Vector Bundles on Drinfeld Symmetric Spaces

James Taylor

Keble College

University of Oxford

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Abstract

Let F be a finite extension of $\mathbb{Q}_p$, $n \geq 1$, and let Ω be the $(n-1)$-dimensional Drinfeld symmetric space. In this thesis we study various vector bundles on Ω and its covering spaces that are of significance in the representation theory of $\mathrm{GL}_n(F)$.

In the first part, we work with any $n \geq 2$, and consider a geometrically connected component Σ^1 of the first Drinfeld covering of Ω. The main result is that the canonical homomorphism determined by the second Drinfeld covering

$$\widehat{(\mathbb{F},+)} \to \mathrm{Pic}(\Sigma^1)[p]$$

is injective. Here $\mathbb{F}$ is the residue field of the unique degree n unramified extension of F. In particular, $\mathrm{Pic}(\Sigma^1)[p] \neq 0$. We also show that when $n = 2$, all vector bundles on Ω are trivial, which extends the classical result that $\mathrm{Pic}(\Omega) = 0$.

In the second part we take $n = 2$, and study any affinoid open subset Σ^1_v of Σ^1 that lies above a vertex v of the Bruhat-Tits tree for $\mathrm{GL}_2(F)$. Our main result is that $\mathrm{Pic}(\Sigma^1_v)[p] = 0$, which we establish by showing that $\mathrm{Pic}(\mathbf{Y})[p] = 0$ for $\mathbf{Y}$ the *Drinfeld curve* - the Deligne-Lusztig variety of $\mathrm{SL}_2(\mathbb{F}_q)$.

In third part, we work with any $n \geq 1$, and show that the action of $D^\times$ on the Drinfeld tower induces an equivalence of categories from finite dimensional smooth representations of $D^\times$ to G^0-finite $\mathrm{GL}_n(F)$-equivariant vector bundles with connection on Ω. Here D is the division algebra over F of invariant $1/n$ and G^0 is the subgroup of $\mathrm{GL}_n(F)$ of elements with norm 1 determinant.

Contents

Introduction

Let p be a prime, let F be a finite extension of $\mathbb{Q}_p$ and let $n \geq 1$. Let K be a complete field extension of L, the completion of the maximal unramified extension of F. The *Drinfeld tower* is a system of $(n-1)$-dimensional rigid analytic spaces over K,

$$\mathcal{M} \leftarrow \mathcal{M}_1 \leftarrow \mathcal{M}_2 \leftarrow \cdots ,$$

for which each space is equipped with a compatible action of $D^\times \times \mathrm{GL}_n(F)$, where D is the division algebra of invariant $1/n$ over F [17, 28, 52]. These spaces play an important role in the representation theory of both $\mathrm{GL}_n(F)$ and $D^\times$. For example, this tower has been shown provide a geometric realisation of both the local Langlands and Jacquet-Langlands correspondences for $\mathrm{GL}_n(F)$ [19, 20, 37, 38].

These correspondences are realised in the cohomology of the spaces $\mathcal{M}_m$ for an appropriate cohomology theory. In this framework, one can also consider the coherent cohomology groups $\mathrm{H}^0(\mathcal{M}_m, \mathcal{O}_{\mathcal{M}_m}) = \mathcal{O}(\mathcal{M}_m)$, for which the topological dual $\mathcal{O}(\mathcal{M}_m)^*$ is naturally a *locally analytic* representation of $\mathrm{GL}_n(F)$. These representations lie in the image of the functor

$$\mathrm{Hom}_{D^\times}(-, \mathcal{O}(\mathcal{M}_\infty))^* \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) \to \mathbf{Rep}_{\mathrm{la}}(\mathrm{GL}_n(F)),$$

defined as the direct limit of the functors

$$\mathrm{Hom}_{D^\times}(-, \mathcal{O}(\mathcal{M}_m))^* \colon \mathbf{Rep}^{\mathbf{fd}}(D^{(m)}) \to \mathbf{Rep}_{\mathrm{la}}(\mathrm{GL}_n(F)),$$

where $D^{(m)} := D^\times/(1 + \Pi^m \mathcal{O}_D)$. When $F = \mathbb{Q}_p$ and $n = 2$, Dospinescu and Le Bras have shown that for any irreducible (necessarily finite dimensional) smooth representation V of $D^\times$ with trivial central character and dimension strictly greater than one, the corresponding locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is *admissible* and *topologically irreducible*. This is deduced as a consequence of much deeper results relating this locally analytic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ to the Jacquet-Langlands and p-adic local Langlands correspondences [27, Thm. 1.2].

One would like to deduce similar admissibility and topological irreducibility results beyond $\mathrm{GL}_2(\mathbb{Q}_p)$, where a p-adic Langlands correspondence is not yet currently formulated. One natural approach is through the use of p-adic $\mathcal{D}$-modules. The functor $\mathrm{Hom}_{D^\times}(-, \mathcal{O}(\mathcal{M}_\infty))^*$ above admits a natural factorisation

$$\mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) \to \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega) \to \mathbf{Rep}(\mathrm{GL}_n(F))$$

through the category $\mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega)$ of $\mathrm{GL}_n(F)$-equivariant vector bundles with connection on Ω, where the second functor is defined by taking the dual of the global sections $\Gamma(\Omega, -)^*$. In this thesis we study the first functor, which we denote by

$$\mathrm{Hom}_{D^\times}(-, f_*\mathcal{O}_{\mathcal{M}_\infty})\colon\ \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) \to \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega),$$

which similarly to above is defined as the direct limit of functors

$$\mathrm{Hom}_{D^\times}(-, f_{m,*}\mathcal{O}_{\mathcal{M}_m})\colon\ \mathbf{Rep}^{\mathbf{fd}}(D^{(m)}) \to \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega),$$

where $f_m\colon \mathcal{M}_m \to \Omega$ is the composition of the Galois covering map $\phi_m\colon \mathcal{M}_m \to \mathcal{M}$ with the Grothendieck-Messing period morphism $\pi_{\mathrm{GM}}\colon \mathcal{M} \to \Omega$.

This functor is also related to the Jacquet-Langlands correspondence: the composition

$$\mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) \to \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega) \xrightarrow{\mathrm{H}^{n-1}_{\mathrm{dR},c}(-)} \mathbf{Rep}_{\mathrm{sm}}(\mathrm{GL}_n(F))$$

should send an irreducible representation ρ of $D^\times$ with $\dim(\rho) > 1$ to the direct sum of n copies of $\mathrm{JL}(\rho)$. This is known in dimension 1 for $\mathrm{GL}_2(F)$ [22, Thm. 0.4], and in any dimension for certain n-dimensional representations of $D^\times$ corresponding to the first Drinfeld covering $\mathcal{M}_1$ [41, Thm. A].

Our first main result is the following.

Theorem A. *The functor*

$$\mathrm{Hom}_{D^\times}(-, f_*\mathcal{O}_{\mathcal{M}_\infty})\colon\ \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) \to \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega)$$

is exact, monoidal, fully faithful, and the essential image is closed under sub-quotients.

The essential image is intrinsically described as the full subcategory

$$\mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega)_{G^0-\mathrm{fin}}$$

with objects those that are finite when viewed as G^0-equivariant vector bundles with connection.

Here G^0 is the subgroup of elements of $\mathrm{GL}_n(F)$ with determinant of norm 1, and for the definition of a finite equivariant vector bundle with connection we direct the reader to Section 4.3. For now, let us simply remark that an equivariant *line bundle* with connection is *finite if and only if it is torsion*, and therefore finiteness can be viewed as a natural generalisation of the notion of a torsion line bundle to vector bundles of arbitrary rank. We also remark that the functor of Theorem A *preserves irreducibility*, as the essential image is closed under sub-objects.

In order to explain the appearance of the group G^0, let $\mathcal{N}$ be a connected component of $\mathcal{M}$, and consider the induced sub-tower

$$\mathcal{N} \leftarrow \mathcal{N}_1 \leftarrow \mathcal{N}_2 \leftarrow \cdots,$$

of $(\mathcal{M}_m)_{m\geq 1}$ defined by $\mathcal{N}_m = \phi_m^{-1}(\mathcal{N})$ for $m \geq 1$. This is stabilised by $\mathcal{O}_D^\times \times G^0$, and in the same way as above one obtains a functor

$$\mathrm{Hom}_{\mathcal{O}_D^\times}(-, f_*\mathcal{O}_{\mathcal{N}_\infty})\colon\ \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) \to \mathbf{VectCon}^{G^0}(\Omega)$$

which is compatible with the functor of Theorem A in the sense that the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{D^\times}(-, f_*\mathcal{O}_{\mathcal{M}_\infty})\colon\ \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) & \longrightarrow & \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathcal{O}_D^\times}(-, f_*\mathcal{O}_{\mathcal{N}_\infty})\colon\ \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) & \longrightarrow & \mathbf{VectCon}^{G^0}(\Omega) \end{array}$$

with vertical forgetful maps commutes. Our second main result is the following, which we use to deduce the description of the essential image of Theorem A above.

Theorem B. *The functor*

$$\operatorname{Hom}_{\mathcal{O}_D^\times}(-, f_*\mathcal{O}_{\mathcal{N}_\infty})\colon\ \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) \to \mathbf{VectCon}^{G^0}(\Omega)$$

is exact, monoidal, fully faithful and the essential image is closed under sub-quotients. The essential image is intrinsically described as the full subcategory

$$\mathbf{VectCon}^{G^0}(\Omega)_{\mathrm{fin}}$$

of finite G^0-equivariant vector bundles with connection.

We prove the description of the essential image by establishing analogous results to Nori's [51] relating finite vector bundles to Galois coverings (Section 4.3), and applying these in conjunction with the factorisation theorem of Scholze-Weinstein [56, Thm. 7.3.1]. The other properties of the functors of Theorem A and Theorem B follow from general results which we describe below regarding functors of this type associated to Galois coverings.

For now, let us describe instead how these results are related to and have the potential to lead to admissibility and topological irreducibility results. The main result of the recent work of Ardakov and Wadsley stated in our context is the following (in which $n = 2$).

Theorem ([3, Thm. A])**.** *Suppose that $\mathcal{L} \in \mathbf{VectCon}^{G^0}(\Omega)_{\mathrm{fin}}$ has rank 1. Then the locally analytic representation $\Gamma(\Omega, \mathcal{L})^*$ of G^0 is admissible and has length at most 2.*

One of the main ingredients that goes into the proof of this theorem is an explicit classification of torsion G^0-equivariant line bundles with connection on Ω: the main result of [2] is an explicit group isomorphism

$$\operatorname{PicCon}^{G^0}(\Omega)_{\mathrm{tors}} \xrightarrow{\sim} \operatorname{Hom}_{\mathrm{sm}}(\mathcal{O}_D^\times, K^\times),$$

where the left-hand side is the group of isomorphism classes of torsion G^0-equivariant line bundles with connection on Ω. Therefore, in this context, Theorem A and Theorem B can be viewed as generalisations of this isomorphism in four different directions:

- To representations of of arbitrary dimension and vector bundles of arbitrary rank,
- To Drinfeld spaces of any dimension,
- From the pair $(\mathcal{O}_D^\times, G^0)$ to both pairs $(D^\times, \mathrm{GL}_n(F))$ and $(\mathcal{O}_D^\times, G^0)$,

- From an isomorphism to a functorial correspondence.

For $\mathcal{L} \in \mathbf{VectCon}^{G^0}(\Omega)$ which is rank 1 and torsion, the results of [3] regarding the length of the admissible representation $\Gamma(\Omega, \mathcal{L})^*$ are actually more precise: $\Gamma(\Omega, \mathcal{L})^*$ is also shown to be topologically irreducible if and only if $\mathcal{L}$ is non-trivial when viewed as an object of $\mathbf{VectCon}(\Omega)$ (if $\mathcal{L} \ncong \mathcal{O}_\Omega$ as vector bundles with connection once we forget the equivariant structure). We would like to similarly understand the restriction map

$$\mathbf{VectCon}^{G^0}(\Omega) \to \mathbf{VectCon}(\Omega)$$

for objects in the image of $\mathrm{Hom}_{\mathcal{O}_D^\times}(-, f_*\mathcal{O}_{\mathcal{N}_\infty})$. Whenever K contains F^{ab} (the maximal abelian extension of F) the spaces $\mathcal{M}_m$ are disjoint unions of geometrically connected components, and we may fix a compatible sequence of geometrically connected components Σ^m at each level to obtain a sub-tower

$$\mathcal{N} \leftarrow \Sigma^1 \leftarrow \Sigma^2 \leftarrow \cdots$$

which similarly defines a functor

$$\mathrm{Hom}_{\mathrm{SL}_1(D)}(-, f_*\mathcal{O}_{\Sigma^\infty}) \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathrm{SL}_1(D)) \to \mathbf{VectCon}(\Omega)$$

where $\mathrm{SL}_1(D) = \ker(\mathrm{Nrd} \colon D^\times \to F^\times)$. Our third main result is the following.

Theorem C. *Suppose that K contains F^{ab}. Then the functor*

$$\mathrm{Hom}_{\mathrm{SL}_1(D)}(-, f_*\mathcal{O}_{\Sigma^\infty}) \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathrm{SL}_1(D)) \to \mathbf{VectCon}(\Omega)$$

is exact, monoidal, fully faithful, the essential image is closed under sub-quotients and

$$\begin{array}{rccc}
\mathrm{Hom}_{\mathcal{O}_D^\times}(-, f_*\mathcal{O}_{\mathcal{N}_\infty}) : & \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) & \longrightarrow & \mathbf{VectCon}^{G^0}(\Omega) \\
 & \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{SL}_1(D)}(-, f_*\mathcal{O}_{\Sigma^\infty}) : & \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathrm{SL}_1(D)) & \longrightarrow & \mathbf{VectCon}(\Omega)
\end{array}$$

commutes.

Therefore, for any finite dimensional smooth representation V of $D^\times$, the underlying $\mathcal{D}$-module structure of $\mathrm{Hom}_{D^\times}(V, f_*\mathcal{O}_{\mathcal{M}_\infty})$ is completely determined by the restriction of V to a representation of $\mathrm{SL}_1(D)$. In particular, with the results of Ardakov and Wadsley described above, this tells us that for any smooth character χ of $\mathcal{O}_D^\times$, the locally analytic representation $\mathrm{Hom}_{\mathcal{O}_D^\times}(\chi, \mathcal{O}(\mathcal{N}_\infty))^*$ of G^0 is topologically irreducible precisely when $\chi|_{\mathrm{SL}_1(D)} \neq 1$.

We also note in passing that all the functors described above send a representation V to a vector bundle of rank $\dim(V)$ (Remark 4.5.22), and commute with taking the determinant representation and determinant line bundle on either side (Remark 4.5.23).

Functors Associated to Galois Coverings

We now give an overview of how the functors of Theorems A, B and C are defined, and how their properties are established. Each functor is constructed as the direct limit of certain functors attached to each of the finite level Galois coverings of each respective tower. We now describe how we define each of these functors at finite level. Suppose in what follows that k is any characteristic 0 field.

Suppose first that X is a smooth scheme or rigid space over k, with an action of an abstract group H. Suppose moreover that X has an additional action of some abstract group G, and that this action commutes with the action of H. In this situation, we can consider the category $\mathbf{VectCon}^{G\times H}(X)$, and there are canonically defined functors

$$\mathcal{O}_X \otimes_k -\colon\ \mathbf{Mod}^{\mathbf{fd}}_{k[H]} \to \mathbf{VectCon}^{G\times H}(X),$$
$$\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, -)\colon\ \mathbf{VectCon}^{G\times H}(X) \to \mathbf{Mod}^{\mathbf{fd}}_{k[H]},$$

The following result is the main technical ingredient.

Theorem D. *Suppose that $c_X(X)^G = k$. Then:*

1. *The functor*
$$\mathcal{O}_X \otimes_k -\colon\ \mathbf{Mod}^{\mathbf{fd}}_{k[H]} \to \mathbf{VectCon}^{G\times H}(X)$$
is exact and fully faithful.

2. *For any $\mathcal{M} \in \mathbf{VectCon}^{G\times H}(X)$,*
$$\dim_k(\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})) \leq \mathrm{rank}(\mathcal{M}).$$
The essential image of $\mathcal{O}_X \otimes_k -$ is the full subcategory with objects $\mathcal{M}$ for which this is an equality. On this full subcategory the solution functor $\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, -)$ *is a quasi-inverse for $\mathcal{O}_X \otimes_k -$.*

3. *If X satisfies $(*)$, then the essential image of $\mathcal{O}_X \otimes_k -$ is closed under subquotients.*

Here c_X is the sheaf of *constant functions*

$$c_X = \ker(d\colon \mathcal{O}_X \to \Omega^1_{X/k}),$$

and for the definition of the condition $(*)$ we direct the reader to Definition 4.2.4. For now, let us remark that this condition is true in many cases of interest, such as when k is algebraically closed, or when X is either a scheme or quasi-Stein rigid space whose connected components are geometrically connected (Lemma 4.2.6).

Remark 0.0.1. When G is trivial, the assumption of the theorem is equivalent to the assumption that X is geometrically connected. This can easily be shown to be a necessary condition for each conclusion of the theorem to hold (Remark 4.4.5).

Suppose now that $f\colon X \to Y$ is a G-equivariant finite étale Galois morphism of smooth schemes or rigid spaces over k, with Galois group H. In this situation there is an equivalence

$$(-)^H\colon\ \mathbf{VectCon}^{G\times H}(X) \to \mathbf{VectCon}^G(Y),$$

and the functor

$$\mathrm{Hom}_{k[H]}(-, f_*\mathcal{O}_X)\colon\ \mathbf{Mod}^{\mathbf{fd}}_{k[H]} \to \mathbf{VectCon}^G(Y)$$

can be shown to be equal to the composition

$$(\mathcal{O}_X \otimes_k (-)^*)^H\colon\ \mathbf{Mod}^{\mathbf{fd}}_{k[H]} \to \mathbf{VectCon}^G(Y)$$

and thus inherits all the good properties of $\mathcal{O}_X \otimes_k -$ from Theorem D. Furthermore, this functor sends the regular representation $\mathcal{O}(H)$ to the pushforward $f_*\mathcal{O}_X$ (Theorem 4.4.2), and in this way we obtain a complete description of $f_*\mathcal{O}_X$ as a semi-simple G-$\mathcal{D}_Y$-module (Theorem 4.4.4).

For example, we deduce the relevant properties of the functor of Theorem B from the properties of the functors at each finite level $m \geq 1$, which are deduced from the above by taking $X = \mathcal{N}_m$, $H = \mathcal{O}_D^{(m)} = \mathcal{O}_D^\times/(1 + \Pi^m\mathcal{O}_D)$ and $G = G^0$.

Representations Arising from the Second Drinfeld Covering

We are also interested in establishing properties of the locally analytic representations $\Gamma(\Omega, \mathcal{V})^*$ of G^0 for $\mathcal{V} \in \mathbf{VectCon}^{G^0}(\Omega)_{\text{fin}}$. Of particular interest are the representations $\mathcal{O}(\mathcal{N}_m)^*$ for $m \geq 1$ (which under direct sums and direct summands generate the category $\mathbf{VectCon}^{G^0}(\Omega)_{\text{fin}}$ by Theorem B above). When $m \geq 2$ these remain poorly understood; for example, the only representations of the family $\mathcal{O}(\mathcal{N}_m)^*$ for which the length is known are when $m = 0, 1$.

For any Galois covering $f\colon X \to Y$ with abelian Galois group H of exponent e over a field which contains a primitive eth root of 1, there is a decomposition

$$f_*\mathcal{O}_X = \bigoplus_{\chi\in\widehat{H}} \mathcal{L}_\chi, \qquad \mathcal{L}_\chi := e_\chi \cdot f_*\mathcal{O}_X,$$

where e_χ is the central primitive idempotent corresponding to the character χ. Furthermore, $\mathcal{L}_\chi \in \mathrm{Pic}(Y)[e]$ for any $\chi \in \widehat{H}$, and the association

$$\widehat{H} \to \mathrm{Pic}(Y)[e], \qquad \chi \mapsto \mathcal{L}_\chi,$$

is a group homomorphism (cf. Proposition 2.1.2).

In particular, applying this to $\mathcal{N}_1 \to \Omega$ (which has abelian Galois group) and taking the global sections we obtain a decomposition

$$\mathcal{O}(\mathcal{N}_1) = \bigoplus_{\chi\in\widehat{H}} \Gamma(\Omega, \mathcal{L}_\chi),$$

and as a consequence the results of Ardakov and Wadsley described above provide a complete description of the locally analytic G^0-representation $\mathcal{O}(\mathcal{N}_1)^*$ [3, Cor. B].

Due to the recent explicit description of $\mathcal{N}_1$ due to Junger [42], $\mathcal{O}(\mathcal{N}_2)^*$ has the potential to be approachable using similar methods to [3], and one might hope to establish a Theorem similar to [3, Thm. A] described above in this context.

Writing $f\colon \mathcal{N}_2 \to \Omega$ for the second Drinfeld covering map, the representation $\mathcal{O}(\mathcal{N}_2)^*$ is the dual of the global section of $f_*\mathcal{O}_{\mathcal{N}_2} \in \mathbf{VectCon}^{G^0}(\Omega)_{\text{fin}}$. The Galois group $\Gamma := \text{Gal}(\mathcal{N}_2/\mathcal{N}_1)$ is abelian, and as above there is a decomposition

$$f_*\mathcal{O}_{\mathcal{N}_2} = \bigoplus_{\chi \in \widehat{\Gamma}} \mathcal{L}_\chi,$$

in $\mathbf{VectCon}^{G^0}(\mathcal{N}_1)$ where, viewed as a $\mathcal{O}_{\mathcal{N}_1}$-module, each $\mathcal{L}_\chi$ is a p-torsion line bundle. The methods of [3] *can be applied to exactly those $\mathcal{L}_\chi$ which are trivial as a line bundle*, and thus it is natural to ask which of the line bundles $\mathcal{L}_\chi$ are trivial.

Theorem E. *Suppose that K contains $L(\varpi)$ and a primitive pth root of 1. Then the group homomorphism,*

$$\widehat{\Gamma} \to \text{Pic}(\mathcal{N}_1)[p], \qquad \chi \mapsto \mathcal{L}_\chi,$$

is injective.

Here ϖ is a $(q-1)$st root of $-\pi$, for π a uniformiser of F, and the extension $L(\varpi)/L$ is the first Lubin-Tate extension of L. In particular, this shows that the methods of [3] cannot be applied directly to understand this the representations obtained from Drinfeld coverings of level greater than 1. Another consequence is that the Picard groups of the first Drinfeld covering spaces $\mathcal{N}_1$ and $\mathcal{M}_1$ are non-trivial, a result that was expected but not previously known to hold.

When $n = 2$, the Drinfeld upper half plane admits a retraction map to $\mathcal{T}$, the Bruhat-Tits tree for $\text{GL}_2(F)$. For any vertex v of $\mathcal{T}$, we can consider the preimage $\mathcal{N}_1^v$ of v in $\mathcal{N}_1$.

Theorem F. *For any vertex v,* $\text{Pic}(\mathcal{N}_1^v)[p] = 0$.

This shows that that whilst the line bundles $\mathcal{L}_\chi$ can be non-trivial globally, they are trivialisable when restricted to each open subset $\mathcal{N}_1^v$. In particular, this opens up the possibility of applying the methods of [3] locally.

As an alternative solution to adressing the non-triviality of the line bundles $\mathcal{L}_\chi$, one might try to directly study the pushforward of $\mathcal{L}_\chi$ to Ω as a G^0-equivariant vector bundle with connection on Ω (of rank q^2-1). Our next result (also for $n = 2$) says that, once we forget the connection and equivariant structure, this will be trivialisable.

Theorem G. *Any vector bundle on Ω is of the form $\mathcal{O}_\Omega^r$, for some $r \geq 0$.*

Other Results

We briefly mention some other results we establish in this thesis which are of independent interest.

The first is Proposition 4.5.5, which in particular implies that the action of $\text{GL}_n(F)$ on each covering space $\mathcal{M}_m$ of the Drinfeld tower is continuous in the sense of [1] (Corollary 4.5.10).

The second is Theorem 4.5.11 which gives a description of the geometrically connected components of $(\mathcal{N}_m)_{m\geq 1}$ (at least for the cofinal system where n divides m). This description can already be found in the unpublished work of Boutot and Zink [18, Thm. 0.20], which uses global methods and p-adic uniformisation of Shimura curves. In contrast, we deduce this by completely elementary methods. Precisely, this description is a simple application of the theory we develop in Section 4.1 regarding the sheaf c_X of constant functions, coupled with a result of Kohlhaase on maximal fields contained inside the global sections of $\mathcal{N}_m$ [44, Prop. 2.7].

The third (Corollary 4.5.14) is a proof that any p-torsion $\mathrm{SL}_n(F)$-equivariant line bundle with connection on Σ^1 is uniquely determined by its underlying line bundle. We use this to give a more conceptual proof of Theorem E (Corollary 4.5.15) - Theorem E is first proven in Chapter 2 using an explicit form of the Kummer exact sequence, but this new proof shows how Theorem E follows from the general theory we develop regarding how the pushforward of the structure sheaf of a Galois covering decomposes as a $\mathcal{D}$-module (Theorem 4.4.10).

Outline of The Thesis

This thesis is organised as follows.

In Chapter 1 we consider the necessary facts we need concerning Drinfeld spaces, $\mathcal{D}$-modules, equivariant sheaves and Galois coverings. This chapter may be safely skipped and referred back to when necessary. This chapter corresponds to the first half of the paper [62].

In Chapter 2, we prove Theorem E (Theorem 2.3.6) and Theorem G (Corollary 2.4.4). The results of this chapter are contained in the paper [60].

In Chapter 3 we prove Theorem F (Theorem 3.2.2). The results of this chapter are contained in the paper [61].

In Chapter 4, which corresponds to the second half of the paper [62], we prove Theorems A, B, C and D.

More precisely, in Section 4.1 we introduce and consider the sheaf of constant functions and relate this sheaf to the notion of geometric connectivity. In Section 4.2 we introduce the functor $\mathcal{O}_X \otimes_k -$, the solution functor $\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, -)$, and prove the main technical result, Theorem D (Theorem 4.2.7). In Section 4.3 we consider finite equivariant vector bundles with connection, and relate these to Galois coverings. In Section 4.4 we bring everything together, and establish the main properties of the functor $(\mathcal{O}_X \otimes_k -)^H$ (Theorem 4.4.2) and its consequences, including a decomposition theorem for the $\mathcal{D}_Y$-module $f_*\mathcal{O}_X$ (Theorem 4.4.4), and a strengthening of Theorem 4.4.2 in the case that the Galois group is abelian (Theorem 4.4.10). In Section 4.5 we apply the preceeding results of this chapter to the Drinfeld tower. The proof of the main results, Theorem A, Theorem B and Theorem C occupies Section 4.5.5 to Section 4.5.11.

Chapter 1

Preliminary Notions

In this section we collect together all the relevant technical notions we will make use of and refer back to when necessary throughout the rest of this thesis.

As an overview, in this thesis we will be concerned with certain categories of equivariant $\mathcal{D}$-modules (Section 1.8) which are defined in terms of the sheaf of differential operators (Section 1.7) and equivariant sheaves (Section 1.6). We are particularly interested in how for a Galois extension (Section 1.9) the various categories we obtain are related to one another (Section 1.10). Throughout, we work with $\mathcal{D}$-modules, and the universal property of the sheaf of differential operators comes from its description as a particular example of a Lie-algebroid (Section 1.5). Lie-algebroids are objects which are locally described by Lie-Rinehart algebras (Section 1.1), and so this is where we begin.

1.1 Lie-Rinehart Algebras

In this section, let R be a commutative ring and let A be a commutative R-algebra. An (R, A)-*Lie algebra* is a pair (L, ρ), where L is simultaneously a R-Lie algebra and A-module, and $\rho : L \to \mathrm{Der}_R(A)$ is a homomorphism of both R-Lie algebras and A-modules such that

$$[x, ay] = a[x, y] + \rho(x)(a)y$$

for any $x, y \in L$ and $a \in A$. Often we write only L when $\rho : L \to \mathrm{Der}_R(A)$ is implicit.

Example 1.1.1. $\mathrm{Der}_R(A)$ is an (R, A)-Lie algebra, with ρ as the identity.

From such a pair, one can construct the universal enveloping algebra of L [54]. This is an R-algebra $U(L)$, together with structure maps

$$\iota_L : L \to U(L),$$
$$\iota_A : A \to U(L),$$

which are morphisms of R-lie algebras and of R-algebras respectively, satisfying

$$\iota_L(ax) = \iota_A(a)\iota_L(x), \qquad [\iota_L(x), \iota_A(a)] = \iota_A(\rho(x)(a)),$$

for all $a \in A, x \in L$. The triple $(U(L), \iota_A, \iota_L)$ satisfies the following universal property.

Lemma 1.1.2 ([54, §2])**.** *Suppose that S is a unital associative R-algebra,*

$$\eta_L : L \to S,$$
$$\eta_A : A \to S,$$

are morphisms of R-lie algebras and of R-algebras respectively, and that for any $a \in A, x \in L$,

$$\eta_L(ax) = \eta_A(a)\eta_L(x), \qquad [\eta_L(x), \eta_A(a)] = \eta_A(\rho(x)(a)).$$

Then there is a unique homomorphism of R-algebras $\varphi : U(L) \to S$ with

$$\begin{array}{ccccc} A & \xrightarrow{\iota_A} & U(L) & \xleftarrow{\iota_L} & L \\ & \searrow^{\eta_A} & \downarrow{\scriptstyle\varphi} & \swarrow_{\eta_L} & \\ & & S & & \end{array}$$

Example 1.1.3. Using the universal property with $S = \mathrm{End}_k(A)$, we see that A is canonically a $U(L)$-module where A acts by left multiplication and L acts by $\rho \colon L \to \mathrm{Der}_k(A) \subset \mathrm{End}_k(A)$.

The morphism $\iota_A : A \to U(L)$ is injective, and by construction $U(L)$ is generated as an R-algebra by the images $\iota_A(A)$ and $\iota_L(L)$ [54, §2]. There is a increasing exhaustive filtration on $U(L)$, where $F_0U(L) = A$, $F_1U(L) = A + \iota_L(L)$, and

$$F_nU(L) = F_1U(L) \cdot F_{n-1}U(L) = A + \sum_{i=1}^{n} \iota_L(L)^i$$

for any $n \geq 2$. A is central in $\mathrm{gr}\, U(L)$, and there is a natural surjection $\mathrm{Sym}_A(L) \to \mathrm{gr}\, U(L)$, which is an isomorphism whenever L is projective [54, Thm. 3.1]. When $\mathrm{Sym}_A(L) \to \mathrm{gr}\, U(L)$ is an isomorphism the natural map $\iota_A \oplus \iota_L : A \oplus L \to U(L)$ is injective, and in this case we will identify both A and L with their images in $U(L)$.

1.1.1 Functoriality

In this section we describe in what sense the construction of the universal enveloping algebra is functorial. Fixing the commutative ring R, we have the following notion from [1, §2.1].

Definition 1.1.4. Suppose that $\varphi : A \to B$ is a morphism of commutative R-algebras. Suppose that L is an (R, A)-Lie algebra, and L' is an (R, B)-Lie algebra. Then $\tilde{\varphi} : L \to L'$ is a *φ-morphism* if

- $\tilde{\varphi}$ is a homomorphism of R-Lie algebras,
- $\tilde{\varphi}(a \cdot x) = \varphi(a) \cdot \tilde{\varphi}(x)$,
- $\varphi(\rho(x)a) = \rho'(\tilde{\varphi}(x))\varphi(a)$,

for all $a \in A$, $x \in L$.

Given a commutative ring R, we denote by $\mathbf{LR}_R$ the category with objects consisting of pairs (A, L), where A is a commutative R-algebra L is an (R, A)-Lie algebra. Morphisms $(A, L) \to (B, L')$ are pairs $(\varphi, \tilde{\varphi})$, where $\varphi : A \to B$ is a homomorphism of R-algebras, and $\tilde{\varphi} : L \to L'$ is a φ-morphism. It is straightforward to check that the composition of morphisms is again a morphism.

Lemma 1.1.5 ([1, Lem. 2.1.7]). *Suppose that $\varphi : A \to B$ is a morphism of commutative R-algebras. Suppose that L is an (R, A)-Lie algebra, L' is an (R, B)-Lie algebra. Then every φ-morphism $\tilde{\varphi} : L \to L'$ extends uniquely to a filtration preserving R-algebra homomorphism $U(\varphi, \tilde{\varphi}) : U(L) \to U(L')$ that makes the following diagram commute*

$$\begin{array}{ccc} A \oplus L & \xrightarrow{\varphi \oplus \tilde{\varphi}} & B \oplus L' \\ {\scriptstyle \iota_A \oplus \iota_L}\downarrow & & \downarrow{\scriptstyle \iota_B \oplus \iota_{L'}} \\ U(L) & \xrightarrow[U(\varphi, \tilde{\varphi})]{} & U(L') \end{array}$$

Therefore, $U(-)$ is a functor from the category $\mathbf{LR}_R$ to the category of positively filtered R-algebras.

1.1.2 Base Change

We discuss the notion of base change for Lie-Rinehart Algebras, following [2, §2.2]. In this section, suppose that L is an (R, A)-Lie algebra.

Lemma 1.1.6 ([2, Lem. 2.2]). *Suppose $\varphi : A \to B$ is a homomorphism of commutative R-algebras, and that $\sigma : L \to \mathrm{Der}_R(B)$ is a φ-morphism. Then the B-module $B \otimes_A L$ has a unique structure of an R-Lie algebra such that*

$$(B \otimes_A L, 1 \otimes \sigma : B \otimes_A L \to \mathrm{Der}_R(B))$$

is an (R, B)-Lie algebra.

Remark 1.1.7. As a consequence of Lemma 1.1.6, if $\varphi : A \to B$ is a homomorphism of commutative R-algebras, and $\psi \colon \mathrm{Der}_R(A) \to \mathrm{Der}_R(B)$ is a φ-morphism, then $\sigma := \psi \circ \rho \colon L \to \mathrm{Der}_R(B)$ is a φ-morphism, and we have a well defined functor

$$B \otimes_A - : (L, \rho) \mapsto (B \otimes_A L, 1 \otimes (\psi \circ \rho))$$

from (R, A)-Lie algebras to (R, B)-Lie algebras. With respect to this structure, the natural map $L \to B \otimes_A L$ is a φ-morphism.

In the case when L is projective as an A-module, we can describe the universal enveloping algebra of the base change.

Lemma 1.1.8. *Suppose $\varphi : A \to B$ is a homomorphism of commutative R-algebras, $\sigma : L \to \mathrm{Der}_R(B)$ is a φ-morphism, and L is is projective. Then the natural map*

$$U(L) \to U(B \otimes_A L)$$

of filtered R-algebras induces isomorphisms

$$B \otimes_A U(L) \to U(B \otimes_A L), \qquad U(L) \otimes_A B \to U(B \otimes_A L),$$

and left and right B-modules respectively.

Proof. This is [2, Prop. 2.3], the proof of which applies whenever the natural map $\mathrm{Sym}_A(L) \to \mathrm{gr}\, U(L)$ is an isomorphism, and therefore in particular whenever L is projective, by Rinehart's Theorem [54, Thm. 3.1]. $\square$

Lemma 1.1.9. *Suppose that $\varphi : A \to B$ is a morphism of commutative R-algebras, ψ: $\mathrm{Der}_R(A) \to \mathrm{Der}_R(B)$ is a φ-morphism, and $\tilde{\varphi} : L \to L'$ is a φ-morphism. Then the natural map*

$$U(\varphi, \tilde{\varphi})_B \colon B \otimes_A U(L) \to U(L')$$

factors as

$$B \otimes_A U(L) \to U(B \otimes_A L) \to U(L'),$$

where we consider $B \otimes_A L$ as an (R, B)-Lie algebra as in Remark 1.1.7. In particular, $U(\varphi, \tilde{\varphi})_B$ is an isomorphism whenever L is projective and $\tilde{\varphi}_B : B \otimes_A L \to L'$ is an isomorphism.

Proof. In the category $\mathbf{LR}_R$, $(\varphi, \tilde{\varphi}) : L \to L'$, factors as the composition of

$$(\varphi, i_L) : L \to B \otimes_A L, \qquad i_L(m) = 1 \otimes m,$$

and

$$(\mathrm{id}_B, \tilde{\varphi}_B) : B \otimes_A L \to L', \qquad \tilde{\varphi}_B(b \otimes m) = b\tilde{\varphi}(m),$$

and therefore $U(L) \to U(L')$ factorises as

$$U(L) \to U(B \otimes_A L) \to U(L').$$

Then the factorisation result follows from the B-linearity of $U(B \otimes_A L) \to U(L')$, and the final claim follows from Lemma 1.1.8. $\square$

1.2 Geometric Setup

From now on in this chapter, k will denote a field. Throughout we will work with a quadruple $(X, \mathcal{B}, \mathcal{B}', \Omega_{X/k})$ which is as in one of the following settings.

- **(A)** X is a scheme over k, $\mathcal{B} = \mathcal{B}'$ is the set of affine open subsets of X, and $\Omega_{X/k}$ is the sheaf of relative differentials of X over k.
- **(B)** X is a rigid space over k, $\mathcal{B}$ is the set of admissible affinoid open subsets of X, $\mathcal{B}'$ is the set of quasi-Stein admissible open subsets of X, and $\Omega_{X/k}$ is the sheaf of relative differentials of X over k (as described in [15] and [30, §4.4]).

In order to give a uniform approach to both cases, we define a *space over* k to be an X as in either case (A) or case (B), and a *morphism of spaces* over k to be a morphism $X \to Y$, where X and Y are either both in case (A) or both in case (B).

1.3 Sheaves on a Basis

For any X as in Section 1.2, we view X as a G-topological space in the sense of [14, §9.1.1]. Specifically, in case (A) we view X with the Zariski topology, and in case (B) we view X with its natural G-topology. With respect to this G-topology on X, the sets $\mathcal{B}$ and $\mathcal{B}'$ are each a basis for the G-topology of X, meaning that every open subset of X has an admissible open covering by elements of $\mathcal{B}$. Note there is no assumption that a basis in this sense is closed under intersections. For a basis $\mathcal{A}$, there is a notion of a sheaf on a basis [2, Def. 9.1], and the restriction functor induces an equivalence of categories from sheaves on X to sheaves on $\mathcal{A}$ [2, Thm. 9.1].

Suppose in what follows that $\mathcal{A} = \mathcal{B}$ or $\mathcal{A} = \mathcal{B}'$. Given $\mathcal{F}$ a sheaf on $\mathcal{A}$, the extension $\mathcal{F}^{\text{ext}}$ to a sheaf on X is defined as follows [2, Prop. A.2]. For any admissible open subset $U \subset X$ and any covering $\mathcal{U} = \{U_i\}_i$ of U, for each pair i, j let $\{W_{ijk}\}_k$ be an admissible open covering of $U_i \cap U_j$ by elements of $\mathcal{A}$. Set

$$\mathrm{H}^0(\mathcal{U}, \mathcal{F}) := \mathrm{eq}\left(\prod_i \mathcal{F}(U_i) \to \prod_{i,j,k} \mathcal{F}(W_{ijk})\right),$$

which is independent of the choice of the coverings $\{W_{ijk}\}_k$, and set

$$\mathcal{F}^{\text{ext}}(U) := \varinjlim_{\mathcal{U}} \mathrm{H}^0(\mathcal{U}, \mathcal{F}).$$

Now suppose that $\mathcal{F}$ is a presheaf on X, such that $\mathcal{F}|_{\mathcal{A}}$ is a sheaf on the basis $\mathcal{A}$. In this case $\mathcal{F}^{\text{ext}}$ is a sheaf, and there is a natural morphism of presheaves $\mathcal{F} \to \mathcal{F}^{\text{ext}}$ defined by

$$\mathcal{F}(U) \to \mathrm{H}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{F}^{\text{ext}}(U),$$

where we choose any covering $\mathcal{U}$ of U, and $\mathcal{F}(U) \to \mathrm{H}^0(\mathcal{U}, \mathcal{F})$ is the natural restriction map. This morphism is independent of the choice of $\mathcal{U}$. The following is direct to verify.

Lemma 1.3.1. *Suppose that $\mathcal{F}$ is a presheaf on X such that $\mathcal{F}|_{\mathcal{A}}$ is a sheaf on the basis $\mathcal{A}$. Then the morphism $\mathcal{F} \to \mathcal{F}^{\text{ext}}$ defined above is a sheafification of the presheaf $\mathcal{F}$.*

1.4 Tangent Sheaf

In the following lemma, which will be fundamental to our constructions, we work with a pair $(\varphi\colon A \to B, \Omega_{A/k})$ which is as in one of the following cases, each the affine version of the corresponding geometric framework of Section 1.2.

(A) $\varphi : A \to B$ is an étale morphism of commutative k-algebras, $\Omega_{A/k}$ is the module of Kähler differentials of A over k,

(B) $\varphi : A \to B$ is an étale morphism of affinoid algebras over k, $\Omega_{A/k}$ is the universal finite differential module of A over k [30, §3.6].

Lemma 1.4.1. *Let $\varphi : A \to B$ be as in (A) or (B) above. Then any $\partial \in \mathrm{Der}_k(B)$ is determined by its restriction to A, and there exists a φ-morphism*

$$\psi\colon\ \mathrm{Der}_k(A) \to \mathrm{Der}_k(B)$$

uniquely determined as a function $\mathrm{Der}_k(A) \to \mathrm{Der}_k(B)$ by the property that for any $\partial \in \mathrm{Der}_k(A)$

$$\psi(\partial) \circ \varphi = \varphi \circ \partial.$$

If furthermore $\Omega_{A/k}$ is finitely generated projective over A, then the natural map

$$B \otimes_A \mathrm{Der}_k(A) \to \mathrm{Der}_k(B), \qquad b \otimes \partial \mapsto b\psi(\partial),$$

is an isomorphism.

Proof. In either case (A) or case (B), because $\varphi : A \to B$ is étale, $B \otimes \Omega_{A/k} \to \Omega_{B/k}$ is an isomorphism. In case (B) this is [8, Prop. 3.5.3(i)], and in case (A), locally in the Zariski topology ψ is standard étale [57, Lem. 02GT], and for standard étale extensions the map is an isomorphism [45, Example 6.1.12]. Taking the B-linear dual, we obtain the following commutative diagram,

$$\begin{array}{ccc}
\mathrm{Hom}_B(\Omega_{B/k}, B) & \xrightarrow{\ \sim\ } & \mathrm{Der}_k(B) \\
{\scriptstyle\sim}\downarrow & & \Big\downarrow \\
\mathrm{Hom}_B(B \otimes_A \Omega^1_{A/k}, B) & & \\
{\scriptstyle\sim}\downarrow & & \Big\downarrow \\
\mathrm{Hom}_A(\Omega_{A/k}, B) & \xrightarrow{\ \sim\ } & \mathrm{Der}_k(A, B) \\
\uparrow & & \uparrow \\
B \otimes_A \mathrm{Hom}_A(\Omega_{A/k}, A) & \xrightarrow{\ \sim\ } & B \otimes_A \mathrm{Der}_k(A) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_A(\Omega_{A/k}, A) & \xrightarrow{\ \sim\ } & \mathrm{Der}_k(A)
\end{array}$$

In particular, we see that the restriction map,

$$\mathrm{Der}_k(B) \to \mathrm{Der}_k(A, B),$$

is an isomorphism. The composite A-linear map,

$$\psi : \mathrm{Der}_k(A) \to \mathrm{Der}_k(B),$$

satisfies,

$$\psi(\partial) \circ \varphi = \varphi \circ \partial,$$

for any $\partial \in \mathrm{Der}_k(A)$, and consequently is the unique function $\mathrm{Der}_k(A) \to \mathrm{Der}_k(B)$ with this property. For any $\partial_1, \partial_2 \in \mathrm{Der}_k(A)$, $\psi([\partial_1, \partial_2])$ and $[\psi(\partial_1), \psi(\partial_2)]$ agree on $\varphi(A)$, and thus are equal, hence $\psi : \mathrm{Der}_k(A) \to \mathrm{Der}_k(B)$ is also a homomorphism of k-Lie algebras.

