The need for closure

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When defining a group, do we need to include closure? This is a detail that is often touched upon when the notion of a group is introduced to undergraduates. Should closure be listed as an axiom in its own right, or should it be regarded as an inherent property of the binary operation? There is no clear answer to this question, although there are firm opinions on both sides. Indeed, a very brief survey of group theory textbooks found in [1, pp. 458-459] suggests that there is a rough 50 : 50 split between authors who include closure explicitly and those who do not. In this Article, we go back to the beginning of the twentieth century to provide some historical perspective on this problem.*

It may be desirable, in the interests of mathematical tidiness, to exclude closure from our list of axioms, but since it is one of the properties that must be verified in determining whether a particular structure is a group, there seem to be sound pedagogical reasons for explicitly retaining it. Thus, the 'standard' group definition that is often given to students is the following, which we include here so that we can refer to it below (see, for example, [2, p. 41]). Let G be a set upon which there is defined a binary operation† whose result is denoted by juxtaposition of elements. We say that G forms a group with respect to this binary operation if the following conditions are satisfied:

G1. the binary operation is *closed* : whenever $g, h \in G$, $gh \in G$;

G2. the binary operation is *associative*: for any $g, h, k \in G$, $(gh)k = g(hk)$;

G3. there exists an *identity element* for the binary operation: an element $e \in G$ such that, for any $g \in G$, $eg = g = ge$;

G4. every element of G has an *inverse element* with respect to the identity element named in (G3): to each $g \in G$ there corresponds $g^{-1} \in G$ such that $gg^{-1} = e = g^{-1}g$.

Despite this being a much-used definition, there are other problems with it besides the question of closure. For instance, (G4) employs the identity element e from (G3) without our having shown that e is in fact unique. We will return to this point below.

Wide-ranging axiomatic questions of this type were addressed at the beginning of the twentieth century by the so-called *postulate analysts*, a small collection of American mathematicians who were clustered loosely around E. H. Moore (1862-1932) in Chicago, with significant contributions

* Parts of the present article are drawn from [1, pp. 411–412].

† This should of course also be defined properly (see, for example, the discussion in [1]), but in the interests of saving space, we think of a binary operation simply in the intuitive way as a procedure that takes two inputs and gives a single well-defined output.

also from E. V. Huntington (1874-1952) at Harvard. Inspired by the treatment of axioms (a.k.a. postulates) in David Hilbert's *Grundlagen der Geometrie* [3], the postulate analysts took sets of axioms as their objects of study. In considering sets of defining axioms for groups, fields, Boolean algebras, and a host of other structures, the postulate analysts sought to ensure that the sets contained no redundancies, and that they were in some sense minimal, the latter being a problematic notion, as we will see. The axioms for groups received special attention; in the course of their investigations, the postulate analysts derived a series of alternative definitions of groups, some quite intuitive, others rather less so, involving exotic axioms. It is the treatment of closure in one such definition that is of interest to us here.

The most prolific of the postulate analysts was the above-mentioned Huntington. Roughly half of the 134 publications that are listed under his name on zbMATH are on a postulational theme. Among his earliest contributions is a paper entitled 'Simplified definition of a group', presented to the American Mathematical Society in February 1902, and published in its *Bulletin* later that same year [4]. At the beginning of the paper, Huntington was quite explicit about what he had set out to achieve:

> Up to the present time no attempt seems to have been made to prove the independence of the postulates employed to define a group, and as a matter of fact the definition usually given contains several redundancies. These redundancies are removed in the following note, the number of necessary postulates being reduced to three, and the independence of these three being established.
>
> [4, p. 296]

The 'usual' definition to which Huntington referred was that given by Heinrich Weber in his *Lehrbuch der Algebra*, in whose second volume a group is defined to be a set together with a binary operation that is subject to the following conditions (see [5, vol. II, pp. 3–4] or [1, pp. 421–422]):

W1. given two elements a, b of the set, ab is uniquely determined as an element of the set;

W2. the associative law holds;

W3. if either $ab = ab'$ or $ab = a'b$, then, in the first case, $b = b'$, and in the second, $a = a'$;

W4. given three elements a, b, c of the set such that $ab = c$, each one is uniquely determined by the other two.

