SOME PROBLEMS IN HARMONIC ANALYSIS AND PROBABILISTIC POTENTIAL THEORY

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To Barbara. Here we go again!

CONTENTS

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INTRODUCTION

In 1917 Borel wrote a monograph entitled "Lessons on Monogenic functions". In it he described a class of functions which behave, in many ways, like analytic functions. The one big difference between Cauchy's analytic functions and Borel's monogenic functions is the following. The domain of an analytic function is by definition open; the domain of a monogenic function need have no interior. We propose examining a similar phenomenon here.

The metric topology is not the only useful topology on $\mathbb{C}$. An examination of potential theory leads one, sooner or later, to study the fine topology. This topology is strictly finer than the metric topology and it is quite easy to construct sets with empty metric interior and non-empty fine interior. The fine topology is the correct one for the theory of harmonic functions, and it makes sense to say a function is harmonic on a finely open set. B.Fuglede proved that these finely harmonic functions have many of the properties of harmonic functions(maximum modulus theorem etc.), and used some of their more subtle properties to prove that the fine topology is locally connected.

In this thesis we study a special class of finely harmonic functions which we call finely holomorphic. Initially we know that these functions, in common with all finely harmonic functions, have a gradient defined in an L^2 sense. By definition the $\frac{\partial}{\partial \bar{z}}$ component of this gradient is zero almost everywhere.

Ultimately we will conclude that a finely holomorphic function f has a fine derivative f' defined everywhere on the domain of f. Moreover the fine derivative is finely holomorphic and so the procedure can be iterated.

Cauchy had Cauchy's theorem to help him. We do not. Three ideas are fundamental to our approach. First, a connected finely open set is polygonally connected. Second, it is possible to obtain a fine local bound on the size of the derivative of a finely holomorphic function. And third, we apply the techniques of rational approximation. Within this framework it is possible to rearrange the proofs slightly.

The reader can judge whether the route taken by the author gives a reasonably unified approach to the subject matter. In order to keep the approach clear we have avoided mentioning some interesting but tangential results concerning representing measures for $A_{\mathfrak{F}}(K)$ (see (12a)).

CHAPTER 1 : A FEW ASPECTS OF PROBABILITY AND POTENTIAL THEORY

In this thesis we deal with one particular interaction between potential theory, complex function theory, and stochastic integration. We have assumed the reader is familiar with the elements of these subjects. The reader should have also seen a probabilistic interpretation of classical potential theory. These subjects are more than adequately covered by (10),(6a),(13),and (16).

In this chapter we propose, firstly, to give a brief description of the fine topology and its probabilistic connections. Secondly, we wish to introduce Itô's formula in a form suited to complex valued martingales. Neither of these topics are dealt with in the basic references mentioned above.

THE FINE TOPOLOGY ON $\mathbb{C}$

Superharmonic functions on $\mathbb{C}$ need not be continuous. However, because they are lower semi-continuous and satisfy a super mean value property, they cannot be too badly discontinuous. For example, if f is superharmonic then $\liminf_{x \to y} f(x)$ is equal to $f(y)$, and this is more than lower semi-continuity alone would imply. There is also the more subtle regularity property : if a lower bounded superharmonic function v is composed with a Brownian motion Z_t then the resulting supermartingale $v(Z_t)$ has continuous paths. At first sight one would expect the paths to be only lower semi-continuous.

Sometime during the 40's, Brelot introduced the idea of a set being thin at a point. A set E is said to be thin at a point $x \notin E$ if E can be separated from x using a superharmonic function. In other words, given x and E one must find an open set $\mathcal{U}$ containing x and a superharmonic function v defined on $\mathcal{U}$ such that $v(x) < \inf_{y \in E \cap \mathcal{U}} v(y)$. Cartan observed, in a letter to Brelot, that if we define $\mathcal{F}$ to be the coarsest topology which makes all superharmonic functions continuous then to say E is thin at x is simply to say that the complement of E is an $\mathcal{F}$ - neighbourhood of x. The topology $\mathcal{F}$ is known as the <u>fine</u> topology.

The fine topology is, by its construction, completely regular and Hausdorff. It also has properties of the Lindelof and Baire types, although it is not first countable. The only sets which are compact for the fine topology on $\mathbb{C}$ are finite. One of the deepest results about the fine topology on $\mathbb{C}$ is that it is locally connected.

However, it is the relationship between finely open sets and probability (or balayage) which really makes the fine topology interesting. It enabled B. Fuglede (5b) to sensibly define harmonic functions whose domains were only finely open.

Throughout this thesis we shall use the following notation. Brownian motion in $\mathbb{C}$ will always be denoted by Z_t, the probability measure which makes it start at x will be $\mathbb{P}^x$ and the associated expectation will be $\mathbb{E}^x$.

The relationship between Brownian motion and the fine topology is well known but proofs are hard to find.

Therefore we will go into some detail. Let x be a point in $\mathbb{C}$ and let the set $E \subset \mathbb{C}$ be thin at x. Choose a superharmonic function v which separates x from E. Compose v with a Brownian motion Z started from x. Certainly $v(Z)$ has right continuous paths. If follows that Z almost surely stays inside the complement of E for a non-zero time. In fact a set F is an $\mathcal{F}$neighbourhood of x if and only if Z almost surely stays inside F for a non-zero time. We have just seen why this is true in one direction. We now tackle the other direction.

Let $Q_F = \inf\{t \geq 0 : Z_t \notin F\}$ be the first time that Z leaves a set $F \subset \mathbb{C}$. Suppose that Brownian motion started at x runs for a non-zero time in F, that is $\mathbb{P}^x\{Q_F > 0\} = 1$. Then the same is true for some compact subset K of F. For we can choose an $\varepsilon > 0$ and a compact set of paths L such that each path in L does not leave F before time ε and $\mathbb{P}^x(L) > 0$. Then we may take K to be the projection of $L \times [0, \varepsilon]$. The superharmonic functions which we shall use to separate x from $\complement F$ are normalised conductor potentials. For an open set $\mathcal{U} \subset \mathbb{C}$ and sufficiently nice sets $W \subset \mathcal{U}$ the normalised conductor potential ${}^{\mathcal{U}}q_W$ for W on $\mathcal{U}$ can be defined probabilistically (16;6.5) as

$${}^{u}q_W(x) = \mathbb{P}^x\{\exists\, t \in (0, Q_u) \quad \text{such that } Z_t \in W\} .$$

Take W to be $\complement K$ and choose an open neighbourhood $\mathcal{U}$ of x for which ${}^{u}q_W(x) < 1$. Since ${}^{u}q_W$ is 1 on $\complement K \cap \mathcal{U}$ it establishes the result we seek.

We know that if $\mathcal{U}$ is finely open then Brownian motion started in it will not leave instantly. So we may ask if functions on a finely open set satisfy a mean value property with respect to this residual bit of Brownian motion.

1.1 DEFINITION: Let U be a finely open set. A function f is finely superharmonic if, firstly, f is a finely lower semi-continuous map of U into $\mathbb{R} \cup \{+\infty\}$, and secondly, $f(Z)$ is a local supermartingale on $[0, Q_u)$. A function f is finely subharmonic if $-f$ is finely superharmonic. A function f is finely harmonic if it is both finely superharmonic and finely subharmonic.

Finely harmonic functions have many good properties analogous to those possessed by ordinary harmonic functions. The main reference for finely harmonic functions is B.Fuglede's book(5b), which contains most of his fundamental work on the subject. In it he proves that finely superharmonic functions are finely continuous. Hence the "fine fine topology" is just the fine topology.

For the rest of this thesis we shall adopt the following convention. If we say that a set is open or that a function is continuous or make any other unqualified reference to a topology we will always be referring to the usual (or initial) topology. All our references to the fine topology will be explicit. It will be convenient to have a notation for the interior of a set: E' will denote the fine interior of the set E, and E° the usual interior.

The following approximation theorem goes a long way towards justifying a look at finely harmonic and finely superharmonic functions. This result and analogous ones for arbitary harmonic spaces have been proved by Bliedtner and Hansen(1). Debiard and Gavean (4a) were first to state Theorem 1.2 (in the harmonic case).

1.2 THEOREM: Let K be a compact set and f be a continuous function on K. Then f is finely harmonic on the fine interior K' if and only if it can be uniformly approximated on K by functions, each of which extends to be harmonic on a neighbourhood of K. If we replace "harmonic" by "continuous and superharmonic" throughout, the modified statement is also true.

We will often use Theorem 1.2 in conjunction with the following localisation result.

1.3 THEOREM: Let E be a finely open set, let X be a topological space with a countable base, and let f be a finely continuous function mapping E into X. Then for any x in E there is a fine neighbourhood of x, which we may take to be compact, on which f is continuous in the usual topology.

Combining 1.2 and 1.3 we see that any finely harmonic function can be finely locally uniformly approximated by ordinary harmonic functions. To look at the local properties of finely harmonic functions, it therefore suffices to study the functions which are continuous on a compact set K and finely harmonic on K'. We will call the set of such functions $\underline{H(K)}$.

A proof of 1.3 can be found in Fuglede(5a). It relies on an old result of Choquet(3).

WIENER'S CRITERION AND EXAMPLES OF FINELY OPEN SETS

It is easy to construct a fine neighbourhood of zero which has no ordinary interior. For example, let D denote the closed unit disc and let $(z_j)_{j=1}^{\infty}$ denote a dense subset of $D \setminus \{0\}$. Remove a small open disc D_j centred at each of the z_j . Choose the radius of

$\mathcal{D}_j$ in such a way the $\mathbb{P}^0$-probability of Z hitting the j'th disc before leaving $\mathcal{D}$ is $2^{-(j+1)}$. By construction the probability that Z first leaves $K = \mathcal{D}\setminus\bigcup_{j=1}^{\infty}\mathcal{D}_j$ through the unit circle is at least a half. Therefore zero is in the fine interior of K. But K has no interior.

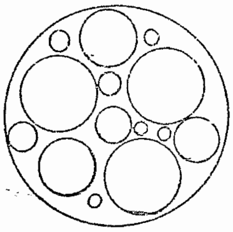

Figure 1
A Swiss Cheese.

Sets like K are often referred to as Swiss Cheeses.

Wiener gave a characterisation of when a set E is thin at a point x in terms of the logarithmic capacity Cap. Let E_n be that part of E which lies in the annulus $\{z : 2^{-n} \leq |z-x| < 2^{-(n-1)}\}$ centred at x.

1.4 THEOREM (WIENER'S TEST): E is thin at x if and only if

$$\sum_0^{\infty} \frac{-n}{\log(\mathrm{Cap}\, E_n)} < \infty .$$

A probabilistic proof of this result can be found in Itô and McKean(11). To apply it one needs to know how to evaluate capacities. The standard reference for such matters is Tsuji(20). We can use Wiener's test to give another example of a fine neighbourhood of zero.

