \overset{s}{\simeq} \bigvee_{k \geq 1} D^k(M, M_0; X)$$

induced by a G-equivariant map $\bar{f} : \Gamma^+(\mathscr{C}) \to \Gamma^+(V)$ provided X has the homotopy type of a CW-complex and (M, M_0) or X is 0-connected.

Proof. Restricting f to the filtrations $\mathscr{C}^k$ induces Γ^+ maps $\bar{f} : \Gamma^+(\mathscr{C}^k) \longrightarrow \Gamma^+(V^k)$ for each k. Moreover, f restricts to well defined maps $f : \mathscr{D}^k \to \Gamma^+(D^k)$ on the filtration quotients. In fact the image is contained in $D^k = E\Sigma_1 \times_{\Sigma_1} D^k$ and f factors as $\mathscr{D}^k \to D^k \hookrightarrow \Gamma^+(D^k)$ where the second map is the canonical inclusion and the first is precisely the homotopy equivalence in Proposition 5.12. Extending this map to a Γ^+ map as in the previous proposition gives the following composition.

$$\Gamma^+(\mathscr{D}^k) \xrightarrow{\Gamma^+ f} \Gamma^+(D^k) \xhookrightarrow{\Gamma^+ i} \Gamma^+\Gamma^+(D^k) \xrightarrow{h_{D^k}} \Gamma^+(D^k).$$

The first map is a homotopy equivalence since Γ^+ is a homotopy functor and the composition of the second two maps is the identity on $\Gamma^+(D^k)$ by [BE74, Proposition 3.6] so we have a homotopy equivalence $\bar{f} : \Gamma^+(\mathscr{D}^k) \to \Gamma^+(D^k)$.

For each k we can form a commutative diagram, in which the vertical sequences are fibrations.

$$\begin{tikzcd}
\Gamma^+(\mathscr{C}^{k-1}) \arrow[r, "\bar{f}"] \arrow[d] & \Gamma^+(V^{k-1}) \arrow[d] \\
\Gamma^+(\mathscr{C}^{k}) \arrow[r, "\bar{f}"] \arrow[d] & \Gamma^+(V^{k}) \arrow[d] \\
\Gamma^+(\mathscr{D}^{k}) \arrow[r, "\bar{f}"] & \Gamma^+(D^{k})
\end{tikzcd}$$

By the above discussion the bottom arrow is always a homotopy equivalence. Noting that $\mathscr{C}^1 = \mathscr{D}^1$ and $C^1 = D^1 = V^1$ we pass to the homotopy groups and proceed by induction on k to obtain a homotopy equivalence $\Gamma^+(\mathscr{C}) \simeq \Gamma^+(V)$. Since $\bar{f}$ is $\mathrm{Diff}(M; M_0 \cup \partial M)$-equivariant it is certainly equivariant under the action of any group acting by a homomorphism $G \to \mathrm{Diff}(M; M_0 \cup \partial M)$.

By our assumptions on (M, M_0) and X, each of the configuration spaces, filtrations, filtration quotients and splittings are path connected and of the homotopy type of a CW-complex. For such spaces Γ^+ is a model for free infinite loop spaces, $\Omega^\infty\Sigma^\infty$, and a Γ^+ map is an infinite loop map so the above homotopy equivalence is a stable homotopy equivalence $\mathscr{C}^k(M, M_0; X) \overset{s}{\simeq} \bigvee_{k\geq 1} D^k(M, M_0; X)$. □

Remark 5.22. With this stable homotopy equivalence we have completed a zigzag of maps

$$\mathrm{Map}(M; \Sigma^n X) \longleftarrow E(M; X) \longrightarrow C(M; X) \longleftarrow \mathscr{C}(M; X) \longrightarrow \bigvee_{i\geq 1} D^i(M; X)$$

each of which is equivariant under smooth group actions and a (stable) homotopy equivalence.

6 Configuration spaces with twisted labels

In this section we alter the definition of configuration spaces and tubular configuration spaces over a manifold M to admit *twisted labels*, i.e. labels in a fibre bundle over

M. The objective is to prove the following further generalisation of Bödigheimer and Madsen's stable splitting result.

Theorem 6.1. *Let G be a group acting smoothly on a closed n-manifold M. Let X be a well-pointed, connected G-space. Suppose M is stably parallelisable, then there is a natural number $N \geq n$ and a collection of pointed G-spaces $D^k(M;X)$ such that*

$$\Omega^\infty\Sigma^\infty\left(EG \ltimes_G \mathrm{Map}(M;\Sigma^N X)\right) \simeq \prod_{k\geq 1} \Omega^\infty\Sigma^\infty\left(EG \ltimes_G D^k(M;X)\right).$$

6.1 Definitions

Throughout this section we make use of fibre bundles with a distinguished section. Such fibre bundles will be labelling spaces for configuration spaces and the base will be the underlying manifold. We denote a bundle with fibre X and projection π together with a section s_0 by the triple $(\widehat{X}, \pi, s_0)$ or simply $\widehat{X}$ when this is unambiguous. We refer to s_0 as the *zero section*. Here $\widehat{X}$ is the total space and the fibre over a point b is denoted X_b. We require a zero section so that we have a well defined choice of basepoint in each fibre. We impose the additional restriction that the zero section should make each fibre X_b well-pointed with basepoint $s_0(b)$.

Definition 6.2. Let M be a compact manifold, let $(\widehat{X}, \pi, s_0)$ be a fibre bundle over M and for each k let $\tilde{C}_k(M)$ be the ordinary ordered configuration space over M. We define the space $\tilde{C}_k(M;\widehat{X})$ of k ordered particles over M with *twisted labels* in $\widehat{X}$ to be the following subspace of $\tilde{C}_k(M) \times \widehat{X}^k$.

$$\tilde{C}_k(M;\widehat{X}) \coloneqq \{(\boldsymbol{m}, \boldsymbol{x}) : x_i \in X_{m_i}\}$$

This space has an obvious action by Σ_k and we define the space $C_k(M;\widehat{X})$ of k *unordered* particles over M with twisted labels in $\widehat{X}$ to be the orbit space, $\tilde{C}_k(M;\widehat{X})/\Sigma_k$.