It is an easy exercise, often given to students, to prove that in the finite case we may make do with just (W1)–(W3). We may also note, for example, that the uniqueness included by Weber in (W4) is not in fact necessary — this is just one of the points of redundancy to which Huntington referred.

The alternative definition offered by Huntington was the following. A group is a set (or 'assemblage', as Huntington had it — a word often used at this time in place of 'set') upon which there is defined a 'rule of combination' $\circ$ that satisfies the following conditions (see [4, p. 297] or [1, p. 430]):*

H1. given any two elements a, b of the set, there is an element x in the set such that $a \circ x = b$;

H2. given any two elements a, b of the set, there is an element y in the set such that $y \circ a = b$;

H3. if $a, b, c, a \circ b, b \circ c$ are elements of the set, and either $(a \circ b) \circ c$ or $a \circ (b \circ c)$ is also, then $(a \circ b) \circ c = a \circ (b \circ c)$.

We notice immediately that Huntington made no explicit reference to closure in his definition. But this was not because he considered it to be part of his notion of a 'rule of combination'. On the contrary, the 'rule' was not *a priori* assumed to be closed:

> A *rule of combination* in an assemblage is any rule or agreement by which, when any two elements (whether the same or different) are given, in a definite order, some object (which may or may not itself belong to the assemblage) is uniquely determined.
>
> [4, p. 296]

Thus, for sets A and B, Huntington's 'rule of combination' was a function $A \times A \to B$, where, at least at first glance, B may be entirely unrelated to A. Indeed, we can see from condition (H3) that something peculiar is afoot, for Huntington had to assume in the first half of the condition that $a \circ b$ and $b \circ c$ are elements of the set — this does not follow immediately from the fact that a, b, c are elements of the set. Nevertheless, we have asserted that Huntington's specification does indeed define a group (and therefore that we may take A for B above), so the operation must be closed, and in fact Huntington showed that this follows from his three stated axioms. After his definition comes a sequence of deductions from the defining conditions, of which the tenth is as follows [4, p. 298]:

> Whatever elements a and b may be, $a \circ b$ is also an element of the assemblage; that is, there is an element c such that $a \circ b = c$.

The proof is in fact quite simple and builds upon Huntington's earlier deductions from his axioms. To see this, we must first note his fourth deduction: that if $a \circ x = a$ for some x and any a, then $b \circ x = b$ for any b. This is proved by using (H2) to take y such that $y \circ a = b$. It follows from the hypothesis that $y \circ (a \circ x) = b$. But then by (H3), $y \circ (a \circ x) = (y \circ a) \circ x$, hence $b \circ x = b$. This result may next be used to show that the operation is left cancellative: that if $a \circ b = a \circ b'$, then $b = b'$ (cf. (W3)). To establish this, we use (H1) to take x such that

* For concrete examples of sets and operations satisfying all or some of the conditions listed here and elsewhere in this article, we refer the interested reader to the original papers cited.

$b' \circ x = b$. Then, by the hypothesis, $a \circ (b' \circ x) = a \circ b'$, from which it follows by (H3) that $(a \circ b') \circ x = a \circ b'$. Using what we proved at the beginning of this paragraph, we have $b' \circ x = b$, hence $b = b'$. It may be shown in a very similar way that the operation is also right cancellative.

With these basic properties established, Huntington was able to turn next to the proof of the closure property, which proceeds like this. Using (H1), we take e such that

$$a \circ e = a \tag{1}$$

and b' such that

$$b \circ b' = e. \tag{2}$$

By (H2), we take c such that

$$c \circ b' = a. \tag{3}$$

By (H1) again, we take β such that

$$a \circ \beta = c \tag{4}$$

and β' such that

$$\beta \circ \beta' = e. \tag{5}$$

It follows from (3) and (4) that $(a \circ \beta) \circ b' = a$ and from (1) and (5) that $a \circ \left(\beta \circ \beta'\right) = a$. By (H3), $(a \circ \beta) \circ \beta' = (a \circ \beta) \circ b'$. Therefore, by left cancellation, $\beta' = b'$. Using this, (5) becomes $\beta \circ b' = e$, so that, by (2), $\beta \circ b' = b \circ b'$. Therefore, by right cancellation, $\beta = b$. It follows from (4) that $a \circ b = c$, as required.