For the second claim, if $\Omega_{A/k}$ is finitely generated projective over A, then the map,

$$B \otimes_A \mathrm{Hom}_A(\Omega_{A/k}, A) \to \mathrm{Hom}_B(B \otimes_A \Omega_{A/k}, B),$$

is an isomorphism [16, Chapter II, §5, Prop. 7]. Therefore, the composite,

$$\gamma\colon B \otimes_A \mathrm{Der}_k(A) \to \mathrm{Der}_k(B), \qquad \gamma(b \otimes \partial) = b\psi(\partial)$$

is an isomorphism. □

We consider the tangent sheaf Θ_X on X as the sheaf extension of the sheaf on the basis $\mathcal{B}$ given by $\Theta_X(U) = \mathrm{Der}_k(\mathcal{O}_X(U))$. For $U, V \in \mathcal{B}$ with $V \subset U$ the restriction map

$$\Theta_X(U) = \mathrm{Der}_k(\mathcal{O}_X(U)) \to \mathrm{Der}_k(\mathcal{O}_X(V)) = \Theta_X(V)$$

is the morphism defined by Lemma 1.4.1 for the étale extension $\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_X(V)$.

Definition 1.4.2. If $\mathcal{F}$, $\mathcal{G}$ are sheaves of k-algebras on X, then a *k-derivation* is a morphism $\partial : \mathcal{F} \to \mathcal{G}$ of sheaves of k-vector spaces on X such that for any admissible open subset U of X and $x, y \in \mathcal{F}(U)$,

$$\partial_U(xy) = x\partial_U(y) + \partial_U(x)y.$$

Remark 1.4.3. For any k-derivation $\mathcal{F} \to \mathcal{G}$ and for any admissible open subset $U \subset X$, the induced map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a k-linear derivation.

The following lemma is direct to verify from the definitions.

Lemma 1.4.4. *The natural map $\Theta_X \to \underline{\mathrm{End}}_k(\mathcal{O}_X)$ induces an isomorphism from Θ_X to the subsheaf*

$$\mathcal{T}_X(U) = \{f \in \mathrm{End}_k(\mathcal{O}_X|_U) \mid f \textit{ is a } k\textit{-derivation}\}.$$

From this we see that for any admissible open subset $U \subset X$ (not necessarily in $\mathcal{B}$), the natural action of $\Theta_X(U)$ on $\mathcal{O}_X(U)$ is an action by k-linear derivations. Henceforth, we will implicitly identify Θ_X with $\mathcal{T}_X$.

1.5 Lie-Algebroids

We follow [5] and [2] in the following definition.

Definition 1.5.1. A *Lie algebroid* on X is a pair $(\rho, \mathcal{L})$ where,

- $\mathcal{L}$ is a coherent sheaf of $\mathcal{O}_X$-modules,
- $\mathcal{L}$ has the structure of a sheaf of k-Lie algebras,
- $\rho : \mathcal{L} \to \Theta_X$ is an $\mathcal{O}_X$-linear morphism of sheaves of k-Lie algebras such that,

  $$[x, ay] = a[x, y] + \rho_U(x)(a)y,$$

  for any open subset U, $x, y \in \mathcal{L}(U)$, and $a \in \mathcal{O}_X(U)$.

The pair $(\rho, \mathcal{L})$ is called *smooth* if $\mathcal{L}$ is locally free of finite rank.

Example 1.5.2. The pair (id, Θ_X) is a Lie algebroid when Θ_X is coherent, and smooth if and only if Θ_X is locally free of finite rank.

Remark 1.5.3. If $(\rho, \mathcal{L})$ is a Lie-algebroid on X, then for any admissible open subset U of X, $(\rho'_U, \mathcal{L}(U))$ is an $(k, \mathcal{O}_X(U))$-Lie algebra, where ρ'_U is the composition

$$\rho'_U : \mathcal{L}(U) \xrightarrow{\rho_U} \Theta_X(U) \to \mathrm{Der}_k(\mathcal{O}_X(U)).$$

Definition 1.5.4. A morphism of Lie algebroids is a morphism of sheaves which is a morphism of $(k, \mathcal{O}_X(U))$-Lie algebras for any admissible open subset U of X.

In the following, we call a (k, A)-Lie algebra L finitely presented if it is finitely presented as an A-module, and smooth if L is finitely generated projective as an A-module. The next proposition shows that Lie algebroids are globalisations of Lie-Rinehart algebras. This is [2, Lem. 9.2.] in case (B), the proof of which also generalises to case (A).

Proposition 1.5.5 ([2, Lem. 9.2.])**.** *Suppose that $X \in \mathcal{B}$, and let $A = \mathcal{O}_X(X)$. Then the global sections functor defines an equivalence of categories from the category of Lie-algebroids on X to the category of finitely presented (k, A)-Lie algebras. This restricts to an equivalence from smooth Lie-algebroids on X to smooth (k, A)-Lie algebras.*

1.5.1 Universal Enveloping Algebra of a Lie Algebroid

Suppose that $\mathcal{L}$ is a Lie algebroid on X, $V \subset U$ are admissible open subsets of X, and let $\varphi : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$, $\tilde{\varphi} : \mathcal{L}(U) \to \mathcal{L}(V)$ be the restriction maps. It is straightforward to verify that $\tilde{\varphi}$ is a φ-morphism, and therefore we have an induced map

$$U(\varphi, \tilde{\varphi}) : U(\mathcal{L}(U)) \to U(\mathcal{L}(V))$$

by Lemma 1.1.5. Because $\mathcal{L}(U)$ is a presheaf, so is $U(\mathcal{L}(-)) \colon V \mapsto U(\mathcal{L}(V))$.

Definition 1.5.6. We define $\mathcal{U}(\mathcal{L})$ to be the sheafification of the presheaf $U(\mathcal{L}(-))$.

There are canonical morphisms

$$\iota_{\mathcal{O}} : \mathcal{O}_X \to \mathcal{U}(\mathcal{L}), \qquad \iota_{\mathcal{L}} : \mathcal{L} \to \mathcal{U}(\mathcal{L}),$$

of $\mathcal{O}_X$-modules and sheaves of k-Lie algebras respectively. For any admissible open subset $U \subset X$, $a \in \mathcal{O}_X(U)$, and $x \in \mathcal{L}(U)$, these satisfy

$$\iota_{\mathcal{L}}(ax) = \iota_{\mathcal{O}}(a)\iota_{\mathcal{L}}(x), \qquad [\iota_{\mathcal{L}}(x), \iota_{\mathcal{O}}(a)] = \iota_{\mathcal{O}}(\rho'_U(x)a).$$

As for Lie-Rinehart algebras, the triple $(\mathcal{U}(\mathcal{L}), \iota_{\mathcal{O}} : \mathcal{O}_X \to \mathcal{U}(\mathcal{L}), \iota_{\mathcal{L}} : \mathcal{L} \to \mathcal{U}(\mathcal{L}))$ enjoys the following universal property.

Lemma 1.5.7. *Suppose that $\mathcal{S}$ is a sheaf of unital associative k-algebras on X, and*

$$\eta_{\mathcal{O}} : \mathcal{O}_X \to \mathcal{S}, \qquad \eta_{\mathcal{L}} : \mathcal{L} \to \mathcal{S},$$

are morphisms of $\mathcal{O}_X$-modules and sheaves of k-Lie algebras respectively which satisfy

$$\eta_{\mathcal{L}}(ax) = \eta_{\mathcal{O}}(a)\eta_{\mathcal{L}}(x), \qquad [\eta_{\mathcal{L}}(x), \eta_{\mathcal{O}}(a)] = \eta_{\mathcal{O}}(\rho'_U(x)a).$$

for any admissible open subset $U \subset X$, $a \in \mathcal{O}_X(U)$, and $x \in \mathcal{L}(U)$. Then there is a unique morphism of sheaves of k-algebras $\mathcal{U}(\mathcal{L}) \to S$ such that

$$\begin{array}{ccccc} \mathcal{O}_X & \xrightarrow{\iota_{\mathcal{O}}} & \mathcal{U}(\mathcal{L}) & \xleftarrow{\iota_{\mathcal{L}}} & \mathcal{L} \\ & \underset{\eta_{\mathcal{O}}}{\searrow} & \downarrow{\scriptstyle \varphi} & \underset{\eta_{\mathcal{L}}}{\swarrow} & \\ & & \mathcal{S} & & \end{array}$$

Proof. Given such a triple $(\mathcal{S}, \iota_{\mathcal{O}} : \mathcal{O}_X \to \mathcal{S}, \iota_{\mathcal{L}} : \mathcal{L} \to \mathcal{S})$, we define a morphism of presheaves $\varphi : U(\mathcal{L}) \to \mathcal{S}$, for any admissible open subset $U \subset X$ using Lemma 1.1.2 and setting

$$\varphi_U : U(\mathcal{L}(U)) \to \mathcal{S}(U)$$

to be the unique morphism induced from $\eta_{\mathcal{O},U} : \mathcal{O}_X(U) \to \mathcal{S}(U)$, $\eta_{\mathcal{L},U} : \mathcal{L}(U) \to \mathcal{S}(U)$. It is direct to check using Lemma 1.1.2 that this is a morphism of presheaves, and we set $\varphi : \mathcal{U}(\mathcal{L}) \to \mathcal{S}$ to be the unique morphism determined by the universal property of the sheafification. The uniqueness follows directly from the uniqueness of Lemma 1.1.2. □

As for Lie-Rinehart algebras, we can consider the universal enveloping algebra as a functor in the following manner. In the following, for a Lie algebroid $\mathcal{L}$ on X, we write $\tau : \mathcal{L} \to \mathcal{T}_X$ for the composition of $\rho \colon \mathcal{L} \to \Theta_X$ with $\Theta_X \to \mathcal{T}_X \subset \underline{\mathrm{End}}_k(\mathcal{O}_X)$.

Definition 1.5.8. Suppose that $\varphi : X \to Y$ is a morphism of spaces over k, and that $\mathcal{L}'$, $\mathcal{L}$ are Lie algebroids over X and Y respectively. Then a φ-morphism is a morphism of sheaves of $\mathcal{O}_Y$-modules,

$$\tilde{\varphi} : \mathcal{L} \to \varphi_* \mathcal{L}',$$

which is also a morphism of sheaves of k-Lie algebras, such that

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\tau} & \underline{\mathrm{End}}_k(\mathcal{O}_Y) & \xrightarrow{-\circ\varphi^\sharp} & \underline{\mathrm{Hom}}_k(\mathcal{O}_Y, \varphi_*\mathcal{O}_X) \\ {\scriptstyle \tilde{\varphi}}\downarrow & & & & \uparrow{\scriptstyle \varphi^\sharp\circ-} \\ \varphi_*\mathcal{L}' & \xrightarrow{\varphi_*\tau'} & \varphi_*\underline{\mathrm{End}}_k(\mathcal{O}_X) & \longrightarrow & \underline{\mathrm{End}}_k(\varphi_*\mathcal{O}_X) \end{array}$$

commutes.

Remark 1.5.9. If $\varphi : X \to Y$ is a morphism of spaces over k, and $\tilde{\varphi} : \mathcal{L} \to \varphi_*\mathcal{L}'$ is a morphism of sheaves of sets, then $\tilde{\varphi}$ is a φ-morphism if and only if

$$\tilde{\varphi}_U : \mathcal{L}(U) \to \mathcal{L}'(\varphi^{-1}(U))$$

is a $\varphi^\sharp_U$-morphism for any $U \in \mathcal{B}_Y$, where $\varphi^\sharp_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$.

Write $\mathcal{LR}_k$ for the category of pairs $(\varphi, \tilde{\varphi})$, where $\varphi \colon X \to Y$ is a morphism of spaces over k and $\tilde{\varphi} \colon \varphi_*\mathcal{L} \to \mathcal{L}'$ is a φ-morphism. Given $(\varphi, \tilde{\varphi}) : (X, \mathcal{L}) \to (Y, \mathcal{L}')$ and $(\psi, \tilde{\psi}) : (Y, \mathcal{L}') \to (Z, \mathcal{L}'')$, then $\tilde{\psi} \circ \psi_*\tilde{\varphi}$ is a $(\psi \circ \varphi)$-morphism, and the composition is defined as $(\psi \circ \varphi, \tilde{\psi} \circ \psi_*\tilde{\varphi})$.

Lemma 1.5.10. *Suppose that $\varphi : X \to Y$ is a morphism of spaces over k, and $\mathcal{L}', \mathcal{L}$ are Lie algebroids on X and Y respectively. Then every φ-morphism $\tilde{\varphi} : \mathcal{L} \to \varphi_*\mathcal{L}'$ extends uniquely to a filtration preserving morphism $\mathcal{U}(\varphi, \tilde{\varphi}) : \mathcal{U}(\mathcal{L}) \to \varphi_*\mathcal{U}(\mathcal{L}')$ of sheaves of k-algebras on Y that makes the following diagram commute*

$$\begin{array}{ccc} \mathcal{O}_Y \oplus \mathcal{L} & \xrightarrow{\varphi^\sharp \oplus \tilde{\varphi}} & \varphi_*\mathcal{O}_X \oplus \varphi_*\mathcal{L}' \\ {\scriptstyle \iota_{\mathcal{O}_Y} \oplus \iota_{\mathcal{L}}} \downarrow & & \downarrow {\scriptstyle \varphi_*\iota_{\mathcal{O}_X} \oplus \varphi_*\iota_{\mathcal{L}'}} \\ \mathcal{U}(\mathcal{L}) & \xrightarrow[\mathcal{U}(\varphi, \tilde{\varphi})]{} & \varphi_*\mathcal{U}(\mathcal{L}') \end{array}$$

Proof. We define a morphism of presheaves,

$$U(\varphi, \tilde{\varphi}) : U(\mathcal{L}) \to \varphi_* U(\mathcal{L}'),$$

by setting for each admissible open $U \subset Y$,

$$U(\varphi, \tilde{\varphi})_U \coloneqq U(\varphi^\sharp, \tilde{\varphi}_U) \colon U(\mathcal{L}(U)) \to U(\mathcal{L}'(\varphi^{-1}(U))),$$

which is well-defined by Remark 1.5.9. Then we define the morphism $\mathcal{U}(\varphi, \tilde{\varphi})$ by the universal property of sheafification to be the unique morphism making the diagram

$$\begin{array}{ccc} \mathcal{U}(\mathcal{L}) & \xrightarrow{\mathcal{U}(\varphi, \tilde{\varphi})} & \varphi_*\mathcal{U}(\mathcal{L}') \\ \uparrow & & \uparrow \\ U(\mathcal{L}) & \xrightarrow[U(\varphi, \tilde{\varphi})]{} & \varphi_* U(\mathcal{L}') \end{array}$$

commute. $\square$

Therefore, we can view $\mathcal{U}(-)$ as a functor from $\mathcal{LR}_k$ to the category of pairs $(X, \mathcal{S})$, where X is a space over k and $\mathcal{S}$ is a sheaf of unital associative filtered k-algebras on X. Morphisms are pairs (φ, ϕ) for $\varphi : X \to Y$ a morphism of schemes over k, and $\phi : \mathcal{S} \to \varphi_*\mathcal{S}'$ a morphism of sheaves of unital associative filtered k-algebras on Y.

We have a more explicit description of the sheaf $\mathcal{U}(\mathcal{L})$ when $\mathcal{L}$ is smooth. In the following, for an admissible open subset $U \in \mathcal{B}'$ and an $\mathcal{O}_X(U)$-module M, we write $\widetilde{M}$ for the sheaf on U associated to M, which for all $V \subset U$ with $V \in \mathcal{B}$ is given by

$$\widetilde{M}(V) = \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} M.$$

Lemma 1.5.11. *Let $(\rho, \mathcal{L})$ be a smooth Lie algebroid on X. Then $\mathcal{U}(\mathcal{L}) \coloneqq U(\mathcal{L}(-))^{\text{ext}}$ is a sheafification of $U(\mathcal{L})$, and for any $U \in \mathcal{B}'$ there is a canonical isomorphism*

$$\widetilde{U(\mathcal{L}(U))} \to \mathcal{U}(\mathcal{L})|_U$$

of sheaves of k-algebras.

Proof. For any pair $U \in \mathcal{B}'$ and $V \in \mathcal{B}$ with $V \subset U$, the natural map

$$\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} U(\mathcal{L}(U)) \to U(\mathcal{L}(V))$$

is an isomorphism by Lemma 1.1.9, as $\mathcal{L}$ is smooth and coherent as an $\mathcal{O}_X$-module. The fact that $\mathcal{L}(U)$ is projective as an $\mathcal{O}(U)$-module follows in the rigid case from [6, Prop. 1.1.13]. Therefore, for any $U \in \mathcal{B}'$, this defines an isomorphism

$$\widetilde{U(\mathcal{L}(U))}|_{\mathcal{B}_U} \xrightarrow{\sim} U(\mathcal{L}(-))|_{\mathcal{B}_U}, \tag{1.1}$$

where $\mathcal{B}_U$ is the basis of U consisting of $V \in \mathcal{B}$ such that $V \subset U$. As this holds for any $U \in \mathcal{B}'$, $U(\mathcal{L}(-))$ defines a sheaf on the basis $\mathcal{B}$, and by Lemma 1.3.1 we see that $\mathcal{U}(\mathcal{L}) := U(\mathcal{L}(-))^{\text{ext}}$ is a sheafification of the presheaf $U(\mathcal{L}(-))$. Furthermore, the isomorphism (1.1) extends uniquely to the required isomorphism. $\square$

For example, if X is a scheme, then Lemma 1.5.11 says that $\mathcal{U}(\mathcal{L})$ is a quasi-coherent sheaf whenever $\mathcal{L}$ is smooth.

1.6 Equivariant Sheaves

Throughout this section 1.6, let G be a group and let X be a set with a G-topology in the sense of [14, §9.1.1]. Suppose also that G acts on X, by which we mean there is a group homomorphism $\rho : G \to \text{Homeo}(X)$, where $\text{Homeo}(X)$ is the group of continuous bijections from X to itself. For each $g \in G$, g_* and g^{-1} are inverse functors from $\text{Sh}(X)$ to itself, where

$$(g^{-1}\mathcal{F})(U) = \mathcal{F}(gU),\ (g_*\mathcal{F})(U) = \mathcal{F}(g^{-1}U),$$

for all admissible open subsets U of X. We summarise some notions from [34, §5.1].

Definition 1.6.1. A *k-linear G-equivariant sheaf* on X, is a pair $(\mathcal{F}, \{g^{\mathcal{F}}\}_{g\in G})$, such that $\mathcal{F}$ is a sheaf of k-vector spaces on X, and for $g \in G$,

$$g^{\mathcal{F}} : \mathcal{F} \to g^{-1}\mathcal{F},$$

is an isomorphism of sheaves of k-vector spaces, and for all $g, h \in G$,

$$(gh)^{\mathcal{F}} = h^{-1}(g^{\mathcal{F}}) \circ h^{\mathcal{F}}.$$

A morphism $\psi : (\mathcal{F}, \{g^{\mathcal{F}}\}_{g\in G}) \to (\mathcal{G}, \{g^{\mathcal{G}}\}_{g\in G})$ is a morphism of sheaves of k-vector spaces, such that

$$g^{-1}(\psi) \circ g^{\mathcal{F}} = g^{\mathcal{G}} \circ \psi,$$

for all $g \in G$.

Definition 1.6.2. A *G-equivariant sheaf of k-algebras* on X is a pair $(\mathcal{A}, \{g^{\mathcal{A}}\}_{g\in G})$, such that

- $(\mathcal{A}, \{g^{\mathcal{A}}\}_{g\in G})$ is a k-linear G-equivariant sheaf,
- $\mathcal{A}$ is a sheaf of k-algebras,

- For all $g \in G$,
$$g^{\mathcal{A}} : \mathcal{A} \to g^{-1}\mathcal{A},$$
is a morphism of sheaves of k-algebras.

Remark 1.6.3. If U is a G-stable admissible open subset of X, then for $a \in \mathcal{A}(U)$,
$$g \cdot a := g^{\mathcal{A}}(a)$$
defines an action of G on $\mathcal{A}(U)$ by k-algebra automorphisms. Therefore, for such a subset U we can form the skew group ring $\mathcal{A}(U) \rtimes G$.

Definition 1.6.4. Let $\mathcal{A}$ be a G-equivariant sheaf of k-algebras on X. A *G-equivariant sheaf of $\mathcal{A}$-modules* or *$\mathcal{G}$-$\mathcal{A}$-module* on X is a sheaf $\mathcal{M}$ of left $\mathcal{A}$-modules, together with a k-linear G-equivariant structure $(\mathcal{M}, \{g^{\mathcal{M}}\}_{g\in G})$ such that for any admissible open subset $U \subset X$,
$$g_U^{\mathcal{M}}(a \cdot m) = g_U^{\mathcal{A}}(a) \cdot g_U^{\mathcal{M}}(m),$$
for all $g \in G$, $a \in \mathcal{A}(U)$ and $m \in \mathcal{M}(U)$. A morphism of $\mathcal{G}$-$\mathcal{A}$-modules is a morphism of sheaves of $\mathcal{A}$-modules which is also a morphism of k-linear G-equivariant sheaves. We write $\mathbf{Mod}(G\text{-}\mathcal{A})$ for the category of G-$\mathcal{A}$-modules on X.

Remark 1.6.5. Suppose that U is a G-stable admissible open subset of X. Then $\Gamma(U, -)$ is a functor from G-$\mathcal{A}$-modules to $\mathcal{A}(U) \rtimes G$-modules [1, Prop. 2.3.5]. For $a \in \mathcal{A}(U)$ and $g \in G$, $ag \in \mathcal{A}(U) \rtimes G$ acts on $\mathcal{M}(U)$ by
$$ag \cdot m := a \cdot g^{\mathcal{M}}(m).$$
for any $m \in \mathcal{M}(U)$.

Remark 1.6.6. Suppose that $\mathcal{A}$ is a G-equivariant sheaf of k-algebras, and $\mathcal{M}$ and $\mathcal{N}$ are G-$\mathcal{A}$-modules. Then it is straightforward to show that $\mathcal{F} := \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ is a G-equivariant sheaf, where
$$g^{\mathcal{F}} : \mathcal{F} \to g^{-1}\mathcal{F}$$
is defined by
$$g_U^{\mathcal{F}}(f) = g_*(g^{\mathcal{N}}|_U \circ f \circ (g^{\mathcal{M}}|_U)^{-1})$$
for any $g \in G$ and $f \in \mathcal{F}(U)$. In particular, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ is a $k[G]$-module where $g \in G$ acts on $f \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ by
$$g \cdot f = g_*(g^{\mathcal{N}} \circ f \circ (g^{\mathcal{M}})^{-1}).$$

Example 1.6.7. We will often find ourselves in the following situation. Suppose that $\mathcal{A}$ is a G-equivariant sheaf of k-algebras on X, and suppose that X_0 is an admissible open subset of X such that X is the disjoint union
$$X = \bigsqcup_{g \in G/G^0} g(X_0),$$
indexed by the set of left cosets G/G^0, where $G^0 := \mathrm{Stab}_G(X_0) \le G$. Then it is straightforward to show that the restriction functor
$$\mathbf{Mod}(G\text{-}\mathcal{A}) \to \mathbf{Mod}(G^0\text{-}\mathcal{A}|_{X_0}),$$
is an equivalence of categories.

Example 1.6.8. We will also later need to make use of the following equivalence between equivariant sheaves and sheaves on the quotient space. Suppose that X has an action of a semi-direct product $H \rtimes G$ and X_0 is an admissible open subset of X such that X is the disjoint union

$$X = \bigsqcup_{h \in H} h(X_0).$$

In this situation, we may form the quotient G-topological space X/H as follows. As a set, X/H is the quotient of X by H, and writing $p\colon X \to X/H$ for the quotient map, a subset $U \subset X/H$ is defined to be admissible open if and only if $p^{-1}(U)$ is an admissible open subset of X. A collection $\{U_i\}_i$ of admissible open subsets of X/H which covers the admissible open subset U is defined to be an admissible open covering of U if $\{p^{-1}(U_i)\}_i$ forms an admissible open covering of $p^{-1}(U)$ in X. It is routine to check that this defines a G-topology on X/H, for which the quotient map $p\colon X \to X/H$ is continuous. Furthermore, because H is normal in $H \rtimes G$, the action of $H \rtimes G$ on X induces an action of $H \rtimes G$ on X/H for which the quotient map $p\colon X \to X/H$ is $H \rtimes G$-equivariant.

Suppose now that $\mathcal{A}$ is a $H \rtimes G$-equivariant sheaf of k-algebras on X. For example, $\mathcal{A}$ could just be the constant sheaf $\underline{k}$ (in which case a $(H \rtimes G)$-$\mathcal{A}$-module is just a $H \rtimes G$-equivariant sheaf), or X could be a locally ringed G-topological space and $\mathcal{A} = \mathcal{O}_X$ the structure sheaf.

For any $H \rtimes G$-equivariant sheaf $\mathcal{F}$ on X, the presheaf $\mathcal{F}^H$ on X/H defined by

$$\mathcal{F}^H(U) := \mathcal{F}(p^{-1}(U))^H$$

is a G-equivariant sheaf on X/H, where the G-equivariant structure $g^{\mathcal{F}^H}\colon \mathcal{F}^H \to g^{-1}\mathcal{F}^H$ is defined as the restriction of $p_* g^{\mathcal{F}}\colon p_*\mathcal{F} \to p_*(g^{-1}\mathcal{F}) = g^{-1}(p_*\mathcal{F})$ to $\mathcal{F}^H$. The fact that $p_*(g^{-1}\mathcal{F}) = g^{-1}(p_*\mathcal{F})$ follows from the $(H \rtimes G)$-equivariance of $p\colon X \to X/H$, and the fact that this restricts to a map $\mathcal{F}^H \to g^{-1}\mathcal{F}^H$ is because H is normal in $H \rtimes G$. In particular, this applies to the $H \rtimes G$-equivariant sheaf $\mathcal{A}$ on X, and we obtain the G-equivariant sheaf of k-algebras $\mathcal{A}^H$ on X/H. Therefore, if $\mathcal{F}$ is a $(H \rtimes G)$-$\mathcal{A}$-module, then $\mathcal{F}^H$ naturally acquires the structure of a G-$\mathcal{A}^H$-module, and this defines a functor

$$(-)^H\colon \mathbf{Mod}((H \rtimes G)\text{-}\mathcal{A}) \to \mathbf{Mod}(G\text{-}\mathcal{A}^H).$$

In the other direction, we have the pullback functor

$$p^{-1}\colon \mathbf{Mod}(G\text{-}\mathcal{A}^H) \to \mathbf{Mod}((H \rtimes G)\text{-}\mathcal{A}),$$

defined (for simplicity because p is an open map) by

$$p^{-1}\mathcal{G}(V) := \mathcal{G}(p(V))$$

for $\mathcal{G} \in \mathbf{Mod}(G\text{-}\mathcal{A}^H)$. The $H \rtimes G$-equivariant structure on $p^{-1}\mathcal{G}$ is defined by

$$g^{p^{-1}\mathcal{G}} := p^{-1}\gamma^{\mathcal{G}}\colon p^{-1}\mathcal{G} \to p^{-1}(g^{-1}\mathcal{G}) = g^{-1}(p^{-1}\mathcal{G}),$$

for $g \in H \rtimes G$. It is direct to show that, because our assumption that X is the disjoint union of copies of X_0 indexed by H, that these functors are mutually inverse quasi-equivalences of categories.

We also note that under our assumption on X and the action of H, the natural map $X_0 \hookrightarrow X \twoheadrightarrow X/H$ is an isomorphism of G-topological spaces, and the pushforward of $\mathcal{A}|_{X_0}$ to X/H is canonically identified with $\mathcal{A}^H$. In particular, if X is a rigid space, then taking $\mathcal{A} = \mathcal{O}_X$ we see that X/H is also a rigid space, canonically identified with X_0.

1.7 $\mathcal{D}$-Modules

In this section we specialise the geometric framework of Section 1.2 in which we work, and additionally assume that *the characteristic of k is zero*, and

(A) X is a smooth scheme over k ($X \to \operatorname{Spec}(k)$ is smooth),

(B) X is a smooth rigid space over k ($X \to \operatorname{Spec}(k)$ is smooth [15, Def. 2.1]).

In both cases, this assumption implies that $\Omega_{X/k}$ is locally free of rank $\dim_x X$ at any $x \in X$ (which in case (B) is actually an equivalent condition [15, Lem. 2.8]). In particular, the dual sheaf Θ_X is locally free (and hence coherent). We also note that in case (A), as $\operatorname{char}(k) = 0$, then $X \to \operatorname{Spec}(k)$ is smooth if and only if $\Omega_{X/k}$ is locally free and X is locally of finite type over k [57, Lem. 04QN].

Definition 1.7.1. We define the sheaf of differential operators $\mathcal{D}_X \coloneqq \mathcal{U}(\Theta_X)$.

We write $\mathbf{Mod}(\mathcal{D}_X)$ for the category of $\mathcal{D}_X$-modules, and $\mathbf{VectCon}(X)$ for the full subcategory integrable connections: $\mathcal{D}_X$-modules for which the underlying $\mathcal{O}_X$-module is a vector bundle. When X is a smooth rigid space over a field of characteristic zero, $\mathcal{D}_X$ is by definition the sheaf of differential operators on X [2]. When X is a smooth scheme over a field of characteristic zero, the sheaf $\mathcal{D}_X$ we have defined coincides with the usual sheaf of Grothendieck differential operators in the sense that the natural morphism $\mathcal{D}_X \to \underline{\operatorname{End}}_k(\mathcal{O}_X)$ induced from $\mathcal{O}_X \to \underline{\operatorname{End}}_k(\mathcal{O}_X)$ and $\Theta_X \to \mathcal{T}_X$ is injective, and has image the subsheaf of $\underline{\operatorname{End}}_k(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\mathcal{T}_X$. This can be seen by taking the associated graded of this morphism, and using Rinehart's Theorem [54, Thm. 3.1] and (for example) [40, §1.1].

We now record some results which will be useful later.

Lemma 1.7.2. *If X is connected then $\mathcal{O}_X$ is an irreducible $\mathcal{D}_X$-module.*

Proof. In case (B), this follows from [3, Prop. 3.1.3]. In case (A), we can argue as in the proof of *loc. cit.* to reduce to the case when X is affine, which follows from [47, Thm. 15.3.8]. □

Lemma 1.7.3. *Any $\mathcal{D}_X$-module which is coherent as an $\mathcal{O}_X$-module is locally free.*

Proof. This follows from exactly the same proof that is given for [40, Thm. 1.4.10]. □

1.7.1 Inverse Image and Direct Image

We now describe the inverse and direct image functors of $\mathcal{D}$-modules for an étale morphism $f\colon X \to Y$. We describe these very explicitly, as we will make use of this explicit description later.

The Direct Image

Let us first describe the direct image

$$f_*\colon \mathbf{Mod}(\mathcal{D}_X) \to \mathbf{Mod}(\mathcal{D}_Y),$$

Because $f\colon X \to Y$ is étale, we can use Lemma 1.4.1 to define a morphism $\Theta_Y \to f_*\Theta_X$ as follows. Suppose first that $U \in \mathcal{B}_Y$ and $V \in \mathcal{B}_X$ with $V \subset f^{-1}(U)$. Then the composition

$$\Theta_Y(U) \to f_*\Theta_X(U) = \Theta_X(f^{-1}(U)) \to \Theta_X(V)$$

is defined to be the morphism of Lemma 1.4.1 determined by the étale extension $\mathcal{O}_Y(U) \to \mathcal{O}_X(V)$. For any $V, V_1, V_2 \in \mathcal{B}_X$ with $V \subset V_1 \cap V_2$, by the uniqueness of the morphism of Lemma 1.4.1, the diagram

$$\begin{array}{ccccc} \Theta_X(V_2) & \longleftarrow & \Theta_Y(U) & \longrightarrow & \Theta_X(V_1) \\ & \searrow & \downarrow & \swarrow & \\ & & \Theta_X(V) & & \end{array}$$

commutes. Therefore, covering $f^{-1}(U)$ with $V \in \mathcal{B}_X$ we obtain a well defined $\mathcal{O}_Y(U)$-linear map,

$$\Theta_Y(U) \to f_*\Theta_X(U).$$

Again using the uniqueness of Lemma 1.4.1, this defines a morphism of $\mathcal{O}_Y$-modules $\Theta_Y \to f_*\Theta_X$ on the basis $\mathcal{B}_Y$, which extends uniquely to a morphism $\Theta_Y \to f_*\Theta_X$ of $\mathcal{O}_Y$-modules on Y.

Now suppose that $\mathcal{N} \in \mathbf{Mod}(\mathcal{D}_X)$. As an $\mathcal{O}_Y$-module, the direct image is the usual direct image of $\mathcal{O}_Y$-modules $f_*\mathcal{N}$. Because $\mathcal{N}$ is a $\mathcal{D}_X$-module, there is a morphism of sheaves of k-Lie algebras,

$$\Theta_X \to \mathcal{D}_X \to \underline{\mathrm{End}}_k(\mathcal{N}),$$

and using this we obtain an action of Θ_Y on $f_*\mathcal{N}$,

$$\Theta_Y \to f_*\Theta_X \to f_*\underline{\mathrm{End}}_k(\mathcal{N}) \to \underline{\mathrm{End}}_k(f_*\mathcal{N}).$$

One can check that this is appropriately compatible with the $\mathcal{O}_Y$-module structure on $f_*\mathcal{N}$ and so both actions extend uniquely by Lemma 1.5.7 to a $\mathcal{D}_Y$-module structure on $f_*\mathcal{N}$.

The Inverse Image

Let us now describe the inverse image

$$f^*\colon \mathbf{VectCon}(Y) \to \mathbf{VectCon}(X).$$

Suppose that $\mathcal{M} \in \mathbf{VectCon}(Y)$. Then

$$f^*\mathcal{M} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M},$$

is a coherent $\mathcal{O}_X$-module. As above, to give $f^*\mathcal{M}$ the structure of a $\mathcal{D}_X$-module, by Lemma 1.5.7 we need to give a morphism of sheaves of k-Lie algebras

$$\Theta_X \to \underline{\mathrm{End}}_k(f^*\mathcal{M})$$

which is appropriately compatible with the $\mathcal{O}_X$-module structure on $f^*\mathcal{M}$. Locally for any $U \in \mathcal{B}_X$, the map on sections

$$\Theta_X(U) \to \mathrm{End}_k(f^*\mathcal{M}|_U)$$

is described as follows. For any $W \in \mathcal{B}_Y$ with $f(U) \subset W$, there is a canonical identification

$$f^*\mathcal{M}|_U \equiv (\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(W)} \mathcal{M}(W))^{\sim},$$

and so for any $V \in \mathcal{B}_X$ with $V \subset U$,

$$(f^*\mathcal{M})(V) \equiv \mathcal{O}_X(V) \otimes_{\mathcal{O}_Y(W)} \mathcal{M}(W).$$

Because $\Omega_{W/k}$ is finitely generated projective, taking the dual of

$$\mathcal{O}_X(V) \otimes_{\mathcal{O}_Y(W)} \Omega_{W/k} \to \Omega_{\mathcal{O}_X(V)/k}$$

as in the proof of Lemma 1.4.1 induces a map

$$\Theta_X(V) \to \mathcal{O}_X(V) \otimes_{\mathcal{O}_Y(W)} \Theta_Y(W).$$

The action of $\partial \in \Theta_X(U)$ on $(f^*\mathcal{M})(V)$,

$$\partial_V \colon \mathcal{O}_X(V) \otimes_{\mathcal{O}_Y(W)} \mathcal{M}(W) \to \mathcal{O}_X(V) \otimes_{\mathcal{O}_Y(W)} \mathcal{M}(W),$$

is defined by setting

$$\partial_V(s \otimes m) := \partial(s) \otimes m + \sum_i s s_i \otimes \partial_i(m),$$

where $\sum_i s_i \otimes \partial_i$ is the image of ∂ under

$$\Theta_X(U) \to \Theta_X(V) \to \mathcal{O}_X(V) \otimes_{\mathcal{O}_Y(W)} \Theta_Y(W).$$

It can be checked that this is is independent of the choice of W, and glues to a morphism of sheaves of k-Lie algebras which satisfies the conditions of Lemma 1.5.7 and so extends to a $\mathcal{D}_X$-module structure.

Remark 1.7.4. We note that in our description of the inverse image we did not use the fact that $f \colon X \to Y$ is étale. Indeed, the morphism $\Theta_X \to f^*\Theta_Y$, which we used implicitly, exists whenever Y is smooth, and our description still holds in this generality.

1.8 Equivariant $\mathcal{D}$-Modules

Suppose in this section that X is as Section 1.2, and that G is a group that acts on X (on the right), given by the data of a group homomorphism

$$\rho : G^{\text{op}} \to \text{Aut}_k(X), \qquad g \mapsto (\rho(g) : X \to X, \rho(g)^\sharp : \mathcal{O}_X \to \rho(g)_*\mathcal{O}_X).$$

Then we have a group homomorphism

$$G \to \text{Homeo}(X), \qquad g \mapsto \rho(g^{-1}),$$

and we can consider G-equivariant sheaves on X. For example, the structure sheaf $\mathcal{O}_X$ is naturally a G-equivariant sheaf of k-algebras, with the G-equivariant structure

$$g^{\mathcal{O}_X} := \rho(g)^\sharp : \mathcal{O}_X \to \rho(g)_*\mathcal{O}_X = g^{-1}\mathcal{O}_X.$$

Suppose now that $(\rho, \mathcal{L})$ is a Lie-algebroid on X, and that $\mathcal{L}$ also has the structure of a G-$\mathcal{O}_X$-module such that each $g^{\mathcal{L}} \colon \mathcal{L} \to g^{-1}\mathcal{L}$ for $g \in G$ is a g^{-1}-morphism in the sense of Definition 1.5.8. In this situation $\mathcal{U}(\mathcal{L})$ is a G-equivariant sheaf of k-algebras via

$$g^{\mathcal{U}(\mathcal{L})} := \mathcal{U}(g^{\mathcal{O}_X}, g^{\mathcal{L}}) \colon \mathcal{U}(\mathcal{L}) \to g^{-1}\mathcal{U}(\mathcal{L}),$$

and $\mathcal{O}_X$ with its natural structure of an $\mathcal{U}(\mathcal{L})$-module and a G-equivariant sheaf is a G-$\mathcal{U}(\mathcal{L})$-module.

Example 1.8.1. If X is additionally as in Section 1.7 (i.e. X is smooth over a field of characteristic 0) then the tangent sheaf Θ_X is naturally a G-$\mathcal{O}_X$-module, where we define $g^{\Theta_X} : \Theta_X \to g^{-1}\Theta_X$ as follows. On each $U \in \mathcal{B}$, $\Theta_X(U) = \text{Der}_k(\mathcal{O}_X(U))$ and we define

$$g_U^{\Theta_X} : \text{Der}_k(\mathcal{O}_X(U)) \to \text{Der}_k(\mathcal{O}_X(g(U)))$$

by

$$g_U^{\Theta_X} : \partial \mapsto g_U^{\mathcal{O}_X} \circ \partial \circ (g_U^{\mathcal{O}_X})^{-1}.$$

Each g^{Θ_X} is a g^{-1}-morphism by Remark 1.5.9 and the description of g^{Θ_X} on $U \in \mathcal{B}$ above. Therefore taking $\mathcal{L} = \Theta_X$ above, $\mathcal{D}_X$ is naturally a G-equivariant sheaf of k-algebras for which $\mathcal{O}_X$ is a G-$\mathcal{D}_X$-module.

In the following, for $U \in \mathcal{B}'$, we call an $\mathcal{O}(U)$-module M *coherent* if as an $\mathcal{O}(U)$-module,

(A) M is finitely presented,

(B) M is coadmissible (in the sense of [55]).

Remark 1.8.2. In case (B), for $U \in \mathcal{B}'$, we are viewing $\mathcal{O}(U)$ as a Fréchet-Stein algebra (in the sense of [55]) with respect to the k-Banach algebras $(\mathcal{O}(U_i))_i$ for any admissible open quasi-Stein covering $(U_i)_i$ of X. We also note that when U is in fact affinoid, M is coadmissible if and only if M is finitely generated.

We write $\mathbf{Mod}_{\text{c}}(G\text{-}\mathcal{U}(\mathcal{L}))$ for the full subcategory of $\mathbf{Mod}(G\text{-}\mathcal{U}(\mathcal{L}))$ consisting of objects for which the underlying $\mathcal{O}_X$-module is coherent. Each $g \in G$ acts on $U(\mathcal{L}(X))$ via

$$U(g_X^{\mathcal{O}_X}, g_X^{\mathcal{L}}) \colon U(\mathcal{L}(X)) \to U(\mathcal{L}(X)),$$

and we write $\mathbf{Mod}_{\text{c}}(U(\mathcal{L}(X)) \rtimes G)$ for the full subcategory of $\mathbf{Mod}(U(\mathcal{L}(X)) \rtimes G)$ consisting of objects for which the underlying $\mathcal{O}_X(X)$-module is coherent.

Proposition 1.8.3. *Suppose that $X \in \mathcal{B}'$, let $\mathcal{L}$ be a smooth Lie-algebroid on X. Then the global sections functor defines an equivalence of categories*

$$\Gamma(X, -)\colon \mathbf{Mod}_{\mathrm{c}}(G\text{-}\mathcal{U}(\mathcal{L})) \xrightarrow{\sim} \mathbf{Mod}_{\mathrm{c}}(U(\mathcal{L}(X)) \rtimes G).$$

Proof. Because $X \in \mathcal{B}'$, the global sections incudes an equivalence

$$\Gamma(X, -)\colon \mathbf{Mod}_{\mathrm{c}}(\mathcal{O}_X) \xrightarrow{\sim} \mathbf{Mod}_{\mathrm{c}}(\mathcal{O}_X(X)),$$

with inverse sending an A-module M to $\widetilde{M}$. If $\mathcal{M} \in \mathbf{Mod}_{\mathrm{c}}(G\text{-}\mathcal{U}(\mathcal{L}))$, then $\mathcal{M}(X)$ is naturally a $U(\mathcal{L}(X)) \rtimes G$-module via the natural map

$$\mathcal{U}(\mathcal{L}(X)) \to U(\mathcal{L})(X)$$

and the action of $g \in G$ by $g_X^{\mathcal{M}}\colon \mathcal{M}(X) \to \mathcal{M}(g(X)) = \mathcal{M}(X)$. Furthermore, for any G-$\mathcal{U}(\mathcal{L})$-morphism $f\colon \mathcal{M} \to \mathcal{N}$ the map on global sections will be $U(\mathcal{L}(X)) \rtimes G$-linear.