The logarithmic capacity of a disc of radius r is in fact r. Put $E = \bigcup_{j=1}^{\infty}\mathcal{D}_j$ where $\mathcal{D}_j$ is a disc of radius r_j centred a distance $\sqrt{2}/2^j$ from zero. Then E is thin at zero if and only if $\sum_{j=0}^{\infty} \frac{-j}{\log(r_j)} < \infty$.

It follows that the r_j must get small quickly if E is to be thin at zero. For example $r_j = 2^{-j^3}$ will do.

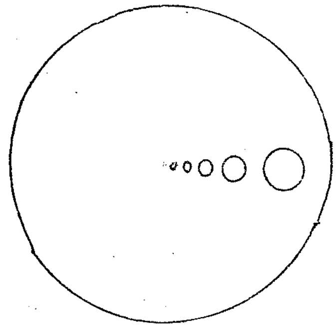

Figure 2
A road runner set.

The complement of a set like E in the unit disc is often known as <u>a road runner set</u>. A road runner set K which has zero in its fine interior is a typical example of a subset of $\mathbb{C}$ on which the Dirichlet problem cannot be solved. In other words not every continuous function g on the boundary of K has a continuous harmonic extension to K°. This is because there are two conflicting ways of giving a value to g at zero. If the harmonic extension of g to K° is to be continuous at zero then one must have $g(0) = E^0(g(Z_{Q_{K'}}))$ and because zero is in K' this will not in general be true. The observation that one cannot always solve the classical Dirichlet problem provided the original motivation for studying thinness.

<u>MARTINGALES AND STOCHASTIC INTEGRATION</u>

Let M be a real local martingale with continuous paths. Then $\langle M,M\rangle$ will denote the unique increasing process such that $M^2 - \langle M,M\rangle$ is a local martingale and $\langle M,M\rangle_0 = 0$. We can extend $\langle\ ,\ \rangle$ to be bilinear by defining $\langle M,N\rangle$ to be the unique bounded variation process, with $\langle M,N\rangle_0 = 0$, for which $MN - \langle M,N\rangle$ is a martingale. We will often deal with complex valued martingales. The definition we have just given for $\langle\ ,\ \rangle$ still holds; however it differs from the definition given in Getoor and Sharpe (8). They define

$\langle\, ,\rangle$ to make $M\bar{N} - \langle M, N\rangle$ a local martingale.

With our definition, complex valued martingales M for which $\langle M, M\rangle$ is identically zero are of special interest. They are the complex martingales which can be time changed into complex Brownian motion and they are known as <u>conformal martingales</u>. (In fact $\langle Z, Z\rangle \equiv 0$ and $\langle Z, \bar{Z}\rangle_t = 2t$.)

We assume the reader is familiar with the general theory of stochastic integration (such as can, for example, be found in Meyer (13)).

For convenience, we will state the full complex form of Itô's lemma. The algebra involved in checking it is rather grotesque (but the form of it comes from Taylor's theorem).

1.5 THEOREM (ITÔ'S FORMULA): Let F be a twice continuouly differentiable function on a domain $\mathcal{D}$ in $\mathbb{C}^d$. Let $T^1, \ldots, T^d$ be complex valued martingales. Suppose $(T^1, \ldots, T^d)$ takes its values in $\mathcal{D}$. Then

$$
\begin{aligned}
F(T^1, \ldots, T^d)\Big|_t = {} & F(T^1, \ldots, T^d)\Big|_0 \\
& + \sum_{j=1}^{d} \left(\int_0^t \frac{\partial F}{\partial z_j}\, dT^j + \int_0^t \frac{\partial F}{\partial \bar{z}_j}\, d\bar{T}^j \right) \\
& + \frac{1}{2} \sum_{j=1}^{d} \sum_{k=1}^{d} \left(\int_0^t \frac{\partial F}{\partial z_j \partial z_k}\, d\langle T^j, T^k\rangle + \int_0^t \frac{\partial F}{\partial \bar{z}_j \partial z_k}\, d\langle \bar{T}^j, T^k\rangle \right. \\
& \left. + \int_0^t \frac{\partial F}{\partial z_j \partial \bar{z}_k}\, d\langle T^j, \bar{T}^k\rangle + \int_0^t \frac{\partial F}{\partial \bar{z}_j \partial \bar{z}_k}\, d\langle \bar{T}^j, T^k\rangle \right).
\end{aligned}
$$

There is one rather special case when the formula simplifies dramatically. Suppose $T^1, \ldots, T^d$ are orthogonal (wrt. $\langle\, ,\rangle$) conformal martingales and F is an analytic function. Then the formula reads

$$
F(T^1, \ldots, T^d)\Big|_t = F(T^1, \ldots, T^d)\Big|_0 + \sum_{j=1}^{d} \int_0^t \frac{\partial F}{\partial z_j}\, dT^j ,
$$

and so we have the usual functional calculus for analytic functions. If we only knew that F was pluri-harmonic we would still get some simplification and the formula would then read:

$$F(T^1, \ldots, T^d)\Big|_t = F(T^1, \ldots, T^d)\Big|_0 + \sum_{j=0}^{d} \left(\int_0^t \frac{\partial F}{\partial z_j}\, dT^j + \int_0^t \frac{\partial F}{\partial \bar{z}_j}\, d\bar{T}^j \right).$$

CHAPTER TWO: STOCHASTIC DIFFERENTIATION OF FINELY HARMONIC FUNCTIONS

In (4b) Debiard and Gavean show, how, given a finely harmonic function f on a finely open set V, one can construct two functions g and h on V. These functions may loosely be though of as $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$. These "derivatives" of f are however only defined in an almost everywhere sense on V. We shall call them stochastic derivatives because they can be used to construct $f(Z)$ as a stochastic integral:

$$f(Z_t) - f(Z_0) = \int_0^t g(Z_s)\, dZ_s + \int_0^t h(Z_s)\, d\bar{Z}_s \quad , (t < Q_V).$$

In future we will use the notation $\frac{\partial f}{\partial z}$ for g and $\frac{\partial f}{\partial \bar{z}}$ for h and trust that this will cause no confusion. To construct these derivatives we first look at smooth functions and establish the inequality Theorem 2.1; then we apply Theorems 1.3 and 1.2.

2.1 THEOREM: Let $K \subset \mathbb{C}$ be a compact set and f a harmonic function defined on a neighbourhood of K. Then for all x in K

$$\mathbb{E}^x\left(\int_0^{Q_K} \left[\left|\frac{\partial f}{\partial z}\right|^2(Z_t) + \left|\frac{\partial f}{\partial \bar{z}}\right|^2(Z_t)\right] dt\right) \leqslant 2\,\|f\|_\infty^2 .$$

PROOF: Apply Itô's formula to the function f and the martingale Z. One obtains

$$f(Z_t) - f(Z_0) = \int_0^t \frac{\partial f}{\partial z}\, dZ + \int_0^t \frac{\partial f}{\partial \bar{z}}\, d\bar{Z} \quad , (t < Q_K).$$

Use the L^2 isometry for stochastic integrals to obtain:

$$2\,\mathbb{E}^x\left(\int_0^{Q_K}\left[\left|\frac{\partial f}{\partial z}\right|^2(Z_t) + \left|\frac{\partial f}{\partial \bar{z}}\right|^2(Z_t)\right] dt\right) = \mathbb{E}^x\left(|f(Z_{Q_K}) - f(Z_0)|^2\right) \leqslant 4\,\|f\|_\infty^2 .$$

***

Denote by G_x the measure on K which assigns to a set $A \subset K$ the $\mathbb{P}^x$ expected time that Z spends in A before leaving K. We may rewrite the inequality at 2.1 in terms of G_x.

For all x,

$$\int_K \left[\left|\frac{\partial f}{\partial z}\right|^2(w) + \left|\frac{\partial f}{\partial \bar{z}}\right|^2(w) \right] G_x(dw) \leqslant 2 \|f\|_\infty^2 .$$

Now suppose that f is a continuous function on K and finely harmonic on K'. We may use 1.2 and 2.1 to construct "derivatives" $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ contained in $L^2(G_x)$. Moreover by the construction of stochastic integrals using the L^2 isometry and functional completion it is clear that these derivatives are the unique elements of $L^2(G_x)$ for which

$$f(Z_t) - f(Z_0) = \int_0^t \frac{\partial f}{\partial z}\, dZ + \int_0^t \frac{\partial f}{\partial \bar{z}}\, d\bar{Z} , \qquad (t < Q_K).$$

It is not immediately clear that the derivatives do not depend on the choice of x. However, it can be proved that if x is in K' and V is in fine component[1] of K' containing x then G_x and Lebesgue measure restricted to V are mutually absolutely continuous. Therefore no problems.[2]

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1) The fine topology on $\mathbb{C}$ is locally connected, and hence every point is contained in a maximal finely connected finely open set.

2) The proof of this is not edifying and so we do not include details. However all one really needs for the derivatives to be well defined is that G_x and G_y should be mutually absolutely continuous if x and y are in the same fine component of K'. It is easy to prove (use the strong Markov property) that if E is Borel then the sets

$$\{ x : G_x(E) = 0 \}$$

and

$$\{ x : G_x(E) > 0 \}$$

are both finely open. It follows that if x and y are in the same fine component of K then G_x and G_y have the same null sets.

It was hoped for some while that versions of these stochastic derivatives could be constructed as other finely harmonic functions. If this had been the case then all finely harmonic functions would be infinitely differentiable. Debiard and Gaveau (4a) gave an example of a Road Runner set K with zero in K' and a finely harmonic function on K' whose gradient could not have a finely continuous version which was finite at zero. More recently B.Øksendal (14) gave a more sophisticated example. He constructed a finely harmonic function f defined on a finely open set U and a subset E of U, with $\mathrm{Cap}(E)>0$ such that the gradient of f could not have a finely continuous version finite at any point of E. This example wrecks any hopes of constructing multiple derivatives of finely harmonic functions.

On the positive side B. Fuglede pointed out the following to me. In (5c) he proves that any finely harmonic function f is finely locally the differences of two potentials p and q. Put $p = -\int \log|x-y|\mu(dy) + h$ where μ is a positive measure and h is a harmonic function. Then the derivative $p'(x)$ of p at x is controlled by $\int \frac{1}{|x-y|}\mu(dy)$. This integral (and hence the derivative of f) is defined except on a set of three dimensional capacity zero. This remark asserts more than the existence of the integral off a null set. Many null sets have non-zero three dimensional capacity.

Let U be a finely open set and let f be a finely harmonic function on U. Theorem 1.3 can be used to extend the derivative of f to the whole of U (except for a Lebesgue null set).

Having obtained this derivative, Debiard and Gaveau then went on to look at the finely holomorphic functions on $\mathcal{U}$, denoted by $\mathcal{O}(\mathcal{U})$. Define $\mathcal{O}(\mathcal{U})$ to be the finely harmonic functions f on $\mathcal{U}$ which satisfy $\frac{\partial f}{\partial \bar{z}} = 0$ a.e. on $\mathcal{U}$. As in the harmonic case, if we wish to study local properties we need only look at the continuous functions on a compact set K which are finely holomorphic on K. We call this Banach algebra (Cf. 2.2) $A_{\mathfrak{F}}(K)$ in analogy with the standard notation $A(K)$ for the algebra of those continuous functions on K which are analytic on K°. It will often pay us to work with $A_{\mathfrak{F}}(K)$ rather than $\mathcal{O}(\mathcal{U})$.