But what was the purpose of treating closure in this way? It seems that Huntington simply sought to condense the group definition as far as possible. He went on to demonstrate the independence of his three axioms, and was evidently quite satisfied that he had reduced Weber's definition via four axioms to one involving just three. What is rather peculiar is that he does not appear to have noticed initially that his (H1) and (H2) are in fact contained in Weber's (W4), and so, had he been so inclined, Huntington could have combined these two into a single axiom, and thus given an even more streamlined definition in terms of just two axioms. We are beginning to see here the problems surrounding the notion of minimality.

A discussion connected with the desirable form of group axioms was opened up later that same year when Moore authored a response to Huntington's paper [6]. It was at this point that a slightly different style of postulate analysis was brought to bear: whereas Huntington appears to have been motivated by formalist, or even aesthetic, ideals, Moore's approach was rather more pragmatic. For Moore, pedagogical considerations appear to have been more prominent: there are indications in the paper that he had tried out different definitions on students [6, p. 488], and he wrote of the need for a definition to be suitable from the 'group-theoretic' point of view, by which he seems to have meant a definition that is useful for doing group theory. Huntington's definition may be quite neat and compact, but it is not necessarily an easy definition with which to work — at least not until further properties of the group (such as closure) have been developed, in which case, why not adopt a more transparent definition from the start?

The issue of minimality of sets of axioms was raised almost immediately in Moore's paper by his claim that Huntington's three-axiom definition is in fact a four-axiom one, for the condition (H3) was, in Moore's view, a 'double-statement' which should have been broken down into two parts: one for the case when $(a \circ b) \circ c$ is an element of the set, and one for $a \circ (b \circ c)$. Indeed, Moore asserted that

> every postulate of a desirably simple definition shall be a simple statement, that is, a single and not a multiple statement.
>
> [6, p. 488]

However, he acknowledged the difficulty of giving a precise definition to the term 'simple statement'. Nevertheless, his own list of group axioms (of which he needed five) might be seen as a demonstration of the principle towards which he was striving: each axiom is quite terse. We will not give the full definition here (see instead [6, pp. 485-486] or [1, pp. 434-435]), but note merely that closure appears explicitly as an axiom. Beyond the discussion of 'simple statements', Moore's paper proceeded in a very similar style to Huntington's: the bulk of it is taken up by a proof of the independence of the five axioms.

Further contributions from Huntington followed, with the occasional indication that he had taken Moore's criticisms to heart:

> Professor Moore's criticism of 'multiple statements' suggested the present form of postulates . . . [7, p. 27]

On the whole, however, Huntington's style remained more formal than practical: in a paper of 1903, for example, closure was still a deduction from axioms, rather than an assumption in its own right. The result was the use of axioms like (H3) that might be considered a little awkward. Nevertheless, there does seem to have been a gradual move towards shorter axioms in Huntington's papers, even though the desired form of postulates was still not entirely clear:

> each postulate should be as nearly as possible a simple statement, not decomposable into two or more parts; but the idea of a simple statement is a very elusive one, which has not yet been satisfactorily defined, much less attained. [8, p. 290]

Indeed, this notion of simplicity, and the allied concept of minimality, were never 'satisfactorily defined' by the postulate analysts.

Although the postulate analysts continued to study the axioms of other structures for some years to come, the concentration on groups (later described by E. T. Bell as having been 'somewhat feverish' [9, p. 225]) had largely burnt itself out by the end of 1905. In that year, Huntington made one final contribution that might be seen as marking a compromise between his own approach to the subject and that advocated by Moore. Apparently no longer seeking an ultra-condensed group definition, Huntington now gave a list of eleven axioms, each being ‘as nearly a simple statement as seems possible’ [10, p. 34]. Some very straightforward and familiar axioms appear in his list, but sometimes in a rather over-elaborate form. For instance, although closure now appeared explicitly, it was handled in a manner that appears to modern eyes to be unnecessarily complicated: Huntington's first axiom was the demand that if a is an element of the set in question, then so too is a^2; his second axiom states that if a, b are distinct elements of the set, then ab belongs to the set. The closure property in its standard form (as, for example, in our (G1)) is an immediately stated consequence of these first axioms. Thus, even in this later paper, there was still a trace of the formalism over pragmatism that had characterised Huntington's earlier contributions to this topic. It is at this stage, however, that we find Huntington making the following remark:

> Further analysis of the postulates . . . does not seem likely to lead to practical advantage . . . [11, p. 183]

And with this, the postulate analysis of groups largely dried up. Elsewhere in this final paper, in connection with the restoration of closure as an explicit axiom, Huntington referred to this as ‘the fundamental postulate of the whole theory’ [11, p. 183].