The only difference between this definition and the ordinary configuration space is that we require that the label associated to any particle sits in the fibre over that particle. We define the configuration spaces with vanishing particles in the same way as for ordinary configuration space, however we have a slightly different equivalence relation.

Definition 6.3. The configuration space of particles in M modulo a compact submanifold M_0 with twisted labels in $\widehat{X}$ is the space

$$C(M, M_0; \widehat{X}) := \left(\coprod_{k \geq 0} C_k(M; \widehat{X}) \right) \Big/ \sim$$

where $(m_1, \ldots, m_k; x_1, \ldots, x_k) \sim (m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1})$ if $m_k \in M_0$ or if $x_k = s_0(m_k)$. That is, particles vanish in M_0 or when their label is at the basepoint in the fibre over the particle.

Remark 6.4. If X is a well-pointed space with basepoint x_0, then the ordinary configuration spaces with labels in X are precisely the corresponding configuration spaces with twisted labels in the trivial bundle $\widehat{X} = M \times X$ with the zero section $m \mapsto (m, x_0)$.

As was the case with ordinary configuration spaces, the twisted variants defined so far are not flexible enough for our purposes. We construct tubular configuration spaces with twisted labels as fibre bundles over configuration spaces with twisted labels. However, the construction requires a few additional notions.

Definition 6.5. Let Y and Z be topological spaces. A *partial map* $f : Y \to Z$ is a map $A \to Z$ where $A \subseteq Y$. We call A the *domain* of f and denote it by $\mathcal{D}(f)$. A *paro map* or *partial map with open domain* is a partial map f such that $\mathcal{D}(f)$ is open in Y.

Definition 6.6. Let $P_o(Y, Z)$ be the set of paro maps $Y \to Z$. The *paro mapping space* is the set of paro maps together with the *paro compact-open* topology. A subbase for the topology is given by sets of the form

$$W(C, U) := \{f \in P_o(Y, Z) : C \subseteq \mathcal{D}(f),\ f(C) \subseteq U\}$$

where $C \subseteq Y$ is compact and $U \subseteq Z$ is open [AAB80].

Remark 6.7. This topology has some nice properties. Given an open set $A \in Y$, the subspace of $P_o(Y, X)$ which consists of maps defined on A is the ordinary mapping space $\mathrm{Map}(A, Z)$.

Definition 6.8. Let M and $\widehat{X}$ be as above and let $\tilde{E}_k(M)$ be the ordinary tubular configuration space over M. Recall that the definition of $\tilde{E}_k(M)$ involved a manifold W without boundary and containing M as a codimension 0 submanifold. Assume $\widehat{X}$ is extended over W, for example if W is M with a collar $\partial M \times [0, 1)$ attached to the boundary, then $\widehat{X}$ is the pullback along the map $W \to M$ which collapses the collar onto ∂M. We define the space $\tilde{E}_k(M; \widehat{X})$ of ordered tubular configurations over M with *twisted labels* in $\widehat{X}$ to be the following subspace of $\tilde{E}_k(M) \times P_o(W, \widehat{X})$.

$$\tilde{E}_k(M; \widehat{X}) := \{(f, s) : \mathcal{D}(s) = Im(f),\ s \text{ is a section}\}$$

This space admits a free action by Σ_k and we define the unordered variant $E_k(M; \widehat{X})$ to be the orbit space, $\tilde{E}_k(M; \widehat{X})/\Sigma_k$.

It is immediate from the discussion in Section 3 that this space, equipped with the obvious projection, is a fibre bundle over $C_k(M; \widehat{X})$. The fibre over a configuration $(\boldsymbol{m}; \boldsymbol{x})$ is not easy to describe precisely. As a set the fibre is a disjoint union of section spaces parameterised by the space of tubular neighbourhoods, $\mathrm{Tub}^{Id}(m_1, \ldots, m_k)$.

$$\coprod_{f \in \mathrm{Tub}^{Id}(\boldsymbol{m})} \Gamma(\widehat{X}|_{Im(f)}, \boldsymbol{x})$$

Here $\Gamma(\widehat{X}|_{Im(f)}, \boldsymbol{x})$ is the space of sections $s : Im(f) \to \widehat{X}|_{Im(f)}$ such that $s(m_i) = x_i$. We denote the restriction of s to $Im(f_i)$ by s_i.

Proposition 6.9. *The projection $p_k : E_k(M; \widehat{X}) \to C_k(M; \widehat{X})$ is a homotopy equivalence.*

Proof. Since $\widehat{X}$ is a fibre bundle and any tubular neighbourhood $f \in \mathrm{Tub}^{Id}(\boldsymbol{m})$ is a chart for M, the section space $\Gamma(\widehat{X}|_{Im(f)}, \boldsymbol{x})$ is homeomorphic to a product of based mapping spaces, $\prod \mathrm{Map}(\mathring{D}^n, 0; X_{m_i}, x_i)$. We have made use of the fact that a homotopy between zero sections of $\widehat{X}$ induces a homotopy equivalence between tubular configuration spaces. The fibre over a configuration $(\boldsymbol{m}, \boldsymbol{x})$ is homeomorphic to the product $\mathrm{Tub}^{Id}(\boldsymbol{m}) \times \prod \mathrm{Map}(\mathring{D}^n, 0; X_{m_i}, x_i)$ which is contractible since it is a product of contractible spaces. A homotopy inverse is given by any continuous section $C_k(M; \widehat{X}) \to E_k(M; \widehat{X})$. □

We define the space of tubular configurations on M modulo a compact submanifold M_0 with twisted labels in $\widehat{X}$, denoted by $E(M, M_0; \widehat{X})$, by allowing components of embeddings to vanish when their midpoints do. This causes a small problem with scanning, which we will address later.