First we establish some elementary properties of $\mathcal{O}(\mathcal{U})$ and $A_{\mathfrak{F}}(K)$.

2.2 THEOREM: The sum and the product of two finely holomorphic functions are both finely holomorphic functions. A uniform limit of finely holomorphic functions is also finely holomorphic. Suppose f and $z.f$ are both finely harmonic; then f is finely holomorphic.

PROOF: The L^2 isometry mentioned in 1.5 implies that there is a unique way of expressing a finely harmonic function as a stochastic integral:

$$f(Z_t) - f(Z_o) = \int_o^t u\, dZ + \int_o^t v\, d\bar{Z} .$$

If and only if the term in $d\bar{Z}$ is zero then f is finely holomorphic. It follows trivially that the sum of two finely holomorphic functions is finely holomorphic.

We may apply Itô's formula to the function multiplication and the martingales $f(Z)$ and $g(Z)$ to see that the product of two finely holomorphic functions is finely holomorphic:

$$\begin{aligned}(f.g)(Z_t) - (f.g)(Z_o) &= \int_o^t f \; dg(Z) + \int_o^t g \; df(Z) \\ &= \int_o^t \left(f \frac{\partial g}{\partial z} + g \frac{\partial f}{\partial z}\right) dZ .\end{aligned}$$

Suppose that f_n converge to f finely locally uniformly and $\frac{\partial f_n}{\partial \bar z}=0$ almost everywhere. Then Theorem 2.1 implies that $\frac{\partial f}{\partial \bar z}=0$ almost everywhere.

Suppose $z f$ is finely harmonic; using definition of $\langle\ ,\ \rangle$ we see that $\langle Z, f(Z)\rangle \equiv 0$. The Kunita-Watanabe characterisation of stochastic integral tell us that

$$\langle Z, f(Z)\rangle_t = 2\int_0^t \frac{\partial f}{\partial \bar z}\, ds ,$$

and therefore $\frac{\partial f}{\partial \bar z}$ is zero almost everywhere. ***

The object of the remainder of this thesis is to prove that finely holomorphic functions have much better regularity properties than finely harmonic functions. I hope the reader will be surprised by how closely the properties of these functions model those of the usual analytic functions. Our strategy will be to look at a function in $A_{\mathcal{F}}(K)$ and try to approximate it by functions in $A_{\mathcal{F}}(K)$ which are genuinely analytic on a neighbourhood of a fixed subset E of $\mathbb{C}$.

CHAPTER III: ALGEBRAS OF CONTINUOUS FUNCTIONS

Let K be a compact subset of $\mathbb{C}$. Define $\underline{C(K)}$ to be the complex valued continuous functions on K, and $\underline{\mathcal{R}(K)}$ to be the functions that can be uniformly approximated on K by functions each of which is analytic in a neighbourhood of K.

We have the following inclusions:

$$C(K) \supset A(K) \supset A_{\mathcal{F}}(K) \supset \mathcal{R}(K).$$

It is difficult to decide when a function in $A(K)$ is contained in $\mathcal{R}(K)$. This problem has generated a substantial literature containing some deep theorems. Some of these theorems, although technical, will be invaluable in later chapters.

DIFFERENTIATION OF FUNCTIONS IN $\mathcal{R}(K)$

Before we introduce the reader to the theorems from analytic approximation theory, we look at a result concerning differentiation of functions in $\mathcal{R}(K)$. One purpose of this chapter will be to extend its scope.

Let $\mathcal{R}_0(K)$ denote the functions that are analytic on a neighbourhood of K. By definition $\mathcal{R}_0(K)$ is dense in $\mathcal{R}(K)$. If k is in K then we can define a linear functional $\mathcal{D}_k$ on $\mathcal{R}_0(K)$ by putting

$$\mathcal{D}_k(f) = f'(k), \qquad (f \in K).$$

When does $\mathcal{D}_k$ extend to be a continuous linear functional on $\mathcal{R}(K)$? In other words, for which points of K can we differentiate every function in $\mathcal{R}(K)$? Hallstrom(9) gave a complete solution to this problem in terms of analytic capacity.

Let F be a subset of $\mathbb{C}$ with analytic capacity $\gamma(F)$. Its complement will be called E and, as in Theorem 1.4, E_n will be that part of E which lies in the annulus $\{x : 2^{-n} \leq |x-k| < 2^{-(n-1)}\}$.

3.1 THEOREM(HALLSTROM): $\mathcal{D}_k$ extends continuously to $\mathcal{R}(K)$ and $\|\mathcal{D}_k\|$ is finite if, and only if,

$$\sum_{j=1}^{\infty} 2^{2j}\,\gamma(E_j) < \infty .$$

One can find a description of analytic capacity in Zalcman (21), where the following theorem is proved.

3.2 THEOREM: The analytic capacity of a set E is always dominated by its logarithmic capacity:

$$\gamma(E) \leq \mathrm{Cap}(E) .$$

By using 3.2 , Debiard and Gaveau (4a) proved the following stimulating result.

3.3 THEOREM(DEBIARD AND GAVEAU): Let K be a compact subset of $\mathbb{C}$. If k is in the fine interior K' of K then $\mathcal{D}_k$ extends to $\mathcal{R}(K)$ and $\|\mathcal{D}_k\|$ is finite.

PROOF: Let E be the complement of K and let k be in K'. By definition, E is thin at k and so $\sum_{n=1}^{\infty} \frac{-n}{\log \mathrm{Cap}\, E_n}$ exists and is finite. It follows that for any positive ε we may find an N so that if $n > N$ then

$$\frac{-n}{\log \mathrm{Cap}\, E_n} < \varepsilon .$$

In other words $\mathrm{Cap}\, E_n$ is less than $e^{-n/\varepsilon}$ for all n greater than N . We use the inequality in 3.2 to see that $\gamma(E_n) < e^{-n/\varepsilon}$. Choose ε to be less than $(\log 2)/4$. Then Hallstrom's criterion is satisfied:

$$\sum_{n=1}^{\infty} 2^{2n}\,\gamma(E_n) < \infty . \qquad ***$$

Debiard and Gaveau have shown, therefore, that a function in $\mathcal{R}(K)$ has a derivative defined everywhere on the fine interior of K . It is our intention to extend this result to $A_{\mathfrak{F}}(K)$ and hence to prove that the derivative of a finely holomorphic function exists everywhere. In fact we will be able to prove more than mere existence.

T_φ-OPERATIONS AND T_φ-INVARIANT ALGEBRAS

We need to approximate functions in $A_{\mathfrak{F}}(K)$ by functions with better analytic properties. The main tool we have available is the T_φ-operator developed by Vitushkin, Davie and others. It allows one to localize approximation problems. Let $\mathcal{B}$ denote the Banach algebra consisting of the bounded Borel functions on $\mathbb{C}$ with the uniform norm. Let φ be any smooth function of compact support on $\mathbb{C}$. We define a linear map T_φ from $\mathcal{B}$ to $\mathcal{B}$.

3.4 DEFINITION: If f is in $\mathcal{B}$ then $T_\varphi f \in \mathcal{B}$ is given by the identity:

$$(T_\varphi f)(w) = \varphi(w)f(w) + \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z)}{z-w} \frac{\partial\varphi}{\partial\bar{z}}(z)\,dx\,dy .$$

By Green's Theorem we also have:

$$(T_\varphi f)(w) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z)-f(w)}{z-w} \frac{\partial\varphi}{\partial\bar{z}}(z)\,dx\,dy .$$

In fact, T_φ is a continuous linear map from $\mathcal{B}$ into the subspace of $\mathcal{B}$ consisting of functions analytic in a neighbourhood of ∞ and zero at ∞ . The operator has certain other important but elementary properties which we now list.

3.5 LEMMA: The operator T_φ has the following characteristics.

(i) $\frac{\partial}{\partial\bar{z}}T_\varphi f = \varphi\frac{\partial}{\partial\bar{z}}f$ in the sense of distributions.

(ii) If $f|_K$ is continuous then $T_\varphi f|_K$ is also continuous.

(iii) $T_\varphi f$ is analytic on any open set where f is analytic.

(iv) If $f|_K$ is zero then $T_\varphi f|_K$ is in $R(K)$.

(v) $T_\varphi f$ is analytic off the closed support of φ .

(vi) $T_\varphi f - f$ is analytic on the interior of the set where $\varphi = 1$.

A proof of this simple result, and of the other theorems which we mention in this chapter, may be found in Gamelin(6b). Many of the results are quite deep. We will not attempt an explanation; to give one would be a lengthy business.

It is a simple consequence of (ii) and (iii) in the lemma that if an element $f \in \mathcal{B}$ is in $A(K)$ when restricted to K then so is $T_\varphi f$. A similar result holds for functions in $\mathcal{R}(K)$ (to see this use the continuity of T_φ and (iv)). The T_φ operators enable one to decompose a function into a sum of nicer functions. For example, choose an element f of $\mathcal{B}$ with compact support and such that $f|_K$ is in $A(K)$. Let $(\varphi_n)_1^\infty$ be a smooth partition of unity. Then $f = \sum_{n=1}^{\infty} T_{\varphi_n} f$. For each n the function $T_{\varphi_n} f$ is analytic off the support of φ_n, and in $A(K)$. By choosing the functions φ_n to have small support we can localise any problem of approximation. The same sort of technique can be applied to any subalgebra B of $\mathcal{B}$ which is mapped to itself by all the T_φ operators. For this reason the following definition is worthwhile.

3.6 DEFINITION: A closed subalgebra B of $C(K)$ is said to be T-invariant if it satisfies the following two conditions. Firstly, B contains $\mathcal{R}(K)$. Secondly, for all smooth functions φ with compact support and all f in $\mathcal{B}$, if $f|_K$ is in B then $T_\varphi f|_K$ also lies in B.

In the next chapter we will prove that $A_{\mathfrak{F}}(K)$ is a T-invariant algebra. In the rest of this chapter we shall give a catalogue of those results on T-invariant algebras which we intend to apply to $A_{\mathfrak{F}}(K)$.

3.7 THEOREM: Let $\mathcal{B}$ be a T-invariant subalgebra of $C(K)$. Let Γ be a point, or an arc of a circle, or a straight line segment. Denote by $\underline{\mathcal{B}^{\Gamma}}$ the subalgebra of $\mathcal{B}$ consisting of those functions in $\mathcal{B}$ which can be extended to be analytic on a neighbourhood of Γ. Then $\mathcal{B}^{\Gamma}$ is uniformly dense in $\mathcal{B}$.