The postulate analysis of groups was an interesting episode in the early twentieth-century development of understanding of the still quite new abstract group notion. Among other things, it served to reinforce the central position of the closure property in the definition, and perhaps pointed towards the desirability of having it as an explicitly stated axiom; as L. E. Dickson (1874-1954), who also contributed to these postulational studies, commented in 1904:

> Most readers, I think, would find it more natural to have this property as a postulate, as is the case in Moore's definitions.
>
> [12, p. 160]

Indeed, Moore had written of the need for definitions that ‘reveal more immediately the fundamental properties of the object of definition’ [6, p. 489], and as Huntington came to realise, closure is indeed a ‘fundamental property’.

It is worth noting that other unsatisfactory aspects of the group definition were also addressed by the postulate analysts: for instance, the difficulty over the uniqueness of the identity that was mentioned above. Their solution was to take an axiom such as (G3) that provides for the existence of *at least one* identity element, and then to combine this with an axiom that demands that each element has at least one inverse *with respect to any given identity*. It was then a matter of showing that the identity element is in fact unique, and therefore that all the inverses for a given element are indeed one and the same. This is all quite straightforward, but a bit unwieldy, which perhaps explains why we typically choose instead to define a group via (G1)–(G4), and to brush the problems with (G3) and (G4) under the carpet (but see below).

We bring this Article to a close by considering an intriguing additional element to Moore's treatment of group axioms in 1902. In an attempt to bring a greater precision to the study of axiom systems, Moore confined his attention to the case of a finite group of order N, and determined the number of calculations that would be necessary, for a particular set of axioms, to determine whether a given structure is indeed a group. Thus, for example, the verification of the closure property requires N^2 calculations; to check associativity, we need to perform $4N^3$ calculations (N^3 ways of choosing three elements a, b, c, followed by the calculation of the four products ab, bc, $(ab)c$, $a(bc)$) — but of course if we have already done the N^2 calculations to check closure, then this reduces the work needed for associativity. The number of calculations needed to verify any axiom that asserts the *existence* of an element will fall into a range, depending on how long it takes us to find the required element. Moore applied this analysis to the sets of axioms put forward by Weber, by Huntington, by himself, and by others, and was able to compare them for efficiency. The definition that turned out to be the best under this scheme was one of his own, which Moore determined would require somewhere between $4N^3 + 3N - 2$ and $4N^3 + N^2 + 2N - 2$ calculations to verify (see [6, p. 491] or [1, p. 439]). The definition is question is not a commonly used one nowadays, but we end by listing Moore's axioms here so that the reader can make up their own mind as to its usefulness!

M1. For every two elements a, b, the product ab is an element of the set;

M2. the associative law is fulfilled;

M3. there exists at least one idempotent element in the set: an element i such that $i^2 = i$;

M4. every idempotent element is a left identity for all other elements;

M5. every idempotent element is a right identity for all other elements;

M6. every element a has a left inverse with respect to each idempotent element i: there exists an element $a_l^{(i)}$ such that $a_l^{(i)}a = i$.

The interested reader may find an overview of the postulate analysts' investigation of group axioms in [13], additional commentary in [14], and a more detailed exploration in [1]. On the postulate analysts more generally, see [15, 16, 17] or [18, §3.5], and for a focus on Huntington specifically, see [19, 20].

Acknowledgement

Thanks must go to the anonymous referee for enthusiastic comments about this article.

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Nemo, continued from page 247.

4. I am not here to bandy quibbles and paradoxes with a girl who quibbles over the greatest names in history.

5. "Yet start some paradox, that we may laugh. Say a woman is a man, or you yourself a stork." At this they smiled.

6. I was prepared for paradoxes from what Malachi Mulligan told us but I may as well warn you that if you want to shake my belief that Shakespeare is Hamlet you have a stern task before you.