Remark 6.10. As was the case for ordinary configuration spaces, $C(M, M_0; \widehat{X})$ and $E(M, M_0; \widehat{X})$ have filtrations induced by the cardinalities of configurations. Each of the spaces defined has the homotopy type of a CW-complex whenever $\widehat{X}$ does by a similar argument to Proposition 3.27. There is an obvious projection $p : E(M, M_0; \widehat{X}) \to C(M, M_0; \widehat{X})$ which is a homotopy equivalence when $\widehat{X}$ has the homotopy type of a CW-complex and a weak homotopy equivalence otherwise. Note that $\widehat{X}$ has the homotopy type of a CW-complex if and only if the fibre X does.

6.2 Twisted scanning

We construct a scanning map for our configuration spaces with twisted labels in precisely the same way as the ordinary configuration space. Let $\Gamma(W \setminus M_0, W \setminus M; \widehat{X})$ be the space of sections of $\overset{\infty}{T}W \wedge_W \widehat{X}$ which are defined outside of M_0 and agree with the canonical section $*$ outside of M. Here the fibrewise smash product takes the image of the zero section as the basepoint in each fibre.

We define the scanning map with a sequence of maps

$$\gamma_k : E_k(M;\widehat{X}) \to \Gamma(W \smallsetminus M_0, W \smallsetminus M; \widehat{X}).$$

Given a tubular configuration $(f;s) \in E_k(M;\widehat{X})$ we define a section

$$\begin{aligned}\gamma_k(f;s) : W \smallsetminus M_0 &\longrightarrow \overset{\infty}{T}W \wedge_W \widehat{X} \\ z &\longmapsto \begin{cases} *_z & \text{if } z \notin Im(f) \\ (d_{f_i^{-1}(z)} f_i \circ r_{f_i} \circ f_i^{-1}(z)) \wedge s(z) & \text{if } z \in Im(f_i). \end{cases}\end{aligned}$$

where $r_{f_i} : \overset{\infty}{T}_{m_i} M \to \overset{\infty}{T}_{m_i} M$ are the stretching maps defined in Section 4.

We would like to define the scanning map

$$\gamma_\infty : E(M, M_0;\widehat{X}) \to \Gamma(W \smallsetminus M_0, W \smallsetminus M; \widehat{X})$$

using the maps γ_k as in the untwisted case. Indeed we can define an *interior* configuration space and a homotopy equivalence $\alpha : E(M, M_0;\widehat{X}) \to \mathring{E}(M, M_0;\widehat{X})$ in the same way as in Section 4.1. Then given a tubular configuration $[(f;s)] \in E(M, M_0;\widehat{X})$ the scanning map should be $\gamma_\infty[(f;s)] := \gamma_k \circ \alpha(f;s)$ where $(f;x) \in E_k(M;\widehat{X})$ is any representative of the class $[(f;x)]$. Recall that a component f_i of an embedding f vanishes when $s(m_i) = s_0(m_i)$. However, there is no restriction on the image of s on $Im(f_i) \smallsetminus m_i$ so the scanning map is not well defined on equivalence classes. To resolve this problem we introduce a different but homotopy equivalent definition for tubular configuration spaces.

Definition 6.11. We define a new tubular configuration space as the space of tubular configurations in a compact manifold M whose components vanish when their midpoint lies inside a compact submanifold M_0 or when their labelling section agrees with the zero section s_0. That is,

$$\mathcal{E}(M, M_0; X) := \coprod_{k \geq 0} E_k(M;\widehat{X}) \Big/ \sim$$

where the equivalence relation is $(f_1, \ldots, f_k; s_1, \ldots, s_k) \sim (f_1, \ldots, f_{k-1}; s_1, \ldots, s_{k-1})$ if $f_k(0) \in M_0$ or $s_k = s_0|_{Im(f_k)}$.

Proposition 6.12. *The obvious quotient map $\mathcal{E}(M, M_0; \widehat{X}) \to E(M, M_0; \widehat{X})$ is a (weak) homotopy equivalence.*

Proof. The map is defined by sending an equivalence class in $\mathcal{E}(M, M_0; \widehat{X})$ to the corresponding equivalence class in $E(M, M_0; \widehat{X})$. This is certainly well defined since if a component vanishes in $\mathcal{E}(M, M_0; \widehat{X})$ it must also vanish in $E(M, M_0; \widehat{X})$. Now $\mathcal{E}(M, M_0; \widehat{X})$ admits filtrations by the length of configurations, which we denote $\mathcal{E}^k(M, M_0; \widehat{X})$. The filtration of length k can be obtained from the filtration of length $k-1$ by attaching $E_k(M; \widehat{X})$ along the inclusion of a subspace $\mathcal{B}_k$ and similarly $E^k(M, M_0; \widehat{X})$ is obtained from $E^{k-1}(M, M_0; \widehat{X})$ by attaching $E_k(M; \widehat{X})$ along the inclusion of a subspace B_k where

$$\mathcal{B}_k := \{(f; s) : f_k(0) \in M_0 \text{ or } s_k = s_0|_{Im(f_k)}\}$$
$$B_k := \{(f; s) : f_k(0) \in M_0 \text{ or } (s_k \circ f_k)(0) = (s_0 \circ f_k)(0)\}.$$