Suppose now that $M \in \mathbf{Mod}_{\mathrm{c}}(U(\mathcal{L}(X)) \rtimes G)$. For any $U \in \mathcal{B}$, using Lemma 1.5.11 we can factorise the natural map

$$\begin{aligned}\widetilde{M}(U) &= \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(X)} \mathcal{M}(X),\\ &\xrightarrow{\sim} \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(X)} U(\mathcal{L}(X)) \otimes_{U(\mathcal{L}(X))} \mathcal{M}(X),\\ &\xrightarrow{\sim} U(\mathcal{L}(U)) \otimes_{U(\mathcal{L}(X))} \mathcal{M}(X)\end{aligned}$$

as the composition of isomorphisms. Therefore, letting $U(\mathcal{L}(U))$ act via left multiplication on the left factor $U(\mathcal{L}(U))$, the sheaf $\widetilde{M}$ naturally has the structure of a sheaf of $\mathcal{U}(\mathcal{L})$-modules. This is further a G-$\mathcal{U}(\mathcal{L})$-module with respect to the G-equivariant structure

$$g^{\widetilde{M}}\colon \widetilde{M} \to g^{-1}\widetilde{M}$$

defined on $U \in \mathcal{B}$ by

$$g_U^{\widetilde{M}} = g_U^{\mathcal{O}_X} \otimes g\colon \mathcal{O}_X(U) \otimes_A \mathcal{M}(X) \to \mathcal{O}_X(g(U)) \otimes_A \mathcal{M}(X).$$

Any morphism $\lambda\colon M \to N$ naturally induces a morphism $1 \otimes \lambda\colon \widetilde{M} \to \widetilde{N}$, and this is easily shown to be a morphism of G-$\mathcal{U}(\mathcal{L})$-modules. It is straightforward to check that with these definitions, the natural isomorphisms

$$\widetilde{\mathcal{M}(X)} \xrightarrow{\sim} \mathcal{M}, \qquad M \xrightarrow{\sim} \widetilde{M}(X),$$

are isomorphisms of G-$\mathcal{U}(\mathcal{L})$-modules and $U(\mathcal{L}(X)) \rtimes G$-modules respectively. □

Remark 1.8.4. In case (A) (so X is an affine scheme), the same proof shows that taking the global sections also defines an equivalence

$$\Gamma(X, -)\colon \mathbf{Mod}_{\mathrm{qc}}(G\text{-}\mathcal{U}(\mathcal{L})) \xrightarrow{\sim} \mathbf{Mod}(\mathcal{U}(\mathcal{L}(X)) \rtimes G).$$

where $\mathbf{Mod}_{\mathrm{qc}}(G\text{-}\mathcal{U}(\mathcal{L}))$ is the full subcategory of $\mathbf{Mod}(G\text{-}\mathcal{U}(\mathcal{L}))$ consisting of objects for which the underlying $\mathcal{O}_X$-module is quasi-coherent.

Suppose now that X is as in Section 1.7 (X is smooth over a field of characteristic 0). In this case, we write $\mathbf{VectCon}^G(X)$ for the category $\mathbf{Mod}_c(G\text{-}\mathcal{D}_X)$ which in light of Lemma 1.7.3 is the category of G-equivariant vector bundles with connection on X. Because coherent sheaves are closed under kernels and cokernels, $\mathbf{VectCon}^G(X)$ is abelian and furthermore $\mathbf{VectCon}^G(X)$ is a rigid abelian tensor category in the sense of [26]. The tensor product $\mathcal{V} \otimes \mathcal{W}$ of $\mathcal{V}, \mathcal{W} \in \mathbf{VectCon}^G(X)$ is defined to be the tensor product of $\mathcal{O}_X$-modules, with $\mathcal{D}_X$-module structure

$$\partial \cdot (x \otimes y) = x \otimes \partial(y) + \partial(x) \otimes y,$$

for a local section ∂ of Θ_X, and G-equivariant structure

$$g^{\mathcal{V}\otimes\mathcal{W}} := g^{\mathcal{V}} \otimes g^{\mathcal{W}} \colon \mathcal{V} \otimes \mathcal{W} \to g^{-1}(\mathcal{V} \otimes \mathcal{W}).$$

With this tensor structure, then $\underline{\mathrm{Hom}}(\mathcal{V}, \mathcal{W})$ is given by the internal hom of $\mathcal{O}_X$-modules with $\mathcal{D}_X$-module structure

$$(\partial \cdot f)(x) = \partial \cdot f(x) - f(\partial \cdot x)$$

for a local section ∂ of Θ_X, and G-equivariant structure as described in Remark 1.6.6.

1.9 Galois Extensions

Let G be a group, and X be as in Section 1.2. We write $\underline{G}$ for the corresponding constant group over k. When G is finite, this given by $\underline{G} = \mathrm{Spec}(\mathcal{O}(G))$ (resp. $\underline{G} = \mathrm{Sp}(\mathcal{O}(G))$), where $\mathcal{O}(G)$ is the k-algebra of functions $f : G \to k$, and in this case we write δ_g for the indicator function of $g \in G$.

Recall that that a (right) *action* of $\underline{G}$ on a space X over k is a morphism $a\colon X \times \underline{G} \to X$ of spaces over k such that the diagrams

$$\begin{array}{ccc} X \times \underline{G} \times \underline{G} & \xrightarrow{a \times p_2} & X \times \underline{G} \\ {\scriptstyle p_X \times m}\downarrow & & \downarrow{\scriptstyle a} \\ X \times \underline{G} & \xrightarrow[a]{} & X \end{array} \qquad \begin{array}{ccc} X \times \underline{1} & \hookrightarrow & X \times \underline{G} \\ & {\scriptstyle p_X}\searrow \quad \swarrow{\scriptstyle a} & \\ & X & \end{array}$$

commute. Such a morphism $a\colon X \times \underline{G} \to X$ is equivalent to the data of a group homomorphism $\rho\colon G^{\mathrm{op}} \to \mathrm{Aut}(X)$. If in addition $f\colon X \to Y$ is a morphism of spaces over k, and $\underline{G}$ acts on both X and Y, then f is called $\underline{G}$-*equivariant* if

$$\begin{array}{ccc} X \times \underline{G} & \xrightarrow{a_X} & X \\ {\scriptstyle f \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ Y \times \underline{G} & \xrightarrow[a_Y]{} & Y \end{array}$$

commutes. In terms of the corresponding homomorphisms $\rho\colon G^{\mathrm{op}} \to \mathrm{Aut}(X)$, $\sigma\colon G^{\mathrm{op}} \to \mathrm{Aut}(Y)$ this translates to the condition that

$$\begin{array}{ccc} X & \xrightarrow{\rho(g)} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\sigma(g)]{} & Y \end{array}$$

commutes for every $g \in G$. In particular, if $\underline{G}$ acts on Y trivially, then we say that the action of $\underline{G}$ on X is *equivariant with respect to the trivial action on* Y. Such an action is equivalent to the data of a group homomorphism $\rho\colon G^{\mathrm{op}} \to \mathrm{Aut}_Y(X)$, where $\mathrm{Aut}_Y(X)$ is the group of automorphisms of X which respect the morphism $f\colon X \to Y$. In this situation, using the notation of Section 1.8, the sheaf of $\mathcal{O}_Y$-modules $f_*\mathcal{O}_X$ has a (left) action of G,

$$G \to \mathrm{Aut}_k(f_*\mathcal{O}_X), \qquad g \mapsto (g^{\mathcal{O}_X}_{f^{-1}(U)} : f_*\mathcal{O}_X(U) \to f_*\mathcal{O}_X(U))_{U\subset Y},$$

which is well-defined as $g^{\mathcal{O}_X} : \mathcal{O}_X \to g^{-1}\mathcal{O}_X$, and $g(f^{-1}(U)) = f^{-1}(U)$ for any admissible open subset $U \subset Y$. Therefore we can consider the sheaf of $\mathcal{O}_Y$-modules $(f_*\mathcal{O}_X)^G$ defined by

$$(f_*\mathcal{O}_X)^G(U) = \mathcal{O}_X(f^{-1}(U))^G,$$

for any admissible open subset U of Y, which is a sheaf because $(-)^G$ preserves products and equalisers.

Definition 1.9.1. Suppose that G is a finite group, $a\colon X \times \underline{G} \to X$ is an action of $\underline{G}$ on X, and $f\colon X \to Y$ is a finite étale morphism which is equivariant with respect to the trivial action of $\underline{G}$ on Y. Then $f\colon X \to Y$ is a *Galois covering with Galois group* G if the natural map $\mathcal{O}_Y \to (f_*\mathcal{O}_X)^G$ is an isomorphism of $\mathcal{O}_Y$-modules and

$$p_X \times a\colon X \times \underline{G} \to X \times_Y X$$

is an isomorphism of spaces over k.

We will also make use of the notion of a Galois extension of commutative k-algebras, which is the affine version of Definition 1.9.1.

Definition 1.9.2 ([21, Def. 1.4])**.** Suppose that $\varphi : A \hookrightarrow B$ is an injective homomorphism of commutative k-algebras, and G is a finite subgroup of $\mathrm{Aut}_{\mathbf{Alg}_k}(B)$. Then $\varphi : A \hookrightarrow B$ is called a Galois extension with Galois group G if $A = B^G$, and

$$\beta : B \otimes_A B \to B \otimes_k \mathcal{O}(G),$$
$$\beta(x \otimes y) = \sum_{g\in G} x(g \cdot y) \otimes \delta_g,$$

is an isomorphism.

Remark 1.9.3. The requirement that G is a subgroup of $\mathrm{Aut}_{\mathbf{Alg}_k}(B)$ can be weakened to the assumption of a group homomorphism $G \to \mathrm{Aut}_{\mathbf{Alg}_k}(B)$, as any such homomorphism will automatically be injective. Indeed, if $g(b) = b$ for all $b \in B$, then the coefficient of δ_g and δ_e would be equal for any element in the image of β, and thus β would not be surjective.

Example 1.9.4. Suppose that $\varphi\colon A \to B$ is a morphism of commutative k-algebras (resp. of affinoid algebras over k), and suppose that $G \to \mathrm{Aut}_{\mathbf{Alg}_k}(B)$ is a group homomorphism. Then the action of G on $X = \mathrm{Spec}(B)$ (resp. $X = \mathrm{Sp}(B)$) is equivariant with respect to the trivial action of G on $Y = \mathrm{Spec}(A)$ (resp. $Y = \mathrm{Sp}(A)$) if and only if $\varphi(A) \subset B^G$, and $f\colon X \to Y$ is a finite étale Galois covering with Galois group G if and only if $\varphi\colon A \to B$ is injective and a Galois extension with Galois group G.

Remark 1.9.5. We note that in case (A) Definition 1.9.1 agrees with the more commonly used definition of a Galois extension, where the condition that $\mathcal{O}_Y \to (f_*\mathcal{O}_X)^G$ is an isomorphism is replaced by the condition that $f\colon X \to Y$ is faithfully flat. Indeed, if $\mathcal{O}_Y \to (f_*\mathcal{O}_X)^G$ is an isomorphism, then for any $U \in \mathcal{B}_Y$, the extension $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is Galois with Galois group G, and thus faithfully flat [33, Lem. 1.9]. Conversely, if $f\colon X \to Y$ is faithfully flat, then for any $U \in \mathcal{B}_Y$ let $A := \mathcal{O}_Y(U)$, $B := \mathcal{O}_X(f^{-1}(U))$. The map β of Definition 1.9.2 is an isomorphism, and therefore because $\beta(b \otimes 1) = \beta(1 \otimes b)$, $b \otimes 1 = 1 \otimes b$ in $B \otimes_A B$. But then because the composition $A \to B^G \to B$ is faithfully flat, the fact that $A = B^G$ follows from the following elementary lemma.

Lemma 1.9.6. *Suppose that $R \subset S \subset T$ are commutative rings with $R \hookrightarrow T$ faithfully flat and $s \otimes 1 = 1 \otimes s$ in $T \otimes_R T$ for any $s \in S$. Then $R = S$.*

Proof. Because $R \hookrightarrow T$ is faithfully flat it is sufficient to show that $T \otimes_R S/R = 0$. By the exact sequence,

$$T \otimes_R R \to T \otimes_R S \to T \otimes_R S/R \to 0,$$

it is sufficient to show that $T \otimes_R R \to T \otimes_R S$ is surjective. We can view $T \otimes_R S \subset T \otimes_R T$, because T is flat over R, and therefore for any pure tensor $t \otimes s \in T \otimes_R S$,

$$ts \otimes 1 \mapsto ts \otimes 1 = (t \otimes 1)(s \otimes 1) = (t \otimes 1)(1 \otimes s) = t \otimes s,$$

and so the map is surjective. □

We note here the following lemma for later use, which says that derivations lifted along a Galois extension commute with the Galois action.

Lemma 1.9.7. *Suppose that G is a finite group and $\varphi\colon A \hookrightarrow B$ is a Galois extension of commutative k-algebras with Galois group G. Then any $\partial \in \mathrm{Der}_k(A)$ and $g \in G$,*

$$g \circ \psi(\partial) = \psi(\partial) \circ g,$$

where $\psi : \mathrm{Der}_k(A) \to \mathrm{Der}_k(B)$ is the A-linear homomorphism of Lemma 1.4.1.

Proof. For any $g \in G$, both ψ and,

$$\psi_g : \partial \mapsto g \circ \psi(\partial) \circ g^{-1},$$

are A-linear maps $\mathrm{Der}_k(A) \to \mathrm{Der}_k(B)$. Furthermore,

$$\psi_g(\partial) \circ \varphi = \varphi \circ \partial,$$

as for any $a \in A$, the left-hand side is,

$$\begin{aligned}(\psi_g(\partial) \circ \varphi)(a) &= g(\psi(\partial)(g^{-1}(\varphi(a)))),\\ &= g(\psi(\partial)(\varphi(a))),\\ &= g(\varphi(\partial(a))),\\ &= \varphi(\partial(a)),\end{aligned}$$

which is exactly the right-hand side. Therefore, by the uniqueness of ψ, $\psi = \psi_g$. □

We will also make use of the following basic fact, for which we were unable to find a suitable reference.

Lemma 1.9.8. *Suppose that $f\colon X \to Y$ is a finite étale Galois covering with Galois group G. Suppose that Y_0 is a connected component of Y, and X_0 is a connected component of $f^{-1}(Y_0)$. Then*

$$f|_{f^{-1}(Y_0)}\colon f^{-1}(Y_0) \to Y_0$$

is a finite étale Galois covering with Galois group G, and

$$f_0 \coloneqq f|_{X_0}\colon X_0 \to Y_0$$

is a finite étale Galois covering with Galois group $H \coloneqq \mathrm{Stab}_G(X_0)$.

Proof. This is based on [33, Prop. 3.8]. It is direct to see that $f\colon f^{-1}(Y_0) \to Y_0$ is finite étale Galois with Galois group G. For the second claim, first note that the morphism $f : X \to Y$ is finite locally free of constant rank $|G|$. Indeed, for any $U \in \mathcal{B}_Y$ and $V \coloneqq f^{-1}(U)$, $\mathcal{O}_X(V)$ is finitely generated projective over $\mathcal{O}_Y(U)$ [21, Thm. 1.3 (c)] and the isomorphism β for the Galois extension $\mathcal{O}_Y(U) \to \mathcal{O}_X(V)$ shows that $\mathrm{rank}_{\mathcal{O}_Y(U)}(\mathcal{O}_X(V)) = |G|$. Therefore, because Y_0 is connected, $f^{-1}(Y_0)$ has at most $|G|$ connected components, and we let S be the corresponding set of primitive orthogonal idempotents of $\mathcal{O}_X(f^{-1}(Y_0))$. G acts on S, and this action is transitive: if $\{e^1, ..., e^m\}$ is an orbit of S, then the sum $e^1 + \cdots + e^m$ is G-invariant, hence is an non-zero idempotent of $\mathcal{O}_Y(Y_0)$. Therefore, $e^1 + \cdots + e^m = 1$, and the orbit is the whole of S.

If e is the idempotent of $\mathcal{O}_X(f^{-1}(Y_0))$ corresponding to the connected component X_0, then the stabiliser H of e in G acts on X_0. Furthermore, G acts on S transitively with stabiliser H, and thus $|S| = |G|/|H|$. The group G provides isomorphisms between all connected components of $f^{-1}(Y_0)$ and these isomorphisms respect the morphism to Y. Therefore

$$|G| = \deg(f : X \to Y) = (|G|/|H|) \cdot \deg(f_0 : X_0 \to Y_0),$$

and so $|H| = \deg(f_0 : X_0 \to Y_0)$. We will make use of this fact below.

To show that $f_0 : X_0 \to Y_0$ is Galois with Galois group H, it is sufficient to show that for any $U \in \mathcal{B}_Y$, if $V_0 \coloneqq f_0^{-1}(U) \subset X_0$, that $f|_{V_0}\colon V_0 \to U$ is Galois with Galois group H.

Set $A \coloneqq \mathcal{O}_Y(U), B_0 \coloneqq \mathcal{O}_X(V_0)$, and $B \coloneqq \mathcal{O}_X(f^{-1}(U))$. By Remark 1.9.4, we know that $A \hookrightarrow B$ is a G-Galois extension of k-algebras, and we are reduced to showing that $A \hookrightarrow B_0$ is a H-Galois extension of k-algebras. To this end, we first show that $B_0^H = A$. For $b \in B_0^H$, consider

$$s \coloneqq \sum_{g \in G/H} g(b) \in B^G = A.$$

Because the action of G on S is transitive with stabiliser H, $eg(b) = 0$ whenever $g \notin H$, thus

$$es = \sum_{g \in G/H} eg(b) = eb = b,$$

where $eb = b$ because $b \in B_0$. Therefore $b \in A$, as $es \in A$, which follows from the fact that $s \in A$ and $ea = a$ for any $a \in A \subset B_0$.

The A-module B_0 is finitely generated projective, being a direct summand of B, and thus to show that $A \hookrightarrow B_0$ is a H-Galois extension of k-algebras, by [21, Thm. 1.3] it is sufficient to show that the natural map

$$j_0 : B_0 \rtimes H \to \mathrm{End}_A(B_0)$$

from the skew group ring $B_0 \rtimes H$ to the ring of A-module endomorphisms $\mathrm{End}_A(B_0)$ is an isomorphism. In fact, because these are both finitely generated projective of rank $|H|^2$ over A (because $|H| = \deg(f_0 : X_0 \to Y_0)$), it is sufficient for us to show this is surjective.

Given any $\psi \in \mathrm{End}_A(B_0)$, then $\psi e \in \mathrm{End}_A(B)$ (where we are identify any $b \in B$ with the left multiplication map by b). Therefore, because

$$j : B \rtimes G \to \mathrm{End}_A(B)$$

is surjective, there are elements $b_g \in B$ such that

$$\sum_{g \in G} b_g g = \psi e.$$

If we multiply by e on both sides we obtain

$$\sum_{g \in H} e b_g g e = e \psi e$$

because $ege = 0$ if $g \notin H$. When we restrict these functions to B_0, as e acts as the identity on B_0,

$$\sum_{g \in H} e b_g g = \psi.$$

Therefore j_0 is surjective, and $f_0 : X_0 \to Y$ is Galois as required. □

1.10 Equivalence of Categories

In this section we specialise the geometric framework of Section 1.2 and Section 1.7 further and suppose that $\mathrm{char}(k) = 0$,

(A) $f\colon X \to Y$ is a Galois covering of smooth schemes over k with Galois group H,

(B) $f\colon X \to Y$ is a Galois covering of smooth rigid spaces over k with Galois group H.

We suppose further that G is a group which contains H as a normal subgroup, and that the actions of H on X and Y extend to actions of G for which the morphism $f\colon X \to Y$ is G-equivariant. For example, in this generality G could be infinite or simply H.

In this section, we show there is a canonically defined equivalence between $\mathbf{VectCon}^{G/H}(Y)$ and $\mathbf{VectCon}^{G}(X)$ (Proposition 1.10.1). This equivalence is completely formal, and taking $G = H$ and forgetting the action of $\mathcal{D}$ restricts to the well-known equivalence between $\mathbf{Coh}(Y)$ and $\mathbf{Coh}^{H}(X)$. We briefly describe both functors of the equivalence explicitly, as we shall later make use of these descriptions.

Inverse Image Functor

First, we show that the inverse image functor $f^*\colon \mathbf{VectCon}(Y) \to \mathbf{VectCon}(X)$ described in Section 1.7.1 extends to a functor

$$f^*\colon \mathbf{VectCon}^{G/H}(Y) \to \mathbf{VectCon}^{G}(X).$$

Suppose that $\mathcal{M}$ is a (G/H)-$\mathcal{D}_Y$-module, coherent as an $\mathcal{O}_Y$-module. Then $f^*\mathcal{M}$ is naturally a G-equivariant sheaf via

$$g^{f^*\mathcal{M}} := g^{\mathcal{O}_X} \otimes f^{-1}g^{\mathcal{M}}\colon f^*\mathcal{M} \to g^{-1}(f^*\mathcal{M}),$$

using that $g^{-1}(f^{-1}\mathcal{M}) = f^{-1}(g^{-1}\mathcal{M})$ as $f\colon X \to Y$ is G-equivariant. It is direct to check that this makes $f^*\mathcal{M}$ a G-$\mathcal{D}_X$-module.

Invariants Functor

Next, we use the direct image functor $f_*\colon \mathbf{Mod}(\mathcal{D}_X) \to \mathbf{Mod}(\mathcal{D}_Y)$ of Section 1.7.1 to define a functor

$$(-)^H\colon \mathbf{VectCon}^{G}(X) \to \mathbf{VectCon}^{G/H}(Y).$$

as follows. Suppose that $\mathcal{N} \in \mathbf{VectCon}^G(X)$. Because $f\colon X \to Y$ is equivariant with respect to the trivial action of H on Y,

$$H \to \mathrm{Aut}_k(f_*\mathcal{N}), \qquad h \mapsto f_*h^{\mathcal{N}},$$

is a well-defined group homomorphism. We set

$$\mathcal{N}^H(U) := \mathcal{N}(f^{-1}(U))^H,$$

which is a sheaf because taking H-invariants commutes with products and equalisers.

The sheaf $f_*\mathcal{N}$ is naturally a $\mathcal{O}_Y$-module via $\mathcal{O}_Y \to f_*\mathcal{O}_X$. In fact, the action of $\mathcal{O}_Y(U)$ on $\mathcal{N}(f^{-1}(U))$ preserves $\mathcal{N}(f^{-1}(U))^H$ because $\mathcal{O}_Y \to f_*\mathcal{O}_X$ has image $\mathcal{O}_X^H$ and $\mathcal{N}$ is a H-$\mathcal{O}_X$-module. From the $\mathcal{D}_Y$-module structure on $f_*\mathcal{N}$ we have

$$\Theta_Y \to \underline{\mathrm{End}}_k(f_*\mathcal{N}),$$

and in fact this action preserves the subsheaf $\mathcal{N}^H$ and induces:

$$\Theta_Y \to \underline{\mathrm{End}}_k(\mathcal{N}^H).$$

Indeed, for any $U, V \in \mathcal{B}_Y$ with $V \subset U$ and $\partial \in \Theta_Y(U)$,

$$\partial_V|_{\mathcal{N}^H(V)} : \mathcal{N}^H(V) \to \mathcal{N}(V)$$

factors as

$$\partial_V|_{\mathcal{N}^H(V)} : \mathcal{N}^H(V) \to \mathcal{N}^H(V) \to \mathcal{N}(V),$$

by Lemma 1.9.7. Using Lemma 1.5.7 this extends to give $\mathcal{N}^H$ the structure of a $\mathcal{D}_Y$-module. For $g \in G/H$, because H is normal in G, we have a well defined morphism of sheaves

$$g^{\mathcal{N}^H} := f_*g^{\mathcal{N}}\colon \mathcal{N}^H \to g^{-1}\mathcal{N}^H,$$

which gives $\mathcal{N}^H$ the structure of a G-$\mathcal{D}_Y$-module. Finally, note that $\mathcal{N}^H$ is coherent, being the kernel of the morphism of coherent sheaves

$$\bigoplus_{h\in H}(f_*h^{\mathcal{N}} - \mathrm{id})\colon f_*\mathcal{N} \to \bigoplus_{h\in H} f_*\mathcal{N}.$$

Proposition 1.10.1. *The functors,*

$$f^*\colon \mathbf{VectCon}^G(Y) \to \mathbf{VectCon}^{H\rtimes G}(X),$$
$$(-)^H\colon \mathbf{VectCon}^{H\rtimes G}(X) \to \mathbf{VectCon}^G(Y),$$

are quasi-inverse equivalences of monoidal categories.

Proof. Let $\mathcal{M} \in \mathbf{VectCon}^{G/H}(Y)$ and $\mathcal{N} \in \mathbf{VectCon}^G(X)$. We define natural transformations

$$\phi\colon f^*\mathcal{N}^H \to \mathcal{N}, \qquad \psi\colon \mathcal{M} \to (f^*\mathcal{M})^H,$$

and show that they are isomorphisms. We first consider ϕ. Let $V \in \mathcal{B}_Y$, $U := f^{-1}(V) \in \mathcal{B}_X$, and define $\phi|_U\colon f^*\mathcal{N}^H|_U \to \mathcal{N}|_U$ on global sections to be the natural multiplication map

$$\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(V)} \mathcal{N}(U)^H \to \mathcal{N}(U).$$

It is straightforward to check that this is $\mathcal{D}_X(U)$-linear using our local description of the $\mathcal{D}$-module structure on $f^*\mathcal{N}$ of Section 1.7.1, and an isomorphism because $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is a Galois extension of k-algebras [21, Thm. 1.3(d)]. For different choices of V, the morphisms $\phi|_U$ agree on their intersection, and thus glue to define a morphism of sheaves $\phi\colon f^*\mathcal{N}^H \to \mathcal{N}$. Because $\{f^{-1}(V)\}_{V\in\mathcal{B}_Y}$ is an admissible open cover of X, ϕ is $\mathcal{D}_X$-linear. Similarly, to show that ϕ is G-linear, because ϕ is a morphism of coherent sheaves it is sufficient to show that for any $g \in G$ and $V \in \mathcal{B}_Y$, $U = f^{-1}(V)$ as above, that

$$\begin{array}{ccc}
\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(V)} \mathcal{N}(U)^H & \xrightarrow{\phi_U} & \mathcal{N}(U) \\
\downarrow{\scriptstyle g_U^{f^*\mathcal{N}^H}} & & \downarrow{\scriptstyle g_U^{\mathcal{N}}} \\
\mathcal{O}_X(g(U)) \otimes_{\mathcal{O}_Y(g(V))} \mathcal{N}(g(U))^H & \xrightarrow[\phi_{g(U)}]{} & \mathcal{N}(g(U))
\end{array}$$

commutes, which follows directly from the definition of $g^{f^*\mathcal{N}^H}$.

Now let us consider $\psi\colon \mathcal{M} \to (f^*\mathcal{M})^H$. For any $V \in \mathcal{B}_Y$, setting $U := f^{-1}(V)$ we define $\psi|_V\colon \mathcal{M}|_V \to (f^*\mathcal{M})^H|_V$ on global sections to be the natural inclusion

$$\mathcal{M}(V) \to (\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(V)} \mathcal{M}(V))^H,$$

which is $\mathcal{D}_Y(V)$-linear (again using our description of the $\mathcal{D}$-module structure on $f^*\mathcal{M}$ of Section 1.7.1), and glues to a well-defined morphism $\psi\colon \mathcal{M} \to (f^*\mathcal{M})^H$. Each restriction ψ_V is an isomorphism, because the H-action on M is trivial and $\mathcal{O}_X(U)$ is projective over $\mathcal{O}_Y(V)$, and therefore ψ is an isomorphism. To show that ψ is G-equivariant, it is sufficient to show that

$$\begin{array}{ccc}
\mathcal{M}(V) & \longrightarrow & (\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(V)} \mathcal{M}(V))^H \\
\downarrow{\scriptstyle g^{\mathcal{M}}} & & \downarrow{\scriptstyle g^{(f^*\mathcal{M})^H}} \\
\mathcal{M}(g(V)) & \longrightarrow & (\mathcal{O}_X(g(U)) \otimes_{\mathcal{O}_Y(g(V))} \mathcal{M}(g(V)))^H
\end{array}$$

which similarly follows directly from the definition of the morphism $g^{(f^*\mathcal{M})^H}$. Finally, given $\mathcal{V}, \mathcal{W} \in \mathbf{VectCon}^G(Y)$, the canonical $\mathcal{O}_X$-linear isomorphism

$$f^*\mathcal{V} \otimes f^*\mathcal{W} \xrightarrow{\sim} f^*(\mathcal{V} \otimes \mathcal{W})$$

can be checked to be G-$\mathcal{D}_X$-linear, using the explicit description of the $\mathcal{D}_X$-module structure on the inverse image, and therefore the equivalences are equivalences of monoidal categories. □

Remark 1.10.2. If X is a scheme, then the same proof shows that the same equivalences hold when one more generally considers equivariant $\mathcal{D}$-modules for which the underlying $\mathcal{O}$-module is quasi-coherent.

1.11 The Drinfeld Tower

In this section, suppose that K is a complete non-archimedean field which contains $L := \breve{F}$, the completion of the maximal unramified extension of F. Let $n \geq 1$, and write Ω for the $(n-1)$-dimensional Drinfeld symmetric space over K. Let D be the division algebra over F of invariant $1/n$ with ring of integers $\mathcal{O}_D$, let Π denote a uniformiser of $\mathcal{O}_D$, and write $\mathrm{Nrd}\colon D^\times \to F^\times$ for the reduced norm of D. The Drinfeld tower is a system of rigid analytic spaces over K,

$$\mathcal{M} \leftarrow \mathcal{M}_1 \leftarrow \mathcal{M}_2 \leftarrow \cdots,$$

for which each space has an action of $\mathrm{GL}_n(F) \times D^\times$ such that the transition morphisms are equivariant. Background material on these spaces is contained in [17, 28, 52]. The connected components of the space $\mathcal{M}$ are canonically identified with $\mathbb{Z}$, and under this identification $(g, \delta) \in \mathrm{GL}_n(F) \times D^\times$ acts on this set of connected components by addition by $\nu(\det(g)\,\mathrm{Nrd}(\delta^{-1}))$. In particular, the connected components of $\mathcal{M}$ are permuted simply transitively by the action of Π. Let $\mathcal{N}$ be a connected component of $\mathcal{M}$. The Grothendieck-Messing period morphism

$$\pi_{\mathrm{GM}}\colon \mathcal{M} \to \Omega$$

is an étale $\mathrm{GL}_n(F) \times D^\times$-equivariant morphism, where Ω is considered with the trivial action of $D^\times$, which induces a $\mathrm{GL}_n(F)$-equivariant isomorphism

$$\pi_{\mathrm{GM}}\colon \mathcal{M}/H \xrightarrow{\sim} \Omega,$$

where $H := D^\times/\mathcal{O}_D^\times$. In particular, the composition

$$\mathcal{N} \hookrightarrow \mathcal{M} \xrightarrow{p} \mathcal{M}/H \to \Omega$$

is a G^0-equivariant isomorphism, where

$$G^0 := \{g \in \mathrm{GL}_n(F) \mid \nu(\det(g)) = 0\},$$

which we will use to identify $\mathcal{N}$ with Ω.

Considering the preimage $\mathcal{N}_m$ of $\mathcal{N}$ in each covering space $(\mathcal{M}_n)_{n\geq 1}$, we obtain a sub-tower,

$$\mathcal{N} \leftarrow \mathcal{N}_1 \leftarrow \mathcal{N}_2 \leftarrow \cdots .$$

Because $\mathcal{N}$ is stable under the action of $G^0 \times \mathcal{O}_D^\times$, for each $n \geq 1$, $\mathcal{N}_m \subset \mathcal{M}_m$ is $G^0 \times \mathcal{O}_D^\times$-stable. The subgroup $1 + \Pi^m \mathcal{O}_D \leq \mathcal{O}_D^\times$ acts trivially on $\mathcal{M}_m$, and the morphisms $\mathcal{M}_m \to \mathcal{M}$, $\mathcal{N}_m \to \mathcal{N}$ are Galois with Galois group $\mathcal{O}_D^\times/(1 + \Pi^m \mathcal{O}_D)$ [44, Thm. 2.2].

Each of the spaces $(\mathcal{N}_m)_{m\geq 1}$ is connected over L [44, Thm. 2.5], but not geometrically connected. The following result is due to Boutot and Zink, and describes the connected components of $(\mathcal{N}_m)_{m\geq 1}$ and $(\mathcal{M}_m)_{m\geq 1}$ over $\mathbb{C}_p$. For the cofinal system of spaces $(\mathcal{N}_{nm})_{m\geq 1}$, we give an independent proof of this result in Chapter 4 (Theorem 4.5.11).

Proposition 1.11.1. *There is a family of bijections*

$$\begin{array}{ccc} \pi_0\left(\mathcal{N}_{m,\mathbb{C}_p}\right) & \xrightarrow{\sim} & \frac{\mathcal{O}_F^\times}{1+\pi^{\lceil \frac{m}{n} \rceil}\mathcal{O}_F} \\ \big\downarrow & & \big\downarrow \\ \pi_0\left(\mathcal{M}_{m,\mathbb{C}_p}\right) & \xrightarrow{\sim} & \frac{F^\times}{1+\pi^{\lceil \frac{m}{n} \rceil}\mathcal{O}_F} \end{array}$$

for any $m \geq 1$, compatible with the natural restriction maps on both sides for different m. These are $\mathrm{GL}_n(F) \times D^\times$-equivariant (resp. $G^0 \times \mathcal{O}_D^\times$-equivariant for $\mathcal{N}_m$) where $(g, x) \in \mathrm{GL}_n(F) \times D^\times$ acts on the right by multiplication by $\det(g)\,\mathrm{Nrd}(x)^{-1} \in F^\times$.

Proof. This is [18, Thm. 0.20], noting that $\mathrm{Nrd}(1 + \Pi^m \mathcal{O}_D) = 1 + \pi^{\lceil \frac{m}{n} \rceil}\mathcal{O}_F$ [53, Lem. 5]. □

Chapter 2

Line Bundles on The First Drinfeld Covering

In this chapter we prove Theorem E. Throughout, F denotes a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$, uniformiser π, and residue field $\mathbb{F}_q$. K is a complete field extension of F, L is the completion of the maximal unramified extension of F, and $\mathbb{C}_p$ is the completion of $\overline{F}$. We let $n \geq 2$, and write D for the division algebra over F of invariant $1/n$, and (in this chapter only) set $G = \mathrm{SL}_n(\mathcal{O}_F)$.

2.1 Abelian Galois Coverings

In this section, we describe how the pushforward of the structure sheaf of an abelian Galois covering decomposes into line bundles. The approach we take here is influenced by the work of Borevič for Kummer extensions of rings [13].

In this section, we suppose that Γ is a finite abelian group, $f \colon X \to Y$ is a Galois covering with Galois group Γ (cf. Section 1.9), and K contains a primitive $e(\Gamma)$th root of 1, where $e(\Gamma)$ is the exponent of Γ. For each $\chi \in \widehat{\Gamma}$ we write e_χ for the corresponding central primitive idempotent

$$e_\chi = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(\gamma^{-1})\gamma \in K[\Gamma].$$

Definition 2.1.1. For any $\chi \in \widehat{\Gamma}$, we define the $\mathcal{O}_Y$-module

$$\mathcal{L}_\chi \coloneqq e_\chi \cdot f_* \mathcal{O}_X.$$

Proposition 2.1.2. *There is a direct sum decomposition of $\mathcal{O}_Y$-modules*

$$f_* \mathcal{O}_X = \bigoplus_{\chi \in \widehat{\Gamma}} \mathcal{L}_\chi,$$

and multiplication in $f_* \mathcal{O}_X$ induces an isomorphism

$$\mathcal{L}_\chi \otimes_{\mathcal{O}_Y} \mathcal{L}_\psi \xrightarrow{\sim} \mathcal{L}_{\chi\psi}.$$

In particular, each $\mathcal{L}_\chi$ is an $e(\Gamma)$-torsion invertible $\mathcal{O}_Y$-module, and the association

$$\widehat{\Gamma} \to \mathrm{Pic}(Y)[e(\Gamma)], \qquad \chi \mapsto \mathcal{L}_\chi,$$

is a group homomorphism.

Proof. The sheaf $f_*\mathcal{O}_X$ is an $\mathcal{O}_Y[\Gamma]$-module, and the direct sum decomposition of $f_*\mathcal{O}_X$ follows from the fact that the e_χ are central orthogonal idempotents which sum to 1. Suppose now that U is an affinoid open subset of Y, and let

$$V := f^{-1}(U) = U \times_Y X \hookrightarrow X.$$

Then $\underline{\Gamma}$ acts on V, $f\colon V \to U$ is equivariant with respect to the trivial action of $\underline{\Gamma}$ on U, and we have a commutative diagram of isomorphisms,

$$\begin{array}{ccc} U \times_Y (X \times \underline{\Gamma}) & \longrightarrow & U \times_Y (X \times_Y X) \\ \downarrow & & \downarrow \\ V \times \underline{\Gamma} & \longrightarrow & (U \times_Y X) \times_U (U \times_Y X) \end{array}$$

Write $A := \mathcal{O}(U)$ and $B := \mathcal{O}(V)$. Since $f\colon X \to Y$ is finite, V is affinoid, and because $\mathcal{O}_Y \to (f_*\mathcal{O}_X)^\Gamma$ is an isomorphism, $A \to B$ is injective and has image B^Γ. Furthermore, because B is finitely generated over A, the natural inclusion

$$B \otimes_A B \to B \widehat{\otimes}_A B,$$

is an isomorphism [14, 3.7.3(6)]. Therefore, the composition of this inclusion with the global sections of $p_X \times a$ induces an isomorphism,

$$B \otimes_A B \xrightarrow{\sim} B \otimes_K \mathcal{O}(\Gamma), \qquad x \otimes y \mapsto \sum_{\gamma \in \Gamma} x(\gamma \cdot y) \otimes \delta_\gamma.$$

Now B is a right $\mathcal{O}(\Gamma)$-comodule algebra for the Hopf algebra $\mathcal{O}(\Gamma)$ via

$$\rho\colon B \to B \otimes_K \mathcal{O}(\Gamma), \qquad \rho\colon b \mapsto \sum_{\gamma \in \Gamma} \gamma(b) \otimes \delta_\gamma,$$

and the above isomorphism says exactly that $A \to B$ is a $\mathcal{O}(\Gamma)$-Galois extension in the sense of [49, Def. 8.1.1]. Because K contains a primitive $e(\Gamma)$th root of 1, the natural map $K[\widehat{\Gamma}] \to \mathcal{O}(\Gamma)$ is an isomorphism of Hopf algebras over K. Therefore, using this identification we can view B as an $K[\widehat{\Gamma}]$-comodule algebra. We have that for $b \in B$,

$$\rho(b) = \sum_{\gamma \in \Gamma} \gamma(b) \otimes \delta_\gamma = \sum_{\chi \in \widehat{\Gamma}} b_\chi \otimes \chi, \tag{2.1}$$

for some unique $b_\chi \in B$, and for each $\chi \in \widehat{\Gamma}$ we define

$$B_\chi := \{b_\chi \mid b \in B\}.$$

Because $A \to B$ is $K[\widehat{\Gamma}]$-Galois, these B_χ make B a strongly graded $\widehat{\Gamma}$-algebra by a result of Ulbrich [49, Thm. 8.1.7], meaning that

$$B = \bigoplus_{\chi \in \widehat{\Gamma}} B_\chi, \qquad \text{and} \qquad B_\chi \cdot B_\psi = B_{\chi\psi} \qquad \text{for all} \qquad \chi, \psi \in \widehat{\Gamma}.$$

In fact, $e_\chi \cdot B = B_\chi$. Indeed, by column orthogonality

$$\delta_\gamma = \frac{1}{|\Gamma|} \sum_{\chi \in \widehat{\Gamma}} \chi(\gamma^{-1})\chi,$$

and therefore substituting this into equation (2.1) and comparing the coefficient of χ shows that,

$$e_\chi \cdot b = b_\chi.$$

There is a natural surjective morphism of A-modules,

$$m_{\chi,\psi} \colon B_\chi \otimes_A B_\psi \to B_\chi \cdot B_\psi = B_{\chi\psi}.$$

In order to show that this is injective, first note that each B_χ is direct summand of B, and thus is finitely generated projective as an A-module. From decomposition of B as the direct sum of B_χ,

$$\sum_{\chi \in \widehat{\Gamma}} \operatorname{rank}_A(B_\chi) = \operatorname{rank}_A(B) = |\Gamma|. \tag{2.2}$$

On the other hand, we have a surjection

$$B_\chi \otimes_A B_{\chi^{-1}} \to B_\chi \cdot B_{\chi^{-1}} = B_1 = e_1 \cdot B = A,$$

and therefore $\operatorname{rank}_A(B_\chi) \operatorname{rank}_A(B_{\chi^{-1}}) \geq 1$, hence $\operatorname{rank}_A(B_\chi) \geq 1$, and from equation (2.2) above, $\operatorname{rank}_A(B_\chi) = 1$. As a consequence, $m_{\chi,\psi}$ is a surjective homomorphism between finitely generated rank 1 A-modules, and as such it is injective.

Now, returning to the global situation, multiplication induces a morphism of $\mathcal{O}_Y$-modules,

$$\mathcal{L}_\chi \otimes_{\mathcal{O}_Y} \mathcal{L}_\psi \to f_* \mathcal{O}_Y. \tag{2.3}$$

Because $f_*\mathcal{O}_Y$ is coherent, then locally over an affinoid open subset U as above this is identified with the morphism of sheaves associated under the associated sheaf construction to the A-module homomorphism

$$e_\chi \cdot B \otimes_A e_\psi \cdot B \to B.$$

We have shown above that this has image $e_{\chi\psi} \cdot B$, and therefore the morphism of $\mathcal{O}_Y$-modules (2.3) above induces an isomorphism,

$$\mathcal{L}_\chi \otimes_{\mathcal{O}_Y} \mathcal{L}_\psi \xrightarrow{\sim} \mathcal{L}_{\chi\psi}.$$

□

Remark 2.1.3. In fact one can show that if Y is connected and Γ_0 is the stabiliser of any connected component X_0 of X, then $f \colon X_0 \to Y$ is a Galois extension with Galois group Γ_0, and the homomorphism

$$\widehat{\Gamma} \to \operatorname{Pic}(Y)[e(\Gamma)]$$

factors as the composition

$$\widehat{\Gamma} \twoheadrightarrow \widehat{\Gamma_0} \to \operatorname{Pic}(Y)[e(\Gamma_0)] \hookrightarrow \operatorname{Pic}(Y)[e(\Gamma)].$$

2.2 Invariant Mod-p Global Units of Drinfeld Symmetric Spaces

In this section, we want to give a description of the G-invariant mod-p global units of Ω, the $(n-1)$-dimensional Drinfeld symmetric space for F (cf. Section 1.11), which we view as a rigid space over K. We will make use of this description in the next section. Recall that if R is a commutative ring and $d \geq 1$, $\mathbb{P}^d(R)$ is the set of tuples $(r_0, ..., r_d) \in R^{d+1}$ such that $R = Rr_0 + \cdots + Rr_d$, up to the scaling action $u \cdot (r_0, ..., r_d) = (ur_0, ..., ur_d)$ of $R^\times$.

Definition 2.2.1. For each $m \geq 1$, let $\mathcal{H}_m := \mathbb{P}^{(n-1)}(\mathcal{O}_F/\pi^m\mathcal{O}_F)$.