This result (with Γ taken to be a single point) makes it sensible to consider point derivations on $\mathcal{B}$. For each point k in K we have a dense linear subspace $\mathcal{B}^{\{k\}}$ of $\mathcal{B}$. We can define a linear functional $\mathcal{D}_k$ on $\mathcal{B}^{\{k\}}$ by putting $\mathcal{D}_k f = f'(k)$. If $\mathcal{D}_k$ is bounded linear functional on $\mathcal{B}^{\{k\}}$ then it has a unique continuous extension to $\mathcal{B}$. Later on, the full strength of Theorem 3.7 will be valuable. For it means we can use the same dense subspace of $A_{\mathfrak{F}}(K)$ in the definition of $\mathcal{D}_k$ for all the points k of a line segment. This enables us to obtain global information about the derivative of a finely holomorphic function.

Hallstrom's theorem on the existence of bounded point derivations has an extension to T-invariant algebras. Analytic capacity, however, is no longer appropriate. We must construct a capacity tailored to the function algebra under study.

Fix a T-invariant subalgebra $\mathcal{B}$ of $A(K)$. Let $\mathcal{Q}_{\mathcal{B}}$ be those continuous functions f on $\mathbb{C}_\infty$ with $f(\infty)=0$ and $f|_K$ in $\mathcal{B}$. We can define a capacity $\alpha_{\mathcal{B}}$ depending on $\mathcal{B}$.

3.8 DEFINITION: Define the $\mathcal{B}$-capacity $\alpha_{\mathcal{B}}(E)$ of a set E as follows: $\alpha_{\mathcal{B}}(E) = \sup\{|f'(\infty)| : f \in \mathcal{Q}_{\mathcal{B}}, f$ is analytic off a compact subset of E, and $\|f\|_\infty \leq 1\}$, where by $f'(\infty)$ we mean $\lim_{z\to\infty} zf(z)$.

If E is open then $\alpha_{\mathcal{R}(K)}(E) = \gamma(E\setminus K)$. So the capacity associated with $\mathcal{R}(K)$ is, more or less, ordinary analytic capacity. We can now state the extension to Hallstrom's theorem.

3.9 THEOREM: Let $\mathcal{B}$ be a T-invariant algebra; choose a point k in K and put $E_n = \{z : 2^{-n} \leqslant |z-k| < 2^{-(n-1)}\}$. The following identity holds:

$$\|\mathcal{D}_k\| \leqslant c \sum_{n=1}^{\infty} 2^{2n} \alpha_{\mathcal{B}}(E_n).$$

The constant c does not depend on the particular T-invariant algebra $\mathcal{B}$ or on which point k is chosen from its maximal ideal space K.

Every T-invariant algebra has the Banach approximation property and, as a consequence, we have the following very useful result.

3.10 THEOREM: Suppose A and $\mathcal{B}$ are T-invariant subalgebras of $C(K)$ and $C(L)$ respectively. The following is a sufficient condition for a continuous function g in $C(K\times L)$ to be in the uniform closure of the algebraic tensor product $A\otimes\mathcal{B}$: Firstly, $g(\cdot,\ell)$ should be in A for all ℓ in L, and secondly, $g(k,\cdot)$ should be in $\mathcal{B}$ for all k in K.

These apparently technical results from the theory of rational approximation will be essential to our study of finely holomorphic functions. They do not, however, adequately represent the theory of rational approximation. Gamelin (6b) has written a comprehensive survey of this area, which includes proofs of these results and many other besides.

CHAPTER 4 : $A_{\mathfrak{F}}(K)$ IS T-INVARIANT

We now prove that $A_{\mathfrak{F}}(K)$ is T-invariant. We then establish an inequality between $\alpha_{A_{\mathfrak{F}}(K)}$ and C_{op}. This will enable us to copy the argument Debiard and Gaveau used to prove Theorem 3.3.

4.1 THEOREM: $A_{\mathfrak{F}}(K)$ is T-invariant.

PROOF: Let φ be any smooth function of compact support and g be any bounded Borel function on $\mathbb{C}$ whose restriction to K is in $A_{\mathfrak{F}}(K)$.
We must prove that $T_\varphi g$ when restricted to K also lies in $A_{\mathfrak{F}}(K)$.

Choose a continuous function $\tilde{g}$ of compact support coinciding with g on K. Lemma 3.5 (iv) implies that the restriction of $T_\varphi(\tilde{g}-g)$ to K is in $\mathcal{R}(K)$ and hence in $A_{\mathfrak{F}}(K)$. It therefore suffices to consider only the case where g is a continuous function of compact support.

We do not know that we can approximate g uniformly by functions which are analytic on a neighbourhood of K. However, using Theorem 1.2., we can find functions g_n converging to g uniformly and such that each g_n is harmonic on a neighbourhood of K. The problem to be overcome is that $T_\varphi g_n$ need not be harmonic on K.

Here are the bones of the proof. We would like to construct $T_\varphi g$ as a limit of holomorphic or harmonic functions. In fact, we construct $(T_\varphi g)(Z_t)$ as a limit of conformal martingales (for $t < Q_K$). We do this by splitting $(T_\varphi g_n)(Z)$ into three pieces using Itô's lemma. One of the pieces is a holomorphic martingale, the second is an antiholomorphic martingale and the third part is of bounded variation. One can prove that the holomorphic martingale converges to $(T_\varphi f)(Z)$ in L^1 norm. It will follow that $T_\varphi f$ is finely holomorphic on K' and that $A_{\mathcal{F}}(K)$ is T-invariant.

We now hang some flesh on the skeleton. Recall from Lemma 3.5(i) that $T_\varphi g_n$ satisfies, in a distributional sense, the following differential equation:

$$\frac{\partial}{\partial \bar{z}}(T_\varphi g_n) = \varphi \frac{\partial}{\partial \bar{z}} g_n .$$

Now $\frac{\partial}{\partial \bar{z}}$ is elliptic and $\varphi \frac{\partial}{\partial \bar{z}} g_n$ is smooth on a neighbourhood of K. Therefore we may apply the elliptic regularity theorems (Rudin(17a; p 201)) to deduce that $T_\varphi g_n$ is smooth in a neighbourhood of K and properly satisfies the differential equation there.

Next we express $(T_\varphi g_n)(Z)$ as a stochastic integral and use the differential equation to simplify the terms:

$$\begin{aligned}(T_\varphi g_n)(Z_{t\wedge Q_K}) - (T_\varphi g_n)(Z_0) &= \int_0^{t\wedge Q_K} \frac{\partial}{\partial z}(T_\varphi g_n)\,dZ + \int_0^{t\wedge Q_K} \frac{\partial}{\partial \bar{z}}(T_\varphi g_n)\,d\bar{Z} + 2\int_0^{t\wedge Q_K} \frac{\partial^2}{\partial z\,\partial \bar{z}}(T_\varphi g_n)\,ds \\ &= \int_0^{t\wedge Q_K} \frac{\partial}{\partial z}(T_\varphi g_n)\,dZ + \int_0^{t\wedge Q_K} \varphi \frac{\partial g_n}{\partial \bar{z}}\,d\bar{Z} + 2\int_0^{t\wedge Q_K} \left(\frac{\partial \varphi}{\partial z}\right)\left(\frac{\partial g_n}{\partial \bar{z}}\right) ds .\end{aligned}$$

To obtain the last term both the differential equation for T_φ and the harmonicity of g_n are needed.

We have achieved our first objective: a decomposition of $T_\varphi g_n(Z_{t\wedge Q_K}) - T_\varphi g_n(Z_0)$ into three pieces. For the sake of brevity we call the holomorphic martingale piece R^n, the antiholomorphic martingale S^n, and the bounded variation part T^n. We put L equal to $T_\varphi g(Z_{t\wedge Q_K}) - T_\varphi g(Z_0)$ and L^n equal to $(T_\varphi g_n)(Z_t) - (T_\varphi g_n)(Z_0)$. Therefore L^n is equal to $R^n + S^n + T^n$. We wish to prove that, for each k in K' the expectation $E^k(|L - R^n|)$ tends to zero as n increases.

The triangle inequality implies that

$$\mathbb{E}^k(|L_t-\mathcal{R}_t^n|) \leqslant \mathbb{E}^k(|L_t^n-L_t|) + \mathbb{E}^k(|S_t^n|) + \mathbb{E}^k(|T_t^n|).$$

We prove that each term on the right hand side converges to zero. Now T_φ is continuous and the (g_n) converge uniformly to g. Therefore L^n tends to L uniformly. and $\mathbb{E}^k(|L_t^n-L_t|)$ tends to zero. To deal with the second term observe that $\mathbb{E}^k(|S_t^n|) \leqslant \mathbb{E}^k(|S_t^n|^2)^{\frac{1}{2}}$. We may apply the L^2 isometry for stochastic integrals to $\mathbb{E}^k(|S_t^n|^2)^{\frac{1}{2}}$ and obtain:

$$\begin{aligned}\mathbb{E}^k(|S_t^n|^2)^{\frac{1}{2}} &= \mathbb{E}^k\left(\int_0^{t\wedge Q_K} |\varphi|^2 \left|\frac{\partial g_n}{\partial \bar z}\right|^2 ds\right)^{\frac{1}{2}}, \\ &\leqslant \|\varphi\|_\infty \, \mathbb{E}^k\left(\int_0^{Q_K} \left|\frac{\partial g_n}{\partial \bar z}\right|^2 ds\right)^{\frac{1}{2}}.\end{aligned}$$

The last term in this inequality tends to zero by Theorem 2.1 because g is in $A_{\mathcal{F}}(K)$. The final term, in T_n, requires an L^1 estimate:

$$\mathbb{E}^k(|T_t^n|) \leqslant 2\,\mathbb{E}^k\left(\int_0^{Q_K} \left|\frac{\partial \varphi}{\partial z}\right|\left|\frac{\partial g_n}{\partial \bar z}\right| ds\right).$$

Applying the Cauchy-Schwartz inequality to the right hand side of the estimate above we obtain:

$$\mathbb{E}^k(|T_t^n|) \leqslant 2\,\mathbb{E}^k\left(\int_0^{Q_K} \left|\frac{\partial \varphi}{\partial \bar z}\right|^2 ds\right)^{\frac{1}{2}} \mathbb{E}^k\left(\int_0^{Q_K} \left|\frac{\partial g_n}{\partial \bar z}\right|^2\right)^{\frac{1}{2}}.$$

Let n tend to infinity; the term which involves φ is finite and does not depend on n, and by theorem 2.1 the other term tends to zero because g is in $A_{\mathcal{F}}(K)$. If a sequence of martingales converges in L^1 norm for each time t then the limit is a martingale; it follows that $T_\varphi g$ is finely harmonic on K'. It is not quite clear in general that an L^1 limit on conformal martingales is also conformal; however we can see directly that $T_\varphi g$ is finely holomorphic. Because Z_t is bounded on $t \leqslant Q_K$ it follows that $Z_{t\wedge Q_K}\cdot\mathcal{R}_t^n$ converges in L^1 to $Z_{t\wedge Q_K}\left[T_\varphi g(Z_{t\wedge Q_K})-T_\varphi g(Z_0)\right]$. Now $Z_{t\wedge Q_t}\cdot\mathcal{R}_t^n$ is a martingale, consequently so is $Z_{t\wedge Q_K}\cdot(T_\varphi g)(Z_{t\wedge Q_K})$. Therefore we may use 2.2 to see that $T_\varphi g$ is finely holomorphic. ***

We wish to prove that a finely holomorphic function has a derivative defined everywhere. Because $A_{\mathcal{F}}(K)$ is T-invariant we can use Theorem 3.9; to do this we must obtain an estimate, similar to that in Theorem 3.2, for $\alpha_{A_{\mathcal{F}}(K)}$ in terms of logarithmic capacity. Our methods require a working knowledge of logarithmic capacity such as can be found in Tsuji(20). We will ultimately prove the following theorem.