We can see from the definition that $E_k(M; \widehat{X})$ is a fibre bundle over $E_k(M)$ with fibre $\Gamma(\widehat{X}|_{Im(f)})$ over a tubular neighbourhood f. We define $E_k^{M_0}(M) \subset E_k(M)$ to be the subspace of tubular configurations f such that $f_k(0) \in M_0$. We also define a pair of subbundles $E_k^{s_0}(M; \widehat{X}) \subset E_k^*(M; \widehat{X}) \subset E_k(M; \widehat{X})$ as the bundles whose fibres over f are the subspaces of sections such that $s_k = s_0|_{Im(f_k)}$ and $s_k \circ f_k(0) = s_0 \circ f_k(0)$ respectively. There is a commutative diagram

$$\begin{tikzcd}[column sep=small]
 & E_k^{s_0}(M; \widehat{X}) \arrow[rr] \arrow[dd] & & E_k^*(M; \widehat{X}) \arrow[dd] \\
E_k^{s_0}(M; \widehat{X})|_{E_k^{M_0}(M)} \arrow[ur] \arrow[rr] \arrow[dd] & & E_k^*(M; \widehat{X})|_{E_k^{M_0}(M)} \arrow[ur] \arrow[dd] & \\
 & \mathcal{B}_k \arrow[rr] & & B_k \\
E_k(M; \widehat{X})|_{E_k^{M_0}(M)} \arrow[ur] \arrow[rr] & & E_k(M; \widehat{X})|_{E_k^{M_0}(M)} \arrow[ur] &
\end{tikzcd}$$

in which the left and right hand faces are pushout squares. The upper maps from front to back are cofibrations so the squares are homotopy pushouts, thus a weak homotopy equivalence on the first three horizontal maps induces a weak homotopy equivalence on the fourth.

Note that the fibres of the bundles $E_k^{s_0}(M;\widehat{X}) \subset E_k^*(M;\widehat{X}) \subset E_k(M;\widehat{X})$ are homeomorphic to $\prod_{k-1}\mathrm{Map}(D;X) \subset \mathrm{Map}_*(D;X) \times \prod_{k-1}\mathrm{Map}(D;X) \subset \prod_k \mathrm{Map}(D;X)$ respectively. The first inclusion is a homotopy equivalence since the based mapping space is contractible, therefore the inclusion $E_k^{s_0}(M;\widehat{X}) \hookrightarrow E_k^*(M;\widehat{X})$ is a homotopy equivalence and so are the two upper horizontal maps in the diagram above. The lower front horizontal map is the identity so it follows that the inclusion $\mathcal{B}_k \hookrightarrow B_k$ is a (weak) homotopy equivalence.

Similarly there is a commutative diagram

$$
\begin{tikzcd}
 & E_k(M;\widehat{X}) \arrow[rr] \arrow[dd] & & E_k(M;\widehat{X}) \arrow[dd] \\
\mathcal{B}_k \arrow[ur] \arrow[rr] \arrow[dd] & & B_k \arrow[ur] \arrow[dd] & \\
 & \mathcal{E}^k(M,M_0;\widehat{X}) \arrow[rr] & & E^k(M,M_0;\widehat{X}) \\
\mathcal{E}^{k-1}(M,M_0;\widehat{X}) \arrow[ur] \arrow[rr] & & E^{k-1}(M,M_0;\widehat{X}) \arrow[ur] &
\end{tikzcd}
$$

in which the left and right hand faces are homotopy pushout squares. The upper rear horizontal map is the identity and we have just seen that the upper front horizontal map is a (weak) homotopy equivalence. Now the lower front horizontal map is the identity when $k = 1$ since each space is just a point. We proceed by induction on k and get the result by taking the colimit over k. $\square$

As before we can define an interior configuration space and a homotopy equivalence $\alpha : \mathcal{E}(M,M_0;\widehat{X}) \to \mathring{\mathcal{E}}(M,M_0;\widehat{X})$. We define the scanning map by

$$
\begin{aligned}
\gamma_\infty : \mathcal{E}(M,M_0;\widehat{X}) &\longrightarrow \Gamma(W \smallsetminus M_0, W \smallsetminus M;\widehat{X}) \\
[(f;s)] &\longmapsto \gamma_k \circ \alpha(f;s)
\end{aligned}
$$

where $(f;s) \in E_k(M;\widehat{X})$ is any representative of the class $[(f;s)]$.

Example 6.13. Let X be a pointed topological space and let $\widehat{X}$ be the trivial X-bundle over W with the obvious section. Let $\nu \to W$ be the normal bundle of W embedded in the euclidean space $\mathbb{R}^N$ for some sufficiently large N. Form the fibrewise smash product $\overset{\infty}{\nu} \wedge_W \widehat{X}$ and equip it with the canonical section $*$. Now the scanning map gives sections of $\overset{\infty}{T}W \wedge_W (\overset{\infty}{\nu} \wedge_W \widehat{X}) = (TW \oplus \nu)^\infty \wedge_W (W \times X)$ which is a trivial $\Sigma^N X$-bundle over W. Since the section space must agree with $*$ outside of M and the bundle is trivial we have a scanning map

$$\mathcal{E}(M, \emptyset; \overset{\infty}{\nu} \wedge_W \widehat{X}) \longrightarrow \mathrm{Map}(M, \Sigma^N X).$$

Now that we have a scanning map it is natural to ask whether it is equivariant under the action of the diffeomorphism group of the underlying manifold and whether it is a homotopy equivalence. We begin by looking at equivariance.

In order to discuss equivariance we need a well defined action on the tubular configuration spaces and the section spaces. Given a diffeomorphism ϕ, the obvious actions to define are $\phi \cdot (f; s) = (\phi \circ f \circ d\phi^{-1}; \phi^{-1} \circ s \circ \phi)$ and $\phi \cdot \sigma = (d\phi \wedge_W \phi) \circ \sigma \circ \phi^{-1}$ respectively. We have assumed that there is a well defined action of the diffeomorphism group on the twisted labelling space. A well defined action must act on the total space in such a way that it projects to the given diffeomorphism on the base, that is, by automorphism. Unfortunately, for a general bundle there is no lift of the diffeomorphism group of the base to the automorphism group of the bundle. We discuss the relationship between the automorphism group of a bundle and the diffeomorphism group of the base.