Lemma 2.2.2. *For all $m \geq 1$, the action of G on $\mathcal{H}_m$ is transitive.*

Proof. For notational simplicity, set $R := \mathcal{O}_F/\pi^m\mathcal{O}_F$. The natural map $G \to \mathrm{SL}_n(R)$ is surjective because $\mathcal{O}_F$ and R are local rings so both groups are generated by elementary matrices [35, Thm. 4.3.9]. The action of $\mathrm{GL}_n(R)$ on $\mathcal{H}_m$ is transitive because any element $\mathbf{r} = (r_0, ..., r_{n-1})$ with $[\mathbf{r}] \in \mathcal{H}_m$ can be extended to a basis of R^n, which can be seen by reducing mod-π. Then the action of $\mathrm{SL}_n(R)$ on $\mathcal{H}_m$ is transitive, as the stabiliser subgroup of the element $x = [(1\colon 0\colon \cdots : 0)]$,

$$\mathrm{Stab}_{\mathrm{SL}_n(R)}(x) \leq \mathrm{Stab}_{\mathrm{GL}_n(R)}(x)$$

is of index $|R^\times|$, the same as the index of $\mathrm{SL}_n(R)$ in $\mathrm{GL}_n(R)$. $\square$

Definition 2.2.3. For an abelian group A and $m \geq 1$, we write $A[\mathcal{H}_m]$ for the abelian group

$$A[\mathcal{H}_m] := \{f\colon \mathcal{H}_m \to A\},$$

of all functions from $\mathcal{H}_m$ to A, and

$$A[\mathcal{H}_m]^0 := \left\{ f\colon \mathcal{H}_m \to A \;\middle|\; \sum_{x \in \mathcal{H}_m} f(x) = 0 \right\} \subset A[\mathcal{H}].$$

For any $m \geq 1$, there is a natural map

$$\rho_m\colon \mathcal{H}_{m+1} \to \mathcal{H}_m,$$

which induces

$$\rho_{m,*}\colon A[\mathcal{H}_{m+1}] \to A[\mathcal{H}_m],$$

defined by

$$\rho_{m,*}(f)(x) = \sum_{y \in \rho_m^{-1}(x)} f(y),$$

for all $x \in \mathcal{H}_m$. This restricts to $\rho_{m,*}\colon A[\mathcal{H}_{m+1}]^0 \to A[\mathcal{H}_m]^0$.

Definition 2.2.4. We set

$$A[[\mathcal{H}]]^0 := \varprojlim_{m \geq 1} A[\mathcal{H}_m]^0.$$

Because each $A[\mathcal{H}_m]^0$ is a $\mathbb{Z}[G]$-module in compatible way, so is $A[[\mathcal{H}]]^0$. Taking $A = \mathbb{Z}$, we have the following description of the global units of Ω due to Junger.

Proposition 2.2.5 ([42, Thm. 4.5(2)])**.** *There is an isomorphism of* $\mathbb{Z}[G]$*-modules,*

$$\mathcal{O}(\Omega)^\times / K^\times \xrightarrow{\sim} \mathbb{Z}[[\mathcal{H}]]^0.$$

For any $m \geq 1$,

$$|\mathcal{H}_m| = q^{(m-1)d}(q^n - 1)/(q-1),$$

and the restriction map

$$\rho_m \colon \mathcal{H}_{m+1} \to \mathcal{H}_m,$$

is surjective with each fibre of size q^{n-1}.

In the proof of the next lemma, we will make use of the following element.

Definition 2.2.6. For each $m \geq 1$, let $\Theta_m \in \mathbb{Z}/p\mathbb{Z}\,[\mathcal{H}_m]$ be defined by,

$$\Theta_m(x) = 1,$$

for all $x \in \mathcal{H}_m$.

Lemma 2.2.7. $(\mathbb{Z}/p\mathbb{Z}\,[[\mathcal{H}]]^0)^G = 0$.

Proof. For any $m \geq 1$, we have projection maps,

$$\phi_m \colon (\mathbb{Z}/p\mathbb{Z}\,[[\mathcal{H}]]^0)^G \to (\mathbb{Z}/p\mathbb{Z}\,[\mathcal{H}_m]^0)^G.$$

Suppose that we have some G-invariant function, $f \in (\mathbb{Z}/p\mathbb{Z}\,[[\mathcal{H}]]^0)^G$. Then for any $m \geq 1$, because $\mathcal{H}_{m+1}$ is a finite set with a transitive action of G (by Lemma 2.2.2),

$$\phi_{m+1}(f) = \lambda\Theta_{m+1},$$

for some $\lambda \in \mathbb{Z}/p\mathbb{Z}$. Now,

$$\rho_m \colon \mathcal{H}_{m+1} \to \mathcal{H}_m,$$

is surjective with each fibre of size q^{n-1}, hence $\phi_m(f) = q^{n-1}\lambda\Theta_m = 0$, as $p \mid q$ and $n \geq 2$. Therefore, $\phi_m(f) = 0$ for all $m \geq 1$, and hence $f = 0$. □

We can now use Lemma 2.2.7 to prove the main technical result of this section.

Corollary 2.2.8. *The inclusion* $K^\times \to \mathcal{O}(\Omega)^\times$ *induces an isomorphism,*

$$K^\times / K^{\times p} \xrightarrow{\sim} \left(\mathcal{O}(\Omega)^\times / \mathcal{O}(\Omega)^{\times p}\right)^G.$$

Proof. We have a short exact sequence of $\mathbb{Z}[G]$-modules,

$$0 \to K^\times \to \mathcal{O}(\Omega)^\times \to \mathbb{Z}[[\mathcal{H}]]^0 \to 0,$$

and applying $- \otimes \mathbb{Z}/p\mathbb{Z}$, we obtain an exact sequence of abelian groups,

$$\mathbb{Z}[[\mathcal{H}]]^0[p] \to K^\times / K^{\times p} \to \mathcal{O}(\Omega)^\times / \mathcal{O}(\Omega)^{\times p} \to \frac{\mathbb{Z}[[\mathcal{H}]]^0}{p\mathbb{Z}[[\mathcal{H}]]^0} \to 0. \tag{2.4}$$

Because p-torsion commutes with taking the inverse limit,

$$\mathbb{Z}[[\mathcal{H}]]^0[p] = \varprojlim_{m\geq 1} \mathbb{Z}[\mathcal{H}_m]^0[p] = 0.$$

Furthermore, we have an exact sequence of inverse systems

$$0 \to (\mathbb{Z}[\mathcal{H}_m]^0)_{m\geq 1} \xrightarrow{\times p} (\mathbb{Z}[\mathcal{H}_m]^0)_{m\geq 1} \to (\mathbb{Z}/p\mathbb{Z}\,[\mathcal{H}_m]^0)_{m\geq 1} \to 0,$$

and,

$$\varprojlim_{m\geq 1}{}^1 \mathbb{Z}[\mathcal{H}_m]^0 = 0,$$

because each transition map is surjective, thus the natural map

$$\frac{\mathbb{Z}[[\mathcal{H}]]^0}{p\mathbb{Z}[[\mathcal{H}]]^0} \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}\,[[\mathcal{H}]]^0,$$

is an isomorphism. Therefore, taking the G-invariants of the exact sequence (2.4) above,

$$0 \to K^\times/K^{\times p} \to \left(\mathcal{O}(\Omega)^\times/\mathcal{O}(\Omega)^{\times p}\right)^G \to (\mathbb{Z}/p\mathbb{Z}\,[[\mathcal{H}]]^0)^G.$$

Then the conclusion follows by Lemma 2.2.7. □

2.3 Line Bundles on the First Drinfeld Covering

Recall that we write L for the completion of the maximal unramified extension of F, and that K is a complete field extension of F. Let $\varpi \in \overline{F}$ be a primitive $(q-1)$st root of $-\pi$. In this section we assume that K contains $L(\varpi)$. The extension $L(\varpi)$ is the first Lubin-Tate extension of L, and as such is independent of the choice of π [46, Thm. 3].

We are interested in the space $\mathcal{N}_1$, which admits the following explicit description due to Junger.

Definition 2.3.1. If X is a rigid space over K, then for any $k \geq 1$, the *Kummer map*,

$$\kappa\colon \mathcal{O}(X)^\times \to \mathrm{H}^1_{\text{ét}}(X, \mu_k)$$

sends $u \in \mathcal{O}(X)^\times$ to

$$X(u^{\frac{1}{k}}) := \underline{\mathrm{Sp}}_X(\mathcal{O}_X[z]/z^k - u).$$

Let $N := q^n - 1$ and $N' := N/(q-1)$. In [42, Thm. 4.9] it is shown that

$$\mathcal{N}_1 \cong \Omega\left(\left(\pi u^{q-1}\right)^{\frac{1}{N}}\right),$$

for some particular $u \in \mathcal{O}(\Omega)^\times$. Note that because L contains all coprime to p roots of 1, $L(\varpi)$ contains a primitive $(q-1)$st root τ of π. Therefore, as K contains $L(\varpi)$,

$$\mathcal{N}_1 \cong \Omega\left(\left((\tau u)^{q-1}\right)^{\frac{1}{N}}\right) \cong \bigsqcup_{\zeta^{q-1}=1} \Sigma^1_\zeta,$$

where

$$\Sigma^1_\zeta := \Omega\left((\zeta\tau u)^{\frac{1}{N'}}\right).$$

Definition 2.3.2. We let $\Sigma^1 := \Sigma_1^1$, and let Σ^2 be the preimage of Σ^1 in $\mathcal{N}_2$.

We recall that a rigid space X over a non-archimedean field k is called *geometrically connected* if for any finite extension k' of k, $X \times_k k'$ is connected.

Corollary 2.3.3. *Σ^1 and Σ^2 are geometrically connected.*

Proof. Let $r \in \{1, 2\}$, and first suppose that $K = L(\varpi)$. The base change $\Sigma^r_{1,\mathbb{C}_p}$ is connected by Proposition 1.11.1, noting that $\lceil \frac{r}{n} \rceil = 1$ because $n \geq 2$. In particular, for any finite extension k of K, $\Sigma_1^r \times_K k$ is connected, and thus Σ_1^r is geometrically connected. Now, to extend from $K = L(\varpi)$ to a general K, Σ_1^r is quasi-Stein and hence quasi-separated by [14, Prop. 9.6.1(7)], and therefore by the discussion after the proof of [23, Thm. 3.2.1], the base change $\Sigma^r = \Sigma_1^r \times_{L(\varpi)} K$ is also geometrically connected. □

Remark 2.3.4. The proof of Corollary 2.3.3 shows that Σ^1_ζ are the geometrically connected components of $\mathcal{N}_1$. We note that these components are all isomorphic, as

$$\mathrm{Nrd}\colon \mathcal{O}_D/(1 + \Pi\mathcal{O}_D) \to \mathcal{O}_F/(1 + \pi\mathcal{O}_F)$$

is surjective so the Galois group of $\mathcal{N}_1 \to \Omega$ acts transitively on these components.

Recall (Section 1.11) that the extension

$$\mathcal{M}_2 \to \mathcal{M}_1$$

is Galois with Galois group

$$H := (1 + \Pi\mathcal{O}_D)/(1 + \Pi^2\mathcal{O}_D).$$

The extension $\Sigma^2 \to \Sigma^1$ is the restriction of this Galois covering to the open subset Σ^1 of $\mathcal{M}_1$, and therefore is also Galois with Galois group H (Lemma 1.9.8). From Proposition 1.11.1 we note that G^0 acts through the determinant on the geometrically connected components of the tower and thus G stabilises both Σ^1 and Σ^2. Furthermore, the action of G on both Σ^1 and Σ^2 commutes with the Galois action.

Proposition 2.3.5. *The inclusion $K^\times \to \mathcal{O}(\Omega)^\times$ induces an isomorphism,*

$$K^\times/K^{\times p} \xrightarrow{\sim} \left(\mathcal{O}(\Sigma^1)^\times/\mathcal{O}(\Sigma^1)^{\times p}\right)^G.$$

Proof. Let σ be a primitive Nth root of π. Then by [42, Thm. 5.1], there is a short exact sequence of abelian groups

$$0 \to \mathcal{O}(\Omega_{K(\sigma)})^\times \to \mathcal{O}(\Sigma^1_{K(\sigma)})^\times \to \mathbb{Z}/(q+1)\mathbb{Z} \to 0.$$

Taking $\mathrm{Gal}(K(\sigma)/K(\varpi))$-invariants and applying $- \otimes \mathbb{Z}/p\mathbb{Z}$, we are left with an isomorphism

$$\mathcal{O}(\Omega)^\times/\mathcal{O}(\Omega)^{\times p} \xrightarrow{\sim} \mathcal{O}(\Sigma^1)^\times/\mathcal{O}(\Sigma^1)^{\times p}.$$

The result then follows from Corollary 2.2.8. □

We now want to prove Theorem E and show that the homomorphism

$$\widehat{H} \to \mathrm{Pic}(\Sigma^1)[p]$$

associated to the Galois covering $f\colon \Sigma^2 \to \Sigma^1$ is injective. In order to prove this, we will make use of the following explicit description of the Kummer exact sequence.

Recall that if X is a rigid space over K, the for any $d \geq 1$ the Kummer exact sequence is the short exact sequence

$$0 \to \mathcal{O}(X)^\times/\mathcal{O}(X)^{\times d} \to \mathrm{H}^1_{\text{ét}}(X, \mu_d) \to \mathrm{Pic}(X)[d] \to 0,$$

arising from the long exact sequence of the functor $\Gamma(X_{\text{ét}}, -)$ applied to the sequence

$$0 \to \mu_d \to \mathbb{G}_m \xrightarrow{\times d} \mathbb{G}_m \to 0,$$

of sheaves of $X_{\text{ét}}$, which is exact because d is invertible in K [24, §3.2]. There is a more explicit description of this sequence, which we summarise now. References in the case of schemes are [57, Tag 03PK], [48, §III.4], from which the case for rigid spaces can be deduced mutatis mutandis.

Let $\{(\mathcal{L}, \alpha)\}/\cong$ be the set of pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L} \in \mathrm{Pic}(X)$ and $\alpha\colon \mathcal{L}^{\otimes d} \xrightarrow{\sim} \mathcal{O}_X$ is an $\mathcal{O}_X$-linear isomorphism, considered up to the natural notion of isomorphism. The set $\{(\mathcal{L}, \alpha)\}/\cong$ forms an abelian group and there is an isomorphism of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(X)^\times/\mathcal{O}(X)^{\times d} & \longrightarrow & \{(\mathcal{L}, \alpha)\}/\cong & \longrightarrow & \mathrm{Pic}(X)[d] & \longrightarrow & 0 \\ & & \Big\downarrow{=} & & \Big\downarrow{\sim} & & \Big\downarrow{=} & & \\ 0 & \longrightarrow & \mathcal{O}(X)^\times/\mathcal{O}(X)^{\times d} & \longrightarrow & \mathrm{H}^1_{\text{ét}}(X, \mu_d) & \longrightarrow & \mathrm{Pic}(X)[d] & \longrightarrow & 0 \end{array}$$

The homomorphism $\{(\mathcal{L}, \alpha)\}/\cong \to \mathrm{Pic}(X)[d]$ is simply $[(\mathcal{L}, \alpha)] \mapsto [\mathcal{L}]$. Given a pair $[(\mathcal{L}, \alpha)]$, then the associated μ_d-torsor in $\mathrm{H}^1_{\text{ét}}(X, \mu_d)$ is $Z := \underline{\mathrm{Sp}}(\mathcal{A})$, where $\mathcal{A}$ is the coherent sheaf of $\mathcal{O}_X$-algebras

$$\mathcal{A} = \bigoplus_{i=0}^{d-1} \mathcal{L}^{\otimes i},$$

with multiplication the natural maps

$$\begin{array}{ll} \mathcal{L}^{\otimes i} \otimes \mathcal{L}^{\otimes j} \to \mathcal{L}^{i+j} & \text{if } i + j \leq d - 1, \\ \mathcal{L}^{\otimes i} \otimes \mathcal{L}^{\otimes j} \to \mathcal{L}^{i+j} \xrightarrow{\alpha} \mathcal{L}^{i+j-d} & \text{if } i + j \geq d, \end{array}$$

for $0 \leq i, j \leq d$. In order to describe the structure of Z as a μ_d-torsor, we first consider this construction locally.

Suppose that $\mathcal{L} = \mathcal{O}_X$. In this case the isomorphism α has the form $\alpha\colon \mathcal{O}_X^{\otimes d} \to \mathcal{O}_X$, and we can use the canonical isomorphism $\psi\colon \mathcal{O}_X \to \mathcal{O}_X^{\otimes d}$ to define $a := \alpha(\psi(1)) \in \mathcal{O}_X(X)^\times$. Then under the construction above,

$$Z = \underline{\mathrm{Sp}}(\mathcal{O}_X[z]/(z^d - a)).$$

For any rigid space Y over X, $Z(Y) = \{s \in \mathcal{O}_Y(Y) \mid s^d = a\}$, which has the structure of a μ_d-torsor via

$$\mu_d(Y) \times Z(Y) \to Z(Y), \qquad (\zeta, s) \mapsto \zeta s.$$

Now for a general pair $[(\mathcal{L}, \alpha)]$, the associated space Z is locally in the rigid topology of the above form, and these structures patch to give Z the structure of a μ_d-torsor.

Suppose now that K contains a primitive dth root of 1. In this case the group scheme μ_d is naturally identified with the constant group scheme $\underline{\mu_d(K)}$, and under this identification there is a correspondence between μ_d-torsors and Galois coverings $Z \to X$ with Galois group $\underline{\mu_d(K)}$ (to use the language of Section 2.1).

We are interested in the homomorphism $\mathrm{H}^1_{\text{ét}}(X, \mu_d) \to \mathrm{Pic}(X)[d]$. From the description of the μ_d-action above, we see that if a Galois covering $f\colon Z \to X$ corresponds to the pair $[(\mathcal{L}, \alpha)]$, we can recover $\mathcal{L}$ as the line bundle

$$\mathcal{L} \cong e_\iota \cdot f_* \mathcal{O}_Z,$$

where ι is the natural inclusion $\iota\colon \mu_d(K) \to K^\times$. More generally, $f\colon Z \to X$ is a Galois covering with Galois group Γ, and $\chi\colon \Gamma \xrightarrow{\sim} \mu_d(K)$ is an isomorphism, then in the induced exact sequence

$$0 \to \mathcal{O}(X)^\times / \mathcal{O}(X)^{\times d} \to \mathrm{H}^1_{\text{ét}}(X, \underline{\Gamma}) \to \mathrm{Pic}(X)[d] \to 0,$$

the image of the Galois covering $f\colon Z \to X$ in $\mathrm{Pic}(X)[d]$ is the line bundle $e_\chi \cdot f_* \mathcal{O}_Z$.

Theorem 2.3.6. *Suppose that K contains $L(\varpi)$ and a primitive pth root of 1. Then the homomorphism*

$$\widehat{H} \to \mathrm{Pic}(\Sigma^1)[p]^G, \qquad \chi \mapsto \mathcal{L}_\chi = e_\chi \cdot f_* \mathcal{O}_{\Sigma^2},$$

is injective.

Remark 2.3.7. The assumption that K contains $L(\varpi)$ is simply to ensure the space Σ^1 is defined, and the assumption that K contains a primitive pth root of 1 is similarly to ensure that the homomorphism is defined. Furthermore, when F is unramified the assumption that K contains a primitive pth root of 1 in the statement of Theorem 2.3.6 is superfluous. Indeed, K contains $L(\varpi)$, and the Lubin-Tate extensions $\mathbb{Q}_p(\zeta_p)$ and $\mathbb{Q}_p((-p)^{1/(p-1)})$ of $\mathbb{Q}_p$ are equal.

Proof. Let $\chi\colon H \to K^\times$ be non-trivial. We want to show that $e_\chi \cdot f_* \mathcal{O}_{\Sigma^2} \in \mathrm{Pic}(\Sigma^1)$ is non-trivial. Because H has exponent p and χ is non-trivial, χ induces an isomorphism

$$\chi'\colon H/H_\chi \xrightarrow{\sim} \mu_p(K),$$

where H_χ is the kernel of χ. From H_χ we may form the quotient

$$f'\colon \Sigma^2/H_\chi \to \Sigma^1.$$

If $U \subset \Sigma^1$ is an admissible open subset, and $V = f^{-1}(U) \subset \Sigma^2$, then above U the quotient Σ^2/H_χ is described by $\mathrm{Sp}(\mathcal{O}(V)^{H_\chi})$. Because H_χ is normal, $f'\colon \Sigma^2/H_\chi \to \Sigma^1$ is Galois with Galois group H/H_χ, which follows from [21, Thm. 2.2] and the fact that each property in the definition of a Galois extension can be checked affinoid locally.

We first note that we have an equality of $\mathcal{O}_{\Sigma^1}$-modules,

$$e_\chi \cdot f_* \mathcal{O}_{\Sigma^2} = e_{\chi'} \cdot f'_* \mathcal{O}_{\Sigma^2/H_\chi}.$$

Indeed, for any admissible open subset U of Σ^1,

$$(e_{\chi'} \cdot f'_* \mathcal{O}_{\Sigma^2/H_\chi})(U) = e_{\chi'} \cdot \mathcal{O}_{\Sigma^2}(f^{-1}(U))^{H_\chi},$$

and

$$(e_\chi \cdot f_* \mathcal{O}_{\Sigma^2})(U) = e_\chi \cdot \mathcal{O}_{\Sigma^2}(f^{-1}(U)).$$

Setting $B \coloneqq \mathcal{O}_{\Sigma^2}(f^{-1}(U))$, we have that

$$e_\chi \cdot B = \{b \in B \mid h(b) = \chi(h)b \text{ for all } h \in H\},$$
$$e_{\chi'} \cdot B^{H_\chi} = \{b \in B^{H_\chi} \mid h(b) = \chi(h)b \text{ for all } h \in H/H_\chi\},$$

and it is direct to check that these are equal. Therefore we are reduced to showing that $e_{\chi'} \cdot f'_* \mathcal{O}_{\Sigma^2/H_\chi}$ is non-trivial.

Now because the action of G on Σ^2 and Σ^1 commutes with the action of H, G acts on Σ^2/H_χ, $f' \colon \Sigma^2/H_\chi \to \Sigma^1$ is G-equivariant, and the G-action commutes with the action of H/H_χ. Therefore the covering $f' \colon \Sigma^2/H_\chi \to \Sigma^1$ defines an element of $\mathrm{H}^1_{\text{ét}}(\Sigma^1, \underline{H/H_\chi})^G$ [42, §4.1], the middle term of the G-invariants of the Kummer exact sequence

$$0 \to \left(\mathcal{O}(\Sigma^1)^\times / \mathcal{O}(\Sigma^1)^{\times p}\right)^G \to \mathrm{H}^1_{\text{ét}}(\Sigma^1, \underline{H/H_\chi})^G \to \mathrm{Pic}(\Sigma^1)[p]^G. \tag{2.5}$$

Suppose now for a contradiction that the line bundle $e_{\chi'} \cdot f'_* \mathcal{O}_{\Sigma^2/H_\chi}$ is trivial. Then from the exact sequence (2.5) above, the space Σ^2/H_χ is given as $\kappa(v) = \Sigma^1(v^{1/p})$ for some

$$v \in \left(\mathcal{O}(\Sigma^1)^\times / \mathcal{O}(\Sigma^1)^{\times p}\right)^G.$$

By Proposition 2.3.5, we actually have $v \in K^\times / K^{\times p}$, and therefore the base change $\Sigma^2/H_\chi \times_K K(\sqrt[p]{v})$ is not connected. Over $K(\sqrt[p]{v})$, the intermediate extension

$$\Sigma^2 \times_K K(\sqrt[p]{v}) \to (\Sigma^2/H_\chi) \times_K K(\sqrt[p]{v})$$

is Galois and hence surjective, and thus $\Sigma^2 \times_K K(\sqrt[p]{v})$ is also not connected. But this is a contradiction, as Σ^2 is geometrically connected by Corollary 2.3.3. □

Remark 2.3.8. If we do not assume that K contains a primitive pth root of 1, then the techniques used in the proof of Theorem 2.3.6 can still be used to show that $\mathrm{Pic}(\Sigma^1)[p]^G \neq 0$. Indeed, if we assume that $\mathrm{Pic}(\Sigma^1)[p]^G = 0$, then the same argument but with H_χ replaced by any index p subgroup H_0 of H will still result in a contradiction.

2.4 Vector Bundles on the Drinfeld Upper Half Plane

Suppose from now until the end of this chapter that $n = 2$. In this section we provide an elementary proof that all vector bundles on Ω are trivial, which extends and uses the result that all line bundles on Ω are trivial [43, Thm. A]. In the context of Theorem 2.3.6, this says that whilst the line bundles $\mathcal{L}_\chi$ on Σ^1 are non-trivial whenever $\chi \neq 1$, the pushforward to Ω^1 will be a trivial vector bundle (of constant rank $q+1$).

Before we state the theorem, we will need the following notions from commutative algebra.

Definition 2.4.1. Let R be an integral domain. R is called a *Prüfer domain* if every finitely generated ideal of R is invertible. R is called a *Bézout domain* if every finitely generated ideal of R is principal.

We provide a proof of the following result, for which we were unable to find a reference.

Lemma 2.4.2. *Suppose that R is a Bézout domain. Then every finitely generated submodule of a free module is free.*

Proof. Suppose that M is finitely generated over R, and M is contained in a free module P. By choosing a basis for P, as M is finitely generated, we have that $M \subset R^n$ for some $n \geq 1$. Let $\pi \colon R^n \to R$ be the projection to the first factor, and let $I \coloneqq \pi(M)$, $K \coloneqq \ker(\pi \colon M \to R)$. Now I is the homomorphic image of M and thus finitely generated, hence I is principal and thus free, because R is a Bézout domain. Therefore, the short exact sequence

$$0 \to K \to M \to I \to 0$$

splits, and $M \cong K \oplus I$. Finally, K is also finitely generated, being a homomorphic image of M, and $K \subset R^{n-1}$, so the result follows by induction. □

We remark that this property actually characterises Bézout domains among integral domains. Indeed, if I is a finitely generated ideal of an integral domain R which satisfies the above property then I is free, but also $I \subset R$, so by passing to the fraction field of R, I must have rank 1, and thus I is principal. This property is analogous to the following property of PID's (which are exactly the Noetherian Bézout domains): a commutative ring R is a PID if and only if every submodule of a free module is free.

Theorem 2.4.3. *Let $\mathfrak{X}$ be a smooth connected one-dimensional quasi-Stein rigid analytic space, with* $\mathrm{Pic}(\mathfrak{X}) = 0$. *Then any vector bundle on $\mathfrak{X}$ is of the form $\mathcal{O}_{\mathfrak{X}}^r$ for some $r \geq 0$.*

Proof. If $\mathfrak{X}$ is as above, the ring $R \coloneqq \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ is an integral domain. The global sections functor defines an equivalence of categories between vector bundles on $\mathfrak{X}$, and finitely generated projective modules over R [6, Prop. 1.1.13]. In particular, $\mathrm{Pic}(R) = 0$, and we are reduced to showing that any finitely generated projective module over R is free. The ring R is a Prüfer domain [6, 1.1.8], and because $\mathrm{Pic}(R) = 0$, R is furthermore a Bézout domain. Then we can conclude, as for such rings any finitely generated projective module is free, by Lemma 2.4.2. □

Corollary 2.4.4. *Any vector bundle on Ω is of the form $\mathcal{O}_\Omega^r$, for some $r \geq 0$.*

Proof. This follows directly from Theorem 2.4.3, as $\mathrm{Pic}(\Omega) = 0$ [43, Thm. A]. □

Chapter 3

Vertex Affinoids in The First Drinfeld Covering

Let $\mathcal{T}$ be the Bruhat-Tits tree for $\mathrm{GL}_2(F)$, let v be the central vertex of $\mathcal{T}$ and let $r : \Sigma^1 \to \Omega \to \mathcal{T}$ the retraction map. Here, by *central vertex*, we mean the unique vertex of $\mathcal{T}$ fixed by $\mathrm{GL}_2(\mathcal{O}_F)$. In this chapter we study the open affinoid subset $\Sigma^1_v := r^{-1}(v)$ of Σ^1. This is stable under the action of $\mathrm{GL}_2(\mathcal{O}_F)$ and after a finite extension of K, Σ^1_v splits up into $q-1$ geometrically connected components, each isomorphic to $\mathrm{Sp}(B)$, where,

$$B = A[z]/(z^{q+1} - (x^q - x)), \text{ for } A = K\left\langle x, \frac{1}{x^q - x}\right\rangle.$$

The group G^0 acts with two orbits on the set of vertices of $\mathcal{T}$, and one can show that for any vertex w adjacent to v, $\Sigma^1_w \cong \Sigma^1_v$. As any such w will be in the other orbit from v, $\Sigma^1_w \cong \Sigma^1_v$ for all vertices $w \in \mathcal{T}$, and consequently this open subset often determines global properties of Σ^1. For example, the first de-Rham cohomology $H^1_{\mathrm{dR}}(\Sigma^1)$ as a representation of $\mathrm{GL}_2(F)$ is determined by $H^1_{\mathrm{dR}}(\Sigma^1_v)$ [41, Thm. 6.1].

The main result of this chapter is that $\mathrm{Pic}(\Sigma^1_v)[p] = 0$ (Theorem 3.2.2). In order to prove this, we consider the affine curve $\mathbf{Y}$ defined by,

$$xy^q - yx^q = 1,$$

over the residue field of K, where $\mathbb{F}_q$ is the residue field of F. This curve was first considered by Drinfeld, who showed that all the discrete series representations of $\mathrm{SL}_2(\mathbb{F}_q)$ can be realised in the cohomology of $\mathbf{Y}$ [12, Pref.]. Inspired by this, these ideas were generalised to all reductive groups $\mathbb{G}$ by Deligne and Lusztig in their landmark paper [25]. They introduce what are now called *Deligne-Lusztig* varieties, which assign to $\mathbb{G}(\mathbb{F}_q)$ and $w \in W$, the Weyl group, a base space $X(w)$ and a finite covering $Y(w)$, and it is in the étale cohomology of $Y(w)$ that the cuspidal representations are realised. These are spaces of considerable interest, and the Picard groups of the base spaces $X(w)$ have been considered in [36]. Here we consider $\mathbf{Y} = Y(w)$ in the special case of $\mathbb{G} = \mathrm{SL}_2$, and $w \neq 1$. It would be interesting to study the Picard groups of $Y(w)$ more generally.

3.1 Deligne-Lusztig Curves

Throughout this section, let $\mathbb{F}$ be an algebraic field extension of $\mathbb{F}_q$. We consider the affine curve,

$$\mathbf{Y} = \operatorname{Spec}\left(\frac{\mathbb{F}[x,y]}{xy^q - yx^q = 1}\right),$$

and its projective closure,

$$\mathbf{Z} = \operatorname{Proj}\left(\frac{\mathbb{F}[X,Y,Z]}{XY^q - YX^q = Z^{q+1}}\right).$$

We also consider the projective curve,

$$\mathbf{W} = \operatorname{Proj}\left(\frac{\mathbb{F}[U,V,W]}{UV^q + VU^q = W^{q+1}}\right).$$

We would first like to show that $\operatorname{Pic}(\mathbf{Z})[p] = 0$.

Lemma 3.1.1. $\mathbf{Z}$ *is a smooth integral projective curve over* $\mathbb{F}$*. Furthermore, if* $\mathbb{F}_{q^4} \subset \mathbb{F}$*, then* $\mathbf{W} \cong \mathbf{Z}$*.*

Proof. The polynomial $P(X,Y,Z) = Z^{q+1} - (XY^q - YX^q) \in \mathbb{F}[X,Y,Z]$ is prime, which follows from Eisenstein's criterion for $P \in \mathbb{F}[X,Y][Z]$, at the prime ideal (X). Therefore $\mathbf{Z}$ is integral. Furthermore, $\mathbf{Z}$ is smooth, because the system $\partial_X P = \partial_Y P = \partial_Z P = 0$ has no solutions over $\mathbf{Z}(\overline{\mathbb{F}})$. For the isomorphism, let $\lambda \in \mathbb{F}_{q^2}$ with $\lambda^{q-1} = -1$, and let $\mu \in \overline{\mathbb{F}}$ with $\mu^{q+1} = \lambda^q$. The element μ lies in $\mathbb{F}_{q^4}$, as,

$$\mu^{q^2} = (\lambda^q)^{q-1}\mu = -\mu,$$

so,

$$\mu^{q^4} = (-\mu)^{q^2} = -(-\mu) = \mu.$$

Then the claimed isomorphism is given by,

$$U = X, \quad V = \lambda Y, \quad W = \mu Z.$$

Indeed,

$$\begin{aligned} X(\lambda Y)^q + (\lambda Y)X^q &= \lambda^q(XY^q - YX^q), \\ &= \lambda^q Z^{q+1} = (\mu Z)^{q+1}, \end{aligned}$$

and similarly $U(\lambda^{-1}V)^q - (\lambda^{-1}V)U^q = (\mu^{-1}W)^{q+1}$. □

Proposition 3.1.2. $\operatorname{Pic}(\mathbf{Z})[p] = 0$.

Proof. By Lemma 3.1.1, $\mathbf{Z}_{\overline{\mathbb{F}}} \cong \mathbf{W}_{\overline{\mathbb{F}}}$, and thus the group $\operatorname{Pic}(\mathbf{Z}_{\overline{\mathbb{F}}})[p] \cong \operatorname{Pic}(\mathbf{W}_{\overline{\mathbb{F}}})[p] \cong J(\overline{\mathbb{F}})[p]$, where J is the Jacobian of $\mathbf{W}$. $\mathbf{W}$ is known as the Hermitian curve, defined by affine equation $w^{q+1} = v^q + v$, and is maximal over $\mathbb{F}_{q^2}$ [58, Lem. 6.4.4], hence $J(\overline{\mathbb{F}})[p] = 0$ by [31, Cor. 2.5]. Then, because pullback induces an exact sequence $0 \to \operatorname{Pic}(\mathbf{Z}) \to \operatorname{Pic}(\mathbf{Z}_{\overline{\mathbb{F}}})$ [57, Tag 0CC5], and p-torsion is left exact, $\operatorname{Pic}(\mathbf{Z})[p] = 0$. □

Our next goal is to establish that $\mathrm{Pic}(\mathbf{Y})[p] = 0$.

Lemma 3.1.3. $\mathbf{Z}(\overline{\mathbb{F}}) \setminus \mathbf{Y}(\overline{\mathbb{F}})$ *consists of the* $q+1$ *points,*

$$\mathcal{P} := \{(a : b : 0) \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}.$$

Furthermore, $\mathcal{P} = \mathbf{Z}(\mathbb{F}_q)$.

Proof. If $(a : b : c) \in \mathbf{Z}(\overline{\mathbb{F}})$ with $c = 0$, then $b^q a - a^q b = 0$, so $b^q a = a^q b$. If $a \neq 0$, then $\left(\frac{b}{a}\right)^q = \frac{b}{a}$, so $\frac{b}{a} \in \mathbb{F}_q$, and $(a : b) \in \mathbb{P}^1(\mathbb{F}_q)$. Similarly, if $b \neq 0$, $(a : b) \in \mathbb{P}^1(\mathbb{F}_q)$. Thus $\mathbf{Z}(\overline{\mathbb{F}}) \setminus \mathbf{Y}(\overline{\mathbb{F}}) = \mathcal{P}$. To see $\mathbf{Z}(\overline{\mathbb{F}}) \setminus \mathbf{Y}(\overline{\mathbb{F}}) = \mathbf{Z}(\mathbb{F}_q)$, there are no points $(a : b : c) \in \mathbf{Z}(\mathbb{F}_q)$ with $c = 1$, because if so then $1 = ab^q - ba^q = ab - ba = 0$, as $a, b \in \mathbb{F}_q$. □

Therefore the closed points of $\mathbf{Z} \setminus \mathbf{Y}$ are $\mathcal{P}$ [32, Prop. 5.4], which we enumerate by $\mathcal{P} = \{P_0, ..., P_q\}$. From [63, Ex. 5.12 (a)] we have an exact sequence,

$$\mathbb{Z}^{q+1} \to \mathrm{Cl}(\mathbf{Z}) \to \mathrm{Cl}(\mathbf{Y}) \to 0,$$

where the first map sends,

$$(m_0, ..., m_q) \mapsto \sum_{i=0}^{q} m_i [P_i],$$

and the second sends, for I a finite set of closed points of $\mathbf{Z}$,

$$\sum_{P \in I} n_P [P] \mapsto \sum_{P \in I \setminus \mathcal{P}} n_P [P].$$

Let $\Gamma = \langle [P_0], ..., [P_q] \rangle \subset \mathrm{Cl}(\mathbf{Z})$ be the image of $\mathbb{Z}^{q+1}$ in $\mathrm{Cl}(\mathbf{Z})$. The resulting exact sequence,

$$0 \to \Gamma \to \mathrm{Cl}(\mathbf{Z}) \to \mathrm{Cl}(\mathbf{Y}) \to 0,$$

yields the long exact sequence,

$$0 \to \Gamma[p] \to \mathrm{Cl}(\mathbf{Z})[p] \to \mathrm{Cl}(\mathbf{Y})[p] \to \Gamma/p\Gamma \to \mathrm{Cl}(\mathbf{Z})/p\,\mathrm{Cl}(\mathbf{Z}) \to \mathrm{Cl}(\mathbf{Y})/p\,\mathrm{Cl}(\mathbf{Y}) \to 0,$$

from the right derived functors of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, -)$. Then from Proposition 3.1.2 and the above discussion we have the following.

Proposition 3.1.4. *There is an exact sequence*

$$0 \to \mathrm{Cl}(\mathbf{Y})[p] \to \Gamma/p\Gamma \to \mathrm{Cl}(\mathbf{Z})/p\,\mathrm{Cl}(\mathbf{Z}),$$

where the map $\Gamma/p\Gamma \to \mathrm{Cl}(\mathbf{Z})/p\,\mathrm{Cl}(\mathbf{Z})$ *is that induced by the inclusion* $\Gamma \hookrightarrow \mathrm{Cl}(\mathbf{Z})$.

Remark 3.1.5. We note that if $\mathbf{Z} \setminus \mathbf{Y}$ contained exactly one degree 1 closed point Q, then we could establish that $\mathrm{Cl}(\mathbf{Y})[p] = 0$ almost immediately in the following way. In the exact sequence,

$$\mathbb{Z} \to \mathrm{Cl}(\mathbf{Z}) \to \mathrm{Cl}(\mathbf{Y}) \to 0,$$

the map $\mathbb{Z} \to \mathrm{Cl}(\mathbf{Z})$ is actually injective and split by the degree homomorphism, hence $\mathrm{Cl}(\mathbf{Z}) \cong \mathbb{Z} \times \mathrm{Cl}(\mathbf{Y})$ so,

$$0 = \mathrm{Cl}(\mathbf{Z})[p] \cong \mathbb{Z}[p] \times \mathrm{Cl}(\mathbf{Y})[p] = \mathrm{Cl}(\mathbf{Y})[p].$$

In particular, this can be applied to show that the class groups of affine dehomogenisations of $\mathbf{Z}$ with respect to both X and Y both have no p-torsion.

We want to show that $\mathrm{Cl}(\mathbf{Y})[p] = 0$, and so in light of Proposition 3.1.4, we want to show that,

$$\Gamma/p\Gamma \to \mathrm{Cl}(\mathbf{Z})/p\,\mathrm{Cl}(\mathbf{Z}),$$

is injective. In order to do so, we now examine the structure of Γ. First we compute the principal divisors of some rational functions on $\mathbf{Z}$.

Definition 3.1.6. For $(a:b) \in \mathbb{P}^1(\mathbb{F}_q)$, we let $P_{(a:b)}$ be the closed point of $\mathbf{Z}$ defined by $(a:b:0) \in \mathbb{P}^1(\overline{\mathbb{F}})$.

Lemma 3.1.7. *Let $(a:b), (c:d) \in \mathbb{P}^1(\mathbb{F}_q)$ with $(a:b) \neq (c:d)$. Then the rational function,*

$$f := \frac{bX - aY}{dX - cY},$$

has associated principal divisor,

$$(f) = (q+1)[P_{(a:b)}] - (q+1)[P_{(c:d)}].$$

Proof. Consider the morphism $\zeta : \mathbf{Z} \to \mathbb{P}^1$ corresponding to the extension of function fields $\mathbb{F}(\mathbb{P}^1) \to \mathbb{F}(\mathbf{Z})$, which sends,

$$\frac{S}{T} \mapsto \frac{bX - aY}{dX - cY},$$

where $\mathbb{P}^1 = \mathrm{Proj}(\mathbb{F}[S,T])$, and $\mathbb{F}(\mathbb{P}^1) = \mathbb{F}(S/T)$. On $\overline{\mathbb{F}}$-points, $\zeta : \mathbf{Z} \to \mathbb{P}^1$ is given by,

$$\zeta(x:y:z) = (bx - ay : dx - cy).$$

This extension $\mathbb{F}(\mathbb{P}^1) \to \mathbb{F}(\mathbf{Z})$ has degree $q+1$ because it differs by an automorphism of $\mathbb{P}^1$ from the extension $\mathbb{F}(\mathbb{P}^1) \to \mathbb{F}(\mathbf{Z})$, defined by,

$$\frac{S}{T} \mapsto \frac{X}{Y},$$

which clearly has degree $q+1$. Let Q_0, Q_∞ be the closed points of $\mathbb{P}^1$ defined by $(0:1), (1:0) \in \mathbb{P}^1(\overline{\mathbb{F}})$ respectively. By [45, Cor. 3.9], we have that,

$$(f) = \zeta^*(S/T) = \zeta^*([Q_0]) - \zeta^*([Q_\infty]),$$

and $\deg(\zeta^*([Q_0])) = \deg(\zeta^*([Q_\infty])) = [\mathbb{F}(\mathbb{P}^1) : \mathbb{F}(\mathbf{Z})] = q+1$. But $\zeta^*([Q_0])$ is some integer multiple of $[P_{(a:b)}]$ and $\zeta^*([Q_\infty])$ some integer multiple of $[P_{(c:d)}]$, hence,

$$(f) = (q+1)[P_{(a:b)}] - (q+1)[P_{(c:d)}]. \qquad \square$$

Let $\Gamma^0 \subset \Gamma$ be the degree 0 subgroup of Γ, and $\mathrm{Cl}^0(\mathbf{Z}) \subset \mathrm{Cl}(\mathbf{Z})$ the degree 0 subgroup of $\mathrm{Cl}(\mathbf{Z})$.

Lemma 3.1.8. *The function $\phi : \mathbb{Z} \times (\mathbb{Z}/(q+1)\mathbb{Z})^q \to \Gamma$,*

$$\phi : (n_0, ..., n_q) \mapsto n_0[P_0] + n_1([P_1] - [P_0]) + ... + n_q([P_q] - [P_0])),$$

is a surjective homomorphism. In particular, Γ^0 is a quotient of $(\mathbb{Z}/(q+1)\mathbb{Z})^q$.

Proof. For each $P_k \in \mathcal{P}$, we can write $P_k = P_{(a_k:b_k)}$ for some $a_k, b_k \in \mathbb{F}_q$. For each $0 \leq i \neq j \leq q$, consider the rational function,

$$f = \frac{b_i X - a_i Y}{b_j X - a_j Y}.$$

Taking the divisor of f,

$$0 = (f) = (q+1)[P_i] - (q+1)[P_j],$$

in Γ, by Lemma 3.1.7. Therefore, ϕ is a well-defined homomorphism, which is surjective because $\{[P_0], ..., [P_q]\}$ generate Γ. Finally, as $\Gamma^0 = \langle [P_1] - [P_0], ..., [P_q] - [P_0] \rangle$, then Γ^0 is a quotient of $(\mathbb{Z}/(q+1)\mathbb{Z})^q$. □

We are finally in a position to prove the main result of this section.

Theorem 3.1.9. $\mathrm{Pic}(\mathbf{Y})[p] = 0$.

Proof. We can split the degree homomorphism with $[P_0]$, as $[P_0]$ has degree 1 and $\langle [P_0] \rangle$ is free [63, Ex. 5.12 (b)]. Then,

$$\begin{aligned} \psi : \mathrm{Cl}(\mathbf{Z}) &\to \mathrm{Cl}^0(\mathbf{Z}) \times \mathbb{Z}, \\ Q &\mapsto (Q - \deg(Q)[P_0], \deg(Q)), \end{aligned}$$

is an isomorphism, which restricts to,

$$\Gamma \cong \Gamma^0 \times \mathbb{Z}.$$

We then obtain the following commutative diagram,

$$\begin{array}{ccccc} \frac{\Gamma}{p\Gamma} & \xrightarrow{\sim} & \frac{\Gamma^0 \times \mathbb{Z}}{p(\Gamma^0 \times \mathbb{Z})} & \xrightarrow{\sim} & \frac{\Gamma^0}{p\Gamma^0} \times \frac{\mathbb{Z}}{p\mathbb{Z}} \\ \downarrow & & \downarrow & & \downarrow \\ \frac{\mathrm{Cl}(\mathbf{Z})}{p\,\mathrm{Cl}(\mathbf{Z})} & \xrightarrow{\sim} & \frac{\mathrm{Cl}^0(\mathbf{Z}) \times \mathbb{Z}}{p(\mathrm{Cl}^0(\mathbf{Z}) \times \mathbb{Z})} & \xrightarrow{\sim} & \frac{\mathrm{Cl}^0(\mathbf{Z})}{p\,\mathrm{Cl}^0(\mathbf{Z})} \times \frac{\mathbb{Z}}{p\mathbb{Z}}. \end{array}$$

Here, the vertical maps are induced from the inclusions of Γ into $\mathrm{Cl}(\mathbf{Z})$ and of Γ^0 into $\mathrm{Cl}^0(\mathbf{Z})$, the left horizontal maps are induced by ψ, and the right horizontal maps are the standard identifications.