4.2 THEOREM: Let G be any compact set with a smooth boundary and interior dense in G. (for example an annulus). The following inequality holds:

$$\alpha_{A_{\mathcal{F}}(K)}(G) \leqslant \operatorname{Cap}(G\setminus K').$$

In fact Theorem 4.2 is still true if G is only compact. This is because firstly, $\alpha_{A_{\mathcal{F}}(K)}$ is monotone, and secondly, $\operatorname{Cap}(G\setminus K')$ can be obtained[1] as the infimum of $\operatorname{Cap}(F\setminus K')$ over all smooth compact sets F containing G.

We must postpone the proof of 4.2 until we have established a probabilistic characterisation of logarithmic capacity.

In order to keep the statement of the lemma simple we restrict our attention to Borel sets.

4.3 LEMMA: Let E be any bounded Borel subset of $\mathbb{C}$ with zero in its interior. The logarithmic capacity of E is given by:

$$-\log \operatorname{Cap} E = \lim_{w\to\infty} \mathbb{E}^w\left(\log\left|\frac{1}{Z_H}\right|\right) = \mathbb{E}^\infty\left(\log\left|\frac{1}{Z_H}\right|\right),$$

here H is the time at which Z first hits E.

1) This is not obvious and relies on Choquet's result(3) that one can cover a finely open set E by an ordinary open set O so that $\operatorname{Cap}(O\setminus E)$ is small.

PROOF: Recall from Tsuji(20) that the logarithmic capacity of a compact set E is obtained by the following process. Let μ be a probability measure on E. We can define the logarithmic potential u of μ by

$$u(z) = \int_E - \log|z-a| \, \mu(da),$$

and the energy $I(\mu)$ of μ by

$$I(\mu) = \int_E\int_E - \log|b-a| \, \mu(da)\, \mu(db).$$

Among all probability measures on E there is one called "the equilibrium distribution" which we shall denote by μ_E. It is the measure with the least energy supported on E. Let V_E be the energy of the equilibrium distribution on E. The logarithmic capacity of E is defined by

$$\mathrm{Cap}(E) = e^{-V_E}.$$

The logarithmic potential u_E of μ_E is known as the <u>conductor potential</u>. It is bounded above by V_E everywhere and, except on a set of capacity zero, equals V_E on E. We can now prove the lemma.

We first deal with the compact case. Choose $E \subset \mathbb{C}$ a compact neighbourhood of zero. Consider the function g defined by

$$g(w) = u_E(w) - \log|\tfrac{1}{w}| .$$

It is bounded and harmonic off E, and tends to zero at ∞. Let H denote the first time Z hits E. Observe that $u_E(Z_H)$ is a.s. equal to the constant V_E providing that Z does not start in E. Fix an element w contained in $\mathbb{C}\backslash E$. Now $u_E(Z_t)$ has continuous paths and therefore

$$\begin{aligned} g(w) &= E^w(g(Z_H)) \\ &= V_E - E^w(\log|\tfrac{1}{Z_H}|). \end{aligned}$$

Let w tend to infinity. This proves the lemma in the case where E is compact.

The general case follows from more abstract considerations.

Consider a bounded open set E. We may find a sequence (E_n) of compact neighbourhoods of zero increasing to E with the following properties. Firstly, the time H_n at which Z first hits E_n decreases to H pointwise and secondly, $\mathrm{Cap}\, E_n$ increases to $\mathrm{Cap}\, E$. We may apply the bounded convergence theorem to see that

$$\lim_n \mathbb{E}^w\left(\log\left|\frac{1}{Z_{H_n}}\right|\right) = \mathbb{E}^w\left(\log\left|\frac{1}{Z_H}\right|\right).$$

Therefore the theorem is true for bounded open sets.

Let F be an arbitary Borel subset of $\mathbb{C}$ containing zero in its interior. Let H_F be the first time Z hits F. Then $F \mapsto \mathbb{E}^w\left(\log\left|\frac{1}{Z_{H_F}}\right|\right)$ is a monotone set function. Let O be any open set containing F, and K be any compact subset of F with zero in its interior.

We have the following inequality:

$$\begin{aligned} -\log(\mathrm{Cap}(K)) &= E^\infty\left(\log\frac{1}{|Z_{H_K}|}\right) \\ &\leqslant E^\infty\left(\log\frac{1}{|Z_{H_F}|}\right) \\ &\leqslant E^\infty\left(\log\frac{1}{|Z_{H_O}|}\right) = -\log(\mathrm{Cap}\, O). \end{aligned}$$

But Borel sets are capacitable and so we have

$$\begin{aligned} \mathrm{Cap}\, F &= \sup_K \mathrm{Cap}(K) \\ &= \inf_O \mathrm{Cap}(O). \end{aligned}$$

It follows that $-\log \mathrm{Cap}(F) = \mathbb{E}^\infty\left(\log\frac{1}{|Z_{H_F}|}\right)$. ***

We can now prove Theorem 4.2.

PROOF OF THEOREM 4.2:

Let $\mathcal{A}$ denote the set of those continuous functions on $\mathbb{C}_\infty$ that have the following properties:

a) they are analytic off G and zero at ∞;

b) they are uniformly bounded in absolute value by 1;

c) they are finely holomorphic on K'.

We see, from Definition 3.8, that the $A_{\mathcal{F}}(K)$ capacity of G is given by:

$$\alpha_{A_{\mathcal{F}}(K)}(G) = \sup_{f \in \mathcal{A}} |f'(\infty)| .$$

(Here, as always, $f'(\infty)$ is defined to be $\lim_{z \to \infty} z f(z)$.) we are required to prove that, if f is in $\mathcal{A}$, then

$$|f'(\infty)| \leqslant \operatorname{Cap}(G \setminus K').$$

In the case where $K \supset G^{\circ}$, the theorem is easy to prove. If $K \supset G^{\circ}$ then any function f contained in $\mathcal{A}$ is analytic in G° and in $\complement G$. Since G has a smooth boundary, Morera's theorem tells us that f is analytic across ∂G and hence identically zero. Therefore the $A_{\mathcal{F}}(K)$ capacity of G is zero and the conclusion of the theorem is trivially satisfied.

We now deal with the case where $G^{\circ} \setminus K$ is non-empty. Because the problem is translation invariant we shall assume that zero is in $G^{\circ} \setminus K$. We will use Lemma 4.3 to obtain an estimate of $\log|f'(\infty)|$ for functions in $\mathcal{A}$. We mention a small technical point. If K is compact then K' is an F_{σ}. This means the measurability condition in 4.3 will never bother us.

Choose an element f of $\mathcal{A}$. Then $z f(z)$ is a bounded finely holomorphic function off $G \setminus K'$. Let H be the first time Z hits $G \setminus K'$. It follows from the submartingale convergence theorem that $\log|Z_{t\wedge H} f(Z_{t \wedge H})|$ is a submartingale.

We have the following string of inequalities:

$$\begin{aligned} \log|f'(\infty)| &\leqslant E^{\infty}(\log|Z_H f(Z_H)|), \\ &\leqslant E^{\infty}(\log|Z_H|), \\ &= \log(\operatorname{Cap}(G \setminus K')). \end{aligned}$$

Therefore $\alpha_{A_{\mathfrak{F}}(K)}(G)$ is less than or equal to $\mathrm{Cap}(G\setminus K')$. At least in spirit, this proof and Zalcman's proof of 3.2 are the same. ***

4.4 COROLLARY: If k is in K' then D_k has a unique continuous extension from $A_{\mathfrak{F}}(K)^{\{k\}}$ to $A_{\mathfrak{F}}(K)$.

PROOF: We may repeat the argument used in 3.3, replacing 3.1 by 3.9 and 3.2 by 4.2. ***

We have shown that a finely holomorphic function has a "derivative" defined at each point of its finely open domain. However we have obtained it in a rather nonconstructive way. We want to give a better characterisation of the derivative. To do this we must prove some results about the fine topology. These results are interesting in their own right.

CHAPTER 5: PATHS IN FINELY OPEN SETS AND A RADIAL PROJECTION THEOREM

In this chapter we will prove that a connected finely open set in $\mathbb{R}^d$ is arcwise connected. The key step of the proof will be a radial projection theorem for Brownian motion. These results settle a relatively important question posed in Fuglede(5d). For this reason we include details for the higher dimensional cases although our application only requires the 2-dimensional version of the theorem.

THE RADIAL PROJECTION THEOREM

During this section many constants will appear; it will be convenient to let the symbol c do for all of them. We must also introduce some more notation. Let $\mathcal{B}$ denote the unit ball $\{\|x\|<1\}$ in $\mathbb{R}^d$, and S denote its boundary the unit sphere. For convenience, we let W_t denote Brownian motion in $\mathcal{B}$ stopped when it first hits S. We define π to be the radial projection that maps $\mathcal{B}\setminus\{0\}$ onto S by taking x to $x/_{\|x\|}$. By $|F|$, where F is a subset of S, we will mean the $(d-1)$ - dimensional Lebesgue measure of F normalised to make $|S| = 1$.

We can now state the radial projection theorem.

5.1 THEOREM: Fix $r<1$. There is a constant $c>0$, depending only on r and d, such that if E is any Borel subset of $r\mathcal{B}\setminus\{0\}$ then the following inequality holds:

$$\mathbb{P}^0(W \text{ hits } E) \geqslant c\,|\pi(E)|.$$

There are two approaches to the proof of this theorem. The first uses probability and while it has an intuitive backbone the details are quite technical. The second proof, due to B. Fuglede after he had seen the author's original proof, is undoubtedly shorter and gives better constants. However it is less clear why it should work. We will give the probabilistic proof.

The ideawhich lies behind McKean's proof of Wiener's criterion(11) also lies behin 5.1. The trick is to spilt $\mathcal{B}$ up into annuli $\mathcal{B}_i$, where $\mathcal{B}_i = \{x : r^i \leqslant \|x\| < r^{(i-1)}\}$. Fix a positive integer n. The probability that W hits $E \cap \mathcal{B}_i$ is roughly independent of the event that W hits $E \cap \mathcal{B}_{i+4n}$.

We prove Theorem 5.1 in two stages. First we establish a simple case of the theorem.