By a bundle morphism between smooth fibre bundles $\pi : E_1 \to B_1$ and $\pi : E_2 \to B_2$ we mean a smooth map $\phi_E : E_1 \to E_2$ such that there exists a smooth map on the bases $\phi_B : B_1 \to B_2$ making the obvious square commutes. We call a bundle morphism an *isomorphism* if there exists a bundle morphism $\psi_E : E_2 \to E_1$ such that $\psi_E \circ \phi_E = Id_{E_1}$ and $\phi_E \circ \psi_E = Id_{E_2}$ and we denote an isomorphism of bundles by

$E_1 \cong E_2$. We call a bundle morphism $\phi : E_1 \to E_2$ between two bundles over the same base B an *equivalence* if it is an isomorphism and $\phi_B = Id_B$. We denote an equivalence of bundles by $E_1 \sim E_2$.

Lemma 6.14. *Let $\pi : E \to B$ be a fibre bundle and let $\mathrm{Aut}_B(E)$ be the group of automorphisms of E, by which we mean the group of bundle isomorphisms $E \to E$. There is a canonical homomorphism $\mathrm{Aut}_B(E) \to \mathrm{Diff}(B)$ whose image is precisely $\mathrm{Diff}_E(B) \coloneqq \{\phi \in \mathrm{Diff}(B) : \phi^* E \sim E\}$.*

Proof. The homomorphism is simply given by

$$\begin{aligned} \sigma : \mathrm{Aut}_B(E) &\longrightarrow \mathrm{Diff}(B) \\ \phi_E &\longmapsto \phi_B. \end{aligned}$$

It is clear that ϕ_B must be a diffeomorphism and σ is a homomorphism. Recall that given a map $\psi : C \to B$ between the bases there is a canonical map $\tilde{\psi} : \psi^* E \to E$ which makes the following square commute.

$$\begin{tikzcd} \psi^* E \arrow[r, "\tilde{\psi}"] \arrow[d] & E \arrow[d] \\ C \arrow[r, "\psi"] & B \end{tikzcd}$$

To see that $\mathrm{Diff}_E(B) \subseteq Im(\sigma)$ suppose we have $\psi \in \mathrm{Diff}_E(B)$ and let $f : E \to \psi^* E$ be an equivalence, then $\tilde{\psi} \circ f$ is an automorphism of E whose image under σ is ψ. For the opposite inclusion let $\phi_E \in \mathrm{Aut}_B(E)$ and note that $\phi_E^{-1} \circ \widetilde{(\phi_B)} : (\phi_B)^* E \to E$ is an equivalence. $\square$

Remark 6.15. The homomorphism in Lemma 6.14 is part of a short exact sequence

$$0 \longrightarrow \Gamma(P_{\mathrm{Aut}(F)}) \hookrightarrow \mathrm{Aut}_B(E) \longrightarrow \mathrm{Diff}_E(B) \longrightarrow 0$$

where F is the fibre of E and $P_{\mathrm{Aut}(F)}$ is the associated principal $\mathrm{Aut}(F)$-bundle.

Remark 6.16. In general $\mathrm{Diff}_E(B) \subsetneq \mathrm{Diff}(B)$. Consider the Hopf bundle over S^2 and the antipodal map on the base. The Chern class of the Hopf bundle is $+1$, however the the Chern class of the pullback is -1 so they cannot be equivalent. When E is a trivial bundle $\mathrm{Diff}_E(B) = \mathrm{Diff}(B)$ and there is a canonical lift $\mathrm{Diff}(B) \to \mathrm{Aut}_B(E)$ splitting the above exact sequence. For any bundle $\mathrm{Diff}_E(B)$ contains $\mathrm{Diff}_0(B)$, the connected component of the identity, since homotopic maps induce equivalent pullbacks.

Definition 6.17. A fibre bundle $E \to B$ is called *natural* if $\mathrm{Diff}_E(B) = \mathrm{Diff}(B)$ and there is a lift $\mathrm{Diff}(B) \to \mathrm{Aut}_B(E)$.

Returning to the question of equivariance of the scanning map we define

$$\mathrm{Aut}_W^{s_0}(\widehat{X}) := \{\psi_E \in \mathrm{Aut}_W(\widehat{X}) : \psi_E \circ s_0 \circ \psi_B^{-1} = s_0\}$$

to be the subgroup of automorphisms of the labelling bundle which fix the zero section. In order for the action on the tubular configuration spaces to be well defined we need this additional property.

Proposition 6.18. *Let* $\mathrm{Aut}_W^{s_0}(\widehat{X}; M_0 \cup (W \smallsetminus M))$ *be the subgroup of automorphisms* $\psi_E \in \mathrm{Aut}_W^{s_0}(\widehat{X})$ *such that* ψ_B *fixes* M_0 *and* $W \smallsetminus M$*. The scanning maps* γ_k *and* γ_∞ *are equivariant under the action of this group.*

Proof. The result follows immediately from the proofs of Proposition 4.8 and Corollary 4.9 by making the appropriate alterations to the labels. $\square$

Remark 6.19. If the labelling bundle $\widehat{X}$ is natural, for example if $\widehat{X}$ is trivial or a tangent bundle, then the scanning map is $\mathrm{Diff}(M, M_0 \cup \partial M)$-equivariant since diffeomorphism of M lift to automorphisms of the labelling bundle.

As for homotopy equivalence of the scanning map, the result is entirely analogous to Theorem 4.10.

Theorem 6.20. *Let M be a compact manifold and let M_0 be a compact submanifold. Let $\widehat{X}$ be a twisted labelling space. The scanning map*

$$\gamma_\infty : \mathcal{E}(M, M_0; \widehat{X}) \longrightarrow \Gamma(W \smallsetminus M_0, W \smallsetminus M; \widehat{X})$$

is a weak homotopy equivalence provided (M, M_0) or the typical fibre X is 0-connected. Moreover, γ_∞ is a homotopy equivalence if $\widehat{X}$ has the homotopy type of a CW-complex.