Now, by Lemma 3.1.8, Γ^0 is a quotient of $(\mathbb{Z}/(q+1)\mathbb{Z})^q$, thus $\Gamma^0/p\Gamma^0 = 0$. Consequently, $\Gamma/p\Gamma \to \mathrm{Cl}(\mathbf{Z})/p\,\mathrm{Cl}(\mathbf{Z})$ is an injection. Therefore, $\mathrm{Cl}(\mathbf{Y})[p] = 0$, by the exact sequence of Proposition 3.1.4. □

3.2 Rigid Curves

Let F be a finite extension of $\mathbb{Q}_p$ with uniformiser π and residue field $\mathbb{F}_q$. Let K be a complete field extension of F with residue field $\mathbb{F}$, such that $\mathbb{F}$ is an algebraic extension of $\mathbb{F}_q$. Let R be the ring of integers of K and $\varpi \in K$ an element with $0 < |\varpi| < 1$.

Let A be the affinoid algebra,

$$A = K\left\langle x, \frac{1}{x^q - x}\right\rangle,$$

for which the associated rigid space $\mathrm{Sp}(A)$ has admissible formal model $\mathrm{Spf}(A_0)$, where,

$$A_0 = R\left\langle x, \frac{1}{x^q - x}\right\rangle.$$

Let $u \coloneqq x^q - x \in A_0^\times \subset A^\times$, and let B be the affinoid algebra,

$$B \coloneqq A[z]/(z^{q+1} - u).$$

Consider the ring extension,

$$B_0 \coloneqq A_0[z]/(z^{q+1} - u).$$

B_0 is ϖ-torsion free, and the natural map,

$$B_0 = A_0[z]/(z^{q+1} - u) \to R\left\langle x, \frac{1}{x^q - x}, z\right\rangle \Big/ \left(z^{q+1} - u\right),$$

is an isomorphism (because $u \in A_0^\times$ is a unit), hence B_0 is an admissible R-algebra. The special fibre of $\mathrm{Spf}(B_0)$ is,

$$\mathrm{Spec}(B_0 \otimes_R \mathbb{F}) = \mathrm{Spec}\left(\mathbb{F}\left[y, 1/v, t\right]/(t^{q+1} - v)\right),$$

where $v = y^q - y$, and the generic fibre of $\mathrm{Spf}(B_0)$ is $\mathrm{Sp}(B_0 \otimes_R K) = \mathrm{Sp}(B)$.

Lemma 3.2.1. $\mathrm{Pic}(\mathrm{Sp}(B)) \cong \mathrm{Pic}(\mathbf{Y})$.

Proof. First note that there is an isomorphism of $\mathbb{F}$-algebras,

$$\mathbb{F}[r, s]/(rs^q - sr^q - 1) \xrightarrow{\sim} \mathbb{F}\left[y, 1/v, t\right]/(t^{q+1} - v),$$

given by $r \mapsto 1/t$, $s \mapsto y/t$, with inverse $y \mapsto s/r$, $t \mapsto 1/r$. Thus $\mathrm{Spec}(B_0 \otimes_R \mathbb{F}) \cong \mathbf{Y}$, and $\mathrm{Spf}(B_0)$ is a smooth admissible formal model of $\mathrm{Sp}(B)$. Therefore by [39, Lem. 3.6], the natural maps,

$$\mathrm{Pic}(\mathrm{Sp}(B)) \xleftarrow{\sim} \mathrm{Pic}(\mathrm{Spf}(B_0)) \xrightarrow{\sim} \mathrm{Pic}(\mathrm{Spec}(B_0 \otimes_R \mathbb{F})),$$

are isomorphisms and we're done. □

We can now state our main results. If K contains $\breve{F}$ the completion of the maximal unramified extension of F, then we can consider the rigid analytic space $\mathcal{N}_1$ defined over any such K (cf. Section 1.11). If $v \in \mathcal{T}$ is the central vertex of the Bruhat-Tits tree, then the open affinoid subset $\mathcal{N}_1^v \coloneqq r^{-1}(v) \subset \mathcal{N}_1$ has coordinate ring isomorphic to,

$$\mathcal{O}(\mathcal{N}_1^v) \cong A[z]/(z^{q^2-1} - (\pi u^{q-1})),$$

by [42, Thm. 2.7].

Let ω be a primitive (q^2-1)st root of π in $\overline{F}$. From now on we strengthen our assumption on the complete field extension K of F and assume that,

$$K \text{ contains } \breve{F}(\omega) \text{ and } \mathbb{F} \text{ is an algebraic extension of } \mathbb{F}_q.$$

We note that this forces $\mathbb{F}$ to be an algebraic closure of $\mathbb{F}_q$, and that this assumption holds for any complete field extension K of $\breve{F}(\omega)$ which is contained in $\mathbb{C}_p$.

Theorem 3.2.2. $\mathrm{Pic}(\mathcal{N}_1^v)[p] = 0$.

Proof. Because K contains ω,

$$\mathcal{O}(\mathcal{N}_1^v) \cong B^{q-1},$$

and therefore,

$$\mathrm{Pic}(\mathcal{N}_1^v) \cong \mathrm{Pic}(\mathrm{Sp}(B^{q-1})) = \mathrm{Pic}(\mathrm{Sp}(B))^{q-1} \cong \mathrm{Pic}(\mathbf{Y})^{q-1},$$

by Lemma 3.2.1. But then $\mathrm{Pic}(\mathcal{N}_1^v)[p] \cong \mathrm{Pic}(\mathbf{Y})[p]^{q-1}$, which is zero by Theorem 3.1.9. □

Recall that $\mathcal{N}_1^v = r^{-1}(v)$ is the pre-image of v, the central vertex of the Bruhat-Tits tree. The vertex v is fixed by $\mathrm{GL}_2(\mathcal{O}_F)$, and because r is equivariant, $\mathrm{GL}_2(\mathcal{O}_F)$ acts on $\mathcal{N}_1^v$.

Corollary 3.2.3. *The natural map,*

$$\mathcal{O}(\mathcal{N}_1^v)^\times/\mathcal{O}(\mathcal{N}_1^v)^{\times p^n} \to H^1_{\text{ét}}(\mathcal{N}_1^v, \mu_{p^n}),$$

arising from the Kummer exact sequence is an isomorphism of $\mathrm{GL}_2(\mathcal{O}_F)$*-modules.*

Proof. Because K has characteristic 0, we can consider the Kummer exact sequence for rigid analytic spaces [24, Sect. 3.2]. Then the result follows from Theorem 3.2.2 after taking the long exact sequence in étale cohomology, using that $\mathrm{Pic}(\mathcal{N}_1^v) \cong H^1_{\text{ét}}(\mathcal{N}_1^v, \mathbb{G}_m)$ [24, Prop. 3.2.4]. □

As a consequence, we may now compute $H^1_{\text{ét}}(\mathcal{N}_1^v, \mathbb{Z}_p(1))$ as the p-adic completion of $\mathcal{O}(\mathcal{N}_1^v)^\times$. This is completely explicit, as the group $\mathcal{O}(\mathcal{N}_1^v)^\times$ has been computed by Junger [42, Thm. 5.1].

Theorem 3.2.4. *There is an isomorphism of* $\mathbb{Z}_p$*-linear representations of* $\mathrm{GL}_2(\mathcal{O}_F)$,

$$H^1_{\text{ét}}(\mathcal{N}_1^v, \mathbb{Z}_p(1)) \cong \varprojlim_{n\geq 1} \mathcal{O}(\mathcal{N}_1^v)^\times/\mathcal{O}(\mathcal{N}_1^v)^{\times p^n}.$$

Proof. For all $n \geq 1$ the diagram,

$$\begin{array}{ccc}
\mathcal{O}(\mathcal{N}_1^v)^\times/\mathcal{O}(\mathcal{N}_1^v)^{\times p^{n+1}} & \longrightarrow & H^1_{\text{ét}}(\mathcal{N}_1^v, \mu_{p^{n+1}}) \\
\downarrow & & \downarrow \\
\mathcal{O}(\mathcal{N}_1^v)^\times/\mathcal{O}(\mathcal{N}_1^v)^{\times p^n} & \longrightarrow & H^1_{\text{ét}}(\mathcal{N}_1^v, \mu_{p^n})
\end{array}$$

commutes. Then by the definition of $H^1_{\text{ét}}(\mathcal{N}_1^v, \mathbb{Z}_p(1))$ and Corollary 3.2.3,

$$H^1_{\text{ét}}(\mathcal{N}_1^v, \mathbb{Z}_p(1)) = \varprojlim_{n\geq 1} H^1_{\text{ét}}(\mathcal{N}_1^v, \mu_{p^n}) \xleftarrow{\sim} \varprojlim_{n\geq 1} \mathcal{O}(\mathcal{N}_1^v)^\times/\mathcal{O}(\mathcal{N}_1^v)^{\times p^n}.$$ □

Chapter 4

Equivariant Vector Bundles with Connection on Drinfeld Symmetric Spaces

In this chapter we prove Theorem A, Theorem B, Theorem C and Theorem D.

More precisely, in Section 4.1 we introduce and consider the sheaf of constant functions and relate this sheaf to the notion of geometric connectivity. In Section 4.2 we introduce the functor $\mathcal{O}_X \otimes_k -$, the solution functor $\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, -)$, and prove the main technical result, Theorem D (Theorem 4.2.7). In Section 4.3 we consider finite equivariant vector bundles with connection, and relate these to Galois coverings. In Section 4.4 we bring everything together, and establish the main properties of the functor $(\mathcal{O}_X \otimes_k -)^H$ (Theorem 4.4.2) and its consequences, including a decomposition theorem for the $\mathcal{D}_Y$-module $f_*\mathcal{O}_X$ (Theorem 4.4.4), and a strengthening of Theorem 4.4.2 in the case that the Galois group is abelian (Theorem 4.4.10). In Section 4.5 we apply the preceeding results of this chapter to the Drinfeld tower. The proof of the main results, Theorem A, Theorem B and Theorem C occupies Section 4.5.5 to Section 4.5.11.

4.1 The Sheaf of Constant Functions

In this section we work in the same the geometric framework of Section 1.7 and assume that k has characteristic zero and

(A) X is a smooth scheme over k ($X \to \mathrm{Spec}(k)$ is smooth),

(B) X is a smooth rigid space over k ($X \to \mathrm{Spec}(k)$ is smooth [15, Def. 2.1]).

In this section we are interested in the following sheaf, known as the sheaf of constant functions, and how it relates to geometric connectivity.

Definition 4.1.1. Let $c_X \coloneqq \ker(d\colon \mathcal{O}_X \to \Omega_{X/k})$.

This sheaf of k-algebras has been considered by Berkovich in the setting of Berkovich spaces [9, 10]. We can give a more explicit description of c_X on the basis $\mathcal{B}$.

Lemma 4.1.2. *For any $U \in \mathcal{B}$,*

$$c_X(U) = \mathcal{O}_X(U)^{\Theta_X(U)=0} = \{f \in \mathcal{O}_X(U) \mid \partial(f) = 0 \text{ for all } \partial \in \Theta_X(U)\}.$$

In particular, c_X is the sheaf extension of the sheaf $U \mapsto \mathcal{O}(U)^{\Theta_X(U)=0}$ on the basis $\mathcal{B}$.

Proof. For $U \in \mathcal{B}$ and $A := \mathcal{O}_X(U)$, we want to show that

$$\ker(d\colon A \to \Omega_{A/k}) = A^{\mathrm{Der}_k(A)},$$

where $\Omega_{A/k}$ has the same meaning in cases (A) and (B) as it does in Section 1.4. Composition with d induces an isomorphism,

$$\mathrm{Hom}_A(\Omega_{A/k}, A) \xrightarrow{\sim} \mathrm{Der}_k(A),$$

and so if $a \in \ker(d)$, then $a \in A^{\mathrm{Der}_k(A)}$. On the other hand, if $\partial(a) = 0$ for all $\partial \in \mathrm{Der}_k(A)$, then for any A-linear $f\colon \Omega_{A/k} \to A$, $f(d(a)) = 0$. Therefore $d(a) = 0$, which can be seen by picking a dual basis for the module $\Omega_{A/k}$, which is projective because X is smooth. □

We also have a third description of c_X, which will be the most relevant for us.

Lemma 4.1.3. *The isomorphism of sheaves of k-algebras,*

$$\mathcal{O}_X \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{O}_X)$$

restricts to an isomorphism

$$c_X \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathcal{D}_X}(\mathcal{O}_X).$$

Proof. It is sufficient to show that for any $U \in \mathcal{B}$, the isomorphism ϕ,

$$\mathcal{O}_X(U) \xrightarrow{\phi} \mathrm{End}_{\mathcal{O}_X|_U}(\mathcal{O}_X|_U) \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_X(U)}(\mathcal{O}_X(U))$$

restricts to an isomorphism,

$$c_X(U) \xrightarrow{\phi} \mathrm{End}_{\mathcal{D}_X|_U}(\mathcal{O}_X|_U) \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}_X(X)}(\mathcal{O}_X(U)).$$

Here we are using the equivalence of Proposition 1.8.3 in the case that G is trivial. Given $f \in \mathcal{O}_X(U)$, the corresponding $\mathcal{O}_X(U)$-linear endomorphism ϕ_f of $\mathcal{O}_X(U)$ is defined by $\phi_f(x) = fx$. For any $x \in \mathcal{O}_X(U)$ and $\partial \in \Theta_X(U)$,

$$\begin{aligned}\partial(l_f(x)) &= \partial(fx),\\ &= f\partial(x) + x\partial(f),\\ &= \phi_f(\partial(x)) + x\partial(f).\end{aligned}$$

Therefore, ϕ_f is $\mathcal{D}_X(U)$-linear if and only if $\partial(f) = 0$ for all $\partial \in \Theta_X(U)$. □

The sheaf c_X is a sheaf of k-algebras. The next lemma shows that for any admissible open subset U of X, $c_X(U)$ is always a product of finite field extensions of k.

Lemma 4.1.4. *Suppose that $U \subset X$ is a connected admissible open subset of X. Then $c_X(U)$ is a finite field extension of k.*

Proof. If $U \in \mathcal{B}$, using the description of Lemma 4.1.2, then in either case (A) or (B) the proof of [3, Prop. 3.1.6] shows that $c_X(U) = \mathcal{O}_X(U)^{\Theta_X(U)=0}$ is a finite field extension of k. Now suppose that U is any connected admissible open subset of X. Let $\mathcal{V}$ be any admissible open covering of U by connected elements of $\mathcal{B}$. In order to see that $c_X(U)$ is a field, suppose that $f \in c_X(U)$ with $f \neq 0$. Then there is some $V_0 \in \mathcal{V}$ with $f_0 := f|_{V_0} \neq 0$. For any element $V \in \mathcal{V}$ of the covering, there is some finite sequence $V_1, ..., V_n \in \mathcal{V}$ with $V_k \cap V_{k+1} \neq \emptyset$ for all $0 \leq k \leq n-1$, and $V_n = V$. Setting $f_k := f|_{V_k}$, because each $c_X(V_k)$ is a field, the restriction maps $c_X(V_k) \to c_X(V_k \cap V_{k+1})$ are injective, and hence by induction $f|_V = f_n \neq 0$. Therefore $f \in c_X(U)$ is non-zero and has an inverse when restricted to any $V \in \mathcal{V}$, and therefore $f \in c_X(U)^\times$. Finally, because $c_X(U)$ is a field, $c_X(U) \to c_X(V)$ is injective for any $V \in \mathcal{V}$, and therefore $c_X(U)$ is also a finite extension of k. □

We now relate the sheaf c_X the behaviour of how the connectivity of X changes under field extension.

Definition 4.1.5. We call X *geometrically connected* if for any finite extension L of k, $X \times_k L$ is connected.

Remark 4.1.6. If X is a scheme, then X is geometrically connected if and only if $X \times_k L$ is connected for any field extension L of k [57, Lem. 0389].

For rigid spaces, the base change functor is more subtle. When L is a finite extension of k, then just as for schemes $-\times_k L$ is defined as the fibre product functor $-\times_{\mathrm{Sp}(k)} \mathrm{Sp}(L)$ on the category of rigid spaces over k. When L/k is an infinite extension of complete fields, this definition no longer makes sense, as L is no longer a k-affinoid algebra. Nevertheless, one can still define a base change functor $- \times_k L$ for quasi-separated rigid spaces Y over k (see [14, §9.3.6] and [23, §3.1] for more details). In this case, it is shown in [23, §3.2] that (just like for schemes), Y is geometrically connected if and only if $Y \times_k L$ is connected for any complete field extension L of k. In particular, this holds for quasi-Stein Y [14, Prop. 9.6.1(7)].

Corollary 4.1.7. *Suppose that $U \subset X$ is a geometrically connected admissible open subset of X. Then $c_X(U) = k$.*

Proof. Let L be the Galois closure of $c_X(U)$ over k in some fixed algebraic closure of $c_X(U)$. L is a finite extension of k by Lemma 4.1.4, and $c_X(U) \otimes_k L \hookrightarrow \mathcal{O}_X(U) \otimes_k L = \mathcal{O}_{X_L}(U_L)$. Because L/k is Galois, $L \otimes_k L$ is isomorphic to the product of $[L:k]$ copies of $c_X(U)$, and thus if $[c_X(U) : k] > 1$, then $[L:k] > 1$, and this would yield non-trivial idempotents in $\mathcal{O}_{X_L}(U_L)$. □

We would now like to show the converse. We first study how c_X behaves under base change.

Lemma 4.1.8. *For any finite field extension L/k, there is a canonical isomorphism*

$$c_X(X) \otimes_k L \xrightarrow{\sim} c_{X_L}(X_L),$$

where X_L denotes the base change of X to L.

Proof. Suppose first that $U \in \mathcal{B}_X$, and set $A := \mathcal{O}_X(U)$. By Lemma 1.4.1, as $A_L := A \otimes_k L$ is étale over A, there is an isomorphism

$$\mathrm{Der}_k(A) \otimes_k L \to \mathrm{Der}_L(A_L),$$

which by the uniqueness part of Lemma 1.4.1 is explictly given by letting $\partial \otimes \lambda$ act on A_L by

$$(\partial \otimes \lambda)(a \otimes \mu) = \partial(a) \otimes \lambda\mu.$$

In particular, if $a \in A^{\mathrm{Der}_k(A)}$ and $\mu \in L$, then for any $\partial \otimes \lambda \in \mathrm{Der}_L(A_L)$,

$$(\partial \otimes \lambda)(a \otimes \mu) = \partial(a) \otimes \lambda\mu = 0,$$

and so the identity of A_L induces a well-defined map

$$A^{\mathrm{Der}_k(A)} \otimes_k L \hookrightarrow A_L^{\mathrm{Der}_L(A_L)}.$$

This is actually an isomorphism, which can be seen directly by picking a basis for L over k.

Writing $\phi \colon X_L \to X$ for the projection, the morphism of sheaves

$$\mathcal{O}_X \to \phi_* \mathcal{O}_{X_L}$$

maps the subsheaf c_X into the subsheaf $\phi_* c_{X_L}$, this being true for any $U \in \mathcal{B}_X$ by the above. Similarly, being true on $\mathcal{B}_X$, the canonical morphism of sheaves

$$c_X \otimes_k L \to \phi_* c_{X_L},$$

is an isomorphism, and taking global sections we obtain the desired result. □

Corollary 4.1.9. *Suppose that $U \subset X$ is a connected admissible open subset of X. Let L be the Galois closure of $c_X(U)$ over k in some fixed algebraic closure of $c_X(U)$. Then U_L is a disjoint union of $\dim_k c_X(U)$ geometrically connected components.*

In particular, if $c_X(U) = k$, then U is geometrically connected.

Proof. The space X_L has at least $\dim_k c_X(U)$ connected components U_i, arguing as in the proof of Corollary 4.1.7, and each has $c_{X_L}(U_i) \supset L$. Therefore, by Lemma 4.1.8,

$$\dim_k c_X(U) = \dim_L(c_{X_L}(U_L)) = \sum_{i=1}^{\dim_k c_X(U)} \dim_L c_{X_L}(U_i) \geq \dim_k c_X(U),$$

and thus each $c_{X_L}(U_i) = L$. Therefore it is sufficient to show that if $c_X(U) = k$, then U is geometrically connected. If there was some finite extension K of k with U_K disconnected, then $c_{X_K}(U_K)$ contains at least $K \times K$, the K-span of each idempotent. However,

$$\dim_K c_{X_K}(U_K) = \dim_k c_X(U) = 1$$

by Lemma 4.1.8, a contradiction. □

Remark 4.1.10. One can also prove Corollary 4.1.9 more geometrically, following the argument of [9, Lem. 8.1.4] and using Lemma 4.1.3, Lemma 1.9.8 and Proposition 1.10.1.

We can also be more precise about the value of c_X on connected admissible open subsets.

Corollary 4.1.11. *Suppose that $U \subset X$ is a connected admissible open subset of X. Then $c_X(U)$ is the unique maximal finite field extension of k contained in $\mathcal{O}_X(U)$.*

Proof. Let L be any finite field extension of k contained in $\mathcal{O}_X(U)$, and let $L \cdot c_X(U)$ be the k-algebra generated by L and $c_X(U)$. If $\{e_i\}$ is a basis of L and $\{f_j\}$ is a basis of $c_X(U)$ then this is generated as a k-vector space by $\{e_i f_j\}$. In particular, because $\mathcal{O}_X(U)$ is an integral domain [3, Lem. 3.1.5] and $c_X(U)$ is a finite field extension of k by Lemma 4.1.4, $L \cdot c_X(U)$ is a finite dimensional field extension of $c_X(U)$. Therefore, it is sufficient to show that if L is a finite field extension of k containing $c_X(U)$, then $L = c_X(U)$. Suppose that $L \supset c_X(U)$ is such a field extension. By Lemma 4.1.8,

$$\dim_k c_X(U) = \dim_L c_{X_L}(U_L).$$

However, U_L has at least $\dim_k L$ connected components, which can be seen as in the proof of Corollary 4.1.7, and so the right-hand side is at least $\dim_k L$. Therefore, $\dim_k c_X(U) \geq \dim_k L$, and $c_X(U) = L$ as required. □

Our collected facts about the sheaf c_X have the following interesting consequence.

Remark 4.1.12. Any connected space X which for which $\mathcal{O}_X(X)$ contains a proper non-trivial finite field extension of k provides an "obvious" example of a connected but not geometrically connected space (cf. the proof of Corollary 4.1.7).

In fact, we now see that this is the only way that X can fail be to geometrically connected: if X is connected but not geometrically connected, then $\mathcal{O}_X(X)$ contains a proper field extension of k, namely $c_X(X)$, by Corollary 4.1.9.

In the rigid case, case (B), the sheaf c_X can be very far from a constant sheaf. This is shown in the following example, due to Jérôme Poineau.

Example 4.1.13. Let $X = \mathbb{D} = \mathrm{Sp}(\mathbb{Q}_p\langle x \rangle)$ be the rigid analytic unit disk over $\mathbb{Q}_p$. Then $\mathbb{D}$ is geometrically connected, but we can construct open subsets $U \subset \mathbb{D}$ which are connected but not geometrically connected, and in fact have $c_X(U)$ being an extension of $\mathbb{Q}_p$ of arbitrarily large degree. Indeed, let $w \in \overline{\mathbb{Q}_p} \setminus \mathbb{Q}_p$ with $|w| \leq 1$, and let $f(x)$ be the minimal polynomial of w over $\mathbb{Q}_p$. For $r \in p^{\mathbb{Q}}$, we can consider the affinoid open subset

$$U = \{z \in \mathbb{D} \mid |f(z)| \leq r\} \subset \mathbb{D}.$$

If L is a splitting field for $f(x)$ over $\mathbb{Q}_p$, then for r small enough, $U_L \subset \mathbb{D}_L$ is the disjoint union of $\dim_{\mathbb{Q}_p} L$ closed disks. By Lemma 1.9.8, the Galois covering $U_L \to U$ over $\mathbb{Q}_p$ restricts to an isomorphism from each connected component to U, and thus U is connected. Therefore U is connected but not geometrically connected, and in fact $\dim_{\mathbb{Q}_p} c_X(U) = \dim_{\mathbb{Q}_p} L$ by Lemma 4.1.8.

4.2 The Functors $\mathcal{O}_X \otimes_k -$ and $\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, -)$

In this section, we have the following running assumptions.

1. X is as in Section 1.7 with k of characteristic 0,
2. X has an action by a product of abstract groups $G \times H$,
3. There are no global non-trivial G-invariant constant functions: $c_X(X)^G = k$.

For example, both G and H could be trivial, or infinite. Assumption (3) is always satisfied whenever the connected components of X are geometrically connected and the action of G on the set of connected components is transitive.

Remark 4.2.1. In the case that G is trivial, then condition (3) is equivalent to the assumption that X is geometrically connected by Corollary 4.1.7 and Corollary 4.1.9.

We will make use of the following consequence of our assumptions on X and G.

Lemma 4.2.2. *Suppose that X and G are as above. Then $\mathcal{O}_X$ is irreducible as a G-$\mathcal{D}_X$-module.*

Proof. First note that X is the disjoint union of its connected components. When X is a rigid space this follows by definition (see [23, §2.1]), and when X is a scheme this follows because X is locally noetherian. Furthermore, G acts transitively on the set of connected components of X. Indeed, if $X = X_1 \sqcup X_2$ were a G-stable disjoint union, then we would have $k \times k \subset c(X_1)^G \times c(X_2)^G \subset c_X(X)^G$. Suppose now that $\mathcal{F}$ is a proper non-trivial G-$\mathcal{D}_X$ submodule of $\mathcal{O}_X$. Then there is some admissible open subset $U \subset X$ with $\mathcal{F}(U) \neq 0$, and so we may find some non-zero $x \in \mathcal{F}(U)$. As $x \neq 0$ in $\mathcal{F}(U)$, and the connected components $\{X_i\}_i$ form an admissible open cover of X, there is some X_i with

$$0 \neq x|_{U \cap X_i} \in \mathcal{F}(U \cap X_i) \subset \mathcal{O}_{X_i}(U \cap X_i).$$

Therefore $\mathcal{F}|_{X_i}$ is a non-trivial $\mathcal{D}_{X_i}$-submodule of $\mathcal{O}_{X_i}$, and thus $\mathcal{F}|_{X_i} = \mathcal{O}_{X_i}$ by Lemma 1.7.2 because X_i is connected. By the transitivity of the G action on the set of connected components and the fact that $\mathcal{F} \hookrightarrow \mathcal{O}_X$ is G-linear, then for any other connected component X_j we also have that $\mathcal{F}|_{X_j} = \mathcal{O}_{X_j}$. Therefore, as this is true for any X_j, $\mathcal{F} = \mathcal{O}_X$. ☐

From the action of $G \times H$ on X, $\mathcal{D}_X$ is a $G \times H$-equivariant sheaf, and we can consider the category $\mathbf{VectCon}^{G\times H}(X)$ (see Section 1.8). In this section we define and study properties of a pair of functors between $\mathbf{Mod}^{\mathbf{fd}}_{k[H]}$ and $\mathbf{VectCon}^{G\times H}(X)$.

The Functor $\mathcal{O}_X \otimes_k -$

In one direction, we define,

$$\mathcal{O}_X \otimes_k -\colon\ \mathbf{Mod}^{\mathbf{fd}}_{k[H]} \to \mathbf{VectCon}^{G\times H}(X)$$

by setting $\mathcal{O}_X \otimes_k V$ to be the sheaf of $\mathcal{D}_X$-modules with

$$(\mathcal{O}_X \otimes_k V)(U) = \mathcal{O}_X(U) \otimes_k V$$

for any admissible open subset $U \subset X$, where $x \in \mathcal{D}_X(U)$ acts by

$$x \cdot (s \otimes v) \coloneqq xs \otimes v.$$

This has a $G \times H$-equivariant structure defined by

$$(g,h)^V \coloneqq (g,h)^{\mathcal{O}_X} \otimes h(-)\colon \mathcal{O}_X \otimes_k V \to (g,h)^{-1}(\mathcal{O}_X \otimes_k V),$$

and with this $G \times H$-equivariant structure $\mathcal{O}_X \otimes_k V$ is a $G \times H$-equivariant sheaf of $\mathcal{D}_X$-modules. Given a $k[H]$-module homomorphism $V \to W$, the induced morphism

$$1 \otimes f : \mathcal{O}_X \otimes_k V \to \mathcal{O}_X \otimes_k W$$

is defined in the obvious manner. It is furthermore direct to verify that the functor $\mathcal{O}_X \otimes_k -$ is monoidal.

The Functor $\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, -)$

In the other direction, we have the *solution functor*,

$$\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, -)\colon\ \mathbf{VectCon}^{G\times H}(X) \to \mathbf{Mod}^{\mathbf{fd}}_{k[H]}.$$

Here, for $\mathcal{M} \in \mathbf{VectCon}^{G\times H}(X)$, H acts on $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$ as in Remark 1.6.6, and this restricts to an action of H on $\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$ because the actions of G and H commute.

We first show that $\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$ is well-defined: that if $\mathcal{M}$ is a $(G \times H)$-$\mathcal{D}_X$-module which is coherent as an $\mathcal{O}_X$-module, then $\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$ is finite dimensional as a k-vector space. This is a consequence of the following lemma.

Lemma 4.2.3. *Suppose that X, G and H are as described at the start of this section, and that $\mathcal{M} \in \mathbf{VectCon}^{G\times H}(X)$. Then the natural map*

$$\mathcal{O}_X \otimes_k \mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}) \to \mathcal{M}$$

is $(G\times H)$-$\mathcal{D}_X$-linear and injective. In particular, $\dim_k(\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})) \leq \mathrm{rank}(\mathcal{M})$.

Proof. The $(G \times H)$-$\mathcal{D}_X$-linearity is direct to verify from the definitions. For the injectivity, we proceed as follows. Suppose that $f_1, ..., f_k \in \mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$ are k-linearly independent, and define $e_1, ..., e_k \in \mathcal{M}(X)^G$ by $e_i \coloneqq f_i(1_X)$. These are also k-linearly independent by [3, Lem. 3.1.4]. Because $\mathcal{O}_X$ is irreducible as a G-$\mathcal{D}_X$-module by Lemma 4.2.2, it is sufficient for us to show that the sum

$$\sum_{i=1}^{k} \mathcal{O}_X \cdot e_i \hookrightarrow \mathcal{M}$$

is direct. We prove this by induction on $k \geq 1$. When $k = 1$ this is trivially true, so suppose that the statement is true for some fixed $k \geq 1$, and consider

$$\sum_{i=1}^{k+1} \mathcal{O}_X \cdot e_i.$$

After rearranging the factors if necessary, it is sufficient to show that

$$\left(\bigoplus_{i=1}^{k} \mathcal{O}_X \cdot e_i\right) \bigcap \mathcal{O}_X \cdot e_{k+1} = 0.$$

If this intersection were non-zero, then

$$\left(\bigoplus_{i=1}^{k} \mathcal{O}_X \cdot e_i\right) \bigcap \mathcal{O}_X \cdot e_{k+1} = \mathcal{O}_X \cdot e_{k+1},$$

by the irreducibility of $\mathcal{O}_X \cdot e_{k+1}$ (Lemma 4.2.2). We can therefore write

$$e_{k+1} = \lambda_1 e_1 + \cdots + \lambda_k e_k$$

for unique $\lambda_i \in \mathcal{O}_X(X)$. We have that for any admissible open subset U and $\partial \in \Theta_X(U)$,

$$\begin{aligned} 0 = \partial(e_{k+1}) &= \partial(\lambda_1 e_1 + \cdots + \lambda_k e_k), \\ &= \partial(\lambda_1)e_1 + \cdots + \partial(\lambda_k)e_k, \end{aligned}$$

in $\mathcal{M}(U)$, and therefore as the sum is direct, $\partial(\lambda_i) = 0$ for all $i = 1, ..., n$. Therefore, as this holds for each admissible open $U \in \mathcal{B}$, each λ_i is a global section of the sheaf c_X by Lemma 4.1.2. Furthermore, each $\lambda_i \in c_X(X)^G = k$ because each $e_i \in \mathcal{M}(X)^G$. However this is a contradiction, as we know that $e_1, ..., e_k, e_{k+1}$ are linearly independent over k. □

We are interested in spaces which satisfy the following condition.

Definition 4.2.4. We say that X satisfies $(*)$ if

There is an admissible open covering $\mathcal{U}$ of X by elements of $\mathcal{B}'$ such that $c_X(U)^{G_U} = k$ for any $U \in \mathcal{U}$.

Here G_U is the stabiliser of U in G.

Remark 4.2.5. Suppose for simplicity that G is trivial, and so X is geometrically connected, and the above condition is that there is an admissible open covering by geometrically connected elements of $\mathcal{B}'$. Example 4.1.13 shows that the existence of a such a covering as in $(*)$ is not obvious for a general geometrically connected rigid space. By passing to an connected open subset the value of the sheaf c_X can only get larger, hence a covering as in $(*)$ is middle ground between two opposing constraints: we need elements of the covering to be large enough that so that c_X takes value k, but small enough that they are still in $\mathcal{B}'$.

Nevertheless, we can always find such a covering in the following examples.

Lemma 4.2.6. *The space X satisfies $(*)$ in any of the following cases:*

- *k is algebraically closed,*
- *X is a rigid space and each connected component Y of X is quasi-Stein with $c_X(Y)^{G_Y} = k$,*

- *X is a scheme and each connected component of X is geometrically connected.*

Proof. If k is algebraically closed then c_X is simply the constant sheaf $\underline{k}$ any admissible open covering by connected elements of $\mathcal{B}$ will suffice. In the second case we can take $\mathcal{U}$ to be the set of connected components of X. If X is a scheme with geometrically connected connected components, then it is sufficient for us to find such a covering for each connected component, so let us assume that X is geometrically connected. Then because X is smooth, X is geometrically normal, and hence geometrically irreducible. Therefore, any connected open subset is also geometrically irreducible, hence c_X is the constant sheaf $\underline{k}$ and any open cover by connected affine open subsets will suffice. $\square$

Theorem 4.2.7. *Suppose that X, G and H are as described at the start of this section. Then:*

1. *The monoidal functor*

$$\mathcal{O}_X \otimes_k -\colon \mathbf{Mod}^{\mathbf{fd}}_{k[H]} \to \mathbf{VectCon}^{G\times H}(X)$$

is exact and fully faithful.

2. *The essential image of $\mathcal{O}_X \otimes_k -$ is the full subcategory with objects those $\mathcal{M}$ which satisfy*

$$\dim_k(\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})) = \mathrm{rank}(\mathcal{M}),$$

and on this subcategory $\mathrm{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, -)$ is a quasi-inverse for $\mathcal{O}_X \otimes_k -$.

3. *If X satisfies $(*)$ then the essential image of $\mathcal{O}_X \otimes_k -$ is closed under subquotients.*

Proof. To prove point (1), first note that the functor $\mathcal{O}_X \otimes_k -$ is clearly exact and faithful. Let $\mathcal{U}$ be an admissible open covering by elements of $\mathcal{B}$. Given any $(G \times H)$-$\mathcal{D}_X$-linear morphism

$$f\colon \mathcal{O}_X \otimes_k V \to \mathcal{O}_X \otimes_k W,$$

then for each $U \in \mathcal{U}$, $v \in V$ and $\partial \in \Theta_U(U)$,

$$\partial(1_U \otimes v) = \partial(1_U) \otimes v = 0,$$

and hence

$$f_U(1_U \otimes v) \in (\mathcal{O}_X(U) \otimes_k W)^{\Theta_U(U)=0} = c_X(U) \otimes_k W,$$

by Lemma 4.1.2 so

$$f_X(1_X \otimes v) \in c_X(X) \otimes_k W.$$

Now because f is a morphism of G-equivariant sheaves,

$$f_X(1_X \otimes v) \in c_X(X)^G \otimes_k W = k \otimes_k W,$$

which allows us to define a morphism $\lambda\colon V \to W$ uniquely determined by the property that

$$f_X(1_X \otimes v) = 1 \otimes \lambda(v).$$

Because f is H-$\mathcal{D}_X$-linear, λ is $k[H]$-linear, and $f = 1 \otimes \lambda$. Therefore $\mathcal{O}_X \otimes_k -$ is full.

For point (2), by Lemma 4.2.3 we have an $(G \times H)$-$\mathcal{D}_X$-linear injection

$$\mathcal{O}_X \otimes_k \operatorname{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}) \hookrightarrow \mathcal{M} \tag{4.1}$$

for any $\mathcal{M} \in \mathbf{VectCon}^{G\times H}(X)$, and by taking the quotient we can extend this to a short exact sequence,

$$0 \to \mathcal{O}_X \otimes_k \operatorname{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}) \to \mathcal{M} \to \mathcal{N} \to 0.$$

The quotient $\mathcal{N}$ is coherent as an $\mathcal{O}_X$-module, and therefore by Lemma 1.7.3, $\mathcal{N}$ is in fact locally free. Therefore, if $\operatorname{rank}(\mathcal{O}_X \otimes_k \operatorname{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})) = \operatorname{rank}(\mathcal{M})$, then $\operatorname{rank}(\mathcal{N}) = 0$ and thus $\mathcal{N} = 0$, so (4.1) is an isomorphism. Consequently, if $\dim_k \operatorname{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}) = \operatorname{rank}(\mathcal{M})$ then $\mathcal{M}$ is in the essential image of $\mathcal{O}_X \otimes_k -$ and $\operatorname{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, -)$ provides a right quasi-inverse to $\mathcal{O}_X \otimes_k -$ on this full subcategory.

On the other hand, if $\mathcal{M} = \mathcal{O}_X \otimes_k V$ is in the image of $\mathcal{O}_X \otimes_k -$, then

$$\operatorname{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X \otimes_k V) = \operatorname{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X \otimes_k k, \mathcal{O}_X \otimes_k V) = V,$$

which follows from the fully faithfulness of $\mathcal{O}_X \otimes_k -$ (part (1)) in the case that H is trivial. Therefore, any $\mathcal{M}$ in the essential image satisfies $\dim_k \operatorname{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}) = \operatorname{rank}(\mathcal{M})$, and $\operatorname{Hom}_{G\text{-}\mathcal{D}_X}(\mathcal{O}_X, -)$ provides a left quasi-inverse to $\mathcal{O}_X \otimes_k -$.

For point (3), suppose now that X satisfies $(*)$. We would like to show that the essential image is closed under sub-quotients. Because the functor is exact and full, it is sufficient to show that the essential image is closed under sub-objects. Given $V \in \mathbf{Mod}^{\mathbf{fd}}_{k[H]}$, suppose that $\mathcal{F} \subset \mathcal{O}_X \otimes_k V$ is a coherent $(G \times H)$-$\mathcal{D}_X$-submodule. Define

$$W := \{v \in V \mid s \otimes v \in \mathcal{F}(U) \text{ for any open } U \subset X, s \in \mathcal{O}_X(U)\}.$$

We first show that W is a $k[H]$-submodule of V. Indeed, suppose that $v \in W$, U is an open subset of X, $h \in H$, and $s \in \mathcal{O}_X(U)$. In order to show that $s \otimes h \cdot v \in \mathcal{F}(U)$, define

$$t := (h^{-1})^{\mathcal{O}_X}_U(s) \in \mathcal{F}(h^{-1}(U)).$$

Because $v \in W$, $t \otimes v \in \mathcal{F}(h^{-1}(U)))$, and because $\mathcal{F}$ is a H-$\mathcal{D}_X$-submodule $h^V(t \otimes v) \in \mathcal{F}(U)$, and thus $h \cdot v \in W$ because

$$s \otimes h \cdot v = h^{\mathcal{O}_X}_{h^{-1}(U)}(t) \otimes h \cdot v = h^V(t \otimes v) \in \mathcal{F}(U).$$

We now claim that

$$\mathcal{F} = \mathcal{O}_X \otimes_k W.$$

By definition $\mathcal{O}_X \otimes_k W \hookrightarrow \mathcal{F}$. Let $\mathcal{U}$ be an admissible open covering of X as given by condition $(*)$. Because $\mathcal{F}$ is coherent it is sufficient to show that for each $U \in \mathcal{U}$,

$$\mathcal{O}_X(U) \otimes_k W \hookrightarrow \mathcal{F}(U)$$

is surjective. Therefore, suppose that $U \in \mathcal{U}$ and

$$m = \sum_i s_i \otimes v_i \in \mathcal{F}(U) \subset \mathcal{O}_X(U) \otimes_k V,$$

is any element of $\mathcal{F}(U)$. Without loss of generality we may assume that the elements $\{s_i\}_i$ are k-linearly independent, by choosing a basis of the k-span of $\{s_i\}_i$. Now $\mathcal{O}_X(U)$ is a simple $\mathcal{D}_X(U) \rtimes G_U$-module by Proposition 1.8.3 and Lemma 1.7.2. Furthermore

$$\operatorname{End}_{\mathcal{D}_X(U)\rtimes G_U}(\mathcal{O}_X(U)) = c_X(U)^{G_U} = k$$

by Lemma 4.1.3 and the condition (∗), and so for any index j we may apply the Jacobson Density Theorem to assert the existence of some $x_j \in \mathcal{D}_X(U) \rtimes G_U$ such that x_j acts on the k-span of $\{s_i\}_i$ by the projection to the basis element s_j. Therefore for all j,

$$x_j \left(\sum_i s_i \otimes v_i \right) = s_j \otimes v_j \in \mathcal{F}(U),$$

thus $1 \otimes v_j \in \mathcal{F}(U)$, using again that $\mathcal{O}_X(U)$ is a simple $\mathcal{D}_X(U) \rtimes G_U$-module and the fact that $\mathcal{D}_X(U) \rtimes G_U$ only acts on the first factor. But then $\mathcal{O}_X \otimes \langle v_j \rangle \subset \mathcal{F}$ by Lemma 4.2.2, as G-$\mathcal{D}_X$ only acts on the first factor. Therefore, each $v_j \in W$, and thus $m \in \mathcal{O}_X(U) \otimes_k W$ and $\mathcal{O}_X(U) \otimes_k W \hookrightarrow \mathcal{F}(U)$ is surjective. □

4.3 Finite Equivariant Vector Bundles with Connection

Throughout this section X and G are as described at the start of Section 4.2, with H taken to be trivial, and we assume that X has a k-rational point $z \in X(k)$. In this section we consider the full (rigid tensor) subcategory

$$\mathbf{VectCon}^G(X)_{\text{fin}} \subset \mathbf{VectCon}^G(X)$$

of G-equivariant vector bundles with connection on X which are *finite*. This is a notion that was introduced by André Weil [64] in the context of vector bundles on complex projective varieties, who showed that the pushforward of the structure sheaf along a finite étale morphism is a finite vector bundle. The converse was shown by Nori for proper integral schemes over a field with a rational point [51]. In this section we prove the corresponding version of their results in our context, for the category $\mathbf{VectCon}^G(X)$. The idea to remove the properness assumption and replace this by the extra data of an integrable connection was first considered by Esnault and Hai [29]. Our method of proof is most similar in style to the approach of Biswas and O'Sullivan [11], who prove the analogue of Nori's result when X is a complex analytic space and G is a complex Lie group.