5.2 LEMMA: Suppose E is a compact subset of the annulus

$$\mathcal{B}_2 = \{x : r^2 \leqslant \|x\| < r\}.$$

There is a constant $c > 0$, such that for w in $\mathcal{B}$ with $\|w\| < r^4$ we have

$$\mathbb{P}^w(W \text{ hits } E) \geqslant c\, |\pi(E)|.$$

PROOF: Let g_B and Cap_B denote the Green function and capacity relative to $\mathcal{B}$. Recall that if μ_E is the equilibrium probability measure on E then:

$$\mathbb{P}^w(W \text{ hits } E) = g_B(\mu_E)(w) \,.\, Cap_B(E).$$

Now $g_B(\cdot,\cdot)$ is a jointly continuous positive function off the diagonal in $\mathcal{B} \times \mathcal{B}$. Therefore g_B is bounded below by $c > 0$ on $\mathcal{B}_2 \times r^4\mathcal{B}$. We have the following inequality:

$$\mathbb{P}^w(W \text{ hits } E) \geqslant c\, Cap_B(E), \qquad (\|w\| < r^4).$$

We now treat the cases $d>2$ and $d=2$ separately. Suppose first that d is greater than 2. Let Cap denote the usual capacity on $\mathbb{R}^d$, and g the Green function. Now, $Cap(E)$ is always less than $Cap_B(E)$ because $g > g_B$. Let μ be the equilibrium measure for $\Pi(E)$ on $\mathbb{R}^d$. We may find a probability measure $\tilde{\mu}$, supported on E, which Π projects onto μ. It is an easy calculation with the Green function to see that

$$g\tilde{\mu}(x) \leqslant c/_{r^{2(d-2)}}\, g\mu(\Pi x) \leqslant c/_{r^{2(d-2)}} \left[\mathrm{Cap}(\Pi(E))\right]^{-1}.$$

Therefore $\mathrm{Cap}(E)$ is greater than $c.\,\mathrm{Cap}(\Pi(E))$. However (see for example Sario and Oikawa (18;p 285)); $\mathrm{Cap}(\Pi(E))$ is greater than $c\,|\Pi(E)|$.

Put these observations together. For w in $\mathcal{B}$ with $\|w\| < r^4$, the following inequalities are valid:

$$\begin{aligned} \mathbb{P}^w(W \text{ hits } E) &\geqslant c\ \mathrm{Cap}_{\mathcal{B}}(E) \\ &\geqslant c\ \mathrm{Cap}(E) \\ &\geqslant c\ \mathrm{Cap}(\Pi(E)) \\ &\geqslant c\ |\Pi(E)|. \end{aligned}$$

Suppose now that $d=2$, and that Cap denotes logarithmic capacity. We can still obtain an estimate relating $\mathrm{Cap}_{\mathcal{B}}$ and Cap:

$$-\log \mathrm{Cap}(E) - \log(1-r) \geqslant \pi \left[\mathrm{Cap}_{\mathcal{B}}(E)\right]^{-1}.$$

However E is a subset of $\mathcal{B}_2$ and so $\mathrm{Cap}E$ is less than r. Thus we may find a $c>0$ so that for all $E \subset \mathcal{B}_2$ the following inequality holds: $\mathrm{Cap}_{\mathcal{B}}(E) \geqslant c/_{-\log \mathrm{Cap}E}$. If ε is a positive real number less than 1 then $\frac{1}{-\log\varepsilon} > \varepsilon$. So we have the following inequality:

$$\mathrm{Cap}_{\mathcal{B}}(E) \geqslant c\ \mathrm{Cap}(E).$$

Because the logarithmic capacity of a compact set is equal to its transfinite diameter (see for example Tsuji (20, p 71)) we see that $\mathrm{Cap}(E)$ is greater than $\frac{1}{r^2}\mathrm{Cap}(\Pi(E))$. The logarithmic capacity of a subset of S dominates a constant multiple of its surface measure (see Sario and Oikawa (18; p 285)).
Putting these inequalities together we have

$$\begin{aligned} \mathbb{P}^w(W \text{ hits } E) &\geqslant c\ \mathrm{Cap}_{\mathcal{B}}(E) \\ &\geqslant c\ \mathrm{Cap}(E) \end{aligned}$$

$$\geq c \, \mathrm{Cap}(\pi(E))$$
$$\geq c \, |\pi(E)| \quad .$$

PROOF OF THEOREM 5.1: Recall that B_i is the annulus $\{x : r^i \leq \|x\| < r^{i-1}\}$. If E^i is defined by $E^i = \bigcup_m (B_{4m+i} \cap E)$, then for at least one integer i in $\{1, \ldots, 4\}$

$$\mathbb{P}^0(W_t \text{ hits } E^i) \geq \frac{1}{4} \mathbb{P}^0(W_t \text{ hits } E).$$

Further $|\pi(E)|$ is a continuous subadditive capacity on the subsets of B. So whenever E is Borel, one can always find compact sets K_n increasing to E such that $|\pi(K_n)|$ increases to $|\pi(E)|$. Combining these two observations we see that we only need to prove 5.2 in the case where E is compact and satifies $|\pi(E)| = \sum_{m=0}^{\infty} |\pi(E_m)|$, when $E_m = B_{4m} \cap E$.

Let T_m denote the first time W leaves $r^{4m}B$ and let A_m be the set of paths on which W manages to miss E_m until T_{m-1} . We can use Lemma 5.2 to estimate from above the probability of missing E completely. Let $\mathcal{R}_{T_i}$ denote the σ- field generated by W_t on $t \leq T_i$, and note that A_m is $\mathcal{R}_{T_m}$ measurable. We have the following estimate:

$$\begin{aligned} \mathbb{P}^0(W \text{ hits } E) &\geq 1 - \mathbb{P}^0\Big(\bigcap_1^\infty A_m\Big) \\ &= 1 - \mathbb{E}^0\Big(\prod_1^\infty \chi_{A_m}\Big) \\ &= 1 - \mathbb{E}^0\Big(\prod_2^\infty \chi_{A_m} \mathbb{E}^0(\chi_{A_1} \mid \mathcal{R}_{T_1})\Big). \end{aligned}$$

Lemma 5.2 gives an upper bound $1 - c|\pi(E_1)|$ for $\mathbb{E}^0(\chi_{A_1}|\mathcal{R}_{T_1})$. We may use the scale invariance of Theorem 5.2 to obtain, for each i, an upper bound for $\mathbb{E}(\chi_{A_i}|\mathcal{R}_{T_i})$. We deduce that:

$$\begin{aligned} \mathbb{P}^0(W \text{ hits } E) &\geq 1 - \mathbb{E}^0\Big(\prod_2^\infty \chi_{A_i}\Big)(1 - c\,|\pi(E_1)|) \\ &\geq 1 - \prod_1^\infty (1 - c\,|\pi(E_i)|) \\ &\geq 1 - \exp\Big(-c \sum_1^\infty |\pi(E_i)|\Big) \\ &= 1 - \exp - c\,|\pi(E)| . \end{aligned}$$

Because $|\pi(E)|$ is bounded independently of E we finally obtain:

$$\mathbb{P}^0(W \text{ hits } E) \geq c \cdot |\pi(E)|.$$

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CONNECTED FINELY OPEN SETS ARE POLYGONAL PATH CONNECTED

In (5d) B.Fuglede asked the question: is the fine topology in $\mathbb{R}^d$ locally arcwise connected? In the case $d=2$ he gave an affirmative answer. In higher dimensions the question remained open. In that paper B.Fuglede gave a very pretty proof of Iverson's theorem. This theorem gives a sufficient condition on a subharmonic function u for there to exist a path γ along which u converges to its supremum. The question which remained concerned the nature of the path. Fuglede proved that a piecewise Brownian path would do. L.Carleson proved that in the special case where u was defined on the whole of $\mathbb{R}^d$ then γ could be chosen to be a polygonal arc. It was clear that if connected finely open sets were polygonal path connected then γ could always be chosen to be a polygonal arc.

Theorem 5.1 provides a simple proof that the fine topology is arcwise connected and gives metric information as well. This metric information will be essential in the next section.

5.3 THEOREM: Let $\mathcal{U} \subset \mathbb{R}^d$ be a fine neighbourhood of x . Then for any $\theta > 1$ there exists a compact fine neighbourhood K of x such that any two points y and z in K are connected by a polygonal arc in $\mathcal{U}$ of total length at most $\theta \| x - y \|$.

PROOF: In fact the arc need consist of only two line segments.

We may as well assume that $\mathcal{U}$ is compact. Let C be the constant corresponding to $r=\frac{1}{2}$ in Theorem 5.1. Now choose a small ball I of radius δ centre at x so that

$$\mathbb{P}^x(Z \text{ quits } \mathcal{U} \text{ before } I) < \varepsilon/2C \ .$$

The function $x \mapsto \mathbb{P}^x(Z$ quits $\mathcal{U}$ before I) is a superharmonic function on $\mathcal{B}$ and so finely continuous. Therefore we can find a (compact) fine neighbourhood K of x such that if y is in K then

$$\mathbb{P}^y(Z \text{ hits } \mathcal{U} \text{ before } I) < \varepsilon/c .$$

Theorem 5.1 implies that $\mathcal{U}$ contains rays from y of length equal to $\frac{1}{2}$ the distance from y to ∂B in all but a small (ε) proportion of directions. In particular suppose y and z are in K and within a distance $\delta/3$ of x. There will be rays from y and from z which intersect and make a small angle ($\frac{\varepsilon}{2\pi}$ in the case $d=2$) with the straight line from y to z, and which also lie in $\mathcal{U}$. ***

CHAPTER 6 $\|\mathcal{D}_k\|$ IS FINELY LOCALLY BOUNDED

We are now able to study the global properties of the derivative of a finely holomorphic function. Remember how we defined $\mathcal{D}_k$. Using Theorem 3.7 we found a dense subspace $[A_{\mathfrak{F}}(K)]^{\{k\}}$ on which $\mathcal{D}_k$ was naturally defined, and then extended it to $A_{\mathfrak{F}}(K)$. The difficulty with obtaining global information is that the dense subspace on which $\mathcal{D}_k$ was defined varies from point to point.

In the last chapter we proved that finely open sets contain lots of polygonal paths. Let γ be one of them. Theorem 3.7 tells us that the algebra of functions in $A_{\mathfrak{F}}(K)$ which have an analytic extension to a neighbourhood of γ, i.e. $[A_{\mathfrak{F}}(K)]^{\gamma}$, is dense in $A_{\mathfrak{F}}(K)$. Therefore we have a dense subspace on which $\mathcal{D}_k$ is naturally and simultaneously defined for all points on the arc γ. This will suffice to prove that finely holomorphic functions have a much more classical sort of derivative.

We wish to prove that if f is in $A_{\mathfrak{F}}(K)$ and k is in K' then $\mathcal{D}_k f = \lim_{y \overset{\mathfrak{F}}{\to} k} \frac{f(y)-f(k)}{y-k}$. In other words we wish to prove that $F(y,k) = \frac{f(y)-f(k)}{y-k}$ can be made into a finely continuous (in fact finely holomorphic) function of y for each fixed k. It will be sufficient if we prove that the finely holomorphic function $F(\cdot,k)$ is bounded on a fine neighbourhood of k. For Fuglede (5b;Thm 9.15) proves that points are removable singularities for bounded finely harmonic functions; by Theorem 2.2 they are also removable for bounded finely holomorphic functions. With this in mind we prove the following theorem.