The proof of this theorem is entirely analogous to the version with ordinary labels. All the results throughout the proof hold with the only adjustment that the condition on the labelling space changes to 0-connectedness on the fibre. This is to be expected since the ordinary configuration spaces are the configuration spaces we have just defined with labels in a trivial bundle.

Corollary 6.21. *Let M be a compact manifold and let $M_0 \subseteq M$ be a compact submanifold. Let $\widehat{X}$ be a fibre bundle over M with zero section s_0. Suppose G is a group acting on $\widehat{X}$ and M by homomorphisms $\psi : G \to \mathrm{Aut}_W^{s_0}(\widehat{X}; M_0 \cup (W \smallsetminus M))$ and $\sigma \circ \psi : G \to \mathrm{Diff}_{\widehat{X}}(M; M_0 \cup (W \smallsetminus M))$. Then*

(i) the scanning map $\gamma_\infty : \mathcal{E}(M, M_0; \widehat{X}) \to \Gamma(W \smallsetminus M_0; W \smallsetminus M; \widehat{X})$ induces a (weak) homotopy equivalence of pointed Borel constructions

$$EG \ltimes_G \mathcal{E}(M, M_0; \widehat{X}) \xrightarrow{\simeq} EG \ltimes_G \Gamma(W \smallsetminus M_0; W \smallsetminus M; \widehat{X})$$

provided (M, M_0) or the typical fibre X is 0-connected;

(ii) the projection $p : \mathcal{E}(M, M_0; \widehat{X}) \to C(M, M_0; \widehat{X})$ induces a (weak) homotopy equivalence on pointed Borel constructions

$$EG \ltimes_G \mathcal{E}(M, M_0; \widehat{X}) \xrightarrow{\simeq} EG \ltimes_G C(M, M_0; \widehat{X})$$

for any (M, M_0) and X.

Corollary 6.22. *There is a homotopy equivalence of the pointed Borel constructions*

$$EG \ltimes_G C(M, M_0; \widehat{X}) \simeq EG \times_G \Gamma(W \smallsetminus M_0; W \smallsetminus M; \widehat{X})$$

provided (M, M_0) or the typical fibre X is 0-connected.

6.3 Stable splittings

Many of the stable splitting results established for ordinary configuration spaces generalise to configuration spaces with twisted labels. Throughout this section we use the notation $\mathrm{Diff}_{\widehat{X}}$ and $\mathrm{Aut}_W^{s_0}$ for the groups $\mathrm{Diff}_{\widehat{X}}(M; M_0 \cup (W \smallsetminus M))$ and $\mathrm{Aut}_W^{s_0}\left(\widehat{X}; M_0 \cup (W \smallsetminus M)\right)$ whenever it is clear which manifolds and labelling bundles we are referring to.

Theorem 6.23. *Let $C^k := C^k(M, M_0; \widehat{X})$ be the filtrations of the configuration space with twisted labels $C(M, M_0; \widehat{X})$ and let $D^k = D^k(M, M_0; \widehat{X}) := C^k/C^{k-1}$ be quotients of successive filtrations. Then there is a (weak) homotopy equivalence of suspension spectra*

$$\Sigma^\infty \left(E\,\mathrm{Diff}_{\widehat{X}} \ltimes_{\mathrm{Diff}_{\widehat{X}}} C(M, M_0; \widehat{X})\right) \simeq \Sigma^\infty \left(\bigvee_{k\geq 1} E\,\mathrm{Diff}_{\widehat{X}} \ltimes_{\mathrm{Diff}_{\widehat{X}}} D^k(M, M_0; X)\right).$$

Proof. Since $\mathrm{Diff}_{\widehat{X}}$ is a subgroup of the diffeomorphism group $\mathrm{Diff}(M)$ the space of embeddings $\mathrm{Emb}(M, \mathbb{R}^\infty)$ is also a universal space for $\mathrm{Diff}_{\widehat{X}}$. The proof is now the same as the proof of Theorem 5.5, with small changes to account for the twisted labels. □

Corollary 6.24. *Let G be a group which acts on $\widehat{X}$ and M via homomorphisms $\psi : G \to \mathrm{Aut}_W^{s_0}$ and $\sigma \circ \psi : G \to \mathrm{Diff}_{\widehat{X}}$. Then there is a (weak) homotopy equivalence of suspension spectra*

$$\Sigma^\infty \left(EG \ltimes_G C(M, M_0; \widehat{X})\right) \simeq \Sigma^\infty \left(\bigvee_{k\geq 1} EG \ltimes_G D^k(M, M_0; X)\right).$$

Proof. This corollary follows from Theorem 6.23 in the same way that Corollary 5.7 follows from Theorem 5.5 for ordinary configuration spaces. □

Putting this result together with Corollary 6.22 we have a splitting of sections spaces,

$$\Sigma^\infty \left(EG \ltimes_G \Gamma(W \smallsetminus M_0; W \smallsetminus M; \widehat{X})\right) \simeq \Sigma^\infty \left(\bigvee_{k\geq 1} EG \ltimes_G D^k(M, M_0; \widehat{X})\right).$$

provided (M, M_0) or the fibre X is 0-connected. We can now prove Theorem 6.1.

Proof of Theorem 6.1. Let us begin in more generality. Let M be a stably parallelisable compact manifold and let G be a group acting on M by diffeomorphism, fixing the boundary. Let M_0 be a G-invariant compact submanifold, let W be a manifold without boundary containing M as a codimension 0 submanifold and let $\nu \to W$ be the normal bundle of W embedded in $\mathbb{R}^N$ for some sufficiently large N. Let $\widehat{X}$ be the trivial X-bundle over W and form the fibrewise smash product with the fibrewise one point compactification of the normal bundle, $\overset{\infty}{\nu} \wedge_W \widehat{X}$, equipped with the canonical section $*$.