$\mathbf{VectCon}^G(X)$ as a Tannakian Category

As described in Section 1.8 $\mathbf{VectCon}^G(X)$ has a natural tensor structure and as such is a rigid abelian tensor category. An identity object of $\mathbf{VectCon}^G(X)$ (in the sense of [26]) is given by $\mathcal{O}_X$, and by our assumption that $c_X(X)^G = k$ and Lemma 4.1.3, $\operatorname{End}(\mathcal{O}_X) = k$ in $\mathbf{VectCon}^G(X)$. Furthermore, the k-rational point $z \in X(k)$ allows us to define a fibre functor

$$\omega_z \colon \mathbf{VectCon}^G(X) \to \mathbf{Vect}_k, \qquad \omega_z \colon \mathcal{V} \mapsto \mathcal{V}(z) := \mathcal{V}_z \otimes_{\mathcal{O}_{X,z}} k(z),$$

which is exact because the sheaves in $\mathbf{VectCon}^G(X)$ are locally free. By [26, Prop. 1.19], ω_z is faithful, and thus $\mathbf{VectCon}^G(X)$ is in this way a neutral Tannakian category [26, Prop. 1.20].

For now, let $\mathcal{C}$ be any neutral Tannakian category in the sense of [26, Def. 2.19]. We will later specialise to $\mathcal{C} = \mathbf{VectCon}^G(X)$. We have the following immediate property of $\mathcal{C}$.

Lemma 4.3.1. *The Krull-Remak-Schmidt Theorem holds in $\mathcal{C}$: any non-zero object $\mathcal{V} \in \mathcal{C}$ can be written as a direct sum*

$$\mathcal{V} \cong \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n,$$

of indecomposable objects, and any such description is unique up to isomorphism and permutation.

Proof. This follows from [4, Thm. 1], using [4, §3 Cor.] and the fact that $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is finite dimensional over k for any $\mathcal{V}, \mathcal{W} \in \mathcal{C}$ because the fibre functor is faithful. □

Definition 4.3.2. For $f = \sum_{n \geq 0} a_n x^n \in \mathbb{Z}_{\geq 0}[x]$ and $\mathcal{V} \in \mathcal{C}$, we define

$$f(\mathcal{V}) := \bigoplus_{n \geq 0} (\mathcal{V}^{\otimes n})^{\oplus a_n} \in \mathcal{C}.$$

Definition 4.3.3. $\mathcal{V} \in \mathcal{C}$ is called *finite* if there exist $f, g \in \mathbb{Z}_{\geq 0}[x]$ with $f \neq g$ and $f(\mathcal{V}) \cong g(\mathcal{V})$. We write $\mathcal{C}_{\mathrm{fin}}$ for the full subcategory of finite objects.

Definition 4.3.4. For $\mathcal{V} \in \mathcal{C}$, we define $I(\mathcal{V})$ to be the set of indecomposable direct summands of $\mathcal{V}$ in $\mathcal{C}$, and

$$S(\mathcal{V}) = \bigcup_{k \geq 0} I(\mathcal{V}^{\otimes k}).$$

Lemma 4.3.5. *Suppose that $\mathcal{V} \in \mathcal{C}$. Then:*

- *$I(\mathcal{V})$ is a finite set,*
- *$S(\mathcal{V})$ is a finite set if and only if $\mathcal{V}$ is finite.*

In particular, if $\mathcal{L} \in \mathbf{VectCon}^G(X)$ has rank 1, then $\mathcal{L}$ is finite if and only if $\mathcal{L}$ is torsion.

Proof. Let $\mathcal{V} \in \mathcal{C}$. That $I(\mathcal{V})$ is finite is a direct consequence of the Krull-Remak-Schmidt Theorem (Lemma 4.3.1 above). For the second point we follow the proof of [59, Prop. 6.7.4], which we include for the readers convenience. Suppose that $S(\mathcal{V})$ is finite, and consider the free abelian group A generated by the isomorphism classes $[\mathcal{W}]$ of indecomposable objects of $\mathcal{C}$, with subgroup $A(\mathcal{V})$ generated by the set $S(\mathcal{V})$. There is a well defined $\mathbb{Z}$-linear map $m_{\mathcal{V}} \colon A \to A$ defined on each generator $[\mathcal{W}]$ to be

$$m_{\mathcal{V}}([\mathcal{W}]) := [\mathcal{V} \otimes \mathcal{W}] := [\mathcal{W}_1] + \cdots + [\mathcal{W}_n],$$

where

$$\mathcal{V} \otimes \mathcal{W} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n$$

is the unique decomposition into indecomposable objects of Lemma 4.3.1, and this linear map preserves $A(\mathcal{V})$. By the assumption that $S(\mathcal{V})$ is finite, $A(\mathcal{V})$ is a finitely generated free abelian group, and we may therefore consider the characteristic polynomial $\chi \in \mathbb{Z}[x]$ of $m_{\mathcal{V}}$, which is monic and satisfies $\chi(m_{\mathcal{V}}) = 0$. If we write $\chi = f - g$, where $f, g \in \mathbb{Z}_{\geq 0}[x]$, then applying $\chi(m_{\mathcal{V}})$ to $[\mathcal{O}_X]$ we see that $f(\mathcal{V}) \cong g(\mathcal{V})$.

Conversely, suppose that we have $f, g \in \mathbb{Z}_{\geq 0}[x]$ with $f \neq g$, and that $f(\mathcal{V}) \cong g(\mathcal{V})$. If $d > 0$ is the degree of $f - g$, then by the uniqueness of Lemma 4.3.1 we may write any element of $I(\mathcal{V}^{\otimes d})$ as a sum of elements of $I(\mathcal{V}^{\otimes k})$ for $k < d$. Similarly, for any $i \geq 0$, using the isomorphism $(x^i f)(\mathcal{V}) \cong (x^i g)(\mathcal{V})$, we can write, for any $m \geq d$, $\mathcal{V}^{\otimes m}$ as the sum of elements of $I(\mathcal{V}^{\otimes k})$ for $k < d$, and thus

$$S(\mathcal{V}) \subset \bigcup_{k=0}^{d-1} I(\mathcal{V}^{\otimes k})$$

and thus $S(\mathcal{V})$ is finite. For the final claim, note that because $\mathrm{End}(\mathcal{O}_X) = k$, the rank of any object of $\mathbf{VectCon}^G(X)$ is $\mathbb{N}$-valued, and so if $\mathcal{L} \in \mathbf{VectCon}^G(X)$ has rank 1 then each $\mathcal{L}^{\otimes k}$ is indecomposable. In particular, $S(\mathcal{L}) = \{\mathcal{L}^{\otimes k}\}_{k \geq 0}$ and thus $\mathcal{L}$ is finite if and only if $\mathcal{L}$ is torsion. □

Corollary 4.3.6. *The subcategory $\mathcal{C}_{\mathrm{fin}}$ is closed under duals, direct sums, direct summands, and tensor products.*

Proof. Each statement follows using the characterisation of Lemma 4.3.5 that an object $\mathcal{V}$ is finite if and only if $S(\mathcal{V})$ is finite. For example, in the case of duals, if $\mathcal{V}$ is finite then $\mathcal{V}^*$ is too, as we have a bijection $(-)^* \colon S(\mathcal{V}) \xrightarrow{\sim} S(\mathcal{V}^*)$. □

Now we specialise to the case where $\mathcal{C} = \mathbf{VectCon}^G(X)$.

Proposition 4.3.7. *Suppose that $f \colon Z \to X$ is a G-equivariant finite étale Galois covering with Galois group H, and that the actions of G and H commute. Then $f_*\mathcal{O}_Z \in \mathbf{VectCon}^G(X)_{\mathrm{fin}}$.*

Proof. Let $\mathcal{A} \coloneqq f_*\mathcal{O}_Z$. Writing H for the Galois group of $Z \to X$, the Galois isomorphism

$$Z \times H \xrightarrow{\sim} Z \times_X Z$$

induces an isomorphism of $\mathcal{A} \otimes \mathcal{A}$ with the direct sum of $|H|$ copies of $\mathcal{A}$ in $\mathbf{VectCon}^G(X)$ (cf. Lemma 4.4.1), this isomorphism being G-equivariant because the actions of G and H commute. Therefore, $I(\mathcal{A}^{\otimes k}) \subset I(\mathcal{A})$ for all $k \geq 1$, and thus $\mathcal{A}$ is finite by Lemma 4.3.5. □

Now we would like to show the converse. In the following we will make use of the following notion. For a set $\mathcal{S}$ of objects of a rigid abelian tensor category $\mathcal{C}$, we write $\mathcal{C}(\mathcal{S})$ for the full subcategory of $\mathcal{C}$ with objects V/W for pairs of objects V, W of $\mathcal{C}$ with

$$W \subset V \subset \bigoplus_{i=1}^{k} S_i$$

for some $k \geq 1$ and $S_1, ..., S_k \in \mathcal{S}$. This is easily checked to be an abelian subcategory of $\mathcal{C}$, which is rigid whenever S is closed under duality. This is further a tensor subcategory of $\mathcal{C}$ whenever the tensor product of any two objects of S is a sub-object of a direct sum of elements of S.

Proposition 4.3.8. *Suppose that $\mathcal{V} \in \mathbf{VectCon}^G(X)_{\mathrm{fin}}$ and k is algebraically closed. Then there is some G-equivariant finite étale Galois covering $f\colon Z \to X$ such that the action of G commutes with the Galois action and $\mathcal{V}$ is a direct summand of $f_*\mathcal{O}_Z$.*

Proof. Suppose that $\mathcal{V} \in \mathbf{VectCon}^G(X)_{\mathrm{fin}}$, and write $\mathcal{C}$ for the full abelian subcategory $\mathcal{C}(S(\mathcal{V}) \cup S(\mathcal{V}^*))$ of $\mathbf{VectCon}^G(X)$ defined above. The set $S(\mathcal{V}) \cup S(\mathcal{V}^*)$ is closed under duality and the tensor product of any two objects of S is a sub-object of a direct sum of elements of S because $\mathcal{V}$ is finite, hence $\mathcal{C}$ is a rigid abelian tensor subcategory. In particular, with the fibre functor ω_z, $\mathcal{C}$ is a neutral Tannakian category in its own right. Therefore, by [26, Thm. 2.11], there is an equivalence of categories

$$F\colon \mathbf{Rep}_k(H) \to \mathcal{C}$$

for some affine group scheme H over k, which is finite because every object of $\mathcal{C}$ is a sub-quotient of a direct sum of copies of X, the direct sum of all elements of $S(\mathcal{V}) \cup S(\mathcal{V}^*)$ [26, Prop. 2.20(a)]. Note further, that because k has characteristic 0 and is algebraically closed, H is a constant group. Because $\mathcal{V}$ is in the essential image of F, $\mathcal{V}$ is a direct summand of a finite direct sum of copies of $\mathcal{A} \coloneqq F(\mathcal{O}(H))$. Therefore, it is sufficient for us to show that there is a G-equivariant finite étale Galois covering $f\colon Z \to X$ such that $\mathcal{A} = f_*\mathcal{O}_Z$. Indeed, given such a covering $f\colon Z \to X$ with Galois group H, then for any $n \geq 1$ we can consider the G-equivariant Galois covering given by

$$h\colon Y \coloneqq \bigsqcup_{i=1}^{n} Z \to X,$$

with Galois group $C_n \times H$ where C_n permutes the disjoint union, which has $h_*\mathcal{O}_Y = \mathcal{A}^{\oplus n}$.

We construct such a covering $f\colon Z \to X$ as follows. First, interpreting $\mathbf{Rep}_k(\mathcal{O}(H))$ as the category of finite dimensional $\mathcal{O}(H)$-comodules, we note that the algebra multiplication

$$m\colon \mathcal{O}(H) \otimes \mathcal{O}(H) \to \mathcal{O}(H)$$

is an $\mathcal{O}(H)$-comodule homomorphism, and therefore we may apply F to obtain

$$F(m)\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A},$$

which gives $\mathcal{A}$ the structure of a finite sheaf of $\mathcal{O}_X$-algebras. We may therefore define $f\colon Z \to X$ to be the unique finite covering corresponding to $\mathcal{A}$ with $f_*\mathcal{O}_Z = \mathcal{A}$ (as sheaves of $\mathcal{O}_X$-algebras). From the G-equivariant structure on $\mathcal{A} = F(\mathcal{O}(H))$, as $F(m)$ is a morphism of G-equivariant sheaves of $\mathcal{O}_X$-modules we obtain an action of the abstract group G on Z for which the morphism $f\colon Z \to X$ is G-equivariant, and $f_*\mathcal{O}_Z = \mathcal{A}$ as G-$\mathcal{O}_X$-modules.

We further define an action of H on Z as follows. For any $h \in H$, we have the $\mathcal{O}(H)$-comodule homomorphism

$$r_h\colon \mathcal{O}(H) \to \mathcal{O}(H), \qquad l_h(\phi) = \phi(-h),$$

which over all H defines a group homomorphism $H \to \mathrm{Aut}_{\mathrm{Comod}}(\mathcal{O}(H))$. Therefore, we obtain a left action of H on $\mathcal{A}$ by letting $h \in H$ act by $F(r_h)$. Because each r_h is furthermore an k-algebra homomorphism of $\mathcal{O}(H)$, this action gives an action of $H \to \mathrm{Aut}(Z)$ on the scheme Z, which commutes with the action of G and for which the morphism $f\colon Z \to X$ is equivariant with respect to the trivial action of H on X. To check that this is finite étale Galois, first note that the morphism $\mathcal{O}_X \to \mathcal{O}_Z^H$ is an isomorphism because this is the image under F of the morphism $k \to \mathcal{O}(H)^H$ which is itself an isomorphism. The morphism

$$p_Z \times a\colon Z \times H \to Z \times_X Z$$

corresponds to the morphism

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes_k \mathcal{O}(H)$$

in $\mathbf{VectCon}^G(X)$, which is an isomorphism precisely because

$$\mathcal{O}(H) \otimes \mathcal{O}(H) \to \mathcal{O}(H) \otimes \mathcal{O}(H), \qquad \phi \otimes \psi \mapsto \sum_{h \in H} \phi\psi(-h) \otimes \delta_h$$

is an isomorphism. The fact that $f\colon Z \to X$ is finite étale follows from the fact that p_Z is an isomorphism as, working locally with $U \in \mathcal{B}_X$, the Galois extension of commutative k-algebras $\mathcal{O}_X(U) \to \mathcal{O}_Z(f^{-1}(U))$ is automatically finite étale by [21, Thm. 1.3(a)].

Finally, we need to verify that the equality $f_*\mathcal{O}_Z = \mathcal{A}$ of G-$\mathcal{O}_X$-modules is actually an equality of G-$\mathcal{D}_X$-modules, for which it suffices to show that for any $U \in \mathcal{B}_X$, the action of $\mathcal{T}(U)$ on $\mathcal{A}(U)$ is the same as the natural action of $\mathcal{T}(U)$ on $(f_*\mathcal{O}_Z)(U) = \mathcal{O}(V)$, where $V \coloneqq f^{-1}(U)$.

By definition of the $\mathcal{D}$-module pushforward (Section 1.7.1), $\mathcal{T}(U)$ acts on $\mathcal{O}(V)$ via the $\mathcal{O}(U)$-linear map $\psi\colon \mathcal{T}(U) \to \mathcal{T}(V)$, which is uniquely characterised amongst functions $\mathcal{T}(U) \to \mathcal{T}(V)$ by the property that $\iota \circ \partial = \psi(\partial) \circ \iota$ for all $\partial \in \mathcal{T}(U)$, where $\iota\colon \mathcal{O}(U) \hookrightarrow \mathcal{O}(V)$ denotes the inclusion map (Lemma 1.4.1).

For the action of $\mathcal{T}(U)$ on $\mathcal{A}(U)$, note that $\mathcal{T}(U)$ acts via derivations on $\mathcal{A}(U)$ because the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a morphism in $\mathbf{VectCon}^G(X)$, and write $\phi\colon \mathcal{T}(U) \to \mathrm{Der}_k(\mathcal{A}(U))$ for the induced map. Then, noting that $F(k) = \mathcal{O}_X$, applying F to the inclusion $k \hookrightarrow \mathcal{O}(H)$ we see that

$$\mathcal{O}_X \hookrightarrow \mathcal{A}$$

is a morphism in $\mathbf{VectCon}^G(X)$ and thus

$$\iota\colon \mathcal{O}(U) \hookrightarrow \mathcal{A}(U)$$

is $\mathcal{D}(U)$-linear. In particular, $\iota \circ \partial = \phi(\partial) \circ \iota$ for all $\partial \in \mathcal{T}(U)$, and hence $\phi = \psi$. □

4.4 $\mathcal{D}$-Modules and Galois Coverings

In this section we prove our main results concerning Galois coverings and $\mathcal{D}$-modules, Theorem 4.4.2 and Theorem 4.4.4. We specialise our geometric framework of the previous sections further and suppose that $\mathrm{char}(k) = 0$, G, H and N are groups, $N \triangleleft H$ a normal subgroup of H,

(A) $f\colon X \to Y$ is a Galois covering of smooth schemes over k with Galois group N,

(B) $f\colon X \to Y$ is a Galois covering of smooth rigid spaces over k with Galois group N,

and X and Y have an action of the abstract group $G \times H$ which extends the action of N and for which $f\colon X \to Y$ is equivariant.

We also fix a section $s\colon H/N \to H$, and write $H_s = s(H/N)$ for the corresponding subgroup of H with $N \cap H_s = 1$ and $NH_s = H$. For example, we could have $H = N$, in which case the only choice for s is the inclusion of the identity into H.

Because $f\colon X \to Y$ is Galois, we have the following commutative diagram,

$$\begin{array}{ccccc} X \times \underline{N} & \xrightarrow{p_X \times a} & X \times_Y X & \xrightarrow{p_2} & X \\ & \searrow^{p_X} & \downarrow{\scriptstyle p_1} & \lrcorner & \downarrow{\scriptstyle f} \\ & & X & \xrightarrow[f]{} & Y \end{array}$$

and the diagram

$$\begin{array}{ccc} X \times \underline{N} & \xrightarrow{\phi} & X \\ {\scriptstyle p_X}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback, where $\phi \coloneqq p_2 \circ (p_X \times a)$. The natural morphism

$$f^* f_* \mathcal{O}_X \to p_{X,*} \phi^* \mathcal{O}_X$$

of sheaves of $\mathcal{O}_X$-modules is an isomorphism because f is finite [32, Cor. 12.7]. Writing

$$X \times \underline{N} = \bigsqcup_{n \in N} X^{(n)},$$

as the disjoint union of copies of X indexed by N, then there is a canonical identification of $\mathcal{O}_X$-modules

$$p_{X,*} \mathcal{O}_{X \times \underline{N}} \xrightarrow{\sim} \mathcal{O}_X \otimes_k \mathcal{O}(N),$$

where for an admissible open subset $U \subset X$,

$$(p_{X,*} \mathcal{O}_{X \times \underline{N}})(U) = \prod_{n \in N} \mathcal{O}_{X^{(n)}}(U) \xrightarrow{\sim} \mathcal{O}_X(U) \otimes_k \mathcal{O}(N), \qquad (s_n)_{n \in N} \mapsto \sum_{n \in N} s_n \otimes \delta_n.$$

We give the sheaf $\mathcal{O}_X \otimes_k \mathcal{O}(N)$ a $(G \times H)$-$\mathcal{D}_X$-module structure using the functor $\mathcal{O}_X \otimes_k -$, where we view $\mathcal{O}(N)$ as a $k[H]$-module via the semi-direct product decomposition $H = N \rtimes H_s$, and letting $(n * \phi)(x) \coloneqq \phi(n^{-1}x)$ for $n \in N$ and $(h * \phi)(x) \coloneqq \phi(h^{-1}xh)$ for $h \in H_s$. Note this this is canonical (independent of the section s) when $H = N$.

We view the direct image $f_* \mathcal{O}_X$ as a $(G \times H/N)$-$\mathcal{D}_Y$-module, through the $(G \times H)$-$\mathcal{D}_Y$-module structure on $f_* \mathcal{O}_X$ described in the construction of the invariant functor of Section 1.10, and composing along the section $s\colon H/N \to H$. The sheaf $f_* \mathcal{O}_X$ is coherent as $f\colon X \to Y$ is finite, and therefore we can naturally consider $f^* f_* \mathcal{O}_X$ as a $(G \times H)$-$\mathcal{D}_X$-module through the equivalence of Section 1.10. Again, we note that the $(G \times H/N)$-$\mathcal{D}_Y$-module structure on $f_* \mathcal{O}_X$ and the $(G \times H)$-$\mathcal{D}_X$-module structure on $f^* f_* \mathcal{O}_X$ are both canonical (independent of the section s) when $H = N$.

Lemma 4.4.1. *The isomorphism*

$$f^* f_* \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X \otimes_k \mathcal{O}(N)$$

is an isomorphism of $(G \times H)$-$\mathcal{D}_X$*-modules.*

Proof. We first show this is a morphism of H-$\mathcal{D}_X$-modules. For any $V \in \mathcal{B}_Y$ and $U \coloneqq f^{-1}(V) \subset X$, then this morphism is given above U by the morphism

$$\beta\colon B \otimes_A B \xrightarrow{\sim} B \otimes_k \mathcal{O}(N), \qquad \beta(x \otimes y) = \sum_{n \in N} x(n \cdot y) \otimes \delta_n,$$

of the Galois extension of commutative k-algebras $A \coloneqq \mathcal{O}_Y(V) \hookrightarrow B \coloneqq \mathcal{O}_X(U)$. It is sufficient to show that β is a morphism of $\mathcal{D}(B) \rtimes H$-modules. For any $h \in N$,

$$\begin{aligned}
\beta(h \cdot (x \otimes y)) &= \beta(h(x) \otimes y),\\
&= \sum_{n \in N} h(x) n(y) \otimes \delta_n,\\
&= \sum_{n \in N} h(x(h^{-1}n \cdot y)) \otimes \delta_n,\\
&= \sum_{n \in N} h(x(h^{-1}n \cdot y)) \otimes h * \delta_{h^{-1}n},\\
&= h \cdot \beta(x \otimes y),
\end{aligned}$$

and for any $h \in H_s$,

$$\begin{aligned}
\beta(h \cdot (x \otimes y)) &= \beta(h(x) \otimes h(y)),\\
&= \sum_{n \in N} h(x) n(h(y)) \otimes \delta_n,\\
&= \sum_{n \in N} h(x(h^{-1}nh \cdot y)) \otimes \delta_n,\\
&= \sum_{n \in N} h(x(n \cdot y)) \otimes \delta_{hnh^{-1}},\\
&= \sum_{n \in N} h(x(n \cdot y)) \otimes h * \delta_n,\\
&= h \cdot \beta(x \otimes y).
\end{aligned}$$

Therefore β is a H-equivariant. Now to show that β is $\mathcal{D}(B)$-equivariant, it suffices by Lemma 1.4.1 to show that for any $b \in B$ and $\partial \in \mathrm{Der}_k(A)$, that β commutes with the action of $b\partial$. Indeed,

$$\begin{aligned}
\beta(b\partial \cdot (x \otimes y)) &= \beta(b\partial(x) \otimes y + bx \otimes \partial(y)),\\
&= \sum_{n \in N} b(\partial(x) n(y) + x n(\partial(y))) \otimes \delta_n,\\
&= \sum_{n \in N} b(\partial(x) n(y) + x \partial(n(y))) \otimes \delta_n,\\
&= \sum_{n \in N} b\partial(x n(y)) \otimes \delta_n,\\
&= b\partial \cdot \beta(x \otimes y).
\end{aligned}$$

Here we have used that the action of N and $\mathrm{Der}_k(A)$ on B commute by Lemma 1.9.7. For the G-equivariance, let $g \in G$, and consider the diagram,

$$\begin{array}{ccccc}
\mathcal{O}_X(U) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U) & \xrightarrow{\equiv} & (f^*f_*\mathcal{O}_X)(U) & \xrightarrow{\beta} & \mathcal{O}_X(U) \otimes_k \mathcal{O}(N) \\
\downarrow{\scriptstyle g_U^{\mathcal{O}_X} \otimes g_U^{\mathcal{O}_X}} & & \downarrow{\scriptstyle g_U^{f^*f_*\mathcal{O}_X}} & & \downarrow{\scriptstyle g_U^{\mathcal{O}_X \otimes_k \mathcal{O}(N)} = g_U^{\mathcal{O}_X} \otimes 1} \\
\mathcal{O}_X(g(U)) \otimes_{\mathcal{O}_Y(g(V))} \mathcal{O}_X(g(U)) & \xrightarrow{\equiv} & (f^*f_*\mathcal{O}_X)(g(U)) & \xrightarrow[\beta]{} & \mathcal{O}_X(g(U)) \otimes_k \mathcal{O}(N)
\end{array}$$

The left-hand square commutes by the definition of the G-equivariant structure on $f^*f_*\mathcal{O}_X$, and because $\{f^{-1}(V) \mid V \in \mathcal{B}_Y\}$ is an affinoid cover of X and the morphism is between coherent sheaves, it is sufficient to show that the outer square commutes. For this, given $x, y \in \mathcal{O}_X(U)$ we can compute,

$$\begin{aligned}
(g_U^{\mathcal{O}_X} \otimes 1)(\beta(x \otimes y)) &= \sum_{n \in N} g_U^{\mathcal{O}_X}(x(n \cdot y)) \otimes \delta_n, \\
&= \sum_{n \in N} g_U^{\mathcal{O}_X}(x)(n \cdot g_U^{\mathcal{O}_X}(y)) \otimes \delta_n, \\
&= \beta(g_U^{\mathcal{O}_X}(x) \otimes g_U^{\mathcal{O}_X}(y)),
\end{aligned}$$

where we have used that $g_U^{\mathcal{O}_X} \colon \mathcal{O}_X(U) \to \mathcal{O}_X(g(U))$ is a ring homomorphism and that the actions of G and N on X commute. $\square$

For the next theorem we will use that, viewing $\mathcal{O}(N)$ as a $k[H]$-module as above,

$$k[N] \to \mathcal{O}(N), \qquad n \mapsto \delta_n$$

is an isomorphism of $k[H]$-modules, where we view $k[N]$ as a $k[H]$ via left multiplication by N and where $h \in H_s$ acts on $n \in N$ by $h * n = hnh^{-1}$. Now

$$k[N] \to k[N]^{\mathrm{op}}, \qquad n \mapsto n^{-1},$$

an isomorphism of k-algebras and in particular,

$$k[N] \xrightarrow{\sim} k[N]^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{k[N]}(k[N]) \xrightarrow{\sim} \mathrm{End}_{k[N]}(\mathcal{O}(N)).$$

Explicitly, if we compose this with the natural inclusion,

$$k[N] \xrightarrow{\sim} \mathrm{End}_{k[N]}(\mathcal{O}(N)) \hookrightarrow \mathrm{End}_k(\mathcal{O}(N)),$$

this is the $k[N]$-module structure $(n \star f)(x) = f(xn)$ on $\mathcal{O}(N)$. Furthermore, it is direct to verify that this induces an isomorphism between the subrings

$$k[N]^{H/N} \xrightarrow{\sim} \mathrm{End}_{k[H]}(\mathcal{O}(N)),$$

where H/N acts on $k[N]$ via the section $s \colon H/N \to H$ and the action of $H_s = s(H/N)$ on $k[N]$.

Theorem 4.4.2. *Suppose that $c_X(X)^G = k$. Then:*

1. *The functor*

$$(\mathcal{O}_X \otimes_k -)^N : \mathbf{Mod}^{\mathbf{fd}}_{k[H]} \to \mathbf{VectCon}^{G\times H/N}(Y),$$

is exact and fully faithful.

2. *For any* $\mathcal{M} \in \mathbf{VectCon}^{G\times H/N}(Y)$,

$$\mathrm{Hom}_{(G\times H)\text{-}\mathcal{D}_X}(\mathcal{O}_X, f^*\mathcal{M}) \leq \mathrm{rank}_Y(\mathcal{M}),$$

and $\mathcal{M}$ *is in the essential image of* $(\mathcal{O}_X \otimes_k -)^H$ *if and only if this is an equality.*

3. *The regular representation* $\mathcal{O}(N)$ *is sent to the* $(G \times H/N)$-$\mathcal{D}_Y$*-module* $f_*\mathcal{O}_X$.

4. *The natural map,*

$$k[N]^{H/N} \to \mathrm{End}_{(G\times H/N)\text{-}\mathcal{D}_Y}(f_*\mathcal{O}_X),$$

is an isomorphism.

5. *If* $H = N$*, then* $(\mathcal{O}_X \otimes_k V)^N \in \mathbf{VectCon}^G(Y)_{\mathrm{fin}}$ *for any* $V \in \mathbf{Mod}^{\mathbf{fd}}_{k[H]}$.

If furthermore X *satisfies* $(*)$*, then:*

(6) *The essential image is closed under sub-quotients.*

(7) *If* $H = N$*, then the essential image can also be described as the full subcategory of objects which admit a* G-$\mathcal{D}_Y$*-linear embedding into* $(f_*\mathcal{O}_X)^{\oplus n}$ *for some* $n \geq 1$.

Remark 4.4.3. When $H = N$, the $(G \times H/N)$-$\mathcal{D}_Y$-module structure on $f_*\mathcal{O}_X$ and the $K[H]$-module structures on both $\mathcal{O}(N)$ and $k[N]$ are all canonical. In general, they each depend on the choice of the section $s\colon H/N \to H$. We note however that the functor $(\mathcal{O}_X \otimes_k -)^N$ is independent of the choice of section $s\colon H/N \to H$.

Proof. The functor $(\mathcal{O}_X \otimes_k -)^N$ is the composition

$$\mathbf{Mod}^{\mathbf{fd}}_{k[H]} \to \mathbf{VectCon}^{G\times H}(X) \xrightarrow{\sim} \mathbf{VectCon}^{G\times H/N}(Y),$$

of the functors of Theorem 4.2.7 and Proposition 1.10.1 and therefore is exact, fully faithful, and when X satisfies $(*)$ the essential image is closed under sub-quotients. This gives statements (1), (6) and (7). Statement (2) is simply a restatement of Theorem 4.2.7(2), noting that $\mathrm{rank}_Y(\mathcal{M}) = \mathrm{rank}_X(f^*\mathcal{M})$.

For (3) and (4), the equivalence of categories of Proposition 1.10.1 provides an isomorphism of $(G \times H/N)$-$\mathcal{D}_Y$-modules

$$f_*\mathcal{O}_X \xrightarrow{\sim} (f^*f_*\mathcal{O}_X)^N,$$

and composing with $(-)^N$ applied to the isomorphism of Lemma 4.4.1,

$$f_*\mathcal{O}_X \xrightarrow{\sim} (\mathcal{O}_X \otimes_k \mathcal{O}(N))^N$$

is an isomorphism of $(G \times H/N)$-$\mathcal{D}_Y$-modules.

There is an action of N on $f_*\mathcal{O}_X$ with commutes with the action of $\mathcal{D}_Y$ on $f_*\mathcal{O}_X$ as explained in the construction of the functor $(-)^N$, and thus we have a natural map,

$$k[N]^{H/N} \to \mathrm{End}_{(G\times H/N)\text{-}\mathcal{D}_Y}(f_*\mathcal{O}_X).$$

This is an isomorphism, as it forms part of the commutative diagram,

$$\begin{array}{ccccc} k[N]^{H/N} & & \xrightarrow{\sim} & & \mathrm{End}_{k[H]}(\mathcal{O}(N)) \\ \downarrow & & & & \downarrow\wr \\ \mathrm{End}_{(G\times H/N)\text{-}\mathcal{D}_Y}(f_*\mathcal{O}_X) & \xrightarrow{\sim} & \mathrm{End}_{(G\times H)\text{-}\mathcal{D}_X}(f^*f_*\mathcal{O}_X) & \xrightarrow{\sim} & \mathrm{End}_{(G\times H)\text{-}\mathcal{D}_X}(\mathcal{O}_X\otimes_k\mathcal{O}(N)) \end{array}$$

by the above discussion. Statement (5) follows directly from Lemma 4.3.7. □

For each $\rho \in \mathrm{Irr}(N)$, we let

$$e_\rho = \frac{\dim(\rho)}{d_\rho \cdot |N|} \sum_{h\in N} \chi_\rho(h^{-1})h \in k[N],$$

be the corresponding central primitive idempotent, where

$$d_\rho \coloneqq \dim_k \mathrm{End}_{k[N]}(\rho).$$

From Theorem 4.4.2, taking $H = N$ we have the following explicit decomposition of $f_*\mathcal{O}_X$ as an object of $\mathbf{VectCon}^G(Y)$.

Theorem 4.4.4. *Suppose that $c_X(X)^G = k$ and X satisfies $(*)$. Then:*

1. *$f_*\mathcal{O}_X$ is a semisimple object of $\mathbf{VectCon}^G(Y)$, with isotypic decomposition,*

$$f_*\mathcal{O}_X = \bigoplus_{\rho\in\mathrm{Irr}(N)} e_\rho \cdot f_*\mathcal{O}_X.$$

2. *For any $\rho \in \mathrm{Irr}(N)$, there is a one-to-one correspondence between decompositions of e_ρ as a sum of primitive orthogonal idempotents in $e_\rho \cdot k[N]$ and decompositions of $e_\rho \cdot f_*\mathcal{O}_X$ as a direct sum of irreducible objects of $\mathbf{VectCon}^G(Y)$. Here we send,*

$$e_\rho = \sum_{i=1}^{\dim(\rho)/d_\rho} e_i \qquad \longmapsto \qquad e_\rho \cdot f_*\mathcal{O}_X = \bigoplus_{i=1}^{\dim(\rho)/d_\rho} e_i \cdot f_*\mathcal{O}_X.$$

3. *In particular, $e_\rho \cdot f_*\mathcal{O}_X$ has length $\dim(\rho)/d_\rho$ in $\mathbf{VectCon}^G(Y)$.*

4. *For any irreducible $\mathcal{O}_X$-coherent G-$\mathcal{D}_Y$-submodule $\mathcal{M}$ of $e_\rho \cdot f_*\mathcal{O}_X$, $\mathcal{M} = e \cdot f_*\mathcal{O}_X$ for some primitive idempotent e of $e_\rho \cdot k[N]$, the natural k-algebra homomorphisms*

$$\begin{aligned} e \cdot k[N] \cdot e &\to \mathrm{End}_{G\text{-}\mathcal{D}_Y}(e \cdot f_*\mathcal{O}_X), \\ e_\rho \cdot k[N] &\to \mathrm{End}_{G\text{-}\mathcal{D}_Y}(e_\rho \cdot f_*\mathcal{O}_X), \end{aligned}$$

   are isomorphisms, and

5. *$e \cdot f_*\mathcal{O}_X$ has rank $\dim(\rho)$ as a vector bundle over Y.*

Proof. Because the idempotents e_ρ are orthogonal and sum to 1, we have the decomposition,

$$f_*\mathcal{O}_X = \bigoplus_{\rho \in \mathrm{Irr}(N)} e_\rho \cdot f_*\mathcal{O}_X. \tag{4.2}$$

Because of the isomorphism,

$$k[N] \to \mathrm{End}_{G\text{-}\mathcal{D}_Y}(f_*\mathcal{O}_X), \tag{4.3}$$

the natural map,

$$e_\rho \cdot k[N] \to \mathrm{End}_{G\text{-}\mathcal{D}_Y}(e_\rho \cdot f_*\mathcal{O}_X),$$

is an isomorphism, because e_ρ is a central idempotent. Therefore, irreducible $\mathcal{O}_X$-coherent G-$\mathcal{D}_Y$ submodules of $e_\rho \cdot f_*\mathcal{O}_X$ are exactly the $e \cdot f_*\mathcal{O}_X$ for $e \in e_\rho \cdot k[N]$ a primitive idempotent, and for such an element e the natural map,

$$e \cdot k[N] \cdot e \to \mathrm{End}_{G\text{-}\mathcal{D}_Y}(e \cdot f_*\mathcal{O}_X),$$

is an isomorphism. From the isomorphism (4.3) and the fact that the idempotents e_ρ are orthogonal, there are no non-zero homomorphisms between the factors of the decomposition (4.2). In order to see that e_ρ is a direct sum of isomorphic irreducible $\mathcal{O}_X$-coherent G-$\mathcal{D}_Y$-modules, take any decomposition,

$$e_\rho = \sum_{i=1}^{\dim(\rho)/d_\rho} e_i,$$

in $e_\rho \cdot k[N]$ into primitive idempotents. Now for all $1 \leq i, j \leq \dim(\rho)$ the right $k[N]$-modules $e_i k[N]$ and $e_j k[N]$ are isomorphic, and so there are $u_{ij} \in (e_\rho \cdot k[N])^\times$ with,

$$u_{ij} e_i = e_j u_{ij},$$

which we may assume satisfy $u_{ij} = u_{ji}^{-1}$. We can define maps,

$$\phi_{ij} : e_i \cdot f_*\mathcal{O}_X \to e_j \cdot f_*\mathcal{O}_X, \qquad \phi_{ij}(e_i b) := u_{ij} e_i b = e_j u_{ij} b,$$

which are homomorphisms of G-$\mathcal{D}_Y$-modules as the action of G-$\mathcal{D}_Y$ commutes with the action of N, and further are isomorphisms because,

$$(\phi_{ji} \circ \phi_{ij})(e_i b) = u_{ji} u_{ij} e_i b = e_\rho e_i b = e_i b.$$

Therefore, the decomposition,

$$e_\rho \cdot f_*\mathcal{O}_X = \bigoplus_{i=1}^{\dim(\rho)/d_\rho} e_i \cdot f_*\mathcal{O}_X,$$

has isomorphic factors, and thus the decomposition (4.2) is isotypical. On the other hand, because of the isomorphism (4.3) any direct sum decomposition of $e_\rho \cdot f_*\mathcal{O}_X$ into irreducible submodules corresponds to a decomposition of e_ρ into primitive orthogonal idempotents in $k[N]$. Finally, in the manner described before Theorem 4.4.2, $\mathcal{O}(N)$ is isomorphic to the regular representation, and if we write $\tilde{\rho}$ for the dual representation of ρ, then $e_\rho \star \mathcal{O}(N) = e_{\tilde{\rho}} \cdot \mathcal{O}(N)$, which has dimension $\dim(\tilde{\rho})^2 = \dim(\rho)^2$. Now

$e_\rho \cdot f_*\mathcal{O}_X$ is the direct sum of $\dim(\rho)/d_\rho$ G-$\mathcal{D}_Y$-modules each isomorphic to $e \cdot f_*\mathcal{O}_X$ and thus,

$$(\dim(\rho)/d_\rho)\,\mathrm{rank}_Y(e \cdot f_*\mathcal{O}_X) = \mathrm{rank}_Y(e_\rho \cdot f_*\mathcal{O}_X),$$

and,

$$\begin{aligned}\mathrm{rank}_Y(e_\rho \cdot f_*\mathcal{O}_X) &= \mathrm{rank}_X(f^*(e_\rho \cdot f_*\mathcal{O}_X)),\\ &= \mathrm{rank}_X(\mathcal{O}_X \otimes_k e_\rho \star \mathcal{O}(N)),\\ &= \dim(\rho)^2/d_\rho,\end{aligned}$$

hence $\mathrm{rank}_Y(e \cdot f_*\mathcal{O}_X) = \dim(\rho)$. □

Remark 4.4.5. The condition that $c_X(X)^G = k$ is in general necessary for Theorem 4.2.7, Theorem 4.4.2 and Theorem 4.4.4. For example, suppose that G is trivial and $H = N$, so the hypothesis is simply that $f\colon X \to Y$ is a finite étale Galois morphism between smooth geometrically connected spaces over k, and consider L a finite Galois extension of k of degree n with Galois group N. Then the corresponding extension (which also makes sense in case (B)),

$$f\colon X = \mathrm{Spec}(L) \to \mathrm{Spec}(k) = Y,$$

is a Galois extension with Galois group N, with X connected but not geometrically connected whenever $n > 1$, as then $c_X(X) = L > k$. Because L/k is separable, $\mathrm{Der}_k(L) = 0$ and $\mathcal{D}_X(X) = L$, hence the functors in this situation become

$$\begin{array}{ccc}\mathbf{Mod}^{\mathbf{fd}}_{k[N]} & \xrightarrow{L\otimes_k -} & \mathbf{Mod}^{\mathbf{fd}}_{L\rtimes N}\\ & & {\scriptstyle (-)^N}\big\downarrow\big\uparrow{\scriptstyle L\otimes_k -}\\ & & \mathbf{Vect}^{\mathbf{fd}}_k\end{array}$$

The map

$$k[N] \to \mathrm{End}_{\mathcal{D}_Y}(f_*\mathcal{O}_X) = \mathrm{End}_k L$$

will never be an isomorphism when $n > 1$, as the left-hand side has k-dimension n and the right-hand side k-dimension n^2. The essential image of $(\mathcal{O}_X \otimes_k -)^N$ also need not be closed under sub-objects, as this functor need not preserve irreducibility: when N is non-abelian $k[N]$ has length strictly less than n as a $k[N]$-module, but L has length n as a k-vector space.

4.4.1 The Functor $\mathrm{Hom}_{k[N]}(-, f_*\mathcal{O}_X)$

We can give an contravariant description of the functor

$$(\mathcal{O}_X \otimes_k -)^N : \mathbf{Mod}^{\mathbf{fd}}_{k[H]} \to \mathbf{VectCon}^{G\times H/N}(Y).$$

For any representation $V \in \mathbf{Mod}^{\mathbf{fd}}_{k[H]}$, we can consider the $\mathcal{O}_Y$-module,

$$\mathrm{Hom}_{k[N]}(V, f_*\mathcal{O}_X)\colon U \mapsto \mathrm{Hom}_{k[N]}(V, f_*\mathcal{O}_X(U))$$

for any admissible open subset $U \subset Y$, where we view $f_*\mathcal{O}_X$ with action of N as described in Section 1.9. In order to describe its structure as a $(G \times H/N)$-$\mathcal{D}_Y$-module, we first note that the analogously defined sheaf $\mathrm{Hom}_k(V, f_*\mathcal{O}_X)$ has a natural structure as a $(G \times H)$-$\mathcal{D}_Y$-module, where the $\mathcal{D}_Y$-module structure is defined, for $U \in \mathcal{B}_Y$, $\partial \in \Theta_Y(U)$, and k-linear $\phi\colon V \to f_*\mathcal{O}_X(U)$, by

$$\partial \cdot \phi := \partial \circ \phi,$$

and the equivariant structure on $\mathcal{F} := \mathrm{Hom}_k(V, f_*\mathcal{O}_X)$ is defined by viewing V as a $k[G \times H]$-module with trivial action of G, and setting, for any admissible open subset $U \subset Y$ and $g \in G \times H$,

$$g_U^{\mathcal{F}}(\phi\colon V \to f_*\mathcal{O}_X(U)) := g_U^{f_*\mathcal{O}_X} \circ \phi \circ g^{-1}.$$

Because N acts trivially on Y, we obtain an action of N on the sheaf $\mathrm{Hom}_k(V, f_*\mathcal{O}_X)$, which has N-invariants the sub-sheaf $\mathrm{Hom}_{k[N]}(V, f_*\mathcal{O}_X)$. In this way $\mathrm{Hom}_{k[N]}(V, f_*\mathcal{O}_X)$ is canonically equipped with the structure of a $(G \times H/N)$-$\mathcal{D}_Y$-module, noting the the $\mathcal{D}_Y$-action on $\mathrm{Hom}_k(V, f_*\mathcal{O}_X)$ restricts to an action of $\mathcal{D}_Y$ on $\mathrm{Hom}_{k[N]}(V, f_*\mathcal{O}_X)$ by Lemma 1.9.7.