6.1 THEOREM: Fix a compact set K and a point k in K'. There is a fine neighbourhood L of k with $\|D_\ell\|$ uniformly bounded for ℓ in L.

PROOF: This result is a significant refinement of Theorem 4.4. Recall from 3.9 that

$$\|D_k\| \leqslant c \sum_1^\infty 2^{2n} \alpha_{A_{\mathfrak{F}}(K)}(E_n^k),$$

where E_n^k is defined to be the set $\{x : 2^{-k} \leqslant |x-k| < 2^{-(k-1)}\}$.

In Theorem 4.2 we obtained the inequality:

$$\alpha_{A_{\mathfrak{F}}(K)}(E_n^k) \leqslant Cap(E_n^k \setminus K').$$

Because k was in the fine interior of K', Wiener's criterion implied that $\sum_1^\infty \frac{-n}{\log Cap(E_n^k \setminus K')}$ is finite. This implied that $\frac{-n}{\log Cap(E_n^k \setminus K')}$ tends to zero, which in turn implied that $Cap(E_n^k \setminus K')$ converges to zero faster than any geometric series.

We wish to see that $\|D_y\|$ is bounded on some fine neighbourhood L of k. It suffices to prove that for any $\varepsilon > 0$ there is an integer N and a fine neighbourhood L of k such that if n is greater than N and ℓ is in L then

$$\frac{-n}{\log Cap(E_n^\ell \setminus K')} < \varepsilon .$$

For then we can obtain an upper bound for $\|D_\ell\|$:

$$\|D_\ell\| \leqslant c \left[\sum_1^N 2^{2n} 2^{-n} + \sum_{N+1}^\infty 2^{2n} e^{-n/\varepsilon} \right].$$

The next lemma proves that for $r > 0$ and $c > 1$, there is an N such that for any $n > N$ the following inequalities hold independently of the choice of y:

$$\frac{-n}{c \log Cap(E_n^y \setminus K')} \leqslant \mathbb{P}^y(Z \text{ hits } E_n^y \setminus K' \text{ before } \{|w-y| = r\})$$
$$\leqslant \mathbb{P}^y(Z \text{ leaves } K' \text{ before } \{|w-y| < r\}).$$

Choose r so that $\mathbb{P}^k(Z \text{ leaves } K' \text{ before } \{|w-x| < 2r\})$ is less than $\varepsilon/2c$. The function taking y to $\mathbb{P}^y(Z \text{ leaves } K'$ before $\{|w-x| < 2r\})$ is superharmonic on $\{|w-x| < 2r\}$ and hence finely continuous there. Therefore we may construct a fine neighbourhood L of x so that if ℓ is in L then

$$\mathbb{P}^{\ell}(Z \text{ leaves } K' \text{ before } \{|z-\ell|<r\}) < \varepsilon/c \,. \qquad ***$$

6.2 LEMMA: Fix $r>0$ and $c>1$. Let F_n be any Borel subset of the annulus $\{z: 2^{-n} \leqslant |z| < 2^{-(n-1)}\}$. There is an integer N, not depending on F_n, such that if $n>N$ then the following inequality holds:

$$\mathbb{P}^{0}(Z \text{ hits } F_n \text{ before } \{|z|=r\}) \geqslant \frac{-n\log 2}{C\log \operatorname{Cap} F_n}\,.$$

PROOF: Let B be the ball $\{|z|<r\}$; Cap_B and g_B are, as usual, the associated capacity and Green function. Let H denote the first time Z leaves B. Suppose that $2^{-(n-1)}<r$. Denote by μ the equilibrium measure on F_n obtained from g_B, and ν that obtained from the logarithmic Green function $-\log|y-z|$. We have the following inequalities:

$$\begin{aligned}
-\log \operatorname{Cap} F_n &= \sup_z \int -\log|z-w|\,\nu(dw)\\
&\geqslant \sup_z \int\left[-\log|z-w| + \mathbb{E}^z(\log|Z_H-w|)\right]\nu(dw)\\
&\qquad\qquad - \sup_{\substack{z\in B\\ w\in F_n}} |\mathbb{E}^z(\log|Z_H-w|)|\\
&\geqslant \pi \sup_z g_B\nu(z) - \log(r-2^{-n})\\
&\geqslant \frac{\pi}{\operatorname{Cap}_B(F_n)} - \log(r-2^{-n}).
\end{aligned}$$

Now $-\log\operatorname{Cap}F_n$ is greater than $(n-1)\log 2$. Consequently, for any $c_1>1$ there is an N_1 so that for $n \geqslant N_1$,

$$\operatorname{Cap}_B(F_n) \geqslant \frac{\pi}{-c_1\log\operatorname{Cap}F_n}\,.$$

We can relate this to the probability that we hit F_n:

$$\begin{aligned}
\mathbb{P}^{0}(Z \text{ hits } F_n \text{ before } \{|z|=r\}) &= \operatorname{Cap}_B(F_n)\, g_B(\mu)(0)\\
&\geqslant \operatorname{Cap}_B(F_n)\,\tfrac{1}{\pi}(-\log 2^{-n}+\log r).
\end{aligned}$$

Therefore, for any $c>c_1$ there is an $N>N_1$ so that if $n\geqslant N$ then the following inequality holds:

$$\mathbb{P}^{0}(Z \text{ hits } F_n \text{ before } \{|z|=r\}) \geqslant \frac{n\log 2}{-c\log\operatorname{Cap}F_n}\,. \qquad ***$$

We have all we need to prove the following theorem.

6.3 THEOREM: Let k be in K'. There is a compact fine neighbourhood L of k in K', such that if f is in $A_{\mathfrak{F}}(K)$ then

$$\left|\frac{f(x) - f(y)}{x - y}\right| , \qquad (x \neq y) ,$$

is uniformly bounded on $L \times L$.

PROOF: Choose a fine neighbourhood M of k on which $\|D_m\| < S$. Choose a compact fine neighbourhood L of k in M such that if x and y are in L then there is a polygonal arc γ from x to y in M of length at most $2|x-y|$. We can make these choices because of Theorem 6.1 and 5.3.

Choose two points x and y in L and such an arc γ joining them. If g is an element of $[A_{\mathfrak{F}}(K)]^{\gamma}$ then g is analytic on a neighbourhood of γ and has derivative bounded by some constant S there. The usual mean value theorem implies the following inequality:

$$\left| \frac{g(x) - g(y)}{x - y} \right| \leqslant 2S .$$

By Theorem 3.7, $[A_{\mathfrak{F}}(K)]^{\gamma}$ is dense in $A_{\mathfrak{F}}(K)$. Consequently, the inequality holds for any element of $A_{\mathfrak{F}}(K)$.

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6.4 COROLLARY: If f is in $A_{\mathfrak{F}}(K)$ and k is in K' then $D_k f$ has the following natural interpretation:

$$D_k f = \lim_{w \xrightarrow{\mathfrak{F}} k} \frac{f(w) - f(k)}{w - k}$$

PROOF: Define $F(k_1, k_2)$ to be $\dfrac{f(k_1) - f(k_2)}{k_1 - k_2}$. By Theorem 6.3, $F(\cdot, k)$ is bounded on a fine neighbourhood of k. The function $F(\cdot, k)$ is finely harmonic on $K' \setminus \{k\}$. Points are removable singularities for bounded finely harmonic functions. Therefore $F(w,k)$ has a limit as w tends finely to k. Elementary checking shows that the limit is $D_k f$.

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It will be noted that in the proof of Theorem 6.4 we used only the fact that $F(\cdot, k)$ was bounded for fixed k. Is the derivative $D_\circ f$ a finely holomorphic function? The full strength of Theorem 6.3 has an important role to play in answering this question.

CHAPTER 7: THE DERIVATIVE OF A FINELY HOLMORPHIC FUNCTION IS FINELY HOLMORPHIC

In Chapter 3 we stated a Theorem (3.10) about tensor products of T-invariant algebras which we now use. Let f be an element of $A_{\mathcal{F}}(K)$, and let k be in K'. As in Chapter 6, let F be the function from $K' \times K'$ into $\mathbb{C}$ defined by:

$$F(x,y) = \frac{f(x)-f(y)}{x-y} \qquad , \ (x \neq y)$$

$$F(x,x) = D_x f.$$

We saw in Theorem 6.2 that there is a compact fine neighbourhood L of k such that $F|_{L\times L}$ is a bounded function. We shall see in Theorem 7.7 that L can be chosen so as to make $F|_{L\times L}$ continuous as well. We will take 7.7 on trust for the moment and use it to prove one of the main results in this thesis.

7.1 THEOREM: Let f be in $A_{\mathcal{F}}(K)$. Then $D.f$ is finely holomorphic on K'.

PROOF: Fix k in K', and a compact fine neighbourhood L of k. Suppose that F is a continuous function on $L\times L$. Then for each ℓ in L, the functions $F(\cdot,\ell)$ and $F(\ell,\cdot)$ are both in $A_{\mathcal{F}}(L)$. Theorem 3.10 implies that F can be uniformly approximated on $L\times L$ by elements of $A_{\mathcal{F}}(L)\otimes A_{\mathcal{F}}(L)$. In other words, for any $\varepsilon>0$ we can find $(g_i)_1^m$ and $(h_j)_1^m$ in $A_{\mathcal{F}}(L)$ such that

$$\sup_{(x,y)\in L\times L} \left| F(x,y) - \sum_{i=1}^{m} g_i(x)h_i(y) \right| < \epsilon.$$

By restricting F to the diagonal and letting ε tend to zero; we see that the derivative of f is in $A_{\mathcal{F}}(L)$. ***

7.2 COROLLARY: Finely holomorphic functions are infinitely finely differentiable.

A natural question poses itself here. Is a finely holomorphic function determined by its derivatives at single point? Fuglede has recently proved that it is. He uses the fine Green function to do it. He also proves, by an easier argument of the same kind, that a non-constant finely holomorphic function can have at most countably many zeros.

A FINE HARTOGS THEOREM

A classical theorem of Hartogs states that a bounded function of several variables, harmonic in each variable separately, is jointly continuous. We will prove a fine version of this theorem.

An old result of Keldych[1] goes as follows. Let $K \subset \mathbb{C}$ be a compact set, and let E be the subset of K consisting of outer regular boundary points. If k is any element of K, then there is a unique measure m_k on E, called the Keldych measure, such that $m_k(f) = f(k)$ for all f in $H(K)$. A standard argument from Choquet theory now shows that E is a G_δ subset of K. In our language E is the fine boundary of K. The strong Markov property implies that Z first leaves K through E, and $\mathbb{E}^k(f(Z_{Q_k})) = f(k)$. In other words the $\mathbb{P}^k$-law of Z_{Q_k} is just m_k.