Since M is stably parallelisable the normal bundle ν restricted to M is trivial, hence so is $\big(\overset{\infty}{\nu} \wedge_W \widehat{X}\big)$. Any diffeomorphism of M lifts to an automorphism of $\big(\overset{\infty}{\nu} \wedge_W \widehat{X}\big)|_M$ so we can assume G acts on this bundle by the lift. Since the diffeomorphisms in question fix ∂M they can be extended trivially over W and over all of $\overset{\infty}{\nu} \wedge_W \widehat{X}$.

There is a sequence of (weak) homotopy equivalences between the configuration space $C(M \smallsetminus M_0, \partial M \smallsetminus M_0; \overset{\infty}{\nu} \wedge_W \widehat{X})$ and the section space $\Gamma(M, M_0; \overset{\infty}{\nu} \wedge_W \widehat{X})$. As noted in Example 6.13, this is a space of sections of $\overset{\infty}{TW} \wedge_W \big(\overset{\infty}{\nu} \wedge_W \widehat{X}\big) = (TW \oplus \nu)^\infty \wedge_W \widehat{X}$ which is a trivial $\Sigma^N X$-bundle over W. Thus there is a (weak) homotopy equivalence

$$C(M \smallsetminus M_0, \partial M \smallsetminus M_0; \overset{\infty}{\nu} \wedge_W \widehat{X}) \simeq \mathrm{Map}(M, M_0; \Sigma^N X, *)$$

which is induced by a sequence of G-equivariant maps. Putting the induced map on the suspension spectra of the pointed Borel constructions together with the stable splitting of section spaces obtained in the preceding discussion we have a (weak) homotopy equivalence of suspension spectra

$$\Sigma^\infty\big(EG \ltimes_G \mathrm{Map}(M, M_0; \Sigma^N X, *)\big) \simeq \Sigma^\infty\left(\bigvee_{k \geq 1} EG \ltimes_G D^k(M \smallsetminus M_0, \partial M \smallsetminus M_0; \widehat{X})\right).$$

The theorem is then the result of applying the infinite loop space functor to this equivalence when M is a closed manifold and $M_0 = \emptyset$. $\square$

Appendix A Tubular neighbourhoods

Definition A.1. Let M be an n-dimensional manifold. Let $X \subset M$ be a k-dimensional compact submanifold and $\nu = \frac{TM|_X}{TX}$ be its normal bundle. A *weak tubular neighbourhood* of X in M is an embedding $f : \nu \to M$ so that the restriction $f|_X : X \to M$, where X is identified with the zero section in ν, is the inclusion.

A *tubular neighbourhood* of X in M is a weak tubular neighbourhood f with the additional property that the differential induces the identity on ν via the composition

$$\nu \xhookrightarrow{i} \nu \oplus TX = T\nu|_X \xrightarrow{df} TM|_X \xrightarrow{\text{proj.}} \nu$$

We denote by $\mathrm{Tub}(X)$ the space of weak tubular neighbourhoods of X topologised as a subspace of $\mathrm{Emb}(\nu, M)$ with the weak Whitney C^∞ topology and by $\mathrm{Tub}^{Id}(X) \subset \mathrm{Tub}(X)$ the space of tubular neighbourhoods.

Proposition A.2. *Let* $f_0 \in \mathrm{Tub}(X)$*. Then* $\mathrm{Tub}(X)$ *is homotopy equivalent to* $\mathrm{Tub}_{f_0}(X)$*, the subspace of weak tubular neighbourhoods contained in the image of* f_0*.*

Proof. Fix a Riemannian metric $\langle \cdot, \cdot \rangle$ on ν. Let $V_0 = f_0(\nu)$ and for each $f \in \mathrm{Tub}(X)$ define an open neighbourhood of the zero section by $U_f = f^{-1}(V_0 \cap f(\nu)) \subset \nu$. For each f and U_f there exists a smooth function $\varepsilon_f : X \to \mathbb{R}_{>0}$ so that $B_{\varepsilon_f}\nu := \{y \in \nu : |y| < (\varepsilon_f \circ p)(y)\} \subset U_f$ where $p : \nu \to X$ is the projection [BJ82]. These functions can be chosen so that they are a continuous family in f.

Following Godin [God08] we define a homotopy $H : \mathrm{Tub}(X) \times I \to \mathrm{Tub}(X)$ by

$$\begin{aligned} H(f,t) : \nu &\longrightarrow M \\ y &\longmapsto f\left((1-t)y + th\left(\frac{\pi|y|}{2(\varepsilon_f \circ p)(y)}\right)y\right) \end{aligned}$$

where $h : \mathbb{R} \to \mathbb{R}$ is defined by the smooth map

$$h(x) := \begin{cases} \frac{1}{x}\arctan(x) & \text{for } x \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

So H is continuous in f and smooth in t. It is easy to check that $H(f,t)$ is an embedding. Since $h(0) = 1$ and $h'(0) = 0$ we can see that $H(f,t)(y) = f(y)$ and $d_y H(f,t) = d_y f$ when $|y| = 0$. Then for each $(f,t) \in \mathrm{Tub}(X) \times I$, $H(f,t) \in \mathrm{Tub}(X)$ and indeed if $f \in \mathrm{Tub}^{Id}(X)$ then $H(f,t) \in \mathrm{Tub}^{Id}(X))$.

For any positive c, $0 \leq \frac{2c}{\pi} \arctan\left(\frac{\pi x}{2c}\right) < c$ so for any $y \in \nu$,

$$h\left(\frac{\pi|y|}{2(\varepsilon_f \circ p)(y)}\right) y = \frac{2(\varepsilon_f \circ p)(y)}{\pi} \arctan\left(\frac{\pi}{2(\varepsilon_f \circ p)(y)}\right) \frac{y}{|y|} \in B_{\varepsilon_f}\nu$$

That is, for any $f \in \mathrm{Tub}(X)$ and $y \in \nu$, $H(f,1)(y) \in f(B_{\varepsilon_f}\nu) \subset f_0(\nu)$.