In this way we obtain a contravariant functor,

$$\mathrm{Hom}_{k[N]}(-, f_*\mathcal{O}_X) : \mathbf{Mod}^{\mathbf{fd}}_{k[H]} \to \mathbf{VectCon}^{G \times H/N}(Y).$$

Proposition 4.4.6. *There is an isomorphism of* $(G \times H/N)$-$\mathcal{D}_Y$*-modules,*

$$(\mathcal{O}_X \otimes V^*)^N \xrightarrow{\sim} \mathrm{Hom}_{k[N]}(V, f_*\mathcal{O}_X),$$

natural in V*.*

Proof. For any $U \in \mathcal{B}_Y$, because V is finite dimensional there is a $\mathcal{O}_Y(U) \times N$-module isomorphism,

$$\mathcal{O}_X(f^{-1}(U)) \otimes_k V^* \xrightarrow{\sim} \mathrm{Hom}_k(V, \mathcal{O}_X(f^{-1}(U))), \qquad x \otimes \lambda \mapsto \lambda(-)x.$$

When we take N-invariants, we obtain an isomorphism of $\mathcal{O}_Y(U)$-modules,

$$(\mathcal{O}_X(f^{-1}(U)) \otimes_k V^*)^N \xrightarrow{\sim} \mathrm{Hom}_{k[N]}(V, \mathcal{O}_X(f^{-1}(U))),$$

which is furthermore $\mathcal{D}_Y(U)$-linear as for any $\partial \in \Theta_Y(U)$,

$$\partial \cdot (x \otimes \phi) = \partial(x) \otimes \phi \mapsto \phi(-)\partial(x) = \partial \cdot (\phi(-)x),$$

because the action of ∂ on $\mathcal{O}_X(f^{-1}(U))$ is k-linear. For $G \times H/N$-linearity, we see that

$$\begin{array}{ccc} (\mathcal{O}_X(f^{-1}(U)) \otimes_k V^*)^N & \longrightarrow & \mathrm{Hom}_{k[N]}(V, \mathcal{O}_X(f^{-1}(U))) \\ \downarrow{\scriptstyle g^{\mathcal{O}_X}_{f^{-1}(U)} \otimes g^*} & & \downarrow{\scriptstyle g^{\mathcal{O}_X}_{f^{-1}(U)} \circ - \circ g^{-1}} \\ (\mathcal{O}_X(f^{-1}(g(U))) \otimes_k V^*)^N & \longrightarrow & \mathrm{Hom}_{k[N]}(V, \mathcal{O}_X(f^{-1}(g(U)))) \end{array}$$

commutes, because $g^{\mathcal{O}_X}_{f^{-1}(U)}$ is k-linear. ∎

4.4.2 Compatibility with Intermediate Coverings

We are also interested in the compatibility of each of the functors $\mathrm{Hom}_{k[N]}(-, f_*\mathcal{O}_X)$ and $(\mathcal{O}_X \otimes_k -)^N$ when passing to Galois sub-covers. Suppose that $N_0 \lhd H$ with N_0 contained in N, and let Z be the intermediate covering of Y defined by N_0, with Galois group N/N_0. We have Galois coverings

$$X \to Z \xrightarrow{\phi} Y$$

with Galois groups N_0 and N/N_0 respectively. Note that because the action of G commutes with the action of N, there is an induced action of $G \times H/N_0$ on Z for which $\phi\colon Z \to Y$ is $G \times H/N_0$-equivariant, and the Galois action of N/N_0 on Z is through this action and the inclusion $N/N_0 \hookrightarrow H/N_0$. We have an inflation functor

$$\iota\colon \mathbf{Mod}^{\mathbf{fd}}_{k[H/N_0]} \to \mathbf{Mod}^{\mathbf{fd}}_{k[H]}\,.$$

The next lemma relates the functors $\mathrm{Hom}_{k[N]}(-, f_*\mathcal{O}_X)$ and $\mathrm{Hom}_{k[N/N_0]}(-, \phi_*\mathcal{O}_Z)$.

Lemma 4.4.7. *Suppose that $N_0 \lhd H$ is a normal subgroup of H which is contained in N, and $\phi\colon Z \to Y$ is the intermediate covering defined by N_0, with Galois group N/N_0. Then there is an isomorphism of $(G \times H/N)$-$\mathcal{D}_Y$-modules*

$$\mathrm{Hom}_{k[N]}(\iota(W), f_*\mathcal{O}_X) \xrightarrow{\sim} \mathrm{Hom}_{k[N/N_0]}(W, \phi_*\mathcal{O}_Z)$$

which is natural in $W \in \mathbf{Mod}^{\mathbf{fd}}_{k[H/N_0]}$.

Proof. For any admissible open subset U of Y, $\phi_*\mathcal{O}_Z(U) = \mathcal{O}_X(f^{-1}(U))^{N_0}$, and therefore

$$\mathrm{Hom}_{k[N/N_0]}(W, \phi_*\mathcal{O}_Z(U)) = \mathrm{Hom}_{k[N/N_0]}(W, \mathcal{O}_X(f^{-1}(U))^{N_0}) = \mathrm{Hom}_{k[N]}(\iota(W), \mathcal{O}_X(f^{-1}(U))).$$

It is direct to check this the morphism of sheaves this defines is natural in W and $(G \times H/N)$-$\mathcal{D}_Y$-linear. □

4.4.3 Compatibility with Connected Components

From now on, we work in a slightly weaker situation to that described at the start of Section 4.4 (taking $H = N$). In this section we show that the functors $\mathrm{Hom}_{k[H]}(-, f_*\mathcal{O}_X)$ and $(\mathcal{O}_X \otimes_k -)^H$ are compatible when passing to a connected component of a Galois covering.

Lemma 4.4.8. *Suppose that $f\colon X \to Y$ is a finite étale Galois morphism of spaces over k with Galois group H, and*

- *G is an abstract group that acts on X and Y,*
- *The action of G commutes with the action of H,*
- *$f\colon X \to Y$ is G-equivariant,*
- *Y is connected,*
- *X_0 is a connected component of X stabilised by G.*

Write $f_0 \colon X_0 \to Y$ for the induced Galois extension with Galois group $H_0 := \mathrm{Stab}_H(X_0)$ (cf. Lemma 1.9.8). Then the diagram

$$\begin{array}{ccc} & \mathbf{Mod}^{\mathbf{fd}}_{k[H]} & \\ \swarrow & & \searrow \\ \mathbf{Mod}^{\mathbf{fd}}_{k[H_0]} & \longrightarrow & \mathbf{VectCon}^G(Y) \end{array}$$

commutes up to natural isomorphism. Explicitly, if $e_0 \in \mathcal{O}_X(X)$ is the idempotent corresponding to X_0,

$$\mathrm{Hom}_{k[H]}(V, f_*\mathcal{O}_X) \to \mathrm{Hom}_{k[H_0]}(V|_{H_0}, f_{0,*}\mathcal{O}_{X_0}), \qquad \phi \mapsto e_0 \cdot \phi(-),$$

is an isomorphism of G-$\mathcal{D}_Y$-modules, natural in V.

Proof. Because G preserves the connected component X_0, this is an isomorphism of G-equivariant sheaves. To see that this morphism is $\mathcal{D}_Y$-linear, let $U \in \mathcal{B}_Y$, $A := \mathcal{O}_Y(U)$, $B := \mathcal{O}_X(f^{-1}(U))$, $B_0 := \mathcal{O}_X(f_0^{-1}(U))$, so $B_0 \subset B$, and $A \hookrightarrow B_0$, $A \hookrightarrow B$ are Galois extensions of commutative k-algebras with Galois groups H_0, H respectively. It is sufficient for us to show that

$$\mathrm{Hom}_{k[H]}(V, B) \to \mathrm{Hom}_{k[H_0]}(V|_{H_0}, B_0), \qquad \phi \mapsto \pi_0(\phi(-)),$$

is an isomorphism of $\mathcal{D}(A)$-modules, where $\pi_0 \colon B \to B_0$ is the projection. First, we show that this is an isomorphism. There is an isomorphism of $k[H]$-modules,

$$B \to \mathrm{Ind}_{H_0}^H B_0 = \mathrm{Hom}_{k[H_0]}(k[H], B_0), \qquad b \mapsto \psi_b, \quad \psi_b(h) := \pi_0(hb)$$

for $h \in H$. Here $\mathrm{Ind}_{H_0}^H B_0$ is a left $k[H]$-module via $(g \cdot \sigma)(h) = \sigma(hg)$. The inverse is given by choosing left coset representatives $h_1, ..., h_k$ of H/H_0 and mapping

$$\sigma \mapsto \sum_{i=1}^{k} h_i \sigma(h_i^{-1}) \in B,$$

and is independent of the choice of coset representatives. Frobenius reciprocity gives us isomorphisms,

$$\mathrm{Hom}_{k[H]}(V, \mathrm{Ind}_{H_0}^H B_0) \xleftrightarrow{\sim} \mathrm{Hom}_{k[H_0]}(V|_{H_0}, B_0),$$

explicitly given by

$$[f \colon V \to \mathrm{Ind}_{H_0}^H B_0] \mapsto \Phi(f), \qquad \Phi(f)(v) := f(v)(1),$$

and

$$[\lambda := V|_H \to B_0] \mapsto \Pi(\lambda), \qquad \Pi(\lambda)(v)(k) := \lambda(kv).$$

Then our map of interest is simply the composition of the induced isomorphisms,

$$\mathrm{Hom}_{k[H]}(V, B) \xrightarrow{\sim} \mathrm{Hom}_{k[H]}(V, \mathrm{Ind}_{H_0}^H B_0) \xrightarrow{\sim} \mathrm{Hom}_{k[H_0]}(V|_{H_0}, B_0).$$

We therefore are left with showing that the isomorphism is $\mathcal{D}(A)$-linear. First note that for e_0 the idempotent of B defining B_0, any $\partial \in \mathrm{Der}_k(B)$ satisfies $\partial(e_0) = 0$, and therefore, $\partial \mapsto \partial_{B_0}$ is a well-defined B_0-linear restriction map. There are A-linear maps $\psi\colon \mathrm{Der}_k(A) \to \mathrm{Der}_k(B)$, $\psi_0\colon \mathrm{Der}_k(A) \to \mathrm{Der}_k(B_0)$ provided by Lemma 1.4.1, and by the uniqueness part of Lemma 1.4.1 we see that the composition

$$\mathrm{Der}_k(A) \xrightarrow{\psi} \mathrm{Der}_k(B) \to \mathrm{Der}_k(B_0)$$

is equal to ψ_0. Therefore, the projection $\pi_0\colon B \to B_0$ is $\mathcal{D}(A)$-linear. Finally, we see that the isomorphism is $\mathcal{D}(A)$-linear, as given $x \in \mathcal{D}(A)$ and $f \in \mathrm{Hom}_{k[H]}(V, B)$,

$$\pi_0(x \cdot f(-)) = x \cdot \pi_0(f(-)). \qquad \square$$

4.4.4 Abelian Galois Coverings

In the section, we continue with the assumptions from the start of Section 4.4, and add the additional assumption that $H = N$ and H, the Galois group of $f\colon X \to Y$, is abelian.

Definition 4.4.9. We define $\mathrm{PicCon}^G(X)$ to be the set of isomorphism classes of G-equivariant line bundles connection on X (i.e. rank 1 elements of $\mathbf{VectCon}^G(X)$).

In the case that G is trivial, we write $\mathrm{PicCon}(X)$ for $\mathrm{PicCon}^G(X)$, the set of isomorphism classes of line bundles with connection on X. The tensor product (as described in Section 1.8) induces an abelian group structure on $\mathrm{PicCon}^G(X)$ for which the natural forgetful map

$$\mathrm{PicCon}^G(X) \to \mathrm{Pic}(X)$$

is a group homomorphism. We can now state the improvement of Theorem 4.4.4 in the case that the Galois group is abelian.

Corollary 4.4.10. *Suppose that in addition to the assumptions of Section 4.4,*

1. *$H = N$, and H is abelian,*
2. *There are no global non-trivial G-invariant constant functions: $c_X(X)^G = k$,*
3. *k contains a primitive eth root of 1, where e is the exponent of H.*

Then the map

$$\widehat{H} \to \mathrm{PicCon}^G(Y)[e], \qquad \chi \mapsto \mathcal{L}_\chi \coloneqq e_\chi \cdot f_*\mathcal{O}_X,$$

is an injective group homomorphism.

Proof. This map is the restriction of the functor

$$(\mathcal{O}_X \otimes_k -)^H\colon \mathbf{Mod}^{\mathbf{fd}}_{k[H]} \to \mathbf{VectCon}^G(Y)$$

to the 1-dimensional representations because for $\chi \in \widehat{H}$ there is a canonical isomorphism

$$(\mathcal{O}_X \otimes_k \chi)^H \xrightarrow{\sim} e_\chi \cdot f_*\mathcal{O}_X,$$

and thus this map is a group homomorphism because $(\mathcal{O}_X \otimes_k -)^H$ is monoidal (Theorem 4.4.2). Alternatively, one can note that the proof of Proposition 2.1.2 (which is written in the setting of case (B), but also applies in case (A)) establishes a group homomorphism $\widehat{H} \to \mathrm{Pic}(Y)[e]$, and is direct to check that the isomorphism $\mathcal{L}_\chi \otimes \mathcal{L}_\psi \to \mathcal{L}_{\chi\psi}$ described there is an isomorphism of G-$\mathcal{D}_Y$-modules, hence the group homomorphism factors through the forgetful maps

$$\widehat{H} \to \mathrm{PicCon}^G(Y)[e] \to \mathrm{Pic}(Y)[e].$$

Finally, the injectivity follows directly from the isotypical decomposition of Theorem 4.4.4. □

4.5 Vector Bundles on Drinfeld Spaces

From now until the end of the thesis we work in the rigid situation (Case (B)). Write K for a characteristic 0 complete non-archimedean field, and let $\mathcal{R}$ denote the ring of integers of K.

In this section we first prove some general results which hold for any rigid space over K (Section 4.5.1). Next, we describe the geometrically connected components of Drinfeld covering spaces (Section 4.5.3). We then use the general theory of Section 4.4 to give a more conceptual proof Theorem E concerning line bundles on the first Drinfeld covering (Section 4.5.4). Finally, in the remainder of this chapter, we prove Theorem A, Theorem B and Theorem C (Section 4.5.5 onwards).

4.5.1 Generalities

We first recall the notion of a continuous group action on a rigid space X, as defined in [1, Def. 3.1.8]. Let X be a qcqs rigid space over K. Then using a formal model for X, [1, Thm. 3.1.5] defines a hausdorff topology τ_X on $\mathrm{Aut}_K(X, \mathcal{O}_X)$, which is independent of the formal model chosen for X.

Definition 4.5.1. Let G be a topological group, X a rigid analytic variety over K. Then G *acts continuously on* X, if there is a group homomorphism $\rho : G \to \mathrm{Aut}_K(X, \mathcal{O}_X)$, such that for all qcqs admissible open subsets $U \subset X$,

- $G_U = \rho^{-1}(\mathrm{Stab}(U))$ is open in G,
- The induced group homomorphism $\rho_U : G_U \to \mathrm{Aut}_K(U, \mathcal{O}_U)$ is continuous with respect to the topology τ_U on $\mathrm{Aut}_K(U, \mathcal{O}_U)$ described above, and the subspace topology of G_U.

Here $\mathrm{Stab}(U) \leq \mathrm{Aut}_K(X, \mathcal{O}_X)$ are those automorphisms for which the underlying map of sets f has $f(U) = U$.

Suppose henceforth that G is a topological group which acts on X continuously, where X is a rigid space over K. We now want to define a subgroup of $\mathrm{PicCon}^G(X)$ of G-equivariant line bundles with flat connection which satisfy an appropriate continuity condition.

For any K-Banach space V, it is shown in [3] during the preamble to Definition 3.2.3 that the unit group $\mathcal{B}(V)^\times$ of $\mathcal{B}(V)$, the K-algebra of K-linear endomorphisms of V, forms a topological group such that the congruence subgroups

$$\Gamma_n(\mathcal{V}) := \{\gamma \in \mathcal{B}(V)^\times \mid (1-\gamma)(\mathcal{V}) \subset \pi^n \mathcal{V}\}$$

form a system of open neighbourhoods of the identity, where $\mathcal{V}$ is the unit ball for some choice of norm on V.

Suppose now that $\mathcal{L} \in \mathrm{PicCon}^G(X)$. Then *loc. cit.* also shows that for any affinoid subdomain U of X the action map induces

$$G_U \to \mathcal{B}(\mathcal{L}(U))^\times,$$

where $\mathcal{L}(U)$ is provided with its canonical K-Banach space structure as a finitely generated $\mathcal{O}_X(U)$-module.

Definition 4.5.2. We define $\mathrm{PicCon}^G_{\mathrm{cts}}(X) \subset \mathrm{PicCon}^G(X)$ be the subgroup of $\mathcal{L} \in \mathrm{PicCon}^G(X)$ such that for any affinoid open subset U of X,

$$G_U \to \mathcal{B}(\mathcal{L}(U))^\times$$

is continuous. We also define

$$\mathrm{Con}^G_{\mathrm{cts}}(X) := \ker(\mathrm{PicCon}^G_{\mathrm{cts}}(X) \to \mathrm{Pic}(X)).$$

Remark 4.5.3. The group we denote by $\mathrm{PicCon}^G_{\mathrm{cts}}(X)$ is written as $\mathrm{PicCon}^G(X)$ in [3], and similarly what we call $\mathrm{Con}^G_{\mathrm{cts}}(X)$ is written as $\mathrm{Con}^G(X)$.

The key result of [3] that we will make use of is the following.

Proposition 4.5.4. *Suppose X is quasi-Stein and geometrically connected, and that G acts continuously on X. Then for any $d \geq 1$ there is an exact sequence,*

$$0 \to \mathrm{Hom}_{\mathrm{cts}}(G, \mu_d(K)) \to \mathrm{Con}^G_{\mathrm{cts}}(X)[d] \to (\mathcal{O}(X)^\times / K^\times \mathcal{O}(X)^{\times d})^G.$$

Proof. We have the commutative diagram,

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{\mathrm{cts}}(G, K^\times) & \longrightarrow & \mathrm{Con}^G_{\mathrm{cts}}(X) & \longrightarrow & \mathrm{Con}_{\mathrm{cts}}(X)^G \\
& \downarrow{=} & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathrm{Hom}_{\mathrm{cts}}(G, K^\times) & \longrightarrow & \mathrm{PicCon}^G_{\mathrm{cts}}(X) & \longrightarrow & \mathrm{PicCon}_{\mathrm{cts}}(X)^G
\end{array}$$

where the second row is exact by [3, Prop. 3.2.14]. Because the second and third vertical maps are injective, a simple diagram chase shows that the first row is also exact. Now because d-torsion is left exact, this first row remains exact after applying $(-)[d]$. Finally, we obtain the required exact sequence using [3, Prop. 3.1.16], which exhibits a G-equivariant isomorphism

$$\theta_d \colon \mathrm{Con}_{\mathrm{cts}}(X)[d] \xrightarrow{\sim} \mathcal{O}_X(X)^\times / K^\times \mathcal{O}_X(X)^{\times d}. \qquad \square$$

In order to apply the previous result to the Drinfeld tower, we will need the following general lemma, which is of interest in its own right.

Proposition 4.5.5. *Suppose that $f\colon X \to Y$ is a finite étale Galois cover of quasi-Stein rigid spaces with Galois group H. Let G be a topological group with an open topologically finitely generated profinite subgroup acting on X and Y such that $f\colon X \to Y$ is G-equivariant, and G acts continuously on Y. Then G acts continuously on X.*

Proof. Let $(U_m)_{m\geq 0}$ be a quasi-Stein rigid cover of Y, and let $(V_m := f^{-1}(U_m))_{m\geq 0}$ be the corresponding quasi-Stein cover of X. In order to show that G acts continuously on X, it is sufficient to show that $G_m \coloneqq G_{V_m}$ is open in G, and acts continuously on V_m for all $m \geq 0$. Indeed, if so, then for an arbitrary qcqs admissible open $V \subset X$, $V = \cup_{m\geq 0}(V \cap V_m)$, hence because V is quasi-compact, $V \subset V_m$ for some $m \geq 0$. Then $(G_m)_V \subset G_m$ is open (hence open in G), and thus G_V is open in G. Furthermore, $(G_m)_V \to \mathrm{Aut}_K(V, \mathcal{O}_V)$ is continuous by [1, Thm. 3.1.10], hence $G_V \to \mathrm{Aut}_K(V, \mathcal{O}_V)$ is continuous.

Fix $m \geq 0$. In order to show that G_m is open and G_m acts continuously on V_m, because V_m is affinoid, hence qcqs, by [1, Thm. 3.1.10] it is sufficient to show that G_m is open and $\rho : G_m \to \mathrm{Aut}_K(V_m, \mathcal{O}_{V_m})$ is continuous. First, because the action of G on Y is continuous, G_{U_m} is open, and so because $G_{U_m} \subset G_m$, G_m is open in G. The extension $V_m \to U_m$ is Galois group H, and therefore the group $\mathrm{Aut}_{\mathcal{O}(U_m)}(\mathcal{O}(V_m))$ is finite, which follows from [21, Thm. 3.5] and Lemma 1.9.8. Therefore, we are done after applying the following Lemma 4.5.6. □

Lemma 4.5.6. *Suppose that $A \to B$ is an étale morphism of affinoid algebras over k, which is G-equivariant with respect to actions $\sigma\colon G \to \mathrm{Aut}_k(A)$, $\rho\colon G \to \mathrm{Aut}_k(B)$ of some topological group G. Suppose that $\mathrm{Aut}_A(B)$ is finite, G has an open topologically finitely generated profinite subgroup, and σ is continuous. Then ρ is continuous.*

Proof. Let $\mathcal{A}$ be a formal model for A. Because σ is continuous, by [1, Lem. 3.2.4], we may after possibly enlarging $\mathcal{A}$ assume that $\mathcal{A}$ is G-equivariant. Any étale morphism of affinoid algebras is *standard étale* [24, Observation 3.1.2], which means that there is a presentation of the morphism $A \to B$ by

$$\iota\colon A \to B = A\langle T_1, ..., T_n\rangle/(f_1, ..., f_n), \text{ where } \det\Delta \in B^\times, \text{ for } \Delta \coloneqq \left(\frac{\partial f_i}{\partial T_j}\right)_{ij}.$$

We take our formal model of B to be

$$\mathcal{B} \coloneqq \iota(\mathcal{A}\langle T_1, ..., T_n\rangle) \subset B.$$

First consider an $\mathcal{R}$-linear derivation $\tilde{\partial}\colon \mathcal{A} \to \mathcal{A}$. This has a unique extension to a K-linear derivation $\tilde{\partial}\colon A \to A$. Then because $A \to B$ is étale, by Lemma 1.4.1 there is a unique $\partial\colon B \to B$ such that for any $a \in A$,

$$\partial(\iota(a)) = \iota(\tilde{\partial}(a)).$$

We first show the following claim: for any $N > 0$, there is some $M > 0$ such that for any $\tilde{\partial}\colon \mathcal{A} \to \mathcal{A}$ with $\tilde{\partial}(\mathcal{A}) \subset p^M\mathcal{A}$, the extension ∂ satisfies

$$\partial(\mathcal{B}) \subset p^N\mathcal{B}.$$

Fix $N > 0$. We first consider the elements $T_1, ..., T_n \in \mathcal{B}$. Write each

$$f_i = \sum_\alpha c^i_\alpha T^\alpha,$$

for some $c^i_\alpha \in \iota(\mathcal{A})$. Suppose now that $\tilde{\partial}\colon \mathcal{A} \to \mathcal{A}$ with $\tilde{\partial}(\mathcal{A}) \subset p^M\mathcal{A}$, for some $M > 0$ to be chosen later. Then because any k-derivation of B is continuous [7, Satz. 2.1.5],

$$\begin{aligned} 0 &= \partial(f_i), \\ &= \partial\left(\sum_\alpha c^i_\alpha T^\alpha\right), \\ &= \sum_\alpha \partial(c^i_\alpha T^\alpha), \\ &= \sum_\alpha \partial(c^i_\alpha)T^\alpha + c^i_\alpha \partial(T^\alpha). \end{aligned}$$

We can simplify the right-hand term to

$$\begin{aligned} c^i_\alpha \partial(T^\alpha) &= \sum_{j=1}^n c^i_\alpha T^{\alpha_1} \cdots \partial(T^{\alpha_j}) \cdots T^{\alpha_n}, \\ &= \sum_{i=1}^n \frac{\partial f_i}{\partial T_j} \partial(T_j). \end{aligned}$$

For the left-hand term, each $c^i_\alpha \in \iota(\mathcal{A})$ and thus $c^i_\alpha = \iota(a^i_\alpha)$ for some $a^i_\alpha \in \mathcal{A}$, hence

$$\sum_\alpha \partial(c^i_\alpha)T^\alpha = \sum_\alpha \iota(\tilde{\partial}(a^i_\alpha))T^\alpha \in p^M\mathcal{B},$$

because $\tilde{\partial}(a^i_\alpha) \in \pi^M\mathcal{A}$, so $\iota(\tilde{\partial}(a^i_\alpha))T^\alpha \in p^M\mathcal{B}$, and $p^M\mathcal{B}$ is closed. Combined, this shows that,

$$\Delta \begin{pmatrix} \partial(T_1) \\ \vdots \\ \partial(T_n) \end{pmatrix} \in (p^M\mathcal{B})^n.$$

Now because $\det(\Delta) \in B^\times$, Δ is invertible, and we set $M_0 := \min\{r \geq 0 \mid p^r\Delta^{-1} \in M_n(\mathcal{B})\}$. Then for $k = 1, ..., n$,

$$\partial(T_k) \in p^{M - M_0}\mathcal{B}.$$

Let us now fix $M := N + M_0$, and consider a general element of $\mathcal{B}$, which will be of the form,

$$b = \sum_\alpha \iota(a_\alpha)T^\alpha$$

for some $a_\alpha \in \mathcal{A}$. When we apply ∂ we obtain, similarly to above,

$$\partial(b) = \sum_\alpha \iota(\tilde{\partial}(a_\alpha))T^\alpha + \iota(a_\alpha)\partial(T^\alpha) \in p^N\mathcal{B}.$$

For each α, the left-hand term is in $p^N\mathcal{B}$, and the right-hand term is too, as

$$\partial(T^\alpha) = \sum_{k=1}^{n} \alpha_k T_1^{\alpha_1} \cdots T_k^{\alpha_k - 1} \cdots T_n^{\alpha_n} \cdot \partial(T_k). \tag{4.4}$$

Therefore, as $p^N\mathcal{B}$ is closed, $\partial(b) \in p^N\mathcal{B}$ and we have shown the claim.

Now we want to show that the morphism $\rho\colon G \to \mathrm{End}_k(B)$ is continuous. For any $g \in G$ with $(\sigma(g) - 1)(\mathcal{A}) \subset p^2\mathcal{A}$ we can consider the logarithm $u_g := \log(\sigma(g)) \in p^2 \operatorname{Der}_R(\mathcal{A})$ of $\sigma(g)$ (see [1, Lem. 3.2.5] for background on the logarithm and exponential in this context). As described above, this $\mathcal{R}$-linear derivation u_g has a unique extension to a K-linear derivation of A (which we also denote by u_g) and by Lemma 1.4.1 there is a unique $v_g \in \mathrm{Der}_k(B)$ which extends u_g.

Because $\mathrm{Aut}_A(B)$ is finite, by [1, Lem. 3.2.4] we may enlarge $\mathcal{B}$ to obtain an $\mathrm{Aut}_A(B)$-stable formal model $\mathcal{B}'$ of B which contains $\mathcal{B}$. To show that $\rho\colon G \to \mathrm{End}_k(B)$ is continuous we need to show that for any $n \geq 0$, $\rho^{-1}(\mathrm{Aut}_k^n(B))$ is open in G, where

$$\mathrm{Aut}_k^n(B) := \{\phi \in \mathrm{Aut}_k(B) \mid (\phi - 1)(\mathcal{B}') \subset p^n\mathcal{B}'\}.$$

To this end, fix $n \geq 2$. Because $\mathcal{B}$ and $\mathcal{B}'$ are both formal models of B, the argument of the proof of [1, Thm. 3.1.5 (b)], the claim established above and the fact that σ is continuous together show the existence of an open subgroup G_0 of G for which $v_g(\mathcal{B}') \subset p^n\mathcal{B}'$ for any $g \in G_0$.

Define $\tau\colon G_0 \to \mathrm{Aut}_k^n(B)$ by $\tau(g) = \exp(v_g)$. We would like to show that, like τ, $\rho(g) \in \mathrm{Aut}_k^n(B)$ for any $g \in G_0$. Firstly, we note that τ is a group homomorphism. Indeed, $\sigma(g)$ is a group homomorphism, and $\sigma(g) = \exp(\log(\sigma(g))) = \exp(u_g)$, so for any $g, h \in G_0$

$$\exp(u_g) \circ \exp(u_h) = \exp(u_{gh}).$$

Taking the logarithm,

$$\log(\exp(u_g) \circ \exp(u_h)) = u_{gh}.$$

The derivation v_{gh} is an extension of u_{gh} to an $\mathcal{R}$-derivation of $\mathcal{B}'$, and $\log(\exp(v_g) \circ \exp(v_h))$ is an extension of $\log(\exp(u_g) \circ \exp(u_h))$ to $\mathcal{B}'$, and therefore by the uniqueness of Lemma 1.4.1

$$\log(\exp(v_g) \circ \exp(v_h)) = v_{gh}.$$

Applying the exponential, we see that $\tau(g) \circ \tau(h) = \tau(gh)$.

Consider the difference of ρ and τ, denoted by $\lambda\colon G_0 \to \mathrm{Aut}_A(B)$, defined by

$$\lambda(g) = \rho(g)\tau(g)^{-1} \in \mathrm{Aut}_A(B).$$

This lies in $\mathrm{Aut}_A(B)$ because both $\rho(g)$ and $\tau(g)$ restrict to $\sigma(g)$ on A. Consider also the restriction homomorphism

$$r_n\colon \mathrm{Aut}_{\mathcal{R}}(\mathcal{B}') \to \mathrm{Aut}_{\mathcal{R}/p^n\mathcal{R}}(\mathcal{B}'/p^n\mathcal{B}'),$$

the kernel of which is $\mathrm{Aut}_k^n(B)$. Because $\tau(g)$ and $\lambda(g)$ both stabilise $\mathcal{B}'$, $\rho(g)$ does too. Therefore, we may apply r_n to $\rho(g)$, and note that as $\tau(g) \in \mathrm{Aut}_k^n(B) = \ker(r_n)$,

$$r_n(\rho(g)) = r_n(\lambda(g)) \in r_n(\mathrm{Aut}_A(B)).$$

Now $r_n \circ \rho \colon G_0 \to r_n(\mathrm{Aut}_A(B))$ is a homomorphism from G_0 to the finite group $r_n(\mathrm{Aut}_A(B))$, and thus because finite index subgroups of G are open [50], $\ker(r_n \circ \rho) \leq G_0$ is open. Therefore, $\ker(r_n \circ \rho) \subset \rho^{-1}(\mathrm{Aut}^n_k(B))$ is an open subgroup, and we're done. □

We also note the following lemma, which with Proposition 4.5.5 above implies that the action of $\mathrm{GL}_n(F)$ on the Drinfeld tower is continuous (Corollary 4.5.10).

Lemma 4.5.7. *Suppose that X is a rigid space with an action of a topological group G, and H is an open subgroup of G which stabilises and acts continuously on each connected component of X. Then the action of G on X is continuous.*

Proof. Suppose that $U \subset X$ is a qcqs admissible open subset of X. Then the stabiliser $G_U \subset G$ is open, as G_U contains the open subgroup H_U of G. Furthermore, the morphism of topological groups $G_U \to \mathrm{Aut}(U)$ is continuous, as the restriction to the subgroup H_U is continuous, which follows from the commutative diagram

$$\begin{array}{ccc} G_U & \longrightarrow & \mathrm{Aut}(U) \\ \uparrow & & \uparrow \\ H_U & \longrightarrow & \prod_i \mathrm{Aut}(U_i) \end{array}$$

where the product is over the connected components X_i, and $U_i := X_i \cap U$. □

In the presence of a continuous group action, we can say more about the image of the homomorphism of Corollary 4.4.10.

Lemma 4.5.8. *Suppose that $f \colon X \to Y$ are as in Proposition 4.5.5, and that H is abelian with exponent e. Then the image of the homomorphism*

$$\widehat{H} \to \mathrm{PicCon}^G(Y)[e]$$

of Corollary 4.4.10 is contained inside the subgroup $\mathrm{PicCon}^G_{\mathrm{cts}}(Y)[e]$.

Proof. Let $\chi \in \widehat{H}$ and recall that $\mathcal{L}_\chi = e_\chi \cdot f_*\mathcal{O}_X$. We want to show that for any affinoid open subset $U \subset Y$, that the natural map

$$G_U \to \mathcal{B}(\mathcal{L}_\chi(U))^\times \tag{4.5}$$

is continuous. By Proposition 4.5.5 the action of G on X is continuous, and therefore by [3, Lemma 3.2.4] we have that $\mathcal{O}_X \in \mathrm{PicCon}^G_{\mathrm{cts}}(X)$, and thus

$$G_V \to \mathcal{B}(\mathcal{O}_X(V))^\times$$

is continuous, where $V = f^{-1}(U)$. Now because $f \colon X \to Y$ is G-equivariant, $G_U \subset G_V$, and thus

$$G_U \to \mathcal{B}(\mathcal{O}_X(V))^\times = \mathcal{B}((f_*\mathcal{O}_X)(U))^\times$$

is continuous. The image of G_U is contained in the subgroup

$$\prod_{\psi \in \widehat{H}} \mathcal{B}(\mathcal{L}_\psi(U))^\times \hookrightarrow \mathcal{B}((f_*\mathcal{O}_X)(U))^\times,$$

and this inclusion is continuous by definition of the topology on the groups $\mathcal{B}(-)^\times$. Therefore,

$$G_U \to \prod_{\psi \in \widehat{H}} \mathcal{B}(\mathcal{L}_\psi(U))^\times$$

is continuous, and when this is composed with the projection to $\mathcal{B}(\mathcal{L}_\chi(U))^\times$ we obtain (4.5). □

4.5.2 Continuous Action on The Drinfeld Tower

As a consequence of the general results established in Section 4.5.1, in particular Proposition 4.5.5, we have the following.

Corollary 4.5.9. *The action of G^0 on each of the spaces $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2, \ldots$ is continuous.*

Proof. The natural action of $\mathrm{GL}_n(\mathcal{O}_F)$ on Ω is continuous by [1, Prop. 3.1.12(b)] and [1, Lem. 3.1.9(b)], and thus the action of $\mathrm{GL}_n(F)$ on Ω is continuous by [1, Lem. 3.1.9(c)]. In particular, the restriction to the subgroup G^0 is continuous, and therefore by the G^0-equivariant isomorphism above $\mathcal{N} \xrightarrow{\sim} \Omega$ the action of G^0 on $\mathcal{N}$ is continuous. Then the result follows from Proposition 4.5.5. □

Corollary 4.5.10. *The action of $\mathrm{GL}_n(F)$ on each of the spaces $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2, \ldots$ is continuous.*

Proof. This follows directly from Lemma 4.5.7 and Corollary 4.5.9. □

4.5.3 Geometrically Connected Components of The Drinfeld Tower

Each of the spaces $(\mathcal{N}_m)_{m\geq 1}$ is connected over L [44, Thm. 2.5], but none of these spaces are geometrically connected. In this section we describe how these spaces break up into geometrically connected components over some finite extension of L. We write L_m for the compositum of the mth Lubin-Tate extension of F with L.

The connected components of the covering spaces $\mathcal{M}_m$ over $\mathbb{C}_p$ have been described by Boutot and Zink [18, Thm. 0.20] using global methods and p-adic uniformisation of Shimura curves. In this section, we describe the geometrically connected components of the cofinal system $(\mathcal{M}_{nm})_{m\geq 1}$ of the covering spaces $(\mathcal{M}_m)_{m\geq 1}$, this being sufficient for our purposes. The result we obtain is not as strong as that of Boutot and Zink, but on the other hand the proof is elementary and self-contained. The proof uses the theory of the sheaf c_X developed in Section 4.1 together with a result of Kohlhaase on the maximal sub-field of the global sections of $\mathcal{N}_{nm}$ [44, Prop. 2.7].

Theorem 4.5.11. *For any $m \geq 1$ and complete field extension K of L_m, there is an isomorphism of $\mathrm{GL}_n(\mathcal{O}_F) \times \mathcal{O}_D^\times$-sets,*

$$\pi_0(\mathcal{N}_{nm}) \xrightarrow{\sim} \mathcal{O}_F^\times/(1 + \pi^m \mathcal{O}_F),$$

where $(g,\delta) \in \mathrm{GL}_n(\mathcal{O}_F)\times\mathcal{O}_D^\times$ *acts on* $\mathcal{O}_F^\times/(1+\pi^m\mathcal{O}_F)$ *by multiplication by* $\det(g)\,\mathrm{Nrd}(\delta^{-1}) \in \mathcal{O}_F^\times$. *These are compatible in the sense that for any* $1 \le r \le m$, *the diagram*

$$\begin{array}{ccc} \pi_0(\mathcal{N}_{nm}) & \xrightarrow{\sim} & \mathcal{O}_F^\times/(1+\pi^m\mathcal{O}_F) \\ \downarrow & & \downarrow \\ \pi_0(\mathcal{N}_{nr}) & \xrightarrow{\sim} & \mathcal{O}_F^\times/(1+\pi^r\mathcal{O}_F) \end{array}$$

commutes.

Proof. Let $m \ge 1$. By Lemma 4.1.8 and its proof, because $\mathcal{N}_{nm}$ is quasi-Stein, there is a $G^0 \times \mathcal{O}_D^\times$-equivariant isomorphism of L_m-algebras,

$$c(\mathcal{N}_{nm,L}) \otimes_L L_m \xrightarrow{\sim} c(\mathcal{N}_{nm,L_m}).$$

The field $c(\mathcal{N}_{nm,L})$ is the maximal field extension of L contained in $\mathcal{O}(\mathcal{N}_{nm})$ by Corollary 4.1.11, and the result [44, Prop. 2.7] of Kohlhaase shows that this is a copy of L_m. Furthermore, it is shown that the action of $(g,\delta) \in \mathrm{GL}_n(\mathcal{O}_F)\times\mathcal{O}_D^\times$ on L_m is by multiplication by $\det(g)\,\mathrm{Nrd}(\delta^{-1}) \in \mathcal{O}_F^\times$ [44, Thm. 2.8(ii)], where this element of $\mathcal{O}_F^\times$ is viewed as an automorphism of L_m over L under the identification $\mathcal{O}_F^\times/(1+\pi^m\mathcal{O}_F) \xrightarrow{\sim} \mathrm{Gal}(L_m/L)$ of Lubin-Tate theory. Because L_m/L is Galois, we have canonical isomorphisms of L_m-algebras

$$c(\mathcal{N}_{nm,L}) \otimes_L L_m = L_m \otimes_L L_m \xrightarrow{\sim} \prod_{\sigma\in\mathrm{Gal}(L_m/L)} L_m.$$

This isomorphism maps $x \otimes y \mapsto (\sigma(x)y)_\sigma$. Composing these isomorphisms, we have a $\mathrm{GL}_n(\mathcal{O}_F)\times\mathcal{O}_D^\times$-equivariant L_m-algebra isomorphism,

$$c(\mathcal{N}_{nm,L_m}) \xrightarrow{\sim} \prod_{\mathcal{O}_F^\times/(1+\pi^m\mathcal{O}_F)} L_m,$$

and thus a $\mathrm{GL}_n(\mathcal{O}_F)\times\mathcal{O}_D^\times$-equivariant bijection,

$$\pi_0(\mathcal{N}_{nm,L_m}) \xrightarrow{\sim} \mathcal{O}_F^\times/(1+\pi^m\mathcal{O}_F).$$

The compatibility follows from the compatibility of the isomorphisms $\mathcal{O}_F^\times/(1+\pi^m\mathcal{O}_F) \xrightarrow{\sim} \mathrm{Gal}(L_m/L)$. Finally, because $\dim_{L_m} c(\mathcal{N}_{nm,L_m}) = |\pi_0(\mathcal{N}_{nm,L_m})|$, each connected component is geometrically connected by Corollary 4.1.9. We obtain the result for general complete extensions K of L_m by Remark 4.1.6. □

Corollary 4.5.12. *For any* $m \ge 1$, $c(\mathcal{N}_m)^{G^0} = K$ *and* $c(\mathcal{M}_m)^{\mathrm{GL}_n(F)} = K$.

Proof. Because the action of $\mathrm{GL}_n(F)$ on the connected components of $\mathcal{M}_m$ is transitive and G^0 is normal in $\mathrm{GL}_n(F)$, projection $\mathcal{O}(\mathcal{M}_m) \to \mathcal{O}(\mathcal{N}_m)$ induces an isomorphism

$$c(\mathcal{M}_m)^{\mathrm{GL}_n(F)} \xrightarrow{\sim} c(\mathcal{N}_m)^{G^0},$$

and so it sufficient to show the claim for $\mathcal{N}_m$. Furthermore, if $r \ge 1$ is such that $rn \ge m$, then in light of the inclusion

$$c(\mathcal{N}_m)^{G^0} \hookrightarrow c(\mathcal{N}_{rn})^{G^0},$$

it is sufficient to show that $c(\mathcal{N}_{rn})^{G^0} = K$, which follows directly from the proof of Theorem 4.5.11 above, as

$$c(\mathcal{N}_{rn}) = L_m,$$

and G^0 acts on L_m through the surjection

$$G^0 \xrightarrow{\det} \mathcal{O}_F^\times \twoheadrightarrow \mathrm{Gal}(L_m/L).$$

□

4.5.4 Equivariant Line Bundles with Connection on the First Drinfeld Covering

We now give an application of Theorem 4.4.10 to understanding equivariant line bundles with connection on the first Drinfeld covering.

In this section we shall assume that $n \geq 2$, and K contains both L_1 and a primitive pth root of 1. Let Σ^1 be a geometrically connected component of of $\mathcal{N}_1$, and let Σ^2 be the preimage of Σ^1 in $\mathcal{N}_2$, which, because $\lfloor \frac{0}{n} \rfloor = 0 = \lfloor \frac{1}{n} \rfloor$, is also geometrically connected by Proposition 1.11.1. The covering,

$$f\colon \Sigma^2 \to \Sigma^1,$$

is a finite étale Galois covering of rigid spaces over K, with abelian Galois group

$$\Gamma := \frac{1 + \Pi\mathcal{O}_D}{1 + \Pi^2\mathcal{O}_D} \cong (\mathbb{F}_{q^n}, +)$$

of exponent p. Furthermore, by Proposition 1.11.1 the spaces Σ^1, Σ^2 are stable under the action of $\mathrm{SL}_n(F)$. This action commutes with the Galois action, and $f\colon \Sigma^2 \to \Sigma^1$ is $\mathrm{SL}_n(F)$-equivariant.

As a consequence of the work at the start of this section, we can deduce the following statement about torsion $\mathrm{SL}_n(F)$-equivariant line bundles with connection on Σ^1.

Corollary 4.5.13. $\mathrm{Con}_{\mathrm{cts}}^{\mathrm{SL}_n(F)}(\Sigma^1)[p] = 0$.