To establish the fine Hartogs theorem we will prove the following. For any k in K' there is a compact fine neighbourhood L of k on which the map m is continuous from L with the usual topology to the measures on E with the norm topology.

1) For a proof of this see Gamelin and Rossi(7).

We must prove this in stages.

7.3 LEMMA: The map m from K' into the measures $\mathcal{M}(E)$ on E is (fine, norm) continuous.

PROOF: Fix a point k in K'. For any $\varepsilon > 0$, we may find a $\delta > 0$, such that if $\mathcal{B}$ is a ball of radius δ about k then

$$\mathbb{P}^k(Z \text{ leaves } K' \text{ before } Z \text{ leaves } \mathcal{B}) < \varepsilon .$$

But, keeping $\mathcal{B}$ fixed, the left hand side of the above inequality is finely super-harmonic on K'. Therefore it is finely continuous and one may find a compact fine neighbourhood L of k so that if ℓ is in L then

$$\mathbb{P}^\ell(Z \text{ leaves } K' \text{ before } Z \text{ leaves } \mathcal{B}) < \varepsilon .$$ [1]

Because K is compact there is a much smaller ball C, centred at k, such that for any ℓ in L

$$m_\ell(C) < 2\varepsilon .$$

Let n_ℓ denote the law of Z when it first leaves $K' \cup C$. By construction we have $\|m_\ell - n_\ell\| < 4\varepsilon$ whenever ℓ is in L. However it is clear from the strong Markov property that n is a (usual, norm) continuous function on $L \cap C$. Therefore, there is a ball $\mathcal{D}$ centered on k such that if ℓ is in $\mathcal{D} \cap L$ then $\|n_\ell - n_k\| < \varepsilon$. It follows that $\|m_k - m_\ell\| < 9\varepsilon$. We have proved the lemma. ***

We should like to use Theorem 1.3 to strengthen 7.3 and obtain (usual, norm) continuity of m on L. However $\mathcal{M}(E)$ does not have a countable base. We can deal with this by using some theorems from measure theory. Let $\mathcal{P}(X)$ denote the probability measures on X and $C_b(X)$ the

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1) The astute reader will have seen this little trick used before.

bounded continuous functions on X. The following theorems relate the structure of X to the structure of $\mathcal{P}(X)$.

7.4 THEOREM: $\mathcal{P}(X)$ is a separable metrizable space, when given the weak* (or $\sigma(\mathcal{P}(X), C_b(X))$ topology, if and only if X is separable.

7.5 THEOREM: Let X be a Polish Space (i.e. homeomorphic to a separable complete metric space), and let Γ be a subset of $\mathcal{P}(X)$. A necessary and sufficient condition for Γ to be relatively weak* compact is that Γ be a uniformly tight set of measures. That is to say, for any $\varepsilon > 0$, there exists a compact set $K \subset X$ such that if μ is in Γ then $\mu(K) > 1 - \varepsilon$.

These results can be found, for example, in Partharsarathy(15; Thms 6.2 & 6.7) We can use theorems 7.4 and 7.5 to prove the result we want.

7.6 THEOREM: Let k be a point in K'. There exists a fine neighbourhood L of k, compact in the usual topology, such that $m|_L$ is a continuous function when L is given the usual topology and $M(E)$ the norm topology. (In other words $m|_L$ is (usual, norm) continuous).

PROOF: We use an approximation argument. Let $\mathcal{U}$ be a relatively compact open subset of $\mathbb{C}$ and n_ℓ be the $\mathbb{P}^\ell$-law of Z on its first exit from $\mathcal{U}$. It is easy to prove that the map n is continuous from the usual topology on $\mathcal{U}$ into the norm topology on $M(\partial\mathcal{U})$. We will uniformly approximate $m|_L$ by functions like n.

The function m maps K' into the set $\mathcal{P}(E)$ of probability measures on E. We know, by Lemma 7.3, that m is (fine, norm) continuous; it is therefore (fine, weak*) continuous. By Theorem 7.4 the set $\mathcal{P}(E)$ is separable and metrizable and so Theorem 1.3 allows us to choose a compact fine neighbourhood L of k on which m is (usual, weak*) continuous. It follows that $\{m_\ell : \ell \in L\}$ is a weak* compact subset of $\mathcal{P}(E)$. We remarked before Lemma 7.3 that E was a G_δ-set; we may therefore apply Theorem 7.5 to deduce that $m(L)$ is a uniformly tight family of measures. We are nearly home.

Choose a positive number ε. There is a compact subset N of E such that if ℓ is in L then $m_\ell(N)$ is at least $1-\varepsilon$. Fix a large sphere S containing K and let n_ℓ be the $\mathbb{P}^\ell$-law of Z when it first hits $N \cup S$. As we have observed n is a (usual, norm) continuous function off $N \cup S$ and so it is continuous on L. However by construction if ℓ is in L then

$$\| n_\ell - m_\ell \| \leqslant 2\varepsilon .$$

The number ε was arbitary and so we have completed our approximation. ***

As an immediate corollary we may prove the extension of Hartog's Theorem.

7.7 THEOREM: Let U and V be finely open sets and let F be a bounded function taking $U \times V$ into $\mathbb{C}$. Suppose that, for each point x in U and y in V, the functions $F(x,.)$ and $F(.,y)$ are finely continuous. Then for any fixed pair of points (x_0, y_0) in $U \times V$ there are compact fine neighbourhoods L_{x_0} of x_0 in U and L_{y_0} of y_0 in V such that $F|_{L_{x_0} \times L_{y_0}}$ is continuous.

PROOF: Fix x_0 and choose a compact fine neighbourhood K_{x_0} for x_0 in U. Similarly choose y_0 and K_{y_0} in V. Let x be a generic point of K_{x_0} and m_x its Keldych measure on the fine boundary of K_{x_0}, similarly y and m_y for K_{y_0}. Because $F(\cdot, y)$ is bounded and finely harmonic on $\mathcal{U}$ it follows that

$$F(x,y) = \int_{K_{x_0}} F(x',y)\, m_x(dx') ,$$

with a similar identity for y. Let L_{x_0} be the fine neighbourhood of x_0 on which m_x is (usual, norm) continuous and L_{y_0} the neighbourhood of y_0 on which m_y is (usual, norm) continuous. It is now a trivial application of the following inequality to see that F is jointly continuous on $L_{x_0} \times L_{y_0}$:

$$| F(x_1, y_1) - F(x_2, y_2) | \leqslant \| F \|_\infty (\| m_{x_1} - m_{x_2} \| + \| m_{y_1} - m_{y_2} \|)$$

Some remarks are in order here. There are two stages in proving Theorem 7.6. The first stage is the proof of 7.3. The second stage consists of the arguments we have included in the proof of 7.6. In fact a greater knowledge of fine potential theory allows one to dispense with either one of the two stages (but not both) The reader is unlikely to possess this knowledge (any more than the author did when he proved the result originally). Therefore we have given this direct proof.

After seeing these results in manuscript B. Fuglede pointed out that 7.7 is true with slightly relaxed conditions. It is essential that F is a bounded function; however it does not need to be harmonic in both variables. It suffices that it be finely continuous in one variable and finely harmonic in the other. The result follows simply from two observations.

a) The Arzela-Ascoli Theorem and 7.6 can be applied to prove that harmonic sections $F(x,\cdot)$ are finely locally a relatively compact family of continuous functions.

b) Theorem 1.3 implies that one may choose a compact set K on which $F(\cdot,y)$ is a continuous function for countably many y simultaneously (because $\mathbb{C}^{N}$ has a countable base).

CHAPTER 8: THE FINE LOCAL APPROXIMATION OF FINELY HOLOMORPHIC FUNCTION BY CLASSICAL ANALYTIC FUNCTIONS

When people started looking at finely holomorphic functions it was not clear what was the correct definition.[1] B.Fuglede made the following definition.

8.1 DEFINITION: Let V be a finely open subset of $\mathbb{C}$. We say that f is finely holomorphic in Fuglede's sense if for each x in V there is a compact fine neighbourhood K of x such that $f|_K$ is in $\mathcal{R}(K)$. We call this class of functions $\mathcal{O}_F(V)$.

It is clear that $\mathcal{O}_F(V)$ is always a subset of $\mathcal{O}(V)$ However it was an open question whether there was always equality. It was not even known whether $\mathcal{O}_F(V)$ was closed under taking uniform limits. This constituted a major disadvantage of the definition. On the positive side, the better approximation properties of functions in $\mathcal{O}_F(V)$ allow one to avoid T-invariance. We can now prove that Fuglede's definition coincides with the one we gave in Chapter 2. The proof is the joint work of A.G. O'Farrell and the author.

8.2 THEOREM: $\mathcal{O}(V) = \mathcal{O}_F(V)$.

PROOF: We have seen how to construct a compact fine neighbourhood K of x such that $\frac{f(z)-f(w)}{z-w}$ is continuous on $K \times K$. We could do the same for $\frac{f'(z)-f'(w)}{z-w}$ and so find a compact fine neighbourhood K of x on which f is in $Lip2$. (in the sense of Stein (19)). Any $Lip2$ function on a compact set extends to a $Lip2$ function on $\mathbb{C}$. Therefore $f|_K$ has a continuously differentiable extension $\tilde{f}$ from K to $\mathbb{C}$.

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1) It still isn't. But for different (pedagogic) reasons.

However $\frac{\partial \tilde{f}}{\partial \bar{z}} = 0$ on K' because of Corollary 6.4.

Suppose g is a continuously differentiable function on $\mathbb{C}$. If $\frac{\partial g}{\partial \bar{z}} = 0$ on a compact set L then a simple Cauchy transform argument (see for example Gamelin (6b;p 12)) implies that $g|_L$ is in $\mathcal{R}(L)$. It follows that if L is any compact fine neighbourhood of x in K' then $f|_L$ is in $\mathcal{R}(L)$. ***

If a direct proof of this result could be found then the main results of this thesis would follow in a very straightfoward manner.

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ABSTRACT OF D.PHIL THESIS

TITLE: SOME PROBLEMS IN HARMONIC ANALYSIS AND PROBABILISTIC POTENTIAL THEORY

This thesis studies the properties of finely holomorphic functions. Using methods from fine potential theory, stochastic processes, and rational approximation, one can make sense of the notion of holomorphy for functions defined on anfinely open set in $\mathbb{C}$. Such a set need have no interior for the usual metric topology.

In this thesis a weak definition of fine holomorphy is adopted. It is then proved that a finely holomorphic function has a fine derivative defined everywhere by

$$f'(z) = \lim_{\substack{w \to z \\ \text{in the fine} \\ \text{topology}}} \frac{f(z) - f(w)}{z - w},$$

and f' is another finely holomorphic function.

The proof rest on showing that a certain algebra of functions is T-invariant using stochastic integration, and on showing that a finely connected finely open set in $\mathbb{R}^d$ is polygonal path connected. This latter result is of independent interest and settles an open problem related to the Iverson theorem.

A radial projection theorem for harmonic measure and a fine Hartogs theorem are also proved.