Then the map $f \mapsto H(f,1)$ and the inclusion $\mathrm{Tub}_{f_0}(X) \hookrightarrow \mathrm{Tub}(X)$ define a homotopy equivalence. $\square$

Corollary A.3. $\mathrm{Tub}^{Id}(X)$ *is homotopy equivalent to* $\mathrm{Tub}^{Id}_{f_0}(X)$.

Remark A.4. Given any two (weak) tubular neighbourhoods f, f_0, the homotopy H defines an isotopy $H(f,-)(-) : I \times \nu \to M$ from f to a (weak) tubular neighbourhood f_1 such that $f_1(\nu) \subset f_0(\nu)$.

Proposition A.5. *Given two weak tubular neighbourhoods $f, f_0 \in \mathrm{Tub}(X)$ of X in M there exists an isomorphism of vector bundles $\Phi : \nu \to \nu$ such that f and $f_0 \circ \Phi$ are isotopic.*

In order to prove this proposition we will require the following high-dimensional version of Taylor's Theorem [Lan72].

Theorem A.6. *Let E, F be Banach spaces. Let U be open in E. Let x, y be two points in U such that the segment $x + ty$ lies in U for $t \in [0,1]$. Let $f : U \to F$ be a*

C^1-map. Then the function $Df(x+ty)\cdot y$ is continuous in t and we have

$$f(x+y) = f(x) + \int_0^1 Df(x+ty)\cdot y dt.$$

Proof of A.5. First note that by Remark A.4 we may assume $f \in \mathrm{Tub}_{f_0}(X)$. We follow Hirsch [Hir76] and Lang [Lan72]. Let $f \in \mathrm{Tub}_{f_0}(X)$. Since $f(\nu) \subset f_0(\nu)$ the map $g = f_0^{-1} f : \nu \to \nu$ is well defined and the identity on the zero section. It is enough to construct an isotopy $G : \nu \times I \to \nu$ from Φ to g, then composition with f_0 gives the required isotopy.

The idea of the isotopy is to linearise g in the fibre direction. This is achieved by scaling the fibres of ν by t, mapping under g and rescaling the image by t^{-1}. As t approaches zero we reach the fibre derivative of g at the zero section, that is, the component along the fibres of the differential $dg : \nu \oplus TX = T\nu|_X \to T\nu|_X = \nu \oplus TX$. This is an isomorphism of vector bundles since $g|_X$ is a diffeomorphism.

Define Φ to be the fibre derivative of g and define the isotopy from Φ to g by

$$\begin{aligned} G : \nu \times I &\longrightarrow \nu \\ (y,t) &\longmapsto \begin{cases} t^{-1} g(ty) & \text{for } t \neq 0 \\ \Phi(y) & \text{otherwise} \end{cases} \end{aligned}$$

where ty denotes multiplication of $y \in \nu$ by t in the fibre. For each fixed t, $G(-,t) : \nu \to \nu$ is an embedding since it is composed of embeddings. We need to check G is smooth at $t = 0$. This is a local condition so we can compose with charts on X and local trivialisations on ν to get local representations of g and Φ as maps $U \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ and similarly a local representation of G as a map $U \times \mathbb{R}^{n-k} \times I \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ where U is an open ball in $\mathbb{R}^k$. The local representation of g is given by

$$\bar{g}(u,v) = (\sigma(u,v), \tau(u,v))$$

with $\sigma(u,0) = u$ and $\tau(u,0) = 0$ since g is the identity on the zero section. The local representation of Φ should be the identity on the first component, corresponding to the zero section, and the derivative of the second factor so we get

$$\bar{\Phi}(u, v) = (u, D_2\tau(u, 0) \cdot v)$$

where $D_2\tau$ is the Jacobian of the second component of τ with respect to the second variable.

The local representation of the isotopy G is defined in the same way as above.

$$\bar{G}(u, v, t) = \begin{cases} (\sigma(u, tv), t^{-1}\tau(u, tv)) & \text{for } t \neq 0 \\ \bar{\Phi}(u, v) & \text{otherwise.} \end{cases}$$

It is immediately clear that the first component of $\bar{G}$ is smooth in all three variables, even for $t = 0$. By fixing $u \in U$ we can apply Theorem A.6 to the second component to get

$$t^{-1}\tau(u, tv) = t^{-1} \int_0^1 D_2\tau(u, stv) \cdot (tv) ds = \int_0^1 D_2\tau(u, stv) \cdot (v) ds.$$

which is smooth for all t and agrees with $\bar{\Phi}$ when $t = 0$. Then G is a smooth isotopy and an isotopy from f to $f_0\Phi$ is given by $F(y, t) = f_0G(y, 1 - t)$. □

It is not hard to see that in the particular case of $f, f_0 \in \mathrm{Tub}^{Id}_{f_0}(X)$, the additional restraint ensures that the fibre derivative of $g_f := f_0^{-1}f : \nu \to \nu$ as defined above is in fact the identity and we have an isotopy from f to f_0. Since the scaling in the fibres which defines the isotopy does not alter the derivative at the zero section or the image we have $f_0G(-, t) \in \mathrm{Tub}^{Id}_{f_0}(X)$ for all t so we have an isotopy through tubular neighbourhoods. For f_0 fixed it is clear that g_f depends continuously on f so the isotopies define a deformation retract of $\mathrm{Tub}^{Id}_{f_0}(X)$ onto f_0.

Corollary A.7. *The space* $\mathrm{Tub}^{Id}(X)$ *of tubular neighbourhoods of a compact submanifold* X *in a manifold* M *is contractible.*

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