Proof. Because Σ^1 is geometrically connected, we can apply Proposition 4.5.4 and thus the group $\mathrm{Con}_{\mathrm{cts}}^{\mathrm{SL}_n(F)}(\Sigma^1)[p]$ fits into the exact sequence,

$$0 \to \mathrm{Hom}_{\mathrm{cts}}(\mathrm{SL}_n(F), \mu_p(K)) \to \mathrm{Con}_{\mathrm{cts}}^{\mathrm{SL}_n(F)}(\Sigma^1)[p] \to (\mathcal{O}(\Sigma^1)^\times / K^\times \mathcal{O}(\Sigma^1)^{\times p})^{\mathrm{SL}_n(F)}.$$

Now $\mathrm{Hom}_{\mathrm{cts}}(\mathrm{SL}_n(F), \mu_p(K)) = 0$ because $\mathrm{SL}_n(F)$ is perfect, and

$$(\mathcal{O}(\Sigma^1)^\times / K^\times \mathcal{O}(\Sigma^1)^{\times p})^{\mathrm{SL}_n(F)} = 0$$

by Proposition 2.3.5, and therefore the result follows. □

Corollary 4.5.14. *The forgetful map* $\mathrm{PicCon}_{\mathrm{cts}}^{\mathrm{SL}_n(F)}(\Sigma^1)[p] \to \mathrm{Pic}(\Sigma^1)[p]$ *is injective.*

As an immediate consequence of Corollary 4.5.14 and Corollary 4.4.10, we can deduce the main result of Chapter 2.

Corollary 4.5.15 (Theorem 2.3.6)**.** *The group homomorphism*

$$\widehat{\Gamma} \to \mathrm{Pic}(\Sigma^1)[p]$$

is injective.

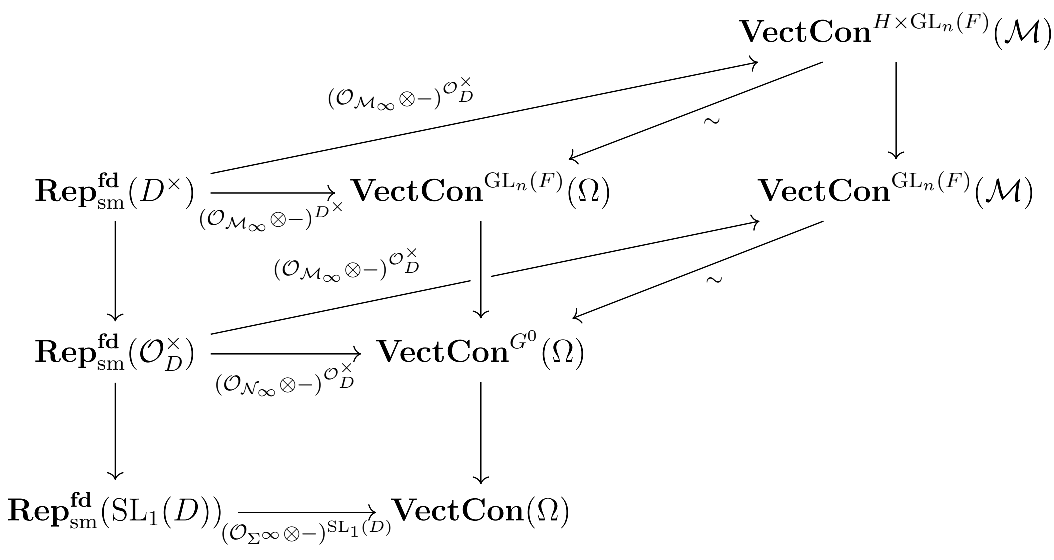
Proof. The natural action of $\mathrm{GL}_n(\mathcal{O}_F)$ on Ω is continuous by [1, Prop. 3.1.12(b)] and [1, Lem. 3.1.9(b)], and thus the action of $\mathrm{GL}_n(F)$ on Ω is continuous by [1, Lem. 3.1.9(c)]. In particular, the action of $\mathrm{SL}_n(F)$ on $\mathcal{N}$ is continuous as the isomorphism $\mathcal{N} \xrightarrow{\sim} \Omega$ is G^0-equivariant and $\mathrm{SL}_n(F) \subset G^0$. Therefore, the action of $\mathrm{SL}_n(F)$ on Σ^1 is continuous by Proposition 4.5.5, and thus the homomorphism of interest factors as the composition

$$\widehat{\Gamma} \to \mathrm{PicCon}_{\mathrm{cts}}^{\mathrm{SL}_n(F)}(\Sigma^1)[p] \to \mathrm{Pic}(\Sigma^1)[p]$$

by Lemma 4.5.8. The first is injective by Corollary 4.4.10, because Σ^2 is geometrically connected, and the second is injective by Corollary 4.5.14 above. □

4.5.5 Equivariant Vector Bundles with Connection on Drinfeld Symmetric Spaces

From now on until the end of the thesis, we will assume that K contains contains L, the completion of the maximal unramified extension of F. In the remaining sections we will define and establish properties the labelled functors of the diagram below, show it is commutative up to natural isomorphism, and use this to prove Theorem A, Theorem B and Theorem C. Vertical arrows below indicate the canonical forgetful maps.



4.5.6 Notation

Here in the above diagram we have used the notation that $H = D^\times/\mathcal{O}_D^\times$.

For $m \geq 1$ we also set

- $D^{(m)} := D^\times/(1 + \Pi^m \mathcal{O}_D)$,
- $\mathcal{O}_D^{(m)} := \mathcal{O}_D^\times/(1 + \Pi^m \mathcal{O}_D)$.

4.5.7 Representations of $D^\times$

In this section we define and establish properties of the functor

$$(\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{D^\times} : \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) \to \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega).$$

We first define the functor

$$(\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{\mathcal{O}_D^\times} \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) \to \mathbf{VectCon}^{H\times \mathrm{GL}_n(F)}(\mathcal{M}).$$

as follows. Suppose that V is a smooth finite dimensional representation of $D^\times$. There is a minimal $m \geq 1$ such that V is inflated from $D^{(m)}$, and we set

$$(\mathcal{O}_{\mathcal{M}_\infty} \otimes V)^{\mathcal{O}_D^\times} \coloneqq (\mathcal{O}_{\mathcal{M}_m} \otimes_k V)^{\mathcal{O}_D^{(m)}} \in \mathbf{VectCon}^{H\times \mathrm{GL}_n(F)}(\mathcal{M})$$

using the functor of Theorem 4.4.2. To make this functorial, given a morphism $\phi \colon V \to W$ let r be minimal such that W is inflated from $D^{(r)}$. We define

$$(\mathcal{O}_{\mathcal{M}_\infty} \otimes \phi)^{\mathcal{O}_D^\times} \colon (\mathcal{O}_{\mathcal{M}_m} \otimes_k V)^{\mathcal{O}_D^{(m)}} \to (\mathcal{O}_{\mathcal{M}_r} \otimes V)^{\mathcal{O}_D^{(r)}}$$

as follows. Let $k \coloneqq \max(m, r)$, and let $(\mathcal{O}_{\mathcal{M}_\infty} \otimes \phi)^{\mathcal{O}_D^\times}$ be defined by the commutative diagram

$$\begin{array}{ccc} (\mathcal{O}_{\mathcal{M}_m} \otimes W)^{\mathcal{O}_D^{(m)}} & \xrightarrow{\sim} & (\mathcal{O}_{\mathcal{M}_k} \otimes \iota^k_m(W))^{\mathcal{O}_D^{(k)}} \\ \downarrow & & \downarrow{\scriptstyle (\mathcal{O}_{\mathcal{M}_k} \otimes \lambda)^{\mathcal{O}^\times_{D,k}}} \\ (\mathcal{O}_{\mathcal{M}_r} \otimes V)^{\mathcal{O}_D^{(r)}} & \xleftarrow[\sim]{} & (\mathcal{O}_{\mathcal{M}_k} \otimes \iota^k_r(V))^{\mathcal{O}_D^{(k)}} \end{array}$$

constructed using Proposition 4.4.6 the natural isomorphisms of Lemma 4.4.7. By the naturality of the isomorphisms of Lemma 4.4.7,

$$(\mathcal{O}_{\mathcal{M}_\infty} \otimes (\phi_1 \circ \phi_2))^{\mathcal{O}_D^\times} = (\mathcal{O}_{\mathcal{M}_\infty} \otimes \phi_1)^{\mathcal{O}_D^\times} \circ (\mathcal{O}_{\mathcal{M}_\infty} \otimes \phi_2)^{\mathcal{O}_D^\times}$$

and $(\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{\mathcal{O}_D^\times}$ is a well-defined functor. We then define $(\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{D^\times}$ as the composition of $(\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{\mathcal{O}_D^\times}$ with the equivalences

$$\mathbf{VectCon}^{H\times \mathrm{GL}_n(F)}(\mathcal{M}) \xrightarrow{\sim} \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M}/H) \xrightarrow{\sim} \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega).$$

of Example 1.6.8 and that induced by the isomorphism $\pi_{\mathrm{GM}} \colon \mathcal{M}/H \xrightarrow{\sim} \Omega$.

We can also give a contravariant version of this functor,

$$\mathrm{Hom}_{D^\times}(-, f_* \mathcal{O}_{\mathcal{M}_\infty}) \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) \to \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega),$$

defined as follows. For any $m \geq 1$, write $\phi_m \colon \mathcal{M}_m \to \mathcal{M}$ for the Galois covering map, and $f_m \colon \mathcal{M}_m \to \Omega$ for the composition

$$\mathcal{M}_m \to \mathcal{M} \xrightarrow{p} \mathcal{M}/H \xrightarrow{\sim} \Omega.$$

Just as in Section 4.4.1, for $V \in \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times)$ which is inflated from $D^{(m)}$, we can define

$$\mathrm{Hom}_{D^\times}(V, f_* \mathcal{O}_{\mathcal{M}_\infty}) \coloneqq \mathrm{Hom}_{D^\times}(V, f_{m,*} \mathcal{O}_{\mathcal{M}_m}) \in \mathbf{Mod}(\mathrm{GL}_n(F)\text{-}\mathcal{D}_\Omega)$$

as in Section 4.4.1. Similarly to above, given a homomorphism $\phi \colon V \to W$ this defines a functor by passing to a high enough level and using Lemma 4.4.7.

Lemma 4.5.16. *There is an isomorphism in* $\mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega)$*,*

$$(\mathcal{O}_{\mathcal{M}_\infty} \otimes V^*)^{D^\times} \xrightarrow{\sim} \mathrm{Hom}_{D^\times}(V, f_*\mathcal{O}_{\mathcal{M}_\infty})$$

natural in $V \in \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times)$*.*

Proof. If V has level m, then by Proposition 4.4.6 there is a natural isomorphism

$$(\mathcal{O}_{\mathcal{M}_m} \otimes V^*)^{\mathcal{O}_D^\times} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_D^\times}(V, \phi_{m,*}\mathcal{O}_{\mathcal{M}_m})$$

in $\mathbf{VectCon}^{H\times\mathrm{GL}_n(F)}(\mathcal{M})$, and when we apply the equivalence

$$\pi_{\mathrm{GM},*} \circ (-)^H \colon \mathbf{VectCon}^{H\times\mathrm{GL}_n(F)}(\mathcal{M}) \xrightarrow{\sim} \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega),$$

we exactly obtain

$$\begin{aligned}(\mathcal{O}_{\mathcal{M}_m} \otimes V^*)^{D^\times} &= \pi_{\mathrm{GM},*}(p_*((\mathcal{O}_{\mathcal{M}_m} \otimes V^*)^{\mathcal{O}_D^\times})^H),\\ &\xrightarrow{\sim} \pi_{\mathrm{GM},*}(p_*(\mathrm{Hom}_{\mathcal{O}_D^\times}(V, \phi_{m,*}\mathcal{O}_{\mathcal{M}_m}))^H),\\ &= \pi_{\mathrm{GM},*}(\mathrm{Hom}_{\mathcal{O}_D^\times}(V, p_*\phi_{m,*}\mathcal{O}_{\mathcal{M}_m})^H),\\ &= \pi_{\mathrm{GM},*}\,\mathrm{Hom}_{D^\times}(V, p_*\phi_{m,*}\mathcal{O}_{\mathcal{M}_m}),\\ &= \mathrm{Hom}_{D^\times}(V, f_{m,*}\mathcal{O}_{\mathcal{M}_m})\end{aligned}$$

as required. □

4.5.8 Representations of $\mathcal{O}_D^\times$

The functor

$$(\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{\mathcal{O}_D^\times} \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) \to \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M})$$

is defined exactly as above, as the direct limit of the functors

$$(\mathcal{O}_{\mathcal{M}_m} \otimes -)^{\mathcal{O}_D^\times} \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^{(m)}) \to \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M})$$

at each finite level. The sub-tower $\mathcal{N} \leftarrow \mathcal{N}_1 \leftarrow \cdots$ is stabilised by G^0 we we may similarly define

$$(\mathcal{O}_{\mathcal{N}_\infty} \otimes -)^{\mathcal{O}_D^\times} \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) \to \mathbf{VectCon}^{G^0}(\Omega),$$

as the inverse limit of the functors

$$(\mathcal{O}_{\mathcal{N}_m} \otimes -)^{\mathcal{O}_D^\times} \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^{(m)}) \to \mathbf{VectCon}^{G^0}(\mathcal{N})$$

composed with the equivalence

$$\pi_{\mathrm{GM},*} \colon \mathbf{VectCon}^{G^0}(\mathcal{N}) \xrightarrow{\sim} \mathbf{VectCon}^{G^0}(\Omega)$$

using the G^0-equivariant isomorphism $\pi_{\mathrm{GM},*} \colon \mathcal{N} \to \Omega$. Just as with Lemma 4.5.16 above, we have the following contravariant version of this functor.

Lemma 4.5.17. *There is an isomorphism in* $\mathbf{VectCon}^{G^0}(\Omega)$,

$$(\mathcal{O}_{\mathcal{N}_\infty} \otimes V^*)^{\mathcal{O}_D^\times} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_D^\times}(V, f_*\mathcal{O}_{\mathcal{N}_\infty})$$

natural in $V \in \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times)$.

By construction, the functor $(\mathcal{O}_{\mathcal{N}_\infty} \otimes -)^{\mathcal{O}_D^\times}$ is equal to the composition of $(\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{\mathcal{O}_D^\times}$ with the (monoidal) restriction equivalence

$$\mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M}) \xrightarrow{\sim} \mathbf{VectCon}^{G^0}(\Omega).$$

of Example 1.6.7. Therefore we have that the two horizontal triangles of the diagram of Section 4.5.5 commute. It is also direct to see that the diagram

$$\begin{array}{rccc} (\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{\mathcal{O}_D^\times} : & \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) & \longrightarrow & \mathbf{VectCon}^{H\times\mathrm{GL}_n(F)}(\mathcal{M}) \\ & \downarrow & & \downarrow \\ (\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{\mathcal{O}_D^\times} : & \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) & \longrightarrow & \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M}) \end{array}$$

commutes by construction. In order to show that the square

$$\begin{array}{rccc} (\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{D^\times} : & \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) & \longrightarrow & \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega) \\ & \downarrow & & \downarrow \\ (\mathcal{O}_{\mathcal{N}_\infty} \otimes -)^{\mathcal{O}_D^\times} : & \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) & \longrightarrow & \mathbf{VectCon}^{G^0}(\Omega) \end{array}$$

commutes, we can either see this directly using the contravariant functors of Lemma 4.5.16 and Lemma 4.5.17 above, or deduce this once we have shown the commutativity of

$$\begin{array}{ccc} \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega) & \xleftarrow{\sim} & \mathbf{VectCon}^{H\times\mathrm{GL}_n(F)}(\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathbf{VectCon}^{G^0}(\Omega) & \xleftarrow{\sim} & \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M}) \end{array}$$

The lower equivalence is defined as the composition of equivalences,

$$\mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M}) \xrightarrow{\sim} \mathbf{VectCon}^{G^0}(\mathcal{N}) \xrightarrow{\sim} \mathbf{VectCon}^{G^0}(\Omega)$$

where the first is restriction (cf. Example 1.6.7), and the second is induced by the G^0-equivariant isomorphism

$$\mathcal{N} \overset{\iota}{\hookrightarrow} \mathcal{M} \xrightarrow{p} \mathcal{M}/H \xrightarrow{\pi_{\mathrm{GM}}} \Omega,$$

and therefore it is sufficient to show the commutativity of

$$\begin{array}{ccc} \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M}/H) & \xleftarrow{\sim} & \mathbf{VectCon}^{H\times\mathrm{GL}_n(F)}(\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathbf{VectCon}^{G^0}(\mathcal{M}/H) & & \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M}) \\ \sim\downarrow & \swarrow{\scriptstyle\sim} & \\ \mathbf{VectCon}^{G^0}(\mathcal{N}) & & \end{array}$$

But this follows directly from the fact that any $\mathcal{F} \in \mathbf{VectCon}^{H \times \mathrm{GL}_n(F)}(\mathcal{M})$ satisfies

$$\mathcal{F}|_{\mathcal{N}} \cong (p \circ \iota)^* \mathcal{F}^H$$

as elements of $\mathbf{VectCon}^{G^0}(\mathcal{N})$. Indeed, the isomorphism is given on any admissible open subset $U \subset \mathcal{N}$ by

$$\begin{aligned} ((p \circ \iota)^* \mathcal{F}^H)(U) &= \mathcal{F}(p^{-1}(p \circ \iota(U)))^H, \\ &= \mathcal{F}(H \cdot U)^H, \\ &\xrightarrow{\sim} \mathcal{F}(U), \end{aligned}$$

with the last isomorphism that induced by sheaf restriction.

4.5.9 Representations of $\mathrm{SL}_1(D)$

Suppose in this section (Section 4.5.9) that K contains L_m for all $m \geq 1$, or equivalently (as K is already assumed to contain the maximal unramified extension of F) that K contains F^{ab}, the maximal abelian extension of F. Under this assumption, by Theorem 4.5.11 each space $\mathcal{N}_m$ is the disjoint union of copies of isomorphic geometrically connected components Σ^m. We can choose these components in such a way that the image of Σ^{m+1} is contained inside Σ^m, and we thus have a tower of geometrically connected rigid spaces,

$$\mathcal{N} \leftarrow \Sigma^1 \leftarrow \Sigma^2 \leftarrow \cdots .$$

Definition 4.5.18. Let $m \geq 1$. We write

- $\mathrm{SL}_1(D) \coloneqq \ker(\mathrm{Nrd}\colon \mathcal{O}_D^\times \to \mathcal{O}_F^\times)$,
- $\mathrm{SL}_1^m(D) \coloneqq \mathrm{SL}_1(D) \cap (1 + \Pi^m \mathcal{O}_D)$ for any $m \geq 1$,
- $\mathcal{O}_F^{[m]} \coloneqq 1 + \pi^{\lceil \frac{m}{n} \rceil} \mathcal{O}_F$.

By Lemma 1.9.8, each extension $f_m\colon \Sigma^m \to \mathcal{N}$ is Galois, with Galois group $\mathrm{Stab}(\Sigma^m) \subset \mathrm{Gal}(\mathcal{N}_m/\mathcal{N})$. By [60, Prop. 3.1] this stabiliser is equal to the kernel of the reduced norm map

$$\mathrm{Nrd}_m\colon \mathcal{O}_D^{(m)} \to \mathcal{O}_F^\times / \mathcal{O}_F^{[m]}.$$

Therefore, using Lemma 4.5.19 below together with the fact that $\mathrm{Nrd}(1 + \Pi^m \mathcal{O}_D) = \mathcal{O}_F^{[m]}$ [53, Lem. 5], we obtain an isomorphism

$$\mathrm{SL}_1(D)/\,\mathrm{SL}_1^m(D) \xrightarrow{\sim} \mathrm{Gal}(\Sigma^m/\mathcal{N}) \hookrightarrow \mathrm{Gal}(\mathcal{N}_m/\mathcal{N}) = \mathcal{O}_D^\times/(1 + \Pi^m \mathcal{O}_D).$$

Lemma 4.5.19. *Suppose that $\phi\colon H_1 \to H_2$ is a surjective group homomorphism and*

$$\begin{aligned} H_1 \geq H_{1,1} \geq H_{1,2} \geq \cdots , \\ H_2 \geq H_{2,1} \geq H_{2,2} \geq \cdots \end{aligned}$$

are chains of normal subgroups with $\phi(H_{1,m}) = H_{2,m}$ for all $m \geq 1$. Then for all $m \geq 1$, ϕ induces an isomorphism,

$$\ker(\phi)/\ker(\phi) \cap H_{1,m} \xrightarrow{\sim} \ker(\phi_m).$$

where ϕ_m is the induced map

$$\phi_m\colon H_1/H_{1,m} \to H_2/H_{2,m}.$$

Proof. Consider the commutative diagram,

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \ker(\phi) \cap H_{1,m} & \longrightarrow & \ker(\phi) & \longrightarrow & \ker(\phi_m) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H_{1,m} & \longrightarrow & H_1 & \longrightarrow & H_1/H_{1,m} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \phi|_{H_{1,m}}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_m} & & \\
1 & \longrightarrow & H_{2,m} & \longrightarrow & H_2 & \longrightarrow & H_2/H_{2,m} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

The right two columns are exact, as ϕ is surjective. The first column is surjective by the assumption that $\phi(H_{1,m}) = H_{2,m}$. Then the result follows by the Nine Lemma. □

In exactly the same way as in Section 4.5.5, taking the direct limit over the functors defined by each covering $\Sigma^m \to \mathcal{N}$ and then applying the isomorphism $\pi_{\mathrm{GM}} \colon \mathcal{N} \xrightarrow{\sim} \Omega$ yields

$$(\mathcal{O}_{\Sigma^\infty} \otimes -)^{\mathrm{SL}_1(D)} \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathrm{SL}_1(D)) \to \mathbf{VectCon}(\Omega).$$

This is compatible with restriction from $\mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times)$ in the sense that the diagram

$$\begin{array}{rccc}
(\mathcal{O}_{\mathcal{N}_\infty} \otimes -)^{\mathcal{O}_D^\times} : & \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) & \longrightarrow & \mathbf{VectCon}^{G^0}(\Omega) \\
 & \downarrow & & \downarrow \\
(\mathcal{O}_{\Sigma^\infty} \otimes -)^{\mathrm{SL}_1(D)} : & \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathrm{SL}_1(D)) & \longrightarrow & \mathbf{VectCon}(\Omega)
\end{array}$$

commutes, which follows directly from Lemma 4.4.8. Furthermore, just as with Lemma 4.5.16 and Lemma 4.5.17 above, we have the following.

Lemma 4.5.20. *There is an isomorphism in* $\mathbf{VectCon}(\Omega)$*,*

$$(\mathcal{O}_{\Sigma^\infty} \otimes V^*)^{\mathrm{SL}_1(D)} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{SL}_1(D)}(V, f_*\mathcal{O}_{\Sigma^\infty})$$

natural in $V \in \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathrm{SL}_1(D))$*.*

4.5.10 Main Theorem

As a result of the work of the previous sections, together with Corollary 4.5.12 and Theorem 4.4.2 we have the following, using the contravariant form of the functors. Note that whenever we consider the functor from $\mathrm{SL}_1(D)$-representations there is an implicit assumption that K contains F^{ab} in order for the spaces Σ^m to be defined.

Theorem 4.5.21. *Each labelled functor of the commutative diagram*

$$\begin{array}{rccc}
\mathrm{Hom}_{D^\times}(-, f_*\mathcal{O}_{\mathcal{M}_\infty}): & \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) & \longrightarrow & \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega) \\
 & \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{O}_D^\times}(-, f_*\mathcal{O}_{\mathcal{N}_\infty}): & \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) & \longrightarrow & \mathbf{VectCon}^{G^0}(\Omega) \\
 & \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{SL}_1(D)}(-, f_*\mathcal{O}_{\Sigma^\infty}): & \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathrm{SL}_1(D)) & \longrightarrow & \mathbf{VectCon}(\Omega)
\end{array}$$

is exact, monoidal, fully faithful, and has its essential image closed under sub-quotients.

This proves all but the essential image parts of Theorem A, Theorem B and Theorem C.

Remark 4.5.22. For each functor of Theorem 4.5.21, the dimension of a representation is equal to the rank of the corresponding vector bundle. This follows from part (5) of Theorem 4.4.4 and the construction of each functor.

Remark 4.5.23. For any representation V of any of the categories of Theorem 4.5.21 one may form the 1-dimensional determinant representation $\det V$. On the other hand, given a an equivariant vector bundle with connection $\mathcal{V}$ of constant rank r then we may form the determinant $\det(\mathcal{V}) = \wedge^r \mathcal{V}$, which is an equivariant line bundle with connection. In fact, each functor forming part of the commutative diagram of Theorem 4.5.21 will commute with taking the determinant on either side, as each functor is exact and monoidal, and the exterior power of an object X can be described in this context as the image of the antisymmetrisation map

$$\sum_{\sigma \in S_r} (-1)^{\mathrm{sgn}(\sigma)} \sigma \colon X^{\otimes r} \to X^{\otimes r}.$$

This similarly applies to symmetric and exterior powers.

Remark 4.5.24. Suppose that G is any subgroup of $\mathrm{GL}_n(F)$ which stabilises the geometrically connected sub-tower $(\Sigma_m)_{m \geq 1}$, such as $\mathrm{SL}_n(F)$ (in fact the subgroups G which stabilise this sub-tower are exactly the subgroups of $\mathrm{SL}_n(F)$ by Theorem 4.5.11). Then in both Theorem 4.5.21 and throughout the previous Section 4.5.9 we may replace $\mathbf{VectCon}(\Omega)$ by $\mathbf{VectCon}^G(\Omega)$ everywhere and all statements remain true. We have chosen to state the theorem with $\mathbf{VectCon}(\Omega)$, as the fully faithfulness and statement that the essential image is closed under sub-quotients for $\mathbf{VectCon}(\Omega)$ directly imply the corresponding statements for $\mathbf{VectCon}^G(\Omega)$.

Remark 4.5.25. Similarly, if G is any group with $\mathrm{GL}_n(\mathcal{O}_F) \leq G \leq G^0$, we may replace G^0 with G in Theorem 4.5.21 and all statements remain true. They key point is that for such a group G, $c(\mathcal{N}_m)^G = K$, which follows from the same proof as given in Corollary 4.5.12 and was the only property we used of G^0. However, we will use G^0 in an essential way in the next section, and in our description of the essential image it is no longer true that G^0 can be replaced with any such G, which is why we choose to state the above theorem with G^0.

Example 4.5.26. For $m \geq 1$,

$$\mathrm{Hom}_{\mathcal{O}_D^\times}(K[\mathcal{O}_D^{(m)}], f_*\mathcal{O}_{\mathcal{N}_\infty}) = f_{m,*}\mathcal{O}_{\mathcal{N}_m} \in \mathbf{VectCon}^{G^0}(\Omega).$$

The representation $V_m \coloneqq K[\mathcal{O}_D^{(m)}]$ of $\mathcal{O}_D^\times$ admits many extentions to a representation of $D^\times$, and each corresponds to an extension of $f_{m,*}\mathcal{O}_{\mathcal{N}_m}$ to a $\mathrm{GL}_n(F)$-equivariant vector bundle with connection. For example, for any choice of uniformiser Π of $\mathcal{O}_D^\times$ (or equivalently any choice of section $s\colon H \to D^\times$), V_m extends to a $D^\times$-representation V_m^s where Π acts by

$$\Pi * n = \Pi n \Pi^{-1}, \qquad n \in \mathcal{O}_D^{(m)},$$

as in Section 4.4. These extensions are typically non-isomorphic for different choices of Π, and

$$\mathrm{Hom}_{D^\times}(V_m^s, f_*\mathcal{O}_{\mathcal{M}_\infty}) = (f_{m,*}\mathcal{O}_{\mathcal{M}_m})^\Pi \in \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega)$$

by Theorem 4.4.2. This can directly be seen to extend $f_{m,*}\mathcal{O}_{\mathcal{N}_m}$, as there is an isomorphism in $\mathbf{VectCon}^{G^0}(\Omega)$, given on an admissible open subset $U \subset \Omega$ by

$$(f_*\mathcal{O}_{\mathcal{N}_m})(U) = \mathcal{O}_{\mathcal{M}_m}(f_m^{-1}(U))^\Pi \xrightarrow{\sim} \mathcal{O}_{\mathcal{M}_m}(f_m^{-1}(U) \cap \mathcal{N}_m) = (f_*\mathcal{O}_{\mathcal{N}_m})(U).$$

4.5.11 The Essential Image

It is possible to give a description of the essential image of each of the functors we have defined, by using part (2) of Theorem 4.4.2. For example, for the functor

$$(\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{D^\times} \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) \to \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega),$$

then $\mathcal{V} \in \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega)$ is in the essential image of $(\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{D^\times}$ if and only if the corresponding $\mathcal{W} \coloneqq (\pi_{\mathrm{GM}} \circ p)^{-1}\mathcal{V} \in \mathbf{VectCon}^{H \times \mathrm{GL}_n(F)}(\mathcal{M})$ is in the essential image of

$$(\mathcal{O}_{\mathcal{M}_\infty} \otimes -)^{\mathcal{O}_D^\times} \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) \to \mathbf{VectCon}^{H \times \mathrm{GL}_n(F)}(\Omega),$$

which using Theorem 4.4.2 is the case if and only if the inequality

$$\dim_K(\mathrm{Hom}_{(\mathrm{GL}_n(F) \times D^{(m)})\text{-}\mathcal{D}_{\mathcal{M}_m}}(\mathcal{O}_{\mathcal{M}_m}, \phi_m^* \mathcal{W})) \leq \mathrm{rank}_{\mathcal{M}}(\mathcal{W}),$$

is an equality for some $m \geq 1$ (equiv. eventually in $m \geq 1$). However this is not an intrinsic description that depends only on the space Ω.

In this section we give an intrinsic description of the essential image of the functor

$$(\mathcal{O}_{\mathcal{N}_\infty} \otimes -)^{\mathcal{O}_D^\times} \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) \to \mathbf{VectCon}^{G^0}(\Omega).$$

Theorem 4.5.27. *The essential image of*

$$(\mathcal{O}_{\mathcal{N}_\infty} \otimes -)^{\mathcal{O}_D^\times} \colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) \to \mathbf{VectCon}^{G^0}(\Omega)$$

is the full subcategory $\mathbf{VectCon}^{G^0}(\Omega)_{\mathrm{fin}}$ *of finite equivariant vector bundles with connection.*

Proof. Suppose first that K is algebraically closed, and $\mathcal{V} \in \mathbf{VectCon}^{G^0}(\Omega)_{\mathrm{fin}}$ is finite. By Proposition 4.3.8, $\mathcal{V}$ is a sub-object of $f_*\mathcal{O}_Z$ for some G^0-equivariant finite étale Galois covering $f\colon Z \to \Omega$, and thus by Theorem 4.5.21 it is sufficient for us to show that $f_*\mathcal{O}_Z$ is in the essential image. If we write $Z = Z_1 \sqcup \cdots \sqcup Z_r$ as a disjoint union of G^0-orbits of connected components, then we have a decomposition

$$f_*\mathcal{O}_Z = f_*\mathcal{O}_{Z_1} \oplus \cdots \oplus f_*\mathcal{O}_{Z_r}$$

in $\mathbf{VectCon}^{G^0}(\Omega)$ and thus we may assume that G^0 acts transitively on the connected components of Z. Furthermore, writing

$$G^n \coloneqq \{g \in \mathrm{GL}_n(F) \mid \nu(\det(g)) \in n\mathbb{Z}\} = G^0 \times \pi^{\mathbb{Z}} I_n,$$

then as $\pi^{\mathbb{Z}} I_n$ acts trivially on Ω, we may view $f\colon Z \to \Omega$ as G^n-equivariant by letting $\pi^{\mathbb{Z}} I_n$ act trivially on Z. For any space X with an action of G^n we can form the space

$$X \times_{G^n} \mathrm{GL}_n(F) \coloneqq (X \times \mathrm{GL}_n(F))/G^n,$$

where G^n acts diagonally through its (right) action on X and its left multiplication action on $\mathrm{GL}_n(F)$. This naturally has a (right) action of $\mathrm{GL}_n(F)$ by right multiplication. If additionally X has an action of $\mathrm{GL}_n(F)$, then there is a natural $\mathrm{GL}_n(F)$-equivariant morphism

$$X \times_{G^n} \mathrm{GL}_n(F) \to X, \qquad (x, g) \mapsto xg,$$

which is finite étale because G^n is finite index in $\mathrm{GL}_n(F)$. Furthermore, there is a natural G^n equivariant open embedding

$$\iota\colon X \hookrightarrow X \times_{G^n} \mathrm{GL}_n(F), \qquad x \mapsto (x, 1).$$

In particular, for $f\colon Z \to \Omega$, we obtain a $\mathrm{GL}_n(F)$-equivariant finite étale covering

$$h\colon Y \coloneqq Z \times_{G^n} \mathrm{GL}_n(F) \to \Omega \times_{G_n} \mathrm{GL}_n(F) \to \Omega,$$

for which $h \circ \iota = f$, where $\iota\colon Z \hookrightarrow Y$ as above. As K is algebraically closed, we may apply the factorisation theorem of Scholze-Weinstein [56, Thm. 7.3.1] to obtain a commutative diagram of $\mathrm{GL}_n(F)$-equivariant morphisms of rigid spaces

$$\begin{array}{ccc} \mathcal{M}_m & \xrightarrow{\phi} & Y \\ & \searrow^{f_m} & \downarrow h \\ & & \Omega \end{array}$$

As Y is the disjoint union of copies of $\iota(Z)$ which are transitively permuted by $\mathrm{GL}_n(F)$ and ϕ is $\mathrm{GL}_n(F)$-equivariant, there is some $i \in \mathbb{Z}$ with $\phi(\mathcal{M}_m^i) \subset \iota(Z)$. As $\mathcal{M}_m^i$ is connected, $\phi(\mathcal{M}_m^i) \subset \iota(Z_0)$ for some connected component Z_0 of Z. Because $f\colon Z_0 \to \Omega$ is finite étale, Z_0 is smooth and therefore normal, and thus $\iota_*\mathcal{O}_{Z_0} \to \phi_*\mathcal{O}_{\mathcal{M}_m^i}$ is injective [44, Prop. A.5]. Then because $\phi\colon \mathcal{M}_m^i \to Z$ is G^0-equivariant and G^0 acts transitively on the connected components of Z, $\iota_*\mathcal{O}_Z \to \phi_*\mathcal{O}_{\mathcal{M}_m^i}$ is injective. In particular, as h_* is left-exact,

$$f_*\mathcal{O}_Z = h_*\iota_*\mathcal{O}_Z \hookrightarrow h_*\phi_*\mathcal{O}_{\mathcal{M}_m^i} = f_{m,*}\mathcal{O}_{\mathcal{M}_m^i}.$$

Now, as f_m is Π-equivariant with respect to the trivial action of Π on Ω and $\Pi^i(\mathcal{M}^i_m) = \mathcal{N}_m$,

$$\begin{aligned} f_{m,*}\mathcal{O}_{\mathcal{M}^i_m} &= f_{m,*}\Pi^i_*\mathcal{O}_{\mathcal{M}^i_m}, \\ &= f_{m,*}\mathcal{O}_{\mathcal{N}_m} \end{aligned}$$

and therefore, as $f_{m,*}\mathcal{O}_{\mathcal{N}_m}$ is in the essential image (Example 4.5.26), we are done because the essential image is closed under sub-objects by Theorem 4.5.21.

Now let K be general, and for any complete field extension L of K write

$$F_L \coloneqq \mathrm{Hom}_{L[\mathcal{O}_D^\times]}(-, f_*\mathcal{O}_{\mathcal{M}_{\infty,L}})\colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm},L}(\mathcal{O}_D^\times) \to \mathbf{VectCon}^{G^0}(\Omega_L)$$

for the base change. Note that for any $W \in \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm},K}(\mathcal{O}_D^\times)$, we have a canonical isomorphism

$$F(W)_L \xrightarrow{\sim} F_L(W_L).$$

Indeed, if m is the level of W, then the inclusion

$$F(W)_L = \mathrm{Hom}_{K[\mathcal{O}_D^\times]}(W, f_{m,*}\mathcal{O}_{\mathcal{M}_m}) \otimes_k L \hookrightarrow \mathrm{Hom}_{L[\mathcal{O}_D^\times]}(W_L, f_{m,*}\mathcal{O}_{\mathcal{M}_{m,L}}) = F_L(W_L)$$

is an isomorphism, as these are both $\mathcal{D}_{\Omega_L}$-modules of the same rank (using the same argument as that of the proof of Theorem 4.2.7(2)).

Suppose now that $\mathcal{V} \in \mathbf{VectCon}^{G^0}(\Omega)$ is finite, and let C be an complete algebraically closed field containing K. Then by the above, there is some $V \in \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm},C}(\mathcal{O}_D^\times)$ with $F_C(V) \cong \mathcal{V}_C$. Because V is inflated from a representation of a finite group, there is some finite extension L of K and $W \in \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm},L}(\mathcal{O}_D^\times)$ with $V \cong W_C$, and thus

$$F_L(W)_C \cong F_C(W_C) \cong \mathcal{V}_C.$$

Furthermore, because $F_L(W)$ and $\mathcal{V}_L$ are coherent, the isomorphism $F_L(W)_C \xrightarrow{\sim} (\mathcal{V}_L)_C$ is defined over some finite extension of L, and so without loss of generality, after potentially enlarging L, we may assume that $F_L(W) \xrightarrow{\sim} \mathcal{V}_L$ over L. Then $\mathcal{V}$ is a direct summand of a finite vector bundle,

$$\mathcal{V} \hookrightarrow (\mathcal{V}_L)|_K \xrightarrow{\sim} F_L(W)|_K \hookrightarrow F_L((W|_K)_L)|_K \xrightarrow{\sim} (F_K(W|_K)_L)|_K = \bigoplus_{i=1}^{[L:\,K]} F_K(W|_K),$$

using that $W \hookrightarrow (W|_K)_L$, and thus $\mathcal{V}$ is itself finite. □

This finishes the proof of Theorem B. We now use this to finish the proof of Theorem A.

Corollary 4.5.28. *The essential image of*

$$\mathrm{Hom}_{D^\times}(-, f_*\mathcal{O}_{\mathcal{M}_\infty})\colon \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) \to \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega)$$

is the full subcategory $\mathbf{VectCon}^{\mathrm{GL}_n(F)}(\Omega)_{G^0\text{-fin}}$ *with objects those that are finite when viewed as* G^0*-equivariant vector bundles with connection.*

Proof. Let us first fix a uniformiser Π of $\mathcal{O}_D^\times$, and identify H with $\Pi^{\mathbb{Z}}$. By the commutative diagram of Section 4.5.5, because all functors are monoidal then in the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_D^\times}(-,\phi_*\mathcal{O}_{\mathcal{M}_\infty})\colon\ \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(D^\times) & \longrightarrow & \mathbf{VectCon}^{\Pi^{\mathbb{Z}}\times \mathrm{GL}_n(F)}(\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{O}_D^\times}(-,\phi_*\mathcal{O}_{\mathcal{M}_\infty})\colon\ \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times) & \longrightarrow & \mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M}) \end{array}$$

the bottom arrow is an equivalence onto $\mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M})_{\mathrm{fin}}$, and it is sufficient for us to show that the essential image the top arrow is the full subcategory

$$\mathbf{VectCon}^{\Pi^{\mathbb{Z}}\times \mathrm{GL}_n(F)}(\mathcal{M})_{\mathrm{GL}_n(F)\text{-}\mathrm{fin}}$$

of $\mathrm{GL}_n(F)$-finite equivariant vector bundles with connection. By the commutativity of the diagram the essential image is contained in this full subcategory, and so suppose conversely that

$$\mathcal{V} \in \mathbf{VectCon}^{\Pi^{\mathbb{Z}}\times \mathrm{GL}_n(F)}(\mathcal{M})$$

is finite when viewed as a $\mathrm{GL}_n(F)$-equivariant vector bundle with connection. Then there is some $V \in \mathbf{Rep}^{\mathbf{fd}}_{\mathrm{sm}}(\mathcal{O}_D^\times)$ of level m and an isomorphism

$$\Phi\colon \mathcal{V} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_D^\times}(V,\phi_*\mathcal{O}_{\mathcal{M}_m})$$

in $\mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M})$. We may view the action of Π on $\mathcal{V}$ as a morphism

$$\Pi^{\mathcal{V}}\colon \mathcal{V} \to \Pi^{-1}\mathcal{V},$$

in $\mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M})$ where $\Pi^{-1}\mathcal{V}$ is viewed as a $\mathrm{GL}_n(F)$-equivariant vector bundle with connection with action of $\mathcal{D}_{\mathcal{M}}$ via $\Pi^{\mathcal{D}_{\mathcal{M}}}\colon \mathcal{D}_{\mathcal{M}} \to \Pi^{-1}(\mathcal{D}_{\mathcal{M}})$ and $\mathrm{GL}_n(F)$-equivariant structure

$$g^{\Pi^{-1}\mathcal{V}} := \Pi^{-1}g^{\mathcal{V}} = \Pi^{-1}(\Pi g\Pi^{-1})^{\mathcal{V}}\colon \Pi^{-1}\mathcal{V} \to g^{-1}(\Pi^{-1}\mathcal{V}).$$

We note that there is an isomorphism

$$\Pi^{f_*\mathcal{O}_{\mathcal{M}}}\circ(-)\colon\ \mathrm{Hom}_{\mathcal{O}_D^\times}({}^{\Pi}V,\phi_*\mathcal{O}_{\mathcal{M}_m}) \xrightarrow{\sim} \Pi^{-1}\,\mathrm{Hom}_{\mathcal{O}_D^\times}(V,\phi_*\mathcal{O}_{\mathcal{M}_m})$$

in $\mathbf{VectCon}^{\mathrm{GL}_n(F)}(\mathcal{M})$, where ${}^{\Pi}V$ is V viewed with action of $x \in \mathcal{O}_D^\times$ by $\Pi x\Pi^{-1}$. By the fully faithfulness of Theorem 4.5.21, there is a unique $\mathcal{O}_D^\times$-linear isomorphism $\varphi\colon {}^{\Pi}V \to V$ for which the functorially induced morphism

$$((-)\circ\varphi)\colon\ \mathrm{Hom}_{\mathcal{O}_D^\times}(V,\phi_*\mathcal{O}_{\mathcal{M}_m}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_D^\times}({}^{\Pi}V,\phi_*\mathcal{O}_{\mathcal{M}_m})$$

is equal to $(\Pi^{f_*\mathcal{O}_{\mathcal{M}}}\circ(-))^{-1}\circ\Phi(\Pi^{\mathcal{V}})$. Because $\varphi\colon {}^{\Pi}V \to V$ is $\mathcal{O}_D^\times$-linear we may extend V to a representation of $D^\times$ by setting Π^{-1} to act by φ, and obtain

$$\mathcal{F} := \mathrm{Hom}_{\mathcal{O}_D^\times}(V,\phi_*\mathcal{O}_{\mathcal{M}}) \in \mathbf{VectCon}^{\Pi^{\mathbb{Z}}\times \mathrm{GL}_n(F)}(\mathcal{M})$$

with action (by definition of the functor) of Π by

$$\Pi^{\mathcal{F}} \coloneqq \Pi^{f_*\mathcal{O}_X} \circ - \circ \Pi^{-1} \colon \mathcal{F} \xrightarrow{\sim} \Pi^{-1}\mathcal{F}.$$

Therefore, as Π^{-1} acts on V by φ,

$$\begin{aligned}
\Pi^{\mathcal{F}} &= \Pi^{f_*\mathcal{O}_X} \circ - \circ \varphi, \\
&= (\Pi^{f_*\mathcal{O}_X} \circ (-)) \circ ((-) \circ \varphi), \\
&= (\Pi^{f_*\mathcal{O}_X} \circ (-)) \circ (\Pi^{f_*\mathcal{O}_{\mathcal{M}}} \circ (-))^{-1} \circ \Phi(\Pi^{\mathcal{V}}), \\
&= \Phi(\Pi^{\mathcal{V}}),
\end{aligned}$$

and therefore Φ is furthermore an isomorphism in $\mathbf{VectCon}^{\Pi^{\mathbb{Z}} \times \mathrm{GL}_n(F)}(\mathcal{M})$. $\square$

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